ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 93, Nos. 1-2, January - June (2024)
(Issued: May, 2024)

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
Website: https://indianmathsoc.org

THE MATHEMATICS STUDENT

Edited by G. P. Youvaraj

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Published by Prof. M. M. Shikare for the Indian Mathematical Society, type set by Prof. G. P. Youvaraj, Former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, Chepauk, Chennai - 600005, India and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune-411041 (India).

Printed in India.

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024)

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REMEMBERING THE LEGACY OF PROFESSOR J. R. PATADIA: A PILLAR OF DEDICATION TO THE IMS

PROF. M. M. SHIKARE


Prof. J. R. Patadia

In memoriam of the late Professor J. R. Patadia, esteemed former faculty member at Maharaj Sayajirao University, Vadodara, Gujarat, and former Editor-in-Chief of The Mathematics Student, it is with heavy hearts that we acknowledge his passing on November 1, 2023, after battling illness. I had the privilege of hosting Prof. Patadia and his daughter at my home in Pune back in June 2023, where we shared pleasant conversations on various familial and IMS-related matters. His vitality during our meeting gave no indication of the forthcoming loss we would all mourn. As a tribute to his memory, I reflect on his profound contributions to the Indian Mathematical Society (IMS).

Among his myriad achievements, Prof. Patadia's stewardship as Editor-in-Chief of The Mathematics Student and his instrumental role in developing and maintaining the IMS website stand out. Assuming the editorship in 2004, he dedicated 15 years to this role, despite initial reluctance. Prof. Patadia inherited a publication in disarray, epitomized by the disconcerting delay in releasing volumes; however, through his relentless efforts, including addressing copyright hurdles and transitioning to online publishing, he not only cleared the backlog but also ensured timely releases. The arduous task of typesetting, once outsourced, became his personal endeavor, showcasing his dedication to the cause. His selfless act of reimbursing the IMS for his typesetting services underscores his commitment.

Prof. Patadia's editorial acumen was not confined to logistics but extended to enhancing the scholarly quality of The Mathematics Student. His discerning decision to discontinue abstracts of conference papers elevated the journal's stature as a research publication, a sentiment echoed by esteemed mathematicians like Professor Bruce Berndt. Under his guidance, the journal's format evolved into a model of international standards, featuring distinct sections catering to various academic interests.

His tireless advocacy for IMS extended beyond print to digital realms, exemplified by his pivotal role in securing recognition for The Mathematics Student in esteemed databases like SCOPUS. Moreover, his meticulous curation of the IMS website, featuring historical archives and vital updates, ensures his legacy endures.

Outside his professional endeavors, Prof. Patadia's unwavering pursuit of justice, as exemplified by his protracted legal battle for teachers' rights, speaks volumes of his character. Even in the face of personal adversity, his altruism shone through, epitomized by his generous donation to the IMS amid unresolved pension disputes.

In honoring Prof. Patadia's memory, we acknowledge the profound loss to the IMS community. His refusal of a dedicated volume in his honor epitomizes his humility and underscores his legacy as a cherished friend, esteemed office-bearer, and unwavering supporter of the IMS.

May his soul find eternal peace.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January-June (2024), 03–13

A KDV-TYPE EQUATION : LAX PAIR AND TRAVELLING WAVE SOLUTION

R. RANGARAJAN AND BHARATHA K.

(Received : 13 - 08 - 2021 ; Revised : 02 - 02 - 2023)

Abstract. In the present paper, a KdV-type equation is derived with the help of scaling technique applied to the dependent variable as well as the Lax pair of the KdV equation. The equation is shown to admit a Miura-type transformation which yields an m-KdV-type equation. Standard travelling wave solution is obtained by direct integration method as well as Adomian decomposition series method. Also lines on which nonlinear and dispersive terms get equated are computed.

1. Introduction

KdV equation is one of the model equations well studied in the field of Partial Differential Equations, Nonlinear Dynamics and their applications. For more details please refer [1, 2, 5, 7, 8, 12]. The KdV equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1}$$

is an integrable equation with the standard Lax pair:

$$L = \frac{\partial^2}{\partial x^2} - uI \text{ and } M = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}I, \tag{1.2}$$

where I is the identity operator.

The equation can be derived in the form

$$\frac{\partial L}{\partial t} + (LM - ML) = 0,$$

where one has to make use of the expansion

$$\frac{\partial}{\partial x} \times (f(x,t)I) = f(x,t)\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}I.$$

2010 Mathematics Subject Classification: 35C07, 35C10, 35Q53
Key words and phrases: KdV-type equation, travelling wave solution, the Adomian decomposition series

For more details on Lax pair please refer [6, 9].

It is well known result that the Miura transformation [10] $u = v^2 + \dfrac{\partial v}{\partial x}$ when applied to KdV equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \text{ yields}$$

$$2v\left[\frac{\partial v}{\partial t} - 6v^2\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right] + \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} - 6v^2\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right] = 0.$$

The modified equation is called mKdV equation.

Given an evolution equation, which is a typical nonlinear PDE of the form

$$\frac{\partial u}{\partial t} + F\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \cdots\right) = 0,$$

the travelling wave solution is given by

$$u(x,t) = U(z), z = x - ct, \ c > 0,$$

where c is called velocity of the wave. The term $U(z)$ can be computed as exact or approximate solution of the corresponding nonlinear ODE. There are many travelling wave solutions such as solitary waves, kinks, periodic waves, compactons and so on. For more details please refer [12].

Adomian decomposition series guided by linear and nonlinear terms of a differential or integral equation provides a lot of flexibility to use series with powers of well known elementary functions such as identity function, exponential function and so on. Many times it is more simpler than Maclaurin series. For more details please refer [3, 4, 11, 12].

In the present paper, section 2 mainly has the following results:

Theorem 1.1. *The KdV-type equation*

$$\frac{\partial u}{\partial t} - \frac{3}{2}\alpha u\frac{\partial u}{\partial x} + \frac{\beta}{4}\frac{\partial^3 u}{\partial x^3} = 0 \tag{1.3}$$

has Lax pair

$$L = \frac{\partial^2}{\partial x^2} - \frac{\alpha}{\beta}uI \ \textit{and}\ M = -\beta\frac{\partial^3}{\partial x^3} + \frac{3}{2}\alpha u\frac{\partial}{\partial x} + \frac{3}{4}\alpha\frac{\partial u}{\partial x}I, \tag{1.4}$$

where α, β are non-zero real parameters.

Corollary 1.2. *For $\alpha = \beta = 4$ in the Theorem 1.1, one gets back the original KdV equation* (1.1) *and its Lax pair* (1.2).

Theorem 1.3. *The Miura-type transformation* $u = \dfrac{\beta}{\alpha}\left(v^2 + \dfrac{\partial v}{\partial x}\right)$ *transforms KdV-type equation* (1.3) *into mKdV-type equation*

$$\frac{\partial v}{\partial t} - \frac{3}{2}\beta v^2 \frac{\partial v}{\partial x} + \frac{\beta}{4}\frac{\partial^3 v}{\partial x^3} = 0, \tag{1.5}$$

where α, β *are non-zero real parameters.*

By employing direct integration in section 3, the exact solitary wave solutions are derived for KdV-type equation (1.3) as well as mKdV-type equation (1.5). In section 4, hypergeometric expansions of solutions of (1.3) and (1.5) equations are obtained. In the ensuing section, Adomian decomposition technique is applied to derive solutions in the series form for both (1.3) and (1.5). In the concluding section, balancing of nonlinear terms with dispersive terms are discussed.

2. Proofs of the Main Theorems

In order to do scaling manipulations, it is convenient to take the dependent variable $\tilde{u}$ and the Lax pair $(\tilde{L}, \tilde{M})$ so that the KdV equation

$$\frac{\partial \tilde{u}}{\partial t} - 6\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial^3 \tilde{u}}{\partial x^3} = 0$$

has the Lax-Pair

$$\tilde{L} = \frac{\partial^2}{\partial x^2} - \tilde{u}I \text{ and}$$

$$\tilde{M} = -4\frac{\partial^3}{\partial x^3} + 6\tilde{u}\frac{\partial}{\partial x} + 3\frac{\partial \tilde{u}}{\partial x}I.$$

The scalings $u = k\tilde{u},\ L = l\tilde{L}$ and $M = m\tilde{M}$ where k, l, m are nonzero real parameters help to derive a KdV-type equations as follows:

$$L = l\frac{\partial^2}{\partial x^2} - kluI,$$

$$M = -4m\frac{\partial^3}{\partial x^3} + 6kmu\frac{\partial}{\partial x} + 3km\frac{\partial u}{\partial x}I,$$

$$\frac{\partial L}{\partial t} = -kl\frac{\partial u}{\partial t}I,$$

$$\begin{aligned} LM = lm&\left[-4\frac{\partial^5}{\partial x^5} + 10ku\frac{\partial^3}{\partial x^3} + 15k\frac{\partial u}{\partial x}\frac{\partial^2}{\partial x^2} + 12k\frac{\partial^2 u}{\partial x^2}\frac{\partial}{\partial x} - 6k^2u^2\frac{\partial}{\partial x}\right] \\ &+ klm\left[3\frac{\partial^3 u}{\partial x^3} - 3ku\frac{\partial u}{\partial x}\right], \end{aligned}$$

$$ML = ml\left[-4\frac{\partial^5}{\partial x^5} + 10ku\frac{\partial^3}{\partial x^3} + 15k\frac{\partial u}{\partial x}\frac{\partial^2}{\partial x^2} + 12k\frac{\partial^2 u}{\partial x^2}\frac{\partial}{\partial x} - 6k^2u^2\frac{\partial}{\partial x}\right]$$
$$+ mlk\left[4\frac{\partial^3 u}{\partial x^3} - 9ku\frac{\partial u}{\partial x}\right],$$

and

$$\frac{\partial L}{\partial t} + (LM - ML) = 0 \text{ yields}$$

$$\frac{\partial u}{\partial t} - 6mku\frac{\partial u}{\partial x} + m\frac{\partial^3 u}{\partial x^3} = 0. \tag{2.1}$$

By taking $\alpha = 4mk$ and $\beta = 4m$ in (2.1), we have proved Theorem 1.1. As a special case, for $\alpha = \beta = 4$ one gets back the original KdV equation, which proves Corollary 1.2.

The derived KdV-type equation admits a Miura-type transformation $u = \frac{\beta}{\alpha}\left(v^2 + \frac{\partial v}{\partial x}\right)$.

It transforms the KdV-type equation (1.3) $\frac{\partial u}{\partial t} - \frac{3}{2}\alpha u\frac{\partial u}{\partial x} + \frac{\beta}{4}\frac{\partial^3 u}{\partial x^3} = 0$ into

$$\frac{\beta}{\alpha}(2v)\left[\frac{\partial v}{\partial t} - \frac{3}{2}\beta v^2\frac{\partial v}{\partial x} + \frac{\beta}{4}\frac{\partial^3 v}{\partial x^3}\right] + \frac{\beta}{\alpha}\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} - \frac{3}{2}\beta v^2\frac{\partial v}{\partial x} + \frac{\beta}{4}\frac{\partial^3 v}{\partial x^3}\right] = 0.$$

Hence the resulting mKdV-type equation (1.5) is

$$\frac{\partial v}{\partial t} - \frac{3}{2}\beta v^2\frac{\partial v}{\partial x} + \frac{\beta}{4}\frac{\partial^3 v}{\partial x^3} = 0,$$

which proves Theorem 1.3.

3. Travelling Wave Solution

There are many methods available in the literature to derive travelling wave solutions of KdV and mKdV equations [1, 2, 5, 7, 8, 12].

3.1. **KdV-type equation.**

Following direct integration method, the KdV-type equation (1.3)

$$\frac{\partial u}{\partial t} - \frac{3}{2}\alpha u\frac{\partial u}{\partial x} + \frac{\beta}{4}\frac{\partial^3 u}{\partial x^3} = 0$$

transforms into

$$\frac{\beta}{4}\left(\frac{dU}{dz}\right)^2 = U^2\left(c + \frac{\alpha}{2}U\right)$$

when applied to the transformation $u(x,t) = U(z), z = x - ct, c > 0$ and setting the arbitrary integration constants to zero.

Case 1 $\alpha > 0,\ \beta > 0$**:** Final equation reduces to $\dfrac{dU}{dz} = \sqrt{\dfrac{2\alpha}{\beta}}U\sqrt{\dfrac{2c}{\alpha} + U}$. The change of variable $U = \dfrac{-2c}{\alpha}\ \mathrm{sech}^2(V)$ readily gives the solution

$$U = \frac{-2c}{\alpha}\ \mathrm{sech}^2\left(\sqrt{\frac{c}{\beta}}z\right).$$

Case 2 $\alpha < 0, \beta > 0$ **:** The final equation reduces to $\dfrac{dU}{dz} = \sqrt{\dfrac{2|\alpha|}{\beta}}\sqrt{\dfrac{2c}{|\alpha|} - U}$. Again, by the change of variable $U = \dfrac{2c}{|\alpha|}\ \mathrm{sech}^2(V)$ readily gives the solution

$$U = \frac{2c}{|\alpha|}\sec h^2\left(\sqrt{\frac{c}{\beta}}z\right).$$

Case 3 $\beta < 0, \alpha > 0$**:** Writing $\beta = -|\beta|$ and changing U to $(-V)$, one can recast the final form

$$\frac{dV}{dz} = \sqrt{\frac{2\alpha}{|\beta|}}V\sqrt{V - \frac{2c}{\alpha}}.$$

The change of variable $V = \dfrac{2c}{\alpha}\sec^2(U)$ readily gives the solution $U = -\dfrac{2c}{\alpha}\sec^2\left(\sqrt{\dfrac{c}{|\beta|}}z\right)$.

Case 4 $\beta < 0, \alpha < 0$**:** The final form of the equation is

$$\frac{dU}{dz} = \sqrt{\frac{2|\alpha|}{|\beta|}}U\sqrt{U - \frac{2c}{|\alpha|}}$$

and the solution is

$$U = \frac{2c}{|\alpha|}\sec^2\left(\sqrt{\frac{c}{|\beta|}}z\right).$$

By observing $\sec(x) = \ \mathrm{sech}(ix),\ i = \sqrt{-1}$ one may express all the four cases simply as

$$U = \frac{-2c}{|\alpha|}\ \mathrm{sech}^2\left(\sqrt{\frac{c}{\beta}}z\right),\ \alpha \neq 0,\ \beta \neq 0.$$

3.2. **mKdV-type equation.** The standard application of the transformation

$$u(x,t) = V(z),\ z = x - ct,\ c > 0$$

to the mKdV-type equation (1.5)

$$\frac{\partial v}{\partial t} - \frac{3}{2}\beta v^2 \frac{\partial v}{\partial x} + \frac{\beta}{4}\frac{\partial^3 v}{\partial x^3} = 0$$

yields the corresponding ODE

$$-c\frac{dV}{dz} - \frac{3}{2}\beta V^2 \frac{dV}{dz} + \frac{\beta}{4}\frac{d^3 V}{dz^3} = 0.$$

Following standard integration method and setting arbitrary integration constants to zero, one can finally obtain

$$\left(\frac{dV}{dz}\right)^2 = V^2\left[\frac{4c}{\beta} + V^2\right].$$

Case 1 $\beta > 0$**:** put $V = iW,\ i = \sqrt{-1}$.
The equation simplifies to

$$\frac{dW}{dz} = W\sqrt{\frac{4c}{\beta} - W^2}.$$

The change of variable $W = \sqrt{\frac{4c}{\beta}}\ \mathrm{sech}(s)$ and it has the solution

$$z = -\sqrt{\frac{\beta}{4c}}s, \text{ which gives}$$

$$W(z) = \sqrt{\frac{4c}{\beta}}\ \mathrm{sech}\left(\sqrt{\frac{4c}{\beta}}z\right).$$

Case 2 $\beta < 0$**:** put $V(z) = W(iz),\ i = \sqrt{-1}$.
The equation simplifies to

$$\frac{dW}{dz} = W\sqrt{\frac{4c}{|\beta|} - W^2}$$

and its solution is

$$\begin{aligned} W(iz) &= \sqrt{\frac{4c}{|\beta|}}\sec h\left(i\sqrt{\frac{4c}{|\beta|}}z\right) \\ &= \sqrt{\frac{4c}{|\beta|}}\ \mathrm{sech}\left(\sqrt{\frac{4c}{\beta}}z\right). \end{aligned}$$

Hence in both the cases, the final solution is

$$V(z) = i\sqrt{\frac{4c}{\beta}}\ \mathrm{sech}\left(\sqrt{\frac{4c}{\beta}}z\right).$$

4. Hypergeometric expansions of solutions of KdV-type and mKdV-type equations

By looking at same form in all the four cases of KdV-type equation (1.3) it is enough to work with Case 1, namely $\alpha > 0$ and $\beta > 0$. Similarly, it is enough to work with $\beta > 0$ for mKdV-type equation (1.5).

4.1. **KdV-type equation.** By writing

$$U(z) = -\frac{2\beta k^2}{\alpha} \operatorname{sech}^2\left(\frac{k}{2}z\right);$$

where $k = 2\sqrt{\frac{c}{\beta}}$, one can express

$$U(z) = \begin{cases} \dfrac{-2\beta k^2}{\alpha}\dfrac{e^{-kz}}{(1+e^{-kz})^2}; & if\ z \geq 0 \\ \dfrac{-2\beta k^2}{\alpha}\dfrac{e^{kz}}{(1+e^{kz})^2}; & if\ z \leq 0 \end{cases}$$
$$= \frac{-2\beta k^2}{\alpha}\frac{e^{-k|z|}}{(1+e^{-k|z|})^2}.$$

Put $\xi = e^{-k|z|}$ and note that $0 < \xi < 1$.

$$U(\xi) = -\frac{2\beta k^2}{\alpha}[\xi - 2\xi^2 + 3\xi^3 - 4\xi^4 + \cdots]$$
$$= -\frac{2\beta k^2}{\alpha}\xi\ {}_2F_1(2,1;1;-\xi)$$

where ${}_2F_1(\alpha,\beta;\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}$ with $(a)_n = a(a+1)\cdots(a+n-1)$, $a = \alpha, \beta, \gamma$, $|z| < 1$ is the standard Gauss Hypergeometric series.

4.2. **mKdV-type Equation.** Similarly, for $\beta > 0$

$$W(z) = |V(z)| = k \operatorname{sech}(kz)$$
$$= 2k\frac{e^{-k|z|}}{1+(e^{-k|z|})^2}$$
$$= 2k\ [\xi - \xi^3 + \xi^5 - + \cdots]$$
$$= 2k\xi\ {}_2F_1(1,1;1;-\xi^2),$$

where $\xi = e^{-k|z|}$, $k = 2\sqrt{\frac{c}{\beta}}$ and $0 < \xi < 1$.

5. Adomian Decomposition Series Method

5.1. **KdV-type equation.** For executing the method for KdV-type equation (1.3) the suitable nonlinear equation is

$$\frac{d^2U}{dz^2} = k^2U + 3\frac{\alpha}{\beta}U^2;$$

$$k = 2\sqrt{\frac{c}{\beta}},\ \alpha > 0,\ \beta > 0,\ o < z < \infty,\ U(0) = -\frac{\beta}{2\alpha}k^2,\ U(\infty) = 0.$$

The Adomian decomposition series is

$$U = U_0 + U_1 + U_2 + \cdots + U_n + \cdots$$

$$\text{where } \frac{d^2U_0}{dz^2} - k^2U_0 = 0,$$

$$\frac{d^2U_n}{dz^2} - k^2U_n - 3\frac{\alpha}{\beta}\sum_{m=0}^{n-1} U_mU_{n-1-m} = 0,\ n = 1, 2, 3, \ldots$$

$$\text{Choose } U_0 = -2\frac{\beta}{\alpha}k^2e^{-kz} \text{ and } U_1 = 4\frac{\beta}{\alpha}k^2e^{-2kz}.$$

Let us assume that $U_m = (-1)^m(m+1)\left(-2\frac{\beta}{\alpha}k^2\right)e^{-(m+1)kz}$ and work-out the major step of principle of mathematical induction.

$$\text{Consider } \frac{d^2U_{m+1}}{dz^2} - k^2U_{m+1} - 3\frac{\alpha}{\beta}\sum_{l=0}^{m} U_lU_{m-l} = 0.$$

By choosing $U_{m+1} = a_{m+1}\ e^{-(m+2)kz}$, we have

$$\begin{aligned}
a_{m+1}[(m+2)^2 - 1] =&(-1)^{m+1}6\left(-2\frac{\beta}{\alpha}\right)k^2\left[\sum_{i=0}^{m}(l+1)(m+1-l)\right]\\
=&(-1)^{m+1}\left(-2\frac{\beta}{\alpha}k^2\right)6\left[\frac{m^2(m+1)}{2} + (m+1)^2\right.\\
&\left. - \frac{m(m+1(2m+1))}{6}\right]\\
=&(-1)^{m+1}\left(-2\frac{\beta}{\alpha}k^2\right)(m+1)(m+2)(m+3)
\end{aligned}$$

Hence

$$a_{m+1} =(-1)^{m+1}\left(-2\frac{\beta}{\alpha}k^2\right)(m+2)$$

and

$$\begin{aligned} U(z) &= \left(-2\frac{\beta}{\alpha}k^2\right) e^{-kz} \sum_{n=0}^{\infty} (-1)^n (n+1) e^{-(n+1)kz} \\ &= -\frac{\beta}{2\alpha}k^2 \operatorname{sech}^2\left(\frac{k}{2}z\right), \text{ which is the desired solution.} \end{aligned}$$

5.2. **mKdV-type Equation.** For solving mKdV-type equation (1.5) by Adomian decomposition series method, the convenient ODE is

$$\frac{d^2W}{dz^2} - k^2W + 2W^3 = 0,\ k = 2\sqrt{\frac{c}{\beta}},\ \beta > 0,\ 0 < z < \infty,\ W(0) = k,\ W(\infty) = 0.$$

The Adomian series $W = W_0 + W_1 + \cdots + W_n + \cdots$ decomposes into the following equations :

$$\frac{d^2W_0}{dz^2} - k^2W_0 = 0$$

$$\frac{d^2W_n}{dz^2} - k^2W_n + 2\sum_{j=0}^{n-1} W_l \left(\sum_{j=0}^{l} W_j W_{n-1-l-j}\right) = 0,\ n = 1, 2, 3, \ldots$$

Choose $W_0 = 2ke^{-kz}$ and $W_1 = -2ke^{-3kz}$. Let us assume that $W_m = (-1)^m\ 2k\ e^{-(2m+1)kz}$ and workout the major steps of principle of mathematical induction.

$$\frac{d^2W_{m+1}}{dz^2} - k^2W_{m+1} + 2\sum_{i=0}^{m} W_l \left(\sum_{j=0}^{l} W_j W_{m-l-j}\right) = 0.$$

Choose $W_{m+1} = a_{m+1}\ e^{-(2m+3)kz}$ then

$$\begin{aligned} a_{m+1}[(2m+3)^2 - 1] =& (-1)^{m+1}(2k)\ 8\sum_{l=0}^{m}(l+1) \\ =& (-1)^{m+1}(2k)\ 4(m+1)(m+2). \\ \text{Hence, } a_{m+1} =& (-1)^{m+1}2k \ \text{ and} \\ W =& 2ke^{-kz}\sum_{n=0}^{\infty}(e^{-2kz})^n \\ =& k\ \operatorname{sech}(kz),\ \text{which is the desired solution.} \end{aligned}$$

6. Balancing property of travelling wave solutions

6.1. **KdV-type equation.** Consider the travelling wave solution of the KdV-type equation (1.3)

$$U(z) = -\frac{2c}{|\alpha|}\ \text{sech}^2\sqrt{\frac{c}{\beta}}z,\ \ \alpha \neq 0,\ \ \beta \neq 0$$

which satisfies the following two relations:

$$\frac{\beta}{4}\left(U\frac{dU}{dz}\right)^2 = U^4\left[c + \frac{\alpha}{2}U\right]$$

and

$$\frac{\beta^3}{64}\left(\frac{d^3U}{dz^3}\right)^2 = U^2[c + \frac{3}{2}\alpha U]^2[c + \frac{\alpha}{2}U].$$

As a result,

$$\left(U\frac{dU}{dz}\right)^2 = \left(\frac{d^3U}{dz^3}\right)^2 \text{ whenever,}$$

$$\frac{\beta^2}{16}U^4[c + \frac{\alpha}{2}U] = U^2[c + \frac{3}{2}\alpha U]^2[c + \frac{\alpha}{2}U]$$

That is,

$$U = 0, U = \frac{-2c}{\alpha} \text{ or } \frac{\beta^2}{16}U^2 = [c + \frac{3}{2}\alpha U]^2.$$

Hence

$$\left(U\frac{dU}{dz}\right)^2 = \left(\frac{d^3U}{dz^3}\right)^2 \text{ is possible in the following cases :}$$

i) $z = 0$
ii) $\beta > 0, z = \pm\infty$
iii) $0 < |\beta| < 6c$ *and* $z = \ \text{sech}^{-1}\sqrt{\frac{2|\alpha|}{6c \mp \beta}}$

6.2. **mKdV-type equation.** Consider the travelling wave solution of mKdV-type equation (1.5)

$$W(z) = k\ \text{sech}(kz),\ \ k = 2\sqrt{\frac{c}{\beta}},\ \ \beta > 0$$

which satisfies two important relations

$$\left(W^2\frac{dW}{dz}\right)^2 = W^6(k^2 - W^2)$$

$$\text{and } \left(\frac{d^3W}{dz^3}\right)^2 = \left(k^2 - 6W^2\right)^2 W^2(k^2 - W^2).$$

It is straight forward to deduce that

$$\left(W^2\frac{dW}{dz}\right)^2 = \left(\frac{d^3W}{dz^3}\right)^2, \text{which is possible whenever}$$

$$z = 0,\ \pm\infty,\ \frac{1}{k}\ \mathrm{sech}^{-1}\frac{1}{\sqrt{5}} \text{ or } \frac{1}{k}\ \mathrm{sech}^{-1}\frac{1}{\sqrt{7}}.$$

In conclusion, the paper demonstrates that all the standard results of KdV and mKdV equations are passed on to KdV-type and mKdV-type equations, respectively in a nontrivial manner.

7. Acknowledgement

The authors are thankful to University Grants Commission (UGC), Govt. of India for financial support under the grant UGC-SAP-DRS-II, No.F.510/12/DRS-II/2018(SAP-I) dated 9 April 2018. We thank the anonymous referee for the valuable comments, corrections and suggestions for the improvement of the paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January-June (2024), 14–23

ON GOSPER'S ACCELERATED SERIES FOR π

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(Received : 24 - 11 - 2021 ; Revised : 22 - 06 - 2022)

Abstract. This paper gives alternative derivation of Gosper's accelerated series for π obtained by him via transformation of slow series. Binomial expansions and Euler's beta function are used for this.

1. Introduction

1.1. **Background.** The circular solar disc, the full moon and the semicircular rainbow must have fascinated our ancient ancestors. Pythagoras considered circle as the most perfect of all plane figures. Circularity also had practical applications. The invention of the wheel stimulated the advance of civilization. Circle is defined as a plane figure enclosed by a curved line every point on which is equidistant from some fixed interior point. The line forming the edge of a circle is called its circumference (C), the common distance the radius (r), and the fixed point the centre. Any straight-line passing through the centre and joining two points on the circumference is called a diameter (D) of the circle. Numerical estimates of the ratio $\frac{C}{D}$ found in the records of most ancient civilizations (3 in the Old Testament of the Bible, $3\frac{1}{8}$ in Babylon, $3\frac{13}{81}$ in Egypt) point to an early realization that $\frac{C}{D}$ is constant irrespective of the circle's size. It is proved in Euclid's *Elements* (Bk. XII, Prop. 2): *Circles are to one another as the squares on (their) diameters.* The ratio is denoted by the Greek letter π, introduced (most likely as an abbreviation for 'perimeter' of circle with unit diameter) by William Jones in his *Synopsis Palmariorum Matheseos: or, a New Introduction to the Mathematics* (1706). The notation became a standard symbol once adopted by Euler in his paper *Variae observationes circa series infinitas* (1744) and his book *Introductio in Analysin Infinitorum* (1748).

2010 Mathematics Subject Classification: 11Y60, 33B15, 40A25, 65B10
Key words and phrases: Gosper's Pi series, convergence acceleration, Beta function

It is not known how the ancients estimated the value of π. Archimedes (287–212 BC) demonstrated rigorously, by means of Eudoxus' method of exhaustion using an inscribed regular polygon and a circumscribing polygon, that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Afterward, mathematicians used polygons with more sides to refine Archimedes' calculations. The method remained in vogue until the discovery (around 1670) of the series for the inverse sine function by Isaac Newton (1642–1727) and for the arctangent function by James Gregory (1638–1675) which revolutionized the process of computing π.

1.2. **Two examples of transformed series.** G. W. Leibniz (1646–1716) discovered in 1673 an alternating series that provided a practical tool to approximate π.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + - \cdots . \tag{1.1}$$

(1.1) can be obtained by putting $x = 1$ in the series for $\arctan x$. The Leibniz series converges very slowly making it useless for computational purpose.

L. Euler (1707–1783) introduced in [4, Part II, Ch. 1] a technique to enhance the rate of convergence of an alternating series. We reproduce here Euler's transform of (1.1) in [4, Part II, Ch. 1, p.295].

CAPUT I **295**

II. ***Sit propofita ifta feries pro circulo***

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ \&c.}$$

$$\text{Diff. } 1 = -\frac{2}{1.3};\ -\frac{2}{3.5};\ -\frac{2}{5.7};\ -\frac{2}{7.9};\ -\frac{2}{9.11} \text{ \&c.}$$

$$\text{Diff. } 2 = +\frac{2.4}{1.3.5};\ \frac{2.4}{3.5.7};\ \frac{2.4}{5.7.9};\ \frac{2.4}{7,9.11} \text{ \&c.}$$

$$\text{Diff. } 3 = \ -\frac{2.4.6}{1.3.5.7};\ -\frac{2.4.6}{3.5.7.9};\ - \text{ \&c.}$$

&c.

Hinc ergo concluditur fore fummam feriei :

$$S = \frac{1}{2} + \frac{1}{3.2} + \frac{1.2}{3.5.2} + \frac{1.2.3}{3.5.7.2} + \text{ \&c.}$$

Euler took a series $S = a_0 - a_1 + a_2 - a_3 + - \ldots$ $(a_i > 0)$ and computed successive rows of differences using his difference operator.

Let $\Delta_k^n = (\Delta_{k+1}^{n-1} - \Delta_k^{n-1})$ with $\Delta_k^0 = (-1)^k a_k$. Then

$$\Delta_0^0 = a_0; \quad \Delta_0^1 = -a_0 + a_1; \quad \Delta_0^2 = \Delta_1^1 - \Delta_0^1 = a_0 - 2a_1 + a_2;$$

and in general,

$$\Delta_0^n = \Delta_1^{n-1} - \Delta_0^{n-1} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

He thus showed that

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n \Delta_0^n}{2^{n+1}}.$$

Euler's series with rate of convergence $\frac{1}{2}$ can be written as

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{2^{k-1}}{(2k+1)\binom{2k}{k}}. \tag{1.2}$$

The following series is ascribed to Newton by Euler in [3, §14]:

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + + - - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{2n+1} = \frac{\pi}{2\sqrt{2}}.$$

Applying what he calls 'the Nilakantha transformation' to this series, Brink [2, eq.(A.5)] obtained a series with rate of convergence $\frac{2}{27}$:

$$\sum_{n=0}^{\infty} \frac{(5n+4)\ (2n)!^2\ (3n)!\ 2^{3n}}{(6n+1)(6n+5)\ n!\ (6n)!} = \frac{3\pi}{8\sqrt{2}}.$$

2. Gosper's accelerated series for π

Let us now take up R. W. Gosper's series [1] with rate of convergence $\frac{2}{27}$:

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{25n-3}{2^n \binom{3n}{n}}. \tag{2.1}$$

Not knowing how Gosper obtained (2.1), we will derive it with the help of Euler's beta function which is defined by the integral [6, p.263]:

$$B(p,q) = \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt \qquad (\Re(p), \Re(q) > 0). \tag{2.2}$$

The beta function is related to the Gamma function (a generalization of the factorial function) by: $B(p,q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$. [6, p.264]

2.1. **Derivation of $\sum_{n=0}^{\infty} \frac{1}{2^n\binom{3n}{n}}$.** Differentiating the geometric series:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \qquad (|y|<1).$$

results in

$$\frac{1}{(1-y)^2} = \sum_{n=0}^{\infty} n\, y^{n-1} \qquad (|y|<1). \tag{2.3}$$

One can easily see using the definition of beta function that

$$\frac{1}{\binom{3n}{n}} = \frac{n!(2n)!}{(3n)!} = \frac{n(n-1)!(2n)!}{(3n)!} = n\, B(n, 2n+1),$$

so that

$$\frac{1}{\binom{3n}{n}} = n \int_0^1 t^{n-1}(1-t)^{2n}\, dt.$$

Multiplying both sides by x^n and taking the sum from $n = 0$ to ∞ yields

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x^n}{\binom{3n}{n}} &= \sum_{n=0}^{\infty} nx^n \int_0^1 t^{n-1}(1-t)^{2n}\, dt \\ &= \int_0^1 x(1-t)^2 \sum_{n=0}^{\infty} nx^{n-1}t^{n-1}\{(1-t)^2\}^{n-1}\, dt\end{aligned}$$

interchanging the order of summation and integration.

Now setting $y = xt(1-t)^2$ in (2.3) we get:

$$\frac{1}{\{1-xt(1-t)^2\}^2} = \sum_{n=0}^{\infty} nx^{n-1}t^{n-1}(1-t)^{2n-2},$$

where $|xt(1-t)^2| < 1$ and $0 < t < 1$.

Thus on taking $x = \frac{1}{2}$, we have:

$$\sum_{n=1}^{\infty} \frac{1}{2^n\binom{3n}{n}} = 2\int_0^1 \frac{(1-t)^2}{\{2-t(1-t)^2\}^2}\, dt.$$

We may write the integrand on RHS as

$$\frac{(1-t)^2}{\{2-t(1-t)^2\}^2} = \frac{(1-t)^2}{(t-2)^2(1+t^2)^2}$$

whose partial fraction expansion is:

$$\frac{-2t-9}{125(t^2+1)} - \frac{2(3t-4)}{25(t^2+1)^2} + \frac{2}{125(t-2)} + \frac{1}{25(t-2)^2}.$$

The resulting integrals are easy to calculate and equal:

$$-\left(\frac{9\pi}{500}+\frac{\log(2)}{125}\right)+\left(\frac{\pi}{25}+\frac{1}{50}\right)-\frac{2\log(2)}{125}+\frac{1}{50}=\frac{11\pi-12\log(2)+20}{500}.$$

We thus get:

$$\sum_{n=0}^{\infty}\frac{1}{2^n\binom{3n}{n}}=\frac{11\pi-12\log(2)+20}{250}. \tag{2.4}$$

2.2. **Derivation of** $\sum_{n=0}^{\infty}\frac{n}{2^n\binom{3n}{n}}$**.** Multiplying both sides of (2.3) by y, and then differentiating gives

$$-\frac{1+y}{(1-y)^3}=\sum_{n=0}^{\infty}n^2\ y^{n-1} \qquad (|y|<1). \tag{2.5}$$

Proceeding as earlier, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n\ x^n}{\binom{3n}{n}}&=\int_0^1\sum_{n=0}^{\infty}n^2\ x^n\ t^{n-1}(1-t)^{2n}\,dt\\&=\int_0^1 x(1-t)^2\sum_{n=0}^{\infty}n^2x^{n-1}t^{n-1}\{(1-t)^2\}^{n-1}\,dt.\end{aligned}$$

Now setting $y=xt(1-t)^2$ in (2.5) we get:

$$-\frac{1+xt(1-t)^2}{(1-xt(1-t)^2)^3}=\sum_{n=0}^{\infty}n^2x^{n-1}t^{n-1}(1-t)^{2n-2},$$

where $|xt(1-t)^2|<1$ and $0<t<1$.

Thus on taking $x=\frac{1}{2}$, we have:

$$\sum_{n=1}^{\infty}\frac{n}{2^n\binom{3n}{n}}=-2\int_0^1\frac{(1-t)^2\ (2+t(1-t)^2)}{\{2-t(1-t)^2\}^3}\,dt.$$

Since $\{2-t(1-t)^2\}^3=-(t-2)^3(t^2+1)^3$, we may write the RHS as

$$2\int_0^1\frac{(1-t)^2(t^3-2t^2+t+2)}{(t-2)^3(t^2+1)^3}.$$

The partial fraction expansion of the integrand is:

$$\begin{aligned}&\frac{8(2t-11)}{125(t^2+1)^3}-\frac{2(41t-148)}{625(t^2+1)^2}+\frac{6t-73}{3125(t^2+1)}\\&+\frac{4}{125(t-2)^3}+\frac{17}{625(t-2)^2}-\frac{6}{3125(t-2)}.\end{aligned}$$

The resulting integrals can be calculated with the help of standard integral tables and hence the sum:

$$\sum_{n=0}^{\infty} \frac{n}{2^n \binom{3n}{n}} = \frac{79\pi - 18\log(2) + 405}{3125}. \tag{2.6}$$

$25 \times (2.6) - 3 \times (2.4)$ yields (2.1).

3. An akin series

We now give a series with two linear factors in the denominator.

$$\frac{\pi}{2} = 1 + \sum_{n=1}^{\infty} \frac{10n+3}{2^n\, n\,(3n+1)\binom{3n}{n}}. \tag{3.1}$$

Proof. To establish (3.1), we split the sum on the RHS into two parts:

$$\sum_{n=1}^{\infty} \frac{3}{2^n\, n\binom{3n}{n}} + \sum_{n=1}^{\infty} \frac{1}{2^n\,(3n+1)\binom{3n}{n}}.$$

Sum 1. It is easy to compute the sum $\sum_{n=1}^{\infty} \frac{1}{2^n\, n\binom{3n}{n}}$ by proceeding as in the last section. Using the binomial expansion and the Beta function, we have

$$\frac{x^n}{n\binom{3n}{n}} = \int_0^1 x^n t^{n-1}(1-t)^{2n}$$

and so

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x^n}{n\binom{3n}{n}} &= \int_0^1 x(1-t)^2 \left[\sum_{n=0}^{\infty} x^{n-1}t^{n-1}(1-t)^{2n-2}\right] dt \\ &= \int_0^1 \frac{x(1-t)^2}{1-xt(1-t)^2}\, dt.\end{aligned}$$

Taking $x = \frac{1}{2}$, we get

$$\sum_{n=1}^{\infty} \frac{1}{2^n\, n\binom{3n}{n}} = \int_0^1 \frac{(1-t)^2}{2-t(1-t)^2}\, dt = \int_0^1 -\frac{1-t)^2}{(t-2)(t^2+1)}\, dt$$

and using the partial fraction expansion

$$\sum_{n=1}^{\infty} \frac{1}{2^n\, n\binom{3n}{n}} = \int_0^1 -\frac{2(2t-1)}{5(t^2+1)}\, dt + \int_0^1 -\frac{1}{5(t-2)}\, dt$$

where the two integrals are easy to calculate. We thus obtain:

$$\sum_{n=1}^{\infty} \frac{1}{2^n\, n\binom{3n}{n}} = \frac{\pi}{10} - \frac{\log(2)}{5}. \tag{3.2}$$

Sum 2. For the second sum, we proceed as follows.

$$\begin{aligned}\frac{1}{\binom{3n}{n}} &= (3n+1)\frac{\Gamma(n+1)\Gamma(2n+1)}{\Gamma(3n+2)}\\ &= (3n+1)B(n+1,2n+1) = (3n+1)\int_0^1 t^n(1-t)^{2n}\,dt.\end{aligned}$$

Hence, we have: $\displaystyle\frac{1}{2^n(3n+1)\binom{3n}{n}} = \int_0^1 \left(\frac{t(1-t)^2}{2}\right)^n dt.$

$$\sum_{n=0}^{\infty}\frac{1}{2^n(3n+1)\binom{3n}{n}} = \int_0^1 \frac{2}{2-t(1-t)^2}\,dt = -\int_0^1 \frac{2}{(t-2)(t^2+1)}\,dt$$

$$= \frac{2}{5}\int_0^1\left(\frac{t+2}{t^2+1}-\frac{1}{t-2}\right)dt = \frac{1}{5}\int_0^1\frac{2t}{t^2+1}+\frac{4}{5}\int_0^1\frac{1}{t^2+1}-\frac{2}{5}\int_0^1\frac{1}{t-2}\,dt,$$

$$= \frac{\log(2)}{5}+\frac{\pi}{5}+\frac{2\log(2)}{5} = \frac{\pi}{5}+\frac{3\log(2)}{5}.$$

We took Sum 2 from $n=0$. So subtract 1 for the sum from $n=1$. Multiplying Sum 1 by 3 and adding that to Sum 2, we get:

$$\sum_{n=1}^{\infty}\frac{10n+3}{2^n\,n\,(3n+1)\binom{3n}{n}} = \frac{3\pi}{10}-\frac{3\log(2)}{5}-1+\frac{3\log(2)}{5}+\frac{\pi}{5} = -1+\frac{\pi}{2}.$$

□

4. Two more series of Gosper

Gosper illustrates in [5] how the rate of convergence of infinite series can be accelerated by a suitable splitting of each term into two parts and then combining the second part of the n-th term with the first part of the $n+1$-th term and leaving the first part of the first term. Repeated application of this process yields a new series which approaches 0 and the series of the left out first parts ('orphans') that converges faster than the original series. We will now discuss two other series obtained by Gosper via transformation of slow series.

Series I. It is easy to see that $\displaystyle\frac{1}{3n+1}+\frac{1}{3n+2} = \frac{3(2n+1)}{(3n+1)(3n+2)}$. Further,

$$\frac{1}{\binom{3n}{n}} = \frac{(3n+1)(3n+2)\Gamma(n+1)\ \Gamma(2n+2)}{(2n+1)\Gamma(3n+3)}$$

or,

$$\frac{2n+1}{(3n+1)(3n+2)\binom{3n}{n}} = \frac{\Gamma(n+1)\ \Gamma(2n+2)}{\Gamma(3n+3)} = B(n+1, 2n+2).$$

Multiplying both sides by x^n and using the integral for the beta function as earlier, we have

$$\frac{(2n+1)x^n}{(3n+1)(3n+2)\binom{3n}{n}} = \int_0^1 x^n t^n (1-t)^{2n+1}\ dt$$

and taking $x = \frac{1}{2}$ and summing the series from 0 to infinity as earlier, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2n+1}{(3n+1)(3n+2)2^n\ \binom{3n}{n}} &= \int_0^1 \frac{2(1-t)}{2-t(1-t)^2}\ dt \\ &= \int_0^1 \frac{2}{5(t-2)}\ dt - \int_0^1 \frac{2(t-3)}{2(1+t)^2}\ dt \\ &= \frac{3}{10}(\pi - 2\log(2))\end{aligned}$$

and hence we deduce

$$\sum_{n=0}^{\infty} \frac{1}{(3n+2)\ 2^n \binom{3n}{n}} = \frac{7\pi}{10} - \frac{12\log(2)}{5}. \tag{4.1}$$

Combining this result with **Sum 1**, we obtain the series which occurs in Gosper's paper [5, p.32]:

$$\sum_{n=0}^{\infty} \frac{5n+3}{(3n+1)(3n+2)} \frac{1}{\binom{3n}{n} 2^n} = \frac{\pi}{2}. \tag{4.2}$$

Brink calls it the Nilakantha transform of the Leibniz series [2, eq(3)].

We also get:

$$\sum_{n=0}^{\infty} \frac{5n+4}{(3n+1)(3n+2)\ 2^n\ \binom{3n}{n}} = 3\log(2). \tag{4.3}$$

By a shift of the index n from 0 to 1, the last formula becomes

$$\sum_{n=1}^{\infty} \frac{5n-2}{n(2n-1)\binom{3n}{n}2^n} = \sum_{n=1}^{\infty} \frac{2}{n\binom{3n}{n}2^n} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)\binom{3n}{n}2^n} = \frac{\pi}{6}.$$

Using value of the first sum obtained in the previous section we deduce

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)\ 2^n\ \binom{3n}{n}} = \frac{2\log(2)}{5} - \frac{\pi}{30}. \tag{4.4}$$

Combining this result and the earlier result gives a series for log(2)

$$\sum_{n=1}^{\infty} \frac{5n-1}{n(2n-1)\ 2^n \binom{3n}{n}} = \log(2). \tag{4.5}$$

Series II and associated series. Lastly, we take up another series of Gosper [5, p.33]:

$$\sum_{n=0}^{\infty} \frac{5}{(2n+1)\ 2^n \binom{3n+4}{n+2}} = 4 - \pi. \tag{4.6}$$

The LHS can be decomposed and the formula can be written as

$$\sum_{n=0}^{\infty} \left[\frac{100}{81(3n+1)} - \frac{20}{27(3n+2)} - \frac{10}{81(3n+4)}\right] \frac{1}{2^n \binom{3n}{n}} = 4 - \pi.$$

Using the known values, we deduce from the last expression:

$$\sum_{n=0}^{\infty} \frac{1}{(3n+4)\ 2^n \binom{3n}{n}} = \frac{59\pi}{10} + \frac{102\log(2)}{5} - \frac{162}{5}. \tag{4.7}$$

Combining this with earlier results gives

$$\sum_{n=0}^{\infty} \frac{11n+15}{(3n+1)(3n+4)\ 2^n \binom{3n}{n}} = \frac{\pi}{10} + \frac{18}{5}. \tag{4.8}$$

$$\sum_{n=0}^{\infty} \frac{19n+24}{(3n+2)(3n+4)\ 2^n \binom{3n}{n}} = \frac{79\pi}{10} - \frac{108}{5}. \tag{4.9}$$

And $19 \times (4.8) - 11 \times (4.9)$ yields

$$\sum_{n=0}^{\infty} \frac{8n+9}{(3n+1)(3n+2)(3n+4)\ 2^{n-1} \binom{3n}{n}} = 18 - 5\pi. \tag{4.10}$$

Concluding remarks

We derived here Gosper's series for π by means of Euler's beta function. We first developed the binomial coefficient $\binom{3n}{n}$ into associated integral, then evaluated the resulting integral and lastly eliminated certain constants, such as logarithms that arose during the process, to obtain the desired series involving only π. We touched upon Euler's transformation of the Leibniz series which involved the central binomial coefficient $\binom{2n}{n}$. However, other binomial coefficients, like $\binom{4n}{2n}$ and $\binom{6n}{3n}$, may not be that easy to handle and may require much more effort. The technique demonstrated here is readily applicable to sums with small binomial coefficient in the denominator and the associated integral easy to manipulate and evaluate.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 24–29

PROOFS OF TWO FORMULAS OF VLADETA JOVOVIC

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(Received : 15 - 12 - 2021 ; Revised : 09 - 09 - 2022)

Abstract. In this paper, we first provide an analytic and a bijective proof of a formula stated by Vladeta Jovovic in the OEIS sequence A117989. We also provide a bijective proof of another interesting result stated by him on the same page concerning integer partitions with fixed differences between the largest and smallest parts.

1. Introduction

A partition π of a positive integer n is a non-increasing sequence of natural numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. $\lambda_1, \lambda_2, \ldots, \lambda_r$ are called the *parts* of the partition π. We call λ_1 and λ_r to be the *largest* and *smallest* parts of the partition π respectively and denote $p(n)$ to be the number of (unrestricted) partitions of n. For example, $p(5) = 7$ where the seven partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. Their corresponding largest parts are 5, 4, 3, 3, 2, 2, and 1 respectively and smallest parts are 5, 1, 2, 1, 1, 1, and 1 respectively.

Throughout the paper, we consider $|q| < 1$ and adopt the usual notation for the conventional q-Pochammer symbols:

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k),$$

$$(a)_\infty = (a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

We define $(a)_n$ for all real numbers n by

$$(a)_n := \frac{(a)_\infty}{(aq^n)_\infty}.$$

2010 Mathematics Subject Classification: 05A17, 05A19, 11P81, 11P82
Key words and phrases: partition, part, largest part, smallest part, fixed difference

By convention, we take $(a)_0 = 1$ and $(0)_\infty = 1$.

2. Main Formulas

Our starting point is the sequence A117989 in the On-Line Encyclopedia of Integer Sequences [3]. The sequence in question, $a(n)$, counts the number of partitions of n where the smallest part occurs at least twice. For example, $a(6) = 7$ where the relevant partitions are $4+1+1$, $3+3$, $3+1+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$, and $1+1+1+1+1+1$.

Also, on the page of A117989, we find the following formula of $a(n)$ given by Vladeta Jovovic:

Formula 1: (Vladeta Jovovic, July 21 2006)

$$a(n) = 2p(n) - p(n+1) \ \forall\, n \geq 1 \tag{2.1}$$

where $p(n)$ denotes the partition function.

We denote $b(n) = 2p(n) - p(n+1)$. By (2.1), we have $a(6) = 7 = 22 - 15 = 2p(6) - p(7) = b(6)$.

We also find another interesting result on the same page posted by Vladeta Jovovic which states:

Formula 2: (Vladeta Jovovic, May 09 2008)

$$a(n) = p(2n, n) \ \forall\, n \geq 1 \tag{2.2}$$

where $p(2n, n)$ denotes the number of partitions of $2n$ with fixed difference equal to n between the largest and smallest parts.

In sections 3 and 4, we give a q-theoretic and a bijective proof of (2.1) respectively and finally we provide a bijective proof of (2.2) in section 5.

3. Analytic Proof of Formula 1

In this section, we prove (2.1) using generating functions and elementary infinite series-product identities from the theory of q-hypergeometric series.

To begin with, we define the generating functions for $a(n)$ and $b(n)$ to be

$$A(q) := \sum_{n=1}^{\infty} a(n) q^n \quad \text{and} \quad B(q) := \sum_{n=1}^{\infty} b(n) q^n \tag{3.1}$$

respectively. We then have

$$
\begin{aligned}
A(q) &= \sum_{k=1}^{\infty} q^{k+k}(1+q^k+q^{2k}+\cdots)(1+q^{k+1}+q^{2(k+1)}+\cdots)\cdots \\
&= \sum_{k=1}^{\infty} \frac{q^{2k}}{(q^k)_\infty} \\
&= \frac{1}{(q)_\infty}\sum_{k=1}^{\infty}(q)_{k-1}q^{2k} \\
&= \frac{q^2}{(q)_\infty}\sum_{k=0}^{\infty}\frac{(q)_k(q)_k q^{2k}}{(q)_k} \\
&= \frac{q^2(q^3)_\infty}{(q^2)_\infty}\sum_{k=0}^{\infty}\frac{(q^2)_k q^k}{(q)_k(q^3)_k} \qquad (3.2) \\
&= \sum_{k=0}^{\infty}\frac{q^{k+2}}{(q)_k(1-q^{k+2})} \\
&= \sum_{k=0}^{\infty}\frac{q^{k+2}(1-q^{k+1})}{(q)_{k+2}} \\
&= \sum_{k=0}^{\infty}\frac{q^{k+2}}{(q)_{k+2}} - \sum_{k=0}^{\infty}\frac{q^{2k+3}}{(q)_{k+2}} \\
&= \sum_{k=0}^{\infty}\frac{q^{k+2}}{(q)_{k+2}} - \frac{1}{q}\sum_{k=0}^{\infty}\frac{(q^2)^{k+2}}{(q)_{k+2}} \\
&= \left(\frac{1}{(q)_\infty}-\frac{1}{1-q}\right)-\frac{1}{q}\left(\frac{1}{(q^2)_\infty}-\frac{1-q+q^2}{1-q}\right) \qquad (3.3) \\
&= \frac{1}{(q)_\infty}-\frac{1}{1-q}-\frac{1-q}{q(q)_\infty}+\frac{1-q+q^2}{q(1-q)} \\
&= \frac{2}{(q)_\infty}-\frac{1}{q(q)_\infty}+\frac{1}{q}-1
\end{aligned}
$$

which is equal to $B(q)$, the generating function for $b(n) = 2p(n) - p(n+1)$ $\forall\ n \geq 1$.

Note that (3.2) follows by replacing $a = q$, $b = q$, $c = 0$, and $t = q^2$ in Heine's transformation [1], p.19, Corollary 2.3] and (3.3) follows by replacing $a = 0$, $t = q$ and $a = 0$, $t = q^2$ respectively in Cauchy's identity [[1], p.17, Theorem 2.1]. $\square$

4. Bijective Proof of Formula 1

In this section, we provide a bijective proof of (2.1).

From (2.1), we have

$$\begin{aligned} a(n) &= 2p(n) - p(n+1) \\ &= p(n) - (p(n+1) - p(n)) \end{aligned}$$

which implies

$$p(n) - a(n) = p(n+1) - p(n). \tag{4.1}$$

Let us now define $c(n) = p(n) - a(n)$ and $d(n) = p(n+1) - p(n)$. Thus, from (4.1), it suffices to prove that $c(n) = d(n)\ \forall\ n \geq 1$.

We note that $c(n)$ denotes the number of partitions of n where the smallest part occurs exactly once. This follows straightforward from the definition of $a(n)$. We also note that $d(n)$ denotes the number of partitions of $n+1$ which do not contain 1 as a part because every partition of $n+1$ which contains 1 as a part can be obtained by adjoining 1 as a part to every partition of n.

Let $\mathcal{C}_n$ be the set of all partitions of n where the smallest part occurs exactly once and $\mathcal{D}_n$ be the set of all partitions of $n+1$ not containing 1 as a part. So, $\#\mathcal{C}_n = c(n)$ and $\#\mathcal{D}_n = d(n)$. Thus, it is clear that we will now produce a bijection between the sets $\mathcal{C}_n$ and $\mathcal{D}_n$ to obtain the desired result.

Firstly, we consider a partition $\pi \in \mathcal{C}_n$ and consider two cases pertaining to π: If 1 is a part of π, since it is the smallest part, it occurs exactly once. Now, add 1 to the 1 already in π to get a new partition $\pi' \in \mathcal{D}_n$ whose smallest part now is 2. Hence, π' does not contain 1 as a part. Now, if 1 is not a part of π, then the smallest part of π is greater than or equal to 2 and hence it does not contain 1 as a part. On adding 1 to the smallest part of π, we get a new partition π' of $n+1$ which does not contain 1 as a part. Hence, $\pi' \in \mathcal{D}_n$.

Now, we consider a partition $\pi' \in \mathcal{D}_n$. Thus, π' does not contain 1 as a part which implies that the smallest part of π' is greater than or equal to 2. Again, we consider two cases concerning π': If the smallest part of π' occurs exactly once, we subtract 1 from it to get a new partition $\pi \in \mathcal{C}_n$ and we are done. On the other hand, if the smallest part of π' occurs at least twice, subtract 1 from any one of the smallest parts to get a new partition $\pi \in \mathcal{C}_n$.

Thus, the process is reversible and hence $\mathcal{C}_n$ is bijection with $\mathcal{D}_n\ \forall\ n \geq 1$. So, we have our desired result. $\square$

5. Bijective Proof of Formula 2

In this section, we provide a bijective proof of (2.2).

Let $\mathcal{A}_n$ be the set of all partitions of n where the smallest part occurs at least twice and $\mathcal{F}_n$ be the set of all partitions of $2n$ where the difference between the largest and smallest parts is equal to n. So, $\#\mathcal{A}_n = a(n)$ and $\#\mathcal{F}_n = p(2n, n)$. Now, we will provide a bijection between the sets $\mathcal{A}_n$ and $\mathcal{F}_n$ to show that $a(n) = p(2n, n)$.

Firstly, we consider a partition $\pi \in \mathcal{A}_n$. Then, we add n to any one of the smallest parts of π (since the smallest part of π occurs at least twice) to get a new partition π'. $\pi' \in \mathcal{F}_n$ because the largest part of π' now is equal to $n+$ the smallest part of π and the smallest part of π' is equal to the smallest part of π thus making the difference equal to n.

For the other way, we now consider a partition $\pi' \in \mathcal{F}_n$. Note that the largest part of π' occurs exactly once. We then subtract n from the largest part of π' to get a new partition π whose smallest part is equal to the smallest part of π' and consider two cases pertaining to π': If the smallest part of π' occurs at least twice, we are done, i.e., $\pi \in \mathcal{A}_n$ since subtracting n from the largest part of π' does not affect the frequency of the smallest part of π ($=$ the smallest part of π') which still remains at least 2. Lastly, if the smallest part of π' occurs exactly once, subtracting n from the largest part of π' makes it equal to the the smallest part of π' and thus, the new partition π that we obtain has smallest part occuring at least twice since smallest parts of π' and π are equal. Hence, $\pi \in \mathcal{A}_n$.

Thus, the process is reversible and hence, we have a bijection between $\mathcal{A}_n$ and $\mathcal{F}_n$ giving our desired result. $\square$

6. Conclusion

Thus, we have $p(2n, n) = a(n) = 2p(n) - p(n+1)\ \forall\ n \geq 1$. We speculate that there is an interesting proof of (2.2) using the generating function approach. Although in [2], Andrews, Beck, and Robbins gave the generating function for $p(n, t)$ (which is the number of partitions of n with fixed difference equal to t between the largest and smallest parts), the generating function of $p(2n, n)$ does not follow straightforward.

In the same spirit, we define $G_m(q) := \sum_{n=1}^{\infty} a_m(n)q^n$ where $a_m(n)$ denotes the number of partitions of n where the smallest part occurs at least m times. On the page of A117989, we see that $G_m(q) = \sum_{k=1}^{\infty} \frac{q^{mk}}{(q^k)_\infty}$. It will be very interesting to see if there is a closed formula analogous to (2.1) for $a_m(n)$ $\forall$ $m \geq 3$ and if there exists such a formula, then it would be nice to provide a combinatorial proof of it.

Acknowledgement: The author would like to thank George E. Andrews for suggesting him to prove the two formulas of Vladeta Jovovic which came up during an ongoing project with him.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January-June (2024), 30–41

UNIQUENESS OF AN ENTIRE FUNCTION AND ITS LINEAR DIFFERENCE POLYNOMIAL SHARING SMALL FUNCTIONS

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(Received : 16 - 03 - 2022 ; Revised : 27 - 04 - 2023)

Abstract. This article investigates the uniqueness problem of a finite order entire function and its linear difference polynomial sharing a set consisting of two distinct entire functions of smaller orders. The result obtained in this paper generalises the result by Jianming Qi, Yanfeng Wang, and Yongyi Gu [14].

1. Introduction

We adopt the following standard notations of the Nevanlinna theory through out the paper, which can be found in [18].
Let $f(z)$ be a meromorphic function and not a constant in the disc $|z| \leq R(0 < R < \infty)$. For $0 < r < R$, Nevanlinna defined the following functions:

Proximity function of $f(z)$:

$$m(r, f) = m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where $log^+ = max\{logx, 0\}$ for all $x \geq 0$. Here $m(r, f)$ is the average of the positive logarithm of $|f(z)|$ on the circle $|z| = r$.

Counting function of poles of $f(z)$:

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

Corresponding author: Renukadevi S. Dyavanal
2010 Mathematics Subject Classification: 30D35, 32H30
Key words and phrases: Entire function $f(z)$, order of $f(z)$, Linear difference polynomial of $f(z)$, Small function of $f(z)$, Meromorphic function, Nevanlinna theory

Reduced counting function of poles of $f(z)$:

$$\overline{N}(r,f) := \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}dt + \overline{n}(0,f)logr,$$

where $n(t,f)$ denotes the number of poles of $f(z)$ in the disc $\mid z \mid \leq t$, multiple poles are counted according to their multiplicities, and $\overline{n}(t,f)$ denotes the number of distinct poles of $f(z)$ in the disc $\mid z \mid \leq t$.

Characteristic function of $f(z)$:

$$T(r,f) = m(r,f) + N(r,f).$$

The characteristic function $T(r,f)$ is obviously a non-negative function and plays a cardinal role in the whole theory of meromorphic functions.

Let a be a complex number. Obviously, $\frac{1}{f(z)-a}$ is meromorphic in the disk $|z| \leq R$. Similar to the above definitions, R. Nevanlinna defined the following functions:

Proximity function of $\frac{1}{f(z)-a}$:

$$m\left(r, \frac{1}{f-a}\right) = m(r, f=a) = m(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})-a|}d\theta,$$

which is the average of the positive logarithm of $\frac{1}{|f(z)|}$ on the circle $|z| = r$.

Counting function of $f(z)$ at the value a:

$$N\left(r, \frac{1}{f-a}\right) = N(r, f=a) = \int_0^r \frac{n(t,\frac{1}{f-a}) - n(0,\frac{1}{f-a})}{t}\, dt + n\left(0, \frac{1}{f-a}\right)\log r,$$

where $n(t, \frac{1}{f-a})$ denotes the number of zeros of $f(z)-a$ in the disc $|z| \leq t$ counting multiplicities and $n(0, \frac{1}{f-a})$ the multiplicity of zeros of $f(z)-a$ at the origin.

The characteristic function of $\frac{1}{f(z)-a}$:

$$T\left(r, \frac{1}{f-a}\right) = m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right).$$

Order of a function: Let $f(z)$ be a meromorphic function on the whole complex plane. The order $\rho(f)$ and lower order $\mu(f)$ are defined, respectively, by the order and lower order of $T(r, f)$, that is

$$\rho(f) = \overline{\lim}_{r\to\infty} \frac{\log T(r, f)}{\log r},$$
$$\mu(f) = \underline{\lim}_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

The exponent of convergence of zeros $\lambda(f)$ of $f(z)$ is defined by

$$\lambda(f) = \overline{\lim_{r\to\infty}} \frac{\log N(r, \frac{1}{f})}{\log r}.$$

Nevanlinna established the following two fundamental theorems.

First fundamental theorem of Nevanlinna: Let $f(z)$ be a meromorphic function on the whole complex plane, and a is any complex number. Then

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1).$$

This means that, for any complex number a, the difference of $T\left(r, \frac{1}{f-a}\right)$ and $T(r, f)$ is a bounded quantity.

The Second fundamental theorem of Nevanlinna for $q(\geq 3)$ values: Suppose that $f(z)$ is a meromorphic function in the complex plane and $a_1, a_2, \cdots, a_q$ are $q(\geq 3)$ distinct values in $\overline{\mathbb{C}}$. Then

$$(q-2)T(r, f) < \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r, f).$$

The Second fundamental theorem of Nevanlinna for three small functions: Suppose that $f(z)$ is a meromorphic function in the complex plane and $a_1(z)$, $a_2(z)$, $a_3(z)$ are three distinct small functions of $f(z)$.

Then

$$T(r,f) < \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f-a_j(z)}\right) + S(r,f).$$

The error term $S(r,f)$ in the second fundamental theorem of Nevanlinna is given by
$S(r,f) = o(T(r,f))$, $(r \to \infty)$, if the order of $f(z)$ is finite and if the order of $f(z)$ is infinite, then $S(r,f) = o(T(r,f))$, $(r \to \infty,\ r \notin E)$, where E is a set, with a finite linear measure.

Small function of $f(z)$:
If $f(z)$ and $a(z)$ are meromorphic functions in the complex plane, then $a(z)$ is called a small function of $f(z)$ if it satisfies $T(r, a(z)) = o(T(r,f)) = S(r,f)$.

Sharing small functions: Let f and g be two meromorphic functions, and a and b are two small functions of f and g. We say that f and g share a pair of small functions (a,b) CM (resp. IM) if $f-a$ and $g-b$ have the same zeros counting multiplicities (resp. ignoring multiplicities). When $a=b$, we say that f and g share a CM (resp. IM) if $f-a$ and $g-a$ have the same zeros, counting multiplicities (resp. ignoring multiplicities).
Sharing a Set: Let S be a finite set of some functions, and $f(z)$ is a meromorphic function. Then a set $E_f(S)$ is defined as

$$E_f(S) = \cup_{a\in S}\{z \mid f(z)-a(z)=0,\ counting\, multiplicities(CM)\}.$$

Assume that g is another function. We say that f and g share the set S CM, provided that $E_f(S) = E_g(S)$.

For a meromorphic function $f(z)$, we define its shift by $f_c = f(z+c)$ and its difference operators by

$$\triangle_c f = f(z+c) - f(z),$$

$$\triangle_c^n f = \triangle_c^{n-1}(\triangle_c f(z)),\ n \in \mathbb{N},\ n \geq 2.$$

A linear difference polynomial of $f(z)$ is denoted by $L(z,f)$ and defined as

$$L(z,f) = a_0 f(z) + a_1 f(z+t_1) + \cdots + a_k f(z+t_k),$$

where t_i's are finite complex numbers and the coefficients a_i's are small functions of $f(z)$.

In 2009, Liu [16] proved the following result:

Theorem 1.1. *Let a be a non-zero complex number and f be a transcendental entire function with finite order. If f and $\triangle_c f$ share $\{a, -a\}$ CM, then $\triangle_c f(z) = f(z)$ for all $z \in \mathbb{C}$.*

In 2012, Li [15] proved the following theorem:

Theorem 1.2. *Suppose that a, b are two distinct entire functions, and $f(z)$ is a non-constant entire function with $\rho(f) \neq 1$ and $\lambda(f) < \rho(f) < \infty$ such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If $f(z)$ and $\triangle_c f(z)$ share $\{a, b\}$ CM, then $f(z) = \triangle_c f(z)$ for all $z \in \mathbb{C}$.*

In 2019, Jianming Qi, Yanfeng Wang, and Yongyi Gu [14] removed the condition $\rho(f) \neq 1$ in the Theorem and proved the following result:

Theorem 1.3. *Suppose that a, b are two distinct entire functions, and $f(z)$ is a non-constant entire function of finite order with $\lambda(f) < \rho(f)$ such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If $f(z)$ and $\triangle_c f(z)$ share $\{a, b\}$ CM, then $f(z) = Ae^{\mu z}$, where A, μ are two non-zero constants satisfying $e^{\mu c} = 2$. Furthermore, $f = \triangle_c f$*

In this paper, we extend the above Theorem 1.3 for $L(z, f)$, a linear difference polynomial of $f(z)$.
The following is the main result of this paper.

Theorem 1.4. *If $f(z)$ and $L(z, f)(\not\equiv 0)$ share $\{c, d\}$ CM, where c and d are two distinct entire functions, and $f(z)$ is a non-constant entire function of finite order with $\lambda(f) < \rho(f)$ such that $\rho(c) < \rho(f)$ and $\rho(d) < \rho(f)$, then $f(z) = Ke^{Cz}$, where K is a non-zero constant and $a_0 + a_1 e^{Ct_1} + \cdots + a_k e^{Ct_k} = 1$. Furthermore, $f(z) = L(z, f)$.*

2. **Some preliminary results**

We need the following results to prove our main result.

Lemma 2.1. [2]*) Let $f(z)$ be a meromorphic function of finite order, and let ω_1 and ω_2 are two arbitrary complex numbers, such that $\omega_1 \neq \omega_2$. Assume that σ is the order of f, then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+\omega_1)}{f(z+\omega_2)}\right) = O(r^{\sigma-1+\epsilon}). \tag{2.1}$$

Lemma 2.2. *([18] Theorem 1.51) Suppose that $f_1(z), f_2(z) \cdots f_n(z) (n \geq 2)$ are meromorphic functions, and $g_1(z) \cdots g_n(z)$ are entire functions satisfying the following conditions:*
1) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} = 0$
2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$
3) *For* $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k})(r \longrightarrow \infty, r \notin E)$.
Then $f_j(z) = 0 (j = 1, 2, \cdots, n)$.

Lemma 2.3. [1]*) Let $g(z)$ be a transcedental and meromorphic function in the plane of order is less than 1. If $h > 0$, then there exists an ϵ-set E such that*

$$\frac{g(z+\omega)}{g(z)} \to 1, \text{ when } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in ω for $\mid \omega \mid \leqslant h$.

Lemma 2.4. *([18] Theorem 1.44) Let $h(z)$ be a non-constant entire function, and $f(z) = e^{h(z)}$. Let ρ and μ be the order and the lower order of $f(z)$, respectively. Then we have*
(i) If $h(z)$ is a polynomial of degree p, then $\rho = \mu = p$.
(ii) If $h(z)$ is a transcidental entire function, then $\rho = \mu = \infty$.

Lemma 2.5. *([18] Theorem 1.18) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane with, $\rho(f)$ as the order of $f(z)$ and $\mu(g)$ as the lower order of $g(z)$. If $\rho(f) < \mu(g)$, then*

$$T(r, f) = o(T(r, g)), \qquad (r \to \infty).$$

Lemma 2.6. *([18] Theorem 1.14) Suppose $f(z)$ and $g(z)$ are two non-constant meromorphic functions in the complex plane with, $\rho(f)$ and $\rho(g)$ as their orders, respectively. Then $\rho(f \cdot g) \leq max\,\{\rho(f), \rho(g)\}$ and $\rho(f+g) \leq max\,\{\rho(f), \rho(g)\}$.*

3. Proof of the Theorem 1.4

Proof. Since f and $L(z, f)$ share $\{c, d\}$ CM, we can write

$$\frac{(L(z,f) - c)(L(z,f) - d)}{(f - c)(f - d)} = e^{\phi}, \tag{3.1}$$

where ϕ is an entire function. Furthermore, it deduces from (3.1) and $max\,\{\rho(c), \rho(d)\} < \rho(f) < \infty$ that ϕ is polynomial.

By the Hadmard Factorization Theorem, we have $f(z) = h(z)e^{p(z)}$, where $h(\not\equiv 0)$ is an entire function, and $p(z)$ is a polynomial satisfying

$$\lambda(f) = \rho(h) < \rho(f) = \rho(e^{p(z)}) = deg(p(z)). \tag{3.2}$$

Hence

$$L(z,f) = a_0h(z)e^{p(z)} + a_1h(z+t_1)e^{p(z+t_1)} + \cdots + a_kh(z+t_k)e^{p(z+t_k)}$$
$$= e^{p(z)}\Big(a_0h(z) + a_1h(z+t_1)e^{p(z+t_1)-p(z)} + \cdots + a_kh(z+t_k)e^{p(z+t_k)-p(z)}\Big). \tag{3.3}$$

Substituting this in (3.1), we get

$$\Big\{e^{p(z)}[a_0h(z) + a_1h(z+t_1)e^{p(z+t_1)-p(z)} + \cdots + a_kh(z+t_k)e^{p(z+t_k-p(z)}] - c(z)\Big\}$$
$$\Big\{e^{p(z)}[a_0h(z) + a_1h(z+t_1)e^{p(z+t_1)-p(z)} + \cdots + a_kh(z+t_k)e^{p(z+t_k)-p(z)})] - d(z)\Big\}$$
$$= (h(z)e^{p(z)} - c)(h(z)e^{p(z)} - d)e^{\phi}, \tag{3.4}$$

where $a_0, a_1, a_2 \cdots a_k$ are small functions of $f(z)$ and $t_1,\ t_2,\ \cdots, t_k$ are finite complex constants.
Set $w_1 = a_0h(z) + a_1h(z+t_1)e^{p(z+t_1)-p(z)} + \cdots + a_kh(z+t_k)e^{p(z+t_k)-p(z)}$.

If $w_1 \equiv 0$, then $L(z,f) \equiv 0$. It contradicts $L(z,f) \not\equiv 0$. Thus, $w_1 \not\equiv 0$.
Using (3.2), Lemma 2.4, and Lemma 2.5, we have

$$T(r,h) = o(T(r,e^{p(z)})), \quad (r \to \infty).$$

This implies that $h(z)$ is a small function of $e^{p(z)}$.
Using (3.2) and repeated application of Lemma 2.6 to w_1, we obtain that

$$\rho(w_1) \le \rho(e^{p(z+t_i)-p(z)}) = deg(p(z)-1) < degp(z) = \rho(e^{p(z)}).$$

Using this with Lemma 2.4, and Lemma 2.5, we get that

$$T(r,w_1) = o(T(r,e^{p(z)})), \quad (r \to \infty).$$

This implies that w_1 is a small function of $e^{p(z)}$.
From (3.4), following can be written

$$e^{\phi} = \frac{w_1^2[e^p - \frac{c}{w_1}][e^p - \frac{d}{w_1}]}{h^2[e^p - \frac{c}{h}][e^p - \frac{d}{h}]}. \tag{3.5}$$

Note that $c \neq d$. Without loss of generality, we suppose that $c \not\equiv 0$. Assume that z_0 is a zero of $e^p - \frac{c}{h}$ but not a zero of w_1. It follows from (3.5) and the assumption about sharing that z_0 is a zero of $e^p - \frac{c}{w_1}$ or $e^p - \frac{d}{w_1}$. We denote by $N_1(r, e^p)$ the reduced counting function of those common zeros of $e^p - \frac{c}{h}$ and $e^p - \frac{c}{w_1}$. Similarly, we denote by $N_2(r, e^p)$ the reduced counting function of those common zeros of $e^p - \frac{c}{h}$ and $e^p - \frac{d}{w_1}$. Note that h is a small function with respect to e^p. Applying the second fundamental theorem to e^p gives

$$\begin{aligned} T(r, e^p) &\leq \overline{N}\left(r, \frac{1}{e^p - \frac{c}{h}}\right) + S(r, f) \\ &= N_1(r, e^p) + N_2(r, e^p) + S(r, e^p). \end{aligned} \tag{3.6}$$

This implies that either $N_1(r, e^p) \neq S(r, e^p)$ or $N_2(r, e^p) \neq S(r, e^p)$. We consider the following two cases:

Case 1 : $N_1(r, e^p) \neq S(r, e^p)$.
Let c_0 be the common zero of $e^p - \frac{c}{h}$ and $e^p - \frac{c}{w_1}$. Then it is clear that c_0 is a zero of $\frac{c}{h} - \frac{c}{w_1}$. If $\frac{c}{h} - \frac{c}{w_1} \not\equiv 0$, then

$$S(r, e^p) \neq N_1(r, e^p) \leqslant N\left(r\,,\, \frac{1}{\frac{c}{h} - \frac{c}{w_1}}\right) \leqslant T\left(r\,,\, \frac{c}{h} - \frac{c}{w_1}\right) = S(r, e^p),$$

a contradiction. Thus $h \equiv w_1$.
It leads to

$$a_0 h(z) + a_1 h(z+t_1) e^{p(z+t_1)-p(z)} + \cdots + a_k h(z+t_k) e^{p(z+t_k)-p(z)} = h(z). \tag{3.7}$$

That is,

$$a_0 h(z) e^{p(z)} + a_1 h(z+t_1) e^{p(z+t_1)} + \cdots + a_k h(z+t_k) e^{p(z+t_k)} = h(z) e^{p(z)}.$$

This implies that

$$a_0 f(z) + a_1 f(z+t_1) + a_2 f(z+t_2) + ... + a_k f(z+t_k) = f(z).$$

That is, $L(z,f)=f(z)$ and also equation (3.7) can be written as

$$(a_0-1)h(z)+a_1h(z+t_1)e^{p(z+t_1)-p(z)}+\cdots+a_kh(z+t_k)e^{p(z+t_k)-p(z)})=0. \tag{3.8}$$

If $p(z+t_1)-p(z),\ p(z+t_2)-p(z),\cdots,\ p(z+t_k)-p(z)$ are not constants, then by Lemma 2.2, we get that $h(z)\equiv 0$, a contradiction.

So, at least one of $p(z+t_1)-p(z)$, $p(z+t_2)-p(z),\cdots,\ p(z+t_k)-p(z)$ is constant, say $p(z+t_i)-p(z)=d_i\, for\, some\,(i\in\{1,2,\cdots,k\})$, where d_i is a constant.
This gives $p'(z+t_i)-p'(z)=0$ and hence $p'(z)$ is a periodic function of period t_i. Also, $p'(z)$ is a polynomial. Since a polynomial can not be a periodic function unless it is a constant function, we get that $p'(z)$ is a constant function. Thus $p(z)$ is a polynomial of degree 1, say $p(z)=Cz+D$, where C and D are two constants and $C\neq 0$. Hence
$e^{p(z)}=e^{Cz+D}=e^De^{Cz}$ and $e^{p(z+t_i)}=e^{Cz}e^{Ct_i+D}$ for $i=1,2,\cdots,k$.
Hence, equation (3.8) becomes

$$(a_0-1)h(z)e^{Cz}e^D+a_1h(z+t_1)e^{Ct_1+D}e^{Cz}+\cdots+a_kh(z+t_k)e^{Ct_k+D}e^{Cz}=0. \tag{3.9}$$

That is,

$$a_1h(z+t_1)e^{Ct_1+D}e^{Cz}+\cdots+a_kh(z+t_k)e^{Ct_k+D}e^{Cz}=(1-a_0)h(z)e^{Cz}e^D.$$

This gives

$$a_1\frac{h(z+t_1)}{h(z)}e^{Ct_1}+\cdots+a_k\frac{h(z+t_k)}{h(z)}e^{Ct_k}=(1-a_0). \tag{3.10}$$

Using $\rho(h)<\rho(f(z))=\rho(e^{p(z)})=1$ and by Lemma 2.3, we obtain
$\dfrac{h(z+t_1)}{h(z)}\to 1,\cdots,\dfrac{h(z+t_k)}{h(z)}\to 1$ and using this, (3.10) leads to

$$a_0+a_1e^{Ct_1}+\cdots+a_ke^{Ct_k}=1.$$

Here $h(z)$ is a periodic function of periods $t_1,\ t_2,\ \cdots t_k$ and using $h(z)$ as an entire function whose growth is small compared to the growth of $e^{p(z)}=e^{Cz+D}$ gives that $h(z)$ is a non-zero constant function, say α. Using this, finally we get $f(z)=\alpha\, e^{Cz+D}=K\, e^{Cz}$, where $K=\alpha\, e^D$ is a non-zero constant and C satisfies $a_0+a_1e^{Ct_1}+\cdots+a_ke^{Ct_k}=1$.

Case 2: $N_2(r, e^p) \neq S(r, e^p)$.

Let d_0 be the common zero of $e^p - \frac{c}{h}$ and $e^p - \frac{d}{w_1}$. Then, it is clear that d_0 is a zero of $\frac{c}{h} - \frac{d}{w_1}$. If $\frac{c}{h} - \frac{d}{w_1} \not\equiv 0$, then

$$S(r, e^p) \neq N_2(r, e^p) \leqslant N\left(r\,, \frac{1}{\dfrac{c}{h} - \dfrac{d}{w_1}}\right) \leqslant T\left(r\,, \frac{c}{h} - \frac{d}{w_1}\right) = S(r, e^p),$$

a contradiction. Thus

$$\frac{c}{h} - \frac{d}{w_1} \equiv 0. \tag{3.11}$$

If $d \equiv 0$, then $\frac{c}{h} \equiv 0$, which implies $c \equiv 0$, which is in contradiction to the fact that c and d are distinct functions. So $d \not\equiv 0$.

We assume that t_0 is a zero of $e^p - \frac{d}{h}$ but not a zero of w_1. It follows from (3.5), that t_0 is a zero of $e^p - \frac{c}{w_1}$ or $e^p - \frac{d}{w_1}$. We denote by $N_3(r, e^p)$ the reduced counting function of those common zeros of $e^p - \frac{d}{h}$ and $e^p - \frac{c}{w_1}$. Similarly, we denote by $N_4(r, e^p)$ the reduced counting function of those common zeros of $e^p - \frac{d}{h}$ and $e^p - \frac{d}{w_1}$. Again, applying the second fundamental theorem to e^p gives

$$\begin{aligned} T(r, e^p) &\leq \overline{N}\left(r, \frac{1}{e^p - \dfrac{d}{h}}\right) + S(r, e^p) \\ &= N_3(r, e^p) + N_4(r, e^p) + S(r, e^p), \end{aligned} \tag{3.12}$$

which implies that either $N_3(r, e^p) \neq S(r, e^p)$ or $N_4(r, e^p) \neq S(r, e^p)$.

Subcase 2(i): If $N_4(r, e^p) \neq S(r, e^p)$, then as in case 1, we get the conclusion of the Theorem 1

Subcase 2(ii): If $N_3(r, e^p) \neq S(r, e^p)$, then as in the above Case 2, we can deduce that

$$\frac{d}{h} - \frac{c}{w_1} \equiv 0. \tag{3.13}$$

It follows from (3.11), and (3.13)

$$c^2 = d^2. \tag{3.14}$$

Note that $c \not\equiv d$. Thus, $c = -d$. Again, by (3.13), one has $w_1 = -h$, which is written as
$(a_0 + 1)h(z) + a_1 h(z + t_1)e^{p(z+t_1)-p(z)}, \cdots , a_k h(z + t_k)e^{p(z+t_k)-p(z)}) \equiv 0$
By Lemma 2.2, if $p(z + t_1) - p(z), p(z + t_2) - p(z), \cdots , p(z + t_k) - p(z)$ are not constants, then $h(z) \equiv 0$, a contradiction.
So, at least one of $p(z + t_1) - p(z)$, $p(z + t_2) - p(z), \cdots$, $p(z + t_k) - p(z)$ is constant. Continuing as in Case 1, we get the conclusion of the Theorem 1.4. □

Acknowledgement: The authors are grateful to the referees and editors for their valuable comments, and suggestions for improving the manuscript. The University Research Seed Grant Policy at Karnatak University, Dharwad, India, provided financial support for the first author's research project with the reference number KU/PMEB/2021-22/76, and the second author is supported by a URS fellowship at the Department of Mathematics at the same institution with the reference number KU/Sch/URS/2019-2020/616.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January-June (2024), 42–53

DIFFERENCE POLYNOMIALS OF ENTIRE FUNCTIONS SHARING A POLYNOMIAL OF CERTAIN DEGREE

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(Received : 07 - 05 - 2022 ; Revised : 07 - 08 - 2022)

Abstract. In this paper we use the notion of weighted sharing to investigate the uniqueness relation of entire functions when difference products say $f^nP(f)\Delta_c f$ and $g^nP(g)\Delta_c g$ share a non-zero polynomial $p(z)$. We also note the other occurrence for $P(z)$, which generalizes the result of Qi, Yang, and Liu [15].

1. Introduction

$f(z)$ is meromorphic if it is analytic in the complex plane except at isolated poles; if there are no poles, then $f(z)$ reduces to an entire function. In what follows, we assume the reader understands Nevanlinna's basic results and notation [8, 16]. Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$. We say that $f,\ g$ share a counted multiplicities (CM) if $f-a,\ g-a$ have the same zeros with the same multiplicities and we say that $f,\ g$ share a ignoring multiplicities (IM) if we do not consider the multiplicities, where a is a small function of f and g.
The relaxation has been done based on the notion of weighted sharing obtained by I. Lahiri as follows.

Definition 1.1. [7] Let $k \in \mathbb{N}\cup\{0\}\cup\{\infty\}$. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that $f,\ g$ share the value a with weight k.

The definition implies that if $f,\ g$ share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\le k)$ if and only if it is an a-point

2010 Mathematics Subject Classification: 30D35
Key words and phrases: Difference operator, uniqueness, Entire function, weighted sharing

of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $m(>k)$, where m is not necessarily equal to n. We write $f,\ g$ share (a,k) to mean that $f,\ g$ share the value a with weight k. Clearly if $f,\ g$ share (a,k) then $f,\ g$ share (a,p) for any integer $p,\ 0 \leq p < k$. Also we note that $f,\ g$ share a value a IM or CM if and only if $f,\ g$ share $(a,0)$ or (a,∞) respectively.
For a finite value z_0, if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$ then we say that a finite value z_0 is called a fixed point of f.
We use the notion (m^*) defined by

$$m^* = \begin{cases} 0, & \text{if } m = 0, \\ m, & \text{if } m \in \mathbb{N}, \end{cases}$$

for the sake of simplicity.

Let $f(z)$ be a transcendental meromorphic function, n be a positive integer. Many authors have investigated the value distributions of $f^n f'$. In 1959, Hayman [4] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geq 3$. A similar proposition was proved by Mues [12] in 1979 for $n = 2$. Bergweiler and Eremenko [1] showed that $ff' - 1$ has infinitely many zeros.
Laine and Yang [9, Theorem 2], investigated, corresponding to the above results, the value distribution of difference product of entire functions, and obtained the following result.

Theorem 1.2. [9] *Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f^n(z)f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

It is interesting to recall Yang and Hua [14] results corresponding to Theorem 1.2, which may be regarded as a gateway to new research to be conducted on the sharing of values between differential polynomials.

Theorem 1.3. [11] *Let f amd g be two non-constant entire functions, $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz},\ g(z) = c_2 c^{-cz}$, where $c_1,\ c_2,\ c \in \mathbb{C}$ statisifying $4(c_1c_2)^{(n+1)}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2001, Fang and Hong studied the uniqueness of differential polynomials of the form $f^n(f-1)f'$ and $g^n(g-1)g'$ and proved the following uniqueness result.

Theorem 1.4. [3] *Let f amd g be two non-constant entire functions, and let $n \geq 11$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f = g$.*

In 2004, Lin and Yi extended the above result in the fixed point view, and they proving the following.

Theorem 1.5. [10] *Let f amd g be two transcendental entire functions, and let $n \geq 7$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f = g$.*

In 2010, Zhang got an analog result in the difference.

Theorem 1.6. [17] *Let f amd g be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

In 2010, Qi, Yang and Liu [15] proved the following uniqueness theorem regarding shift operator, which is a difference counterpart of Theorem 1.3.

Theorem 1.7. [15] *Let f and g be transcendental entire functoins of finite order, let c be a non-zero complex constant, and let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share z CM, then $f(z) \equiv tg(z)$ for a constant t satisifying $t^{n+1} = 1$.*

Theorem 1.8. [15] *Let f and g be transcendental entire functoins of finite order, let c be a non-zero complex constant, and let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM, then $fg \equiv t_2$ or $f \equiv t_3 g$ for a constant t_2 and t_3 that satisfy $t_3^{n+1} = 1$.*

So we have seen that there are many generalizations of the difference polynomial. The purpose of this paper is to study the uniqueness problem for more general difference polynomials $f^n P(f)\Delta_c f(z)$ and $g^n P(g)\Delta_c g(z)$ sharing a non-zero polynomial $p(z)$. Here we also note the other occurrence for $P(z)$.
We now present the following theorem, which is the main result of the paper.

Theorem 1.9. *Let f and g be transcendental entire functoins of finite order, let $c \in \mathbb{C} \setminus \{0\}$ be a complex constant and let $p(z)$ be a non-zero polynomial with $deg(p) \leq n-1$, $n(\geq 2)$, m^* be two integers such that $n > m^* + 7$. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ be a non-zero polynomial. If $f^n P(f)\Delta_c f(z) - p(z)$ and $g^n P(g)\Delta_c g(z) - p(z)$ share $(0, 2)$ then*

(1) When $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ is a non-zero polynomial, one of the following three cases holds,
 (a) $f(z) = tg(z)$ for a constant t such that $t^d = 1$,
 where $d = \gcd(n+1, \cdots, n+m+1-i, \cdots, n+m+1)$,
 (b) f and g satisify the algebraic equation $R(f,\ g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n P(w_1)\Delta_c w_1(z) - w_2^n P(w_2)\Delta_c w_2(z),$$

 (c) $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$, for $i \in \{0, 1, 2, \cdots, m\}$ if $p(z)$ is a non-zero constant b, then $f(z) = c_1 e^{d_1 z}$, $g(z) = c_2 e^{d_2 z}$ where $d_1,\ d_2,\ c_1,\ c_2$ are non-zero constants such that $a_i (c_1 c_2)^{(n+1+i)} \left(e^{d_1 z} + e^{-d_1 z} - 2\right) = -b^2$;

(2) When $P(z) = z^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;

(3) When $P(z) = (z-1)^m$ $(m \geq 2)$, one of the following two cases holds:
 (a) $f(z) = g(z)$,
 (b) f and g satisify the algebraic equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n P(w_1)\Delta_c w_1(z) - w_2^n P(w_2)\Delta_c w_2(z),$$

(4) When $P(z) \equiv c_0$, then $f \equiv tg$ for some constant t such that $t^{n+1} = 1$.

2. Necessary Lemmas

Lemma 2.1. [2] *Let f be a meromorphic function of finite order σ, and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\sigma-1+\epsilon}) = S(r, f).$$

The following lemma has a few modifications to the original version [2, Corollary 2.5]

Lemma 2.2. [2] *Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.3. [5] *Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then $N\left(r, 0; f(z+c)\right) \leq N\left(r, 0; f(z)\right) + S(r, f)$,*
$N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f)$, $\overline{N}\left(r, 0; f(z+c)\right) \leq \overline{N}\left(r, 0; f(z)\right) + S(r, f)$,
$\overline{N}(r, \infty; f(z+c)) \leq \overline{N}(r, \infty; f) + S(r, f)$.

Lemma 2.4. [16] *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), ..., a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, ..., n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.5. *Let f be an entire function of finite order ρ, c be a fixed non-zero complex constant and let $n \in \mathbb{N}$ and $P(z)$ be defined as in Theorem 1.9 then for each $\epsilon > 0$, we have*

$$T\left(r, f^n P(f) \Delta_c f(z)\right) = (n + m^* + 1)T(r, f) + O(r^{\rho-1+\epsilon}).$$

Proof. By Lemma 2.1,

$$\begin{aligned} T\left(r, f^n P(f) \Delta_c f\right) &= m\left(r, f^n P(f) \Delta_c f\right), \\ &\leq T\left(r, f^{n+1} P(f)\right) + O\left(r^{\rho-1+\epsilon}\right), \\ &\leq (n + m^* + 1)T(r, f) + s(r, f). \\ (n + m^* + 1)T(r, f) &= T\left(r, f^{n+1} P(f)\right), \\ &\leq m\left(r, f^n P(f) \Delta_c f\right) + m\left(r, \frac{f}{\Delta_c f}\right), \\ &\leq m\left(r, f^n P(f) \Delta_c f\right) + S(r, f). \end{aligned}$$

Therefore,

$$T\left(r, f^n P(f) \Delta_c f(z)\right) = (n + m^* + 1)T(r, f) + S(r, f).$$

□

Lemma 2.6. *Let f be a transcendental entire function of finite order ρ, $c \in \mathbb{C} \setminus \{0\}$ be a complex constant , $n(\geq 1)$, $m^*(\geq 0)$ be two integer and let $a(z)(\not\equiv 0\,, \infty)$ be a small function of f. If $n > 2$, then $f^n P(f) \Delta_c f - a(z)$ has infinitely many zero's.*

Proof. Let $\Psi = F^n P(f)\Delta_c f$, now from Lemma 2.3, and [13], we get

$$\begin{aligned}
T(r,\psi) &\leq \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}(r,\psi) + \overline{N}\left(r,\frac{1}{\psi - a(z)}\right) + (\epsilon + o(1)),\\
&\leq 3\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{P(f)}\right) + \overline{N}\left(r,\frac{1}{\psi - a(z)}\right) + (\epsilon + o(1)),\\
&\leq (3+m^*)T(r,f) + \overline{N}\left(r,\frac{1}{\psi - a(z)}\right) + (\epsilon + o(1)),
\end{aligned}$$

from Lemma 2.4 and 2.5,

$$(n + m^* + 1)\, T(r,f) \leq (3+m^*)T(r,f) + \overline{N}\left(r,\frac{1}{\psi - a(z)}\right).$$

Take $\epsilon < 1$. Since $n > 2$ from above, one can easily say that $\psi - a(z)$ has infinitely many zero's. $\square$

Lemma 2.7. *[Hadamard Factorization Theorem] Let f be an entire function of finite order ρ with zero's a_1, $a_2, \cdots, a_n$ each zero is counted as often as its multiplicity. Then f can be expressed in the form $f(z) = Q(z)e^{\alpha(z)}$, where $\alpha(z)$ is a polynomial of degree not exceeding $[\rho]$ and $Q(z)$ is the canonical product formed with the zeros of f.*

Lemma 2.8. [6] *Let f and g be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following holds:*

(1) $T(r,f) \leq N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + N_2(r,f) + N_2(r,g) + s(r,f) + s(r,g)$,

(2) $fg = 1$,

(3) $f \equiv g$.

Lemma 2.9. *Let f, g be two transcendental entire functions of finite order, $c \in \mathbb{C}\setminus\{0\}$ be a complex constant and $n \in \mathbb{N}$ and p be a non-zero polynomial such that deg $(p) \leq (n-1)$ where $n \in \mathbb{N}$. Let $P(z)$ be a non-zero polynomial de?ned as in Theorem 1.9. Suppose*

$$(f^n P(f)\Delta_c f(z))\,(g^n P(g)\Delta_c g(z)) \equiv p^2,$$

then $P(z)$ reduces to a non-zero monomial namely $P(z) = a_i z^i \neq 0$. For $i \in \{0,1,2,\cdots,m\}$. If $p(z) = b \in \mathbb{C}\setminus\{0\}$ then $f(z) = c_1 e^{d_1(z)}$, $g(z) = c_2 e^{d_2(z)}$ where d_1, d_2, c_1, and c_2 are non-zero constants such that,

$$a_i^2\,(c_1 c_2)^{(n+1+i)}\left(e^{d_1 c} + e^{-d_1 c} - 2\right) = -b^2$$

Proof. Let $F = f^nP(f)\Delta_c f$ and $G = g^nP(g)\Delta_c g$,
suppose
$$FG = p^2, \tag{2.1}$$
we consider the following cases,
Case I. Let deg $p(z) = l(\geq 1)$, since F and G are transcendental entire functions, we deduce by 2.1 that $N\left(r, \frac{1}{f}\right) = O(logr) = N\left(r, \frac{1}{g}\right)$.
Suppose $P(z)$ is not a non-zero monomial. For the sake of simplicity let $P(z) = z - a$ where $a \in \mathbb{C} \setminus \{0\}$, clearly $\Theta(0, f) + \Theta(a, f) = 2$, which is impossible for a entire function. Thus $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \neq 0$ for some $i \in \{0, 1, 2, ..., m\}$ and so 2.1 reduces to
$$a_i^2 f^{n+i}\Delta_c f g^{n+i}\Delta_c g \equiv p^2. \tag{2.2}$$
From 2.2 it follows that $N\left(r, \frac{1}{f}\right) = O\left(logr\right) = N\left(r, \frac{1}{g}\right)$. From Lemma 2.7, we obtain that $f = h_1e^{\alpha_1}$ and $g = h_2e^{\beta_1}$, where h_1, h_2 are two nonconstant ploynomials and α_1, β_1 are two non constant polynomials. By virtue of the polynomial $p(z)$, from 2.2 we arrive at a contradiction.
Case II. Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Then from 2.1, we have
$$\left(f^nP(f)\Delta_c f(z)\right)\left(g^nP(g)\Delta_c g(z)\right) \equiv b^2, \tag{2.3}$$
Now from the assumption that f and g are two nonconstant entire functions, we deduce by 2.3 that $f^nP(f) \neq 0$ and $g^nP(g) \neq 0$. By Picard's theorem, we claim that $P(z) = a_1z^i \not\equiv 0$ for $i \in \{0, 1, 2, \cdots, m\}$, otherwise the Picard's exception values are atleast three which is contradiction. Then 2.3 reduces to
$$(a_i^2 f^{n+i}\Delta_c f)(g^{n+i}\Delta_c g) \equiv b^2. \tag{2.4}$$
Hence by Lemma 2.7, we obtain that
$$f = e^{\alpha},\ g = e^{\beta} \tag{2.5}$$
$$a_i^2 e^{(\alpha+\beta)(n+i)}\left(e^{\alpha(z+c)} - e^{\alpha(z)}\right)\left(e^{\beta(z+c)} - e^{\beta(z)}\right) = b^2,$$
$$\left(e^{\alpha(z+c)-\alpha(z)} - 1\right)\left(e^{\beta(z+c)-\beta(z)} - 1\right) = d^2e^{(\alpha(z)+\beta(z))(n+i+1)} \tag{2.6}$$
we conclude that from 2.6 that $e^{\alpha(z+c)-\alpha(z)} - 1$ has no zero's.
Let $\psi = e^{\alpha(z+c)-\alpha(z)}$, then $\psi \neq 0, 1, \infty$ for any $z \in \mathbb{C}$. By picard theorem, ψ is constant, so deg$(\alpha(z)) = 1$. Similarly we can prove that deg$(\beta) = 1$.
Assume now that $f(z) = c_1e^{d_1(z)}$, $g(z) = c_2e^{d_2(z)}$ where d_1, d_2, c_1, c_2 are non-zero constants.

From 2.3, we get $d_1 = -d_2$ and $a_i^2\,(c_1c_2)^{(n+1+i)}\left(e^{d_1c} + e^{-d_1c} - 2\right) = -b^2$. Hence the proof. $\square$

Lemma 2.10. *Let f and g be two transcendental entire functions of finite order $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ with $n > 1$. If $f^nP(f)\Delta_c f \equiv g^nP(g)\Delta_c g$, where $P(z)$ is defined as in the Theorem 1.9 then*

(1) When $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$ is a non-zero polynomial, one of the following three cases holds,
- *(a) $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = \gcd(n+1, \cdots, n+m+1-i, \cdots, n+m+1)$,*
- *(b) f and g satisify the algebraic equation $R(f,\ g) \equiv 0$, where*

$$R(w_1, w_2) = w_1^nP(w_1)\Delta_c w_1(z) - w_2^nP(w_2)\Delta_c w_2(z),$$

(2) When $P(z) = z^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;

(3) When $P(z) = (z-1)^m (m \geq 2)$, one of the following two cases holds:
- *(a) $f(z) = g(z)$,*
- *(b) f and g satisify the algebraic equation $R(f, g) \equiv 0$, where*

$$R(w_1, w_2) = w_1^nP(w_1)\Delta_c w_1(z) - w_2^nP(w_2)\Delta_c w_2(z),$$

(4) When $P(z) \equiv c_0$, then $f \equiv tg$ for some constant t such that $t^{n+1} = 1$.

Proof. Suppose

$$f^nP(f)\Delta_c f \equiv g^nP(g)\Delta_c g. \tag{2.7}$$

Since g is transcendental entire function, hence $g(z),\ g(z+c) \neq 0$ we consider following cases,

Case 1. $P(z) = c_0$,

let $h = \frac{f}{g}$, if h is a constant by putting $f = hg$ in 2.7, we get

$$a_mg^m\left(h^{(n+m+1)} - 1\right) + a_{m-1}g^{m-1}\left(h^{(n+m)} - 1\right) + \cdots + a_0\left(h^{(n+1)} - 1\right) \equiv 0,$$

which implies that $h^d = 1$, where d is such that $t^d = 1$, where d is the gcd of the elements of $J = \{p \in I : a_p \neq 0\}$ and $I = \{n+1, n+2, \cdots, n+m+1-i, \cdots, n+m+1\}$.

Thus $f = tg$ for a constant t such that $t^d = 1$, where d is the GCD of the elements of $J = \{p \in I : a_p \neq 0\}$ and $I = \{n+1, n+2, \cdots, n+m+1-i, \cdots, n+m+1\}$ $i \in \{0, 1, 2, \cdots, m\}$.

If h is not a constant then we know that by 2.7 that f and g satisifying the algebraic equation $R(f,g)=0$, where $R(w_1,w_2)=w_1^nP(w_1)\Delta_c w_1(z)-w_2^nP(w_2)\Delta_c w_2(z)$. We now discuss the following subcases,

Subcase 1.1. $P(z)=z^m-1$, Then from 2.7, we have

$$f^n(f^m-1)\Delta_c f\equiv g^n(g^m-1)\Delta_c g. \tag{2.8}$$

Let $h=\frac{f}{g}$ and from 2.8, we get

$$h^{n+1}=\frac{(g^m-1)\Delta_c g}{g}.\frac{(f^m-1)\Delta_c f}{f}$$

If h is not constant, then we have

$$\begin{aligned}(n+1)T(r,h)\leq T\left(r,\frac{(g^m-1)\Delta_c g}{g}\right)+T\left(r,\frac{(f^m-1)\Delta_c f}{f}\right)+S(r,f)\\ +S(r,g),\end{aligned}$$

$$(n+1)T(r,h)\leq(m+1)\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g).$$

Combining the above inequality with,

$$T(r,h)=T\left(r,\frac{f}{g}\right)=T(r,f)+T(r,g)+S(r,f)+S(r,g),$$

we obtain,

$$(n-m)\left(T(r,f)+T(r,g)\right)\leq S(r,f)+S(r,g),$$

which is impossible. Therefore h is a constant, then by substituting $f=gh$ into 2.8, we get

$$(gh)^n((gh)^n-1)\Delta_c(gh)=g^n(g^n-1)\Delta_c g,$$

$$g^m\left(h^{(n+m+1)}-1\right)=h^{n+1}-1. \tag{2.9}$$

Since h is constant and g is transcendental entire function, from 2.9, we have $h^{(n+m+1)}-1=0$ if and only if $h^{n+1}-1=0$ and so $h^m=1$. Thus $f(z)=tg(z)$ for a constant t such that $t^m=1$.

Subcase 1.2. Let $P(z)=(z-1)^m$. Then from 2.7, we have

$$f^n(f-1)^m\Delta_c f\equiv g^n(g-1)^m\Delta_c g. \tag{2.10}$$

Let $h=\frac{f}{g}$, if $m=1$ then the result follows from subcase 1.1

For $m\geq 2$, first we suppose that h is non-constant then from 2.10, we can say that f and g satisify the algebraic equation $R(f,g)=0$,

where $R(w_1,w_2)=w_1(w_1-1)^m\Delta_c w_2-w_2(w_2-1)^m\Delta_c w_2$. Next we suppose that h is constant, then from 2.10,we get

$$f^n\Delta_c f(z)\sum_{i=0}^{m}(-1)^i\binom{m}{m-i}f^{m-i}=g^n\Delta_c g(z)\sum_{i=0}^{m}(-1)^i\binom{m}{m-i}g^{m-i} \tag{2.11}$$

Now substitute $f=gh$ in 2.11, we get

$$\sum_{i=0}^{m}(-1)^m\binom{m}{m-i}g^{m-i}\left[h^{n+m+1-i}-1\right]\equiv 0,$$

which implies that $h=1$. Hence $f\equiv g$.

Case.2 $P(z)=c_0$.

Let $h=\frac{f}{g}$, then from 2.7, we have

$$\begin{aligned}f^n c_0\Delta_c f &= g^n c_0\Delta_c g,\\ h^{n+1} &= \frac{\Delta_c g}{g}.\frac{f}{\Delta_c f}.\end{aligned} \tag{2.12}$$

Therefore from Lemma 2.3 and 2.4,

$$\begin{aligned}(n+1)T(r,h) &= T\left(r,\frac{\Delta_c g}{g}\right)+T\left(r,\frac{f}{\Delta_c f}\right),\\ &\leq 2\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g),\end{aligned}$$

$$(n+1)\left(T(r,f)+T(r,g)\right)\leq 2\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g).$$

Contradicts with $n\geq 2$. Hence h must be constant, which implies that $h^{n+1}=1$, thus $f\equiv tg$ and $t^{n+1}=1$. Which completes the proof. □

3. Proof of Theorem

Proof. Let $F=\frac{f^nP(f)\Delta_c f}{p(z)}$ and $G=\frac{g^nP(g)\Delta_c g}{p(z)}$ Then F, G share $(1,2)$ except the zero's of $p(z)$. Now applying Lemma 2.8, we see that one of the following three cases holds

Case 1. Suppose

$$\begin{aligned}T(r,F) &\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G),\\ &\leq 2\left(N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right)+N\left(r,\frac{1}{P(f)}\right)\\ &\quad+N\left(r,\frac{1}{\Delta_c f}\right)+N\left(r,\frac{1}{P(g)}\right)+N\left(r,\frac{1}{\Delta_c g}\right)\\ &\quad+S(r,f)+S(r,g),\end{aligned}$$

$$\begin{aligned} T(r,F) &\leq (m^*+4)\left(T(r,f)+T(r,g)\right)+S(r,f)+S(r,g),\\ &\leq (2m^*+8)T(r)+S(r). \end{aligned}$$

From Lemma 2.5,

$$(n+m^*+1)T(r,f) \leq (2m^*+8)T(r)+S(r). \tag{3.1}$$

Similarly, we have

$$(n+m^*+1)T(r,g) \leq (2m^*+8)T(r)+S(r). \tag{3.2}$$

Combining 3.1 and 3.2, we get

$$(n+m^*+1)T(r) \leq (2m^*+8)T(r)+S(r)$$

which contradicts with $n > m^*+7$.
Case 2. $F \equiv G$,
$f^nP(f)\Delta_c f = g^nP(g)\Delta_c g$ and so the result follows from Lemma 2.10.
Case 3. $FG \equiv 1$.
Then we have $(f^nP(f)\Delta_c f)(g^nP(g)\Delta_c g) \equiv p^2$ and so the result follows from Lemma 2.9.

Which completes the proof.

□

Acknowledgement: The author is also grateful to the anonymous referee for his/her valuable suggestions which considerably improved the presentation of the paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2 , January-June (2024), 54–66

NEW METHOD OF LINES SOLUTION FOR THE NONLINEAR SINE GORDON EQUATION WITH REPRODUCING KERNEL HILBERT SPACE METHOD

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(Received : 28 - 08 - 2022 ; Revised : 15 - 06 - 2023)

Abstract. The paper reports a method of lines (MOL) for generating a solution for the sine Gordon equation in one dimension with suitable initial and boundary conditions based on the reproducing kernel Hilbert space method (RKHSM). Two valid test examples are given to determine the validity and effectiveness of the current technique. We observed that the presented method has better accuracy and efficiency compared to the other methods in the literature.

1. Introduction

Nonlinearity appear in many branches of scientific fields such as plasma physics, solid state physics, mathematical biology, fluid dynamics and chemical kinetics, can be modeled by partial differential equations (PDEs). A broad class of analytical and numerical solution methods have been used to handle these problems.

In this article, we consider the following one dimensional sine Gordon equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - sin(u), (x,t) \in \Omega \times \Gamma, \tag{1.1}$$

where $u = u(x,t)$ represents the wave displacement at position x and time t and $sin(u)$ is the nonlinear force, with initial and boundary conditions

$$u(x,0) = \Theta_1(x), \quad u_t(x,0) = \Theta_2(x), \quad x \in \overline{\Omega}, \tag{1.2}$$

$$u(a,t) = \Phi_1(t), \quad u(b,t) = \Phi_2(t), \quad t \in \overline{\Gamma}, \tag{1.3}$$

2010 Mathematics Subject Classification: 46C99, 46E22, 37N99, 37N25
Key words and phrases: Mathematical model, Nonlinear system of ODEs, Reproducing kernel Hilbert space method, Method of Lines

where $\Omega = (a, b)$, $\Gamma = (0, T]$ with $0 < T < \infty$.

The sine Gordon equation is a nonlinear hyperbolic equation, which is a model of soliton wave. It was first introduced by Edmond Bour [6] in the study of surfaces of constant negative curvature and rediscovered by Frenkel and Kontorova [12] in the study of the Frenkel-Kontorova model. Also, the sine Gordon equation appears in many applications like stability of fluid motion, differential geometry, applied sciences and nonlinear physics [3], propagation of fluxion in Josephson junctions [25].

The study of the solution of nonlinear sine Gordon wave equation has been gained much attention in the past several decades. It has been solved by Ablowitz et al. [1] using the inverse-scattering method. Ben-Yu et al. [4] presented a numerical solution by two difference schemes. Bratsos [7] solved the equation using a fourth-order rational approximation to the matrix exponential term in a three time level recurrence relation for the numerical solution. Dehghan and Shokri [11] used radial basis functions with collocation points to approximate the solution. Uddin et al. [27] evaluated numerical solution using a meshfree approach based on radial basis function. Mittal and Bhatia [19] discussed for the numerical solution using modified cubic B-spline collocation method based on collocation of modified cubic B-splines over finite elements.

In this paper we present a semi analytical method of lines (MOL) solution. The MOL were applied to solve the PDEs in fully numerical based by Schiesser et al. [26]. In 2004, Koto [15] applied this technique to approximations of delay differential equations using Runge Kutta method. Hamdi et al. [14] gave basic idea of MOL. The semi analytic MOL is used actively for solving linear PDEs. For example see [21, 22, 23].

In our work a different approach is used, the usual finite difference scheme is employed for spatial discretization in the nonlinear initial boundary valued PDE to convert nonlinear initial valued system of ordinary differential equations (ODEs) and then using RKHSM to find a solution. The theory of reproducing kernels [24] dates to the first half of the 20th century, and its roots go back to the pioneering papers by S. Zeremba [28], Mercer [18], and Bergman [5]. In 1950, N. Aronszajn [2] outlined the past works and gave a systematic reproducing kernel theory and laid a good foundation for the research of each special case and greatly simplified the proof. This theory has been successfully applied on linear and nonlinear applications with

different type conditions by many researchers [8, 9, 13, 16]. The analytic solution of RKHSM is represented in the form of series. The RKHSM is easily implemented, grid free and without time discretization. Also, we can evaluate the solution for finite number of points and use it often.
The paper is laid out as follows. In the next section, we show how we use MOL to solve the sine Gordon equation. The results of numerical experiments are presented in Section 3. Final Section is dedicated to a brief conclusion. Finally the references are listed at the end.

2. Method of Lines

In this section, we derive MOL to solve the sine Gordon equation using RKHSM. To do this, we divide the section into two subsections. In the first subsection, we discretize the spatial derivatives to obtain a system of ODEs in the time variable. In the second subsection, we explain RKHSM to solve the system of ODEs.

2.1. **Discretization.** To use the MOL for solving (1.1)-(1.3), we discretize the spatial coordinate x with $m-1$ grid points $x_i = x_{i-1} + h, h = (b-a)/m, x_0 = a, x_m = b, i = 1, 2, \ldots, m-1$. We apply a second-order difference approximation for the second-order derivative in x in grid points x_1, x_{m-1} and a fourth-order difference approximation to the second-order derivative in x in grid points $x_i, i = 2, 3, \ldots, m-2$.
Let us consider $u_i(t)$ approximate $u(x_i, t)$. Here, using the central difference approximations in second-order and forth-order for the second order derivative in x of (1.1), we get

$$\begin{aligned}
\frac{d^2u_1}{dt^2} &= \frac{u_0 - 2u_1 + u_2}{h^2} - sin(u_1),\\
\frac{d^2u_{m-1}}{dt^2} &= \frac{u_{m-2} - 2u_{m-1} + u_m}{h^2} - sin(u_{m-1}),\\
\frac{d^2u_i}{dt^2} &= \frac{-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}}{12h^2} - sin(u_i),\\
& i = 2, 3, \ldots, m-2.
\end{aligned} \tag{2.1}$$

Further, the conditions (1.2) and (1.3) become

$$\begin{aligned}
&u_i(0) = \Theta_0(x_i), \quad \frac{du_i(0)}{dt} = \Theta_1(x_i), \quad i = 1, 2, 3, \ldots, m-1,\\
&u_0 = \Phi_1(t), \quad u_m = \Phi_2(t).
\end{aligned}$$

We change system (2.1) to a system of the first-order ODEs by using

$$v_i = \frac{du_i}{dt}, \quad i = 1, 2, \ldots, m-1.$$

Consequently, we get the following initial values system of $2m-2$ equations

$$\begin{aligned}
\frac{du_i}{dt} =& v_i \quad i = 1, 2, \ldots, m-1, \\
\frac{dv_1}{dt} =& \frac{\Phi_1(t) - 2u_1 + u_2}{h^2} - sin(u_1), \\
\frac{dv_2}{dt} =& \frac{-\Phi_1(t) + 16u_1 - 30u_2 + 16u_3 - u_4}{12h^2} - sin(u_2), \\
\frac{dv_i}{dt} =& \frac{-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}}{12h^2} - sin(u_i), \\
& i = 3, 4, \ldots, m-3, \\
\frac{dv_{m-2}}{dt} =& \frac{-u_{m-4} + 16u_{m-3} - 30u_{m-2} + 16u_{m-1} - \Phi_2(t)}{12h^2} \\
& - sin(u_{m-2}), \\
\frac{dv_{m-1}}{dt} =& \frac{u_{m-2} - 2u_{m-1} + \Phi_2(t)}{h^2} - sin(u_{m-1}),
\end{aligned} \tag{2.2}$$

with initial conditions

$$u_i(0) = \Theta_0(x_i), \quad v_i(0) = \Theta_1(x_i), \quad i = 1, 2, \ldots, m-1. \tag{2.3}$$

In the next section, we will discuss the RKHSM to solve the first-order nonlinear system of ODEs with homogeneous initial conditions.

2.2. **The Reproducing Kernel Hilbert Space Method.** In this section, firstly we construct a homogeneous initial values system of ODEs from the equations (2.2) and (2.3).
To do this, we take $u_i = \Theta_0(x_i)(1-t) + w_i$, $v_i = \Theta_1(x_i)(1-t) + w_{m-1+i}$, $i = 1, 2, \ldots, m-1$, we get

$$\begin{aligned}
\frac{dw_i}{dt} =& \Theta_0(x_i) + \Theta_1(x_i)(1-t) + w_{m-1+i} \quad i = 1, 2, \ldots, m-1, \\
\frac{dw_m}{dt} =& \Theta_1(x_1) + \frac{\Phi_1(t) - 2(\Theta_0(x_1)(1-t) + w_1) + \Theta_0(x_2)(1-t) + w_2}{h^2} \\
& - sin(\Theta_0(x_1)(1-t) + w_1), \\
\frac{dw_{m+1}}{dt} =& \Theta_1(x_2) + \frac{-\Phi_1(t) + 16(\Theta_0(x_1)(1-t) + w_1) - 30(\Theta_0(x_2)(1-t) + w_2)}{12h^2} \\
& + \frac{16(\Theta_0(x_3)(1-t) + w_3) - (\Theta_0(x_4)(1-t) + w_4)}{12h^2} - sin(\Theta_0(x_2)(1-t) + w_2), \\
\frac{dw_{m-1+i}}{dt} =& \Theta_1(x_i) + \frac{-(\Theta_0(x_{i-2})(1-t) + w_{i-2}) + 16(\Theta_0(x_{i-1})(1-t) + w_{i-1})}{12h^2} \\
& + \frac{-30(\Theta_0(x_i)(1-t) + w_i) + 16(\Theta_0(x_{i+1})(1-t) + w_{i+1}) - (\Theta_0(x_{i+2})(1-t) + w_{i+2})}{12h^2}
\end{aligned}$$

$$
\begin{aligned}
&- sin(\Theta_0(x_i)(1-t)+w_i), i=3,4,\ldots,m-3, \qquad (2.4)\\
\frac{dw_{2m-3}}{dt} =&\Theta_1(x_{m-2})+\frac{-(\Theta_0(x_{m-4})(1-t)+w_{m-4})+16(\Theta_0(x_{m-3})(1-t)+w_{m-3})}{12h^2}\\
&+\frac{-30(\Theta_0(x_{m-2})(1-t)+w_{m-2})+16\Theta_{m-1}-\Phi_2(t)}{12h^2}\\
&- sin(\Theta_0(x_{m-2})(1-t)+w_{m-2}),\\
\frac{dw_{2m-2}}{dt} =&\Theta_1(x_{m-1})+\frac{\Theta_0(x_{m-2})(1-t)+w_{m-2}-2(\Theta_0(x_{m-1})(1-t)+w_{m-1})+\Phi_2(t)}{h^2}\\
&- sin(\Theta_0(x_{m-1})(1-t)+w_{m-1}),
\end{aligned}
$$

with homogeneous initial conditions

$$w_i(0)=0, i=1,2,\ldots,2m-2. \tag{2.5}$$

Now, we introduce the reproducing kernel Hilbert spaces $\mathcal{W}_2^2[0,T]$ and $\mathcal{W}_2^1[0,T]$ with corresponding reproducing kernel functions $\mathcal{R}(t,s)$ and $\mathcal{G}(t,s)$, respectively, to generate the algorithm of the RKHSM.

Definition 2.1. [10] Consider $\mathcal{H}$ is a Hilbert space of real or complex valued functions defined on a nonempty set X endowed with $\langle f,g\rangle_{\mathcal{H}}$.
A function $\mathcal{R}(\cdot,\cdot): X\times X\to\mathbb{F}$ ($\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$) is called *a reproducing kernel* of the Hilbert space $\mathcal{H}$ if it satisfies,

$$\langle f,\mathcal{R}(\cdot,s)\rangle_{\mathcal{H}}=f(s), \tag{2.6}$$

for each fixed $s\in X$.

The equation (2.6) is known as “the reproducing property”.

Definition 2.2. [20] The inner product space $\mathcal{W}_2^2[0,T]$ is defined as $\mathcal{W}_2^2[0,T]=\{w: w,w'$ are absolutely continuous real valued functions on $[0,T]$, $w''\in\mathcal{L}^2[0,T]$, and $w(0)=0\}$ with the inner product and the norm of $\mathcal{W}_2^2[0,T]$ are defined, respectively, by

$$\langle w,y\rangle_{\mathcal{W}_2^2}=w(0)y(0)+\frac{dw(0)}{dt}\frac{dy(0)}{dt}+\int_0^T\frac{d^2w(t)}{dt^2}\frac{d^2y(t)}{dt^2}dt,$$

$$\|w\|_{\mathcal{W}_2^2}=\sqrt{\langle w,w\rangle_{\mathcal{W}_2^2}}.$$

Theorem 2.3. [20] *The Hilbert space $\mathcal{W}_2^2[0,T]$ is a reproducing kernel Hilbert space and its reproducing kernel function $\mathcal{R}(\cdot,\cdot)$ can be written as*

$$\mathcal{R}(t,s)=\begin{cases}\frac{s}{6}(6t+3ts-s^2), & s\le t,\\ \frac{t}{6}(6s+3ts-t^2), & s>t.\end{cases}$$

Definition 2.4. [17] The inner product space $\mathcal{W}_2^1[0,T]$ is defined as $\mathcal{W}_2^1[0,T] = \{w : w \text{ is absolutely continuous real valued function on } [0,T],\ w' \in \mathcal{L}^2[0,T]\}$ with the inner product and the norm of $\mathcal{W}_2^1[0,T]$ are defined, respectively, by

$$\langle w, y\rangle_{\mathcal{W}_2^1} = w(0)y(0) + \int_0^T \frac{dw(t)}{dt}\frac{dy(t)}{dt}dt,$$

$$\|w\|_{\mathcal{W}_2^1} = \sqrt{\langle w, w\rangle_{\mathcal{W}_2^1}}.$$

Theorem 2.5. [17] *The Hilbert space $\mathcal{W}_2^1[0,T]$ is a reproducing kernel Hilbert space and its reproducing kernel function $\mathcal{G}(\cdot,\cdot)$ can be written as*

$$\mathcal{G}(t,s) = \begin{cases} 1+s, & s \le t, \\ 1+t, & s > t. \end{cases}$$

We interpret a differential operator for $Lw_i(t) = \frac{dw_i}{dt}$, $i = 1, 2, \dots, 2m-2$ as $L : \mathcal{W}_2^2[0,T] \to \mathcal{W}_2^1[0,T]$, such that equations (2.4) and (2.5) become $Lw_i(t) = f_i(t, w_1(t), w_2(t), \dots, w_{2m-2}(t)), 0 < t < T$, subject to the initial conditions, $w_i(0) = 0, i = 1, 2, \dots, 2m-2$, where $w_i(\cdot) \in \mathcal{W}_2^2[0,T]$ and $f_i(\cdot, w_1(\cdot), w_2(\cdot), \dots, w_{2m-2}(\cdot)) \in \mathcal{W}_2^1[0,T]$.

Theorem 2.6. *The operator $L : \mathcal{W}_2^2[0,T] \to \mathcal{W}_2^1[0,T]$ is bounded and linear.*

Proof. For the proof, we refer to [10]. □

Now, we consider function in the form $\psi_j(\cdot) = L^*\mathcal{G}(\cdot, t_j)$, $j = 1, 2, 3, \dots$, where $\{t_j\}_{j=1}^{\infty}$ is dense on $[0,T]$ and L^* is the adjoint operator of L. Like,

$$\begin{aligned}\langle w_i(\cdot), \psi_j(\cdot)\rangle_{\mathcal{W}_2^2} &= \langle w_i(\cdot), L^*\mathcal{G}(\cdot, t_j)\rangle_{\mathcal{W}_2^2} \\ &= \langle Lw_i(\cdot), \mathcal{G}(\cdot, t_j)\rangle_{\mathcal{W}_2^1} \\ &= Lw_i(t_j), j = 1, 2, \dots, i = 1, \dots, 2m-2.\end{aligned}$$

Since,

$$\begin{aligned}\psi_j(\cdot) &= L^*\mathcal{G}(\cdot, t_j) \\ &= \langle L^*\mathcal{G}(\cdot, t_j), \mathcal{R}(t,s)\rangle_{\mathcal{W}_2^2} \\ &= \langle \mathcal{G}(\cdot, t_j), L_s\mathcal{R}(\cdot, s)\rangle_{\mathcal{W}_2^1} \\ &= \langle L_s\mathcal{R}(\cdot, s), \mathcal{G}(\cdot, t_j)\rangle_{\mathcal{W}_2^1}\end{aligned}$$

$$= L_s\mathcal{R}(\cdot, s)|_{s=t_j}, j = 1, 2, \dots.$$

Thus, $\psi_j(\cdot)$ can be evaluated by $\psi_j(\cdot) = L_s\mathcal{R}(\cdot, s)|_{s=t_j}, j = 1, 2, 3, \dots$, where L_s is a differential operator with respect to s.

Theorem 2.7. *If $\{t_j\}_{j=1}^{\infty}$ is dense on $[0, T]$, then $\{\psi_j(\cdot)\}_{j=1}^{\infty}$ is a complete system of the space $\mathcal{W}_2^2[0, T]$.*

Proof. For the proof, we refer to [10]. □

Now, we will derive the method of analytical solution of the equations (2.4) and (2.5) in the reproducing kernel Hilbert space $\mathcal{W}_2^2[0, T]$.
The orthonormal system $\Psi_j(\cdot)$ of $\mathcal{W}_2^2[0, T], j = 1, 2, 3, \dots$ can be constructed from Gram-Schmidt orthogonalization process of $\psi_j(\cdot), j = 1, 2, 3, \dots$ with coefficients β_{jk}, as

$$\Psi_j(\cdot) = \sum_{k=1}^{j} \beta_{jk}\psi_k(\cdot), j = 1, 2, 3, \dots. \tag{2.7}$$

Theorem 2.8. *If $\{t_j\}_{j=1}^{\infty}$ is dense on $[0, T]$, then the analytic solution of* (2.4) *and* (2.5) *represented by*

$$w_i(\cdot) = \sum_{j=1}^{\infty}\sum_{k=1}^{j} \beta_{jk} f_i(\cdot, w_1(\cdot), w_2(\cdot), \dots, w_{2m-2}(\cdot))\Psi_j(\cdot), i = 1, 2, \dots, 2m-2. \tag{2.8}$$

Proof. Let $w_i(\cdot), i = 1, 2, \dots, 2m-2$ be the solution of (2.4) and (2.5) in $\mathcal{W}_2^2[0, T]$. Therefore, $\sum_{j=1}^{\infty}\langle w_i(\cdot), \Psi_j(\cdot)\rangle_{\mathcal{W}_2^2}\Psi_j(\cdot), i = 1, 2, \dots, 2m-2$ are the expansion of about orthonormal system $\Psi_j(\cdot)$ and convergent in the sense of $\|.\|_{\mathcal{W}_2^2}$. On the other hand, using (2.7) yields that

$$\begin{aligned}
w_i(\cdot) &= \sum_{j=1}^{\infty}\langle w_i(\cdot), \Psi_j(\cdot)\rangle_{\mathcal{W}_2^2}\Psi_j(\cdot) \\
&= \sum_{j=1}^{\infty}\langle w_i(\cdot), \sum_{k=1}^{j}\beta_{jk}\psi_k(\cdot)\rangle_{\mathcal{W}_2^2}\Psi_j(\cdot) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\beta_{jk}\langle w_i(\cdot), \psi_k(\cdot)\rangle_{\mathcal{W}_2^2}\Psi_j(\cdot) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\beta_{jk}\langle w_i(\cdot), L^*\mathcal{G}(\cdot, t_k)\rangle_{\mathcal{W}_2^2}\Psi_j(\cdot)
\end{aligned}$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \beta_{jk} \langle L w_i(\cdot), \mathcal{G}(\cdot, t_k) \rangle_{\mathcal{W}_2^1} \Psi_j(\cdot)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \beta_{jk} \langle f_i(\cdot, w_1(\cdot), w_2(\cdot), \ldots, w_{2m-2}(\cdot)), \mathcal{G}(\cdot, t_k) \rangle_{\mathcal{W}_2^1} \Psi_j(\cdot)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \beta_{jk} f_i(t_k, w_1(t_k), w_2(t_k), \ldots, w_{2m-2}(t_k)) \Psi_j(\varrho),$$

$$i = 1, 2, \ldots, 2m-2.$$

Thus, (2.8) is the analytical solution of (2.4) and (2.5). □

Here, equation (2.8) is the analytic solution of (2.4) and (2.5). Now, for the solution, we can define initial conditions as $w_{i,0}(t_1) = w_i(t_1)$ and set n terms approximations to $w_i(t), i = 1, 2, \ldots, 2m-2$ as $w_{i,n}(t) = \sum_{j=1}^{n} A_j^{\{i\}} \Psi_j(t), i = 1, 2, \ldots, 2m-2$, where the coefficients $A_j^{\{i\}}$ of $\Psi_j(t)$ are given as $A_j^{\{i\}} = \sum_{k=1}^{j} \beta_{jk} f_i(t_k, w_{1,k-1}(t_k), w_{2,k-1}(t_k), \ldots, w_{2m-2,k-1}(t_k))$, $i = 1, 2, \ldots, 2m-2$.

3. Numerical Experiments

In this section, we consider two nonlinear time dependent sine Gordan type equations on a finite interval that are implemented to demonstrate the accuracy and capability of the proposed algorithm, and all of them were performed on the computer using a program written in Matlab. To show the efficiency of the presented scheme we calculate the error norms L_2 and L_∞ as

$$L_2 = \|u_{exact} - u_{MOL}\|_2 = \sqrt{h \sum_{i=1}^{m} |(u_{exact})_i - (u_{MOL})_i|^2},$$

$$L_\infty = \|u_{exact} - u_{MOL}\|_\infty = \max_i |(u_{exact})_i - (u_{MOL})_i| .$$

Example 3.1. To test the MOL in the domain $[-1, 1]$, we consider the initial conditions $\Theta_0(x) = 0, \Theta_1(x) = 4\,\mathrm{sech}(x)$ and boundary conditions $\Phi_1(t) = 4\arctan(t\,\mathrm{sech}(-1)), \Phi_2(t) = 4\arctan(t\,\mathrm{sech}(1))$. The analytic solution [19] is given as $u(x,t) = 4\arctan(t\,\mathrm{sech}(x))$.

In Table 1, we introduce the L_2 and L_∞ errors between the MOL and analytic solutions in the domain $[-1, 1]$ with $h = 0.04$ and $\Delta t = 0.0001$ for

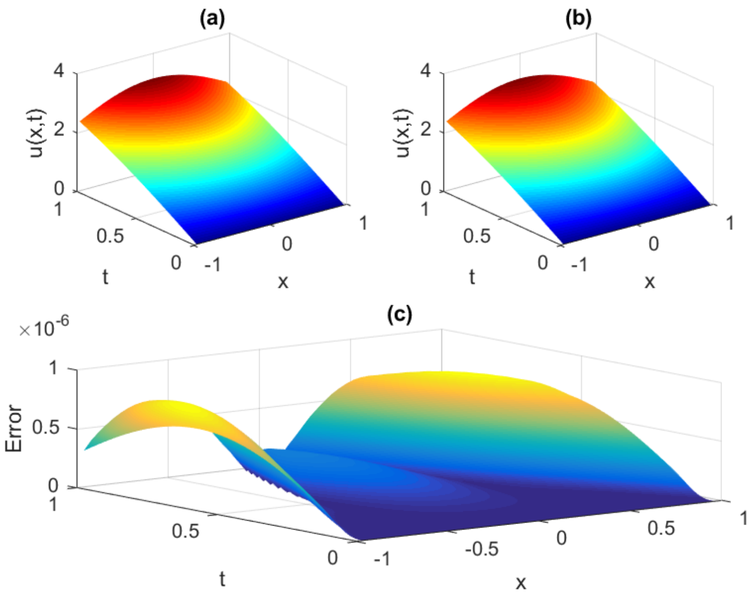

FIGURE 1. The result of the MOL with $h = 0.04$. (a) The analytic solution. (b) The MOL solution. (c) The absolute error.

TABLE 1. L_2 and L_∞ errors of Example 3.1 with $h = 0.04$ at $t = 0.25, 0.50, 0.75, 1.00$.

	MOL		Mittal and Bhatia [19]		Dehghan and Shokri [11]	
t	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.25	1.7478×10^{-7}	3.7809×10^{-7}	1.18×10^{-5}	2.32×10^{-5}	3.91×10^{-5}	5.89×10^{-6}
0.50	4.8636×10^{-7}	6.8728×10^{-7}	4.19×10^{-5}	4.11×10^{-5}	1.30×10^{-4}	2.01×10^{-5}
0.75	7.0941×10^{-7}	7.7246×10^{-7}	7.78×10^{-5}	1.02×10^{-4}	2.35×10^{-4}	3.63×10^{-5}
1.00	6.7936×10^{-7}	6.7041×10^{-7}	1.30×10^{-4}	1.64×10^{-4}	3.27×10^{-4}	5.07×10^{-5}

Example 3.1. Also, a comparison of some past works is given in the table, which gives that the proposed method is accurate. The analytic solution, the MOL solution and the absolute errors of Example 3.1 with $h = 0.04$ are displayed in Figure 1.

Example 3.2. The MOL solution of (1.1) is calculated in the computational domain $[-3, 3]$ with initial conditions $\Theta_0(x) = 4\arctan\left(exp\left(\frac{x}{\sqrt{1-c^2}}\right)\right)$, $\Theta_1(x) = \frac{-4c\gamma exp(\gamma x)}{1+exp(2\gamma x)}$ and boundary conditions $\Phi_1(t) = 4\arctan(exp(\gamma(-3-$

$ct)))$, $\Phi_2(x) = 4\arctan(exp(\gamma(3-ct)))$. The analytic solution [19] is given as $u(x,t) = 4\ \arctan(exp(\gamma(x-ct)))$, where $\gamma = 1/\sqrt{1-c^2}$.

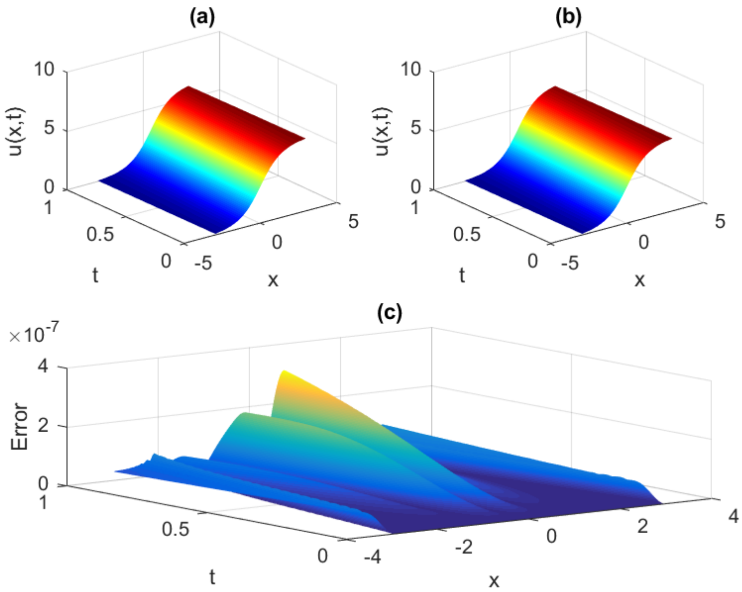

FIGURE 2. The result of the MOL with $h = 0.04$. (a) The analytic solution. (b) The MOL solution. (c) The absolute error.

TABLE 2. L_2 and L_∞ errors of Example 3.2 with $h = 0.04$ at $t = 0.25, 0.50, 0.75, 1.00$.

	MOL		Mittal and Bhatia[19]		Dehghan and Shokri [11]	
t	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.25	5.7347×10^{-8}	6.4305×10^{-8}	3.66×10^{-5}	4.90×10^{-5}	1.76×10^{-5}	4.95×10^{-6}
0.50	1.4646×10^{-7}	1.5131×10^{-7}	9.00×10^{-5}	7.55×10^{-5}	4.31×10^{-5}	8.42×10^{-6}
0.75	2.2560×10^{-7}	2.4219×10^{-7}	1.60×10^{-4}	1.43×10^{-4}	8.25×10^{-5}	1.65×10^{-5}
1.00	2.7964×10^{-7}	3.0977×10^{-7}	2.27×10^{-4}	2.10×10^{-4}	1.27×10^{-4}	2.51×10^{-5}

We consider $h = 0.04$, $\Delta t = 0.0001$ and $c = 0.5$. The analytic solution, the MOL solution and the absolute error are plotted in Figure 2. Table 2 illustrates the L_2 and L_∞ errors and a comparison with the method in [11, 19] and it is clear that our method is accurate.

4. Conclusion

In this paper, we proposed an efficient algorithm to solve a nonlinear time dependent sine Gordon equation. The equation is reduced to a system of ODEs and the system is solved by RKHSM. The main benefits of the method are simplicity of performance, high-order accuracy, and low computational cost. The efficiency and superiority of the scheme for a solution have been shown through experimental examples. It may be extended to solve different types of PDEs.

Acknowledgement: We wish to thank the editor-in-chief and the anonymous referees for their valuable suggestions to improve the quality of the paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 93 , Nos. 1-2, January-June (2024), 67–70

A CHARACTERIZATION OF CHAOTIC GROUP ACTIONS

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(Received : 12 - 08 - 2022 ; Revised : 03 - 04 - 2023)

Abstract. In this paper, we prove that a group acts chaotically on an infinite Hausdorff space X if and only if any finite collection of nonempty open subsets of X shares infinitely many periodic orbits.

1. Introduction

Chaotic dynamical systems have been the topic of intensive research. Most of the work (see [2]) in chaotic dynamics has been concerned with the iteration of self-maps. In [1], the authors developed a theory in its most general setting as an action of a group on some topological space. Though the groups acting chaotically on Hausdorff spaces have been characterized earlier in [1], no further characterizations were done related to the space on which the group action takes place. The objective of this paper is to provide further characterizations of chaotic group actions in terms of the underlying space and the periodic orbits shared by open subsets of the underlying space.

In this paper, we prove that the following are equivalent for an action of a group G on an infinte Hausdorff space X.

(1) action is chaotic.
(2) any two nonempty open subsets of X share a periodic orbit.
(3) any finite collection of nonempty open subsets of X share infinitely many periodic orbits.

We prove these results with the help of basic ideas involved in the areas of group theory and topology. These characterizations will be helpful for further studies of the dynamics of continuous group actions.

2020 Mathematics Subject Classification: 37B05, 37B02
Key words and phrases: group actions, periodic orbits, topological transitivity, periodic point, chaotic

Definition 1.1. Let G be a group acting on a Hausdorff topological space X. The ***orbit*** of a point $x \in X$ under G is defined as $Gx := \{gx : g \in G\}$.

Definition 1.2. Let G be a group acting on a Hausdorff topological space X. The action of G on X is ***continuous*** if for each $g \in G$, the map $g : X \to X$ is continuous.

Throughout this paper, we denote G for a group and X for a Hausdorff topological space. The group action here is always considered as continuous.

Definition 1.3. The action of G on X is called ***topologically transitive*** if for any two nonempty open sets U and V of X, there exists an element $g \in G$ such that $gU \cap V \neq \phi$.

Definition 1.4. A point $x \in X$ is called a ***periodic point*** of the action if the orbit of x under G is finite. The number of elements of the orbit is called the ***period*** of the orbit.

Definition 1.5. An action is called ***chaotic*** if it is topologically transitive and the set of periodic points forms a dense subset of X.

In [3], a collection of nonempty open sets shares a periodic orbit for a self-map is defined. Analogous to this we define the following for a group action.

Definition 1.6. A finite collection of nonempty open sets $V_1, V_2, ..., V_n$ share a periodic orbit if there exists a periodic point whose orbit meets each V_i for $i = 1, 2, ..., n$.

2. Main Results

The following proposition can be considered as another definition of chaotic group actions.

Proposition 2.1. *The action of G on X is chaotic if and only if any two nonempty open subsets of X share a periodic orbit.*

Proof. If any two nonempty open subsets share a periodic orbit, then every open set contains a periodic point and therefore periodic points are dense in X. To prove the topological transitivity, let U and V be any two nonempty open subsets of X. By assumption, U and V share a periodic orbit. That is, there exists $x \in X$ such that the orbit $Gx = \{gx : g \in G\}$ intersects

U and V. Therefore there exists $g_1, g_2 \in G$ such that $g_1x \in U \cap Gx$ and $g_2x \in V \cap Gx$. Let $g = g_2{g_1}^{-1}$. This implies $gU \cap V \neq \emptyset$ and hence the action is topologically transitive. Conversely, if the action is chaotic, for any two nonempty open sets U, V there exists $x \in U$ and $g \in G$ such that $gx \in V$. Then $W := g^{-1}V \cap U$ is nonempty, open, and $gW \subset V$. This W must contain a periodic point p since the periodic points are dense in X. Observe that $p \in U$ and $gp \in V$. Hence U and V share a periodic orbit. $\square$

The following lemma proves the existence of a periodic orbit of a chaotic action of a group G on a Hausdorff topological space X shared by finitely many non-empty open sets.

Lemma 2.2. *The action of G on X is chaotic if and only if any finite collection of nonempty open sets shares a periodic orbit.*

Proof. The proof follows by Proposition 2.1 and by applying induction on the number of nonempty open sets. $\square$

Now we are ready to consider our main theorem.

Theorem 2.3 (Main Theorem)**.** *The action of G on an infinite X is chaotic if and only if every finite collection of nonempty open subsets of X shares an infinite number of periodic orbits.*

Proof. Let $U_1, U_2, ..., U_n$ be nonempty open subsets of X. Suppose this collection of open sets share only finitely many periodic orbits. Let P be the set of all points in the periodic orbits shared by $U_1, U_2, ..., U_n$. Then P is finite since it is the finite union of periodic orbits that are finite. We define $V_i = U_i \backslash P$ for $i = 1, ..., n$. Then each V_i is a nonempty open subset of X since X is Hausdorff, P is finite and the action is chaotic. Observe that $V_i \subset U_i$ for all i. Now $V_1, V_2, ..., V_n$ forms a finite collection of nonempty open subsets of X. Therefore by the assumption, they share a periodic orbit under the action of G. But this orbit is not contained in P and the orbit intersects the finite collection $\{U_1, U_2, ..., U_n\}$, a contradiction. The converse follows from Lemma 2.2. $\square$

We conclude our short article by providing the following remark.

Remark 2.4. Let G be a group acting chaotically on an infinite Hausdorff topological space X. Because of Proposition 2.1 and Theorem 2.3, we can conclude that the following are equivalent for any two non-empty subsets U and V of X.

(1) U and V of X share a periodic orbit.
(2) U and V of X share finitely many periodic orbits.
(3) U and V share an infinite number of periodic orbits.

Acknowledgement: The authors are grateful to the referee for valuable suggestions. The first author acknowledges CSIR, India and the second author acknowledges SERB-MATRICS Grant No. MTR/2018/000256 for financial support.

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The Mathematics Student ISSN: 0025-5742
Vol. 93 (1-2), January-June (2024), 71–75

A PROOF OF $\zeta(2) = \frac{\pi^2}{6}$ RELATED TO FRACTIONAL CALCULUS

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(Received : 24 - 10 - 2022 ; Revised : 23 - 01 - 2023)

Abstract. We offer a new and appealingly simple proof related to fractional calculus of the famous equality $\zeta(2) = \frac{\pi^2}{6}$.

1. Introduction

Euler's closed-form formula

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots = \frac{\pi^2}{6} \tag{1.1}$$

is one of the most celebrated and most beautiful formulas in all of mathematics. It appears that this Euler formula is the one classical formula with the most published research articles specifically devoted to new and different proofs of it. For example, the lengthy list of References in our recent contribution [2], in which Euler's formula was proved using the Wilf–Zeilberger method [8], lists many articles based on proofs of Euler's formula, including the recent *Mathematics Student* article [6] that has inspired our current article. Our new proof of Euler's formula relies on a very recently published identity [1] that was derived using an area in mathematics that is known as *fractional calculus* and that is described below, and our new proof also relies on a special function known as the *dilogarithm*, which is reviewed below.

1.1. **Fractional calculus.** The repeated application of differential or of differential-type operators is used in virtually every form of mathematical analysis. The use of the phrase "repeated application" suggests that one's applying such operators over and over again is of a discrete nature, in terms of an integer number of times such operators may be applied. Fractional

2010 Mathematics Subject Classification: 11Y60, 33B30
Key words and phrases: Basel problem, Riemann zeta function, fractional calculus

calculus may be thought of as being given by how one may generalize this notion using non-integer values.

If we let D denote a linear, differential operator such as the operator $\frac{d}{dx}$ (defined on some suitable domain), then expressions such as $\left(\frac{d}{dx}\right)^n$ or D^n would typically refer to self-iterations of the form

$$\underbrace{D \circ D \circ \cdots \circ D}_{n}.$$

If we consider how such self-iterations may be generalized and applied using analytic functions, this leads us to consider the relation

$$\underbrace{D \circ D \circ \cdots \circ D}_{n}(x^n) = n! \tag{1.2}$$

for $D = \frac{d}{dx}$ and how it is typically generalized in the field of fractional calculus.

To generalize (1.2) for non-integer values n, one is led toward the use of the Γ-function (see Rainville [9, §8]) given by the Euler integral $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du$. For example, for $\alpha > -\frac{1}{2}$, we let the *Caputo operators* $D^{1/2}$ and $D^{-1/2}$ be such that

$$D^{1/2} x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)} x^{\alpha-\frac{1}{2}}$$

and

$$D^{-1/2} x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} x^{\alpha+\frac{1}{2}}, \tag{1.3}$$

following the references [1, 3] that are to be later involved in our work. In view of (1.2), we are led to find that

$$\left(D^{1/2} \circ D^{1/2}\right) x^\alpha = \alpha x^{\alpha-1}$$

and similarly with respect to (1.3), and hence a standard way of generalizing identities such as (1.2) using non-integer values n. The recent article [1] concerning Caputo operators introduced an identity via such operators that we briefly review below and that we are to prove in a self-contained way to provide a new proof of $\frac{1}{1^2} + \frac{1}{2^2} + \cdots = \frac{\pi^2}{6}$.

1.2. **The dilogarithm function.** The study of higher logarithm functions constitutes a large area in special functions theory, with reference to the 33B30 Mathematics Subject Classification. Something of a prototype or main instance of a so-called higher logarithm function is the famous dilogarithm function, which we are to review below. For excellent and relevant exposition on the dilogarithm and the history concerning this special function, we refer to Stewart's recent article [10], and the following background material is partly inspired by [10].

We write $\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ to denote the dilogarithm function, letting this function be defined for $|x| \leq 1$. As in [10], we provide [5] as a reference for the history of the use of the function Li_2 in mathematics. For a succinct review of many of the fundamental properties concerning Li_2, such as its usual integral formula

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt$$

and the Euler reflection formula

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x),$$

we again refer to Stewart's article [10]. For our present purposes, we are more interested in an integral identity involving Li_2 given in [1] that is to be proved and applied in a self-contained way.

Let the inner product $\langle \cdot, \cdot \rangle$ be such that $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\, dx$ for (Riemann) integrable functions f and g. The method of *semi-integration by parts* (SIBP) was introduced in [3] and formulated in the following way:

$$\langle f, g \rangle = \left\langle \left(D^{1/2}\tau\right) f, \left(\tau D^{-1/2}\right) g \right\rangle,$$

provided that both sides of this equality are well-defined, and where τ denotes the operator such that $h(x) \mapsto h(1-x)$. We let f and g respectively denote the analytic functions given by the power series expansions with summands given by $a_n = \left(\frac{\alpha^2}{16}\right)^n \binom{2n}{n}^2$ and $b_n = \left(\frac{1}{16}\right)^n \binom{2n}{n}^2$; we may then apply SIBP, as in [1], to prove that

$$\int_0^1 \frac{\sin^{-1}\left(\sqrt{x}\right)}{\sqrt{1-x}\sqrt{\alpha^2(x-1)+1}}\, dx = \frac{\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(-\alpha)}{\alpha}. \tag{1.4}$$

We are to prove this identity in a self-contained way, using "Feynman's favorite trick" [7], to prove that $\zeta(2) = \frac{\pi^2}{6}$, writing $\zeta(x) = \frac{1}{1^x} + \frac{1}{2^x} + \cdots$ to denote the Riemann zeta function.

1.3. **Feynman's favorite trick.** The interchange of limiting operations arises in virtually all areas of classical and modern analysis. For example, exchanging the order of integration and infinite summation is a common kind of "trick" used in the determination of closed-form evaluation of infinite series, and may be justified, for reasonably well-behaved integrands, subject to the conditions of famous results in mathematical analysis such as the Monotone Convergence Theorem; see classic texts such as [4, p. 317]. To justify differentiating under the integral sign, one would typically appeal to the famous *Leibniz integral rule.* According to this rule, for integrals with upper and lower limits that are constant, and with an integrand $f(\alpha, x)$ such that $f(\alpha, x)$ and $\frac{\partial}{\partial\alpha} f(\alpha, x)$ are continuous in α and x (for the closed interval given by the upper and lower limits of the integral), one may differentiate under the integral sign (with respect to α). Although the integrand in (2.1) is only continuous with respect to x for $[0, 1)$ and not for $[0, 1]$, one may use a standard generalization of the Leibniz integral rule for improper integrals.

2. A new proof of $\zeta(2) = \frac{\pi^2}{6}$

Although (1.4) had initially been discovered via fractional calculus, we are to prove (1.4) in a self-contained way, to provide a new and self-contained proof that $\zeta(2) = \frac{\pi^2}{6}$. The famous problem of evaluating $\zeta(2)$ in closed form is referred to as the *Basel problem*, as below.

Theorem 2.1. *The solution to the Basel problem whereby $\zeta(2) = \frac{\pi^2}{6}$ holds true.*

Proof. To prove that (1.4) holds for $|\alpha| \leq 1$, we instead prove the equivalent formulation

$$\int_0^1 \frac{\sin^{-1}\left(\sqrt{x}\right)\alpha}{\sqrt{1-x}\sqrt{\alpha^2(x-1)+1}}\,dx = \mathrm{Li}_2(\alpha) - \mathrm{Li}_2(-\alpha). \tag{2.1}$$

By differentiating with respect to α on both sides of the purported identity in (2.1) (using the generalization of the Leibniz integral rule indicated above), we can show that (2.1) is equivalent to

$$\int_0^1 \frac{\sin^{-1}\left(\sqrt{x}\right)}{\sqrt{1-x}\left(1+(x-1)\alpha^2\right)^{3/2}}\,dx = \frac{\log(\alpha+1)}{\alpha} - \frac{\log(1-\alpha)}{\alpha}. \tag{2.2}$$

However, we may easily check that an antiderivative of the integrand of the left-hand side of the above equality is

$$\frac{2\log\left(\alpha\left(\sqrt{\alpha^2(x-1)+1}+\alpha\sqrt{x}\right)\right)}{\alpha} - \frac{2\sqrt{1-x}\sin^{-1}\left(\sqrt{x}\right)}{\sqrt{\alpha^2(x-1)+1}},$$

so taking the required limits then gives us a proof of (2.1). From (2.1), we obtain that $\int_0^1 \frac{\sin^{-1}(\sqrt{x})}{\sqrt{1-x}\sqrt{x}}\, dx = \mathrm{Li}_2(1) - \mathrm{Li}_2(-1)$. We may easily check that an antiderivative for the integrand on the left-hand side is $\sin^{-1}\left(\sqrt{x}\right)^2$, so this gives us a proof that $\frac{\pi^2}{4} = \mathrm{Li}_2(1) - \mathrm{Li}_2(-1)$. So, we have proved that $\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, and the use of series bisections easily gives us that this is equivalent to $\zeta(2) = \frac{\pi^2}{6}$. $\square$

Acknowledgement: The anonymous reviewer of this article offered many suggestions that significantly improved this article.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January-June (2024), 76–82

ON KAKUTANI'S CHARACTERIZATION OF THE CLOSED LINEAR SUBLATTICES OF $C(X)$—REVISITED

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(Received : 18 - 01 - 2023 ; Revised : 27 - 05 - 2023)

Abstract. In his paper [Concrete representation of abstract (M)-spaces (A characterization of the space of continuous functions), *Annals of Mathematics*, No. (4), **42** (1941), 994–1024.], S. Kakutani gave an interesting representation of the closed linear sublattices of the space of real-valued continuous functions on a compact Hausdorff space, which is determined by a set of algebraic relations. In this short note, we present a simple proof of this representation without using any profound lattice theory or functional analysis results, making this proof accessible even to undergraduate students.

1. Preliminaries

Let X be a compact Hausdorff space. We denote the Banach space (in other words, a complete normed linear space) of all real-valued continuous functions on X, equipped with the supremum norm, by $C(X)$. It is well-known that $C(X)$ is a lattice under the operation of pointwise maximum or minimum of a pair of functions in $C(X)$ and an algebra under the operation of pointwise multiplication of a pair of functions in $C(X)$. All the subspaces in this note are assumed to be closed with respect to the norm topology. We say that a closed linear subspace $\mathcal{A}$ of $C(X)$ is a *sublattice* of $C(X)$ if $\mathcal{A}$ is a lattice in its own right under the same lattice operation as in $C(X)$. A closed linear subspace $\mathcal{A}$ of $C(X)$ is said to be a *subalgebra* of $C(X)$ if $\mathcal{A}$ is an algebra in its own right under the same algebra operation as in $C(X)$. Kakutani presented an algebraic characterization of the closed linear sublattices of $C(X)$ as follows:

2010 Mathematics Subject Classification: 46E15, 46B42
Key words and phrases: space of real-valued continuous functions, compact Hausdorff space, sublattices, subalgebras

Theorem 1.1 ([2, Theorem 3, pg. 1005])**.** *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a closed linear subspace of $C(X)$. Then $\mathcal{A}$ is a sublattice of $C(X)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in X \times X \times [0,1]$, for each $i \in I$ such that*

$$\mathcal{A} = \{f \in C(X) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{1.1}$$

In Section 2, we provide an elementary proof of the result above. In fact, we exploit the structure of the closed linear sublattices of the two-dimensional vector space, $\mathbb{R}^2$, in this proof. Moreover, in this proof, we apply a few auxiliary results whose proofs assume the knowledge of a basic course in real analysis and point-set topology.

Let us first understand what the sublattices look like in $\mathbb{R}^2$ under the lattice operation of co-ordinate-wise maximum or minimum of a pair of vectors in $\mathbb{R}^2$. For simplicity, we denote the linear span of a vector $(a, b) \in \mathbb{R}^2$ as *span*$\{(a, b)\}$. Let $\mathcal{A}$ be a closed linear subspace of $\mathbb{R}^2$. Basic knowledge of linear algebra tells us that the possible dimensions of $\mathcal{A}$ are $0, 1$ or 2. If the dimension of $\mathcal{A}$ is either 0 or 2, then $\mathcal{A} = \{(0, 0)\}$ or $\mathcal{A} = \mathbb{R}^2$ respectively; obviously, these subspaces are sublattices of $\mathbb{R}^2$. Assume that $\mathcal{A}$ is a one-dimensional subspace of $\mathbb{R}^2$. We know that if $\mathcal{A} = span\{(a, b)\}$ for some $(a, b) \in \mathbb{R}^2$, then $\mathcal{A} = span\{(\lambda a, \lambda b)\}$ for each $\lambda \in \mathbb{R}\backslash\{0\}$. Therefore, without loss of generality, there exists $a, b \in [-1, 1]$ such that $\mathcal{A} = span\{(a, b)\}$. It is easy to see that if $-1 \leq a, b \leq 0$ or $0 \leq a, b \leq 1$, then $\mathcal{A}$ is a sublattice of $\mathbb{R}^2$. Furthermore, if $-1 \leq a < 0 < b \leq 1$ then the minimum of (a, b) and $(2a, 2b)$ is $(2a, b) \notin \mathcal{A}$; hence $\mathcal{A}$ is not a sublattice of $\mathbb{R}^2$. Using a similar argument, if $-1 \leq b < 0 < a \leq 1$ then $\mathcal{A}$ is not a sublattice of $\mathbb{R}^2$. Therefore, without loss of generality, we arrive at the following conclusion.

Lemma 1.2. *Consider $\mathbb{R}^2$ as a lattice under the operation of co-ordinate-wise maximum or minimum of a pair of vectors in $\mathbb{R}^2$. Then the only closed linear sublattices of $\mathbb{R}^2$ are $\{(0, 0)\}$, $\mathbb{R}^2$ and span$\{(a, b)\}$, for those $(a, b) \in \mathbb{R}^2$ satisfying $0 \leq a, b \leq 1$.*

We can also list all possible closed linear subalgebras of $\mathbb{R}^2$ as follows:

Lemma 1.3 ([1, Lemma 4.46])**.** *Consider $\mathbb{R}^2$ as an algebra under co-ordinate-wise addition and multiplication of a pair of vectors in $\mathbb{R}^2$. Then the only subalgebras of $\mathbb{R}^2$ are $\{(0, 0)\}$, $\mathbb{R}^2$, span$\{(1, 0)\}$, span$\{(0, 1)\}$ and span$\{(1, 1)\}$.*

We now recall a few known facts and definitions needed to prove Theorem 1.1. For a compact Hausdorff space X, we say a closed linear subspace $\mathcal{A}$ of $C(X)$ *separates the points of* X if for every two distinct points $x, y \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

The following result shows the interconnection between a subalgebra and a sublattice of $C(X)$.

Lemma 1.4 ([1, Lemma 4.48])**.** *Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed linear subalgebra of $C(X)$, then $\mathcal{A}$ is a sublattice of $C(X)$.*

Furthermore, for a compact Hausdorff space X and a closed linear sublattice $\mathcal{A}$ of $C(X)$, a sufficient condition for a function in $C(X)$ to be in $\mathcal{A}$ is

Lemma 1.5 ([1, Lemma 4.49])**.** *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a closed linear sublattice of $C(X)$ and $f \in C(X)$. If for every $x, y \in X$ there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy}(x) = f(x)$ and $g_{xy}(y) = f(y)$, then $f \in \mathcal{A}$.*

Let T be a locally compact Hausdorff space. We denote the Banach space of real-valued continuous functions on T vanishing at infinity by $C_0(T)$. Further, let us denote T_∞ as the one-point compactification of T and t_∞ be the "point at infinity". We consider the subspace $J_{t_\infty} := \{f \in C(T_\infty) \colon f(t_\infty) = 0\}$ of $C(T_\infty)$. Now, consider the restriction map $\Phi \colon J_{t_\infty} \to C_0(T)$ defined as $\Phi(f) = f|_T$, for $f \in J_{t_\infty}$. Then the map Φ is an isometry and lattice, algebra and linear isomorphism (in other words, a map which preserves the distances as well as the lattice, algebraic and linear structures). The identification above and Theorem 1.1 help us to algebraically characterize the closed linear sublattices and subalgebras of the space $C_0(T)$ in Section 2.

2. Main Results

Let us now prove our main result.

Proof of Theorem 1.1. We assume that X contains at least two points because if X is a singleton set, then $C(X)$ is simply $\mathbb{R}$ and the only closed linear sublattices or subalgebras are $\{0\}$ and $\mathbb{R}$. If $\mathcal{A}$ has the description as in (1.1), then clearly $\mathcal{A}$ is a sublattice of $C(X)$.

Now, assume that $\mathcal{A}$ is a sublattice of $C(X)$. For every two distinct points $x, y \in X$, define

$$\mathcal{A}_{xy} = \{(g(x), g(y)) \in \mathbb{R}^2 \colon g \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is a lattice, $\mathcal{A}_{xy}$ is a sublattice of $\mathbb{R}^2$ (under the operation of co-ordinate-wise maximum of a pair of vectors in $\mathbb{R}^2$).

Case 1: Assume $\mathcal{A}$ separates the points of X.

If for every $x, y \in X$, $\mathcal{A}_{xy} = \mathbb{R}^2$ then by Lemma 1.5, $\mathcal{A} = C(X)$. Hence $\mathcal{A}$ has the description as in (1.1). Otherwise, there exists two distinct points $x_0, y_0 \in X$ such that $\mathcal{A}_{x_0y_0}$ is a proper sublattice of $\mathbb{R}^2$. Consider the following collection:

$$I = \{(t, s, \lambda) \in X \times X \times [0, 1] \colon f(t) = \lambda f(s), \text{ for each } f \in \mathcal{A}\}.$$

We now show that $I \neq \emptyset$. Since $\mathcal{A}$ separates the points of X, $\mathcal{A}_{x_0y_0}$ cannot be $\{(0, 0)\}$ or $span\{(1, 1)\}$. Thus $\mathcal{A}_{x_0y_0} = span\{(a, b)\}$ for some $0 \leq a, b \leq 1$ and $a \neq b$. If $a = 0$ and $b > 0$ then for each $g \in \mathcal{A}$, $g(x_0) = 0$ and hence $(x_0, y_0, 0) \in I$. If $a > 0$ and $b = 0$ then for each $g \in \mathcal{A}$, $g(y_0) = 0$ and hence $(y_0, x_0, 0) \in I$. If without loss of generality $0 < a < b$, then for each $g \in \mathcal{A}$, there exists $r_g \in \mathbb{R}$ such that $(g(x_0), g(y_0)) = (r_g a, r_g b)$. It follows that $g(x_0) = \frac{a}{b} g(y_0)$. Thus $(x_0, y_0, \frac{a}{b}) \in I$.

We index I by I itself, that is, each element of I is indexed by itself. Further, define

$$\mathcal{A}' = \{f \in C(X) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}.$$

By the definition of I, it is clear that $\mathcal{A} \subseteq \mathcal{A}'$.

We next show that $\mathcal{A}' \subseteq \mathcal{A}$. Let $f \in \mathcal{A}'$. In order to show $f \in \mathcal{A}$, by Lemma 1.5, it suffices to show that for each $x, y \in X$, there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy}(x) = f(x)$ and $g_{xy}(y) = f(y)$. Therefore, it suffices to show that for each $x, y \in X$, $(f(x), f(y)) \in \mathcal{A}_{xy}$.

Let $x, y \in X$. Since $\mathcal{A}$ separates the points of X, $\mathcal{A}_{xy}$ cannot be $\{(0, 0)\}$ or $span\{(1, 1)\}$. We remark here that in this case, we use the assumption that $\mathcal{A}$ separates the points of X only to prove that $I \neq \emptyset$ and to rule out the above two possibilities of $\mathcal{A}_{xy}$.

If $\mathcal{A}_{xy} = \mathbb{R}^2$, then clearly $(f(x), f(y)) \in \mathbb{R}^2 = \mathcal{A}_{xy}$. If $\mathcal{A}_{xy} = span\{(0, 1)\}$ then $(0, f(y)) \in \mathcal{A}_{xy}$ and for each $g \in \mathcal{A}$, $g(x) = 0$. Thus $(x, y, 0) \in I$. Since $f \in \mathcal{A}'$, $f(x) = 0$. Hence $(f(x), f(y)) \in \mathcal{A}_{xy}$. Similar arguments hold true if $\mathcal{A}_{xy} = span\{(1, 0)\}$.

Without loss of generality, let $0 < a < b < 1$. Consider $\mathcal{A}_{xy} = span\{(a, b)\}$. Then for each $g \in \mathcal{A}$, $g(x) = \frac{a}{b} g(y)$. Thus $(x, y, \frac{a}{b}) \in I$. Since $f \in \mathcal{A}'$, let $\frac{f(x)}{a} = \frac{f(y)}{b} = r$ (say). It follows that $(f(x), f(y)) \in span\{(a, b)\} = \mathcal{A}_{xy}$.

Case 2: Assume that $\mathcal{A}$ does not separate the points of X. Thus there exists two distinct points $x_0, y_0 \in X$ such that $f(x_0) = f(y_0)$, for each $f \in \mathcal{A}$. Consider the following collection:

$$I = \{(t, s, \lambda) \in X \times X \times [0, 1] \colon f(t) = \lambda f(s), \text{ for each } f \in \mathcal{A}\}.$$

Since $(x_0, y_0, 1), (y_0, x_0, 1) \in I$, clearly $I \neq \emptyset$. We index I by I itself, that is, each element of I is indexed by itself. Let us define

$$\mathcal{A}' = \{f \in C(X) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}.$$

Clearly $\mathcal{A} \subseteq \mathcal{A}'$. In order to show $\mathcal{A}' \subseteq \mathcal{A}$, by Lemma 1.5, it suffices to show that for each $f \in \mathcal{A}'$ and each $x, y \in X$, $(f(x), f(y)) \in \mathcal{A}_{xy}$.

Let $x, y \in X$. We first consider the following possibilities of $\mathcal{A}_{xy}$: $\mathbb{R}^2$, $span\{(0, 1)\}$, $span\{(1, 0)\}$ and $span\{(a, b)\}$ for those $a, b \in \mathbb{R}$ satisfying $0 < a < b < 1$ (without loss of generality). For each of the above possibilities, we apply arguments similar to that used in Case 1 to show that $(f(x), f(y)) \in \mathcal{A}_{xy}$.

In this case, due to our assumption that $\mathcal{A}$ does not separate the points of X, we need to consider the two possibilities which we rule out in Case 1. They are as follows: If $\mathcal{A}_{xy} = \{(0, 0)\}$ then for each $g \in \mathcal{A}$, $g(x) = 0 = g(y)$. Hence, $(x, y, 0), (y, x, 0) \in I$. Since $f \in \mathcal{A}'$, $(f(x), f(y)) = (0, 0) \in \mathcal{A}_{xy}$. If $\mathcal{A}_{xy} = span\{(1, 1)\}$ then for each $g \in \mathcal{A}$, $g(x) = g(y)$. Thus $(x, y, 1) \in I$. Since $f \in \mathcal{A}'$, $(f(x), f(y)) = (f(x), f(x)) \in \mathcal{A}_{xy}$. □

The closed linear subalgebras of the $C(X)$ space has a representation similar to that in (1.1) with the value of the coefficients λ_i being either 0 or 1. With the help of Lemmas 1.3, 1.4 and 1.5, the following result is proved using a similar argument as in Theorem 1.1, and hence we omit it.

Theorem 2.1. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a closed linear subspace of $C(X)$. Then $\mathcal{A}$ is a subalgebra of $C(X)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in X \times X \times \{0, 1\}$, for each $i \in I$ such that*

$$\mathcal{A} = \{f \in C(X) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{2.1}$$

The following characterizations of the sublattices and subalgebras of $C_0(T)$ for a locally compact Hausdorff space T follows directly from Theorems 1.1 and 2.1 and the fact that $C_0(T)$ is isometrically lattice, algebra and linear isomorphic to the subalgebra J_{t_∞} of $C(T_\infty)$.

Corollary 2.2. *Let T be a locally compact Hausdorff space. Let $\mathcal{A}$ be a closed linear subspace of $C_0(T)$. Then*

(1) $\mathcal{A}$ is a sublattice of $C_0(T)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in T \times T \times [0,1]$, for each $i \in I$ such that

$$\mathcal{A} = \{f \in C_0(T) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{2.2}$$

(2) $\mathcal{A}$ is a subalgebra of $C_0(T)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in T \times T \times \{0,1\}$, for each $i \in I$ such that

$$\mathcal{A} = \{f \in C_0(T) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{2.3}$$

Remark 2.3. Kakutani provided a precise identification of a much general class of Banach spaces, namely abstract (M)-spaces (see [2, pg. 994] for the definition), with a subspace of $C(X)$ for some compact Hausdorff space X. Let X be a compact Hausdorff space and $\mathcal{A}$ be a closed subspace of $C(X)$. If $\mathcal{A}$ is an abstract (M)-space, then by [2, Theorem 1, pg. 998], there exists a compact Hausdorff space Ω such that $\mathcal{A}$ is isometrically lattice and linear isomorphic to the subspace, $\{f \in C(\Omega) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}$, of $C(\Omega)$ for some index set I and co-ordinates $(t_i, s_i, \lambda_i) \in \Omega \times \Omega \times [0,1]$ for each $i \in I$. However, it is not necessary that $\mathcal{A}$, being a subspace of $C(X)$, has a description in $C(X)$ as given in (1.1). For example, consider $\mathcal{A}$ to be the space of real-valued affine continuous functions on $[0,1]$. Now, $\mathcal{A}$ is a closed subspace of $C([0,1])$ but not a sublattice of $C([0,1])$. Therefore, by Theorem 1.1, $\mathcal{A}$ does not have a description as in (1.1), for any given subfamily of co-ordinates in $[0,1] \times [0,1] \times [0,1]$. Nevertheless, it is easy to see that $\mathcal{A}$ is isometrically lattice and linear isomorphic to $C(\{0,1\})$ and hence is an abstract (M)-space.

Acknowledgement: The author would like to thank Prof. Dirk Werner for indicating Remark 2.3. The author is also grateful to the editor and the referee for their helpful suggestions to improve the quality of this manuscript.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January-June (2024), 83–100

A NOTE ON FLEXIBLE IMMERSIONS

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Abstract. In this note, we introduce the notion of a flexible immersion of a compact surface in a 4–manifold and give its applications. As a first application, we show that every open book of every closed 3–manifold open book immerses in the 5–sphere S^5 with the trivial open book. As a second application, we show that given a second integral homology class d of $\mathbb{C}P^2$ and given a closed oriented surface Σ, there is a flexible immersion $f : \Sigma \looparrowright \mathbb{C}P^2$ such that $[f] = d$. We also show that each of the classes 0 and $\pm 2 \in \mathbb{Z} = H_2(\mathbb{C}P^2, \mathbb{Z})$ can be represented by a flexible embedding $f : \Sigma \hookrightarrow \mathbb{C}P^2$ of any closed oriented surface Σ into $\mathbb{C}P^2$.

1. Introduction

Hirose and Yasuhara [8] discussed the notion of flexible embeddings of closed surfaces in compact 4–manifolds. A proper embedding $f : \Sigma \hookrightarrow X$ of a compact surface Σ in a compact 4–manifold X is said to be flexible if for given a diffeomorphism $\phi : \Sigma \to \Sigma$, there exists an isotopy $\psi_t : X \to X$ of X such that ψ_1 maps $f(\Sigma)$ to itself and $f^{-1} \circ \psi_1 \circ f = \phi$. We consider the diffeomorphisms and isotopies of a 4–manifold which are the identity near the boundary.

Proper flexible embeddings of compact surfaces in compact 4–manifolds are used in [4], [13] and [6] to study open book embeddings of closed 3–manifolds in $S^2 \times S^3$ as well as in $S^2 \widetilde{\times} S^3$. Proper flexible embeddings of compact surfaces in compact 4–manifolds are also used in [5] and [14] to study Lefschetz fibration embeddings of 4–manifolds in 6–manifolds.

An open book of a closed manifold M is a way to express M as a locally trivial fiber bundle $\pi : M \setminus B \to S^1$ in the complement of a co-dimension–2 submanifold B having trivial normal bundle such that the closure of the boundary of each fiber is B. Alexander [1] proved that every

2010 Mathematics Subject Classification: 57R45, 57R15, 57R65
Key words and phrases: 3–manifolds, open book, immersion

closed orientable 3–manifold admits an open book. The existence of an open book for a closed non-orientable 3–manifold is proved in [2] as well as in [7]. For various proofs of the existence of an open book for closed 3–manifolds, refer [3].

Roughly speaking, we say that a smooth manifold M with a given open book admits an open book embedding in an open book of a smooth manifold N, provided there is an embedding of M in N such that –as a submanifold of N– the given open book decomposition on M is compatible with the open book of N. It is not known that whether every closed 3–manifold admits an open book which open book embeds in an open book of S^5.

In this exposition, we define the notion of a flexible immersion of a compact surface as well as the notion of an open book immersion, see Definition 3.1 and Definition 4.2. We show that given a compact surface Σ, there is a proper flexible immersion $f : \Sigma \hookrightarrow D^4$ of Σ into D^4, see Lemma 3.7 and Lemma 3.8. As a first application of this, we show that every open of every closed 3–manifold open book immerses in the 5–sphere S^5 with the trivial open book, see Theorem 4.4.

As a second application, we show that given a second integral homology class d of $\mathbb{C}P^2$ and given a closed oriented surface Σ, there is a flexible immersion $f : \Sigma \looparrowright \mathbb{C}P^2$ such that $[f] = d$, see Theorem 5.3. We also show that each of the classes 0 and $\pm 2 \in \mathbb{Z} = H_2(\mathbb{C}P^2, \mathbb{Z})$ can be represented by a flexible embedding $f : \Sigma \hookrightarrow \mathbb{C}P^2$ of any closed oriented surface into $\mathbb{C}P^2$, see Theorem 5.5.

2. Preliminaries

In this section, we recall the necessary definitions and the results required for this article. Unless otherwise specified, we assume all the diffeomorphisms and the isotopies of a compact manifold are the identity near the boundary. Let us begin with the definition of the mapping class group.

Definition 2.1. The mapping class group $\mathcal{MCG}(\Sigma, \partial\Sigma)$ of a compact oriented surface Σ is a group of orientation preserving diffeomorphisms of Σ upto isotopy. In case, the surface Σ is non-orientable, the mapping class group is a group of diffeomorphisms of Σ up to isotopy.

Let us recall a few well-known facts about the generators of the mapping class group of a compact surface which we require for this article. Lickorish [10], [12] showed that the mapping class group of a compact oriented

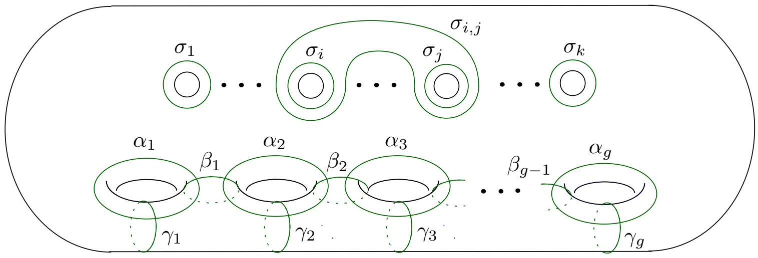

FIGURE 1. The Dehn twists along the curves shown in the above figure generates the mapping class group $\mathcal{MCG}(\Sigma, \partial\Sigma)$.

surface Σ is generated by the Dehn twists along the simple closed curves on Σ and the mapping class group of a compact non-orientable surface N is generated by the Dehn twists along the two-sided simple closed curves on N and the y–homeomorphisms along embedded Y–spaces in N. A Dehn twist along a two-sided simple closed curve c on a surface S, we mean a diffeomorphism $d_c : S \to S$ supported in a regular neighborhood of c in S obtained by cutting S along c, twisting 2π to right or left and regluing. Recall that a Y–space can be obtained from a Möbius band μ by removing one disc $\mathcal{D}$ from the interior and attaching a Möbius band $\mathcal{M}$ along $\partial\mathcal{D}$. A y–homeomorphism of a Y–space is a homeomorphism $y : Y \to Y$ which is obtained by sliding the Möbius band $\mathcal{M}$ on Y once along the core of the Möbius band μ.

In fact, Lickorish showed that the mapping class group $\mathcal{MCG}(\Sigma_{g,k})$ of a compact oriented surface of genus g with k boundary components is generated by the set of all the Dehn twists along the simple closed curves α_i's, β_i's, γ_i's, σ_i's and $\sigma_{i,j}$'s as shown in Figure 1. We call these curves Lickorish generating curves.

Now, we describe a finite generating set for the mapping class group of a compact non-orientable surface given by Omori and Kobayashi [11]. To do this, we first describe a compact non-orientable surface $N_{g,k}$ of genus g with k–boundary components as follows. Consider the surface S obtained by removing $g+k$ disjoint open discs $D_1, D_2, \ldots, D_{g+k}$ from the 2–sphere. Then, the non-orientable surface $N_{g,k}$ is obtained from S by attaching g Möbius bands $\mu_1, \mu_2, \ldots, \mu_g$ along $\partial D_1, \partial D_2, \ldots \partial D_g$, respectively as shown in Figure 2. The cross marks in Figure 2 depicts the g Möbius bands $\mu_1, \mu_2, \ldots, \mu_g$. Kobayashi and Omori [11] proved that the mapping class

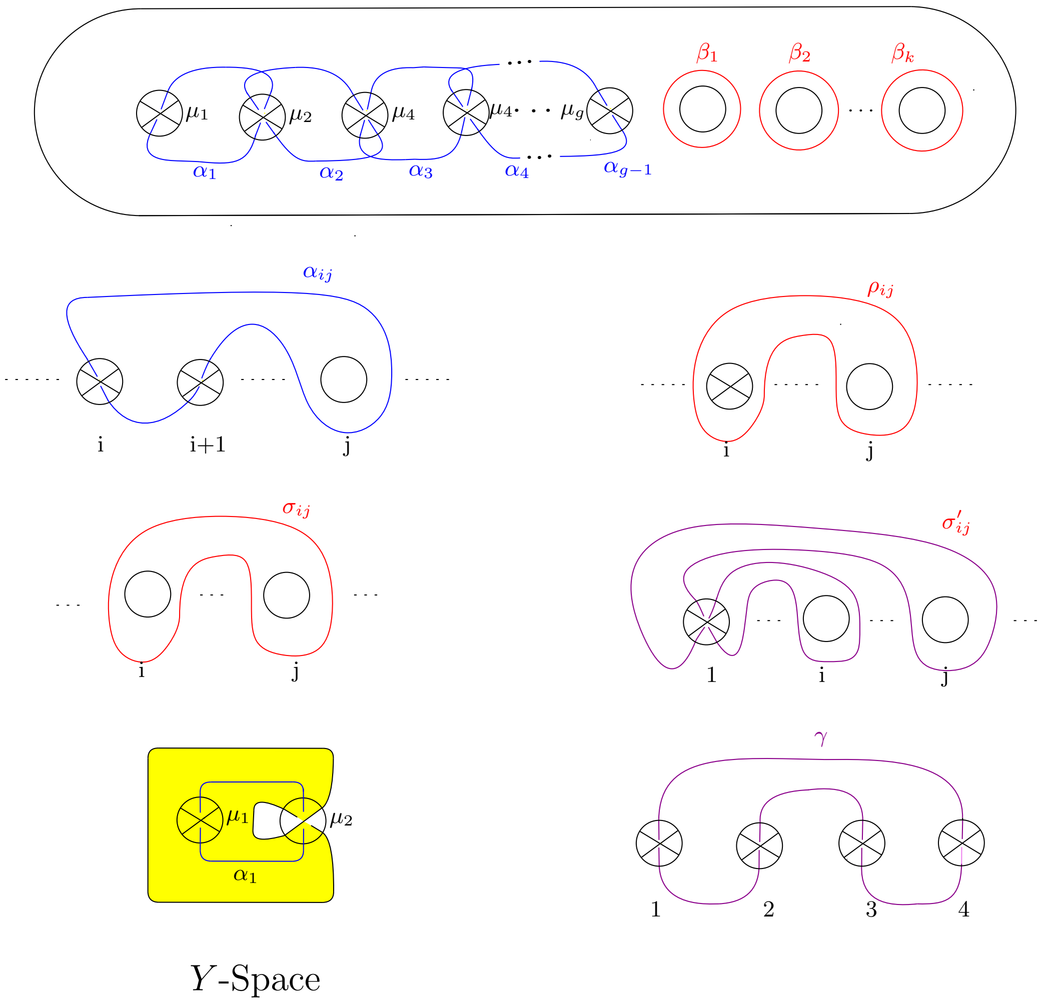

FIGURE 2. Kobayashi and Omori generators for the mapping class group of a compact non-orientable surface with boundary.

group $\mathcal{MCG}(N_{g,k}, \partial N_{g,k})$ is generated by the Dehn twists along the simple closed curves $\alpha_i, \beta_i, \alpha_{i,j}, \rho_{i,j}, \sigma_{i,j}, \sigma'_{i,j}, \gamma$ as shown in Figure 2 and the y–homeomorphism obtained by sliding the Möbius band along α_1 once on the Y–space as shown in Figure 2. We call these curves Kobayashi-Omori generating curves.

3. Flexible Immersions in D^4

In this section, we define the notion of a flexible immersion of a compact surface in a compact 4–manifold and show that the 4–ball D^4 admits a proper flexible immersion of every compact surface. Let us begin by the following definition.

Definition 3.1. Let S be a compact surface and X be a compact 4–manifold. A proper immersion $f : (S, \partial S) \looparrowright (X, \partial X)$ is said to be a flexible immersion if given any diffeomorphism $\phi : S \to S$, there exists a family $f_t : S \looparrowright X$ of proper immersions such that $f_0 = f$, $f_1 = f \circ \phi$ and all the immersions f_t agree on a collar neighborhood of $\partial\Sigma$. In this case, we say that the diffeomorphism ϕ is flexible with respect to the immersion f.

Remark 3.2. In the above definition, if each f_t is an embedding, then the embedding f is called a flexible embedding in the sense of Hirose and Yasuhara [8].

Definition 3.3. Let $f : S \looparrowright X$ be a proper immersion of a compact surface S into a compact 4–manifold X. We say that a two-sided simple closed curve c on S is in standard position with respect to the immersion f if

(1) the map f restricted to an annular neighborhood $\mathcal{A}$ of c in S is an embedding.
(2) the curve $c' = f(c)$ bounds a disc D in X which intersects $f(S)$ only in c',
(3) a tubular neighborhood $\mathcal{N}(D)$ of D in X intersects $f(S)$ in a regular neighborhood $\nu(c') = f(\mathcal{A})$ of c' in $f(S)$,
(4) there is a coordinate chart $\phi : \mathbb{C}^2 \to \mathcal{N}(D)$ such that $\phi(g^{-1}(1)) = \nu(c')$ and $\partial(\phi \circ g^{-1}(z)) = \partial(\phi \circ g^{-1}(w))$ for all $z, w \in \mathbb{C}$, where the map $g : \mathbb{C}^2 \to \mathbb{C}$ is given by $g(z_1, z_2) = z_1 z_2$.

Remark 3.4. The standard position of a curve c on a compact surface S with respect to an embedding f of S into a compact 4–manifold is defined in [5].

Now, we have the following Lemma:

Lemma 3.5. *Let f be a proper immersion of a compact surface S into a compact 4–manifold X. Let c be a two-sided simple closed curve in S which* *is in standard position with respect to the immersion f. Then, the Dehn twist d_c along c is flexible with respect to the immersion f.*

Proof. We show there is a smooth family f_t of immersions of S in X such that $f_0 = f$ and $f_1 = f \circ d_c$.

As c is in standard position, it bounds a disc D such that the disc D and a tubular neighborhood $\mathcal{N}(D)$ of D satisfies all the properties in Definition 3.3. Let $\mathcal{A}$ be an annular neighborhood of c in S as in Definition 3.3. Note that the regular fiber of the Lefschetz fibration $\pi = g \circ \phi^{-1} : \mathcal{N}(D) \to \mathbb{C}$ is an annulus with $\partial \pi^{-1}(y) = h_1 \cup h_2$, for all $y \in \mathbb{C}$ in X, where g and ϕ are as in Definition 3.3. The monodromy of the fibration π along any simple closed curve $\gamma : [0,1] \to \mathbb{C}$ which goes around the origin O of $\mathbb{C}$ anticlockwise is a Dehn twist along the central curve of the annulus $\pi^{-1}(\gamma(0))$. Therefore, there exists a smooth map $\psi : [0, 2\pi] \times \mathcal{N}(D) \to \mathcal{N}(D)$ such that

- for each θ, we have a diffeomorphism $\psi_\theta : \mathcal{N}(D) \to \mathcal{N}(D)$ given by $\psi_\theta(x) = \psi(\theta, x)$.
- $\psi_\theta(\pi^{-1}(1)) = \pi^{-1}(e^{i\theta})$.
- ψ_θ is the identity on $h_1 \cup h_2$.
- $\psi_{2\pi}|_{\pi^{-1}(1)} : \pi^{-1}(1) \to \pi^{-1}(1)$ is a Dehn twist along the central curve of the annulus $\pi^{-1}(1)$.

Now, we consider the family $f_t : S \looparrowright D^4$ of proper immersions of S into D^4 as follows.

$$f_t(x) = \begin{cases} f(x) & \text{if } x \in S \setminus \mathcal{A} \\ \psi_{2\pi t} \circ f(x) & \text{if } x \in \mathcal{A} \end{cases}$$

Note that $f_0 = f$ and $f_1 = \psi_{2\pi} \circ f = f \circ d_c$. Thus, f_t is the required family of immersions.

□

Remark 3.6. In the definition of the family f_t described in the above lemma, if we use $\psi_{2\pi t}^{-1}$ instead of $\psi_{2\pi t}$, then we get flexibility of the Dehn twist d_c^{-1} with respect to f.

Now, we prove the following important lemma of this article.

Lemma 3.7. *Let Σ be a compact orientable surface. Then, there exists a proper flexible immersion $f : \Sigma \looparrowright D^4$ of Σ into D^4.*

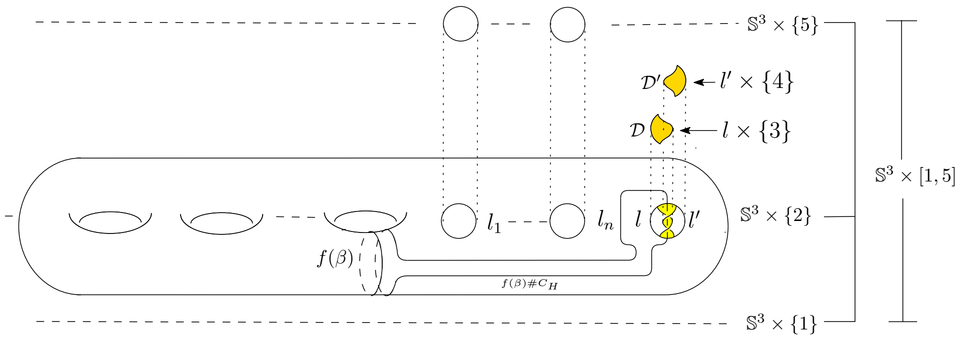

FIGURE 3. Proper flexible immersion f of $\Sigma_{g,n}$ into D^4

Proof. We regard $S^3 \times [1,5]$ as a collar of ∂D^4 in D^4 with $\partial D^4 = S^3 \times \{5\}$. Let Σ_g be the boundary of a standardly embedded genus g handlebody in $S^3 \times \{2\}$. Now, we remove the interiors of $n+1$ disjoint discs $D_1, D_2, \dots D_{n+1}$ from Σ_g and attach one full twisted band along ∂D_{n+1}. This produces an embedded genus g surface $\Sigma_{g,n+2}$ with $n+2$ boundary components $l_1, l_2, \dots l_n, l, l'$ in $S^3 \times \{2\}$ as shown in Figure 3. Finally, the embedded surface $\Sigma_{g,n+1}$ together with n cylinders $l_1 \times [2,5], \dots l_n \times [2,5]$ and the properly embedded discs $\mathcal{D}$ and $\mathcal{D}'$ in $S^3 \times [2,3)$ and $S^3 \times [2,4)$, respectively with $\partial\mathcal{D} = l$ and $\partial\mathcal{D}' = l'$ produces a proper immersion of the genus g surface $\Sigma_{g,n}$ with n boundary components in D^4. We denote this immersion by the map $f : \Sigma_{g,n} \hookrightarrow D^4$. After a small perturbation, we can make f smooth.

Now, we show that the map f is a flexible immersion. Recall that the mapping class group $\mathcal{MCG}(\Sigma_{g,n})$ is generated by the Dehn twists along the Lickorish generating curves $\alpha_i, \beta_i, \gamma_i, \sigma_i$ and $\sigma_{i,j}$ as shown in Figure 1. Therefore, it is enough to show that Dehn twists along the Lickorish generating curves are flexible with respect to f. Let γ be a Lickorish generating curve. Since the simple closed curve l in $f(\Sigma_{g,n})$ bounds the disc $\mathcal{D}$ in $f(\Sigma_{g,n})$, the curve γ can be isotoped in $\Sigma_{g,n}$ to a simple closed curve β such that $f(\beta)$ is the band connected sum $f(\gamma)\#C_H$ of $f(\gamma)$ and the curve $C_H \subset f(\Sigma_{g,n}) \cap S^3 \times \{2\}$, where C_H is a simple closed curve parallel to l and passes once through the full twisted band as shown in Figure 3. Notice that a regular neighborhood of $f(\beta)$ in $f(\Sigma_{g,n})$ is a Hopf annulus $\mathcal{H}$ in $S^3 \times \{2\}$. Note that $f(\beta)$ is an unknot in $S^3 \times \{2\}$. Hence, it bounds a disc D in $S^3 \times (1,2]$. Since, the regular neighborhood of $f(\beta)$ in $f(\Sigma_{g,n})$ is a Hopf annulus $\mathcal{H}$ in $S^3 \times \{2\}$, the disc D and a tubular neighborhood $\mathcal{N}(D)$ of D in D^4 satisfy all the properties required for the curve β to be in standard position with respect to the immersion f. Let ϕ_t be an isotopy of $\Sigma_{g,n}$ which sends γ to β. Now, it is clear that the curve γ is in standard position with respect to the immersion $f \circ \phi_1$.

Therefore, by Lemma 3.5, there is a family $h_t : \Sigma_{g,n} \looparrowright D^4$ of $\Sigma_{g,n}$ into D^4 of immersions with $h_0 = f \circ \phi_1$ and $h_1 = f \circ \phi_1 \circ d_\gamma$.

Now, we consider the family $f_t : \Sigma_{g,n} \looparrowright D^4$ of $\Sigma_{g,k}$ into D^4 as follows.

$$f_t = \begin{cases} f \circ \phi_t & 0 \le t \le 1 \\ h_{t-1} & 1 \le t \le 2 \\ f \circ \phi_{3-t} \circ d_\gamma & 2 \le t \le 3 \end{cases}$$

Observe that $f_0 = f$ and $f_3 = f \circ d_\gamma$. This completes the proof of the Lemma.

□

Now, we discuss the similar Lemma for compact non-orientable surfaces.

Lemma 3.8. *Let N be a compact non-orientable surface. Then, there exists a proper flexible immersion $f : N \looparrowright D^4$ of N into D^4.*

Proof. We regard $S^3 \times [1,5]$ as a collar of ∂D^4 in D^4 with $\partial D^4 = S^3 \times \{5\}$. Let S be the boundary of a 3–ball B^3 in $S^3 \times \{2\}$. Now, we remove the interiors of $g+k+1$ disjoint discs $D_1, D_2, \ldots D_{g+k+1}$ from S and attach g half twisted bands along $\partial D_1, \partial D_2, \ldots, \partial D_g$ and one full twisted band along ∂D_{g+k+1}. This produces an embedded genus surface S' with $g+k+2$ boundary components $b_1, b_2, \ldots, b_g, l_1, l_2, \ldots l_k, l, l'$ in $S^3 \times \{3\}$ as shown in Figure 3. Finally, the embedded surface S' together with n cylinders $l_1 \times [2,5], \ldots l_k \times [2,5]$, the properly embedded discs $\Delta_2, \ldots \Delta_g, \mathcal{D}'$ in $S^3 \times [2,4)$ with $\partial \Delta_i = b_i, \partial \mathcal{D}' = l'$ and the properly embedded discs $\Delta_1, \mathcal{D}'$ in $S^3 \times [2,3)$ with $\partial \Delta_1 = b_1, \partial \mathcal{D} = l$ produces a proper immersion of the non-orientable genus g surface $N_{g,k}$ with k boundary components in D^4. We denote this proper immersion by the map $f : N_{g,k} :\hookrightarrow D^4$. After a small perturbation, we can make f smooth.

Now, we show that the map f is a flexible immersion. Recall that the mapping class group $\mathcal{MCG}(N_{g,k})$ is generated by the Dehn twists along the Kobayashi-Omori generating curves $\alpha_i, \beta_i, \alpha_{i,j}, \rho_{i,j}, \sigma_{i,j}, \sigma'_{i,j}, \gamma$ as shown in

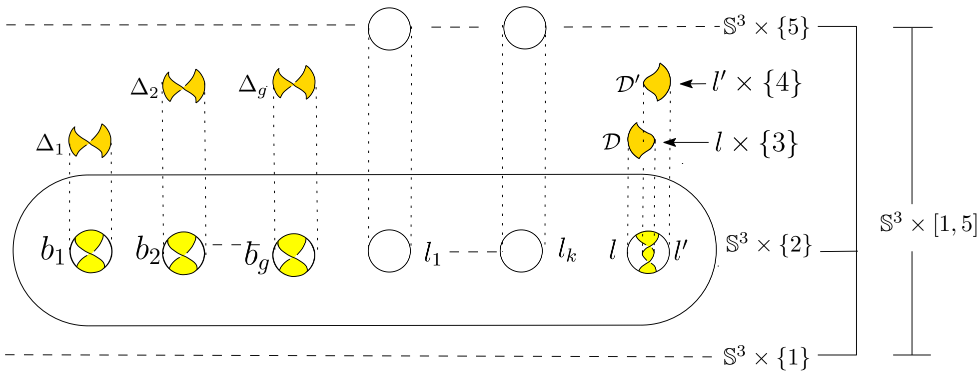

FIGURE 4. Proper flexible immersion f of $N_{g,k}$ into D^4

Figure 2 and the y–homeomorphism obtained by sliding the Möbius band μ_1 along α_1 once. Therefore, it is enough to show that Dehn twists along the Kobayashi-Omori generating curves and the y–homeomorphism are flexible with respect to f. Notice that each Kobayashi-Omori generating curve can be isotped to a curve on $N_{g,k}$ which is in standard position with respect to f. Therefore, by similar arguments used in the proof of Lemma 3.7, we can see that the Dehn twists along Kobayashi-Omori generators are flexible with respect to f.

Now, we show that the y–homeomorphism is flexible with respect to f. Let D^3 be a 3–disc in $S^3 \times \{2\}$ containing only the first half twisted band and ∂D_1 from the immersed surface $f(N_{g,k})$. One can easily see that there is an isotopy ϕ_t of $S^3 \times \{2\}$ which slides B^3 once along α_1. We use this isotopy ϕ_t to define the diffeomorphism $\psi : S^3 \times [1,5] \to S^3 \times [1,5]$ given by

$$\psi(x,t) = \begin{cases} (\phi_{t-1}(x), t), & 1 \leq t \leq 2 \\ (\phi_1(x), t), & 2 \leq t \leq 3 \\ (\phi_{4-t}(x), t), & 3 \leq t \leq 4 \end{cases}$$

Note the diffeormphism ψ is identity on $S^3 \times \{1\}$ and $S^3 \times \{4\}$. Therefore, the diffeomorphism ψ can be extended by the identity map in the complement of $S^3 \times [1,4]$ in D^4 to a diffeomorphism $\Psi : D^4 \to D^4$ which is isotopic to the identity map. Let ψ_t be an isotopy of D^4 with $\Psi_0 = Id$ and $\Psi_1 = \Psi$. Now, we consider the family $f_t : N_{g,k} \to D^4$ of proper immersions given by $f_t = \Psi_t \circ f$. One can easily see that $f_0 = f$ and $f_1 = \Psi_1 \circ f = f \circ y$. This completes the proof of the Lemma.

$\square$

4. Open book immersions of all closed 3–manifolds in the trivial open book of S^5

In this section, we recall the definition of an open book decomposition and define the notion of an open book immersion. Then, we show that all closed 3–manifolds open book immerses in the trivial open book of S^5. Let us begin with the following definition.

Definition 4.1. An open book decomposition of a closed manifold M is a pair (B, π), where

(1) B is a closed co-dimension 2 submanifold of M with trivial normal bundle,
(2) $\pi : M \setminus B \to S^1$ is a fiber bundle,
(3) there is a trivialization of a tubular neighborhood $\mathcal{N}(B) = B \times D^2$ of B in Msuch that the map $\pi : B \times D^2 \setminus B \times \{0\} \to S^1$ sends (x, r, θ) to θ, where x is a coordinate on B and (r, θ) are the polar coordinates on D^2.

Here, B is called the binding, and the closure of $\pi^{-1}(\theta)$ is called the page of the open book (B, π).

Note that an open book (B, π) of M is completely determined upto diffeomorphism of M by the page $S = \pi^{-1}(\theta)$ of the open book (B, π) and the monodromy ϕ of the fibration $\pi : M \setminus B \to S^1$. i.e. there is diffeomorphism $F : M_{S,\phi} \to M$, where

$$M_{S,\phi} = \mathcal{M}T(S, \phi) \bigcup_{Id} \partial S \times D^2, \text{ where } \mathcal{M}T(S, \phi) = \frac{S \times [0, 1]}{(x, 1) \sim (\phi(x), 0)}$$

such that

(1) the map $\pi \circ F : \mathcal{M}T(S, \phi) \to S^1$ is given by $(\pi \circ F)(y, \theta) = e^{i\theta}$,
(2) the map $\pi \circ F : \partial S \times D^2 \setminus \partial S \times \{0\} \to S^1$ is given by $(\pi \circ F)(x, r, \theta) = \theta$, where x is a coordinate on ∂S and (r, θ) are the polar coordinates on D^2.

In this case, we call the pair (S, ϕ) an *abstract open book* of M associated to the open book of (B, π). Now, we state the notion of an open book immersion.

Definition 4.2. Let M and N be closed manifolds with the open books (B, π) and (B', π'). We say that an immersion $f : M \to N$ is an open book immersion if

(1) the map f restricted to B is an embedding of B into B',
(2) the following diagram commutes:

$$\begin{array}{ccc} M \setminus B & \xrightarrow{f} & N \setminus B' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ S^1 & \xrightarrow{Id} & S^1 \end{array}$$

Remark 4.3. Note that the condition 2 in the above definition implies for each $\theta \in S^1$, $f : \overline{\pi^{-1}(\theta)} \to \overline{\pi'^{-1}(\theta)}$ is a proper immersion, i.e. f properly immerses the page of the open book (B, π) into the page of the open book (B', π').

Recall that the n–sphere S^n admits the trivial open book (B', π') with pages the $n-1$ disc D^{n-1} and the monodromy the identity map, where

$$B' = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_1 = x_2 = 0\}$$

and the map $\pi' : S^n \setminus B \to S^1$ is given by

$$\pi'((x_1, x_2, \ldots, x_{n+1})) = \frac{(x_1, x_2)}{x_1^2 + x_2^2}.$$

Now, we prove the following theorem as a first application of Lemma 3.7.

Theorem 4.4. *Every open book of every closed* 3*–manifold open book immerses in the* 5*–sphere* S^5 *with the trivial open book.*

Proof. Let M be a closed 3–manifold with an open book (B, π). Let (S, ϕ) be an abstract open book of M associated to (B, π). Let $F : M_{S,\phi} \to M$ be a diffeomorphism, where

$$M_{S,\phi} = \mathcal{M}T(S, \phi) \bigcup_{Id} \partial S \times D^2$$

such that

(1) the map $\pi \circ F : \mathcal{M}T(S, \phi) \to S^1$ is given by $(\pi \circ F)(y, \theta) = e^{i\theta}$,

(2) the map $\pi \circ F : \partial S \times D^2 \setminus \partial S \times \{0\} \to S^1$ is given by $(\pi \circ F)(x, r, \theta) = \theta$, where x is a coordinate on ∂S and (r, θ) are the polar coordinates on D^2.

Similarly, the trivial open book decomposition (B', π') of S^5 decomposes S^5 as

$$D^4 \times S^1 \cup_{Id} S^3 \times D^2$$

such that

(1) the map $\pi' : D^4 \times S^1 \to S^1$ is given by $\pi'(y, \theta) = \theta$,
(2) the map $\pi' : S^3 \times D^2 \setminus S^3 \times \{0\} \to S^1$ is given by $\pi(x, r, \theta) = \theta$, where x is a coordinate on S^3 and (r, θ) are the polar coordinates on D^2.

Our first aim is to construct a proper immersion $G : \mathcal{M}T(S, \phi) \looparrowright D^4 \times S^1$ such that $\pi' \circ F = \pi$. By Lemma 3.7 for orientable surfaces and by Lemma 3.8 for non-orientable surfaces, there exists a proper flexible immersion $g : S \looparrowright D^4$. Therefore, there exists a family $g_t : S \looparrowright D^4$ of proper immersions such that $g_0 = g$ and $g_{2\pi} = g \circ \phi$. This family g_t allows us to define the desired map G as $G(y, \theta) = g_\theta(y)$.

Note that the map G restricted to the boundary $\partial \mathcal{M}T(S, \phi)$ of $\mathcal{M}T(S, \phi)$ is an embedding. Now, it is easy to see that the proper immersion G can be extended to an open book immersion of $M_{S,\phi}$ into the trivial open book of S^5. Thus, $G \circ F^{-1} : M \to S^5$ is the required open book immersion of M into S^5. $\square$

5. Representing the homology classes by flexible immersions

It is well known that every $d \in H_2(\mathbb{C}P^2, \mathbb{Z})$ can be represented by an embedding of a closed-oriented surface. Therefore, it is interesting to know that given $d \in H_2(\mathbb{C}P^2, \mathbb{Z})$, can we represent it by a flexible embedding of a closed-oriented surface? In [9], Kronheimer and Mrowka showed that if a closed oriented smoothly embedded genus g surface Σ in $\mathbb{C}P^2$ represents the same homology class as an algebraic curve of degree d then $g \geq \frac{(d-1)(d-2)}{2}$. Therefore, it is not possible to represent a given $d \in H_2(\mathbb{C}P^2, \mathbb{Z})$ by a closed-oriented genus g embedded surface for every g. Hence, it is not possible to represent a given $d \in H_2(\mathbb{C}P^2, \mathbb{Z})$ by a flexible embedding of the closed oriented surface of genus g for every g. Here, we ask the same question for flexible immersions.

In this section, we give a second application of flexible immersions of compact surfaces in 4–manifolds. We show that given a second integral homology class d of $\mathbb{C}P^2$ and given a closed oriented surface Σ, there is a flexible immersion $f : \Sigma \looparrowright \mathbb{C}P^2$ such that f represents the class d. We also show that each of the classes 0 and $\pm 2 \in \mathbb{Z} = H_2(\mathbb{C}P^2, \mathbb{Z})$ can be represented by a flexible embedding $f : \Sigma \hookrightarrow \mathbb{C}P^2$ of any oriented surface Σ into $\mathbb{C}P^2$. Let us begin by the following Lemma.

Lemma 5.1. *Let $g : S^2 \looparrowright X$ be an immersion of the 2–sphere S^2 into a closed oriented 4-manifold X such that g represents the second integral homology class $d \in H_2(X, \mathbb{Z})$. Suppose g is injective in the complement of finitely many points on S^2. Then, given a closed-oriented surface Σ, there is a flexible immersion $f : \Sigma \looparrowright X$ which represents the class d.*

Proof. Let D^4 be a 4–disc in X disjoint from the immersion g, i.e. $D^4 \cap g(S^2) = \emptyset$. Let $h : \Sigma \looparrowright D^4 \subset X$ be a flexible immersion of the closed oriented surface Σ in D^4 as constructed in Lemma 3.7. Therefore, the immersion h represents the class $0 \in H_2(X, \mathbb{Z})$. Note that that the the flexible immersion h is in a collar $S^3 \times [1, 5]$ of ∂D^4 in D^4 with $\partial D^4 = S^3 \times \{5\}$. Also note that the immersion h is injective in the complement of two points on Σ. Therefore, there is a point p_1 on S^2 and a point p_2 on Σ such that the $g^{-1}(g(p_1)) = p_1$ and $h^{-1}(h(p_2)) = p_2$. Let α be an embedded arc in $(X \setminus D^4) \cup S^3 \times [2, 5]$ with the end points $g(p_1)$ and $g(p_2)$ on $g(S^2)$ and $h(\Sigma)$, respectively such that the interior of the arc α is disjoint from both the immersions g and h. Now, we take appropriate ambient connected sum of the immersions g and h in X along a tube contained in a tubular neighborhood $\mathcal{N}(\alpha)$ of the arc α in X to get the immersion $f = g \# h : \Sigma \approx S^2 \# \Sigma \looparrowright X$ such that the immersion f represents the class d. Now, it is easy to see that the flexibility of the immersion f follows by the similar arguments used in Lemma 3.7. □

Remark 5.2. Any generic immersion $g : S^2 \looparrowright X^4$ is injective in the complement of finitely many points of S^2.

Now, we prove the following theorem.

Theorem 5.3. *Let $d \in H_2(\mathbb{C}P^2, \mathbb{Z})$ be a second integral homology class of the complex projective 2–space $\mathbb{C}P^2$ and Σ be a closed oriented surface. Then, there exists a flexible immersion $f : \Sigma \looparrowright \mathbb{C}P^2$ which represents the class d.*

Proof. By Lemma 5.1, it is enough to construct an immersion $g : S^2 \looparrowright \mathbb{C}P^2$ which is injective in the complement of finitely many points of S^2 and represents the class d.

To construct the required immersion g, we consider the following convenient handle decomposition of $\mathbb{C}P^2$. The handle decomposition consists of one 0–handle H_0, one 2–handle H_2 and one 4–handle H_4. In this decomposition, the 2–handle H_2 is attached to the 0–handle along an unknot in ∂H_0 with the framing -1 and finally the 4–handle H_4 is attached to $H_0 \cup H_2$ along $\partial(H_0 \cup H_2) = S^3$ to get $\mathbb{C}P^2$.

Now, we describe the desired immersion g of S^2 in $\mathbb{C}P^2$. We regard $S^3 \times [1, 5]$ as a collar of $\partial H_0 = \partial D^4$ in the 0–handle $H_0 = D^4$ with $\partial D^4 = S^3 \times \{5\}$. Consider an embedded 2–sphere S as the boundary of a 3–disc in $S^3 \times \{2\}$. Then, we remove the interiors of d disjoint discs $D_1, D_2, \ldots, D_d$ from S and attach d cylinder $\partial D_1 \times [2, 5], \ldots, \partial D_d \times [2, 5]$ along $\partial D_1, \ldots, \partial D_d$, respectively to get a properly embedded surface S' of genus 0 in H_0 with d boundary components $l_1, l_2, \ldots, l_d$. Let K be the attaching circle of the 2–handle H_2 with the framing -1 and let $\mathcal{N}(K) = S^1 \times D^2$ be an attaching region of H_2 in ∂H_0 . Notice that each l_i is an unknot in $\partial H_0 = S^3$ without linking each other. Therefore, without loss of generality, we can assume that $l_i = S^1 \times \{(\frac{1}{i}, 0)\} \subset S^1 \times D^2 = \mathcal{N}(K)$. Now, it is easy to see that each l_i bounds a disc $\mathcal{D}_i$ in the 2–handle H_2 and intersects the co-core of H_2 once. Since the 2–handle H_2 is attached along K with -1 framing, all the discs $\mathcal{D}_i$'s intersect at the origin of the co-core of H_2. Thus, the embedded surface S' together with the d discs $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_d$ produces an immersion of S^2 in $\mathbb{C}P^2$ which is injective on S^2 in the complement of $2d$ points. We denote this immersion by the map $g : S^2 \to \mathbb{C}P^2$. By choosing the correct orientation on S^2, we can make sure that $[g] = d \in \mathbb{Z} = H_2(\mathbb{C}P^2, \mathbb{Z})$. $\square$

Remark 5.4. Let $d \in H_2(\#_k\mathbb{C}P^2, \mathbb{Z})$ be a second integral homology class of the connected sum $\#_k\mathbb{C}P^2$ of k copies of the complex projective 2–spaces $\mathbb{C}P^2$'s and Σ be a closed oriented surface. Then, there exists a flexible immersion $f : \Sigma \looparrowright \#_k\mathbb{C}P^2$ which represents the class d.

Finally, we prove the following theorem.

Theorem 5.5. *Let Σ_g be a closed-oriented surface of genus g. Then, each of the second integral homology classes 0 and $\pm 2 \in \mathbb{Z} = H_2(\mathbb{C}P^2, \mathbb{Z})$ can be represented by a flexible embedding $f : \Sigma_g \hookrightarrow \mathbb{C}P^2$.*

Proof. We consider the same handlebody decomposition of $\mathbb{C}P^2$ as described in the proof of Theorem 5.3 and regard $S^3 \times [1,5]$ as a collar of $\partial H_0 = \partial D^4$ in the 0–handle $H_0 = D^4$ with $\partial D^4 = S^3 \times \{5\}$.

Representing the class $0 \in H_2(\mathbb{C}P^2, \mathbb{Z})$ **by a flexible embedding of** Σ_g**:**

Consider the embedded surface Σ_g as a boundary of standardly embedded genus g handlebody H_g in $S^3 \times \{2\}$. We denote this embedding of Σ_g in $\mathbb{C}P^2$ by the map $f : \Sigma_g \hookrightarrow \mathbb{C}P^2$. In order to show the embedding f is flexible, it is enough to show that the Dehn twists along each of the Lickorish generating curves on Σ_g is flexible with respect to the embedding f. Let γ be a Lickorish generating curve on Σ_g and A be a regular neighborhood of γ in Σ_g. Observe that if there is an isotopy $\psi_t : \mathbb{C}P^2 \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ such that $\psi_0 = Id$ and the curve γ is in standard position with respect to the embedding $\psi_1 \circ f$, then the curve γ is also in standard position with respect to the embedding f. Then, the flexibility of the Dehn twist d_γ with respect to f follows by Lemma 3.5.

Let us show that such an isotopy ψ_t exists. Notice that a regular neighborhood $f(A)$ of $f(\gamma)$ in $f(\Sigma_g)$ is a planar annulus $\mathcal{A}$ in $S^3 \times \{2\}$, i.e. $\partial\mathcal{A}$ is a link consisting of two unknots with the linking number 0 in $S^3 \times \{2\}$. Let K be the attaching circle of the 2–handle H_2 with the framing -1 and let $\mathcal{N}(K) = S^1 \times D^2$ be an attaching region of H_2 in ∂H_0. Now, one can observe the following:

(1) There is family $f_t : \Sigma_g \to H_0 \subset \mathbb{C}P^2$, $0 \leq t \leq 1$ of embeddings such that $f_0 = f$ and $f_1(A) = S^1 \times [-1,1] \subset S^1 \times D^2 = \mathcal{N}(K) \subset \partial H_0$ and all f_t agree with f in the complement of a small neighborhood of A in Σ_g. This can be achieved by pushing the planar annulus $\mathcal{A}$ towards the attaching region $\mathcal{N}(K)$ of the 2–handle H_2.
(2) Since, the 2–handle H_2 is attached to H_0 along the unknot K with the framing -1, $f_1(A) = S^1 \times [-1,1]$ becomes a Hopf annulus $\mathcal{H} \subset \partial H_2$ in the boundary of the 2–handle H_2.
(3) Therefore, there is family $h_t : \Sigma_g \to H_0 \cup H_2 \subset \mathbb{C}P^2$ of embeddings such that $h_0 = f_1$ and $h_1(A)$ is a properly embedded annulus in H_2 with $h_1(A) = g^{-1}(1)$, where $g : H_2 = \mathbf{D}^4 \to \mathbb{C}$ is Lefschetz fibration given by $g(z_1, z_2) = z_1 z_2$ with complex coordinates (z_1, z_2) on $\mathbf{D}^4$. The family h_t can be achieved by pushing the interior of $\mathcal{H} \subset \partial H_2$ into the interior of the 2–handle H_2.

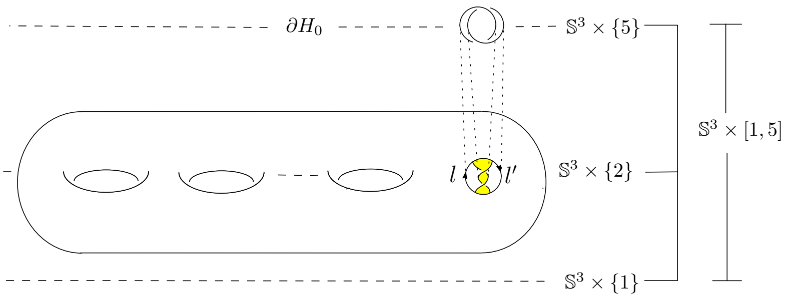

FIGURE 5. Figure depicts the proper embedding g of $\Sigma_{g,2}$ into the 0–handle H_0.

Now, we consider the following family of embeddings $g_t : \Sigma_g \to \mathbb{C}P^2$ defined by.

$$g_t = \begin{cases} f_t & 0 \leq t \leq 1 \\ h_{t-1} & 1 \leq t \leq 2 \end{cases}$$

Note that the curve γ is in standard position with respect to the embedding $g_2 = h_1$. Since g_t is a family of embeddings of Σ_g in $\mathbb{C}P^2$, there is an isotopy $\psi_t : \mathbb{C}P^2 \to \mathbb{C}P^2$ such that $g_t = \psi_t \circ f$. Thus, as the curve γ is in standard position with respect to $\psi_1 \circ f = g_2$, we have the required isotopy ψ_t.

Representing the classes $\pm 2 \in H_2(\mathbb{C}P^2, \mathbb{Z})$ by a flexible embeddings of Σ_g into $\mathbb{C}P^2$:

Let Σ_g be the boundary of a standardly embedded genus g handlebody in $S^3 \times \{2\}$. Now, we remove the interior of a disc D from Σ_g and attach one full twisted band along ∂D. This produces an embedded genus g surface $\Sigma_{g,2}$ with 2 boundary components l, l' in $S^3 \times \{2\}$ as shown in Figure 5. Finally, the embedded surface $\Sigma_{g,2}$ together with 2 cylinders $l \times [2,5], l' \times [2,5]$ produces a proper embedding of the genus g surface $\Sigma_{g,2}$ with 2 boundary components in D^4. We denote this embedding by the map $g : \Sigma_{g,2} \hookrightarrow \mathbb{C}P^2$. Now, we describe a flexible embedding $f_2 : \Sigma_g \hookrightarrow \mathbb{C}P^2$ which represents the class $2 \in H_2(\mathbb{C}P^2, \mathbb{Z})$.

Since, the link $l \cup l'$ is a Hopf link in ∂H_0, we can assume that the 2–handle H_2 is attached to H_0 along the unknot l with the framing -1 and the unknot l' corresponds to the framing -1 on the attaching region $\mathcal{N}(l)$ of the 2–handle in ∂H_0. As the unknot l' links once to the attaching circle l

of the 2–handle H_2 in ∂H_0 and the 2–handle H_2 is attached to the 0–handle H_0 along l with the framing -1, the unknot l bounds a disc $\mathbf{D}$ (the core of the 2–handle H_0) in the 2–handle and the unknot l' bounds a disc $\mathbf{D}'$ which is parallel to the core disc $\mathbf{D}$. Notice that the discs $\mathbf{D}$ and $\mathbf{D}'$ are disjoint and each of them intersects once positively to the co-core of the 2–handle H_0. Thus, the embedding $g : \Sigma_{g,2} \hookrightarrow \mathbb{C}P^2$ together with the discs $\mathbf{D}$ and $\mathbf{D}'$ gives an embedding $f_2 : \Sigma_g \hookrightarrow \mathbb{C}P^2$ which represents the class $2 \in H_2(\mathbb{C}P^2, \mathbb{Z})$. The flexibility of the embedding f_2 follows by the similar arguments used in the Lemma 3.7.

Let $\phi : \Sigma_g \to \Sigma_g$ be an orientation reversing diffeomorphism of Σ_g. Then, the flexible embedding $f_2 \circ \phi$ represents the class -2 in $H_2(\mathbb{C}P^2, \mathbb{Z})$. $\square$

Remark 5.6. Let Σ_g be a closed oriented surface of genus g. Then, each of the second integral homology classes $(d_1, d_2, \ldots, d_k) \in \mathbb{Z}^k = H_2(\#_k\mathbb{C}P^2, \mathbb{Z})$ can be represented by a flexible embedding $f : \Sigma \hookrightarrow \#_k\mathbb{C}P^2$ of the oriented surface Σ_g in $\#_k\mathbb{C}P^2$, where each $d_i \in \{0, \pm 2\}$.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 101–115

A SHIMIZU-TYPE THEOREM FOR SUBGROUPS IN $\mathrm{Sp}(N,1)$

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(Received : 23 - 04 - 2023 ; Revised : 13 - 07 - 2024)

Abstract. In this work we shall provide a proof of the Shimizu type inequality for two generator subgroups of $\mathrm{Sp}(n,1)$ with a unipotent parabolic generator. We will also compare our result with some existing Shimizu type results and as an application discuss the extremality of the inequality. Our main result is useful in proving discreteness criteria for Zariski dense subgroups of $\mathrm{Sp}(n,1)$ and $\mathrm{SU}(n,1)$ such as [GMT19].

1. Introduction

To motivate the reader, for Shimizu lemma in $\mathrm{Sp}(n,1)$, we will start with the classical Shimizu lemma and other discreteness criteria for subgroups in $\mathrm{SL}(2,\mathbb{C})$. We also mention some application of related ideas in classical case.

1.1. **Historical Background.**

The study of discrete subgroups of the group of orientation preserving isometries of hyperbolic spaces is closely related with the study of geometry and topology of corresponding orbit spaces i.e. of hyperbolic manifolds. Here a group G is said to be discrete if it is a discrete set in the usual matrix topology or equivalently by discreteness we mean that for a sequence of transformations $T_n \to I$ in the group G, we must have $T_n = I$ $\forall$ large n.

The classical Jørgensen inequality, see [Jor76], gives a necessary criterion to check discreteness of a two generator subgroup (say G) of $\mathrm{SL}(2,\mathbb{C})$ that acts by Möbius transformations on the Riemann sphere. Jørgensen in 1976 proved that the question of discreteness of arbitrary groups in $\mathrm{PSL}(2,\mathbb{C})$ can be reduced to the question of discreteness of two generator subgroups.

2010 Mathematics Subject Classification: Primary 20H10; Secondary 51M10
Key words and phrases: hyperbolic space, Jørgensen inequality, discreteness, quaternions.

Now question arises when the two generator group $G = \langle f, g \rangle$ considered as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is discrete?

Only necessary or only sufficient conditions for the discreteness have been obtained so far. For the necessary condition, we have the remarkable Jørgensen inequality, the inequality of Shimizu-Leutbecher and their analogues. A sufficient condition has been discussed in [GMMR97].

The Jörgensen inequality is a celebrated result concerning the classical problem to understand discreteness of subgroups of $\mathrm{SL}(2, \mathbb{C})$. It provides the necessary condition of discreteness for a two generator subgroup of $\mathrm{SL}(2, \mathbb{C})$ (see [Jor76]). Jörgensen's inequality, which generalises the classical Shimizu's lemma, for two generator subgroups of $\mathrm{PSL}(2, \mathbb{C})$ is among the most important results and tools in the study of three dimensional manifolds.

An element $f \in \mathrm{SL}(2, \mathbb{C})$ is elliptic if it has a fixed point on the hyperbolic 3-space. It is parabolic, respectively loxodromic, if it is non-elliptic and has exactly one, respectively two fixed points on the boundary $\hat{\mathbb{C}}$. For elliptic transformations canonical form of f is given by $z \mapsto \lambda z, |\lambda| = 1$ such that $tr(f) \in \mathbb{R}$ with $tr^2(f) < 4$. For parabolic transformations canonical form is given by $z \to z + k, k \in \mathbb{C}$. such that $tr(f) = \pm 2$. For loxodromic transformations canonical form is given by $z \mapsto \lambda z, \lambda \in \mathbb{C}$ and $|\lambda| \neq 1$. From the canonical form, of loxodromic element, note that, $tr(f) = \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}$ with $tr(f) \neq \pm 2$. This subdivides loxodromic elements into: (purely) hyperbolic when $\lambda \geq 0$ such that $tr(f) \in \mathbb{R}$, with $tr^2(f) > 4$ and loxodromic: $tr(f) \notin \mathbb{R}$. With this background we will now state Jörgensen inequality:

Theorem 1.1. [Jor76]. *If $G =< f, g >$ is discrete group of Möbius transformations, acting on Riemann Sphere $\hat{\mathbb{C}}$ then we have*

$$\|tr^2(f) - 4\| + \|tr(fgf^{-1}g^{-1}) - 2\| \geq 1 \tag{1.1}$$

except in the following three cases, which are elementary groups:

(A) G cyclic or a finite abelian extension of a cyclic group and $|tr^2(f) - 4| < 1$.

(B) f is loxodromic or elliptic with $|tr^2(f) - 4| < 1/2$, while g interchanges the fixed points of f.

(C) f is parabolic while g is parabolic or elliptic of order 2, 3, 4 or 6 and fixes the fixed point of f.

Remark 1.2. If one of the generators, of two generator subgroup for which Jörgensen's inequality holds, is parabolic then the discreteness condition given above is just a reformulation of the classical Shimizu's lemma. In this sense Jörgensen's inequality generalises Shimizu's lemma and deals also with groups with loxodromic and elliptic generators.

More precisely classical Shimizu's lemma [Shi63] gives a necessary condition for a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ containing a parabolic element to be discrete. This result is also very important in studying the structure of cusp(s)in Riemann surfaces. Shimizu proved that:

Theorem 1.3. [Shi63] *Let G be a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *containing the parabolic map $T(z) = z + t$ for some $t > 0$ and any $S(z) = \frac{az+b}{cz+d} \in G$ with $c \neq 0$ then $|c| \geq 1/t$.*

This result Shimizu obtained, as a by-product, in his work of computing the dimension of the space of cusp forms corresponding to any irreducible, discrete subgroup of a product of copies of $\mathrm{SL}(2,\mathbb{R})$. Geometrically Shimizu's lemma says that the radius r_S of the isometric sphere of isometry S satisfies $r_S \leq t$, where t is the translational length of the T. Therefore Shimizu lemma provides a uniform bound on the radii of isometric spheres of those elements of the group not having infinity as their fixed point. This uniform bound is the translation length (in Euclidean metric) of the parabolic map fixing infinity. For the modular group $\mathrm{PSL}(2,\mathbb{Z})$, Shimizu lemma is sharp with $t = 1$, and $S(z) = -1/z$ has $r_S = 1$. There is a cusp neighborhood C which cannot be enlarged. It has area equal to 1, which is large compared to the area $\mathbf{H^2}/\mathrm{PSL}(\mathbf{2},\mathbb{Z})$. In another way other hand this can also be considered as a special case of another important result known as Collar lemma. Later these discreteness criteria have been vastly generalized; a striking example is Margulis's lemma for non-positively curved manifolds. We will mention a couple of generalisations:

1.2. **Some Applications.** The Margulis lemma (named after Grigory Margulis) is a result about discrete subgroups of isometries of a non-positively curved Riemannian manifold (e.g. the hyperbolic n-space). Roughly, it states that within a fixed radius, usually called the Margulis constant, the structure of the orbits of such a group cannot be too complicated. More precisely, within this radius around a point all points in its orbit are in fact in the orbit of a nilpotent subgroup (in fact a bounded finite number of such). The Margulis lemma can be formulated as follows.

Theorem 1.4. [Rat2006] *Let X be a simply-connected manifold of non-positive bounded sectional curvature. There exist constants $C, \varepsilon > 0$ with the following property. For any discrete subgroup Γ of the group of isometries of X and any $x \in X$, if F_x is the set:*

$$F_x = \{g \in \Gamma : d(x, gx) < \varepsilon\}$$

then the subgroup generated by F_x contains a nilpotent subgroup of index less than C.

Here d is the distance induced by the Riemannian metric. A particularly well studied family of examples of negatively curved manifolds are given by the symmetric spaces associated to semisimple Lie groups. In this case the Margulis lemma can be given the following more algebraic formulation which dates back to Hans Zassenhaus.

Theorem 1.5. [Rag72] *If G is a semisimple Lie group there exists a neighbourhood Ω of the identity in G and a $C > 0$ such that any discrete subgroup Γ which is generated by $\Gamma \cap \Omega$ contains a nilpotent subgroup of index $\leq C$.*

Such a neighbourhood Ω is called a Zassenhaus neighbourhood in G.

1.3. **Shimizu Lemma in** $\mathrm{Sp}(n, 1)$**.** In this paper we prove a generalisation of the Shimizu lemma in $\mathrm{Sp}(n, 1)$. A complex hyperbolic version of this lemma was obtained by Hersonsky and Paulin in (proposition (A.1)[HP96]). More precisely, in complex case, Kamiya [Kam83] generalised Shimizu's lemma for vertical Heisenberg translations and Parker [Par92, Par97] generalised Shimizu's lemma for non-vertical Heisenberg translation or some special type of screw parabolic map. Jiang, Parker and Kamiya [JP03, KP08] generalised Shimizu's lemma to groups containing screw parabolic map. In quaternionic case, Kim and Parker [KP03] generalised Shimizu's lemma to groups containing non-vertical Heisenberg translation. In case $n = 2$, Daeyong Kim [Kim04] extended the results of [JKP03] to quaternionic case. Cao-Parker in [CP18] proved a general version of the Shimizu's lemma that involves an arbitrary parabolic map. Other related problems and results on Shimizu type inequality and discreteness see [BM98, HP96, Par98, GMT19, GMT21] and references therein.

Though the methods of Cao and Parker works for an arbitrary parabolic map, it is pretty technical and we find it difficult for application e.g. to obtain discreteness criteria for Zariski dense subgroups in $\mathrm{Sp}(n,1)$ as in [GMT19] the version of Shimizu type lemma proved by Cao-Parker is not suitable. Hence, here we provide a much simpler form of the quaternionic Shimizu's lemma that is quite analogous to the result of Hersonsky and Paulin. We must remark that though this version is simpler, it is weaker than the versions of Kim and Parker, and Cao and Parker mentioned above. We have also noted a corollary that gives us some information about the extremality of the inequality.

Let $\mathbb{H}$ denote the division ring of Hamilton's quaternions. Let $\mathbf{H}^n_{\mathbb{H}}$ denote the n-dimensional quaternionic hyperbolic space. Let $\mathrm{Sp}(n,1)$ be the linear group that acts on $\mathbf{H}^n_{\mathbb{H}}$ by isometries. Up to conjugacy, we assume that (see following section for the details) an Heisenberg translation fixes the boundary point 0, i.e. it is of the form

$$T_{s,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix}, \tag{1.2}$$

where ζ is column vector known as translation length of $T_{s,\zeta}$, s is a scalar with $Re(s) = \frac{1}{2}|\zeta|^2$ and I is $(n-1)\times(n-1)$ identity matrix. ζ^* denotes the conjugate transpose of ζ. Now we state a version of Shimizu type lemma:

Theorem 1.6. *Suppose $T_{s,\zeta}$ is a Heisenberg translation in $\mathrm{Sp}(n,1)$ and $A \in \mathrm{Sp}(n,1)$ is given by*

$$A = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & U \end{pmatrix}$$

Suppose A does not fix 0. Set

$$t = Sup\{|b|, |\beta|, |\gamma|, |U - I|\},\ M = |s| + 2|\zeta|.$$

If

$$Mt + 2|\zeta| < 1,$$

then the two generator group $\langle A, T_{s,\zeta}\rangle$ is either non-discrete or fixes the point 0.

After some preliminaries in section 2, we will prove the above theorem in section 3 and in section 4 we will note some applications of this theorem.

2. Preliminaries

We begin with some background material on quaternionic hyperbolic geometry. Much of this can be found in [CG74, KP03].

Let $\mathbb{H}^{n,1}$ be the right vector space over $\mathbb{H}$ of quaternionic dimension $(n+1)$ (so real dimension $4n+4$) equipped with the quaternionic Hermitian form for $z = (z_0, ..., z_n),\ w = (w_0, ..., w_n)$,

$$\langle z, w\rangle = -(\bar{z}_0 w_1 + \bar{z}_1 w_0) + \Sigma_{i=2}^{n} \bar{z}_i w_i$$

Thus the Hermitian form is defind by the matrix

$$J = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}$$

Following Section 2 of [CG74], let

$$V_0 = \Big\{\mathbf{z} \in \mathbb{H}^{n,1} - \{0\} : \langle \mathbf{z}, \mathbf{z}\rangle = 0\Big\},\ V_- = \Big\{\mathbf{z} \in \mathbb{H}^{n,1} : \langle \mathbf{z}, \mathbf{z}\rangle < 0\Big\}.$$

It is obvious that V_0 and V_- are invariant under $\mathrm{Sp}(n,1)$. We define an equivalence relation $\sim$ on $\mathbb{H}^{n,1}$ by $\mathbf{z} \sim \mathbf{w}$ if and only if there exists a non-zero quaternion λ so that $\mathbf{w} = \mathbf{z}\lambda$. Let $[\mathbf{z}]$ denote the equivalence class of $\mathbf{z}$. Let $\mathbb{P} : \mathbb{H}^{n,1} - \{0\} \longrightarrow \mathbb{HP}^n$ be the *right projection* map given by $\mathbb{P} : \mathbf{z} \longmapsto z$, where $z = [\mathbf{z}]$. The n dimensional quaternionic hyperbolic space is defined to be $\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_-)$ with boundary $\partial\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_0)$.

In our chosen model there are two distinct points 0 and ∞ on $\partial\mathbf{H}^n_{\mathbb{H}}$. For $z_1 \neq 0$, the projection map $\mathbb{P}$ is given by

$$\mathbb{P}(z_1, z_2, \ldots, z_{n+1}) = (z_2 z_1^{-1}, \ldots, z_{n+1} z_1^{-1}),$$

and accordingly we choose boundary points

$$\mathbb{P}(0, 1, \ldots, 0, 0)^t = 0. \tag{2.1}$$

$$\mathbb{P}(1, 0, \ldots, 0, 0)^t = \infty. \tag{2.2}$$

2.1. **Bergmann and Cygan Metric.** The Bergmann metric on $\mathbf{H}^n_{\mathbb{H}}$ is given by the distance formula

$$\cosh^2\frac{\rho(z,w)}{2}=\frac{\langle \mathbf{z},\,\mathbf{w}\rangle\langle \mathbf{w},\,\mathbf{z}\rangle}{\langle \mathbf{z},\,\mathbf{z}\rangle\langle \mathbf{w},\,\mathbf{w}\rangle},\ \text{ where } z,w\in\mathbf{H}^n_{\mathbb{H}},\ \mathbf{z}\in\mathbb{P}^{-1}(z),\mathbf{w}\in\mathbb{P}^{-1}(w).$$

The above forumula is independent of the choice of $\mathbf{z}$ and $\mathbf{w}$.

The finite points in the boundary of $\mathbf{H}^n_{\mathbb{H}}$ naturally carry the structure of the generalised Heisenberg group $\mathfrak{N}_n$, which is defined to be $\mathfrak{N}_n=\mathbf{H}^{n-1}_{\mathbb{H}}\times\Im(\mathbb{H})$, with the group law

$$(\zeta_1,v_1)(\zeta_2,v_2)=(\zeta_1+\zeta_2,v_1+v_2+2\Im(\zeta_2^*\zeta_1)). \tag{2.3}$$

We define *horospherical coordinates* on quaternionic hyperbolic space as follows.

$$\psi:\mathfrak{N}_n\times\mathbb{R}_+\to V_0\cup V_-$$

$$\psi(\zeta,v,u)=\begin{pmatrix}(-|\zeta|^2-u+v)/2\\ \zeta\\ 1\end{pmatrix},\quad \psi(\infty)=\begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix},\quad \psi(o)=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} \tag{2.4}$$

The *Cygan metric* on Heisenberg group corresponding to the norm

$$|(\zeta,v)|_H=||\zeta|^2+v|^{\frac{1}{2}}=(|\zeta|^4+|v|^2)^{\frac{1}{4}}$$

is given by

$$\begin{aligned}d_H((\zeta_1,v_1),(\zeta_2,v_2)) &= |(\zeta_1,v_1)^{-1}(\zeta_2,v_2)|_H=|2\langle\psi(\zeta_1,v_1,0),\psi(\zeta_2,v_2,0)\rangle|^{\frac{1}{2}}\\ &= |(\zeta_2-\zeta_1,v_2-v_1-2\Im(\zeta_2^*\zeta_1)|_H\\ &= ||\zeta_2|^2+|\zeta_1|^2-2\zeta_2^*\zeta_1+v_2-v_1|^{\frac{1}{2}}.\end{aligned}$$

Now consider the non-compact linear Lie group

$$\mathrm{Sp}(n,1)=\{A\in\mathrm{GL}(n+1,\mathbb{H}):A^*JA=J\}.$$

An element $g\in\mathrm{Sp}(n,1)$ acts on $\overline{\mathbf{H}_{\mathbb{H}}}^n=\mathbf{H}^n_{\mathbb{H}}\cup\partial\mathbf{H}^n_{\mathbb{H}}$ as $g(z)=\mathbb{P}g\mathbb{P}^{-1}(z)$. Thus the isometry group of $\mathbf{H}^n_{\mathbb{H}}$ is given by $\mathrm{PSp}(n,1)=\mathrm{Sp}(n,1)/\{I,-I\}$.

2.2. **Classification of Isometries.** As with real and complex hyperbolic isometries, non-trivial element g of $\mathrm{Sp}(n,1)$ are classified as:

(i) *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{H}}$.
(ii) *parabolic* if it has exactly one fixed point which lies in $\partial\mathbf{H}^n_{\mathbb{H}}$.
(iii) *loxodromic* if it has exactly two fixed points which lie in $\partial\mathbf{H}^n_{\mathbb{H}}$.

Definition 2.1. Let H_s be a parabolic element of $\mathrm{Sp}(n,1)$. Up to conjugacy, we may assume it to fix a point 0 on the boundary. Using Jordan decomposition, we can see it to be of the form, up to conjugacy,

$$H_s = R_{(\mu,U)}T_{(\zeta,s)}$$

The element with the form $T_{(\zeta,s)}$ is called a Heisenberg translation and it fixes the boundary point 0. Whereas $R_{(\mu,U)}$ is an elliptic element fixing 0.

Up to conjugacy, Heisenberg translation $T_{\zeta,s} \in \mathrm{Sp}(n,1)$ is given by (see [CG74, p. 70])

$$T_{s,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix}. \tag{2.5}$$

here ζ is column vector known as translation length of $T_{s,\zeta}$, s is a scalar with $Re(s) = \frac{1}{2}|\zeta|^2$ and I is $(n-1)\times(n-1)$ identity matrix. ζ^* denotes the conjugate transpose of ζ. If $\zeta = 0$, it is a *vertical* Heisenberg translation, otherwise it is a *non-vertical* Heisenberg translation.

Identifying the boundary $\partial\mathbf{H}^n_{\mathbb{H}}$ with $\mathfrak{N}_{4n-1}$, the $(4n-1)$-dimensional generalized Heisenberg group with 3-dimensional center. There is a natural metric, the Cygan metric, on $\mathfrak{N}_{4n-1}$. Let T be a parabolic element fixing ∞. It is a Cygan isometry on $\mathfrak{N}_{4n-1}$. The natural projection from $\mathfrak{N}_{4n-1}$ to $\mathbb{H}^{n-1}$ defines the vertical projection of T, which is a Euclidean isometry of $\mathbb{H}^{n-1}$.

Definition 2.2. Let $g \in \mathrm{Sp}(n,1)$ not fixing ∞, then, there exists a Cygan sphere, called the *isometric sphere*, in $\mathfrak{N}_{4n-1}$ centered at $g^{-1}(\infty)$,

$$I_g = \{z \in \mathbf{H}_{\mathbb{H}} : |\langle\psi(z), \psi(\infty)\rangle| = \langle\psi(z), g^{-1}(\psi(\infty))\rangle|\}$$

which is sent by g to the Cygan sphere centered at $g(\infty)$ having the same radius.

2.3. **Some Computation.** Now we shall use the Hermitian form J to note down a few equations to be used later. Letting $\mathbf{z}$ and $\mathbf{w}$ vary over a basis for $\mathbb{H}^{n,1}$, we see that $J = A^*JA$. From this we find that $A^{-1} = J^{-1}A^*J$ for $A \in \mathrm{Sp}(n,1)$ given by

$$A = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & U \end{pmatrix} \tag{2.6}$$

$$A^{-1} = \begin{pmatrix} \bar{d} & \bar{b} & -\beta^* \\ \bar{c} & \bar{a} & -\alpha^* \\ -\delta & -\gamma & U^* \end{pmatrix}$$

where a, b, c, d are scalars, $\gamma, \delta, \alpha, \beta$ are column matrices and U is an element in $U(n-1, \mathbb{H})$.

Since $AA^{-1} = I$, equating both sides we have the following relations:

$$\begin{aligned}
a\bar{d} + b\bar{c} - \gamma^*\delta &= 1 \\
a\bar{b} + b\bar{a} - \gamma^*\gamma &= 0 \\
-a\beta^* - b\alpha^* + \gamma^*U^* &= 0 \\
c\bar{d} + d\bar{c} - \delta^*\delta &= 0 \\
c\bar{b} + d\bar{a} - \delta^*\gamma &= 1 \\
-c\beta^* - d\alpha^* + \delta^*U^* &= 0 \\
\alpha\bar{d} + \beta\bar{c} - U\delta &= 0 \\
\alpha\bar{b} + \beta\bar{a} - U\gamma &= 0 \\
-\alpha\beta^* - \beta\alpha^* + UU^* &= I \\
\bar{d}a + \bar{b}c - \beta^*\alpha &= 1 \\
\bar{d}b + \bar{b}d - \beta^*\beta &= 0 \\
\bar{d}\gamma^* + \bar{b}\delta^* - \beta^*U &= 0 \\
\bar{c}a + \bar{a}c - \alpha^*\alpha &= 0 \\
\bar{c}b + \bar{a}d - \alpha^*\beta &= 1 \\
\bar{c}\gamma^* + \bar{a}\delta^* - \alpha^*U &= 0 \\
-\delta a - \gamma c + U^*\alpha &= 0 \\
-\delta b - \gamma d + U^*\beta &= 0 \\
-\delta\gamma^* - \gamma\delta^* + U^*U &= I
\end{aligned}$$

2.4. **Some Shimizu type Theorems.** Before proving our main result, we will note here some other Shimizu type theorems.

Theorem 2.3. [KP08] *Let A be a positively oriented screw parabolic element of $PU(2,1)$ fixing ∞. Let $e^{i\theta} \in U(1)$ denote the rotational part of A and suppose that $|e^{i\theta} - 1| < 1/4$. Let B be any element of $PU(2,1)$ not protectively fixing ∞ and let r_B denote the radius of the isometric sphere of B. If*

$$\frac{\rho_0(B(\infty), AB(\infty))\rho_0(B^{-1}(\infty), AB^{-1}(\infty))}{r_B^2} < \left(\frac{1+\sqrt{1-4|e^{i\theta}-1|}}{2}\right)^2$$

then $\langle A, B\rangle$ is not discrete.

Theorem 2.4. [KP03] *Let G be a discrete subgroup of* $\mathrm{PSp}(n,1)$ *that contains a Heisenberg translation g by $(\tau, t) \in \mathfrak{N}_n$. Let h be any element of G not fixing ∞ with isometric sphere of radius r_h. Then*

$$r_h^2 \leq d_H(h^{-1}(\infty), gh^{-1}(\infty))d_H(h(\infty), gh(\infty)) + 4|\tau|^2$$

3. A Shimizu-Type Theorem for Subgroups in $\mathrm{Sp}(n,1)$

The following theorem follows essentially mimicking the argument of Hersonsky and Paulin in Appendix of [HP96].

Theorem 3.1. *Suppose $T_{s,\zeta}$ be a Heisenberg translation in* $\mathrm{Sp}(n,1)$ *and A be an element in* $\mathrm{Sp}(n,1)$ *of the form (2.6). Suppose A does not fix* 0. *Set*

$$t = Sup\{|b|, |\beta|, |\gamma|, |U-I|\},\ M = |s| + 2|\zeta|. \tag{3.1}$$

If

$$Mt + 2|\zeta| < 1, \tag{3.2}$$

then the group generated by A and $T_{s,\zeta}$ is either non-discrete or fixes 0.

When $\zeta = 0$, as a corollary to the above theorem we obtain theorem (3.2)[Kam83].

Proof. Consider the sequence $A_0 = A$, $A_{k+1} = A_k T_{s,\zeta} A_k^{-1}$. Thus

$$A_{k+1} = \begin{pmatrix} a_{k+1} & b_{k+1} & {\gamma_{k+1}}^* \\ c_{k+1} & d_{k+1} & {\delta_{k+1}}^* \\ \alpha_{k+1} & \beta_{k+1} & U_{k+1} \end{pmatrix} \tag{3.3}$$

$$= \begin{pmatrix} a_k & b_k & {\gamma_k}^* \\ c_k & d_k & {\delta_k}^* \\ \alpha_k & \beta_k & U_k \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix} \begin{pmatrix} \bar{d}_k & \bar{b}_k & -\beta_k^* \\ \bar{c}_k & \bar{a}_k & -\alpha_k^* \\ -\delta_k & -\gamma_k & U_k^* \end{pmatrix} \tag{3.4}$$

$$\tag{3.5}$$

Using the above set of relations, equating both sides we have:

$$a_{k+1} = 1 + b_k s\bar{d}_k - b_k\zeta^*\delta_k + \gamma_k^*\zeta\bar{d}_k \tag{3.6}$$
$$b_{k+1} = b_k s\bar{b}_k + 2Im(\gamma_k^*\zeta\bar{b}_k) \tag{3.7}$$
$$c_{k+1} = d_k s\bar{d}_k + 2Im(\delta_k^*\zeta\bar{d}_k) \tag{3.8}$$
$$d_{k+1} = 1 + d_k s\bar{b}_k + \delta_k^*\zeta\bar{b}_k - d_k\zeta^*\gamma_k \tag{3.9}$$
$$\gamma_{k+1}^* = -b_k s\beta_k^* - \gamma_k^*\zeta\beta_k^* + b_k\zeta^* U_k^* \tag{3.10}$$
$$\delta_{k+1}^* = -d_k s\beta_k^* - \delta_k^*\zeta\beta_k^* + d_k\zeta^* U_k^* \tag{3.11}$$
$$\alpha_{k+1} = \beta_k s\bar{d}_k - \beta_k\zeta^*\delta_k + U_k\zeta\bar{d}_k \tag{3.12}$$
$$\beta_{k+1} = \beta_k s\bar{b}_k - \beta_k\zeta^*\gamma_k + U_k\zeta\bar{b}_k \tag{3.13}$$
$$U_{k+1} = I - \beta_k s\beta_k^* - U_k\zeta\beta_k^* + \beta_k\zeta^* U_k^*. \tag{3.14}$$

Set

$$t_k = Sup\{|b_k|, |\beta_k|, |\gamma_k|, |U_k - I|\},\ M = |s| + 2|\zeta|.$$

Now it follows from the expressions above that

$$|b_{k+1}| \leq Mt_k^2 \tag{3.15}$$
$$|\gamma_{k+1}| \leq Mt_k^2 + |b_k||\zeta| \tag{3.16}$$
$$|\beta_{k+1}| \leq Mt_k^2 + |b_k||\zeta| \tag{3.17}$$
$$|U_{k+1} - I| \leq Mt_k^2 + 2|\beta_k||\zeta|. \tag{3.18}$$

These relations imply that

$$t_{k+1} \leq Mt_k^2 + 2|\zeta|t_k, . \tag{3.19}$$

Choose $\epsilon > 0$ such that

$$Mt_0 + 2|\zeta| \leq \epsilon < 1.$$

From (3.19) it follows that $t_1 \leq \epsilon t_0$, $t_2 \leq \epsilon t_0(Mt_1 + 2|\zeta|)$. Now note that $Mt_1 \leq M\epsilon t_0 < Mt_0$, hence $Mt_1 + 2|\zeta| \leq Mt_0 + 2|\zeta| < \epsilon$, hence $t_2 \leq \epsilon^2 t_0$. Repeating this proces we have

$$t_k \leq \epsilon^k t_0.$$

Since $\epsilon < 1$, hence $t_k \to 0$. In particular, $U_k \to I$, $\beta_k \to 0$, $\gamma_k \to 0$, $b_k \to 0$.

Now note that for $k \geq 3$

$$|d_{k+1} - 1| < kt_k(|s| + |\zeta| + |\zeta|^2)C_k,$$

$$|\delta_{k+1} - \zeta| \leq kt_k(|s| + |\zeta| + |\zeta|^2)C_k + |\zeta||d_k - 1|,$$

where $C_k = Sup\{|d_k - 1|, |\delta_k - \zeta|, 1\}$. If required, after passing to a subsequence, we have

$$d_k \to 1, \ \delta_k \to \zeta.$$

Similarly,

$$a_k \to 1, \ c_k \to s, \ \alpha_k \to \zeta.$$

If $t_0 \neq 0$, this implies that $A_k \to A_*$, where $A_* \in \mathrm{Sp}(n,1)$. Thus the group $\langle A, T_{s,\zeta}\rangle$ can not be discrete.

Suppose $t_0 = 0$. Then $b = 0$, $\gamma = 0$, $\beta = 0$. This implies that A fixes the fixed point $f_1 = (1, 0, ..., 0)^t$ of $T_{s,\zeta}$.

This proves the theorem. □

4. Application of the main result

Now we will discuss some applications and comparison of the main result we just proved.

Corollary 4.1. *Let $T_{s,\zeta}$ be a Heisenberg translation in $\mathrm{Sp}(n,1)$ and $A \in \mathrm{Sp}(n,1)$. Suppose $A(\infty) \neq \infty$ set M, t, as in the previous theorem. Suppose $\langle A, T_{s,\zeta}\rangle$ is non-elementary and discrete and*

$$Mt + 2|\zeta| = 1$$

Consider Shimizu-Leutbecher sequence

$$A_0 = A, A_{k+1} = A_k T_{s,\zeta} A_k^{-1}$$

then for each k, $\langle A_k, T_{s,\zeta}\rangle$ is non-elementary discrete subgroup of $\mathrm{Sp}(n,1)$ and

$$Mt_k + 2|\zeta| = 1 \ \forall \ k.$$

In particular, $t_k = t$ for all k.

Proof. Note that

$$A_k T_{s,\zeta} A_k^{-1}(A_k(\infty)) = A_k(\infty)$$

So, the limit set of $\langle A_k, T_{s,\zeta}\rangle$ can form $\{A_k(\infty), \infty\}$. Now, if $\langle A_1, T_{s,\zeta}\rangle$ is elementary, it must preserve $\{A(\infty), \infty\}$. But $T_{s,\zeta}$ can not fix $A(\infty)$. This implies that $\langle A_1, T_{s,\zeta}\rangle$ is non-elementary. By induction, $\langle A_k, T_{s,\zeta}\rangle$ is non-elementary. They are also discrete. So,

$$Mt_k + 2|\zeta| \geq 1.$$

Note that if $Mt_0 + 2|\zeta| = 1$, then

$$Mt_k + 2|\zeta| \leq Mt_0 + 2|\zeta| = 1.$$

$$\Rightarrow Mt_k + 2|\zeta| = 1 = Mt + 2|\zeta| \;\forall\; k.$$

In particular, $t_k = t$. □

Remark 4.2. Observe that theorem (4.8) of [KP03] says that if,

$$1 > \frac{d_H(A^{-1}(\infty), T_{s,\zeta}A^{-1}(\infty))d_H(A(\infty), T_{s,\zeta}A(\infty)) + 4|\zeta|^2}{r_A^2}$$

then $\langle T_{s,\zeta}, A\rangle$ is not discrete. Here d_H is a Cygan metric and r_A is the radius of the isometric sphere of A. If A is of the form (2.6), then $r_A = \sqrt{\frac{2}{|b|}}$ and the above inequality becomes

$$1 > |b||s + \zeta^*\beta b^{-1} - \bar{b}^{-1}\beta^*\zeta|^{\frac{1}{2}}|s - \zeta^*\gamma\bar{b}^{-1} - b^{-1}\gamma^*\zeta|^{\frac{1}{2}} + 2|\zeta|^2|b| \qquad (4.1)$$

Suppose now that

$$Mt + 2|\zeta| < 1$$

then

$$t < \frac{1 - 2|\zeta|}{|s| + 2|\zeta|}$$

now right hand side of the (4.1) can be seen (thorough computations) to be less than 1. Hence whenever our theorem holds, Kim and Parker's theorem also holds, accordingly the theorem of Cao and Parker also holds. Therefore Theorem (3.1) is weaker than the theorem of Kim and Parker [KP03], or Cao and Parker [CP11], though it is simpler.

Remark 4.3. Similarly one can show that our theorem implies Cao and Parker's Shimizu's lemma for Heisenberg translation.

We will conclude the paper by stating the main application of the result (3.1). A subgroup G of $\mathrm{Sp}(n,1)$ is called *Zariski dense* if it does not fix a point on $\mathbf{H}^n_{\mathbb{H}} \cup \partial \mathbf{H}^n_{\mathbb{H}}$, and neither it preserves a totally geodesic subspace of $\mathbf{H}^n_{\mathbb{H}}$. With the above notations, we apply our main result to prove the following part (3) of the following theorem in [GMT19]:

Theorem 4.4. [GMT19] *Let G be a Zariski dense subgroup of* $\mathrm{Sp}(n,1)$.

(1) Let $g \in \mathrm{Sp}(n,1)$ be a regular elliptic element such that $\delta(g) < 1$. If $\langle g, hgh^{-1} \rangle$ is discrete for every loxodromic element $h \in G$, then G is discrete.

(2) Let $g \in \mathrm{Sp}(n,1)$ be loxodromic element such that $M_g < 1$. If $\langle g, hgh^{-1} \rangle$ is discrete for every loxodromic element $h \in G$, then G is discrete.

(3) Let $g \in \mathrm{Sp}(n,1)$ be a Heisenberg translation such that $|\zeta| < \frac{1}{2}$. If $\langle g, hgh^{-1} \rangle$ is discrete for every loxodromic h in G, then G is discrete.

Acknowledgement: The author thanks the anonymous referee for many useful comments and suggestions, which helped improving the manuscript. During the work the author was supported by NBHM postdoctoral fellowship.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 116–131

CHARACTERIZATION OF ROUGH 3-PRIME IDEALS AND 3-PRIME FUZZY IDEALS IN NEAR RINGS

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(Received : 01 - 06 - 2023 ; Revised : 06 - 03 - 2024)

Abstract. Our aim in this paper is to introduce and study rough prime ideals and semiprime ideals of a near ring $\mathcal{N}$ as a generalization of rough prime and semiprime ideals in commutative rings. We also define rough fuzzy ideal of $\mathcal{N}$ and carry out properties of rough 3-prime fuzzy ideals. In addition, we examine the conditions under which the upper and lower rough 3-prime ideals and upper and lower approximations of their homomorphic images are related.

1. Introduction

The idea of fuzzy sets offers a useful mechanism for explaining the behaviour of systems that are either too complicated or too poorly specified to allow for precise mathematical study using conventional tools and techniques. It has demonstrated great potential for managing uncertainties to a manageable degree, particularly in decision-making models under various types of risks, subjective assessment, ambiguity, and vagueness. Expert systems, control systems, pattern recognition, and image processing are just a few of the many disciplines in which this idea has already been widely applied. Zadeh [8] was the first to present the theory of fuzzy sets. After that Rosenfeld [1] and Zaid [17] introduced and studied fuzzy group and fuzzy ring respectively. In [5, 7, 10, 16, 19, 20, 21, 22], researchers introduced and studied fuzzy prime, semiprime, maximal and radical of a fuzzy ideal of a ring $\mathscr{R}$.

Pawlak [27] was the first to present the theory of rough sets. Pawlak introduced rough set theory as a potent mathematical tool for handling data

2010 Mathematics Subject Classification: 03E72, 16N60, 16W25, 16Y30.
Key words and phrases: 3-prime ideals, 3-semiprime ideals, rough 3-prime ideals, fuzzy ideals, rough 3-prime fuzzy ideal

uncertainty. Rough set theory begins with the premise that components of a universe with the same characterization are indistinguishable in light of the information at hand. An extension of set theory known as rough set theory describes a subset of the universe as being defined by two classical sets known as the lower and upper approximations. The foundation for creating the lower and the upper approximations are the equivalence classes [23, 26]. For basic ideas without clear boundaries, such as those that are hazy and ambiguous, rough sets offer an appropriate mathematical representation. Rough set theory is becoming into a potent tool for working with unreliable data. Recently, it has drawn a lot of attention from the research community in both the theory and practical applications. Rough subgroups and rough ideals in semigroup discussed by Biswas et al. [15] and Kuroki [11] respectively. Some characteristics of the lower and upper approximations with regard to the normal subgroups were represented by Kuroki and Wang[12]. Kazanci and Davvaz [13] brought up the ideas of rough prime and rough fuzzy prime ideals of commutative rings in 2008.
In this paper we define rough fuzzy 3-prime and 3-semiprime ideals in a near ring $\mathcal{N}$. We also discuss some of their features. This in-depth investigation could perhaps offer a useful tool for approximative reasoning, in our opinion. When it comes to the theory as well as applications of fuzzy sets and rough sets, we think the rough near rings presented here will be more beneficial. As a generalisation of rings, Pilz[6] proposed the idea of near rings in 1983. In which, only one distributive law is required, hence the addition operation is not required to be commutative. It would be unfair to characterise near ring theory as merely a collection of unimportant findings pertaining to particular pathological systems with little relevance to other areas of mathematics. In contrast to applications in axioms and geometry, novel and extremely effective classes of balanced incomplete block designs are provided by particular classes of finite near rings (finite planar near ring).

2. Preliminaries

Definition 2.1. A full congruence relation (F.C.R.) $\mathscr{R}$ on a near ring $\mathcal{N}$ is an equivalence relation that preserves the algebraic operations on that set i.e., for $p, q \in \mathcal{N}$, $(p, q) \in \mathscr{R}$ implies that $(p+r, q+r), (r+p, r+q), (pr, qr)$ and $(rp, rq) \in \mathscr{R}$, for all $r \in \mathcal{N}$.

Lemma 2.2. *If $\mathscr{R}$ is F.C.R. on a near ring $\mathcal{N}$, then $(p,q),(r,s) \in \mathscr{R}$ imply $(p+r,q+s),(pr,qs)$ and $(-p,-q) \in \mathscr{R}$.*

Proof. Consider $\mathscr{R}$ is a F.C.R. on a near ring $\mathcal{N}$ and $(p,q),(r,s) \in \mathscr{R}$. Then $(p+r,q+r),(q+r,q+s) \in \mathscr{R}$. By definition of F.C.R. $(p+r,q+s) \in \mathscr{R}$. In similar manner, we can see that $(pr,qs),(-p,-q) \in \mathscr{R}$. □

Lemma 2.3. *Let $\mathscr{R}$ be F.C.R. on a near ring $\mathcal{N}$. If $\mathfrak{m},\mathfrak{n} \in \mathcal{N}$, then*

(*i*) $[\mathfrak{m}]_{\mathscr{R}} + [\mathfrak{n}]_{\mathscr{R}} = [\mathfrak{m}+\mathfrak{n}]_{\mathscr{R}}$
(*ii*) $[-\mathfrak{m}]_{\mathscr{R}} = -[\mathfrak{m}]_{\mathscr{R}}$
(*iii*) $\{ab \mid a \in [\mathfrak{m}]_{\mathscr{R}}, b \in [\mathfrak{n}]_{\mathscr{R}}\} \subseteq [\mathfrak{m}\mathfrak{n}]_{\mathscr{R}}$.

Proof. (*i*) Let $x = \mathfrak{p}+\mathfrak{q} \in [\mathfrak{m}]_{\mathscr{R}}+[\mathfrak{n}]_{\mathscr{R}}$, $\mathfrak{p} \in [\mathfrak{m}]_{\mathscr{R}}$, $\mathfrak{q} \in [\mathfrak{n}]_{\mathscr{R}}$. Then $(\mathfrak{p},\mathfrak{m}),(\mathfrak{q},\mathfrak{n}) \in \mathscr{R}$. By lemma 2.2, $(\mathfrak{p}+\mathfrak{q},\mathfrak{m}+\mathfrak{n}) \in \mathscr{R}$ i.e., $\mathfrak{p}+\mathfrak{q} = x \in [\mathfrak{m}+\mathfrak{n}]_{\mathscr{R}}$. Now, take $z \in [\mathfrak{m}+\mathfrak{n}]_{\mathscr{R}}$ i.e., $(z,\mathfrak{m}+\mathfrak{n}) \in \mathscr{R}$ or $(-\mathfrak{m}+z,\mathfrak{n}) \in \mathscr{R}$ i.e., $(-\mathfrak{m}+z) \in [\mathfrak{n}]_{\mathscr{R}}$ entails that $z \in [\mathfrak{m}]_{\mathscr{R}} + [\mathfrak{n}]_{\mathscr{R}}$. Thus $[\mathfrak{m}]_{\mathscr{R}} + [\mathfrak{n}]_{\mathscr{R}} = [\mathfrak{m}+\mathfrak{n}]_{\mathscr{R}}$.
(*ii*) For any $a \in [-\mathfrak{m}]_{\mathscr{R}}$ or equivalently

$$\begin{aligned}(a,-\mathfrak{m}) \in \mathscr{R} &\iff (0,-\mathfrak{m}-a) \in \mathscr{R}\\ &\iff (\mathfrak{m},-a) \in \mathscr{R}\\ &\iff -a \in [\mathfrak{m}]_{\mathscr{R}}\\ &\iff a \in -[\mathfrak{m}]_{\mathscr{R}}.\end{aligned}$$

This show that $[-\mathfrak{m}]_{\mathscr{R}} = -[\mathfrak{m}]_{\mathscr{R}}$.
(*iii*) Let $c = ab$ such that $a \in [\mathfrak{m}]_{\mathscr{R}}$, $b \in [\mathfrak{n}]_{\mathscr{R}}$. Then $(a,\mathfrak{m}) \in \mathscr{R}$ and $(b,\mathfrak{n}) \in \mathscr{R}$. Since $\mathscr{R}$ is F.C.R., hence $(ab,\mathfrak{m}\mathfrak{n}) \in \mathscr{R}$ i.e., $ab = c \in [\mathfrak{m}\mathfrak{n}]_{\mathscr{R}}$. This implies that $[\mathfrak{m}]_{\mathscr{R}}[\mathfrak{n}]_{\mathscr{R}} \subseteq [\mathfrak{m}\mathfrak{n}]_{\mathscr{R}}$. □

Definition 2.4. A F.C.R. $\mathscr{R}$ on $\mathcal{N}$ is said to be complete (C.C.R.) if $[\mathfrak{m}\mathfrak{n}]_{\mathscr{R}} = \{\mathfrak{p}\mathfrak{q} \mid \mathfrak{p} \in [\mathfrak{m}]_{\mathscr{R}}, \mathfrak{q} \in [\mathfrak{n}]_{\mathscr{R}}\}$, for all $\mathfrak{p},\mathfrak{q} \in \mathcal{N}$.

Definition 2.5. Let $\mathscr{R}$ be a F.C.R. on a near ring $\mathcal{N}$ and $\mathcal{S} \subseteq \mathcal{N}$. Then the sets $\mathscr{R}_{-}(\mathcal{S}) = \{n \in \mathcal{N} \mid [n]_{\mathscr{R}} \subseteq \mathcal{S}\}$ and $\mathscr{R}^{-}(\mathcal{S}) = \{n \in \mathcal{N} \mid [n]_{\mathscr{R}} \cap \mathcal{S} \neq \phi\}$ are referred to $\mathscr{R}$-lower and $\mathscr{R}$-upper approximation of the set $\mathcal{S}$.

$\mathscr{R}(\mathcal{S}) = (\mathscr{R}_{-}(\mathcal{S}),\mathscr{R}^{-}(\mathcal{S}))$ is called a rough set with respect to $\mathscr{R}$ if $\mathscr{R}_{-}(\mathcal{S}) \neq \mathscr{R}^{-}(\mathcal{S})$. $\mathcal{S}$ is said to be upper rough ideal if $\mathscr{R}^{-}(\mathcal{S})$ is an ideal of $\mathcal{N}$.

Lemma 2.6. *Let $\mathscr{R}$ be a F.C.R. on $\mathcal{N}$. Then $\mathscr{R}^-(\mathcal{J})$ is an ideal of $\mathcal{N}$, if $\mathcal{J}$ is an ideal of $\mathcal{N}$.*

Proof. Let $x, y \in \mathscr{R}^-(\mathcal{J})$ and $n \in \mathcal{N}$. Then $[x]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$ and $[y]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$. So, there exist $u \in [x]_{\mathscr{R}} \cap \mathcal{J}$ and $v \in [y]_{\mathscr{R}} \cap \mathcal{J}$. This implies that $u - v \in \mathcal{J}$ and $nu \in \mathcal{J}$. By lemma 2.3, $u - v \in [x]_{\mathscr{R}} - [y]_{\mathscr{R}} = [x-y]_{\mathscr{R}}$. Thus $u - v \in [x-y]_{\mathscr{R}}$ i.e., $[x-y]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$. Which implies that

$$x - y \in \mathscr{R}^-(\mathcal{J}). \tag{2.1}$$

Since $(u, x) \in \mathscr{R}$, hence $(nu, nx) \in \mathscr{R}$. Thus $nu \in [nx]_{\mathscr{R}}$ also $nu \in \mathcal{J}$. So, $[nx]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$. Which implies that

$$nx \in \mathscr{R}^-(\mathcal{J}). \tag{2.2}$$

Moreover, $n + x - n \in \mathscr{R}^-(\mathcal{J})$. Since $x \in \mathscr{R}^-(\mathcal{J})$, then $[x]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$. So, there exist $w \in [x]_{\mathscr{R}}$ and $w \in \mathcal{J}$. Since $\mathcal{J}$ is an ideal and $\mathscr{R}$ is F.C.R., hence $n - w + n \in [n - x + n]_{\mathscr{R}}$ and $n - w + n \in \mathcal{J}$. Thus $[n - x + n]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$ or

$$n - x + n \in \mathscr{R}^-(\mathcal{J}). \tag{2.3}$$

Consider $n_1, n_2 \in \mathcal{N}$ and $i \in \mathscr{R}^-(\mathcal{J})$. Then $[i]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$. So, there exist $k \in [i]_{\mathscr{R}} \cap \mathcal{J}$ i.e., $(i, k) \in \mathscr{R}$ and $k \in \mathcal{J}$. Since $\mathcal{J}$ is an ideal of $\mathcal{N}$ and $\mathscr{R}$ is F.C.R., hence $((n_1+i)n_2 - n_1n_2, (n_1+k)n_2 - n_1n_2) \in \mathscr{R}$ and $(n_1+k)n_2 - n_1n_2 \in \mathcal{N}$. Thus $(n_1 + k)n_2 - n_1n_2 \in [(n_1 + i)n_2 - n_1n_2]_{\mathscr{R}} \cap \mathcal{J}$. Which implies that $[(n_1 + i)n_2 - n_1n_2]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$. Therefore

$$(n_1 + i)n_2 - n_1n_2 \in \mathscr{R}^-(\mathcal{J}). \tag{2.4}$$

Equations (2.1)-(2.4) show that $\mathscr{R}^-(\mathcal{J})$ is an ideal of $\mathcal{N}$. □

Remark 2.7. The converse statement of the previous lemma is not true.

Example 2.8. Consider $(\mathcal{N}, +, \cdot)$ be a near ring under following operations

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

and

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	0
3	0	3	2	3.

Let $\mathscr{R}$ be a F.C.R. on $\mathcal{N}$ with equivalence classes $[0]_{\mathscr{R}} = \{0, 2\}$ and $[1]_{\mathscr{R}} = \{1, 3\}$. For $\mathcal{J} = \{0, 1, 3\}$, we can see that $\mathcal{J}$ is not an ideal of $\mathcal{N}$. But, $\mathscr{R}^-(\mathcal{J}) = \mathcal{N}$ is an ideal of $\mathcal{N}$.

Lemma 2.9. *Let $\mathscr{R}$ be a F.C.R. on a near ring $\mathcal{N}$ and $\mathcal{J}$ be an ideal of $\mathcal{N}$. If $\mathscr{R}_-(\mathcal{J})$ is a nonempty set, then $\mathscr{R}_-(\mathcal{J}) = \mathcal{J}$.*

Proof. Assume that $\mathscr{R}_-(\mathcal{J})$ is nonempty set. Then there exists $j \in \mathscr{R}_-(\mathcal{J})$. Clearly, we can see that $\mathscr{R}_-(\mathcal{J}) \subseteq \mathcal{J}$. Now, we will only show that $\mathcal{J} \subseteq \mathscr{R}_-(\mathcal{J})$. Suppose that $i, j \in \mathcal{J}$. Then $[0]_{\mathscr{R}} = [j + (-j)]_{\mathscr{R}} = [j]_{\mathscr{R}} + [-j]_{\mathscr{R}} \subseteq \mathcal{J} + \mathcal{J} = \mathcal{J}$ and

$$\begin{aligned} j \in i + [0]_{\mathscr{R}} &\iff j - i \in [0]_{\mathscr{R}} \\ &\iff (j - i, 0) \in \mathscr{R} \\ &\iff (j, i) \in \mathscr{R} \\ j \in [i]_{\mathscr{R}}. \end{aligned}$$

We get $[i]_{\mathscr{R}} \subseteq \mathcal{J}$. This implies that $i \in \mathscr{R}_-(\mathcal{J})$. □

Definition 2.10. Let $\mathcal{X}$ be a subset of a near ring $\mathcal{N}$ and $(\mathscr{R}_-(\mathcal{X}), \mathscr{R}^-(\mathcal{X}))$ a rough set. If $\mathscr{R}_-(\mathcal{X})$ and $\mathscr{R}^-(\mathcal{X})$ are ideals of $\mathcal{N}$, then we call $(\mathscr{R}_-(\mathcal{X}), \mathscr{R}^-(\mathcal{X}))$ a rough ideal.

Corollary 2.11. *If $\mathcal{J}$ is an ideal of $\mathcal{N}$ and $\mathscr{R}_-(\mathcal{J})$ is a nonempty set, then $(\mathscr{R}_-(\mathcal{J}), \mathscr{R}^-(\mathcal{J}))$ is a rough ideal.*

Proof. From lemmas 2.6 and 2.9. □

Corollary 2.12. *If $\mathcal{J}$ and $\mathcal{K}$ are ideals of $\mathcal{N}$ such that $\mathscr{R}_-(\mathcal{J} \cap \mathcal{K}) \neq \phi$, then $(\mathscr{R}_-(\mathcal{J} \cap \mathcal{K}), \mathscr{R}^-(\mathcal{J} \cap \mathcal{K}))$ is a rough ideal of $\mathcal{N}$.*

Proof. From corollary 2.11, $(\mathscr{R}_-(\mathcal{J} \cap \mathcal{K}), \mathscr{R}^-(\mathcal{J} \cap \mathcal{K}))$ is a rough ideal of $\mathcal{N}$. □

Proposition 2.13. *Let $f : \mathcal{N}_1 \to \mathcal{N}_2$ be an epimorphism of a near ring $\mathcal{N}_1$ to a near ring $\mathcal{N}_2$. Then*

- *(i) $\mathscr{R}_1 = \{(n_1, n_2) \in \mathcal{N}_1 \times \mathcal{N}_2 \mid (f(n_1), f(n_2)) \in \mathscr{R}_2\}$ is F.C.R. on $\mathcal{N}_1$, if $\mathscr{R}_2$ is F.C.R.*
- *(ii) If $\mathscr{R}_2$ is complete and f is injective, then $\mathscr{R}_1$ is complete.*
- *(iii) For any subset $\mathcal{X}$ of $\mathcal{N}_1$, $f(\mathscr{R}_1^-(\mathcal{X})) = \mathscr{R}_2^-(f(\mathcal{X}))$.*
- *(iv) $f(\mathscr{R}_{1-}(\mathcal{X})) \subseteq \mathscr{R}_{2-}(f(\mathcal{X}))$. Equality holds if f is injective.*

Proof. (*i*) Let $(n_1, n_2) \in \mathscr{R}_1$ and $n \in \mathcal{N}_1$ and $n \in \mathcal{N}_1$. Then $(f(n_1), f(n_2)) \in \mathscr{R}_2$. Since $\mathscr{R}_2$ is F.C.R., hence $(f(n)+f(n_1), f(n)+f(n_2)), (f(n_1)+f(n), f(n_2)+f(n)), (f(n)f(n_1), f(n)f(n_2))$ $(f(n_1)f(n), f(n_2)f(n)) \in \mathscr{R}_2$. As f is homomorphism, $(f(n + n_1), f(n + n_2)), (f(n_1 + n), f(n_2 + n)), (f(nn_1), f(nn_2)),$

$(f(n_1n), f(n_2n)) \in \mathscr{R}_2$. This implies that $(n+n_1, n+n_2), (n_1+n, n_2+n)$,$(nn_1, nn_2), (n_1n, n_2n) \in \mathscr{R}_1$.

(ii) Let $l \in [mn]_{\mathscr{R}_1}$. Then $(l, mn) \in \mathscr{R}_1$. By (i), $(f(l), f(mn) \in \mathscr{R}_2$. So,

$$\begin{aligned} f(l) \in [f(mn)]_{\mathscr{R}_2} &= [f(m)f(n)]_{\mathscr{R}_2} \\ &= \{f(m^{'})f(n^{'}) \mid f(m^{'}) \in [f(m)]_{\mathscr{R}_2}, f(n^{'}) \in [f(n)]_{\mathscr{R}_2}\}. \end{aligned}$$

Thus there exist $a, b \in \mathcal{N}_1$ such that $f(l) = f(a)f(b) = f(ab)$ and $f(a) \in [f(m)]_{\mathscr{R}_2}, f(b) \in [f(n)]_{\mathscr{R}_2}$. $f(l) = f(a)f(b)$ implies that $l = ab$, since f is injective and $a \in [m]_{\mathscr{R}_1}$, $b \in [n]_{\mathscr{R}_1}$. Hence

$$l \in \{ab \mid a \in [m]_{\mathscr{R}_1}, b \in [n]_{\mathscr{R}_1}\}. \tag{2.5}$$

From equation 2.5 and (iii) part of lemma 2.3, equality holds.

(iii) Let $z \in f(\mathscr{R}_1^-(\mathcal{X}))$. Then one can find $y \in \mathscr{R}_1^-(\mathcal{X}))$ so that $f(y) = z$. Since $[y]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi$, hence there exists $x \in [y]_{\mathscr{R}_1} \cap \mathcal{X}$, which implies that $(x, y) \in \mathscr{R}_1$ and $x \in \mathcal{X}$ or $(f(x), f(y)) \in \mathscr{R}_2$. So, $[f(y)]_{\mathscr{R}_2} \cap f(\mathcal{X}) \neq \phi$ implies that $z = f(y) \in \mathscr{R}_2^-(f(\mathcal{X}))$ i.e.,

$$f(\mathscr{R}_1^-(\mathcal{X})) \subseteq \mathscr{R}_2^-(f(\mathcal{X})) \tag{2.6}$$

Conversely, assume that $a \in \mathscr{R}_2^-(f(\mathcal{X}))$. Then there is $b \in \mathcal{N}_1$ so that $f(b) = a$. So, $[f(b)]_{\mathscr{R}_2} \cap f(\mathcal{X}) \neq \phi$. Therefore, one can find $c \in \mathcal{X}$ in a way that $f(c) \in f(\mathcal{X})$, $f(c) \in [f(b)]_{\mathscr{R}_2}$ and $c \in [b]_{\mathscr{R}_1}$. Thus, $[b]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi$ implies that $b \in \mathscr{R}_1^-(\mathcal{X})$. So, $a = f(b) \in f(\mathscr{R}_1^-(\mathcal{X}))$ i.e.,

$$\mathscr{R}_2^-(f(\mathcal{X})) \subseteq f(\mathscr{R}_1^-(\mathcal{X})). \tag{2.7}$$

From equations (2.6) and (2.7), $f(\mathscr{R}_1^-(\mathcal{X})) = \mathscr{R}_2^-(f(\mathcal{X}))$.

(iv) Suppose that $a \in f(\mathscr{R}_{1-}(\mathcal{X}))$, then there is $b \in \mathscr{R}_{1-}(\mathcal{X})$ in such a manner that $f(b) = a$. So, we have $[b]_{\mathscr{R}_1} \subseteq \mathcal{X}$. Now, we assume that $c \in [a]_{\mathscr{R}_2}$, then there is $d \in \mathcal{N}_1$ so that $f(d) = c$ and $f(d) \in [f(b)]_{\mathscr{R}_2}$. Hence, $d \in [b]_{\mathscr{R}_1} \subseteq \mathcal{X}$ and so $c = f(d) \in f(\mathcal{X})$. Thus $[a]_{\mathscr{R}_2} \subseteq f(\mathcal{X})$ which yields that $a \in \mathscr{R}_{2-}(f(\mathcal{X}))$. This demonstrate that

$$f(\mathscr{R}_{1-}(\mathcal{X})) \subseteq \mathscr{R}_{2-}(f(\mathcal{X})). \tag{2.8}$$

Now assume that f is one to one and $a \in \mathscr{R}_{2-}(f(\mathcal{X}))$, then there exists $b \in \mathcal{N}_1$ such that $f(b) = a$ and $[f(b)]_{\mathscr{R}_2} \subseteq f(\mathcal{X})$. Suppose $c \in [b]_{\mathscr{R}_1}$, then $f(c) \in [f(b)]_{\mathscr{R}_2} \subseteq f(\mathcal{X})$ and so $c \in \mathcal{X}$. Thus $[b]_{\mathscr{R}_1} \subseteq \mathcal{X}$, which implies that $b \in \mathscr{R}_{1-}(\mathcal{X})$. Then $a = f(b) \in f(\mathscr{R}_{1-}(\mathcal{X}))$, so,

$$\mathscr{R}_{2-}(f(\mathcal{X})) \subseteq f(\mathscr{R}_{1-}(\mathcal{X})). \tag{2.9}$$

From equations (2.8) and (2.9), $f(\mathscr{R}_{1-}(\mathcal{X})) = \mathscr{R}_{2-}(f(\mathcal{X}))$. □

3. Rough 3-prime ideals

Definition 3.1. An ideal $\mathcal{J}$ in a near ring $\mathcal{N}$ is 3-prime if $a, b \in \mathcal{N}$ and $a\mathcal{N}b \subseteq \mathcal{J}$ implies that $a \in \mathcal{J}$ or $b \in \mathcal{J}$.

Definition 3.2. Let $\mathscr{R}$ be a C.R. on a near ring $\mathcal{N}$. Then a subset $\mathcal{X}$ of $\mathcal{N}$ is called a lower rough 3-prime ideal of $\mathcal{N}$, if $\mathscr{R}_-(\mathcal{X})$ is 3-prime ideal of $\mathcal{N}$.

In similar manner, upper rough 3-prime ideal can be defined.

Definition 3.3. An ideal $\mathcal{J}$ in near ring $\mathcal{N}$ is 3-semiprime if $a \in \mathcal{N}$ and $a\mathcal{N}a \subseteq \mathcal{J}$ implies that $a \in \mathcal{J}$.

Definition 3.4. An ideal $\mathcal{J}$ of a near ring $\mathcal{N}$ is rough 3-prime (semiprime) ideal of $\mathcal{N}$ if it is both a lower and an upper rough 3-prime (semiprime) ideal of $\mathcal{N}$.

Lemma 3.5. *Let $\mathscr{R}$ be C.C.R. on near ring $\mathcal{N}$ and $\mathcal{L}$ is a 3-prime ideal of $\mathcal{N}$ such that $\mathscr{R}^-(\mathcal{L}) \neq \mathcal{N}$. Then $\mathcal{L}$ is an upper rough prime ideal of $\mathcal{N}$.*

Proof. By lemma2.6, $\mathscr{R}^-(\mathcal{L})$ is an ideal. Now, we will only show that $\mathscr{R}^-(\mathcal{L})$ is 3-prime. Assume that $a, b \in \mathcal{N}$ such that $anb \in \mathscr{R}^-(\mathcal{L})$, for all $n \in \mathcal{N}$. Then $[anb]_{\mathscr{R}} \cap \mathcal{L} \neq \phi$. Since $\mathscr{R}$ is complete, hence $\{xmy \mid x \in [a]_{\mathscr{R}}, m \in [n]_{\mathscr{R}}, y \in [b]_{\mathscr{R}}\} \cap \mathcal{L} \neq \phi$. Suppose that there exist $x \in [a]_{\mathscr{R}}, m \in [n]_{\mathscr{R}}, y \in [b]_{\mathscr{R}}$ such that $xmy \in [anb]_{\mathscr{R}} \cap \mathcal{L}$. Since $\mathcal{L}$ is 3-prime, hence $xmy \in \mathcal{L}$ implies that $x \in \mathcal{L}$ or $y \in \mathcal{L}$. Therefore, $[a]_{\mathscr{R}} \cap \mathcal{L} \neq \phi$ or $[b]_{\mathscr{R}} \cap \mathcal{L} \neq \phi$. This implies that $a \in \mathscr{R}^-(\mathcal{L})$ or $b \in \mathscr{R}^-(\mathcal{L})$. □

Lemma 3.6. *Let $\mathscr{R}$ be C.C.R. on a near ring $\mathcal{N}$ and $\mathcal{L}$ is a 3-prime ideal of $\mathcal{N}$ such that $\mathscr{R}_-(\mathcal{L}) \neq \mathcal{N}$. Then $\mathcal{L}$ is a lower rough prime ideal of $\mathcal{N}$.*

Proof. Direct from lemma 2.9. □

Lemma 3.7. *Let $\mathscr{R}$ be a C.C.R. on a near ring $\mathcal{N}$ and $\mathcal{J}$ be a 3-semiprime ideal of $\mathcal{N}$ such that $\mathscr{R}^{-}(\mathcal{J}) \neq \mathcal{N}$. Then $\mathcal{J}$ is an upper rough 3-semiprime ideal of $\mathcal{N}$.*

Proof. Let $a \in \mathcal{N}$ such that $ana \in \mathscr{R}^{-}(\mathcal{J})$, for all $n \in \mathcal{N}$. Then $[ana]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$. Since $\mathscr{R}$ is C.C.R., hence $\{xmx \mid x \in [a]_{\mathscr{R}}, m \in [n]_{\mathscr{R}}\}$. Thus there exist $x \in [a]_{\mathscr{R}}, m \in [n]_{\mathscr{R}}$ such that $xmx \in \mathcal{J}$. As $\mathcal{J}$ is 3-semiprime, $x \in \mathcal{J}$. This implies thta $[a]_{\mathscr{R}} \cap \mathcal{J} \neq \phi$ i.e., $a \in \mathscr{R}^{-}(\mathcal{J})$. □

Lemma 3.8. *Let $\mathscr{R}$ be a C.C.R. on $\mathcal{N}$ and $\mathcal{J}$ a 3-semiprime ideal of $\mathcal{N}$ such that $\mathscr{R}_{-}(\mathcal{J}) \neq \mathcal{N}$. Then $\mathcal{J}$ is a lower rough 3-semiprime ideal of $\mathcal{N}$.*

Proof. Direct from lemma 2.9 □

Theorem 3.9. *Let $f : \mathcal{N}_1 \to \mathcal{N}_2$ be an epimorphism from a near ring $\mathcal{N}_1$ to a near ring $\mathcal{N}_2$ and $\mathscr{R}_2$ be a F.C.R. on $\mathcal{N}_2$. Let $\mathcal{X}$ be a subset of $\mathcal{N}_1$. If $\mathscr{R}_1 = \{(n_1, n_2) \in \mathcal{N}_1 \times \mathcal{N}_2 \mid (f(n_1), f(n_2)) \in \mathscr{R}_2\}$, then*

(i) *$\mathscr{R}_1{}^{-}(\mathcal{X})$ is an ideal of $\mathcal{N}_1$ if and only if $\mathscr{R}_2{}^{-}(f(\mathcal{X}))$ is an ideal of $\mathcal{N}_2$.*

(ii) *$\mathscr{R}_1{}^{-}(\mathcal{X})$ is 3-prime ideal of $\mathcal{N}_1$ if and only if $\mathscr{R}_2{}^{-}(f(\mathcal{X}))$ is 3-prime ideal of $\mathcal{N}_2$, provided $\mathscr{R}_2$ is complete.*

(iii) *$\mathscr{R}_1{}^{-}(\mathcal{X})$ is 3-semiprime ideal of $\mathcal{N}_1$ if and only if $\mathscr{R}_2{}^{-}(f(\mathcal{X}))$ is 3-semiprime ideal of $\mathcal{N}_2$, provided $\mathscr{R}_2$ is complete.*

Proof. (i) Suppose that $\mathscr{R}_2{}^{-}(f(\mathcal{X}))$ is an ideal of $\mathcal{N}_2$. Let $x_1, x_2 \in \mathscr{R}_1{}^{-}(\mathcal{X})$, $n \in \mathcal{N}_1$. Then $f(x_1), f(x_2) \in f(\mathscr{R}_1{}^{-}(\mathcal{X}))$, $f(n) \in \mathcal{N}_2$. By proposition 2.13, $f(x_1), f(x_2) \in \mathscr{R}_2{}^{-}(f(\mathcal{X}))$. As $\mathscr{R}_2{}^{-}(f(\mathcal{X}))$ is an ideal of $\mathcal{N}_2$, $f(x_1) - f(x_2), f(n) - f(x_1) + f(n)$, $f(n)f(x_1) \in \mathscr{R}_2{}^{-}(f(\mathcal{X}))$, also for $f(n_1) \in \mathcal{N}_2$, $(f(n) + f(x_1))f(n_1) - f(n)f(n_1) \in \mathscr{R}_2{}^{-}(f(\mathcal{X}))$. Since f is homomorphism, hence $f(x_1 - x_2), f(n - x_1 + n), f(nx_1), f((n + x_1)n_1 - nn_1) \in \mathscr{R}_2{}^{-}(f(\mathcal{X}))$ or $f(x_1 - x_2), f(n - x_1 + n), f(nx_1), f((n + x_1)n_1 - nn_1) \in f(\mathscr{R}_1{}^{-}(\mathcal{X}))$. So, there exist $a, b, c, d \in \mathscr{R}_1{}^{-}(\mathcal{X})$ such that $f(x_1 - x_2) = f(a), f(n - x_1 + n) = f(b), f(nx_1) = f(c), f((n + x_1)n_1 - nn_1) = f(d)$ and we get $[a]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi, [b]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi, [c]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi, [d]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi$ and $x_1 - x_2 \in [a]_{\mathscr{R}_1}, n - x_1 + n \in [b]_{\mathscr{R}_1}, nx_1 \in [c]_{\mathscr{R}_1}, (n + x_1)n_1 - nn_1 \in [d]_{\mathscr{R}_1}$. Thus $[x_1 - x_2]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi, [n - x_1 + n]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi, [nx_1]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi, [(n + x_1)n_1 - nn_1]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi$. This implies that $x_1 - x_2, n - x_1 + n, nx_1, (n + x_1)n - nn_1 \in \mathscr{R}_1{}^{-}(\mathcal{X})$. Thus $\mathscr{R}_1{}^{-}(\mathcal{X})$ is an ideal of $\mathcal{N}_1$.

Conversely, suppose that $\mathscr{R}_1{}^{-}(\mathcal{X})$ is an ideal of $\mathcal{N}_1$. Let $y_1, y_2 \in \mathscr{R}_2{}^{-}(f(\mathcal{X}))$

and $n_2 = f(n_2{}') \in \mathcal{N}_2$. Given that f is epimorphism, so by proposition 2.13, $y_1, y_2 \in \mathscr{R}_2{}^-(f(\mathcal{X})) = f(\mathscr{R}_1{}^-(\mathcal{X}))$ i.e., there exist $a, b \in \mathscr{R}_1{}^-(\mathcal{X})$ such that $y_1 = f(a), y_2 = f(b)$. Then $y_1 - y_2 = f(a-b)$, $n_2 y_1 = f(n_2{}')f(a) = f(n_2{}'a)$, $n_2 + y_1 - n_2 = f(n_2{}' + a - n_2{}')$, and $((n+y_1)n_2 - nn_2) = f((n' + a)n_2' - n'n_2{}')$. Since $\mathscr{R}_1{}^-(\mathcal{X})$ is an ideal of $\mathcal{N}_1$, hence $a-b, n_2{}'a, n_2{}' + a - n_2{}', (n' + a)n_2{}' - n'n_2{}' \in \mathscr{R}_1{}^-(\mathcal{X})$. This show that $\mathscr{R}_2{}^-(f(\mathcal{X}))$ is an ideal of $\mathcal{N}_2$.

(ii) Suppose that $\mathscr{R}_2{}^-(f(\mathcal{X}))$ is a 3-prime ideal of $\mathcal{N}_2$ and for $a, b \in \mathcal{N}_1$, $an_1b \in \mathscr{R}_1{}^-(\mathcal{X})$, for all $n_1 \in \mathcal{N}_1$. Then proposition 2.13, $f(an_1b) \in f(\mathscr{R}_1{}^-(\mathcal{X}))$ i.e., $f(a)f(n_1)f(b) \in \mathscr{R}_2{}^-(f(\mathcal{X}))$. As $\mathscr{R}_2{}^-(f(\mathcal{X}))$ is 3-prime ideal, $f(a) \in \mathscr{R}_2{}^-(f(\mathcal{X}))$ or $f(b) \in \mathscr{R}_2{}^-(f(\mathcal{X}))$ i.e., $f(a) \in f(\mathscr{R}_1{}^-(\mathcal{X}))$ or $f(b) \in f(\mathscr{R}_1{}^-(\mathcal{X}))$. Then there exist $x, y \in \mathscr{R}_1{}^-(\mathcal{X})$ such that $f(a) = f(x)$ and $f(b) = f(y)$. Thus $[x]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi$ and $a \in [x]_{\mathscr{R}_1}$ or $[y]_{\mathscr{R}_1} \cap \mathcal{X} \neq \phi$ and $b \in [y]_{\mathscr{R}_1}$. Which implies that $a \in \mathscr{R}_1{}^-(\mathcal{X})$ or $b \in \mathscr{R}_1{}^-(\mathcal{X})$.
Conversely, suppose that $\mathscr{R}_1{}^-(\mathcal{X})$ is a 3-prime ideal of $\mathcal{N}_1$. Let $a, b \in \mathcal{N}_2$ such that $an_2b \in \mathscr{R}_2{}^-(f(\mathcal{X}))$, for all $n_2 \in \mathcal{N}_2$. Then there exist $n_1{}', n_1{}'', n_2{}' \in \mathcal{N}_1$ such that $f(n_1{}') = a$, $f(n_1{}'') = b$ and $f(n_2{}') = n_2$. So, $f(n_1{}')f(n_2{}')f(n_1{}'') \in \mathscr{R}_2{}^-(f(\mathcal{X}))$ or $f(n_1{}'n_2{}'n_1{}'') \in \mathscr{R}_2{}^-(f(\mathcal{X}))$. Thus $[f(n_1{}')f(n_2{}')f(n_1{}'')]_{\mathscr{R}_2} \cap f(\mathcal{X}) \neq \phi$. Since $\mathscr{R}_2$ is complete, hence there exist $f(m_1{}') \in [f(n_1{}')]_{\mathscr{R}_2}$, $f(m_2{}') \in [f(n_2{}')]_{\mathscr{R}_2}$, $f(m_1{}'') \in [f(n_1{}'')]_{\mathscr{R}_2}$ such that $f(m_1{}')f(m_2{}')f(m_1{}'') = f(m_1{}'m_2{}'m_1{}'') \in f(\mathcal{X})$. Then by proposition 2.13, $m_1{}' \in [n_1{}']_{\mathscr{R}_1}, m_2{}' \in [n_2{}']_{\mathscr{R}_1}, m_1{}'' \in [n_1{}'']_{\mathscr{R}_1}$ and there exist $z \in \mathcal{X}$ such that $f(m_1{}'m_2{}'m_1{}'') = f(z)$. Hence $m_1{}'m_2{}'m_1{}'' \in [n_1{}'n_2{}'n_1{}'']_{\mathscr{R}_1}$ and $z \in [m_1{}'m_2{}'m_1{}'']_{\mathscr{R}_1}$ or $z \in [n_1{}'n_2{}'n_1{}'']_{\mathscr{R}_1}$. Thus $[n_1{}'n_2{}'n_1{}''] \cap \mathcal{X} \neq \phi$. This implies that $n_1{}'n_2{}'n_1{}'' \in \mathscr{R}_1{}^-(\mathcal{X})$. Since $\mathscr{R}_1{}^-(\mathcal{X})$ is 3-prime ideal, hence

$$n_1{}' \in \mathscr{R}_1{}^-(\mathcal{X}) \text{ or } n_1{}'' \in \mathscr{R}_1{}^-(\mathcal{X}). \tag{3.1}$$

By equation 3.1 and proposition 2.13,

$$a = f(n_1{}') \in f(\mathscr{R}_1{}^-(\mathcal{X})) = \mathscr{R}_2{}^-(f(\mathcal{X}))$$

$$or$$

$$b = f(n_1{}'') \in f(\mathscr{R}_1{}^-(\mathcal{X})) = \mathscr{R}_2{}^-(f(\mathcal{X})).$$

This show that $\mathscr{R}_2{}^-(f(\mathcal{X}))$ is 3-prime ideal.
(iii) Proof running parallel to (ii) □

Theorem 3.10. *Let $f : \mathcal{N}_1 \to \mathcal{N}_2$ be an isomorphism from a near ring $\mathcal{N}_1$ to a near ring $\mathcal{N}_2$ and $\mathscr{R}_2$ be a C.C.R. on $\mathcal{N}_2$. Let $\mathcal{X}$ be a subset of $\mathcal{N}_1$. If $\mathscr{R}_1 = \{(n_1, n_2) \in \mathcal{N}_1 \times \mathcal{N}_1 \mid (f(n_1), f(n_2)) \in \mathscr{R}_2\}$, then*

(i) *$\mathscr{R}_{1-}(\mathcal{X})$ is an ideal of $\mathcal{N}_1$ if and only if $\mathscr{R}_{2-}(f(\mathcal{X}))$ is an ideal of $\mathcal{N}_2$.*
(ii) *$\mathscr{R}_{1-}(\mathcal{X})$ is 3-prime ideal of $\mathcal{N}_1$ if and only if $\mathscr{R}_{2-}(f(\mathcal{X}))$ is 3-prime ideal of $\mathcal{N}_2$, provided $\mathscr{R}_2$ is complete.*
(iii) *$\mathscr{R}_{1-}(\mathcal{X})$ is 3-semiprime ideal of $\mathcal{N}_1$ if and only if $\mathscr{R}_{2-}(f(\mathcal{X}))$ is 3-semiprime ideal of $\mathcal{N}_2$, provided $\mathscr{R}_2$ is complete.*

Proof. Firstly, from proposition 2.13, we get $f(\mathscr{R}_{1-}(\mathcal{X})) = \mathscr{R}_{2-}(f(\mathcal{X}))$. After that proof is running parallel to theorem 3.9. □

4. Rough 3-prime fuzzy ideals

Definition 4.1. Let $\mathscr{R}$ be a F.C.R. on a near ring $\mathcal{N}$ and ζ be fuzzy ideal (F.I.) of $\mathcal{N}$. Then we define the fuzzy sets $\mathscr{R}_-(\zeta)$ and $\mathscr{R}^-(\zeta)$ as follow:

$$\mathscr{R}_-(\zeta)(n) = \inf_{a \in [n]_{\mathscr{R}}} \{\zeta(a)\} \text{ and } \mathscr{R}^-(\zeta)(n) = \sup_{b \in [n]_{\mathscr{R}}} \{\zeta(a)\}.$$

$\mathscr{R}_-(\zeta)$ and $\mathscr{R}^-(\zeta)$ are known as $\mathscr{R}$-lower and $\mathscr{R}$-upper approximation of fuzzy set ζ.

Definition 4.2. Let $\mathscr{R}_-(\zeta)$ and $\mathscr{R}^-(\zeta)$ are $\mathscr{R}$-lower and $\mathscr{R}$-upper approximation of a fuzzy ideal ζ. Then $\mathscr{R}(\zeta) = (\mathscr{R}_-(\zeta), \mathscr{R}^-(\zeta))$ is said to be a rough fuzzy set (R.F.S) if $\mathscr{R}_-(\zeta) \neq \mathscr{R}^-(\zeta)$.

Definition 4.3. Let ζ be a fuzzy set of a near ring $\mathcal{N}$. Then ζ is called upper rough F.I., if $\mathscr{R}^-(\zeta)$ is a F.I. of $\mathcal{N}$.

Definition 4.4. A F.I. ζ of $\mathcal{N}$ is said to be 3-prime if for $n_1, n_2 \in \mathcal{N}$,

$$\vee\{\zeta(n_1), \zeta(n_2)\} \geq \underset{n \in \mathcal{N}}{\wedge} \zeta(n_1 n n_2).$$

Definition 4.5. A rough fuzzy set $(\mathscr{R}_-(\zeta), \mathscr{R}^-(\zeta))$ is said to be a rough F.I., if $\mathscr{R}_-(\zeta)$ and $\mathscr{R}^-(\zeta)$ is a F.I. of a near ring $\mathcal{N}$.

Example 4.6. Consider $(\mathcal{N}, +, \cdot)$ be a near ring under following operations

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

and

$+$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	1	0
2	0	2	0	2	0	2	2	0
3	0	3	0	3	0	3	3	0
4	4	4	4	4	4	4	4	4
5	4	5	4	5	4	5	5	4
6	4	6	4	6	4	6	6	4
7	4	7	4	7	4	7	7	4

Let $\mathscr{R}$ be a F.C.R. on $\mathcal{N}$ with equivalence classes $[0]_{\mathscr{R}} = \{0,2,4,6\}$ and $[1]_{\mathscr{R}} = \{1,3,5,7\}$. Take 3-prime F.I. ζ on $\mathcal{N}$ which is defined by $\zeta(0) = \zeta(1) = \zeta(2) = \zeta(3) = 0.1$, $\zeta(4) = \zeta(5) = \zeta(6) = \zeta(7) = 0.2$. We can easily find that $\mathscr{R}_{-}(\zeta)(a) = 0.1 \neq 0.2 = \mathscr{R}^{-}(\zeta)(b)$ for all $a, b \in \mathcal{N}$. Since $\mathscr{R}^{-}(\zeta)$ and $\mathscr{R}_{-}(\zeta)$ are F.I., hence rough fuzzy set $\mathscr{R}(\zeta)$ is rough F.I..

Lemma 4.7. *Let $\mathscr{R}$ be a F.C.R. on a near ring $\mathcal{N}$. If ζ is a F.I. of $\mathcal{N}$, then*

(*i*) *$\mathscr{R}^{-}(\zeta)$ is a F.I. of $\mathcal{N}$, and*
(*ii*) *$\mathscr{R}_{-}(\zeta)$ is a F.I. of $\mathcal{N}$.*

Proof. (*i*) For $i, j \in \mathcal{N}$,

$$\begin{aligned}
\mathscr{R}^{-}(\zeta)(i-j) &= \sup_{k\in[i-j]_{\mathscr{R}}}\{\zeta(k)\} = \sup_{k\in[i]_{\mathscr{R}}-[j]_{\mathscr{R}}}\{\zeta(k)\} \\
&= \sup_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_1-k_2)\} \geq \sup_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_1)\wedge\zeta(k_2)\} \\
&= \sup_{k_1\in[i]_{\mathscr{R}}}\{\zeta(k_1)\} \wedge \sup_{k_2\in[j]_{\mathscr{R}}}\{\zeta(k_2)\} \\
&= \mathscr{R}^{-}(\zeta)(i) \wedge \mathscr{R}^{-}(\zeta)(j). \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
\mathscr{R}^{-}(\zeta)(ij) &= \sup_{k\in[ij]_{\mathscr{R}}}\{\zeta(k)\} \geq \sup_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_1k_2)\} \\
&= \sup_{k_1\in[i]_{\mathscr{R}}}\{\zeta(k_1)\} \wedge \sup_{k_2\in[j]_{\mathscr{R}}}\{\zeta(k_2)\} \\
&= \mathscr{R}^{-}(\zeta)(i) \wedge \mathscr{R}^{-}(\zeta)(j). \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}\mathscr{R}^{-}(\zeta)(j+i-j) &= \sup_{k\in[j+i-j]_{\mathscr{R}}}\{\zeta(k)\} = \sup_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_2+k_1-k_2)\}\\ &= \sup_{k_1\in[i]_{\mathscr{R}}}\{\zeta(k_1)\}\\ &= \mathscr{R}^{-}(\zeta)(i).\end{aligned}\tag{4.3}$$

$$\begin{aligned}\mathscr{R}^{-}(\zeta)(ij) &= \sup_{k\in[ij]_{\mathscr{R}}}\{\zeta(k)\} \geq \sup_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_1k_2)\}\\ &\geq \sup_{k_2\in[j]_{\mathscr{R}}}\{\zeta(k_2)\} = \mathscr{R}^{-}(\zeta)(j).\end{aligned}\tag{4.4}$$

$$\begin{aligned}\mathscr{R}^{-}(\zeta)((i+l)j-ij) &= \sup_{k\in[(i+l)j-ij]_{\mathscr{R}}}\{\zeta(k)\} = \sup_{k\in[(i+l)j]_{\mathscr{R}}-[ij]_{\mathscr{R}}}\{\zeta(k)\}\\ &\geq \sup_{k_1\in[i]_{\mathscr{R}},k_2\in[l]_{\mathscr{R}},k_3\in[j]_{\mathscr{R}}}\{\zeta((k_1+k_2)k_3-k_1k_2)\}\\ &\geq \sup_{k_2\in[l]_{\mathscr{R}}}\{\zeta(k_2)\} = \mathscr{R}^{-}(\zeta)(l).\end{aligned}\tag{4.5}$$

Equations (4.1)-(4.5) show that $\mathscr{R}^{-}(\zeta)$ is a F.I. of $\mathcal{N}$.

(ii) For $i,j\in\mathcal{N}$,

$$\begin{aligned}\mathscr{R}_{-}(\zeta)(i-j) &= \inf_{k\in[i-j]_{\mathscr{R}}}\{\zeta(k)\} = \inf_{k\in[i]_{\mathscr{R}}-[j]_{\mathscr{R}}}\{\zeta(k)\}\\ &\geq \inf_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_1)\wedge\zeta(k_2)\}\\ &= \mathscr{R}_{-}(\zeta)(i)\wedge\mathscr{R}_{-}(\zeta)(j).\end{aligned}\tag{4.6}$$

$$\begin{aligned}\mathscr{R}_{-}(\zeta)(ij) &= \inf_{k\in[ij]_{\mathscr{R}}}\{\zeta(k)\} \geq \inf_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_1k_2)\}\\ &= \inf_{k_1\in[i]_{\mathscr{R}}}\{\zeta(k_1)\}\wedge\inf_{k_2\in[j]_{\mathscr{R}}}\{\zeta(k_2)\}\\ &= \mathscr{R}_{-}(\zeta)(i)\wedge\mathscr{R}_{-}(\zeta)(j).\end{aligned}\tag{4.7}$$

$$\begin{aligned}\mathscr{R}_{-}(\zeta)(j+i-j) &= \inf_{k\in[j+i-j]_{\mathscr{R}}}\{\zeta(k)\} = \inf_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_2+k_1-k_2)\}\\ &= \inf_{k_1\in[i]_{\mathscr{R}}}\{\zeta(k_1)\} = \mathscr{R}_{-}(\zeta)(i).\end{aligned}\tag{4.8}$$

$$\begin{aligned}\mathscr{R}^{-}(\zeta)(ij) &= \inf_{k\in[ij]_{\mathscr{R}}}\{\zeta(k)\} \geq \inf_{k_1\in[i]_{\mathscr{R}},k_2\in[j]_{\mathscr{R}}}\{\zeta(k_1k_2)\}\\ &\geq \inf_{k_2\in[j]_{\mathscr{R}}}\{\zeta(k_2)\} = \mathscr{R}_{-}(\zeta)(j).\end{aligned} \tag{4.9}$$

$$\begin{aligned}\mathscr{R}_{-}(\zeta)((i+l)j-ij) &= \inf_{k\in[(i+l)j-ij]_{\mathscr{R}}}\{\zeta(k)\} = \inf_{k\in[(i+l)j]_{\mathscr{R}}-[ij]_{\mathscr{R}}}\{\zeta(k)\}\\ &\geq \inf_{k_1\in[i]_{\mathscr{R}},k_2\in[l]_{\mathscr{R}},k_3\in[j]_{\mathscr{R}}}\{\zeta((k_1+k_2)k_3-k_1k_2)\}\\ &\geq \inf_{k_2\in[l]_{\mathscr{R}}}\{\zeta(k_2)\} = \mathscr{R}_{-}(\zeta)(l).\end{aligned} \tag{4.10}$$

Equations (4.6)-(4.10) show that $\mathscr{R}^{-}(\zeta)$ is a F.I. of $\mathcal{N}$. □

Corollary 4.8. *A rough fuzzy set* $(\mathscr{R}_{-}(\zeta),\mathscr{R}^{-}(\zeta))$ *is F.I. if* ζ *is F.I..*

Proof. By lemma 4.7, $(\mathscr{R}_{-}(\zeta),\mathscr{R}^{-}(\zeta))$ is F.I.. □

Theorem 4.9. *Let* ζ *be a 3-prime F.I. of a near ring* $\mathcal{N}$. *If* $\mathscr{R}$ *is C.C.R. on* $\mathcal{N}$, *then*

(*i*) $\mathscr{R}_{-}(\zeta)$ *is 3-prime F.I. of* $\mathcal{N}$.
(*ii*) $\mathscr{R}^{-}(\zeta)$ *is 3-prime F.I. of* $\mathcal{N}$.

Proof. By lemma 4.7, $\mathscr{R}_{-}(\zeta)$ is F.I. of $\mathcal{N}$. Now, we will only show that 3-primeness of $\mathscr{R}_{-}(\zeta)$. Let $a,b\in\mathcal{N}$. Then

$$\begin{aligned}\inf_{n\in\mathcal{N}}\mathscr{R}_{-}(\zeta)(anb) &= \inf_{n\in\mathcal{N}}\{\bigwedge_{z\in[anb]_{\mathscr{R}}}\zeta(z)\}\\ &= \inf_{n\in\mathcal{N}}\{\bigwedge_{z_1\in[a]_{\mathscr{R}},m\in[n]_{\mathscr{R}},z_2\in[b]_{\mathscr{R}}}\zeta(z_1mz_2)\}\\ &= \bigwedge_{z_1\in[a]_{\mathscr{R}},m\in[n]_{\mathscr{R}},z_2\in[b]_{\mathscr{R}}}\{\inf_{m\in\mathcal{N}}\zeta(z_1mz_2)\}\\ &\leq \bigwedge_{z_1\in[a]_{\mathscr{R}},z_2\in[b]_{\mathscr{R}}}\{\max(\zeta(z_1),\zeta(z_2))\}\\ &= \max\{(\bigwedge_{z_1\in[a]_{\mathscr{R}}}\zeta(z_1),\bigwedge_{z_2\in[b]_{\mathscr{R}}}\zeta(z_2))\}\\ &= \max\{\mathscr{R}_{-}(\zeta)(a),\mathscr{R}_{-}(\zeta)(b)\}.\end{aligned} \tag{4.11}$$

This shows that $\mathscr{R}_{-}(\zeta)$ is 3-prime F.I. of $\mathcal{N}$.
(ii) By lemma 4.7, $\mathscr{R}^{-}(\zeta)$ is F.I. of $\mathcal{N}$. Now, we will only show that 3-primeness of $\mathscr{R}^{-}(\zeta)$. Let $a, b \in \mathcal{N}$. Then

$$\begin{aligned}
\inf_{n\in\mathcal{N}} \mathscr{R}^{-}(\zeta)(anb) &= \inf_{n\in\mathcal{N}}\{\bigvee_{z\in[anb]_{\mathscr{R}}} \zeta(z)\} \\
&= \inf_{n\in\mathcal{N}}\{\bigvee_{z_1\in[a]_{\mathscr{R}}, m\in[n]_{\mathscr{R}}, z_2\in[b]_{\mathscr{R}}} \zeta(z_1 m z_2)\} \\
&= \bigvee_{z_1\in[a]_{\mathscr{R}}, m\in[n]_{\mathscr{R}}, z_2\in[b]_{\mathscr{R}}} \{\inf_{m\in\mathcal{N}} \zeta(z_1 m z_2)\} \\
&\leq \bigvee_{z_1\in[a]_{\mathscr{R}}, z_2\in[b]_{\mathscr{R}}} \{\max(\zeta(z_1), \zeta(z_2))\} \\
&= \max\{(\bigvee_{z_1\in[a]_{\mathscr{R}}} \zeta(z_1), \bigvee_{z_2\in[b]_{\mathscr{R}}} \zeta(z_2))\} \\
&= \max\{\mathscr{R}^{-}(\zeta)(a), \mathscr{R}^{-}(\zeta)(b)\}. \qquad (4.12)
\end{aligned}$$

This shows that $\mathscr{R}^{-}(\zeta)$ is 3-prime F.I. □

5. Conclusion

Rough fuzzy ideals give a useful and effective tool for modeling complex systems with ambiguity, imprecision, and uncertainty. The rough fuzzy ideal of a near ring is a generalization of an ideal of near ring. So, we replaced a universe set by a near ring and introduced the notion of rough 3-prime (3-semiprime) ideals and rough fuzzy 3-prime ideal of near rings. We discussed the conditions under which the upper and lower rough 3-prime ideals and upper and lower approximations of their homomorphic images are related.

Acknowledgement: We are grateful to the referee for the comments which improved the quality of the paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 132–137

A SOLUTION TO AN INTERESTING SUM INVOLVING CLASSICAL HARMONIC NUMBER AND CENTRAL BINOMIAL COEFFICIENT

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(Received : 09 - 07 - 2023 ; Revised : 14 - 10 - 2023)

Abstract. In this paper we solve an interesting sum recently considered by A.S.Nimbran concerning the calculation of a series involving the classical harmonic number and central binomial coefficient.The key ingredients for obtaining our infinite series result are some of the difficult and obscure definite integral formulas due to K.S.Kölbig and Cornel Ioan Vălean.

1. Introduction

Amrik Singh Nimbran left the following problem of evaluating a series involving the classical harmonic number and central binomial coefficient as an interesting sum in [4, p. 134], which we will provide a solution to in this paper:

$$S=\sum_{n=1}^{\infty}\frac{H_n\binom{2n}{n}}{n^2 2^{2n}}$$

Throughout this paper H_n denotes the nth-classical harmonic number de?ned by $H_n=\sum_{k=1}^{n}\frac{1}{k}$, $\binom{2n}{n}$ denotes the central binomial coefficient, which is defined for $n\geq 1$ by $\binom{2n}{n}=\frac{(2n)!}{(n!)^2}$, $\zeta\left(s\right)$ denotes the Riemann zeta function, which is defined by $\zeta\left(s\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}$, $\Re(s)>1$ and

2010 Mathematics Subject Classification: Primary 40A25, 65B10, 05A10; Secondary 11M06
Key words and phrases: Harmonic numbers, central binomial coefficients, Riemann zeta function, infinite summation formulas

$\mathrm{Li}_n(x)$ denotes the polylogarithm function, which is defined for $|x| \leq 1$ by $\mathrm{Li_n}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}, \quad n \in \mathbb{N}, \quad n \geq 2.$

To evaluate S, we shall establish a lemma.

Lemma 1.1. *The following identity holds:*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n \left(\cos^2\theta\right)^n}{n^2} = &-\mathrm{Li}_3\left(\sin^2\theta\right) + \mathrm{Li}_3\left(\cos^2\theta\right) + \\ &\mathrm{Li}_2\left(\sin^2\theta\right)\ln\left(\sin^2\theta\right) + \\ &\frac{1}{2}\ln\left(\cos^2\theta\right)\ln^2\left(\sin^2\theta\right) + \zeta(3).\end{aligned}$$

Proof. In the recent article [1, Lemma 4.5], Seán Mark Stewart had evaluated the following generating function involving classical harmonic number:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n x^n}{n^2} = &-\mathrm{Li}_3(1-x) + \mathrm{Li}_3(x) + \mathrm{Li}_2(1-x)\ln(1-x) + \\ &\frac{1}{2}\ln(x)\ln^2(1-x) + \zeta(3). \end{aligned} \tag{1.1}$$

Substituting $x = \cos^2\theta$ on both sides of (1.1), we attain the following equation:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n \left(\cos^2\theta\right)^n}{n^2} &= -\mathrm{Li}_3\left(1-\cos^2\theta\right) + \mathrm{Li}_3\left(\cos^2\theta\right) + \\ &\quad \mathrm{Li}_2\left(1-\cos^2\theta\right)\ln\left(1-\cos^2\theta\right) + \\ &\quad \frac{1}{2}\ln\left(\cos^2\theta\right)\ln^2\left(1-\cos^2\theta\right) + \zeta(3) \\ &= -\mathrm{Li}_3\left(\sin^2\theta\right) + \mathrm{Li}_3\left(\cos^2\theta\right) + \\ &\quad \mathrm{Li}_2\left(\sin^2\theta\right)\ln\left(\sin^2\theta\right) + \\ &\quad \frac{1}{2}\ln\left(\cos^2\theta\right)\ln^2\left(\sin^2\theta\right) + \zeta(3).\end{aligned}$$

□

Theorem 1.2. *The infinite series*

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{n^2 2^{2n}}$$

admits the symbolic form

$$\frac{9}{2}\zeta(3) - 4\ln 2\zeta(2).$$

Proof. $\displaystyle\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{n^2 2^{2n}} = \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\frac{1}{2^{2n}} \binom{2n}{n} \right)$

Invoking the Wallis' well-known integral formula, namely,

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n}\, d\theta = \frac{1}{2^{2n}} \binom{2n}{n}. \text{ we have,}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\frac{1}{2^{2n}} \binom{2n}{n} \right) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n}\, d\theta \right)$$

$$= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} \frac{H_n \left(\cos^2\theta\right)^n}{n^2} \right) d\theta \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \left(-\operatorname{Li}_3\left(\sin^2\theta\right) + \operatorname{Li}_3\left(\cos^2\theta\right) + \operatorname{Li}_2\left(\sin^2\theta\right) \ln\left(\sin^2\theta\right) \right) d\theta \right] +$$

$$\frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} \ln\left(\cos^2\theta\right) \ln^2\left(\sin^2\theta\right) + \zeta(3) \right) d\theta \right]$$

$$= \frac{2}{\pi} \left[-\int_0^{\frac{\pi}{2}} \operatorname{Li}_3\left(\sin^2\theta\right) d\theta + \int_0^{\frac{\pi}{2}} \operatorname{Li}_3\left(\cos^2\theta\right) d\theta \right] +$$

$$\frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \operatorname{Li}_2\left(\sin^2\theta\right) \ln\left(\sin^2\theta\right) d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln\left(\cos^2\theta\right) \ln^2\left(\sin^2\theta\right) d\theta \right] +$$

$$\frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \zeta(3)\, d\theta \right].$$

In this expression, due to symmetry, recognizing that,

$$\int_0^{\frac{\pi}{2}} \operatorname{Li}_3\left(\sin^2\theta\right) d\theta = \int_0^{\frac{\pi}{2}} \operatorname{Li}_3\left(\cos^2\theta\right) d\theta$$

and $\int_0^{\frac{\pi}{2}} \ln\left(\cos^2\theta\right) \ln^2\left(\sin^2\theta\right) d\theta = \int_0^{\frac{\pi}{2}} \ln\left(\sin^2\theta\right) \ln^2\left(\cos^2\theta\right) d\theta$, we

conclude:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{n^2 2^{2n}} &= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \operatorname{Li}_2\left(\sin^2\theta\right) \ln\left(\sin^2\theta\right) d\theta\right] + \\
&\quad \frac{2}{\pi}\left[\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\sin^2\theta\right) \ln^2\left(\cos^2\theta\right) d\theta + \int_0^{\frac{\pi}{2}} \zeta(3)\, d\theta\right] \\
&= \frac{2}{\pi}\left[2\int_0^{\frac{\pi}{2}} \operatorname{Li}_2\left(\sin^2\theta\right) \ln(\sin\theta)\, d\theta\right] + \\
&\quad \frac{2}{\pi}\left[4\int_0^{\frac{\pi}{2}} \ln(\sin\theta) \ln^2(\cos\theta)\, d\theta + \frac{\pi}{2}\zeta(3)\right]
\end{aligned}$$

$$= \frac{4}{\pi}\left(\int_0^{\frac{\pi}{2}} \operatorname{Li}_2\left(\sin^2\theta\right) \ln(\sin\theta)\, d\theta\right) + \frac{8}{\pi}\left(\int_0^{\frac{\pi}{2}} \ln(\sin\theta) \ln^2(\cos\theta)\, d\theta\right)$$

$$+\zeta(3)$$

Using Landen's identity, $\operatorname{Li}_2(-a) = -\operatorname{Li}_2\left(\frac{a}{1+a}\right) - \frac{1}{2}\ln^2(1+a)$ for $a \geq 0$ [2], Cornel Ioan Vălean [3, (1.97)], had recently evaluated the definite integral

$$\int_0^{\frac{\pi}{2}} \ln(\sin\theta) \operatorname{Li}_2\left(\sin^2\theta\right) d\theta = \frac{5\pi}{8}\zeta(3) - \pi\ln 2\zeta(2) + \pi\ln^3 2$$

and

using the derivative of the Euler's beta function and the Leibniz formula for the differentiation of products, K.S.Kölbig [5, p. 25], had evaluated the definite integral

$$\int_0^{\frac{\pi}{2}} \ln(\sin\theta)\ln^2(\cos\theta)\,d\theta = \frac{\pi}{8}\left(\zeta(3) - 4\ln^3 2\right).$$

giving us that the following equality holds:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n\binom{2n}{n}}{n^2 2^{2n}} &= \frac{4}{\pi}\left(\frac{5\pi}{8}\zeta(3) - \pi\ln 2\zeta(2) + \pi\ln^3 2\right) + \\ &\quad \frac{8}{\pi}\left(\frac{\pi}{8}\left(\zeta(3) - 4\ln^3 2\right)\right) + \zeta(3) \\ &= \frac{5}{2}\zeta(3) - 4\ln 2\zeta(2) + 4\ln^3 2 + \\ &\quad \zeta(3) - 4\ln^3 2 + \zeta(3) \\ &= \frac{9}{2}\zeta(3) - 4\ln 2\zeta(2).\end{aligned}$$

Here in the proof of Theorem 1.2, Bernstein's theorem [6, Thm. 9.30, p. 243] justifies interchanging the order of integration and summation because of the positivity of the coefficients.

□

Acknowledgement: I express my sincere thanks to the anonymous referee for providing constructive comments and invaluable suggestions on the first version of this paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 138–148

NIVEN NUMBERS AND A UNIQUE PROPERTY OF 2023

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Abstract. A positive integer n is called a Niven number if n is divisible by the sum of its decimal digits. In this article, we study Niven numbers of special types connecting with the values of generalized happy functions, and show that 2023 is a unique Niven number of a particular special type.

1. Introduction, literature review, and main results

At the end of the year 2022, the authors thought about finding some unique properties of the integer 2023. We came up with an idea connecting 2023, Niven numbers, divisors, the sum of digits, and the happy and Alladi-Erdős functions.

For each $n \in \mathbb{N}$, let $s_1(n)$ be the sum of the decimal digits of n and let $s_2(n)$ be the sum of the squares of the decimal digits of n. A positive integer n is called a Niven number or a harshad number if n is divisible by $s_1(n)$ and n is called a happy number if

$$s_2^{(k)}(n) \text{ converges to 1 as } k \to \infty,$$

where $s_2^{(k)}$ is the k-fold composition of s_2. While the history of happy numbers is unclear, harshad numbers can be traced back to D. R. Kaprekar-an Indian mathematician, and they were made more popular in the paper by Ivan Niven [11].

Niven and happy numbers and the functions s_1 and s_2 have been studied by many mathematicians. For example, if $N(x)$ is the number of Niven numbers not exceeding x, then Kennedy and Cooper [12] showed in 1984

2010 Mathematics Subject Classification: 11A63, 11A25, 11A41, 11B83
Key words and phrases: Niven number, happy number, digital problem, prime, factorization
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that $N(x)/x$ converges to zero as $x \to \infty$, which was later generalized to other integer sequences by the same authors [13] in 1989. Cooper and Kennedy [3] also proved in 1992 that there does not exist a sequence of more than 20 consecutive integers which are Niven numbers, and 20 is sharp in the sense that there are infinitely many sequences of 20 consecutive Niven numbers. Cooper and Kennedy's result [3] was extended by Grundman [7], Cai [1], and Wilson [22] to Niven numbers in the general bases $b \geq 2$ during 1994 to 1997. Then De Koninck, Doyon, and Kátai [14] obtained in 2003 an asymptotic formula $N(x) \sim cx/\log x$ as $x \to \infty$, where $c = (14 \log 10)/27$. Results concerning the distribution of the function s_1 on arithmetic progressions were obtained by Gelfond [5] and investigated further by several researchers, which were collected and explained in Morgenbesser's thesis [15]. Answering a question in Guy's book [9], El-Sedy and Siksek [4] proved in 2000 that there are arbitrarily long consecutive happy numbers. Other mathematicians have also made some contributions to happy numbers; see for example in the work of Chase [2], Gilmer [6], Grundman and Teeple [8], Hargreaves and Siksek [10], Noppakeaw, Phoopha, and Pongsriiam [16], Pan [17], Phoopha, Pongsriiam, and Phunphayap [18], Styer [20], Subwattanachai and Pongsriiam [21], and Zhou and Cai [23]. For more information on Niven and happy numbers, we refer the reader to the sequences A005349 and A007770 in OEIS [19].

However, in this article, we consider a more elementary problem that connects the integer 2023, Niven numbers, prime factorizations, the sum of digits, the sum of divisors, and the happy and Alladi-Erdős functions. We have $s_1(2023) = 2+0+2+3 = 7$ is a prime, $s_2(2023) = 2^2+0^2+2^2+3^2 = 17$ is also a prime, and the canonical factorization of 2023 into powers of primes is $2023 = 7 \times 17^2 = s_1(2023)s_2(2023)^2$, and we show in Corollary 1.2 that 2023 is the only positive integer with this property. In addition, we show in Theorem 1.4 that the only positive integer n satisfying $n = s_1(n)s_1(\sigma(n))^2$ or $n = s_1(n)s_2(A(n))^2$ where $s_1(n)$ and $s_2(A(n))$ are prime is $n = 2023$. Here $\sigma(n)$ is the sum of positive divisors of n and $A(n)$ is the sum of prime divisors (with multiplicity) of n.

We can extend the investigation to a larger set of Niven numbers of similar types by connecting them with the values of generalized happy functions. For each integers $\alpha \geq 1$ and $b \geq 2$, let $s_{\alpha,b} : \mathbb{N} \to \mathbb{N}$ be defined by

$$s_{\alpha,b}(n) = a_k^\alpha + a_{k-1}^\alpha + \cdots + a_0^\alpha$$

if $n = (a_k a_{k-1} \cdots a_0)_b = \sum_{i=0}^{k} a_i b^i$ is the b-adic expansion of n with $a_k \neq 0$ and $0 \leq a_i < b$ for all $i = 0, 1, 2, \ldots, k$. We say that n is a Niven number in base b or a b-adic Niven number if n is divisible by $s_{1,b}(n)$, and n is a b-adic Niven number of type 2 if n can be written as

$$n = s_{\alpha_1,b}(n)^{a_1} s_{\alpha_2,b}(n)^{a_2} \cdots s_{\alpha_k,b}(n)^{a_k}, \tag{1.1}$$

where $1 = \alpha_1 < \alpha_2 < \cdots < \alpha_k$ and $a_1, a_2, \ldots, a_k$ are positive integers. In addition, we call n a b-adic Niven number of type 3 or a special b-adic Niven number if (1.1) is the canonical factorization of n into powers of primes, that is, n satisfies (1.1) and $s_{\alpha_i,b}(n)$ is a prime number for each i. Furthermore, if $a_1 + a_2 + \cdots + a_k = d$, then we say that n is of degree d. If the $s_{\alpha_i,b}(n)$ are distinct and $\alpha_k = \ell$, then n is said be at level ℓ. Finally, we remark that b-adic Niven numbers are usually called b-Niven numbers for short, but for clarity, we do not use the short name in this article.

For simplicity, we focus only on the case of special Niven numbers where the base $b = 10$ and the degree d and the level ℓ are small. We also write s_α instead of $s_{\alpha,10}$. Then we obtain the following theorem.

Theorem 1.1. *Let n be a positive integer. Then the following statements hold.*

(i) *(Special Niven numbers of degree 1)*
$n = s_1(n)$ *if and only if* $n \leq 9$.
Consequently, n is a special Niven number of degree 1 if and only if $n = 2, 3, 5, 7$.

(ii) *(Special Niven numbers of degree 2 at level $\ell \leq 3$)*
$n = s_1(n)^2$ *if and only if* $n = 1$ *or* $n = 81$;
$n = s_1(n)s_2(n)$ *if and only if* $n = 1, 133, 315, 803, 1148, 1547, 2196$;
$n = s_1(n)s_3(n)$ *if and only if* $n = 1, 1215, 3700, 11680, 13608, 87949$.
Among these integers, 133 *and* 803 *are the special Niven numbers with degree 2 at level 2, and* 87949 *is the only special Niven number of degree 2 at level 3.*

(iii) *(Special Niven numbers of degree 3 at level $\ell \leq 2$)*
$n = s_1(n)^3$ *if and only if* $n = 1, 512, 4913, 5832, 17576, 19683$;
$n = s_1(n)^2 s_2(n)$ *if and only if* $n = 1, 2511, 24624$;
$n = s_1(n)s_2(n)^2$ *if and only if* $n = 1, 2023, 2400, 52215, 615627,$ 938600, 1648656.

Consequently, 4913 *is the only special Niven number of degree* 3 *at level* 1*, and* 2023 *is the only special Niven number of degree* 3 *at level* 2.

(iv) *(Special Niven numbers of degree* 3 *at level* 3*)*

$n = s_1(n)^2 s_3(n)$ *if and only if* $n = 1, 32144, 37000, 111616, 382360$;

$n = s_1(n)s_2(n)s_3(n)$ *if and only if* $n = 1, 3163617, 3822147$;

$n = s_1(n)s_3(n)^2$ *if and only if* $n = 1, 4147200, 12743163$, 21147075, 39143552, 52921472, 156754936, 205889445, 233935967.

In particular, there is no special Niven number of degree 3 *at level* 3.

(v) *(Special Niven numbers of degree* 2 *at level* $\ell = 4, 5, 6$*)*

$n = s_1(n)s_4(n)$ *if and only if* $n = 1, 182380, 444992$;

$n = s_1(n)s_5(n)$ *if and only if* $n = 1, 415000$, 3508936, 3828816, 4801896, 5659875;

$n = s_1(n)s_6(n)$ *if and only if* $n = 1, 21268584$.

In particular, there is no special Niven number of degree 2 *at level* $\ell = 4, 5, 6$.

Before proving Theorem 1.1, let us record some arithmetic curiosities that can be obtained straightforwardly from our theorem.

Corollary 1.2. *The only positive integer* n *satisfying* $n = s_1(n)s_2(n)^2$ *where* $s_1(n)$ *and* $s_2(n)$ *are prime is* $n = 2023$.

Corollary 1.3. *We have*

(A) $81 = (8+1)^2, \quad 133 = (1+3+3)(1^2+3^2+3^2),$
$315 = (3+1+5)(3^2+1^2+5^2), \quad 803 = (8+0+3)(8^2+0^2+3^2),$
$1148 = (1+1+4+8)(1^2+1^2+4^2+8^2),$
$1547 = (1+5+4+7)(1^2+5^2+4^2+7^2),$
$2196 = (2+1+9+6)(2^2+1^2+9^2+6^2),$

$1215 = (1+2+1+5)(1^3+2^3+1^3+5^3),$
$3700 = (3+7+0+0)(3^3+7^3+0^3+0^3),$
$11680 = (1+1+6+8+0)(1^3+1^3+6^3+8^3+0^3),$
$13608 = (1+3+6+0+8)(1^3+3^3+6^3+0^3+8^3),$
$87949 = (8+7+9+4+9)(8^3+7^3+9^3+4^3+9^3),$

(B) $512 = (5+1+2)^3, \quad 4913 = (4+9+1+3)^3,$
$5832 = (5+8+3+2)^3, \quad 17576 = (1+7+5+7+6)^3,$

$$19683 = (1+9+6+8+3)^3,$$

$$\begin{aligned} 2511 &= (2+5+1+1)^2(2^2+5^2+1^2+1^2), \\ 24624 &= (2+4+6+2+4)^2(2^2+4^2+6^2+2^2+4^2), \end{aligned}$$

$$\begin{aligned} 2023 &= (2+0+2+3)(2^2+0^2+2^2+3^2)^2, \\ 2400 &= (2+4+0+0)(2^2+4^2+0^2+0^2)^2, \\ 52215 &= (5+2+2+1+5)(5^2+2^2+2^2+1^2+5^2)^2, \\ 615627 &= (6+1+5+6+2+7)(6^2+1^2+5^2+6^2+2^2+7^2)^2, \\ 938600 &= (9+3+8+6+0+0)(9^2+3^2+8^2+6^2+0^2+0^2)^2, \\ 1648656 &= (1+6+4+8+6+5+6)(1^2+6^2+4^2+8^2+6^2+5^2+6^2)^2, \end{aligned}$$

(C) $$\begin{aligned} 32144 &= (3+2+1+4+4)^2(3^3+2^3+1^3+4^3+4^3), \\ 37000 &= (3+7+0+0+0)^2(3^3+7^3+0^3+0^3+0^3), \\ 111616 &= (1+1+1+6+1+6)^2(1^3+1^3+1^3+6^3+1^3+6^3), \\ 382360 &= (3+8+2+3+6+0)^2(3^3+8^3+2^3+3^3+6^3+0^3), \end{aligned}$$

(D) $$\begin{aligned} 182380 &= (1+8+2+3+8+0)(1^4+8^4+2^4+3^4+8^4+0^4), \\ 444992 &= (4+4+4+9+9+2)(4^4+4^4+4^4+9^4+9^4+2^4) \end{aligned}$$

and this is the complete list of positive integers larger than 1 *with this property.*

Finally, let us record one more theorem connecting 2023, Niven numbers, the sum of divisors function $\sigma(n)$, and Alladi-Erdős function $A(n)$.

Theorem 1.4. *Let n be a positive integer. Then*

$n = s_1(n)s_1(\sigma(n))^2$ *if and only if* $n = 1, 100, 2023, 7500, 23328$;

$n = s_1(n)s_2(A(n))^2$ *if and only if* $n = 48, 1521, 2023, 3600, 154652, 798950$.

In particular, a positive integer n satisfies the first equation with $s_1(n)$ being a prime if and only if $n = 2023$. The only positive integer n satisfying the second equation where $s_1(n)$ and $s_2(A(n))$ are prime is $n = 2023$.

We give some lemmas and prove Theorems 1.1 and 1.4 in the next section.

2. Lemmas and the proof of the main theorems

Throughout, for each $a, b \in \mathbb{Z}$, $a \geq 1$, $b \geq 0$, $c \geq 4$, let $f_{a,b}$, F_c, and $g_{a,b}$ be the functions defined by

$$f_{a,b}(n) = s_1(n)^a s_2(n)^b, \quad F_c(n) = s_1(n)s_c(n), \quad g_{a,b} = s_1(n)^a s_3(n)^b$$

for all $n \in \mathbb{N}$. In addition, let $h(n) = s_1(n)s_2(n)s_3(n)$ for all $n \in \mathbb{N}$. Then we have the following result.

Lemma 2.1. *For each $n \in \mathbb{N}$, we have*

(i) $f_{2,0}(n) \leq f_{1,1}(n) \leq f_{3,0}(n) \leq f_{2,1}(n) \leq f_{1,2}(n)$;
(ii) $g_{2,1}(n) \leq h(n) \leq g_{1,2}(n)$;
(iii) $g_{1,1}(n) \leq F_4(n) \leq F_5(n) \leq F_6(n)$.

Proof. This follows from the fact that $s_1(n) \leq s_2(n) \leq s_1(n)^2$, and $s_2(n) \leq s_3(n) \leq s_4(n) \leq s_5(n) \leq s_6(n)$ for all $n \in \mathbb{N}$. □

Lemma 2.2. *We have $9(k+1) < 10^k$ for all $k \geq 2$, $9^5(k+1)^3 < 10^k$ for all $k \geq 8$, $9^7(k+1)^2 < 10^k$ for all $k \geq 9$, and $9^7(k+1)^3 < 10^k$ for all $k \geq 10$.*

Proof. It is easy to verify that the second inequality holds when $k = 8$. If $k \geq 8$ and the second inequality holds for k, then

$$9^5(k+2)^3 = 9^5(k+1)^3\left(1 + \frac{1}{k+1}\right)^3 \leq 9^5(k+1)^3\left(\frac{10}{9}\right)^3 < 10^{k+1},$$

which proves the second inequality by induction. The others are similar. □

From this point, we apply Lemma 2.2 without further reference.

Lemma 2.3. *We have $s_1(n) < n$ for all $n \geq 10$.*

Proof. If $n \geq 100$ and $n = (a_k a_{k-1} \cdots a_0)_{10}$ where $k \geq 2$, then

$$s_1(n) = \sum_{i=1}^{k} a_i \leq 9(k+1) < 10^k \leq n.$$

It is easy to check that $s_1(n) < n$ for $10 \leq n \leq 99$. So the proof is complete. □

Lemma 2.4. *We have $f_{1,2}(n) < n$ for all $n \geq 2 \times 10^7$ and 2×10^7 is sharp in the sense that if $m = 2 \times 10^7 - 1$, then $f_{1,2}(m) > m$. In addition, $g_{1,2}(n) < n$ for all $n \geq 3 \times 10^9$, and if $m = 3 \times 10^9 - 1$, then $g_{1,2}(m) > m$. Furthermore, $F_6(n) < n$ for all $n \geq 3 \times 10^8$ and if $m = 3 \times 10^8 - 1$, then $F_6(m) > m$.*

Proof. Let $n \geq 10^8$. Then $n = (a_k a_{k-1} \cdots a_0)_{10}$ where $k \geq 8$, $a_k \neq 0$, and $0 \leq a_i \leq 9$ for all i. For convenience, we write f instead of $f_{1,2}$. Then we obtain

$$f(n) = \left(\sum_{i=0}^{k} a_i\right)\left(\sum_{i=0}^{k} a_i^2\right)^2 \leq 9(k+1)\left(9^2(k+1)\right)^2 < 10^k \leq n.$$

Therefore $f(n) < n$ for all $n \geq 10^8$. Next, if $30233088 < n \leq 10^8 - 1$, then

$$f(n) \leq s_1(10^8 - 1)s_2(10^8 - 1)^2 = 30233088 < n.$$

If $21192665 < n \leq 30233088$, then

$$f(n) \leq s_1(3 \times 10^7 - 1)s_2(3 \times 10^7 - 1)^2 = 21192665 < n.$$

Finally, if $2 \times 10^7 \leq n \leq 21192665$, then

$$f(n) \leq s_1(20999999)s_2(20999999)^2 = 13445600 < n.$$

Thus $f(n) < n$ for all $n \geq 2 \times 10^7$. It is easy to check that $f(m) > m$ if $m = 2 \times 10^7 - 1$. The inequalities for $g_{1,2}$ and F_6 can be proved similarly. So the proof is complete. $\square$

With these lemmas, we are now ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. It is straightforward to verify that the converse of each statement in (i) to (v) of Theorem 1.1 holds. By Lemma 2.3, it is easy to see that $s_1(n) = n$ if and only if $n \leq 9$, and so (i) is proved. Solving the equations $n = s_1(n)^2$ and $n = s_1(n)s_2(n)$ in (ii) are equivalent to solving $f_{2,0}(n) = n$ and $f_{1,1}(n) = n$. By Lemmas 2.1 and 2.4, the integers larger than or equal to 2×10^7 are not the solutions. The number of positive integers less than 2×10^7 is finite and not too large, and we can therefore use a computer to check whether or not an integer $n < 2 \times 10^7$ is a solution.

Similarly, $n = s_1(n)s_3(n)$ if and only if $g_{1,1}(n) = n$, and by Lemmas 2.1 and 2.4, we only need to search for the solutions in positive integers $n < 3 \times 10^8$. This leads to the solutions to each equation in (ii). Then it is easy to check that among these solutions, both $s_1(n)$ and $s_2(n)$ are prime numbers only when $n = 133, 803, 87949$. This proves (ii).

Similarly, by Lemmas 2.1 and 2.4, the proof of the other parts can be reduced to finding positive integers $n < M$ such that $f(n) = n$ where M is 2×10^7, 3×10^8, or 3×10^9 as appropriate and $f = f_{3,0}, f_{2,1}$, $f_{1,2}$, F_4, F_5, F_6, $g_{2,1}$, h, $g_{1,2}$, respectively. This completes the proof. $\square$

Theorem 1.4 can be proved by using the same idea as follows.

Proof of Theorem 1.4. Let $f(n) = s_1(n)s_1(\sigma(n))^2$ and $g(n) = s_1(n)s_2(A(n))^2$ for all $n \in \mathbb{N}$. We first observe that for each $n \in \mathbb{N}$, we have

$$\sigma(n) = \sum_{d|n} d \leq n \sum_{d|n} 1 \leq n^2. \tag{2.1}$$

If $n \geq 10^7$, then $n = (a_k a_{k-1} \cdots a_0)_{10}$ where $k \geq 7$, $a_k \neq 0$, and $0 \leq a_i \leq 9$ for every i, which implies $\sigma(n) \leq 10^{2k+2}$, and therefore $s_1(\sigma(n)) \leq 9(2k+2)$. Thus if $n \geq 10^7$, then

$$f(n) \leq 9(k+1) \cdot 9^2(2k+2)^2 < 10^k \leq n,$$

where the second inequality can be proved by induction. So the solutions to $f(n) = n$ can be found only in the range $n < 10^7$, which can be done by using a computer search. This proves the first statement.

Next, let $n > 1$ and write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and $a_1, a_2, \ldots, a_k$ are positive integers. We observe that if $a \geq 1$ and $q \geq 2$ are integers, then $q^{a-1} \geq 2^{a-1} \geq a$, and so $aq \leq q^a$. Therefore

$$A(n) = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k \leq p_1^{a_1} + p_2^{a_2} + \cdots + p_k^{a_k}. \tag{2.2}$$

We also observe that if $c_1, c_2 \geq 2$, then $(c_1 - 1)(c_2 - 1) \geq 1$, which implies $c_1 + c_2 \leq c_1 c_2$. So if $c_1, c_2, \ldots, c_k \geq 2$, then $c_1 + c_2 + c_3 + \cdots + c_k \leq c_1 c_2 c_3 \cdots c_k$. From this and (2.2), we see that $A(n) \leq n$. Now if $n \geq 10^8$, $n = (a_k a_{k-1} \cdots a_0)_{10}$, $k \geq 8$, $a_k \neq 0$, and $0 \leq a_i \leq 9$ for every i, then $s_2(A(n)) \leq 81(k+1)$, and so

$$g(n) \leq 9(k+1)(81(k+1))^2 < 10^k \leq n.$$

Since $g(n) < n$ for all $n \geq 10^8$, we can use a computer to search for the solution to $g(n) = n$ in the range $n < 10^8$. This proves the second statement. The rest follows immediately. This completes the proof. □

It is possible to solve the equation $n = f_{a,b}(n)$ and $n = g_{a,b}(n)$ when a, b are large, but the upper bound may be much larger than 3×10^9, which takes a longer computational time and requires a more powerful computer and a better skill in computer programming. More generally, if $a_1, a_2, \ldots, a_k$, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are fixed, it is possible to solve the equation

$$n = s_{\alpha_1}(n)^{a_1} s_{\alpha_2}(n)^{a_2} \cdots s_{\alpha_k}(n)^{a_k}.$$

We leave this problem to the interested reader.

Table 1 shows special Niven numbers of degree d at level ℓ. By the definition, when $d = 1$, we automatically have $\ell = 1$. So there are no result for $d = 1$ and $\ell \geq 2$. The case $d = 3$ and $\ell = 4, 5, 6$ is not included in Theorem 1.1, so we put $*$ in the table to mean that it is open. It seems that there may be an infinite number of Niven numbers of type 2 but the Niven numbers of type 3 are very rare.

d, ℓ	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$d = 1$	$2, 3, 5, 7$	-	-	-
$d = 2$	None	133 and 803	87949	None
$d = 3$	4913	2023	None	$*$

TABLE 1

Problem Determine whether or not there are infinitely many Niven numbers of type 2. Prove or disprove that the number of Niven numbers of type 3 is finite. If d and ℓ are very large, is there a special Niven number of degree d at level ℓ? Are there infinitely many n such that $n = s_1(n)^a s_\alpha(\sigma(n))^b$ or $n = s_1(n)^a s_\alpha(A(n))^b$ for some $a, b, \alpha \in \mathbb{N}$ distinct from those in Theorem 1.4?

ACKNOWLEDGEMENT

Prapanpong Pongsriiam's research project is supported jointly by the Faculty of Science Silpakorn University and the National Research Council of Thailand (NRCT), grant number NRCT5-RSA63021-02. He was also supported by the Tosio Kato Fellowship given by the Mathematical Society of Japan during his visit at Nagoya University in July 2022 to July 2023. This work was started when he was in Japan and finished when he came back to Thailand. He is thankful to Professor Kohji Matsumoto for being his host professor and finding a very convenient and comfortable accommodation and office for him during his visit at Nagoya University.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 149–159

SOLVING WIENER-HOPF EQUATION INVOLVING XOR-OPERATION

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Abstract. In this study, we explore a novel application of the Wiener-Hopf equation by incorporating the XOR-operation. Our findings reveal that the Wiener-Hopf equation involving XOR-operation is mathematically equivalent to a variational inequality problem. To solve this equation, we propose an iterative algorithm specifically tailored for XOR-based Wiener-Hopf equations. By utilizing the Bounded Inverse Theorem, we successfully obtain the solution. Furthermore, we discussed the convergence criteria associated with this approach.

1. Introduction

The Wiener-Hopf method is a well-established mathematical technique that finds extensive applications in applied mathematics. It is particularly useful for solving two-dimensional partial differential equations with mixed boundary conditions on a shared boundary. The introduction of Wiener-Hopf equations can be attributed to Robinson [21] and Shi [26, 27], who initially employed them in specific contexts. These equations are employed in various scenarios to analyze the properties of solutions for different classes of variational inequalities and their extensions.

Variational inequalities find applications in a wide range of fields, including fluid flow through porous media, moving boundary value problems, traffic assignment problems, optimization problems, economic equilibrium and more. For detailed information on variational inequalities and Wiener-Hopf equations, one can refer to works such as [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 22, 23, 24, 25] and the references provided therein.

In logic, there exists a connective called 'exclusive or' also known as exclusive disjunction. It evaluates to true when exactly one (but not both) of two conditions is true. The XOR operation often represented as $\oplus$ is used

2010 Mathematics Subject Classification: 49J40, 47H05
Key words and phrases: Inequality, equation, convergence, solution, algorithm

to express this connective. By applying XOR operation to the values of n and m, the binomial coefficient $\binom{m}{n}$ mod 2 can be efficiently calculated. This property simplifies the construction of Pascal's triangle mod 2. The XOR Cipher utilizes the XOR logical operation for data encryption. The data is encrypted by performing XOR operations with a specific key resulting in encrypted data. To decrypt the data, the same key is used and XOR operation is applied again. Notably, the XOR operation employs the same key for both encryption and decryption processes.

Motivated by the aforementioned considerations, this paper presents an introduction to a novel Wiener-Hopf equation that incorporates the XOR operation. Additionally, we discuss the corresponding variational inequality problem and establish an equivalence between the two. To solve the Wiener-Hopf equation involving XOR operation we propose an iterative algorithm. Finally, we demonstrate the existence and convergence of a solution by employing the Bounded Inverse Theorem.

2. Preliminaries

In this paper, we consider a real ordered Hilbert space denoted as H equipped with a norm $\|.\|$ and an inner product $\langle .,. \rangle$. We also assume the existence of a closed convex subset K and a cone C within H. To establish the main result of this paper, we draw upon several established concepts and results many of which can be found in references such as [4, 14, 15] by Du and Li.

Definition 2.1. Consider a cone C in the set H. For any elements l and m belonging to H, the inequality $l \leq m$ is true if and only if $l - m$ belongs to the cone C. This relationship denoted by "$\leq$" is referred to as a partial ordered relation.

Definition 2.2. Any two elements l and m belonging to the set H are considered comparable if either $l \leq m$ or $m \leq l$ holds (represented as $l \propto m$).

Definition 2.3. For any elements l and m belonging to the set H, we use the notations $\text{lub}\{l, m\}$ and $\text{glb}\{l, m\}$ to represent the least upper bound and the greatest lower bound of the set $\{l, m\}$, respectively. If both $\text{lub}\{l, m\}$ and $\text{glb}\{l, m\}$ exist, we can define certain binary operations as follows:

(i) $l \vee m = lub\{l, m\}$,
(ii) $l \wedge m = glb\{l, m\}$,
(iii) $l \oplus m = (l - m) \vee (m - l)$,
(iv) $l \odot m = (l - m) \wedge (m - l)$.

The symbols $\vee$, $\wedge$, $\oplus$, and $\odot$ refer to the OR, AND, XOR, and XNOR operations, respectively.

Proposition 2.4. *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR-operation. Then the following relations hold:*

(i) $l \odot l = 0,\ l \odot m = m \odot l = -(l \oplus m) = -(m \oplus l)$,
(ii) *if* $l \propto 0$, *then* $-l \oplus 0 \leq l \leq l \oplus 0$,
(iii) $(\lambda l) \oplus (\lambda m) = |\lambda|(l \oplus m)$,
(iv) $0 \leq l \oplus m$, *if* $l \propto m$,
(v) *if* $l \propto m$, *then* $l \oplus m = 0$ *if and only if* $l = m$,
(vi) $\|0 \oplus 0\| = \|0\| = 0$,
(vii) $\|l \vee m\| \leq \|l\| \vee \|m\| \leq \|l\| + \|m\|$,
(viii) $\|l \oplus m\| \leq \|l - m\|$,
(ix) *if* $l \propto m$, *then* $\|l \oplus m\| = \|l - m\|$.

Lemma 2.5. *Given $t \in H$, $s \in K$ satisfies the inequality*

$$\langle s - t, v - s \rangle \geq 0, \ \textit{for all } v \in K \tag{2.1}$$

if and only if

$$u = P_K(t)$$

where K is a nonempty closed convex set in H and P_K is the projection operator.

Theorem 2.6 (Bounded Inverse Theorem)**.** *A bijective, bounded, linear operator T from one Banach space to another has bounded inverse T^{-1}.*

3. Formulation of the problem and equivalence result

Consider the mappings $T, g : H \to H$, where g is bijective, bounded, linear, Lipschitz continuous and T is linear. We explore a novel Wiener-Hopf equation involving XOR-operation as:

Find $s, t \in H$ such that

$$\rho T g^{-1} P_K(t) \oplus (Q_K(t)) = 0, \tag{3.1}$$

Here, $t = g(s) - \rho T(s)$, $Q_K(t) = (P_K - I)(t)$, I denotes the identity operator, $\rho > 0$ is a constant and P_K represents the projection of H onto K.

We now introduce its equivalent variational inequality problem as:

Find $s \in H$ such that $g(s) \in K$ satisfying the inequality:

$$\langle T(s), g(v) - g(s)\rangle \geq 0, \text{ for all } g(v) \in K. \tag{3.2}$$

Both equations (3.1) and (3.2) are interconnected and the study of the Wiener-Hopf equation involving XOR-operation reveals its equivalence with the variational inequality problem (3.2). This connection offers new perspectives and opportunities for analyzing and solving complex mathematical challenges in various applications. The following Lemma ensures that the variational inequality problem (3.2) is equivalent to a fixed point equation.

Lemma 3.1. *The variational inequality problem* (3.2) *admits a solution* $s \in K$, $g(s) \in K$ *if and only if it satisfies the following equation*

$$g(s) = P_K[g(s) - \rho T(s)],$$

where $\rho > 0$ *is a constant.*

Proof. Proof is easy and hence omitted. □

Now in following Lemma we show that Wiener-Hopf equation inolving XOR-operation (3.1) and variational inequality problem (3.2) are equivalent.

Lemma 3.2. *The Wiener-Hopf equation involving XOR-operation* (3.1) *admits a solution* $s, t \in H$ *if and only if* $s \in H$, $g(s) \in K$ *satisfies the variational inequality problem* (3.2)*, given that* $\rho T g^{-1} P_K(t) \propto Q_K(t)$*, where* C *denotes a cone in* H*,*

$$g(s) = P_K(t),\ t = g(s) - \rho T(s),\ Q_K(t) = (P_K - I)(t) \tag{3.3}$$

and $\rho > 0$ *is a constant.*

Proof. Let $s, t \in H$ be a solution of Wiener-Hopf equation inolving XOR-operation (3.1). Then for some $\rho > 0$, we have

$$\rho T g^{-1} P_K(t) \oplus Q_K(t) = 0.$$

Since $\rho T g^{-1} P_K(t) \propto Q_K(t)$, by (v) of Proposition 2.4, we have

$$\begin{aligned} \rho T g^{-1} P_K(t) &= Q_K(t) \\ &= P_K(t) - t. \end{aligned} \tag{3.4}$$

By using Lemma 2.5, Lemma 3.1, (3.4) and the fact that $t = g(s) - \rho T(s)$, we have

$$\begin{aligned} \langle g(s) - t, g(v) - g(s) \rangle &\geq 0, \\ \langle P_K(t) - t, g(v) - g(s) \rangle &\geq 0, \\ \langle \rho T g^{-1} P_K(t), g(v) - g(s) \rangle &\geq 0, \\ \rho \langle T g^{-1} P_K(t), g(v) - g(s) \rangle &\geq 0, \end{aligned}$$

which gives us

$$\langle T(s), g(v) - g(s) \rangle \geq 0,$$

the required variational inequality problem (3.2). Conversely, suppose that the variational inequality problem (3.2) holds, that is $s \in H,\ g(s) \in K$ such that

$$\langle T(s), g(v) - g(s) \rangle \geq 0,\ \ for\ all\ g(v) \in K.$$

Using Lemma 3.1, we have

$$g(s) = P_K[g(s) - \rho T(s)] = P_K(t), \tag{3.5}$$

where $t = g(s) - \rho T(s)$.

Using (3.5), we have

$$\begin{aligned} t &= g(s) - \rho T(s) \\ &= P_K(t) - \rho T g^{-1} P_K(t), \end{aligned}$$

so,

$$\begin{aligned} \rho T g^{-1} P_K(t) &= P_K(t) - t \\ &= Q_K(t). \end{aligned}$$

Thus, we have

$$\begin{aligned} \rho T g^{-1} P_K(t) \oplus Q_K(t) &= Q_K(t) \oplus Q_K(t) = 0, \\ \text{that is, } \rho T g^{-1} P_K(t) \oplus Q_K(t) &= 0, \end{aligned}$$

the required Wiener-Hopf equation involving XOR-operation 3.1. □

4. Algorithm, Existence and Convergence

In this section, we present an original iterative algorithm and establish both the existence and convergence results for the Wiener-Hopf equation involving XOR-operation (3.1). By simple manipulation equation (3.1) can be rewritten as:

$$\begin{aligned}
\rho T g^{-1} P_K(t) \oplus (P_K(t) - t) &= 0, \text{ where } t = g(s) - \rho T(s) \\
\text{or } \rho T g^{-1} P_K(t) &= P_K(t) - (g(s) - \rho T(s)) \\
\rho T g^{-1} P_K(t) &= P_K(t) + \rho T(s) - g(s) \\
\rho T g^{-1} P_K(t) \oplus (P_K(t) + \rho T(s)) &= (P_K(t) + \rho T(s)) \oplus (P_K(t) + \rho T(s)) - g(s) \\
\rho T g^{-1} P_K(t) \oplus (P_K(t) + \rho T(s)) &= -g(s),
\end{aligned}$$

thus, we have

$$g(s) = \rho T g^{-1} P_K(t) \odot (P_K(t) + \rho T(s)), \text{ as } l \odot m = -(l \oplus m). \tag{4.1}$$

This fixed point formulation 4.1 enables us to develop the following iterative algorithm.

Algorithm 4.1. For $s_0, t_0 \in H$, compute the sequences $\{s_n\}$ and $\{t_n\}$ by the following scheme:

$$g(s_{n+1}) = \rho T g^{-1} P_K(t_n) \odot (P_K(t_n) + \rho T(s_n)). \tag{4.2}$$

$$\text{where } t_n = g(s_n) - \rho T(s_n),\ n = 0, 1, 2, \ldots . \tag{4.3}$$

Theorem 4.1. *Let H be a real ordered Hilbert space, K be a closed convex subset of H and C be a cone in H with partial ordering " $\leq$ ". Let $T, g : H \to H$ be the single-valued mappings such that g is bijective, bounded, linear, Lipschitz continuous with constant λ_g, strongly monotone with constant δ_g and T is linear, Lipschitz continuous with constant λ_T. Suppose that $g(s_{n+1}) \propto g(s_n)$, $\rho T g^{-1} P_K(t) \propto Q(t)$, where $Q_K(t) = (P_K - I)(t)$, $t = g(s) - \rho T(s)$, $g(s) \in K$, n= 0,1,2,... .*
If the following condition is satisfied

$$|\lambda_g + \rho\lambda_T| < \sqrt{\rho^2\lambda_T^2 + \Big\{\frac{(\xi_g - \rho\lambda_T)^2}{(1 + \rho\lambda_T\delta_g)^2} - \rho^2\lambda_T^2\Big\}}. \tag{4.4}$$

Then, there exist $s, t \in H$ the solution of Wiener-Hopf equation involving XOR-operation (3.1) and the sequences $\{s_n\}$ and $\{t_n\}$ generated by Algorithm 4.1 converge strongly to s and t, respectively.

Proof. Using 4.2 of Algorithm 4.1, (i) and (iii) of proposition 2.4, we have

$$\begin{aligned} 0 \le g(s_{n+1}) \oplus g(s_n) &= [\rho T g^{-1} P_K(t_n) \odot (P_K(t_n) + \rho T(s_n)] \\ &\quad \oplus [\rho T g^{-1} P_K(t_{n-1}) \odot (P_K(t_{n-1}) + \rho T(s_{n-1})] \\ &= [-(\rho T g^{-1} P_K(t_n) \oplus (P_K(t_n) + \rho T(s_n))] \\ &\quad \oplus [-(\rho T g^{-1} P_K(t_{n-1}) \oplus (P_K(t_{n-1}) + \rho T(s_{n-1}))] \\ &= |-1| [\rho T g^{-1} P_K(t_n) \oplus (P_K(t_n) + \rho T(s_n)] \\ &\quad \oplus [\rho T g^{-1} P_K(t_{n-1}) \oplus (P_K(t_{n-1}) + \rho T(s_{n-1})]. \end{aligned} \tag{4.5}$$

From 4.5, $(viii)$ of proposition 2.4, we have

$$\begin{aligned} \|g(s_{n+1}) \oplus g(s_n)\| &= \|[\rho T g^{-1} P_K(t_n) \oplus (P_K(t_n) + \rho T(s_n)] \\ &\quad \oplus [\rho T g^{-1} P_K(t_{n-1}) \oplus (P_K(t_{n-1}) + \rho T(s_{n-1})]\| \\ &= \|[\rho T g^{-1} P_K(t_n) \oplus \rho T g^{-1} P_K(t_{n-1})] \\ &\quad \oplus [(P_K(t_n) + \rho T(s_n)) \oplus (P_K(t_{n-1}) + \rho T(s_{n-1}))]\| \\ &\le \|[\rho T g^{-1} P_K(t_n) - \rho T g^{-1} P_K(t_{n-1})]\| \\ &\quad + \|[(P_K(t_n) + \rho T(s_n)) - (P_K(t_{n-1}) + \rho T(s_{n-1}))]\| \\ &= \rho \|T g^{-1} P_K(t_n) - T g^{-1} P_K(t_{n-1})]\| \\ &\quad + \|[(P_K(t_n) + \rho T(s_n)) - (P_K(t_{n-1}) + \rho T(s_{n-1}))]\| \\ &= \rho \|T g^{-1} P_K(t_n) - T g^{-1} P_K(t_{n-1})]\| \\ &\quad + \|(P_K(t_n) - P_K(t_{n-1})) + \rho(T(s_n) - T(s_{n-1})\|. \end{aligned} \tag{4.6}$$

Using λ_T-Lipschitz continuity of T, non-expansiveness of the projection operator and Cauchy-Bunyakovsky-Schwarz inequality, we have
$\|(P_K(t_n) - P_K(t_{n-1})) + \rho(T(s_n) - T(s_{n-1}))\|^2$

$$\begin{aligned} &= \|P_K(t_n) - P_K(t_{n-1})\|^2 + 2\langle \rho(T(s_n) - T(s_{n-1})), P_K(t_n) - P_K(t_{n-1})\rangle \\ &\quad + \rho^2 \|T(s_n) - T(s_{n-1})\|^2 \\ &\le \|t_n - t_{n-1}\|^2 + 2\rho\lambda_T \|s_n - s_{n-1}\| \|t_n - t_{n-1}\| + \rho^2 \lambda_T^2 \|s_n - s_{n-1}\|^2 \\ &= (\|t_n - t_{n-1}\| + \rho\lambda_T \|s_n - s_{n-1}\|)^2, \end{aligned}$$

thus, we have

$$\|(P_K(t_n) - P_K(t_{n-1})) + \rho(T(s_n) - T(s_{n-1}))\| \le \|t_n - t_{n-1}\| + \rho\lambda_T \|s_n - s_{n-1}\|. \tag{4.7}$$

Since T is λ_T-Lipschitz continuous, the projection operator is non-expansive and using Bounded Inverse Theorem 2.6 for g, we have

$$\begin{aligned} \|Tg^{-1}(P_K(t_n)) - Tg^{-1}(P_K(t_{n-1}))\| &\leq \lambda_T\|g^{-1}(P_K(t_n) - P_K(t_{n-1})\| \\ &\leq \lambda_T\|g^{-1}\|\|P_K(t_n) - P_K(t_{n-1})\| \\ &\leq \lambda_T\delta_g\|t_n - t_{n-1}\|. \qquad (4.8) \end{aligned}$$

Combining (4.7) and (4.8) with (4.6), we have

$$\begin{aligned} \|g(s_{n+1}) \oplus g(s_n)\| &\leq \rho\lambda_T\delta_g\|t_n - t_{n-1}\| + \|t_n - t_{n-1}\| + \rho\lambda_T\|s_n - s_{n-1}\| \\ &= (1 + \rho\lambda_T\delta_g)\|t_n - t_{n-1}\| + \rho\lambda_T\|s_n - s_{n-1}\|. \qquad (4.9) \end{aligned}$$

Using (4.3) of Algorithm 4.1, we have

$$\begin{aligned} \|t_n - t_{n-1}\| &= \|[g(s_n) - \rho T(s_n)] - [g(s_{n-1}) - \rho T(s_{n-1})]\| \\ &= \|g(s_n) - g(s_{n-1}) - \rho(T(s_n) - T(s_{n-1}))\|. \qquad (4.10) \end{aligned}$$

Since g is λ_g-Lipschitz and T is λ_T-Lipschitz, we have

$$\begin{aligned} &\|g(s_n) - g(s_{n-1}) - \rho(T(s_n) - T(s_{n-1}))\|^2 \\ &= \|g(s_n) - g(s_{n-1})\|^2 + 2\rho\langle g(s_n) - g(s_{n-1}), \\ &\quad T(s_n) - T(s_{n-1})\rangle + \rho^2\|T(s_n) - T(s_{n-1})\|^2 \\ &\leq \lambda_g^2\|s_n - s_{n-1}\|^2 + 2\rho\|g(s_n) - g(s_{n-1})\| \\ &\quad \|T(s_n) - T(s_{n-1})\| + \|T(s_n) - T(s_{n-1})\|^2 \\ &\leq \lambda_g^2\|s_n - s_{n-1}\|^2 + 2\rho\lambda_g\lambda_T\|s_n - s_{n-1}\|^2 \\ &\quad +\rho^2\lambda_T^2\|s_n - s_{n-1}\| \\ &= (\lambda_g^2 + 2\rho\lambda_g\lambda_T + \rho^2\lambda_T^2)\|s_n - s_{n-1}\|^2. \end{aligned}$$

That is

$$\|g(s_n) - g(s_{n-1}) - \rho(T(s_n) - T(s_{n-1}))\| \leq P(\theta)\|s_n - s_{n-1}\|, \qquad (4.11)$$

where $P(\theta) = \sqrt{\lambda_g^2 + 2\rho\lambda_g\lambda_T + \rho^2\lambda_T^2}$.
Combining (4.10) with (4.11), we have

$$\|t_n - t_{n-1}\| \leq P(\theta)\|s_n - s_{n-1}\|. \qquad (4.12)$$

Using (4.12), (4.9) becomes

$$\|g(s_{n+1}) \oplus g(s_n)\| \leq (1 + \rho\lambda_T\delta_g)P(\theta)\|s_n - s_{n-1}\| + \rho\lambda_T\|s_n - s_{n-1}\|. \quad (4.13)$$

Since $g(s_{n+1}) \propto g(s_n)$, from (4.13), we have

$$\|g(s_{n+1})-g(s_n)\| \leq (1+\rho\lambda_T\delta_g)P(\theta)\|s_n-s_{n-1}\|+\rho\lambda_T\|s_n-s_{n-1}\|. \quad (4.14)$$

where $P(\theta)$ is defined in (4.11).

Since g is strongly monotone with constant ξ_g, we have

$$\begin{aligned} \|g(s_{n+1})-g(s_n)\| &\geq \xi_g\|s_{n+1}-s_n\| \\ \text{that is } \|s_{n+1}-s_n\| &\leq \frac{1}{\xi_g}\|g(s_{n+1})-g(s_n)\|. \end{aligned} \quad (4.15)$$

Using (4.15), (4.14) becomes

$$\begin{aligned} \|s_{n+1}-s_n\| &\leq \frac{1}{\xi_g}[(1+\rho\lambda_T\delta_g)P(\theta)+\rho\lambda_T]\|s_n-s_{n-1}\|, \\ \|s_{n+1}-s_n\| &\leq \xi(\theta)\|s_n-s_{n-1}\|, \end{aligned} \quad (4.16)$$

where $\xi(\theta)=\frac{1}{\xi_g}[(1+\rho\lambda_T\delta_g)P(\theta)+\delta\lambda_T]$ and $P(\theta)=\sqrt{\lambda_g^2+2\rho\lambda_g\lambda_T+\rho^2\lambda_T^2}$. By (4.4), we have $\xi(\theta)<1$, thus from (4.16) it follows that $\{s_n\}$ is a Cauchy sequence. Since H is complete, we can suppose that $s_n \to s \in H$. From (4.12), it follows that $\{t_n\}$ is also a Cauchy sequence, thus there exist $t \in H$ such that $t_n \to t$. Thus by Lemma 3.1 and Lemma 3.2, we conclude that $s, t \in H$ is a solution of Wiener-Hopf equation involving XOR-operation (3.1). □

Conclusion

In conclusion, this study introduces a novel application of the Wiener-Hopf equation by incorporating the XOR-operation. The research demonstrates that the Wiener-Hopf equation involving XOR-operation is mathematically equivalent to a variational inequality problem. To effectively address this equation, a dedicated iterative algorithm is proposed, specifically designed for XOR-based Wiener-Hopf equations. The application of the Bounded Inverse Theorem enables the successful derivation of the solution.

Furthermore, the study extensively discusses the convergence criteria associated with this tailored approach. These findings not only advance the understanding of the Wiener-Hopf method in XOR-based scenarios but also establish a powerful framework for solving intricate mathematical challenges in diverse applications. The proposed iterative algorithm and convergence analysis contribute significantly to the potential application of the Wiener-Hopf equation in various domains, promising new avenues for future research and practical implementations.

Acknowledgement: The authors are grateful to the anonymous refree for his favourable evaluation and suggestions for improvements.

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The Mathematics Student ISSN: 0025 - 5742
Vol. 93, Nos. 1-2, January - June (2024), 160 - 172

WHAT IS THE VOLUME OF THE CHAMBERED NAUTILUS SHELL ?

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(Received: 25 - 08 - 2023; Revised 25 - 02 - 2024)

Abstract. Elementary calculus and geometry are used to compute an approximation of the volume of the chambered nautilus shell.

1. Introduction

The chambered nautilus (CN) (and the nautilus in general) is a member of the marine mollusc family Nautilidae and is found in the Indian and Pacific oceans. The CN has a hard shell. As the CN matures it periodically seals off the previous shell component and creates a new, continuing, and larger living chamber. The growth process yields a spiral structure which is consistently characterized in relation to a logarithmic or equiangular spiral.

Descriptions of the nautilus in general and of the CN and its shell can be found in many articles which are available on-line. See articles on-line using the computer with phrases such as "nautilus", "chambered nautilus", "nautilus shell", "mathematics of the nautilus shell", "geometry of the nautilus shell", "logarithmic spiral", and "equiangular spiral". Sketches and discussion of the CN shell and of logarithmic spirals are given in many articles. In particular [1], [2], and [3] of this paper contain relevant sketches and discussion of the CN shell in relation to the logarithmic spiral. We further discuss the logarithmic spiral concept and state a formula in polar coordinates with corresponding discussion for logarithmic spiral below in this paper.

Recently a high school teacher of one-variable calculus inquired of this author if a value of the volume of the CN shell could be determined mathematically. The teacher was using this question and its possible solution as a project for the students to consider in the calculus class. Because of the shape of the CN shell, this inquiry possed an interesting problem to

2020 Mathematics Subject Classification: 26A06, 51M04, 00A05, 00A08
Key words and phrases: logarithmic spiral, nautilus shell, volume, cone

this author. The purpose of this paper is to obtain a technique to mathematically compute a good approximation to the volume of a CN shell using one-variable calculus and intuitive geometry. The author has related the contents of this paper to the above noted inquiring high school teacher.

2. Volume of a CN Shell

A logarithmic spiral in the Euclidean plane is defined by the rule that for a given rotation angle, such as one revolution or 2π radians, the distance from the pole (origin or spiral origin) to a point on the spiral is multiplied by a fixed real constant. When this fixed multiplication amount is the "golden ratio" of $(1+5^{1/2})/2$ a particular type of logarithmic spiral is formed. See [3] and references in [3] for a discussion of the questioned relationship between the shape of the CN shell and the logarithmic spiral based on the "golden ratio".

As described in analysis [2, p. 733] the general formula for the logarithmic spiral in the plane in polar coordinates (r, θ) is $r = \alpha e^{k\theta}, k \neq 0$. To be specific with respect to the CN shell we take the polar equation for the associated logarithmic spiral to be

$$r = \alpha e^{cot(\beta)\theta} \tag{1}$$

as in reference [1]. Here $\alpha > 0$ and $\beta,\ 0 < \beta < \pi/2$, are arbitrary but fixed real constants. As α and β appropriately vary as fixed constants, different logarithmic spirals are sketched in the plane. Since we take $\beta,\ 0 < \beta < \pi/2$, then the cotangent of β is greater than 0, $cot(\beta) > 0$, and the logarithmic spiral is outward or a counter clockwise spiral. As the constant $\alpha > 0$ increases the associated spiral expands. For a fixed constant $\alpha > 0$ the fixed constant $\beta,\ 0 < \beta < \pi/2$, is the angle between the radial line outward from the origin of the spiral (center of the plane) and the tangent lines to each associated spiral (for all points on the spirals corresponding to the fixed $\alpha > 0$) with the view of the angle from above the spiral. (We emphasize that the angle β will be determined by the contour of the nautilus shell. In a given logarithmic spiral the angle β is constant at all points on the spiral; see [2, problem 37, p. 784] and [1]. For a fixed value of α, different values of β can occur for different logarithmic spirals. See the sketches in [1].) See reference [1] for various sketches of logarithmic spirals associated with given specific values of α and β. See also sketches in many articles noted previously that can be obtained on-line.

Before beginning the description of our process to calculate the volume of the CN shell we suggest that the reader note the illustrations given at the end of this paper after the references; these illustrations will be helpful in the visualization of the descriptions of our process. In the discussion that follows in this paper Illustration I refers to the second sheet of illustrations after the references; Illustration II refers to the third sheet of illustrations; Illustration III refers to the fourth sheet of illustrations. The first sheet of illustrations presents a summary of Illustrations I, II, and III. Also we suggest that the reader note the picture of the nautilus shell in reference [3] if possible. This picture clearly displays the spiral form of the nautilus shell, the form of the constructed chambers, and the relationship of the boundary of the shell in relation to the spirals.

Using the model of equation (1) for logarithmic spiral we now obtain an approximation of the volume of the CN shell. To begin our process to calculate the volume first we visualize a line (curved line or arc) starting at the spiral origin of the CN shell and proceeding through the spirals at the center of the spirals until reaching the end of the chambers (Illustration I). Another manner to visualize this curved line (or arc) is to cut the shell in half thus displaying the logarithmic spiral nature of the CN shell. The curved line (or arc) previously described is equivalently obtained by again starting at the spiral origin of one-half of the CN shell and following the spiral of the shell through the center of the spiral structure until reaching the end of the last one-half chamber. This curved line (or arc) is then a logarithmic spiral against the plane face of the cutaway of the one-half of the CN shell. Let (r, θ) be the polar coordinates of this plane face of the cutaway of the shell. Thus for the described curved line (or arc), which is a logarithmic spiral, there is a fixed real number $\alpha > 0$ and a fixed real number β, $0 < \beta < \pi/2$, such that the curved line (or arc) is given in (r, θ) form on the plane face as equation (1). A given CN shell could have numerous spirals of 2π radians each before the end of the shell is reached. We call the positive integer n to be the number of spirals of 2π radians each of the CN shell; thus the θ variable in the equation (1) will vary over $0 \leq \theta \leq 2\pi n$. The opening at the end of the CN shell is approximately circular in nature; we approximate this ending as a circle of radius fixed

$R > 0$ (Illustration I). This circular opening at the end of the CN shell is where the nautilus emerges.

The question now is how to put all of this information together concerning the logarithmic spiral $r = \alpha e^{cot(\beta)\theta},\ 0 \leq \theta \leq 2\pi n$, with $\alpha > 0$ and $\beta,\ 0 < \beta < \pi/2$, both fixed and with the positive integer n fixed and with the fixed radius $R > 0$ at the end of the spirals in order to compute the volume of the CN shell. Envision the boundary of the winding chambers as they are constructed on top of the previous chamber as a type of clay or putty. By the construction of the CN shell in the form of a spiral, if the chambers are unwound (Illustration II) and the object is placed upright with the circular opening of the shell at the bottom and the spiral origin at the top, the resulting object is approximately a right circular cone (Illustration III). We know that the volume V of such a cone is $V = (\pi/3)q^2h$ where q is the radius of the base and h is the height of the cone.

Before proceeding with the visualization of the resulting right circular cone in the preceding paragraph let us further describe the process to "unwind" the chambers in order to obtain a right circular cone as stated in the preceding paragraph. To visualize this process let us "unwind" the logarithmic spiral that defines the shape of the CN shell and that is a line (curved line or arc) starting at the spinal origin of the CN shell and proceeding through the chambers at the center of the chambers until reaching the end of the chambers. This line (curved line or arc) is a logarithmic spiral which we assume as above to be an outward spiral or a counter clockwise spiral. Assume now that we have such a spiral which is constructed from flexible and bendable wire. Place the wire on a plane surface and fix the end point of the wire that coincides with the spinal origin of the curved line. Now grasp the other end of the wire and turn the end in the clockwise direction on the plane surface until all of the spirals are "unwound" (Illustration II); the result will be a straight line wire with one end at the spinal origin and the other end coinciding with the tip end of the spiral. This is the exact process to "unwind" the chambers of the CN shell as described in the preceding paragraph. (The spinal origin of the CN shell corresponds to the spinal origin of the curved line. The opening end of the CN shell corresponds to the other end of the wire that is to be grasped.) When the CN shell is "unwound" by clockwise rotation parallel to a plane the resulting object will be approximately a right circular cone with the spinal origin as the tip of the cone and the circular end of the CN shell as the base of the cone (Illustration III).

Let us now proceed in our analysis of the relationship between the CN shell and the right circular cone. In the present case the radius of the base of our constructed approximate cone from the CN shell is the above stated fixed $R > 0$, the radius of the opening at the end of the CN spirals as noted above (Illustrations I and III). The height of our constructed approximate cone from the CN shell is the arc length of the above described logarithmic spiral curved line of form (1) for the fixed $\alpha > 0$ and fixed $\beta,\ 0 < \beta < \pi/2$, with $0 \leq \theta \leq 2\pi n$ for a fixed positive integer n. That is, in transforming the CN shell to the approximate right circular cone as described, the described logarithmic spiral becomes the height line of the cone (Illustrations I and III). This height line has length equal to the arc length of the logarithmic spiral following the middle of the chambers from the spiral origin to the end of the chambers of the CN shell as described above (Illustrations I and III). (The spirals of the boundary of a CN shell will not be smooth like that of the boundary of a right circular cone. Thus in transforming the CN shell to the cone by the unwinding process as described above we call the resulting object an approximate cone. The resulting object will be in the shape of a right circular cone, and the volume of the resulting object will be a close approximation to the volume of the CN shell. Thus we use the formula for the volume of a right circular cone below to approximate the volume of the resulting cone like object and hence of the CN shell.) h The arc length of the above described logarithmic spiral of form (1) can be computed by calculus using, for example, the integral formula in [2, p. 740] or [4, p. 697] for arc length in polar coordinates. Thus the height h of our constructed approximate cone from the CN shell can be computed as follows and is the arc length of the described logarithmic spiral $r = \alpha e^{cot(\beta)\theta},\ 0 \leq \theta \leq 2\pi n$, where $\alpha > 0$ and $\beta,\ 0 < \beta < \pi/2$, and the positive integer n are fixed for the specific CN shell. We have from [2, p. 740] or [4, p. 697]

$$h = \text{arc length} = \int_0^{2\pi n} (r^2 + (\frac{dr}{d\theta})^2)^{1/2} d\theta \tag{2}$$
$$= \int_0^{2\pi n} (\alpha^2 e^{2cot(\beta)\theta} + \alpha^2 (cot(\beta))^2 e^{2cot(\beta)\theta})^{1/2} d\theta$$
$$= \int_0^{2\pi n} (\alpha^2 e^{2cot(\beta)\theta}(1 + (cot(\beta))^2))^{1/2} d\theta$$
$$= \alpha(1 + (cot(\beta))^2)^{1/2} \int_0^{2\pi n} e^{cot(\beta)\theta} d\theta$$
$$= \frac{\alpha(1 + (cot(\beta))^2)^{1/2}}{cot(\beta)}(e^{2\pi n cot(\beta)} - 1).$$

Thus the volume V of a given CN shell having logarithmic spiral form $r = \alpha e^{cot(\beta)\theta}$ for fixed $\alpha > 0$ and $\beta,\ 0 < \beta < \pi/2$; having positive integer n spirals of 2π radians each; and having an approximate circle of radius $R > 0$ at the end (or base) of the spirals is thus approximately

$$V = \frac{\pi R^2 \alpha(1 + (cot(\beta))^2)^{1/2}}{3cot(\beta)}(e^{2\pi n cot(\beta)} - 1) \tag{3}$$

cubic units.

Since $0 < \beta < \pi/2$ and $n = 1, 2, 3, 4, ...$ in (3), the formula for V yields a positive number; and both intuitively and through the formula (3), the volume V increases in value as n increases.

As stated in section 2 we chose β in (1) to be such that $0 < \beta < \pi/2$ thus making $cot(\beta) > 0$ in (1) and yielding the logarithmic spiral to be outward or a counter clockwise spiral. We then proceeded to obtain the volume formula in (3) for the CN shell. Mathematically we can obtain a value as in (3) from the logarithmic spiral equation (1) for $\pi/2 < \beta < \pi$ and hence $cot(\beta) < 0$; in this case the logarithmic spiral (1) is inward or a clockwise spiral. Now assuming $\pi/2 < \beta < \pi$ so that $cot(\beta) < 0$ we start at (1) and proceed with the same procedure and analysis to compute the corresponding values as in (2) and (3). In this case corresponding to (2) we obtain

$$h = \text{arc length} = \frac{\alpha(1 + (cot(\beta))^2)^{1/2}}{-cot(\beta)}(1 - e^{2\pi n cot(\beta)}); \tag{4}$$

and corresponding to (3) we have

$$V = \frac{\pi R^2 \alpha (1 + (cot(\beta))^2)^{1/2}}{-3cot(\beta)} (1 - e^{2\pi n cot(\beta)}) \tag{5}$$

cubic units.

For the case $\pi/2 < \beta < \pi$ and thus $cot(\beta) < 0$ notice that

$$\lim_{n\to\infty} h = \frac{\alpha(1 + (cot(\beta))^2)^{1/2}}{-cot(\beta)} \tag{6}$$

and

$$\lim_{n\to\infty} V = \frac{\pi R^2 \alpha (1 + (cot(\beta))^2)^{1/2}}{-3cot(\beta)}. \tag{7}$$

These limits yield mathematical upper bounds on the height (arc length) h and the volume V for the case $\pi/2 < \beta < \pi$ and $cot(\beta) < 0$. (In the case that $\pi/2 < \beta < \pi$ and hence $cot(\beta) < 0$, the CN shell and its associated logarithmic spiral is inward or a clockwise spiral. In this case the unwinding process described in Section 2 would be done in the counter clockwise direction to transform the CN shell into a right circular cone.)

3. Computation Examples

We consider the three logarithmic spirals in [1, p. 22] in which three sets of values for α and β are stated with an accompanying sketch for each of the α, β pairs. We recommend observing these sketches in [1, p. 22] as we have the following discussion. For each of these three α, β pairs the logarithmic spiral is of the type leading to formula (3) for the volume of the associated CN shell; each of these spirals defines the shape of the associated CN shell. In each of these cases the stated β value, which is stated in radians, satisfies $0 < \beta < \pi/2$ in radians; hence the sprial rotation is counter clockwise in each case. For each of these three α, β pairs we state the number n of rotations of 2π radians that we consider and a value of the radius R of the CN shell opening at the bottom of the corresponding right circular cone and use equation (3) to compute the approximating volume of the associated CN shell for each of the three pairs of α, β. We leave the computational details to compute each of the volumes to the reader if desired; we have stated the three approximate volumes that we have obtained below.

For $\alpha = 1.2,\ \beta = 1.4$ radians, $n = 4$, and $R = 7$ centimeters (cm) we have $V = 29403.776\ \text{cm}^3$.

For $\alpha = 0.8,\ \beta = 1.4$ radians, $n = 4$, and $R = 6$ cm we have $V = 14401.692$ cm^3.

For $\alpha = 0.8,\ \beta = 1.45$ radians, $n = 4$, and $R = 5$ cm we have $V = 3606.460$ cm^3.

Comparing the first two spirals we have $\beta = 1.4$ radians is the same for both , but $\alpha = 1.2$ is larger than $\alpha = 0.8$ for the second spiral. For the same β, as α increases the associated spiral expands as we have mentioned before; hence the volume in the associated CN shell expands as α increases for the same β. Because of this the radius at the end of the CN shell gets larger as α increases for the same number n of 2π radian revolutions. This is the reason for choosing $R = 7$ cm for the first spiral and $R = 6$ cm for the second spiral. The volume for the associated first CN shell is larger than that for the second associated CN shell for the same $n = 4$ revolutions of 2π radians each. (For each of the three sets of values for $\alpha,\ \beta,\ n$, and R the corresponding spiral is the mid line for the corresponding CN shell as described before.)

In comparing the second and third spirals, $\alpha = 0.8$ in each whereas $\beta = 1.4$ radians in the second spiral while $\beta = 1.45$ radians in the third spiral. When β increases for the same α value the spirals are compressed (or flattened) and the amount of volume in the associated CN shell is decreased. Seen otherwise, as the angle β decreases for the same α the tangent lines to the spiral have increasing positive slope which creates larger volume in the CN shell. For this reason $R = 5$ cm was chosen corresponding to $\beta = 1.45$ radians while $R = 6$ cm was chosen corresponding to $\beta = 1.4$ radians for the same number $n = 4$ revolutions of 2π radians each. The volume of the CN shell corresponding to the second spiral is larger than that corresponding to the third spiral for the same $n = 4$ revolutions of 2π radians each.

In comparing the first spiral with the third spiral, the first spiral has the advantage for larger volume of the CN shell in both the value of α and the value of β as seen in the commentary of the previous two paragraphs. Thus the volume of the associated CN shell for the first spiral would naturally be much larger than that for the third spiral for the same number of 2π radian revolutions in each.

Dedication: This paper is dedicated in remembrance of the author's research colleague Dr. Ram Shankar Pathak, Banaras Hindu University, Varanasi, India.

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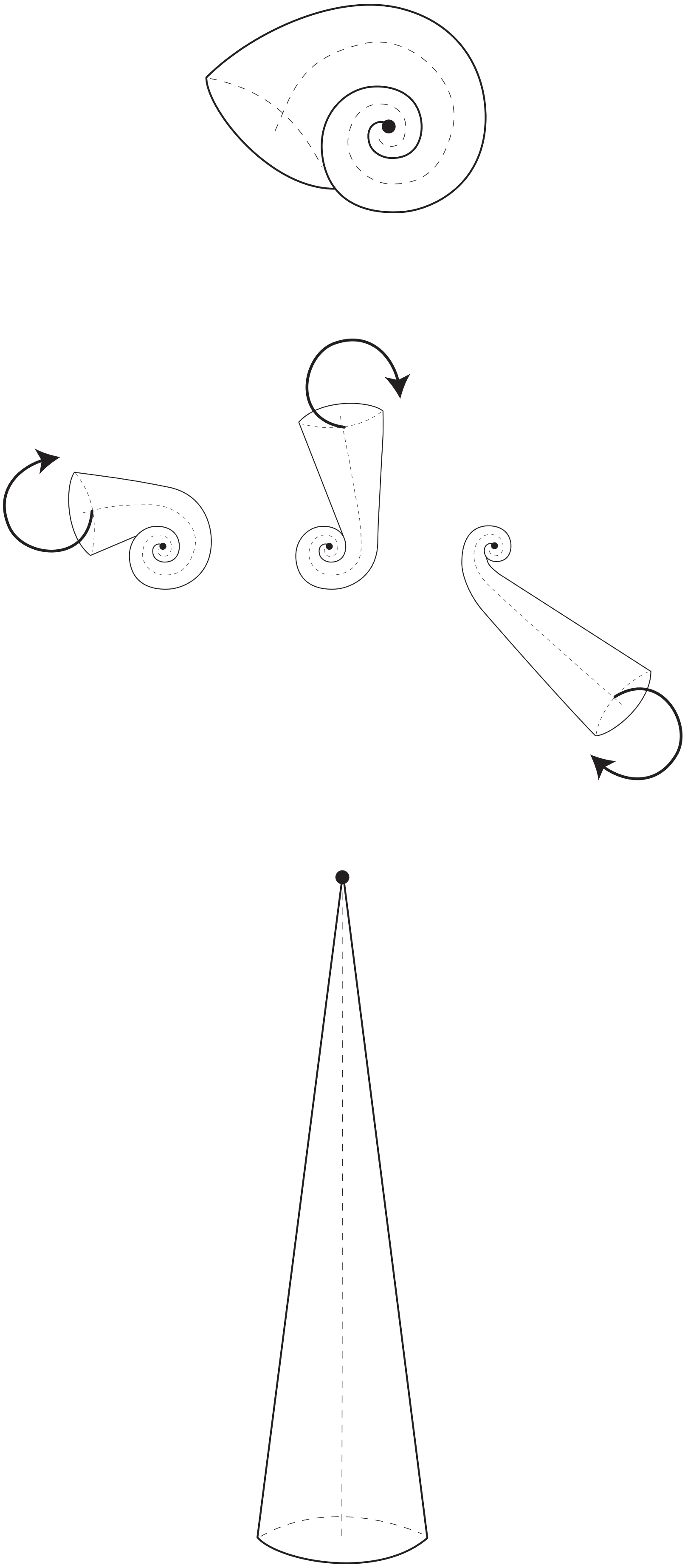

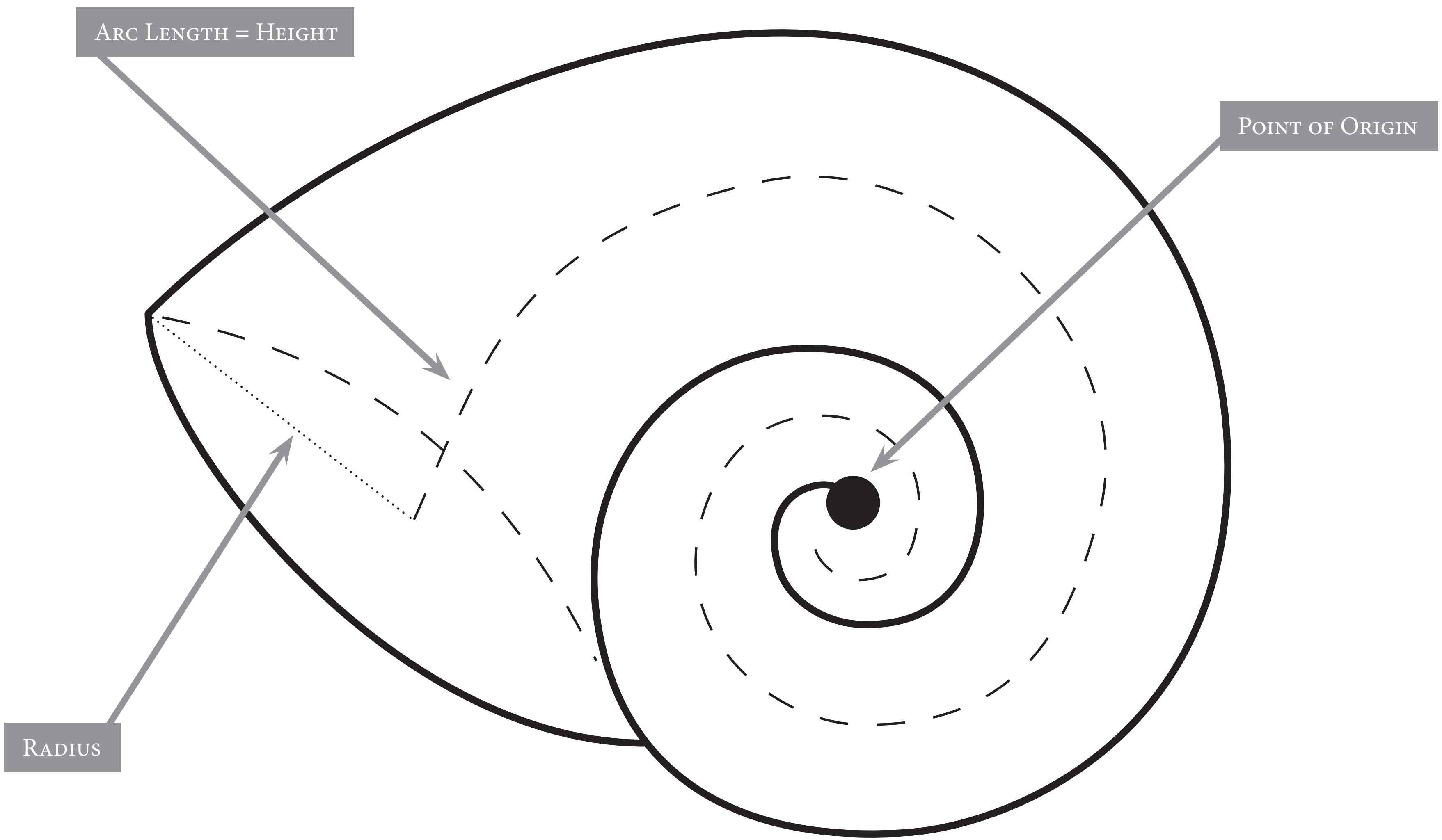

Illustration I

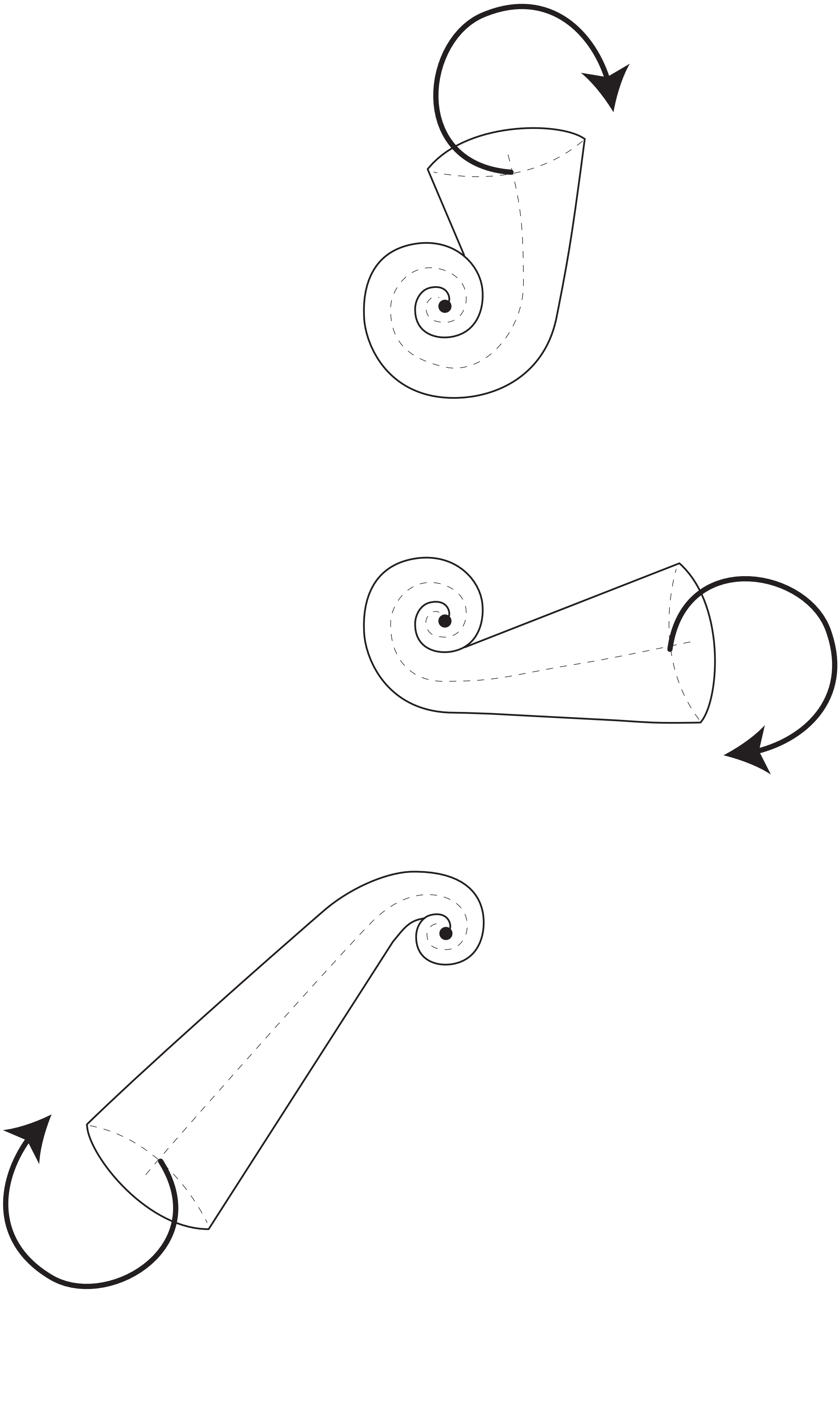

Illustration II

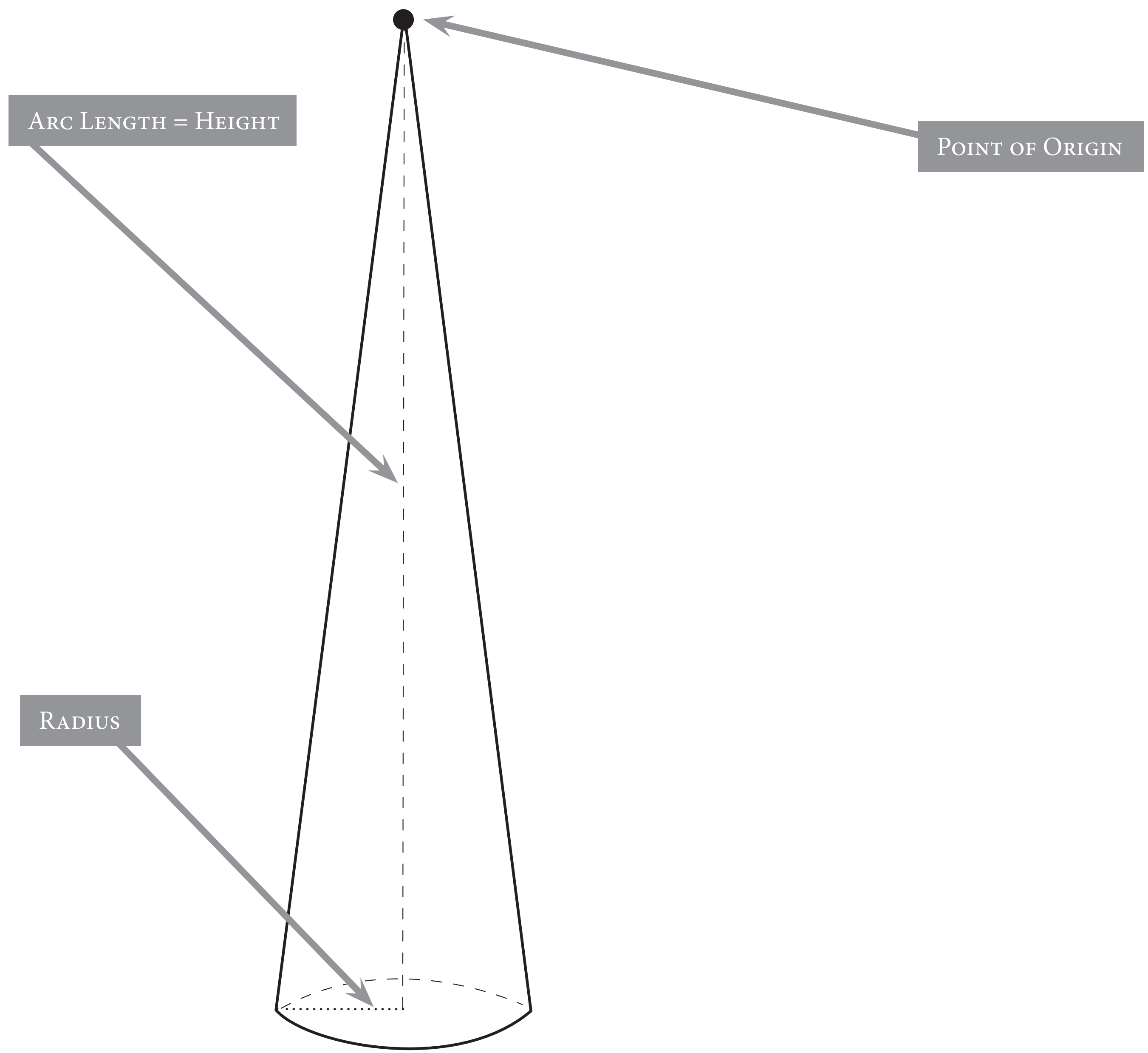

Illustration III

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 173–180

UNIQUENESS OF MEMORYLESS DISTRIBUTIONS ON PERIODIC TIME SCALES

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Abstract. The classical memoryless property induces probability distributions for both discrete and continuous random variables. On the positive real line, the only continuous random variables with the memoryless property are the exponentials, while on $\mathbb{N}$ the geometric random variables are the only memoryless ones. In their recent paper, Cuchta and Niichel explored various memoryless properties, leaving an open problem as to whether a certain type of "memoryless" criterion on periodic time scales implies a unique family of distributions. We answer that question in the affirmative herein.

1. Introduction

The memoryless property in the classical literature defines a random variable X to be "memoryless" if whenever $t, s > 0$ and t, s are in the sample space,

$$P(X > s+t | X > t) = P(X > s), \tag{1.1}$$

or equivalently,

$$P(X > s+t) = P(X > t)P(X > s). \tag{1.2}$$

This property gives rise to two different families of distributions. On $\mathbb{R}^+$, the exponential is the only continuous random variable with the memoryless property, while on $\mathbb{N}_0$, the geometric random variables are the only memoryless ones.

Time scales calculus has developed significantly over the past several decades, and a description of the most salient results can be found in [?]. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The function $g_X(t) = g(t) = P(X > t)$ is called the *survival function* of X. Throughout this paper, we will assume that $\inf \mathbb{T} = 0$, and similar to the exponential

2010 Mathematics Subject Classification: 60a05
Key words and phrases: Memoryless property, time scales, commensurable

and geometric, the survival function $g(t)$ always satisfies $g(0) = 1$. A time scale is said to be periodic with period $\omega > 0$ if whenever $t \in \mathbb{T}$, $t + \omega \in \mathbb{T}$. We say that $\mathbb{T}$ is n-periodic with periods $\omega_1, \ldots \omega_n > 0$ if whenever $t \in \mathbb{T}$ and $m_1, \ldots, m_n \in \mathbb{N}_0$,

$$t + \sum_{j=1}^{n} m_j \omega_j \in \mathbb{T}.$$

With multiple periods in play, there is the possibility that some of the periods will have common multiples. We say that two periods ω_1 and ω_2 are *commensurable* if $\omega_1/\omega_2 \in \mathbb{Q}$; otherwise, ω_1 and ω_2 are said to be *non-commensurable*.

The main problem with the memoryless property on time scales is the potential for lacuna in the time scale itself. That is, $t, s \in \mathbb{T}$ does *not* imply that $t + s \in \mathbb{T}$. A number of alternative memoryless properties have been introduced, including those by Poulsen et al. [**?**, Definition V.2], and Matthews [**?**, Definition 34]. The definition by Poulsen et al. is pertinent to this paper:

$$\forall t \in \mathbb{T} \quad P(X > t + \omega | X > t) = P(X > \omega). \tag{1.3}$$

Note that this definition of memorylessness eliminates the freedom of choice of s in the classical definition, (1.1); now only ω is allowed in this position. It is straightforward to show that (1.3) implies

$$\forall t \in \mathbb{T}, k \in \mathbb{N}_0 \quad P(X > t + k\omega | X > t) = P(X > \omega)^k. \tag{1.4}$$

Cuchta and Niichel [**?**] have recently explored the implications of these new definitions. They also presented a modified version of the memoryless property (1.3) with (1.4) for n-periodic time scales:

$$P\left(X > t + \sum_{k=1}^{n} m_k \omega_k\right) = P(X > t) \prod_{k=1}^{n} P(X > \omega_k)^{m_k}, \tag{1.5}$$

where $t \in \mathbb{T}$ and $m_1, \ldots, m_n \in \mathbb{N}_0$. In that paper, they also left open the following question:

> Let $\omega_1 < \omega_2$ be non-commensurable real numbers and consider the time scale $\mathbb{T} = \{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{N}_0\}$. Suppose that the random variable X has the memoryless property (1.5), so defining $g(x) = P(X > x)$ yields $g(n_1\omega_1 + n_2\omega_2) =$

> $g(\omega_1)^{n_1} g(\omega_2)^{n_2}$. If we choose $g(\omega_1) \in (0,1)$, then what restrictions are forced upon the value of $g(\omega_2)$?

They conjectured that $g(\omega_2) = g(\omega_1)^{\omega_2/\omega_1}$ is necessary because of the memoryless property (1.5) together with the non-commensurability of ω_1 and ω_2, but did not prove it. We have taken up that task and shown it to be true. We also explore a few other related questions.

2. The Commensurable Case

Our first result pertains to commensurable periods:

Theorem 2.1. *If $g(t)$ is the survival function of a memoryless random variable on a 2-periodic (or "doubly" periodic) time scale $\mathbb{T} = \{m_1\omega_1 + m_2\omega_2 | m_1, m_2 \in \mathbb{N}_0\}$ with commensurable periods ω_1 and ω_2, then $g(\omega_2) = g(\omega_1)^{\omega_2/\omega_1}$*

Proof. Since ω_1 and ω_2 are commensurable, there exist $m_1, m_2 \in \mathbb{N}$ such that $m_1\omega_1 = m_2\omega_2$, implying that $m_1/m_2 = \omega_2/\omega_1$. Hence by (1.5),

$$\begin{aligned} g(m_2\omega_2) &= g(m_1\omega_1), \\ g(\omega_2)^{m_2} &= g(\omega_1)^{m_1}, \\ g(\omega_2) &= g(\omega_1)^{m_1/m_2}, \\ g(\omega_2) &= g(\omega_1)^{\omega_2/\omega_1}, \end{aligned}$$

completing the proof. □

If $\mathbb{T} = \left\{ \sum_{j=1}^n m_j\omega_j | m_1, \dots, m_n \in \mathbb{N}_0 \right\}$ is an n-periodic time scale with periods $0 < \omega_1 < \omega_2 < \cdots < \omega_n$, all of which are commensurable with ω_1 (and hence with each other), by the same proof as in Theorem 2.1, it must be the case that the survival function g satisfies $g(\omega_j) = g(\omega_1)^{\omega_j/\omega_1}$. This implies part of the following result.

Theorem 2.2. *If $\mathbb{T} = \left\{ \sum_{j=1}^n m_j\omega_j | m_1, \dots, m_n \in \mathbb{N}_0 \right\}$ is an n-periodic time scale with periods $0 < \omega_1 < \omega_2 < \cdots < \omega_n$, all of which are commensurable with ω_1 then*

$$g(\omega_j) = g(\omega_1)^{\omega_j/\omega_1} \tag{2.1}$$

and therefore $g(t) = g(\omega_1)^{t/\omega_1}$ for all $t \in \mathbb{T}$.

Proof. The discussion before the theorem proves that $g(\omega_j) = g(\omega_1)^{\omega_j/\omega_1}$. It must be shown that $g(t) = g(\omega_1)^{t/\omega_1}$ for all $t \in \mathbb{T}$. To that end, suppose that $t = \sum_{j=1}^{n} m_j\omega_j$. By (1.5),

$$\begin{aligned} g(t) &= g(0)\prod_{j=1}^{n} g(\omega_j)^{m_j} \\ &= \prod_{j=1}^{n} g(\omega_1)^{\omega_j m_j/\omega_1} \\ &= g(\omega_1)^{\omega_1^{-1}\sum_{j=1}^{n}\omega_j m_j} \\ &= g(\omega_1)^{t/\omega_1}, \end{aligned} \tag{2.2}$$

which completes the proof. □

Theorem 2.2 implies the following:

Corollary 2.3. *Given an n-periodic time scale*

$$\mathbb{T} = \left\{ \sum_{j=1}^{n} m_j\omega_j | m_1, \ldots, m_n \in \mathbb{N}_0 \right\}$$

with pairwise commensurable periods $\omega_1 < \cdots < \omega_n$*, if a* $\mathbb{T}$*-valued random variable* X *satisfies* (1.5)*, then it is part of a one-parameter family of distributions on* $\mathbb{T}$*.*

3. The Non-Commensurable Case

Turning now to a doubly-periodic time scale $\mathbb{T} = \{m_1\omega_1 + m_2\omega_2 | m_1, m_2 \in \mathbb{N}_0\}$ with ω_1 and ω_2 non-commensurable, is it still true that $g(\omega_2) = g(\omega_1)^{\omega_2/\omega_1}$? This is the question posed by Cuchta and Niichel. If so, it would imply that Theorem 2.3 can be extended to encompass non-commensurable periods as well. Indeed, the proof of Theorem 2.2 did not require anything in particular about the periods, just that the survival function satisfied (2.1).

The proof of the non-commensurable case proved more slippery. However, there were good reasons to believe it was true. We ran simulations in Python using Tom Cuchta's `timescalescalculus` package.[1] We first generated the values of a doubly-periodic time scale using the two periods $\omega_1 = \sqrt{1/10}$ and $\omega_2 = \sqrt{1/2}$. We then assigned different values to $g(\omega_1)$ and $g(\omega_2)$; values of g on the rest of the time scale were then determined

[1]Commit 5252076. See `https://github.com/tomcuchta/timescalecalculus`

using the property (1.5). We then graphed $g(t)$, assigning the color orange to a point t_i if g increased from the previous point in the time scale (thus indicating that g fails to be non-increasing, and so cannot be a survival function). Several of these graphs are below:

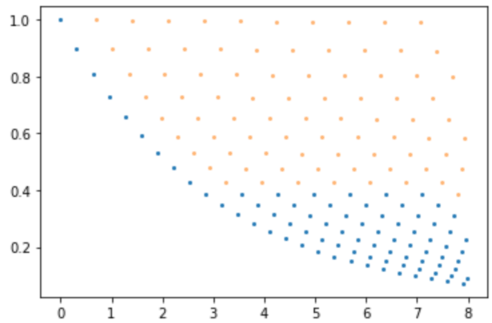

FIGURE 1. $g(\omega_1) = 0.9$; $g(\omega_2) = 0.99$. The graph fails to be non-increasing. Clearly, $g(\omega_2)$ should not be larger than $g(\omega_1)$.

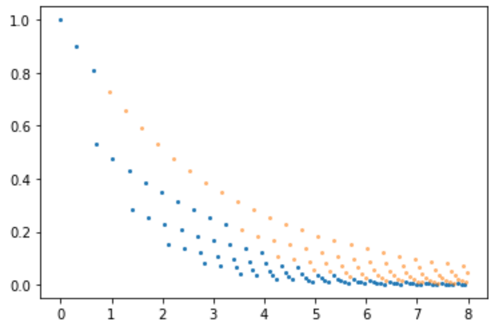

FIGURE 2. $g(\omega_1) = 0.9$; $g(\omega_2) = 0.53$. The graph fails to be non-increasing, even though $g(\omega_1) > g(\omega_2)$. Arbitrary values of $g(\omega_2)$ that are less than $g(\omega_1)$ are not guaranteed to work.

These graphs appear to indicate that when $g(\omega_2)$ is *not* the conjectured value, different "threads" appear in the graph. When $g(\omega_1)$ and $g(\omega_2)$ are closer a similar phenomenon occurs, though it is more subtle and it can take longer for the first increasing value to appear.

Theorem 3.1. *Let $\mathbb{T} = \{m_1\omega_1 + m_2\omega_2 | m_1, m_2 \in \mathbb{N}_0\}$ be a doubly-periodic time scale with non-commensurable periods $\omega_1 < \omega_2$. If a random variable*

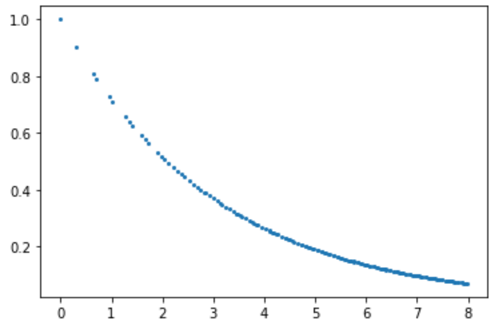

FIGURE 3. $g(\omega_1) = 0.9$; $g(\omega_2) \approx 0.9^{\sqrt{5}}$. This is the conjectured value for $g(\omega_2)$. The graph appears to be non-increasing.

satisfies (1.5) *on* $\mathbb{T}$*, then the survival function* $g(t)$ *of the random variable satisfies* $g(\omega_2) = g(\omega_1)^{\omega_2/\omega_1}$.

Proof. By way of contradiction, assume $g(\omega_2) \neq g(\omega_1)^{\frac{\omega_2}{\omega_1}}$. For $x \in \{n\omega_1 | n \in \mathbb{N}_0\}$, $g(x) = \left(g(\omega_1)^{\frac{1}{\omega_1}}\right)^x$ and similarly for $y = m\omega_2$, $g(y) = \left(g(\omega_2)^{\frac{1}{\omega_2}}\right)^y$. Since $g(\omega_2) \neq g(\omega_1)^{\frac{\omega_2}{\omega_1}}$, $\left(g(\omega_1)^{\frac{1}{\omega_1}}\right) \neq \left(g(\omega_2)^{\frac{1}{\omega_2}}\right)$. We now have need of the following lemma:

Lemma 3.2. *Let* $0 < a < b < 1$ *and consider the curves* a^x *and* b^x *on* $[0, \infty)$*. Let* $G(x) = x^* - x$*, where* x^* *is such that* $b^{x^*} = a^x$*. Then* $\lim_{x\to\infty} G(x) = \infty$ *monotonically.*

Proof of lemma 5. Compute:

$$\begin{aligned} b^{x^*} &= a^x \\ \ln(b^{x^*}) &= \ln(a^x) \\ x^* \ln(b) &= x \ln(a) \\ x^* &= x\frac{\ln(a)}{\ln(b)} \end{aligned}$$

Since $0 < a < b < 1$, $\ln a < \ln b < 0$. This implies that $\ln a / \ln b > 1$, hence,

$$\begin{aligned}\lim_{x\to\infty} G(x) &= \lim_{x\to\infty} x^* - x \\ &= \lim_{x\to\infty} x\frac{\ln(a)}{\ln(b)} - x \\ &= \lim_{x\to\infty} \left(\frac{\ln(a)}{\ln(b)} - 1\right) x \\ &= \infty,\end{aligned}$$

completing the proof of lemma 5. $\square$

Suppose that $\left(g(\omega_1)^{\frac{1}{\omega_1}}\right) > \left(g(\omega_2)^{\frac{1}{\omega_2}}\right)$. Since the limit in lemma 3.2 is monotonic, and using the notation from that theorem, there is an $M \in \mathbb{N}$ such that $G(M\omega_2) > \omega_1$. There is therefore a multiple of ω_1, say $N\omega_1$, such that $M\omega_2 < N\omega_1 < (M\omega_2)^*$. However, this implies that $g(N\omega_1) = \left(g(\omega_1)^{\frac{1}{\omega_1}}\right)^{N\omega_1} > \left(g(\omega_2)^{\frac{1}{\omega_2}}\right)^{M\omega_2} = g(M\omega_2)$, which in turn implies that g increases, a contradiction to the fact that g is a survival function.

The argument for $\left(g(\omega_1)^{\frac{1}{\omega_1}}\right) < \left(g(\omega_2)^{\frac{1}{\omega_2}}\right)$ is symmetric, and the proof is complete. $\square$

We now know that for a 2-periodic time scale $\mathbb{T}$, whether or not a period is commensurable or non-commensurable with ω_1, it must be the case that $g(\omega_j) = g(\omega_1)^{\omega_j/\omega_1}$. Thus we have the following extension of Corollary 2.3:

Corollary 3.3. *Given an n-periodic time scale*

$$\mathbb{T} = \left\{ \sum_{j=1}^{n} m_j\omega_j | m_1, \ldots, m_n \in \mathbb{N}_0 \right\}$$

with periods $\omega_1 < \cdots < \omega_n$, if the $\mathbb{T}$-valued random variable X satisfies (1.5)*, then it is part of a one-parameter family of distributions on $\mathbb{T}$.*

Proof. The proof follows from Theorems 2.1 and 3.1; each period is considered individually in terms of its relationship to ω_1. Each of them, whether commensurable or non-commensurable, must satisfy $g(\omega_j) = g(\omega_1)^{\omega_j/\omega_1}$. Using a similar argument as that in Theorem 2.2 (i.e., the progression (2.2)), this implies that for *every* element $t \in \mathbb{T}$, $g(t)$ can be expressed in terms of $g(\omega_1)$ alone. $\square$

Acknowledgement: We are grateful to the referee for the comments which improved the quality of the paper.This project was supported by National Science Foundation grant DMS-2150226.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 181–190

GENERALIZED RICCI SOLITONS

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(Received : 13 - 09 - 2023 ; Revised : 04 - 04 - 2024)

Abstract. In this paper, we have investigated the condition for which the generalized Ricci soliton will not be an Einstein manifold and will be Ricci-semi-symmetric whenever the vector field is a conformal killing vector field. We studied the effects of the electromagnetic field, perfect fluids and charged fluids of the energy-momentum tensor on the generalized Ricci soliton without cosmological constant. Finally, we derive the isotropic pressure without cosmological constant fluid in the generalized Ricci soliton from a 3-dimensional example.

1. Introduction

The importance and application of soliton in manifolds (M) are growing day by day in the physics of relativity and cosmology and it is an important chapter in modern geometry. In modern times, the solution of solitons, the singularity model of geometric flow, has been considered an important topic. Authors who have worked on solitons are Hamilton [5], Barman ([1], [2]), Duggal and Sharma [3], Erken [4], Nurowski and Randall [8], O'Neill [9], Ozen [10] and many others.

The Ricci tensor is a constant multiple of the Riemannian metric g and the Ricci soliton [7] is a generalization of the Einstein metric. A Ricci soliton can be written as a manifold such that

$$L_V g(X,Y) + 2S(X,Y) + 2\mu g(X,Y) = 0,$$

where L denotes the Lie derivative of Riemannian metric g along a vector field V, μ is a constant and S is a Ricci tensor of type $(0,2)$ of the manifolds.

2020 Mathematics Subject Classification: 53C15, 53C25
Key words and phrases: Ricci soliton, generalized Ricci soliton, Conformal killing vector field, Ricci-semi-symmetric, Einstein's field equation, Electromagnetic field, Perfect fluid.

Depending on whether it (μ) is negative, zero, or positive, a Ricci soliton is said to be shrinking, stable, or expanding. Recent progress has been made in the study of the generalized Ricci soliton [8]. The generalized Ricci soliton of an n-dimensional semi-Riemannian manifold can be defined as

$$L_V g(X,Y) = -2C_1 X_1^b(X) \odot X_1^b(Y) + 2C_2 S(X,Y) + 2\lambda g(X,Y) \tag{1.1}$$

on a vector field X and X_1^b is a non-zero 1-form such that

$$< X, X_1^b >= g(X,X)$$

and C_1, C_2 and λ are arbitrary real constants and $\odot$ is the Tensor products.

The generalized Ricci soliton equation (1.1) contains some important equations in differential geometry. The equations are as follows:

1. Killing's equation:- If $C_1 = C_2 = \lambda = 0$ in equation (1.1), then the generalized Ricci soliton equation is known as Killing's equation.

2. Equation for homotheties:- If $C_1 = C_2 = 0, \lambda \neq 0$ in equation (1.1), then the generalized Ricci soliton equation is known as an equation for homotheties.

3. Ricci Solitons:- If $C_1 = 0, C_2 = -1, \lambda \neq 0$ in equation (1.1), then the generalized Ricci soliton equation is known as the Ricci Solitons.

4. Cases of Einstein-Weyl:- If $C_1 = 1, C_2 = -\frac{1}{n-2}, \lambda \neq 0$ in equation (1.1), then the generalized Ricci soliton equation is known as Cases of Einstein-Weyl.

5. Metric projective structures with skew-symmetric Ricci tensor in projective class:- If $C_1 = 1, C_2 = -\frac{1}{n-1}, \lambda = 0$ in equation (1.1), then the generalized Ricci soliton equation is known as Metric projective structures with skew-symmetric Ricci tensor in projective class.

6. Vacuum near-horizon geometry equation:- If $C_1 = 1, C_2 = \frac{1}{2}, \lambda \neq 0$ in equation (1.1), then the generalized Ricci soliton equation is known as

the Vacuum near-horizon geometry equation.

Let (M^n, g) be a n-dimensional spacetime of the generalized Ricci soliton which admits a conformal killing vector field [11] V^* with conformal function ρ^* satisfying

$$L_V g(X,Y) = 2\rho^* g(X,Y), \tag{1.2}$$

which reduces to a homothetic or killing vector field whenever ρ^* is a non-zero constant or zero respectively.

The General relativity given by Einstein's equation [9] of the lower form of flow is as follows:

$$S(X,Y) - \frac{1}{2}r^*_{sc}g(X,Y) + \lambda^*_{cc}g(X,Y) = \kappa^*_{gc}T(X,Y),$$

where r^*_{sc} is the scalar curvature, $T(X,Y)$ is the energy-momentum tensor of type $(0,2)$, λ^*_{cc} is the cosmological constant and κ^*_{gc} is the gravitational constant. Einstein's equation gave the equation without the cosmological constant as follows:

$$S(X,Y) - \frac{1}{2}r^*_{sc}g(X,Y) = \kappa^*_{gc}T(X,Y). \tag{1.3}$$

We now present a brief description of two energy-momentum tensor fields.

(i) Electromagnetic field :- The electric and magnetic fields can be combined into a single tensor field $F(X,Y)$ on M, that is, $F(X,Y)$ is a skew-symmetric (2-form) tensor of type $(0,2)$. The energy-momentum tensor of an electromagnetic field [3] is

$$T(X,Y) = \frac{1}{4}g(X,Y) - F(X,Y). \tag{1.4}$$

(ii) Perfect fluids :- The energy-momentum tensor describes a Perfect fluid [6] if

$$T(X,Y) = (\sigma^*_{ed} + p^*_{ip})A(X)A(Y) + p^*_{ip}g(X,Y);$$
$$\sigma^*_{ed} + p^*_{ip} \neq 0;\ \ \sigma^*_{ed} > 0, \tag{1.5}$$

where σ^*_{ed} is the energy density and p^*_{ip} is the isotropic pressure of the fluid.

A perfect fluid is said to be isentropic if it is subject to a barotropic equation of state $\sigma^*_{ed} = \sigma^*_{ed}(p^*_{ip})$. Cases which are of particular physical interest are the dust matter field ($p^*_{ip} = 0$), the radiation field ($p^*_{ip} = \frac{\sigma^*_{ed}}{3}$) and the stiff matter ($p^*_{ip} = \sigma^*_{ed}$).

The energy-momentum tensor of the charged fluid [3] is then given

$$T(X,Y) = (\sigma^*_{ed} + p^*_{ip})A(X)A(Y) + (p^*_{ip} + \frac{1}{4})g(X,Y) - F(X,Y). \tag{1.6}$$

The present paper is organised as follows: In introduction, we briefly discussed the Ricci soliton, generalized Ricci soliton, conformal killing vector field and two energy-momentum tensor fields which are the electromagnetic field and perfect fluids of the Einstein field equation. The condition for which the generalized Ricci soliton will not be an Einstein manifold and will be Ricci-semi-symmetric whenever the vector field is a conformal killing vector field is defined in Section 2. In Section 3, we studied the generalized Ricci soliton of the energy-momentum tensor without cosmological constants and formed the relation of the electromagnetic field operator and also the non-zero 1-form is positive of the perfect fluid with the corresponding vector field. Finally, we derive the isotropic pressure from a 3-dimensional example in a generalized Ricci soliton without cosmological constant fluid.

2. Conformal motions in generalized Ricci soliton

Theorem 2.1. *For killing equations and equation for homothetes, the generalized Ricci soliton does not exist in Einstein manifolds whenever the vector field is a conformal killing vector field.*

Proof. Combining equation (1.1) and equation (1.2), the resulting representation is

$$2\rho^* g(X,Y) = -2C_1 X_1^b(X) \odot X_1^b(Y) + 2C_2 S(X,Y) + 2\lambda g(X,Y). \tag{2.1}$$

Since X_1^b is a non-zero 1-form and U^* is a corresponding vector field defined by

$$X_1^b(Y) = g(Y, U^*). \tag{2.2}$$

If we substitute equation (2.2) into equation (2.1), we get exactly this

$$2C_2S(X,Y) = (2\rho^* + 2C_1U_1^{*^b} - 2\lambda)g(X,Y). \tag{2.3}$$

Equation (2.3) can be rearranged and written like this

$$S(X,Y) = \frac{(\rho^* + C_1U_1^{*^b} - \lambda)}{C_2}g(X,Y). \tag{2.4}$$

The proof of Theorem is completed. □

Theorem 2.2. *For cases of Einstein-Weyl, metric projective structures with skew-symmetric Ricci tensor in projective class and Vacuum near-horizon geometry equation, the Ricci tensor of the generalized Ricci soliton is flat whenever the vector field is a conformal killing vector field.*

Proof. If $\rho^* + C_1U_1^{*^b} - \lambda = 0$ or $U_1^{*^b} = \frac{\lambda - \rho^*}{C_1}$ in equation (2.4), then this equation will be

$$S(X,Y) = 0. \tag{2.5}$$

That means the Ricci tensor of the generalized Ricci soliton is flat, for cases of Einstein-Weyl, metric projective structures with skew-symmetric Ricci tensor in projective class and Vacuum near-horizon geometry equation. The Theorem has proven. □

Definition 2.3. A manifold is said to be locally Ricci symmetric if it satisfies

$$\nabla S = 0,$$

where ∇ denotes the Levi-Civita connection.

Definition 2.4. A manifold is said to be Ricci-semi-symmetric if the relation

$$R(X,Y) \cdot S = 0$$

holds, where R is the curvature operator.

Theorem 2.5. *For cases of Einstein-Weyl, metric projective structures with skew-symmetric Ricci tensor in projective class and Vacuum near-horizon geometry equation, the generalized Ricci soliton is a Ricci-semi-symmetric whenever the vector field is a conformal killing vector field.*

Proof. Since every Ricci flat manifold is locally Ricci symmetric and every locally Ricci symmetric manifold is Ricci-semi-symmetric.

Using the above analysis and Theorem 2.2, we can be written the Theorem 2.5. □

3. Einstein's field equation in generalized Ricci soliton

Theorem 3.1. *The generalized Ricci soliton of the energy-momentum tensor without the cosmological constant for the electromagnetic tensor field, perfect fluid and charged fluid will be equation* (3.2), *equation* (3.3) *and equation* (3.4) *respectively.*

Proof. From equation (1.1) and equation (1.3) it follows that

$$L_V g(X,Y) = -2C_1 X_1^b(X) \odot X_1^b(Y) + (C_2 r_{sc}^* + 2\lambda) g(X,Y) + 2C_2 \kappa_{gc}^* T(X,Y), \tag{3.1}$$

which is the equation of generalized Ricci soliton of the energy-momentum tensor without the cosmological constant.

Combining equation (1.4) and equation (3.1) shows that

$$L_V g(X,Y) = -2C_1 X_1^b(X) \odot X_1^b(Y) + (C_2 r_{sc}^* + 2\lambda + \frac{1}{2} C_2 \kappa_{gc}^*) g(X,Y) - 2C_2 \kappa_{gc}^* F(X,Y), \tag{3.2}$$

which is the equation of generalized Ricci soliton of the electromagnetic tensor field without the cosmological constant.

Adding equation (1.5) and equation (3.1) together gives

$$L_V g(X,Y) = -2C_1 X_1^b(X) \odot X_1^b(Y) + (C_2 r_{sc}^* + 2\lambda + 2C_2 \kappa_{gc}^* p_{ip}^*) g(X,Y) + 2C_2 \kappa_{gc}^* (\sigma_{ed}^* + p_{ip}^*) A(X) A(Y), \tag{3.3}$$

which is the equation of generalized Ricci soliton of perfect fluid without the cosmological constant.

Substituting equation (1.6) into equation (3.1) implies that

$$L_V g(X,Y) = -2C_1 X_1^b(X) \odot X_1^b(Y) + (C_2 r_{sc}^* + 2\lambda + \frac{1}{2}C_2\kappa_{gc}^*$$
$$+2C_2\kappa_{gc}^* p_{ip}^*)g(X,Y) + 2C_2\kappa_{gc}^*(\sigma_{ed}^* + p_{ip}^*)A(X)A(Y)$$
$$-2C_2\kappa_{gc}^* F(X,Y), \qquad (3.4)$$

which is the equation of generalized Ricci soliton of the energy-momentum tensor of the charged fluid without the cosmological constant.
The proof is completed. □

Theorem 3.2. *If the generalized Ricci soliton resides in the energy-momentum tensor without the cosmological constant, then the electromagnetic field operator will have the relation* $fY = (\frac{1}{4} - p_{ip}^*))Y$.

Proof. Subtracting equation (3.4) from equation (3.3), we obtain

$$F(X,Y) = (\frac{1}{4} - p_{ip}^*)g(X,Y). \qquad (3.5)$$

Since $F(X,Y) = g(X, fY)$, where f be an electromagnetic field operator.

Equation (3.5) can improve this

$$g(X, fY) = (\frac{1}{4} - p_{ip}^*)g(X,Y).$$

Or

$$fY = (\frac{1}{4} - p_{ip}^*)Y.$$

□

Theorem 3.3. *If generalized Ricci soliton lies in the energy-momentum tensor without the cosmological constant, then a perfect fluid is isentropic.*

Proof. Again subtracting Equation (3.4) from Equation (3.2), we have

$$p_{ip}^* g(X,Y) + (\sigma_{ed}^* + p_{ip}^*)A(X)A(Y) = 0. \qquad (3.6)$$

Since $A(X)$ is a non-zero 1-form of the perfect fluid and W_1 is the corresponding vector field such that $A(X) = g(X, W_1)$.

Putting $X = W_1$ in equation (3.6), it follows that

$$\{p_{ip}^* + \sigma_{ed}^* A(W_1) + p_{ip}^* A(W_1)\}A(Y) = 0. \qquad (3.7)$$

Since $A(X) \neq 0$, then the equation (3.7) can written as

$$\sigma^*_{ed} = -\frac{[1 + A(W_1)]p^*_{ip}}{A(W_1}, \tag{3.8}$$

which is a barotropic equation ($\sigma^*_{ed} = \sigma^*_{ed}(p^*_{ip})$).
The proof of the Theorem is completed. □

Theorem 3.4. *If the dust-matter field of a perfect fluid lies in the generalized Ricci soliton of the energy-momentum tensor without the cosmological constant, then the non-zero* 1*-form of the perfect fluid with the corresponding vector field is positive.*

Proof. Since $\sigma^*_{ed} > 0$, then equation (3.8) will be

$$p^*_{ip} < \frac{A(W_1)}{1 + A(W_1)}.$$

For the dust matter field $p^*_{ip} = 0$, the above equation is $A(W_1) > 0$.
The Theorem has proven here. □

4. Example

In this section, we consider a 3-dimensional example and let M be a 3-dimensional manifold defined by

$$M = \{(X^*, Y^*, V^*) \in \mathbb{R}^3\},$$

where (X^*, Y^*, V^*) are the standard coordinates in $\mathbb{R}^3$. Let's consider 3 vector fields on M as follows;

$$E{*}_1 = E*^{-2V^*} \frac{\partial}{\partial X^*}, \quad E{*}_2 = \frac{\partial}{\partial Y^*}, \quad E{*}_3 = \frac{\partial}{\partial V^*}.$$

Define a Riemannian metric g by $g(E{*}_i, E{*}_j) = 0$ for $i \neq j, 1 \leq i, j \leq 3$ and $g(E{*}_i, E{*}_i) = -1$. Thus, the vector fields $(E{*}_1, E{*}_2, E{*}_3)$ have formed an orthonormal vector field set which is the basis of M.

Then we obtain

$$[E{*}_1, E{*}_3] = 2E{*}_1, \quad [E{*}_1, E{*}_2] = [E{*}_2, E{*}_3] = 0.$$

Recall the classical Kozsul's formula from the Differential geometry,

$$2g(\nabla_{X^*} Y^*, X^*) = X^* g(Y^*, Z^*) + Y^* g(X^*, Z^*) - Z^* g(X^*, Y^*) - g(X^*, [Y^*, Z^*])$$
$$- g(Y^*, [X^*, Z^*]) + g(Z^*, [X^*, Y^*]),$$

for all X^*, Y^*, Z^* on M. Using Koszul's formula we get the following

$$\nabla_{E*_1} E*_1 = -2E*_3, \quad \nabla_{E*_1} E*_2 = 0, \quad \nabla_{E*_1} E*_3 = 2E*_1;$$

$$\nabla_{E*_2} E*_1 = 0, \qquad \nabla_{E*_2} E*_2 = 0, \qquad \nabla_{E*_2} E*_3 = 0;$$

$$\nabla_{E*_3} E*_1 = 0, \qquad \nabla_{E*_3} E*_2 = 0, \qquad \nabla_{E*_3} E*_3 = 0.$$

By using the above Levi-Civita connection results, we can find the non-zero components of the curvature tensor (R) on the manifolds as follows:

$$R(E*_1, E*_3)E*_1 = 4E*_3,$$

and

$$R(E*_3, E*_1)E*_3 = 4E*_1 .$$

With the help of the above curvature tensor (R) results, we can conclude the non-zero Ricci tensor (S) as follows:

$$S(E*_1, E*_1) = S(E*_3, E*_3) = -4.$$

The scalar curvature (r^*_{sc}) of the manifolds will be $r^*_{sc} = S(E*_1, E*_1) + S(E*_2, E*_2) + S(E*_3, E*_3) = -8$.

Since the gravitational constant κ^*_{gc} is 9.81 and using the non-zero Ricci tensor (S) and scalar curvature (r^*_{sc}), we get the energy-momentum tensor T as follows

$$T(E*_1, E*_1) = T(E*_3, E*_3) = -0.815494.$$

From the energy-momentum tensor T, we have the electromagnetic tensor field F as follows:

$$F(E*_1, E*_1) = F(E*_3, E*_3) = 0.565494$$

With the help of the above electromagnetic tensor field F results and equation (3.5), the isotropic pressure p^*_{ip} without cosmological constant fluid in the generalized Ricci soliton is 0.815494.

Acknowledgement: We are grateful to the referee for the comments which improved the quality of the paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 191–199

AN INTRODUCTION TO THE MATHEMATICS OF HILLEL FURSTENBERG

ANISH GHOSH

Abstract. This is a popular exposition of some of the beautiful mathematics of Hillel Furstenberg. It is based on a talk given by the author at the Bangalore Probability Seminar.

1. Introduction

As a first year undergraduate student with an interest in mathematics but no background to speak of, I came across the book “Proofs from the Book" by Aigner and Ziegler. There, I found an enchanting proof of the infinitude of primes. This proof, which Hillel Furstenberg published [2] while he was an undergraduate, uses a beautiful ‘topological’ idea[1]. I liked it so much that I gave a talk about it to my fellow students. Little was I to know then that I would have the privilege to read many beautiful and deep works of mathematics by Furstenberg.

The Abel prize for the year 2020 was awarded to two mathematical giants, and heroes of mine, Hillel Furstenberg and Gregory Margulis for “pioneering the use of methods from probability and dynamics in group theory, number theory and combinatorics". This article is meant to be an introduction to the work of Hillel Furstenberg on the occasion of his winning the Abel prize. To summarize all or indeed a representative portion of Furstenberg’s work in a few pages is a daunting task, well beyond the abilities of the author of this article. Even if one were to make the difficult choice of discussing one or two papers, it is still a steep uphill climb. A quote from the Abel prize website aptly captures the author’s predicament: “When Hillel Furstenberg published one of his early papers, a rumor circulated that he was not an individual but instead a pseudonym for a group of

2010 Mathematics Subject Classification: 00A09, 11B30, 37B20
Key words and phrases: Ergodic Theory, Topological Dynamics, Ramsey theory.

[1]Much later, I recognized this to be the profinite topology.

mathematicians. The paper contained ideas from so many different areas, surely it could not possibly be the work of one man?". Accordingly, I will limit myself and focus on two themes in this article which illustrate Furstenberg's seminal contributions to the interactions between dynamics, combinatorics and number theory. Namely I will discuss commuting maps on the torus, and ergodic Ramsey theory. The article ends with a conjecture of Furstenberg's which he has popularized in the last few years and which I personally find very appealing. I have tried to make the article accessible to a broad scientific audience and included a bibliography with suggestions for future reading; any omissions should be blamed on my own intellectual limitations and to a lesser extent, on space constraints.

2. TIMES 2 ($\times_2$) AND TIMES 3 ($\times_3$)

We will discuss Furstenberg's work in an area of mathematics called *ergodic theory*, which is closely related to the field of *dynamical systems*. Very loosely speaking, the latter involves the mathematical study of moving objects, whilst the former comprises the study of statisticial or measure theoretic properties of dynamical systems. Thus in the study of dynamical systems we have a space, for example a metric or topological space, along with a rule which tells us how points in the space move, while in ergodic theory, we are also given a measure on the space and we are interested in the long term behaviour of typical as well as exceptional orbits with respect to the given measure.

Consider the circle doubling map

$$\times_2 : x \to 2x \bmod 1.$$

Here we write the circle additively as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and denote by $\mathcal{B}$ its Borel sigma algebra. The idea is to iterate this map and study the resulting dynamics. It turns out that the dynamics is quite rich, namely this map is a classical example of what one would call a "chaotic" system. It has periodic points, i.e. points which return to themselves after finitely many iterations, as well as points with dense orbits. In fact, it is an easy exercise to check that there are $2^n - 1$ points of period n and moreover that the set of periodic points is dense in $\mathbb{T}$. A periodic orbit is an example of an *invariant* set. A subset X of $\mathbb{T}$ is said to be invariant under $\times_2$ if $\times_2(X) \subset X$. There are many (closed) invariant subsets for such maps. For instance, the circle is invariant, as is a periodic orbit. A more interesting example is the middle third Cantor set which is invariant for $\times_3$ [2]. This brings us to a fundamental question: is it possible to concretely describe the dynamics of such maps? For instance, can one list all the possible closed invariant subsets? It turns out that one can model $\times_2$ (and also $\times_3$ and so on) using *symbolic* dynamical systems. In the case at hand, this relationship is easy to describe. Every real number $x \in [0,1]$ has a base n expansion

$$x = 0.x_1x_2\cdots = \sum_{i=1}^{\infty} x_i n^{-i}, x_i \in \{0,1,\ldots,n-1\}.$$

This expansion is unique (except for a countable set of numbers which have two). If we take $x \in [0,1]$ with base n expansion $x = 0.x_1x_2x_3\ldots$ then $\times_n(x) = 0.x_2x_3x_4\ldots$. Let $\Sigma = \{0,\ldots,n-1\}^{\mathbb{N}}$ denote the set of sequences of elements in $\{0,\ldots,n-1\}$. On this space, we define the *shift* map as follows:

$$\sigma((x_1,x_2,x_3,\ldots)) = (x_2,x_3,\ldots).$$

In other words, σ takes as input a sequence in Σ and outputs another sequence obtained by removing the first element of the input sequence and shifting the remaining elements one place to the left. The base n expansion connects our circle map to the shift in the following natural manner. Consider the map $\phi : \Sigma \to [0,1]$ given by

$$\phi((x_1,x_2,\ldots)) = \frac{x_1}{n} + \frac{x_2}{n^2} + \frac{x_3}{n^3} + \ldots$$

We can think of ϕ as a map from Σ to $\mathbb{T}$ by identifying 0 and 1 and it provides us with a *semi-conjugacy* between the circle map $\times_n$ and the shift σ. This means that $\phi \circ \sigma = \times_n \circ \phi$. Moreover, ϕ is onto and $1-1$.[3] The upshot is that one can try and understand the dynamics of a complicated system like $(\mathbb{T}, \times_n)$ using a (hopefully) easier dynamical system (Σ, σ) via the semi-conjugacy ϕ.

We now turn to the ergodic theory of the doubling map. A map $T : \mathbb{T} \to \mathbb{T}$ is called a measure-preserving transformation if

$$\mu(T^{-1}B) = \mu(B) \text{ for all } B \in \mathcal{B}.$$

Alternatively, one says that T preserves μ; it turns out that $\times_n$ preserves Lebesgue measure. We say that a measure preserving transformation T of

[2]For the purposes of our discussion, one can similarly consider the $\times_n$ map for any integer $n \geq 2$.

[3]provided of course, that one ignores a countable subset of sequences.

a probability space $(X, \mathcal{B}, \mu)$ is ergodic, or equivalently that μ is an ergodic measure for T, if it is "indecomposable" namely if

$$T^{-1}B = B \text{ implies that } \mu(B) = 0 \text{ or } 1.$$

One could ask how many ergodic measures a dynamical system has? Is it possible to list them? As the reader might have guessed already, it turns out that the $\times_2$ map has infinitely many (in fact uncountably many) ergodic measures.

Let us now consider two maps together, say $\times_2$ and $\times_3$. We have seen that each of them has many invariant subsets. Are there any subsets of $\mathbb{T}$ which are jointly invariant under both? In other words, do expansions in base 2 and base 3 have anything in common? The answer is provided in a beautiful theorem of Furstenberg which can be found in his seminal work [3]. Two integers $p, q > 1$ are multiplicatively independent if they are not both rational powers of a single integer, i.e. $\log p / \log q \notin \mathbb{Q}$. For instance, 2 and 3 are multiplicatively independent but 4 and 8 are not. Then $S_{p,q}$ be the semigroup of the natural numbers defined as follows

$$S_{p,q} = \{p^m q^n \ : \ m, n \in \mathbb{Z}, m, n \geq 0\}.$$

Theorem 2.1. *(Furstenberg 1967): Let p, q be multiplicatively independent integers and suppose that $X \subset \mathbb{T}$ is closed and invariant under $S_{p,q}$. Then either $X = \mathbb{T}$ or X is finite (and so $X \subset \mathbb{Q}$).*

Furstenberg's theorem is an example of a (topological) *rigidity* result in mathematics. Rigidity results in ergodic theory often have striking consequences in other fields such as number theory and geometry. Here is an interesting Diophantine consequence of the Theorem above, see [1] for an elementary proof of this result.

Theorem 2.2. *(Furstenberg): If $S_{p,q}$ is as above and $\alpha \in \mathbb{R}$ is an irrational, then*

$$\{s\alpha \ mod\ 1 \ : \ s \in S_{p,q}\}$$

is dense in $[0, 1]$.

The reader will recognize this to be a generalization of Weyl's classical equidistribution theorem. In [3], Furstenberg introduced a new property of dynamical systems called *disjointness*. This idea has since proved to be very influential in dynamics. Furstenberg introduced this notion both in the topological as well as the measure theoretic setting. We will not go into the precise definition here but remark that it can be thought of as the opposite of isomorphism in some sense. Furstenberg showed that dynamical systems with many periodic points are disjoint from those which have the property that every orbit is dense.

What about measure theoretic rigidity? This turns out to be deeper and more subtle than the topological question. Furstenberg made the following influential conjecture in the same paper.

Conjecture 2.3. *(Furstenberg 1967): Let p, q be two multiplicative independent positive integers. Any Borel measure μ on $\mathbb{T}$ ergodic under the action of $S_{p,q}$ is either Lebesgue measure or an atomic measure supported on finitely many rational points.*

Furstenberg's conjecture is possibly the most famous open problem in ergodic theory. His work has had far reaching consequences in the dynamics of group actions on homogeneous spaces. We refer the reader to [10, 11] for details and references.

3. Ergodic Ramsey Theory

Let us begin with a basic result in dynamical systems, namely the Poincaré Recurrence Theorem. It states that if $(X, \mathcal{B}, \mu, T)$ is a measure preserving system and $A \in \mathcal{B}$ has positive measure, then there exists $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A) > 0$. In his seminal paper [4], Furstenberg proved a far reaching generalization of the Poincaré Recurrence Theorem, namely he considered *multiple* recurrences. Furstenberg proved:

Theorem 3.1. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and k be a positive integer. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n > 0$ such that*

$$\mu(A \cap T^n(A) \cap \cdots \cap T^{(k-1)n}A) > 0.$$

Furstenberg has spectacularly demonstrated how recurrence results are related to deep problems in additive number theory. The branch of additive number theory we are interested in is called Ramsey theory and starts with van der Waerden's famous result.

Theorem 3.2. *(van der Waerden 1927): Let k and r be given. There exists a number $N = N(k, r)$ such that if the integers in $[1, N]$ are coloured* *using r colours, then there is a non-trivial monochromatic k term arithmetic progression.*

In 1936, Erdős and Turán made a famous conjecture generalizing van der Waerden's theorem above. This conjecture was eventually settled in a landmark work by Szemerédi in 1974. In part for this work, Szemerédi was awarded the Abel prize in the year 2012. In order to state his result we need the notion of upper density of a subset of the integers. The upper density of a subset S of the integers is defined as

$$d^*(S) := \limsup_{n\to\infty} \frac{1}{2n+1}|A \cap \{-n, -n+1, \dots, n-1, n\}|.$$

Theorem 3.3. *(Szemerédi 1974):Any subset of $\mathbb{Z}$ with positive upper density contains arbitrarily long arithmetic progressions.*

The first non-trivial case of the conjecture is the case $k = 3$ which was settled by Roth [12] using analytic methods, namely exponential sums. The case $k = 4$ was then settled by Szemerédi before he went on to settle the general case. Szemerédi's proof is based on intricate combinatorial methods. Subsequently, an analytic proof was found by Gowers [7] in a significant breakthrough.

Furstenberg [4] in 1977 gave a new, completely different proof of Szemerédi's theorem using ergodic theory and thereby introduced a new branch of mathematics called *ergodic Ramsey theory*. The main idea in Furstenberg's proof is to translate the problem into dynamics via the *Furstenberg correspondence principle*. What is the dynamical system to consider? It turns out to be the symbolic system considered above. Namely, Furstenberg's multiple recurrence theorem applied to a Bernoulli shift implies Szemerédi's theorem. For a beautiful discussion of these topics, I refer the reader to Furstenberg's article [6] and his book [5]. Furstenberg's work has had very far reaching consequences and it's influence can be seen in diverse important breakthroughs in the area including, for instance, the Green-Tao theorem which states that the primes[4] contain arbitrarily long non-trivial arithmetic progressions. We refer the reader to [9] and [16] for an overview of the many exciting recent developments, many of which can be traced back to this seminal paper of Furstenberg.

[4]which have zero density.

4. Well approximable numbers and homogeneous dynamics

I'll end this article with a beautiful problem that Furstenberg has posed on many occasions in recent times and has to do with the interaction between Diophantine approximation and dynamics on homogeneous spaces of Lie groups. The study of Diophantine approximation deals with the approximation of real numbers by rational numbers. This is an ancient subject with connections to many branches of mathematics and the sciences. Yet many basic questions in the area remain unanswered and pose a fundamental challenge to modern mathematics. In particular, understanding the Diophantine properties of individual numbers is a very difficult problem in general. A basic and indispensable tool in this regard is the continued fraction expansion of a real number. We point the reader to [8] for a classic exposition. We all know from our school days that $22/7$ is a good approximation for π. In fact, every real number has a *continued fraction expansion.* For example, the one for π is

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \dots],$$

which is shorthand for

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

The entries in the continued fraction expansion are called partial quotients. One can read off rational approximations for π by looking at truncations of the continued fraction expansion, for example

$$[3; 7] = 3 + \frac{1}{7} = \frac{22}{7}.$$

Similarly, the continued fraction expansion for $\sqrt{2}$ is

$$\sqrt{2} = [1; 2, 2, 2, \dots].$$

This is an instance of a theorem of Lagrange which states that α is a quadratic irrational number if and only if its continued fraction expansion is eventually periodic. Similarly, one can prove that a real number α is *badly* *approximable* by rational numbers if and only if it has bounded partial quotients, $\sqrt{2}$ presents an example. Badly approximable numbers have zero Lebesgue measure but are nevertheless plentiful in that they form a set of full Hausdorff dimension. On the other end of the spectrum, a number $\alpha \in \mathbb{R}$ is called well approximable if it has unbounded partial quotients. These form a full Lebesgue measure subset of the real line. One might then wonder about concrete examples of well approximable numbers. This turns out to be very difficult problem as illustrated by the following folklore conjecture:

Conjecture 4.1. *$\sqrt[3]{2}$ is well approximable.*

In fact, see [15] questions 2.9, 2.10, it is unknown whether any algebraic number of degree greater than 2 has bounded partial quotients or whether any algebraic number of degree greater than 2 has unbounded partial quotients. Here are the first few terms in the continued fraction expansion of $\sqrt[3]{2}$.

$$\sqrt[3]{2} = [1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, \dots]$$

In fact, Furstenberg has presented the above problem as a challenge to the ergodic theory community by converting it into a problem about the dynamics of certain flows on homogeneous spaces. The precise formulation will carry us too far from this article, so I'll refer the interested reader to appendix A in [13] for the statement and a discussion of Furstenberg's conjecture.

5. Concluding Remarks

This article presents the tip of the iceberg as far as Furstenberg's mathematical achievements are concerned. Thanks to giants like Furstenberg, who have devoted a lifetime to developing and nurturing mathematics at the very highest level, Israel is now the centre of the world as far as ergodic theory is concerned. As a mathematician from a developing country, with some understanding of the serious issues and difficulties involved in such matters, this amazing achievement fills me with admiration.

Acknowledgement: This article is an expanded version of a talk I first gave at the Bangalore Probability Seminar. I am grateful to the organizers of the seminar for the invitation to give the talk. I thank C.S. Aravinda for his kind invitation to write this article, and for his patience as I subjected the deadline to a particularly chaotic dynamical system.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 200–212

A BRIEF INTRODUCTION TO CERTAIN DYNAMICAL SYSTEMS RELATED TO NUMBER THEORY

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Abstract. The area of homogeneous dynamics has been one of the rapidly developing areas in recent decades, It intertwines ideas from dynamics, group theory, ergodic theory and number theory, and has led to several new insights and resolution of long-standing problems. The aim of this article is to give a first introduction to the topic, with minimal background requirements.

1. Introduction

One way of thinking about what 'intuitively' constitutes a 'dynamical system' is to realize it as a space, say X, together with a family of transformations of X into itself, encoding the changes over time. The main aspect that we shall concern ourselves here is the cumulative effect of repeated changes over unit time over a (relatively) large time scale, asymptotically. For convenience, we shall choose time to shift discretely and the change to be governed by the same rule at each stage. We shall further assume the space to have a topology in which it is locally compact and the transformation to be a homeomorphism. Thus a dynamical system in our context is nothing but a homeomorphism of a locally compact space, except that our main interest is in the asymptotic behaviour over a large number of iterations of the homeomorphism (and not in the topological aspects of the homeomorphism). We shall denote such a system as (X, T), where X is the space and T is the homeomorphism.

Let us begin with the following example. Let $X = \mathbb{S}^1$, the circle, say realised as $\{z \in \mathbb{C} \mid |z| = 1\}$, where $\mathbb{C}$ stands for complex numbers. Let

2010 Mathematics Subject Classification: 37A17, 11J25.
Key words and phrases: Homogeneous dynamics, periodic and dense orbits, recurrence, uniform distribution, Diophantine approximation.

$\theta \in \mathbb{R}$, namely a real number. Let $T : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $T(z) = e^{i\theta}z$, for all $z \in \mathbb{S}^1$; it is easy to see that T is a homeomorphism of $\mathbb{S}^1$. Thus we have a dynamical system (X, T). The transformation at each stage consists of 'rotating' a wheel (the circle) by an angle θ; if θ is positive then the rotation is considered, by convention, to be anticlockwise by an angle θ and if θ is negative the rotation is considered to be clockwise by an angle $|\theta|$.

Suppose first that $\frac{\theta}{2\pi}$ is rational, say $\frac{\theta}{2\pi} = \frac{p}{q}$, where p and q are integers and q is positive. Then we see that, for any $z \in \mathbb{S}^1$, $T^q(z) = e^{iq\theta}z = e^{2\pi ip}z = z$, which shows that if we keep applying the transformation T, after q steps we reach the same point where we started. The whole behaviour (no matter what the starting point) then repeats after q iterations. Such a transformation is said to be *periodic*. The minimum number of times after which the repetition occurs is called the *period* of the dynamical system. In the above case it can be verified that q would be the period, provided p and q are co-prime.

Next suppose that $\frac{\theta}{2\pi}$ is not rational. Let us start from a point $z \in \mathbb{S}^1$ and consider the sequence of points $z, T(z), T^2(z), ..., T^k(z), ...,$ to which the point will get transformed after successive iterations. Since $T^k(z) = e^{ik\theta}z$ and $k\theta$ is not a multiple of 2π, $T^k(z)$ is not equal to z, for any k. Thus the system is not periodic. We shall next see that this system has the following interesting property:

Proposition 1.1. *Let θ be as above. Then for any z and w in $\mathbb{S}^1$ and $\epsilon > 0$ there exists a natural number k such that $|T^k(z) - w| < \epsilon$.*

Proof: It is enough to show that there exists a j such that T^j is a rotation by an angle ψ such that $|\psi| < \epsilon$, since for any z and w the desired assertion would then hold for a suitable multiple k of j, depending on the relative positions of z and w; according to the sign of ψ the rotation would be clockwise or anticlockwise, but the preceding contention holds in either case. There is no loss of generality in assuming θ to be such that $|\theta| < \pi$. Let m be the unique positive integer $m|\theta| < 2\pi < (m+1)|\theta|$. Then at least one of $(m+1)|\theta| - 2\pi$ and $2\pi - m|\theta|$ has absolute value less than $|\frac{\theta}{2}|$ and we choose θ_1 to be one of them such that $|\theta_1| < |\frac{\theta}{2}|$. On account of the choice as above there exists j_1, viz. $m \operatorname{sign}\theta$ or $(m+1)\operatorname{sign}\theta$, such that T^{j_1} is, in effect (setting aside the full angle 2π), a rotation by an angle θ_1, with $|\theta_1| < \frac{\theta}{2}$. Repeating the procedure we find $j_2, j_3, \ldots$ such that $T^{j_1 j_2 \cdots j_r}$ is a rotation by an angle θ_r such that $|\theta_r| < |\frac{\theta}{2^r}|$. Choosing $j = j_1 j_2 \dots j_r$ with r such that $|\frac{\theta}{2^r}| < \epsilon$ we get the assertion as desired above. This completes the proof.

2. Recurrence and uniform distribution

Given a dynamical system (X, T), a point $x \in X$ is said to be *recurrent* if there exists a sequence $\{n_i\}$ tending to infinity such that $T^{n_i}x$ converges to x as $i \to \infty$; in other words, for such a point the iterates will keep coming arbitrarily close to the starting point. A point $x \in X$ is said to be *periodic* if there exists a positive integer p such that $T^p x = x$. A periodic point is always recurrent. In the above examples if $\frac{\theta}{\pi}$ is rational all points are periodic and if $\frac{\theta}{\pi}$ is irrational then no point is periodic but all points are recurrent.

Let (X, T) be a dynamical system. For $x \in X$ the sequence of points $x, Tx, ..., T^j x, ...$ is called the *orbit* of x under T; often the set underlying the sequence is also called the orbit. Proposition 1.1 shows that for a rotation of the circle by an angle which is an irrational multiple of 2π, all orbits are dense in the circle. In an intuitive sense, they fill up the space as we pick more and more points from any given orbit. It turns out that they even fill it quite uniformly, as we go along: we mean the following: consider any angular arc say A in $\mathbb{S}^1$. Start with a $z \in \mathbb{S}^1$ and consider the orbit $z, T(z), T^2(z), ..., T^j(z), ...$ where T is the rotation by an angle θ. For any positive integer k let $O_k = \{z, T(z), ..., T^{k-1}(z)\}$ be the part of the orbit considering of the first k elements. Let n_k be the number of elements of O_k contained in A. Then as $k \to \infty$, the average $\frac{n_k}{k}$ tends to $\frac{\alpha}{2\pi}$ where α is the angular width of the arc A. This means in particular that if we take two different arcs of the same angular width, then the average frequency of visiting them approaches the same value, as we consider longer and longer parts of the orbit. To show that this holds it is enough to show that for any continuous function f on $\mathbb{S}^1$

$$\frac{1}{k}\sum_{j=0}^{k-1} f(T^j(z)) \longrightarrow \frac{1}{2\pi}\int_0^{2\pi} f(\cos t + i \sin t)\, dt. \tag{2.1}$$

To deduce the preceding assertion one may apply (2.1) to functions which are 1 on most part of the arc, taper off to 0 at the ends and vanish on the complement of the arc, and also to similar functions with respect to the complementary arc.

Before going to the proof of the above, for continuous functions, we formulate a notion of uniform distribution of orbits in a general dynamical system. Let (X,T) be a dynamical system and let $x \in X$. Let $C(X)$ denote the space of all bounded continuous complex-valued functions on X. The orbit of x is said to be *uniformly distributed* if for any $f \in C(X)$ the sequence $\frac{1}{k}\sum_{j=0}^{k-1} f(T^j x)$ converges as $k \to \infty$ and the limit is positive for all non-negative functions $f \in C(X)$ for which $f(x) > 0$; if c_f is the limit of the sequence for $f \in C(X)$, then the function $I : C(X) \to \mathbb{C}$ defined by $I(f) = c_f$ will be called the asymptotic integral corresponding to the (uniformly distributed) orbit. We note that a point whose orbit is uniformly distributed in this sense is necessarily recurrent. It may however not be dense in the whole space; the latter will be ensured if the invariant integral is positive for all nonzero non-negative functions in $C(X)$. By the Lebesgue integral on $\mathbb{S}^1$ we mean the function $I : C(\mathbb{S}^1) \to \mathbb{C}$ defined by setting, for $f \in C(\mathbb{S}^1)$, $I(f)$ to be the right hand side of 2.1. Our claim above can now be restated as follows:

Proposition 2.1. *Let $X = \mathbb{S}^1$ and let T be the rotation by an angle θ such that $\frac{\theta}{\pi}$ is irrational. Then the orbit of any $z \in \mathbb{S}^1$ is uniformly distributed and the corresponding asymptotic integral is the Lebesgue integral on $\mathbb{S}^1$.*

Proof: We have to show that (2.1) holds for all continuous functions on $\mathbb{S}^1$. First consider any function of the form $f(z) = z^m$, where $m \in \mathbb{Z} - (0)$. Then

$$\frac{1}{k}\sum_{j=0}^{k-1} f\big(T^j(z)\big) = \frac{1}{k}\sum_{j=0}^{k-1} e^{ijm\theta} z^m = \frac{z^m}{k}\frac{1-e^{ikm\theta}}{1-e^{im\theta}},$$

where the denominator in the last term is non-zero since $m\theta$ is not a multiple of 2π. As $k \to \infty$ the right hand side clearly tends to 0, which is the same as the Lebesgue integral of the function $z \to z^m$. Also equation (2.1) evidently holds for all constant functions. It follows therefore that equation (2.1) holds for all trigonometric polynomials, namely functions of the form $\sum_{m=-n}^{n} a_m z^m$, where n is any positive integer and a_m, $-n \le m \le n$, are complex numbers. Now let $f \in C(X)$ and $\epsilon > 0$ be arbitrary. By the Weierstrass approximation theorem there exists a trigonometric polynomial

φ such that $|f(x) - \varphi(x)| < \frac{\epsilon}{3}$ for all $x \in X$. Then

$$\Big|\frac{1}{k}\sum_{j=0}^{k-1} f(T^j(z)) - \frac{1}{k}\sum_{j=0}^{k-1} \varphi(T^j(z))\Big| < \epsilon/3.$$

Since equation (2.1) holds for φ in the place of f and since $|I(f)-I(\varphi)| < \frac{\epsilon}{3}$, this implies that $\big|\frac{1}{k}\sum_{j=0}^{k-1} f\big(T^j(z)\big) - I(f)\big| < \epsilon$ for all large k, I being the Lebesgue integral. Since ϵ is arbitrary this means that equation (2.1) holds for f. This completes the proof.

For any real number t we denote by $\langle t \rangle$ the fractional part of t, namely the unique $s \in [0,1)$ such that $t = s + n$ for some integer n. A sequence $\{t_j\}$ in $\mathbb{R}$ is said to be uniformly distributed mod 1 if for any interval (a,b) contained in $(0,1)$

$$\frac{1}{k}\Big|\big\{j \mid 1 \leq j \leq k \text{ and } < t_j > \in (a,b)\big\}\Big| \to (b-a),$$

as $k \to \infty$; here $|\cdot|$ stands for the cardinality.

Corollary 2.2. *Let $\alpha \in \mathbb{R}$ be an irrational number. Then the sequence $\{j\alpha\}$ is uniformly distributed mod 1.*

Proof: follows immediately from Proposition 2.1 and the fact that for $t \in \mathbb{R}$ and $a, b \in (0,1), \langle t \rangle \in (a,b)$ if and only if $e^{2\pi i t}$ belongs to the arc $\{e^{2\pi i s} \mid a < s < b\}$ in $\mathbb{S}^1$.

We mention the following interesting consequence of the Corollary 2.2. Let $\{d_j\}$ be the sequence formed by taking the leading digit in the (usual) expression for 2^j in base 10; $d_1 = 2, d_2 = 4, d_3 = 8, d_4 = 1, d_5 = 3, \cdots$. How frequently will we see the digit 1 in this sequence? Specifically this question would mean the following: if we count the number of 1's in $\{d_1, \ldots, d_k\}$ and divide it by k to get the average, will the ratio have a limit as k tends to infinity and if so what is the limit? The answer is that it indeed tends to $\log_{10} 2$; so we will see 1's a little over 30% of the time if we follow the sequence long enough. This readily follows from the above corollary 2.2 applied to $\alpha = \log_{10} 2$.

The rotations as above form a rather special class of dynamical systems and such 'neat' behaviour does not occur in general. However a large class of dynamical systems come close to having such properties in some weak ways and some results of this kind constitute an important theme in ergodic theory and dynamical systems. We now sketch some of these properties. A crucial role is played in this by finite invariant measures for the dynamical systems. (From this point on it would be necessary to have some familiarity with general measure theory).

Let (X, T) be a dynamical system. By a measure on X we shall mean a Borel measure, namely a measure defined on the σ-algebra of all Borel sets. A measure μ is said to be finite if $\mu(X)$ is finite, and it is said to be T-invariant if $\mu(T^{-1}(E)) = \mu(E)$ for any Borel set E; $T^{-1}(E)$ denotes the set $\{x \mid Tx \in E\}$).

Theorem 2.3. (Poincaré recurrence lemma) *Let (X, T) be a dynamical system and let R be the set of its recurrent points. Suppose that the topology of X has a countable base. Let μ be any finite T-invariant measure on X. Then for the complement $X \backslash R$ of R in X we have $\mu(X \backslash R) = 0$.*

When the space X is noncompact a transformation T need not admit a finite invariant measure; e.g. $X = \mathbb{R}$ and $Tx = x+1$ for all $x \in \mathbb{R}$. However, when X is compact the existence of such a measure is assured, namely we have the following theorem.

Theorem 2.4. (Krylov and Bogoliubov) *Let (X, T) be a dynamical system where X is compact. Then there exist finite T-invariant measures on X*

The two results together imply in particular that any dynamical system on a compact second countable space admits recurrent points. This may not hold on non-compact spaces; this is again illustrated by the example $X = \mathbb{R}$ and $T(x) = x + 1$ for all $x \in \mathbb{R}$, for which there are no recurrent points.

3. Some ergodic theory

We next recall a fundamental theorem from ergodic theory. In the formulation we shall restrict to dynamical systems as above, though the theorem holds in the generality of measure-preserving transformations of measure spaces. We recall that $L^1(X, \mu)$ denotes the space of all measurable functions f on X such that $\int_X |f| d\mu < \infty$.

Theorem 3.1. (Birkhoff's ergodic theorem) *Let (X, T) be a dynamical system and let μ be a T-invariant measure on X. Let $f \in L^1(X, \mu)$. Then as $k \to \infty$, $\frac{1}{k} \sum_{j=0}^{k-1} f(T^j x)$ converges μ-almost everywhere.*

We note that in general the set of points where the convergence holds is a proper subset and depends on the function f.

Given a dynamical system (X, T), a T-invariant measure on X is said to be *ergodic* if for any Borel subset E such that $T^{-1}(E) = E$ either $\mu(E) = 0$ or $\mu(X - E) = 0$. The condition means that the system can not be 'partitioned' in to two measure-theoretically nontrivial parts. Typically finer and finer partitions into invariant sets may be possible.

In general the class of ergodic invariant measures of a dynamical system can be large, and even uncountable. A general T-invariant measure μ such that $\mu(X) = 1$ can be expressed, in a certain canonical way, as a 'continuous sum' (or 'integral') or ergodic invariant measures each with total measure 1.

It is not difficult to deduce that if μ as in the hypothesis of Birkhoff's ergodic theorem is an ergodic T-invariant measure and $\mu(X) = 1$, then the limit in the conclusion coincides with $\int_X f d\mu$ for μ-almost all points. Together with the results recalled above one can deduce from this the following result about uniformly distributed orbits.

Theorem 3.2. *Let (X, T) be a dynamical system and suppose that X has a countable base. Let μ be a T-invariant measure on X such that $\mu(X) < \infty$. Then for μ-almost all x the orbit of x is uniformly distributed. If μ is ergodic then the associated integral is the integral corresponding to μ, for μ-almost all x.*

The asymptotic integral corresponding to the uniformly distributed orbits as in the theorem would of vary with the measure under consideration. We note in particular that if μ is an invariant measure and there is a compact invariant subset A such that $\mu(A) > 0$ and $\mu(X - A) > 0$, there would be $x \in A$ whose orbit is uniformly distributed, for which the corresponding asymptotic integral would have to be different from the integral with respect to μ.

4. Some simple dynamical systems related to Number theory

The above discussion shows that in some sense 'most' orbits in a general dynamical system on a compact second countable space behave like those of the rotations of the circle. In many problems, especially those related to

Number Theory, it is important to know precisely which orbits have this property. We shall now discuss various examples and describe the situation in this respect.

Let n be a positive integer and let $X = \mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ (n copies), the n-dimensional torus. Let $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$ and let $T : X \to X$ be defined by $T(z_1, z_2, , \ldots, z_n) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n)$ for all $z_1, z_2, \ldots, z_n \in \mathbb{S}^1$. It is a homeomorphism consisting of (possibly different) rotations in each coordinate. These systems are called translations of the torus. This case is somewhat similar to the rotation of the circle.

Proposition 4.1. *Let the notation be as above. Then all orbits are uniformly distributed. If there do not exist any integers $k_1, \ldots, k_n$ with at least one k_j nonzero such that $\sum_{j=1}^n k_j\theta_j$ is a multiple of π, then the asymptotic integral corresponding to the orbits is the integral with respect to the product measure $\lambda \times \lambda \times \ldots \times \lambda$, where λ is $\frac{1}{2\pi}$ times the angle measure on $\mathbb{S}^1$. There exist dense orbits if and only if the preceding condition holds and in that case all orbits are dense.*

Unlike in the one-dimensional case where the systems are either periodic or have all orbits dense, in higher dimensions there are 'intermediate' possibilities, where the closure of the orbit could be a finite union of tori of some intermediate dimension. Using the above Proposition 4.1 one can get the following result on simultaneous Diophantine approximation with linear forms; we recall that by a linear form on $\mathbb{R}^n$ one means a function of the form $L(x_1, \ldots, x_n) = \sum a_j x_j$ for all $x_1, \ldots, x_n \in \mathbb{R}$, where $a_1, \ldots, a_n$ are real constants (coefficients).

Corollary 4.2. *Let $L_1, L_2, \ldots, L_k$ be linear forms on $\mathbb{R}^n$, where $1 \leq k \leq n-1$. Suppose that there do not exist any $c_1, \ldots, c_k \in \mathbb{R}$, except all 0's, such that $c_1L_1 + c_2L_2 + \ldots + c_kL_k$ is a linear form with rational coefficients. Then for any $a_1, a_2, \ldots a_k \in \mathbb{R}$ $\epsilon > 0$ there exist integers $x_1, \ldots x_n$ such that for all $j = 1, 2, \ldots, k$*

$$|L_j(x_1, \ldots, x_n) - a_j| < \epsilon.$$

Next let $X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Let $A = (a_{ij})$ be a 2×2 matrix with integer entries and determinant 1. Let $T : X \to X$ be defined by

$$T(z_1, z_2) = (z_1^{a_{11}} z_2^{a_{12}}, z_1^{a_{21}} z_2^{a_{22}}) \qquad \forall z_1, z_2 \in \mathbb{S}^1.$$

Then T is a homeomorphism of X and thus (X, T) is a dynamical system. These systems are automorphisms of the group $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$; (similar systems are studied in higher dimensions and also on more general groups, but for simplicity we restrict ourselves to $\mathbb{T}^2$). In this case the behaviour depends crucially on the eigenvalues of the matrix A. If A is unipotent (namely 1 is the only eigenvalue or, equivalently, $(A - I)^2 = 0$ where I denotes the identity matrix), then T consists of rotations along certain embedded family of circles (at different angles) and some fixed points; when $a_{11} = a_{12} = a_{22} = 1$ and $a_{21} = 0$, the second coordinate is unaltered and in the circle along the first coordinate the transformation is a rotation by as much as the second coordinate. In this case all orbits are uniformly distributed but no orbit is dense in $\mathbb{T}^2$. If both eigenvalues of A are roots of unity other than 1 then T is periodic.

When there exists an eigenvalue which is not a root of unity then there have to be two real eigenvalues, one of absolute value greater than 1 and other of absolute value less than 1; such an automorphism (when there is no eigenvalue of absolute value 1) is called a hyperbolic automorphism. In this case there is great variety of orbits. Some orbits are periodic, some are dense, while there are also others whose closures are totally disconnected Cantor-like sets. While there have to exist uniformly distributed orbits, there are also orbits which are not uniformly distributed and even points which are not recurrent. For these transformations explicit description of the set of points for which recurrence or density or uniform distribution etc. holds seems to be a hopeless task.

One can also consider composites of translations and automorphisms. When the automorphism is hyperbolic, the behaviour of the composite is like the hyperbolic automorphism. However when the automorphism is defined by a unipotent matrix we can have composites all whose orbits are dense and uniformly distributed. For simplicity we note only the following illustrative example.

Proposition 4.3. (Furstenberg) *Let $X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Let $\theta \in \mathbb{R}$ be such that $\frac{\theta}{\pi}$ is irrational and let $T : X \to X$ be the homeomorphism defined by $T(z_1, z_2) = (z_1 z_2, e^{i\theta} z_2)$ for all $z_1, z_2 \in \mathbb{S}^1$. Then every orbit of T is uniformly distributed and the corresponding asymptotic integral is the integral with respect to the product measure $\lambda \times \lambda$, where λ is $\frac{1}{2\pi}$ times the angle measure on $\mathbb{S}^1$. In particular every orbit is dense in $\mathbb{T}^2$.*

A similar result holds in greater generality and in all dimensions and it has the following interesting consequence.

Corollary 4.4. (H. Weyl) *Let* $P(t) = \sum_{j=0}^{n} a_j t^j$ *be a polynomial with real coefficients* $a_j, j = 0, 1, \cdots, n$, *and suppose that at least one of* $a_1, \cdots, a_n$ *is irrational. Then the sequence* $\{P(k)\}$ *is uniformly distributed mod* 1.

5. Homogeneous dynamics and Diophantine approximation

Next let X be the space of lattices in $\mathbb{R}^n$, where $n \geq 2$ is fixed; (a lattice in $\mathbb{R}^n$ is a subgroup generated by a set of n vectors which form a basis of $\mathbb{R}^n$); the subgroup would consist of integral combinations of the vectors. We define a topology on X as follows: a subset $\mathscr{O}$ of X is said to be open if for any $\Lambda \in \mathscr{O}$ and any basis B of $\mathbb{R}^n$ generating Λ there exists an $\epsilon > 0$ such that for any basis B' of $\mathbb{R}^n$ formed by choosing one point each from the ϵ-neighbourhoods of the points of B, the lattice generated by B' belongs to $\mathscr{O}$. It can be verified that this indeed defines a topology on X. Further, X is a locally compact space with respect to this topology; in fact it is a manifold of dimensions n^2 (that is, each point has a neighbourhood which is homeomorphic to $\mathbb{R}^{n^2}$).

On X there is a canonical class of dynamical systems arising from invertible linear transformations of $\mathbb{R}^n$: for any invertible linear transformation A we define $T_A : X \to X$ by setting, for any lattice Λ, $T_A(\Lambda)$ to be the lattice $A(\Lambda) = \{A(v) \mid v \in \Lambda\}$. It is easy to see that each T_A is a homeomorphism of X.

The study of the dynamics of these systems has some very important applications to questions in Diophantine approximation. A longstanding conjecture due to A. Oppenheim about values of quadratic forms at integral points was settled by G. A. Margulis in 1987 via such a study, proving the following.

Theorem 5.1. (G.A.Margulis) *Let* $Q(x_1, \cdots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$, *where* $n \geq 3$, *and* (a_{ij}) *is a symmetric non-singular matrix. Suppose that there exist* i, j, k *and* l *such that* $a_{kl} \neq 0$ *and* a_{ij}/a_{kl} *is irrational. Then the set* $\{Q(p_1, \cdots, p_n) \mid p_1, \cdots, p_n \in \mathbb{Z}\}$ *is dense in* $\mathbb{R}$; *namely for any* $\alpha \in \mathbb{R}$ *and* $\epsilon > 0$ *there exist* $p_1, \cdots, p_n \in \mathbb{Z}$ *such that*

$$|Q(p_1, \cdots, p_n) - \alpha| < \epsilon.$$

In the 1990s some remarkable results were proved by Marina Ratner, on the theme of invariant measures and uniform distribution of orbits of actions on 'homogeneous spaces' by translation by 'unipotent elements'; the subspace X_1 of the space X of lattices in $\mathbb{R}^n$ (as above), consisting of those lattices whose discriminant (namely the volume of the paralleloped in $\mathbb{R}^n$ corresponding to the generating vectors together with the zero vector, or equivalently the determinant of the matrix corresponding to the n vectors) is 1, is a typical illustrative example of a homogeneous space as involved in Ratner's theory, and the translating 'unipotent' elements in the general case correspond to unipotent linear transformations (namely those for which 1 is the only complex eigenvalue) in this case. The space X_1 is a manifold and admits a natural *finite* measure which is invariant under actions of all linear transformations with determinant 1, enabling study of orbits via methods discussed above. Ratner's results yield in particular the following.

Theorem 5.2. (M. Ratner) *If A is a unipotent linear transformation then all orbits of T_A on X (notation as above) are uniformly distributed.*

Ratner's results also give a description of all invariant measures of the systems and in particular of the asymptotic integrals of the orbits and settle a conjecture of M.S. Raghunathan, about closures of the orbits. Note that the orbits of the systems as above are not dense in X. For $\Lambda \in X$ the set of lattices with the same discriminant as Λ is a closed set invariant under T_A as above; in particular if $\Lambda \in X_1$ then the closure is contained in X_1; while for most $\Lambda \in X_1$ the closure is X_1 itself, for a class of exceptional lattices it could be still smaller.

To give a description of orbits of T_A in X_1 we first recall the following definition: a subgroup H of $\mathrm{GL}(n, \mathbb{R})$ is said to be an *algebraic subgroup defined over rationals* if there exist some polynomials $P_1, \cdots P_l$ in the n^2 variables having rational coefficients and such that $H = \big\{x = (x_{ij}) \in \mathrm{GL}(n, \mathbb{R}) \mid P_k(x_{ij}) = 0 \text{ for all } k = 1, \cdots, l\big\}$.

Corollary 5.3. *Let $\Lambda \in X_1$, B be a basis of $\mathbb{R}^n$ contained in Λ and let (a_{ij}) be the matrix of the transformation A with respect to the basis B. The T_A-orbit of Λ is dense in X_1 if and only if any algebraic subgroup H of $\mathrm{GL}(n, \mathbb{R})$ which is defined over rational and contains (a_{ij}) contains all matrices of determinant* 1.

Subsequently Margulis and I (jointly) strengthened Ratner's results, where we study also the dependence on the starting point while considering uniform distribution, and applied to get lower estimates for the number of solutions in large balls, for the inequalities as in Theorem 5.1. We will not go in to the details here.

There has been a great deal of activity in the last three decades following Ratner's results spurred by Ratner's milestone results and its applications in various areas, in Number theory, Geometry as also certain more applied areas. The area has now been christened as "Homogeneous Dynamics". We will not go into more details here but mention one of the problems in Number theory that now stands as a challenge: a conjecture of Littlewood.

For any $t \in \mathbb{R}$ let $\delta(t)$ denote the distance of t from the nearest integer. Littlewood conjectures that for any $s, t \in \mathbb{R}$, $\liminf_{n\to\infty} n\delta(s)\delta(t) = 0$. Consider the set, say $\mathcal{E}$, of $(s,t) \in \mathbb{R} \times \mathbb{R}$ for which this is not true; conjecturally $\mathcal{E}$ is empty. There have been results asserting various specific of pairs (s,t) not being contained in $\mathcal{E}$. It is relatively straightforward to show that $\mathcal{E}$ is a set of (planar) Lebesgue measure 0. Considerable effort has now gone into understanding $\mathcal{E}$ by techniques of homogeneous dynamics. One of the notable results, by M. Einsiedler, A. Katok and E. Lindenstrauss in this respect (which was featured in the Fields medal citation of the last named author!) has been to show that $\mathcal{E}$ is a countable union of compact sets of 'box dimension' 0; the box dimension is a measure of size of a set and the smallness established in the result is a remarkable stride in showing the set to be very small (if nonempty), as also in terms of the techniques used; if the dynamical results involved in proving it could be strengthened to avoid a certain 'entropy condition' used, that would prove the full conjecture.

6. Suggestions for follow up

The following books are recommended basic material in the area:

(1) P. R. Halmos, Lectures in Ergodic Theory, The Mathematical Society of Japan, 1956.

(2) Peter Walters, An introduction to ergodic theory. Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

(3) M. Bachir Bekka and Matthias Mayer, Ergodic theory and topological dynamics of group actions on homogeneous spaces. London Mathematical Society Lecture Note Series, 269. Cambridge University Press, Cambridge, 2008.

(4) Manfred Einsiedler and Thomas Ward, Ergodic theory with a view towards number theory, Graduate Texts in Mathematics, 259. Springer-Verlag London, Ltd., London, 2011.

I would also suggest the following paper, giving a proof of the Oppenheim conjecture (Theorem 12 above) involving relatively little background.

(5) Shrikrishna G. Dani, On the Oppenheim conjecture on values of quadratic forms. Essays on geometry and related topics, Vol. 1, 2, 257–270, Monogr. Enseign. Math., 38, Enseignement Math., Geneva, 2001.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 213–215

ELEMENTARY PRIME COUNTING

DINESH S. THAKUR

Abstract. This is a short note describing some fun examples of elementary counting of primes.

1. Introduction

The first two chapters of [HW75], my first and fond interaction with number theory, give several proofs of the infinitude of primes together with lower bounds for the prime counting function $\pi(x)$ (being the number of primes $\leq x$) arising from making the proofs constructive. For example, the Euclid's argument producing a new prime as a divisor of $p_1 \cdots p_n + 1$, or the relative primality of Fermat numbers argument gives the lower bound (essentially) $\log\log(x)$ and a little more sophisticated proof [HW75, Sec. 2.6] (parts credited to Euler 1737 and Erdos 1938) gives $\log(x)/\log(4)$ lower bound.

While Tchebyshef's 1852 famous argument leading to essentially optimal lower bound $cx/\log(x)$ has now been simplified to just a few lines, this note describes intermediate bounds the author derived (in 1979-1980) essentially from the Euler product (from 1737) for the Riemann zeta (which essentially leads to infinitude of primes by divergence of harmonic series, or perversely by irrationality of π^2 and Euler's evaluation of $\zeta(2)$).

The author has not seen these simple arguments anywhere in the literature, and hopes that they might still be of some interest.

2. Euler product arguments

(1) Since positive integers are made up of primes, we see (**without even using unique factorization into primes**) that, for $s, x > 1$,

2010 Mathematics Subject Classification: 11N05, 11A41
Key words and phrases: Primes, Euler product

$$\frac{(x+1)^{1-s}}{1-s} - \frac{1}{1-s} = \int_1^{x+1} \frac{dx}{x^s} \leq \sum_{n \leq x} \frac{1}{n^s} \leq \prod_{p \leq x} (1 - \frac{1}{p^s})^{-1}$$

$$\leq \prod_{2 \leq t \leq \pi(x)+1} (1 - \frac{1}{t^s})^{-1} \leq e^{e \sum 1/t^s} \leq e^{e((\pi(x)+1)^{1-s}/(1-s) - 1/(1-s))}.$$

With $1 - s = e/\sqrt{\log(x)}$, we get $\pi(x) + 1 \geq (1+e)^{\sqrt{\log(x)}/e}$, for large x.

Details and variants of the manipulations: (i) In more detail, with $P := \pi(x) + 1$ and $1 - s = 1/z$, the inequality, for large z, is $ez(P^{1/z} - 1) > \log(z((x+1)^{1/z} - 1)$, so that $ezP^{1/z} > ez + (\log(x)/z)$ equivalent to the claim.

(ii) For simplicity, we have used $e^{bu} \geq 1/(1-u)$, for $b = e$, $0 \leq u \leq 1/2$, but could have improved the bound by a constant power by optimising for $b > 1$ when it works for sufficiently small positive u, by splitting the sum over t's into small and large t's. We leave it to the interested reader.

(2) **A different elementary argument**: Taking logarithm of the Euler product inequality shows $1 + \sum_{p \leq x}(1/p) > \log\log(x)$. Raise this to the n-th power and note that in the expanded product there are $(\pi(x)+1)^n = P^n$ reciprocals, and none can occur more than $n!$ times (by unique factorization), giving

$$n!(1 + \frac{1}{2} + \cdots + \frac{1}{P^n}) > (\log\log(x))^n$$

so that roughly $n! n \log(P) > (\log\log(x))^n$, which with $n = \log\log(x)$, by the Stirling approximation of $n!$ gives $\pi(x) > e^{\log(x)/(\log\log(x))^{3/2+\epsilon}}$, for large x.

Details of the manipulations: Writing ℓ_k for the k-th iterated logarithm, we have $n!n(\ell_1(P) + (\gamma + \epsilon)/n) > (\ell_2(x))^n$. So

$$C + n\ell_1(n) - n + (3/2)\ell_1(n) + \ell_2(P) > \ell_1(n!) + \ell_1(n) + \ell_2(P) > n\ell_3(x).$$

So, $n = \ell_2(x)$ gives $\ell_1\ell_1(P) = \ell_2(P) > \ell_2(x) - (3/2)\ell_3(x) - C > \ell_1(\ell_1(x)/(\ell_2(x))^{3/2+\epsilon})$

Remarks The Euler product heuristically (under very strong regularity assumptions) leads to the prime number theorem itself.

Acknowledgements and dedication I dedicate this paper (on his 75th birthday) to B. Godse, who introduced [HW75] book to me in school, to Mangesh Rege, to S. S. Rangachari and to the memory of S. Srinivasan, K. G. Ramanathan and K. Ramachandra from TIFR for their encouragement in my school and undergraduate years. In fact, S. Srinivasan had pointed out to me then that my first counting did not even use the unique factorization into primes.

References

[HW75] G. H. Hardy and E. M. Wright. *Introduction to the theory of numbers*, The English Language Book Society and Oxford University Press (1975), 4th edition.

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The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 1-2, January - June (2024), 216–224

PROBLEM SECTION

In the Mathematics Student, volume 92(3-4) 2023, we had invited solutions for Problems 5, 6 from MS 92(1-2) 2023 and for the eight new problems of MS 92(3-4) 2023 till February, 2024.

We have received solutions to Problems 5, 6 of MS 92 (1-2) 2023 by Dr. Andrés Ventas, Santiago de Compostela, Spain; these problems were proposed by Dr. Anup Dixit, The Institute of Mathematical Sciences, India. For the new problems from MS 92(3-4) 2023, we have received solutions for Problems 1, 3, 4, 5. Problem 1 was suggested by Dr. Shpetim Rexhepi and Dr. Ilir Demiri, Mother Teresa University, North Macedonia and the solution was provided by Dr. Henry Ricardo, Westchester Area Math Circle, New York, USA. Problems 2-5 were proposed by Dr. B. Sury, Indian Statistical Institute, India and Problems 6-8 were proposed by Dr. Chudamani Pranesachar Anil Kumar, KREA University, India. We didn't receive any solutions for Problems 2, 6, 7, 8. Below we present the solutions received based on the recommendations of the proposers and the experts. We appreciate the contributions from the proposers and sincerely acknowledge all solutions received from the readers.

First we present new problems for this volume. We invite solutions for these and for problems 2, 6, 7, 8 of MS 92 (3-4) 2023 from the readers till September 30, 2024. Correct solutions received by this date will be published in volume 93 (3-4) 2024 of The Mathematics Student, which is scheduled to be published in October 2024.

New Problems

Problems 1-3. are proposed by **Dr. Chudamani Pranesachar Anil Kumar**, KREA University, India.

MS 93(1-2) 2024 : Problem 1.

Let m, n be two positive integers such that $2 \leq m \leq n$. For a permutation $\sigma \in S_n, 0 \leq i \leq m-1$, let

$$T_i = \{\sigma \in S_n \mid inv(\sigma) \equiv i \mod m\}.$$

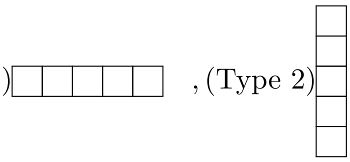
Show that $| T_0 |=| T_1 |= \cdots =| T_{m-1} |$, that is, the sets T_i have equal cardinality for $0 \leq i \leq m-1$ and find this cardinality.

MS 93 (1-2) 2024 : Problem 2.

In a 2023×2023 chess-board (not the usual 8×8), the four corner squares are removed.

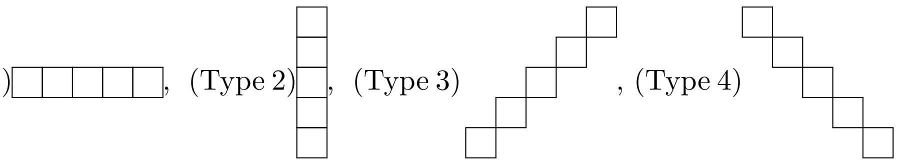
(1) Can the rest be covered by a combination of 5×1 dominoes, (that is, 5 square boxes) by putting them horizontally, vertically on the board?

(Type 1), (Type 2)



(2) Can the rest be covered by a combination of 5×1 dominoes, (that is, 5 square boxes) by putting them horizontally, vertically or diagonally in both ways (as shown in the figure) on the board?

(Type 1), (Type 2), (Type 3), (Type 4)



MS 93(1-2) 2024 : Problem 3.

Let $P = \{p_1 = 2 < p_2 < \ldots < p_l\}$ be a finite set of l primes. Let

$$T = \{n \in \mathbb{Z} \mid \gcd(n, p_i) = 1,\ 1 \leq i \leq l\} = \{\ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots\}.$$

Then prove the following:

(1) The sequence $x_j = a_{j+1} - a_j$, $j \in \mathbb{Z}$ is bounded.
(2) There exists $j \in \mathbb{Z}$ such that $2l \leq x_j = a_{j+1} - a_j$.

Problems 4-6 are proposed by **Dr. B. Sury**, ISI, Bengaluru, India.

MS 93(1-2) 2024 : Problem 4.

Let $f : [0,\, 1] \to \mathbb{R}$ be differentiable. Suppose $f,\, f'$ have no common zeroes. Prove that the zero set of f :$\{x \in [0,1] \mid\ f(x) = 0\}$ must be finite.

MS 93(1-2) 2024 : Problem 5.

Find all triangles with vertices $A = (0,0)$, $B = (4,3)$ and $C = (u,v)$ where u, v are integers and AB, AC have integer lengths.

MS 93(1-2) 2024 : Problem 6.

Let G be a group on which the m-th power map and n-th power,$m, n \in \mathbb{N} \setminus \{1\}$, map are both homomorphisms. If $m(m-1)/2$ and $n(n-1)/2$ are relatively prime, prove that G must be abelian. Conversely, if the greatest common divisor of $m(m-1)/2$ and $n(n-1)/2$ is > 1, show there exist non-abelian groups G on which the m-th power map and the n-th power maps are homomorphisms.

MS 93(1-2) 2024 : Problems 7. (Proposed by **Dr. Shpetim Rexhepi** and **Dr. Ilir Demiri**, Mother Teresa University, North Macedonia.)

Let α, β and γ be any real numbers satisfying $\alpha^2(\beta+\gamma) + \beta^2(\gamma+\alpha) + \gamma^2(\alpha+\beta) = 0$. Let

$$A := \sin^4\alpha + \sin^4\beta - 16\cos\left(\frac{4\alpha+\beta+2\gamma}{2}\right)\ \sin^4\left(\frac{\alpha+\beta}{2}\right),$$

$$B := \sin^3\alpha + \sin^3\beta - 8\cos^3\left(\frac{3\alpha+\beta+2\gamma}{2}\right)\ \sin^3\left(\frac{\alpha+\beta}{2}\right)$$

and

$$C := \sin^7\alpha + \sin^7\beta - 128\cos^7\left(\frac{\alpha+\beta+2\gamma}{2}\right)\ \sin^7\left(\frac{\alpha+\beta}{2}\right).$$

Prove that $\dfrac{AB}{C} = \dfrac{6}{7}$.

MS 93(1-2) 2024 : Problems 8. (Proposed by **Dr. Andrés Ventas** Santiago de Compostela, Spain.)

Given the following two constants defined by continued fractions with all their coefficients repeated, one with a positive sign, the Golden Section, $\varphi = [1, 1, 1, 1, \cdots]$, and the other with a negative sign, the Pena Trevinca constant, τ,

$$\tau = 3 - \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{3 - \cdots}}}$$

Prove that $\tau = \varphi + 1$.

Solutions to the New Problems

MS 92 (3-4) 2023 : Problem 1. (Proposed by **Dr. Shpetim Rexhepi** and **Dr. Ilir Demiri**, Mother Teresa University, Skopje, North Macedonia)

If d_1 and d_2 are metrics on the metric space X and $x_1, x_2, ..., x_n$ are in X, then prove that:

$$\frac{\sum_{i=1}^{n-1} d_1^2\left(x_i, x_{i+1}\right)}{1 + d_2(x_1, x_n) + n} \;\geq\; \frac{d_1^2(x_1, x_n)}{(n-1)\left(1 + n + \sum_{i=1}^{n-1} d_2\left(x_i, x_{i+1}\right)\right)}.$$

Solution: (by **Dr. Henry Ricardo**, Westchester Area Math Circle, New York, USA).

We use the Cauchy-Schwarz inequality in the form

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \;\geq\; \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i} \tag{1}$$

and the generalized triangle inequality for metrics d:

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n). \tag{2}$$

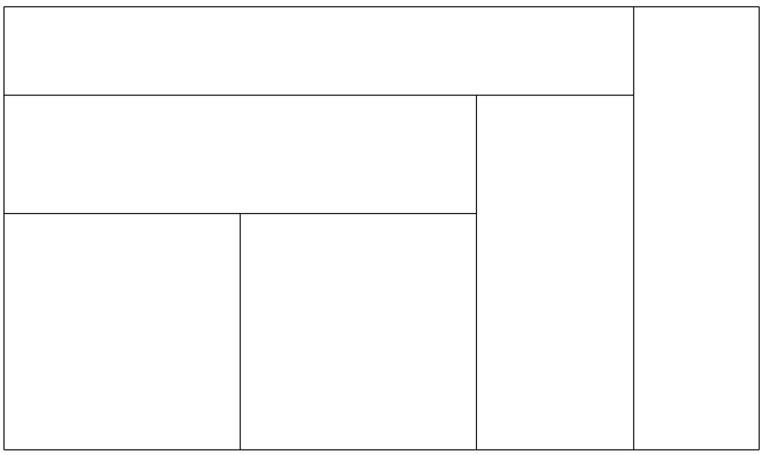
Now we have

$$\begin{aligned}
\frac{\sum_{i=1}^{n-1} d_1^2(x_i, x_{i+1})}{1 + d_2(x_1, x_n) + n} &= \sum_{i=1}^{n-1} \frac{d_1^2(x_i, x_{i+1})}{1 + d_2(x_1, x_n) + n} \\
&\overset{(1)}{\geq} \frac{\left(\sum_{i=1}^{n-1} d_1(x_i, x_{i+1})\right)^2}{(n-1) + (n-1)d_2(x_1, x_n) + n(n-1)} \\
&\overset{(2)}{\geq} \frac{d_1^2(x_1, x_n)}{(n-1)\left(1 + n + \sum_{i=1}^{n-1} d_2(x_i, x_{i+1})\right)}.
\end{aligned}$$

Problems 3, 4, 5 of 92(3-4) were proposed by **Dr. B. Sury**, ISI, Bengaluru, India.

92 (3-4) 2023 Problem 3.

Start with a unit square. To its right, adjoin a square of unit area. Then, we have a rectangle of base 2 and height 1. Now, adjoin a rectangle on top of the earlier one which has the same base (2 in this case) and area 1. Thus, the new rectangle would have height 1/2. In this manner, recursively adjoin rectangles to the right and on the top to the previous one, each time with unit area as in the figure.



Find the limit of the ratio of the base to the height of the large rectangle formed at each stage as $n \to \infty$.

Solution: (by **Dr. Andrés Ventas**, Santiago de Compostela, Spain).

We start with a rectangle with base 2 and height 1. Next we build a small rectangle on top of this with base 2 and area 1. So its height must be 1/2. Now the new larger rectangle has base 2 and height $3/2 = 1 + 1/2$. Next we construct a small rectangle on the side of the new large rectangle so that the area of this small rectangle is 1; hence its base must be equal to the inverse of 3/2, which is 2/3. Thus the base of the new larger rectangle at this stage is equal to $2 + 2/3$. Next we construct a small rectangle on top of this with area 1 and calculate the height of the new larger rectangle. So

$$\begin{aligned}\text{height of the large rectangle} &= \text{height of the large rectangle two stages before}\\ &+ \text{inverse of the base of the previous large rectangle.}\end{aligned}$$

So starting with base 2, at each stage we calculate height or the base of the new larger rectangle alternatively. Let us calculate first elements of this sequence and here it is preferable not to simplify:

$a_0 = 2;$

$a_1 = \dfrac{1}{2} + 1 = \dfrac{3}{2};$

$a_2 = \dfrac{2}{3} + 2 = \dfrac{8}{3} = \dfrac{2\cdot 4}{3};$

$a_3 = \dfrac{3}{2\cdot 4} + \dfrac{3}{2} = \dfrac{3\cdot 5}{2\cdot 4};$

$a_4 = \dfrac{2\cdot 4}{3\cdot 5} + 2\cdot 43 = \dfrac{2\cdot 4\cdot 6}{3\cdot 5};$

$a_5 = \dfrac{3\cdot 5}{2\cdot 4\cdot 6} + \dfrac{3\cdot 5}{2\cdot 4} = \dfrac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6};$

$\cdots$

Looking for a pattern in the above we see that the double factorial notation will be useful. Denoted by $n!!$, the double factorial is the product of all the positive integers up to n that have the same parity (odd or even) as n. That is,

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k) = n(n-2)(n-4)\cdots$$

. Thus we have $a_n = \dfrac{n!!}{(n-1)!!}$, with n even for the base and n odd for the height. We need to calculate $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n-1}}$. For which, we need the Stirling asymptotic approximation,

$$n!! \sim \begin{cases} \sqrt{\pi n}\left(\dfrac{n}{e}\right)^{n/2}, & \text{if } n \text{ is even,} \\ \\ \sqrt{2n}\left(\dfrac{n}{e}\right)^{n/2}, & \text{if } n \text{ is odd.} \end{cases}$$

Using the above we calculate,

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{\dfrac{\sqrt{\pi n}\left(\dfrac{n}{e}\right)^{n/2}}{\sqrt{2n}\left(\dfrac{n}{e}\right)^{n/2}}}{\dfrac{\sqrt{2n}\left(\dfrac{n}{e}\right)^{n/2}}{\sqrt{\pi n}\left(\dfrac{n}{e}\right)^{n/2}}} = \lim_{n\to\infty} \frac{\left(\sqrt{\pi n}\left(\dfrac{n}{e}\right)^{n/2}\right)^2}{\left(\sqrt{2n}\left(\dfrac{n}{e}\right)^{n/2}\right)^2}$$

$$= \lim_{n\to\infty} \frac{\pi n\left(\dfrac{n}{e}\right)^{n}}{2n\left(\dfrac{n}{e}\right)^{n}} = \frac{\pi}{2}.$$

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92 (3-4) 2023 Problem 4.

For each real number $x \geq 0$, let $N_x = \{n : \lfloor nx \rfloor \text{ is even}\}$. If $f(x) = \sum_{n\in N_x} \frac{1}{2^n}$ for $x \in [0,1)$, prove that $\text{Inf}\{f(x) : x \in [0,1)\} = \frac{4}{7}$.

Solution: (by **Dr. Andrés Ventas**, Santiago de Compostela, Spain).

We shall show $\quad \text{Inf}\{f(x) : x \in [0,1)\} = \frac{2}{3}$.

When x goes to 0, the sum has all members for each n, because $\lfloor n \cdot 0 = 0 \rfloor = 0$ is even, then we get the supremum value of the sum.

When x goes to 1, we tend to alternate the members between odd and even, because $\lfloor n \cdot 1 \rfloor = n$. We would have $N_x = \{2, 4, 6, 8, \cdots\}$, starting at $1/4$, thus $\sum_{n\in N_x} \frac{1}{2^{2n}} = \frac{1/4}{1-1/4} = \frac{1}{3}$.

But 1 is excluded from the interval, so our infimum sum, the greater lower bound, has alternating members $N_x = \{1, 3, 5, 7, \cdots\}$, starting at $1/2$, this is $\sum_{n=1}^{\infty} \frac{1}{2^{2n-1}}$.

Thus, our geometric series has $a = \dfrac{1}{2}$ and $r = \dfrac{1}{4}$.

$$\text{Inf}\{f(x) : x \in [0,1)\} = \sum_{n \in 2n-1} \frac{1}{2^{2n-1}} = \frac{1/2}{1-1/4} = \frac{2}{3}.$$

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92 (3-4) 2023 Problem 5.

If f is a function on $\mathbb{R}$ satisfying $f(U) = \mathbb{R}$ for every nonempty open set U, then show that f is discontinuous at EVERY point, Further, show there exist such functions f.

Solution: (by **Dr. Henry Ricardo**, Westchester Area Math Circle, New York.)
Solution 1. The function in the hypothesis of the problem is said to be *strongly Darboux*. The "Conway base 13 function" is a Darboux function, an everywhere surjective function, that is discontinuous at every point of $\mathbb{R}$ (https://en.wikipedia.org/wiki/Conway_base_13_function)

Solution 2. In 2018, A. Bergfeldt devised a simpler example: If $\{x_i\}_i \in \mathbb{Z}^+$ is the binary expansion of x, so that each $x_i \in \{0,1\}$, define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{(-1)^{x_k}}{k} \text{ if the series converges, and } f(x) = 0 \text{ otherwise.}$$

This function is open and (according to Bergfeldt)
"clearly not continuous at any point".

https://en.wikipedia.org/wiki/Conway_base_13_function

Proposer's Notes:

(1) In addition to the two nice solutions above, the following third solution is due essentially to **Brian Scott**.
Consider a basis B of the $\mathbb{Q}$-vector space $\mathbb{R}/\mathbb{Q}$. Map B bijectively onto $\mathbb{R}$ and extend it $\mathbb{Q}$-linearly to a surjection from $\mathbb{R}/\mathbb{Q}$ to $\mathbb{R}$. Composing this surjection with the natural quotient map from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Q}$ of $\mathbb{Q}$-vector spaces, one has a function $f : \mathbb{R} \to \mathbb{R}$ with the asserted property.

(2) For Conway's base-13 function, readers may also refer to an article by **A. Ayyer, B. Sury** entitled *John Horton Conway,*

The Magical Genius Who Loved Games, Resonance, May 2021, p. 595-601.

(3) Also, for further aspects of this problem, readers are advised to refer to a paper by **Israel Halperin** entitled *Discontinuous functions with the Darboux property*, published in *Canadian Mathematical Bulletin, Vol. 2, No. 2, May 1959, pp. 111-118.*

FORM IV

1.	Place of Publication:	PUNE
2.	Periodicity of publication:	QUARTERLY
3.	Printer's Name: Nationality: Address:	DINESH BARVE INDIAN PARASURAM PROCESS 38/8, ERANDWANE PUNE-411 004, INDIA
4.	Publisher's Name: Nationality: Address:	M. M. SHIKARE INDIAN GENERAL SECRETARY THE INDIAN MATHEMATICAL SOCIETY EMERITUS PROFESSOR DEPARTMENT OF MATHEMATICS JSPM UNIVERSITY, PUNE 412207 MAHARASHTRA, INDIA
5.	Editor's Name: Nationality: Address:	G. P. YOUVARAJ INDIAN RAMANUJAN INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, UNIVERSITY OF MADRAS, CHENNAI 600005, TN, INDIA
6.	Names and addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital:	THE INDIAN MATHEMATICAL SOCIETY

I, M. M. Shikare, the General Secretary of the IMS, hereby declare that the particulars given above are true to the best of my knowledge and belief.

M. M. Shikare
Signature of the Publisher

Dated: May 15, 2024

Published by Prof. M. M. Shikare for the Indian Mathematical Society, type set by Prof. G. P. Youvaraj, Former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, Chepauk, Chennai - 600005, India and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune 411 041 (India).

Printed in India

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos.1-2, January - June (2024)

Edited by G. P. Youvaraj and published by M. M. Shikare, the General Secretary of the IMS, for the Indian Mathematical Society.

Typeset by Prof. G. P. Youvaraj, Former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, Chepauk, Chennai - 600005, India and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No.129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune - 411041, Maharashtra, India. Printed in India