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IMS President's Speech

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Discrete stories: Stray facts related to some early Ramsey-type theorems

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Abstract

I would like to thank the members of the Indian Mathematical Society for electing me the President of the Society for the year 2022-2023. It is a great honor and privilege to address the mathematicians and the guests attending the conference.

I looked into some of the articles in Math. Student based on the past Presidential Addresses (General) in Annual Conferences of the IMS. One sees valuable articles: emphasizing the role of history in learning and teaching mathematics; informing us about mathematicians, institutes devoted to mathematics and conferences on mathematics; suggestions to the policy makers and advises to teachers and students; reminding us about our mathematical heritage; explaining the role of various prizes and other recognitions towards the promotion of mathematics; and so on.

Discussion on mathematics teaching at the school level would be a topic of interest; however even if one has been involved in training students at that level, it requires much more research before one embarks on talking on this theme. Though some institutes and individuals are doing their best through many training programmes etc., to avoid looking into school teaching in general, will be like looking for food supplements as a substitute for whole foods. It is a rather complex topic which involves several socio-economic issues. So I decided to eschew this issue in today's address, and instead took up a theme for my address, issuing from a point raised in the article by Prof. B. Sury, "View of mathematics by our society and what our role could be"; this incidentally was based on his talk at the 86th Annual Conference of the IMS, and it drew my attention to the need for popular writings to create general awareness about mathematics.

I decided to take up the present theme, which will require only high school mathematics and can be explained by simple examples drawn from 'common sense' situations as Prof. Sury would suggest. At the same time, the origin of many important recent developments in mathematics can be traced back to these results.

The classical Ramsey theorem (1930) can be said to be a generalization of the pigeonhole principle. The pigeonhole principle says that if $kn+1$ objects are put in n pigeonholes, then there will be a pigeonhole containing at least $k+1$ objects; it is a consequence of a simple counting argument.

As a first case of Ramsey's theorem, one has the following well-known high school exercise:

If you have a party of at least 6 people, you can guarantee that there will be a group of 3 people who all know each other, or a group of 3 people who all do not know each other. In the language of graph theory it says that, if you have a complete graph with six vertices and colour the edges with two colours, say red and blue, then there will be a monochromatic triangle.

The above statement follows easily from the pigeonhole principle, stated above.

Ramsey's theorem, a generalization of the party problem above, appeared as a lemma in a paper (1930) of Frank Plumpton Ramsey on Mathematical logic. Ramsey passed away in January 1930 at the age of 26. It should be mentioned that Ramsey was mainly interested in philosophy and in spite of his passing away at a very young age, he made remarkable contributions to mathematical economics.

Three years after the publication of Ramsey's theorem, a different proof was given by Skolem. While Skolem was aware of the result, Paul Erdős and George Szekeres were led independently to this result while solving a problem brought to them by their friend Esther Klein; Paul, George and Esther were about 19, 21 and 22 years old respectively at that time. Soon afterwards, Erdős and Szekeres ran into the paper of Ramsey. The problem of Esther Klein, which was solved by Erdős and Szekeres was about ascertaining the existence of an integer $ES(n)$, such that any $ES(n)$ points in the plane in general position, will have a subset of n points forming a convex polygon. Erdős named the theorem the "happy ending problem" as it led to the marriage of Esther Klein and George Szekeres. As Soifer has mentioned in his book 'Ramsey Theory: Yesterday, Today, and Tomorrow", after enjoying a life full of mathematics and cheer, George Szekeres and Esther Klein Szekeres passed away on the same day (on August 28, 2005). Paul Erdős, who was slightly younger among the three, had already passed away in 1996.

Subsequently, the branch of combinatorics called Ramsey Theory grew in stature and significance and with hindsight we can now see the unifying feature of the early Ramsey-type theorems which are seemingly unrelated. Some results due to Schur, van der Waerden and Hilbert, share the credit of preceding the result of Ramsey in the class of Ramsey-type theorems.

Commenting on Ramsey's being called as 'eponymous' by Mellor (1983), Harary in his tribute (1983) to Frank P. Ramsey says:

" Of course! The study of ramsey theory has become so important that his name has become an adjective, along with other immortal mathematical lower case adjectives, including abelian, boolean, cartesian, ..."

Existence of regular substructures in general combinatorial structures is the phenomena which can be said to characterize the subject of Ramsey theory. Most often, we come across results saying that:

"If a large structure is divided into finitely many parts, at least one of the parts will retain certain regularity properties of the original structure."

In some results in Ramsey theory, 'Large' substructures are seen to have certain regularities.

The statement of Theodore Motzkin that "Complete disorder is impossible" is perhaps the best to describe the philosophy of Ramsey Theory in a nutshell.

Instead of taking up the developments involving the generalizations of the early Ramsey-type theorems (due to Schur, van der Waerden and Hilbert), I will confine myself to making some remarks on the theorem of van der Waerden.

The theorem of van der Waerden is one among the 'pearls' that Khinchin presented in his 'Three pearls of Number Theory'; apart from van der Waerden's theorem, this small book also talks about the Landau-Schnirelman hypothesis and Waring's problem. Incidentally, this book was written by Khinchin for a soldier (Seryozha) in the second world war, who, lying in a hospital, had written to Khinchin for something mathematical to study and pass his time.

The theorem of van der Waerden has led to many interesting developments in Combinatorics and Number Theory. It says:

Given positive integers k and r, there exists a positive integer $W(k,r)$ such that for any r-colouring of $\{1, 2, \ldots W(k,r)\}$, there is a monochromatic arithmetic progression of k terms.

Regarding van der Waerden's theorem, we are lucky to have van der Waerden's personal account (1971): 'How the proof of Baudet's conjecture was found". It contains the formulation of the problem with the valuable suggestions due to Emil Artin and Otto Schreier and depicts how the sequence of basic ideas occurred as an elaboration of the psychology of invention. It seems the result was independently conjectured by Schur and Baudet; since van der Waerden came to know it through Baudet, he calls it Baudet's conjecture.

I add some comments before I close. A theorem due to Hales and Jewett (1963) revealed the combinatorial nature of van der Waerden's theorem, showing that this 'pearl of number theory' belongs to the ancient shore of Combinatorics. Today we see that van der Waerden's theorem was a prelude to a very important theme where interplay of several areas of Mathematics would be seen. One had the results of Roth and Szemerédi and a number of different proofs of these results including the ergodic proof of Szemeredi's theorem due to Furstenberg, which started the subject of Ergodic Ramsey Theory. Then came the results of Gowers and the Green-Tao Theorem. Many challenging open questions are still remaining to be answered.