SOME RECENT PROGRESS IN ANALYTIC NUMBER THEORY

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1. Introduction

Let me start by expressing my gratitude to the council and the members of Indian Mathematics society for this opportunity of delivering the presidential address.

Many important issues have been discussed earlier by the past presidents in their address. This includes:-

(a) Mathematics in the post independence India
(b) View of mathematics by our society and what our role could be
(c) The role of history in learning and teaching of mathematics
(d) Road to mathematical Sciences in India
(e) Prizes, recognations and promotion of mathematics
(f) The universal appeal of mathematics
(g) The importance of mathematics at the school level
(h) Unsinkable mathematics that is sinking
(i) The mathematics education in india

I will urge the students here to go through these articles which have been published in the journal Math Student of Indian Math. society.

Today I will confine myself to some recent progress in Analytic Number theory.

2. sphere packing

Let us consider the Euclidean space $\mathbb{R}^d$ with standard L_2 norm and fill it up with congruent balls of radius r. We assume that the balls do not intersect except along the boundaries. What is the best way to arrange so that maximum number of balls can be packed and what is the density $c(d)$ of the part that is covered by the balls. You may intrepret this as follows :-
Take a big ball of radius R and see the density covered. Take $\limsup$ as $R \to \infty$.

In each dimension, we know some way of placing the balls and this gives a lower bound for $c(d)$. The problem is solved if we can get a matching upper bound. The case $n = 1$ and

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$n = 2$ are clear. The case $n = 3$ is known as Kepler's problem and was solved by T.Hales. Practically nothing was known till 2003 for $n \geq 4$.

In 2003, Cohn and Elkies developed an upper bound. This upper bound turned out to be tantalisingly close to the lower bound known for the case $d = 8$ and $d = 24$.

Finally Maryana Viazovska settled the case $d = 8$ in 2017 and soon thereafter also settled with her coauthors, the case $d = 24$.

The students should undertake to read the articles at least to understand why this comes under number theory.

3. Small difference between primes

We shall start with a question on primes. We know that the number of primes upto N is around $\frac{N}{\log N}$. Thus, the average value of the difference between two consecutive primes around N is $\log N$. Is it possible there are consecutive primes which are much smaller than $\log N$ or consecutive primes which are larger than $\log N$. Let us try with few examples. We see that 29 and 31 are consecutive primes with a difference 2. Other examples are $(101, 103)$ and $(107, 109)$. As of now, the largest such pair known to us is $2996863034895 2^{1290000} \pm 1$ which has 388, 342 decimal digits. It was a part of PrimeGrid's Sophie Germain Prime Search and was discovered in September 2016 by Tom Greer of the United States using an Intel(R) Xeon(R) CPU E5-2623 v3 @ 3GHz with 32 GB RAM running Windows 10 Professional. This computer, using LLR, took approximately 23 minutes to complete the primality tests of both primes.

The Twin prime conjecture asserts that there are infinitely many such examples. In fact a conjecture of Hardy and Littelwood asserts that the number of prime twins upto N is around $\frac{cN}{(\log N)^2}$ for a constant $c > 0$.

This conjecture dates back thousands of years. Yet, not much progress could be made on this until recently. For example, in the year 2007, all that was known is that that there are many examples where the difference between consecutive primes is at most $0.25 \times \log N$

The first breakthrough in this problem is due to Goldston, Pintz and Yildrim in 2009 followed by Yitang Zhang in 2014.

Finally in 2015, James Maynard announced the startling result that there are infinitely many examples where consecutive primes differ by a constant less than 600. In fact he proved the following stronger statement :-
Let m be a natural number. Then there exists a constant c depending only on m such that, for infinitely many integers N, the interval $[N, N + c]$ contains atleast m primes.

This was a triumph for Sieve Methods developed by Brun, Davenport, Selberg, Iwaniec and others.

Learning more about this will also lead you to the Polymath project, a fascinating tale of crowdsourcing and online collaborations in modern mathematics, leading the number appearing in Maynard's work to go down from 600 to 246.

4. Ternary Goldbach problem

Goldbach's conjecture states that every even number greater than or equal to 6 can be written as a sum of two prime numbers. This occurs in a letter from Goldbach to Euler in 1742. In the margin of the letter, he also states a weaker conjecture known as the Ternary Goldbach problem. This states that every odd integer bigger than or equal to 7 is a sum of three primes numbers.

It was observed by Hardy and Littlewood in 1920s that such problems can be attacked by a variant of circle method, a technique developed by Hardy and Ramanujan for tackling the asymptotic formula for the partition function.

Regarding binary Goldbach conjecture, it was proved in 1975 by H.L. Montgomery and R.C. Vaughan that "most" even numbers are expressible as a sum of two prime numbers.

Regarding the ternary conjecture, it was proved by Vinogradov that every sufficiently large odd number (which has more than 1347 digits) is a sum of three prime numbers. In 1997, Deshouillers, Effinger, te Riele, Zinoviev proved the conjecture for every odd number assuming Riemann Hypothesis.

In 2013, Harald Helfgott announced an unconditional proof of Ternary Goldbach conjecture.

5. Arithmetic progressions in primes

One would believe that if you take a "large" subset of natural numbers, then the set would mimic the properties of all natural numbers. In this connection we have the following conjecture of Erdős.

We say that $A \subset \{1 \ldots, N\}$ has positive upper density if there exists $\alpha > 0$ such that $\{a \in A : a \geq N\}$ has more than αN elements for infinitely many natural numbers N.

Then the conjecture states that given $k \geq 3$, if A has a positive upper density, then it has a k-term arithmetic progression.

After the initial work for $k = 3$ by Roth and $k = 4$ by Szemerédi, the general case was settled by Szemerdi in 1975. Later different proofs were given by Furstenberg and Gowers.

In fact Erdős conjectured more. He conjectured that instead of assuming positive upper density, it is sufficient to assume that the sum of the reciprocals of the elements of A diverges.

A classic example of a set A which is not of positive density and the reciprocals diverge is given by the set of prime numbers.

In 2004, Terence Tao and Ben Green proved the remarkable result that primes contain a k term arithmetic progression.

6. Conclusion

Some great progress has happened in the last twenty years in Number theory and the other branches of Mathematics. Let me conclude by reminding the young students here that they live in a period where exciting developments are taking place and they should utilise this opportunity well.

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