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# THE MATHEMATICS STUDENT

**Edited by G. P. Youvaraj**

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# EXISTENCE, UNIQUENESS AND NUMERICAL RESULTS ON $\psi-$HILFER FRACTIONAL DIFFERENTIAL EQUATIONS

RANJINI, M. C., SHABNA, M. S., SHAMEEMA, V.
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Abstract. This study addresses the fractional order system in terms of $\psi-$Hilfer fractional differential equations. The Banach Contraction principle and the non-compactness measure are used to analyse the existence and uniqueness of the mild solution. The generalization of Leibnitz type rule is applied to find the series solution of $\psi-$Hilfer Riccati fractional differential equations. Further, an approximate solution to Hilfer-Katugampola Ricatti fractional differential equation (HKR-FDE), based on operational matrix and collocation techniques, is presented using fractional order Euler Polynomials (FEPs). The convergence and error bounds of the proposed method are also evaluated. Numerical examples are demonstrated to show the accuracy of the suggested method.

## 1. Introduction

Since the introduction of fractional calculus, numerous definitions and proofs for fractional integrals and derivatives, employing various kernels and addressing diverse applications have been developed. Due to its widespread use in both pure and applied mathematics over the past few decades, fractional calculus has attracted significant attention from researchers [?, ?, ?, ?, ?, ?, ?, ?]. However, selecting appropriate operators can be challenging for those interested in studying fractional calculus.

One approach to addressing this issue is the adoption of more inclusive and generalized terminology. A notable contribution in this regard is the $\psi$-Hilfer fractional derivative, introduced by Sousa and Oliveira [?].

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One of the key advantages of the $\psi$-Hilfer fractional operator lies in its flexibility with respect to the order and type of classical differential operators. This adaptability enables the definition of a broader class of fractional derivatives. In particular, the $\psi$-Hilfer derivative generalizes several known operators as special cases, including the $\psi$-Caputo, $\psi$-Riemann–Liouville, Chen, Weyl, Hilfer, and Hilfer–Katugampola derivatives.

Although the $\psi$-Hilfer derivative offers these advantages, researchers often face challenges when defining associated operators such as the fractional differential operator or determining suitable kernel functions. A major difficulty arises from the arbitrariness of the kernel, which complicates the derivation of fundamental properties typically associated with classical derivatives. One way to mitigate this issue is by employing specific kernel types within the Caputo-type fractional derivative defined with respect to another function [3]. Nevertheless, even with such refinements, the selection of appropriate kernels for the $\psi$-Hilfer derivative remains an open area for further research. Several authors have contributed to the advancement of $\psi$-Hilfer-type fractional derivatives. For instance, Vanterler and Capelas de Oliveira [12, 13] introduced a Leibniz-type rule and a generalized Gronwall inequality involving fractional integrals with respect to another function. Nelson Vieira et al. [18] studied time-fractional diffusion equations involving the $\psi$-Hilfer derivative. Moreover, researchers such as [15, 16] have explored the use of the $\psi$-Hilfer derivative in the study of hybrid differential equations. Norouzi et al. [17] investigated the existence and uniqueness of mild solutions to $\psi$-Hilfer fractional differential equations using the Banach contraction principle and the measure of non-compactness. As an application, they analyzed a financial crisis model.

Furthermore, numerous studies have addressed existence, uniqueness, and approximate solutions of fractional equations involving the $\psi$-Hilfer operator using various fixed-point theorems [10, 11, 12, 17].
In our study, we consider the $\psi-$Hilfer fractional differential equation given by:

$$\begin{cases} {}^{H}\mathbb{D}_{a_0+}^{\alpha,\nu;\psi} p(u) = q(u, p(u)), \quad a.e \quad u \in J = [a_0, T], \\ p(u_0) = \phi_0 \end{cases} \tag{1.1}$$

where ${}^{H}\mathbb{D}_{a_0}^{\alpha,\nu;\psi}(\cdot)$ is the $\psi-$Hilfer fractional derivative with $0 < \alpha < 1, 0 \leq \nu \leq 1$, $\alpha \leq \Upsilon = \alpha + \nu - \alpha\nu < 1$ and $q \in \mathcal{C}_{1-\Upsilon;\psi}(J \times \mathbb{R}, \mathbb{R})$. $I = [a_0, a_0 + a]$

is a bounded interval in $\mathbb{R}$ for some $a_0$.

We employ the measure of non-compactness and the Banach contraction principle to establish the existence and uniqueness of solutions, respectively. As the application of fractional differential equations (FDEs) in modeling real-world phenomena continues to grow [19], there is a rising demand for accurate and efficient numerical techniques for their analysis. In our previous works [20, 21, 22, 23], we developed numerical methods based on Fractional Euler Polynomials (FEPs) to solve a variety of FDEs involving operators such as Caputo, conformable, Caputo–Fabrizio, and related derivatives. Building on this foundation, we now extend the methodology to encompass the generalized $\psi$-Hilfer fractional derivative. Specifically, we propose an operational matrix-based approach for the numerical solution of the Hilfer–Katugampola Riccati fractional differential equation (HKR-FDE) with initial conditions, thereby broadening the applicability of the FEP framework. The proposed method transforms the FDE into a system of equations using operational matrix and collocation techniques.
This paper is set up as follows: Section 2 presents several fundamental lemmas and definitions. Additionally, it contains a few findings that are necessary to support our main findings. In Section 3 we analyse the existence and uniqueness of the mild solution for the given $\psi-$ Hilfer fractional differential equation (1.1), with initial condition and provided a series solution using the generalisation of the Leibniz type rule. In Section 4, an operational matrix for the Hilfer-Katugampola fractional differential operator is constructed. In Section 5, we present an approximation method based on operational matrices to solve HKR-FDE with initial condition. We estimate the error bounds for the numerical solution in Section 6. In Section 7, we solve a few examples and compare the outcomes for various $\rho$ values. In Section 8, a conclusion is provided.

## 2. Preliminaries

A few definitions, properties, and lemmas of the fractional calculus theory are provided in this section.
Let $I = [a, b]$, $(0 < a < b < \infty)$ be finite interval on the half axis $\mathbb{R}^+$ and let $C[a, b]$ be the space of continuous functions, $AC^k[a, b]$ be the space of

all $k-$times absolutely continuous functions and $C^k[a,b]$ be k-times continuously differentiable functions on $[a,b]$ with the norm.

$$\text{Define, } ||p||_{C[a,b]} = \max_{u\in[a,b]} |p(u)|. \tag{2.1}$$

The weighted space $C^k_{\Upsilon;\psi}[a,b]$ of continuous $p$ on $(a,b]$ is defined by:

$$C^k_{\Upsilon;\,\psi}[a,b] = \{p:(a,b]\to\mathbb{R}; (\psi(u)-\psi(a))^{\Upsilon}p(u)\in C[a,b]\}, \tag{2.2}$$

$0\le\Upsilon<1$ with the norm

$$||p||_{C_{\Upsilon;\,\psi}}[a,b] = ||(\psi(u)-\psi(a))^{\Upsilon}p(u)||_{C[a,b]} = \max_{u\in[a,b]}|(\psi(u))-\psi(a))^{\Upsilon}p(u)|.$$

The weighted space $C^n_{\Upsilon;\psi}[a,b]$ of continuous $p$ on (a,b] is defined by:

$$C^k_{\Upsilon;\,\psi}[a,b] = \{p:(a,b]\to\mathbb{R}; p(u)\in C^{k-1}[a,b]; p^{(k)}(u)\in C_{\Upsilon,\,\psi}[a,b]\},$$

$$0<\Upsilon<1\text{with the norm } \|p\|_{C^k_{\Upsilon;\,\psi}[a,b]} = \sum_{i=0}^{k-1}\|p^{(i)}\|_{C[a,b]} + \|p^{(k)}\|_{C_{\Upsilon;\,\psi}[a,b]}.$$

**Definition 2.1.** [1] Let $(a,b)\,(-\infty\le a<b\le\infty)$, be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha>0$. Also let $\psi(u)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(u)$ on $(a,b)$. The left-sided fractional integral of a function $p$ with respect to a function $\psi$ on [a,b] is defined by:

$$I^{\alpha;\,\psi}_{a+}p(u) = \frac{1}{\Gamma(\alpha)}\int_a^u \psi'(s)(\psi(u)-\psi(s))^{\alpha-1}p(u)ds. \tag{2.3}$$

**Definition 2.2.** [1] Let $\psi'(u)\neq 0$, $(-\infty\le a<b\le\infty)$ and $\alpha>0$, $k\in\mathbb{N}$. The Rieman-Liouville derivative of a function with respect to $\psi$ of order $\alpha$ is defined by:

$$\mathcal{D}^{\alpha;\,\psi}p(u) = \left(\frac{1}{\psi'(u)}\frac{d}{du}\right)^k I^{k-\alpha;\,\psi}p(u).$$

**Definition 2.3.** [10] Let $k-1<\alpha<k$ with $k\in\mathbb{N}$, let $I=[a,b]$ be an interval such that $-\infty\le a<b\le\infty$ and let $p,\psi\in C^k[a,b]$ be two functions such that $\psi$ is increasing and $\psi'(u)\neq 0$, for all $u\in I$. The left-sided $\psi-$Hilfer fractional derivative ${}^H\mathbb{D}^{\alpha,\,\nu;\,\psi}_{a+}(\cdot)$ of a function $p$ of order $\alpha$ and type $0\le\nu\le 1$, is defined by

$${}^H\mathbb{D}^{\alpha,\,\nu;\,\psi}_{a+}p(u) = I^{\nu(k-\alpha);\,\psi}_{a+}\left(\frac{1}{\psi'(u)}\frac{d}{du}\right)^k I^{(1-\nu)(k-\alpha);\,\psi}_{a+}p(u). \tag{2.4}$$

**Definition 2.4.** For the class of all bounded subsets $\mathcal{M}_U$ of the metric space $U$, a map $\mu : \mathcal{M}_U \to [0,\infty)$ is called a measure of non-compactness on $U$, if the following properties are satisfied for $M_1, M_2 \in \mathcal{M}_U$.

$\mathcal{C}_1$**:** : $\mu(M) = 0$ if and only if $M$ is pre compact,
$\mathcal{C}_2$**:** :$\mu(M) = \mu(\overline{M})$,
$\mathcal{C}_3$**:** :$\mu(M_1 \cup M_2) = \max\{\mu(M_1), \mu(M_2)\}$.

**Lemma 2.5.** [10] *If* $p \in C^k[a,b]$, $0 < \alpha < 1$ *and* $0 \le \nu \le 1$, *then*

$$I_{a+}^{\alpha;\psi}\, {}^{H}\mathbb{D}_{a+}^{\alpha,\nu;\psi} p(u) = p(u) - \sum_{i=1}^{k} \frac{(\psi(u)-\psi(a))^{\Upsilon-i}}{\Gamma(\Upsilon-i+1)} p_{\psi}^{[k-i]} I_{a+}^{(1-\nu)(k-\alpha);\psi} p(a).$$

**Lemma 2.6.** [10] *If* $p \in C^1[a,b]$, $0 < \alpha < 1$ *and* $0 \le \nu \le 1$, *then*

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\nu;\psi} I_{a+}^{\alpha;\psi}\, p(u) = p(u). \tag{2.5}$$

**Lemma 2.7.** [12] *Let* $(a,b]$ *with* $-\infty \le a < b \le \infty$ *an interval in real line,* $\alpha > 0$ *and* $\psi(u)$ *a monotone growing and positive function in* $[a,b]$*, whose derivative is continuous in (a,b). Thus,*

$$I_{a+}^{\alpha;\psi} p(u) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} p^{(k)}(u) \frac{(\psi(u)-\psi(a))^{\alpha+k}}{\Gamma(\alpha+k+1)},$$

*and the left fractional integral of a two-function product is given by:*

$$I_{a+}^{\alpha;\psi}(pf)(u) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} p^{(k)}(u) I_{a+}^{\alpha+k;\psi},$$

*where* $p^{(k)}(u)$ *is the* $k-th$ *derivative of the entire order and* $u > a$.

**Theorem 2.8.** ***(Banach contraction principle)*** *For a non-empty complete metric space* $(E,d)$*, let* $0 \le K < 1$ *and let the mapping* $A : E \to F$ *satisfy the inequality* $d(Ax, Ay) \le Kd(x,y)$ *for every* $x, y \in E$*. Then,* $A$ *has a uniquely determined fixed point* $x^*$*. Furthermore, for any* $x_0 \in E$*, the sequence* $(A^j(x_0))_{j=1}^{\infty}$ *converges to this fixed point* $x^*$.

**Theorem 2.9.** ***(Monch's Fixed Point Theorem)*** *Let* $\mathrm{Y}$ *be a bounded, closed and convex subset of a Banach space* $\mathrm{X}$ *such that which contains* $0$ *and let* $\mathrm{A}$ *be a continuous mapping of* $\mathrm{Y}$ *into itself. If the implication* $E = conv\mathrm{A}(\mathrm{E})$ *or* $\mathrm{E} = \mathrm{A}(\mathrm{E}) \cup \{0\} \implies \mathrm{E}$ *is compact holds for every subset* $\mathrm{E}$ *of* $\mathrm{Y}$*, then* $\mathrm{A}$ *has a fixed point.*

The Hilfer-Katugampola is a special case of $\psi-$Hilfer operator. We denote ${}^{H}\mathbb{D}_{a+}^{\alpha,\nu;\psi}$, $\psi = u^{\rho}$ and $\nu \to 0$ as ${}^{HK}\mathbb{D}_{a+}^{\alpha,\rho}$.

**Definition 2.10.** [24] Let $v \in C$, $Re(v) \geq 0$ and $\rho > 0$ then, for all $0 \leq a < u < b \leq \infty$, the definition of the Hilfer-Katugampola derivative is defined by

$$^{HK}\mathbb{D}_{a+}^{\alpha,\rho}p(u) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_{a}^{u}\frac{p'(s)}{(u^{\rho}-s^{\rho})^{\alpha}}ds. \tag{2.6}$$

The Hilfer-Katugampola fractional operator $^{HK}\mathbb{D}_{a+}^{\alpha,\rho}$ holds the following properties:

**1.:** $^{HK}\mathbb{D}_{a+}^{\alpha,\rho}(ap+bq) = a\ ^{HK}\mathbb{D}_{a+}^{\alpha,\rho}(p) + b\ ^{HK}\mathbb{D}_{a+}^{\alpha,\rho}(q)$, for all $a,\, b \in \mathbb{R}$,
**2.:** $^{HK}\mathbb{D}_{a+}^{\alpha,\rho}p(u) = 0$, for all constant functions $p(u) = \lambda,\ \lambda \in \mathbb{R}$,
**3.:** $^{HK}\mathbb{D}_{a+}^{\alpha,\rho}(u^{r\alpha}) = \frac{r\alpha\rho^{\alpha-1}\Gamma(\frac{r\alpha}{\rho})}{\Gamma(\frac{r\alpha}{\rho}+1-\alpha)}u^{\alpha(r-\rho)}$.

**Proposition 2.11.** [24] *Let $0 < a < b < \infty$ and $0 < \alpha \leq 1$. Let $m$ be an arbitrary non negative integer and $n = \lfloor (m+1)\alpha \rfloor$. Let $p \in AC_{\Upsilon}^{n}[a,\, b]$ and suppose $^{HK}\mathbb{D}_{a+}^{(m+1)\alpha,\rho}p \in C[a,\, b]$. Then the Taylor-Lagrange formula involving the generalized Caputo type fractional derivatives writes*

$$p(u) = \sum_{j=0}^{m}\left(\frac{u^{\rho}-a^{\rho}}{\rho}\right)^{j\alpha}\frac{^{HK}\mathbb{D}_{a+}^{j\alpha,\rho}p(a)}{\Gamma(j\alpha+1)} + \left(\frac{u^{\rho}-a^{\rho}}{\rho}\right)^{(m+1)\alpha}\frac{^{HK}\mathbb{D}_{a+}^{(m+1)\alpha,\rho}p(\epsilon)}{\Gamma((m+1)\alpha+1)}. \tag{2.7}$$

**Definition 2.12.** [25] The fractional order Euler polynomial (FEP) with order $j\alpha$, $\alpha > 0$ is defined as follows:

$$E_{j}^{\alpha}(u) = \frac{1}{j+1}\sum_{k=1}^{j+1}e_{(j,k)}u^{(j+1-k)\alpha},\ j = 0,\, 1,\, ...,\, m, \tag{2.8}$$

where $e_{j,\,k} = (2-2^{k+1})\begin{pmatrix} j+1 \\ k \end{pmatrix}B_k$ and $B_k = B_k(0)$ is the Bernoulli number for each $k = 0, 1, ..., j$.

Let $E^{\alpha}(u) = [E_0^{\alpha}(u),\, E_1^{\alpha}(u),\, ...E_m^{\alpha}(u)]^T$ and $U^{\alpha} = \left[u^{\alpha}, u^{2\alpha}, \cdots u^{m\alpha}\right]^T$.

Then $E^{\alpha}$can be expressed as [26] $E^{\alpha}(u) = B^{\alpha}U^{\alpha}$, (2.9)

$$\begin{bmatrix} \frac{2-2^2}{1}\begin{pmatrix}1\\1\end{pmatrix}B_1(0) & 0 & \cdots & 0 \\ \frac{2-2^3}{2}\begin{pmatrix}2\\2\end{pmatrix}B_2(0) & \frac{2-2^2}{2}\begin{pmatrix}2\\1\end{pmatrix}B_1(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2-2^{m+2}}{m+1}\begin{pmatrix}m+1\\m+1\end{pmatrix}B_{m+1}(0) & \frac{2-2^{m+1}}{m+1}\begin{pmatrix}m+1\\m\end{pmatrix}B_m(0) & \cdots & \frac{2-2^2}{m+1}\begin{pmatrix}m+1\\1\end{pmatrix}B_1(0) \end{bmatrix}. \tag{2.10}$$

$$\text{Let } C = [c_0,\, c_1,\, ...,\, c_m],\; p(u) \in L^2[0,\, 1] \text{ and } f(u) = \sum_{i=0}^{\infty} c_i E_i^{\alpha}. \tag{2.11}$$

Then $C = [c_0,\, c_1,\, \ldots,\, c_m]$ can be evaluated from the following relation [26],

$$C^T = A^T D^{-1}, \tag{2.12}$$

$$\text{where } A^T = [a_0,\, a_1,\, a,\, \ldots,\, a_m], \quad a_j = \int_0^1 E_j^{\alpha}(u) f(u) u^{1-\alpha} du,$$

$$D = [d_{ij}^{\alpha}], \quad d_{ij} = \int_0^1 E_i^{\alpha}(u) E_j^{\alpha}(u) u^{1-\alpha} du.$$

## 3. Existence and Uniqueness results

**Lemma 3.1.** *Any function satisfies the IVP given by*

$$\begin{cases} {}^H\mathbb{D}_{a_0+}^{\alpha,\nu;\psi} p(u) = q(u, p(u)), \quad a.e \quad u \in J = [a_0, T], \\ p(u_0) = \phi_0, \end{cases} \tag{3.1}$$

*satisfies the Volterra integral equation*

$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} q(s, x(s)) ds. \tag{3.2}$$

*Proof.* Let us assume to have the $\psi-$Hilfer fractional differential equation (1.1) and prove that $p(u) \in C([a_0, T], E)$ satisfies the Volterra integral equation (3.2). By properties of $\psi-$Hilfer fractional derivative, for $\Upsilon = \alpha + \nu(n - \alpha)$ we have

$${}^H\mathbb{D}^{\alpha,\nu;\psi} p(u) = I^{\nu(n-\alpha);\psi} \mathcal{D}^{\Upsilon,\psi} p(u) = I^{\Upsilon-\alpha;\psi} \mathcal{D}^{\Upsilon;\psi} p(u), \tag{3.3}$$

Using equation (3.3), we have
${}^H\mathbb{D}^{\alpha,\nu;\psi} p(u) = q(u, p(u))$, $I^{\Upsilon-\alpha;\psi} \mathcal{D}^{\Upsilon;\psi} p(u) = q(u, p(u))$. Applying $I^{\alpha;\psi}$ on both sides of the equation, $I^{\alpha;\psi} I^{\Upsilon-\alpha;\psi} \mathcal{D}^{\Upsilon;\psi} p(u) = I^{\alpha;\psi} q(u, p(u))$

$$I^{\Upsilon;\psi} \mathcal{D}^{\Upsilon;\psi} p(u) = I^{\alpha;\psi} q(u, p(u))$$

$$p(u) - p(u_0) = \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s) \left(\psi(u) - \psi(s)\right)^{\alpha-1} q(s, p(s))ds$$
$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s) \left(\psi(u) - \psi(s)\right)^{\alpha-1} q(s, p(s))ds.$$

Now let us assume that $p(u) \in C([a_0, T], E)$ satisfies the volterra integral equation (3.2) and prove that the $\psi-$Hilfer fractional differential equation (1.1) holds.

By applying ${}^{H}\mathbb{D}^{\alpha,\nu;\psi}$ to both sides of the equation (3.2), we can obtain:

$$
\begin{aligned}
&{}^{H}\mathbb{D}^{\alpha,\nu;\psi}p(u)\\
&={}^{H}\mathbb{D}^{\alpha,\nu;\psi}\phi_0 + {}^{H}\mathbb{D}^{\alpha,\nu;\psi}\left[\frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s) \left(\psi(u) - \psi(s)\right)^{\alpha-1} q(s, p(s))ds\right]\\
&={}^{H}\mathbb{D}^{\alpha,\nu;\psi}\phi_0 + I^{\Upsilon-\alpha;\psi}D^{\Upsilon,\psi}\left[\frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s) \left(\psi(u) - \psi(s)\right)^{\alpha-1} q(s, p(s))ds\right]\\
&= I^{\Upsilon-\alpha;\psi}\left(\frac{1}{\psi'(u)}\frac{d}{du}\right)^{n} I^{(1-\nu)(n-\alpha);\psi}\times\\
&\qquad\qquad\left[\frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s) \left(\psi(u) - \psi(s)\right)^{\alpha-1} q(s, p(s))ds\right]\\
&= I^{\Upsilon-\alpha;\psi}\left(\frac{1}{\psi'(u)}\frac{d}{du}\right)^{n} I^{n-\nu n+\alpha,\nu;\psi}q(u, p(u)).
\end{aligned}
$$

Hence, we obtain: ${}^{H}\mathbb{D}^{\alpha,\nu;\psi}p(u) = I^{\Upsilon-\alpha;\psi}\mathcal{D}^{\Upsilon-\alpha;\psi}q(u, p(u)) = q(u, p(u))$. □

**Definition 3.2.** A function $p(u) \in C([a_0, T], E)$ is said to be a mild solution of the system

$$\begin{cases} {}^{H}\mathbb{D}^{\alpha,\nu;\psi}_{a_0+}p(u) = q(u, p(u)), \quad a.e \quad u \in J = [a_0, T],\\ p(u_0) = \phi_0 \end{cases}$$

if it satisfies the equation

$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} q(s, p(s))ds, \quad u \in [a_0, T].$$

**Theorem 3.3.** *Under the assumptions:*

- **($\mathcal{A}_1$):** *For the Banach space $X$ (real or complex), $q : I \times X \to X$, the function $u \to q(u, p(u))$ is measurable on $I$, $\forall p \in X$ and the function $p \to q(u, p(u))$ is continuous on $X$ for a.e $u \in [a_0, T]$,*

**($\mathcal{A}_2$):** *There exist $K > 0$ such that*

$$\|q(u,p_1) - q(u,p_2)\| < K\|p_1 - p_2\|, \ \forall\, u \in [a_0,T] \ \forall p_1, p_2 \in X,$$

*the IVP* (1.1) *has a unique mild solution on* $[a_0,T]$*, provided* $K \leq \frac{\Gamma(\alpha)}{\psi(T)^{1-\alpha}}$.

*Proof.* Consider the operator $T : C_{1-\alpha}(J,X) \to C_{1-\alpha}(J,X)$ on $[a_0,T]$ by:

$$(Fp)(u) = \psi(u)^{\alpha-1}p(u_0) + I^{\alpha,\psi}q(u,p(u)).$$

Choose $M_1 = \sup\limits_{u\in[a_0,T]} \|q(u_0,p(u_0))\|$ and $\psi(a_0) = 0$.

Let $\mathcal{B}_r = \{p \in C_{1-\alpha}(J,X), \|p\|_\alpha \leq r\},$ where $r \geq 2\left[|\phi_0| + \dfrac{M_1\psi(t)}{\Gamma(\alpha+1)}\right].$

Then $\psi(u)^{1-\alpha}\|(Fp)(u)\| \leq$

$$|\phi_0| + \frac{\psi(u)^{1-\alpha}}{\Gamma(\alpha)}\int_{a_0}^{u} \psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\|q(s,p(s))\|ds$$

$\leq |\phi_0|+$

$$\frac{\psi(T)^{1-\alpha}}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\left[\|q(s,p(s)) - q(s,\phi_0)\| + \|q(s,\phi_0)\|\right]ds$$

$$\leq |\phi_0| + \frac{\psi(T)^{1-\alpha}}{\Gamma(\alpha)}\|q(s,p(s)) - q(s,\phi_0)\|_\alpha + \frac{M_1\psi(T)}{\Gamma(\alpha+1)}$$

$$\leq |\phi_0| + \frac{M_1\psi(T)}{\Gamma(\alpha+1)} + \frac{\psi(T)^{1-\alpha}}{\Gamma(\alpha)}\mathcal{B}_{(\alpha,\alpha)}r \ \leq r,$$

which proves that $F(\mathcal{B}_r) \subset \mathcal{B}_r$. Now take $p_1, p_2 \in C_{1-\alpha(J,X)}$, then we obtain

$$\|(Fp_1)(u) - (Fp_2)(u)\| =$$

$$\frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\|q(s,p_1(s)) - q(s,p_2(s))\|ds.$$

Multiplying $\psi(u)^{1-\alpha}$ on both sides,

$$\psi(u)^{1-\alpha}\|(Fp_1)(u) - (Fp_2)(u)\|$$

$$= \psi(u)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\|q(s,p_1(s)) - q(s,p_2(s))\|ds$$

$$\leq \frac{K}{\Gamma(\alpha)}\psi(T)^{1-\alpha}\|p_1 - p_2\|_\alpha.$$

Which implies $F$ is a contraction mapping. Hence by Banach fixed point theorem, $F$ has a unique fixed point, which is given by the mild solution to the IVP (1.1). □

**Theorem 3.4.** *The IVP* (1.1) *has at least one mild solution defined on $I$, under the assumptions*

- **($\mathcal{A}_1$):** *For the Banach space $X$ (real or complex), $q : J \times X \to X$, the function $u \to q(u, p(u))$ is measurable on $J$, $\forall p \in X$ and the function $p \to q(u, p(u))$ is continuous on $X$ for a.e $u \in [a_0, T]$,*
- **($\mathcal{A}_2$):** *There exist a function $l \in L^{\infty}(I)$ such that :* $\|q(u, p(u))\| \leq l(u)\,(1 + \|u\|); \quad \forall u \in J; \quad$ *and* $\quad p \in X,$
- **($\mathcal{A}_3$):** $\mu(p(u, B)) \leq m(u)\,\mu(B); \quad \forall B \subset X,$
- **($\mathcal{A}_4$):** *The function $\psi(\eta)$ is uniformly continuous and $\psi(a_0) = 0$,*

*hold and* $k = T\frac{\psi(T)^{\alpha}}{\Gamma(\alpha+1)}m^* < 1.$

*Proof.* Let us transform the IVP (1.1) into a fixed point theorem.
Consider the operator $F : BC \to BC$ on $[a_0, T]$ defined by,

$$(Fu)(\eta) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{\eta} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1} q(s, u(s))ds.$$

Let $\tau > 0$ such that $\tau \geq \frac{\Gamma(\alpha+1)\|\phi_0\| + Tl^*\psi(T)^{\alpha}}{1 - l^*\psi(T)^{\alpha}}$.
Consider the ball $\mathcal{B}_{\tau} = \mathcal{B}(a_0, \tau) = \{u \in BC : \|u - a_0\|_{BC} \leq \tau\}$.
For any $\mathcal{B}_{\tau}$ and $t \in J$,

$$\begin{aligned} \|(Fu)(\eta)\| &\leq \|\phi_0\| + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{\eta} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1} \|q(s, u(s))\| ds \\ &\leq \|\phi_0\| + \frac{Tl^*(1+\tau)}{\Gamma(\alpha)} \int_{a_0}^{\eta} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1} ds \\ &\leq \|\phi_0\| + \frac{Tl^*(1+\tau)\psi(T)^{\alpha}}{\Gamma(\alpha+1)} \leq \tau. \end{aligned}$$

Thus, $\|(Fu)(\eta)\|_{BC} \leq \tau$, which proves that $F$ transforms the ball $\mathcal{B}_{\tau}$ into itself.
Now we will demonstrate that the operator $F : \mathcal{B}_{\tau} \to \mathcal{B}_{\tau}$ complies with the Monch's theorem's assumptions. So, the succeeding three steps need to be proven.

**Step (1) :** $F : \mathcal{B}_{\tau} \to \mathcal{B}_{\tau}$ is continuous:
Let $\{\zeta_m\}_{m \in \mathbb{N}}$ be a sequence such that $\zeta_m \to \zeta$ as $m \to \infty$ in $\mathcal{B}_{\tau}$.

Then $\forall \eta \in J$, we'll obtain:

$$\|(F\zeta_m)(\eta) - (F\zeta)(\eta)\| = \frac{1}{\Gamma(\alpha)} \int_{a_0}^{\eta} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1} \|q(s, \zeta_m(s)) - q(s, \zeta(s))\| ds.$$

Since $\zeta_m \to \zeta$ as $m \to \infty$ in $\mathcal{B}_\tau$ and $q$ is continuous, then by the Lebesgue dominated convergence theorem, $\|(F\zeta_m)(\eta) - (F\zeta)(\eta)\|_{BC} \to 0$ as $m \to \infty$. Then, $F : \mathcal{B}_\tau \to \mathcal{B}_\tau$ is continuous.

**Step (2) :** $F(\mathcal{B}_\tau)$ is bounded and equicontinuous.
$F(\mathcal{B}_\tau)$ bounded and $F(\mathcal{B}_\tau) \subset \mathcal{B}_\tau$, implies that $F(\mathcal{B}_\tau)$ is bounded.
Let $\eta, x \in I$ and let $x < \eta, \zeta \in \mathcal{B}_\tau$,

$$\begin{aligned}
\psi(\eta)^{1-\alpha}\|(F\zeta)(\eta) - (F\zeta)(x)\| &= [\psi(\eta) - \psi(x)]^{1-\alpha}\|\phi_0\| + \\
&\quad \frac{[\psi(\eta) - \psi(x)]^{1-\alpha}}{\Gamma(\alpha)} \int_{a_0}^{x} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1}\|q(s, \zeta(s))\| ds \\
&\leq [\psi(\eta) - \psi(x)]^{1-\alpha}\|\phi_0\| + \frac{[\psi(\eta) - \psi(x)]^{1-\alpha}}{\Gamma(\alpha+1)} l^*(1+\tau)[\psi(\eta) - \psi(x)]^{\alpha} \\
&\leq [\psi(\eta) - \psi(x)]^{1-\alpha}\|\phi_0\| + \frac{[\psi(\eta) - \psi(x)]}{\Gamma(\alpha+1)} l^*(1+\tau).
\end{aligned}$$

Since $\psi(\eta)$ is uniformly continuous implies that the right hand side of the above inequality converges to zero, as $\eta \to x$.
Similarly the case $\eta, x \in I, \eta < x$ can be proved, which implies that $F(\mathcal{B}_\tau)$ is equicontinuous.

**Step (3) :** The implication $E = conv\,\mathrm{F}\,(\mathrm{E})$ *or* $\mathrm{E} = \mathrm{F(E)} \cup \{0\} \implies$ E is compact holds:
Let $E$ be a subset of $B_\tau$ such that $\mathrm{E} \subset \mathrm{F(E)} \cup \{0\}$. From step (2), it is clear that $E$ is bounded and equicontinuous, hence $u \to e(u) = \mu(E(u))$ is continuous on $J$. Using the properties of measure $\mu$, for each $u \in J$, we obtain:

$$\begin{aligned}
e(u) &\leq \mu((FE)(u) \cup \{0\}) \leq \mu((FE)(u)) \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} m(s)\mu(E(s)) ds \\
&\leq \frac{\psi(T)^\alpha}{\Gamma(\alpha+1)} m^* \int_{a_0}^{u} e(s) ds \leq T\frac{\psi(T)^\alpha}{\Gamma(\alpha+1)} m^* \|e\|_\infty. \\
&\implies e(u) \leq k\|e\|_\infty, \text{where, } k = T\frac{\psi(T)^\alpha}{\Gamma(\alpha+1)} m^* < 1.
\end{aligned}$$

Now, $\|e\|_\infty = 0 \to e(s) = \mu(E(u)) = 0$, $\forall u \in J$.
Hence $E(s)$ is relatively compact in $X$ and $E$ is relatively compact in $\mathcal{B}_\tau$. By Ascoli-Arzela theorem, $E$ is relatively compact in $\mathcal{B}_\tau$. Hence in view of Monch's fixed point theorem, we conclude that $F$ has a fixed point which is a mild solution of our $IVP(1.1)$. □

**Lemma 3.5.** *Consider the $IVP(1.1)$ with $q(u,p(u)) = f(u)p^2(u)+g(u)p(u)$ $+q(u)$. The $\psi-$Hilfer Riccati fractional differential equation,*

$$\begin{cases} {}^H\mathbb{D}_{1+}^{\alpha,\nu;\psi} p(u) = f(u)p^2(u) + g(u)p(u) + q(u) \\ u(a_0) = \phi_0 \end{cases} \tag{3.4}$$

*will be equivalent to the volterra integral equation:*

$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u)-\psi(s))^{\alpha-1}(f(u)p^2(u)+g(u)p(u)+q(u))ds. \tag{3.5}$$

*Also by using lemma(2.7), the series solution is given by:*

$$p(u) = \frac{(\psi(u)-\psi(a_0))^{\Upsilon-1}}{\Gamma(\Upsilon)} I_{a_0+}^{(1-\nu)(1-\alpha);\psi}\phi_0 + \sum_{k=0}^{\infty} \binom{-\alpha}{k} f^{(k)}(u)\, I_{a_0+}^{\alpha+k;\psi} p^2(u) + \sum_{k=0}^{\infty} \binom{-\alpha}{k} g^{(k)}(u)\, I_{a_0+}^{\alpha+k;\psi} p(u) + I_{a_0+}^{\alpha+k;\psi} q(u). \tag{3.6}$$

## 4. Operational Matrix

In this section, we approximate ${}^{HK}\mathbb{D}_{a+}^{\alpha,\rho} E^\alpha(u)$ with an $(m+1)\times(m+1)$ operational matrix $\mathbb{P}^{(m,\alpha,\rho)}$ for $E^\alpha(u)$.
We start with the third property of ${}^{HK}\mathbb{D}_{a+}^{\alpha,\rho} E^\alpha(u)$.

$${}^{HK}\mathbb{D}_{a+}^{\alpha,\rho}(u^{r\alpha}) = \frac{r\alpha\rho^{\alpha-1}\Gamma(\frac{r\alpha}{\rho})}{\Gamma(\frac{r\alpha}{\rho}+1-\alpha)} u^{\alpha(r-\rho)}. \tag{4.1}$$

Now applying (2.11) and (2.12), for $r = 0, 1, \ldots, m$

$${}^{HK}\mathbb{D}_{a+}^{\alpha,\rho}(u^{r\alpha}) = \frac{r\alpha\rho^{\alpha-1}\Gamma(\frac{r\alpha}{\rho})}{\Gamma(\frac{r\alpha}{\rho}+1-\alpha)} u^{\alpha(r-\rho)} = n_r E^\alpha(u).$$

where

$$n_r = \overline{a}_r D^{-1},$$

$$\overline{a}_r = [\ a_{r0},\, a_{r1},\, \dots,\, a_{rm}]\ , \qquad D = [d_{ij}],$$

$$a_{ri} = \frac{r\alpha\rho^{\alpha-1}\Gamma(\frac{r\alpha}{\rho})}{\Gamma(\frac{r\alpha}{\rho}+1-\alpha)}\int_0^1 E_i^{\alpha}(u)u^{\alpha(r-\rho-1)+1}du, \qquad i=0,\,1,\,\dots m,$$

$$d_{ij} = \int_0^1 E_i^{\alpha}(u)E_j^{\alpha}(u)u^{1-\alpha}du, \qquad i=0,\,1,\,\dots m, \qquad j=0,\,1,\,\dots m.$$

Then for $U^{\alpha} = [1,\, u^{\alpha},\, u^{2\alpha},\, \dots,\, u^{m\alpha}]$.

$$^{HK}\mathbb{D}_{a+}^{\alpha,\,\rho}\, U^{\alpha}(u) = [n_0,\, n_1,\, \dots n_m]E^{\alpha}(u) = Q^{\alpha,\,\rho}E^{\alpha}(u), \tag{4.2}$$

where $Q^{m,\,\alpha,\,\rho} = [n_0,\, n_1,\, \dots n_m]E^{\alpha}(u)$. Using equations (2.9) and (4.2) and the properties of the fractional operator $^{HK}\mathbb{D}_{a+}^{\alpha,\,\rho}$, we have

$$\begin{aligned} ^{HK}\mathbb{D}_{a+}^{\alpha,\,\rho}E^{\alpha}(u) &=^{HK}\mathbb{D}_{a+}^{\alpha,\,\rho}B^{\alpha}U^{\alpha}(u) \\ &= B^{\alpha\,HK}\mathbb{D}_{a+}^{\alpha,\,\rho}U^{\alpha}(u) \\ &= B^{\alpha}Q^{(m,\,\alpha,\rho)}E^{\alpha}(u). \end{aligned} \tag{4.3}$$

Hence $\mathbb{P}^{(m,\,\alpha,\,\rho)} = B^{\alpha}Q^{(m,\,\alpha,\,\rho)}$ is the operational matrix of $E^{\alpha}(u)$.

## 5. Approximation Method

In this section, we develop an approximate solution for the Hilfer-Katugampola Riccati differential equation (HKR-FDE) of the form,

$$^{HK}\mathbb{D}_{a+}^{\alpha,\rho}p(u) = F(u) + G(u)p(u) + H(u)p^2(u), \qquad 0 \le v \le 1. \tag{5.1}$$

$$p(0) = p_0, \tag{5.2}$$

where $p_0$ is given real number. Let $p_{\alpha}(u) \approx C^T E^{\alpha}(u)$ be the numerical solution for the above mentioned HKR-FDE. To determine $C^T$, we approximate $F(u)$, $G(u)$ and $H(u)$ as follows:

$$\begin{aligned} p(u) &\approx C^T E^{\alpha}(u), \\ F(u) &\approx F^T E^{\alpha}(u), \\ G(u) &\approx G^T E^{\alpha}(u), \\ H(u) &\approx H^T E^{\alpha}(u), \end{aligned} \tag{5.3}$$

where

$$F^T = \begin{bmatrix} f_0 & f_1 & ... & f_m \end{bmatrix}, \quad G^T = \begin{bmatrix} g_0 & g_1 & ... & g_m \end{bmatrix}$$
$$\text{and } H^T = \begin{bmatrix} h_0 & h_1 & ... & h_m \end{bmatrix}.$$

Then

$$^{HK}\mathbb{D}^{\alpha,\rho}_{a+}p(u) = ^{HK}\mathbb{D}^{\alpha,\rho}_{a+}(C^T E^\alpha(u)) = C^T(\mathbb{P}^{(m,\alpha,\rho)}E^\alpha(u)). \tag{5.4}$$

By applying equations (5.3) and (5.4) into equations (5.1) and (5.2), we get

$$C^T(\mathbb{P}^{(m,\alpha,\rho)}E^\rho(u)) - F^T E^\alpha(x) - G^T E^\alpha(u)(C^T E^\alpha(u)) \\ - H^T E^\alpha(u)(C^T E^\alpha(u)(E^\alpha(u))^T C) = 0. \tag{5.5}$$

and

$$[C^T E^\rho(x)]_{x=0} = y_0. \tag{5.6}$$

By using equation (5.6), express one of the $m+1$ unknown elements of $C^T = \begin{bmatrix} c_0 & c_1 & ... & c_m \end{bmatrix}$ in terms of the others and substituting in equation (5.5), we get

$$C^T(J^{(v,\rho)}E^\rho(x)) - F^T E^\rho(x) - G^T E^\rho(x)(C^T E^\rho(x)) \\ - H^T E^\rho(x)(C^T E^\rho(x)(E^\rho(x))^T C) = 0. \tag{5.7}$$

Now we substitute $m$ different values for $u$, $\{\frac{1}{m}, \frac{2}{m}, \dots, \frac{m}{m}\}$ in equation (5.7), we obtain a system of algebraic equations from which the unknown elements of the vector $C$ can be determined.

## 6. Error Analysis

A function $f(u) \in L^2[0,1]$ can be expressed as

$$f(u) \approx \sum_{i=0}^{m} C_i E_i(u) = C^T E^\alpha(u) = f_m(u). \tag{6.1}$$

The error analysis of the proposed approximation is given by the following theorem.

**Theorem 6.1.** *Suppose that* $^{HK}\mathbb{D}^{k\alpha,\rho}_{a+} \in C(0,1]$ *for* $k = 1, 2, ..., m$ *and* $P^\alpha_m = span\{E^\alpha_0, E^\alpha_1, ..., E^\alpha_m\}$ *is a vector space. If* $f_m(u)$ *is the best approximation of* $f$ *out of* $P^\alpha_m$*, then the mean error bound is presented as follows:*

$$\|f - f_m\| \leq \frac{M_{\alpha,\rho}}{\rho^{(m+1)\alpha}\left(\Gamma((m+1)\alpha+1)\right) 2\rho(2\rho(m+1)\alpha+1)},$$

*where*

$$M_{\alpha,\rho} \geq | \sup_{\epsilon\in[0,1]} {}^{HK}\mathbb{D}_{a+}^{(m+1)\alpha,\rho} p(\epsilon)|.$$

*Proof.* Consider the the generalized Taylor formula (2.7) with $a = 0$,

$$p(u) = \sum_{j=0}^{m} \left(\frac{u^\rho}{\rho}\right)^{j\alpha} \frac{{}^{HK}\mathbb{D}_{a+}^{j\alpha,\rho} p(0)}{\Gamma(j\alpha+1)},$$

$$|f(u) - p(u)|_2^2 \leq M_{\alpha,\rho} \frac{(x^\rho - a^\rho)^{(m+1)\alpha}}{\rho^{(m+1)\alpha}\Gamma((m+1)\alpha+1)}.$$

Since $f_m$ is the best approximation of $f$ from $P_m^\alpha$ and $p \in P_m^\alpha$, we have,

$$\begin{aligned}
\|f - f_m\|_2^2 &\leq \|f - p\|^2 \\
&= \int_0^1 (f(u) - p(u))^2\, du \\
&= \int_0^1 \frac{M_{\alpha,\rho}^2}{\rho^{2(m+1)\alpha}\left(\Gamma((m+1)\alpha+1)\right)^2} u^{2\rho(m+1)\alpha}\, du \\
&= \frac{M_{\alpha,\rho}^2}{\rho^{2(m+1)\alpha}\left(\Gamma((m+1)\alpha+1)\right)^2} \int_0^1 u^{2\rho(m+1)\alpha}\, du \\
&= \frac{M_{\alpha,\rho}^2}{\rho^{2(m+1)\alpha}\left(\Gamma((m+1)\alpha+1)\right)^2 2\rho(2\rho(m+1)\alpha+1)}.
\end{aligned}$$

The theorem is proved by taking the square roots. Therefore the proposed approximations of $f(x)$ are convergent. □

## 7. Numerical Examples

In this section, we apply the proposed technique to solve some examples for Hilfer-Katugampola riccati FDE numerically.

**Example 7.1.** Consider the following fractional differential equation:

$$ {}^{HK}\mathbb{D}_{a+}^{0.8,\rho} p(u) = \frac{5}{8}e^{-4u} - \frac{5}{8} + \frac{5}{2}u, \tag{7.1}$$

$$p(0) = 0. \tag{7.2}$$

The exact solution to this equation is

$$p(u) = \frac{1}{\rho^{\alpha}} \int_a^u \rho s^{\rho-1} (u^{\rho} - s^{\rho})^{\alpha-1} \left( \frac{5}{8} e^{-4s} - \frac{5}{8} + \frac{5}{2} s \right) ds.$$

| **u** | $\rho = 0.1$ | | | $\rho = 0.2$ | | | $\rho = 0.3$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\alpha = 4/5$ | $\alpha = 3/5$ | $\alpha = 2/5$ | $\alpha = 4/5$ | $\alpha = 3/5$ | $\alpha = 2/5$ | $\alpha = 4/5$ | $\alpha = 3/5$ | $\alpha = 2/5$ |
| 0.2 | 1.346635E-5 | 8.025498E-7 | 3.081248E-5 | 8.408785E-6 | 5.355401E-7 | 1.852883E-5 | 4.876254E-6 | 3.160649E-7 | 1.117184E-5 |
| 0.4 | 1.822872E-5 | 3.657922E-7 | 2.779037E-5 | 1.288457E-5 | 1.291565E-7 | 1.623101E-5 | 9.097952E-6 | 5.580321E-8 | 9.492119E-6 |
| 0.6 | 2.537227E-5 | 1.724002E-7 | 2.646559E-5 | 1.977184E-5 | 3.871871E-7 | 1.497033E-5 | 1.574126E-5 | 5.48041E-7 | 8.508192E-6 |
| 0.8 | 3.401546E-5 | 7.8E261E-7 | 2.485556E-5 | 2.830253E-5 | 9.845266E-7 | 1.387569E-5 | 2.415765E-5 | 1.131335E-6 | 7.557012E-6 |
| 1.0 | 4.363535E-5 | 1.440134E-6 | 2.378988E-5 | 3.799188E-5 | 1.641212E-6 | 1.313355E-5 | 3.390728E-5 | 1.783623E-6 | 7.108784E-6 |

TABLE 1. Absolute errors of the Example 7.1
for $\rho = 0.1$, $\rho = 0.2$, $\rho = 0.3$.

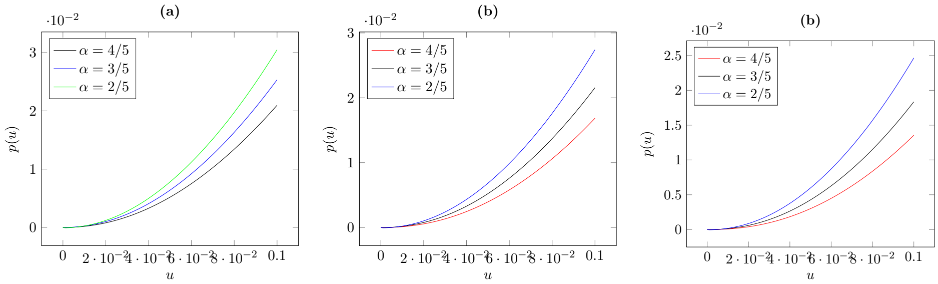


FIGURE 1. Numerical solutions for Example 7.1 :
**(a)** $\rho = 0.1$ **(b)** $\rho = 0.2$ **(c)** $\rho = 0.3$.

We take $\alpha = 4/5,\ 3/5,\ 2/5$ and $\rho = 0.1,\ 0.2,\ 0.3$ into consideration. Figure 1 displays the numerical solutions for each case, obtained with $m = 16$. The absolute errors are presented in Table 1. The comparison of the solution for various values of $m = 8,\ 10,\ 12,\ 16$ is also shown in Table 2. For, we choose $\alpha = 4/5$ and $\rho = 0.1$. The comparison of the results shows that as the values of $m$ increase, the error decreases.

| $u$ | $m = 8$ | $m = 10$ | $m = 12$ | $m = 16$ |
|---|---|---|---|---|
| 0.2 | 2.952702E-4 | 1.052324E-4 | 4.628915E-5 | 1.346635E-5 |
| 0.4 | 2.842574E-4 | 1.063239E-4 | 4.996402E-5 | 1.822872E-5 |
| 0.6 | 2.872725E-4 | 1.112992E-4 | 5.665746E-5 | 2.537227E-5 |
| 0.8 | 2.907455E-4 | 1.187983E-4 | 6.505289E-5 | 3.401546E-5 |
| 1.0 | 2.979202E-4 | 1.278021E-4 | 7.454548E-5 | 4.363535E-5 |

TABLE 2. Absolute errors of the Example 7.1
for $m = 8$, $m = 10$, $m = 12$, $m = 16$

**Example 7.2.** Consider the following fractional differential equation:

$$^{HK}\mathbb{D}_{a+}^{\alpha,\rho}p(u) = \rho^{\alpha}p(u), \tag{7.3}$$

$$p(0) = 1. \tag{7.4}$$

The exact solution to this equation is

$$p(u) = E_{\alpha}(u^{\rho\alpha}).$$

| **u** | $\rho = 0.5$ | | | | | $\rho = 0.75$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha = 0.25$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.75$ | $\alpha = 0.2$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.75$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.2 | 0.022052 | 0.036947 | 0.069861 | 0.099729 | 0.143020 | 0.000083 | 0.004681 | 0.0118424 | 0.020492 | 0.038279 |
| 0.4 | 0.025349 | 0.042490 | 0.080471 | 0.115199 | 0.167575 | 0.000094 | 0.005270 | 0.0133898 | 0.023322 | 0.044804 |
| 0.6 | 0.028640 | 0.048019 | 0.091022 | 0.130485 | 0.191045 | 0.001091 | 0.006088 | 0.0154987 | 0.027072 | 0.052621 |
| 0.8 | 0.031916 | 0.053523 | 0.101511 | 0.145648 | 0.214066 | 0.001265 | 0.007046 | 0.0179546 | 0.031412 | 0.061405 |
| 1.0 | 0.035205 | 0.059046 | 0.112032 | 0.160845 | 0.237033 | 0.001461 | 0.008131 | 0.0207340 | 0.036312 | 0.071339 |

TABLE 3. Absolute errors of the Example 7.2 for $\rho = 0.5$, $\rho = 0.75$.

The solutions based on FEFs are shown in Figure 2 for $\alpha = 0.2,\ 0.3,\ 0.4,\ 0.5,$ $0.75$ with $m = 16$ and $\rho = 0.5$ fixed values. We determine the absolute error between the analytic solution and approximate solution for distinct values of $\rho$ and $\alpha$ and present it in Table 3.

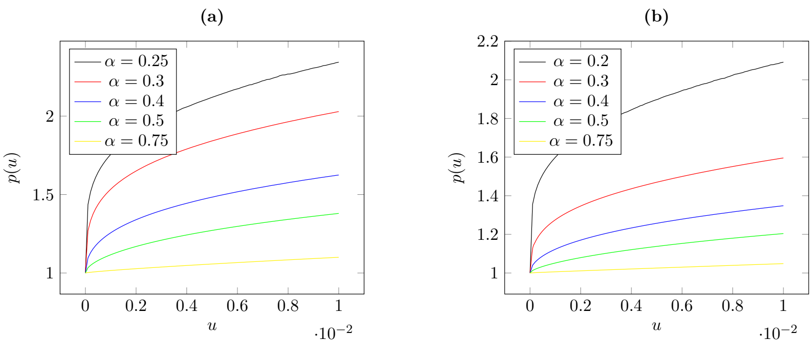


FIGURE 2. Numerical solutions for Example 7.2 :
**(a)** $\rho = 0.5$ **(b)** $\rho = 0.75$.

**Robustness with respect to $\rho > 1.5$:** To examine the robustness of the proposed numerical method, we extended our analysis to include values of the parameter$\rho > 1.5$. Table 4 presents the absolute errors for the numerical solution of the Example 7.2 based on fractional Euler polynomials, for $\rho = 1.6, 1.8,$ and $2.0$. For all these cases, the numerical results remained highly accurate and stable. No changes to the algorithm or parameter tuning were necessary, indicating that the method is robust with respect to variations in $\rho$, even beyond the commonly tested interval. This further confirms the reliability and adaptability of the proposed scheme across a wider class of fractional problems.

| u | $\rho = 1.6$ | $\rho = 1.8$ | $\rho = 2.0$ |
|---|---|---|---|
| 0.2 | 7.955275E-5 | 2.798311E-5 | 3.504801E-5 |
| 0.4 | 6.345917E-5 | 2.231967E-5 | 6.602228E-6 |
| 0.6 | 6.522533E-5 | 2.424771E-5 | 1.224123E-5 |
| 0.8 | 7.542861E-5 | 3.045374E-5 | 3.586186E-5 |
| 1.0 | 9.420751E-5 | 4.197421E-5 | 7.345388E-5 |

TABLE 4. Absolute errors for large values of $\rho$ at $\alpha = 0.4$.

**Example 7.3.** Consider the following fractional differential equation:

$$^{HK}\mathbb{D}^{\alpha,\rho}_{a+}p(u) = 1 - p^2(u), \tag{7.5}$$

$$p(0) = 0. \tag{7.6}$$

The exact solution to this equation for $\rho = 1$ is

$$p(u) = \frac{e^{2u} - 1}{e^{2u} + 1}.$$

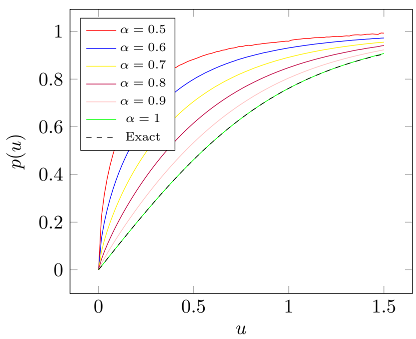


FIGURE 3. Exact and approximated solutions for Example 7.3.

$y(x) = \frac{e^{2x}-1}{e^{2x}+1}$ is the exact solution to this initial value problem for $\rho = 1$.

For $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$, the approximate solutions are evaluated and shown in Table 5.

A graphical representation of the solution is presented in Figure 3. The tabulated results demonstrate the applicability of the presented method for fractional differential equations of the Riccati type involving Hilfer-Katugampola derivatives.

| u | $\alpha = 0.5$ | $\alpha = 0.6$ | $\alpha = 0.7$ | $\alpha = 0.8$ | $\alpha = 0.9$ | $\alpha = 1$ | Exact |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.54731535 | 0.55916665 | 0.43264376 | 0.33199126 | 0.25529144 | 0.19738357 | 0.19737532 |
| 0.4 | 0.75531541 | 0.74377191 | 0.63653001 | 0.53754341 | 0.45193649 | 0.37995631 | 0.37994896 |
| 0.6 | 0.85404854 | 0.84076339 | 0.76125687 | 0.68090898 | 0.60541095 | 0.53705567 | 0.53704957 |
| 0.8 | 0.90667486 | 0.89665463 | 0.84026737 | 0.7801570 | 0.72064492 | 0.66404156 | 0.66403677 |
| 1.0 | 0.9386879 | 0.93071876 | 0.89139412 | 0.84832065 | 0.80446828 | 0.76159757 | 0.76159416 |

TABLE 5. Absolute errors of the Example 7.1
for $\alpha = 0.5$, $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 1$.

**Example 7.4.** Consider the following fractional differential equation:

$$^{HK}\mathbb{D}_{a+}^{\alpha,\rho} p(u) = 2p(u) - p^2(u) + 1, \tag{7.7}$$

$$p(0) = 0. \tag{7.8}$$

We solve this initial value problem numerically by considering $\alpha = 0.7,\ 0.8,\ 0.9,\ 0.95,\ 1.0$, $m = 16$. The results are presented in Table 6 and Figure 4.

| u | $\alpha = 0.7$ | $\alpha = 0.8$ | $\alpha = 0.9$ | $\alpha = 0.95$ | $\alpha = 1$ | Exact |
|---|---|---|---|---|---|---|
| 0.2 | 0.69875433 | 0.45499423 | 0.3301928 | 0.2820445 | 0.24170381 | 0.2419768 |
| 0.4 | 1.269093 | 0.93662822 | 0.72763514 | 0.64183817 | 0.56746495 | 0.56781217 |
| 0.6 | 1.6990768 | 1.3867159 | 1.1526226 | 1.0481063 | 0.95318379 | 0.95356622 |
| 0.8 | 1.9824445 | 1.743872 | 1.5392232 | 1.4402822 | 1.3460041 | 1.3463637 |
| 1.0 | 2.1560589 | 1.994476 | 1.8433818 | 1.7659131 | 1.6892167 | 1.6894984 |

TABLE 6. Absolute errors of the Example 7.4
for $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 0.95$, $\alpha = 1$.

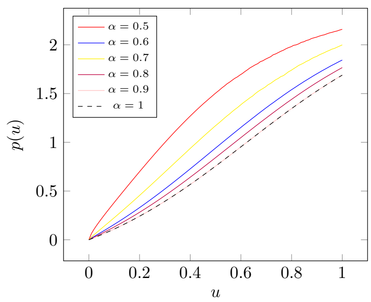


FIGURE 4. Exact and approximated solutions for Example 7.4.

The exact solution to this equation for $\rho = 1$ is

$$p(u) = 1 + \sqrt{2}\tanh(\sqrt{2}u + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})).$$

## 8. Conclusion

In this work, we have established new analytical and numerical frameworks for $\psi-$Hilfer fractional differential equations. By employing various fixed point theorems, we demonstrated the existence and uniqueness of mild solutions, and further derived exact series solutions for $\psi-$Hilfer Riccati equations using a generalized Leibniz rule. Besides that, a numerical method to solve Hilfer-Katugampola Riccati FDE with an initial condition is also presented. The solution is approximated in terms of FEPs and the FDE is converted into a set of algebraic equations. The error analysis of the solution is also discussed. Further, several numerical examples are presented to show the efficiency of the suggested method.

To the best of our knowledge, there is currently no numerical method in the literature that addresses $\psi-$Hilfer fractional differential equations for arbitrary kernel functions $\psi$. Our approach not only fills this gap but is also inherently adaptable, allowing for extension to even more generalized fractional operators. Our method uniquely combines operational matrices with direct comparisons to exact solutions, enabling a more comprehensive evaluation of numerical performance. Future work could extend these techniques to variable order $\psi-$Hilfer operators, integrate machine learning strategies for high dimensional or data driven problems, and $\psi-$Hilfer generalized proportional fractional operators, as recent research has begun to explore their properties and applications.

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## References

[1] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J., Theory and applications of fractional differential equations, **204**, elsevier, 2006.

[2] Zhou, Y., Wang, J., & Zhang, L., Basic theory of fractional differential equations, *World scientific*, 2016.

[3] Almeida, R., A Caputo fractional derivative of a function with respect to another function. *Communications in Nonlinear Science and Numerical Simulation* **44** (2017), 460 – 481.

[4] Almeida, R., Malinowska, A. B., & Monteiro, M. T. T., Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Mathematical Methods in the Applied Sciences* **41(1)** (2018), 336 – 352.

[5] Ricardo Almeida., Fractional differential equations with mixed boundary conditions, *The Bulletin of the Malaysian Mathematical Society* **2** (2018).

[6] Tenreiro Machado, J. A., Silva, M. F., Barbosa, R. S., Jesus, I. S., Reis, C. M., Marcos, M. G., & Galhano, A. F., Some applications of fractional calculus in engineering, *Mathematical problems in engineering* (2010).

[7] Mehdi Dalir., Majid Bashour., Applications of Fractional Calculus, *Applied Mathematical Sciences* **4** (2010), 1021 – 1032.

[8] Zhang, Y., Sun, H., Stowell, H. H., Zayernouri, M., & Hansen, S. E., A review of applications of fractional calculus in Earth system dynamics, *Chaos, Solitons & Fractals* **102** (2017), 29 – 46.

[9] Bashiri T., Vaezpour S. M., & Park C., Existence results for fractional hybrid differential systems in Banach algebras, *Advances in Difference Equations* **1** (2016), 1 -13.

[10] Sousa, J. V. D. C., & De Oliveira, E. C., On the $\psi-$Hilfer fractional derivative, *Communications in Nonlinear Science and Numerical Simulation* **60** (2018), 72-91.

[11] Sousa, J. V. D. C., & de Oliveira, E. C., On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the $\psi-$Hilfer operator, *Journal of Fixed Point Theory and Applications* **20(3)** (2018), 1 – 21.

[12] Sousa, J. V. D. C., & De Oliveira, E. C., Leibniz type rule: $\psi-$Hilfer fractional operator, *Communications in Nonlinear Science and Numerical Simulation* **77** (2019), 305 – 311.

[13] Vanterler da Costa Sousa, J., & Capelas de Oliveira, E., A Gronwall inequality and the Cauchy-type problem by means of $\psi$-Hilfer operator, arXiv e-prints, arXiv-1709, 2017.

[14] Vieira, N., Rodrigues, M. M., & Ferreira, M., Time-fractional diffusion equation with $\psi-$Hilfer derivative, *Computational and Applied Mathematics* **41(6)** (2022), 230.

[15] Shabna, M. S., & Ranjini, M C., On existence of $\psi-$Hilfer hybrid fractional differential equations, *South East Asian Journal of Mathematics & Mathematical Sciences* **16(2)** (2020).

[16] Kucche, K. D., & Mali, A. D., On the nonlinear $\psi-$Hilfer hybrid fractional differential equations, *Computational and Applied Mathematics* **41(3)** (2022), 86.

[17] Norouzi, F., & N'Guérékata, G. M., A study of $\psi-$Hilfer fractional differential system with application in financial crisis, *Chaos, Solitons & Fractals* **X 6** (2021), 100056.

[18] Vieira, N., Rodrigues, M. M., & Ferreira, M., Time-fractional diffusion equation with $\psi-$Hilfer derivative, *Computational and Applied Mathematics* **41(6)** (2022), 230.

[19] Debnath, L., Recent applications of fractional calculus to science and engineering, *International Journal of Mathematics and Mathematical Science*, **54** (2003), 3413 – 3442.

[20] Shameema, V., & Ranjini, M. C., A Numerical Method for Fractional Variations Problems based on Fractional order Euler Functions, *Advances in Mathematics: Scientific Journal* **9** (2020), 1867 – 1874.

[21] Shameema, V., & Ranjini, M. C., A Numerical Solution to Conformable Fractional Order Logistic Equation, *AIP Conference Proceedings*, 2261, 1 – 8, 2020.

[22] Shameema, V., & Ranjini, M. C., Logarithmic fractional order Euler functions for solving Caputo type modified Hadamard fractional integro differential equations, *Malaya Journal of Mathematik* **1** (2019), 610 – 614.

[23] Shameema, V., & Ranjini, M. C., An Operational Matrix method for Fractional Differential Equations with Non-Singular Kernel, *Journal of Fractional Calculus and Applications* **14 (1)** (2023), 157 – 170.

[24] Benjemaa, M., Taylor's formula involving generalized fractional derivatives. *Applied Mathematics and Computation* **335** (2018), 182 – 195.

[25] Cheon, G.S., A note on the Bernoulli and Euler polynomials. *Applied Mathematics Letters* **16 (3)** (2003), 365 – 368.

[26] Wang, Y., Zhu, L., & Wang, Z., Fractional-order Euler functions for solving fractional integro-differential equations with weakly singular kernel. *Advances in Difference Equations* **1** (2018), 1 – 13.

[27] Granas A., & Dugundji J., Fixed point theory, *Science* & *Business Media* (2013), Springer.

[28] Krishnarajulu, K., Sevugan, R., & Sivaramakrishnan, G., A new approach to space fractional differential equations based on fractional order Euler polynomials, *Publications de l'Institut Mathematique*, **104 (118)** (2018), 157–168.

[29] Almeida, R., Malinowska, A. B., & Odzijewicz, T., Fractional differential equations with dependence on the Caputo–Katugampola derivative, *Journal of Computational and Nonlinear Dynamics* **11 (6)** (2016), 061017.

Ranjini, M. C., Shabna, M. S., Shameema, V.
E-mail: ranjiniprasad@gmail.com
E-mail: shabnasaidh@gmail.com (corresponding author)
E-mail: shameemavelari@gmail.com

Department of Mathematics
MES Mampad College, Mampad
University of Calicut
Kerala, India.

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# A BRIEF HISTORY OF TOPOLOGICAL COMPLEXITY AND ITS VERSIONS

EKANSH JAUHARI

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Abstract. The field of topological complexity is relatively new, still, it has undergone tremendous progress since its inception. Having applications in robotics as well as deep connections with classical theories and problems in algebraic and geometric topology, it has attracted several brilliant minds to contribute to its progress and explore its beauty. This paper aims to chronologically survey the step-by-step development in the field of topological complexity. We explain how the theory of topological complexity was built over the years, what motivated the foundation of new ideas, how these ideas created new theories, and how these theories were used to solve problems and further research in the subject. We highlight various important advancements seen in this field by tracing some of the major contributions made all the way from its outset to this point (April 2024) and attempt to explain their significance in a broader sense.

## 1. Introduction

In the field of robotics, the ultimate aim is to develop autonomously functioning mechanical systems that can understand well-defined descriptions of tasks and execute them without any further human intervention. One of the simplest tasks is moving the system from a given initial position to a given final position. Of course, such positions are given in some "space", and the motion of the system must happen inside that space. Let us call this the *configuration space of the system*. For this task, the input for the system is the data of the initial position $A$ and the final position $B$, and the desired output is the "continuous" motion of the system from $A$ to $B$.

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Constructing a robot that can accomplish this task autonomously is called the *motion planning problem*.

Roughly speaking, the motion planning problem is equivalent to finding a “nice” algorithm that assigns to each given pair of positions $(A, B)$ a continuous motion of the system from $A$ to $B$. Ideally, such an algorithm with the least possible *complexity* is desired. Clearly, the configuration space inside which the system is required to move in a continuous fashion can influence the complexity of the above algorithm. That is where the motion planning problem becomes interesting to a topologist. Based on various topological features of the configuration space in question, one can hope to provide a topological measure of the complexity of the motion planning problem. This topological viewpoint on the motion planning problem was first given at the dawn of the 21-st century by Michael Farber. His work was inspired by the earlier work of Stephen Smale [64] and Victor Vasiliev [66] from the 1980s on the topological complexity of algorithms for solving polynomial equations. Farber introduced the notion of “topological complexity of a topological space”, developed its theory, and provided some examples and applications of his notion in [23].

Since then, topological complexity has become an active area of research in topology, and the field has seen significant progress in the last 24 years, with remarkable contributions from both theoretical topologists and engineers. The reasons are twofold. First, it provides a robust approach to attacking the general motion planning problem in robotics. This has led to progress in the fields of artificial intelligence and the control and kinematics of mechanical systems. Second, and perhaps more important to mathematicians, is the fact that topological complexity has deep connections with many classical subjects and problems in algebraic, geometric, and differential topology. Computing the topological complexity of a space turns out to be a very challenging task. As we will see in this paper, various techniques from algebraic topology are needed to estimate it. In fact, topological complexity has pushed a great deal of research in many branches of algebra, geometry, and topology, and has also encouraged the use of novel techniques coming from outside of topology.

This paper is organized as follows. We start by defining the notion of topological complexity ($\mathsf{TC}$) of spaces in Section 2 using the classical motion planning problem. Then, we recall a classical invariant, the Lusternik–Schnirelmann category, in Section 3 and explain how it relates to $\mathsf{TC}$. In Section 4, we mention some primary topological tools that are often helpful in computing $\mathsf{TC}$. We discuss some advanced motion planning problems in Section 5 and explain how the notion of $\mathsf{TC}$ is suitably modified for each of these problems. In Section 6, we survey some newer tools and techniques that were employed in computations of $\mathsf{TC}$. We then discuss a few more versions of $\mathsf{TC}$ in Section 7 that were motivated primarily by mathematical curiosity. In Section 8, we comprehensively survey the work between 2014 and 2019 on computing $\mathsf{TC}$ of several interesting classes of spaces. After this, we proceed to review various remarkable generalizations and extensions of the notion of $\mathsf{TC}$ in Section 9. We end this paper in Section 10 by summarizing some of the most recent advancements in the field of topological robotics that happened in the last year.

**Declaration.** The list of authors and their "major" contributions to the field of topological complexity given in this paper is not exhaustive. We do not claim that the historical account we give is complete (in fact, it is not)! It is impossible to include all important contributions to this field, so an attempt has been made to list some of the most remarkable ones that fit the narrative and flow of this article. The author apologizes for not mentioning any names or contributions that he should have mentioned.

As a convention, when introducing an author for the first time in this paper, we use their full (official) name. In subsequent mentions, we only use their last name.

## 2. Topological complexity

Smale's topological approach to the study of the complexity of algorithms for solving polynomial equations involved measuring the number of paths of computation that a "real number" machine requires to find a polynomial root. A few years after this, Jean-Claude Latombe provided a general theory of robot motion planning algorithms in 1991. Let $X$ denote the configuration space[1] of a robot and $X \times X$ be the space of all ordered pairs of positions $(x, y)$ in $X$. A *motion planning algorithm* is a function that assigns to each $(x, y)$ a continuous motion of the system from $x$ to $y$ inside $X$. Inspired, in part, by Latombe's work [50], Farber provided the

[1]All the topological spaces considered in this paper are path-connected.

modern formulation of the theory that formalizes the above motion planning problem for robots in the following way.

Let $P(X) = \{f \mid f : [0,1] \to X\}$ denote the path space of $X$. A motion planning algorithm on $X \times X$ is a continuous map $s : X \times X \to P(X)$ such that $s(x,y)(0) = x$ and $s(x,y)(1) = y$ for each $(x,y) \in X \times X$. A natural question is: when does such an algorithm exist? Unfortunately, not most of the time!

**Theorem 2.1** (Farber)**.** *A motion planning algorithm on $X \times X$ exists if and only if $X$ is contractible.*

Hardly any mechanical system in the real world has a contractible configuration space. Hence, a certain amount of discontinuity is inevitable in motion planning. So, instead of hoping for a *full* motion planning algorithm on $X \times X$, we can partition $X \times X$ into finitely many "nice" patches and hope for a partial algorithm on each one of them. To obtain the minimal degree of discontinuity or instability in the motion of a system with configuration $X$, we want to minimize the number of such partial algorithms. This idea brings to life Farber's topological complexity.

**Definition 2.2.** The *topological complexity* of a space $X$, denoted $\mathsf{TC}(X)$, is the smallest integer $n$ such that there exist open sets $U_1, U_2, \ldots, U_{n+1}$ that cover $X \times X$ and continuous maps $s_i : U_i \to P(X)$ satisfying $s_i(x,y)(0) = x$ and $s_i(x,y)(1) = y$ for each $(x,y) \in U_i$.

**Definition 2.3** (Relative version)**.** Given a subspace $A \subset X \times X$, the topological complexity of $A$ *relative to* $X$, denoted $\mathsf{TC}_X(A)$, is the smallest integer $n$ such that there exist sets $V_1, V_2, \ldots, V_{n+1}$ that are open in $A$ and cover $A$ and continuous maps $s_i : V_i \to P(X)$ satisfying $s_i(a,b)(0) = a$ and $s_i(a,b)(1) = b$ for each $(a,b) \in V_i$.

It is easy to see that $\mathsf{TC}_X(X \times X) = \mathsf{TC}(X)$.

**Remark 2.4.** We note that in the original definition [23], it was said that $\mathsf{TC}(X) \leq n$ if and only if there exist $n$ number of open sets $U_i$ that cover $X \times X$ and their corresponding partial motion planners. However, the above definition is the new convention in the modern study of $\mathsf{TC}$ where $n+1$ open sets are taken instead of $n$ open sets to achieve the criterion that $\mathsf{TC}(X) = 0$ if and only if $X$ is contractible. This is often described as the *normalized* form of topological complexity.

Intuitively, $\mathsf{TC}(X)$ is a measure of the discontinuity of motion planners in $X$. Unless $X$ is contractible, there is some discontinuity, i.e., $\mathsf{TC}(X) \geq 1$.

After this definition, the mathematical interest was to study this number $\mathsf{TC}$ and determine it for various classes of spaces. In his paper [23], Farber proved that $\mathsf{TC}$ is a homotopy invariant of spaces, meaning, if $X$ and $Y$ are homotopy equivalent spaces, then $\mathsf{TC}(X) = \mathsf{TC}(Y)$. This showed that $\mathsf{TC}$ depends only on the homotopy type of spaces and hence satisfies the basic desire of an algebraic topologist. But it turns out, it is *not* easy to determine $\mathsf{TC}(X)$ precisely, even for standard spaces.

**Example 2.5.** Consider $X = S^1$, the unit circle. Since $S^1$ is not contractible, $\mathsf{TC}(S^1) \geq 1$. Farber demonstrated in [23] that the open sets $U_1 = \{(x, y) \mid x \neq -y\}$ and $U_2 = \{(x, y) \mid x \neq y\}$ admit partial motion planners and hence satisfy the conditions of Definition 2.2, thereby implying $\mathsf{TC}(S^1) \leq 1$. This determines $\mathsf{TC}(S^1) = 1$. This idea works for all odd spheres $S^{2n-1}$ due to the existence of nowhere-vanishing vector fields. Therefore, $\mathsf{TC}(S^{2n-1}) = 1$ is obtained for all $n$.

In general, the above technique does not work, even for all spheres. For instance, Farber came up with three open sets that admit partial motion planners in the case of $X = S^2$. This gives the estimates $1 \leq \mathsf{TC}(S^2) \leq 2$ but does not determine the exact value. Explicitly constructing the smallest open cover of $X \times X$ that satisfies the conditions of Definition 2.2 is very difficult. So, tools and techniques from algebraic topology are needed to estimate $\mathsf{TC}(X)$ by bounding it from above and below.

## 3. Lusternik–Schnirelmann category

Let us take a small detour in history so that we can see the bigger picture later. In the 1920s, while studying the solutions of differential equations on manifolds, Lazar Lyusternik (1899–1981) and Lev Schnirelmann (1905–1938) aimed to provide a lower bound on the number of critical points of any smooth function on a given manifold. For this purpose, they introduced an invariant of a manifold in 1929, which they called its *category* [53]. The following is the modern formulation of category given by Ralph Fox (1913–1973) in 1941.

**Definition 3.1** ([35])**.** The *Lusternik–Schnirelmann category* (or LS-category) of space $X$, denoted $\mathsf{cat}(X)$, is the minimal number $n$ such that there exist open sets $U_1, U_2, \dots, U_{n+1}$ that cover $X$ and whose inclusion into $X$ is null-homotopic.

Let $\mathrm{Crit}(M)$ denote the minimum number of critical points of any smooth function on a manifold $M$. Lusternik and Schnirelmann proved in [54] that $\mathsf{cat}(M) + 1 \leq \mathrm{Crit}(M)$. Originally introduced due to motivations coming from analysis, LS-category had strong connections with geometry and algebraic topology. Soon after, it became the focus of the so-called "numerical invariants" movement in the 1950s. Historical development of the theory of LS-category has been traced down nicely in the classical text [10]. Indeed, computing $\mathsf{cat}(X)$ itself is not so easy. However, in many cases, it is slightly easier than computing $\mathsf{TC}(X)$.

**Example 3.2.** If $X = S^n$ is any sphere, then $X$ can be covered by its two open hemispheres that are contractible in $S^n$. So, $\mathsf{cat}(S^n) \leq 1$. On the other hand, since $S^n$ is not contractible, $\mathsf{cat}(S^n) \geq 1$. This gives $\mathsf{cat}(S^n) = 1$ for each $n$.

Farber realized that LS-category could be used to estimate topological complexity. He proved that $\mathsf{cat}(X) \leq \mathsf{TC}(X) \leq \mathsf{cat}(X \times X) \leq 2\,\mathsf{cat}(X)$, where the last inequality was shown in [35]. This can make things easy sometimes. For instance, if $X = S^n$ for any $n$, we get $1 \leq \mathsf{TC}(S^n) \leq 2$.

The example of odd spheres and the above inequality pose a natural question. For what classes of spaces $X$ does the equality $\mathsf{TC}(X) = \mathsf{cat}(X)$ hold? One such class was obtained by Farber in a subsequent work in 2002. It follows from [24, Lemma 8.2] that if $X$ is a topological group, then $\mathsf{TC}(X) = \mathsf{cat}(X)$. This gives a plethora of spaces whose $\mathsf{TC}$ can be computed, such as some Lie groups, because $\mathsf{cat}$ has been determined for a few classes of Lie groups.

Relating $\mathsf{cat}$ with $\mathsf{TC}$ was an important step. On one hand, it made it easier to estimate $\mathsf{TC}(X)$ for various spaces $X$, on the other hand, it emphasized how certain ideas from topology can turn out to be helpful in robotics. The analogues of some nice results that hold for $\mathsf{cat}$ were proven for $\mathsf{TC}$. For instance, $\mathsf{cat}(X \times Y) \leq \mathsf{cat}(X) + \mathsf{cat}(Y)$ holds. Motivated by this, it was shown in [23, 24] that $\mathsf{TC}(X \times Y) \leq \mathsf{TC}(X) + \mathsf{TC}(Y)$. Throughout this text, we will find various other such instances. But there was still a basic problem. While $\mathsf{cat}$ provided easier estimates, computing the precise value $\mathsf{TC}(X)$, even for standard spaces like the even spheres, was still a challenge.

## 4. Tools from algebraic and differential topology

To overcome the theoretical obstacles that occur in the computation of $\mathsf{TC}$ for standard spaces, certain tools and techniques from topology were employed by Farber.

4.1. **A lower bound to $\mathsf{TC}$.** Given a space $X$ and a ring $R$ with unity, the *cup-length* of the cohomology ring $H^*(X;R)$, denoted $c\ell_R(X)$, is the *maximum* length $k$ of a non-zero cup product $\alpha_1 \smile \cdots \smile \alpha_k$ of positive degree cohomology classes $\alpha_i \in H^*(X;R)$.

It is well-known in the literature that $c\ell_R(X) \leq \mathsf{cat}(X)$, see [10, Proposition 1.5]. If an analogous lower bound can be obtained for $\mathsf{TC}(X)$, then that might make things simpler. Motivated by this, Farber developed the following theory.

Let $\Delta : X \to X \times X$ denote the diagonal map. Call the cohomology classes of the ideal $\mathrm{Ker}(\Delta^* : H^*(X \times X;R) \to H^*(X;R))$ the zero-divisors of $X$ and let $zc\ell_R(X)$ denote the cup-length of $\mathrm{Ker}(\Delta^*)$. This is also called the *zero-divisor cup-length* of $X$.

**Theorem 4.1** (Farber)**.** *For any space $X$ and ring $R$, $zc\ell_R(X) \leq \mathsf{TC}(X)$.*

A proof can be found in [23, Theorem 7]. The idea for the proof is similar to that of the result $c\ell_R(X) \leq \mathsf{cat}(X)$. The significance of this result is that the knowledge of the cohomology rings $H^*(X)$ and $H^*(X \times X)$ from algebraic topology can help us get an estimate for $\mathsf{TC}(X)$ from below. Farber used this to get the following examples in [23].

**Example 4.2.**
(1) We have $zc\ell_{\mathbb{Q}}(S^{2n}) = 2$. Thus, $\mathsf{TC}(S^{2n}) = 2$.
(2) For closed orientable manifolds $\Sigma_g$ of genus $g \geq 2$, $zc\ell_{\mathbb{Q}}(\Sigma_g) = 4$. Using this, we get $4 \leq \mathsf{TC}(\Sigma_g) \leq 2\,\mathsf{cat}(\Sigma_g) = 4$.
(3) For the $n$-torus $T^n$, $n = zc\ell_{\mathbb{Q}}(T^n) \leq \mathsf{TC}(T^n) \leq n\,\mathsf{TC}(S^1) = n$.

Now, suppose one wants to compute $\mathsf{TC}(\mathbb{C}P^n)$ for the complex projective space $\mathbb{C}P^n$. Clearly, $2n = \mathsf{cat}(\mathbb{C}P^n) \leq \mathsf{TC}(\mathbb{C}P^n) \leq 2\,\mathsf{cat}(\mathbb{C}P^n) = 4n$. Furthermore, $zc\ell_{\mathbb{Q}}(\mathbb{C}P^n) = 2n$. So, the lower bound is not improved, and still the exact value $\mathsf{TC}(\mathbb{C}P^n)$ can't be determined. This suggests that more techniques need to be employed in the study of $\mathsf{TC}$.

4.2. **Schwarz genus.** To start this subsection, we take a step back to look at an invariant which generalizes the notion of $\mathsf{cat}$ and $\mathsf{TC}$. In the 1960s, Albert S. Schwarz was studying fibrations and fiber spaces. He came up with the notion of something called the *genus* of a fibration and developed that theory extensively in a series of papers. The following definition captures Schwarz's idea.

**Definition 4.3** ([62]). Given a fibration $p : E \to B$, its *Schwarz genus*, denoted $\mathsf{secat}(p)$, is the smallest integer $n$ such that there exist open sets $U_1, U_2, \dots, U_{n+1}$ that cover $B$ and over each of which there exists a partial section $\psi_i : U_i \to E$ of $p$.

The reason for the notation $\mathsf{secat}(p)$ is that in the modern convention, the term "Schwarz genus" is replaced by the term *sectional category* to prevent overuse of the word "genus".

While Schwarz's idea was coined decades before $\mathsf{TC}$, it turns out that it generalizes the notion of $\mathsf{TC}$ in the following way. Take the path fibration $\pi^X : P(X) \to X \times X$ defined as $\pi^X : \phi \mapsto (\phi(0), \phi(1))$. Then, $\mathsf{TC}(X) = \mathsf{secat}(\pi^X)$ by Definition 2.2, since in the case of $\pi^X$, the maps $\psi_i$ are just the partial motion planners. Therefore, general results from the work of Schwarz can be used to improve the theory of $\mathsf{TC}$. One such result is the following.

**Theorem 4.4** ([62, Theorem 5], [24, Theorem 5.2]). *If $X$ is an $r$-connected CW polyhedron, then*

$$\mathsf{TC}(X) < \frac{2\dim(X) + 1}{r + 1},$$

*where* $\dim(X)$ *denotes the dimension of* $X$.

For $X = \mathbb{C}P^n$ above, since $r = 1$ and $\dim(\mathbb{C}P^n) = 2n$, we immediately get $\mathsf{TC}(\mathbb{C}P^n) \leq 2n$. Combining with the lower bound, $\mathsf{TC}(\mathbb{C}P^n) = 2n$ is obtained. More generally, this gives the equality $\mathsf{TC}(M) = \dim(M)$ for closed, simply connected symplectic manifolds, see [33]. Furthermore, Theorem 4.1 is obtained as a corollary of a general result of Schwarz from [62]. So, a specific case of Schwarz's theory turned out to be instrumental in topological robotics. Subsequently, $\mathsf{TC}$ values were estimated for various spaces by several mathematicians.

4.3. **The peculiar case of $\mathbb{R}P^n$.** One of the most elementary problems in topological robotics is to study the motion planning problem of rotating a line in $\mathbb{R}^{n+1}$ that is fixed by a revolving joint at a base point, say the origin. This is equivalent to calculating $\mathsf{TC}(\mathbb{R}P^n)$. The obvious bounds are $n = \mathsf{cat}(\mathbb{R}P^n) \leq \mathsf{TC}(\mathbb{R}P^n) \leq 2\,\mathsf{cat}(\mathbb{R}P^n) = 2n$. In general, better lower bounds are difficult to obtain because $\mathbb{R}P^n$ does not have a long zero-divisor cup-length. Similar is the problem with the upper bounds because $\mathbb{R}P^n$ is not highly connected. In 2002, Farber, Sergei Tabachnikov, and Sergey Yuzvinsky determined that computing $\mathsf{TC}(\mathbb{R}P^n)$ is very difficult. Using Stiefel–Whitney classes and the theory of nonsingular and axial maps, they showed in [33] that for any $n$, the number $\mathsf{TC}(\mathbb{R}P^n)$ is the smallest integer $k$ such that there exists a nonsingular map $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$. For $n \neq 1, 3,$ and 7, this means that $\mathsf{TC}(\mathbb{R}P^n)$ is equal to the smallest $k$ such that $\mathbb{R}P^n$ can be *immersed* in $\mathbb{R}^k$. So, this deep result suggests that finding $\mathsf{TC}(\mathbb{R}P^n)$ is equivalent to finding the immersion dimension of $\mathbb{R}P^n$. The latter is a fairly challenging problem in differential topology. However, as corollaries, the authors succeeded in determining $\mathsf{TC}(\mathbb{R}P^n)$ for various special cases. In particular, they found $\mathsf{TC}(\mathbb{R}P^n)$ for all $n \leq 23$. This is just one of the many instances where one can see a close, perhaps somewhat unexpected, connection of $\mathsf{TC}$ with well-known problems in topology.

## 5. Some advanced motion planning algorithms

5.1. **Collision-free motion.** Soon after the notion of $\mathsf{TC}$ was developed for the most simplistic motion planning problem, mathematicians started looking for algorithms for more complex motion planning problems that are more likely to occur in physical scenarios for robots. One such situation is when $n$ number of robots are required to move simultaneously inside a configuration space from one point to another by avoiding collisions with each other. Farber and Yuzvinsky were the first to tackle this problem in [34] for Euclidean configuration spaces. Their approach was as follows. Consider $n$ ordered robots that are supposed to move inside $\mathbb{R}^m$. Define $C_n(\mathbb{R}^m)$ to be the configuration space of ordered sequences of $n$ *distinct* points in $\mathbb{R}^m$. Then the problem of constructing an algorithm for a collision-free motion of $n$ robots in $\mathbb{R}^m$ boils down to finding $\mathsf{TC}(C_n(\mathbb{R}^m))$. Seeing the union of the hyperplanes of $\mathbb{R}^m$ as the obstacles to be avoided, the authors proved that $\mathsf{TC}(C_n(\mathbb{R}^2)) = 2n-2$ and $\mathsf{TC}(C_n(\mathbb{R}^m)) = 2n-1$ whenever $m$ is odd. For even $m$, the generalization $\mathsf{TC}(C_n(\mathbb{R}^m)) = 2n-2$ was obtained later in 2008 by Farber and Mark Grant [26].

Further work on the collision-free motion for points on graphs and trees was continued by Farber in [25]. But perhaps what's more interesting is that in this paper, he introduced a *probabilistic* version of $\mathsf{TC}$. Given a pair $(x, y)$ of positions, Farber defined a random $n$-valued path from $x$ to $y$ as an ordered formal linear combination $\sigma = p_1\gamma_1 + \cdots + p_n\gamma_n$, where $\gamma_i \in P(X)$ is a path from $x$ to $y$ for each $i$ and $p_i \in [0, 1]$ are such that $p_1 + \cdots + p_n = 1$. The paths $\gamma_i$ can be viewed as *states* of $\sigma$ and the numbers $p_i$ represent the *probability* of $\sigma$ being in $\gamma_i$. If $P_n(X)$ denotes the space of all random $n$-valued paths, then an *$n$-valued motion planning algorithm* on $X$ can be defined as a continuous map $s : X \times X \to P_n(X)$ such that $s(x, y)$ is a random $n$-valued path from $x$ to $y$. The smallest integer $n$ such that an $(n+1)$-valued motion planning algorithm exists on $X$ can be called the *probabilistic topological complexity* of $X$. Farber showed [25, Theorem 6] that probabilistic $\mathsf{TC}$ agrees with the classical $\mathsf{TC}$ on many natural classes of spaces and used that, and the theory of subspace arrangements, to get some estimates for the number $\mathsf{TC}(F(\Gamma, n))$, which represents the complexity of collision-free motion of $n$ ordered points in the configuration graph $\Gamma$.

In an effort to extend the work on collision-free motion to more well-known configuration spaces, Daniel C. Cohen and Farber considered the case of closed orientable surfaces $\Sigma_g$ of genus $g \geq 0$ in 2009. As before, if $F(\Sigma_g, n)$ denotes the configuration space of $n$ distinct points in $\Sigma_g$, then using a certain description of the cohomology of configuration spaces of algebraic varieties, they showed in [3] that

$$\mathsf{TC}(F(\Sigma_g, n)) = \begin{cases} 2 & : g = 0 \text{ and } n \leq 2 \\ 2n-3 & : g = 0 \text{ and } n \geq 3 \\ 2n & : g = 1 \text{ and } n \geq 1 \\ 2n+2 & : g \geq 2 \text{ and } n \geq 2. \end{cases}$$

Some additional results for some suitably-punctured 2-spheres occurring as the configuration spaces were also obtained [3, Theorem 6.1] for collision-free motion.

5.2. **Symmetric motion.** To obtain a more controlled motion of the robots, mathematicians started imposing additional conditions on the motion planning algorithms. One such condition was imposed to get symmetry in the motion. The idea was that the motion of the system from $x$ to $y$ must be the motion of the system of $y$ to $x$ traversed in the opposite direction, so that the motion from $x$ to $x$ must be constant at $x$, i.e., the system should not move at all when its initial and final positions coincide. The study of *symmetric motion planners* that could achieve this task was initiated by Farber and Grant in 2006. Their theory is developed in [27] as follows.

Let $P'X$ denote the space of paths $\phi$ for which $\phi(0) \neq \phi(1)$ and let $F(X;2) = X \times X \setminus \Delta X$. Then $\mathbb{Z}_2$ acts on $F(X;2)$ by permutation of coordinates and on $P'X$ by sending a path to its reverse. Hence, the restriction of the evaluation fibration $\pi : P(X) \to X \times X$, denoted $\pi' : P'X \to F(X;2)$, is a $\mathbb{Z}_2$-equivariant map. If $\widetilde{\pi} : P''X \to B(X;2)$ is the map obtained after quotient by $\mathbb{Z}_2$, where $P''X := P'X/\mathbb{Z}_2$ and $B(X;2) := F(X;2)/\mathbb{Z}_2$, then the *symmetric topological complexity* of $X$, denoted $\mathsf{TC}^S(X)$, can be defined as $\mathsf{secat}(\widetilde{\pi})$.

While $\mathsf{TC}^S(X)$ calculates the complexity of the symmetric motion, it comes with a cost: $\mathsf{TC}^S$ is *not* a homotopy invariant, and $\mathsf{TC}^S(X) \geq \mathsf{TC}(X)$ holds, as is intuitively clear since symmetric algorithms are more difficult to obtain than arbitrary algorithms. The computation of $\mathsf{TC}^S(X)$ required usable lower bounds, like in the case of $\mathsf{TC}(X)$. By studying the cohomology rings of $P'X$ and $B(X;2)$ and using Schwarz's theory, Farber and Grant obtained a general cohomological lower bound to $\mathsf{TC}^S(X) - 1$ for closed smooth manifolds $X$. Using that, the authors proved that $\mathsf{TC}^S(S^n) = 2$ for each $n$, and in [27, Example 28] that $\mathsf{TC}^S(T^n) = 2n$ for the $n$-torus $T^n$, thereby showing that the notion of $\mathsf{TC}^S$ is different than that of $\mathsf{TC}$ because $\mathsf{TC}(T^n) = \mathsf{cat}(T^n) = n$.

Soon after, more computations for $\mathsf{TC}^S$ were done. In 2008, Jesús González and Peter Landweber decided to determine $\mathsf{TC}^S(\mathbb{R}P^n)$. They found that just as $\mathsf{TC}(\mathbb{R}P^n)$ relates to the immersion dimension of $\mathbb{R}P^n$, the symmetric version $\mathsf{TC}^S(\mathbb{R}P^n)$ relates to the *embedding dimension* of $\mathbb{R}P^n$. More precisely, using the concept of symmetric axial maps, the authors showed in [40] that $\mathsf{TC}^S(\mathbb{R}P^n)$ is equal to the Euclidean embedding dimension of $\mathbb{R}P^n$ when $n \geq 16$ or $n \in \{1, 2, 4, 8, 9, 13\}$. Here, the cases in $n$ occur because of André Haefliger's *metastable range* $n \geq 16$, while the other individual cases were proven case-by-case by the authors, with no definite conjecture for the left-out cases. This turned out to be yet another instance where motion planning in robotics turned out to be intimately connected with classical problems in topology.

5.3. **Sequential motion.** Before 2009, the motion planning problem and its versions were only considered for situations in which a robot has to move from an initial position to a final position. However, in more realistic scenarios, the robots are actually required to move between multiple pre-specified positions. In 2009, Yuli B. Rudyak (1948–2024) considered the following sequential motion planning problem. Given an autonomous mechanical system with configuration space $X$, a number $m \geq 2$, and positions $x_1, x_2, \dots, x_m$ in $X$, a "nice" algorithm needs to be constructed to get from $x_1$ to $x_m$ via the $m-2$ intermediate positions $x_2, \dots, x_{m-1}$ attained in that order for each ordered tuple $(x_1, x_2, \dots, x_m) \in X^m$ in an uniformly-timed motion. As a natural generalization to Farber's original idea, Rudyak's idea [59] for this sequential motion can be described as follows.

**Definition 5.1.** The $m$-th *sequential topological complexity* of a space $X$, denoted $\mathsf{TC}_m(X)$, is the smallest integer $n$ such that there exist open sets $U_1, U_2, \dots, U_{n+1}$ that cover $X^m$ and continuous maps $s_i : U_i \to P(X)$ that satisfy for all $1 \leq j \leq m$ the equality

$$s_i\left(x_1, x_2 \dots, x_m\right)\left(\frac{j-1}{m-1}\right) = x_j$$

for each $(x_1, \dots, x_m) \in U_i$.

We note that Rudyak's original definition was stated in a slightly different (equivalent) way, and he called $\mathsf{TC}_m(X)$ the $m$-th *higher* topological complexity of $X$. However, in modern convention, the term "higher" has been replaced by the term "sequential" to emphasize the sequential motion in robotics. When $m = 2$, this notion coincides with Farber's $\mathsf{TC}$. Thus, $\mathsf{TC}_2(X) = \mathsf{TC}(X)$ for any space $X$.

Equivalently, $\mathsf{TC}_m(X)$ can also be defined in terms of the sectional category (Schwarz genus). Let $\pi_m^X : P(X) \to X^m$ denote the fibration that evaluates at the points $(j-1)/(m-1)$ for all $1 \leq j \leq m$. Then, $\mathsf{TC}_m(X) = \mathsf{secat}(\pi_m^X)$. It is known that the fibration $\pi_m^X$ is homotopy equivalent to the diagonal map $\Delta_m : X \to X^m$. Hence, using Schwarz's result [62, Theorem 4], it can be proved that the cup-length of the ideal $\mathrm{Ker}(\Delta_m^* : H^*(X^m; R) \to H^*(X; R))$, called the $m$-th zero-divisor cup-length of $X$, is a lower bound to $\mathsf{TC}_m(X)$. Furthermore, [62, Theorem 5] can be used to deduce that if $X$ is $r$-connected, then

$$\mathsf{TC}_m(X) < \frac{m \dim(X) + 1}{r + 1}.$$

These generalize Theorems 4.1 and 4.4, respectively. In light of these results, it is natural to expect $\mathsf{TC}_m$ to satisfy most of the nice properties like $\mathsf{TC}$. In particular, $\mathsf{TC}_m$ is a homotopy invariant of spaces. In [59], Rudyak showed that $\mathsf{TC}_m(X) \leq \min\{\mathsf{cat}(X^m), \mathsf{TC}_{m+1}(X)\}$, and that if $X$ is a non-contractible CW complex of finite type, then $\mathsf{TC}_m(X) \geq m - 1$. Furthermore, by generalizing Farber's zero-divisor computation technique for spheres, Rudyak found that $\mathsf{TC}_m(S^{2n}) = m$ and $\mathsf{TC}_m(S^{2n-1}) = m - 1$ for all $n$ and $m$. The following year, the theory of $\mathsf{TC}_m$ was further developed by Ibai Basabe, Gonzalez, Rudyak, and Dai Tamaki in [2]. As was expected, it turned out that the theory of $\mathsf{TC}_m$ gives the panoramic view of the theory of $\mathsf{TC}$. The following is a summary of the new results that they proved for each $m \geq 2$.

- $\mathsf{cat}(X^{m-1}) \leq \mathsf{TC}_m(X)$. Equality holds if $X$ is a topological group.
- If $(X \times Y)^m$ is normal, then $\mathsf{TC}_m(X \times Y) \leq \mathsf{TC}_m(X) + \mathsf{TC}_m(Y)$.
- $\mathsf{TC}_m(S^{k_1} \times S^{k_2} \times \cdots \times S^{k_n}) = n(m-1) + l_n$, where $l_n$ is the number of even spheres.
- If $M$ is a closed simply connected symplectic manifold of (even) dimension $\dim(M)$, then $\mathsf{TC}_m(X) = m \dim(M)/2$.

In [2], the authors also extended the sequential versions to symmetric motion planning in an obvious way. Using the theory of equivariant sections and braid spaces, [2] provided an equivalent definition of $\mathsf{TC}^S_m$ in terms of $\mathsf{secat}$, and as a result, they proved the generalization $\mathsf{TC}^S_m(S^1) = 2(m-1)$.

## 6. Advancements in computing $\mathsf{TC}$

While new versions of $\mathsf{TC}$ were introduced and theoretically developed to accommodate different, more realistic motion problems in robotics, the development of more novel tools and techniques to determine the classical $\mathsf{TC}$ for various classes of spaces was ongoing.

### 6.1. $\mathsf{TC}$-weight.

At this point, we can agree that $\mathsf{cat}$ is a close cousin of $\mathsf{TC}$. So, one can obtain analogues of ideas that are used to determine $\mathsf{cat}(X)$ and hope to use them to determine $\mathsf{TC}(X)$. One such idea is something known as the *category weight* of a cohomology class of $X$. This was introduced by Edward Fadell (1926–2018) and Sufian Husseini (1929–2017) in 1992 and developed further, independently, by Rudyak in 1996 and Jeffrey Strom in 1997. Using category weight, Fadell and Husseini computed the

LS-category of lens spaces [22]. Motivated by this and a closely related notion of Rudyak's *strict* category weight [58] and Strom's *essential* category weight [65], Farber and Grant developed the theory of $\mathsf{TC}$-weight in 2007.

**Definition 6.1** ([28])**.** For an abelian group $G$ and a fibration $p : E \to B$, the *weight of a cohomology class* $u \in H^*(B;G)$ with respect to $p$, denoted $\mathsf{wgt}_p(u)$, is the smallest integer $k$ such that for any $f : Y \to B$ with $\mathsf{secat}(f^\star p) \leq k$, the image $f^*(u) = 0 \in H^*(Y;B)$.

Here, $f^\star$ denotes the pullback map of $f$ along $p$. This generalizes Rudyak's strict category weight for $\mathsf{cat}$. In particular, when the fibration is $\pi : P(X) \to X \times X$, one gets $\mathsf{wgt}_\pi(u)$, called the $\mathsf{TC}$-weight of $u \in H^*(X \times X;G)$ and denoted $\mathsf{wgt}(u)$. In [28], the authors proved various properties of $\mathsf{wgt}$, all of which are similar to those of (strict) category weight [22, 58].

- $\mathsf{wgt}(u) \geq 1$ if and only if $u$ is a zero-divisor.
- If $u \neq 0$ has positive degree, then $\mathsf{TC}(X) \geq \mathsf{wgt}(u)$.
- If $u, v \in H^*(X \times X)$, then $\mathsf{wgt}(u \smile v) \geq \mathsf{wgt}(u) + \mathsf{wgt}(v)$.

When combined, these properties imply that if $u_i \in H^*(X \times X;G)$ are positive degree zero-divisors for all $1 \leq i \leq n$ for some $n$ such that $u := u_1 \smile u_2 \smile \cdots \smile u_n \neq 0$, then

$$\mathsf{TC}(X) \geq \mathsf{wgt}(u) \geq \sum_{i=1}^{n} \mathsf{wgt}(u_i) \geq n.$$

This recovers Theorem 4.1. Because of these properties, finding "heavier" non-zero cohomology classes becomes the primary objective. Using the concept of *stable cohomology operations* and their *excess*, Farber and Grant found a general criterion for classes $u$ to have $\mathsf{wgt}(u) \geq 2$. The authors wanted to determine $\mathsf{TC}$ of lens spaces. So, first, they studied motion planning in fiber spaces. Using homotopy theoretic arguments, they proved that if $p : E \to B$ is a (Hurewicz) fibration with fiber space $F$, then $\mathsf{TC}(E) \leq (\mathsf{TC}(F)+1)(\mathsf{cat}(B \times B)+1)-1$, which is an analogue of the classical result for $\mathsf{cat}$ (see [10, Theorem 1.41]). In the case of lens spaces $L_m^{2n+1}$ for any $n \geq 1$ and $m \geq 2$, due to the fibration $S^1 \to L_m^{2n+1} \to \mathbb{C}P^n$, the estimate $\mathsf{TC}(L_m^{2n+1}) \leq 4n+1$ is obtained from the above inequality. Using classes of weight at least 2, Farber and Grant proved that the above inequality is equality in several situations depending on the combinatorics of $n$ and $m$, see [28, Theorems 12, 13, 14] for details. In particular, the equality holds when $n = 1$ and $m \geq 3$. This improved a previous work of González.

6.2. $\mathsf{TC}$ **of special spaces.** Let $\Gamma$ be a discrete group and let $B\Gamma$ denote its classifying space, which is unique up to homotopy equivalence. Since $\mathsf{TC}_m$ is a homotopy invariant of spaces, it makes sense to define $\mathsf{TC}_m(\Gamma) := \mathsf{TC}_m(B\Gamma)$. In other terminology, $B\Gamma$ are aspherical spaces, which are the Eilenberg–Mac Lane spaces of type $K(\Gamma, 1)$, where $\Gamma$ is the fundamental group of the space. Aspherical spaces are important objects of study in geometric topology. Many mathematicians studied $\mathsf{TC}(\Gamma)$. In the case of LS-category, there is a nice algebraic description of $\mathsf{cat}(\Gamma)$ in terms of its cohomological dimension[2], denoted $\mathsf{cd}(\Gamma)$, due to the Eilenberg–Ganea theorem: $\mathsf{cat}(\Gamma) = \mathsf{cd}(\Gamma)$. But such a description is not known in the case of $\mathsf{TC}(\Gamma)$. So, to determine $\mathsf{TC}(\Gamma)$ values, work needed to be done to improve the obvious estimates $\mathsf{cd}(\Gamma) \leq \mathsf{TC}(\Gamma) \leq \mathsf{cat}(\Gamma \times \Gamma) \leq 2\,\mathsf{cd}(\Gamma)$. For torsion-free nilpotent groups, the upper bound was improved by Grant. In 2011, he showed [41] that if $\mathcal{Z}$ denotes the center of $\Gamma$, then we have the inequality $\mathsf{TC}(\Gamma) \leq \mathsf{cat}((\Gamma \times \Gamma)/\mathcal{Z}) - 1$. Additionally, if $\Gamma$ is finitely generated, then $\mathsf{TC}(\Gamma) \leq 2\,\mathsf{rank}(\Gamma) - \mathsf{rank}(\mathcal{Z})$.

After better upper bounds were obtained for the above special class of groups $\Gamma$, some good lower bounds for $\mathsf{TC}(\Gamma)$ were needed, ideally for general $\Gamma$. In 2013, Grant, Gregory Lupton, and John Oprea obtained algebraic lower bounds for $\mathsf{TC}(\Gamma)$ that were better than the crude lower bound $\mathsf{cd}(\Gamma)$. They showed that if $A$ and $B$ are subgroups of $\Gamma$ such that $gAg^{-1} \cap B$ is trivial for each $g \in \Gamma$, then $\mathsf{cd}(A \times B) \leq \mathsf{TC}(\Gamma)$, see [43, Section 5]. This happens, in particular, if $\Gamma$ is a semi-direct product of $A$ and $B$. Their work recovered the lower bounds that were obtained earlier in 2007 by Cohen and Goderdzi Pruidze [6] for the special case of *right-angled Artin groups*. Grant–Lupton–Oprea used their result to show in [43, Theorem 4.1] that the topological complexity of *Higman's group* is 4, which is two times its cohomological dimension.

Another special class of spaces consists of $H$-spaces, which generalize the notion of topological groups. While it was known that $\mathsf{TC}_{m+1}(X) = \mathsf{cat}(X^m)$ for topological groups $X$, the same equality for CW $H$-spaces was obtained by Lupton and Jérôme Scherer in 2011. Their arguments [52]

[2]For a discrete group $\Gamma$, its cohomological dimension, denoted $\mathsf{cd}(\Gamma)$, is defined to be the largest integer $k$ such that $H^k(\Gamma, A) \neq 0$ for some $\mathbb{Z}\Gamma$-module $A$.

relied primarily on the interplay of the notion of $\mathsf{secat}$ with pullbacks, and the understanding of the set $[A, X]$ of pointed homotopy classes between any CW complex $A$ and CW $H$-space $X$ as an algebraic loop. They further showed that the equality $\mathsf{TC}_{m+1}(X \times Y) = \mathsf{TC}_{m+1}(X) + \mathsf{TC}_{m+1}(Y)$ holds if and only if $\mathsf{cat}((X \times Y)^m) = \mathsf{cat}(X^m) + \mathsf{cat}(Y^m)$.

Another interesting, broad class is that of smooth manifolds. Grant figured in 2011 that if a closed smooth manifold $X$ admits a locally smooth action of a compact Lie group $G$, then better estimates for $\mathsf{TC}(X)$ can be obtained. Let $F(G, X) = \{a \in X \mid g \cdot a = a \text{ for all } g \in G\}$ denote the fixed-point set of $G$, let $P$ denote the principal orbit, and $G_a$ denote the stabilizer group of $a \in X$. Grant showed in [41, Theorem 5.2] that if for all pairs $(x, y) \in X \times X$, either $F(G_x \cap G_y, X)$ is path-connected or $x, y \in F(G, X)$, then $\mathsf{TC}(X) \leq 2\dim(X) - \dim(P)$. The author also obtained better upper bounds for some *nice* normal spaces $X$. He showed in [41, Theorem 4.3] that if $q : X \times X \to Y$ is closed with $Y$ paracompact and if $\mathsf{TC}_X(q^{-1}(y)) \leq n$ for each $y \in Y$, then $\mathsf{TC}(X) \leq n(\dim(Y) + 1) - 1$.

As the theory of topological complexity was developing and more computations were being done, a natural question emerged. What can be said about a non-contractible space that has the least possible $\mathsf{TC}_m$ values? More precisely, if $\mathsf{TC}_m(X) = m - 1$ for some $m \geq 2$, then what can be said about $X$? This was an interesting question because it was known that if $\mathsf{cat}(X) = 1$ (the least possible for a non-contractible space), then $X$ is homotopic to a co-$H$-space, [10]. In 2012, Grant, Lupton, and Oprea answered this question completely for the case $m = 2$ and partially for other cases. The authors proved that $\mathsf{TC}(X) = 1$ if and only if $X$ is homotopic to an odd sphere [42, Corollary 3.5]. Upon a deeper assessment of certain cup products in the tensor product ring $H^*(X)^{\otimes m}$, the authors characterized spaces $X$ with $\mathsf{TC}_m(X) = m - 1$ based on their fundamental groups. It turns out [42, Theorem 3.4] that if $X$ is simply connected, then $X$ is homotopic to an odd sphere. If not, then in general, nothing can be said. But if additionally, $X$ is a nilpotent space or a co-$H$-space, then $X$ must be homotopic to $S^1$. This meant, in particular, that if $X$ is co-$H$-space that is not homotopic to any odd sphere, then $\mathsf{TC}_m(X) = m$, such as in the case of even spheres.

## 7. More versions of topological complexity

Alongside significant progress in the theory of $\mathsf{TC}$ and more generally, the theory of $\mathsf{TC}_m$, some mathematicians were looking for special versions of topological complexity. While some of these versions were motivated by robot motion planning problems, others came into existence due to pure mathematical curiosity and later turned out to be helpful in finding some features for the classical $\mathsf{TC}$.

7.1. **Monoidal complexity.** In 2012, Norio Iwase and Michihiro Sakai decided to relax the condition in the symmetric motion planning (described in Section 5.2) that the system necessarily moves from position $y$ to $x$ through the same route it traveled from $x$ to $y$ taken in the opposite direction. They defined a more general version that only needed the system to remain stationary when the initial and final positions were the same.

**Definition 7.1** ([46])**.** The *monoidal topological complexity* of a space $X$, denoted $\mathsf{TC}^{\mathcal{M}}(X)$, is the smallest integer $n$ such that $X \times X$ can be covered by open sets $U_1, \ldots, U_{n+1}$ each of which contain the diagonal $\Delta X$ and admit a continuous (partial) section $s_i : U_i \to P(X)$ to $\pi : P(X) \to X \times X$ such that $s_i(x, x) = c_x$, where $c_x$ denotes the constant loop at $x$.

Like $\mathsf{TC}^S$, the new notion $\mathsf{TC}^{\mathcal{M}}$ is also *not* a homotopy invariant of spaces in general. However, the authors argued that if $(X \times X, \Delta X)$ is a neighborhood deformation pair, then unlike $\mathsf{TC}^S$, $\mathsf{TC}^{\mathcal{M}}$ is a homotopy invariant. It is clear from the definition that $\mathsf{TC}(X) \leq \mathsf{TC}^{\mathcal{M}}(X) \leq \mathsf{TC}^S(X)$. What the authors showed in [46, Section 5] is that if $X$ is a finite simplicial complex, then $\mathsf{TC}(X) = \mathsf{TC}^{\mathcal{M}}(X)$. Moreover, they conjectured that this equality holds for all "nice" spaces $X$.

Very soon, a step towards this conjecture was taken by Alexander Dranishnikov. First, he argued in [14] that the condition that the open sets in the cover of $X \times X$ in Definition 7.1 contain the diagonal $\Delta X$ is unnecessary. Then, he proved that $\mathsf{TC}^{\mathcal{M}}(X) \leq \mathsf{TC}(X) + 1$ when $X$ is a Euclidean neighborhood retract (ENR). Dranishnikov obtained a Schwarz-type characterization for $\mathsf{TC}^{\mathcal{M}}$, and using it, he proved that whenever $X$ is an $r$-connected simplicial complex satisfying $\dim(X) \leq (r + 1)\,\mathsf{TC}(X) + r$, then $\mathsf{TC}^{\mathcal{M}}(X) = \mathsf{TC}(X)$, see [14, Theorem 2.5]. Further, he proved using simple arguments that equality also holds for topological groups. The main result of his paper was a nice upper bound to the monoidal complexity of the wedge sum of ENR spaces: $\mathsf{TC}^{\mathcal{M}}(X \vee Y) \leq \mathsf{TC}^{\mathcal{M}}(X) + \mathsf{TC}^{\mathcal{M}}(Y)$. In the case of LS-category, it holds that if $p : E \to B$ is a covering map, then $\mathsf{cat}(E) \leq \mathsf{cat}(B)$, see, for example, [10, Corollary 1.45]. Whether such a relationship exists in the case of $\mathsf{TC}$ was unknown till 2012. Using the upper bound theorem for $\mathsf{TC}^{\mathcal{M}}(X \vee Y)$, Dranishnikov gave examples in [14, Theorem 3.8] to show that $\mathsf{TC}(E)$ and $\mathsf{TC}(B)$ can be unrelated in general. This is one of the many instances where one observes that $\mathsf{TC}$ need not satisfy analogues of all the properties that $\mathsf{cat}$ satisfies.

7.2. **Equivariant and invariant complexities.** In the same year, Hellen Colman and Grant were studying the topological complexity of spaces that admit continuous actions of compact Hausdorff topological groups. To deal with the $\mathsf{TC}$ of such spaces, the authors defined a more general notion in [9]. The *equivariant sectional catgeory* of a $G$-fibration $p : E \to B$, denoted $\mathsf{secat}_G(p)$, is the smallest integer $n$ such that $B$ can be covered by $n+1$ number of invariant open sets $U_i$ over each of which there exists a partial $G$-section $s_i : U_i \to E$ of $p$. Then, the *equivariant topological complexity* of a $G$-space $X$, denoted $\mathsf{TC}_G(X)$, was defined to be $\mathsf{secat}_G(\pi^X)$. Similarly, the *equivariant LS-catgeory* of $G$-space $X$, denoted $\mathsf{cat}_G(X)$, was defined. Colman and Grant developed the theory of $\mathsf{secat}_G$ by establishing in the equivariant setting analogues of classical results and properties that hold for $\mathsf{secat}$, $\mathsf{TC}$, and $\mathsf{cat}$. In particular, they proved that $\mathsf{TC}_G(X) = \mathsf{cat}(X)$ if $G$ acts freely on $X$. The authors provided various examples.

**Example 7.2.** If $G = \mathbb{Z}_2$ acts on $X = S^n$ by multiplying $-1$ in the last coordinate, then $\mathsf{TC}_G(S^1)$ is infinite while $\mathsf{TC}_G(S^n) = 2$ for all $n \geq 2$. So, the gap between $\mathsf{TC}_G$ and $\mathsf{TC}$ can be arbitrary. However, if $X = S^1 = G$ acts on itself by rotations, then $\mathsf{TC}_G(S^1) = 1$.

Wojciech Lubawski and Waclaw Marzantowicz realized that the impact of symmetry on a motion planning algorithm might not be analyzed fully by merely studying equivariant topological complexity. As a result, they developed their own notion in 2013 as follows. Given a $G$-space $X$, define $P^*X = \{(\phi,\psi) \mid G\phi(0) = G\psi(1)\}$ to be a $(G \times G)$-space with componentwise action. Then the map $\widehat{\pi} : P^*X \to X \times X$ given by $\widehat{\pi} : (\phi,\psi) \to (\phi(0),\psi(1))$ is a $G$-fibration. So, the *invariant topological complexity* of $X$, denoted $\mathsf{TC}^G(X)$, can be defined as the smallest integer $n$ such that there exist $n+1$ open $(G \times G)$-sets $U_i$ that cover $X \times X$ and over each of which there exists a partial $(G \times G)$-section of $\widehat{\pi}$. The authors explained in [51] that one of the main advantages $\mathsf{TC}^G$ has over $\mathsf{TC}_G$ is that in the case of a free action of $G$ on $X$, the equality $\mathsf{TC}^G(X) = \mathsf{TC}(X/G)$ holds between the invariant complexity of $X$ and the (non-invariant) complexity of its resulting orbit space $X/G$, whereas in the equivariant setting, $\mathsf{TC}_G(X) = \mathsf{TC}(X/G)$ might *not* hold even in simple cases.

Using a Clapp–Puppe invariant of Lusternik–Schnirelmann-type and a Whitehead-type characterization, the authors developed the theory of $\mathsf{TC}^G$ and showed that it satisfied properties like the homotopy invariance and the product inequality of $\mathsf{TC}^G$, and that it can be estimated suitably from the corresponding *invariant version* of LS-category, denoted $\mathsf{cat}^G$. The authors obtained the following examples in [51] to show where $\mathsf{TC}^G$ and $\mathsf{TC}_G$ disagree.

- If $G$ acts on itself by left-translation, then $\mathsf{TC}^G(G) = \mathsf{TC}(G/G) = 0$ is obtained since the action is free, whereas, $\mathsf{TC}_G(G) = \mathsf{cat}(G)$.
- If $G = \mathbb{Z}_2$ acts on $X = S^n$ by multiplying $-1$ in the last coordinate, then $\mathsf{TC}^G(S^n) = 1$ whenever $n$ is even, whereas $\mathsf{TC}_G(S^n) = 2$ for even $n$.

## 8. Progress between 2014–2019

Remarkable progress has been made in the field of topological complexity in the last 10 years. This started right from the first half of the last 10 years: 2014–2019. Progress was in terms of the development of new theories, the advancement of existing theories, as well as computations of $\mathsf{TC}$ values for various interesting spaces. Most of these developments were happening in parallel, so we might not see them listed here chronologically.

8.1. **$\mathsf{TC}$ of non-orientable surfaces.** Notice how the $\mathsf{TC}$ of closed orientable surfaces was computed easily in Section 4.2, but we never talked about $\mathsf{TC}$ of closed non-orientable surfaces $N_g$ of genus $g \geq 1$. For $g = 1$, $\mathsf{TC}(\mathbb{R}P^2) = 3$ was known and so were the estimates $3 \leq \mathsf{TC}(N_g) \leq 4$ for $g \geq 2$. But over the course of years, it turned out that computing $\mathsf{TC}(N_g)$ for $g \geq 2$ was very challenging. In particular, one could not directly use the orientable covers of a given $N_g$ to determine $\mathsf{TC}(N_g)$, as is typically done in the case of $\mathsf{cat}$ to determine $\mathsf{cat}(N_g) = 2$, because a covering space inequality does not hold for $\mathsf{TC}$ due to the work of Dranishnikov.

In 2011–2012, José Manuel García-Calcines and Lucile Vandembroucq started studying the homotopy cofiber $C_{\Delta X}$ of the diagonal map $\Delta_X : X \to X \times X$. They found in [36] that the LS-category of $C_{\Delta X}$ bounds $\mathsf{TC}(X)$ from below for some CW complexes, and equality holds for $H$-spaces, $\Sigma_g$, finite graphs, $\mathbb{R}P^n$, etc. The number $\mathsf{cat}(C_{\Delta X})$ was also studied by Dranishnikov. He continued on this work for $N_g$ and in [15, Section 3], he showed that $\mathsf{cat}(C_{\Delta N_g}) = 3$. But what's more remarkable was that he proved using advanced obstruction theory and sheaf theory that $\mathsf{TC}(N_g) = 4$ when $g \geq 4$, see [15, Section 5]. On one hand, this provided a counter-example to the conjecture that $\mathsf{cat}(C_{\Delta X}) = \mathsf{TC}(X)$ for finite CW complexes; on the other hand, it left only the values $\mathsf{TC}(N_2)$ and $\mathsf{TC}(N_3)$ undetermined. Soon after, Dranishnikov published a note giving another proof of the fact that $\mathsf{TC}(N_g) = 4$ when $g \geq 4$. In [16], he showed $\mathsf{TC}(M\#N) \geq \mathsf{TC}(N)$ for closed surfaces $M$ and $N$ where $M$ is orientable. His result, that $\mathsf{TC}(N_g) = 4$ when $g \geq 4$, was obtained from this inequality and the primary obstructions of $T \vee K$ for the 2-torus $T$ and the Klein bottle $K$.

The values $\mathsf{TC}(N_2)$ and $\mathsf{TC}(N_3)$ were still undecided. Dranishnikov had already shown that his methods for $g \geq 4$ can't extend to the cases $g = 2, 3$ [16, Section 3]. Partial progress for the interesting case of the Klein bottle ($g = 2$) was made by Donald M. Davis in [11] by appealing directly to the simplicial structure of $K$. But the values $\mathsf{TC}(N_2)$ and $\mathsf{TC}(N_3)$ were finally determined as 4 only by Cohen and Vandembroucq towards the end of 2016. In [7], the authors used obstruction theory and computations of zero-divisors cup-lengths with local coefficients to get their result, thereby finally completing the work on the topological complexity of $N_g$ for all $g \geq 1$.

8.2. **Sequential $\mathsf{TC}$ of surfaces and $\mathbb{R}P^n$.** All this while, mathematicians were equally interested in generalizing their results and computations for the sequential versions of $\mathsf{TC}$. While the $\mathsf{TC}$ values for all closed surfaces were known, work was to be done to compute their $\mathsf{TC}_m$ values for $m \geq 3$. In 2015, González, Bárbara Gutiérrez, Aldo Guzmán, Cristhian Hidber, María Mendoza, and Christopher Roque determined $\mathsf{TC}_m(\Sigma_g)$ values for all $m$ and $g$ [38]. They improved the work of [6] and found that $\mathsf{TC}_m(\Sigma_g) = 2m$ for $g, m \geq 2$, thereby recovering the result for $\mathsf{TC}(\Sigma_g)$. In the same year, González, Bárbara Gutiérrez, Darwin Gutiérrez, and Adrian Lara were studying the topological complexity of real-flag manifolds. Using their mod-2 cohomology, the authors gave a minimal presentation of the cohomology ring for *semi-complete* flag manifolds and used it to determine $\mathsf{TC}$ for some particular cases. But more importantly, they extended their work to bound $\mathsf{TC}_m$ values of some semi-complete real flag manifolds from below [37, Section 5] and to fully compute $\mathsf{TC}_m(N_g)$. They proved in [37, Proposition 6.1] that $\mathsf{TC}_m(N_g) = 2m$ for all $m \geq 3$ and $g \geq 1$. Surprisingly, computing $\mathsf{TC}_m(N_g)$ for $m \neq 2$ turned out to be much easier than computing $\mathsf{TC}_2(N_g)$.

While the exact values $\mathsf{TC}_m(\mathbb{R}P^2)$ and a good description of $\mathsf{TC}(\mathbb{R}P^n)$ were known, it was still difficult to estimate $\mathsf{TC}_m(\mathbb{R}P^n)$ for $n, m \geq 3$, let alone compute them. Building upon earlier works on real projective spaces and their topological complexities, in 2017, Davis obtained an explicit formula for the best lower bound for $\mathsf{TC}_m(\mathbb{R}P^n)$. This was implied by the classical mod-2 cohomology and helped obtain precise values of the $m$-th zero-divisor cup-lengths of $\mathbb{R}P^n$ for various small values of $m$ and $n$, see [12, Table 1]. Davis also gave the examples when $\mathsf{TC}_m(\mathbb{R}P^n)$ had their maximum possible values $mn$. In a subsequent work in 2018, he obtained the lower bounds using Brown–Peterson cohomology of $\mathbb{R}P^n$. Davis showed [13] that these new lower bounds were significantly stronger than those obtained earlier from the mod-2 cohomology. This led him to compute $\mathsf{TC}_m(\mathbb{R}P^n)$ for many more cases depending on the combinatorics of $n$ and $m$.

8.3. **(Sequential) topological complexity of aspherical spaces.** After improvements in the estimates of $\mathsf{TC}$ of aspherical spaces, more computations were done and theories were developed. Some sub-classes of particular interest have been those of right-angled Artin groups (RAAG) and braid groups. In 2016, González, Gutiérrez, and Hugo Mas studied sequential motion planning on random RAAG. For a random RAAG $\Gamma$, they obtained precise values $\mathsf{TC}_m(K(\Gamma, 1))$ in [39, Section 3]. Their work also described $\mathsf{TC}_m(K(\Gamma, 1))$ *asymptotically* for any $K(\Gamma, 1)$ associated to random graph groups. In the same year, Grant and David Recio-Mitter focused on the topological complexity of some subgroups of Artin's braid groups. For any $n$, the *full* braid group $B_n$, the *pure* braid group $P_n$, and the symmetric group $S_n$ fit into $1 \to P_n \to B_n \xrightarrow{\tau} S_n \to 1$. If $G \subset S_n$ is any subgroup of $S_n$, then $B_n^G := \tau^{-1}(G)$ has cohomological dimension $n-1$. So, $(m-1)(n-1) \leq \mathsf{TC}_m(B_n^G) \leq m(n-1)$. But, the authors showed that for some *special* subgroups $G$ that make $B_n^G$ the subgroups of *mixed* braids, the upper bound can be improved to $\mathsf{TC}_m(B_n^G) \leq m(n-1) - 1$, see [45, Theorems 1.1 and 6.3] for details. In fact, it turns out that equality holds if $G < S_{n-2} \times \{1\}^2$. Furthermore, if $G < S_{n-k} \times S_k$ for any $k \geq 2$, then $m(n-1) - k + 1 \leq \mathsf{TC}(B_n^G) \leq m(n-1) - 1$.

Among all this progress for special aspherical spaces $X$, the problem of describing $\mathsf{TC}(X)$ in terms of the algebraic invariants of their fundamental group $\Gamma := \pi_1(X)$, discussed briefly in Section 6.2, was not yet settled. While it was known that $\mathsf{cd}(\Gamma) \leq \mathsf{TC}(\Gamma) \leq 2\,\mathsf{cd}(\Gamma)$, Rudyak [60] had shown in 2013 that obtaining better lower or upper bounds is not always possible in general since $\mathsf{TC}(\Gamma) := \mathsf{TC}(X)$ can attain any integer value between $\mathsf{cd}(\Gamma)$ and $2\,\mathsf{cd}(\Gamma)$. A major step in describing $\mathsf{TC}(X)$ was taken in 2017 by Farber and Stephen Mescher. For finite CW aspherical spaces $X$, this can be done if $\Gamma$ is a hyperbolic group (in the sense of Gromov). In that case, the lower bound $\mathsf{cd}(\Gamma) \leq \mathsf{TC}(\Gamma)$ can be significantly improved [30, Section 9] to get $\mathsf{cd}(\Gamma \times \Gamma) - 1 \leq \mathsf{TC}(\Gamma) \leq \mathsf{cd}(\Gamma \times \Gamma)$. A few years later, using further tools from geometric topology and geometric group theory for torsion-free hyperbolic groups, Dranishnikov proved in [17] that the above upper bound is sharp. The result of Farber and Mescher led to another important computation. Till 2017, $\mathsf{TC}$ was computed only for simply connected symplectic manifolds. Good estimates of $\mathsf{TC}(M)$ were not known for *symplectically aspherical manifolds* that are well-known, important objects in differential topology and geometry. Using their above result and the notion of $\mathsf{TC}$-weight, the authors proved in [30, Theorem 8] that $2\dim(M) - 1 \leq \mathsf{TC}(M) \leq 2\dim(M)$ for such manifolds $M$.

While mathematicians were gradually getting to know more about $\mathsf{TC}(\Gamma)$, little was known as to what happens to the $\mathsf{TC}$ values under some well-known operations of discrete groups. In particular, how does the topological complexity of the free product $\Gamma_1 * \Gamma_2$ depend on the numerical invariant values of $\Gamma_1$ and $\Gamma_2$? In 2017, Dranishnikov and Rustom Sadykov tackled this problem. They proved in [20, Theorem 6] that $\mathsf{TC}(X \vee Y) = k$ if complexes $X$ and $Y$ satisfy $\dim(X \vee Y) < k := \max\{\mathsf{TC}(X), \mathsf{TC}(Y), \mathsf{cat}(X \times Y)\}$. In particular, this means that if $X$ and $Y$ are aspherical with $\pi_1(X) = \Gamma_1$ and $\pi_1(Y) = \Gamma_2$, then $\mathsf{TC}(\Gamma_1 * \Gamma_2) = \max\{\mathsf{TC}(\Gamma_1), \mathsf{TC}(\Gamma_2), \mathsf{cd}(\Gamma_1 \times \Gamma_2)\}$. Thus, for infinite cyclic groups, the authors deduced that $\mathsf{TC}(\mathbb{Z}^m * \mathbb{Z}^n) = m + n$, thereby recovering Rudyak's result from [60].

A breakthrough in the theory of $\mathsf{TC}$ of finite CW aspherical spaces was made in 2018 by Farber, Grant, Lupton, and Oprea. They reformulated the problem of determining $\mathsf{TC}(\Gamma)$ by giving a formal, almost-algebraic description of $\mathsf{TC}(\Gamma)$, thereby allowing the use of tools from group theory and homological algebra in dealing with $\mathsf{TC}(\Gamma)$. Their theory is developed as follows. Let $\Gamma$ be a discrete group and $E_n(\Gamma)$ denote the join of its $(n+1)$-copies such that $E_n(\Gamma)$ is equipped with the canonical left-diagonal action of $\Gamma \times \Gamma$. Since $E_n(\Gamma)$ embeds equivariantly in $E_{n+1}(\Gamma)$, one can define the direct limit $E(\Gamma) = \cup_n E_n(\Gamma)$. Similarly, the classifying space $E(\Gamma \times \Gamma)$ of all the spaces that admit some free $(\Gamma \times \Gamma)$-actions is formed such that it also inherits the above diagonal action of $\Gamma \times \Gamma$. Further, the $(\Gamma \times \Gamma)$-equivariant map $f : \Gamma \times \Gamma \to \Gamma$ defined as $f : (x, y) \mapsto xy^{-1}$ extends naturally to $F : E(\Gamma \times \Gamma) \to E(\Gamma)$. Let $X$ be an aspherical finite CW complex with fundamental group $\Gamma$. Then the following two characterizations were obtained for $\mathsf{TC}(\Gamma)$.

**Theorem 8.1** ([29, Theorem 2.1]). $\mathsf{TC}(\Gamma)$ *is the smallest integer* $n$ *such that there exists an* $(\Gamma \times \Gamma)$*-equivaraint map* $E(\Gamma \times \Gamma) \to E_n(\Gamma)$.

**Theorem 8.2** ([29, Theorem 3.1]). $\mathsf{TC}(\Gamma)$ *is the smallest integer* $n$ *for which there exists a* $(\Gamma \times \Gamma)$*-CW complex* $L$ *with* $\dim(L) = n$ *such that the induced map* $F : E(\Gamma\times\Gamma) \to E(\Gamma)$ *can be factored as* $E(\Gamma\times\Gamma) \to L \to E(\Gamma)$.

Furthermore, by defining a special class of subgroups of $\Gamma \times \Gamma$ and using Bredon cohomology, the authors obtained stronger lower bounds for $\mathsf{TC}(\Gamma)$ in [29, Section 4].

Most of this work was generalized in the following year by Farber and Oprea. First, they obtained a better lower bound to $\mathsf{TC}_m(\Gamma)$ as follows. Let $K \subset \Gamma^m$ be a subgroup with the property that if $H \subset \Gamma^m$ is any subgroup conjugate to the diagonal subgroup $\Delta \subset \Gamma^m$, then $K \cap H$ is trivial. Then $\mathsf{cd}(K) \leq \mathsf{TC}_m(\Gamma)$ holds in this case, see [31, Section 3.4]. This was the most general lower bound of its time without any assumptions on $\Gamma$ and helped them to conclude that $\mathsf{TC}_m(\mathcal{H}) = 2m$ for all $m \geq 2$, where $\mathcal{H}$ is Higman's group. Second, they found a general analogue of Theorem 8.2 for $\mathsf{TC}_m$ in their paper ([31, Theorem 3.1]) and extended the classical lower bounds from [29] to sequential $\mathsf{TC}$ using Bredon cohomology.

8.4. **$\mathsf{TC}$ values of special manifolds.** Just as in the case of $\mathsf{TC}$ of groups, the theory of $\mathsf{TC}$ of some well-known classes of manifolds was developing; however, not much was known as to how $\mathsf{TC}$ would depend on operations between manifolds. The answer for the wedge of some manifolds was given in [20], but not a lot was known about $\mathsf{TC}$ of the connected sum of manifolds besides a partial answer given for surfaces in [16]. Work in this direction was continued by Dranishnikov and Sadykov in 2017 for simply connected orientable manifolds. The authors showed in [21] that for such manifolds $M$ and $N$, $\max\{\mathsf{TC}(M), \mathsf{TC}(N)\} \leq \mathsf{TC}(M\#N)$. This enabled them to conclude [21, Section 7] that $\mathsf{TC}(M) = 4$ for any smooth closed simply-connected 4-manifold $M \not\simeq S^4$.

But what happens in the case of the connected sum of non-orientable manifolds? In 2018, Cohen and Vandembroucq thought in this direction for $\mathbb{R}P^n$ (recall that $\mathbb{R}P^n$ is non-orientable if and only if $n$ is even). For any $g, n \geq 2$, if $\mathcal{P}_g^n$ denotes the connected sum of $g$-copies of $\mathbb{R}P^n$, then they obtained in [8] that $\mathsf{TC}(\mathcal{P}_g^n) = 2n$, partly by using their obstruction theory calculations from [7]. The following year, this result was generalized to the sequential versions by Jorge Aguilar-Guzmán and González without much difficulty using $\mathbb{Z}_2$-zero-divisor cup-lengths of $\mathbb{R}P^n$ and $\mathcal{P}_g^n$. They showed in [1] that $\mathsf{TC}_m(\mathcal{P}_g^n) = mn$ for all $m, n, g \geq 2$.

As mentioned earlier, symplectic manifolds are interesting objects in topology. The estimate $2\dim(M) - 1 \leq \mathsf{TC}(M) \leq 2\dim(M)$ was only known for symplectically aspherical manifolds, see Section 8.3. In 2018–2019, Grant and Mescher made a giant leap forward by computing the exact values of symplectically atoroidal manifolds, thereby generalizing the previous works in non-simply connected cases. We recall that a symplectic manifold $(M, \omega)$ is called symplectically atoroidal if for all smooth maps $f : T \to M$, where $T$ is the 2-torus, $\int_T f^*(\omega) = 0$. Well-known classes of symplectically hyperbolic and symplectically aspherical manifolds whose fundamental group does not contain a $\mathbb{Z}^2$-subgroup fall under this category. For all such $(M, \omega)$, Grant and Mescher proved in [44] that $\mathsf{TC}(M) = 2\dim(M)$. Again, the notion of $\mathsf{TC}$-weight was employed. Basically, the authors succeeded in finding $\dim(M)$ number of zero-divisors of weight 2 whose cup product was non-zero.

## 9. Some generalizations of topological complexity

While the theory of (sequential) topological complexity of spaces (and groups) was developing from different angles and as more and more computations were being made, the community kept looking for mathematical interpretations of robot motion planning situations that resembled the real-world problems more. Honestly, some motivations were also driven purely by mathematical curiosity.

9.1. **Topological complexity of maps.** One such motivation arises from the parallel theory of LS-category. For a map $f : X \to Y$ between topological spaces, $\mathsf{cat}(f)$ is well defined and studied (see [10, Section 1.7.1]). As a natural analogue of this, and as an extension to $\mathsf{TC}(X)$, Dranishnikov proposed to develop the concept of topological complexity of a map. Through a series of papers between 2017–2018, Peter Pavešic developed this idea. His theory from [56] evolves as follows.

Given a map $f : X \to Y$, define $\pi_f : P(X) \to X \times Y$ as the map $\pi_f : \phi \mapsto (\phi(0), f(\phi(1)))$. Then the *topological complexity of the map* $f$, denoted $\mathsf{TC}(f)$, can be defined as $\mathsf{secat}(\pi_f)$. This idea generalizes the notion of $\mathsf{TC}(X)$ and $\mathsf{cat}(X)$ as follows: $\mathsf{TC}(X) = \mathsf{secat}(\pi_{\mathrm{Id}_X})$, where $\mathrm{Id}_X : X \to X$ is the identity map, and $\mathsf{cat}(X) = \mathsf{secat}(\pi_{\mathrm{ev}})$, where $\mathrm{ev} : P(X) \to X$ is the evaluation map at $t = 1$. Pavešic proved that $\mathsf{TC}(f)$ is a homotopy invariant of fiberwise homotopy equivalent (FHE) maps and satisfies (under additional conditions) many properties that $\mathsf{TC}(X)$ and $\mathsf{cat}(f)$ satisfy. Some are listed from [56].

- If $v : Z \to X$ is a deformation retraction, then $\mathsf{TC}(fv) = \mathsf{TC}(f)$. Similarly, if $u : Y \to Z$ is a deformation, then $\mathsf{TC}(uf) = \mathsf{TC}(f)$.
- If $f$ is a fibration, then $\mathsf{cat}(Y) \le \mathsf{TC}(f) \le \min\{\mathsf{TC}(Y), \mathsf{cat}(X \times Y)\}$. Equality $\mathsf{TC}(f) = \mathsf{TC}(Y)$ holds when $f$ admits a homotopy section. The lower bound is attained when $Y$ is an $H$-space.
- If $f$ admits a section, then $\mathsf{TC}(f) \le \mathsf{TC}(X)$.
- $\mathsf{TC}(f) \ge c\ell_R(\mathrm{Ker}((\mathrm{Id}_X, f)^* : H^*(X \times Y) \to H^*(X)))$ for the map $(\mathrm{Id}_X, f) : x \mapsto (x, f(x))$.
- If $f : SO(n) \to S^{n-1}$ is the standard fibration, $\mathsf{TC}(f) = 1$ if $n$ is even, otherwise $\mathsf{TC}(f) = 2$.

Around this time, Aniceto Murillo and Jie Wu were studying *work maps* associated with mechanical systems. Independent of Pavešic, they developed their notion of topological complexity of maps that could measure the complexity of algorithms that control a specific task that a given system can perform. This, in particular, covers the task of moving the system from an initial point to a final point. In [55], their notion was defined as follows.

For a map $f : X \to Y$, its topological complexity[3] $\mathsf{TC}^*(f)$ is the smallest integer $n$ such that $X \times X$ can be covered by $n+1$ number of open sets $U_i$ over each of which there exists a map $s_i : U_i \to P(X)$ such that the composition $(f \times f)\pi s_i$ is homotopic to $(f \times f)_{|U_i}$. In this situation, if $(f \times f)\pi s_i = (f \times f)_{|U_i}$, the smallest such integer $n$ is called the *naive topological complexity* of $f$ and is denoted by $\mathsf{tc}(f)$. The following properties and examples were given in [55].

- $\mathsf{cat}(f) \leq \mathsf{TC}^*(f) \leq \min\{\mathsf{TC}(X), \mathsf{cat}(f \times f), \mathsf{tc}(f)\}$. The equality $\mathsf{TC}^*(f) = \mathsf{tc}(f)$ holds if $f$ is a fibration.
- $\mathsf{TC}^*(f) \geq c\ell_R(\mathrm{Ker}(\Delta^* : H^*(X \times X) \to H^*(X)) \cap \mathrm{Im}((f \times f)^* : H^*(Y \times Y) \to H^*(X \times X)))$.
- If $f : S^n \to S^m$ is any map, then $\mathsf{TC}^*(f) = 0$ if $n \neq m$ or if $n = m$ and degree of $f$ is 0. Otherwise, $\mathsf{TC}^*(f) = 1$ if $n$ is odd and $\mathsf{TC}^*(f) = 2$ if $n$ is even. So, $\mathsf{TC}(f) \neq \mathsf{TC}^*(f)$.
- $\mathsf{tc}(f) \leq \mathsf{TC}(X)$ and equality holds if $f$ is injective.

In 2020, Jamie Scott developed his idea of the topological complexity of a map in [63]. He gave the most comprehensive survey of properties of his invariant and compared it with those of Pavešic and Murillo–Wu. Scott defined the topological complexity of a map $f : X \to Y$, denoted[4] $\mathsf{TC}^\bullet(f)$, as the smallest integer $n$ such that there exist open sets $U_1, \ldots, U_{n+1}$ that cover $X \times X$ and maps $s_i : U_i \to P(Y)$ satisfying $s_i(x, y)(0) = f(x)$ and $s_i(x, y)(1) = f(y)$. This definition was obtained to parallel that of $\mathsf{cat}(f)$ and implies that $\mathsf{TC}^\bullet(f) = \mathsf{tc}(f)$. In [63, Theorem 3.4], he gave equivalent characterizations of $\mathsf{TC}^\bullet(f)$. Unlike Pavešic's $\mathsf{TC}(f)$ that satisfied nice properties only for fibrations, Scott's $\mathsf{TC}^\bullet(f)$ satisfied all the properties as $\mathsf{cat}(f)$ without additional assumptions. Moreover, due to its definition in terms of pullback, it was slightly easier to work with than Murillo–Wu's $\mathsf{TC}^*(f)$ and $\mathsf{tc}(f)$. Scott obtained the following new properties for his invariant.

- $\mathsf{cat}(f) \leq \mathsf{TC}^\bullet(f) \leq \min\{\mathsf{TC}(X), \mathsf{TC}(Y), \mathsf{cat}(f \times f)\}$.
- If $g : Z \to W$, then $\mathsf{TC}^\bullet(f \times g) \leq \mathsf{TC}^\bullet(f) + \mathsf{TC}^\bullet(g)$.
- If $h : Y \to Z$, then $\mathsf{TC}^\bullet(hf) \leq \min\{\mathsf{TC}^\bullet(f), \mathsf{TC}^\bullet(h)\}$.
- $\mathsf{TC}^\bullet(f) = \mathsf{cat}(f)$ if $X$ is an $H$-space.

[3]Here, we use the notation $\mathsf{TC}^*(f)$ to distinguish from Pavešic's notion.

[4]This is the notation we use to distinguish this notion of complexity of maps from other notions from this section, which we denoted by $\mathsf{TC}(f)$ and $\mathsf{TC}^*(f)$.

After various notions of $\mathsf{TC}$ of maps were presented, a natural next step was to develop a unifying notion that has all the (or, at least, most of the required) nice properties. At the same time, mathematicians were interested in developing sequential versions of the topological complexity of maps. Both these feats were achieved by César A. Ipanaque Zapata and González towards the end of 2022. Their theory was actually motivated by Rudyak's sequential topological complexity of spaces and Pavešic's topological complexity of maps and describes an advanced multitasking motion planning problem in robotics. For a map $f : X \to Y$ and integers $1 \leq s \leq m$ where $m \geq 2$, Zapata and González defined the $(m, s)$-*higher topological complexity of* $f$, denoted $\mathsf{TC}_{m,s}(f)$, as follows in [67]. Let $J_m$ be the wedge of $m$ closed intervals $[0, 1]_i$ where $0_i \in [0, 1]_i$ are identified. Define a map $e^f_{m,s} : X^{J_m} \to X^{m-s} \times X^s$ as $e^f_{m,s} = (\mathrm{Id}_{X^{m-s}} \times f^s)\,\pi^X_m$, where $\pi^X_m$ is the fibration evaluating at each $(j-1)/(m-1)$. Then, $\mathsf{TC}_{m,s}(f) := \mathsf{secat}(e^f_{m,s})$. This recovers $\mathsf{TC}_m(X)$ when $f$ is the identity map for any $s \geq 1$. Also, taking $m = 2$ and $s = 1$, Pavešic's $\mathsf{TC}(f)$ is recovered. In terms of a robot, $s$ is the number of tasks the kinematic map $f$ must take into account, and $m - s$ is the number of configurations (or input positions).

Shortly before this notion, Rudyak and Soumen Sarkar had defined in [61] their version of $\mathsf{TC}_m(f)$ as an extension to $\mathsf{TC}^*(f)$. It was shown in [67, Section 3.4] that their invariant is a special case of $\mathsf{TC}_{m,s}(f)$ for fibrations that admit sections. The following results were shown for $\mathsf{TC}_{m,s}(f)$.

- $\mathsf{TC}_{m,s}(f) \leq \min\{\mathsf{TC}_{m+1,s}(f), \mathsf{TC}_{m+1,s+1}(f)\}$. If $f$ is a fibration, then $\mathsf{TC}_{m,s}(f) \leq \mathsf{cat}(X^{m-s} \times Y^s)$ as well.
- $\mathsf{TC}_{m,s}(f) \geq \max\{\mathsf{secat}(f^s), \mathsf{cat}(X^{m-s-1} \times Y^s)\}$ when $s < m$.
- If $g : Z \to W$ and $(X \times Z)^{m-s} \times (Y \times Z)^m$ is a normal CW complex, then $\mathsf{TC}_{m,s}(f \times g) < \mathsf{TC}_{m,s}(f) + \mathsf{TC}_{m,s}(g)$.
- If $h : W \to X$ and $g : Y \to Z$, then $\mathsf{TC}_{m,s}(fh) \leq \mathsf{TC}_{m,s}(f)$ and $\mathsf{TC}_{m,s}(g) < \mathsf{TC}_{m,s}(gf)$.
- If $\Delta_{m,s} := (\mathrm{Id}_{X^{m-1}} \times f^s)\,\Delta_m : X \to X^{m-s} \times Y^s$, then there is a cohomological lower bound $\mathsf{TC}_{m,s}(f) \geq c\ell_R(\mathrm{Ker}(\Delta^*_{m,s}))$.

9.2. **Parametrized topological complexity.** Typically, in real-world situations, mechanical systems are expected to function in a variety of situations that involve external conditions (such as the presence of obstacles, etc.) that are *not* a priori fixed. To accommodate all such situations in general, and not just the collision-free motion as seen in Section 5.1, where there were no obstacles besides the robots themselves, more "universal" and "flexible" motion planning algorithms were needed. Ideally, such algorithms should be able to take into account all relevant external conditions as parameters, alongside the initial and the final positions, and produce as output the motion of the system that *respects* the external conditions. In particular, when the external parameters are objects with a priori unknown positions, the motion should be collision-free. Clearly, the original concept of topological complexity cannot guarantee to accommodate the complexity of this kind of parametrized motion.

In 2020, Cohen, Farber, and Shmuel Weinberger studied the problem of developing parameterized motion planning algorithms. Basically, they introduced a notion of topological complexity associated with each fibration $p : E \to B$ that will measure the complexity of motion planning algorithms in all configuration spaces that are parametrized by $p : E \to B$. As in the classical case of $\mathsf{TC}$, they aimed to realize their new invariants as the sectional category of some maps. Their theory in [4] develops as follows.

Let $\widehat{P}(E) := \{\phi \in P(E) \mid p\phi \text{ is constant}\}$ and

$$E \times_B E := \{(e, e') \in E \times E \mid p(e) = p(e')\}.$$

Let $\Pi : \widehat{P}(E) \to E \times_B E$ be the end-points evaluation fibration. Then, the *parametrized topological complexity of* $p : E \to B$, denoted $\mathsf{TC}[p : E \to B]$ or $\mathsf{TC}_B(X)$, is defined as $\mathsf{secat}[\Pi : \widehat{P}(E) \to E \times_B E]$. Here, each fiber $p^{-1}(b)$ is the configuration space of the system constrained by external conditions $p \in B$. This recovers the notion of $\mathsf{TC}$: if the external conditions are not changing for a configuration space $X$, then for the trivial fibration $p : X \times B \to B$, one gets $\mathsf{TC}[p : X \times B \to B] = \mathsf{secat}(p \times \mathrm{Id}_B) = \mathsf{TC}(X)$. Also, for principal $G$-bundles $p : E \to B$, $\mathsf{TC}_B(G) = \mathsf{TC}(G) = \mathsf{cat}(G)$. The authors showed that $\mathsf{TC}[p : E \to B]$ is a homotopy invariant of FHE fibrations and also provided some sharp upper bounds and lower bounds for $\mathsf{TC}[p : E \to B]$ in [4, Section 7] to make estimates.

The authors used their invariant to plan the obstacle-avoiding collision-free motion of $n \geq 1$ robots in $\mathbb{R}^k$, for $k \geq 3$ odd, in the presence of $r \geq 2$ obstacles whose positions are not known in advance. The parametrized complexity of such a motion planning algorithm turns out to be $2n + r - 1$. Interestingly, in the space $\mathbb{R}^k$, the (non-parametrized) complexity cannot exceed $2n$. So, as was intuitively obvious, planning a parametrized motion is a much more complex task. In a subsequent work [5], by developing new and more advanced algebraic machinery, the authors solved the above problem when $k \geq 2$ is even. The parametrized complexity in that case turns out to be $2n + r - 2$.

As a natural extension, Farber and Amit Kumar Paul introduced in 2022 the sequential versions of parametrized topological complexity. The idea is similar. Given a fibration $p : E \to B$ and timestamps $0 \leq t_1 < \cdots < t_m \leq 1$, let $E^m_B := \{(e_1, \ldots, e_m) \mid p(e_i) = p(e_j)\}$ and $\Pi_m : \widehat{P}(E) \to E^m_B$ be the fibration evaluating at $t_i$ for all $1 \leq i \leq m$. Then the $m$-th *sequential parametrized topological complexity of* $p : E \to B$, denoted $\mathsf{TC}_m[p : E \to B]$, is defined as $\mathsf{secat}[\Pi_m : \widehat{P}(E) \to E^m_B]$, [32] It is a homotopy invariant and satisfies most of the nice properties as $\mathsf{TC}[p : E \to B]$ and $\mathsf{TC}_m(X)$, see [32] for details. For the obstacle-avoiding collision-free motion in $\mathbb{R}^k$, the authors determined that the $m$-th sequential parametrized complexity values are $mn+r-1$ when $k$ is odd and $mn+r-2$ when $k$ is even, thereby generalizing the known results for the case $m = 2$.

## 10. Latest advancements in motion planning

We end this paper in this section by looking at some of the most recent advancements in motion planning that have taken place in the last year.

Towards the end of 2023, Dranishnikov and Jauhari decided to work on motion planning for some advanced autonomous systems, such as robots like Terminator 2 from the eponymous movie, that are capable of breaking and reassembling. They figured that this advanced feature could help to plan a different kind of motion, which could potentially reduce the complexity of navigation. The idea is as follows. At a given initial position $x$, the system breaks into finitely many weighted pieces. These pieces travel *independently* to a given final position $y$. Then at $y$, the pieces reassemble back into the original system. Such a navigation situation can be found, for example, in the case of huge flocks of identical drones traveling between two given positions. To deal with this situation mathematically, one can put an upper bound $n$ on the number of pieces into which the system can break while traveling between all possible pairs of configurations. The theory is developed as follows in [19] for metric spaces.

Define an $n$-distributed path in $X$ from $x$ to $y$ to be an *unordered* collection of weighted paths in $X$ from $x$ to $y$ whose weights are non-negative and add up to 1. If $P(x, y)$ denotes the space of paths in $X$ that start at $x$ and end at $y$, then the collection of all $n$-distributed paths from $x$ to $y$ is $\mathcal{B}_n(P(x, y))$, i.e., the space of probability measures in $P(x, y)$ having support at most $n$. The algorithm that gives the above motion should be a continuous function that assigns to each pair $(x, y)$ an $n$-distributed path from $x$ to $y$. Note that the breaking of the system into $n$ pieces is like an $n$-th degree discontinuity of its motion. So, naturally, one would want to minimize this number $n$ to obtain the most optimal motion. This gives birth to the following new invariant.

**Definition 10.1** ([19]). The *distributional topological complexity* of a space $X$, denoted $\mathsf{dTC}(X)$, is the smallest integer $n$ such that there exists a continuous map $s : X \times X \to \mathcal{B}_{n+1}(P(X))$ satisfying $s(x, y) \in \mathcal{B}_{n+1}(P(x, y))$ for all $(x, y) \in X \times X$.

Dranishnikov and Jauhari showed that $\mathsf{dTC}$ is a homotopy invariant and is a potential improvement to $\mathsf{TC}$, in the sense that $\mathsf{dTC}(X) \leq \mathsf{TC}(X)$, where the inequality can be strict. To study $\mathsf{dTC}$ better, they also developed a *distributional* version of LS-category, denoted $\mathsf{dcat}$, such that $\mathsf{dcat}(X) \leq \mathsf{cat}(X)$. The authors showed that $\mathsf{dTC}$ has most of the nice properties as $\mathsf{TC}$ and it interacts with $\mathsf{dcat}$ in the same way as $\mathsf{TC}$ interacts with $\mathsf{cat}$. Sharp cohomological lower bounds were also obtained for $\mathsf{dTC}(X)$ and $\mathsf{dcat}(X)$ using the cohomology of the symmetric products of $X \times X$ and $X$, respectively. In particular, the inequality $zc\ell_{\mathbb{Q}}(X) \leq \mathsf{dTC}(X)$ holds, [19, Corollary 4.13]. Furthermore, as analogues to Schwarz's results in the classical case, characterizations of these new invariants were obtained in terms of well-known fibrations. The following important examples and properties were obtained in [19] for $\mathsf{dTC}$.

- $\mathsf{dTC}(\mathbb{R}P^n) = 1$ for all $n$, in stark contrast to the peculiar and challenging case of $\mathsf{TC}(\mathbb{R}P^n)$.
- $\mathsf{dTC}$ values agree with the respective $\mathsf{TC}$ values on $\mathbb{C}P^n$, $\Sigma_g$, $S^n$, and finite graphs.
- If $X, Y$ are aspherical CW complexes, then $\mathsf{dTC}(X \vee Y) = k$ if $\dim(X \vee Y) < k := \max\{\mathsf{dTC}(X), \mathsf{dTC}(Y), \mathsf{dcat}(X \times Y)\}$.
- $\mathsf{dTC}(X) = \mathsf{dcat}(X)$ if $X$ is a topological group.

Around the same time, Ben Knudsen and Weinberger were looking for an "analog" version of Farber's "digital" notion of probabilistic topological complexity (see Section 5.1). Independent of Dranishnikov–Jauhari, in 2023–2024, they developed a new, improved version of $\mathsf{TC}$, which they called *analog topological complexity* and denoted by $\mathsf{ATC}$. In their paper [49], they also defined and studied sequential versions of $\mathsf{ATC}$, denoted $\mathsf{ATC}_m$, and an analog version of LS-category, denoted $\mathsf{acat}$. Roughly speaking, for defining their invariants, they use $\mathcal{B}_n(P(X))$ space equipped with the canonical quotient topology from [48]. However, the topology considered on $\mathcal{B}_n(P(X))$ in [19] was induced by the Lévy–Prokhorov metric [57]. While that could potentially be a reason for $\mathsf{dTC}$ and $\mathsf{ATC}$ to be different notions, [49] conjectured that $\mathsf{ATC}(X) = \mathsf{dTC}(X)$ for metric spaces $X$. In any case, $\mathsf{ATC}$ and $\mathsf{dTC}$ satisfy similar properties, and their introduction around the same time presents strong evidence of the naturality of these ideas and emphasizes the need for their emergence to potentially advance the field of topological robotics. The following important results were proved in [49].

- $\mathsf{ATC}_m(\mathbb{R}P^n) \leq 2m+1$ for all $n$, in contrast to the case of $\mathsf{TC}_m(\mathbb{R}P^n)$.
- If $\Gamma$ is a finite group, then $\mathsf{ATC}_m(\Gamma) \leq |\Gamma| - 1$.
- $\mathsf{acat}(\Gamma) = \mathsf{cd}(\Gamma)$ when $\Gamma$ is a torsion-free discrete group (this is an analogue of the Eilenberg–Ganea theorem, see Section 6.2).

A few weeks after [19], Jauhari introduced the sequential versions of distributional topological complexity, denoted $\mathsf{dTC}_m$. As a natural extension, these invariants cater to the motion planning of advanced systems which have to attain a sequence of positions $x_i$ at given timestamps $t_i \in [0, 1]$ by breaking and reassembling repeatedly into finitely many pieces of the same weights at each position $x_i$, see [47, Section 1.A] for details. He showed that the properties for $\mathsf{dTC}$ can be generalized to $\mathsf{dTC}_m$. In his paper, he developed the theory of *distributional sectional category* (whose idea was introduced implicitly in [19, Section 5]) and obtained sharp cohomological lower bounds to $\mathsf{dTC}_m$ in Alexander–Spanier cohomology that helped greatly in the computations. Furthermore, he showed using simple arguments that for any metric space $X$ and $m \geq 2$, $\mathsf{dTC}_m(X) \leq \mathsf{ATC}_m(X)$. So, $\mathsf{dTC}_m$ is a not-necessarily-strict improvement to $\mathsf{ATC}_m$. New results from [47] can be listed as follows.

- $\mathsf{dTC}_{m+1}(X) = \mathsf{dcat}(X^m)$ if $X$ is a CW $H$-space.
- $\mathsf{dcat}(X^{m-1}) \leq \mathsf{dTC}_m(X) \leq \min\{\mathsf{dcat}(X^m), \mathsf{dTC}_{m+1}(X)\}$ for all $m$.

- $\mathsf{dTC}_m$ values agree with the respective $\mathsf{TC}_m$ values for finite products of spheres, $\Sigma_g$, and product of $\Sigma_g$ with finite products of spheres.
- $\mathsf{dTC}_m(X) \geq m-1$ if $H^d(X;\mathbb{Q}) \neq 0$ for some $d \geq 1$.

Very recently, Dranishnikov contributed further to the theory of $\mathsf{dTC}$ in [18] and obtained many new results. He completed more calculations for $\mathsf{dTC}$ by showing that $\mathsf{dTC}(N_g) = 4$ for all $g \geq 4$ using his work on $\mathsf{TC}(N_g)$ done in [16], and that $\mathsf{dTC}(\mathbb{Z}_p) = p-1$ for all primes $p$ by providing an extension of Singhof's theorem on dimension of categorical sets to arbitrary simplicial complexes. Furthermore, through a deeper study of the connectivity of the space $\mathcal{B}_n(Z)$ for a possibly disconnected space $Z$ (compare with the result of [48] in simply connected case), he removed the assumption of the asphericity of the CW complexes $X$ and $Y$ in the determination of the formula for $\mathsf{dTC}(X \vee Y)$ obtained in [19], and proved that $\mathsf{dTC}(X\#Y) = 2n$ for closed $n$-dimensional manifolds $X$ and $Y$ under some very special conditions, see [18, Theorem 4.7].

## Concluding remarks

In this paper, we saw two major themes: the development of new tools and techniques to settle computations in the field of topological complexity, and the emergence of new ideas that led to the development of nuanced and more advanced versions of topological complexity. These themes intertwine beautifully, and together, they have led to a vast subject where problems of motion planning are dealt with from the perspective of topology. Topological complexity has been a rapidly evolving discipline; still, there are many problems to be solved and advancements to be made. New theories are waiting to be developed, and existing theories are demanding new insights. Hopefully, continued active research will bring the field of topological complexity into the mainstream of topology.

## References

[1] Aguilar-Guzmán, J., González, J., Sequential motion planning in connected sums of real projective spaces, *Morfismos* **23** (2019), no. 2, 17–21.

[2] Basabe, I., González, J., Rudyak, Y. B., Tamaki, D., Higher topological complexity and its symmetrization, *Algebr. Geom. Topol.* **14** (2014), 2103–2124.

[3] Cohen, D. C., Farber, M., Topological complexity of collision-free motion planning on surfaces. *Compos. Math.* **147** (2011), no. 2, 649–660.

[4] Cohen, D. C., Farber, M., Weinberger, S., Topology of parametrized motion planning algorithms. *SIAM J. Appl. Algebra Geom.* **5** (2021), no. 2, 229–249.

[5] Cohen, D. C., Farber, M., Weinberger, S., Parametrized topological complexity of collision-free motion planning in the plane. *Ann. Math. Artif. Intell.* **90** (2022), no. 2, 999–1015.

[6] Cohen, D. C., Pruidze, G., Motion planning in tori. *Bull. London Math. Soc.* **40** (2008), no. 2, 249–262.

[7] Cohen, D. C., Vandembroucq L., Topological complexity of the Klein bottle. *J Appl. Comput. Topology* **1** (2017), 199–213.

[8] Cohen, D. C., Vandembroucq L., Motion planning in connected sums of real projective spaces. *Topology Proc.* **54** (2019), 323–334.

[9] Colman, H., Grant, M., Equivariant topological complexity. *Algebr. Geom. Topol.* **12** (2012), 2299–2316.

[10] Cornea, O., Lupton, G., Oprea, J., Tanré D., *Lusternik–Schnirelmann category*, Math. Surveys Monogr. 103, AMS, Providence, 2003.

[11] Davis, D. M., An approach to the topological complexity of the Klein bottle. *Morfismos* **21** (2017), no. 1, 1–13.

[12] Davis, D. M., A lower bound for higher topological complexity of real projective space. *J. Pure Appl. Algebra* **222** (2018), no. 10, 2881–2887.

[13] Davis, D. M., Bounds for higher topological complexity of real projective space implied by *BP*. *Bol. Soc. Mat. Mexican* **25** (2018), 701–712.

[14] Dranishnikov, A., Topological complexity of wedges and covering maps. *Proc. Amer. Math. Soc.* **142** (2014), no. 12, 4365–4376.

[15] Dranishnikov, A., The topological complexity and the homotopy cofiber of the diagonal map for non-orientable surfaces, *Proc. Amer. Math. Soc.* **144** (2017), no. 11, 4999–5014.

[16] Dranishnikov, A., On topological complexity of non-orientable surfaces, *Topology Appl.* **232** (2017), 61–69.

[17] Dranishnikov, A., On topological complexity of hyperbolic groups, *Proc. Amer. Math. Soc.* **148** (2020), no. 10, 4547–4556.

[18] Dranishnikov, A., Distributional topological complexity of groups. Preprint, arXiv:2404.03041 [math.GT] (2024), pp. 29.

[19] Dranishnikov, A., Jauhari, E., Distributional topological complexity and LS-category. In *Topology and AI*, eds. Farber M., González, J., EMS Ser. Ind. Appl. Math., 4, EMS Press, Berlin, 2024, pp. 363–385.

[20] Dranishnikov, A., Sadykov, R., The topological complexity of the free product, *Math. Z.* **293** (2019), 407–416.

[21] Dranishnikov, A., Sadykov, R., On the LS-category and topological complexity of a connected sum, *Proc. Amer. Math. Soc.* **147** (2019), no. 5, 2235–2244.

[22] Fadell, E., Husseini, S., Cohomology weights and Steenrod operations, *Bol. Soc. Mat. Mexican* **37** (1992), no. 2, 151–161.

[23] Farber, M., Topological complexity of motion planning. *Discrete Comput. Geom.* **29** (2003), no. 2, 211–221.

[24] Farber, M., Instabilities of robot motion. *Topology Appl.* **140** (2004), 245–266.

[25] Farber, M., Collision free motion planning on graphs. In *Algorithmic Foundations of Robotics IV*, eds. Erdmann M., *et al.*, Springer, 2005, pp. 123–138.

[26] Farber, M., Grant, M., Topological complexity of configuration spaces. *Proc. Amer. Math. Soc.* **137** (2008), no. 5, 1841–1847.

[27] Farber, M., Grant, M., Symmetric motion planning. In *Topology and Robotics*, eds. Farber M., *et al.*, Contemp. Math. 438, AMS, Providence, 2007, pp. 85–104.

[28] Farber, M., Grant, M., Robot motion planning, weights of cohomology classes, and cohomology operations. *Proc. Amer. Math. Soc.* **136** (2008), no. 9, 3339–3349.

[29] Farber, M., Grant, M., Lupton, G., Oprea, J., Bredon cohomology and robot motion planning. *Algebr. Geom. Topol.* **19** (2019), 2023–2059.

[30] Farber, M., Mescher, S., On the topological complexity of aspherical spaces. *J. Topol. Anal.* **12** (2020), no. 2, 293–319.

[31] Farber, M., Oprea, J., Higher topological complexity of aspherical spaces. *Topology Appl.* **258** (2019), 148–160.

[32] Farber, M., Paul, A. K., Sequential parametrized motion planning and its complexity. *Topology Appl.* **321** (2022), 108256, pp. 23.

[33] Farber, M., Tabachnikov, S., Yuzvinsky, S., Topological robotics: motion planning in projective spaces. *Intl. Math. Res. Not.* (2003), no. 34, 1853–1870.

[34] Farber, M., Yuzvinsky, S., Topological robotics: subspace arrangements and collision free motion planning. *Transl. of AMS* **212** (2004), 145–156.

[35] Fox, R. H., On the Lusternik–Schnirelmann category. *Ann. of Math* **42** (1941), no. 2, 333–370.

[36] García Calcines, J. M., Vandembroucq, L., Topological complexity and the homotopy cofibre of the diagonal map. *Math. Z.* **274** (2013), 145–163.

[37] González, J., Gutiérrez, B., Gutiérrez, D., Lara, A., Motion planning in real flag manifolds. *Homology Homotopy Appl.* **18** (2016), no. 2, 359–375.

[38] González, J., Gutiérrez, B., Guzmán, A., Hidber, C., Mendoza, M., Roque, C., Motion planning in tori revisited. *Morfismos* **19** (2015), no. 1, 7–18.

[39] González, J., Gutiérrez, B., Mas, H., Pairwise disjoint maximal cliques in random graphs and sequential motion planning on random right angled Artin groups. *J. Topol. Anal.* **11** (2019), no. 2, 371–386.

[40] González, J., Landweber, P., Symmetric topological complexity of projective and lens spaces. *Algebr. Geom. Topol.* **9** (2009), no. 1, 473–494.

[41] Grant, M., Topological complexity, fibrations and symmetry. *Topology Appl.* **159** (2012), no. 1, 88–97

[42] Grant, M., Lupton, G., Oprea, J., Spaces of topological complexity one. *Homology Homotopy Appl.* **15** (2013), no. 2, 73–81.

[43] Grant, M., Lupton, G., Oprea, J., New lower bounds for the topological complexity of aspherical spaces. *Topology Appl.* **189** (2015), 78–91.

[44] Grant, M., Mescher, S., Topological complexity of symplectic manifolds. *Math. Z.* **296** (2020), 667–679.

[45] Grant, M., Recio-Mitter, D., Topological complexity of subgroups of Artin's braid groups. In *Topological Complexity and Related Topics*, eds. Grant M., *et al.*, Contemp. Math. 702, AMS, pp. 12.

[46] Iwase, N., Sakai, M., Erratum to "Topological complexity is a fibrewise L-S category" [*Topology Appl.* **157** (2010), no. 1, 10–21]. *Topology Appl.* **159** (2012), no. 10–11, 2810–2813.

[47] Jauhari, E., On sequential versions of distributional topological complexity. *Topology Appl.* **363** (2025), 109271, pp. 28.

[48] Kallel, S., Karoui, R., Symmetric joins and weighted barycenters. *Adv. Nonlinear Stud.* **11** (2011), no. 1, 117–143.

[49] Knudsen, B., Weinberger, S., Analog category and complexity. *SIAM J. Appl. Algebra Geom.* **8** (2024), no. 3, 713–732.

[50] Latombe, J-C., *Robot Motion Planning*. Springer Intl. SECS 124, Springer, New York, 1991.

[51] Lubawski, W., Marzantowicz, W., Invariant topological complexity. *Bull. London Math. Soc.* **47** (2015), 101–117.

[52] Lupton, G., Scherer, J., Topological complexity of $H$-spaces. *Proc. Amer. Math. Soc.* **141** (2013), no. 5, 1827–1838.

[53] Lusternik, L., Schnirelmann, L., Sur le problème de trois géodesiques fermées sur less surfaces de genre 0 (in French). *C. R. Acad. Sci. Paris* **189** (1929), 269–271.

[54] Lusternik, L., Schnirelmann, L., *Méthodes Topologiques dans les Problemes Variationnels* (in French). Hermann, Paris, 1934.

[55] Murillo, A., Wu, J., Topological complexity of the work map. *J. Topol. Anal.* **13** (2021), no. 1, 219–238.

[56] Pavešic, P., Topological complexity of a map. *Homology Homotopy Appl.* **21** (2019), no. 2, 107–130.

[57] Prokhorov, Y. V., Convergence of random processes and limit theorems in probability theory. *Theory Probab. Appl.* **1** (1956), no. 2, 157–214.

[58] Rudyak, Y. B., On category weight and its applications. *Topology* **38** (1999), no. 1, 37–55.

[59] Rudyak, Y. B., On higher analogs of topological complexity. *Topology Appl.* **157** (2010), no. 5, 916–920.

[60] Rudyak, Y. B., On topological complexity of Eilenberg–Mac Lane spaces. *Topology Proc.* **48** (2016), no. 5, 65–67.

[61] Rudyak, Y. B., Sarkar, S., Relative LS categories and higher topological complexities of maps. *Topology Appl.* **322** (2022), 108317, pp. 16.

[62] Schwarz, A. S., The genus of a fiber space. In *Eleven papers on topology and algebra*, Amer. Math. Soc. Transl. Ser. 2, 55, AMS, Providence, 1966, pp. 49–140.

[63] Scott, J., On the topological complexity of maps. *Topology Appl.* **314** (2022), 108094, pp. 25.

[64] Smale, S., On the topology of algorithms. I. *J. Complexity* **3** (1987), no. 2, 81–89.

[65] Strom, J., Category weight and essential category weight. PhD Thesis, University of Wisconsin–Madison (1997).

[66] Vasiliev, V. A., Cohomology of braid groups and complexity of algorithms. *Funct. Anal. Appl.* **22** (1988), no. 3, 182–190.

[67] Zapata, C. A. I., González, J., Higher topological complexity of a map. *Turk. J. Math* **47** (2023), 1616–1642.

Ekansh Jauhari
Department of Mathematics
University of Florida,
Gainesville, FL 32611, USA.
E-mail: ekanshjauhari@ufl.edu

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# ERRATUM TO "A TOPOLOGICAL PROOF OF THE FUNDAMENTAL THEOREM OF ARITHMETIC"

JHIXON MACÍAS
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In this note, we correct the final proof of the paper [1]. The idea is to show that the set of prime numbers $\mathbb{P}$ is dense in the topological space $X := (\mathbf{N_2}, \tau)$, where $\tau$ is the topology generated by the basis $\beta := \{\mathcal{O}_n : n \geq 2\}$, $\mathcal{O}_n := \{d \in \mathbf{N_2} : \ d \mid n\}$, and $\mathbf{N_2}$ is the set of integers $\geq 2$. The argument we used in [1] was the following:

*Suppose that* $\mathbb{P}$ *is not dense in* $X$. *Then there exists* $n \in \mathbf{N}_2$ *such that* $\mathcal{O}_n \cap \mathbb{P} = \emptyset$. *Thus* $n$ *is not prime and cannot be factored into primes. Likewise, neither do the divisors of* $n$. *Let* $\Delta := \{n \in \mathbf{N}_2 : \mathcal{O}_n \cap \mathbb{P} = \emptyset\}$. *It is clear that* $\mathbf{N}_2 \setminus \Delta$ *is the set of all* $m \in \mathbf{N}_2$ *such that* $m$ *is prime or can be factored into primes. Therefore, for every* $n \in \Delta$ *and* $m \in \mathbf{N}_2 \setminus \Delta$, *we have* $\mathcal{O}_n \cap \mathcal{O}_m = \emptyset$. *Now, let* $A = \bigcup_{n\in\Delta} \mathcal{O}_n$ *and* $B = \bigcup_{m\in\mathbf{N}_2\setminus\Delta} \mathcal{O}_m$. *Then* $A$ *and* $B$ *are disjoint open sets in* $X$, *such that* $\mathbf{N}_2 = A \cup B$, *in other words,* $A$ *and* $B$ *form a separation of* $X$, *which is absurd since* $X$ *is connected.*

However, this argument is not correct because it assumes that for every $n \in \Delta$ and $m \in \mathbf{N}_2 \setminus \Delta$, we have $\mathcal{O}_n \cap \mathcal{O}_m = \emptyset$. To see that this is false, it suffices to take $n \in \Delta$ and $p \in \mathbb{P}$, and let $m = np$. It is easy to see that $m \in \mathbf{N}_2 \setminus \Delta$, but $\mathcal{O}_n \cap \mathcal{O}_m = \mathcal{O}_n \neq \emptyset$.

Fortunately, we found another proof that, although simple, is correct and elegant. The argument is as follows:

*Proof.* Suppose that $\mathbb{P}$ is not dense in $X$. Then there exists $n$ such that $\mathcal{O}_n \cap \mathbb{P} = \emptyset$. Clearly, $n$ is not a prime number, and it cannot be divisible by a prime number either; the same holds for each of its divisors. However, if we consider $d := \min\{x : x \in \mathcal{O}_n\}$, then necessarily $\mathcal{O}_d = \{d\}$, and consequently, $d$ is a prime number. This leads to a contradiction. □

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Key words and phrases: Prime numbers, topology, fundamental theorem of arithmetic

Therefore, $\mathbb{P}$ is dense in $X$, and by virtue of [1, Theorem 2.1], we obtain *the topological proof of the fundamental theorem of arithmetic.*

## References

[1] Macías, J. A topological proof of the fundamental theorem of arithmetic, *The Mathematics Student* **93** (2024), 179-182.

Jhixon Macías
Department of Mathematics
Faculty of Arts and Sciences
University of Puerto Rico at Mayaguez,
00681-9000, Mayaguez, Puerto Rico.
E-mail: jhixon.macias@upr.edu

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# COLOUR EQUIVALENCE PATTERNS AND GROUPS ON RUBIK'S CUBES

RAINER HOLLERBACH

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Abstract. We introduce the concept of colour equivalence patterns on Rubik's cubes. These are patterns where: (a) the six colours are first split into two sets of three, (b) within each set those three colours must be distributed equally, and (c) the distributions of the two sets must be mirror images of one another. The complete set of such patterns is first obtained by computer programs. The patterns are then analysed from both geometrical and group theoretical perspectives, which can differ significantly. The group theory results are presented at a level appropriate for students just learning the subject.

## 1. Introduction

It is 50 years now since the Rubik's cube was invented, with several 100 million cubes sold since then. Mathematicians immediately realised that it is far more than just a puzzle though, and has a deep underlying group theoretical structure [1, 2, 3, 4]. This reveals, for example, that it is not possible to exchange two side pieces while leaving everything else in place. This property is very familiar to anyone who has ever played around with cubes for any length of time, but to provide a rigorous proof that it could never be done requires some rather sophisticated group theory. Another interesting question is the maximum number of turns needed to solve an arbitrary position. Direct computer searches show this to be 11 for the $2 \times 2 \times 2$ cube, and 20 for the $3 \times 3 \times 3$ cube [8]. For the general $n \times n \times n$ cube this so-called group diameter scales as $n^2/\log n$ [5].

There are also enthusiasts who are simply interested in a variety of pretty patterns that can be constructed, without regard to any special mathematical properties they might have. An online search easily yields

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several dozen such patterns, many of them very intricate, but typically presented without any associated mathematical analysis. One recent attempt to bridge this gap between very abstract mathematics on the one hand versus pretty patterns on the other introduced the concept of *colour equality patterns* [6, 7], which were defined to be patterns where all six colours are distributed exactly equally. That is, if you had two identically arranged cubes, you could rotate them with respect to one another such that any given colour on the first cube and any other colour on the second cube would coincide in all of their positions.

Figure 1a shows an example of such a colour equality pattern. We begin by arbitrarily choosing the colours red/yellow/blue (r/y/b) to constitute the front, and orange/white/green (o/w/g) the back. Then take the three side pieces around the r/y/b front corner and the three side pieces around the o/w/g back corner – which will together be known as the '3+3' side pieces – and rotate each set clockwise as shown. All six colours are clearly distributed in exactly the same way. The study of colour equality patterns as a whole is then concerned with finding all such patterns, systematically classifying them according to their various symmetries, etc. One discovery is that there are patterns having different symmetries, so different that they are mutually incompatible. That is, two patterns might each separately satisfy colour equality, but if you try to combine them, colour equality is violated. This is certainly an interesting result, but it does also mean that the set of all colour equality patterns does not constitute a group.

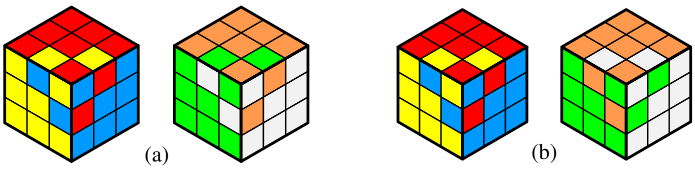


FIGURE 1. (a) A colour equality pattern. (b) A colour equivalence pattern. In each case the left panel shows the three faces on the front, and the right panel the three faces on the back, after the cube has been turned over. Supplementary Material available at `eprints.whiterose.ac.uk/229635/` also includes templates to cut out and tape together to form cubes which can then be rotated as desired to see the equality/equivalence relations between the different colours even more clearly.

Next, suppose we arrange those same '3+3' side pieces around the front and back corners as in Figure 1b. The r/y/b front pieces are still rotated clockwise, but now the o/w/g back pieces are rotated counter-clockwise. Strict colour equality, between all six colours, is then no longer satisfied. However, we can observe two things: First, within each {r,y,b} and {o,w,g} set separately, colour equality is still satisfied. Second, the patterns formed by red and orange, say, are not the same, but they are mirror images of each other. And since yellow and blue were already seen to be equal to red, and white and green equal to orange, we realise that we still have a nice, symmetrical equivalence between the two colour sets.

Let us therefore define *colour equivalence patterns* to be patterns of this type, with {r,y,b} and {o,w,g} satisfying equality within each set, and being mirror images between the two sets. As before, we would then like to find all such patterns, systematically classify them in various ways, etc. One difference that will become apparent below is that the equivalence patterns are far more regular than the equality patterns were, with every pattern having the same symmetries, and thus being mutually compatible. From the point of view of studying different symmetries, this makes the equality patterns the more interesting set. On the other hand, precisely this mutual compatibility means that the set of all colour equivalence patterns does constitute a group.

The purpose of this article is then to derive the full set of equivalence patterns, and analyse them from both geometrical and group theoretical perspectives. At the end I also outline how the $2\times2\times2$ equality patterns can be split into two (partially overlapping) sets that do satisfy the requirements to be groups, with the details left as an exercise for readers. My hope is that both types of patterns might not only be interesting in their own right, in terms of symmetries of three-dimensional patterns, but that the resulting groups could serve as useful examples for students encountering group theory for the first time.

## 2. The Corners

We start with the $2 \times 2 \times 2$ cube consisting only of corners. The corners of all larger cubes follow exactly the same rules. The number of possible configurations, $7! \cdot 3^6 \approx 4 \cdot 10^6$, is sufficiently small that we can scan through them all with a computer and pick out the ones that satisfy the colour equivalence criterion. There turn out to be 18 such patterns, shown in Figure 2. The representation is as before, with three faces that constitute the front, then the cube is flipped over to show the three faces on the back. The mirror symmetry between red versus orange, yellow versus white, and blue versus green is again easily recognisable. The three-fold symmetry among {r,y,b} and {o,w,g} also shows the equality within the given set. Once again though, to fully appreciate the patterns it is best to construct actual 3D cubes, using the templates provided in the Supplementary Material (`eprints.whiterose.ac.uk/229635/`).

The labels in Figure 2 indicate the group structure. First, the standard solved configuration $I$ is the identity element. Let's focus next on the three patterns $X$, $Y$, and $Z$, since these allow us to generate the entire group. $X$ is easiest to describe; it simply rotates the r/y/b front corner one twist

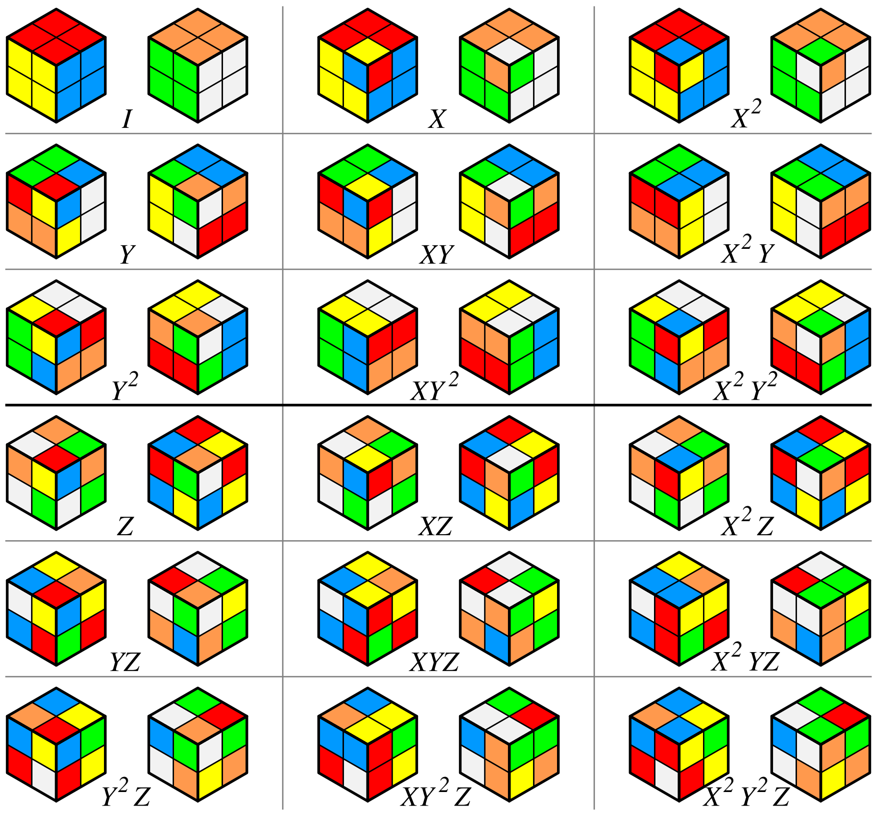


FIGURE 2. The 18 corner patterns, labeled according to how they are constructed from the basic actions $\{X, Y, Z\}$.

clockwise, and the o/w/g back corner one twist counter-clockwise. $Y$ is slightly more complicated, but just like $X$ it also leaves all eight corners in their original positions. The r/y/b front and o/w/g back pieces also retain their original orientation. The three corners around the r/y/b front piece each rotate one twist clockwise, the three corners around the o/w/g back piece one twist counter-clockwise. From these geometrical definitions we can immediately see that $X^3 = Y^3 = I$ and $XY = YX$, that is, $X$ and $Y$ commute.

The third pattern $Z$ has the most complicated geometrical action, but even it is relatively straightforward to understand. The front and back corners again remain in their original positions, and with their original orientation. The other six pieces switch places with their diagonally opposite partners, while maintaining the same relative orientation to one another. That is, they undergo a 180° flip as if they were a single connected object. From this definition we can again easily see that $Z^2 = I$ and also $XZ = ZX$. The commutation relation between $Y$ and $Z$ is less obvious, and indeed they don't commute. By directly carrying out the various steps involved, you should be able to convince yourself that $ZY = Y^2Z$ and similarly $ZY^2 = YZ$.

At this point, make sure also that you fully understand what combinations of this type mean. Think of $\{X, Y, Z\}$ not just as static patterns that are whatever they happen to be, but also think of them as *instructions* telling you what to do next. That is, a combination such as $XYZ$ tells you: First start with the instructions $Z$, which when applied to the original configuration gives you just the pattern called $Z$. Next apply the instructions $Y$ to this pattern $Z$, which gives you something else. Finally apply the instructions $X$ to whatever you just produced, and that is the final pattern called $XYZ$. Interpreting $\{X, Y, Z\}$ as instructions and not just as static patterns is precisely why it was so important to first clarify what exactly it is that they actually do, in terms of rotating and/or moving the various corners. You will probably find it useful also to construct actual cubes, including filling in some colours yourself on the blank templates provided, to really see how the different corners move around under combined actions like these.

Given the results $X^3 = Y^3 = Z^2 = I$ and the various commutation relations, it is straightforward to work out that the 18 elements $X^mY^n$

and $X^mY^nZ$, with $m, n = 0, 1, 2$, are the only possibilities. Any other combinations of these three elements could always be reduced to one of these 18 elements. And sure enough, Figure 2 consists of exactly these 18 elements. Furthermore, from the basic definitions we see that $X$ and $X^{-1} = X^2$ should be mirror images in terms of which corners rotate which way, and similarly $Y$ and $Y^{-1} = Y^2$ should be mirror images. In contrast, $Z$ is its own inverse, and correspondingly has no handedness. If you carefully compare various pairs of patterns, you find that this handedness of $X$ and $Y$ then also exactly carries over to all the combinations; for example $XY^2Z$ and $X^2YZ$ are mirror image partners, as we would expect if their $X$ versus $X^2$ and $Y^2$ versus $Y$ parts separately are mirror images.

There is one further twist to these patterns and how best to represent them: Suppose we define $W = XZ$. From a geometrical perspective this combination is needlessly complicated compared with the relative simplicity of $X$ and $Z$ separately. However, from the group theoretical perspective something rather interesting happens. Using our previous results that $XZ = ZX$ and $X^3 = Z^2 = I$, it is straightforward to obtain $W^2 = X^2$, $W^3 = Z$, $W^4 = X$, $W^5 = X^2Z$, and finally $W^6 = I$. The important ones here are $X = W^4$ and $Z = W^3$. That is, $X$ and $Z$ can both be reconstructed from the single element $W$. From an abstract group theoretical perspective, we recognise therefore that we don't need three generators $\{X, Y, Z\}$; we can also obtain the entire group from just the two generators $\{W, Y\}$.

It is important to emphasise also that this entire geometrical and group theoretical labeling and analysis was done only after the patterns had originally been obtained by the computer search. The computer program conducted its search based entirely on the original geometrical definition of colour equivalence, and simply produced the 18 patterns here. It was only afterwards that a careful examination revealed them to be a group, geometrically best described by the three elements $\{X, Y, Z\}$, but group theoretically most compactly defined by just the two elements $\{W, Y\}$.

## 3. Adding the Centres

Moving on to $3 \times 3 \times 3$ cubes, the first step is to consider how these corner patterns fit together with the centres. The centres form a rigid lattice, so there are only 24 ways it can be oriented with respect to the corner patterns. Three of these are compatible with colour equivalence, corresponding to successive rotations about the r/y/b – o/w/g preferred axis, as shown in Figure 3. In the language of group theory the Figure 3a orientation is the identity element $I_c$, then 3b is the generator $C$, 3c has the orientation $C^2$, and finally 3d is back to $C^3 = I_c$.

If we next examine the side pieces in Figure 3, in 3a and 3b they are all in their original positions. In 3c and 3d though, the same '3+3' side pieces as in Figure 1 have each been exchanged with their diagonally opposite partners. This exchange is *not* a side piece action though, which we will get to below. Instead, it is required by the corner element $Z$, which is present in both 3c and 3d. This requirement to diagonally exchange these '3+3' side pieces for the nine corner patterns containing $Z$ comes about because of the point noted in the introduction, that two side pieces cannot be exchanged while leaving everything else in place. In particular, if you try to construct any corner pattern containing $Z$ along with the side pieces as in 3a or 3b, the last two side pieces will be in the wrong positions. Adding these three diagonal exchanges gets you back to an even number of side piece switches, which can be done.

That is, the nine corner patterns that do not contain $Z$ *must* have their standard side piece arrangement as in Figures 3a and 3b, whereas the

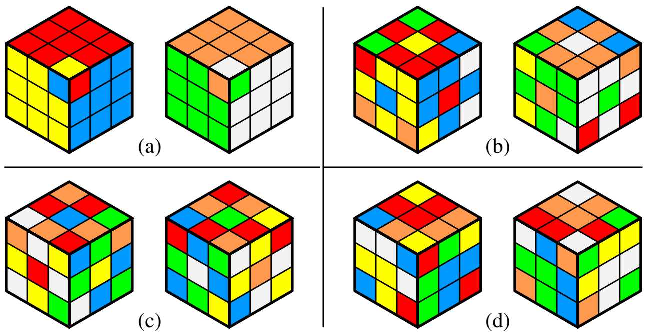


FIGURE 3. The centre and corner labelings for these patterns are: (a) $I_c X$, (b) $C Y$, (c) $C^2 Z$, (d) $I_c XYZ$. As noted also in the text, the labels ($I_c$, $C$, $C^2$, $I_c$) refer to the arrangements of the centres only, and constitute rotations about the r/y/b – o/w/g preferred axis. $X$ and $Y$ apply to the corners only, exactly as before. $Z$ applies to the corners as before, but additionally the arrangement of the side pieces in (c,d) is part of the new definition of $Z$.

nine corner patterns that do contain $Z$ *must* have their standard side piece arrangement as in Figures 3c and 3d. The only remaining question is, how to describe this in the language of group theory? It's quite simple: we just extend the definition of the element $Z$ (or equivalently $W$, if we want to use the $\{W, Y\}$ representation). That is, $X$ and $Y$ remain as before, and act only on the corners, exactly as before. In contrast, $Z$ now acts on the corners exactly as before, but in addition also does this diagonal exchange of the '3+3' side pieces (which very conveniently is also still consistent with colour equivalence).

## 4. The Side Pieces

The number of possible configurations for the side pieces is $12! \cdot 2^{10} \approx 5 \cdot 10^{11}$, far greater than the $4 \cdot 10^6$ possibilities for the corner pieces. Nevertheless, even a number as large as this is still manageable by the direct computational approach, requiring only a few days on a reasonably powerful workstation. The results show that the 12 side pieces can be arranged in 144 different ways that are consistent with colour equivalence. The equivalent of Figure 2, that is, explicitly showing all 144 patterns, would then be rather unwieldy. However, just as the 18 patterns in Figure 2 could be constructed out of a much smaller number of generators, for the side pieces also the 144 patterns can be constructed out of a relatively small number of geometrically simple actions.

Indeed, we have already seen one of these actions before, namely the clockwise/counter-clockwise rotation of the '3+3' side pieces in Figure 1b. This clearly contributes a factor of three to our target of 144. Another action we can do is to flip these same '3+3' side pieces in place. Alternatively, we can take the other six side pieces, constituting a 'ring' that separates the front and back halves of the cube, and flip those in place. At this point we thus have $3 \cdot 2 \cdot 2 = 12$ possibilities. The remaining factor of 12 that we need to achieve our target of 144 comes from two further geometrical actions, one yielding a factor of two, and the other a factor of six. Can you think what they are? Solutions are provided in the Supplementary Material, but perhaps you can already discover these two geometrical actions yourself.

Another interesting point comes about when we consider the associated group structure. For the corners we already saw that geometrically the patterns are best described by the three actions $\{X, Y, Z\}$, but that group theoretically we can actually specify things more compactly in terms of just two generators $\{W, Y\}$. For the side pieces we will similarly find that geometrically these 144 patterns are best described by the three actions already mentioned here, plus the further two actions in the Supplementary Material. However, from the group theoretical perspective, we again do *not* need five generators. There is a certain redundancy among the geometrical actions, with the result that any one of them can be constructed from suitable combinations of the others. The entire group can ultimately be constructed from only three generators. Full details are again presented in the Supplementary Material. Any readers though who have deduced the missing two geometrical actions can further think about which actions might then be constructible from some of the others.

## 5. Conclusion

Starting from the previous concept of colour equality patterns on Rubik's cubes [6, 7], we constructed the new category of colour equivalence patterns. By direct computational searches, we found that there are 18 corner patterns and 144 side piece patterns. Together also with three possible arrangements of the centres, there are therefore $18 \cdot 144 \cdot 3 = 7776$ different equivalence patterns. For the 18 corner patterns we analysed the group structure in full detail, and showed that it can be obtained from just three generators $\{X, Y, Z\}$, corresponding to relatively simple geometrical actions, or even from just two generators $\{W, Y\}$, if we are willing to accept that $W = XZ$ is geometrically more complicated than either $X$ or $Z$ alone. For the 144 side piece patterns we left most of the details for the Supplementary Material, but here also there are relatively simple geometrical actions, and ultimately just three generators are needed to create the entire group.

Let us return also to our starting point, the colour equality patterns. As already noted in the introduction, equality patterns cannot constitute one single group, since there are mutually incompatible patterns. However, it turns out that the $2 \times 2 \times 2$ corner patterns can be suitably split up into two separate groups. The Supplementary Material includes two sets $\{x, y, z\}$ and $\{U, V\}$ that allow you to construct the entire groups. We then compare the $\{X, Y, Z\}$ equivalence group with the $\{x, y, z\}$ equality group, and show that the underlying group structure is the same, although curiously enough the way the elements have to be matched up is not quite as simple as $X \leftrightarrow x$, $Y \leftrightarrow y$, $Z \leftrightarrow z$.

Finally, it is interesting to note how colour equality is simpler than colour equivalence, in terms of the original geometrical definitions, but in terms of the group structures, equivalence is simpler than equality. For equivalence all of the patterns nicely constitute one single group, whereas for equality they require this somewhat awkward splitting into separate groups. Is there some underlying reason for this? I don't know, but if anyone can discover something here, I would certainly be very interested to hear about it!

**Acknowledgement:** I thank the referee for their useful comments.

## References

[1] Bandelow, C., *Inside Rubik's Cube and Beyond*, Birkhäuser, (1982).

[2] Chen, J., *Group theory and the Rubik's cube* (online lecture notes, 2004), `tinyurl.com/Chen-cube`

[3] Cornock, C., Teaching group theory using Rubik's cubes, *Int. J. Math. Ed. Sci. Tech.* **46** (2015), 957-967.

[4] Davis, T., *Group theory via Rubik's cube* (online lecture notes, 2006), `geometer.org/rubik/group.pdf`

[5] Demaine, E. D., Demaine, M. L., Eisenstat, S., Lubiw, A. and Winslow, A., Algorithms for solving Rubik's cubes, *European Symposium on Algorithms* (2011), 689-700. `dspace.mit.edu/handle/1721.1/73771`

[6] Hollerbach, R., Colour equality patterns on Rubik's cubes, *Mathematics Today* **59** (2023), 115–117. `eprints.whiterose.ac.uk/199822/`

[7] Hollerbach, R., Farbengleichheitsmuster auf verdrehten $2 \times 2 \times 2$ und $3 \times 3 \times 3$ Rubik-Würfeln, *Mitteilungen der Deutschen Mathematiker-Vereinigung* **32** (2024), 34-37. [Paper in German. For an English translation see `eprints.whiterose.ac.uk/211314/`]

[8] Rokicki, T., Kociemba, H., Davidson, M. and Dethridge, J., The diameter of the Rubik's cube group is twenty, *SIAM Review* **56** (2014), 645-670.

Rainer Hollerbach
School of Mathematics
University of Leeds
Leeds LS2 9JT, UNITED KINGDOM
E-mail: R.Hollerbach@leeds.ac.uk

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# NOTE ON THE DIFFERENTIAL EQUATION SATISFIED BY A DAMPED HARMONIC OSCILLATOR

DAVID M. BRADLEY

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Abstract. We exhibit a simple formula that unifies all three cases of the solution to the differential equation satisfied by an undriven harmonic oscillator damped by a frictional force proportional to the velocity. A brief elementary proof that every solution can be so expressed is also provided.

## 1. Introduction

It is well known that if $\alpha$ and $\omega$ are real constants, then the real-valued function $x : \mathbf{R} \to \mathbf{R}$ satisfies the second order homogeneous linear differential equation

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega^2 x = 0 \tag{1.1}$$

if and only if there exist real constants $c_1$ and $c_2$ such that for all real $t$,

$$x(t) = e^{-\alpha t}\begin{cases} c_1\cos(t\sqrt{\omega^2-\alpha^2}) + c_2\sin(t\sqrt{\omega^2-\alpha^2}), & \text{if } \alpha^2 < \omega^2, \\ c_1 + c_2 t, & \text{if } \alpha^2 = \omega^2, \\ c_1\cosh(t\sqrt{\alpha^2-\omega^2}) + c_2\sinh(t\sqrt{\alpha^2-\omega^2}), & \text{if } \alpha^2 > \omega^2, \end{cases} \tag{1.2}$$

and that the equation (1.1) models the behavior of an undriven harmonic oscillator subject to a frictional damping force proportional to the velocity. Of course, the two cases of the solution (1.2) for which $\alpha^2 \neq \omega^2$ can be unified using complex numbers. Thus, if $\beta$ is any non-zero complex root of the equation $\alpha^2 + \beta^2 = \omega^2$, then there exist complex constants $c_1$ and $c_2$ such that for all real $t$,

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Key words and phrases: damped harmonic oscillator, existence-uniqueness

$$x(t) = e^{-\alpha t}\big(c_1\cos(\beta t) + c_2\sin(\beta t)\big) = e^{-\alpha t}\left(A_1 e^{i\beta t} + A_2 e^{-i\beta t}\right), \qquad (1.3)$$

where $A_1 = (c_1 - ic_2)/2$, $A_2 = (c_1 + ic_2)/2$, and $i^2 = -1$. However, since $x$ is assumed to be real-valued, not all combinations of complex constants are permissible in (1.3). For example, if $\beta$ is non-zero and purely imaginary (so $\alpha^2 > \omega^2$ and consequently $i\beta$ is non-zero and real) then $ic_2$ must also be real. But (see Theorem 1.1 below) by introducing the unnormalized sinc function, an entire function defined for all complex $z$ by

$$\operatorname{sinc}(z) := \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n+1)!} = \begin{cases} (\sin z)/z, & \text{if } z \neq 0, \\ 1, & \text{if } z = 0, \end{cases} \qquad (1.4)$$

all three cases of the solution (1.2) can be unified via the single formula

$$x(t) = e^{-\alpha t}\big(\kappa\cos(\beta t) + \lambda t\operatorname{sinc}(\beta t)\big), \qquad (1.5)$$

where $\beta$ is any root of the equation $\alpha^2 + \beta^2 = \omega^2$, and $\kappa$ and $\lambda$ are arbitrary real constants.

Treatments of the differential equation (1.1) usually derive the three cases (1.2) of the solution separately [13], often by employing either the Laplace transform, the exponential ansatz, or operational methods [2, 6, 7]. Alternatively [10, 12], one can first make a change of variable that reduces the differential equation (1.1) to the frictionless case $\alpha = 0$. See [14] for a brief outline of several of the aforementioned approaches.

The solution in the critically damped case $\alpha^2 = \omega^2$ is almost universally derived either ab initio, or as a limiting case of one of the other two cases [1, 2, 3, 8, 13, 17]. The fact that all solutions of (1.1) necessarily have the form (1.2) often goes unmentioned, or is briefly stated without proof as a consequence of more general uniqueness results—either for the inverse Laplace transform or for a wider class of linear differential equations with specified initial conditions. In light of this state of affairs, and the relative simplicity of the compact form (1.5) of the solution—which permits treating all three cases simultaneously on an equal footing, with no need to distinguish separate cases or resort to limiting arguments—we believe there may be some pedagogical value in providing a brief elementary proof of the following result.

**Theorem 1.1.** *Let $\alpha \in \mathbf{R}$, let $\omega \in \mathbf{R}$, and let $\beta \in \mathbf{C}$ be such that $\alpha^2 + \beta^2 = \omega^2$. Then the function $x : \mathbf{R} \to \mathbf{R}$ satisfies the differential equation* (1.1) *if and only if there exist constants $\kappa \in \mathbf{R}$ and $\lambda \in \mathbf{R}$ such that for all $t \in \mathbf{R}$, $x(t)$ is given by the formula* (1.5).

## 2. The Proof

*Proof.* Let $x : \mathbf{R} \to \mathbf{R}$ satisfy the differential equation (1.1). Define functions $\kappa = \kappa(t)$ and $\lambda = \lambda(t)$ for $t \in \mathbf{R}$ by the equations

$$\begin{aligned} \kappa &= \big(x\cos(\beta t) \;-\; \big(x' + \alpha x\big)\, t \operatorname{sinc}(\beta t)\big)\, e^{\alpha t}, \\ \lambda &= \big(\beta x \sin(\beta t) + \big(x' + \alpha x\big)\cos(\beta t)\big)\, e^{\alpha t}, \end{aligned} \tag{2.1}$$

in which the customary abbreviations $x = x(t)$, $x' = x'(t) = dx/dt$ are employed to reduce parenthetical clutter. Note that since $\alpha$ and $\omega$ are both real and $\alpha^2 + \beta^2 = \omega^2$, it follows that $\beta$ is either real or purely imaginary; i.e. $\beta \in \mathbf{R}$ or $i\beta \in \mathbf{R}$. Thus, as $\cos(\beta t)$, $\operatorname{sinc}(\beta t)$, and $\beta \sin(\beta t)$ are all even functions of $\beta$, and as $\alpha$ is real and $x$ is real-valued, it follows that $\kappa$ and $\lambda$ as given by (2.1) are both real-valued functions of the real variable $t$. In fact, we shall presently see that $\kappa$ and $\lambda$ are constant functions; i.e. $\kappa \in \mathbf{R}$ and $\lambda \in \mathbf{R}$. To this end, first observe that for any complex constant $\beta$,

$$\begin{aligned} \frac{d}{dt}\big(t \operatorname{sinc}(\beta t)\big) &= \frac{d}{dt}\sum_{n=0}^{\infty}(-1)^n \frac{\beta^{2n} t^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(\beta t)^{2n}}{(2n)!} = \cos(\beta t), \end{aligned} \tag{2.2}$$

and

$$\frac{d}{dt}\cos(\beta t) = -\beta \sin(\beta t) = -\beta^2 t \operatorname{sinc}(\beta t). \tag{2.3}$$

With the help of the derivative formulas (2.2) and (2.3), the defining equations (2.1) for $\kappa$ and $\lambda$ imply that the derivatives $\kappa' = \kappa'(t) = d\kappa/dt$ and $\lambda' = \lambda'(t) = d\lambda/dt$ are given by

$$\begin{aligned} \kappa' &= \big(x'\cos(\beta t) - \beta^2 x t \operatorname{sinc}(\beta t) - \big(x'' + \alpha x'\big)\, t \operatorname{sinc}(\beta t) \\ &\qquad - \big(x' + \alpha x\big)\cos(\beta t)\big)\, e^{\alpha t} + \big(x\cos(\beta t) - \big(x' + \alpha x\big)\, t \operatorname{sinc}(\beta t)\big)\, \alpha e^{\alpha t} \\ &= -\big(x'' + 2\alpha x' + (\alpha^2 + \beta^2)x\big)\, t e^{\alpha t} \operatorname{sinc}(\beta t) \end{aligned}$$

and

$$\begin{aligned}\lambda' &= \big(\beta x' \sin(\beta t) + \beta^2 x \cos(\beta t) + \big(x'' + \alpha x'\big)\cos(\beta t)\\ &\qquad - \big(x' + \alpha x\big)\,\beta \sin(\beta t)\big)\,e^{\alpha t} + \big(\beta x \sin(\beta t) + \big(x' + \alpha x\big)\cos(\beta t)\big)\,\alpha e^{\alpha t}\\ &= \big(x'' + 2\alpha x' + (\alpha^2 + \beta^2)x\big)\,e^{\alpha t}\cos(\beta t).\end{aligned}$$

In view of the relationship $\alpha^2 + \beta^2 = \omega^2$ and the differential equation (1.1), it follows that $\kappa'$ and $\lambda'$ both vanish identically. Consequently, $\kappa$ and $\lambda$ are both independent of $t$. Having seen already that $\kappa$ and $\lambda$ are both real-valued, we may regard $\kappa$ and $\lambda$ as real constants by identifying each with its unique value as a constant function. Eliminating $x'$ from the equations (2.1) now yields the representation (1.5).

For the converse, let $\kappa \in \mathbf{R}$ and $\lambda \in \mathbf{R}$ be constants, and suppose that the function $x$ is defined for all real $t$ by the formula (1.5). Again, since $\alpha$ and $\omega$ are both real and $\alpha^2 + \beta^2 = \omega^2$, $\beta$ is real or purely imaginary. Thus, as $\cos(\beta t)$ and $\operatorname{sinc}(\beta t)$ are both even functions of $\beta$, it follows that $x$ is real-valued. Clearly $x$ as given by (1.5) is also differentiable to all orders, and one easily verifies that it satisfies the differential equation (1.1). □

## 3. Examples

**Example 3.1** (Frictionless mass-spring system)**.** A frictionless mass-spring system with positive mass $m$ and non-negative spring constant $k$ is modeled by the differential equation $mx'' + kx = 0$, which can be rewritten in the form (1.1) with $\alpha = 0$ and $\omega = \sqrt{k/m}$. By the principle of superposition, every solution can be written as a linear combination of any two linearly independent solutions. If $\omega > 0$, then the solutions $x_1(t) = \cos(\omega t)$, $x_2(t) = \sin(\omega t)$ are linearly independent and are typically employed for this purpose. With this choice, one writes $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, where $c_1$ and $c_2$ are constants that are determined by the initial conditions via the equations $c_1 = x(0)$, $\omega c_2 = x'(0)$. If $\omega = 0$, then the latter equation fails to determine $c_2$, and moreover the solutions $x_1$ and $x_2$ are no longer linearly independent. Thus a separate treatment of the degenerate case $\omega = 0$ is necessary with this approach. While this is simple enough to carry out, we nevertheless advocate instead for employing the solutions $x_1(t) = \cos(\omega t)$, $x_2(t) = t \operatorname{sinc}(\omega t)$, which are linearly independent regardless of the value of $\omega$. By Theorem 1.1 we can now write $x(t) = c_1 \cos(\omega t) + c_2\, t \operatorname{sinc}(\omega t)$. It is easy to see that $c_1 = x(0)$, $c_2 = x'(0)$, and no separate treatment of the degenerate case $\omega = 0$ is required.

**Example 3.2** (Forced harmonic oscillator)**.** Let $\gamma$ and $\omega$ be positive constants. An undamped harmonic oscillator with natural frequency $\omega$ driven by an external force of unit magnitude and frequency $\gamma$ satisfies the inhomogeneous linear differential equation $x'' + \omega^2 x = \cos(\gamma t)$. For simplicity, suppose that the initial conditions $x(0) = x'(0) = 0$ are satisfied. If the frequencies $\gamma$ and $\omega$ are distinct, then the solution can be written in either of the two equivalent forms

$$\begin{aligned} x(t) &= \frac{\cos(\gamma t) - \cos(\omega t)}{\omega^2 - \gamma^2} \\ &= \frac{2}{\omega^2 - \gamma^2} \sin\left(\left(\frac{\omega - \gamma}{2}\right) t\right) \sin\left(\left(\frac{\omega + \gamma}{2}\right) t\right). \end{aligned} \tag{3.1}$$

From a numerical point of view, both representations (3.1) are somewhat awkward to deal with in the case of practical resonance (i.e., when $\gamma$ is nearly equal to $\omega$). The alternative representation

$$x(t) = \frac{t}{\omega + \gamma} \operatorname{sinc}\left(\left(\frac{\omega - \gamma}{2}\right) t\right) \sin\left(\left(\frac{\omega + \gamma}{2}\right) t\right) \tag{3.2}$$

seems preferable, since it is valid even if $\gamma = \omega \neq 0$, and makes the phenomenon of practical resonance readily apparent. Since $\operatorname{sinc}(0) = 1$, the representation (3.2) (as opposed to either representation in (3.1)) also exposes the form of the solution in the case of pure resonance ($\gamma = \omega$) as

$$x(t) = \frac{t \sin(\omega t)}{2\omega}$$

without the need for computing a limit. One could go even further and rewrite (3.2) in the equivalent form

$$x(t) = \frac{1}{2} t^2 \operatorname{sinc}\left(\left(\frac{\omega - \gamma}{2}\right) t\right) \operatorname{sinc}\left(\left(\frac{\omega + \gamma}{2}\right) t\right),$$

so that the degenerate case

$$x(t) = \frac{1}{2} t^2$$

(in which $\gamma = \omega = 0$) can be readily ascertained, again without the need for computing a limit.

## Concluding comments

The nomenclature "sinc" for a normalized version of (1.4) was apparently first proposed in [16, eq. (24)]. Our unnormalized sinc function coincides with the zeroth-order spherical Bessel function of the first kind [9, Eq. 10.49.3], and may also be denoted by sinc in the literature [11]. (See also [9, Eq. 3.3.44] and [15].)

In [4] and [5], Fourier transforms of products of sinc functions are evaluated (respectively using the inclusion-exclusion principle and repeated integration by parts) to determine the distribution of the sum of $n$ non-identically distributed uniform random variables. Several additional problems in which the sinc function naturally arises are described in [11].

It is easy to see that $\kappa = x(0)$ and $\lambda = x'(0) + \alpha x(0)$ in (1.5). One can motivate the definition of the functions $\kappa$ and $\lambda$ in (2.1) by differentiating (1.5) with respect to $t$ while assuming $\kappa$ and $\lambda$ are both constant, and then solving the resulting system of two equations for $\kappa$ and $\lambda$.

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## References

[1] Balasubramanian, S., Note on the equation $\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0$, *Am. J. Phys.* **43** (1975), 830, https://doi.org/10.1119/1.9695

[2] Balasubramanian, S., Fatehally, R., Comment on 'On the solution of the equation for damped oscillation,' *Am. J. Phys.* **44** (1976), 705, https://doi.org/10.1119/1.10167

[3] Baslaw, R. S., Hastings, H. M., On the critically damped oscillator, *Math. Mag.* **48** (1975), 105–106, https://doi.org/10.2307/2689831

[4] Bradley, D. M., Some remarks on sinc integrals and their connection with combinatorics, geometry and probability, *Analysis*, **22** (2002), no. 2, 219–224, https://doi.org/10.1524/anly.2002.22.2.219

[5] Bradley, D. M., Gupta, R. C., On the distribution of the sum of $n$ non-identically distributed uniform random variables, *Ann. Inst. Statist. Math.* **54** (2002), no. 3, 689–700, https://doi.org/10.1023/A:1022483715767

[6] Garrison, J. D., On the solution of the equation for damped oscillation, *Am. J. Phys.* **42** (1974), 694, http://doi.org/10.1119/1.1987808

[7] Garrison, J. D., Erratum: On the solution of the equation for damped oscillation, *Am. J. Phys.* **43** (1975), 463, https://doi.org/10.1119/1.10110

[8] Maor-Metzger, E., Solomon, A., Some remarks on the equation $\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0$, *Am. J. Phys.* **40** (1972), 1699–1700, https://doi.org/10.1119/1.1987023

[9] Olver, F. W. J. et al, eds. *NIST Digital Library of Mathematical Functions*, Release 1.2.4 of 2025-03-15, https://dlmf.nist.gov/

[10] Gauthier, N., Novel approach for solving the equation of motion of a simple harmonic oscillator, *Int. J. Math. Educ. Sci. Technol.* **35** (2004), 446–452, https://doi.org/10.1080/00207390410001686580

[11] Gearhart, W. B., Shultz, H. S., The function $\frac{\sin x}{x}$, *College Math. J.* **21** (1990), 90–99, https://doi.org/10.1080/07468342.1990.11973290

[12] Greenberg, H., Further remarks concerning the solution of the equation $\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0$, *Am. J. Phys.* **44** (1976), 1135–1136, https://doi.org/10.1119/1.10209

[13] Dutta Roy, S. C., Transient response of RLC networks revisited, *IETE J. Educ.* **44** (2003), 207–211, https://doi.org/10.1080/09747338.2003.11689766

[14] Dutta Roy, S. C., Many ways of solving an ordinary linear second order differential equation with constant coefficients, *IETE J. Educ.* **48** (2007), 73–76, https://doi.org/10.1080/09747338.2007.11657868

[15] Stenger, F., Numerical methods based on Whittaker cardinal, or sinc functions, *SIAM Review*, **23** (1981), 165–224, https://doi.org/10.1137/1023037

[16] Woodward, P. M., Davies, I. L., Information theory and inverse probability in telecommunication, *IEE Proceedings Part III: Radio and Communication Engineering* **99** (1952), 37–44, https://doi.org/10.1049/pi-3.1952.0011

[17] Zrnic, D. S., Additional remarks on the equation: $\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0$, *Am. J. Phys.* **41** (1973), 712, https://doi.org/10.1119/1.1987338

David M. Bradley
Department of Mathematics and Statistics
University of Maine
5752 Neville Hall
Orono, ME 04469, U.S.A.
E-mail: david.bradley@maine.edu

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# ON AN IDENTITY FOR CLASSICAL HARMONIC NUMBERS

NANDAN SAI DASIREDDY

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Abstract. The remarkable harmonic series formula (proved by Furdui and Sîntămărian in 2023) and the first part of open problem 3.105 in the recent monograph *Sharpening mathematical analysis skills*, concerns the calculation of a series involving consecutive classical harmonic numbers

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+1} H_{n+2}}{n\,(n+1)\,(n+2)} = \frac{5}{8}\zeta\,(4) + \frac{5}{4}\zeta\,(3) - \frac{1}{2}\zeta\,(2)\,.$$

In this paper we provide a new proof of this harmonic series formula. Our proof is reminiscent of author's proof in his paper [2].

## 1. Introduction

The main concern of this article is the following remarkable harmonic series formula:

**Theorem 1.1.** $$\sum_{n=1}^{\infty} \frac{H_n H_{n+1} H_{n+2}}{n\,(n+1)\,(n+2)} = \frac{5}{8}\zeta\,(4) + \frac{5}{4}\zeta\,(3) - \frac{1}{2}\zeta\,(2)\,.$$

The classical proof of this result can be found in [6]. Our goal in this paper is to give a new proof of this result and this new approach is robust and depends on the partial fraction decomposition.

Throughout this paper, $H_n$ denotes the $nth$-classical harmonic number de?ned by $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and $\zeta\,(s)$ denotes the Riemann zeta function, which is defined by $\zeta\,(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) \geq 1.$

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1.1. **Organization of the paper.** The paper is organized as follows: In Section 2 several linear and nonlinear harmonic sums are established, while in Section 3 we treat the proof of Theorem 1.1.

In the following, the linear harmonic sums and nonlinear harmonic sums will be applied extensively in the proofs of lemma and theorem.

(i) $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$ [4, pp. 471-472];

(ii) $\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4)$ [5, pp. 207-208];

(iii) $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4)$ [3, pp. 267-268];

(iv) $\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)} = 3\zeta(3)$ [7, Corollary 2, p. 111];

(v) $\sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)(n+2)} = 1 + \zeta(2)$ [8, (8)];

(vi) $\sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{11}{4}\zeta(4)$ [9, Equation (2)];

(vii) $\sum_{n=1}^{\infty} \frac{H_n^3}{n(n+1)(n+2)} = 5\zeta(4) - 2\zeta(3) - \zeta(2) - \frac{1}{2}$ [10, Remark 2.1].

## 2. Preliminaries

In this section we focus on establishing the required intermediary lemma.

**Lemma 2.1.** *The following identities hold:*

*(a)* $\sum_{n=1}^{\infty} \left(\frac{H_{n+2}}{n+2}\right)^2 = \frac{17}{4}\zeta(4) - \frac{25}{16}$;

*(b)* $\sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)^2}\right)^2 = \zeta(4) - 2\zeta(3) + \frac{23}{16}$;

*(c)* $\sum_{n=1}^{\infty} \frac{H_{n+2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right)}{(n+2)^2} = -2\zeta(3) + \frac{5}{4}\zeta(4) + \frac{23}{16}$;

*(d)* $\sum_{n=1}^{\infty} \frac{H_n^2}{(n+2)^2} = \frac{11}{4}\zeta(4) + 2\zeta(3) - 3$;

*(e)* $\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)(n+2)} = \frac{3}{2}\zeta(3) - \frac{1}{2}\zeta(2) - \frac{1}{2}$;

*(f)* $$\sum_{n=1}^{\infty}\frac{2H_n^2}{n\left(n+1\right)^2\left(n+2\right)}+\sum_{n=1}^{\infty}\frac{H_n^2}{n\left(n+1\right)\left(n+2\right)^2} = -\frac{33}{8}\zeta\left(4\right)+\frac{19}{4}\zeta\left(3\right)+\frac{1}{4}\zeta\left(2\right)-\frac{5}{4};$$

*(g)* $$\sum_{n=1}^{\infty}\frac{H_n}{n\left(n+1\right)^3\left(n+2\right)}+\sum_{n=1}^{\infty}\frac{H_n}{n\left(n+1\right)^2\left(n+2\right)^2} = -\frac{1}{4}\zeta\left(4\right)-\frac{3}{2}\zeta\left(3\right)+\frac{1}{4}\zeta\left(2\right)+\frac{7}{4}.$$

*Proof.* (a) $$\sum_{n=1}^{\infty}\left(\frac{H_{n+2}}{n+2}\right)^2=\sum_{i=1}^{\infty}\frac{H_i^2}{i^2}-\frac{25}{16} = \frac{17}{4}\zeta\left(4\right)-\frac{25}{16}.$$

(b) By partial fraction decomposition, we have,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\left(n+1\right)\left(n+2\right)}+\frac{1}{\left(n+2\right)^2}\right)^2=\sum_{n=1}^{\infty}\frac{1}{\left(\left(n+1\right)\left(n+2\right)\right)^2}+ \sum_{n=1}^{\infty}\frac{1}{\left(n+2\right)^4}+\sum_{n=1}^{\infty}\frac{2}{\left(n+1\right)\left(n+2\right)^3}$$

$$=\sum_{n=1}^{\infty}\frac{1}{\left(n+1\right)^2}-\sum_{n=1}^{\infty}\frac{1}{\left(n+2\right)^2}-\sum_{n=1}^{\infty}\frac{2}{\left(n+2\right)^3}+\sum_{n=1}^{\infty}\frac{1}{\left(n+2\right)^4}$$

$$=\left(\sum_{j=1}^{\infty}\frac{1}{j^2}-1\right)-\left(\sum_{j=1}^{\infty}\frac{1}{j^2}-\frac{5}{4}\right)-2\left(\sum_{j=1}^{\infty}\frac{1}{j^3}-\frac{9}{8}\right)+ \left(\sum_{j=1}^{\infty}\frac{1}{j^4}-\frac{17}{16}\right)$$

$$=\zeta(4)-2\zeta(3)+\frac{23}{16}.$$

(c) The infinite series in Lemma 2.1 part (c) is rewritten successively as follows:

$$\sum_{n=1}^{\infty}H_{n+2}\left(\frac{1}{\left(n+1\right)\left(n+2\right)^2}+\frac{1}{\left(n+2\right)^3}\right)$$

$$=\sum_{n=1}^{\infty}H_{n+2}\left(\frac{1}{\left(n+1\right)\left(n+2\right)}-\frac{1}{\left(n+2\right)^2}+\frac{1}{\left(n+2\right)^3}\right)$$

$$=\sum_{n=1}^{\infty}H_{n+2}\left(\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{\left(n+2\right)^2}+\frac{1}{\left(n+2\right)^3}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_{n+2}}{n+1} - \sum_{n=1}^{\infty} \frac{H_{n+2}}{n+2} - \sum_{n=1}^{\infty} \frac{H_{n+2}}{(n+2)^2} + \sum_{n=1}^{\infty} \frac{H_{n+2}}{(n+2)^3}$$

$$= \sum_{n=1}^{\infty} \frac{H_{n+1}}{n+1} + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} - \sum_{n=1}^{\infty} \frac{H_{n+2}}{n+2} - \sum_{n=1}^{\infty} \frac{H_{n+2}}{(n+2)^2} + \sum_{n=1}^{\infty} \frac{H_{n+2}}{(n+2)^3}$$

$$= \left(\sum_{m=1}^{\infty} \frac{H_m}{m} - 1\right) + \frac{1}{2} - \left(\sum_{m=1}^{\infty} \frac{H_m}{m} - \frac{7}{4}\right) - \left(\sum_{m=1}^{\infty} \frac{H_m}{m^2} - \frac{11}{8}\right) \left(+ \sum_{m=1}^{\infty} \frac{H_m}{m^3} - \frac{19}{16}\right)$$

$$= -2\zeta(3) + \frac{5}{4}\zeta(4) + \frac{23}{16}.$$

(d) $$\sum_{n=1}^{\infty} \frac{H_n^2}{(n+2)^2} = \sum_{n=1}^{\infty} \frac{\left(H_{n+2} - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)\right)^2}{(n+2)^2}$$

$$= \sum_{n=1}^{\infty} \left(\frac{H_{n+2}}{n+2}\right)^2 + \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)^2}\right)^2 - 2\sum_{n=1}^{\infty} \frac{H_{n+2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right)}{(n+2)^2}$$

$$= \frac{11}{4}\zeta(4) + 2\zeta(3) - 3.$$

(e) By partial fraction decomposition, we have,

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)(n+2)} = \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)} - \sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)(n+2)}\right] = \frac{3}{2}\zeta(3) - \frac{1}{2}\zeta(2) - \frac{1}{2}.$$

(f) By partial fraction decomposition, we have,

$$2\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)^2(n+2)} + \sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)(n+2)^2}$$

$$= 2\left(\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)^2} - \sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)(n+2)}\right) +$$

$$\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)(n+2)}-\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(n+2)^2}\right)$$
$$=2\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)}-\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}\right)-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)(n+2)}+$$
$$\frac{1}{2}\left(-\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(n+2)}+\sum_{n=1}^{\infty}\frac{H_n^2}{(n+2)^2}\right)$$
$$=\frac{11}{8}\zeta(4)+\frac{19}{4}\zeta(3)+\frac{1}{4}\zeta(2)-\frac{5}{4}-2\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}$$
$$=-\frac{33}{8}\zeta(4)+\frac{19}{4}\zeta(3)+\frac{1}{4}\zeta(2)-\frac{5}{4}.$$

(g) By partial fraction decomposition, we have,

$$\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)^3(n+2)}+\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)^2(n+2)^2}$$
$$=-\frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n}{n+2}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n}{n}-\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{(n+2)^2}$$
$$=\frac{3}{4}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}-\sum_{n=1}^{\infty}\frac{H_{n+2}}{n+2}+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}+\sum_{n=1}^{\infty}\frac{1}{(n+2)^2}\right)$$
$$-\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}-\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^3}+\sum_{n=1}^{\infty}\frac{1}{(n+1)^4}$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_{n+2}}{(n+2)^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{n+1}+\frac{1}{n+2}\right)}{(n+2)^2}$$
$$=\frac{3}{4}\left(\frac{7}{4}+\frac{1}{2}+\zeta(2)-\frac{5}{4}\right)-2\zeta(3)+\zeta(3)-\frac{5}{4}\zeta(4)+\zeta(4)-$$
$$\zeta(3)+\frac{11}{16}+\frac{\zeta(3)}{2}+\frac{5}{16}-\frac{\zeta(2)}{2}$$
$$=-\frac{1}{4}\zeta(4)-\frac{3}{2}\zeta(3)+\frac{1}{4}\zeta(2)+\frac{7}{4}.$$

□

## 3. Proof of Theorem 1.1

*Proof.* The infinite series in Theorem 1.1 is rewritten successively as follows:

$$\sum_{n=1}^{\infty}\frac{\left(H_n^2+\frac{H_n}{n+1}\right)\left(H_n+\frac{1}{n+1}+\frac{1}{n+2}\right)}{n\left(n+1\right)\left(n+2\right)}$$

$$=\sum_{n=1}^{\infty}\frac{H_n^3}{n\left(n+1\right)\left(n+2\right)}+2\sum_{n=1}^{\infty}\frac{H_n^2}{n\left(n+1\right)^2\left(n+2\right)}+\sum_{n=1}^{\infty}\frac{H_n^2}{n\left(n+1\right)\left(n+2\right)^2}$$
$$+\sum_{n=1}^{\infty}\frac{H_n}{n\left(n+1\right)^3\left(n+2\right)}+\sum_{n=1}^{\infty}\frac{H_n}{n\left(n+1\right)^2\left(n+2\right)^2}$$
$$=\sum_{n=1}^{\infty}\frac{H_n^3}{n\left(n+1\right)\left(n+2\right)}-\frac{33}{8}\zeta\left(4\right)+\frac{19}{4}\zeta\left(3\right)+\frac{1}{4}\zeta\left(2\right)-\frac{5}{4}+$$
$$\sum_{n=1}^{\infty}\frac{H_n}{n\left(n+1\right)^3\left(n+2\right)}+\sum_{n=1}^{\infty}\frac{H_n}{n\left(n+1\right)^2\left(n+2\right)^2}\text{ [by Lemma 2.1 part (f)]}$$
$$=\sum_{n=1}^{\infty}\frac{H_n^3}{n\left(n+1\right)\left(n+2\right)}-\frac{33}{8}\zeta\left(4\right)+\frac{19}{4}\zeta\left(3\right)+\frac{1}{4}\zeta\left(2\right)-\frac{5}{4}-$$
$$\frac{1}{4}\zeta\left(4\right)-\frac{3}{2}\zeta\left(3\right)+\frac{1}{4}\zeta\left(2\right)+\frac{7}{4}\qquad\text{[by Lemma 2.1 part (g)]}$$
$$=\sum_{n=1}^{\infty}\frac{H_n^3}{n\left(n+1\right)\left(n+2\right)}-\frac{35}{8}\zeta\left(4\right)+\frac{13}{4}\zeta\left(3\right)+\frac{1}{2}\zeta\left(2\right)+\frac{1}{2}$$
$$=5\zeta\left(4\right)-2\zeta\left(3\right)-\zeta\left(2\right)-\frac{35}{8}\zeta\left(4\right)+\frac{13}{4}\zeta\left(3\right)+\frac{1}{2}\zeta\left(2\right)$$
$$=\frac{5}{8}\zeta\left(4\right)+\frac{5}{4}\zeta\left(3\right)-\frac{1}{2}\zeta\left(2\right),$$

and the theorem is proved.

□

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## References

[1] Furdui, O., Sîntămărian, A., Sharpening mathematical analysis skills, Problem Books in Mathematics, Springer, Cham, First Edition 2021.

[2] Dasireddy, N. S., An alternating sum with three consecutive harmonic numbers, *Gazeta Matematică, Seria A* **XLII(CXXI)**(1-2) (2024), 23–33.

[3] Kneser, M., A summation problem, Solution to problem 4305, *Amer. Math. Monthly* **57**(4) (1950), 265–272. doi: 10.1080/00029890.1950.11999528

[4] Steinberg, R., A summation problem, Solution to problem 4431, *Amer. Math. Monthly* **59**(7) (1952), 469–473. doi: 10.1080/00029890.1952.11988166

[5] Furdui, O., Limits, Series, and Fractional Part Integrals: Problems in Mathematical Analysis, Problem Books in Mathematics, Springer, New York, First Edition 2013.

[6] Furdui, O., Sîntămărian, A., A series involving a product of three consecutive harmonic numbers, *Gazeta Matematică, Seria A* **XLI(CXX)**(1-2) (2023), 35–38.

[7] Sofo, A., Hassani, M., Quadratic harmonic number sums, *Appl. Math. E-Notes* **12** (2012), 110–117. Available from: https://www.math.nthu.edu.tw/~amen/2012/110712-2.pdf

[8] Wang, X., Chu, W., Infinite series identities involving quadratic and cubic harmonic numbers, *Publ. Mat.* **62**(1) (2018), 285–300. doi: 10.5565/PUBLMAT6211813

[9] Borwein, D., Borwein, J. M., On an intriguing integral and some series related to $\zeta(4)$, *Proc. Amer. Math. Soc.* **123**(4) (1995), 1191–1198.

[10] Sofo, A., Cubic Alternating Harmonic Number sums, *Aust. J. Math. Anal. Appl.* **16**(1) (2019), Article 14, 14pp.



Nandan Sai Dasireddy
House No. 3-11-363, Pavani Plaza, Flat No:401,
Road No. 2, Shivaganga Colony, Siris Road,
L.B. Nagar, Hyderabad, Telangana, India.
E-mail: dasireddy.1818@gmail.com

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# ELEMENTARY PROOFS OF TWO CONGRUENCES FOR PARTITIONS WITH ODD PARTS REPEATED AT MOST TWICE

JAMES A. SELLERS

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Abstract. In a recent article on overpartitions, Merca considered the auxiliary function $a(n)$ which counts the number of partitions of $n$ where odd parts are repeated at most twice (and there are no restrictions on the even parts). In the course of his work, Merca proved the following: For all $n \geq 0$,

$$a(4n+2) \equiv 0 \pmod{2}, \quad \text{and}$$
$$a(4n+3) \equiv 0 \pmod{2}.$$

Merca then indicates that a classical proof of these congruences would be very interesting. The goal of this short note is to fulfill Merca's request by providing two truly elementary (classical) proofs of these congruences.

## 1. Introduction

A *partition* of a positive integer $n$ is a finite non–increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We refer to the integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ as the *parts* of the partition. For example, the number of partitions of the integer $n = 4$ is 5, and the partitions counted in that instance are as follows:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

The number of partitions of $n$ is denoted $p(n)$, and the corresponding generating function is given by

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{f_1} \tag{1.1}$$

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where $f_r = (1-q^r)(1-q^{2r})(1-q^{3r})\dots$ for any positive integer $r$.

An *overpartition* of a positive integer $n$ is a partition of $n$ wherein the first occurrence of a part may be overlined. For example, the number of overpartitions of $n=4$ is 14 given the following list of overpartitions of 4:

$$4, \quad \overline{4}, \quad 3+1, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{3}+\overline{1},$$

$$2+2, \quad \overline{2}+2, \quad 2+1+1, \quad \overline{2}+1+1, \quad 2+\overline{1}+1, \quad \overline{2}+\overline{1}+1,$$

$$1+1+1+1, \quad \overline{1}+1+1+1$$

The number of overpartitions of $n$ is often denoted $\overline{p}(n)$, so from the example above we see that $\overline{p}(4)=14$. As noted by Corteel and Lovejoy [2], the generating function for $\overline{p}(n)$ is given by

$$\sum_{n\geq 0}\overline{p}(n)q^n = \frac{f_2}{f_1^2}.$$

In recent work on overpartitions [5], Merca considered the auxiliary function $a(n)$ which counts the number of partitions of weight $n$ wherein no part is congruent to 3 modulo 6. From this definition, it is clear that the generating function for $a(n)$ is given by

$$\sum_{n\geq 0}a(n)q^n = \prod_{i\geq 1}\frac{(1-q^{6i-3})}{(1-q^i)} = \prod_{i\geq 1}\frac{(1-q^{6i-3})}{(1-q^i)}\cdot\frac{(1-q^{6i})}{(1-q^{6i})} = \frac{f_3}{f_1 f_6}. \tag{1.2}$$

The function $a(n)$ also counts the number of partitions of $n$ where odd parts are repeated at most twice (and there are no restrictions on the even parts). See [6, A131945] for more information.

In [5], Merca proved the following two generating function identities:

**Theorem 1.1.** *We have*

$$\sum_{n\geq 0}a(4n+2)q^n = 2\frac{f_3f_4^3f_6^2f_{24}}{f_1^2f_2^3f_8f_{12}^2} + 4q\frac{f_6^5f_{24}^2}{f_1^3f_2^2f_{12}^3} - 2q^2\frac{f_3f_4^3f_{24}^4}{f_1^2f_2^2f_6f_8^2f_{12}^2} + 4q^3\frac{f_6^2f_{24}^5}{f_1^3f_2^5f_8f_{12}^3} - 8q^5\frac{f_{24}^8}{f_1^3f_6f_8^2f_{12}^3}.$$

**Theorem 1.2.** *We have*

$$\sum_{n\geq 0}a(4n+3)q^n = 2\frac{f_4^8f_6^6f_{24}^3}{f_1f_2^7f_8^3f_{12}^7} + 8q\frac{f_4^5f_6^9f_{24}^4}{f_1^2f_2^6f_3f_8^2f_{12}^8} - 2q^2\frac{f_4^8f_6^3f_{24}^6}{f_1f_2^6f_8^4f_{12}^7} - 16q^3\frac{f_4^5f_6^6f_{24}^7}{f_1^2f_2^5f_3f_8^3f_{12}^8} + 8q^5\frac{f_4^5f_6^3f_{24}^{10}}{f_1^2f_2^4f_3f_8^4f_{12}^8}.$$

In [5], Merca relied completely on the work of Radu [7] and Smoot [8] to provide “automated” proofs of Theorem 1.1 and Theorem 1.2. While these proofs are indeed correct, they do not provide any insights into the results themselves. (In particular, the technique provides no guidance as to whether simpler generating function identities exist.) In this case, the “power” of these techniques is, in essence, unnecessary.

As corollaries of Theorem 1 and Theorem 2, Merca then noted the following:

**Corollary 1.3.** *For all $n \geq 0$,*

$$a(4n+2) \equiv 0 \pmod{2}, \quad \textit{and}$$
$$a(4n+3) \equiv 0 \pmod{2}.$$

At the end of [5, Section 4], Merca indicates that a classical proof of these congruences would be very interesting. The goal of this short note is to fulfill Merca's request by providing two truly elementary (classical) proofs of Corollary 1.3.

In Section 2, we give brief elementary proofs of the following generating function results (which are much simpler than those in Theorem 1.1 and Theorem 1.2):

**Theorem 1.4.** *We have*

$$\sum_{n\geq 0} a(4n+2)q^n = 2\frac{f_2 f_6^2 f_8^2}{f_1^4 f_3 f_{12}}.$$

**Theorem 1.5.** *We have*

$$\sum_{n\geq 0} a(4n+3)q^n = 2\frac{f_2^4 f_8^2 f_{12}}{f_1^5 f_4^2 f_6}.$$

Clearly, Corollary 1.3 follows immediately from Theorem 1.4 and Theorem 1.5.

Then, in Section 3, we return to the congruences stated in Corollary 1.3 and recall that there are no squares in the arithmetic progressions $4n+2$ and $4n+3$. With this in mind, we link the generating function for $a(n)$ with a particular theta function which then immediately gives a new proof of these divisibilities (and sheds some light as to why these particular arithmetic progressions arise).

## 2. Proofs via Simplified Generating Function Identities

In order to prove Theorem 1.4 and Theorem 1.5, we require two elementary dissection lemmas.

**Lemma 2.1.** *We have*

$$\frac{f_3}{f_1} = \frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}} + q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}.$$

*Proof.* See da Silva and Sellers [3, Lemma 1, (14)]. □

**Lemma 2.2.** *We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8}.$$

*Proof.* See da Silva and Sellers [3, Lemma 1, (11)]. □

These are all the tools needed to complete our proofs in this section.

*Proof.* (of Theorem 1.4 and Theorem 1.5) We begin by finding the 2–dissection of the generating function for $a(n)$, using (1.2) as our starting point.

$$\begin{aligned}
\sum_{n\geq 0} a(n)q^n &= \frac{f_3}{f_1f_6} \\
&= \frac{1}{f_6}\left(\frac{f_3}{f_1}\right) \\
&= \frac{1}{f_6}\left(\frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}} + q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}\right) \qquad \text{(Lemma 2.1)} \\
&= \frac{f_4f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}} + q\frac{f_8^2f_{48}}{f_2^2f_{16}f_{24}}.
\end{aligned}$$

Thus, we can immediately read off the following:

$$\sum_{n\geq 0} a(2n)q^n = \frac{f_2f_8f_{12}^2}{f_1^2f_4f_6f_{24}},$$

$$\sum_{n\geq 0} a(2n+1)q^n = \frac{f_4^2f_{24}}{f_1^2f_8f_{12}}.$$

We now perform two additional 2–dissections. First,

$$\begin{aligned}\sum_{n\geq 0} a(2n)q^n &= \frac{f_2f_8f_{12}^2}{f_4f_6f_{24}}\left(\frac{1}{f_1^2}\right)\\ &= \frac{f_2f_8f_{12}^2}{f_4f_6f_{24}}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right) \quad \text{(Lemma 2.2)}\\ &= \frac{f_8^6f_{12}^2}{f_2^4f_4f_6f_{16}^2f_{24}}+2q\frac{f_4f_{12}^2f_{16}^2}{f_2^4f_6f_{24}}.\end{aligned}$$

We then see that

$$\sum_{n\geq 0} a(4n+2)q^n = 2\frac{f_2f_6^2f_8^2}{f_1^4f_3f_{12}}$$

and this proves Theorem 1.4. Finally,

$$\begin{aligned}\sum_{n\geq 0} a(2n+1)q^n &= \frac{f_4^2f_{24}}{f_8f_{12}}\left(\frac{1}{f_1^2}\right)\\ &= \frac{f_4^2f_{24}}{f_8f_{12}}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\ &= \frac{f_4^2f_8^4f_{24}}{f_2^5f_{12}f_{16}^2}+2q\frac{f_4^4f_{16}^2f_{24}}{f_2^5f_8^2f_{12}}.\end{aligned}$$

Hence,

$$\sum_{n\geq 0} a(4n+3)q^n = 2\frac{f_2^4f_8^2f_{12}}{f_1^5f_4^2f_6}$$

and this proves Theorem 1.5. □

## 3. Proof via a Particular Theta Function

We begin this section by stating one additional lemma which we need as we complete our generating function manipulations below.

**Lemma 3.1.** *For all positive integers $a$ and $b$, and for all primes $p$,*

$$f_{ap}^b \equiv f_a^{bp} \pmod{p}.$$

*Proof.* This lemma follows immediately from the Binomial Theorem and the fact that, for any prime $p$ and all $1 \leq i \leq p-1$, the binomial coefficients $\binom{p}{i}$ are divisible by $p$. □

We are now in a position to provide an alternative proof of Corollary 1.3. Consider

$$G(q) := \frac{f_2^2f_3f_{12}}{f_1f_4f_6}$$

which appears in Mersmann's list of the 14 primitive eta–products which are holomorphic modular forms of weight 1/2. See [1, page 30], [4, Theorem 1.1], and [6, A089801] for additional details. After some elementary calculations, we find that

$$G(q) = \sum_{k=-\infty}^{\infty} q^{3k^2+2k} = 1+q+q^5+q^8+q^{16}+q^{21}+q^{33}+q^{40}+q^{56}+q^{65}+q^{85}+\dots \tag{3.1}$$

If we write

$$G(q) = \sum_{n\geq 0} g(n)q^n,$$

then we see from (3.1) that

$$g(n) = \begin{cases} 1, & \text{if } n = 3k^2+2k \text{ for some integer } k \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

From (1.1), we see that

$$\frac{1}{f_{12}}G(q) = \left(\sum_{n\geq 0} p(n)q^{12n}\right)\left(\sum_{n\geq 0} g(n)q^n\right).$$

Moreover,

$$\begin{aligned} \frac{1}{f_{12}}G(q) &= \frac{f_2^2 f_3}{f_1 f_4 f_6} \\ &\equiv \frac{f_4 f_3}{f_1 f_4 f_6} \pmod 2 \quad \text{(Lemma 3.1)} \\ &= \frac{f_3}{f_1 f_6} \\ &= \sum_{n\geq 0} a(n)q^n. \end{aligned}$$

Therefore, we know that

$$\sum_{n\geq 0} a(n)q^n \equiv \left(\sum_{n\geq 0} p(n)q^{12n}\right)\left(\sum_{n\geq 0} g(n)q^n\right) \pmod 2,$$

so that, by convolution, we have

$$a(n) \equiv \sum_{k\geq 0} p(k)g(n-12k) \pmod 2 \tag{3.3}$$

where we define $g(m) = 0$ if $m < 0$. In order to prove the first congruence in Corollary 1.3, we want to focus on $a(4n+2)$. Using (3.3), we see that

$$a(4n+2) \equiv \sum_{k\geq 0} p(k)g(4n+2-12k) \pmod 2$$

or

$$a(4n+2) \equiv \sum_{k\geq 0} p(k)g(4(n-3k)+2) \pmod 2.$$

From this congruence, we see that, if we can prove $g(4N+2) = 0$ for all $N$, then we will have proven that, for all $n \geq 0$,

$$a(4n+2) \equiv 0 \pmod 2.$$

So we now focus our attention on $g(4N+2)$.

From (3.2), we need to determine whether $4N + 2 = 3k^2 + 2k$ for some integer $k$. If such a solution exists, then it means that

$$3(4N+2) + 1 = 3(3k^2 + 2k) + 1 = (3k+1)^2$$

which is clearly square. However, $3(4N + 2) + 1 = 12N + 7 \equiv 3 \pmod 4$, and we know that there are no squares that are congruent to 3 modulo 4. Thus, there can be no solutions, which means $g(4N + 2) = 0$ for all $N$.

Similarly, the focus of the second congruence in Corollary 1.3 is $a(4n+3)$, which means we need to show that $g(4N+3) = 0$ for all $N$. Using a similar argument to the above, we can show that this is true because $4N + 3 = 3k^2 + 2k$ for some integer $k$ if and only if

$$3(4N+3) + 1 = 3(3k^2 + 2k) + 1 = (3k+1)^2$$

which is clearly square. However, $3(4N + 3) + 1 = 12N + 10 \equiv 2 \pmod 4$, and there are no squares congruent to 2 modulo 4. So $g(4N+3) = 0$ for all $N$, and this means that

$$a(4n+3) \equiv \sum_{k\geq 0} p(k)g(4(n-3k)+2) \pmod 2 \equiv 0 \pmod 2.$$

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## References

[1] Bruinier, J. H., van der Geer, G., Harder, G., and Zagier, D., The 1-2-3 of modular forms, Universitext, Springer-Verlag, Berlin, 2008.

[2] Corteel, S. and Lovejoy, J., Overpartitions, *Trans. Amer. Math. Soc.* **356**, no. 4 (2004), 1623–1634.

[3] da Silva, R. and Sellers, J. A., Congruences for the coefficients of the Gordon and McIntosh mock theta function $\xi(q)$, *Ramanujan J.* **58**, no. 3 (2022), 815–834.

[4] Lemke Oliver, R. J., Eta–quotients and theta functions, *Adv. Math.* **241** (2013), 1–17.

[5] Merca, M., Overpartitions in terms of 2–adic valuation, *Aequationes Math.* **99** (2025), no. 3, 1153–1173.

[6] Sloane, N. J. A., ed., The Online Encyclopedia of Integer Sequences, https://oeis.org, 2025.

[7] Radu, C.-S., An algorithmic approach to Ramanujan-Kolberg identities, *J. Symbolic Comput.* **68** (2015), 225–253.

[8] Smoot, N. A., On the computation of identities relating partition numbers in arithmetic progressions with eta quotients: an implementation of Radu's algorithm, *J. Symbolic Comput.* **104** (2021), 276–311.

James A. Sellers
Department of Mathematics and Statistics
University of Minnesota Duluth
Duluth, MN 55812, USA.
E-mail: jsellers@d.umn.edu

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# A NICE INTEGRAL REPRESENTATION OF CATALAN'S CONSTANT REVISITED

NANDAN SAI DASIREDDY
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Abstract. This short note contains a simple proof of following integral representation of catalan's constant $C = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2}$

$$\int_0^{\pi/4} \log\left(\frac{\cos u + \sin u}{\cos u - \sin u}\right) du = C.$$

## 1. Introduction

The result

$$\int_0^{\pi/4} \log\left(\frac{\cos u + \sin u}{\cos u - \sin u}\right) du = C, \tag{1.1}$$

was established in 1867 in Bierens de Haan's famous treatise [1, under entry 287.14]. Very Recently, in the November 2020 issue of the Mathematical Gazette [2, Equation (35)], Seán Mark Stewart has revived and brought to light (1.1) by giving a proof of it which was based on the calculation of the logarithmic integral $\int_0^{\pi/4} \log(\cos u - \sin u)\ du = -\frac{C}{2} - \frac{\pi}{8}\log 2$ and a recent proof based on the formula $-\int_0^{\pi/4} \log\tan t\ dt = C$ was given by Raymond Mortini in [3].

In this short note we derive (1.1) in a simple way.

## 2. The proof

Our proof is based on the well-known result

$$C = -\int_0^1 \frac{\log(x)}{1+x^2} dx, \tag{2.1}$$

a proof of which will be recalled below. By using the substitution

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$$x = \frac{\cos u - \sin u}{\cos u + \sin u}, \text{ so that } dx = -\frac{2}{(\cos u + \sin u)^2}\; du$$

(note the monotonicity on $[0, \pi/4] \to [0, 1]$) , we obtain

$$\begin{aligned} C = -\int_0^1 \frac{\log(x)}{1+x^2} dx &= -\int_{\pi/4}^0 \frac{\log\left(\frac{\cos u - \sin u}{\cos u + \sin u}\right)}{\left(\frac{2}{(\cos u + \sin u)^2}\right)} \frac{-2}{(\cos u + \sin u)^2} du \\ &= -\int_0^{\pi/4} \log\left(\frac{\cos u - \sin u}{\cos u + \sin u}\right) du = \int_0^{\pi/4} \log\left(\frac{\cos u + \sin u}{\cos u - \sin u}\right) du. \end{aligned}$$

Now we recall for the readers convenience the standard proof of (2.1):

$$\int_0^1 \frac{\log(x)}{1+x^2} dx = \int_0^1 \log(x) \left(\sum_{j=0}^{\infty} (-1)^j x^{2j}\right) dx.$$

Since,

$$\left|\sum_{j=0}^{N} (-1)^j x^{2j} \log(x)\right| \leq \frac{\left(1 - (-1)^{N+1} \left(x^2\right)^{N+1}\right) |\log(x)|}{x^2+1} \leq 2\,|\log(x)|$$

and that $\log x$ is an $L^1$-function on $]0, 1[$, we can interchange the order of integration and summation, giving us

$$\begin{aligned} & -\sum_{j=0}^{\infty} (-1)^j \left(-\int_0^1 x^{2j} \log(x)\, dx\right) \\ &= -\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} \\ &= -C. \end{aligned}$$

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## References

[1] Bierens de Haan, D., Nouvelles tables d'sintégrales définies. 1st ed. P. Engels, Libraire Éditeur, Leide (1867).

[2] Stewart, Seán. M., A Catalan constant inspired integral odyssey, *Math. Gaz.* **CIV**(561) (2020), 449-459. doi: 10.1017/mag.2020.99

[3] Mortini, R., Yet Another Integral Representation of Catalan's Constant, *Amer. Math. Monthly* **131**(2) (2024), 130. doi: 10.1080/00029890.2023.2274242

Nandan Sai Dasireddy
House No. 3-11-363, Pavani Plaza, Flat No:401,
Road No. 2, Shivaganga Colony, Siris Road,
L.B. Nagar, Hyderabad, Telangana, India.
E-mail: dasireddy.1818@gmail.com

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# SOME NOTES ON PARABOLIC PROJECTILE MOTION

PETER R. MERCER

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Abstract. We use first-semester calculus to make various observations about parabolic projectile motion.

## 1. Introduction

We use first-semester calculus to obtain some interesting geometric aspects of projectile flight paths having launch angle $\theta \in (0, \pi/2)$. The paths are parabolic: their foci trace a semi-circle, their maxima trace a semi-ellipse, and their envelope is part of another parabola.

## 2. Background

The flight of a projectile launched with initial velocity $v > 0$ and angle $\theta \in (0, \pi/2)$ from horizontal, impeded only by gravity $g$, is well-modeled by

$$\begin{aligned} x &= v\cos(\theta)t \\ y &= -\tfrac{1}{2}gt^2 + v\sin(\theta)t \end{aligned} \qquad t \geq 0\,. \tag{2.1}$$

The projectile begins flight when $t = 0$ and returns to the ground when $t = \frac{2v\sin(\theta)}{g}$, at which time

$$x = v\cos(\theta)\frac{2v\sin(\theta)}{g} = \frac{v^2\,2\cos(\theta)\sin(\theta)}{g} = \frac{v^2\sin(2\theta)}{g}\,. \tag{2.2}$$

Maximizing $x$ with respect to $\theta$, the projectile has **maximal horizontal range** when $\theta = \frac{1}{4}\pi = 45°$.

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Key words and phrases: projectile motion, parabola, flight path.

Eliminating $t$ in (2.1), makes it (perhaps) more evident that the trajectory is parabolic:

$$y = -\frac{1}{2}\frac{g}{(v\cos(\theta))^2}x^2 + \tan(\theta)x\,. \tag{2.3}$$

In [4] (also [6]) is an interesting analysis, beginning with (2.3), which shows that the projectile attains **maximal flight path length** when $\theta$ is the solution of

$$\frac{2v^2}{g}\cos(\theta)\left[1 - \sin(\theta)\ln\left(\frac{1+\sin(\theta)}{\cos(\theta)}\right)\right] = 0\,.$$

This gives

$$\theta \approx (0.3137)\pi \approx 56.47^\circ.$$

In [1], beginning with (2.1), it is shown that this $\theta$ (also) solves the more succinct

$$\csc(\theta) = \coth(\csc(\theta)).$$

In [2] it is shown that the projectile achieves **maximal area under the flight path** when $\theta = \frac{1}{3}\pi = 60^\circ$, a fact which appears to be not well known.

In the next section we verify this differently than in [2] for its own sake, but also to serve as a segue to further investigation.

## 3. A Different Parameter

Using (2.2) and (2.3), the area between the flight path and the $x$-axis is

$$A = \int_0^{\frac{v^2\sin(2\theta)}{g}} \left[-\frac{1}{2}\frac{g}{(v\cos(\theta))^2}x^2 + \tan(\theta)x\right]dx.$$

We are interested in when $\frac{dA}{d\theta} = 0$, which seems rather unwieldy. Using instead the parameter $m = \tan(\theta)$ (the slope of the parabola at the origin) and employing $\sec^2(\theta) = \tan^2(\theta) + 1$ and $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)} = \frac{2m}{m^2+1}$, we get

$$A = \int_0^{\frac{v^2}{g}\frac{2m}{m^2+1}} \left[-\frac{g}{2v^2}(m^2+1)x^2 + mx\right]dx\,.$$

Now, $\frac{dA}{d\theta} = \frac{dA}{dm}\frac{dm}{d\theta} = \frac{dA}{dm}\sec^2(\theta)$ and $\sec^2(\theta) > 0$, so we need only be concerned with $\frac{dA}{dm}$. Performing the integration we get

$$A = \frac{2m^3v^4}{3g^2(m^2+1)^2}\,.$$

Then, after some simplification,

$$\frac{dA}{dm} = \frac{-2m^2v^4(m^2-3)}{3g^2(m^2+1)^3}.$$

Therefore $\frac{dA}{dm} = 0$ when $m = 0$ or $m = \pm\sqrt{3}$. But only $m = +\sqrt{3}$ yields $\theta \in (0, \pi/2)$, specifically $\theta = \frac{1}{3}\pi = 60°$. This clearly yields the maximum we seek.

## 4. More with this Parameter

Other interesting aspects can be observed by utilizing $m = \tan(\theta)$ as our parameter. As $\theta$ goes from 0 to $\pi/2$, $m$ goes from 0 to $\infty$.

• **Foci**: The focus of a parabola $y = ax^2+bx+c$ is the point $\left(-\frac{b}{2a}, \frac{4ac-b^2+1}{4a}\right)$. So the foci of our parabolas

$$y = -\frac{g}{2v^2}(m^2+1)x^2 + mx \tag{4.1}$$

are $(x_f, y_f) = \left(\frac{v^2}{g}\frac{m}{m^2+1}, \frac{v^2}{2g}\frac{m^2-1}{m^2+1}\right)$. As $m$ goes from 0 to $\infty$, $x_f$ goes from 0 to $\frac{v^2}{2g}$ (at $m = 1$), and then back to 0. Meanwhile, $y_f$ goes from $\frac{v^2}{2g}$ to 0 ($m = 1$), and then to $-\frac{v^2}{2g}$. So there appears to be symmetry. In fact, it is easily verified that

$$x_f^2 + y_f^2 = \left(\frac{v^2}{g}\frac{m}{m^2+1}\right)^2 + \left(\frac{v^2}{2g}\frac{m^2-1}{m^2+1}\right)^2 = \frac{1}{4}\frac{v^4}{g^2}.$$

Therefore, as $m$ goes from 0 to $\infty$, the foci trace (counterclockwise) the right-half of the **circle** with center $(0,0)$ and radius $r = \frac{v^2}{2g}$. This semicircle has area $\frac{\pi}{8}\frac{v^4}{g^2}$.

A similar analysis yields the following, which can be found in [3]. (It is obtained there by entirely different means.)

• **Maxima**: Our parabolas have maxima ($y' = 0$) also at $x_m = \frac{v^2}{g}\frac{m}{m^2+1}$, at which $y_m = \frac{v^2}{2g}\frac{m^2}{m^2+1}$. Here, as $m$ goes from 0 to $\infty$, $y_m$ goes from 0 to $\frac{v^2}{4g}$ ($m = 1$), and then to $\frac{v^2}{2g}$. So there appears to be symmetry with respect to $y = \frac{v^2}{4g}$. Indeed, a bit of algebra reveals that

$$\left(\frac{v^2}{g}\frac{m}{m^2+1}\right)^2 + \left(\frac{v^2}{2g}\frac{m^2}{m^2+1} - \frac{v^2}{4g}\right)^2 = \left(\frac{v^2}{g}\frac{m}{m^2+1}\right)^2 + \left(\frac{v^2}{4g}\frac{m^2-1}{m^2+1}\right)^2$$

$$= -3\left(\frac{v^2}{4g}\frac{m^2-1}{m^2+1}\right)^2 + \frac{1}{4}\frac{v^4}{g^2}.$$

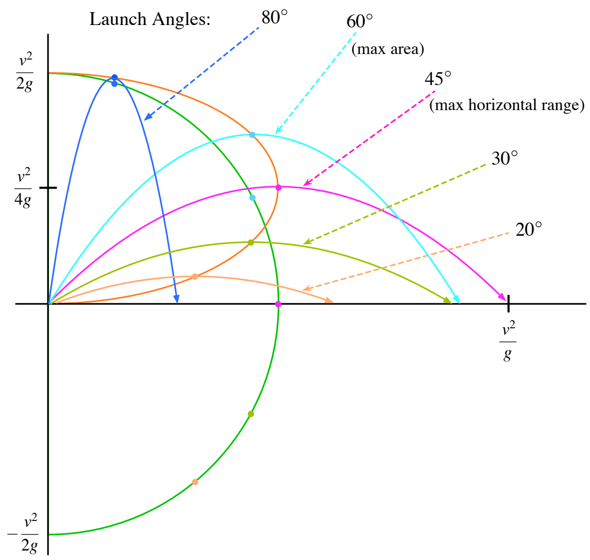


FIGURE 1. Five flight paths, with maxima on the orange ellipse and foci on the green circle.

That is,

$$\left(\frac{v^2}{g}\frac{m}{m^2+1}\right)^2 + 4\left(\frac{v^2}{2g}\frac{m^2}{m^2+1} - \frac{v^2}{4g}\right)^2 = \frac{1}{4}\frac{v^4}{g^2}.$$

Therefore, as $m$ goes from 0 to $\infty$, the maxima trace (counterclockwise) the right-half of the **ellipse**

$$\frac{x^2}{(v^2/2g)^2} + \frac{(y - \frac{v^2}{4g})^2}{(v^2/4g)^2} = 1$$

with center $(0, \frac{v^2}{4g})$. This semi-ellipse has area $\frac{\pi}{16}\frac{v^4}{g^2}$; half the area of the semi-circle of foci obtained above.

See Figure 1, which shows the semi-circle of foci, the semi-ellipse of maxima, along with five possible trajectories.

• **Envelope**: The envelope of all trajectories is the curve consisting of points which share a tangent line with some particular trajectory [5]. (Points on the envelope are precisely those which can be hit in just one way – that is, with exactly *one* $\theta \in (0, \pi/2)$.) This envelope is given by the simultaneous

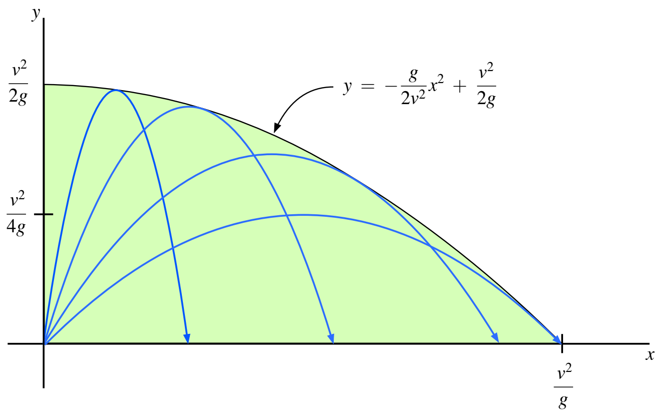


FIGURE 2. The envelope, and four of the trajectories which make it up.

equations

$$\begin{aligned} F(x,y,m) &= y + \frac{g}{2v^2}(m^2+1)x^2 - mx = 0 \\ \frac{\partial F(x,y,m)}{\partial m} &= \frac{mg}{v^2}x^2 - x = 0 \end{aligned} \qquad m \geq 0\,. \qquad (4.2)$$

The second equation gives $m^2 = v^4/x^2g^2$ and $mx = v^2/g$. Substituting these into the first equation yields the **parabola**

$$y = -\frac{g}{2v^2}x^2 + \frac{v^2}{2g}\,.$$

The area between this parabola and the positive $x$-axis is

$$\int_0^{\frac{v^2}{g}} \left(-\frac{g}{2v^2}x^2 + \frac{v^2}{2g}\right)dx = \frac{1}{3}\frac{v^4}{g^2}.$$

See Figure 2, which shows the envelope, and four of the trajectories which make it up. As Figure 2 is meant to suggest, the envelope is made from parabolas having $m \geq 1$, i.e. $\theta \geq 45°$. (Parabolas having $m < 1$ do not contribute to the envelope.) Here's a verification: The envelope has

$$y' = -\frac{g}{v^2}x\,,$$

while our parabolas (4.1) have

$$y' = -\frac{g}{v^2}(m^2+1)x + m\,.$$

These are equal when $x = \frac{1}{m}\frac{v^2}{g}$, but necessarily, $m \geq 1$. For *both* the envelope *and* any parabola (4.1) these $x$-values yield the same $y$-value $\frac{v^2}{2g}\frac{m^2-1}{m^2}$.

## References


[1] Cooper, J. and Swifton, A., Throwing a ball as far as possible, revisited, *Amer. Math. Monthly* **124** (2017), 955-959.

[2] Glaister, P., On the area swept out by projectile, *Math. Gazette* **76** (1992), 384-385.

[3] Opp, R.L., A classification of projectile paths, *Math. Teacher* **69** (1976), 493-494.

[4] Sarafian, H., On projectile motion, *Physics Teacher* **37** (1999), 86-88.

[5] Wikipedia (accessed 2024): Wikipedia, https:wikipedia.org/wiki/Envelope-(mathematics).

[6] Yan-Qing, J., Projectile motion path length and initial projectile angle, *J. Sci. Teachers' College Univ.* **3** (2005), 49-51.



Peter R. Mercer
Department of Mathematics
Buffalo State University
Buffalo, New York, 14222
U.S.A.
E-mail: mercerpr@buffalostate.edu

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# APOLLONIUS'S THEOREM: A PROOF BY MEANS OF THE POWER OF A POINT THEOREM

NANDAN SAI DASIREDDY

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Abstract. We present a geometric solution to the well-known Apollonius's theorem that consists of (geometrically) constructing the circle with one of the sides of the triangle as diameter. The main ingredient of our proof is the power of a point theorem.

## 1. Introduction

In Euclidean geometry, Apollonius's theorem is a relation between the length of a median of a triangle and the lengths of its three sides. The theorem is named after the Greek geometer and astronomer Apollonius of Perga (240 BCE–c. 190 BCE).

In this note we present a new proof of this celebrated Theorem. The proofs available in the literature (some of them can be found in [1–7]) are mostly based on the Pythagorean theorem, dissection, the parallelogram law, extensive use of the cosine and the sine rules with extremely laborious calculations, Heron's formula and Ptolemy's theorem. In contrast, our present proof is quite different; the approach we propose is new and combines the straightforwardness of the Pythagorean theorem with the power of a point theorem.

The statement of the theorem was introduced to the author of this note in his mathematics book in his high school sophomore year in the Fall of 2012, and it has been his dream to work out an elementary proof ever since. Finally, one such proof was discovered: it uses nothing more than the basic properties of circle geometry and the Pythagorean theorem.

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Key words and phrases: Apollonius's Theorem, power of a point theorem, the Pythagorean theorem

## 2. Preliminaries

Thales's Theorem says that if a triangle is inscribed in a circle, with one of the sides of the triangle as the diameter of the circle, then the triangle is always a right triangle.

We are going to use Figures 1 and 2 to describe the power of a point theorem.

Let $S$, $T$, $U$, $V$ be concyclic points in the plane with the secant lines $ST$ and $UV$ intersecting at point $P$. If $P$ lies outside the circle (Figure 1), then the *power of a point theorem* (also known as the *intersecting secants theorem*) states that

$$PS \cdot PT = PU \cdot PV.$$

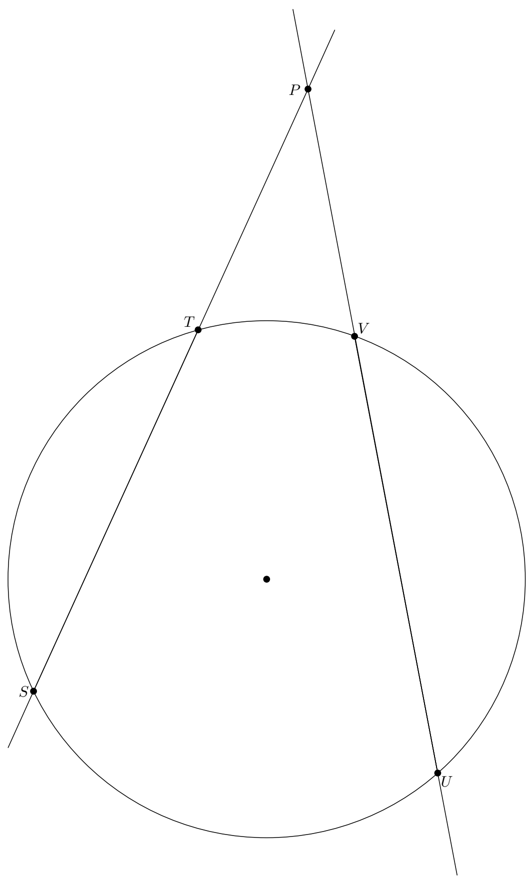


Figure 1. Power of a point ($PS \cdot PT = PU \cdot PV$).

If $P$ lies inside the circle (Figure 2), the power of a point theorem states that

$$PS \cdot PT = PU \cdot PV.$$

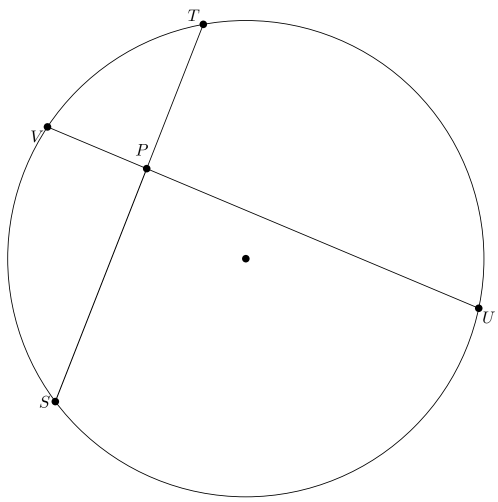


FIGURE 2. Power of a point ($PS \cdot PT = PU \cdot PV$).

## 3. Main Theorem

**Theorem 3.1.** ***(Apollonius's theorem).*** *Let $ABC$ be a triangle where $a$, $b$, and $c$ are the lengths of the sides of the triangle, and suppose $d$ is the length of the median from $A$ to $BC$ bisecting the side $a$. Then*

$$b^2 + c^2 = 2\left(\frac{a^2}{4} + d^2\right).$$

*Proof.* The canonical proof of this theorem is based on the Pythagorean theorem but we can still use the power of a point theorem in a very nice way.

Let $\angle B > \angle A > \angle C$ be the angles of $\triangle$ABC. Let $a$, $b$, $c$ denote $BC$, $AC$, $AB$. We have three cases.

CASE I. $\triangle$ABC is acute.

By condition, let $M$ be the midpoint of $BC$. Construct the circle with diameter $BC$ which intersects the secant through $A$ and $C$ at $C$ and $D$. Thales's theorem tells us that $D$ is the projection of $B$ onto $AC$. Denote $E = (BC) \cap AM$. Let point $F$ be the reflection of $E$ across the midpoint of $BC$. Then point $F$ lies on the circle with diameter $BC$ (Figure 3).

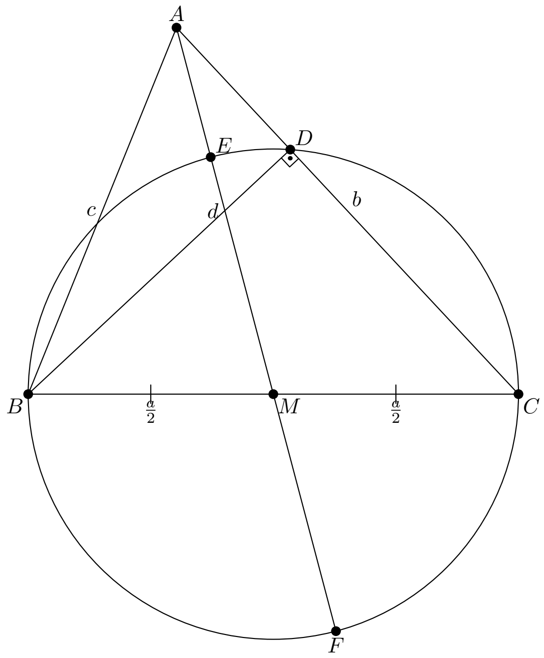


FIGURE 3. Acute triangle.

The remaining steps consist of straightforward length computations. We observe that $FM = EM = a/2$, $AE = AM - EM = d - \frac{a}{2}$ and $AF = AM + FM = d + \frac{a}{2}$. By the Pythagorean theorem applied to triangles BDA and BDC,

$$a^2 - CD^2 = c^2 - (b - CD)^2 ,$$

which reduces to

$$CD = \frac{b^2 + a^2 - c^2}{2b},$$

so $AD = AC - CD = b - \left(\dfrac{b^2 + a^2 - c^2}{2b}\right) = \dfrac{b^2 + c^2 - a^2}{2b}$.

Additionally, the power of a point at $A$ with respect to $(DEFC)$ tells us that $AE \cdot AF = AD \cdot AC \implies \left(d - \frac{a}{2}\right)\left(d + \frac{a}{2}\right) = \left(\frac{b^2+c^2-a^2}{2b}\right)(b)$, so simplifying gives us

$$2d^2 - \frac{a^2}{2} = b^2 + c^2 - a^2.$$

Rearranging, we get

$$b^2 + c^2 = 2\left(\frac{a^2}{4} + d^2\right)$$

as desired.

CASE II. $\triangle$ABC is right.

By condition, let $M$ be the midpoint of $BC$. Construct the circle with diameter $BC$ which intersects the secant through $A$ and $C$ at $C$ and $D$. Thales's theorem tells us that $D$ is the projection of $B$ onto $AC$. Denote $E = (BC) \cap AM$. Let point $F$ be the reflection of $E$ across the midpoint of $BC$. Then point $F$ lies on the circle with diameter $BC$ (Figure 4).

The remaining steps consist of straightforward length computations. We observe that $FM = EM = a/2$, $AE = AM - EM = d - \frac{a}{2}$, $AF = AM + FM = d + \frac{a}{2}$. By the Pythagorean theorem applied to triangles BDA and BDC,

$$a^2 - CD^2 = c^2 - (b - CD)^2,$$

which reduces to

$$CD = \frac{b^2 + a^2 - c^2}{2b},$$

so $AD = AC - CD = b - \left(\dfrac{b^2 + a^2 - c^2}{2b}\right) = \dfrac{b^2 + c^2 - a^2}{2b}$.

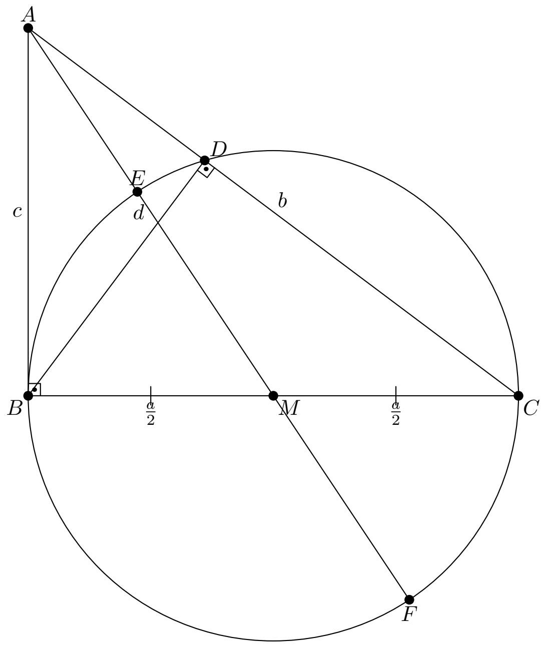


FIGURE 4. Right triangle.

Additionally, the power of a point at $A$ with respect to $(DEFC)$ tells us that $AE \cdot AF = AD \cdot AC \implies \left(d - \frac{a}{2}\right)\left(d + \frac{a}{2}\right) = \left(\frac{b^2+c^2-a^2}{2b}\right)(b)$, so simplifying gives us

$$2d^2 - \frac{a^2}{2} = b^2 + c^2 - a^2.$$

Rearranging, we get

$$b^2 + c^2 = 2\left(\frac{a^2}{4} + d^2\right)$$

as desired.

As a special case, when constructing the circle with diameter $AC$ let $M'$ be the midpoint of $AC$ and let $B'$ be the reflection of $B$ across the midpoint of $AC$. Then point $B'$ lies on the circle with diameter $AC$ as in Figure 5.

The power of a point at $M'$ with respect to $(AB'CB)$ tells us that $AM' \cdot CM' = BM' \cdot B'M' \implies d^2 = \frac{b^2}{4}$, so simplifying gives us

$$4d^2 = 2b^2 - b^2.$$

Additionally, the Pythagorean theorem applied to △ABC gives

$$4d^2 = 2\left(c^2 + a^2\right) - b^2.$$

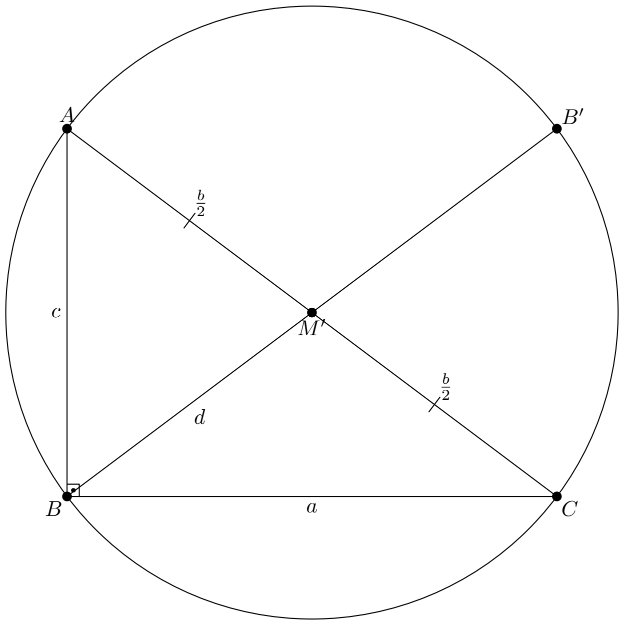


FIGURE 5. Right triangle.

Rearranging, we get

$$a^2 + c^2 = 2\left(\frac{b^2}{4} + d^2\right)$$

as desired.

CASE III. △ABC is an obtuse triangle.

By assumption, let $M$ be the midpoint of $BC$. Construct the circle with diameter $BC$ which intersects the secant through $A$ and $C$ at points $C$ and $D$. Thales's theorem tells us that $D$ is the projection of $B$ onto $AC$. Denote

$E = (BC) \cap AM$. Let point $F$ be the reflection of $E$ across the midpoint of $BC$. Then point $F$ lies on the circle with diameter $BC$ (Figure 6).

The remaining steps consist of straightforward length computations. We observe that $FM = EM = a/2$, $AE = AM - EM = d - \frac{a}{2}$ and $AF = AM + FM = d + \frac{a}{2}$. By the Pythagorean theorem applied to triangles BDA and BDC,

$$a^2 - CD^2 = c^2 - (b - CD)^2 ,$$

which reduces to

$$CD = \frac{b^2 + a^2 - c^2}{2b},$$

so $AD = AC - CD = b - \left(\frac{b^2 + a^2 - c^2}{2b}\right) = \frac{b^2 + c^2 - a^2}{2b}$.

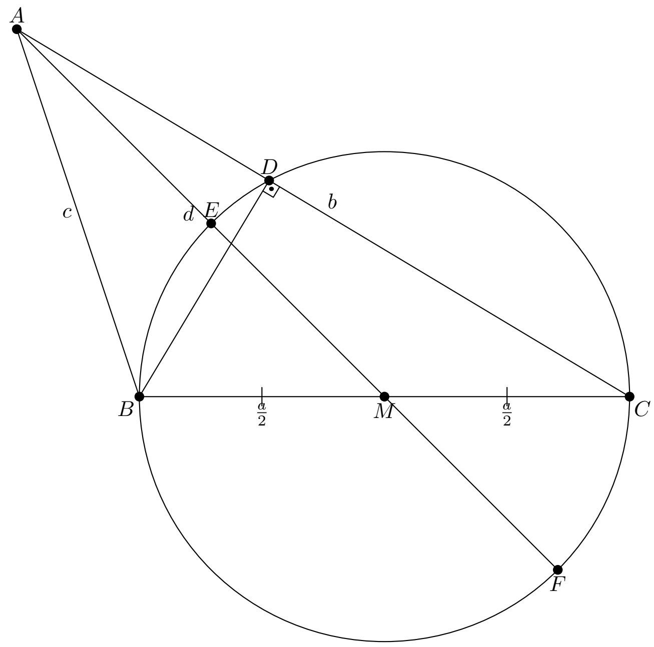


FIGURE 6. Obtuse triangle.

Additionally, the power of a point at $A$ with respect to $(DEFC)$ tells us that $AE \cdot AF = AD \cdot AC \implies \left(d - \frac{a}{2}\right)\left(d + \frac{a}{2}\right) = \left(\frac{b^2+c^2-a^2}{2b}\right)(b)$, so simplifying gives us

$$2d^2 - \frac{a^2}{2} = b^2 + c^2 - a^2.$$

Rearranging, we get

$$b^2 + c^2 = 2\left(\frac{a^2}{4} + d^2\right)$$

as desired.

As a special case, let $M'$ be the midpoint of $AC$. When constructing the circle with diameter $AC$ which intersects the secant through $B$ and $C$ at points $C$ and $D'$, Thales's theorem tells us that $D'$ is the projection of $A$ onto $BC$. Denote $E' = (BC) \cap BM'$. Let point $F'$ be the reflection of $E'$ across the midpoint of $AC$. Then point $F'$ lies on the circle with diameter $AC$ as in Figure 7.

The remaining steps consist of straightforward length computations. We observe that $F'M' = E'M' = b/2$, $BE' = M'E' - BM' = \frac{b}{2} - d$ and $BF' = BM' + F'M' = d + \frac{b}{2}$. By the Pythagorean theorem applied to triangles $BD'A$ and $AD'C$,

$$c^2 - BD'^2 = b^2 - (a + BD')^2,$$

which reduces to

$$BD' = \frac{b^2 - a^2 - c^2}{2a}.$$

Additionally, the power of a point at $B$ with respect to $(CE'D'F')$ tells us that $BE' \cdot BF' = BD' \cdot BC \implies \left(\frac{b}{2} - d\right)\left(\frac{b}{2} + d\right) = \left(\frac{b^2-c^2-a^2}{2a}\right)(a)$, so simplifying gives us

$$c^2 + a^2 - 2d^2 = \frac{b^2}{2}.$$

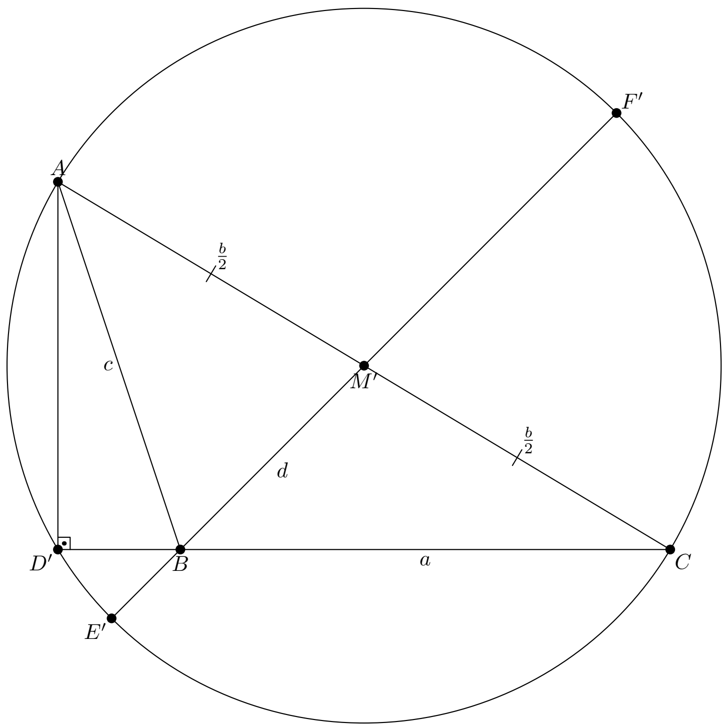


FIGURE 7. Obtuse triangle.

Rearranging, we get

$$a^2 + c^2 = 2\left(\frac{b^2}{4} + d^2\right),$$

as desired.

□

**Acknowledgement:** I want to express my deep gratitude to an anonymous reviewer of this journal who suggested that I should address all three cases (acute-angled, right-angled, and obtuse-angled), and in particular suggested to discuss the case where a point lies inside the circle and for taking the time to offer many sagacious suggestions for correcting and polishing the penultimate draft. Moreover, I also owe a special thanks to the reviewer for shortening the proof using the Pythagorean theorem. My original version relied heavily on law of cosines. It was much longer and more opaque. Figures in the note were drawn using the *METAPOST*.

## References


[1] Stokes, G. D. C., The theorem of Apollonius by dissection, *Edinburgh Math Notes.* **24** January, (1929), xviii. doi: 10.1017/S1757748900001973

[2] Lawes, C. P., Proof Without Words: The Length of a Triangle Median via the Parallelogram Law, *Math. Mag.* **86**(2) (2013), 146.

[3] Lopes, A. B., Apollonius's Theorem via Heron's Formula, *Math. Mag.* **97**(3) (2024), 272–273.

[4] Buysse, M., Theorems of Euclidean Geometry Through Calculus, *Math. Intelligencer* **45**(4) (2023), 338-345.

[5] Nelsen, R. B., Apollonius's Theorem via Ptolemy's Theorem, *Math. Mag.* to appear. doi: 10.1080/0025570X.2024.2385255

[6] Rajpoot, H. C., Proofs of Apollonius Theorem, preprint, (2016). Available from: https://www.researchgate.net/publication/383178077_Three_Proofs_of_Apollonius_Theorem

[7] Celli, M., Proof Without Words: Apollonius's Theorem (Parallelogram Identity) via the Pythagorean Theorem, *Far East Journal of Mathematical Education* **27**(1) (2025), 17-19. doi: 10.17654/0973563125003



Nandan Sai Dasireddy
House No. 3-11-363, Pavani Plaza, Flat No:401,
Road No. 2, Shivaganga Colony, Siris Road,
L.B. Nagar, Hyderabad, Telangana, India.
E-mail: dasireddy.1818@gmail.com

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# ANTIMAGIC LABELINGS OF A COMPLETE GRAPH

A. N. BHAVALE
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Abstract. In 1990, Hartsfield and Ringel introduced antimagic graphs. Hartsfield and Ringel conjectured that every connected graph (and in particular, a tree) except $K_2$ is antimagic. In 2010, Hefetz et al. raised two questions: Is every orientation of any simple connected undirected graph antimagic? and Given any undirected graph $G$, does there exist an orientation of $G$ which is antimagic? They call such an orientation an *antimagic orientation* of $G$. Recently, Bhavale provided an edge labeling for a given graph on $n$ vertices without isolated vertices. In this paper, using the labeling of Bhavale, we prove that a complete graph $K_n$ for $n \geq 3$ is super antimagic as well as totally antimagic total graph. We also prove that there exists an antimagic orientation of $K_n$ for $n \geq 3$.

## 1. Introduction

According to Gallian [3], a *graph labeling* is an assignment of integers to the vertices or edges, or both, subject to certain conditions. A *labeled graph* $G = (V, E)$ is a finite series of graph vertices $V$ with a set of graph edges $E$ of 2-subsets of $V$ (see Weisstein [8]). Formally, given a graph $G = (V, E)$, *a vertex labeling* is a function of $V$ to a set of labels; a graph with such a function defined is called *a vertex-labeled graph*. Likewise, *an edge labeling* is a function of $E$ to a set of labels. In this case, the graph is called *an edge-labeled graph*. A *complete graph* on $n$ vertices, denoted by $K_n$, is a (simple) graph in which each vertex is connected to every other vertex by an edge.

Graph labelings were first introduced in the mid 1960s. The origin of graph labelings can be attributed to Rosa [6]. A general survey of graph labelings is found in Gallian [3]. In the intervening years over 200 graph

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labelings techniques have been studied in over 3000 papers. There are many researchers from all over the world working on different kinds of graph labelings. In 1990, Hartsfield and Ringel [4] introduced antimagic graphs, i.e., graphs having an antimagic labeling. Hartsfield and Ringel [4] conjectured that every tree as well as a connected graph except $K_2$ is antimagic. Recently, Bhavale [1] obtained an edge labeling for a given vertex labeling of a simple graph on $n$ unisolated vertices. In this paper, using the labeling of Bhavale, we prove that a complete graph $K_n$ for $n \geq 3$ is super antimagic as well as totally antimagic total graph. We also prove that there exists an antimagic orientation of $K_n$ for $n \geq 3$. For the other definitions, notation, and terminology see [4, 7].

## 2. Antimagic labelings of graphs/directed graphs

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = l$. An *antimagic labeling* of a graph $G$ is a bijection $f : E \to \{1, 2, \ldots, l\}$ such that all the $n$ vertex sums are pairwise distinct, where a *vertex sum* is the sum of labels of all edges incident to that vertex. Thus, if a vertex sum of a vertex $u$ is denoted by $S_u$ then $S_u = \sum_{e \in I(u)} f(e)$, where $I(u)$ is the set of all edges incident to $u$. A graph is called *antimagic* if it has an antimagic labeling. Hartsfield and Ringel [4] showed that paths $P_n$ ($n \geq 3$), cycles, wheels, and complete graphs $K_n$ ($n \geq 3$) are antimagic. In 2010, Hefetz et al. [5] investigated antimagic labelings of directed graphs. An *antimagic labeling of a directed graph $D$* with $n$ vertices and $l$ arcs (or directed edges) is a bijection from the set of arcs of $D$ to the integers $\{1, 2, \ldots, l\}$ such that all $n$ oriented vertex sums are pairwise distinct, where an *oriented vertex sum* is the sum of labels of all edges entering that vertex minus the sum of labels of all edges leaving it. We define an *antimagic labeling of a directed graph $\overrightarrow{D}$* with vertices $\{v_1, v_2, \ldots, v_n\}$ and $l$ arcs (or directed edges) as a bijection from the set of arcs of $\overrightarrow{D}$ to the integers $\{1, 2, \ldots, l\}$ such that all $n$ vertex sums are pairwise distinct, where a *vertex sum* is the sum of labels of all edges entering as well as leaving that vertex. Let $S^-_{v_i} = \sum_{\overrightarrow{e} \in N^-(v_i)} f(\overrightarrow{e})$, where $N^-(v_i) = \{\overrightarrow{(v_p, v_i)} \in E(\overrightarrow{D}) | p < i\}$. Similarly, let $S^+_{v_i} = \sum_{\overrightarrow{e} \in N^+(v_i)} f(\overrightarrow{e})$, where $N^+(v_i) = \{\overrightarrow{(v_i, v_q)} \in E(\overrightarrow{D}) | i < q\}$. Then the vertex sum of vertex $v_i$ is given by $S_{v_i} = S^-_{v_i} + S^+_{v_i}$, and the oriented vertex sum of vertex $v_i$ is given by $S^o_{v_i} = S^-_{v_i} - S^+_{v_i}$. A digraph is called *antimagic* if it has an antimagic labeling. Let $S = \{(i,j) | 1 \le i < j \le n\}$. Let $J_N = \{k | 1 \le k \le N = \binom{n}{2}\}$ where $n \ge 2$. The following result is due to Bhavale [1].

**Theorem 2.1.** [1] *Let $n \ge 2$. Define $F : S \to J_N$ by $F(i,j) = (i-1)n - \binom{i}{2} + j - i, \forall (i,j) \in S$. Then $F$ is bijective, and hence $|S| = N = \binom{n}{2}$.*

Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let $\overrightarrow{K_n}$ be the directed graph associated with $K_n$ such that whenever $1 \le i < j \le n$ there is a directed edge from $v_i$ to $v_j$, (that is, $\overrightarrow{(v_i, v_j)} \in E(\overrightarrow{K_n})$). Then the following two results follows from Theorem 2.1.

**Corollary 2.2.** [1] *Consider a directed graph $\overrightarrow{K_n}$ with vertex labeling $\{1, 2, \ldots, n\}$. Then for $i < j$, a directed edge $\overrightarrow{(i,j)} \in E(\overrightarrow{K_n})$ can be labeled (uniquely) as $k = (i-1)n - \binom{i}{2} + j - i \in J_N$. Moreover, for any $k \in J_N$ there exists unique directed edge $\overrightarrow{(i,j)} \in E(\overrightarrow{K_n})$.*

**Corollary 2.3.** [1] *Suppose $\overrightarrow{G}$ is a directed subgraph (on $n$ unisolated vertices) of $\overrightarrow{K}_n$ with vertex labeling $\{v_1, v_2, \ldots, v_n\}$. Then for $i < j$, if $\overrightarrow{(v_i, v_j)} \in E(\overrightarrow{G})$ then $\overrightarrow{(v_i, v_j)}$ can be labeled (uniquely) as $k = (i-1)n - \binom{i}{2} + j - i \in J_N$. Moreover, if $k \in J_N$ is a label of an edge $\overrightarrow{e} \in E(\overrightarrow{G})$ then $\overrightarrow{e} = \overrightarrow{(v_i, v_j)}$.*

Note that, $K_n$ is an underlying graph of $\overrightarrow{K_n}$. Therefore if $\overrightarrow{G}$ is a (directed) subgraph of $\overrightarrow{K_n}$ then its underlying graph $G$ is a subgraph of $K_n$. Thus using Corollary 2.3, for a given finite simple graph $G$ on $n$ unisolated vertices, we get an edge labeling for the graph $G$, which is uniquely determined.

**Theorem 2.4.** *A digraph $\overrightarrow{K_n}$ is antimagic for $n \ge 3$.*

*Proof.* Let $V(\overrightarrow{K_n}) = \{v_1, v_2, \ldots, v_n\}$ and let $E(\overrightarrow{K_n}) = \{\overrightarrow{e_k} = \overrightarrow{(v_i, v_j)} | k = (i-1)n - \binom{i}{2} + j - i, 1 \le i < j \le n\}$. Let $f : E(\overrightarrow{K_n}) \to J_N$ be defined as $f(\overrightarrow{e_k}) = k$. Then by Theorem 2.1, $f$ is bijective. Now $S^-_{v_i} = \sum_{\overrightarrow{e} \in N^-(v_i)} f(\overrightarrow{e})$, where $N^-(v_i) = \{\overrightarrow{(v_p, v_i)} \in E(\overrightarrow{K_n}) | p < i\}$.

$$\text{Therefore } S^-_{v_i} = \sum_{p=1}^{i-1} f(\overrightarrow{(v_p, v_i)}) = \sum_{p=1}^{i-1} \left( (p-1)n - \binom{p}{2} + i - p \right)$$
$$= (n+1)\binom{i}{2} - n(i-1) - \frac{i(i-1)(i-2)}{6}.$$

Similarly, $S^+_{v_i} = \sum_{\overrightarrow{e} \in N^+(v_i)} f(\overrightarrow{e})$, where $N^+(v_i) = \{\overrightarrow{(v_i, v_q)} \in E(\overrightarrow{K_n}) | i < q\}$.

Therefore $S^+_{v_i} = \sum_{q=i+1}^{n} f(\overrightarrow{(v_i, v_q)}) = \sum_{q=i+1}^{n} \left( (i-1)n - \binom{i}{2} + q - i \right)$

$$= \binom{n}{2} + (n - (i+1)) \left( n(i-1) - \binom{i}{2} \right).$$

But then the vertex sum of vertex $v_i$ is given by $S_{v_i} = S^-_{v_i} + S^+_{v_i} = \frac{i^3}{3} - (n-1)i^2 + (n^2 - n - \frac{4}{3})i - \frac{n(n-3)}{2}$. We now claim that $S_{v_i} \neq S_{v_j}$ for $i \neq j$, i.e., $S_{v_i} \neq S_{v_j}$ for $v_i \neq v_j$. For if, suppose $S_{v_i} = S_{v_j}$. Without loss of generality, suppose $i < j$. As $S_{v_j} - S_{v_i} = 0$, we have $\frac{(j^3 - i^3)}{3} - (n-1)(j^2 - i^2) + (n^2 - n - \frac{4}{3})(j - i) = 0$. That is, $\frac{(j^2 + ij + i^2)}{3} - (n-1)(j+i) + (n^2 - n - \frac{4}{3}) = 0$, since $j - i > 0$. This implies that $n^2 - nx + y = 0$, where $x = i + j + 1$ and $y = \frac{(i^2 + ij + j^2 - 4)}{3} + (i + j)$. Now $x^2 - 4y = \frac{19}{3} - 2(i + j) - \frac{(i-j)^2}{3} = 0$, if $i = 1, j = 2$, i.e., if $n = \frac{x}{2} = 2$, and $x^2 - 4y < 0$, otherwise. This is a contradiction, since $n$ is an integer $\geq 3$. Thus the vertex sums are pairwise distinct, and hence the proof. □

Note that in the proof of Theorem 2.4, we get $x^2 - 4y < 0$ whenever $S_{v_i} = S_{v_j}$ for $i < j$ and $n \geq 3$. This implies that $n^2 - nx + y > 0$, and hence $S_{v_j} - S_{v_i} > 0$, i.e., $S_{v_i} < S_{v_j}$ whenever $i < j$. Also $\overrightarrow{K_2}$ is not antimagic, and hence $K_2$ is not antimagic. Now $I(v_i) = N^-(v_i) \cup N^+(v_i)$ and $K_n$ is an underlying graph of $\overrightarrow{K_n}$. Therefore the following result immediately follows from Theorem 2.4.

**Corollary 2.5.** *A complete graph $K_n$ is antimagic for $n \geq 3$.*

**Proposition 2.6.** *Let $V(\overrightarrow{K_n}) = \{v_1, v_2, \ldots, v_n\}$ where $n \geq 3$. Then $S^-_{v_i} < S^-_{v_j}$ whenever $i < j$.*

*Proof.* In the proof of Theorem 2.4, we have seen that
$S^-_{v_i} = (n+1)\binom{i}{2} - n(i-1) - \frac{i(i-1)(i-2)}{6}$. Therefore $S^-_{v_j} - S^-_{v_i}$
$= (n+1)\left(\binom{j}{2} - \binom{i}{2}\right) - n(j-i) - \left(\frac{j(j-1)(j-2)}{6} - \frac{i(i-1)(i-2)}{6}\right)$
$= (n+1)\left(\frac{j^2 - i^2}{2} - \frac{j-i}{2}\right) - n(j-i) - \frac{(j^3 - i^3) - 3(j^2 - i^2) + 2(j-i)}{6}$
$= (j-i)\left((n+1)\left(\frac{i+j-1}{2}\right) - n - \frac{(i^2 + ij + j^2) - 3(i+j) + 2}{6}\right)$
$= (j-i)\left(n\left(\frac{i+j-3}{2}\right) + i + j - \frac{i^2 + ij + j^2 + 5}{6}\right)$
$= (j-i)\left(n\left(\frac{i+j-3}{2}\right) - \frac{(i+j-3)^2}{6} + \frac{ij+4}{6}\right) > 0$, since $1 \leq i < j$. □

## 3. Total labelings of graphs/directed graphs

A *total labeling* of a graph $G$ is a bijection $f : V \cup E \to \{1, 2, \ldots, n + l\}$. A total labeling is called *super* if $f(V) = \{1, 2, \ldots, n\}$. For a total labeling $f$, the associated *edge-weight* of an edge $e = \{u, v\}$ is defined by $w(f(e)) = f(u) + f(v) + f(e)$. The associated *vertex-weight* of a vertex $v$ is defined by $w_f(v) = f(v) + S_v$. We extend the above definitions to directed graphs as follows. A *total labeling* of a directed graph $\overrightarrow{D}$ with vertex set $V$ and arc set $E$, is a bijection $f : V \cup E \to \{1, 2, \ldots, n + l\}$. A total labeling is called *super* if $f(V) = \{1, 2, \ldots, n\}$. For a total labeling $f$, the associated *edge-weight* of a directed edge $\overrightarrow{e} = (\overrightarrow{u, v})$ is defined by $w(f(\overrightarrow{e})) = f(u) + f(v) + f(\overrightarrow{e})$. The associated *vertex-weight* of a vertex $v$ is defined by $w_f(v) = f(v) + S_v^- + S_v^+$. A total labeling $f$ is called *edge-antimagic total (vertex-antimagic total)* if all edge-weights (vertex-weights) are pairwise distinct. A graph (digraph) that admits an edge-antimagic total (vertex-antimagic total) labeling is called an *edge-antimagic total (vertex-antimagic total) graph (digraph).* A labeling that is simultaneously edge-antimagic total and vertex-antimagic total is called *totally antimagic total labeling.* A graph (digraph) that admits a totally antimagic total labeling is called a *totally antimagic total graph (digraph).* An edge-antimagic total labeling is called *super* if $f(V) = \{1, 2, \ldots, n\}$ and $f(E) = \{n + 1, n + 2, \ldots, n + l\}$. A graph (digraph) that admits a super edge-antimagic total labeling is called a *super edge-antimagic total graph (digraph).*

**Theorem 3.1.** *For $n \geq 3$, $\overrightarrow{K_n}$ is vertex-antimagic total digraph.*

*Proof.* Let $V(\overrightarrow{K_n}) = \{v_1, v_2, \ldots, v_n\}$ and let $E(\overrightarrow{K_n}) = \{\overrightarrow{e_k} = \overrightarrow{(v_i, v_j)} | k = (i-1)n - \binom{i}{2} + j - i, 1 \leq i < j \leq n\}$. Let $f : V(\overrightarrow{K_n}) \cup E(\overrightarrow{K_n}) \to \{1, 2, \ldots, n + l\}$ be defined as $f(v_i) = i, \forall v_i \in V(\overrightarrow{K_n})$ and $f(\overrightarrow{e_k}) = n + k, \forall \overrightarrow{e_k} \in E(\overrightarrow{K_n})$. Then, by Theorem 2.1, $f$ is bijective. Hence $f$ is not only total labeling but super also. Therefore $w_f(v_i) = f(v_i) + S_{v_i} = i + \frac{i^3}{3} - (n-1)i^2 + (n^2 - n - \frac{4}{3})i - \frac{n(n-3)}{2} = \frac{i^3}{3} - (n-1)i^2 + (n^2 - n - \frac{1}{3})i - \frac{n(n-3)}{2}$. Suppose $v_i, v_j \in V(\overrightarrow{K_n})$ with $v_i \neq v_j$, i.e., $i \neq j$. Without loss, suppose $i < j$. Consider $w_f(v_j) - w_f(v_i) = 0$. That is, $\frac{(j^3 - i^3)}{3} - (n-1)(j^2 - i^2) + (n^2 - n - \frac{1}{3})(j - i) = 0$. Therefore $\frac{(j^2 + ij + i^2)}{3} - (n-1)(j + i) + (n^2 - n - \frac{1}{3}) = 0$, since $j - i > 0$. This implies that $n^2 - nx + y = 0$, where $x = i + j + 1$ and $y = \frac{(i^2 + ij + j^2 - 1)}{3} + (i + j)$. Now $x^2 - 4y = \frac{7}{3} - 2(i + j) - \frac{(i-j)^2}{3} = -(\frac{(i-j)^2 - 7}{3} + 2(i + j)) < 0$, since

$1 \le i < j \le n$. Thus $n^2 - nx + y > 0$, and hence $(j-i)(n^2 - nx + y) > 0$. Therefore $w_f(v_j) - w_f(v_i) > 0$, i.e., $w_f(v_i) < w_f(v_j)$. □

**Theorem 3.2.** *Let $n \ge 3$. Let $\overrightarrow{e_k}, \overrightarrow{e_{k'}} \in E(\overrightarrow{K_n})$ with $\overrightarrow{e_k} \ne \overrightarrow{e_{k'}}$, i.e., $k \ne k'$. If $k = (i-1)n - \binom{i}{2} + j - i$, $k' = (i'-1)n - \binom{i'}{2} + j' - i'$ with $n \ne \frac{2(j-j')}{(i'-i)} + \frac{i'+i-1}{2}$ whenever $1 \le i < i' < j' < j \le n$, then $\overrightarrow{K_n}$ is super edge-antimagic total digraph.*

*Proof.* Let $V(\overrightarrow{K_n}) = \{v_1, v_2, \ldots, v_n\}$ and let $E(\overrightarrow{K_n}) = \{\overrightarrow{e_k} = \overrightarrow{(v_i, v_j)} | k = (i-1)n - \binom{i}{2} + j - i, 1 \le i < j \le n\}$. Let $f : V(\overrightarrow{K_n}) \cup E(\overrightarrow{K_n}) \to \{1, 2, \ldots, n + l\}$ be defined as $f(v_i) = i, \forall v_i \in V(\overrightarrow{K_n})$ and $f(\overrightarrow{e_k}) = n + k, \forall \overrightarrow{e_k} \in E(\overrightarrow{K_n})$. Then by Theorem 2.1, $f$ is bijective. Hence $f$ is not only total labeling but super also. Therefore $w(f(\overrightarrow{e_k})) = f(v_i) + f(v_j) + f(\overrightarrow{e_k}) = i + j + (n + k) = i + j + n + ((i-1)n - \binom{i}{2} + j - i) = ni + 2j - \binom{i}{2}$. Suppose $\overrightarrow{e_k}, \overrightarrow{e_{k'}} \in E(\overrightarrow{K_n})$ with $\overrightarrow{e_k} \ne \overrightarrow{e_{k'}}$, i.e., $k \ne k'$. Without loss, suppose $k < k'$. Using Theorem 2.1, suppose $k = F(i, j)$ and $k' = F(i', j')$ where $(i, j), (i', j') \in S$. Consider $w(f(\overrightarrow{e_{k'}})) - w(f(\overrightarrow{e_k})) = (ni' + 2j' - \binom{i'}{2}) - (ni + 2j - \binom{i}{2}) = n(i' - i) + 2(j' - j) - (\binom{i'}{2} - \binom{i}{2}) = 2(j' - j) + (i' - i)(n - \frac{i'+i-1}{2})$. As $k < k'$, $(i, j) < (i', j')$. Therefore either $i < i'$, or $i = i'$ and $j < j'$. If $i < i'$ then we have either $j \le j'$ or $j > j'$. Now if $i < i'$ with $j \le j'$, and $i = i'$ with $j < j'$ then $w(f(\overrightarrow{e_{k'}})) - w(f(\overrightarrow{e_k})) > 0$, since $n - \frac{i'+i-1}{2} = \frac{n-i'}{2} + \frac{n-i}{2} + \frac{1}{2}$ and $i < i' \le n$. Also if $i < i'$ with $j > j'$ then we have $1 \le i < i' < j' < j \le n$ with $n \ge 4$. Therefore $w(f(\overrightarrow{e_{k'}})) - w(f(\overrightarrow{e_k})) = 0$ if and only if $(i' - i)(n - \frac{i'+i-1}{2}) = 2(j - j')$. Thus if $n \ne \frac{2(j-j')}{(i'-i)} + \frac{i'+i-1}{2}$, where $1 \le i < i' < j' < j \le n$, then $w(f(\overrightarrow{e_{k'}})) - w(f(\overrightarrow{e_k})) \ne 0$. Hence the proof. □

Using Theorem 3.1 and Theorem 3.2, we have the following results.

**Corollary 3.3.** *Let $n \ge 3$. Let $\overrightarrow{e_k}, \overrightarrow{e_{k'}} \in E(\overrightarrow{K_n})$ with $\overrightarrow{e_k} \ne \overrightarrow{e_{k'}}$, i.e., $k \ne k'$. If $k = (i-1)n - \binom{i}{2} + j - i$, $k' = (i'-1)n - \binom{i'}{2} + j' - i'$ with $n \ne \frac{2(j-j')}{(i'-i)} + \frac{i'+i-1}{2}$ whenever $1 \le i < i' < j' < j \le n$, then $\overrightarrow{K_n}$ is totally antimagic total digraph.*

**Corollary 3.4.** *Let $n \ge 3$. Let $e_k, e_{k'} \in E(K_n)$ with $e_k \ne e_{k'}$, i.e., $k \ne k'$. If $k = (i-1)n - \binom{i}{2} + j - i$, $k' = (i'-1)n - \binom{i'}{2} + j' - i'$ with $n \ne \frac{2(j-j')}{(i'-i)} + \frac{i'+i-1}{2}$ whenever $1 \le i < i' < j' < j \le n$, then $K_n$ is totally antimagic total graph.*

## 4. Antimagic orientation of graphs

Recall that, an *antimagic labeling of a directed graph* $D$ with $n$ vertices and $l$ arcs (or directed edges) is a bijection from the set of arcs of $D$ to the integers $\{1, 2, \ldots, l\}$ such that all $n$ oriented vertex sums are pairwise distinct, where an *oriented vertex sum* is the sum of labels of all edges entering that vertex minus the sum of labels of all edges leaving it. Hefetz et al. [5] raised two questions: Is every orientation of any simple connected undirected graph antimagic?, Given any undirected graph $G$, does there exist an orientation of $G$ which is antimagic? They call such an orientation an *antimagic orientation* of $G$. In the following, we prove that there exists an antimagic orientation of $K_n$.

**Theorem 4.1.** *For $n \geq 3$, $K_n$ has an antimagic orientation.*

*Proof.* Let $V(\overrightarrow{K_n}) = \{v_1, v_2, \ldots, v_n\}$ and let $E(\overrightarrow{K_n}) = \{\overrightarrow{e_k} = \overrightarrow{(v_i, v_j)} | k = (i-1)n - \binom{i}{2} + j - i, 1 \leq i < j \leq n\}$. Let $f : E(\overrightarrow{K_n}) \to J_N$ be defined as $f(\overrightarrow{e_k}) = k$. Then by Theorem 2.1, $f$ is bijective. Now $S^-_{v_i} = \sum_{\overrightarrow{e} \in N^-(v_i)} f(\overrightarrow{e})$, where $N^-(v_i) = \{\overrightarrow{(v_p, v_i)} \in E(\overrightarrow{K_n}) | p < i\}$.

$$\text{Therefore } S^-_{v_i} = \sum_{p=1}^{i-1} f(\overrightarrow{(v_p, v_i)}) = \sum_{p=1}^{i-1} \left( (p-1)n - \binom{p}{2} + i - p \right)$$
$$= (n+1)\binom{i}{2} - n(i-1) - \frac{i(i-1)(i-2)}{6}.$$

Similarly, $S^+_{v_i} = \sum_{\overrightarrow{e} \in N^+(v_i)} f(\overrightarrow{e})$, where $N^+(v_i) = \{\overrightarrow{(v_i, v_q)} \in E(\overrightarrow{K_n}) | i < q\}$.

$$\text{Therefore } S^+_{v_i} = \sum_{q=i+1}^{n} f(\overrightarrow{(v_i, v_q)}) = \sum_{q=i+1}^{n} \left( (i-1)n - \binom{i}{2} + q - i \right)$$
$$= \binom{n}{2} + (n - (i+1)) \left( n(i-1) - \binom{i}{2} \right).$$

But then the oriented vertex sum of vertex $v_i$ is given by $S^o_{v_i} = S^-_{v_i} - S^+_{v_i} = -\frac{2}{3}i^3 + (2n+1)i^2 - (n^2 + 2n + \frac{1}{3})i + \binom{n+1}{2}$. We claim that $S^o_{v_i} \neq S^o_{v_j}$ for $v_i \neq v_j$, i.e., $S^o_{v_i} \neq S^o_{v_j}$ for $i \neq j$. Without loss, suppose $i < j$. Now suppose $S^o_{v_j} - S^o_{v_i} = 0$. That is, $-\frac{2}{3}(j^3 - i^3) + (2n+1)(j^2 - i^2) - (n^2 + 2n + \frac{1}{3})(j - i) = 0$. Therefore we have $\frac{2}{3}(i^2 + ij + j^2) - (2n+1)(i+j) + (n^2 + 2n + \frac{1}{3}) = 0$, since $j - i \neq 0$. That is, $n^2 - nx + y = 0$, where $x = 2(i + j - 1)$ and

$y = \frac{1-3(i+j)+2(i^2+ij+j^2)}{3}$. Now $x^2-4y = \frac{4}{3}((i+j)^2+i(j-1)+j(i-1)+2) > 0$. Hence $n = (i+j-1) \pm \sqrt{\frac{1}{3}((i+j)^2+i(j-1)+j(i-1)+2)}$. Thus $n$ is dependent on $i$ and $j$. That is, $n$ is not unique, a contradiction. Therefore $S^o_{v_i} \neq S^o_{v_j}$. Hence the proof. □

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## References

[1] Bhavale, A. N., Equivalence of labeled graphs and lattices, *Congressus Numerantium*, (2025), (communicated), https://arxiv.org/abs/2501.05064

[2] Chang, F.-H., Chin, P., Li, W.-T., and Pan, Z., The strongly antimagic labelings of double spiders, arXiv:1712.09477

[3] Gallian, J. A., A dynamic survey of graph labeling, *Electron J. Combin.*, #DS6 (2022).

[4] Hartsfield, N., and Ringel, G., Pearls in Graph Theory, Academic Press, San Diego, 1990.

[5] Hefetz, D., Mütze, T., and Schwartz, J., On antimagic directed graphs, *J. Graph Theory*, **64** (2010), 219–232.

[6] Rosa, A., On certain valuations of the vertices of a graph, *Theory of graphs* (International Symposium, Rome, July 1996), Dunod Gordon and Breach, Paris, (1967), 349–355.

[7] West, D. B., Introduction to Graph Theory, Prentice Hall of India, 2002.

[8] Weisstein, E. W., Labeled graph, MathWorld, https:// mathworld.wolfram.com/ LabeledGraph.html

Dr. A. N. Bhavale
Department of Mathematics
PES Modern College of Arts, Science and Commerce
(Autonomous), Shivajinagar, Pune 411005,
(affiliated to Savitribai Phule Pune University, Pune 411007),
Maharashtra, India.
E-mail: ashokbhavale@gmail.com, hodmaths@moderncollegepune.edu.in

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# A PROBABILISTIC PROOF OF $\zeta(4) = \dfrac{\pi^4}{90}$

JEWEL MAHAJAN

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Abstract. This article contains a probabilistic proof of $\zeta(4) = \dfrac{\pi^4}{90}$ using the density function of a Cauchy variable.

## 1. Introduction

The Riemann zeta function, named after the German mathematician Bernhard Riemann, was introduced by him in 1859 as an extension of the Euler sum formula. The function is defined for complex numbers $s$ with its real part greater than 1 by the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For integer values of $s$, the zeta function has been of special interest. For $s = 2$, it has a famous connection to the Basel problem, which was first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734, almost a century later. It is important to note that James Bernoulli claimed that he had computed the value of $\zeta(2)$ in 1691, but the first published proof for the formula $\zeta(2) = \frac{\pi^2}{6}$ is due to Euler in 1734. Although Euler's proof was correct, it lacked rigour and had to await further developments in complex analysis. He proved that $\zeta(2n) \in \pi^{2n}\mathbb{Q}$ and $L(2n+1, \chi_4) \in \pi^{2n+1}\mathbb{Q}$, for each natural number $n$, where $L(s, \chi_4) := \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}$. More precisely, he proved the following formula for each natural number $n$,

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} 2^{2n-1} \pi^{2n}}{(2n)!},$$

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where $B_n$ denotes the Bernoulli numbers. Murty has provided a simple proof of the fact that $\zeta(2) = \frac{\pi^2}{6}$ using only first-year calculus (See [3]) and he has also shown $\zeta(2n) \in \pi^{2n}\mathbb{Q}$ using an elementary recursion.

The values $\zeta(2n)$ are significant for their intrinsic mathematical beauty and their applications across various branches of mathematics and physics. The values of $\zeta$ at positive integers are especially interesting due to their close connection with the volumes of unit spheres in higher dimensions. In this regard, the following formula (in Tamagawa measure), due to Siegel, is well-known:

$$\mathrm{vol}\big((SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})\big) = \zeta(2)\zeta(3)\ldots\zeta(n).$$

We also have the following formula (in Tamagawa measure) due to Siegel:

$$\mathrm{vol}\big(Sp_{2n}(\mathbb{Z})\backslash Sp_{2n}(\mathbb{R})\big) = \zeta(2)\zeta(4)\ldots\zeta(2n).$$

When $s = 4$, there are several interesting proofs of the fact $\zeta(4) = \frac{\pi^4}{90}$ in the literature, using different techniques, namely, Fourier analysis, Euler's proof for the Basel problem, contour integration and many others. One of the proofs, which is probabilistic in nature, is due to Frits Beukers, Calabi and Kolk (See [1]). It is to note that Elkies in his paper (See [2]) remarks that the proof using only the formula for change of variables of multiple integrals is due to Calabi alone, which transforms $n$-cube $(0,1)^n$ to the open $n$-dimensional convex polytope. Then one evaluates this integral by introducing four independent random variables $U_1$, $U_2$, $U_3$, and $U_4$, which are uniform $(0,1)$ and observing that the volume of this $n$-dimensional open polytope is equal to $\mathbb{P}(U_1 + U_2,\, U_2 + U_3,\, U_3 + U_4,\, U_4 + U_1 < 1)$.

1.1. **Organisation of the article.** This article is organised as follows: In Section 2.1, we mention some auxiliary lemmas and a proposition which we will use in the final part of the proof. In Section 2.2, we begin with the cumulative distribution function of the product of 4 independent Cauchy variables, which we denote by $F_T$, and we express it in terms of a certain multiple integral in equation (2.4). We then use a simple property of the cumulative distribution function to obtain the iterated integral in equation (2.5). The goal of this article is to evaluate this multiple integral. We evaluate this multiple integral by integrating with respect to $y$, $z$, and $w$ in Lemma 2.1, Lemma 2.4, and Proposition 2.5 respectively. We then integrate three parts of $K(x)$ (obtained in Proposition 2.5) with respect to $x$, in

Lemmas 2.6, 2.7, and 2.8 respectively. Finally, we find the value of $\zeta(4)$ from the equation (2.6) by combining the Lemmas 2.6, 2.7, and 2.8.

## 2. New Results

**2.1. Auxiliary Lemmas and Propositions.** In this section, we prove the following lemmas and a proposition which will be needed in Section 2.2 for the conclusion of the proof.

**Lemma 2.1.** *Let* $I(z,w,x) := \int_{y=0}^{\infty} \frac{y}{y^2 + (\frac{x}{zw})^2} \frac{1}{1+y^2} dy$, *and* $p = \frac{x}{w}$. *Then*

$$I(z,w,x) = \frac{\ln|p| - \ln|z|}{p^2 - z^2} z^2.$$

***Proof.*** We have

$$\begin{aligned}
I(z,w,x) &= \int_{y=0}^{\infty} \frac{y}{y^2 + (\frac{x}{zw})^2} \frac{1}{1+y^2} dy \\
&= \int_{y=0}^{\infty} \frac{y}{y^2 + q^2} \frac{1}{1+y^2} dy \left( q = \frac{x}{zw} \right) \\
&= \frac{1}{2(q^2-1)} \int_{y=0}^{\infty} \left( \frac{2y}{y^2+1} - \frac{2y}{y^2+q^2} \right) dy \\
&= \frac{1}{2(q^2-1)} \lim_{N\to\infty} \left( \int_{y=0}^{N} \frac{d(y^2+1)}{y^2+1} - \int_{y=0}^{N} \frac{d(y^2+q^2)}{y^2+q^2} \right) \\
&= \frac{1}{2(q^2-1)} \lim_{N\to\infty} (\ln|y^2+1| - \ln|y^2+q^2|)|_{y=0}^{N} \\
&= \frac{1}{2(q^2-1)} \lim_{N\to\infty} \left( \ln \frac{|y^2+1|}{|y^2+q^2|} \right)\Bigg|_{y=0}^{N} = \frac{\ln|q|}{(q^2-1)} \\
&= \frac{\ln|\frac{x}{w}| - \ln|z|}{\left(\frac{x}{wz}\right)^2 - 1} = \frac{\ln|p| - \ln|z|}{p^2 - z^2} z^2 \left( p = \frac{x}{w} \right).
\end{aligned}$$

□

**Remark 2.2.** In all the following calculations, wherever we interchange the order of integration, we apply Tonelli's theorem for non-negative measurable function, and, wherever we interchange sum and integral, we apply Fubini's theorem there, as either of $\sum \int |f_n| < \infty$, or, $\int \sum |f_n| < \infty$ holds.

**Remark 2.3.** We will frequently use the following facts in our calculations. This can be proved by using induction on $n$ and integration by parts. For

$k, N \in \mathbb{N} \cup \{0\}$,

$$\int_{t=0}^{1} t^k (\ln t)^n dt = \frac{(-1)^n n!}{(k+1)^{n+1}}.$$

**Lemma 2.4.** *Let* $J(w,x) := \int_{z=0}^{\infty} I(z,w,x) \frac{1}{z(1+z^2)}\, dz$. *Then*

$$J(w,x) = \frac{w^2(\ln|x| - \ln|w|)^2}{(x^2+w^2)} + \int_{z=0}^{\infty} \frac{w^2 z \ln|z|}{(z^2+1)(w^2z^2-x^2)} dz.$$

***Proof***. We use Lemma 2.1 to obtain

$$\begin{aligned}
J(w,x) &= \int_{z=0}^{\infty} I(z,w,x) \frac{1}{z(1+z^2)}\, dz \\
&= \int_{z=0}^{\infty} \frac{\ln|p| - \ln|z|}{p^2 - z^2} \frac{z}{(1+z^2)}\, dz \\
&= \frac{1}{2(p^2+1)} \int_{z=0}^{\infty} (\ln|p| - \ln|z|) \left( \frac{2z}{z^2+1} - \frac{2z}{z^2-p^2} \right) dz \\
&= \frac{\ln|p|}{2(p^2+1)} \lim_{N\to\infty} \left( \ln \frac{|z^2+1|}{|z^2-p^2|} \right) \Bigg|_{z=0}^{N} \\
&\qquad - \frac{1}{2(p^2+1)} \int_{z=0}^{\infty} \ln|z| \left( \frac{2z}{z^2+1} - \frac{2z}{z^2-p^2} \right) dz \\
&= \frac{(\ln|p|)^2}{(p^2+1)} + \int_{z=0}^{\infty} \frac{z \ln|z|}{(z^2+1)(z^2-p^2)} dz \\
&= \frac{w^2(\ln|x| - \ln|w|)^2}{(x^2+w^2)} + \int_{z=0}^{\infty} \frac{w^2 z \ln|z|}{(z^2+1)(w^2z^2-x^2)} dz.
\end{aligned}$$

□

**Proposition 2.5.** *Let* $K(x) := \int_{w=0}^{\infty} J(w,x) \frac{1}{w(1+w^2)}\, dw$. *Then*

$$\begin{aligned}
K(x) =& \frac{(\ln|x|)^3}{x^2-1} + 2 \int_{w=0}^{\infty} \frac{w(\ln|w|)^2}{(x^2+w^2)(1+w^2)} dw \\
& - 3 \int_{w=0}^{\infty} \frac{w \ln|x| \ln|w|}{(x^2+w^2)(1+w^2)} dw.
\end{aligned}$$

***Proof***. We use Lemma 2.4 to obtain

$$\begin{aligned}
K(x) &= \int_{w=0}^{\infty} J(w,x) \frac{1}{w(1+w^2)} dw \qquad (2.1)\\
&= \int_{w=0}^{\infty} \frac{w^2(\ln|x| - \ln|w|)^2}{(x^2+w^2)} \frac{1}{w(1+w^2)} dw
\end{aligned}$$

$$+\int_{w=0}^{\infty}\int_{z=0}^{\infty}\frac{w^2 z\ln|z|}{(z^2+1)(w^2z^2-x^2)}dz\frac{1}{w(1+w^2)}\,dw$$

$$=\int_{w=0}^{\infty}\frac{w(\ln|x|-\ln|w|)^2}{(x^2+w^2)(1+w^2)}dw$$

$$+\int_{z=0}^{\infty}\frac{z\ln|z|}{(z^2+1)}\left(\int_{w=0}^{\infty}\frac{w}{(1+w^2)(w^2z^2-x^2)}dw\right)dz$$

$$=\int_{w=0}^{\infty}\frac{w(\ln|x|-\ln|w|)^2}{(x^2+w^2)(1+w^2)}dw+\int_{z=0}^{\infty}\frac{z\ln|z|}{(z^2+1)}A(z,x)\,dz,\ \text{where}$$

$$\begin{aligned}
A(z,x) &= \int_{w=0}^{\infty}\frac{w}{(1+w^2)(w^2z^2-x^2)}dw\\
&=\frac{1}{z^2}\int_{w=0}^{\infty}\frac{w}{(1+w^2)(w^2-\frac{x^2}{z^2})}dw\\
&=\frac{1}{2z^2(1+\frac{x^2}{z^2})}\int_{w=0}^{\infty}\left(\frac{2w}{(w^2-\frac{x^2}{z^2})}-\frac{2w}{(1+w^2)}\right)dw\\
&=\frac{1}{2(x^2+z^2)}\lim_{N\to\infty}(\ln|w^2-\frac{x^2}{z^2}|-\ln|w^2+1|)\Big|_{w=0}^{N}\\
&=\frac{1}{2(x^2+z^2)}\lim_{N\to\infty}\left(\ln\frac{|w^2-\frac{x^2}{z^2}|}{|w^2+1|}\right)\Bigg|_{w=0}^{N}\\
&=-\frac{\ln|\frac{x}{z}|}{x^2+z^2}=\frac{\ln|z|-\ln|x|}{x^2+z^2}.
\end{aligned}$$

Putting the value of $A(z,x)$ in equation (2.1), we obtain

$$\begin{aligned}
K(x) &= \int_{w=0}^{\infty}\frac{w(\ln|x|-\ln|w|)^2}{(x^2+w^2)(1+w^2)}dw+\int_{z=0}^{\infty}\frac{z\ln|z|(\ln|z|-\ln|x|)}{(x^2+z^2)(1+z^2)}dz\\
&=\int_{w=0}^{\infty}\frac{w(\ln|x|-\ln|w|)^2}{(x^2+w^2)(1+w^2)}dw+\int_{w=0}^{\infty}\frac{w\ln|w|(\ln|w|-\ln|x|)}{(x^2+w^2)(1+w^2)}dw\\
&=\int_{w=0}^{\infty}\frac{w\Big((\ln|x|)^2-2\ln|x|\ln|w|+2(\ln|w|)^2-\ln|x|\ln|w|\Big)}{(x^2+w^2)(1+w^2)}dw\\
&=\int_{w=0}^{\infty}\frac{w\Big((\ln|x|)^2-3\ln|x|\ln|w|+2(\ln|w|)^2\Big)}{(x^2+w^2)(1+w^2)}dw\\
&=(\ln|x|)^2\int_{w=0}^{\infty}\frac{w\,dw}{(x^2+w^2)(1+w^2)}-3\int_{w=0}^{\infty}\frac{w\ln|x|\ln|w|}{(x^2+w^2)(1+w^2)}dw\\
&\quad+2\int_{w=0}^{\infty}\frac{w(\ln|w|)^2}{(x^2+w^2)(1+w^2)}dw
\end{aligned}$$

$$=\frac{(\ln|x|)^3}{x^2-1}+2\int_{w=0}^{\infty}\frac{w(\ln|w|)^2}{(x^2+w^2)(1+w^2)}dw$$
$$-3\int_{w=0}^{\infty}\frac{w\ln|x|\ln|w|}{(x^2+w^2)(1+w^2)}dw.$$

□

We now integrate $K(x)$ with respect to $x$ by combining the integration of each of the three parts appearing in the expression for $K(x)$ with respect to $x$ in the next three lemmas. The following three lemmas will be needed in the final part of the proof.

**Lemma 2.6.** *Let* $M_1 := \int_{x=0}^{\infty}\frac{(\ln|x|)^3}{x^2-1}dx$. *Then* $M_1 = \frac{45}{4}\zeta(4)$.

***Proof.*** We have

$$\begin{aligned}
M_1 &= \int_{x=0}^{\infty}\frac{(\ln|x|)^3}{x^2-1}dx\\
&= -2\int_{x=0}^{1}\frac{(\ln|x|)^3}{1-x^2}dx\\
&= -\int_{x=0}^{1}(\ln|x|)^3\Big(\sum_{n=0}^{\infty}x^{2n}\Big)dx\\
&= -2\sum_{n=0}^{\infty}\Big(\int_{x=0}^{1}x^{2n}(\ln|x|)^3dx\Big)\\
&= -2\sum_{n=0}^{\infty}\frac{3!(-1)^3}{(2n+1)^4} = 12\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = 12\Big(1-\frac{1}{16}\Big)\zeta(4) = \frac{45}{4}\zeta(4).
\end{aligned}$$

□

**Lemma 2.7.** *Let* $M_2 := \int_{w=0}^{\infty}\frac{(\ln|w|)^2}{(1+w^2)}dw$. *Then* $M_2 = \frac{\pi^3}{8}$.

***Proof.*** Considering the product of 3 independent standard Cauchy variables $X$, $Y$, and $Z$ and using a similar technique, we obtain

$$\begin{aligned}
1 &= \frac{2^3}{\pi^3}\int_{x=0}^{\infty}\Big(\int_{z=0}^{\infty}I(z,1,x)\frac{1}{z(1+z^2)}\,dz\Big)dx\\
&= \frac{2^3}{\pi^3}\int_{x=0}^{\infty}J(1,x)dx\\
&= \frac{2^3}{\pi^3}\int_{x=0}^{\infty}\frac{(\ln|x|)^2}{(1+x^2)}dx + \frac{2^3}{\pi^3}\int_{x=0}^{\infty}\int_{z=0}^{\infty}\frac{z\ln|z|}{(z^2+1)(z^2-x^2)}dzdx
\end{aligned}$$

$$=\frac{2^3}{\pi^3}M_2 + \frac{2^3}{\pi^3}\int_{z=0}^{\infty}\frac{\ln|z|}{(z^2+1)}\left(\int_{x=0}^{\infty}\frac{z}{(z^2-x^2)}dx\right)dz$$

$$=\frac{2^3}{\pi^3}M_2 + \frac{2^3}{\pi^3}\int_{z=0}^{\infty}\frac{\ln|z|}{(z^2+1)}\left(\lim_{N\to\infty}\left(\frac{1}{2}\ln\frac{|z+x|}{|z-x|}\right)\Bigg|_{x=0}^{N}\right)dz = \frac{2^3}{\pi^3}M_2.$$

□

**Lemma 2.8.** *Let* $M_3 := \displaystyle\int_{x=0}^{\infty}\int_{w=0}^{\infty}\frac{w\ln|x|\ln|w|}{(x^2+w^2)(1+w^2)}dw\,dx.$
*Then* $M_3 = \frac{\pi^4}{16}$.

***Proof.*** We have

$$\begin{aligned}
M_3 &= \int_{x=0}^{\infty}\int_{w=0}^{\infty}\frac{w\ln|x|\ln|w|}{(x^2+w^2)(1+w^2)}dw\,dx \\
&= \int_{w=0}^{\infty}\frac{w\ln|w|}{(1+w^2)}\left(\int_{x=0}^{\infty}\frac{\ln|x|}{(x^2+w^2)}dx\right)dw \\
&= \int_{w=0}^{\infty}\frac{w\ln|w|}{(1+w^2)}\left(\int_{r=0}^{\infty}\frac{\ln|rw|}{(r^2w^2+w^2)}w\,dr\right)dw \\
&= \int_{w=0}^{\infty}\frac{\ln|w|}{(1+w^2)}\left(\int_{r=0}^{\infty}\frac{\ln|rw|}{(r^2+1)}dr\right)dw \\
&= \int_{w=0}^{\infty}\int_{r=0}^{\infty}\frac{\ln|w|}{(1+w^2)}\frac{\ln|r|}{(r^2+1)}dr\,dw \\
&\qquad + \int_{w=0}^{\infty}\int_{r=0}^{\infty}\frac{\ln|w|}{(1+w^2)}\frac{\ln|w|}{(r^2+1)}dr\,dw \\
&= \left(\int_{w=0}^{\infty}\frac{\ln|w|}{(1+w^2)}dw\right)^2 + \frac{\pi}{2}\int_{w=0}^{\infty}\frac{(\ln|w|)^2}{(1+w^2)}dw \\
&= \left(\int_{w=0}^{1}\frac{\ln|w|}{(1+w^2)}dw + \int_{w=1}^{\infty}\frac{\ln|w|}{(1+w^2)}dw\right)^2 + \frac{\pi}{2}M_2 \\
&= \left(\int_{w=0}^{1}\frac{\ln|w|}{(1+w^2)}dw + \int_{s=1}^{0}\frac{-\ln|s|}{(1+\frac{1}{s^2})}\left(-\frac{1}{s^2}\right)ds\right)^2 + \frac{\pi}{2}\frac{\pi^3}{8} \\
&= \left(\int_{w=0}^{1}\frac{\ln|w|}{(1+w^2)}dw - \int_{s=0}^{1}\frac{\ln|s|}{(1+s^2)}ds\right)^2 + \frac{\pi}{2}\frac{\pi^3}{8} = \frac{\pi^4}{16},
\end{aligned}$$

where we use the fact $\int_{s=0}^{1}\frac{\ln|s|}{(1+s^2)}ds = -\int_{w=1}^{\infty}\frac{\ln|w|}{(1+w^2)}dw < \infty$, because $\ln x = \mathcal{O}(\sqrt{x})$ and hence $\displaystyle\sum_{n=1}^{\infty}\frac{\ln|n|}{1+n^2} < \infty$, using comparison test. □

2.2. **Final Proof.** We start with the following proposition, where we write down the expression for the probability density function for a product of independent continuous random variables.

**Proposition 2.9.** *Let $n \geq 2$, and $t \in \mathbb{R}$. Let $X_1, X_2, \dots, X_n$ be $n$ independent, continuous random variables with probability density functions $f_{X_1}, f_{X_2}, \dots f_{X_n}$ respectively. Then the probability density function for the product random variable $X_1 X_2 \dots X_n$ is given by*

$$f_{X_1 X_2 \dots X_n}(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1}(x_1) \dots f_{X_{n-1}}(x_{n-1}) f_{X_n}\left(\frac{t}{x_1 x_2 \dots x_{n-1}}\right) \frac{dx_1 \dots dx_{n-1}}{|x_1 \dots x_{n-1}|}.$$

***Proof.*** We know that for two independent, continuous random variables $X$ and $Y$ with probability density functions $f_X$ and $f_Y$, the probability density function for the product random variable $XY$ is given by

$$f_{XY}(t) = \int_{-\infty}^{\infty} f_X(x) f_Y\left(\frac{t}{x}\right) \frac{1}{|x|} dx. \tag{2.2}$$

This proves the proposition for $n = 2$.

We now assume that the proposition is true for $n = k - 1$. Since the product of two independent random variables is independent, we have $X = X_1 X_2 \dots X_{n-1}$ and $Y = X_n$ are two independent, continuous random variables. Therefore, using equation (2.2), we obtain

$$\begin{aligned}
& f_{X_1 X_2 \dots X_n}(t) \qquad (2.3) \\
=& f_{XY}(t) \\
=& \int_{-\infty}^{\infty} f_{X_1 X_2 \dots X_{n-1}}(x) f_{X_n}\left(\frac{t}{x}\right) \frac{1}{|x|} dx \\
=& \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1}(x_1) \dots f_{X_{n-2}}(x_{n-2}) \right. \\
& \qquad \left. \times f_{X_{n-1}}\left(\frac{x}{x_1 x_2 \dots x_{n-2}}\right) \frac{dx_1 \dots dx_{n-2}}{|x_1 x_2 \dots x_{n-2}|} \right) f_{X_n}\left(\frac{t}{x}\right) \frac{1}{|x|} dx \\
=& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1}(x_1) \dots f_{X_{n-2}}(x_{n-2}) \\
& \qquad \times f_{X_{n-1}}\left(\frac{x}{x_1 x_2 \dots x_{n-2}}\right) f_{X_n}\left(\frac{t}{x}\right) \frac{dx_1 \dots dx_{n-2} dx}{|x_1 x_2 \dots x_{n-2} x|}.
\end{aligned}$$

We now make the following change of variables:

$$y_1 = x_1,\ y_2 = x_2,\ \ldots\ ,\ y_{n-2} = x_{n-2},\ \text{ and } y_{n-1} = \frac{x}{x_1 x_2 \ldots x_{n-2}}.$$

Therefore, we have $x = y_1 y_2 \ldots y_{n-2} y_{n-1}$, and we denote $y_1 y_2 \ldots y_{n-2} y_{n-1}$ by $y$. Therefore, the Jacobian $J(y_1, y_2, \ldots, y_{n-1})$ is given by

$$\begin{aligned}
&J(y_1, y_2, \ldots, y_{n-1}) \\
=&\frac{\partial(x_1, \ldots, x_{n-2}, x)}{\partial(y_1, \ldots, y_{n-2}, y_{n-1})} \\
=&\begin{vmatrix}
\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} & \cdots & \frac{\partial x_1}{\partial y_{n-1}} \\
\frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} & \cdots & \frac{\partial x_2}{\partial y_{n-1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_{n-2}}{\partial y_1} & \frac{\partial x_{n-2}}{\partial y_2} & \frac{\partial x_{n-2}}{\partial y_3} & \cdots & \frac{\partial x_{n-2}}{\partial y_{n-1}} \\
\frac{\partial x}{\partial y_1} & \frac{\partial x}{\partial y_2} & \frac{\partial x}{\partial y_3} & \cdots & \frac{\partial x}{\partial y_{n-1}}
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & \ldots & 1 & 0 \\
\frac{y}{y_1} & \frac{y}{y_2} & \frac{y}{y_3} & \ldots & \frac{y}{y_{n-2}} & \frac{y}{y_{n-1}}
\end{vmatrix} \\
=&\frac{y}{y_{n-1}}.
\end{aligned}$$

Therefore, using equation (2.3) together with the change of variables formula for integration, we get

$$\begin{aligned}
&f_{X_1 X_2 \ldots X_n}(t) \\
=&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1}(x_1) \ \ldots \ f_{X_{n-2}}(x_{n-2}) \\
&\qquad \times f_{X_{n-1}} \left( \frac{x}{x_1 x_2 \ \ldots \ x_{n-2}} \right) f_{X_n} \left( \frac{t}{x} \right) \frac{dx_1 \ldots dx_{n-2}\, dx}{|x_1 x_2 \ \ldots \ x_{n-2} x|} \\
=&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1}(y_1) \ldots f_{X_{n-1}}(y_{n-1}) \\
&\qquad \times f_{X_n} \left( \frac{t}{y_1 y_2 \ \ldots \ y_{n-1}} \right) \frac{|J(y_1, \ \ldots \ , y_{n-1})|}{|y|} \frac{dy_1 \ldots dy_{n-2}\, dy_{n-1}}{|y_1 y_2 \ \ldots \ y_{n-2}|} \\
=&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1}(y_1) \ldots f_{X_{n-1}}(y_{n-1}) \\
&\qquad \times f_{X_n} \left( \frac{t}{y_1 y_2 \ \ldots \ y_{n-1}} \right) \frac{1}{|y_1 \ldots y_{n-2} y_{n-1}|} dy_1 \ldots \ dy_{n-2}\, dy_{n-1}.
\end{aligned}$$

This proves the proposition for all $n \geq 2$, by the principle of mathematical induction. □

Let $X$, $Y$, $Z$, and $W$ be independent standard Cauchy variables. We consider the new random variable $T = XYZW$, i.e., $T$ is the product of 4 independent standard Cauchy variables. The cumulative distribution function of $T$ is given by $F_T(t) = P(T \le t)$.

Let $f_X$, $f_Y$, $f_Z$, $f_W$ and $f_{XYZW}$ denote the probability density functions of the random variables $X$, $Y$, $Z$, $W$, and $XYZW$ respectively. Then, $f_X(x) = f_Y(x) = f_Z(x) = f_W(x) = \dfrac{1}{\pi(1+x^2)}$.

Therefore, using Proposition 2.9, we obtain

$$\begin{aligned} F_T(t) &= \int_{x=-\infty}^{t} f_{XYZW}(x)\,dx \qquad (2.4)\\ &= \int_{x=-\infty}^{t}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\int_{w=-\infty}^{\infty} f_X\Big(\frac{x}{yzw}\Big) f_Y(y)\\ &\qquad \times f_Z(z) f_W(w)\frac{1}{|yzw|}dw\,dz\,dy\,dx\\ &= \frac{1}{\pi^4}\int_{x=-\infty}^{t}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\int_{w=-\infty}^{\infty} \frac{(yzw)^2}{(yzw)^2+x^2}\frac{1}{1+y^2}\\ &\qquad \times \frac{1}{1+z^2}\frac{1}{1+w^2}\frac{1}{|yzw|}dw\,dz\,dy\,dx\\ &= \frac{1}{\pi^4}\int_{x=-\infty}^{t}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\int_{w=-\infty}^{\infty} \frac{|yzw|}{(yzw)^2+x^2}\frac{1}{1+y^2}\\ &\qquad \times \frac{1}{1+z^2}\frac{1}{1+w^2}dw\,dz\,dy\,dx. \end{aligned}$$

Since $F_T$ is a cumulative distribution function, we have $\lim_{t\to\infty} F_T(t) = 1$, which implies

$$\begin{aligned} 1 &= \lim_{t\to\infty} F_T(t) \qquad (2.5)\\ &= \frac{1}{\pi^4}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\int_{w=-\infty}^{\infty} \frac{|yzw|}{(yzw)^2+x^2}\frac{1}{1+y^2}\frac{1}{1+z^2}\\ &\qquad \times \frac{1}{1+w^2}dw\,dz\,dy\,dx\\ &= \frac{2^4}{\pi^4}\int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=0}^{\infty}\int_{w=0}^{\infty} \frac{yzw}{(yzw)^2+x^2}\frac{1}{1+y^2}\frac{1}{1+z^2}\\ &\qquad \times \frac{1}{1+w^2}dw\,dz\,dy\,dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^4}{\pi^4} \int_{x=0}^{\infty} \int_{z=0}^{\infty} \int_{w=0}^{\infty} \Big( \int_{y=0}^{\infty} \frac{y}{y^2 + (\frac{x}{zw})^2} \frac{1}{1+y^2} dy \Big) \frac{1}{z(1+z^2)} \\
&\qquad \times \frac{1}{w(1+w^2)} dw\, dz\, dx \\
&= \frac{2^4}{\pi^4} \int_{x=0}^{\infty} \int_{w=0}^{\infty} \Big( \int_{z=0}^{\infty} I(z,w,x) \frac{1}{z(1+z^2)}\, dz \Big) \frac{1}{w(1+w^2)} dw\, dx \\
&= \frac{2^4}{\pi^4} \int_{x=0}^{\infty} \Big( \int_{w=0}^{\infty} J(w,x) \frac{1}{w(1+w^2)} dw \Big)\, dx \\
&= \frac{2^4}{\pi^4} \int_{x=0}^{\infty} K(x)\, dx \\
&= \frac{2^4}{\pi^4} \int_{x=0}^{\infty} \left( \frac{(\ln|x|)^3}{x^2-1} + 2 \int_{w=0}^{\infty} \frac{w(\ln|w|)^2}{(x^2+w^2)(1+w^2)} dw \right. \\
&\qquad \left. - 3 \int_{w=0}^{\infty} \frac{w \ln|x| \ln|w|}{(x^2+w^2)(1+w^2)} dw \right) dx,
\end{aligned}
$$

where $I(z,w,x)$, $J(w,x)$, and $K(x)$ are defined in Lemma 2.1, Lemma 2.4, and Proposition 2.5 respectively.

Now, equation (2.5) implies,

$$
\begin{aligned}
\frac{\pi^4}{16} &= \int_{x=0}^{\infty} \frac{(\ln|x|)^3}{x^2-1} dx + 2 \int_{w=0}^{\infty} \frac{(\ln|w|)^2}{(1+w^2)} \left( \int_{x=0}^{\infty} \frac{w}{(x^2+w^2)} dx \right) dw \qquad (2.6) \\
&\qquad - 3 \int_{x=0}^{\infty} \int_{w=0}^{\infty} \frac{w \ln|x| \ln|w|}{(x^2+w^2)(1+w^2)} dw\, dx \\
&= \int_{x=0}^{\infty} \frac{(\ln|x|)^3}{x^2-1} dx + \pi \int_{w=0}^{\infty} \frac{(\ln|w|)^2}{(1+w^2)} dw \\
&\qquad - 3 \int_{x=0}^{\infty} \int_{w=0}^{\infty} \frac{w \ln|x| \ln|w|}{(x^2+w^2)(1+w^2)} dw\, dx \\
&= M_1 + \pi M_2 - 3M_3 \\
&= \frac{45}{4} \zeta(4) + \frac{\pi^4}{8} - \frac{3\pi^4}{16},
\end{aligned}
$$

where $M_1$, $M_2$, and $M_3$ are defined and evaluated in Lemmas 2.6, 2.7, and 2.8 respectively. Hence, we obtain

$$
\zeta(4) = \frac{4}{45} \left( \frac{\pi^4}{16} + \frac{3\pi^4}{16} - \frac{\pi^4}{8} \right) = \frac{4}{45} \left( \frac{\pi^4}{4} - \frac{\pi^4}{8} \right) = \frac{4}{45} \frac{\pi^4}{8} = \frac{\pi^4}{90}.
$$

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## References


[1] Beukers, F., Calabi, E., and Kolk, Johan A. C., Sums of generalized harmonic series and volumes, *Nieuw Arch. Wisk.* **11** (1993), no. 4, 217–224.

[2] Elkies, N. D., On the Sums $\sum_{k=-\infty}^{\infty} (4k+1)^{-n}$, *The American Mathematical Monthly* **110** (2003), no. 7, 561–573.

[3] Murty, M. Ram, A simple proof that $\zeta(2) = \frac{\pi^2}{6}$, *The Mathematics Student* **88** (2019), no. 1-2, 113–115.


Jewel Mahajan
Bhaskaracharya Pratishthana, 56/14, Off Law College Road, Damle Road, Erandwane, Pune 411004.
E-mail: jewelmahajan421@gmail.com

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# THE INTEGRALS FOR POWERS OF TRIGONOMETRIC FUNCTIONS: THE JOY OF REDISCOVERING WELL-KNOWN FORMULAS!

RAYMOND MORTINI, RUDOLF RUPP
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*Dedicated to Venant Wilhelm, the first author's school-teacher in Mathematics, on the occasion of his 80th birthday.*

Abstract. In this survey note we compute the primitives of $(\sin x)^n$, $(\cos x)^n$, $(\tan x)^n$ and $(\cot x)^n$ and revisit the proofs yielding the explicit values of the Wallis integrals $\int_0^{\pi/2}(\sin x)^n dx$ and $\int_0^{\pi/4}(\tan x)^n dx$. Some new formulas are given, too.

## Introduction

The only purpose of this note is to present in a single spot and in various ways the computations of the primitives of the functions $(\sin x)^n$, $(\cos x)^n$, $(\tan x)^n$ and $(\cot x)^n$ and to revisit known proofs for the evaluation of the Wallis integrals $\int_0^{\pi/2}(\sin x)^n dx$, $\int_0^{\pi/2}(\cos x)^n dx$ and the integrals $\int_0^{\pi/4}(\tan x)^n dx$. As a byproduct we obtain some new formulas for certain combinatoric sums. Our hope is to have established with this a convenient way for future referencing those items *with proofs* in scientific papers or in the presentation of solutions to exercises appearing in the problem sections of journals as the College Math. J., the Math. Magazine, the Amer. Math. Monthly, the Elemente der Math., the Math Gazette, Crux Mathematicorum etc. It is always a pleasure to rediscover oneself complicated formulas and to help with this exposition interested students to overcome possible difficulties in finding a/the correct proof. The items themselves (and much, much more) are listed e.g. in [3, p. 152-153, 161]. See also [6, p. 126].

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## 1. The Wallis integrals

In the sequel we use the Bourbaki notation, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Moreover, we fix the following terminology: if $n \in \mathbb{N}$, then

$$\begin{aligned} n¡ &:= 1 \cdot 3 \cdot 5 \cdot 7 \cdots n \text{ if } n \text{ is odd} \\ n¡ &:= 2 \cdot 4 \cdot 6 \cdot 8 \cdots n \text{ if } n \text{ is even, } n \neq 0, \\ 0¡ &:= 1 \text{ and } (-1)¡ := 1. \end{aligned}$$

Usually, our $n¡$ is denoted very ambiguously by $n!!$ and called a double factorial (se [4]), or, which we prefer, a semifactorial.

**Proposition 1.1** (Wallis integrals)**.** *For $n \in \mathbb{N}$ let*

$$I_n := \int_0^{\pi/2} (\sin x)^n dx.$$

*Then $I_0 = \pi/2$, $I_1 = 1$ and for $n \in \mathbb{N}^*$,*

*(1)* $I_{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \frac{\pi}{2} = \frac{(2n-1)¡}{(2n)¡} \frac{\pi}{2} = \frac{(2n)!}{4^n (n!)^2} \cdot \frac{\pi}{2} = \frac{\binom{2n}{n}}{4^n} \cdot \frac{\pi}{2}$.

*(2)* $I_{2n+1} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1} = \frac{(2n)¡}{(2n+1)¡} = \frac{4^n (n!)^2}{(2n+1)!} = \frac{4^n}{(2n+1)\binom{2n}{n}}$.

*(3)* $I_n = \dfrac{\sqrt{\pi}}{2} \dfrac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}$.

*More generally, if $\alpha \in \mathbb{C}$ with* $\operatorname{Re} \alpha > -1$*, then*

*(4)* $I(\alpha) := \displaystyle\int_0^{\pi/2} \sin^\alpha(x) dx = \int_0^{\pi/2} \cos^\alpha(x) dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{\alpha+2}{2})}$.

*Proof.* We first show that for $n \geq 2$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

In fact,

$$\begin{aligned} nI_n - (n-1)I_{n-2} &= \int_0^{\pi/2} \big(n(\sin x)^n - (n-1)(\sin x)^{n-2}\big) dx \\ &= \int_0^{\pi/2} (\sin x)^{n-2} \Big(n \sin^2 x - (n-1)\Big) \, dx \\ &= -\int_0^{\pi/2} (\sin x)^{n-2} (n \cos^2 x - 1) \, dx \\ &= -\big[(\sin x)^{n-1} \cos x\big]_0^{\pi/2} = 0. \end{aligned}$$

Now (1) and (2) follow by finite induction.

(3) This follows from (1) and (2) by noticing that $\Gamma(n+1) = n!$, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(z+1) = z\Gamma(z)$ for $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$.

(4) Using for $x > 0$ and $a \in \mathbb{C}$ the definition $p(x) := x^a = e^{a \log x}$ and $p(0) = 0$, gives the symbol $(\sin x)^a$ for $0 \le x \le \pi/2$ a sense. Making the substitutions $t = \sin x$ and then $s = t^2$, yields

$$\begin{aligned} I(\alpha) &= \int_0^1 t^\alpha \frac{dt}{\sqrt{1-t^2}} = \int_0^1 \frac{s^{\alpha/2}}{\sqrt{1-s}} \frac{ds}{2\sqrt{s}} \\ &= \frac{1}{2} \int_0^1 s^{\frac{\alpha-1}{2}} (1-s)^{-\frac{1}{2}} ds \\ &= \frac{1}{2}\, \beta\left(\frac{\alpha+1}{2}, \frac{1}{2}\right), \end{aligned}$$

where $\beta(z_1, z_2)$ is the Eulerian $\beta$-function, Re $z_j > 0$. Assertion (4) now follows from the well known result due to Euler that

$$\beta(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)},$$

which can easily be deduced from the definition of the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds,$$

and Fubini's theorem. □

Finally let us mention that the equality $\int_0^{\pi/2} (\sin x)^n dx = \int_0^{\pi/2} (\cos x)^n dx$ is trivial as $\cos x = \sin(\pi/2 - x)$.

## 2. Primitives of $(\sin x)^n$ and $(\cos x)^n$

We present two methods to determine the primitives of the powers of sin and cos. We will work here with primitives vanishing at the origin, given by $F_n(x) = \int_0^x f^n(t)\, dt$. The first method derives a relation between $F_{n+2}$ and $F_n$, and the second method is the standard one of linearization.

**Proposition 2.1.** *For $n \in \mathbb{N}$, let $S_n(x) = \int_0^x (\sin t)^n dt$ and $C_n(x) = \int_0^x (\cos t)^n dt$. Then, for $n \ge 2$,*

*(1)* $S_n(x) = \dfrac{n-1}{n} S_{n-2}(x) - \dfrac{(\sin x)^{n-1} \cos x}{n}$.

*(2)* $C_n(x) = \dfrac{n-1}{n} C_{n-2}(x) + \dfrac{(\cos x)^{n-1} \sin x}{n}$.

*Proof.* (1) Using partial integration with $u' = (\sin x)^{n-2}\cos x$ and $v = \cos x$,

$$\begin{aligned} S_n(y) &= \int_0^y (\sin x)^{n-2}(1-\cos^2 x)dx \\ &= S_{n-2}(y) - \int_0^y [(\sin x)^{n-2}\cos x]\cos x\, dx \\ &= S_{n-2}(y) - \left(\frac{(\sin y)^{n-1}}{n-1}\cos y + \frac{1}{n-1}S_n(y)\right). \end{aligned}$$

Hence

$$\frac{n}{n-1}S_n(x) = S_{n-2}(x) - \frac{(\sin x)^{n-1}}{n-1}\cos x.$$

(2) Using partial integration with $u' = (\cos x)^{n-2}\sin x$ and $v = \sin x$,

$$\begin{aligned} C_n(y) &= \int_0^y (\cos x)^{n-2}(1-\sin^2 x)dx \\ &= C_{n-2}(y) - \int_0^y [(\cos x)^{n-2}\sin x]\sin x\, dx \\ &= C_{n-2}(y) - \left(-\frac{(\cos y)^{n-1}}{n-1}\sin y + \frac{1}{n-1}C_n(y)\right). \end{aligned}$$

Hence

$$\frac{n}{n-1}C_n(x) = C_{n-2}(x) + \frac{(\cos x)^{n-1}}{n-1}\sin x. \qquad \square$$

Here are representations of the primitives of $(\sin x)^n$ and $(\cos x)^n$ in terms of lower powers. Note that the empty sum is defined to be 0 in case $n = 0$.

**Theorem 2.2.** *Let $n \in \mathbb{N}$, $\sigma_n := \lceil \frac{n}{2} \rceil - 1$. There exist suitable constants $C(n)$, namely $C(n) = 0$ if $n$ is even and $C(n) = \frac{(n-1)¡}{n¡}$ if $n$ is odd, and positive rational numbers $c_j^{(n)}$ recursively given by*

$$\begin{aligned} c_0^{(n)} &= \frac{1}{n} \quad \textit{for } n \geq 1 \\ c_{j+1}^{(n+2)} &= \frac{n+1}{n+2}c_j^{(n)} \quad \textit{for } n \in \mathbb{N},\ j = 0, \dots, \max\{0, \sigma_n\} \\ c_{\sigma_0+1}^{(0)} &= 1 \quad \textit{and} \quad c_{\sigma_1+1}^{(1)} = 0, \end{aligned}$$

*such that*

*(1)* $S_n(x) = \displaystyle\int_0^x (\sin t)^n dt = c_{\sigma_n+1}^{(n)}x - \sum_{j=0}^{\sigma_n} c_j^{(n)}(\sin x)^{n-1-2j}\cos x + C_n.$

*(2)* $C_n(x) = \displaystyle\int_0^x (\cos t)^n dt = c_{\sigma_n+1}^{(n)}x + \sum_{j=0}^{\sigma_n} c_j^{(n)}(\cos x)^{n-1-2j}\sin x.$

Before we present a proof, here are some remarks.

**Remark 2.3.** • Note that $\sigma_n = n/2 - 1 = m - 1$ if $n = 2m$, $m \in \mathbb{N}$, and $\sigma_n = (n-1)/2 = m$ if $n = 2m+1$, $m \in \mathbb{N}$. E.g. $(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = (-1, 0, 0, 1, 1, 2, 2)$.

FIGURE 1. The inductive procedure

• Figure 1 displays a method mimicking the movement of a knight in chess to construct these coefficients. Note that $c^{(n)}_{\sigma_n+1} = 0$ whenever $n$ is odd, because the integral in (1) represents an even function. Here is now the display of the first few coefficients. Note that for $n$ odd we have $\sigma_n + 1$ summands, and $\sigma_n + 2$ summands if $n$ is even (since in that case a linear factor additionally appears):

| $n=0$ $\sigma_0{=}-1$ | $n=1$ $\sigma_1{=}0$ | $n=2$ $\sigma_2{=}0$ | | $n=3$ $\sigma_3{=}1$ | | $n=4$ $\sigma_4{=}1$ | | | $n=5$ $\sigma_5{=}2$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_0^{(0)}$ | $c_0^{(1)}$ | $c_0^{(2)}$ | $c_1^{(2)}$ | $c_0^{(3)}$ | $c_1^{(3)}$ | $c_0^{(4)}$ | $c_1^{(4)}$ | $c_2^{(4)}$ | $c_0^{(5)}$ | $c_1^{(5)}$ | $c_2^{(5)}$ |
| 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{4}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{5}$ | $\frac{4}{15}$ | $\frac{8}{15}$ |

*Proof.* We proceed via induction using Proposition 2.1. Observe that for $n$ even, $S_n$ is an odd function, and for $n$ odd, $S_n$ is an even function. So we expect our summands $(\sin x)^j \cos x$ to change parity, too.

(1) If $n = 1$, then $S_1(x) = \int_0^x \sin t \, dt = -\cos x + 1$, and if $n = 0$, then $S_0(x) = \int_0^x (\sin t)^0 \, dt = x$. Both items are compatible with the formula. Now suppose that the formula holds for $n$. We show that it holds for $n+2$. Observe that $\sigma_{n+2} = \sigma_n + 1$.

$$S_{n+2}(x) = \frac{n+1}{n+2} S_n(x) - \frac{1}{n+2} (\sin x)^{n+1} \cos x$$

$$\begin{aligned}
&= \frac{n+1}{n+2}\left(c^{(n)}_{\sigma_n+1}x - \sum_{j=0}^{\sigma_n} c_j^{(n)}(\sin x)^{n-1-2j}\cos x + C(n)\right)\\
&\qquad -\frac{1}{n+2}(\sin x)^{n+1}\cos x\\
&\underset{k=j+1}{=} \frac{n+1}{n+2}\big(C(n)+c^{(n)}_{\sigma_n+1}x\big) - \sum_{k=1}^{\sigma_n+1}\left(\frac{n+1}{n+2}c^{(n)}_{k-1}\right)(\sin x)^{n+1-2k}\cos x\\
&\qquad -\frac{1}{n+2}(\sin x)^{n+1}\cos x\\
&= c^{(n+2)}_{\sigma_{n+2}+1}x - \sum_{k=1}^{\sigma_{n+2}} c_k^{(n+2)}(\sin x)^{n+1-2k}\cos x - \frac{1}{n+2}(\sin x)^{n+1}\cos x\\
&\qquad +C(n+2).
\end{aligned}$$

For the value of $C(n)$, see Remark 2.5.

(2) To calculate a primitive of $(\cos x)^n$ we use the identity $\cos x = \sin(\frac{\pi}{2} - x)$, and so

$$\int (\cos x)^n dx = -\int (\sin u)^n du$$

where $u = \frac{\pi}{2} - x$. Now pull in. □

The coefficients in Theorem 2.2 can be given explicitly:

**Proposition 2.4.**

*(1)* $$c^{(n)}_{\sigma_n+1} = \begin{cases} \frac{1}{2^n}\binom{n}{\frac{n}{2}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{n-1}{n} = \frac{(n-1)¡}{n¡} & \textit{if } n \textit{ is even, } n>0\\ 1 & \textit{if } n=0\\ 0 & \textit{if } n \textit{ is odd.}\end{cases}$$

*(2)* $$c_j^{(n)} = \frac{1}{n}\,\frac{(n-1)}{(n-2)}\,\frac{(n-3)}{(n-4)}\cdots\frac{n-2j+1}{n-2j} \quad \textit{for } j=1,\dots,\sigma_n,\ n\geq 3,$$

$$c_0^{(0)} = 1,\ c_0^{(1)} = 1,\ c_0^{(2)} = c_1^{(2)} = \frac{1}{2}.$$

*Proof.* (1) This follows from $c^{(n+2)}_{\sigma_n+1} = \frac{n+1}{n+2}\, c^{(n)}_{\sigma_n}$ and the initial data $1 = c_0^{(0)} = c^{(0)}_{\sigma_0+1}$ as well as $0 = c_1^{(1)} = c^{(1)}_{\sigma_1+1}$.

(2) If $j=1$, then $c_1^{(n)} = \frac{n-1}{n}c_0^{(n-2)} = \frac{n-1}{n}\frac{1}{n-2} = \frac{1}{n}\,\frac{n-1}{n-2}$ for $n>2$. Moreover, by that formula, $c_2^{(n)} = \frac{n-1}{n}c_1^{(n-2)} = \frac{n-1}{n}\,\frac{1}{n-2}\frac{n-3}{n-4} = \frac{1}{n}\frac{n-1}{n-2}\frac{n-3}{n-4}$. By induction, we obtain assertion (2). □

**Remark 2.5.** It follows from Theorem 2.2 and Proposition 2.4 that the coefficient $c_{\sigma_n}^{(n)}$ associated with the lowest sine-power

$$n-1-2\sigma_n = \begin{cases} 0 & \text{for } n=2m+1 \\ 1 & \text{for } n=2m \end{cases}$$

equals

$$c_{\sigma_n}^{(n)} = \frac{(n-1)¡}{n¡},$$

and that $C(n) = \frac{(n-1)¡}{n¡}$, too, if $n$ is odd, respectivley $C(n)=0$ if $n$ is even. Moreover, $c_{\sigma_n}^{(n)}$ coincides with $\int_0^{\pi/2} (\sin x)^n dx$ if $n$ is odd and with $\frac{2}{\pi}\int_0^{\pi/2} (\sin x)^n dx$ if $n$ is even (compare with Proposition 1.1). Also, for $n$ even, $c_{\sigma_n+1}^{(n)} = c_{\sigma_n}^{(n)}$. We recapitulate: $\sigma_{2m+1} = m$ and $\sigma_{2m} = m-1$,

- $n = 2m+1$ :

$$S_n(x) = \frac{(n-1)¡}{n¡} - \cos x \left( \sum_{j=0}^{m} c_{\sigma_n - j}^{(n)} (\sin x)^{2j} \right) =$$

$$\frac{(n-1)¡}{n¡} - \cos x \Big( \frac{(n-1)¡}{n¡} + c_{\sigma_n-1}^{(n)} (\sin x)^2 + c_{\sigma_n-2}^{(n)} (\sin x)^4 + \cdots +$$
$$+ c_1^{(n)} (\sin x)^{2m-2} + \frac{1}{2m+1} (\sin x)^{2m} \Big)$$

- $n = 2m$ :

$$S_n(x) = c_{\sigma_n+1}^{(n)} x - \cos x \Big( \sum_{j=0}^{m-1} c_{\sigma_n-j}^{(n)} (\sin x)^{2j+1} \Big) =$$

$$\frac{(n-1)¡}{n¡} x - \cos x \Big( \frac{(n-1)¡}{n¡} \sin x + c_{\sigma_n-1}^{(n)} (\sin x)^3 + \cdots$$
$$+ c_1^{(n)} (\sin x)^{2m-3} + \frac{1}{2m} (\sin x)^{2m-1} \Big).$$

- $n = 2m+1$ :

$$C_n(x) = \sin x \left( \sum_{j=0}^{m} c_{\sigma_n-j}^{(n)} (\cos x)^{2j} \right) =$$

$$\sin x \Big( \frac{(n-1)¡}{n¡} + c_{\sigma_n-1}^{(n)} (\cos x)^2 + c_{\sigma_n-2}^{(n)} (\cos x)^4 + \cdots$$
$$+ c_1^{(n)} (\cos x)^{2m-2} + \frac{1}{2m+1} (\cos x)^{2m} \Big)$$

- $n = 2m$ :

$$C_n(x) = c_{\sigma_n+1}^{(n)} x + \sin x \left( \sum_{j=0}^{m-1} c_{\sigma_n-j}^{(n)} (\cos x)^{2j+1} \right) =$$

$$\frac{(n-1)¡}{n¡}x + \sin x\Big(\frac{(n-1)¡}{n¡}\cos x + c^{(n)}_{\sigma_n - 1}(\cos x)^3 + \cdots$$
$$+c^{(n)}_1(\cos x)^{2m-3} + \frac{1}{2m}(\cos x)^{2m-1}\Big).$$

We have the following nice formula, which we guessed from the listing in tabular form above:

**Observation 2.6.** *Let $n \in \mathbb{N}$. Then*

$$\sum_{j=0}^{\sigma_n+1} c^{(n)}_j = 1.$$

*Proof.* By Theorem 2.2

$$\int_0^x (\cos t)^n\, dt = c^{(n)}_{\sigma_n+1}x + \sum_{j=0}^{\sigma_n} c^{(n)}_j (\cos x)^{n-1-2j}\sin x.$$

Dividing for $x \in ]0, \pi/2]$ by $\sin x$, and applying de l'Hospital's rule with $x \to 0$, yields

$$1 = \lim_{x\to 0}\frac{\int_0^x (\cos t)^n dt}{\sin x} = c^{(n)}_{\sigma_n+1} + \sum_{j=0}^{\sigma_n} c^{(n)}_j. \qquad \square$$

Yet another well-known representation of $S_n$ for $n$ odd, and in terms of powers of the cosine function, can readily be given (note that for $n = 2m+1$ we have $\sigma_n = m$). See e.g. [2, p. 357].

**Proposition 2.7.** *For $m \in \mathbb{N}$ we have*

$$\int_0^x (\sin u)^{2m+1} du = \sum_{j=0}^{m}(-1)^{j+1}\binom{m}{j}\frac{(\cos x)^{2j+1}}{2j+1} + \kappa_m$$

*for the constant*

$$\kappa_m = \sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^j}{2j+1} = \frac{\sqrt{\pi}m!}{2\Gamma(m+3/2)} = \frac{(2m)¡}{(2m+1)¡}.$$

*Proof.* This is straightforward. We first determine some primitive. First write

$$S(x) := (\sin x)^{2m+1} = (1-\cos^2 x)^m \sin x.$$

Now let $F(t)$ be a primitive of $-(1-t^2)^m$. Then $P(x) := F(\cos x)$ is a primitive of $S(x)$, since

$$P'(x) = F'(\cos x)(-\sin x) = S(x).$$

Next note that

$$\begin{aligned} F(t) &= -\int \left((-t^2)+1\right)^m dt = -\int \sum_{j=0}^{m} \binom{m}{j} (-1)^j t^{2j} dt \\ &= \sum_{j=0}^{m} \binom{m}{j} (-1)^{j+1} \frac{t^{2j+1}}{2j+1}. \end{aligned}$$

Hence, with $t = \cos x$,

$$S_{2m+1}(x) = \int_0^x (\sin u)^{2m+1} du = \sum_{j=0}^{m} (-1)^{j+1} \binom{m}{j} \frac{(\cos x)^{2j+1}}{2j+1} + \kappa_m,$$

where $\kappa_m$ is chosen so that the sums vanish at $t = 1$ resp. $x = 0$. That is

$$\kappa_m = \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{2j+1}.$$

The explicit value will be determined in Corollary 2.8 below. □

Combining Propositions 2.4 and 2.7, we get the following nice formula, which to the best of our knowledge is new:

**Corollary 2.8.** *Let $c_j^{(n)}$ and $C(n)$ be the coefficients in Theorem 2.2. Then, for $n$ odd, i.e. $n = 2m+1$, and $k = 0, 1, \ldots, m$*

$$c_{m-k}^{(2m+1)} \overset{0\le k\le m}{=} \sum_{j=k}^{m} (-1)^{j+k} \frac{\binom{m}{j}\binom{j}{k}}{2j+1} \overset{0\le k<m}{=} \frac{1}{2m+1} \cdot \frac{2m}{2m-1}\,\frac{2m-2}{2m-3} \cdots \frac{2+2k}{1+2k}.$$

*Moreover,*

$$\kappa_m = \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{2j+1} = \frac{(2m)¡}{(2m+1)¡}.$$

*Proof.* That

$$c_{m-k}^{(2m+1)} = \frac{1}{2m+1}\,\frac{2m}{2m-1}\,\frac{2m-2}{2m-3} \cdots \frac{2+2k}{1+2k}$$

for $0 \le k < m$ and $c_0^{(2m+1)} = \frac{1}{2m+1}$ for $k = m$ follows from Proposition 2.4 (2) applied to $n = 2m+1$ and $j$ replaced by $m-k$. As we have no direct proof for the second identity, we derive it via a new calculation of $S_{2m+1}(x)$. Since $\cos^2 x = 1 - \sin^2 x$, we obtain from Proposition 2.7 that

$$\begin{aligned}
S_{2m+1}(x) &= \sum_{j=0}^{m}(-1)^{j+1}\binom{m}{j}\frac{(\cos x)^{2j+1}}{2j+1}+\kappa_m\\
&= \left(\sum_{j=0}^{m}(-1)^{j+1}\binom{m}{j}\frac{((-\sin^2 x)+1)^j}{2j+1}\right)\cos x+\kappa_m\\
&= \left(\sum_{j=0}^{m}\frac{(-1)^{j+1}}{2j+1}\binom{m}{j}\sum_{k=0}^{j}(-1)^k\binom{j}{k}(\sin x)^{2k}\right)\cos x+\kappa_m\\
&= \sum_{k=0}^{m}\underbrace{\left[\sum_{j=k}^{m}\frac{(-1)^{j+1+k}}{2j+1}\binom{m}{j}\binom{j}{k}\right]}_{:=C_k^{2m+1}}(\sin x)^{2k}\cos x+\kappa_m.
\end{aligned}$$

Since Theorem 2.2 and Remark 2.5 (which tells us that $c_{\sigma_n}^{(n)}=\frac{(2m)¡}{(2m+1)¡}$) imply that

$$S_{2m+1}(x)=-\sum_{k=0}^{m}c_{m-k}^{(2m+1)}(\sin x)^{2k}\cos x+\frac{(2m)¡}{(2m+1)¡},$$

and since the system $\{1,\sin^{2k}\cos x : k=0,1,\dots,m\}$ is linearly independent [1] in the vector space of all continuous real-valued functions on $\mathbb{R}$, we conclude that the coefficients are equal; that is

$$C_k^{2m+1}=-c_{m-k}^{(2m+1)}\text{ and }\kappa_m=\frac{(2m)¡}{(2m+1)¡}.\qquad\square$$

**Remark 2.9.** *We also have that*

$$C_k^{2m+1}=\sum_{j=k}^{m}\frac{(-1)^{j+1+k}}{2j+1}\binom{m}{j}\binom{j}{k}=-2^{m-k}\frac{m!}{k!}\frac{(2k-1)¡}{(2m+1)¡}.$$

*Avis aux amateurs!*

As a byproduct, we obtain an explicit representation of the primitives of the rational function $R(t)=(1+t^2)^{-n}$ (see [3, p.77])

**Corollary 2.10.** *For $n\in\mathbb{N}^*$, we have*

$$\int\frac{1}{(1+t^2)^n}dt=c_{\sigma_{2n-2}+1}^{(2n-2)}\arctan t+\sum_{j=0}^{\sigma_{2n-2}}c_j^{(2n-2)}\frac{t}{(1+t^2)^{n-j-1}}+C.$$

*Proof.* Since $(\tan x)'=1+\tan^2 x=\frac{1}{\cos^2 x}$,

$$I:=\int\frac{1}{(1+t^2)^n}dt\ \overset{t=\tan x}{\underset{x=\arctan t}{=}}\ \int\frac{1+\tan^2 x}{(1+\tan^2 x)^n}\,dx=\int(\cos x)^{2n-2}dx,$$

[1] Let $\lambda+\sum_{k=0}^{m}\lambda_k(\sin x)^{2k}\cos x=0$. Then with $x=\pi/2$ we see that $\lambda=0$ and so, for $0\le x<\pi/2$, $\sum_{k=0}^{m}\lambda_k(\sin x)^{2k}=0$. Now put $t:=\sin^2 x$. Then, for $0\le t<1$, $\sum_{k=0}^{m}\lambda_k t^k=0$, implying that all the $\lambda_k=0$ (restriction of a polynomial to $[0,1[$).

we deduce that $I = C_{2n-2}(\arctan t) + C$. Now

$$\cos^2(\arctan t) = (\tan^2(\arctan t) + 1)^{-1} = (t^2 + 1)^{-1}.$$

Hence, by Theorem 2.2,

$$\begin{aligned} I &= c^{(2n-2)}_{\sigma_{2n-2}+1} \arctan t + \sum_{j=0}^{\sigma_{2n-2}} c_j^{(2n-2)} (\cos(\arctan t))^{2n-3-2j} \sin(\arctan t) + C \\ &= c^{(2n-2)}_{\sigma_{2n-2}+1} \arctan t + \sum_{j=0}^{\sigma_{2n-2}} c_j^{(2n-2)} (\cos^2(\arctan t))^{n-1-j} \frac{\sin(\arctan t)}{\cos(\arctan t)} + C \\ &= c^{(2n-2)}_{\sigma_{2n-2}+1} \arctan t + \sum_{j=0}^{\sigma_{2n-2}} c_j^{(2n-2)} \frac{t}{(1+t^2)^{n-j-1}} + C. \end{aligned}$$

□

**Corollary 2.11.** *Let* $n \in \mathbb{N}^*$. *Then*

$$\int_0^\infty \frac{1}{(1+t^2)^n} dt = \frac{\pi}{2} \frac{(2n-3)\text{¡}}{(2n-2)\text{¡}}.$$

*Proof.* This follows from the previous Corollary 2.10 and Remark 2.5. □

Next we derive representations of all the primitives of $(\sin x)^n$ and $(\cos x)^n$ in terms of $\sin(kx)$ and $\cos(kx)$. The way to achieve this, is commonly known as the "linearization method". At first glance, we find the outcome a bit surprising as the formulas are not ‘symmetric’ (two times the sine function appears, but only once the cosine function). The reason is that the primitives vanishing at the origin of the even functions $(\sin x)^{2n}$ and $(\cos x)^n$ are odd, whereas only for $(\sin x)^{2n+1}$ are primitives even.

**Proposition 2.12.**

(1) *The primitives of* $(\cos x)^n$ *can be represented as* $\mathbb{Q}$*-linear combinations of sine terms of the form* $\sin(n-2j)x$.

(2) *The primitives of* $(\sin x)^n$ *can be represented as* $\mathbb{Q}$*-linear combinations of* $x$ *and sine terms of the form* $\sin(n-2j)x$ *if* $n$ *is even, and as* $\mathbb{Q}$*-linear combinations of cosine terms of the form* $\cos(n-2j)x$ *if* $n$ *is odd.*

*Proof.* (2) Using $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, we obtain

$$\int (\sin x)^n dx \quad = \quad \frac{1}{(2i)^n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \int e^{i(2j-n)x} dx.$$

If $n$ is even, $i^n = (-1)^{n/2}$ and so, by taking real parts,

$$\begin{aligned}
S_n(x) &= \frac{1}{(2i)^n}\left(\sum_{\substack{j=0\\ j\neq n/2}}^{n}\frac{(-1)^{n-j}\binom{n}{j}}{i(2j-n)}e^{i(2j-n)x}+(-1)^{n/2}\binom{n}{n/2}x\right)+C\\
&= \frac{(-1)^{n/2}}{2^n}\left(\sum_{\substack{j=0\\ j\neq n/2}}^{n}\frac{(-1)^{n-j}\binom{n}{j}}{2j-n}\sin(2j-n)x+(-1)^{n/2}\binom{n}{n/2}x\right)+C\\
&= \boxed{\frac{(-1)^{(n/2)}}{2^{n-1}}\sum_{j=0}^{(n/2)-1}\frac{(-1)^{j}\binom{n}{j}}{n-2j}\sin(n-2j)x+\frac{1}{2^n}\binom{n}{n/2}x+C,}
\end{aligned}$$

where for the last identity we have used for $n/2 < j \leq n$ the index transformation $k = n - j$, and the symmetry $\binom{n}{k} = \binom{n}{n-k}$. As the first half of the series then reappears, the factor $1/2^n$ is then doubled.

If $n$ is odd, $i^n = i(-1)^{(n-1)/2}$, hence, by taking real parts again,

$$\begin{aligned}
S_n(x) &= \frac{1}{(2i)^n}\left(\sum_{j=0}^{n}\frac{(-1)^{n-j}\binom{n}{j}}{i(2j-n)}e^{i(2j-n)x}\right)+C\\
&= \frac{(-1)^{(n-1)/2}}{2^n}\left(-\sum_{j=0}^{n}\frac{(-1)^{n-j}\binom{n}{j}}{2j-n}\cos(2j-n)x\right)+C\\
&= \frac{(-1)^{(n-1)/2}}{2^{n-1}}\left(-\sum_{j=0}^{(n-1)/2}\frac{(-1)^{n-j}\binom{n}{j}}{2j-n}\cos(n-2j)x\right)+C\\
&= \boxed{\frac{(-1)^{(n+1)/2}}{2^{n-1}}\left(\sum_{j=0}^{(n-1)/2}\frac{(-1)^{j}\binom{n}{j}}{n-2j}\cos(n-2j)x\right)+C},
\end{aligned}$$

where the third identity is achieved as above via the transformation $k = n - j$ with

$$(n+1)/2 \leq j \leq n.$$

(1) To calculate a primitive of $(\cos x)^n$ we again use the standard identity that $\cos x = \sin(\frac{\pi}{2} - x)$, and so

$$\int (\cos x)^n dx = -\int (\sin u)^n du,$$

where $u = \frac{\pi}{2} - x$. Now

$$\begin{aligned}\sin(n-2j)u &= \sin\Big((n-2j)(\frac{\pi}{2}-x)\Big) = \sin\Big((\frac{n}{2}-j)\pi-(n-2j)x\Big)\\ &= -(-1)^{(n/2)-j}\sin(n-2j)x \quad \text{if } n \text{ is even}\end{aligned}$$

$$\begin{aligned}\cos(n-2j)u &= \cos\Big((n-2j)(\frac{\pi}{2}-x)\Big) = \cos\Big((\frac{n}{2}-j)\pi-(n-2j)x\Big)\\ &= (-1)^{((n-1)/2)-j}\sin(n-2j)x \quad \text{if } n \text{ is odd.}\end{aligned}$$

Thus, if $n$ is even,

$$\int(\cos x)^n dx = \frac{1}{2^{n-1}}\sum_{j=0}^{(n/2)-1}\frac{\binom{n}{j}}{n-2j}\sin(n-2j)x + \frac{1}{2^n}\binom{n}{n/2}x + C^*,$$

and if $n$ is odd,

$$\int(\cos x)^n dx = \frac{1}{2^{n-1}}\sum_{j=0}^{(n-1)/2}\frac{\binom{n}{j}}{n-2j}\sin(n-2j)x + C^*. \qquad \square$$

## 3. Integrals of powers of the tangent function

Astonishingly, the calculation of a primitive of $(\tan x)^n$ is simpler than in the previous case of $(\sin x)^n$. We present some rarely viewed formulas here.

**Proposition 3.1.** *The following assertions hold for* $n \in \mathbb{N}^*$ :

*(1) Let* $P_n$ *be the primitive of* $(\tan x)^n$ *on* $]-\pi/2, \pi/2[$ *vanishing at the origin. Then*

$$\begin{aligned}P_{2n+1}(x) &= (-1)^{n+1}\log\cos x + \sum_{j=1}^{n}(-1)^{n-j}\frac{1}{2j}(\tan x)^{2j},\\ P_{2n}(x) &= (-1)^n x + \sum_{j=1}^{n}(-1)^{n-j}\frac{1}{2j-1}(\tan x)^{2j-1}.\end{aligned}$$

*(2) The definite integrals may be evaluated as follows (where for* $m=1$ *only the last summand appears):*

$$\int_0^{\pi/4}(\tan x)^m\,dx =$$

$$= \begin{cases}\dfrac{1}{m-1}-\dfrac{1}{m-3}+\dfrac{1}{m-5}+\cdots+(-1)^{\frac{m-3}{2}}\dfrac{1}{2}+(-1)^{\frac{m-1}{2}}\dfrac{\log 2}{2} & \text{if } m \text{ is odd}\\[2ex] \dfrac{1}{m-1}-\dfrac{1}{m-3}+\dfrac{1}{m-5}+\cdots+(-1)^{\frac{m-2}{2}}+(-1)^{\frac{m}{2}}\dfrac{\pi}{4} & \text{if } m \text{ is even.}\end{cases}$$

*Proof.* Let $m \geq 2$. Then $P_m(x) = \dfrac{(\tan x)^{m-1}}{m-1} - P_{m-2}(x)$, because $(\tan x)' = 1 + \tan^2 x$ implies

$$\begin{aligned}
P_m(x) = \int_0^x (\tan t)^m dt &= \int_0^x (\tan t)^{m-2} \cdot \tan^2 t dt \\
&= \int_0^x (\tan t)^{m-2} (\tan t)' dt - \int_0^x (\tan t)^{m-2} dt \\
&= \frac{(\tan x)^{m-1}}{m-1} - P_{m-2}(x).
\end{aligned}$$

Hence, if $m = 2n+1$,

$$P_{2n+1}(x) = \frac{(\tan x)^{2n}}{2n} - \frac{(\tan x)^{2n-2}}{2n-2} + \cdots + (-1)^{n-1}\frac{(\tan x)^2}{2} + (-1)^n \int_0^x \tan t \, dt,$$

and if $m = 2n$,

$$P_{2n}(x) = \frac{(\tan x)^{2n-1}}{2n-1} - \frac{(\tan x)^{2n-3}}{2n-3} + \cdots + (-1)^{n-1}\frac{\tan x}{1} + (-1)^n \int_0^x 1 \, dt.$$

□

As a byproduct, we obtain the value of the following integral:

**Corollary 3.2.**

$$J_n := \int_0^1 \frac{x^n}{1+x^2} \, dx = \int_0^{\pi/4} (\tan u)^n \, du.$$

*Proof.* Just substitute $x = \tan u$ and use that $du = \frac{dx}{1+x^2}$. A direct way to calculate $J_n$ can be given as follows:

$$\begin{aligned}
\int_0^1 \frac{x^n}{1+x^2} dx &= \int_0^1 \sum_{j=0}^{\infty} (-1)^j x^n x^{2j} \, dx = \sum_{j=0}^{\infty} (-1)^j \int_0^1 x^{n+2j} \, dx \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{n+2j+1} = \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+5} \mp \ldots,
\end{aligned}$$

where $\sum \int = \int \sum$ is guaranteed by Lebesgue's dominated convergence theorem, since the partial sums $\sum_{j=0}^{N-1} (-1)^j x^n x^{2j}$ are bounded by $2x^n/(1+x^2)$. Hence, if $n = 2m$ is even, $(m \in \mathbb{N}^*)$,

$$\begin{aligned}
J_{2m} &\overset{j=k-m}{=} \sum_{k=0}^{\infty} (-1)^{m-k} \frac{1}{2k+1} - \sum_{k=0}^{m-1} (-1)^{m-k} \frac{1}{2k+1} \\
&= (-1)^m \arctan 1 + \frac{1}{2m-1} - \frac{1}{2m-3} \pm \cdots + (-1)^m \frac{1}{3} + (-1)^{m-1} \\
&= \frac{1}{n-1} - \frac{1}{n-3} \pm \cdots + \frac{(-1)^{\frac{n}{2}}}{3} + (-1)^{\frac{n}{2}-1} + (-1)^{\frac{n}{2}} \frac{\pi}{4}.
\end{aligned}$$

And if $n = 2m+1$ is odd ($m \in \mathbb{N}$),

$$
\begin{aligned}
J_{2m+1} &\overset{j=k-m}{=} \sum_{k=0}^{\infty}(-1)^{m-k}\frac{1}{2k+2} - \sum_{k=0}^{m-1}(-1)^{m-k}\frac{1}{2k+2} \\
&= \frac{(-1)^m}{2}\log 2 + \frac{1}{2m} - \frac{1}{2m-2} \pm \cdots + \frac{(-1)^{m-1}}{2} \\
&= \frac{1}{n-1} - \frac{1}{n-3} \pm \cdots + \frac{(-1)^{\frac{n-3}{2}}}{2} + (-1)^{\frac{n-1}{2}}\frac{\log 2}{2}.
\end{aligned}
$$

Of course, by denoting the primitive of $x^n/(1+x^2)$ vanishing at the origin by $J_n^*$, we also have that $J_n^*(x) = \frac{x^{n-1}}{n-1} - J_{n-2}^*(x)$, which can readily be seen by writing

$$\frac{x^n}{x^2+1} = \frac{x^{n-2}(x^2+1) - x^{n-2}}{x^2+1},$$

and from which we regain our result by using that

$$J_0 = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \text{ and } J_1 = \frac{1}{2}\log(1+x^2)\Big|_0^1 = \frac{\log 2}{2}. \qquad \square$$

On a more advanced level, one can also prove that

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{n+2j+1} = \frac{1}{4}\left(\Psi\left(\frac{n+3}{4}\right) - \Psi\left(\frac{n+1}{4}\right)\right),$$

where $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. This is clearly related to the incomplete $\beta$-function defined by (where $\operatorname{Re} z_2 > -1$ only for $u = 1$)

$$\beta(u, z_1, z_2) := \int_0^u t^{z_1-1}(1-t)^{z_2-1}dt, \ 0 \le u \le 1, \ \operatorname{Re} z_1 > 0.$$

In fact, let $a > 0$. By using the transformation $1 - t = 1/(1+x)$,

$$
\begin{aligned}
\beta(a) := \beta\left(\frac{1}{2}, a, 1-a\right) &= \int_0^1 \frac{x^{a-1}}{1+x}dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{a+k} \\
&= \frac{1}{2}\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right],
\end{aligned}
$$

and so

$$\int_0^1 \frac{x^n}{1+x^2}dx = \frac{1}{2}\beta\left(\frac{n+1}{2}\right)$$

(see [5, Chapter 11]). Properties of the Gamma function can be found, e.g., in [1].

Primitives of negative powers of tangent (equivalently primitives of powers of cotangent) are defined on $]0,\pi[$ and are obtained by using (as in Theorem 2.2) the substitution $u = \frac{\pi}{2} - x$, which maps $]0,\pi[$ bijectively onto $]-\pi/2,\pi/2[$, and the formula $\int(\cot x)^n dx = -\int(\tan u)^n du$. Thus we obtain:

**Proposition 3.3.** *Let $m \in \mathbb{N}^*$. A primitive $Q_m$ on $]0,\pi[$ of $(\cot x)^m$ is given by*

$$
\begin{aligned}
Q_{2n+1}(x) &= (-1)^n \log \sin x - \sum_{j=1}^{n} (-1)^{n-j} \frac{1}{2j} (\cot x)^{2j} \text{ if } m = 2n+1 \text{ is odd} \\
Q_{2n}(x) &= (-1)^n x \quad - \sum_{j=1}^{n} (-1)^{n-j} \frac{1}{2j-1} (\cot x)^{2j-1} \text{ if } m = 2n \text{ is even.}
\end{aligned}
$$

## 4. A gap filler

Here we compute the following integrals (statements in [4] e.g.):

**Proposition 4.1.** *Let $n, m \in \mathbb{N}$. Then*

$$
I(n,m) := \int_0^{\pi/2} (\sin x)^n (\cos x)^m \, dx = \begin{cases} \dfrac{(n-1)¡\,(m-1)¡}{(n+m)¡} \dfrac{\pi}{2} & \text{if } n,m \text{ even} \\ \dfrac{(n-1)¡\,(m-1)¡}{(n+m)¡} & \text{otherwise.} \end{cases}
$$

*Proof.* We may assume that $n, m \geq 1$.

$$
\begin{aligned}
I(n,m) &= \frac{1}{2} \int_0^{\pi/2} (\sin x)^{n-1} (\cos x)^{m-1} (2 \sin x \cos x) dx \overset{t=\sin^2 x}{=} \\
&= \frac{1}{2} \int_0^1 t^{(n-1)/2} (1-t)^{(m-1)/2} dt = \frac{1}{2} \beta\left(\frac{n+1}{2}, \frac{m+1}{2}\right) \\
&= \frac{1}{2} \frac{\Gamma(\frac{n+1}{2})\,\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+n}{2}+1)}.
\end{aligned}
$$

The use of the formulas $\Gamma\left(\frac{1}{2}+k\right) = \frac{\sqrt{\pi}}{2^k}\,(2k-1)¡$ and $\Gamma(k+1) = \frac{(2k)¡}{2^k}$ now yields the assertion:

*Case 1.* $n = 2N$ and $m = 2M$. Then

$$
\Gamma\left(\frac{n+1}{2}\right) = \Gamma\left(N + \frac{1}{2}\right)
$$

and so

$$
I(n,m) = \frac{1}{2} \frac{\frac{\pi}{2^{N+M}}\,(2N-1)¡\,(2M-1)¡}{\frac{(2N+2M)¡}{2^{M+N}}} = \frac{(n-1)¡\,(m-1)¡}{(n+m)¡} \frac{\pi}{2}.
$$

*Case 2.* $n = 2N$ and $m = 2M-1$. Then

$$
\begin{aligned}
I(n,m) &= \frac{1}{2} \frac{\frac{\sqrt{\pi}}{2^N}(2N-1)¡\,\Gamma(M)}{\Gamma\left(N+M+\frac{1}{2}\right)} = \frac{1}{2} \frac{\frac{\sqrt{\pi}}{2^N}(2N-1)¡\,\frac{(2M-2)¡}{2^{M-1}}}{\frac{\sqrt{\pi}}{2^{M+N}}(2N+2M-1)¡} \\
&= \frac{(n-1)¡\,(m-1)¡}{(n+m)¡}.
\end{aligned}
$$

*Case 3.* $n = 2N - 1$ and $m = 2M - 1$. Then

$$\begin{aligned} I(n,m) &= \frac{1}{2}\,\frac{\Gamma(N)\,\Gamma(M)}{\Gamma(N+M)} = \frac{1}{2}\,\frac{\frac{(2N-2)\text{¡}}{2^{N-1}}\;\frac{(2M-2)\text{¡}}{2^{M-1}}}{\frac{(2N+2M-2)\text{¡}}{2^{M+N-1}}} \\ &= \frac{(n-1)\text{¡}\,(m-1)\text{¡}}{(n+m)\text{¡}}. \end{aligned}$$

Being symmetric in $n$ and $m$, there is no additional case that has to be considered. □

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## References

[1] Artin, E., The Gamma function, Holt, Rinehart and Winston, New York, 1964.

[2] Bronstein, I.N., Semendjajew, K.A., Taschenbuch der Mathematik, 20. Auflage, Thun, Frankfurt, 1981.

[3] Gradshteyn, I.S., Rhyzik, I.M., Table of integrals, series and products, 7th Edition, Elsevier, 2007

[4] Meserve, B.E., Double factorials, *Amer. Math. Monthly* **55** (1948), 425–426.

[5] Moll, V., Special Integrals of Gradshteyn and Ryzhik, the Proofs – Volume I, Taylor Francis, Boca Raton, London New York, 2015.

[6] Stewart, S.M., Using exponential generating functions to evaluate definite integrals, *Scientia. Series A, Math. Sciences* **32** (2022), 99–133.

Raymond Mortini
Université de Lorraine, Département de Mathématiques et Institut Élie Cartan de Lorraine, CNRS, F-57000 Metz, France
*current address*
Université du Luxembourg, Département de Mathématiques, L-4364 Esch-sur-Alzette, Luxembourg
Email: raymond.mortini@univ-lorraine.fr

Rudolf Rupp
Fakultät für Angewandte Mathematik, Physik und Allgemeinwissenschaften
Technische Hochschule Nürnberg, Georg Simon Ohm
Kesslerplatz 12
D-90489 Nürnberg, Germany
E-email: rudolf.rupp@th-nuernberg.de

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# EXTENSION OF THE GENERALIZED HYPERGEOMETRIC FUNCTION IN LUCAS-UZAWA MODEL WITH DIFFERENT INEQUALITIES

POOJA SINGH

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Abstract. The Lucas-Uzawa model is a popular framework for studying economic growth. In this model, convergence and stability properties can be established using the Cauchy-Bunyakovsky-Schwarz inequality. The power series expression for $P_n(x)$ can be used to study the behavior of the model under different parameter values, with solutions given by linear combinations of Legendre polynomials. Additionally, if the production function is log-convex, an improved inequality holds which can be used to analyze the impact of the production function on economic growth. These results have important implications for understanding economic growth and development.

## 1. Introduction

In recent studies, researchers [9] have explored the Uzawa-Lucas model using Gaussian hypergeometric functions. Their method, which is global and does not rely on dimension reduction, has inspired our work where we independently apply hypergeometric functions to the Lucas-Uzawa model. This particular endogenous growth model has unique properties and has been studied extensively, but an open problem still remains as in work of [15] and [2].

In this paper, we aim to address this open problem by using a chain of inequalities of hypergeometric functions and general power series where we will be refering to [8], [11] and [21]. Our focus is on the equilibrium dynamics of physical and human capital in the Lucas-Uzawa model, and

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we show that Gaussian hypergeometric functions can provide an explicit representation of their equilibrium dynamics.

To structure our paper, we provide a brief overview of convexity and concavity of functions in Section 2. In Section 3, we present the Lucas model using the same notation as [10] and [22], giving the first order conditions and the non-linear dynamic system that drives the economy in equilibrium. Section 4 introduces the Gaussian hypergeometric functions and recalls the inequalities that we will use in the equilibrium dynamics of the model in further sections. Finally, in Sections 5, 6, and 7, we discuss analytical solutions for these inequalities in the Lucas-Uzawa model.

Our research contributes to the ongoing discussion in the field of endogenous growth models by providing new insights into the Lucas-Uzawa model using Gaussian hypergeometric functions.

## 2. Convexity and Concavity of functions

According to [4], in order to define a convex function $f$, one must verify that the following condition is satisfied: for all $\alpha \in [0,1]$ and for any $p, q$ in the interval $[a,b]$, the inequality

$$f(\alpha p + (1-\alpha)q) \leq \alpha f(p) + (1-\alpha) f(q) \tag{2.1}$$

holds. In alternative terms, to define a convex function $f$, one can consider its epigraph, which consists of all points lying on or above the graph. It follows that $f$ is convex iff its epigraph is also convex. The function $f$ is called concave if the inequality in the definition of convexity i.e eqn (2.1) is reversed, i.e., if $-f$ is convex. Further, a function $f$ is said to be strictly convex if the inequality given in equation (2.1) holds for any $p$ not equal to $q$ and $\alpha$ in $(0,1)$. Conversely, we can say that a similar description applies to strictly concave functions.

If a function $f$ is differentiable, its convexity or concavity can be determined by examining the behavior of its first derivative $f'$. Specifically, $f$ is (strict) concave (convex) iff $f'$ is (strictly) decreasing (increasing). Furthermore, when $f$ is differentiable twice, its (strict) concavity (convexity) can be determined by examining the behavior of its second derivative $f''$. That is, $f$ is (strictly) concave or convex iff $f''$ is (strictly) negative or positive. It is important to understand that the addition of (strictly) concave or convex functions results in (strictly) concave or convex functions is significant.

Additionally, a limit function of a convergent sequence of (strictly) concave or convex functions is also (strictly) concave or convex.

Consider a function $g$ which is defined on the closed interval $[a, b]$ and taking its values to positive real numbers is called logarithmically convex if natural logarithm, denoted by $\log g$, is a convex function. In other words, for all $p, q \in [a, b]$ and $\alpha \in [0, 1]$, the following inequality holds:

$$g(\alpha p + (1-\alpha)q) \leq [g(p)]^{\alpha}[g(q)]^{1-\alpha}. \tag{2.2}$$

If this equation is inverted, i.e.,

$$g(\alpha p + (1-\alpha)q) \geq [g(p)]^{\alpha}[g(q)]^{1-\alpha} \tag{2.3}$$

then $g$ is said to be logarithmically concave. Equivalently, $g$ is log-concave if and only if the function $1/g$ exhibits log-convexity. Additionally, if the aforementioned inequality is strict for $p \neq q$ and $\alpha \in (0, 1)$, then $g$ is said to be strictly log-convex. Strictly log-concave functions also fit an analogous description.

A function $g$ is said to be log-convex if its natural logarithm $logg$ is convex on $[a, b]$. Logarithmic convexity is a property of functions $g$ defined on intervals $[a, b] \subset \mathbb{R}$. On the other hand, if $1/g$ is log-convex, then $g$ is said to be log-concave. If the inequality in the definition of log-convexity is strict, then $g$ is said to be strictly log-convex for $r \neq s$ and $\alpha \in (0, 1)$. Similarly, a function is said to be strictly log-concave if the inequality in the definition of log-concavity is strict.

It is common knowledge that if a function $g$ can be differentiated, then $g$ is (strictly) log-convex (log-concave) if and only if its logarithmic derivative $g'/g$ is (strictly) increasing (decreasing). Convex and concave functions are similar, but log-convex and log-concave functions have different characteristics. For instance, the sum of log-convex functions is also log-convex, as is the product of log-convex (or log-concave) functions. Additionally, if the limit is positive, a convergent series of log-convex (or log-concave) functions has a corresponding log-convex (or log-concave) limit function. The sum of log-concave functions, however, is not always log-concave.

Log-convex (log-concave) functions frequently appear in a variety of problems in classical analysis and probability theory because of their intriguing properties. To note, a function that is log-convex may be convex, but not all convex functions are log-convex. Likewise, a function that is concave is also log-concave, but not all log-concave functions are necessarily concave. These definition and its properties are also discussed extensively in [19], [24] and [5].

## 3. Lucas-Uzawa Model

The analysis of the advancement and expansion of economics has been of great interest to economists for many decades. One of the first long-run economic growth models to use neoclassical principles was the Solow-Swan model, developed in the 1950s and 1960s. This model, however, did not account for the role that human capital plays in economic growth. The Lucas-Uzawa model, created in the 1960s by Robert Lucas and Hirofumi Uzawa, included human capital as an additional factor of production to get around this restriction.The Lucas-Uzawa model is a neoclassical growth model that combines elements of the Solow-Swan model and human capital theory to analyze long-run economic growth. This model assumes that the economy produces a single good using two types of inputs: human capital and physical capital. The production function exhibits constant returns to scale which means that doubling both inputs leads to a doubling of output and can be expressed as:

$$Y_t = K_t^{\alpha} H_t^{1-\alpha} \tag{3.1}$$

where $Y_t$ stands for output at time $t$, $K_t$ for the stock of physical capital, $H_t$ for the stock of human capital, and $\alpha$ for the elasticity of output in relation to the stock of physical capital. Using the Cobb-Douglas production function, we can express the production function as follows:

$$Y_t = F(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha} \tag{3.2}$$

where output $Y_t$, capital $K_t$, labor $L_t$, technology $A_t$, and the capital share $\alpha$ of output $Y_t$ are all present. The capital-labor ratio, which is given by:

$$Y/K = f(K/L) = \sum_{n=0}^{\infty} a_n (K/L)^n \tag{3.3}$$

can be used to express the production function in the Lucas-Uzawa model as a power series, where $a_n$ is a coefficient that represents the contribution of the n-th power of the capital-labor ratio to output.

The rate of consumption investment and the level of physical capital both influence the accumulation of human capital. The level of per capita output in a steady state can be found by solving for the level of physical and human capital that lead to a constant rate of increase in output per capita over time. It can be used to analyze the long-run growth rate of output per capita. The model assumes that the economy has a constant savings rate, and that savings are invested in physical capital to generate future output. The investment function can be defined as $I_t = sY_t - \delta K_t$ where $I_t$ is investment at time $t$, $s$ is the savings rate, and $\delta$ is the depreciation rate of physical capital. The investment function implies that savings are allocated between consumption and investment in physical capital. The individuals invest in human capital through education and training. The accumulation of human capital can be modeled as: $\dot{H}_t = \eta \frac{H_t}{C_t} I_t$ where $\eta$ represents the responsiveness (elasticity) of output to changes in human capital., and $C_t$ is consumption at time $t$.

The Lucas-Uzawa model assumes that households maximize their utility function subject to a budget constraint. The budget constraint is given by:

$$c_t + i_t = (1 + r_t)w_t + B_{t+1} \tag{3.4}$$

where $c_t$ denotes consumption at time $t$, $i_t$ signifies investment at time $t$, $r_t$ denotes the interest rate at time $t$, $w_t$ denotes wage at time $t$, and $B_{t+1}$ denotes the household's asset stock at the end of period $t$. The utility function is assumed to be separable and of the form:

$$U(c_t) + \beta U(c_{t+1}) \tag{3.5}$$

where the discount rate is $\beta$. Subject to financial constraints, the individuals seek to maximise intertemporal utility, which depends on consumption and leisure. The budget constraint implies that individuals allocate their time between work and leisure, and that consumption is a function of current and future income, subject to the budget constraint can be expressed as:

$$\int_0^\infty e^{-\rho t} \frac{C_t^\gamma}{1-\gamma} dt \leq \int_0^\infty e^{-\rho t} (W_t - C_t) dt \tag{3.6}$$

where $\rho$ represents rate of discount, $\gamma$ denotes the elasticity of consumption w.r.t income, and $W_t$ is the wage rate. The goal of the household problem is to select a consumption and saving strategy that, while taking the intertemporal budget constraint into account, maximises lifetime utility. The issue facing the home is expressed as follows:

$$\max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} \beta^t U(C_t) \text{ subject to } C_t + K_{t+1} \leq (1-\delta)K_t + F(K_t, A_t L_t) \tag{3.7}$$

where $C_t$ is consumption, $K_t$ is capital, $\delta$ is the depreciation rate of capital, and $U(C_t)$ is the utility function.The representative agent's intertemporal utility function can be written as follows, assuming logarithmic utility for consumers:

$$U = \int_0^\infty e^{-\rho t} \ln(C_t) dt \tag{3.8}$$

where $C_t$ is consumption at time $t$ and $\rho$ is the discount rate.

## 4. The Hypergeometric function

Consider a function as $f(x)$ to be transcendental if it cannot be expressed as a polynomial with $x$ and $f(x)$ as its variables, where the highest powers of $x$ and $f(x)$ are both greater than one. The hypergeometric series is an example of a higher transcendental function. Classic and comprehensive references for special functions, including the Gamma function, Gaussian hypergeometric function, generalized hypergeometric function and Kummer functions, can be found in various sources such as [13], [1], [20], and [6]. For a more comprehensive treatment of hypergeometric functions related to the problems discussed in this paper, interested readers may refer to [18], [16] and [23]. These references provide detailed applications and analysis of hypergeometric functions.

Within this context we are going to use power series

$$f(x) = \sum_{n \geq 0} A_n x^n \tag{4.1}$$

we can relate this power series to hypergeometric functions as follows: Let $f(x)$ be a function defined by the hypergeometric series ${}_2F_1(a, b; c; x)$, which is defined as:

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{4.2}$$

where pochhammer symbol denoted as $(a)_n$, which represents the product $a(a+1)(a+2)\cdots(a+n-1)$.

4.1. **Inequalities Using Hypergeometric functions.** This section focuses on to discuss the generalized theorem presenting some extension of [5] and later in further sections we will point out some applications of these results in Lucus-Uzawa Model. As we are consider power series in (4.1) and (4.2), we get $H(x, y) = \dfrac{x+y}{1+xy}, \quad x, y \in \mathbb{R}$.The following result is based on

Theorem 1.10 of [3], where similar inequalities for zero-balanced hypergeometric functions were proved.

**Theorem 4.1.** *Let us consider the hypergeometric function* $f(x) = {}_2F_1(a, b; c; x)$*, where* $a, b, c > 0, a + b < c$*. Then following hold:*

*(1) If the sequence* $B_{n+1}, n \geq 0$ *is decreasing, where* $B_{n+1} = (n+1)A_{n+1}$ *and* $A_n$ *is given by the hypergeometric series coefficients, then for all* $p, q \in (0, 1)$*:*

$$\begin{aligned} {}_2F_1(a_p, b_q; c_{p,q}; 1) &\leq {}_2F_1(a, b; c; p) \leq {}_2F_1(a, b; c; q) \\ &\leq {}_2F_1(a_p, b_q; cp, q; 1 - H(1 - p, 1 - q)) \end{aligned}$$

*where* $a_p = a + \frac{(c-a-b)r}{1-p}$, $b_q = b + \frac{(c-a-b)s}{1-p}$, *and* $c_{p,q} = c + \frac{(c-a-b)rs}{1-pq}$.

*(2) If the sequences* $B_{n+1} - B_n/A_n$ *and* $(B_{n+1} - B_n)/A_n$ *are decreasing, where* $n \geq 0, B_{n+1} = (n+1)A_{n+1}$ *and* $A_n$ *is given by the hypergeometric series coefficients, then for all* $p, q \in (0, 1)$*:*

$$\begin{aligned} {}_2F_1(a_p, b_q; c_{p,q}; 1) &\leq {}_2F_1(a, b; c; p) \leq {}_2F_1(a, b; c; q) \\ &\leq {}_2F_1(a_p, b_q; c_{p,q}; 1 - H(1 - p, 1 - q)) \\ &\leq {}_2F_1(a, b; c_{1-p,1-q}; 1). \end{aligned}$$

*The equalities in this theorem hold if and only if* $p = q$.

Here, $a_p$, $b_q$, and $c_{p,q}$ are called the Appell parameters of the hypergeometric function, and are related to the variables $p$.

This theorem can be applied in the Lucas-Uzawa model by considering the power series of the hypergeometric function involved in the model and checking whether the coefficients satisfy the conditions of this theorem. If the criteria for the theorem holds, then the inequalities in the theorem can be used to obtain bounds on the value of the hypergeometric function. There are many paper discussing the closed-form solution to the model of Lucas with externalises.

4.2. **The Cauchy-Bunyakovsky-Schwarz.** The CBS inequality states that for any two sequences of real numbers $(a_n)$ and $(b_n)$, we have

$$\left(\sum_{n=0}^{\infty} a_n^2\right)\left(\sum_{n=0}^{\infty} b_n^2\right) \geq \left(\sum_{n=0}^{\infty} a_n b_n\right)^2.$$

In the context of the given statement, we have $a_n = \sqrt{A_n}$ and $b_n = \sqrt{x^n}$ for $n \geq 0$. Using the Cauchy-Bunyakovsky-Schwarz inequality as stated in

[17], we get

$$\left(\sum_{n=0}^{\infty} A_n x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) \geq \left(\sum_{n=0}^{\infty} \sqrt{A_n} x^n\right)^2 .$$

Simplifying the left-hand side gives

$$\left(\sum_{n=0}^{\infty} A_n x^n\right)\left(\frac{1}{1-x}\right) = f(x)\frac{1}{1-x}$$

and simplifying the right-hand side gives

$$\left(\sum_{n=0}^{\infty} \sqrt{A_n} x^n\right)^2 = P_0(x)^2 + 2\sum_{n=1}^{\infty} P_n(x)P_{n-1}(x),$$

where $P_n(x) = \sum_{k=0}^{n} \sqrt{A_k} x^k$. Substituting these expressions back into the inequality gives

$$f(x)\frac{1}{1-x} \geq P_0(x)^2 + 2\sum_{n=1}^{\infty} P_n(x)P_{n-1}(x). \tag{4.3}$$

This is the desired result which will be used in further work. The Cauchy-Bunyakovsky-Schwarz inequality and the expression for the power series as in eqn (4.3) can be used in the Lucas-Uzawa model to analyze the convergence and behavior of the model's solutions. In particular, the inequality can be used to establish bounds on certain terms or integrals in the model, while the power series can be used to express certain functions in the model as a sum of coefficients multiplied by powers of a variable. This can help in simplifying the analysis of the model and in deriving results such as convergence conditions, stability properties, and the behavior of the model under various parameter values.

## 5. Application of Inequalities in Lucas-Uzawa Model

In the context of the Lucas-Uzawa model, the theorem (4.1) can be interpreted as follows:

Suppose we have a production function $F(K_t, A_t L_t)$ as defined in (3.2), where $K_t$ denotes the capital stock at time $t$, $L_t$ denotes labor, $A_t$ denotes productivity, and $F$ exhibits constant returns to scale. The household's optimization problem is to choose a consumption and investment path, denoted by $\{C_t, I_t\}_{t=0}^{\infty}$, to maximize lifetime utility subject to the budget constraint:

$$\sum_{t=0}^{\infty}\frac{C_t}{(1+\rho)^t}+\frac{I_0}{(1+\rho)^{t_0}}\leq\sum_{t=0}^{\infty}\frac{F(K_t,A_tL_t)-K_{t+1}-(1-\delta)K_t}{(1+\rho)^t}+\frac{K_0}{(1+\rho)^{t_0}}, \tag{5.1}$$

where $\rho$ is the rate of time preference, $t_0$ is the current time period, and $\delta$ is the depreciation rate of capital. The capital accumulation equation is given by:

$$K_{t+1}=(1-\delta)K_t+I_t. \tag{5.2}$$

The Lucas-Uzawa model is a dynamic optimization model that describes the optimal path of consumption and investment over time, subject to constraints imposed by a production function and the preferences of the agent. Let us assume that the production function can be expressed in the form of a power series, with positive coefficients, and satisfies the conditions of Theorem (4.1).

Using Theorem (4.1), we can establish bounds on the optimal consumption and investment paths that hold for all time periods. Specifically, let us consider the optimal consumption path $C(t)$ and the optimal investment path $I(t)$ at time $t$ as given in . We can express these paths as power series in $t$ as follows:

$$C(t)=\sum_{n=0}^{\infty}c_nt^n,\ I(t)=\sum_{n=0}^{\infty}i_nt^n \tag{5.3}$$

where $c_n$ and $i_n$ are the coefficients of the power series.

Using the power series representation of the production function, we can express the output at time $t$ as: $Y(t)=\sum_{n=0}^{\infty}y_nt^n$ where $y_n$ are the coefficients of the power series. The optimization problem can be expressed as follows:

$$\max_{C(t),I(t)}\int_0^{\infty}e^{-\rho t}U(C(t))dt,$$

$$\text{subject to: } Y(t)=F(K(t),L(t))$$

$$K(t)=K(0)+\int_0^t(I(s)-\delta K(s))ds$$

$$L(t)=L(0)e^{nt}$$

$$C(t)+I(t)\leq Y(t)$$

where $\rho$ is the discount rate, $U(C(t))$ is the utility function, $\delta$ is the depreciation rate, $n$ is the population growth rate, $K(0)$ and $L(0)$ are the initial capital stock and labor supply, respectively.

Using Theorem (4.1), we can derive inequalities for the optimal consumption and investment paths that hold for all time periods. Specifically, if we assume that the coefficients of the power series satisfy the conditions of Theorem (4.1), then we can obtain the following inequalities:

$$C(t) \leq {}_2F_1(a,b;c;s),\ I(t) \leq Y(t) - C(t) \leq Y(t) - {}_2F_1(a,b;c;s) \qquad (5.4)$$

where $s \in (0,1)$ is a constant that depends on the production function and the preference parameters, and ${}_2F_1(a,b;c;s)$ is the hypergeometric function evaluated at $s$. These inequalities provide upper bounds on the optimal consumption and investment paths that hold for all time periods. By solving the optimization problem subject to these bounds, we can obtain the optimal paths of consumption and investment without having to solve the entire optimization problem.

## 6. Analytical Solution for Lucas-Uzawa Model

**Preposition 6.1.** Convergence of the Lucas-Uzawa model: Let $f(x)$ be the solution of the Lucas-Uzawa model with parameter $x$, and let $A_n$ be the $n$th coefficient of the power series expansion of $f(x)\frac{1}{1-x}$. Suppose there exists a sequence of non-negative constants $(\gamma_n)$ such that $\sum_{n=0}^{\infty}\gamma_n < \infty$ and $|\sqrt{A_n}x^n| \leq \gamma_n$ for all $n$. Then $f(x)\frac{1}{1-x}$ converges absolutely for $|x| < 1$, and the Lucas-Uzawa model converges for $|x| < 1$.
*Proof.* The Lucas-Uzawa model is defined as:

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{(1-x)^{1+\alpha}} \qquad (\alpha > 0).$$

We want to show that the model converges absolutely for $|x| < 1$. To do so, we will use the Cauchy-Bunyakovsky-Schwarz inequality:

$$\left|\sum_{n=0}^{\infty} a_n b_n\right| \leq \sqrt{\sum_{n=0}^{\infty}|a_n|^2}\sqrt{\sum_{n=0}^{\infty}|b_n|^2}.$$

We choose $a_n = \sqrt{|A_n|}$ and $b_n = x^n$. Then, we have:

$$\left|\sum_{n=0}^{\infty}\sqrt{|A_n|}x^n\right| \leq \sqrt{\sum_{n=0}^{\infty}|A_n|}\sqrt{\sum_{n=0}^{\infty}|x^n|^2}.$$

The series $\sum_{n=0}^{\infty}|A_n|$ is assumed to converge, since $\alpha > 0$ implies that $(1-x)^{1+\alpha}$ has a convergent Taylor series. Thus, we only need to show that $\sum_{n=0}^{\infty}|x^n|^2$ converges. We have:

$$\sum_{n=0}^{\infty}|x^n|^2 = \sum_{n=0}^{\infty}x^{2n} = \frac{1}{1-x^2}, \quad \text{for} \quad |x| < 1.$$

Therefore, we have:

$$\left|\sum_{n=0}^{\infty}\sqrt{A_n}x^n\right| \leq \sqrt{\sum_{n=0}^{\infty}|A_n|}\sqrt{\frac{1}{1-x^2}}.$$

Let $\gamma_n = \sqrt{|A_n|}$ for all $n$. Then, we have $|\sqrt{|A_n|}x^n| \leq \gamma_n$. Also, since $\sum_{n=0}^{\infty}|A_n|$ converges, we have: $\displaystyle\sum_{n=0}^{\infty}\gamma_n^2 = \sum_{n=0}^{\infty}|A_n| < \infty$. Thus, we have found a sequence of non-negative constants $(\gamma_n)$ such that $\sum_{n=0}^{\infty}\gamma_n < \infty$ and $|\sqrt{A_n}x^n| \leq \gamma_n$ for all $n$. Therefore, we can use the Cauchy-Bunyakovsky-Schwarz inequality to show that $\sum_{n=0}^{\infty}\sqrt{A_n}x^n$ converges absolutely for $|x| < 1$. This implies that the Lucas-Uzawa model converges for $|x| < 1$. $\square$

**Preposition 6.2.** Stability of the Lucas-Uzawa model: Let $f(x)$ be a continuous function on $[0,1)$ and $A_n$ be a non-negative sequence such that $\sum_{n=0}^{\infty}\beta_n < \infty$ and $|\sqrt{A_n} - \sqrt{A_{n-1}}| \leq \beta_n$ for all $n \geq 1$. Then, if $f(x)\frac{1}{1-x}$ is Lipschitz continuous on $[0,1)$ with Lipschitz constant $\sqrt{\sum_{n=0}^{\infty}\beta_n^2}$, the Lucas-Uzawa model is stable for $|x| < 1$.

*Proof.* To prove the stability of the Lucas-Uzawa model, we need to show that the function $f(x)\frac{1}{1-x}$ is Lipschitz continuous on $[0,1)$ with Lipschitz constant $\sqrt{\sum_{n=0}^{\infty}\beta_n^2}$. To do this, we will use the Cauchy-Bunyakovsky-Schwarz inequality. First, let us define $g(x) = f(x)\frac{1}{1-x}$. Then, we have:

$$\begin{aligned}\big|g(x) - g(y)\big| &= \left|f(x)\frac{1}{1-x} - f(y)\frac{1}{1-y}\right| = \left|\frac{f(x)-f(y)}{1-x} - \frac{f(x)-f(y)}{1-y}\right| \\ &= \big|f(x) - f(y)\big| \cdot \big|\frac{1}{1-x} - \frac{1}{1-y}\big|\end{aligned}$$

$$= \left|f(x) - f(y)\right| \cdot \left|\frac{y-x}{(1-x)(1-y)}\right|$$
$$\leq \left|f(x) - f(y)\right| \cdot \frac{1}{(1-\epsilon)^2} \quad (\text{where } \epsilon = \min(x,y))$$
$$\leq \left|f(x) - f(y)\right| \cdot \frac{1}{(1-x)^2}$$

Now, using the Cauchy-Bunyakovsky-Schwarz inequality, we have:

$$\left|g(x) - g(y)\right| \leq \left|f(x) - f(y)\right| \cdot \frac{1}{(1-x)^2}$$
$$\leq \sqrt{\sum_{n=0}^{\infty}(\beta_n^2)(A_n)} \cdot |x-y| \quad (\text{by 4.2})$$

$$= \sqrt{\sum_{n=0}^{\infty} \beta_n^2 \cdot \frac{A_n}{(1-\rho)^{2n}}} \cdot |x-y| \quad (\text{where } \rho = \min(x,y))$$
$$\leq \sqrt{\sum_{n=0}^{\infty} \beta_n^2} \cdot |x-y|.$$

Therefore, we have shown that $g(x) = f(x)\frac{1}{1-x}$ is Lipschitz continuous on $[0,1)$ with Lipschitz constant $\sqrt{\sum_{n=0}^{\infty} \beta_n^2}$. This implies that the Lucas-Uzawa model is stable for $|x| < 1$. □

Note that this theorem is a generalization of the stability analysis of the Lucas-Uzawa model based on the Cauchy-Bunyakovsky-Schwarz inequality, which is commonly used in the literature. The inequality can be used to derive a Lipschitz condition for the function $f(x)\frac{1}{1-x}$, which is related to the equilibrium dynamics of the model. The Lipschitz constant is then used to establish the stability of the model for a range of values of the endogenous variable $x$. This preposition provides a more general framework for stability analysis based on the Cauchy-Bunyakovsky-Schwarz inequality, which can be applied to a wider range of models and functions.

**Preposition 6.3.** Behavior of the Lucas-Uzawa model under different parameter values: Let $P_n(x)$ be the solution to the recurrence relation $A_nP_n(x) = xP_{n-1}(x) - B_nP_{n-2}(x)$,where $A_n$, $B_n$ are non-negative constants and $P_0(x) = 1$, $P_1(x) = x$. If $A_n = \frac{1}{n+1}$ for all $n$, then $P_n(x)$ is given by the Legendre polynomial sequence, i.e., $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{\sqrt{k+1}}$.

*Proof.* To prove that if we let $A_n = \frac{1}{n+1}$, then $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{\sqrt{k+1}}$ for all $n$, we start with the recurrence relation for $P_n(x)$:

$$P_n(x) = x\sqrt{\frac{A_{n-1}}{A_n}}P_{n-1}(x) - \sqrt{\frac{A_{n-1}}{A_n}}P_{n-2}(x).$$

Substituting $A_n = \frac{1}{n+1}$ gives

$$P_n(x) = x\sqrt{\frac{n}{n+1}}P_{n-1}(x) - \sqrt{\frac{n-1}{n+1}}P_{n-2}(x).$$

Now we use induction to show that $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{\sqrt{k+1}}$ for all $n$. Base case: $n = 0$. In this case, $P_0(x) = 1$, which is equal to $\frac{x^0}{\sqrt{0+1}}$. So, the base case holds.Inductive step: Assume that $P_{n-1}(x) = \sum_{k=0}^{n-1} \frac{x^k}{\sqrt{k+1}}$ and $P_{n-2}(x) = \sum_{k=0}^{n-2} \frac{x^k}{\sqrt{k+1}}$. Then we have

$$\begin{aligned}
P_n(x) &= x\sqrt{\frac{n}{n+1}}P_{n-1}(x) - \sqrt{\frac{n-1}{n+1}}P_{n-2}(x) \\
&= x\sqrt{\frac{n}{n+1}}\sum_{k=0}^{n-1}\frac{x^k}{\sqrt{k+1}} - \sqrt{\frac{n-1}{n+1}}\sum_{k=0}^{n-2}\frac{x^k}{\sqrt{k+1}} \\
&= \sum_{k=1}^{n}\frac{x^k}{\sqrt{k}}\left(x\sqrt{\frac{n}{n+1}}\frac{1}{\sqrt{k}} - \sqrt{\frac{n-1}{n+1}}\frac{1}{\sqrt{k-1}}\right) + \frac{x^0}{\sqrt{1}}\sqrt{\frac{n}{n+1}} \\
&= \sum_{k=1}^{n}\frac{x^k}{\sqrt{k}}\left(\frac{x}{\sqrt{k+1}}\sqrt{\frac{n}{n+1}}\right) + \frac{x^0}{\sqrt{1}}\sqrt{\frac{n}{n+1}} \\
&= \sum_{k=1}^{n}\frac{x^{k+1}}{\sqrt{k(k+1)}}\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+1}} \\
&= \sqrt{\frac{n+1}{n}}\sum_{k=0}^{n}\frac{x^k}{\sqrt{k+1}} = \sum_{k=0}^{n}\frac{x^k}{\sqrt{k+1}}.
\end{aligned}$$

Thus, we have shown that $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{\sqrt{k+1}}$ for all $n$ when $A_n = \frac{1}{n+1}$. This implies that the solutions to the Lucas-Uzawa model are given by linear combinations of Legendre polynomials, and we can use the properties of Legendre polynomials to analyze the behavior of the model under different parameter values. □

**Preposition 6.4.** If the production function is log-convex, then the inequality $f(A(p,q)) \leq H(f(p), f(q))$ holds for all $p, q \in (0,1)$.

*Proof.* The inequality given in Theorem (4.1) can be used in the context of Lucas-Uzawa model to bound the consumption and capital accumulation paths. In particular, it can be used to show that the consumption path is bounded from above and the capital accumulation path is bounded from below.

The function $S(x)$ represents the ratio of output to capital in the Lucas-Uzawa model. The inequality shows that if the sequence of ratios $(C_n)$ is strictly increasing, then the function that maps $x \mapsto S(x)/S'(x)$ is strictly increasing as well. This implies that the output-capital ratio is log-convex, which is a desirable property for economic growth models.

Using this inequality, we can show that the savings rate and the investment rate are bounded from above and below, respectively. This is important for ensuring that the economy does not experience explosive growth or collapse. Moreover, the inequality provides a useful tool for policy analysis, as it can be used to evaluate the effects of changes in various parameters on the long-term growth path of the economy.

Let $u_r(x)$ be a log-concave function on $(0, 1)$, and let $u(x) = u_r(1/x)$. Then, the function $x \mapsto u(\sqrt{x}) = u_r(1/\sqrt{x})$ is log-convex and increasing on $(0, \infty)$.

Similarly, if $I_r(x)$ is log-convex for all $r \geq -1/2$, then for any $c < 0$ and $\kappa \geq 1/2$, the function $\alpha_r(x) = u_r(x^2)$ satisfies $\alpha_p(x)$ is log-convex on $\mathbb{R}$. This implies that for all $p, q \in (0, \infty)$, the inequality $u_r(H(p, q)) \leq u_r(A(p, q)) \leq H(u_r(p), u_r(q))$ holds, where $H$ and $A$ are as defined previously.

This inequality improves upon the inequality given by Baricz {for details refer ([5]) [Remark 3.4]}.

□

## 7. Conclusion

In this research work, we have explored the behavior of the Lucas-Uzawa model using various mathematical tools, including the Cauchy-Bunyakovsky-Schwarz inequality, Gaussian hypergeometric functions, and power series expressions. Our results provide new insights into the model's dynamics of equilibrium for physical and human capital., as well as its stability and convergence properties.

In particular, we have shown that the model can be analyzed using various polynomial sequences, including the Legendre polynomials and the power series expressions for $P_n(x)$. These polynomial sequences can be used to study the behavior of the model under different parameter values, and provide explicit representations of its equilibrium dynamics.

Our results contribute to the ongoing discussion in the field of endogenous growth models, and highlight the importance of mathematical tools and techniques in analyzing economic models. Further research can build on our work by exploring the Lucas-Uzawa model using other mathematical methods, or by applying our approach to other endogenous growth models.

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## References

[1] Abramowitz, M and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, *Dover publications, New York*,55 (1965).

[2] Andrews, G. E., Askey, R., Roy, R., Special functions, *Cambridge University press, Cambridge*, 71 (1999).

[3] Baricz, Á., Functional inequalities involving Special functions II, *Journal of Mathematical Analysis and Applications*, 327 (2007), 1202-1213.

[4] Baricz, Á., Geometric properties of generalized Bessel functions, *Publicationes Mathematicae Debrecen*, 73 (2008), 155-178.

[5] Baricz, Á., Inequalities Involving Bessel and Hypergeometric Functions, *In Generalized Bessel Functions of the First Kind*, (2010), 71-186.

[6] Barnes, E. W., The asymptotic expansion of integral functions defined by generalised hypergeometric series, *Proceedings of the London Mathematical Society*, 2 (1907), 59-116.

[7] Bethmann, D., Reiß, M., Simplifying numerical analyses of Hamilton–Jacobi–Bellman equations, *Journal of Economics*, 107 (2012), 101-128.

[8] Boucekkine, R., Licandro, O., Puch, L. A., Del Rio, F., Vintage capital and the dynamics of the AK model, *Journal of Economic Theory*, 120 (2005), 39-72.

[9] Boucekkine, R., Ruiz-Tamarit, J. R., Special functions for the study of economic dynamics: The case of the Lucas-Uzawa model, *Journal of Mathematical Economics*, 44 (2008), 33-54.

[10] Caballero, R. J., Engel, E. M., Explaining investment dynamics in US manufacturing: a generalized (S, s) approach, *Econometrica*, 67 (1999), 783-826.

[11] Chilarescu, C., A closed-form solution to the transitional dynamics of the Lucas–Uzawa model, *Economic Modelling*, 26 (2009), 135-138.

[12] Chilarescu, C., The effect of externality on the transitional dynamics: The case of Lucas model, *Economics Bulletin*, 38 (2018), 1685-1702.

[13] Erdélyi, A., Higher transcendental functions, *Higher transcendental functions*, 59 (1953).

[14] Hiraguchi, R., A note on the closed-form solution to the Lucas–Uzawa model with externality, *Journal of Economic Dynamics and Control*, 33 (2009), 1757-1760.

[15] Klenow, P. J., Rodriguez-Clare, A., Economic growth: A review essay, *Journal of Monetary Economics*, 40 (1997), 597-617.

[16] Lawford, S., A hypergeometric test for omitted nonlinearity, *Brunel University*, (2003).

[17] Masjed-Jamei, M., Dragomir, S. S., Srivastava, H. M., Some generalizations of the Cauchy–Schwarz and the Cauchy–Bunyakovsky inequalities involving four free parameters and their applications, *Mathematical and Computer Modelling*, 49 (2009), 1960-1968.

[18] Paris, R., A Kummer-type transformation for a ${}_2F_2$ hypergeometric function, *Journal of Computational and Applied Mathematics*, 173 (2005), 379-382.

[19] Silverman, H., Starlike and convexity properties for hypergeometric functions, *Journal of Mathematical Analysis and Applications*, 172 (1993), 574-581.

[20] Singh, K., Saxena, R., Kumar, S., Caputo-based fractional derivative in fractional Fourier transform domain, *IEEE Journal on Emerging and Selected Topics in Circuits and Systems*, 3 (2013), 330-337.

[21] Singh, P., Hypergeometric functions: Application in distribution theory, *International Journal of Mathematics Trends and Technology*, 40 (2016), 157-163.

[22] Singh, P., Discussion on Hypergeometric Functions and Their Applications, *International Journal of Research And Analytical Reviews*, 10 (2023), 444-451.

[23] Singh, P., Certain Integral representation of Appell functions F1, F2 and F3, *International Journal of Emerging Technologies and Innovative Research*, 11 (2024), 272-279.

[24] Srivastava, H., Murugusundaramoorthy, G., Sivasubramanian, S., Hypergeometric functions in the parabolic starlike and uniformly convex domains, *Integral Transforms and Special Functions*, 18 (2007), 511-520.

Dr. Pooja Singh
Assistant professor
Department of Applied Sciences
Maharaja Surajmal Institute of Technology, Delhi 110058
E-mail: pooja.singh@msit.in

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# INFINITUDE OF PRIMES

MITHUN PADINHARE VEETTIL
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Abstract. In this paper, we discuss two well-known proofs of the infinitude of primes in $\mathbb{N}$ and give a new proof using elementary commutative algebra.

## 1. Introduction

A positive integer $n > 1$ is called a prime number if $n$ is divisible only by 1 and itself. Thus, we obtain a subset $\mathcal{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$ of the natural numbers $\mathbb{N}$ that are prime. The infinitude of primes in $\mathbb{N}$ states that the cardinality of the set $\mathcal{P}$ is infinite. There are several existing proofs for this, of which the first and most well-known would be Euclid's proof of the infinitude of primes. In this introductory section, we discuss Euclid's proof and a topological proof by Furstenberg. In Section 2, we give a new proof using commutative algebra notions.

### 1.1. **Euclid's Proof of Infinitude of Primes.**

Euclid's argument relies on the fact that every natural number $n > 1$ has a prime divisor. This can be verified by applying the Principle of Mathematical Induction. The result is clearly true for $n = 2$. Suppose that it has been proved for all natural numbers $< n$. Now we have two cases: If $n$ is a prime, then $n$ itself is a divisor of $n$. If $n$ is not a prime, we can write $n = ab$ for some $a < n$. By inductive hypothesis $a$ has a prime factor and so does $n$.

Now we discuss Euclid's proof of infinitude of primes[2]. Let us consider a finite subset $\{p_1, \dots, p_m\} \subseteq \mathcal{P}$. Set $n = p_1 p_2 \cdots p_m + 1$. From the above discussion, there is a prime divisor, say $q$ for $n$. If $q = p_i$ for some $1 \le i \le m$,

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then we get $q$ divides $n - p_1 p_2 \dots p_m = 1$ which is a contradiction. So, $q$ is not in the subset $\{p_1, \dots, p_m\}$. This suggests that every finite set of prime numbers has another prime number missing. Therefore, the cardinality of $\mathcal{P}$ is infinite.

In the next subsection, we discuss a *topological* proof of the infinitude of primes in $\mathbb{N}$ by Harry Furstenberg[3].

### 1.2. **Furstenberg's Proof of Infinitude of Primes.**

Let $a + b\mathbb{Z} := \{a + bn \mid n \in \mathbb{Z}\}$ where $b \neq 0$. If $x \in a + b\mathbb{Z} \bigcap c + d\mathbb{Z}$, then it follows that $x + bd\mathbb{Z} \subseteq a + b\mathbb{Z} \bigcap c + d\mathbb{Z}$. So the collection of all such arithmetic sequences forms a basis for a topology on $\mathbb{Z}$. If $U$ is open in $\mathbb{Z}$, then it must contain an arithmetic sequence, and hence $U$ must contain an infinite number of elements. Thus, finite subsets of $\mathbb{Z}$ cannot be open. If $x \in \mathbb{Z} \setminus a + b\mathbb{Z}$, then $x + b\mathbb{Z} \subseteq \mathbb{Z} \setminus a + b\mathbb{Z}$. Therefore, $\mathbb{Z} \setminus a + b\mathbb{Z}$ is open and therefore $a + b\mathbb{Z}$ is closed. Let $A_p := p\mathbb{Z}$ be multiples of $p$. Notice that $A_p$ is closed. As every integer except $\pm 1$ has a prime divisor, we must have $\bigcup_{p \in \mathcal{P}} A_p = \mathbb{Z} \setminus \{-1, 1\}$. From the above observation, $\{-1, 1\}$ cannot be open and, hence, $\bigcup_{p \in \mathcal{P}} A_p$ is not closed. A finite union of closed sets is closed; hence, the union above must be an infinite union. This suggests that $\mathcal{P}$ has infinitely many elements.

## 2. Yet Another Proof

In this section, we present a new proof of the infinitude of primes using basic commutative algebra. So, we have the scope of extending this method to larger classes of domains or specifically number rings. But here we will stick to $\mathbb{Z}$. First, we present necessary definitions and facts from commutative algebra that can be found in[1].

### 2.1. **Some Definitions and Facts from Commutative Algebra.**

- A domain $R$ is called a Principal Ideal Domain (PID) if every ideal is generated by a single element of $R$. For example, $\mathbb{Z}$ is a PID. Moreover, an ideal $P$ is prime in $\mathbb{Z}$ if and only if $P = p\mathbb{Z}$ for some prime number $p \in \mathbb{N}$.
- In a PID every nonzero prime ideal is a maximal ideal.
- For a commutative ring $R$, the Jacobson radical of $R$, denoted by $J$ is defined as the intersection of all maximal ideals $\mathfrak{m}$.

- If $x \in J$ then $1+x$ is a unit in $R$. For if $1+x$ is not a unit, then it lies in some maximal ideal $\mathfrak{m}$. Since, $J \subseteq \mathfrak{m}$, we have $(1+x)-x \in \mathfrak{m}$. So, $1 \in \mathfrak{m}$, which is a contradiction.
- Every nonzero ideal in $\mathbb{Z}$ is infinite. For instance, say $I$ is a nonzero ideal in $\mathbb{Z}$. Let $0 \neq x \in I$. Then, the map $n \mapsto nx$ is an injection from $\mathbb{Z}$ into $I$.

Now we move on to the proof of the infinitude of primes in $\mathbb{N}$.

2.2. **Proof.**

Suppose that $\mathbb{N}$ contains only finitely many prime numbers $p_1, \ldots, p_n$. Let $P_i$ be the prime ideal generated by $p_i$ in $\mathbb{Z}$. As $\mathbb{Z}$ is a PID, $P_i$ is a maximal ideal. Let $\mathfrak{m}$ be another maximal ideal of $\mathbb{Z}$. Then $\mathfrak{m}$ is prime, and hence $\mathfrak{m} = p_j\mathbb{Z} = P_j$ for some $1 \leq j \leq n$. Therefore, $P_1, \ldots, P_n$ are all the maximal ideals of $\mathbb{Z}$. Let $J = \bigcap_{i=1}^{n} P_i$ be the Jacobson radical. Clearly $J$ is a nonzero ideal in $\mathbb{Z}$ and thus contains infinitely many elements. Now, notice that for any two distinct elements $x, y \in J$ we have $1+x$ and $1+y$ are distinct units in $\mathbb{Z}$. Thus, $\mathbb{Z}$ has infinitely many units, which is absurd as $\{-1, 1\}$ are the only units of $\mathbb{Z}$. So, we get a contradiction for the initial assumption, and hence there are infinitely many primes in $\mathbb{N}$.

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## References

[1] M.F. Atiyah, I.G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley Series in Mathematics, Chapter 1, Proposition 1.9
[2] Ore, Oystein (1988) [1948], Number Theory and its History, Dover, p. 65
[3] Furstenberg, Harry (1955). "On the infinitude of primes". American Mathematical Monthly. 62 (5): 353.

Mithun Padinhare Veettil
Ph.D Candidate
Department of Mathematics and Statistics
State University of New York at Binghamton
4400 Vestal Parkway, Vestal, NY, USA, 13850.
E-mail: mithunpv333@gmail.com

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# WHETHER THE ADDITIVE INVERSE OF A BAD PRIMITIVE ROOT OF $P$ IS A PRIMITIVE ROOT OF $P^2$?

V. P. RAMESH AND DEVI, S.
(Received : 24 - 04 - 2025 ; Revised : 06 - 06 - 2025)

Abstract. In this article, for any odd prime $p$, we prove that the additive inverse of a bad primitive root of $p$ is not bad only when $p \equiv 1 \pmod 4$, and for the case $p \equiv 3 \pmod 4$, we estimate a primitive root of $p^\ell$ from a bad primitive root of $p$.

## 1. Introduction

Let $a \in (\mathbb{Z}/n\mathbb{Z})^*$, if the order of $a$ (denoted by $ord_n(a)$) is $\phi(n)$, we say that $a$ is a *primitive root of* $n$ [3]. For any odd prime $p$ and a natural number $\ell$, Gauss proved that, *if $a$ is a primitive root of $p$, then there exists an integer $m$ such that $a + mp$ is a primitive root of $p^\ell$. Moreover, if $a$ is a primitive root of $p^2$, then $a$ is a primitive root of $p^\ell$* (see [1, Chapter 8]). In the case $m = 0$ of Gauss's theorem, for any odd prime $p$, *a primitive root of $p$ is said to be bad if it is not a primitive root of $p^2$* [2].

In 1974, Cohen et al. [2] analytically proved that *for each $\epsilon > 0$, there exists a constant $P(\epsilon)$ such that for any prime $p > P(\epsilon)$, the number of bad primitive roots of $p$ is bounded above by $p^{\frac{1}{2}+\epsilon}$*. In 2023, Ramesh and Gowtham [4] proved that *the multiplicative inverse of a bad primitive root of $p$ is not bad.*

A question that arises immediately is whether the additive inverse of a bad primitive root of $p$ is also not bad. This is not always true and can be seen by examining the multiplicative group $(\mathbb{Z}/43\mathbb{Z})^*$. It is easy to verify that 19 is a bad primitive root of 43 since $ord_{43}(19) = 42 = ord_{43^2}(19)$,

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but the additive inverse of 19, which is 24, is not even a primitive root of 43. We may still ask: Is there a modified statement that is true? In other words, we ask: Is the statement true if we restrict the primes to a suitable class? We observed that if $p \equiv 1 \pmod 4$, then the additive inverse of a bad primitive root of $p$ is not bad. For the other class, namely, $p \equiv 3 \pmod 4$, we estimate a primitive root of $p^2$ from a bad primitive root of $p$. Indeed, we have the following observation.

**Theorem 1.1.** *Let $p$ be an odd prime and $\ell$ be any natural number. Let $a$ be a primitive root of $p$.*
*(i) If $p \equiv 1 \pmod 4$, then $a$ or $p - a$ is a primitive root of $p^\ell$.*
*(ii) If $p \equiv 3 \pmod 4$, then $a$ or $\lceil \frac{a^2}{p} \rceil p - a^2$ is a primitive root of $p^\ell$.*

*Proof.* In order to prove this result, by Gauss's theorem (see [1, Chapter 8]), it is enough to prove the result for $\ell = 2$. Let $a$ be a primitive root of $p$, then necessarily $p - 1 \mid ord_{p^2}(a)$. Thus, $ord_{p^2}(a) = p - 1$ or $p(p - 1)$.
**Case 1.** $p \equiv 1 \pmod 4$.

We claim that $ord_p(p-a) = p-1$. For if $ord_p(p-a) = k < p-1$, then, since $1 \equiv (p - a)^k \equiv (-a)^k \pmod p$, we have $k$ is odd, whence $a^{2k} \equiv 1 \pmod p$. Now, since $k$ is odd and $p \equiv 1 \pmod 4$, clearly $2k < p-1$, which is a contradiction to $a$ being a primitive root of $p$. Thus, $ord_p(p-a) = p-1$.

Now to complete the proof, it is enough to show that if $ord_{p^2}(a) = p-1$, then $ord_{p^2}(p-a) = p(p-1)$ (or equivalently, $(p-a)^{p-1} \not\equiv 1 \pmod{p^2}$). Suppose that $(p-a)^{p-1} \equiv 1 \pmod{p^2}$. By the binomial theorem, $pa^{p-2}+a^{p-1} \equiv 1 \pmod{p^2}$, whence $pa^{p-2} \equiv 0 \pmod{p^2}$. Thus, $p \mid a^{p-2}$, a contradiction. Therefore, we have shown that $p - a$ is a primitive root of $p^2$, and hence $p - a$ is a primitive root of $p^\ell$.
**Case 2.** $p \equiv 3 \pmod 4$.

We claim that $ord_p(\lceil \frac{a^2}{p} \rceil p - a^2) = p - 1$. Clearly, $\frac{p-1}{2}$ is odd and $(\lceil \frac{a^2}{p} \rceil p-a^2)^{\frac{p-1}{2}} \equiv (-a^2)^{\frac{p-1}{2}} \equiv -1 \pmod p$, whence $ord_p(\lceil \frac{a^2}{p} \rceil p-a^2)$ is even, say $k$. Then, $1 \equiv (\lceil \frac{a^2}{p} \rceil p-a^2)^k \equiv a^{2k} \pmod p$, and since $k \neq \frac{p-1}{2}$, it follows that $ord_p(\lceil \frac{a^2}{p} \rceil p - a^2) = p-1$. Now, we must show that if $ord_{p^2}(a) = p-1$, then $ord_{p^2}(\lceil \frac{a^2}{p} \rceil p - a^2) = p(p - 1)$. Indeed, as in the proof of case 1, a similar argument for $(\lceil \frac{a^2}{p} \rceil p - a^2)^{p-1} \equiv 1 \pmod{p^2}$ yields $\lceil \frac{a^2}{p} \rceil pa^{2(p-2)} \equiv 0 \pmod{p^2}$. Thus, $p \mid \lceil \frac{a^2}{p} \rceil$ or $p \mid a^{2(p-2)}$, which is a contradiction. This completes the proof. □

**Corollary 1.2.** *Let $p$ be an odd prime and $\ell$ be any natural number. Let $a$ be a bad primitive root of $p$. Then,*
*(i)* $p \equiv 1 \pmod 4$ *if and only if* $p - a$ *is a primitive root of* $p^\ell$.
*(ii)* $p \equiv 3 \pmod 4$ *if and only if* $\lceil \frac{a^2}{p} \rceil p - a^2$ *is a primitive root of* $p^\ell$.

*Proof.* The forward part follows from Theorem 1.1. Conversely, if $p - a$ is a primitive root of $p^\ell$, then $a^{\frac{p-1}{2}} \equiv -1 \equiv (p-a)^{\frac{p-1}{2}} \equiv (-a)^{\frac{p-1}{2}} \pmod p$. Thus, $\frac{p-1}{2}$ is even, and hence $p \equiv 1 \pmod 4$. Similarly, if $\lceil \frac{a^2}{p} \rceil p - a^2$ is a primitive root of $p^\ell$, then $-(a^2)^{\frac{p-1}{2}} \equiv -1 \equiv (\lceil \frac{a^2}{p} \rceil p - a^2)^{\frac{p-1}{2}} \equiv (-a^2)^{\frac{p-1}{2}} \pmod p$. Thus, $\frac{p-1}{2}$ is odd, and hence $p \equiv 3 \pmod 4$. □

**Remark 1.3.** It is easy to see from the proof of the above corollary, that the converse of $(i)$ follows for a primitive root of $p$ that is not necessarily bad and the converse of $(ii)$ follows for any element of a multiplicative group modulo $p$ which is not necessarily a bad primitive root of $p$.

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## References

[1] Burton, D. M., *Elementary Number Theory*, 7th ed., McGraw-Hill, 2012.

[2] Cohen, S. D., Odoni, R. W. K., and Stothers, W. W., On the least primitive root modulo $p^2$, *Bulletin of the London Mathematical Society*, **6** (1974), 42-46.

[3] Gauss, C. F., *Disquisitiones Arithmeticae* (translated by Clarke A. A.), Yale University Press, 1966.

[4] Ramesh, V. P., and Gowtham, R., The inverse of a bad primitive root is not bad, *The American Mathematical Monthly*, **130**(10) (2023), 928.

V. P. Ramesh and Devi, S
Department of Mathematics
Central University of Tamil Nadu
Thiruvarur, Tamil Nadu - 610 005.
E-mail: vpramesh@gmail.com, devisphd22@students.cutn.ac.in

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# SOME BOUNDS ON STANDARD NORMAL PROBABILITIES

GEOFFREY W. BROWN AND ADAM C. BROWN

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Abstract. This paper derives two inequalities related to standard normal probabilites, and it accomplishes this using ideas from conditional probability.

The goal of this paper is to derive two inequalities related to standard normal probabilities. We derive these two inequalities by using conditional probability as applied to the normal distribution. We refer the reader to [1] *Mathematical Statistics with Applications*, Chapter 4, for information and results concerning the normal distribution. We begin with the definition of the standard normal random variable.

**Definition:** A random variable $X$ is said to have a standard normal probability distribution if the density function of $X$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ for } -\infty < x < \infty.$$

It may be shown that $E[X] = 0$, and $Var[X] = 1$. We now suppose that we know that $a \leq X \leq b$, where $a$ and $b$ are real numbers such that $a < b$. Thus, we condition on the event $a \leq X \leq b$, and we let

$$P(a, b) = P(a \leq X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Then $X|a \leq X \leq b$ has density function given by

$$f_{[a,b]}(x) = \frac{1}{P(a, b)} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ for } a \leq x \leq b. \tag{0.1}$$

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Using (0.1), we proceed to calculate the first two moments of $X|a \leq X \leq b$. Thus, we find

$$E[X|a \leq X \leq b] = \frac{1}{P(a,b)} \frac{1}{\sqrt{2\pi}} \int_a^b x e^{-x^2/2} dx. \tag{0.2}$$

We may evaluate (0.2) by direct integration, and we find that

$$\begin{aligned} E[X|a \leq X \leq b] &= \frac{1}{P(a,b)} \frac{1}{\sqrt{2\pi}} \left( -e^{-x^2/2} \right) \Big|_a^b \\ &= \frac{1}{P(a,b)} \frac{1}{\sqrt{2\pi}} \left( e^{-a^2/2} - e^{-b^2/2} \right). \end{aligned} \tag{0.3}$$

For the second moment, we find that

$$E[X^2|a \leq X \leq b] = \frac{1}{P(a,b)} \frac{1}{\sqrt{2\pi}} \int_a^b x^2 e^{-x^2/2} dx. \tag{0.4}$$

We may evaluate (0.4) using integration by parts, and we find that

$$\begin{aligned} E[X^2|a \leq X \leq b] &= \frac{1}{P(a,b)} \frac{1}{\sqrt{2\pi}} \left[ -e^{-x^2/2} x \Big|_a^b - \int_a^b -e^{-x^2/2} \cdot 1 dx \right] \\ &= \frac{1}{P(a,b)} \frac{1}{\sqrt{2\pi}} \left( e^{-a^2/2} a - e^{-b^2/2} b \right) + 1, \end{aligned} \tag{0.5}$$

using the definition of $P(a,b)$ to simplify (0.5).

Our first inequality follows from (0.5) and the fact that

$$E[X^2|a \leq X \leq b] \geq 0,$$

and this gives

$$\begin{aligned} \frac{1}{P(a,b)} \frac{1}{\sqrt{2\pi}} \left( e^{-a^2/2} a - e^{-b^2/2} b \right) + 1 \geq 0 \\ \left( e^{-a^2/2} a - e^{-b^2/2} b \right) + \sqrt{2\pi} P(a,b) \geq 0 \\ P(a,b) \geq \frac{1}{\sqrt{2\pi}} \left( e^{-b^2/2} b - e^{-a^2/2} a \right). \end{aligned} \tag{0.6}$$

We have thus found a lower bound for $P(a,b)$ with (0.6). Our second inequality follows from the fact that for any random variable $Y$, $Var[Y] \geq 0$. We also know that

$$Var[Y] = E\left[(Y - E[Y])^2\right] = E[Y^2] - E[Y]^2. \tag{0.7}$$

Replacing $Y$ by $X|a \leq X \leq b$ in (0.7) and using (0.5) and (0.3), we obtain the following inequality:

$$Var[X|a \leq X \leq b] \geq 0$$

$$E\left[X^2|a \leq X \leq b\right] - E[X|a \leq X \leq b]^2 \geq 0$$

$$\frac{1}{P(a,b)}\frac{1}{\sqrt{2\pi}}\left(e^{-a^2/2}a - e^{-b^2/2}b\right)+1-\left[\frac{1}{P(a,b)}\frac{1}{\sqrt{2\pi}}\left(e^{-a^2/2} - e^{-b^2/2}\right)\right]^2 \geq 0. \tag{0.8}$$

Multiplying (0.8) through by $\left(P(a,b)\sqrt{2\pi}\right)^2$, we obtain

$$P(a,b)\sqrt{2\pi}\left(e^{-a^2/2}a - e^{-b^2/2}b\right) + P(a,b)^2 \cdot 2\pi - \left(e^{-a^2/2} - e^{-b^2/2}\right)^2 \geq 0. \tag{0.9}$$

Re-arranging (0.9), we see that we have a quadratic equation in $P(a,b)$:

$$2\pi P(a,b)^2 + \sqrt{2\pi}\left(e^{-a^2/2}a - e^{-b^2/2}b\right)P(a,b) - \left(e^{-a^2/2} - e^{-b^2/2}\right)^2 \geq 0. \tag{0.10}$$

Now (0.10) is a parabola opening upwards, and we may find its roots using the quadratic formula. Letting $r_1$ and $r_2$ be the roots of (0.10), we find that

$$\begin{aligned} r_1, r_2 =& \frac{-\sqrt{2\pi}\left(e^{-a^2/2}a - e^{-b^2/2}b\right)}{4\pi} \\ &\pm \frac{\sqrt{2\pi\left(e^{-a^2/2}a - e^{-b^2/2}b\right)^2 + 8\pi\left(e^{-a^2/2} - e^{-b^2/2}\right)^2}}{4\pi}. \end{aligned} \tag{0.11}$$

Now the discriminant of (0.11) is a sum of squares and so is always positive, and thus the roots $r_1$ and $r_2$ of (0.11) are both real and distinct because $a < b$. The condition from (0.8) that the left hand side of (0.8) be non-negative implies that $P(a,b) \geq max(r_1, r_2)$ or $P(a,b) \leq min(r_1, r_2)$, with exactly one of these two inequalities holding. Thus our second inequality for $P(a, b)$ is either an upper or a lower bound for $P(a, b)$, and together with the lower bound for $P(a, b)$ given by (0.6) completes our paper.

## References

[1] *Mathematical Statistics with Applications*, 7th Edition, Wackerly D.D., Mendenhall W., Scheaffer R.L., Brooks/Cole Cengage Learning, 2008.

Geoffrey W. Brown
Toronto Metropolitan University
350 Victoria Street
Toronto, ON, M5B 2K3, CANADA
E-mail: geoffrey.brown@torontomu.ca

Adam C. Brown
University of Toronto Schools
371 Bloor Street West
Toronto, ON, M5S 2R7, CANADA
E-mail: abrown@utschools.ca

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# A GEOMETRIC APPROACH TO DIRICHLET'S THEOREM

GAURAV SAWANT
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Abstract. In this article, we use the geometry of the hyperbolic space in the upper half plane model to give an alternate proof of Dirichlet's classical theorem on rational approximation. We then use the unit disk model to describe an application to fixed points.

## 1. Introduction

Let $\Delta$ denote a model of hyperbolic space embedded in the plane:

$$\Delta = \begin{cases} \mathbb{H} = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\} & \text{(Upper half plane model)}, \\ \mathbb{D} = \{z \in \mathbb{C} : |z|^2 < 1\} & \text{(Unit disk model)}. \end{cases}$$

Then

$$\text{Isom}(\Delta) = \begin{cases} \text{SL}_2(\mathbb{R}) & \text{if } \Delta = \mathbb{H}, \\ \text{SU}(1,1) \cong \text{SL}_2(\mathbb{R}) & \text{if } \Delta = \mathbb{D}. \end{cases}$$

A *Fuchsian group* $G$ is a discrete subgroup of $\text{Isom}(\Delta)$ that consists of orientation-preserving isometries. The group $G$ acts on $\Delta$ through Möbius transformations:

$$g(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \in \mathbb{H}; a,b,c,d \in \mathbb{R}, ad-bc = 1, \\ \dfrac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}} & \text{if } z \in \mathbb{D}; \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1. \end{cases}$$

Correspondingly, $g$ has a matrix representation of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{Isom}(\mathbb{H})$ and of the form $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ in $\text{Isom}(\mathbb{D})$. In fact, with an appropriate scaling constant, both these forms are conjugate in $\text{SL}_2(\mathbb{C})$.

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We define $\mu(g)$ to be the square of the matrix norm of $g$:

$$\mu(g) = \begin{cases} a^2 + b^2 + c^2 + d^2 & g \in G \leq \mathrm{Isom}(\mathbb{H}), \\ 2(|\alpha|^2 + |\beta|^2) & g \in G \leq \mathrm{Isom}(\mathbb{D}). \end{cases}$$

The hyperbolic distance between two points $a, b \in \Delta$, denoted by $[a, b]$, is

$$[a, b] = \begin{cases} \log \dfrac{|a - \bar{b}| + |a - b|}{|a - \bar{b}| - |a - b|} & \text{in } \mathbb{H}, \\ \log \dfrac{|1 - a\bar{b}| + |a - b|}{|1 - a\bar{b}| - |a - b|} & \text{in } \mathbb{D}. \end{cases}$$

Then

$$\mu(g) = \begin{cases} 2\cosh[i, g(i)] & \text{in } \mathbb{H}, \\ 2\cosh[0, g(0] & \text{in } \mathbb{D}. \end{cases}$$

## 2. A Geometric Setup through Hedlund's Lemma

Let $G$ be a finitely generated Fuchsian group acting on $\Delta$. There exists a fundamental domain $R$ of $G$ in $\Delta$ which we also denote by $F$ in $\mathbb{H}$ and $D$ in $\mathbb{D}$. The open set $R$ is bounded by a finite number of geodesic arcs $\{u_j, u_{j'}\}_{1\leq j\leq n}$ paired uniquely by the generators $\{\gamma_j\}_{1\leq j\leq n}$ of $G$: $\gamma_j(u_j) = u_{j'}$, and thus $\overline{R}$ has finitely many vertices. If a vertex $p$ among these lies on $\partial\Delta$ such that there are $\gamma_j$-paired arcs $u_j, u_{j'}$ meeting at $p$, we call it a parabolic vertex. Also, let $G_p$ denote the (parabolic) subgroup of $G$ fixing $p$.

Let $L_G$ denote the limit set of $G$, that is, if $L_x$ is the set of limit points of the orbit $\mathrm{Orb}(x) = \{g \cdot x : g \in G\}$ of a point $x \in \Delta$, then

$$L_G = \overline{\{L_x : x \in \Delta\}}.$$

If $L_G = \partial\Delta$, then $G$ is said to be of the first kind. Otherwise, $L_G$ is a proper, closed, nowhere-dense subset of $\partial\Delta$, in which case $G$ is of the second kind. Then $\Omega = \partial\Delta \setminus L_G$ is a countable union of disjoint open intervals: $\Omega = \bigsqcup_{j\in\mathbb{N}} \Omega_j$ (which is empty if $G$ is of the first kind). Note that the $\Omega_j$ are linear on $\partial\mathbb{H}$ and circular on $\partial\mathbb{D}$.

For $0 < \alpha \leq \frac{\pi}{2}$, let $\lambda_j$ be the circular arc in $\Delta$ joining the endpoints of $\Omega_j$ and making an internal angle $\alpha$, and let $\Lambda_j(\alpha)$ denote the open region bounded by $\lambda_j$ and $\Omega_j$. (See Figure 1 for different types of these regions)

Two parabolic vertices $p, p'$ are called *equivalent* if there is a $g \in G$ with $g(p) = p'$; else they are called *inequivalent*. Let $\{p_i\}_{1\leq i\leq r}$ be the collection

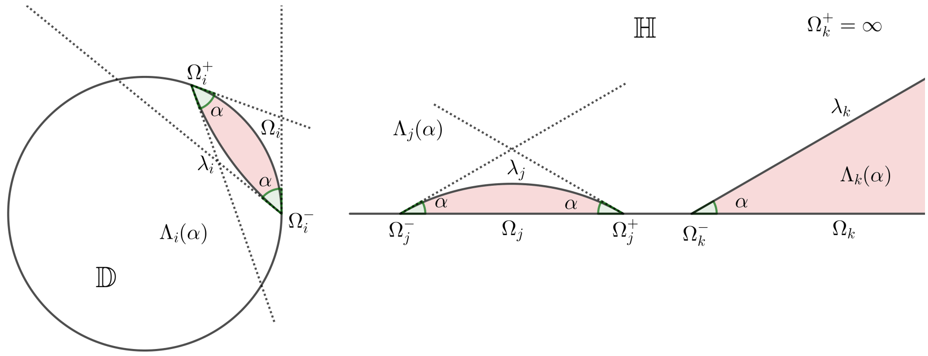



FIGURE 1. Regions $\Lambda_j(\alpha)$ in $\mathbb{D}$ and in $\mathbb{H}$.

of inequivalent parabolic vertices of $F$ (respectively $D$). Then there exist horodisks $C_{p_i}$ at $p_i$, dubbed *admissible*, such that

(a) $\{C_{p_i}, \Lambda_j\}_{1\leq i\leq r; j\in\mathbb{N}}$ are disjoint,
(b) $C_{g(p_i)} = g(C_{p_i})$ for each $g \in G$, for each $1 \leq i \leq r$,
(c) $F \setminus (\bigcup_i C_{p_i}) \setminus \bigcup_j \Lambda_j$ is relatively compact in $\mathbb{H}$ (respectively, $D \setminus (\bigcup_i C_{p_i}) \setminus \bigcup_j \Lambda_j$ is relatively compact in $\mathbb{D}$), and
(d) $F$ (respectively $D$) intersects with only a finite number of $C_{p_i}$ and $\Lambda_j$.

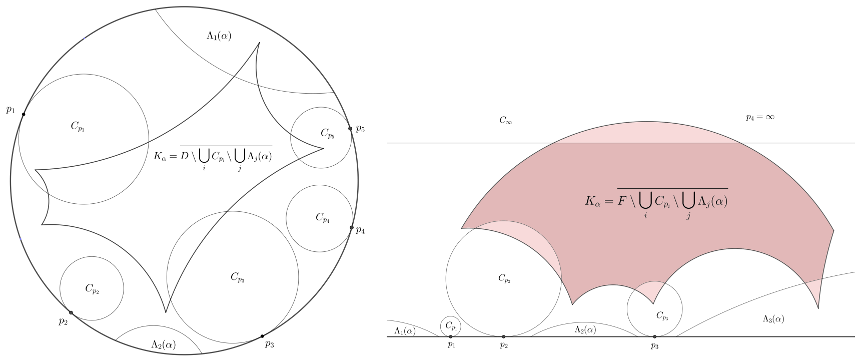
Fix a Fuchsian group $G$. We call a circular arc in $\Delta$ making an internal angle $\beta$ $(0 < \beta \leq \frac{\pi}{2})$ a $\beta$-line. Then we have the following:

**Theorem 2.1** (Hedlund [3, Theorem 2.4],[5, Chapter V, 5G])**.** *Given $\alpha \in \left(0, \frac{\pi}{2}\right]$, there is a compact set $K_\alpha$ in $\Delta$ with the following property: if $x \in L_G$ is non-parabolic and $\lambda$ is a $\beta$-line from $\zeta \in \Delta$ to $x$ with $\alpha \leq \beta$, then there is a sequence of points $(x_n)_{n\in\mathbb{N}}$ on $\lambda$ and $(g_n)_{n\in\mathbb{N}}$ so that $x_n \to x$ and $g_n^{-1}(x_n) \in K_\alpha$. Furthermore $K_\alpha$ is independent of $\alpha$ if and only if $G$ is of the first kind.*

*Proof.* Define the compact set

$$K_\alpha := \overline{\Delta \bigcap \left( R \setminus \bigcup_j \Lambda_j(\alpha) \setminus \bigcup_i C_{p_i} \right)},$$

where $C_{p_i}$ are the admissible horodisks at the parabolic vertices $p_i$. (See figures 2a,2b)




(A) The region $K_\alpha$ in $\mathbb{D}$. (B) The region $K_\alpha$ in $\mathbb{H}$.

FIGURE 2. The region $K_\alpha$ in $\Delta$.

If $\lambda$ intersects with some $\Lambda_j(\alpha)$, it has exactly one endpoint in $\Omega_j$ because $\alpha \leq \beta$. Since one endpoint of $\lambda$, $x$, is a limit point, $\lambda$ can intersect at most one $\Lambda_j(\alpha)$. The figure below shows the picture after a conjugation that sends one endpoint of $\lambda_j$ to 0 and the other to $\infty$ in $\mathbb{H}$.

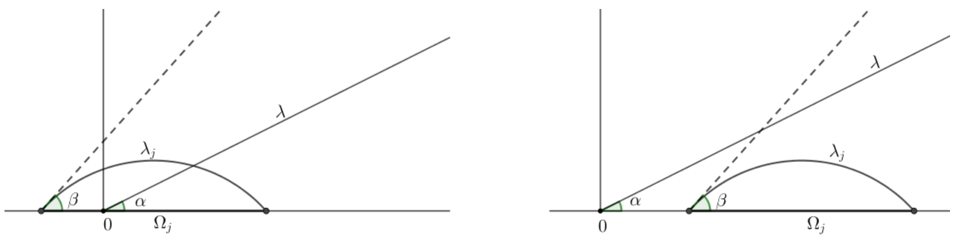



FIGURE 3. $\lambda_j$ has at most one endpoint in $\Omega_j$, and $\Omega_j$ has at lease one endpoint in $[0, \infty)$.

We use the following procedure to generate the sequence of points $x_n$:

($\star$) There is a point $\zeta'$ on $\lambda$ so that the arc segment $\lambda' = [\zeta', x]$ of $\lambda$ does not intersect any $\Lambda_j(\alpha)$. Let $\tilde{x_1}$ be a point on $\lambda'$. Then $\tilde{x_1} \in \Delta \setminus \bigcup_j \Lambda_j(\alpha)$, so there is a $\tilde{g_1} \in G$ so that $\tilde{g_1}^{-1}(\tilde{x_1}) \in \overline{R \setminus \bigcup_j \Lambda_j(\alpha)}$.

Suppose that $\tilde{g_1}^{-1}(\tilde{x_1}) \in C_{p_i}$ for some $1 \leq i \leq r$. Since $x$ is non-parabolic, there is a sub-segment of $[\tilde{x_1}, x]$ on $\lambda'$ that does not belong to $\tilde{g_1}(C_{p_i})$. Consequently there exists an $x_1 \in [\tilde{x_1}, x]$ on $\lambda'$ that lies on $\tilde{g_1}(\partial C_{p_i})$, and there is a $g_1 \in \tilde{g_1}(G_{p_i})$ so that $g_1^{-1}(\lambda')$ intersects $\partial K_\alpha$, hence $K_\alpha$ at the image of a point of $\lambda'$ closer to $x$ than $\tilde{x_1}$ to $x$; call this point $x_1$.

If $\tilde{g_1}^{-1}(\tilde{x_1}) \notin C_{p_i}$ for any $i$, then rename $\tilde{x_1}$ to $x_1$ and $\tilde{g_1}$ to $g_1$.

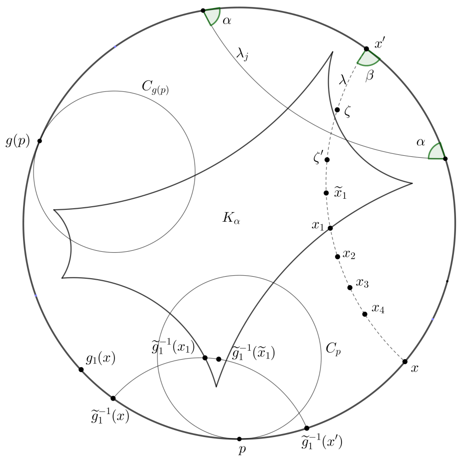



FIGURE 4. The procedure $(\star)$.

Having found $x_1, \dots, x_k$ and the corresponding $g_1, \dots, g_k$, we set $\zeta' = x_k$, and following the procedure $(\star)$, we find the point $x_{k+1}$ on the sub-segment $[x_k, x]$ of $\lambda'$ and a $g_{k+1} \in G$ with $g_{k+1}^{-1}(x_{k+1}) \in K_\alpha$. This yields a sequence of points $(x_n)$ on $\lambda$ and $(g_n)$ in $G$ so that $g_n^{-1}(x_n) \in K_\alpha$ for all $n \in \mathbb{N}$ and $x_n \to x$.

If $G$ is of the first kind, then $L_G = \partial\Delta$, so $K_\alpha = \overline{R \setminus \bigcup_i C_{p_i}}$ does not depend on $\alpha$. If $G$ is of the second kind, let $x$ be one of the endpoints of $\Omega_j$, $\beta = \alpha$ so that $\lambda$ is the $\alpha$-line $\lambda_j$. As $\alpha \searrow 0$, the distance of $g(\lambda)$ from the origin ($i$ in $\mathbb{H}$, 0 in $\mathbb{D}$) approaches infinity. Thus no fixed compact set $K$ can intersect $g(\lambda)$ for all $\alpha$, indicating that such a compact set is dependent on $\alpha$. □

## 3. Interpretation of Theorem 2.1 in the Upper Half Plane

We interpret Theorem 2.1 in case of $G = \mathrm{SL}_2(\mathbb{Z})$, which is a Fuchsian group of the first kind generated by the maps $(z \mapsto z+1)$ and $\left(z \mapsto -\frac{1}{z}\right)$, acting on $\mathbb{H}$. In this case, the fundamental domain $F$ is bounded by the vertical geodesics $\lambda_\pm = \left\{\mathfrak{Re}(z) = \pm\frac{1}{2}\right\}$ and the semicircle $\lambda_0 = \left\{z : |z|^2 = 1\right\}$. The point at $\infty$ is the only parabolic limit point of $F$, and $\mathbb{Q} \cup \{\infty\}$ is the set of all parabolic limit points of $G$. Let $m \in \mathbb{N}$ be large enough, and let $C_m$ be the horodisk $\{\mathfrak{Im}(z) > m^2\}$ at $\infty$. Let $\alpha = \beta = \frac{\pi}{2}$, and $K = K_\alpha = \overline{F \setminus C_m}$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$, and let $\lambda$ be the vertical geodesic from $x$, with $\zeta$ a point on $\lambda$ in $\mathbb{H}$.

Using Theorem 2.1, we obtain a sequence of points $(x_n)$ on $\lambda$ and corresponding $(g_n)$ in $G$ so that $x_n \to x$ and $g_n^{-1}(x_n) \in K$. Observe that there is a subsegment of $\lambda$ that contains $x_n$ inside $g_n(K)$ which intersects $g_n(\lambda^*)$, where $\lambda^* = \{z : \mathfrak{Re}(z) = 0\}$ is the vertical geodesic from 0. Denote this intersection point by $\xi_n$. Then $\xi_n \to x$, $\xi_n$ lie on $\lambda$, and $g_n^{-1}(\xi_n) \in K \cap \lambda^*$.

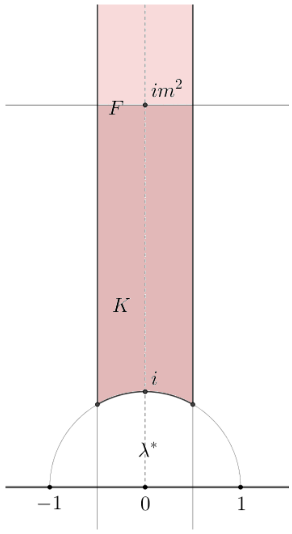



FIGURE 5. The fundamental domain $F$ and its relatively compact subset $K$.

Now let $g_n(z) = \frac{p_n z + r_n}{q_n z + s_n}$, where $p_n, q_n, r_n, s_n \in \mathbb{Z}$ and $p_n s_n - q_n r_n = 1$. Observe that

- $g_n(\lambda^*)$ is a geodesic with endpoints $0 \mapsto \frac{r_n}{s_n}$ and $i\infty \mapsto \frac{p_n}{q_n}$.
-

| As $K$ is bounded by | $g_n(K)$ is bounded by |
|---|---|
| ○ geodesic $\lambda_0 = \{\lvert z\rvert^2 = 1\}$, | ○ geodesic $g_n(\lambda_0)$ joining $g_n(-1)$ and $g_n(1)$, |
| ○ geodesics $\lambda_\pm = \{\mathfrak{Re}(z) = \pm\frac{1}{2}\}$, | ○ geodesics $g_n(\lambda_\pm)$ joining $g_n\left(\pm\frac{1}{2}\right)$ and $\frac{p_n}{q_n}$, |
| ○ horocycle $h_m = \{\mathfrak{Im}(z) = m^2\}$, | ○ horocycle $g_n(h_m)$ tangent to the real axis at $\frac{p_n}{q_n}$ and of Euclidean diameter $\frac{1}{(mq_n)^2}$. |

- As the horocycle $h_1$ is tangent to $\lambda_0$ at $i$. the horocycle $g_n(h_1)$ at $\frac{p_n}{q_n}$ that has Euclidean diameter $\frac{1}{q_n^2}$ is tangent to $g_n(\lambda_0)$ at $g_n(i)$.

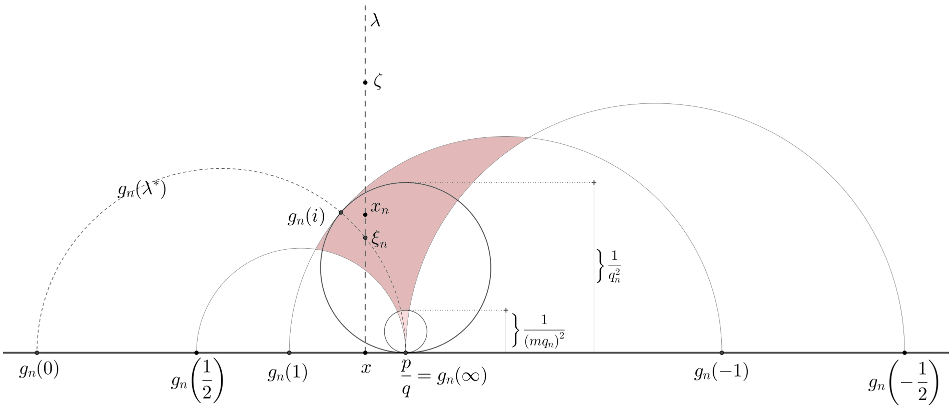



FIGURE 6. Images of $F$ (shaded region) and $K$ (dark gray region) under $g_n \equiv \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix}$.

Since $1 \leq \mathfrak{Im}(g_n^{-1}(\xi_n)) \leq m^2$, the point $\xi_n$ lies on or inside the horocycle $g_n(h_1)$. Consequently,

$$\left|\mathfrak{Re}(\xi_n) - \frac{p_n}{q_n}\right| < \text{EuclideanRadius}(g_n(h_1)) = \frac{1}{2q_n^2};$$

that is,

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2} < \frac{1}{q_n^2}$$

for all $n \in \mathbb{N}$.

On the other hand, when $x \in \mathbb{Q}$, the $g_n$ may be chosen in such a way that $g_n^{-1}(\lambda) = \lambda_0$, which gives $\xi_n = g_n(i)$. In this case,

$$\left|\mathfrak{Re}(\xi_n) - \frac{p_n}{q_n}\right| = \frac{1}{2q_n^2},$$

so that

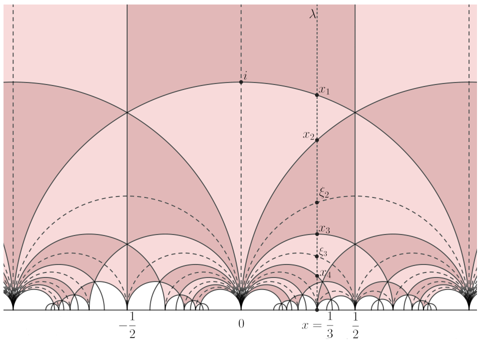
$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2} < \frac{1}{q_n^2}$$

for all $n \in \mathbb{N}$; thereby proving

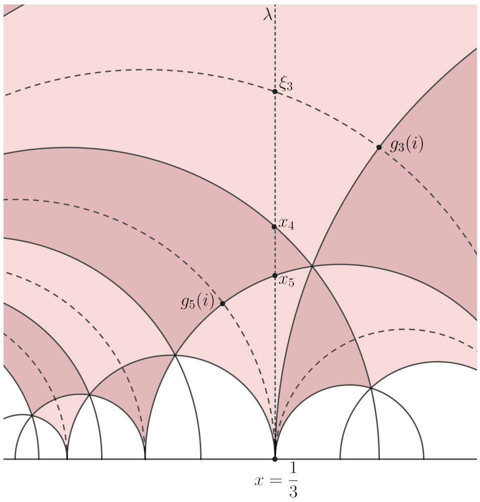
**Corollary** (Dirichlet's theorem). *For any $x \in \mathbb{R}$, there exists a rational number $\frac{p}{q}$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}.$$

*In particular, if $x \in \mathbb{R} \setminus \mathbb{Q}$, then there are infinitely many such rational numbers $\frac{p}{q}$.*




(A) The vertical geodesic $\lambda$ from $x \in \mathbb{Q}$ intersects with only finitely many images of $F$. Here $x = \frac{1}{3}$.




(B) Beyond an $N \in \mathbb{N}$, $\lambda$ is completely contained in $g_N(F)$. Here $N = 5$.

FIGURE 7. For $x \in \mathbb{Q}$, the second conclusion of Dirichlet's theorem fails.

**Remark 3.1.** 1. When $x \in \mathbb{R} \setminus \mathbb{Q}$, the rational numbers $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}}$ explicitly provide the infinitely many numbers mentioned in Corollary 3. The infinitude arises from the fact that $g_n^{-1}(\lambda)$ intersects with two of the three geodesics $\lambda_0, \lambda_\pm$ inside $K$ for all $n \in \mathbb{N}$.

2. When $x \in \mathbb{Q}$, one can write $x = \frac{p}{q}$, and there is an $N \in \mathbb{N}$ with $g_N(z) = \frac{pz+r}{qz+s}$ for suitable $r, s \in \mathbb{Z}$ such that the $g_N$-image of the geodesic segment $[i, i\infty]$ *completely contained in* $F$ is completely contained in $g_N(F)$. In this case, $g_n^{-1}(\lambda)$ coincides with $\lambda^*$ for all

$n \geq N$. Effectively, there are only finitely many rational numbers $\frac{p}{q}$ satisfying the primary conclusion of Corollary 3.

## 4. Applications to Fixed Points [7, Section 6, Theorem 1]

We shall consider the unit disk model $\mathbb{D}$ of the hyperbolic space for this section.

For $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$, define the Poisson kernel as

$$\mathcal{P}(z, \zeta) = \frac{1 - |z|^2}{|z - \zeta|^2}.$$

For $p \in \partial\mathbb{D}$ and $d > 0$, the set $C(p, d) = \left\{z : \mathcal{P}(z, p) > \frac{1}{d}\right\}$ is a horodisk at $p$ with Euclidean diameter $\kappa(d) = \frac{2d}{1+d}$. Also, if $C = C(p, d)$ is a horodisk, we shall write $p = p(C)$ and $d = d(C)$.

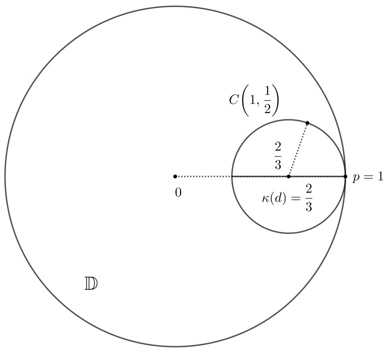



Figure 8. The horodisk $C\left(1, \frac{1}{2}\right)$.

Let $\gamma$ be a conformal (that is, angle-preserving) map of $\mathbb{D}$, written $\gamma \in \text{conf}(\mathbb{D})$. Writing

$$\gamma(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad \left(\alpha, \beta \in \mathbb{C}; |\alpha|^2 - |\beta|^2 = 1\right),$$

the following four properties can easily be verified directly:

(1) $\gamma(C(p, d)) = C(\gamma(p), |\gamma'(p)|d)$,
(2) If $z, w \in \overline{\mathbb{D}}$, then $|\gamma(z) - \gamma(w)| = |z - w|\sqrt{|\gamma'(z)||\gamma'(w)|}$,
(3) If $z \in \mathbb{D}$, then $1 - |\gamma(z)|^2 = |\gamma'(z)|(1 - |z|^2)$,
(4) If $z \in \mathbb{D}$, then $1 - |z| \leq 1 - |z|^2 \leq 2(1 - |z|)$.

We need the following sequence of lemmas to prove the main theorem of this section:

**Lemma 4.1.** *Let $p_1, p_2 \in \partial\mathbb{D}$. Then there are absolute constants $c, c' > 0$ such that*

*(i) If $C(p_1, d_1)$ and $C(p_2, d_2)$ intersect, then $|p_1 - p_2| < c\sqrt{d_1 d_2}$.*
*(ii) If $C(p_1, d_1)$ and $C(p_2, d_2)$ do not intersect, then $|p_1 - p_2| > c'\sqrt{d_1 d_2}$.*

*Proof.* Let $C, C'$ be two horodisks. Define

$$A(C, C') = \frac{|p(C) - p(C')|^2}{d(C)d(C')}.$$

Let $\gamma \in \mathrm{conf}(\mathbb{D})$ be such that $\gamma(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$ with $\alpha, \beta \in \mathbb{C}; |\alpha|^2 - |\beta|^2 = 1$. Then

$$\begin{aligned} A(\gamma(C), \gamma(C')) &= \frac{|p(\gamma(C)) - p(\gamma(C'))|^2}{d(\gamma(C))d(\gamma(C'))} \\ &= \frac{|\gamma(p(C)) - \gamma(p(C'))|^2}{|\gamma'(p(C))|d(C) \cdot |\gamma'(p(C'))|d(C')} \qquad \text{(using property 1)} \\ &= \frac{|p(C) - p(C')| \cdot |\gamma'(p(C))\gamma'(p(C'))|}{|\gamma'(p(C))\gamma'(p(C'))| \cdot d(C)d(C')} \qquad \text{(using property 2)} \\ &= \frac{|p(C) - p(C')|^2}{d(C)d(C')} = A(C, C'), \end{aligned}$$

showing that $A$ is an *invariant.*

Assume that $p(C) \neq p(C')$. Then one can find a $\gamma \in \mathrm{conf}(\mathbb{D})$ such that $\gamma(p(C)) = 1$, $\gamma(p(C')) = -1$, and $\gamma(C) = \left\{z : \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}$, so that $d(\gamma(C)) = 1$. This makes $\gamma(C') = \{z : |z + r| < 1 - r\}$ for some $r \in (0, 1)$ (that is, $\gamma(C')$ is a Euclidean circle with centre at $-r$ and Euclidean radius $1 - r$), so that $d(\gamma(C')) = \frac{r}{1-r}$. Observe that $\gamma(C')$ intersects with $\gamma(C)$ if and only if $r < \frac{1}{2}$. For such $r$,

$$A(C, C') = A(\gamma(C), \gamma(C')) = \frac{|p(\gamma(C)) - p(\gamma(C'))|^2}{d(\gamma(C))d(\gamma(C'))} = \frac{4(1 - r)}{r} < 4.$$

So $C, C'$ intersect if and only if $A(C, C') < 4$. In this case, writing $C = C(p_1, d_1)$ and $C' = C(p_2, d_2)$ gives $|p_1 - p_2|^2 < 4d_1 d_2$; thus $|p_1 - p_2| < 2\sqrt{d_1 d_2}$, showing 4.1(i). On the other hand, if $C, C'$ do not intersect, then $A(C, C') \geq 4$; so that $|p_1 - p_2|^2 \geq 4d_1 d_2$; whereby $|p_1 - p_2| \geq 2\sqrt{d_1 d_2}$, showing 4.1(ii). □

Let $H$ be a subgroup of $G$. Then we define $G\|H$ to be the set of representatives $g$ of the cosets $\{gH : g \in G\}$ chosen so that if $g \in G\|H$ and $h \in H$, then $\mu(g) \leq \mu(gh)$.

Let $P$ be the set of parabolic vertices of $G$. Recall that a set $\{C_p : p \in P\}$ of horodisks is *admissible* if $C_{g(p)} = g(C_p)$ for all $g \in G$. Given an admissible set of horodisks $\{C_p\}$, we define $d_p$ by $C(p, d_p) = C_p$ for each $p$. From the discussion at the beginning of this section, there exists an admissible set of horodisks $\{C_p\}$ such that $d_p \leq 1$; equivalently, $0 \notin C(p, d_p)$.

Suppose that $\{C'_p\}$ is another admissible set of horodisks with $C(p, d'_p) = C'_p$. From the Poisson kernel definition of $C(p, d_p)$ and the identity

$$\frac{\mathcal{P}(z,\zeta)}{\mathcal{P}(w,\zeta)} = \frac{\mathcal{P}(g(z), g(\zeta))}{\mathcal{P}(g(w), g(\zeta))}, \tag{4.1}$$

we deduce that

$$\frac{d_p}{d'_p} = \frac{d_{g(p)}}{d'_{g(p)}}. \tag{4.2}$$

There is a finite set $\{p_1, \ldots, p_r\}$ of *inequivalent* vertices; that is, if $p \in P$, $p = g(p_j)$ for some $g \in G$, $a \leq j \leq r$. Thus, as $d_p \leq 1$, we have

$$d'_{g(p_j)} = \frac{d'_{p_j} d_{g(p_j)}}{d_{p_j}} \leq \frac{d'_{p_j}}{d_{p_j}} \leq \max_{1 \leq j \leq r} \frac{d'_{p_j}}{d_{p_j}}.$$

This shows that, given any admissible set of horodisks $\{C'_p\}$,

$$\{d'_p : p \in P\} \text{ is bounded above.} \tag{4.3}$$

**Lemma 4.2.** *Let $\{C_p = C(p, d_p) : p \in P\}$ be an admissible set of horodisks. For a fixed $p \in P$, there is a constant $c = c(G, p) > 0$ such that*

*(i) $d_p c^{-1} \leq d_{g(p)} \mu(g)$ if $g \in G$,*
*(ii) $d_p c \geq d_{g(p)} \mu(g)$ if $g \in G \| G_p$.*

*Proof.* Let $\{C'_p : p \in P\}$ be a fixed admissible set of horodisks. We shall show that for some constant $c$, depending only on $p$,

(i) $c^{-1} \leq d'_{g(p)} \mu(g)$ if $g \in G$,
(ii) $c \geq d'_{g(p)} \mu(g)$ if $g \in G \| G_p$,

where $d'_q$ is defined by $C'_q = C(q, d'_q)$.

Let $x$ be the point of $\partial C'_p$ hyperbolically closest to 0; we shall call $x$ the *summit* of $C'_p$. As seen in the picture, $x$ is the intersection point of $\partial C'_p$ and the Euclidean diameter of $\overline{\mathbb{D}}$ through $p$. Then $g G_p g^{-1} = G_{g(p)}$ preserves $g C'_p = C'_{g(p)}$, and $g(x) \in \partial C'_{g(p)}$.

Given $y, z \in \partial C'_p$, there is $h \in G_p$ for which $[y, h(z)] \leq c_1$ for a suitable constant $c_1$. Likewise, for $y^*, z^* \in \partial C'_{g(p)}$, there is $h^* \in g G_p g^{-1}$ for which $[y^*, h^*(z^*)] \leq c_1$. In particular, if $x^*$ is the summit of $C'_{g(p)}$, then there is

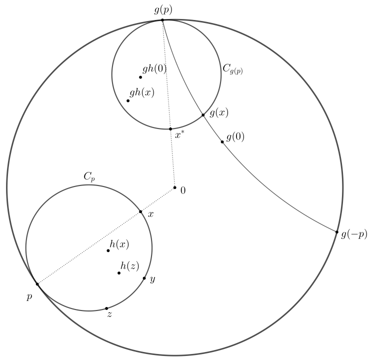



FIGURE 9. Interaction of horodisks $C_p$ and $C_{g(p)}$. Being admissible, they do not intersect.

$h^* \in gG_pg^{-1}$ so that $[x^*, h^*g(x)] \leq c_1$ (Note that $g(x)$ may be different from $x^*$). Let $h = g^{-1}h^*g \in G_p$. Then the previous equality can be rewritten as $[x^*, gh(x)] \leq c_1$. Let $c_2 = [0, x] \left(= \log \frac{1+|x|}{1-|x|}\right)$. Then

$$\begin{aligned} [0, x^*] &\leq [0, g(x)] \leq [0, g(0)] + [g(0), g(x)] \\ &= [0, g(0)] + [0, x] = [0, g(0)] + c_2 \end{aligned}$$

for all $g \in G$. Also

$$\begin{aligned} [0, x^*] &\geq [0, gh(0)] - [gh(0), gh(x)] - [gh(x), x^*] \\ &\geq [0, gh(0)] - ([0, x] + c_1) = [0, gh(0)] - (c_2 + c_1). \end{aligned}$$

These imply

$$[0, x^*] - c_2 \leq \min_{h \in G_p} [0, gh(0)] \leq [0, x^*] + c_1 + c_2. \tag{4.4}$$

Let $C(q, d)$ be a horodisk with summit $\xi$, and $d \leq 1$. Since the Euclidean diameter of $C(q, d)$ is $\kappa(d) = \frac{2d}{1+d}$, we have $|\xi| = 1 - \frac{2d}{1+d} = \frac{1-d}{1+d}$. Then

$$\begin{aligned} 1 + \cosh[0, \xi] &= 1 + \frac{1}{2}\left(e^{\log \frac{1+|\xi|}{1-|\xi|}} + e^{-\log \frac{1+|\xi|}{1-|\xi|}}\right) \\ &= 1 + \frac{1}{2}\left(\frac{1+|\xi|}{1-|\xi|} + \frac{1-|\xi|}{1+|\xi|}\right) \\ &= 1 + \left(\frac{1+|\xi|^2}{1-|\xi|^2}\right) = \frac{2}{1-|\xi|^2} \\ &= \frac{2(1+d)^2}{(1+d)^2 - (1-d)^2} = \frac{1+d^2}{2d}. \end{aligned}$$

Thus $\frac{1}{2}\left(e^{[0,\xi]}+e^{-[0,\xi]}\right)=\frac{1+d^2}{2d}$, giving $e^{[0,\xi]}=d^{\pm 1}$. From (4.3), there is a constant $B>0$ such that $d'_{g(p)}\leq B$. Using (4.2), we have

$$\left|[0,x^*]-\log d^{-1}_{g(p)}\right|\leq 2\log|B|. \tag{4.5}$$

Observe that for $\theta>0$,

$$e^{\theta}<e^{\theta}+e^{-\theta}<2e^{\theta} \quad\Rightarrow\quad \theta<\log(2\cosh\theta)<\theta+\log 2.$$

Putting $\theta=[0,g(0)]$ and recalling $\mu(g)=2\cosh[0,g(0)]$ for $g\in G$, we have

$$|[0,g(0)]-\log\mu(g)|<\log 2. \tag{4.6}$$

From (4.4), (4.5) and (4.6), it follows that

$$\left|\min_{h\in G_p}\log\mu(gh)+\log d_{g(p)}\right|\leq c_3.$$

So

$$e^{-c_3}\leq d_{g(p)}\min_{h\in G_p}\mu(gh)\leq e^{c_3}. \tag{4.7}$$

Since the minimum of $\mu(gh)$, $h\in G_p$ occurs when $gh\in G\|G_p$, (4.7) is just the claim made at the beginning of the proof. Finally, invoking (4.2), the proof is complete. □

We need one more lemma before stating and proving the main theorem of this section:

**Lemma 4.3.** *Let $H_1$ and $H_2$ be parabolic subgroups of the discrete subgroup $G$ of conf($\mathbb{D}$) with fixed points $p_1\neq p_2$. Then there is a constant $c$ such that*

$$\mu(h_1h_2)\geq c\mu(h_1)\mu(h_2)$$

*for $h_j\in H_j$, $j=1,2$.*

*Proof.* There exists a $\gamma\in\mathrm{conf}(\mathbb{D})$ with $\gamma(p_1)=1$, $\gamma(p_2)=-1$. As

$$\frac{\mu(\gamma g\gamma^{-1})}{(\mu(\gamma))^2}\leq\mu(g)\leq(\mu(\gamma))^2\mu(\gamma g\gamma^{-1})$$

for all $g\in\mathrm{conf}(\mathbb{D})$, we prove the lemma for the special case $p_1=1, p_2=-1$.

Let the parabolic elements $g_1,g_2$ fix $1,-1$, respectively. As $H_1,H_2$ are discrete, $g_1,g_2$ are of the form

$$g_1\equiv\begin{pmatrix}1+it & -it\\ it & 1-it\end{pmatrix},\qquad g_2\equiv\begin{pmatrix}1+iu & iu\\ -iu & 1-iu\end{pmatrix},$$

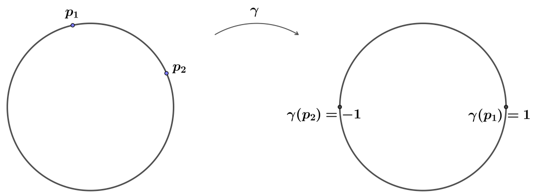



FIGURE 10. Conjugation by $\gamma$.

where $t \in t_0\mathbb{Z}$ and $u \in u_0\mathbb{Z}$ for some fixed nonzero $t_0, u_0 \in \mathbb{R}$ depending only on $H_1, H_2$, respectively. Since $\mu(g) = 2(|\alpha|^2 + |\beta|^2)$ for $g \equiv \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$, we find

$$\mu(g_1) = 4t^2+2, \quad \mu(g_2) = 4u^2+2, \quad \mu(g_1g_2) = 2(8t^2u^2+2(t-u)^2+1). \tag{4.8}$$

We claim that there exists a constant $c > 0$ such that

$$\mu(g_1g_2) - c\mu(g_1)\mu(g_2) \geq 0. \tag{4.9}$$

Let $s \in \mathbb{R}$ be such that $u_0 = st_0$. We put $t = mt_0, u = nu_0 = nst_0$ for $m, n \in \mathbb{Z}$ in (4.8) and substitute the resulting expressions in the left side of (4.9) to get

$$\begin{aligned} &\mu(g_1g_2) - c\mu(g_1)\mu(g_2) \\ &= 2[(8-8c)m^2n^2s^2t_0^4 - 4c(m^2+n^2s^2)t_0^2 + 2(m-ns)^2t_0^2 + (1-2c)] \\ &\geq 2[(m^2t_0^2 - 4c)(n^2s^2t_0^2 - 4c) + (7-8c)m^2n^2s^2t_0^4 + (1-2c-16c^2)]. \end{aligned}$$

Here, the term $2(m-ns)^2t_0^2$ is always nonnegative and thus can be omitted. Choosing

$$c = \min\left\{\frac{t_0^2}{4}, \frac{s^2t_0^2}{4}, \frac{1+\sqrt{5}}{8}\right\}$$

ensures that (4.9) holds for all $m, n \in \mathbb{Z}$, proving the lemma. ☐

**Remark 4.4.** In Patterson's paper [7, Page 542] the inequality

$$1 + 8t^2u^2 \geq \frac{(1+2t^2)(1+2u^2)}{9}$$

is claimed for all $t, u \in \mathbb{R}$ which corresponds to the constant $c = \frac{1}{18}$. However, the inequality does not hold: for example, if we take $t = 3$, then the complementary inequality is true for $|u| < \sqrt{\frac{1}{61}}$. When the group under consideration ($G$) is discrete, the parameters $t, u$ must come from discrete subsets of $\mathbb{R}$ (namely, $t_0\mathbb{Z}$ and $u_0\mathbb{Z}$ for some $t_0, u_0 > 0$), so that neither of $t, u$ can become arbitrarily small.

We can now state and prove the main theorem of this section:

**Theorem 4.5.** *Let $p, q$ be parabolic vertices of $G$. Then there is a constant $c > 0$ so that if $g(p) \neq q$, we have*

$$|g(p) - q| > \frac{c}{\sqrt{\mu(g)}}. \tag{4.10}$$

*Moreover, if for some $k > 0$,*

$$0 < |g_n(p) - q| < \frac{k}{\sqrt{\mu(g_n)}} \tag{4.11}$$

*for a sequence $(g_n)$ in $G$, then the $g_n$ belong to at most a finite number of right cosets of $G_q$.*

*Proof.* By the discussion in Section (1), we can find an admissible set $\{C_p : p \in P\}$ of *disjoint* horodisks. As $g(C_p)$ and $C_q$ are disjoint, lemma 4.1(ii) gives

$$|g(p) - q| > c_1\sqrt{d(C_{g(p)})d(C_q)}.$$

By lemma 4.2(i), we have

$$d(C_{g(p)}) \geq \frac{c_2}{\mu(g)}. \tag{4.12}$$

Combining the two inequalities gives us (4.10).

For proving the second assertion, we claim that there exists a $d$ so that $C(q, d) \bigcap g_n(C_p) \neq \emptyset$ for all $n$. For if $g_n(C_p)$ and $C(q, d)$ are disjoint, by lemma 4.1(ii), we have

$$|g_n(p) - q| \geq c_1\sqrt{d(C_{g(p)})d}.$$

By lemma 4.2(i) we also have

$$d(C_{g_n(p)}) \geq \frac{c_2}{\mu(g_n)}.$$

Thus

$$|g_n(p) - q| \geq c_1\sqrt{c_2}\sqrt{\frac{d}{\mu(g_n)}}.$$

If $d > \frac{k^2}{c_1^2 c_2}$ where $k$ is as from the hypothesis, we get a contradiction with (4.11). So $g_n(C_p)$ must intersect $C(q, d)$. However, there is a compact subset $K$ of $\partial C(q,d)$ containing $q$ so that

$$G_q(K) = \partial C(q,d).$$

Thus for every $g_n$ there is $h_n \in G_q$ such that $h_n g_n(C_p) \bigcap K \neq \emptyset$. Lemma 4.2((ii)) implies that only a finite subset of an admissible set of horodisks has diameter greater than a given quantity. Thus there is a finite subset $B$ of $G$ such that, given $n$, there is $b_n \in B$ for which $h_n g_n(C_p) = b_n(C_p)$. So there is $f_n \in G_p$ with $g_n = h_n^{-1} b_n f_n$.

As $g_n(p) = h_n^{-1} b_n(p)$, (4.10) gives

$$|g_n(p) - q| > \frac{c}{\mu(h_n^{-1} b_n)}.$$

Combining with (4.11), we find

$$\mu(h_n^{-1} b_n f_n) \leq c_3 \mu(h_n^{-1} b_n).$$

Thus, by using $\mu(\gamma_1\gamma_2) \leq \mu(\gamma_1)\mu(\gamma_2)$, we get

$$\mu((b_n^{-1} h_n b_n)^{-1} f_n) \leq (c_3 \mu(b_n)^2)\mu((b_n^{-1} h_n b_n)^{-1}). \tag{4.13}$$

As the $b_n$ run through a finite set, we may take $b_n = b$ for all $n$, without loss of generality. Now, as $h_n \in G_q$, $b^{-1} h_n b \in G_{b^{-1}q}$. Also, if $b^{-1}(q) = p$, then $g_n(p) = h_n^{-1} b(p) = h_n^{-1}(q) = q$, contradicting our hypothesis; so $b^{-1}(q) \neq p$.

Now we use Lemma 4.3 on $H_1 = b^{-1} G_q b$ and $H_2 = G_q$ to find a constant $0 < c_4 = c_4(b; p, q)$ for which

$$\mu((b_n^{-1} h_n b_n)^{-1} f_n) \geq c_4 \mu((b_n^{-1} h_n b_n)^{-1})\mu(f_n).$$

Combining with (4.13), we get

$$\mu(f_n) \leq \frac{c_3 \mu(b)^2}{c_4} =: c_5.$$

Thus, the set of possible $f_n$ is finite. Consequently, $\{g_n = h_n^{-1} b_n f_n\}$ runs through at most a finite number of right cosets of $G_q$. □

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## References


[1] Cassels, J. W. S. *Some metrical theorems in Diophantine approximation: I.* Proc. Camb. Phil. Soc. **46** (1950), 209–218.

[2] Greenberg, L. *Fundamental polygons for Fuchsian groups.* J. d'Analyse **18** (1967), 99–105.

[3] G. Hedlund. *Fuchsian groups and transitive horocycles.* Duke Math. J. **2**(3) (1936), 530–542.

[4] Katok, S. *Fuchsian Groups.* Chicago Lectures in Mathematics, The University of Chicago Press, (1992).

[5] J. Lehner. *Discontinuous Groups and Automorphic Functions.* American Mathematical Society (1964).

[6] Morris, D. W. *Introduction to Arithmetic Groups.* arXiv:math/0106063v6 [Math.DG], (2015).

[7] S. J. Patterson. *Diophantine approximation in Fuchsian groups.* Philos. Trans. Roy. Soc. London Ser. A **282** (1976) 527–563.

[8] Rankin, R. A. *Diophantine approximation and horocyclic groups.* Can. J. Math. **9** (1957), 277–290.

[9] Walkden, C. *Lecture Notes in Hyperbolic Geometry.* The University of Manchester, (2017).



Gaurav Sawant
Department of Mathematics
Veermata Jijabai Technological Institute
H. R. Mahajani Road, Matunga (East)
Mumbai 400019, India
E-mail: gauravsawant.math@gmail.com

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# PRE-S OPEN SETS IN TOPOLOGY

SHIV S. JHA, KUSUM
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Abstract. This paper is concerned to introduce the concept of Pre-S open sets and to investigate some properties of Pre-S open sets along with the interrelationship between the Pre-S open sets and the other existing classes. Moreover, characterizations of $\beta$-disconnectedness with the help of Pre-S open sets are investigated and will discuss the concepts of Pre-S open Continuity and almost Pre-S open Continuity.

## 1. Introduction

Semi-open sets, preopen sets, $\alpha$-open sets, $\beta$-open sets and $\delta$-open sets play an important role in the theory of classical point set topology. In 1963, Levine introduced the concept of semi-open sets which is weaker than the concept of open sets in topological spaces [2]. After the work of Levine on semiopen sets, several mathematicians turned their attention to the generalizations of various concepts of topology by considering semi-open sets instead of open sets. While open sets are replaced by semi-open sets, new results are obtained in some occasions and in other occasions substantial generalizations are exhibited [26]. Veličko[28] introduced $\delta$-open sets, which are stronger than open sets. Park et al.[13] have offered a new notion called $\delta$-semiopen sets which are stronger than semi-open sets but weaker than $\delta$-open sets and investigated the relationships between several types of these open sets. The notion of $\alpha$-open sets (originally called $\alpha$-sets) in topological spaces was introduced by Njȧstad [3]in 1965. By using $\alpha$-open sets, Moshhour et al.[6] defined and studied the notions of $\alpha$-continuity and $\alpha$-openness in topological spaces. In 1982, Mashhour et al. introduced and investigated the concepts of preopen sets and precontinuous functions in topological spaces. In 1983, Abd El-Monsef[7] et al. introduced a weak

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form of open sets called $\beta$-open sets. The concept of $\beta$-open sets is equivalent to that of semi-preopen sets. In 1996, Andrijevic [12] introduced a class of generalized open sets in a topological space, the so-called $b$-open sets. The class of $b$-open sets is contained in the class of $\beta$-open sets and contains all semi-open sets and preopen sets. The purpose of the present paper is to introduce and study a new class of sets called Pre-S open sets , which is independent of both open and closed sets. Priorly, we define Pre-S open sets for topological spaces and then we discuss their properties.

Furthermore, we will discuss the interrelationship between the Pre-S open sets and the other existing classes such as $\alpha$-open, $\beta$-open, pre-open, regular open, regular closed, semi-closed. Moreover, we will discuss the decompositions of regular open, $\beta$-closed, $\alpha$-open with the help of Pre-S open sets. In 1994, semipreconnected or $\beta$-connected, which are equivalent notions, were defined and investigated independently by Aho and Nieminen[9] and Poha and Noiri, respectively[16]. In this paper, we obtain further characterizations of $\beta$-disconnected. In the last section, Pre-S open continuity and almost Pre-S open continuity are defined and interrelationship of almost Pre-S open continuity with other existing mappings such that almost precontinuity, $\beta$-continuity and almost $\alpha$-continuity is investigated.

## Motivation

Topology plays a vital role in understanding the properties of spaces and sets. The concept of open sets is fundamental in topology, and various generalizations of open sets have been introduced to study different topological properties. In 1963, Levine introduced the concept of semi-open sets which is weaker than the concept of open sets. While open sets are replaced by semi-open sets, new results are obtained in some new occasions. Veličko introduced $\delta$-open sets, Park et al. introduced $\delta$- semiopen sets which are stronger than semi-open sets but weaker than $\delta$-open sets. Thus, in this way several mathematicians turned their attention to the generalizations of various concepts of topology by considering semi-open sets instead of open sets. One such generalization is the notion of pre-open sets. In this paper, we introduce the concept of Pre-S open sets. The motivation behind this study lies in the potential applications of Pre-S open sets in understanding the relationships between different topological properties. Our research aims to establish the interrelationships between Pre-S open sets and other existing sets, such as semi-open sets, $\beta$-open sets, $\beta$-closed sets, and $\alpha$-

open sets. By establishing these interrelationships, we aim to gain a deeper understanding of the properties of Pre-S open sets and their potential applications. Furthermore, we investigate the concept of $\beta$-disconnectedness and utilize Pre-S open sets to provide new insights into this topological property.

## 2. Preliminaries

Throughout the present paper, $(X, \tau)$ denotes a topological space. Suppose $A$ be any subset of $(X, \tau)$, then we denote interior of $A$ and closure of $A$ by i$(A)$ and c$(A)$ respectively.

**Definition 2.1.** [17][27][15][25][1][14] Let $(X, \tau)$ be a topological space. Then a subset $A$ of $X$ is called

(1) semiopen if $A \subseteq ci(A)$;
(2) $\alpha$-open if $A \subseteq ici(A)$;
(3) $b$-open if $A \subseteq ci(A) \cup ic(A)$;
(4) preopen if $A \subseteq ic(A)$;
(5) $\beta$-open if $A \subseteq cic(A)$;
(6) regular open (resp., regular closed) if $A = ic(A)$ (resp., $A = ci(A)$ ).
(7) q-set if $ic(A) \subseteq ci(A)$;
(8) Q-set if $ci(A) = ic(A)$;

The complement of a semiopen(respectively, $\alpha$-open, $b$-open, preopen and $\beta$-open) set is called semiclosed(respectively, $\alpha$-closed, preclosed, $\beta$-closed and $b$-closed).

The families of regular open and regular closed sets on $(X, \tau)$ are denoted by $RO(X, \tau)$ and $RC(X, \tau)$ respectively. The intersection of all preclosed sets containing a set $A$ is called the preclosure of $A$ and is denoted by $c_\pi(A)$. Dually, the pre-interior of $A$ is defined to be the union of all preopen sets contained in $A$ and is denoted by $i_\pi(A)$.

**Theorem 2.2.** [14] *Let $A$ be a subset of a space $(X, \tau)$. Then*

*(i)* $i_\pi(A) = A \cap ic(A)$;

*(ii)* $c_\pi(A) = A \cup ci(A)$.

*where $i_\pi$ and $c_\pi$ denote the pre-interior and pre-closure of $A$, respectively.*

**Definition 2.3.** [9][8][10] A topological space $X$ is called semipreconnected or $\beta$-connected(resp., preconnected) if $X$ can not be expressed as the union of two nonempty disjoint semipreopen (resp., preopen) sets of $X$.

**Definition 2.4.** [5][11] Let $X$ and $Y$ be two topological spaces. Then a function $f : X \to Y$ is called

(i) continuous if the inverse image of a closed set in $Y$ is closed in $X$.

(ii) semi-continuous if the inverse image of a closed set in $Y$ is semi-closed in $X$.

(iii) pre-continuous if the inverse image of an open set in $Y$ is pre-open in $X$.

## 3. Pre-S open sets in Topological Spaces

In this section, we define a new family over $(X, \tau)$, study about it with the help of interior and closure. Furthermore, we will discuss relation between Pre-S open sets and the existing classes such as $\alpha$-open, $\alpha$-closed, regular open, regular closed, semi-open, semi-closed, pre-open, pre-closed, $\beta$-open, $\beta$-closed, Q-set and q-set.

**Definition 3.1.** A subset $A \subseteq X$ of a topological space $(X, \tau)$ is said to be a Pre-S open set if,

$$A = i_\pi(c_\pi(A))$$

The class of all Pre-S open sets in $(X, \tau)$ is denoted by PSS$(X, \tau)$. Complement of Pre-S open set is called Pre-S closed.

**Remark 3.2.** For a topological space $(X, \tau)$, one can observe $\phi$ and $X$ both are Pre-S open sets.

**Remark 3.3.** The family of Pre-S open sets PSS$(X, \tau)$ is not closed under union as well as intersection.

We will show this with the help of following examples

**Example 3.4.** Let $X = \mathbb{R}$ be the set of all real numbers with the usual topology. Consider, $A = (1, 2)$, then $c_\pi(A) = [1, 2]$ and $i_\pi([1, 2]) = (1, 2)$. Hence, $A = i_\pi(c_\pi(A)) = (1, 2)$, therefore $A$ is a Pre-S open set. Now consider $B = (2, 3)$, then $c_\pi(B) = [2, 3]$ and $i_\pi([2, 3]) = (2, 3)$. Hence, $B = i_\pi(c_\pi(B)) = (2, 3)$, therefore $B$ is a Pre-S open set.

Consider $S = A \cup B = (1, 2) \cup (2, 3)$, then $c_\pi(S) = [1, 3]$ and $i_\pi([1, 3]) = (1, 3)$. Thus, we get $i_\pi(c_\pi(S)) = (1, 3) \neq S$

Therefore, Pre-S open set is not closed under union.

Next we will show Pre-S open sets are not closed under intersection.

**Example 3.5.** Let $X = \mathbb{R}$ be the set of all real numbers with the cofinite topology. Consider, $A = \mathbb{N}$, then $c_\pi(A) = \mathbb{N}$ and $i_\pi(\mathbb{N}) = \mathbb{N}$, therefore $A$ is a Pre-S open set. Now consider $B = (1,5)$, then similarly one can observe $B$ is also a Pre-S open set.

Now consider $S = A \cap B = \mathbb{N} \cap (1,5) = \{2,3,4\}$, then $c_\pi(S) = \{2,3,4\}$ and $i_\pi(\{2,3,4\}) = \phi$. Thus we get, $i_\pi(c_\pi(S)) = \phi \neq S$. Hence, $S$ is not a Pre-S open set.

Therefore, Pre-S open set is not closed under intersection.

Next we will compare the Pre-S open set with the existing notions of generalized sets.

**Remark 3.6.** $\alpha$-open and Pre-S open set are independent of each other.

We have the following example

First we will illustrate that $\alpha$-open set need not be a Pre-S open set.

**Example 3.7.** Let $X = \{a,b,c,d,e\}$,$\tau = \{\phi, \{a,b,c\}, \{d,e\}, \{c\}, \{d,e,c\}, X\}$ and $A = \{a,c\}$. Then we have, $ici(A) = \{a,b,c\}$. Therefore, $A \subseteq ici(A)$. Hence, $A$ is $\alpha$-open. Next since we have $c_\pi(A) = \{a,b,c\}$ and $i_\pi(\{a,b,c\}) = \{a,b,c\}$. Therefore, $A$ is not a Pre-S open set as $i_\pi(c_\pi(A)) \neq A$. Thus, $A$ is $\alpha$-open but not a Pre-S open set.

Now we will show that Pre-S open set need not be $\alpha$-open set.

**Example 3.8.** From example 3.7 as mentioned above, consider $A = \{a,b,c,d\}$. Then $ici(A) = \{a,b,c\}$ therefore, $A \nsubseteq ici(A)$. Hence, $A$ is not $\alpha$-open but $A$ is Pre-S open set as $c_\pi(A) = \{a,b,c,d\}$ and $i_\pi(\{a,b,c,d\}) = \{a,b,c,d\} = A$.

Hence, we can conclude that $\alpha$-open and Pre-S open set are independent of each other.

**Remark 3.9.** $\alpha$-closed and Pre-S open set are independent of each other.

We have the following example,

Next we will demonstrate that $\alpha$-closed set need not be a Pre-S open set.

**Example 3.10.** Let $X = \mathbb{R}$ be the set of all real numbers with usual topology and $A = [1, 2]$, then we have $cic(A) = [1, 2] \subseteq A$; therefore, $A$ is $\alpha$-closed.

Now we have $c_\pi([1, 2]) = [1, 2]$ and $i_\pi([1, 2]) = (1, 2) \neq A$ and hence $A$ is not a Pre-S open set.

Now we will show that Pre-S open set need not be $\alpha$-closed set.

**Example 3.11.** Let $X = \{a, b, c, d, e, f\}$,

$\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}$ and $A = \{a, b\}$. Then we have $c_\pi(A) = \{a, b, d, e, f\}$ and $i_\pi(\{a, b, d, e, f\}) = \{a, b\}$. Hence, $A$ is Pre-S open set but $A$ is not $\alpha$-closed as $cic(A) = \{a, b, d, e, f\}$ is not subset of $A$.

Hence, we can conclude that $\alpha$-closed and Pre-S open set are independent of each other.

[29] Next we will try to show that semi-open and Pre-S open are independent of each other. It is well-known that every $\alpha$-open is semi-open and $\alpha$-open is independent of Pre-S open as we have showed in remark 3.6 this implies that there is no relation between semi-open and Pre-S open set.

**Remark 3.12.** Semi-open and Pre-S open sets are independent of each other.

**Example 3.13.** [29] From the example 3.7 mentioned above, it can be easily seen that semi-open and Pre-S open sets are independent of each other. Consider $A = \{a, c\}$, then $A$ is semi-open, as every $\alpha$-open is semi-open, but $A$ is not a Pre-S open set.

**Example 3.14.** Now consider $A = \{a, b, c, d\}$, then from example 3.8 mentioned above it is clear that, $A$ is Pre-S open set. But $A$ is not semi-open as $ci(A) = \{a, b, c\}$ and hence $A \nsubseteq ci(A)$.

**Remark 3.15.** Semi-closed and Pre-S open sets are independent of each other.

We have the following example

First we will illustrate that semiclosed is not a Pre-S open set.

**Example 3.16.** Let $X = \{a, b, c, d, e\}$, $A = \{d, e, b\}$ and

$\tau = \{\phi, \{a, b, c\}, \{d, e\}, \{c\}, \{d, e, c\}, X\}$. Then, $ic(A) = \{d, e\}$. Therefore, $ic(A) \subseteq A$. Hence, $A$ is semi-closed. Now we have $c_\pi(A) = \{d, e, b\}$

and $i_\pi(\{d,e,b\}) = \{d,e\}$ and hence $i_\pi(c_\pi(A)) \neq A$. Thus, $A$ is not a Pre-S open set.

Next we will show that every Pre-S open set need not be a semiclosed set.

**Example 3.17.** Consider $A = \{a,b,c,d\}$. Then $ic(A) = X$ therefore, $ic(A) \nsubseteq A$. Hence, $A$ is not semi-closed but from example 3.8 mentioned above, it is clear that $A$ is a Pre-S open set. Thus, $A$ is not semi-closed but Pre-S open set.

Hence, we can conclude that, semi-closed and Pre-S open set are independent of each other.

Next we will try to show that there is no relation between Q-set and Pre-S open set as well as q-set and Pre-S open set.

**Remark 3.18.** Q-set and Pre-S open set are independent of each other.

We have the following example

**Example 3.19.** $X = \{a,b,c,d,e\}$, $\tau = \{\phi, X, \{a,b,c\}, \{c\}, \{d,e\}, \{d,e,c\}\}$, $A = \{c\}$. Then, $ci(A) = \{a,b,c\}$, $ic(A) = \{a,b,c\}$

Now, we have $c_\pi(A) = \{a,b,c\}$ and $i_\pi(\{a,b,c\}) = \{a,b,c\}$; therefore, we have $i_\pi(c_\pi(A)) \neq A$. Hence, $A$ is not a Pre-S open set but Q-set as $ci(A) = \{a,b,c\} = ic(A)$. Hence, $A$ is Q-set but not a Pre-S open set.

**Example 3.20.** From example 3.7 mentioned above, consider $A = \{a,b,c,d\}$. Then, $ci(A) = \{a,b,c\}$, $ic(A) = X$; thus, $ci(A) \neq ic(A)$ and hence $A$ is not a Q-set

Next we have $c_\pi(A) = \{a,b,c,d\}$ and $i_\pi(\{a,b,c,d\}) = \{a,b,c,d\} = A$ and hence $A$ is a Pre-S open set.

Therefore, $A$ is a Pre-S open set but not Q-set.

**Remark 3.21.** q-set and Pre-S open set are independent of each other.

We have the following example

**Example 3.22.** From example 3.7 & 3.8 mentioned above, consider $A = \{a,b,c,d\}$ then it is clear that $A$ is Pre-S open set but not q-set as $ic(A) \nsubseteq ci(A)$.

**Example 3.23.** Now, consider $X = \mathbb{R}$, the set of all real numbers with usual topology and $A = [a,b]$. $A$ is not Pre-S open set as $c_\pi(A) = [a,b]$ and $i_\pi([a,b]) = (a,b)$ and hence we have $i_\pi(c_\pi(A)) = (a,b) \neq [a,b] = A$.

But $A$ is q-set as $ic(A) = (a, b) \subseteq [a, b] = c(i(A))$. Therefore, $A$ is a q-set but not a Pre-S open set.

In the next result, we will show that pre open sets are weaker form of Pre- S open sets.

**Theorem 3.24.** *Every Pre-S open set is preopen set.*

*Proof.* Let $A$ be a Pre-S open set. Then, we have $A = i_\pi(c_\pi(A))$. Thus, $A$ is preopen set. □

But, converse of the above result is not true in general.

We have the following example

**Example 3.25.** [24] From example 3.7 mentioned above, it is clear that $A$ is pre-open set as every $\alpha$-open set is pre-open but $A$ is not a Pre-S open set. Thus, converse is not true.

**Remark 3.26.** Preclosed and Pre-S open set are independent of each other.

We have the following example

**Example 3.27.** From example 3.10 mentioned above, it is clear that $A$ is preclosed but not Pre-S open set.

**Example 3.28.** From example 3.11 mentioned above, it is clear that $A$ is Pre-S open set but not preclosed set.

**Theorem 3.29.** *Every Pre-S open set is b-open set.*

*Proof.* [24] Let $A$ be Pre-S open set. Then, we know that every Pre-S open set is pre-open set and every pre-open set is $b$-open set. Therefore, we have $A$ is $b$-open set. □

But, converse of the above result is not true in general.

We have the following example

**Example 3.30.** From example 3.16 mentioned above, it is clear that $A$ is not a Pre-S open set.

But, as $ic(A) = \{a, b, c\}$ and $ci(A) = \{a, b, c\}$ and also since we have $A = \{c\}$. Therefore, it is clear that, $A \subseteq ci(A) \cup ic(A) = \{a, b, c\}$ and hence, $A$ is $b$-open set. Thus, $A$ is $b$-open but not a Pre-S open set.

**Theorem 3.31.** *Let $(X, \tau)$ be a topological space. Then, a subset $A \subseteq X$ is a Pre-S open set if and only if*

$$A = [A \cup ci(A)] \cap ic(A)$$

*Proof.* For a topological space $X$, we have

$$i_\pi(A) = A \cap ic(A)$$
$$c_\pi(A) = A \cup ci(A)$$

and let, $A$ be a Pre-S open set then we have, $A = i_\pi(c_\pi(A))$; therefore we have, $A = i_\pi(A \cup ci(A))$. Now let,

$$A \cup ci(A) = B(say) \tag{3.1}$$

then, we have $A = i_\pi(B)$; therefore, $i_\pi(B) = B \cap ic(B)$ and hence $A = B \cap ic(B)$. Thus, by equation (3.1) we have, $A = [A \cup ci(A)] \cap i[c(A) \cup cci(A)]$ implies $A = [A \cup ci(A)] \cap i[c(A) \cup ci(A)]$. Hence, $A = [A \cup ci(A)] \cap ic(A)$ Therefore, $A = [A \cup ci(A)] \cap ic(A)$ □

Now we provide some characterizations of Pre-S open set

**Theorem 3.32.** *Every regular open set is Pre-S open set.*

*Proof.* Let $A$ be a regular open set. Then,

$$A = ic(A) \tag{3.2}$$

Now, consider,

$$[A \cup ci(A)] \cap ic(A)$$

and by equation (3.2), we have, $[ic(A) \cup ci(A)] \cap ic(A)$,

$$= ic(A)$$
$$= A$$

Therefore, $A$ is a Pre-S open set. □

**Remark 3.33.** Converse of the above theorem is not true in general that is Pre-S open set need not be regular open.

For this we have the following example

**Example 3.34.** From example 3.7 & 3.8 mentioned above, it is clear that $A = \{a, b, c, d\}$ is a Pre-S open set.

but since, $ic(A) = \{a, b, c, d, e\} = X \neq A$

Therefore, $A$ is not regular open.

**Remark 3.35.** Now we will discuss relation of regular closed sets with Pre-S open sets. The notions of Pre-S open sets and Regular closed sets are independent of each other.

We have the following example

**Example 3.36.** From example 3.10 mentioned above, it is clear that $A$ is regular closed as $ci(A) = [1, 2] = A$ but $A$ is not a Pre-S open set.

**Example 3.37.** Moreover, from example 3.7 & 3.8 mentioned above, it is clear that $A = \{a, b, c, d\}$ is a Pre-S open set but not regular-closed as $ci(A) = \{a, b, c\} \neq A$.

Next we will show that in general open set as well as closed set need not be a Pre-S open set.

We have the following example

**Example 3.38.** Consider $X = \{a, b, c, d, e, f, g, h\}$,

$\tau = \{\phi, \{a, b, c, d, f\}, \{c, d, f\}, \{c, d, e, f, g, h\}, X\}$ and $A = \{c, d, f\}$. Now we have $ci(A) = X$ and $ic(A) = X$, and hence we have $[A \cup ci(A)] \cap ic(A) = [\{c, d, f\} \cup X] \cap X = X \neq A$; therefore $A$ is not a Pre-S open set but $A$ is open and hence open set need not be a Pre-S open set.

Similarly closed set need not be a Pre-S open set.

We have the following example

**Example 3.39.** Consider $X = \mathbb{R}$ be the set of real numbers and $A = [a, b]$, then $ci(A) = [a, b]$ and $ic(A) = (a, b)$. Thus,we have $[A \cup ci(A)] \cap [ic(A)] = [[a, b] \cup [a, b]] \cap (a, b) = (a, b) \neq A$ and hence $A$ is not a Pre-S open set. Thus, $A$ is closed but not a Pre-S open set.

Now we will try to show that if $A$ is Pre-S open set then it is not necessary that $A$ is open or closed.

We have the following example

**Example 3.40.** From example 3.7 & 3.8 mentioned above, it is clear that $A = \{a, b, c, d\}$ is a Pre-S open set, but $A$ is neither open nor closed.

Thus, we can say that Pre-S open set is independent from open set as well as from closed set.

In the next set of results, we try to decompose some strong form of open sets with the help of Pre-S open set.

**Theorem 3.41.** *Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then a set $A$ is regular open if $A$ is semi-closed and Pre-S open set.*

*Proof.* Let $A$ be Pre-S open set and semi-closed. Since, $A$ is Pre-S open set, therefore $A = [A \cup ci(A)] \cap [ic(A)] \subseteq ic(A)$; that is $A \subseteq ic(A)$ and since $A$ is semi-closed, therefore $ic(A) \subseteq A$ Thus, $A = ic(A)$. Hence, $A$ is regular open. □

**Corollary 3.42.** *Every Pre-S open set is regular open if and only if it is semi-closed.*

*Proof.* Let $A$ be a Pre-S open set and hence is regular open. Since $A$ is regular open; therefore, we have $A = ic(A)$ and thus $ic(A) \subseteq A$. Hence, $A$ is semi-closed. Conversely, let $A$ be semi-closed and Pre-S open set. Hence, by using above theorem $A$ is regular open. □

Next we will try to show with the help of following example that semi-closed need not imply regular open.

**Example 3.43.** Let $X = \mathbb{R}$ be the set of all real numbers with usual topology and $A = [a, b]$ and we get $ic(A) = (a, b) \subseteq A$. Thus, $A$ is semi-closed but not regular open as $ic(A) = (a, b) \neq A$.

Thus we can conclude that a semiclosed set $A$ is regular open if it is Pre-S open set. From the last theorem, we can conclude the following

**Remark 3.44.** Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then a set $A$ is open if $A$ is semiclosed and Pre-S open set.

*Proof.* [4] Let $A$ be a Pre-S open set and semiclosed. Then, by theorem mentioned above, $A$ is regular open. Since, every regular open set is open; therefore, $A$ is open. □

**Remark 3.45.** Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then a set $A$ is regular open if $A$ is Pre-S open set and closed.

Closedness and Pre-S open set are necessary conditions for a set to be regular open.

But converse of the above result is not true in general, we have the following example

**Example 3.46.** Consider $X = \mathbb{R}$ be the set of all real numbers with usual topology, $A = (1, 2)$.

Therefore, $A$ is regular open but not closed.

Now, we will try to decompose some other weaker form of sets that is $\alpha$-open set with the help of Pre-S open set. Since, we have discussed above, that $\alpha$-open and Pre-S open set are independent of each other; therefore with the help of decomposition, we can induce some relation between $\alpha$-open and Pre-S open sets. Thus, we have the following theorem

**Theorem 3.47.** *Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then a set $A$ is $\alpha$-open if $A$ is regular closed and Pre-S open set.*

*Proof.* [22] Let $A$ be a regular closed and Pre-S open set and since every regular closed is semi-closed therefore we have $A$ is semi-closed and Pre-S open set and thus by using result mentioned above, we have $A$ is open and since every open set is $\alpha$-open therefore $A$ is $\alpha$-open. □

**Theorem 3.48.** *Every Pre-S open set is $\beta$-closed.*

*Proof.* Let $A$ be a Pre-S open set. Then,

$$A = [A \cup ci(A)] \cap ic(A) \tag{3.3}$$

Now since we know, $ici(A) \subseteq ci(A)$
$\subseteq A \cup ci(A)$

Also we have, $ici(A) \subseteq ic(A)$

Therefore, we have

$$ici(A) \subseteq [A \cup ci(A)] \cap ic(A)$$

Thus, by using equation (3.3) we have, $ici(A) \subseteq A$; therefore $A$ is a $\beta$-closed set.

Hence, every Pre-S open set is $\beta$-closed set. □

**Remark 3.49.** But, converse of the above theorem is not true in general. we have the following example

**Example 3.50.** From example 3.7 mentioned above, consider $A = \{a\}$. Then, we have $ici(A) = \phi \subseteq A$. Therefore, $A$ is $\beta$-closed set but not a Pre-S open set as $c(A) = \{a, b\}$ then $ic(A) = \phi$ and $ci(A) = \phi$. Now we have $[A \cup ci(A)] \cap ic(A) = [\{a\} \cup \phi] \cap \phi = \phi \neq A$.

**Theorem 3.51.** *Every Pre-S open set is $\beta$-open set as well.*

*Proof.* [22] Let $A$ be Pre-S open set. Then, we know that every Pre-S open set is pre-open set and every pre-open set is $\beta$-open set. Therefore, we have $A$ is $\beta$-open set. □

**Remark 3.52.** But, converse of the above result is not true in general.

We have the following example

**Example 3.53.** From example 3.7 mentioned above, we have $A = \{a, c\}$ is $\beta$-open as $cic(A) = \{a, b, c\}$ and hence $A \subseteq cic(A)$. But $A$ is not a Pre-S open set.

Hence, we can conclude that $A$ is $\beta$-open but not a Pre-S open set.

**Remark 3.54.** The following diagram holds for each a subset $A$ of X.

Regular open $\rightarrow$ Pre-S open set $\rightarrow$ pre open $\rightarrow$ $b$-open $\rightarrow$ $\beta$-open

$\swarrow$ $\quad$ $\searrow$ $\quad$ $\downarrow$ $\quad$ $\nearrow$

$\beta$-closed $\quad$ z-open

## 4. Pre-S open sets in Connected and Disconnected Spaces

In this section, we will discuss about some forms of connectedness with the help of Pre-S open set.

Some well-known definitions and results are as follows that will provide a foundation for our subsequent discussion,

**Definition 4.1.** [19] A topological space $(X, \tau)$ is said to be $\beta$-connected (resp., $\gamma$-connected ), if $X$ cannot be expressed as a union of two non-empty and disjoint semi-open (resp., $b$-open) subsets of $(X, \tau)$.

**Lemma 4.2.** [19] *A space $(X, \tau)$ is $\beta$-connected if and only if there is no set $S \in SPR(X, \tau) = SPO(X, \tau) \cap SPC(X, \tau\ )$ such that $\phi \neq S \neq X$, where $SPO$, $SPC$ are class of semi pre-open and semi pre-closed respectively and members of $SPO$ and $SPC$ are $\beta$-open and $\beta$-closed sets and $SPR$ is intersection of class of semi pre-open and semi pre-closed.*

**Fact 4.3.** [20] If a space $(X, \tau)$ is $\gamma$-connected then it is $\beta$-connected.

Now we provide characterization of $\beta$-disconnectedness in the following way

**Theorem 4.4.** *A space $(X, \tau)$ is $\beta$-disconnected if there exists a non-empty proper Pre-S open set.*

*Proof.* Assume there exists a non-empty proper Pre-S open set and since every Pre-S open set is $\beta$-open as well as $\beta$-closed. Therefore, there exists a Pre-S open set which belongs to intersection of $\beta$-open and $\beta$-closed. Hence A space $(X, \tau)$ is $\beta$-disconnected. $\square$

**Fact 4.5.** A space $(X, \tau)$ is $\gamma$-disconnected if there exists a non-empty proper Pre-S open set.

*Proof.* Suppose there exists a non empty proper Pre-S open set then space $X$ is $\beta$-disconnected. Hence, $\gamma$-disconnected. □

**Definition 4.6.** [23] Two subsets $A$ and $B$ in a space $X$ are said to be half $b$-separated (resp., $b$-separated ) if and only if $A \cap bCl(B) = \phi$ or $bCl(A) \cap B = \phi$ ( resp., $A \cap bCl(B) = \phi = bCl(A) \cap B$).

**Theorem 4.7.** *Two subsets $A$ and $B$ in a space $X$ are said to be $b$-separated if $A$ and $B$ are Pre-S open sets.*

*Proof.* Let $A$ and $B$ be Pre-S open sets. Then $A$ and $B$ be $b$-open as every Pre-S open set is $b$-open sets therefore $X \setminus A$ and $X \setminus B$ are $b$-closed sets. Now using definition we get, $A \cap bCl(B) = \phi = bCl(A) \cap B$. Hence, $A$ and $B$ are $b$-separated. □

**Theorem 4.8.** *Two subsets $A$ and $B$ in a space $X$ are said to be half $b$-separated if $A$ and $B$ are Pre-S open sets.*

*Proof.* Let $A$ and $B$ be Pre-S open sets. Then $A$ and $B$ be $b$-open as every Pre-S open set is $b$-open set therefore $X \setminus A$ and $X \setminus B$ are $b$-closed sets. Now using definition we get, $A \cap bCl(B) = \phi$ or $bCl(A) \cap B = \phi$. Hence, $A$ and $B$ are half $b$-separated. □

**Theorem 4.9.** *A space $X$ is half $b$-connected if it cannot be expressed as the disjoint union of a nonempty Pre-S open set and a nonempty $b$-closed set.*

*Proof.* Let $X$ be a half $b$-connected space. If possible suppose that $X = U \cup F$, where $U \cap F = \phi$, $U(\neq \phi)$ is a Pre-S open set and $F(\neq \phi)$ is a $b$-closed set in $X$. Since $F$ is a $b$-closed set in $X$, then $U \cap bCl(F) = \phi$ and hence $U$ and $F$ are half $b$-separated. Therefore $X$ is not a half $b$-connected space. Which is a contradiction,

In case $bCl(A) \cap B = \phi$ we have the similar argument. □

**Definition 4.10.** [18] Let $(L^X, \delta)$ be an $L$-topological space, $A$, $B \in L^X$. Then $A$, $B$ are said to be $\beta$-separated if $\beta cl(A) \bigwedge B = B \bigwedge \beta cl(A) = 0$.

**Proposition 4.11.** [18] *Let $(L^X, \delta)$ be an $L$-topological space, $A \in L^X$. Then*

*(i) A is a $\beta$-open set if and only if $A = \beta int(A)$.*
*(ii) A is a $\beta$-closed set if and only if $A = \beta cl(A)$.*

**Theorem 4.12.** *Let $(L^X, \delta)$ be an L-topological space, A, $B \in L^X$. Then A, B are said to be not $\beta$-separated if $\beta cl(A) \bigwedge B = B \bigwedge \beta cl(A) \neq 0$ where A and B are Pre-S open sets.*

*Proof.* Let $(L^X, \delta)$ be an $L$-topological space. Suppose $A$ and $B$ are Pre-S open sets. We know that every Pre-S open set is $\beta$-closed set therefore $\beta cl(A) \bigwedge B = B \bigwedge \beta cl(A) \neq 0$. Hence, $A$ and $B$ are not $\beta$-separated. □

**Definition 4.13.** [18]
Let $(L^X, \delta)$ be an $L$-topological space, $A \in L^X$. $A$ is called $\beta$-connected if $A$ cannot be represented as a union of two non-null $\beta$-separated $L$-sets. If $A = 1$ is $\beta$-connected, we call $(L^X, \delta)$ a $\beta$-connected $L$-topological space.

**Theorem 4.14.** *Let $(L^X, \delta)$ be an L-topological space, $A \in L^X$. Then A is $\beta$-connected if there exist a Pre-S open set.*

*Proof.* Suppose $B$ and $C$ are Pre-S open sets. By previous theorem then we get, $B$ and $C$ are not $\beta$-separated. By definition 4.13 we get, $A$ is $\beta$-connected. □

## 5. Pre-S Continuity and Almost Pre-S Continuity

**Definition 5.1.** [21] A function $f : X \to Y$ is said to be R-map (resp., almost continuous , almost $\alpha$-continuous, almost semicontinuous, and almost precontinuous) if $f^{-1}(V)$ is regular open, (resp., open, $\alpha$-set, semiopen, and preopen) for every regular open set $V$ in $Y$.

Now we define Pre-S open continuity and almost Pre-S open continuity in the following way,

**Definition 5.2.** A mapping $f : X \to Y$ said to be Pre-S open continuous at a point $x \in X$ if for every open neighbourhood $M$ of $f(x)$ there exist a Pre-S open neighbourhood $N$ of $x$ such that $f(N) \subseteq M$.

**Definition 5.3.** A mapping $f : X \to Y$ said to be almost Pre-S open continuous at a point $x \in$ X if for every neighbourhood $M$ of $f(x)$ there exist a Pre-S open neighbourhood $N$ of $x$ such that $f(N) \subseteq int(cl(M))$.

A mapping $f : X \to Y$ said to be Pre-S open continuous(resp., almost Pre-S open continuous) if it is Pre-S open continuous(resp., almost Pre-S open continuous) at each point $x$ of $X$.

**Theorem 5.4.** *For a mapping $f : X \to Y$, the following are equivalent*

***(i)*** *$f$ is Pre-S open continuous.*

***(ii)*** *inverse image of every open subset of $Y$ is Pre-S open set.*

***(iii)*** *inverse image of every closed subset of $Y$ is Pre-S closed set.*

*Proof.* Firstly ((i) $\implies$ (ii)):

Let $U$ be an open subset of $Y$ and $x \in f^{-1}(U)$ which implies, $f(x) \in U$. Therefore, there exists a Pre-S open set $V$ in $X$ such that $f(V) \subseteq U$. Thus, $x \in V \subseteq f^{-1}(U)$, where $f^{-1}(U)$ is Pre-S open neighbourhood of $x$. Hence, $f^{-1}(U)$ is Pre-S open set.

Now ((ii) $\implies$ (iii)):

Let $F$ be any closed subset of $Y$. Then, $Y \setminus F$ is open and by (ii), $f^{-1}(Y \setminus F)$ is Pre-S open which implies $X \setminus f^{-1}(F)$ Pre-S open. Hence, $f^{-1}(F)$ is Pre-S closed set.

Finally ((iii) $\implies$ (i)):

Let $M$ be any open neighbourhood of $f(x)$ therefore, $Y \setminus M$ is closed which implies $f^{-1}(Y \setminus M)$ is Pre-S closed; therefore $X \setminus f^{-1}(M)$ is Pre-S closed. Thus $f^{-1}(M)$ is Pre-S open, and hence $x \in f^{-1}(M) = N$ ( say ). Then, $N$ is Pre-S neighbourhood of $x$ such that $f(N) \subseteq M$. Hence, $f$ is Pre-S open continuous. □

**Theorem 5.5.** *If $f : X \to Y$ is a Pre-S open continuous (resp., almost Pre-S open continuous) map then it is $\beta$-continuous (resp., almost $\beta$-continuous).*

*Proof.* Let $U$ be any open (resp., regular open) subset in $Y$ then $f^{-1}(U)$ is a Pre-S open and hence $f^{-1}(U)$ is $\beta$-open set which implies $f$ is $\beta$-continuous (resp., almost $\beta$-continuous). □

**Theorem 5.6.** *If $f : X \to Y$ is a Pre-S open continuous (resp., almost Pre-S open continuous) map then it is precontinuous (resp., almost precontinuous).*

*Proof.* Let $U$ be any open (resp., regular open) subset in $Y$ then $f^{-1}(U)$ is a Pre-S open and then $f^{-1}(U)$ is Pre-S open set and hence pre-open set. Hence $f$ is precontinuous (resp., almost precontinous). □

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## References

1. Stone, M. H., Applications of the Theory of Boolean rings to General Topology, Transactions of the American Mathematical Society, 41, 3, 375-481, 1937, JSTOR.
2. Norman, Levine, Semi-open Sets and semi-continuity in Topological Spaces, The American Mathematical Monthly, 70, 1, 36–41, 1963, Taylor & Francis.
3. Olav, N., On Some Classes of Nearly Open Sets, Pacific Journal of Mathematics, 15, 3, 961–970, 1965, Mathematical Sciences Publishers.
4. Willard, S., (1970), General topology, Addison-wesely Publishing Company.
5. Mashhour, A. S., Abd El-Monsef, M. E., and El-Deeb, S. N., On Precontinuous and Weak precontinuous functions Proc. Math. Phys. Soc. Egypt, 53, (1982), 47-53.
6. Mashhour, A. S., Hasanein I. A. and E1-Deeb, S. N., $\alpha$-continuous and $\alpha$-open Mappings, Acta Mathematica Hungarica, 41, 213–218, 1983, Springer.
7. Abdelm, M. E., $\beta$-open Sets and $\beta$-continuous Mappings, Bull. Fac. Sci. Assiut Univ., 12, 77–90, 1983.
8. Popa, V. and Noiri,T., Weakly $\beta$-continuous functions, An. Univ. Timisoara Ser. Mat. Inform., 32, 27-32, 1985.
9. Aho, T., Nieminen, T., Spaces in Which Preopen Subsets are Semiopen, Ricerche di Matematica, 43, 1, 45–60, 1994, Naples: Universita di Napoli, Istituto di matematica.
10. Popa, V., Properties of H-almost continuous functions, Bull. Math. Soc. Sci. Math. R. S. Roumanic(N.S.), 31, 83-92, 1994.
11. Dontchev, J., On generalizing Semipreopen sets, Mem. Fac. Sci. Kochi Uni. Ser A, Math., 16 (1995), 35-48.
12. Andrijević, Dimitrije, On b-open Sets, Matematički Vesnik, 205, 59–64, 1996, Društvo matematičara Srbije.
13. Park, J. H., Lee, B. Y. and Son,M. J., On $\delta$-semiopen Sets in Topological Spaces, J. Indian Acad. Math, 19, 1, 59–67, 1997.
14. Császár, Á. , On the $\gamma$-interior and $\gamma$-closure of a Set, Acta Math. Hunger, 80(1998), 89-93.
15. Thangavelu, P., and Chandrasekhara,K. Rao, q-sets in topological spaces, Bull. Pure and Appl. Sci, 21, 2, 341–345, 2002.
16. Jafari, S. and Noiri, T., Properties of $\beta$-connected Spaces, Acta Mathematica Hungarica, 101, 227–236, 2003, Springer.
17. Mustafa, J. M., Some Separation axioms by b-open Sets, Mu'tah Lil-Buhuth wad-Dirasat, 20, 3, 57-63, 2005.
18. Chen, B., $\beta$-connectedness in L-topological Spaces, Tamkang Journal of Mathematics, 42, 1, 69-77, 2011.
19. Duszyński, Zbigniew, $\beta$-connectedness and S-connectedness of Topological Spaces, Matematički Vesnik, 63, 244, 115–126, 2011.

20. Khalaf, B. Alais and Ali, S. Hardi, On SP-Connected Spaces, Science Journal of University of Zakho, 1, 325–332, 2013.
21. G. Ankit and S. Ratna Dev, PS-Regular Sets in Topology and Generalized Topology, Journal of Mathematics, 2014, 1, 274592.
22. Mubarki, M. Ali and Al-Rshudi, M. Massed and Al-Juhani, $\beta^*$-Open Sets and $\beta^*$-continuity in Topological Spaces, Journal of Taibah University for Science, 8, 2, 142–148, 2014, Elsevier.
23. Noiri, T., and Modak, S., Half b-connectedness in Topological Spaces, Journal of the Chungcheong Mathematical Society, 29, 2, 221-221, 2016.
24. Ameen, Z. A., and Asaad, B. A., A Note on Some Unified Types of Open and Locally Closed Sets, General Letters in Mathematics, 4, 102-106, 2018.
25. Hassan, A. Adam, Semi Generalized Open Sets and Generalized Semi Closed Sets in Topological Spaces, International Journal of Analysis and Application, 18, 6, 1029–1036, 2020.
26. Chawalit, B., and Jeeranunt, K., ($\Lambda$, sp)-open Sets in Topological Spaces, European Journal of Pure and Applied Mathematics, 15, 2, 572–588, 2022.
27. Rashed, R. M., On $\beta$-Continuity in Neutrosophic Topological Spaces, 2023.
28. Pue-on, P., and Boonpok, C., On $\delta$s ($\Lambda$, s)-open Sets in Topological Spaces, Int. J. Math. Comput. Sci, 18, 749–753, 2023.
29. Robert, A. and Missier, S. Pious, A new class of nearly open Sets, Intenational Journal of Mathematical Archive, 3, 7, 2575–2582, 2012.

Shiv S. Jha
Ramjas College
Department of Mathematics
University of Delhi
110007, Delhi, India.
E-mail: shivjha412000@gmail.com

Kusum
Miranda House
Department of Mathematics
University of Delhi
110007, Delhi, India.
E-mail: kusumsharma662003@gmail.com

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# PROBLEM SECTION

In volume 94(1-2) 2025 of the Mathematics Student, we had invited solutions for a set of five new problems, in addition to the eight problems from 93(3-4). We have received solutions for problems 1, 2, 4 and 5 from 93(3-4); and for the new problems 1-4 and part (1) of problem 5 from 94(1-2). For each of these problems, among the solutions including the proposers', one solution is selected for presentation below. We have not received solutions for problems 3, 6-8 from 93(3-4). For these problems we shall present solutions by the proposers. We shall give some more time for the problem 5 from 94(1-2).

We present five new problems in this volume and invite solutions for these problems from the readers till April 15, 2026. Correct solutions received by this date will be published in volume 95 (1-2) 2026 of The Mathematics Student, which is scheduled to be published in May 2025.
We sincerely acknowledge all the cooperation from the people who contribute problems and solutions to the problem section. We also welcome new problems along with detailed solutions for the problem section from the readers.

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**New Problems**

**MS 94(3-4) 2025 : Problem 1.** (by **Dr. Quang Hung Tran**, High School for Gifted Students, Vietnam University of Science, Vietnam.)
Consider a skew polygon $A_1A_2\ldots A_n$, in the Euclidean space $\mathbb{E}^d$, $n, d \in \mathbb{N}\setminus\{1\}$. Prove that for any point $P \in \mathbb{E}^d$, there always exist $\alpha_i \in \{-1, 0, 1\},\ i = 1,\, 2,\, \ldots\, n$ such that

$$\left|\sum_{j=1}^{n} \alpha_j \mathbf{PA_j}\right|^2 \geq \frac{1}{4}\left((A_1A_2)^2 + (A_2A_3)^2 + \cdots + (A_{n-1}A_n)^2 + (A_nA_1)^2\right),$$

where $\mathbf{PA_j}$ is the vector from $P$ to $A_j$, $j = 1,\, 2,\, \cdots, n$.

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**Problems 2-4** (by **Dr. B. Sury**, ISI, Bengaluru, India).

**MS 94(3-4) 2025 : Problem 2.**

For each real number $x \geq 0$, let $N_x = \{n \in \mathbb{N} \mid nx \quad \text{is even}\}$ . If $f(x) = \sum_{n \in N_x} \frac{1}{2^n}$ for $x \in [0, 1)$, prove that $\text{Inf}\{f(x) \mid x \in [0, 1)\} = \frac{4}{7}$.

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**MS 94(3-4) 2025 : Problem 3.**

If we have $n \in \mathbb{N}$, $n > 2$, discs on the plane with the property that any three of them intersect, then prove that the intersection of all discs is non-empty.

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**MS 94(3-4) 2025 : Problem 4.**

Given positive integers $m$, $n$, prove that there are infinitely many positive integers $N$ such that the binomial coefficient $\binom{N}{m}$ is relatively prime to $n$.

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**MS 94(3-4) 2025 : Problem 5.** (by **Dr. Chudamani Pranesachar Anil Kumar**, KREA University, India).

Baker, Heegner and Stark in the middle of the $20^{th}$ century have proved a version of the statement that if $d > 0$ is squarefree and $d \equiv -1 \mod 4$, then the ring

$$R_d = \frac{\mathbb{Z}[X]}{\langle X^2 - X + \frac{(1+d)}{4} \rangle}$$

is a principal ideal domain if and only if $d = 3, 7, 11, 19, 43, 67, 163$. This problem is about proving the following:

Let $p \in \mathbb{N}$ be a prime and $D > 0$ be a squarefree integer such that $D \equiv -1 \mod 4$ and let $E = \frac{D+1}{4}$. Suppose $0 < p < \frac{D}{4}, p \nmid D$ and the polynomial $X^2 - X + E$ modulo $p$ factorizes into linear factors in $\mathbb{F}_p[X]$. Then:

(1) Show that $R_D = \frac{\mathbb{Z}[X]}{\langle X^2 - X + E \rangle}$ is not a principal ideal domain.

(2) Conclude using (1) that, if $D > 3, D$ squarefree and $E = \frac{D+1}{4}$ is not prime, then the ring $R_D$ is not a principal ideal domain.

(3) Conclude that the rings $R_D$ for $D = 51, 91, 115, 123$ are not principal ideal domains. What are the choices of $p$ in these four cases?

(4) Conclude that if $D > 0, D \equiv -1 \mod 4$, $D$ is squarefree, then $R_D$ is a not a principal ideal domain for all $3 \le D \le 163$ and $D \notin \{3, 7, 11, 19, 43, 67, 163\}$.

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**Solutions for the unsolved problems from MS 93(3-4) 2025**

**MS 93(3-4) 2025 : Problem 1.** (by **Dr. Henry Ricardo**, Westchester Area Math Circle, New York, USA).

If an $n \times n$ matrix $A$ commutes with all $n \times n$ nilpotent matrices, must $A$ be nilpotent? Determine the whole class of these matrices. (Recall that a square matrix $M$ is said to be nilpotent whenever $M^k = O$ for some positive integer $k$.)

**Solution:** (by the Proposer)
The answer to the first question is negative, with the identity matrix $A = I_n$ providing a simple counterexample.

Since any scalar multiple of the $n \times n$ identity $I_n$ commutes with every $n \times n$ matrix, the commutant of the set $\mathcal{N}_n$ of nilpotent $n \times n$ matrices contains every scalar multiple of $I_n$—and hence non-nilpotent members. In fact, these are the only elements of the commutant of $\mathcal{N}_n$. Indeed, assuming $n \ge 2$ for non-triviality, each element $A$ of the commutant commutes with every matrix of the form $xy^T$, where $x, y \in \mathbb{C}^n$ are column vectors and $y^T x = 0$. In particular, $(Ax)y^T = x(A^T y)^T$ for any such pair $x, y$. It follows that each nonzero $x \in \mathbb{C}^n$ is an eigenvector of $A$. Hence $A$ is a scalar multiple of $I_n$.

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**MS 93 (1-2) 2024 : Problem 2.** ( by **Mr. Himadri Lal Das**, IIT Kharagpur, India).

Let $\{F_n\}_{n\ge 0}$ be the sequence of *Fibonacci numbers*, where $F_0 = 1$ and $F_1 = 1$. Evaluate the following limit, if exists,

$$\lim_{n\to\infty} \left( \sum_{\substack{k_1+k_2+\cdots+k_n=n,\\ 1\le i\le n}} \frac{k_i F_{i-1}}{k_1!k_2!\ldots k_n!} \right)^{\frac{1}{n}}.$$

This problem was solved by Mr. Santanu Panda, Hooghly, West Bengal, India.

**Solution:** (by the Proposer)
We first show that

$$\sum_{\substack{k_1+k_2+\cdots+k_n=n,\\ 1\le i\le n}} \frac{n!k_iF_{i-1}}{k_1!k_2!\dots k_n!} = n^n\sum_{i=0}^{n-1}F_i. \tag{1}$$

We rewrite the left side of the equation (1), as follows:

$$\sum_{k_1+k_2+\cdots+k_n=n}(k_1F_0+k_2F_1+\cdots+k_nF_{n-1})\times\frac{n!}{k_1!k_2!\dots k_n!}.$$

Now let us consider the set

$$\mathcal{S}=\{(x_1,x_2,\dots,x_n): x_i\in\{F_0,F_1,\dots,F_{n-1}\}, i=1,2,\dots,n\}.$$

For fixed values of $k_i$s with $k_1+k_2+\cdots+k_n=n$, the number of elements of $\mathcal{S}$ in which $F_i$ appears $k_i$ times for $i=1,2,\dots,n$ is $n!/k_1!k_2!\dots k_n!$. Hence

$$\sum_{(x_1,x_2,\dots,x_n)\in\mathcal{S}} x_1+x_2+\cdots+x_n =$$
$$\sum_{k_1+k_2+\cdots+k_n=n}(k_1F_0+k_2F_1+\cdots+k_nF_{n-1})\times\frac{n!}{k_1!k_2!\dots k_n!}.$$

Now, there are $n^{n-1}$ different elements in $\mathcal{S}$ having equal value at the $i$th position for some fixed $i$. Hence, for a fixed $i$,

$$\sum_{(x_1,x_2,\dots,x_n)\in\mathcal{S}} x_i = n^{n-1}\sum_{i=0}^{n-1}F_i, \qquad \text{which implies}$$

$$\sum_{(x_1,x_2,\dots,x_n)\in\mathcal{S}} x_1+x_2+\cdots+x_n = n^n\sum_{i=0}^{n-1}F_i.$$

This establishes the relation (1). Now, one can write $F_i=F_{i+2}-F_{i+1}$ for $i\ge 0$. Hence

$$\sum_{i=0}^{n-1}F_i=\sum_{i=0}^{n-1}(F_{i+2}-F_{i+1})=F_{n+1}-F_1=F_{n+1}-1.$$

Binet's formula (see [1]) gives $F_n=\frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^{n+1}-\left(\frac{1-\sqrt5}{2}\right)^{n+1}\right]$.
Thus $\lim_{n\to\infty}(F_{n+1}-1)^{\frac1n}=\frac{1+\sqrt5}{2}$; also, it is easy to check that $\lim_{n\to\infty}(n/(n!)^{(1/n)})=e$. Hence we have

$$\lim_{n\to\infty}\left(\sum_{\substack{k_1+k_2+\cdots+k_n=n,\\1\le i\le n}}\frac{k_iF_{i-1}}{k_1!k_2!\ldots k_n!}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{(n!)^{\frac{1}{n}}}(F_{n+1}-1)^{\frac{1}{n}}=\frac{e(1+\sqrt{5})}{2}.$$

## References

[1] N. N. Vorobiev, Fibonacci Numbers, Springer, Basel, 2002.

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**MS 93(3-4) 2024 : Problem 3.** (by **Dr. Andrés Ventas**, Santiago de Compostela, Spain).

Let $(x_0, y_0)$ denote the fundamental solution to the negative Pell's equation $x^2 - Dy^2 = -1$ and let $(x, y)$ denote the fundamental solution to the positive Pell's equation $x^2 - Dy^2 = +1$, both of them for the same $D$. Prove that all the infinite solutions to the negative Pell's equation, $x^2 - Dy^2 = -1$ satisfy the following recurrences,

$$x_{n+2} = 2xx_{n+1} - x_n;\ \text{with } x_{-1} = -x_0 \text{ and } x_0 = x_0,$$

$$y_{n+2} = 2xy_{n+1} - y_n;\ \text{with } y_{-1} = y_0,\ \text{and } y_0 = y_0.$$

**Solution:** (by the Proposer)

**Proposition:** If we have $x_0^2 - Dy_0^2 = -1$ then $x_0^2 + Dy_0^2 = x$.

*Proof.* We are going to use Heron's or Newton's method of calculating roots. We begin with the value $\frac{x_0}{y_0}$ and we obtain one more term,

$$\frac{x}{y} = \frac{1}{2}\Big(\frac{x_0}{y_0} + \frac{Dy_0}{x_0}\Big) = \frac{x_0^2 + Dy_0^2}{2x_0y_0}$$

.

Let $N(x, y) = x^2 - Dy^2$ denote the norm of $x + \sqrt{D}y$. We obtain the norm of the new term $N(x, y)$,

$$(x_0^2 + Dy_o^2)^2 - D(2x_0y_0)^2 = x_0^4 + y_0^4 + 2Dx_0^2y_0^2 - 4Dx_0^2y_0^2$$
$$= x_0^4 + x_0^4 - 2Dx_0^2y_0^2 = (x_0^2 - Dy_0^2)^2 = N^2.$$

when $N = -1$, we have $N^2 = 1$ and $(x,\ y) = (x_0^2 + Dy_0^2,\ 2x_0y_0)$. □

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**Problems 4-6** (by **Dr. B. Sury**, ISI, Bengaluru, India). **MS 93(3-4) 2024 : Problem 4.**

Let $Q$ denote the set of all positive integers that are one less than the perfect powers greater than one. That is,

$$Q = \{3, 7, 8, 15, 24, 26, 31, \cdots\} = \{4-1, 8-1, 9-1, 16-1, 25-1, \cdots\}.$$

Evaluate $\displaystyle\sum_{q \in Q} \frac{1}{q}$.

This problem was solved by Dr. Henry Ricardo, Westchester Area Math Circle, New York, USA, Mr. Santanu Panda, Hooghly, West Bengal, India, Dr. Andrés Ventas, Santiago de Compostela, Spain, Dr. Hari Kishan, D. N. College, Meerut, India.

**Solution:** (by the Proposer)

As pointed out by two solvers, this result is known as the Goldbach-Euler theorem, and is the statement $\sum_{k \geq 2} (\zeta(k) - 1) = 1$. Here is a cute proof by Jan Kristian Hauglund. Let $P$ denote the set of all proper powers $\{4, 8, 9, 16, 25, 27, 32, \cdots\}$. The main point to observe is that

$$s := \sum_{n \in P} \frac{1}{n} = \sum_{n \notin P} \left( \frac{1}{n^2} + \frac{1}{n^3} + \cdots \right).$$

These series converge and the right hand side is summed only over $n \geq 2$. Now

$$s = \sum_{n \notin P} \frac{1}{n(n-1)} = 1 - \sum_{n \in P} \frac{1}{n(n-1)} = 1 - \sum_{n \in P} \left( \frac{1}{n-1} - \frac{1}{n} \right)$$
$$= 1 - \sum_{n \in P} \frac{1}{n-1} + s.$$

This gives the required sum $\sum_{n \in P} \frac{1}{n-1} = 1$; interestingly, the value of the sum $s$ seems to be unknown.

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**MS 93(3-4) 2024 : Problems 5.**

For a function $f$ from the set $\mathbb{N}$ of positive integers to itself, define $f^{n+1} = f \circ f^n$ for all $n \geq 1$. For a positive integers $r$, determine all positive integers $d$ such that there exists a function $f$ satisfying $f^r(n) = n + d$ for all $n \in \mathbb{N}$. For such $r, d$ determine the number of such functions.

This problem was solved by Mr. Santanu Panda, Hooghly, West Bengal, India.

**Solution:**( by the Proposer)
We prove that if $r|d$, then the number of maps is $d!/(d/r)!$; and if $r$ does not divide $d$, there are no such maps.
Let $f^r(n) = n + d$ for all $n \in \mathbb{N}$. Then
$f(n+d) = f(f^r(n)) = f^r(f(n)) = f(n) + d$. Hence, $f$ is determined by its values on $D := \{1, 2, \cdots d\}$. So, we define $F : D \to D$ by $F(n) = f(n)$ modulo $d$. Since $f(n-d) = f(n)$ for $n > d$, we have $F^r(n) = n$ for all $n \in D$. Now, for $c \in D$, consider the sequence $\{c_n\}$ where
$c = c_1, F(c_i) = c_{i+1}$; this is periodic whose period is clearly a divisor of $r$. Writing $F(c_i) = c_{i+1} + m_i d$ for non-negative integers $m_i$, we have

$$f^k(c) = c + (m_1 + m_2 + \cdots + m_k)d \ \ \forall\ k.$$

For $k = r$, this gives $c + d = c + (\sum_{i=1}^{r} m_i)d$ which means $m_i = 0$ except for one $i$ for which it is 1. In other words, the period is equal to $r$. Evidently, since the powers of $F$ form a cyclic group acting on $D$, all orbits have cardinality $r$ and, in each orbit, there is an element $t$ with
$f(t) = F(t) + d$, while $f(s) = F(s)$ for the other elements $s$. Thus, we know now that if $r$ does not divide $d$, there are no functions as asserted. Now, let $r|d$. Let $\{c_1, c_2, \cdots, c_r\}$ be an orbit of $F$, where $f(c_i) = c_{i+1}$ for $i \neq r$ and $f(c_r) = c_1 + d$. In this way, the elements of $D$ form $d/r$ sequences, each of length $r$. Note that this can happen in $d!/(d/r)!$ ways. Conversely, if $S$ is a set of $d/r$ sequences of length, say
$S = \{c_1, c_2, \cdots, c_r\}$, we define $f : \mathbb{N} \to \mathbb{N}$ by $f(c_i) = c_{i+1}$ for $i < r$, and $f(c_r) = c_1 + d$, and extend $f$ periodically mod $d$. It is easy to sees that indeed $f^r(n) = n + d$ for all $n \in \mathbb{N}$. So if $r|d$, the number of functions is $d!/(d/r)!$.

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**MS 93(3-4) 2024 : Problems 6.**

Consider the series $\sum_{n\geq 1} \frac{1}{n^3 \sin^2(n)}$. Show that the sequence of partial sums $s_n$ satisfies $s_{354} < 5$ while $s_{355}$ is close to 30. Note that $\pi$ is close to 355/113. Consider the infimum $irr(\pi)$ of all constants $\alpha > 0$ such that there are only finitely many solutions of $|\pi - p/q| \leq \frac{1}{q^\alpha}$. If $irr(\pi) \geq 2.5$, show that the series diverges. It is expected (but unknown) that $irr(\pi) = 2$; if this is so, show that the series converges.

**Solution:** (by the Proposer)

(Condensed from an argument by Wadim Zudilin and David Simmons.) The convergence or divergence of $\sum_n \frac{1}{n^3 \sin^2 n}$ depends on the behavior of $|n \sin n|$ as $n \to \infty$. This depends, in turn, on the irrationality measure of $\pi$. Let us see first why this is so. Indeed, for arbitrary $\epsilon > 0$, one may neglect the terms in the sum where $n|\sin n| \geq n^\epsilon$, as that makes only a convergent contribution. Thus, we need to consider only the series $\sum_{n|\sin n|<n^\epsilon} \frac{1}{n^3 \sin^2 n}$. For such $n$ in the series, if $q = q(n)$ minimizes the distance $|q\pi - n| \leq \pi/2$, then

$$|\sin(q\pi - n)| = |\sin n| < \frac{1}{n^{1-\epsilon}}.$$

Then, using the fact that $\sin x \sim x$ for $x \to 0$, we have for some $a > 0$, that $|q\pi - n| < \frac{a}{n^{1-\epsilon}}$; that is, $|\pi - n/q| < \frac{a}{qn^{1-\epsilon}}$. For all $n$ in the last sum, since $n/q \sim \pi$, we have $|\pi - n/q| < \frac{b}{n^{2-\epsilon}}$ for some $b > 0$ or $|\pi - n/q| < \frac{c}{q^{2-\epsilon}}$ for another constant $c > 0$.

Note in passing that the last inequality holds even in the stronger form $|\pi - n/q| < \frac{1}{q^2}$ for infinitely many pairs $n, q$ by Dirichlet's box principle. Thus, the original series converges if, and only if, the series $\sum_{|\pi-n/q|<b/n^{2-\epsilon}} \frac{1}{n^5|\pi-n/q|^2}$ converges. In the last sum, we could sum over $q$ instead of over $n$ as $n \sim q\pi$. Therefore, we need to consider the convergence of the series $\sum_{q\geq 1} \frac{1}{q^5(\pi-n/q)^2}$ where now $n = n(q)$ is chosen to minimize $|\pi - n/q|$.

Now, if the irrationality constant of $\pi$ is greater than $5/2$, then there are infinitely many pairs $n, q$ satisfying the inequality $|\pi - n/q| < c_0/q^{5/2}$ for some constant $c_0 > 0$. Thus, in the last series $\sum_{q\geq 1} \frac{1}{q^5(\pi-n/q)^2}$, infinitely many terms are bounded below by $1/c_0$ which means this series (and hence, the original series) diverges.

On the other hand, if the irrationality measure of $\pi$ is less than $5/2$ (it is expected to be 2), we have for all but finitely many $n$, an inequality $|\pi - p_n/q_n| \geq c_1/q^{5/2-\epsilon}$, where $p_n/q_n$ is the sequence of convergents of $\pi$. Thus, the series is bounded above by the series $\sum_n \frac{1}{q_n^{2\epsilon}}$. As the $q_n$'s grow at least exponentially fast (this is intuitive but needs a formal proof which we avoid here), the last series converges and so does the original one.

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**Problems 7-8** (by **Dr. Chudamani Pranesachar Anil Kumar**, KREA University, India).

**MS 93(3-4) 2024 : Problem 7.**

Let $\mathbb{F}$ be a field. Let $A, B \subset \mathbb{F}$ be two finite subsets. For any $\xi \in \mathbb{F}^*$, We say $\{(a_1, b_1), (a_2, b_2)\} \subset A \times B$ is a $\xi$-pair in $A \times B$ if $\frac{b_1 - b_2}{a_1 - a_2} = \xi$. Let $F_\xi$ be the set of all $\xi$ pairs in $A \times B$. Let $B - A\xi = \{b - a\xi \mid a \in A, b \in B\}$. A line $L \subset \mathbb{F}^2$ is said to be $k$-rich if $L$ contains exactly $k$ points of the set $A \times B$.

(1) Show that if $\mathbb{F}$ is a finite field then there exists $\xi \in \mathbb{F}^*$ such that

$$| F_\xi | \geq \frac{2}{| \mathbb{F}^* |} \binom{| A |}{2} \binom{| B |}{2}.$$

(2) (Here $\mathbb{F}$ need not be finite). If there are no $k$-rich lines in $A \times B$ with slope $\xi$ for any $k \geq 4$ then show that the number of points in $A \times B$ is bounded below and above as:

$$\min \left( \frac{3}{2} | B - A\xi | + \frac{1}{2} | F_\xi |, | B - A\xi | + | F_\xi | \right) \geq | A || B |$$
$$\geq | B - A\xi | + \frac{2}{3} | F_\xi | .$$

**Solution:**(by the Proposer)

We prove (1). Consider the following map.

$$\phi : A \times B \times A \times B \longrightarrow \mathbb{F} \cup \{\infty\} \cup \{ \text{ undefined } \}$$

defined by

$$\phi(a_1, b_1, a_2, b_2) = \frac{b_1 - b_2}{a_1 - a_2}.$$

We observe that the fibre of $\phi$ corresponding to $\xi$ has cardinality $2 | F_\xi |$. So we have

$$\sum_{\xi \in \mathbb{F}^*} 2 | F_\xi | = | A |^2 | B |^2 - | A |^2 | B | - | A || B |^2 + | A || B |$$
$$= 4 \binom{| A |}{2} \binom{| B |}{2}$$
$$\Rightarrow \sum_{\xi \in \mathbb{F}^*} | F_\xi | = 2 \binom{| A |}{2} \binom{| B |}{2}.$$

So if the field $\mathbb{F}$ is finite then we have that there exists $\xi \in \mathbb{F}^*$ such that

$$| F_\xi | \geq \frac{2}{| \mathbb{F}^* |} \binom{| A |}{2} \binom{| B |}{2}.$$

We prove (2). Consider all the lines with slope $\xi$ in the affine plane $\mathbb{F}^2$. The idea is to find a way to count the minimum number of lines with slope $\xi$ required which in all pass through all the points of $A \times B$.
If $\xi \notin image(\phi) = S$ the slope set of $A \times B$ then the number of lines required with slope $\xi$ passing through all of $A \times B$ is $\mid A \mid\mid B \mid$. If the slope is zero or $\infty$ then we get that the number of lines required with slope 0 or $\infty$ is $\mid B \mid$ or $\mid A \mid$ respectively.
For a general $\xi \in \mathbb{F}^*$ consider the map $\sigma_\xi$ defined below
$\sigma_\xi : F_\xi \longrightarrow B - A\xi \subseteq \mathbb{F}$ defined by
$\sigma_\xi(\{(a_1, b_1), (a_2, b_2)\}) \longrightarrow b_1 - a_1\xi = b_2 - a_2\xi$.
Let $m_k^\xi$ denote the number of $k$-rich lines with slope $\xi$ for $k \geq 1$ containing points of $A \times B$ then we have

$$\mid B - A\xi \mid = m_1^\xi \binom{1}{0} + m_2^\xi \binom{2}{0} + m_3^\xi \binom{3}{0} + m_4^\xi \binom{4}{0} \dots$$

$$\mid A \mid\mid B \mid = m_1^\xi \binom{1}{1} + m_2^\xi \binom{2}{1} + m_3^\xi \binom{3}{1} + m_4^\xi \binom{4}{1} \dots$$

$$\mid F_\xi \mid = m_2^\xi \binom{2}{2} + m_3^\xi \binom{3}{2} + m_4^\xi \binom{4}{2} + \dots$$

If there are no $k$-rich slope $\xi$ lines for $k \geq 4$ then inequalities follow by solving for $m_1^\xi, m_2^\xi, m_3^\xi$. We have upon solving:
The number of one-rich slope $\xi$ lines

$$m_1^\xi = 3 \mid B - A\xi \mid + \mid F_\xi \mid -2 \mid A \mid\mid B \mid.$$

The number of two-rich slope $\xi$ lines

$$m_2^\xi = 3 \mid A \mid\mid B \mid -3 \mid B - A\xi \mid -2 \mid F_\xi \mid.$$

The number of three-rich slope $\xi$ lines

$$m_3^\xi = \mid B - A\xi \mid - \mid A \mid\mid B \mid + \mid F_\xi \mid.$$

The inequalities follow from the fact that $m_i^\xi \geq 0$ for $i = 1, 2, 3$. This solves the problem.

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**MS 93(3-4) 2024 : Problems 8.**

Let $n$ be a positive integer and $S=\{1,2,\cdots,n\}$. We say a partition of $S$ say $\mathcal{X}=\{S_1,S_2,\cdots,S_k\}$ is split adjacent to a partition $\mathcal{Y}=\{T_1,T_2,\cdots,T_l\}$ of $S$ if the following condition holds. There exist $i,r,s$ such that $1\le i\le l, 1\le r\ne s\le k$ and $T_i=S_r\cup S_s$ and $\mathcal{Y}\backslash\{T_i\}=\mathcal{X}\backslash\{S_r,S_s\}$, that is, exactly some two parts of $\mathcal{S}$ combine together to give a part of $\mathcal{T}$ and the remaining parts are the same. We denote this by

$$\mathcal{Y}\longrightarrow\mathcal{X}.$$

A path from the partition $\{S\}$ of $S$ where there is only one part to the finest partition $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$ of $S$ where there are $n$ parts is a sequence of split adjacent partitions $\mathcal{Y}_i$ of $S$ of the form

$$\mathcal{Y}_1=\{S\}\longrightarrow\mathcal{Y}_2\longrightarrow\cdots\longrightarrow\mathcal{Y}_{t-1}\longrightarrow\mathcal{Y}_t=\{\{1\},\{2\},\{3\},\cdots,\{n\}\}.$$

(1) Enumerate the number of paths from $\{S\}$ to $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$.
(2) Let $\mathcal{U}=\{A_1,A_2,\cdots,A_m\}$ be a fixed partition of $S$. Enumerate the number of paths from $\{S\}$ to $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$ passing through $\mathcal{U}$.

**Solution:** (by the Proposer)
We show (1). Note that in any path $\mathcal{Y}_1=\{S\}\longrightarrow\mathcal{Y}_2\longrightarrow\cdots\longrightarrow\mathcal{Y}_{t-1}\longrightarrow\mathcal{Y}_t=\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$ the cardinality of $\mathcal{Y}_i=i$ and $t=n$. Also since $\mathcal{Y}_{i-1}$ is obtained from $\mathcal{Y}_i$ by taking the union of some two parts of $\mathcal{Y}_i$ the total number of possibilities of $\mathcal{Y}_{i-1}$ given $\mathcal{Y}_i$ is $\binom{i}{2}$. Hence the total number of paths is

$$\binom{n}{2}\binom{n-1}{2}\cdots\binom{3}{2}\binom{2}{2}=\frac{n(n-1)}{2}\frac{(n-1)(n-2)}{2}\cdots\frac{3*2}{2}\frac{2*1}{2}$$
$$=\frac{n!(n-1)!}{2^{n-1}}.$$

We show (2). In a similar way the number of paths from $\mathcal{U}$ to $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$ is

$$\frac{\mid A_1\mid!(\mid A_1\mid-1)!}{2^{|A_1|-1}}\frac{\mid A_2\mid!(\mid A_2\mid-1)!}{2^{|A_2|-1}}\cdots\frac{\mid A_m\mid!(\mid A_m\mid-1)!}{2^{|A_m|-1}}$$

and the total number of paths from $\{S\}$ to $\mathcal{U}$ is

$$\frac{m!(m-1)!}{2^{m-1}}.$$

Hence the number of paths from $\{S\}$ to $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$ passing through $\mathcal{U}$ is

$$\frac{m!(m-1)!}{2^{m-1}}\frac{|\,A_1\,|!(|\,A_1\,|-1)!}{2^{|A_1|-1}}\frac{|\,A_2\,|!(|\,A_2\,|-1)!}{2^{|A_2|-1}}\cdots\frac{|\,A_m\,|!(|\,A_m\,|-1)!}{2^{|A_m|-1}}.$$

This solves the problem.

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**Solutions for problems from MS 94(1-2) 2025**

**MS 94(1-2) 2025 : Problem 1.** (proposed by **Dr. Shpetim Rexhepi** and **Dr. Ilir Demiri**, Mother Teresa University, Skopje, North Macedonia).

Prove the following inequalities:

(i) For any $a, b > 1$, $\sqrt{\ln a\ \ln b} - 2\ln(a+b) + \ln\sqrt{ab} \leq -2\ln 2.$

(ii)

$$\sqrt{\ln\sum_{i=0}^{1012}e^{2i}}\sqrt{\ln\sum_{i=0}^{1012}e^{2i+1}} + \ln\sqrt{\sum_{i=0}^{1012}e^{2i}\sum_{i=0}^{1012}e^{2i+1}} \leq -\ln 4 + 2\ln\sum_{i=0}^{2025}e^{i}.$$

where $e$ is Euler's number.

This problem was solved by Dr. Carlos A. Cardozo Delgado, Pontificia Universidad Javeriana, Bogata, Columbia, Dr. Henry Ricardo, Westchester Area Math Circle, New York, USA, Mr. Santanu Panda, Hooghly, West Bengal, India, Dr. Andrés Ventas, Santiago de Compostela, Spain, Dr. Hari Kishan, D. N. College, Meerut, India.

**Solution:** (by Dr. Hari Kishan, D. N. College, Meerut, India)

(i) We have to prove that for any $a, b > 1$,

$$\sqrt{\ln a\ \ln b} - 2\ln(a+b) + \ln\sqrt{ab} \leq -2\ln 2. \tag{1}$$

We know that for any two positive numbers $a, b$, their $GM \leq AM$.
Therefore ,

$$\sqrt{ab} \leq \frac{a+b}{2} \qquad \text{which implies} \qquad \ln(2\sqrt{ab}) \leq \ln(a+b).$$

Thus we have

$$-\ln(a+b) \leq -\ln(2\sqrt{ab}). \tag{2}$$

Also,

$$\sqrt{\ln a \ln b} \leq \frac{\ln a + \ln b}{2} = \frac{\ln(ab)}{2}. \tag{3}$$

Now consider the left hand side of (1),

$$\begin{aligned}
LHS =& \sqrt{\ln a\ \ln b} - 2\ln(a+b) + \ln\sqrt{ab} \\
\leq& \frac{1}{2}\ \ln(ab) - 2\ln(2\sqrt{ab}) + \ln\sqrt{ab} \quad [\text{Using (2) and (3)}] \\
=& \frac{1}{2}\ln(ab) - 2\ln 2 - \ln(ab) + \frac{1}{2}\ln(ab) = -2\ln 2 = RHS.
\end{aligned}$$

(ii) Next we prove

$$\sqrt{\ln\sum_{i=0}^{1012} e^{2i}}\sqrt{\ln\sum_{i=0}^{1012} e^{2i+1}} + \ln\sqrt{\sum_{i=0}^{1012} e^{2i}\sum_{i=0}^{1012} e^{2i+1}} \leq -\ln 4 + 2\ln\sum_{i=0}^{2025} e^{i}.$$

Let $a = \sum_{i=0}^{1012} e^{2i},\ b = \sum_{i=0}^{1012} e^{2i+1}$. Now proving (ii) is same as proving

$$\sqrt{\ln a\ \ln b} + \ln(\sqrt{ab}) \leq -\ln 4 + 2\ln\sum_{i=0}^{2025} e^{i} \tag{4}$$

. Now using $AM \leq GM$ for the above $a, b$ and taking log on both sides we obtain

$$\ln\sqrt{a\,b} \leq \ln\frac{a+b}{2} = \ln\left(\frac{\sum_{i=0}^{2025} e^{i}}{2}\right) = \ln\left(\frac{e^{2026}-1}{e-1}\right) - \ln 2 \tag{5}$$

Also,

$$\begin{aligned}
\sqrt{\ln a\ \ln b} \leq \frac{\ln a + \ln b}{2} &= \frac{\ln\sum_{i=0}^{1012} e^{2i} + \ln\sum_{i=0}^{1012} e^{2i+1}}{2} \\
&= \frac{1}{2}\ln\left(\frac{e^{2026}-1}{e^{2}-1}\right) + \frac{1}{2}\ln\left(e\frac{e^{2026}-1}{e^{2}-1}\right) \\
&= \frac{1}{2}\ln\left(e\left(\frac{e^{2026}-1}{e^{2}-1}\right)^{2}\right) \\
&= \frac{1}{2}\ln e + \ln\left(\frac{e^{2026}-1}{e^{2}-1}\right). \qquad (6).
\end{aligned}$$

Now, using (5) and (6), left hand side of (4) is

$$\begin{aligned} LHS =& \sqrt{\ln a \, \ln b} + \ln(\sqrt{ab}) \\ \leq& \frac{1}{2}\ln e + \ln\left(\frac{e^{2026}-1}{e^2-1}\right) + \ln\left(\frac{e^{2026}-1}{e-1}\right) - \ln 2 \\ =& 2\ln\left(\frac{e^{2026}-1}{e-1}\right) - \ln(e+1) - \ln 2 + \frac{1}{2}\ln e \\ =& 2\ln\sum_{i=0}^{2025} e^i - \ln 2 - \ln\left(\frac{e+1}{\sqrt{e}}\right) \\ \leq& 2\ln\sum_{i=0}^{2025} e^i - \ln 4 = RHS, \quad \text{as} \quad (e+1)/\sqrt{e} < 2. \end{aligned}$$

This completes the proof.

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**Problems 2-4** (proposed by **Dr. B. Sury**, ISI, Bengaluru, India).

**MS 94(1-2) 2025 : Problem 2.**

Recall that a Wieferich prime is a prime number $p$ that satisfies the congruence $2^{p-1} \equiv 1 \bmod p^2$. To this day, the only two Wieferich primes to have been found are 1093 and 3511. Also, recall that Mersenne primes are primes of the form $2^n - 1$ and Fermat primes are primes of the form $2^n + 1$. Prove that neither a Mersenne prime nor a Fermat prime can be a Wieferich prime.

This problem was solved by Mr. Santanu Panda, Hooghly, West Bengal, India, Dr. Hari Kishan, D. N. College, Meerut, India.

**Solution:** (by Mr. Santanu Panda)

Let $p = 2^m - 1$ be a Mersenne prime. Let $2^x \equiv 1 \pmod{p}$ and $x = mq + r$, where $0 \leq r < m$. Then

$$\begin{aligned} 2^{mq+r} \equiv 1 \pmod{p} &\Rightarrow (2^m)^q \cdot 2^r \equiv 1 \pmod{p} \\ &\Rightarrow 2^r \equiv 1 \pmod{p} \quad [\text{since } 2^m \equiv 1 \pmod{p}]. \end{aligned}$$

As $r < m$, $2^r - 1 < p$, so $r = 0$. So, $2^x \equiv 1 \pmod{p} \Rightarrow m$ is a factor of $x$ . Now, if possible, let $2^{p-1} \equiv 1 \pmod{p^2}$. So, $m$ must be a factor of $p-1$; let $p - 1 = ma$. Then

$$\begin{aligned} 2^{ma} \equiv 1 \pmod{p^2} &\Rightarrow 2^{ma} - 1 \equiv 0 \pmod{p^2} \\ \Rightarrow (2^m - 1)\left(1 + 2^m + 2^{2m} + \cdots + 2^{m(a-1)}\right) &\equiv 0 \pmod{p^2}. \end{aligned}$$

Since $2^m - 1 = p$, we get:

$$p \cdot (1 + 1 + \cdots + 1[a \text{ times}]) \equiv 0 \pmod{p^2} \quad \Rightarrow a \equiv 0 \pmod{p}.$$

As $a$ is a factor of $p-1$, $0 < a < p$, that is, $a \not\equiv 0 \pmod{p}$. Hence, a Mersenne prime cannot be a Wieferich prime.

**For the second part,**
Let $p = 2^n + 1$ be a Fermat prime number. Then $n = 2^m$ for some $m \geq 0$, obviously $m < n$. Let, if possible ,

$$2^{p-1} - 1 = 2^{2^n} - 1 = \prod_{i=0}^{n-1} \left(2^{2^i} + 1\right).$$

So, $p = 2^{2^m} + 1$ is a factor of $2^{p-1}$–1. If $p^2$ is a factor of $2^{p-1} - 1$, and since $p$ is a prime number, $p$ must be a factor of some $2^{2^k} + 1$ where $m < k < n$. Now, let $k$ be any positive integer in the range $(m, n)$:

$$\begin{aligned} 2^{2^k} + 1 &= \left(2^{2^m}\right)^{2^{k-m}} + 1 \\ &\equiv (-1)^{2^{k-m}} + 1 \pmod{p} \\ &\equiv (1+1) \pmod{p} \quad [\text{ since } 2^{k-m} \text{ is even as } k > m\ ] \\ &\equiv 2 \pmod{p}. \end{aligned}$$

So, for all $m < k < n$, $2^{2^k} + 1 \equiv 2 \pmod{p}$. That is, there exists no $k$ in the range $(m, n)$ such that $p$ is a factor of $2^{2^k} + 1$. Hence, a Fermat prime cannot be a Wieferich prime.

**Remark.** The only two Wieferich primes to have been found to this day are 1093 and 3511. Note also that

$$1093 = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6;$$

that is, the base 3 expansion consists entirely of 1's. In this sense, it is analogous to a Mersenne prime, which is a prime number whose base 2 expansion consists entirely of 1's. Note also that the base 2 expansion of 1093 is $(10001000101)_2$. Thus, the 1's occur at the 1st, 3rd, 7th and 11th places (from the right). Note that the 1's are NOT occurring in places in an arithmetic progression. A similar thing is true for Fermat primes. Problem 2 actually generalizes to the following question which may be of interest:

**Problem 2 generalized.** *Let $p$ be a prime whose expression in a base $b > 1$ is of the form*

$$1 + b^k + b^{2k} + \cdots + b^{nk}$$

*for some $n, k \geq 1$. Then, $b^{p-1} \equiv 1 + \frac{p-1}{(n+1)k}(b^k - 1)p \not\equiv 1 \bmod p^2$.*

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**MS 94(1-2) 2025 : Problem 3.**

An olympiad examination has $N$ participants. They work on six problems, each of which has three possible answers. At the conclusion of the examination, the judges find that for each pair of students, the number of problems that both have the same answers is either 0 or 2. What is the largest possible value of $N$?

This problem was solved by Mr. Santanu Panda, Hooghly, West Bengal, India.

**Solution:** (by Mr. Santanu Panda)

The largest possible value of $N$ is 18. Let the questions be numbered from 1 to 6, and each question has answer choices in the set $\{1, 2, 3\}$. At most, three students can exist so that no pair of them has the same answer for any question. Since there are only three answer choices per question, by the pigeonhole principle, a fourth student would necessarily repeat an answer with at least one of the previous three students. Suppose that there are at least three students such that all have the same answers for two specific questions. Without loss of generality, assume that these are questions 1 and 2 and that each of those students answers them as $(1, 1)$. If a fourth student answers question 1 with 1, then with each of the first three students, he/she will match on exactly one of the remaining 5 questions. This implies that there are at least two questions in which his/her answer differs from all three others, which is not possible. Similarly, there cannot exist a fourth student who answered question 2 with 1. So, we may conclude that if there are three students and for two specific questions and their answers are identical, then there cannot exist a fourth one who answers any of the two questions the same as those three students.

We explore all possible configurations where there are 3 students, each of them answers the first two questions as $(1, 1)$ and also the configuration does not violate the given conditions. In each configuration, we will try to maximize the possible different answer pairs for the first two questions and also to maximize the number of students per answer pair without violating the given conditions.

| **Case** | **Answers for Q1 and Q2** | **Number of Students** |
|---|---|---|
| 1 | 11, 22, 33 | 3 each (Total = 9) |
| 2 | 11 (3), 22 (2), 23 (2), 32 (2), 33 (2) | Total = 11 |

The remaining configurations are equivalent to them or their subsets. Suppose that there are no such specific student trips. Then, for two specific questions and for each possible answer pair for them, there can be at most two students. The number of possible answer pairs for two questions is $3 \times 3 = 9$ . Each such pair can appear at most twice. Therefore, $N \leq 2 \times 9 = 18$. If there exists at least a valid solution set for $N = 18$, we can conclude that the largest value of N is 18.

**Example of a valid solution set for $N = 18$:**

Let $f(i, j)$ denote the answer of the $i$-th student to the $j$-th question. For $1 \leq i \leq 6$ and $1 \leq j \leq 6$, the answer matrix is given as follows:

| $j \backslash i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 2 | 3 | 3 |
| 3 | 1 | 2 | 1 | 3 | 2 | 3 |
| 4 | 1 | 2 | 3 | 1 | 3 | 2 |
| 5 | 1 | 3 | 2 | 3 | 1 | 2 |
| 6 | 1 | 3 | 3 | 2 | 2 | 1 |

For students $7 \leq i \leq 18$, define:

$$f(i, j) \equiv (f(i-6, j) + 1) \pmod 3 \text{ and } f(i, j) \in \{1, 2, 3\}.$$

This construction satisfies all the conditions given and achieves the maximum value $N = 18$.

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**MS 94(1-2) 2025 : Problem 4.**

Show that there are infinitely many numbers of the form $4^n$ which have an odd digit in their usual decimal expansion.

This problem was solved by Dr. Hari Kishan, D. N. College, Meerut, India, Mr. Santanu Panda, Hooghly, West Bengal, India, Mr. A. Prabha Roy, IISER, Mohali, India, Dr. Andrés Ventas, Santiago de Compostela, Spain.

**Solution:** (by the Proposer)
The last two digits of powers of 4 give the sequence repeated ad infinitum:

$$04, 16, 64, 56, 24, 96, 84, 36, 44, 76$$

Thus note that if a certain power of 4 ends in two even digits, the next power has an odd digit in the tens place. Hence, arbitrary large powers of 4 have an odd digit.

**Remark.** The above idea to consider powers of 4 modulo 100 also reduces to considering powers of 4 modulo 25, and using Euler's congruence. Correct solutions along the same ideas has been given by Arka Prabha Roy and Santanu Panda.

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