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THE UNIVERSAL APPEAL OF MATHEMATICS*

GEETHA S. RAO

Mathematics is the Queen of all Sciences, the King of all Arts and the Master of all that is being surveyed. Such is the immaculate and immense potential of this all-pervasive, fascinating subject, that it transcends all geographical barriers, territorial domains and racial prejudices.

The four pillars that support the growth, development, flowering and fruition of this ever - green subject are analytic thinking, logical reasoning, critical reviewing and decision making.

Every situation in real life can be modeled and simulated in mathematical language. So much so, every human must be empowered with at least a smattering of mathematical knowledge. Indeed, the field of Artificial Intelligence is one where these concepts are implemented and imparted to the digital computers of today.

From times immemorial, people knew how to count and could trade using the barter system. Those who could join primary schools learnt the fundamental arithmetic and algebraic rules. Upon entry into high school and higher secondary classes, the acquaintance with the various branches of this exciting subject commences. It is at this point that effective communication skills of the teacher impact the comprehension and conceptual understanding of the students.

Unfortunately, if the teacher is unsure of the methods and rules involved, then begins a dislike of the subject by the students being taught. To prevent a carcinogenic spread of this dislike, the teacher ought to be suitably oriented and know precisely how to captivate the imagination of the students. If this is the case, the students enjoy the learning process and even start loving the subject, making them eagerly await the Mathematics classes, with bated breath!

* The text of the Presidential Address (general) delivered at the 79th Annual Conference of the Indian Mathematical Society held at Rajagiri School of Engineering and Technology, Rajagiri Valley, Kakkanad, Dist. Ernakulam, Cochin - 682 039, Kerala, during the period December 28 - 31, 2013.

Acquiring necessary knowledge of algebraic operations, permutations and combinations, rudiments of probabilistic methods, persuasive ideas from differential and integral calculus and modern set theory will strengthen the bonds of mathematical wisdom.

From that stage, when one enters the portals of university education, general or technical, the opportunity to expand one's horizon of mathematical initiation is stupendous. Besides, the effective use of Mathematics in Aeronautical, Agricultural, Biological, Chemical, Geographical and Physical Sciences, Engineering, Medicine, Meteorology, Robotics, Social Sciences and other branches of knowledge is indeed mind boggling.

Armed with this mathematical arsenal, the choice of a suitable career becomes very diverse. No two humans need to see eye-to-eye as far as such a choice is concerned, as the variety is staggering! So, it is crystal clear that studying Mathematics, at every level, is not only meaningful and worthwhile but absolutely essential.

A natural mathematical genius like Srinivasa Ramanujan was and continues to be an enigma,and a Swayambhu, who could dream of extraordinary mathematical formulae, without any formal training.

A formally trained mathematician is capable of achieving laudable goals and imminent success in everything that he chooses to learn and if possible, discover for himself, the eternal truths of mathematics, provided he pursues the subject with imagination, passion, vigour and zeal.

Nothing can be so overwhelming as a long standing problem affording a unique solution, by the creation of new tools, providing immense pleasure, a sense of reward and tremendous excitement in the voyage of discovery.

These flights of imagination and intuition form the core of research activities. With the advent of the computers, numerical algorithms gained in currency and greater precision, enabling the mathematical techniques to grow rapidly, by leaps and bounds!

Until the enunciation of the Uncertainty Principle of W. Heisenberg, in 1932, mathematics meant definite rules of certainty. One may venture to say that perhaps this is the origin of Fuzziness. L. A. Zadeh wrote a seminal paper, entitled Fuzzy sets, Information and Control, **8** (1965), 328-353. He must be considered a remarkable pioneer who invented the subject of fuzzy mathematics, which is an amalgam of mathematical rules and methods of probability put together to define domains of fuzziness.

Fuzzy means frayed, fluffy, blurred, or indistinct. On a cold wintry day, haziness is seen around dawn, and a person or an object at a distance, viewed through the mist, will appear hazy. This is a visual representation of fuzziness. The input variables in a fuzzy control system are mapped into sets of membership functions known as fuzzy sets. The process of converting a crisp input value to a fuzzy value is called fuzzification.

A control system may also have various types of switches or on-off inputs along with its analog inputs, and such switch inputs will have a truth value equal to either 0 or 1.

Given mappings of input variables into membership functions and truth values, the micro-controller makes decisions concerning what action should be taken, based on a set of rules. Fuzzy concepts are those that cannot be expressed as true or false, but rather as partially true!

Fuzzy logic is involved in approximating rather than precisely determining the value. Traditional control systems are based on mathematical models in which one or more differential equations that define the system's response to the inputs will be used. In many cases, the mathematical model of the control process may not exist, or may be too expensive, in terms of computer processing power and memory, and a system based on empirical rules may be more effective.

Furthermore, fuzzy logic is more suited to low-cost implementation based on inexpensive sensors, low resolution analog-to-digital converters and 4-bit or 8-bit microcontroller chips. Such systems can be easily upgraded by adding new rules/novel features to improve performance. In many cases, fuzzy control can be used to enhance the power of existing systems by adding an extra layer of intelligence to the current control system. In practice, there are several different ways to define a rule, but the most simple one employed is the max-min inference method, in which the output membership function is given the truth value generated by the underlying premise. It is important to note that rules involved in hardware are parallel, while in software they are sequential.

In 1985, interest in fuzzy systems was sparked by the Hitachi company in Japan, whose experts demonstrated the superiority of fuzzy control systems for trains. These ideas were quickly adopted and fuzzy systems were used to control accelerating, braking and stoppage of electric trains, which led to the historic introduction, in 1987, of the bullet train, with a speed of 200 miles per hour, between Tokyo and Sendai.

During an international conference of fuzzy researchers in Tokyo, in 1987, T. Yamakawa explained the use of fuzzy control, through a set of simple dedicated fuzzy logic chips, in an inverted pendulum experiment. The Japanese soon became infatuated with fuzzy systems and implemented these methods in a wide range of astonishing commercial and industrial applications.

In 1988, the vacuum cleaners of Matsushita used micro-controllers running fuzzy algorithms to interrogate dust sensors and adjust suction power accordingly. The Hitachi washing machines used fuzzy controllers to load-weight, fabric-mix and dirt sensors and automatically set the wash cycle for the optimum use of power, water and detergent.

The renowned Canon camera company developed an auto-focusing camera that uses a charge coupled device to measure the clarity of the image in six regions of its field of view and use the information provided to determine if the image is in focus. It also tracks the rate of change of lens movement during focusing and controls its speed to prevent overshoot.

Work on fuzzy systems is also being done in USA, Europe, China and India. NASA in USA has studied fuzzy control for automated space docking, as simulation showed that a fuzzy control system can greatly reduce fuel consumption. Firms such as Boeing, General Motors, Allen-Bradley, Chrysler, Eaton and Whirlpool have used fuzzy logic to improve on automotive transmissions, energy efficient electric motors, low power refrigerators, etc.

Researchers are concentrating on many applications of fuzzy control systems, have developed fuzzy expert systems and have integrated fuzzy logic, neural networks and adaptive genetic software systems, with the ultimate goal of building self-learning fuzzy control systems.

This, in my opinion, is sufficient reason to induce you to start learning mathematics!

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A NEW TOOL IN APPROXIMATION THEORY*

GEETHA S. RAO

Abstract. The concept of 'best coapproximation' is introduced. The properties of the cometric projection are discussed. The effectiveness of this tool is studied in detail. Connections between bases and best coapproximation in Banach spaces are pointed out.

1. Best coapproximation in a normed linear space

In 1970, I. Singer wrote a treatise on best approximation in normed linear spaces. C. Franchetti and M. Furi (1972) introduced an inequality while characterizing real Hilbert spaces among real reflexive Banach spaces. P.L. Papini and I. Singer (1979) introduced the definition of 'best coapproximation', which is a new tool in Approximation Theory. Let X be a real normed linear space and G a nonempty subset of X.

Definition 1.1. An element $g_0 \in G$ is called an element of best coapproximation (resp. best approximation) of $x \in X$ by the elements of G if

$$\|g_0 - g\| \leq \|x - g\|, \text{ for every } g \in G$$

$$(\text{resp.} \quad \|x - g_0\| \leq \|x - g\|, \text{ for every } g \in G)$$

.
Notation. The set of all the best coapproximations (respectively, best approximations) of $x \in X$ by the elements of G is denoted by $R_G(x)$ ($P_G(x)$).

Definition 1.2. The set-valued map R_G, defined on $D(R_G)$, by $x \to R_G(x)$ is called the cometric projection of X onto G.

Definition 1.3. The set-valued map P_G, defined on $D(P_G)$, by $x \to P_G(x)$ is called the metric projection of X onto G.

* The text of the Presidential Address (technical) delivered at the 79th Annual Conference of the Indian Mathematical Society held at Rajagiri School of Engineering and Technology, Rajagiri Valley, Kakkanad, Dist. Ernakulam, Cochin - 682 039, Kerala, during the period December 28 - 31, 2013.

Some properties of $R_G(x)$ and R_G

(1) G is contained in $D(R_G)$ and $R_G(x) = \{x\}$, if $x \in G$.
(2) If G is closed, $R_G(x)$ is closed.
(3) If G is convex, $R_G(x)$ is convex.
(4) If $x \in D(R_G)$, $R_G(x)$ is bounded.
(5) If $g_0 \in R_G(x)$, then $g_0 \in R_G(tx + (1-t)g_0)$, $t \geq 1$.
(6) If $0 \in G$ or if G is a linear subspace of X, then $\|g_0\| \leq \|x\|,\ g_0 \in R_G(x)$.
(7) If $g_0 \in R_G(x)$, then $\|x - g_0\| \leq 2\|x - g\|,\ g \in G$.
(8) If G' is a linear subspace of X such that $G \subset G'$, then $R_G(R_{G'}(x)) \subset R_G(x), x \in X$.
(9) If $x \in D(R_G)$, then $R_G(R_G(x)) = R_G(x)$.
(10) If $x \in D(R_G)$ and $g \in G$, then $x+g \in D(R_G)$ and $R_G(x+g) = R_G(x)+g$, *i.e.*, R_G is quasi-additive.
(11) If $x \in D(R_G)$ and $\alpha \in R$ then $\alpha x \in D(R_G)$ and $R_G(\alpha x) = \alpha R_G(x)$, *i.e.*, R_G is homogeneous.

2. The Gateaux derivative or the tangent functional

Definition 2.1. The right-hand Gateaux derivative or tangent functional of a pair of elements $x, y \in X$, denoted by $\tau(x, y)$, is defined by

$$\tau(x,y) = \lim_{t \to 0^+} \frac{||x + ty|| - ||x||}{t}.$$

Remarks.

(1) This limit exists always.
(2) The functional $\tau(x, y)$ is increasing with t and $-\tau(x, -y) \leq \tau(x, y)$. Equality holds for every pair x, y when X is a smooth Banach space.

Orthogonal retract map

Definition 2.2. Let G be a linear subspace of X and $x \in X$. An element $g_0 \in G$ is an orthogonal retract to X from G if $\tau(g_0 - g, x - g_0) \geq 0$, for every $g \in G$.

The set of all orthogonal retracts from G is denoted by $R'_G(x)$ and the map

$$R'_G : x \to R'_G(x)$$

is called the orthogonal retract map.

3. Orthogonality in a normed linear space

Definition 3.1 (G.D. Birkhoff, 1935)**.** Let G be a linear subspace of a real normed linear space X and let $x, y \in X$. Then x is orthogonal to y, written as $x \perp y$, if $\|x + \alpha y\| \geq \|x\|$, for every $\alpha \in \mathbb{R}$.

Further, $x \perp G$, if $x \perp g$, for every $g \in G$, and $G \perp x$, if $g \perp x$, for every $g \in G$.

Remarks.

(1) $g_0 \in R_G(x)$ if and only if $G \perp (x - g_0)$.
(2) $g_0 \in P_G(x)$ if and only if $(x - g_0) \perp G$.
(3) $R_G(x) = P_G(x)$ in inner product spaces and Hilbert spaces.
(4) In an arbitrary normed linear space $R_G(x) \neq P_G(x)$, in general.

Some special nomenclature

Let X be a real normed linear space and G a linear subspace of X.

Definition 3.2. G is a proximinal (resp. semi-Chebyshev or Chebyshev) subspace of X if $R_G(x)$ contains at least (resp. at most or exactly one) element.

4. **Continuity of R_G**

P.L. Papini and I. Singer (1979), GSR and K. R. Chandrasekaran (1986)and GSR and S. Muthukuamr (1987) proved results concerning the continuity of R_G.

Definition 4.1. A set G in X is boundedly compact if every bounded sequence in G has a convergent subsequence.

The following are true:

(1) If G is a finite-dimensional Chebyshev subspace of X, then R_G is continuous on X.
(2) If G is a Chebyshev set in X, then R_G is continuous on G.
(3) If for some fixed α, $x_n \in R_\alpha(x)$ and $x_n \to x$, then $R_G(x_n) \to R_G(x)$, where for $\alpha \geq 0$, we define

$$R_\alpha(x) = \{y \in X : \|R_G(x-y) - (R_G(x) - R_G(y))\| \leq \alpha\|x-y\|\}.$$

(4) If for some fixed α, $R_\alpha(x)$ contains a neighbourhood of x, then R_G is continuous at x.
(5) If $R_G^{-1}(0) + R_G^{-1}(0) \subset R_G^{-1}(0)$, then R_G is continuous.
(6) If G is also a hyperplane, then R_G is continuous.
(7) If R_G is continuous at the points of $R_G^{-1}(0)$, then R_G is continuous.
(8) If G is closed and $R_G^{-1}(0)$ is boundedly compact, then R_G is continuous.

5. **Semicontinuity of R_G**

Definition 5.1 (I. Singer)**.** Let X, Y be real normed linear spaces. Let 2^Y denote the collection of all bounded closed subsets of Y. A mapping $u : X \to 2^Y$ is said to be

(1) upper semicontinuous (resp. lower semicontinuous) if the set $\{x \in X : u(x) \cap N \neq \emptyset\}$ is closed for every closed set $N \subset Y$ (resp. open for every open set $N \subset Y$).
(2) upper (K) semicontinuous (resp. lower (K) semicontinuous) if the relations $\lim_{n\to\infty} x_n = 0$, $y_n \in u(x_n)$, $(n = 1, 2, \cdots)$, $\lim_{n\to\infty} y_n = y$ imply that $y \in u(x)$ (respectively, if the relations $\lim_{n\to\infty} x_n = x$, $y \in u(x)$ imply the existence of a sequence $\{y_n\}$ with $y_n \in u(x_n)$, $n = 1, 2, \cdots$, such that $\lim_{n\to\infty} y_n = y$.

6. Radial continuity of R_G and R'_G

W. Pollard, Diplomarbeit, University of Bonn, 1967, introduced the concept of radial continuity of a set-valued operator. This involves the restriction of the concerned set-valued maps to certain prescribed line segments to be upper semicontinuous or lower semicontinuous.

Definition 6.1 ((GSR and K.R. Chandrasekaran, 1981)**.** Let G be a non-empty set in X and $x_0 \in X$. R_G is said to be

(1) outer radially lower (ORL) continuous at x_0 if for each $g_0 \in R_G(x_0)$ and each open set W with $W \cap R_G(x_0) \neq \emptyset$, there exists a neighbourhood U of x_0 such that $R_G(x) \cap W \neq \emptyset$ for every $x \in U \cap \{g_0 + \alpha(x - g_0) : \alpha \geq 1\}$. R_G is ORL continuous if it is ORL continuous at each point.
(2) R_G is Inner Radially Lower (IRL) continuous at x_n if for each $g_0 \in R_G(x_0)$ and each open set W with $W \cap R_G(x_0) \neq \emptyset$, there exists a neighbourhood U of x_n such that $R_G(x) \cap W \neq \emptyset$ for for every $x \in U \cap \{g_0 + \alpha(x - g_0) : 0 \leq \alpha \leq 1\}$. R_G is IRL continuous if it is IRL continuous at each point.
(3) R_G is Outer Radially Upper (ORU) continuous at x_n if for each $g_0 \in R_G(x_0)$ and each open set $W \supset R_G(x_0)$, there exists a neighbourhood U of x_n such that $W \supset R_G(x)$ for every $x \in U \cap \{g + 0 + \alpha(x_0 - g_0 : \alpha \geq 1\}$. R_G is ORU continuous if it is ORU continuous at each point.
(4) R_G is inner radially upper (IRU) continuous at x_0 if for each $g_0 \in R_G(x_0)$ and each open set W with $W \supset R_G(x_0) \neq \emptyset$, there exists a neighbourhood U of x_0 such that $W \supset R_G(x) \neq \emptyset$ for every $x \in U \cap \{g_0 + \alpha(x - g_0) : 0 \leq \alpha \leq 1\}$. R_G is IRU continuous if it is ORU continuous at each point.

Some Results (GSR and K.R. Chandrasekaran)

(1) If G is a non-empty subset of X then R_G is ORL continuous and IRU continuous.
(2) Let G be a non-empty subset of $X, x_0 \in X$. If $\|x_0 - \beta g_0 - (1 - \beta) g_1\| =$ dist (x_0, G), for all $g_0, g_1 \in P_G(x_0)$ and $\beta \geq 1$, then P_G is IRL continuous at x_0.

(3) Let $x_0 \in X$. Let G be a non-empty subset of X such that $\beta g + (1-\beta)g_0 \in G$ for all $\beta \geq 1, g \in G$ and $g_0 \in R'_G(x_0)$. If $R'_G(x_0)$ is such that $\alpha g_0 + (1-\alpha)g_1 \in R'_G(x_0)$ for all $\alpha \geq 1, g_0, g_1 \in R'_G(x_0)$, then R'_G is IRL continuous at x_0.

(4) If G is a convex subset of X then R'_G is ORL continuous.

(5) Let G be a boundedly compact closed subset of X such that $R'_G(x) \neq \emptyset$, for every $x \in X$. Then R'_G is upper semicontinuous.

7. Metric projection bound, $MPB(X)$ and Cometric projection bound, $CMPB(X)$

M. A. Smith (1981) introduced the notion of metric projection bound of a real normed linear space X and proved that $1 \leq MPB(X) \leq 2$. GSR and K. R. Chandrasekaran (1986) introduced the notion of cometric projection bound of X, as follows:

Definition 7.1. The cometric projection bound of X, denoted by $CMPB(X)$, is defined by

$$CMPB(X) = sup\{\|R_G\| : G \text{ is a closed proximinal subspace of X}\},$$

where $\|R_G\| = sup\ \{\|y\| : y \in R_G(x),\ \|x\| \leq 1\}$.

Remark 7.1. $R_G = P_G$ for every subspace G of X if and only if $MPB(X) = 1$.

8. Linear selections for R_G

GSR and K.R. Chandrasekaran (1986): Let G be a proximinal subspace of a real normed linear space X.

Definition 8.1. A selection for R_G is a function $s : X \to G$ such that $s(x) \in R_G(x)$, for every $x \in X$.

A linear (resp. continuous) selection for R_G is a selection which is also linear (resp. continuous).

Theorem 8.1. *Let G be a Chebyshev subspace of X. The following statements are equivalent:*

(1) R_G is linear.

(2) $R_G^{-1}(0)$ is a subspace of X.

(3) $R_G^{-1}(0)$ contains a subspace H of X such that $X = G \oplus H$.

(4) R_G is quasi-linear, i.e. there exists a constant K such that

$$\|R_G(x+y)\| \leq K(\|R_G(x)\| + \|R_G(y)\|).$$

Theorem 8.2. *Let G be a proximinal subspace of X which is also a hyperplane. Then R_G admits a linear selection.*

Remark 8.1.

(1) If R_G admits a linear selection then R_G admits a continuous selection.
(2) R_G admits a linear selection if and only if $R_G^{-1}(0)$ contains a closed subspace H of X such that $X = G \oplus H$.
(3) Following H. K. Hsiao and R. Smarzewski (1995), GSR and R. Saravanan proved that every selection for R_G is a sunny selection, i.e., $s(f_\alpha) = s(f)$, for all $f \in X, \alpha \geq 0$ and $f_\alpha = \alpha f + (1-\alpha)\ s(f)$.

9. Modulus of continuity of R_G

Let X be a real normed linear space. Let A, B be nonempty subsets of X.

Definition 9.1. The deviation of A from B, denoted by $\delta(A,B)$, is defined by

$$\delta(A,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\| = \sup_{x\in A}\ \mathrm{dist}(x,B)$$

Following G. Godini (1981), GSR and K.R. Chandrasekaran defined :

Definition 9.2. Let G be a linear subspace of X. Let $x \in D(R_G), \varepsilon \geq 0$. The modulus of continuity of R_G is defined by

$$w_G(x, \in) = \delta(R_G(B(x,\varepsilon)), R_G(x)).$$

Results concerning modulus of continuity of R_G

(1) As $R_G(x) \subset R_G(B(x,\varepsilon))$, whenever $R_G(x)$ is not empty, $R_G(B(x,\varepsilon))$ is not empty.
(2) $w_G(x,0) = 0,\ w_G(x,\varepsilon_1) \leq w_G(x,\varepsilon_2),\ \varepsilon_1 \leq \varepsilon_2.$
(3) If $x \in D(R_G),\ \varepsilon \geq 0$, then $w_G(x+g,\varepsilon) = w_G(x,\varepsilon), t \in\ R.$
(4) $w_G(x+g,\varepsilon) = w_G(x,\varepsilon), g \in\ G.$
(5) $\varepsilon \leq w_G(x,\varepsilon) \leq \varepsilon + 2\|x\|.$

Cases where equality holds

(6) If G is a proximinal subspace of X which is also a hyperplane, then $w_G(x,\varepsilon) = \varepsilon$.
(7) If $R_G^{-1}(0)$ is a subspace of X, then $w_G(x,\varepsilon) = \varepsilon$.
(8) If X is a real inner product space, G is a linear subspace of X and x $\varepsilon D(R_G)$, then $w_G(x,\varepsilon) = \varepsilon$.

10. Lipschitz and uniform Lipschitz conditions

Let G be a linear subspace of a real normed linear space X.

Definition 10.1. Let $x \in D(R_G)$. R_G satisfies a Lipschitz condition at x if there is a constant C, depending on x and G, such that

$$\delta(R_G(y), R_G(x)) \leq C\|x-y\|, forally \in D(R_G).$$

Each real number C satisfying the inequality is called a Lipschitz constant for R_G at x. R_G satisfies a uniform Lipschitz condition on $M \subset D(R_G)$ if there exists a constant C, depending on M and G, which is a Lipschitz constant for R_G at each $x \in M$.

Theorem 10.1. *Let G be a linear subspace of a real normed linear space X. Then R_G satisfies a Lipschitz condition at $x \in D(R_G)$ if and only if*

$$\sup_{t>0} w_G(tx, 1) < \infty.$$

Remarks.

(1) If $x \in D(R_G)$ and $\sup_{t>0} \ w_G(tx, 1) < \infty$, then $\sup_{t>0} w_G \ (tx, 1)$ is the smallest Lipschitz constant for R_G at x.

(2) If G is a Chebyshev subspace of X and for $x \in X$, there exist constants $C > 0, \delta > 0$ such that

$$\|R_G(y) - R_G(x)\| \ \leq C\|x - y\|$$

whenever $y \in X$ and $\|x - y\| < \delta$, then for all $y \in X$

$$\|R_G(y) - R_G(x)\| \leq max(C, 1 + 2(| \ x\|\delta)\|x - y\|)$$

(3) If R_G satisfies a Lipschitz condition at every point $x \in S(0, 1) \cap R_G^{-1}(0)$, then R_G satisfies a Lipschitz condition at every point $y \in D(R_G)$.

Definition 10.2. Let G be a linear subspace of a real normed linear space X and $M \subset D(R_G)$. The modulus of upper (H) semicontinuity of R_G on M, denoted by $w_G(M; \varepsilon)$, is defined by

$$w_G(M; \varepsilon) = \sup w_G(x; \varepsilon) : x \in M, \varepsilon > 0.$$

Remark. When $M = D(R_G)$, $w_G(M; \varepsilon) = w_G(D(R_G); \varepsilon)$ is denoted by $w_G(\varepsilon)$. When is $w_G(M; \varepsilon)$ finite?

(1) When $M \backslash G$ is a bounded subset of X.

(2) When X is an inner product space.

(3) Let $G_b = \{x \in X : \text{dist}(x, G) \leq b\}$, $b > 0$. Let M $= G_b \cap D(R_G)$. Then $w_G(M; \varepsilon)$ is finite, for each $\varepsilon > 0$.

(4) If M is a cone with vertex at the origin and $M \subset D(R_G)$, then for each $\varepsilon > 0$, $w_G(M; \varepsilon) = \varepsilon w_G(M; 1)$.

Theorem 10.2. *Let G be a linear subspace of a real normed linear space X. Then R_G satisfies a uniform Lipschitz condition on $D(R_G)$ if and only if $w_G(1) < \infty$. Further, R_G satisfies a uniform Lipschitz condition on $D(R_G)$ if and only if there exist $b, r > 0$ such that $R_G(y) \cap B(0; r) \neq \emptyset$ for every $y \in (R_G^{-1}(0)_b) \cap D(R_G)$, where $y \in R_G^{-1}(0)_b$ implies that $dist(y, R_G^{-1}(0)) \leq b$.*

11. **Two fixed point theorems.**

Theorem 11.1. *Let G be a proximinal subspace of X such that*

$$G = G_1 \oplus G_2,$$

where G_1, G_2 are proximinal subspaces of X with

$$R_{G_1}^{-1}(0) \cap R_{G_2}^{-1}(0) = R_G^{-1}(0).$$

If $T_x : x \to G_1, x \in X$ and $T_x(y) = R_{G_1}(x - R_{G_2}(x - y)), y \in X$, then T_x has a unique fixed point.

Theorem 11.2. *Let G be a nonempty subset of X such that $R'_G(y) \neq \emptyset$, for every $y \in X$. Let $x \in X \backslash 0$ be such that $R'_G(x)$ is a singleton. For $r > 0$, let the many-valued mapping $f_{x,r} : X \to 2^X$ be defined by*

$$z \to x + \frac{r}{r + \|x - R'_G(x)\|}(z - R'_G(z))$$

$z \in X$. The following statements are equivalent :

(1) $z \in X$ is a fixed point of $f_{x,r}$.
(2) $\|z - x\| = r$ and there exists $y \in R'_G(z)$ such that $x \in (z, y) = \{tz + (1-t)y : 0 \leq t \leq 1.\}$

12. **Strong best approximation and strong best coapproximation**

M. W. Bartelt and H. W. McLaughlin (1973) studied the problem of strong best approximation. P. L. Papini (1978) introduced strong approximation via tangent functionals. G. Nuernberger (1979) obtained some special results concerning unicity and strong unicity for best approximation.

Definition 12.1. Let G be a subset of a normed linear space $X, f \in X \backslash G$ and $g_f \in G$. Then g_f is called a strongly unique best approximation to f from G, if there exists a constant $k_f > 0$ such that for all $g \in G$,

$$\|f - g_f\| \leq \|f - g\| - k_f \|g - g_f\|. \tag{12.1}$$

g_f is called a strongly unique best coapproximation to f from G, if there exists a constant $k_f > 0$ such that for all $g \in G$

$$\|g - g_f\| \leq \|f - g\| - k_f \|f - g_f\|. \tag{12.2}$$

GSR and R. Saravanan (2003) list the following results:

(1) It is clear that if some $k_f > 0$ satisfies (1) (resp. (1')), then every smaller value of $k_f > 0$ will also satisfy (1) (resp. (1')). The maximum of all such numbers $k_f > 0$ is called the strong unicity constant of f and is denoted by $K(f)$.

(2) Obviously, every strongly unique best approximation is a unique best approximation. But a strongly unique best coapproximation need not imply the uniqueness.

(3) It is clear that g_f is a strongly unique best coapproximation to f from G with the corresponding, strong unicity constant equal to 1 if and only if g_f is a strongly unique best approximation to f from G with the corresponding strong unicity constant equal to 1. Thus, the strongly unique best coapproximation implies the uniqueness if the corresponding strong unicity constant is equal to 1.

(4) In contrast to best coapproxiation, the strongly unique best coapproximation does not coincide with the strongly unique best approximation in inner product spaces. However, in such spaces, every strongly unique bestcoapproximation is unique.

13. **Two important estimates**

Theorem 13.1. *Let G be a finite dimensional subspace of X and $L : X \Rightarrow G$ be a projection with $\|L\| < 1$. Let $f \in X \backslash G$ and $g_f \in G$ be a best coapproximation to f. Then*

$$\|g - g_f\| \leq \frac{\|f - L(f)\|}{1 - \|L\|}.$$

Theorem 13.2. *Let G be a subset of X. Let $f_1, f_2 \in X \backslash G$, $g_{f_1} \in R_G(f_1)$, $g_{f_2} \in R_G(f_2)$, $g_1 \in P_G(f_1)$ and $g_2 \in P_G(f_2)$. Then*

$$max\{\frac{1}{2}\|f_1 - g_{f_1}\|, \frac{1}{2}\{\|g_{f_1} - g_{f_2}\| - \|f_1 - f_2\|\} \leq \|f_1 - g_1\| \tag{13.1}$$

and

$$max\{\frac{1}{2}\|f_2 - g_{f_2}\|, \frac{1}{2}\{\|g_{f_1} - g_{f_2}\| - \|f_1 - f_2\|\} \ \leq \ \|f_2 - g_2\|. \tag{13.2}$$

14. **Kolmogorov criterion for best uniform coapproximation**

GSR and R. Saravanan (1999, 2002) established the following results concerning the Kolmogorov criterion :

For all functions $f \in C[a, b]$, the uniform norm is defined by

$$\|f\| = \sup_{t \in [a,b]} |f(t)|.$$

Best coapproximation with respect to this norm is called best uniform coapproximation.

Definition 14.1. The set $E(f)$ of extreme points of a function $f \in C[a, b]$ is defined by

$$E(f) = \{t \in [a, b] : |f(t)| = \|f\|_\infty\}.$$

Theorem 14.1. *Let G be a subset of $C[a,b]$ such that $\alpha g \in G$, for all $g \in G$ and $\alpha \in [0,\infty)$. Let $f \in C[a,b]\backslash G$ and $g_f \in G$. The following are equivalent :*

(i) The function g_f is a best uniform coapproximation to f from G.

(ii) For every function $g \in G$,

$$\min_{t \in E(g-g_f)} (f(t)-g_f(t))/(g(t)-g_f(t)) \leq 0.$$

If, in addition, g_f is a unique best uniform coapproximation to f from G, then for every function $g \in G\backslash g_f$ and every set U containing $E(g-g_f)$,

$$\inf_{t \in G} (f(t)-g_f(t))(g(t)-g_f(t)) \leq 0.$$

Remark 14.1. Let G be a subset of $C[a,b]$, $f \in C[a,b]\backslash G$ and $g_f \in G$ be a best uniform coapproximation to f from G. It is known that for every $g \in G$,

$$\|f-g_f\|_\infty \leq 2\|f-g\|_\infty.$$

However, if G is a Chebyshev subspace, then a lower bound for $\|f-g_f\|_\infty$ can be obtained in the following way:

Theorem 14.2. *Let G be an n-dimensional Chebyshev subspace of $C[a,b]$, $f \in C[a,b]\backslash G$ and $g_f \in G$. If g_f is a best uniform coapproximation but not a best uniform approximation to f from G, then there exists a non-trivial function $g \in G$ such that*

$$\|g\| \leq \|f-g_f\|_\infty.$$

Following the technique of G. Nuernberger (1989) let $S_n(x_1,\cdots,x_k)$ denote the space of polynomial splines of degree n with k fixed knots $x_1,\cdots,x_k$.

Theorem 14.3. *Let $G = S_n(x_1,\cdots,x_k) \subset C[a,b], f \in C[a,b]\backslash G$ and $g_f \in G$. If g_f is a best uniform coapproximation but a best uniform approximation to f from G, then there exists a non-trivial spline $g \in G$ such that*

$$\|g\|_\infty \leq \|f-g_f\|_\infty.$$

A relation between interpolation and best uniform coapproximation can be obtained.

Theorem 14.4. *Let G be an n-dimensional Chebyshev subspace of $C[a,b], f \in C[a,b]\backslash G$ and $g_f \in G$. If g_f is a best uniform coapproximation to f from G then g_f interpolates f at at least n points of $[a,b)$.*

An upper bound for the error $\|f-g_f\|_\infty$ under some conditions:

Theorem 14.5. *Let G be a space of polynomials of degree n defined on $[a,b]$ and $f \in C^n[a,b]\backslash G$. If $g_f \in G$ is a best uniform coapproximation to f from G, then*

$$\|f-g_f\| \in \frac{1}{n!}\|f^{(n)}\|_\infty \|w\|_\infty,$$

where $w(x) = (x - x_1)\cdots(x - x_n)$ *and* $x_1, \cdots, x_n$ *are the points in* $[a, b]$ *at which* g_f *interpolates* f*.*

Theorem 14.6. *Let* G *be a subspace of* $C[a, b]$*,* $f \in C[a, b]\backslash G$ *and* $g_f \in G$ *be a best uniform coapproximation to* f *from* G*. Then there does not exist a function in* G*, which interpolates* $f - g_f$ *at its extremal points.*

15. Characterization of best L_1- coapproximation

Let $f \in C[a, b]$. The zero set of f is defined by

$$Z(f) = \{t \in [a, b] : f(t) = 0\}.$$

Following B. R. Kripke and T. J. Rivlin (1965). GSR and R. Saravanan (1999), established the following:

Theorem 15.1. *Let* G *be a subspace of* $C[a, b], f \in C[a, b]\backslash G$ *and* $g_f \in G$*. The following statements are equivalent :*

(i) The function g_f *is a best* L_1*-coapproximation to* f *from* G*.*
(ii) For every function $g \in G$*,*

$$\int_a^b [f(t) - g_f(t)]\ sgn(g(t))dt \leq \int_{Z(g)} |f(t) - g_f(t)|\ dt.$$

Theorem 15.2. *Let* G *be a subset of* $C[a, b], f \in C[a, b]\backslash G$ *and* $g_f \in G$*. Then the function* g_f *is a best* L_1*- coapproximation to* f *from* G *if and only if for all* $g \in G$*,*

$$\int_{[a,b]\backslash Z(f-g_f)} |g(t) - g_f(t)|dt \leq \int_{[a,b]\backslash Z(f-g_f)} |f(t) - g(t)|dt.$$

Theorem 15.3. *Let* G *be a subset of* $C[a, b], f \in C[a, b]\backslash G$ *and* $g_f \in G$*. Then each of the following statements implies that* g_f *is a best* L_1*-coapproximation to* f *from* G *:*

(i) For every function $g \in G$ *and all* $t \in [a, b]$*,*

$$(f(t) - g_f(t))\ (g_f(t) - g(t))\ \geq\ 0.$$

(ii) For every function $g \in G$ *and all* $t \in [a, b]$*,*

$$g(t) - |f(t) - g(t)| \leq g_f(t) \leq g(t) + |f(t) - g(t)|.$$

Remark 15.1. Statement (ii) also implies that

(a) g_f is a best L_1-approximation to f from G.
(b) g_f is a strongly unique best L_1-coapproximation and strongly unique best L_1-approximation to f from G.

Theorem 15.4. *Let G be a subset of $C[a,b], f \in C[a,b]\backslash G$ and $g_f \in G$. If for every $g \in G$,*

$$(g_f(t) - f(t)) \perp sgn(g_f(t) - g(t)), t \leq [a,b],$$

then g_f is a best L_1-approximation to f from G.

16. Characterization of best one-sided L_1-coapproximation

GSR and R. Saravanan (1999). Let G be a subset of $C[a,b]$ and $f \in C[a,b]$. A function $g_f \in G$ with $g_f \leq f$ is called a best one-sided L_1-coapproximation to f from G if for every $g \in G$ with $g \leq f$,

$$\|g - g_f\|_1 \leq \|f - g\|_1.$$

Example 16.1. Let $G \subset C[a,b]$ be defined by

$$G = \{g \in C[a,b] : g(t) = 0 \text{ for all } t \in [a,b]\}.$$

Then 0 is a best one-sided L_1-coapproximation to every $f \in C[a,b]\backslash G$.

Remarks.

(1) The set of best one-sided L_1-coapproximations is a convex set.

(2) Let G be a subset of $C[a,b]$ such that $0 \in G$ and $f \in C[a,b]\backslash G$. If 0 and $g_f \neq 0$ are best one-sided L_1-coapproximation to f from G, then

$$Z(f_\alpha - \alpha g_f) \subset Z(g_f), \ 0 < \alpha < 1.$$

Furthermore, if f and g_f are continuously differentiable then

$$Z(f' - \alpha g_f') \cap (Z(f - \alpha g_f) \cap (a,b)) \subset Z(g_f') \cap (Z(g_f) \cap (a,b)).$$

17. Best simultaneous coapproximation

Definition 17.1. (GSR and R. Saravanan, 1998). Let G be a subset of a normed linear space X. An element $g_0 \in G$ is called a best simultaneous coapproximation to $f \in X$ from G if

$$\|g - g_0\| \leq \max(\|f - g\|, \|f - g\|)$$

for every $g \in G$.
Denote by $S_G(f,f)$ the set of all best simultaneous coapproximation to f from G, for $f \in X$.

Properties.

(i) $S_G(f,\tilde{f})$ is closed if G is closed.

(ii) $S_G(f,\tilde{f})$ is convex if G is convex.

(iii) $S_G(f,\tilde{f})$ is bounded.

(iv) If $g_0 \in S_G(f,\tilde{f})$, then

$$g_0 \in S_G(\alpha^n f + (1-\alpha^n)g_0, \alpha^n \tilde{f} + (1-\alpha^n)g_0), \alpha \geq 1, n = 0, 1, 2, \cdots$$

(v) If $0 \in G$ or if G is a subspace of X, then $\|g_0\| \leq \max(\|f\|, \|\tilde{f}\|)$.

(vi) If $g_0 \in S_G(f, \tilde{f})$, then for every $g \in G$,

$$\|f - g_0\| \leq 2\max(\|f - g\|, \|\tilde{f} - g\|)$$

$$\|\tilde{f} - g_0\| \leq 2\max(\|f - g\|, \|\tilde{f} - g\|)$$

(vii) If G is finite dimensional, $S_G(f, \tilde{f})$ is compact.

(viii) If $f \in G$ and $\tilde{f} \in X \backslash G$, then $f \in S_G(f, \tilde{f})$. If $f, \tilde{f} \in G$, then $\alpha f + (1 - \alpha)\ \tilde{f}$, $0 \leq \alpha \leq 1$, is a best simultaneous coapproximation to $f, \tilde{f}$ from G. Moreover, every element in $S_G(f, \tilde{f})$ is of the form $\alpha f + (1 - \alpha)$. Thus, if $f, \tilde{f} \in G$, then

$$S_G(f, \tilde{f}) = \alpha f + (1 - \alpha)\tilde{f} : 0 \leq \alpha \leq 1$$

(ix) If G is a subspace of X, $f, \tilde{f} \in X$, then

$$S_G(f + g, \tilde{f} + g) = S_G(f, \tilde{f}) + g, \text{ for all } g \in G.$$

$$S_G(\alpha f, \alpha\tilde{f}) = S_G(f, \tilde{f}) + g, \text{ for all} g \in \Re.$$

(x) If $f, \tilde{f} \in X \backslash G$ and $g_0 \in G$, then if g_0 is a best coapproximation to f from G, g_0 is a best simultaneous coapproximation to $f, \tilde{f}$ from G. The converse is not true. i.e. if g_0 is a best simultaneous coapproximation to f, $\tilde{f} \in X \backslash G$, then g_0 need not be a best coapproximation to either f or $\tilde{f}$. Thus, $R_G(f) \cup R_G(\tilde{f}) \subset S_G(f, \tilde{f})$.

(xi) If $0 \leq \alpha \leq 1$, and g_0 is a best simultaneous coapproximation to $\alpha f + (1 - \alpha) f$, for some α, then g_0 is a best simultaneous coapproximation to $f, \tilde{f}$ from G.

(xii) A Kolmogorov type criterion can also be proved for best simultaneous coapproximation

18. **Bases in Banach spaces**

The idea of introducing the notion of a basis in a finite or infinite dimensional Banach space was conceived by J. Schauder in 1927.

S. Banach raised the basis problem in 1932 and this problem was solved in the negative by P. Enflo.

So many genearlizations of the notion of a basis in a Banach Space (even non-separable) have evolved since then, like Schauder decompositions, Enflo- Rosenthal sets (1973) and transfinite bases.

Definition 18.1. A sequence $\{x_n\}$ in an infinite dimensional Banach space E is called a basis of E if for every $x \in E$, there exists a unique sequence of scalars $\{\alpha_n\}_\infty \subset \mathcal{K}$, such that

$$x = \sum_{i=1}^{\infty} \alpha_i\ x_i, \text{ i.e. } \lim_{n \to \infty} \|x - \sum_{i=1}^{n} \alpha_i x_i, \| = 0.$$

Definition 18.2. A system of n elements $\{x_i\}_{i=1}^n$ in a Banach space E of dimension $n < \infty$ is called a basis of E if it is a basis of the underlying linear space, i.e., if for every $x \in E$, there exists a unique system of n scalars $\{\alpha_i\}_{i=1}^n \subset \mathcal{K}$ such that $x = \sum_{i-1}^{\infty} \alpha_i x_i$.

Definition 18.3. Let $\{x_n\}$ be a basis of a Banach space E. The sequence of linear functionals $\{f_n\}$ defined by $f_n(x) = \alpha_n$, where

$$x = \sum_{i-1}^{\infty} \alpha_i x_i, \quad n = 1, 2, \cdots,$$

is called the associated sequence of coefficient functionals.

Schauder basis. (a.s.c.f.) are continuous on E.

J. R. Retherford (1970) and P. K. Jain and K. Ahmad (1980, 1981) connected Schauder bases and best approximation very effectively.

GSR and M. Swaminathan (1990) did extensive work on characterizing monotone, strictly monotone, comonotone, and strictly comonotone bases by introducing two spectacular properties (Λ_1) and (Λ_2).

Orthogonal, strictly orthogonal, coorthogonal, strictly coorthogonal and unconditional bases are defined and characterized. Strict polynomial bases are defined and a theorem involving 18 equivalent conditions to characterize these bases in terms of best coapproximation has been provided.

Normal and strongly normal bases are defined and the bases are characterized in terms of best coapproximation.

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ABSTRACTS OF THE PAPERS SUBMITTED FOR PRESENTATION AT THE 79TH ANNUAL CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY HELD AT RAJAGIRI SCHOOL OF ENGINEERING AND TECHNOLOGY, RAJAGIRI VALLEY, KAKKANAD, DIST. ERNAKULAM, COCHIN - 682 039, KERALA DURING DECEMBER 28 - 31, 2013

A: combinatorics, graph theory and discrete mathematics :

A-1: The diameter variability of the strong product of graphs, Chithra M.R. and A. Vijayakumar (Cochin: *vambat@gmail.com*).

The diameter of a graph can be affected by the addition or the deletion of edges. In this paper we examine the strong product of graphs whose diameter increases (decreases) by the deletion (addition) of a single edge. The problems of minimality and maximality of the strong product with respect to its diameter are also solved. These problems are motivated by the fact that a good network must be hard to disrupt and the transmissions must remain connected even if some vertices or edges fail.

A-2: Friendly Index Set of One Point Union of Two Complete Graphs, Pradeep G. Bhat and Devadas Nayak C (Manipal : *devadasnayakc@yahoo.com*).

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Consider the set $A = \{0, 1\}$. A labeling $f : V(G) \to A$ induces a partial edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x)$, if and only if $f(x) = f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. The balance index set of a graph G, denoted by $BI(G)$ is defined as $\{|e_{f^*}(1) - e_{f^*}(0)| :$ where f^* runs over all friendly labeling f of $G\}$. If the labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = |f(x) - f(y)|$ for each edge xy in $E(G)$, then the friendly index set of a graph G, denoted by $FI(G)$

is defined by $\{|e_{f^*}(1) - e_{f^*}(0)|$: where f^* runs over all friendly labeling f of $G\}$. In this paper we obtain the relation between balance index set and friendly index set. Also we obtain friendly index set of one point union of two complete graphs.

A-3: On The Path Eigenvalues of Graphs, S.C. Patekar (Pune : *2008scpatekar@math.unipune.ac.in*).

We introduce the concept of Path Matrix for a graph and explore the eigenvalues of this matrix. We call these eigenvalues as the path eigenvalues of the graph. Some results concerning path eigenvalues have been investigated.

A-4: Removal cycles avoiding two connected subgraphs, Y. M. Borse and B. N. Waphare (Pune : *ymborse@math.unipune.ac.in; ymborse11@gmail.com*).

In this paper, we provide a sufficient condition for the existence of a cycle C in a connected graph G which is edge-disjoint from two connected subgraphs of G such that $G - E(C)$ is connected.

A-5: A Characterization of n-Connected Matroids, P.P. Malavadkar (Pune : *pmalavadkar@gmail.com*).

It is known that if M is n-connected matroid then, its girth and cogirth is at least n. This condition is necessary but not sufficient. In this paper we give a necessary and sufficient condition for a $(n-1)$-connected matroid to be n-connected.

A-6: On 3-connected es-splitting binary matroids, S.B. Dhotre (Pune).

The es-splitting operation on a 3-connected matroid need not pre-serve the 3-connectedness of the matroid. In this paper, we provide a sufficient condition for a 3-connected binary matroid which yields a 3-connected binary matroid by es-splitting operation. We derive a splitting lemma for 3-connected matroids from the results obtained in the process.

A-7: Hexagonal array grammar system, Jismy Joseph, Dersanambika K.S. and Sujathakumari K. (Kerala : *jismykjoseph@gmail.com, dersanapdf@yahoo.com, nksujathakumari@gmail.com*).

Cooperating rectangular array grammar system was introduced in 1995 by J.Dassow, R.Freund and G.Paun. The results of the rectangular array grammar system either contradict the corresponding result for string grammar system or are not even known for the string grammar system. Motivated by their work here we introduce hexagonal array grammar system which is an extension of rectangular array grammar system. Also we investigate about the generating power of hexagonal array grammar system and compare different types of hexagonal array grammar system.

B: Algebra, Number Theory and Lattice Theory:

B-1: Multiplicative generalized derivations in semiprime rings, Asma Ali and Shahoor Khan (Aligarh : *asma_ali2@rediffmail.com, shahoor.khan@rediffmail.com*).

Let R be a semiprime ring and let $F, f : R \to R$ be maps (not necessarily additive) satisfying $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$. The purpose of the paper is to study the following identities: (i) $F(xy) \pm [x, y] \in Z(R)$, (ii) $F(xy) \pm (x \circ y) \in Z(R)$, (iii) $F(x)F(y) \pm [x, y] \in Z(R)$, (iv) $F(x)F(y) \pm (x \circ y) \in Z(R)$, (v) $F([x, y]) \pm [x, y] = 0$, (vi) $F([x, y]) \pm xy = 0$,(vii) $F(x \circ y) \pm (x \circ y) = 0$, (viii) $F(xy) = F(x)F(y)$ and (ix) $F(xy) = F(y)F(x)$ for all x, y in some appropriate subset of R.

B-2: Some Results on Derivations on Semirings, R. Vembu (Tamilnadu).

In the recent past the concept of derivation on many algebraic structures were defined and theories were developed to a certain extent on them. Derivation on semirings, derivations on semi-prime rings and Jordan derivation on semirings are a few among them. The theory available in the literature on these derivations contains many conceptual and logical errors. In this paper we point out some of such errors and correct them. We also prove some results on derivations on semirings in this paper.

B-3: Special Jacobson radicals for near-rings, Ravi Srinivasa Rao and K.J. Lakshmi Narayana (Andhra Pradesh).

For a right near-ring R, right R-groups of type-d_ν are introduced, which is a class of distributively generated right R-groups, $\nu \in \{0, 1, 2\}$. Using them the right Jacobson radicals of type-d_ν, $J^r_{d_\nu}$, are introduced for near-rings which generalize the Jacobson radical of rings. It is proved that $J^r_{d_\nu}$ is a special radical in the class of all near-rings.

B-4: $(\alpha, 1)$ **Derivations on Semirings**, S.P. Nirmala Devi and M. Chandramouleeswaran (Tamil Nadu : *moulee59@gmail.com, spnirmala1980@gmail.com*).

The notion of a semiring was introduced by H.S. Vandiver in 1934. The notion of derivations of rings can be naturally extended in semirings. The theory of derivations on semirings is not well developed as compared to the theory of derivations on rings due to the absence of additive inverse and the lack of some important concepts included by commutators. This motivated Chandramouleeswaran and Thiruveni to discuss in detail the notion of derivations on semirings in 2010. In 2008, Mustaf Kazaz and AkinAlkan introduced the notion of two-sided $\Gamma - \alpha -$ derivations in prime and semiprime $\Gamma -$near-rings. In this paper, we introduce the notion of two sided α derivation and $(\alpha, 1)$ derivation on a semiring and derive some of its properties on prime semirings.

B-5: Left Jordan Derivations on Semirings, V. Thiruveni and M. Chandramouleeswaran (Tamil Nadu : *moulee59@gmail.com, thiriveni2009@gmail.com*).

Based on the works on derivations on rings and near rings, in 2010, we introduced the notion of derivation on semirings. Here, we introduce the notion of Left Jordan derivation on semirings and the main theorem of this paper states that the existence of a nonzero Left Jordan derivation $D : S \to X$ forces S to be commutative.

B-6: On Left Derivations on TM-algebras, T. Ganesh Kumar and M. Chandramouleeswaran (Tamil Nadu : *moulee59@gmail.com, ganeshkumar.wbc@gmail.com*).

It is well known that BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Recently another algebra based on propositional calculi was introduced by Tamilarasi and Mekalai in the year 2010 known as TM-algebras. Motivated by the notion of derivations on rings and near-rings Jun and Xin studied the notion of derivation on BCI-algebras. Recently, in 2012 we have introduced the notion of derivation on TM-algebras. In this paper, we introduce the notion of left derivation on TM-algebras. We study the properties of regular left derivations on TM-algebras and prove that the set of all left derivations on a TM-algebra forms a semi group under a suitable binary composition.

B-7: Level β -subalgebras of β-algebras, M. Abu Ayub Ansari and M. Chandramouleeswaran (Tamil Nadu : *moulee59@gmail.com, ayubansari61@gmail.com*).

In 1966, Y.Imai and K.Iseki introduced two new classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 2002, J.Neggers and H.S.Kim introduced the notion of -algebras. In 2012 Y.H.Kim investigated some properties of -algebras. Lofti A.Zadeh, in 1965 introduced the theory of fuzzy sets . The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971, by Rosenfeld. O.G.Xi applied the concept of fuzzy sets to BCK-algebras and got some results in 1991. In 1993, Y.B.Jun applied it to BCI-algebras. This motivated us to study the fuzzy algebraic structures on -algebras. Recently, we have introduced the notion of fuzzy -subalgebras on -algebras and investigated some of their properties. In this paper, we introduce the notion of level -subalgebras of a -algebra and investigate some of their properties.

B-8: Cocentralizing derivations on prime rings, Asma Ali and Farhat Ali (Aligarh : *asma_ali2@rediffmail.com, 04farhatamu@gmail.com*).

Let R be a ring with center $Z(R)$. An additive mapping $f : R \to R$ is said to be a derivation on R if $f(xy) = f(x)y + xf(y)$, for all $x, y \in R$. We extend the result of Samman and Thaheem to the case of Lie ideals. In the present note, we prove that if R is a semiprime ring, L is a non zero square closed Lie ideal of R such that $L \nsubseteq Z(R)$ and f, g are derivations of R such that $f(x)y + yg(x) = 0$ for all $x, y \in L$; if $f(L) \subseteq L$ and $g(L) \subseteq L$, then $f(u)[x, y] = 0 = [x, y]g(u)$ for all $x, y, u \in L$ and f, g central on L. If R is a prime ring with $char R = 2$, then $f = g = 0$ on R.

B-9: On symmetric generalized biderivations of prime and semiprime rings, Asma Ali and Khalid Ali Hamdin (Aligarh : *asma_ali2@rediffmail.com, hamdinkh@yahoo.com*)

Let R be a ring with center $Z(R)$. A biadditive mapping $D(.,.)$: $R \times R \to R$ is said to be a biderivation if for all $x, y \in R$, the mappings $y \mapsto D(x, y)$ and $x \mapsto D(x, y)$ are derivations of R. A mapping $f : R \to R$ defined by $f(x) = D(x, x)$ for all $x \in R$, is called trace of D. A biadditive mapping $\Delta : R \times R \to R$ is said to be a generalized D-biderivation if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of R associated with the function $y \mapsto D(x, y)$ as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of R associated with the function $x \mapsto D(x, y)$, where D is a biadditive map on R. In this paper, motivated by a result of Vukman, we prove that if L is a noncentral square closed Lie ideal U of a prime ring R admitting a symmetric generalized D-biderivation Δ is commuting on U, then $\Delta = 0$ on U. Moreover, we prove that a nonzero left ideal L of a 2-torsion free semiprime ring is central if it satisfies one of the following properties: (i) $[x, y] = f(xy) - f(yx)$, (ii) $[x, y] = f(yx) - f(xy)$, (iii) $xy - D(x, x) = yx - D(y, y)$, (iv) $xy + D(x, x) = yx + D(y, y)$, (v) $yx - D(x, x) = xy - D(y, y)$ and (vi) $yx + D(x, x) = xy + D(y, y)$, for all $x, y \in L$, where f stands for the trace of symmetric biderivation $D(.,.) : R \times R \to R$.

B-10: On classical prime subtractive subsemimodules of quotient semimodule, Jaypraksh Ninu Chaudhari (Jalgaon : *jnchaudhari@rediffmail.com*).

In this paper, we obtain the relation between classical prime (weakly classical prime) subtractive subsemimodules containing a Q-subsemimodule N of an R-semimodule and classical prime (weakly classical prime) subtractive subsemimodule of the quotient semimodule $M/N_{(Q)}$.

B-11: On generalization of classical prime subsemimodules, Dipak Ravindra Bonde (Dharangaon : *drbonde@rediffmail.com*).

In this paper, the concept of classical prime subsemimodule is generalized to weakly classical prime subsemimodules of semimodules over semirings. We prove that, if N is a weakly classical prime subtractive subsemimodule of a semimodule M over an entire semiring R, then either N is classical prime or $(N : M)(N : M)N = 0$.

B-12: Characterization of strong regularly in near-rings, M.K. Manoranjan (Madhepura : *manojmanoranjan.kumar@gmail.com*).

In this paper, we shall prove some characterizations of the strong regularity in near-rings which are closely related with strongly reduced near-rings. A near-ring R is said to be left regular if for each $a \in R$ there exists $x \in R$ such that $a = xa^2$. A near-ring is called strongly left regular if R

is left regular and regular, similarly we define right regular. A strongly left regular and strongly right near-ring is called strongly regular near-ring. Equivalently, left and right regularity implies strong regularity. Also, the concept of strongly left, strongly right and strong regularities are all equivalent. An idempotent element $e \in R$ is called left semi central if $ea = eae$ for $a \in R$. Similarly, right semi centrality can be defined in a symmetric way. A near-ring in which every idempotent element is left semi central is called left semi central. A near-ring R is reduced if R has no non-zero nilpotent elements. We find that a strongly regular near-ring is reduced and every strongly reduced near-ring is reduced.

B-13: Ideals In Bisemirings, M.D. Suryawanshi (Dhule : *manoharsuryawanshi65@gmail.com*).

In this paper, we introduce the notion of a partitioning biideal in a bisemiring and hence the quotient structure of bisemiring is defined. Also we prove that I is the partitioning biideal of a bisemiring R with respect to two subsets Q and Q', then the quotient bisemiring $\dfrac{R}{l_q}$ and $\dfrac{R}{l_q}$ are isomorphic.

B-14: Posets dismantlable by doubly irreducibles, A. N. Bhavale (Pune : *ashokbhavale@gmail.com*).

Benoit Larose and Lfiszl Zadori introduced the concept of a poset dismantlable by irreducibles. We introduce the concept of a poset dismantlable by doubly irreducibles. In order to study these posets we introduce the operations of 1-sum and 2-sum for posets. Using these operations, we obtain the structure theorem for posets dismantlable by doubly irreducibles.

B-15: Some special classes of near-ring modules, Ravi Srinivasa Rao and K.J. Lakshmi Narayana (Andhra Pradesh : *dr_rsrao@yahoo.com*).

Near-rings considered are right near-rings and R is a near-ring. Recently, special classes of near-ring right modules have been introduced and studied. Also Characterization of some concrete special radicals of near-rings in terms of the special classes of near-ring right modules are presented. In this paper six more classes of near-ring right modules have been studied. It is shown that they are all special classes.

B-16: On $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$ and $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$, Susil Kumar Jena (Bhubaneswar : *susil_kumar@yahoo.co.uk*).

The two related diophantine equations: $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$ and $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$, have infinitely many non-trivial and primitive integral solutions for their parameters. We give two parametric solutions, one for each of these equations.

B-17: S. Chowla and S.S. Pillai The story of two peerless Indian Mathematicians, Dayasankar Gupta (Sant Kabir Nagar).

This paper represents the Chowla Pillai-Correspondence, Warings problem, Least prime-quadratic residue, Chowla's counter-examples to a claim of Ramanujan and a disproof of Chowla's conjecture, problem on consecutive numbers, conjecture, independent values of cotangent function, number of permutation of a given order, closed form for the prime case, applications to finite groups, convenient number and class numbers, matrices and quadratic polynomials, average of Eulers phi-functions.

C: Real and Complex Analysis (Including Special Functions, Summability and Transforms)

C-1 Some results on a class of entire dirichlet series with complex frequencies, Niraj Kumar and Garima Manocha (New Delhi :*nirajkumar2001@hotmail.com, garima89.manocha@gmail.com*).

Let F be a class of entire functions represented by Dirichlet series with complex frequencies for which $e^{k|\lambda^k|}\,|a_k|$ is bounded. Some results for this set are then studied.

C-2: On a class of integral transform of pathway type, Dilip Kumar (Kerala :*dilipkumar.cms@gmail.com*).

The integral transform named P_α-transform introduced in this paper is a binomial type transform containing many class of transforms including the well known Laplace transform. The paper is motivated by the idea of pathway model introduced by Mathai in 2005 [*Linear Algebra and Its Applications*, **396**, 317-328]. The composition of the transform with differential and integral operators are proved along with convolution theorem. Being a new transform, the P_α-transform of some elementary functions are given in the paper. As an illustration of applications of the general theory of differential equations, a simple differential equation is solved by the new transform. P_α-transform of some generalized special functions such as H-function, G-function, Wright generalized hypergeometric function, generalized hypergeometric function and Mittag-Leffler function are also obtained. Also the solutions of fractional kinetic equations are obtained by using P_α-transform. The results for the classical Laplace transform is retrieved by letting $\alpha \to 1$.

C-3: An Extension of α-type polynomial sets, S.J. Rapeli, S.B. Rao and A.K. Shukla (Surat).

In this paper, we discuss α-type polynomial sets and also its generalization in two variables. Some properties of certain polynomials have also been shown for in support of α-type zero in two variables.

C-4: Recurence Relation and Integral Representation of Generalized K-Mittag-Leffer Function $\mathbf{GE}^{\gamma,\mathbf{q}}_{\mathbf{k},\alpha,\beta}(\mathbf{z})$, Kuldeep Singh Gehlot (Rajasthan :*drksgehlot@rediffmail.com*).

In this paper author calculate the recurrence relations and six different integral representation of Generalized K-Mittag-Leffler function introduced by Gehlot and Kuldeep Singh (2012). Also find out six different integral representations of K-Mittag-Leffler function, definded by Dorrego G.A. and Cerutti R.A. (2012). And several special cases have been discussed.

D: Functional Analysis

D-1: Watson transform for boehmians, R. Roopkumar (Karaikudi).

Proving the required auxiliary results, we construct two Boehmian spaces which properly contain $T'(\lambda, \mu)$ and $T'(1-\mu, 1-\lambda)$. Next we prove the convolution theorem for the Watson transform on $T'(\lambda, \mu)$ using both Mellin type convolutions $\vee$ and $\wedge$. Applying the convolution theorem, we extend the Watson transform to the context of Boehmians as a bijective map a Boehmian space onto the other Boehmian space and prove that the extended Watson transform is linear, continuous map with respect to δ-convergence as well as Δ-convergence.

D-2: An iterative algorithm for generalized mixed vector equilibrium problems and relativity non-expansive mapping in Banach spaces, K. R. Kazmi and Mohammad Farid
(Aligarh :*kekazmi@gmail.com; mohdfrd55@gmail.com*).

In this paper, we introduce an iterative schemes for finding a common solution of split generalized vector variational inequality problem and fixed point in a real Hilbert space. We prove that the sequences generated by the proposed iterative scheme converge strongly to the common solution of split generalized vector variational inequality problem and the fixed point problem for nonexpansive mappings. The results presented in this paper are the supplement, extension and generalization of the previously known results in the area.

D-3: An iterative method for split generalized vector equilibrium problem and fixed point problem, K.R. Kazmi, S.H. Rizvi and Mohd. Farid (Aligarh).

In this paper, we introduce and study an explicit iterative method to approximate a common solution of split generalized vector equilibrium problem and fixed point problem for a finite family of nonexpansive mappings in real Hilbert spaces using the viscosity *Cesáro* mean approximation. We prove that the sequences generated by the proposed iterative scheme converge strongly to the common solution of split generalized vector equilibrium problem and fixed point problem for finite family of nonexpansive mappings. Further, we give a numerical a example to justify our main result. The results presented in this paper are the supplement, extension and generalization of the previously known results in this area.

D-4: Common fixed point theorem for cyclic weak (ϕ, ψ)-contraction in Menger space, Sahni Mary Roosevelt and Dersanambika K.S. (Kerala : *sahniroosevelt@gmail.com; dersanapdf@yahoo.com*).

An altering distance function is a control function which alter the metric distances between two points enabling one to deal with relatively new classes of fixed point problems. But, the uniqueness of control function creates difficulties in proving the existence of fixed point under contractive conditions. Cyclic weak (ϕ, ψ)-contraction mapping is extended to Menger space and fixed point theorem for such mappings are studied in Menger space.

D-5: A unique common fixed point theorem in complete G-metric space with six mappings, Anushri A. Aserkar and Manjusha P. Gandhi (Nagpur :*aserkar_aaa@rediffmail.com*).

In the present paper, we have proved a unique common fixed point theorem for six mappings which is an extension of Gregus theorem [1980] in complete symmetric G-metric space. The mappings in pairs satisfy the weakly compatibility condition. Mustafa and Sims [2006] generalized metric spaces to G metric space. This new structure was a great alternative to amend the flaws in the concept of D-metric spaces. It was proved by Mustafa and Sims that every G-metric space is topologically equivalent to a metric space.

D-6: Common fixed point of coincidently commuting mappings in 2 non-archimedean menger PM-space, Bijendra Singh, V. K. Gupta and Jaya Kushwah (Ujjain : *kushwahjaya@gmail.com*).

In the present paper, we prove a fixed point theorem for quasi-contraction pair of coincidentally commuting mappings in a 2 non-Archimedean Menger PM-space usings idea of Achari [1] and Chamola et.al.[2].

D-7: Common fixed points of generalized Meir-Keeler α-contractions, Deepesh Kumar Patel, Thabet Abdeljawad and Dhananjay Gopal (Surat : *deepesh456@gmail.com, gopal.dhananjay@rediffmail.com*).

Motivated by Abdeljawad (Fixed Point Theory and Applications 2013), we establish some common fixed point theorems for three and four self-mappings satisfying generalized Meir-Keeler α-contraction in metric spaces. As a consequence the results of Rao and Rao (Indian J. Pure Appl. Math., 16(1)(1985), 1249-1262), Jungck (Internat. J. Math. Math. Sci., 9(4)(1986), 771-779), and Abdeljawad itself are generalized, extended and improved. Sufficient examples are given to support our main results.

D-8: On Existence of Coincidence and Common Fixed Point for Faintly Compatible Maps, Anita Tomar (Dehradun : *anitatmr@yahoo.com*).

In this paper, we discuss the existence of coincidence and common fixed point for faintly compatibility maps satisfying both contractive and non contractive condition . Our results improve the results of Bisht and Shahzad without containment and continuity requirement of involved maps on metric space. Example to demonstrate the validity of results obtained is also furnished.

E: Differential Equations, Integral Equations and Functional Equations

E-1: On some mixed integral inequalities and applications, S.D. Kendre (Pune : *sdkendre@yahoo.com*).

In this paper, we establish some nonlinear mixed integral inequalities which provide an explicit bound on unknown function, and can be used as a tool in the study of certain nonlinear mixed integral equations. The purpose of this paper is to extend certain results which proved by Pachpatte.

E-2: On the fixed solutions of second order nonlinear delay difference equations with asymptotic and stability behaviors, Dr. B. Selvaraj and Mr. S. Raju (Coimbatore : *rajumurugasamy@gmail.com*).

Some new criteria are obtained for asymptotic and stability behaviors of fixed solutions of the second order nonlinear delay difference equation of the form $\Delta^2(x_n + p_n x_{n-k} - q_n x_{n-1}) + f(x_n) = 0$, $n = 0, 1, 2, 3...$ Examples are inserted to illustrate the results.

E-3: Immovability of a quartic functional equation in Felbin's Type Spaces, M. Arunkumar and S. Karthikeyan (Tiruvannamalai :*annarun2002@yahoo.co.in, karthik.sma204@yahoo.com*).

In this paper, the authors investigate the immovability of a quartic functional equation

$$\sum_{i=1}^{n} f\left(\sum_{i=1}^{n} x_{ij}\right) = 6 \sum_{1\le i<j\le n} f(x_i + x_j + x_k) - (6n-18) \sum_{1\le i<j\le n} f(x_i - x_j)$$
$$+(n-8) f\left(\sum_{i=1}^{n} x_i\right) + \left(\frac{3n^2 - 17n + 22}{16}\right) f\left(\sum_{i=1}^{n} 2x_i\right),$$

where

$$x_{ij} = \begin{cases} -x_j & \text{if } i = j \\ x_j & \text{if } i \neq j \end{cases}$$

in Felbin's type spaces.

E-4: Random stability of an additive quadratic functional equation: a fixed point approach, M. Arun Kumar and P. Agilan (Tiruvannamalai :*annarun2002@yahoo.co.in, agilram@gmail.com*).

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the following additive quadratic functional equation

$$f(3x+2y+z)+f(3x-2y+z)+f(3x+2y-z)+f(3x-2y-z) \\ =12f(x)+12[f(x)+f(-x)]+8[f(y)+f(-y)]+2[f(z)+f(-z)]$$

in random normed spaces.

E-5: Solution and stability of a n-dimensional quadratic functional equation in quasi-beta normed spaces: direct and fixed point methods, M. Arun Kumar, S. Murthy, S. Ramamoorthi and G. Ganapathy

(Tiruvannamalai : *annarun2002@yahoo.co.in, smurthy07@yahoo.co.in, ramsdmaths@yahoo.com, ganagandhi@yahoo.co.in*).

In this paper, the authors has proved the general solution and generalized Ulam-Hyers stability of a n-dimensional quadratic functional equation of the form

$$\sum_{i=0}^{n}\left[f\left(\frac{x_{2i}+x_{2i+1}}{2}\right)+f\left(\frac{x_{2i}-x_{2i+1}}{2}\right)\right]=\frac{1}{2}\sum_{i=0}^{n}f(x_{2i})+f(x_{2i+1})$$

where $n \geq 1$ in Quasi-Beta normed spaces using direct and fixed point methods.

E-6: Permanence of 2-variable additive functional equation in non-archimedean fuzzy ϕ-2-normed space : Hyers direct method, M. Arunkumar, T. Namachivayam

(Tiruvannamalai :*annarun2002@yahoo.co.in, namachi.siva@rediffmail.com*).

In this paper, the authors investigate the solution of a 2-variable additive functional equation

$$g(2x \pm y \pm z, 2u \pm v \pm w) = g(x \pm y, u \pm v) + g(x \pm z, u \pm w).$$

Its generalized Ulam-Hyers stability in non-Archimedian fuzzy ϕ-2-normed spaces using Hyers direct method.

E-7: New Oscillation Criteria for Higher Order Non-Linear Functional Difference Equations, B.Selvaraj and S.Kaleeswari (Tamil Nadu :*professorselvaraj@gmail.com, kaleesdesika@gmail.com*).

In this paper some new criteria for the oscillation of high order functional difference equation of the form

$$\Delta^2\left(r(n)\left[\Delta^{(m-2)}y(n)\right]^{\alpha}\right)+q(n)\,f\left[y(g(n))\right]=0,$$

where $\sum_{s=n_0}^{\infty} r^{-\frac{1}{\alpha}}(s)<\infty$ and $m>1$ are discussed. Examples are given to illustrate the results.

E-8: Battle outcome in a counter insurgency operation by security force under decapitation warfare involving range-dependent attrition-rate coefficients in the regular combat, Lambodara Sahu (Pune : *lsahucme@gmail.com*).

In this paper, a conceptual model dealing with certain operational factors like robustness of forces, undermining effects, maximum effective range, break-points, is being discussed referring the concepts of Lanchester-type equations with range-dependent attrition-rate coefficients to project the effectiveness of forces of regular combat under decapitation warfare, while figuring out the importance of undermining operation in addition to the advantage of closeness to the target by considering a few case studies.

F: Geometry

F-1: Contact CR-submanifolds of an indefinite trans-Sasakian manifold, Bandana Das and Arindam Bhattacharyya.

This paper is based on contact CR-submanifolds of an indefinite trans-Sasakian manifold of type $(\alpha,\ \beta)$. Here some properties of contact CR-submanifolds of an indefinite trans-Sasakian manifold have been studied and also the sectional curvatures of contact CR-submanifolds of an indefinite trans-Sasakian space form are discussed.

F-2: On generalized φ-recurrent trans-sasakian manifolds, D.Debnath and A.Bhattacharyya (*dipankardebnath123@hotmail.com*).

The object of the present paper is to study generalized φ-recurrent trans-Sasakian manifolds. It is proved that a generalized φ-recurrent trans-Sasakian manifold is an Einstein manifold. Also we obtained a relation between the associated 1-forms A and B for a generalized φ-recurrent and generalized concircular φ-recurrent trans-Sasakian manifolds and finally proved that a three dimensional locally generalized φ-recurrent trans-Sasakian manifold is of constant curvature.

F-3: Some properties of slant and pseudo-slant submanifolds of an ϵ-paracontact Sasakian manifold, Barnali Laha and Arindam Bhattacharyya (Kolkata : *barnali.laha87@gmail.com, bhattachar1968@yahoo.co.in*).

In the present note, we have derived some results pertaining to the geometry of slant and pseudo-slant submanifolds of an ϵ-paracontact Sasakian manifold. In particular, we have obtained the necessary and sufficient conditions of a totally umbilical proper slant submanifold to be totally geodesic, provided the mean curvature vector $H \in \mu$. In addition to this, we have obtained the integrability conditions of the distributions of pseudo-slant submanifold.

F-4: Almost pseudo Ricci symmetric viscous fluid spacetime, Buddhadev Pal and Arindam Bhattacharyya
(Kolkata : *buddha.pal@rediffmail.com, bhattachar1968@yahoo.co.in*).

The object of the present paper is to investigate the application of almost pseudo Ricci symmetric manifolds to the General Relativity and Cosmology. Also, we study the space time when the anisotropic pressure tensor in energy momentum tensor of type $(0, 2)$ takes the different form.

F-5: Evolution of $\Im$-functional and ω-entropy functional for the conformal Ricci flow, Nirabhra Basu and Arindam Bhattacharyya (West Bengal: *nirabhra.basu@yahoo.com, bhattachar1968@yahoo.co.in*).

In this paper we define the $\Im$-functional and the ω-entropy functional for the conformal Ricci flow and see how they evolve according to time.

G: Topology

G-1: On b^*-$\mathcal{I}$-open sets in ideal topological spaces, K. Viswanathan, S. Jafari and J. Jayasudha.

In this paper, we introduce and investigate the notions of b^*-$\mathcal{I}$-open sets in ideal topological spaces. Further we have discussed some properties of b^*-$\mathcal{I}$-open sets and obtained decomposition of *semi**-$\mathcal{I}$-continuity.

G-2: Idealization of a Decomposition Theorem, R. Santhi and M. Rameshkumar
(Tamil Nadu : *santhifuzzy@yahoo.co.in, rameshngm@gmail.com*).

In this paper we introduce and investigate the notion of regular-$\mathcal{I}_s$-closed set, $A_{\mathcal{I}_s}$-set, regular-$\mathcal{I}_s$-continuous and $A_{\mathcal{I}_s}$-continuous in ideal topological spaces. Then we show that a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is continuous if and only if it is α-$\mathcal{I}_s$-continuous and $A_{\mathcal{I}_s}$-continuous. Also we proved that regular-$\mathcal{I}_s$-closed set and regular closed are independent.

G-3: Fuzzy generalized minimal continuous maps in fuzzy topological spaces, Suwarnlatha. N. Banasode (Karnataka).

In this paper a new class of fuzzy generalized minimal continuous maps that includes a class of fuzzy generalized minimal irresolute maps are introduced and studied in fuzzy topological spaces. A mapping $f : X \to Y$, from a fts X into a fts Y is said to be fuzzy generalized minimal continuous (briefly f-g-m_i continuous) map if the inverse image of every fuzzy minimal closed set in Y is a fuzzy g-m_i closed set in X.

G-4: Contra functions via b-$\mathcal{I}$-open sets, S.P. Jothiprakash (Tamil Nadu).

In this paper, we introduce contra open, contra closed, irresolute and contra irresolute functions via b-$\mathcal{I}$-open sets and study some of their properties.

G-5: Soft πg-closed set in soft topological spaces, A. Selvi and I. Arockiarani (Tamil Nadu :*selviantony.pc@gmail.com*).

In 1999, Molodtsov initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Later several other authors have developed many areas of soft set theory. Recently, in 2011, Shabir and Naz introduced soft topological spaces. Kannan introduced soft generalized closed sets in soft topological spaces which are defined over an initial universe with a fixed set of parameters. This paper aims to introduce a new concept as soft πg-closed sets in soft topological spaces and give a study of soft πg-closed set and soft πsg-closed sets in soft topological spaces. Moreover, some of the characterizations are obtained. Then we investigate the relationships of soft πg-closed sets with other existing soft closed sets with counter examples. These results will enable us to carry out a general framework for their applications in real life.

G-6: Pre closed sets in bim spaces, A. Dhanis Arul Mary and I. Arockiarani (Tamil Nadu : *dhanisarulmary@gmail.com*).

In 2000, V. Popa and T. Noiri introduced the concept of minimal structure space. They also introduced the notion of m_x-open set and m_x-closed set and characterize those sets using m_x-closure and m_x-interior operators, respectively. In 1969, J.C. Kelly introduced the notion of bitopological spaces and extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. In 2010, C. Boonpok introduced the notion of biminimal structure spaces. In this paper, we introduce the concept of pre closed sets in biminimal structure spaces and studied $m_x^1 m_x^2$- pre closed sets and $m_x^1 m_x^2$-pre open sets and derived some of their properties. Also, we discuss the concept of pre-neighborhood and pre-accumulation points in biminimal structure spaces and obtain some of their characterizations.

G-7: Topologies generated by the a-cuts of a fuzzy set, R. Padmapriya and Dr. P. Thangavelu
(Coimbatore :*priyabharathi28@gmail.com, thangavelu@karunya.edu*).

Zadeh introduced the concept of fuzzy sets in 1965. The a-cuts of a fuzzy subset A of a non-empty set X may generate a topology. Such a topology is called a topology generated by the fuzzy set A of X. The purpose of this paper is to characterize such topologies.

G-8: Generalizations of Pawlaks rough approximation spaces by using $\alpha\beta$-open sets, K. Reena and I. Arockiarani (Coimbatore :*reenamaths1@gmail.com*).

This paper extends Pawlak's rough set theory to a topological space model where the set approximations are defined using the topological notion $\alpha\beta$ open sets. A number of important results using the topological notion $\alpha\beta$-open sets are obtained. We also proved that some of the properties of $\alpha\beta$ Pawlaks rough set model are special instances of those topological generalizations.

G-9: Fuzzy neutrosophic soft matrix theory, I.R. Sumathi and I. Arockiarani (Coimbatore : *sumathi_raman2005@yahoo.co.in*).

Molodtsov introduced the concept of soft set theory which can be used as a mathematical tool for dealing with uncertainty. The parameterizations tool of soft set theory enhances the flexibility of its application. Moreover, Maji extended soft sets to intuitionistic fuzzy soft set which handles only the truth membership and falsity membership; it does not handle the indeterminacy. Neutrosophic set was initiated by Smarandache which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. One of the important theory of mathematics which has a vast application in science and engineering is the theory of matrices. The subject explored in this paper is the matrix representation of fuzzy neutrosophic soft set namely Fuzzy neutrosophic soft matrices. Further, we have defined some basic operations on Fuzzy Neutrosophic soft matrices and have applied in decision making problem. This study provides us an opportunity to go further on fuzzy neutrosophic soft matrices with new operations and this matrix models could be carried out in the real world problems.

G-10: On generalized fuzzy neutrosophic soft sets, J. Martina Jency and I. Arockiarani (Coimbatore :*martinajency@gmail.com*).

Molodtsov proposed the novel concept of soft set theory which provides a completely new approach for modeling vagueness and uncertainty. Combining soft sets with intuitionistic fuzzy sets Maji defined intuitionistic fuzzy soft sets for solving decision making problems. F. Smarandache introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Inspired by the varied applications of neutrosophic set in real life situations, we have introduced a new notion of set namely fuzzy neutrosophic soft set. In the present study, we have introduced the concept of generalized fuzzy neutrosophic soft set and studied some of its properties. We have put forward some propositions based on this new notion. We hope that this paper will promote the future study on generalized fuzzy neutrosophic soft set and generalized fuzzy neutrosophic soft topological spaces to carry out a general framework for their application in practical life.

G-11: A Few Covering Properties of the α-Topology, Shalu (Modinagar : *drshalumath@gmail.com*).

In the present paper we have obtained some results of P-closed properties by using α-topology. A topological space (X, T) is para-rc-Lindelof if every cover of X by regular closed sets has a locally countable refinements by regular closed sets .We also prove that T and T^α share these properties.

H: Measure Theory, Probability Theory and Stochastic Processes, and Information Theory

H-1: Reliability of Machine Repair Problem with Spares and Partial Server Vacation Policies for Repairmen, D.C. Sharma (Rajasthan :*dcsharma_1961@yahoo.co.in*).

In this paper we have taken a machine repairable system with spares and repairmen with the partial server vacation policy. In our system, the first repairman never takes vacations and always available for serving the failed units. The second repairman goes to vacation of random length when number of failed units is less than certain number of machine (say) N. At the end of vacation period, this repairman returns back if there are N or more failed units/machine accumulated in the system. Otherwise this repairman goes for another vacation. Vacation time is exponentially distributed. By using of Markov process theory, we develop the steady state probabilities equations and solve these equations recursively. We present reliability measures. Availability of the system is maintained at certain level and the optimum value of N has been calculated. A Sensitivity analysis is also investigated.

I: Numerical Analysis, Approximation Theory and Computer Science

I-1: On Oscillatory Matrices, Ravinder Kumar and Ram Asrey Rajput (Agra : *ravinder_dei@yahoo.com, ramasreyrajput@yahoo.co.in*).

A real matrix $\mathbf{A}$ is called oscillatory if all its minors are nonnegative and if some power of $\mathbf{A}$ has all minors positive. In this paper we present eigenvalue inequalities that hold for oscillatory matrices. In particular bounds for determinant of an oscillatory matrix are derived.

I-2: A New Technique to Solve Reaction-Diffusion Boundary Value Problems, Surabhi Tiwari (Allahabad :*surabhi@mnnit.ac.in*).

Reaction-diffusion equations are important to a wide range of applied areas such as cell processes, drug release, ecology, spread of diseases, industrial catalytic processes, transport of contaminants in the environment, chemistry in interstellar media, etc. The aim of this paper is to build an efficient initial value technique for solving a third order linear reaction diffusion singularly perturbed boundary value problem. Using this technique, a third order linear reaction diffusion singularly perturbed boundary value problem is reduced to three approximate unperturbed linear initial value problems and then Runge-Kutta fourth order scheme is used to solve these unperturbed linear problems numerically. Numerical examples are solved using this given method. It is observed that the presented method approximates the exact solution very well for crude mesh size h. Error analysis and convergence analysis of the method are also described.

I-3: Simulation of extended spiking neural P systems with astrocytes using petri nets, Rosini B. and Dersanambika K.S. (Kerala :*brosini@gmail.com, dersanapdf@yahoo.com*).

Spiking Neural $P(SNP)$ system characterizes the movement of spikes among neurons. However, in biological nervous system, besides neurons themselves, astrocytes (star-shaped glial cells spanning around neurons) also play an important role on the functioning and interaction of neurons. In this work, we focus on the excitatory and inhibitory role of astrocytes. This paper proposes the concept of translating Extended Spiking Neural P systems with Astrocytes (ESN PA system) to Petri nets. For a given ESN PA system, we are able to model a Petri net, that can be employed to simulate the behavior of an ESN PA system.

I-4: On generalized α-difference operator of third kind and its applications in number theory, G. Britto Antony Xavier, P. Rajiniganth and V. Chandrasekar (Tamilnadu, *shcbritto@yahoo.co.in*).

In this paper, the authors extend the theory of the generalized α-difference operator $\Delta_{\alpha(l)}$ to the Generalized α-difference operator of the third kind $\Delta_{\alpha(l_1,l_2,l_2)}$ for the positive reals l_1, l_2 and l_3. We also present the discrete version of Leibnitz Theorem, Binomial Theorem, Newton's formula with reference to $\Delta_{\alpha(l_1,l_2,l_3)}$. Also, by defining its inverse, we establish a few formulae for the sum of the second partial sums of higher powers of arithmetic-geometric progression in number theory.

I-5: Eigen frequency based solutions focusing vibration isolation system design for industrial applications, S.N. Bagchi (Pune : *design@resistoflex.in; www.resistoflex.in*).

This is an industry-academia oriented presentation related to the mathematical solutions focusing Vibration and Shock isolation system design for industrial applications, covering power plants to transportation. The application of Air springs in deluxe buses and railways for a comfortable journey will be highlighted. The resonance frequency of our flexible body organs like intestine, backbone, shoulder joints, neck and knee joint etc., are all in the low frequency range. Hence during a long journey on a rough road, some of the body organs are excited by road vibrations. A fuzzy model and the statistical data of the human body response is used to optimize the bus suspension design using a low natural frequency Air Spring for a comfortable long journey. The mathematical approach and modalities have an engineering-physics orientation. In view of the inter-disciplinary nature of the subject and wide spectrum of coverage from Molecular vibration to Machine vibration the presentation is based on simple spring mass models of polyatomic molecule to spring supported industrial machines. Based on spring-mass model of a system the eigen-frequency is calculated which is the solution of a differential equation. The response of the spring supported system due to any exciting force of dynamic nature depends on the unique value of eigen-frequency in x, y and z direction. An optimized design calculation of a rotating machine is presented. The seismic isolators used for the Earthquake protection of buildings is briefed.

I-6: A numerical method for weighted low-rank matrix approximation, Aritra Dutta (Florida :*d.aritra2010@knights.ucf.edu*).

In scientific and engineering field, the best approximation by a low-rank matrix has become an important tool in dimensional reduction of data and matrix decomposition. When the Frobenious norm or ℓ_1-norm is used, the best low rank approximation has a closed form formula in terms of Principal Component Analysis [1] or Robust Principal Component Analysis [2]. When a weight is inserted in the norm, the corresponding weighted approximation problem is much harder to solve. Indeed, according to Srebro and

Jaakkola [3], the weighted low-rank approximation problems do not admit a closed form solution in general. In this presentation, I propose a new numerical method to solve the weighted low-rank approximation problem. More precisely, we use the optimization approach using the Augmented Lagrange Method (ALM) together with the penalty terms and then using the alternating directional method to solve the problem numerically. I also present some numerical experiments to show the effectiveness of the new algorithm.

I also show, as a limiting case, how our algorithm for the weighted low-rank approximation is related to the work of Golub-Hoffman-Stewart [4] where they showed how to obtain a best approximation of a low rank matrix in which a specified set of columns of the matrix remains fixed. Finally, I discuss some applications of the weighted low-rank approximation in image and video analysis.

J: Operations Research

No paper

K: Solid Mechanics, Fluid Mechanics, Geophysics and Relativity

K-1: Effects of piezoelectricity on waves in monoclinic poro-elastic materials, Anil K. Vashishth and Vishakha Gupta (Kurukshetra : *akvashishth@kuk.ac.in, vi_shu85@yahoo.co.in*).

Piezoelectric materials are materials which produce electric field when stress is applied and get strained when electric field is applied. Piezoelectric materials are acting as very important functional components in sonar projectors, fluid monitors, pulse generators and surface acoustic wave devices. Wave propagation in porous piezoelectric material having crystal symmetry 2 is studied analytically. The Christoffel equation is derived. The phase velocities of propagation and the attenuation quality factors of all these waves are described in terms of complex wave velocities. The effects of phase direction, porosity, wave frequency and the piezoelectric interaction on phase velocities are studied numerically for a particular model. The results are reduced for the crystal classes $2mm$ and $6mm$ from the class 2.

K-2: Uniqueness theorem and theorem of reciprocity in the Linear Theory of Porous Piezoelectricity, Anil K. Vashishth and Vishakha Gupta (Kurukshetra :*akvashishth@kuk.ac.in, vi_shu85@yahoo.co.in*).

General theorems of classical elasticity are generalized for the linear theory of porous piezoelectric material. The constitutive equations are formulated for the porous piezoelectric materials. The reciprocal relation in the linear theory of porous piezoelectric materials is proved. The uniqueness theorem is established for the three dimensional porous piezoelectric body with assumption of positive definiteness of elastic fields. Alternative proof of the uniqueness theorem, without using the assumption of positive definiteness of elastic fields, is also given.

K-3: Pulsatile flow in carotid artery bifurcation in reference to atherosclerosis with varying frequencies, G. Manjunatha and K.S. Basavarajappa (Manipal : *gudekote_m@rediffmail.com*).

The mathematical model is studied to analyze the plaques of atherosclerosis in the common carotid and internal carotid arteries (CCA and ICA) of the vascular bifurcations. 'Y' model with flow ratio 70%: 30% between internal carotid artery (ICA) and external carotid artery (ECA) from the neck of the common carotid artery (CCA) is employed in the analysis. The dilation of the offspring (the carotid sinus or bulb) is analyzed for plaques. For a pulsatile flow, a representative normal carotid artery bifurcation wave form is imposed at the carotid artery (CCA) inlet of the model with mean and peak flow rates 6.0 - 23.8 m/sec. Peak systolic and diastolic pressure are compared with varying cardiac cycle (< 70 and > 70 beats per min). Series solution method is used to study the governing equations of motions with perturbations on the characterizing physiological flow parameters.

K-4: Numerical simulation of Bödewadt flow of a non-Newtonian fluid, Bikash Sahoo (Rourkela).

Both Newtonian and non-Newtonian flows past rotating disks have drawn the attention of many researchers due to their fundamental immense engineering and industrial applications. The problem arising when a viscous fluid rotates with a uniform angular velocity at a larger distance from a stationary disk is one of the few problems in fluid dynamics for which the Navier-Stokes equations admit an exact solution. This problem was initially studied by *Bödewadt* by making boundary layer approximations.

That is why the flow is well known as *Bödewadt* flow. However, the Navier-Stokes equations for the *Bödewadt* flow for a non-Newtonian fluid do not possess an exact solution and one has to adopt effective numerical method to solve the resulting nonlinear differential equations. In this paper, the steady *Bödewadt* flow of a non-Newtonian Reiner-Rivlin fluid has been considered. The resulting highly nonlinear differential equations are solved by a second order finite difference method. The effects of non-Newtonian cross-viscous parameter K on the velocity field has been studied in detail and shown graphically. One of the important findings of the present investigation is that when the non-Newtonian parameter K is increased, solutions to the boundary value problem tend to approach their far-field asymptotic boundary values more rapidly.

L: Electromagnetic Theory, Magneto-Hydrodynamics Astronomy And Astrophysics

L-1: Effect of Thermal Radiation on MHD flow with variable Viscosity and Thermal Conductivity over a Stretching Sheet in Porous Media, Pentyala Srinivasa Rao, B. Kumbhakar and B.V. Rathish Kumar (Dhanbad : *pentyalasrinivasa@gmail.com*).

An investigation has been made for two dimensional steady stagnation point flow of a viscous incompressible electrically conducting fluid over a linearly stretching sheet in porous media with variable viscosity and thermal conductivity. The viscosity and thermal conductivity are taken as inverse linear and linear functions of temperature respectively. The medium is influenced by a traverse magnetic field and volumetric rate of heat generation or absorption in the presence of radiation effect. The governing boundary layer equations are transformed into ordinary differential equations by taking suitable similarity variables. The resulting coupled nonlinear differential equations are solved numerically by using fourth order Runge Kutta method along with shooting method. The effect of various parameters such as radiation, porosity, viscosity, thermal conductivity, Hartmann number, Prandtl number etc. have been discussed in detail with computer generated figures and tables.

L-2: Effect of nonuniform temperature gradients and DC electric field on thermal convective instability in an Oldroyd-B fluid

saturated porous media, Deepa K Nair, Potluri Geetha Vani and I. S. Shivakumara (Bangalore :*vijaydeepavijay@rediffmail.com, shivakumarais@gmail.com, gipotluri@gmail.com*).

The effect of vertical DC electric field on the onset of convection in a horizontal layer of an Oldroyd-B viscoelastic dielectric fluid saturated Brinkman porous medium heated either from below or from above is investigated. The isothermal boundaries are considered to be either rigid or free. The resulting eigenvalue problem is solved using the Galerkin method for three kinds of velocity boundary conditions namely, free-free, rigid-rigid, and lower rigid and upper free. The results indicate that the instability behavior depends significantly on the nature of boundaries. The effects of Darcy number, the Prandtl number, the ratio of strain-retardation time to the stress-relaxation time and the stress-relaxation parameter are analyzed on the stability of the system. Besides, the similarities and differences between free-free, rigid-rigid and rigid-free boundaries are emphasized in triggering convective instability. The stress-free boundaries are found to be less stable than that of rigid-rigid and rigid-free boundaries. The existing results in the literature are obtained as special cases from the present general study.

L-3: Solution of Flow of Current in Electrical Circuit via Fractional Calculus Approach, P.V. Shah, A.D. Patel and A.K. Shukla (Surat).

The present paper deals with the solution of Flow of Current in Electrical Circuit by using fraction calculus approach. We also discuss different cases of R-I (Resistence-Inductence) circuit. We also analyze the main result.

M: Bio-Mathematics

M-1: Fractional Bioheat Model to Study Effect of Frequency and Blood Perfusion in Skin Tissue with Sinusoidal Heat Flux Condition on Skin Tissue, R.S. Damor, S. Kumar and A.K. Shukla (Surat).

This paper deals with the study of fractional bioheat model for heat transfer in skin tissue with sinusoidal heat flux condition to evaluate effect of different frequency and blood perfusion on skin tissue. Numerical solution is obtained by implicit finite difference method. The effect of anomalous diffusion in skin tissue has been studied with different frequency and blood perfusion respectively, the temperature profiles are obtained for different order fractional bioheat model.

M-2: Study of magnetic fluid hyperthermia, Sonalika Singh and Sushil Kumar (Surat).

Hyperthermia is a type of cancer treatment which kills cancereous cells, usually with minimal damage to normal tissues. The present study has been discussed the numerical methods to solve mathematical model of single phase lag heat transfer in bi-layered spherical tissue during magnetic fluid hyperthermia treatment for tumour.

M-3: A probabilistic model to study the gene expression in a cell, Amit Sharma and Neeru Adlakha (Surat :*amitsharmajrf@gmail.com, neeru.adlakha21@gmail.com*).

Cell is the primordial unit of all living organisms and to understand the mechanism of cell, it is important to understand the transcription and translation process in the cell. A mathematical model of gene expression is developed to understand these processes. In this paper, future state of the model is predicted on the base of present state. The initial state of the system is assumed to be known. Based on the initial state, the successive states are predicted using probabilities. The model is used to predict the final state of central dogma.

M-4: Biomathematics : modular forms and Galois representation of BIS process, Ashwini K. Sinha, M.M. Bajaj and Rashmi Sinha.

$GL_2(\mathbb{R})$ consists of matrices with positive determinant. Let $\mathbb{H}$ be the complex upper half plane endowed with its natural $GL_2^+(\mathbb{R})$ action by linear fractional transformations: $\gamma : z := \dfrac{az+b}{cz+d}$, where $a,\ b,\ c, d$ are the entries of γ as above and $z \in \mathbb{H}$. If f is any $\mathbb{C}$-valued function on $\mathbb{H}$. wstabilizer of ∞ in $SL_2(\mathbb{Z})$. Assume conditions Adding the cusps oince $f \in M_k(SL_2(\mathbb{Z}))$ is a cusp form iff u vanishes infinity there is an exact sequence for even $k \geq 4$

$$0 \to S_k(SL_2(\mathbb{Z})) \to M_k(SL_2(\mathbb{C})) \to \mathbb{C} \to 0$$

induced by the map

$$\sum_{n \geq 0} a_n q^n \to a_0.$$

It follows that in this case there is a unique normalized modular form.

N: History and teaching of mathematics:

No paper.

ABSTRACT OF THE PAPERS FOR IMS PRIZES.
GROUP-1:

IMS-1: Modified lattice paths and Gordon-McIntosh eight order mock theta functions, Rachna Sachdeva (Chandigarh, *rachna.sachdeva.1989@gmail.com*).

We modify the lattice paths introduced by Agarwal and Bressoud in 1989. We use these modified lattice paths to provide combinatorial interpretations of two Gordon-McIntosh eight order mock theta functions.

IMS-2: Number of self-orthogonal negacyclic codes over finite fields, Amita Sahni (Chandigarh : *sahniamita05@gmail.com*).

The main objective of this article is to study self-orthogonal negacyclic codes of length n over a finite field $\mathbb{F}_q$, where the characteristic of $\mathbb{F}_q$ does not divide n. We investigate issues related to their existence, characterization and enumeration. We find the necessary and sufficient conditions for the existence of self-orthogonal negacyclic codes of length n over a finite field $\mathbb{F}_q$. We characterizes the defining sets and the corresponding generator polynomials of these codes. We obtain formulae to calculate the number of self-dual and self-orthogonal negacyclic codes of a given length n over $\mathbb{F}_q$. The enumeration formula for self-orthogonal negacyclic codes involves a two-variable function χ defined by $\chi(d, q) = 0$ if d divides $(q^k + 1)$ for some $k \geq 0$ and $\chi(d, q) = 1$, otherwise. We have found necessary and sufficient conditions when $\chi(d, q) = 0$ holds.

IMS-3: The b-chromatic number of graphs, Aparna Lakshmanan S. (Kerala :*aparnaren@gmail.com*).

Given a coloring of G, a color class of G is the collection of all vertices having the same color. A coloring of G is a b-coloring if every color class contains a vertex that is adjacent to at least one vertex in each of the remaining color classes. The b-chromatic number of G, denoted by $\varphi(G)$, is the largest integer k such that G admits a b-coloring with k colors. In this paper, we find the b-chromatic number of complete-n-partite graphs and the double graph of the complete graphs, cycles and paths. In the last section, the problem of "for what values of $a \in [\chi(G), d+1]$ does there exist a d-regular graph G, with $\varphi(G) = a$" is discussed.

IMS-4: Combinatorial Interpretations of two Gordon-McIntosh eight order mock theta functions,Garima Sood (Chandigarh :*garima-sood18@gmail.com*).

In 2004, A.K. Agarwal gave the combinatorial interpretations of four mock theta functions of Srinivasa Ramanujan using n-color partitions which were introduced by himself and G.E. Andrews in 1987. In this paper we introduce a new class of partitions and call them "split $(n+t)$-color partitons". These new partitions generalize Agarwal-Andrews $(n+t)$-color partitons. We use these new combinatorial objects and give combinatorial meaning to two eight order mock theta functions of Gordon-McIntosh found in 2000. The work done here has great potential for further research.

IMS-5: On Some Modular Equations In The Spirit Of Ramanujan, B.R. Srivatsa Kumar, (Manipal :*sri_ vatsabr@yahoo.com*).

We establish certain new modular equations, by employing Ramanujan's modular equations.

IMS-6: Equitable Δ-coloring conjecture for generalized Mycielskian of graphs, T. Kavaskar (*t_kavaskar@yahoo.com*).

A graph G is equitably k-colorable if G has a proper k-coloring in which any two color classes differ in size by at most 1. The smallest integer k for which G is equitably k-colorable is defined to be the equitable chromatic number of G, denoted by $\chi = (G)$. In 1994, Chen, Lih and Wu conjectured that, every connected graph G, different from complete graph, odd cycle or $K_{2n+1,2n+1}$, is equitably ∇-colorable. Only few families of graphs have been proved to satisfy this conjecture. In this paper, we prove that one more family of graphs satisfies this conjecture, namely, the generalized Mycielskian of some families graphs.

GROUP-2:

IMS-7: Existence Theorems for Solvability of Functional Equations arising in Dynamic Programming, Deepmala (Raipur :*dmrai23@ gmail.com*).

The existence problems of solutions of various functional equations arising in dynamic programming are both theoretical and practical interest. Under certain conditions, we give some sufficient conditions ensuring both the existence and the uniqueness of solutions for functional equation arising in dynamic programming of multistage decision processes. Here, we use Boyd-Wong fixed point theorem to show the solvability of the functional equation arising in dynamic programming. Our main results extend, improve and generalize the results due to several authors. Thus, we can say that the method described in our main section is an important procedure which is helpful for engineers, computer scientists, economists and researchers for finding the existence and uniqueness of the solutions of functional equations arising in dynamic programming. An example is also given to demonstrate the advantage of our results over the existing ones in the literature.

GROUP-3:
No paper.

GROUP-4:

IMS-8: Solution and stability of a functional equation originating from consecutive terms of a geometric progression, M. Arunkumar (Tiruvannamalai :*annarun2002@yahoo.co.in*).

In this paper, the author has proved the generalized Ulam-Hyers stability of a new type of the functional equation

$$l(uv) + l\left(\frac{u}{v}\right) = 2l(u),$$

with $v \neq 0$, which is originating from consecutive terms of an geometric progression. An application of this functional equation is also studied.

IMS-9: Existence and controllability results for mixed functional integrodifferential equations with infinite delay, Kishor D. Kucche (Maharashtra : *kdkucche@gmail.com*).

Sufficient conditions are established for the existence of mixed neutral functional integrodifferential equations with infinite delay. The results are obtained using the theory of fractional powers of operators and the

Sadovskii's fixed point theorem. As an application we prove a controllability result for the system.

GROUP-5:

No paper.

GROUP-6:

IMS-10: Mathematical Modeling of delivery of molecular medicine to solid tumers with chemotherapy, Ram Singh
(Rajouri : singh_ram2008@hotmail.com).

The key aim of this investigation is to develop a pharmacokinetic mathematical model for the localization of anti-cancer agents in the solid tumor tissues and subsequent intra tumoral drug generation associated with two step cancer chemotherapy. The equations governing the diffusion of large anti-cancer molecular cojugate out of vasculature and into the tumor are derived and numerically analyzed. The expressions for the concentration of molecular agents into various compartments have been obtained. The effects of the tumor vasculature, binding kinetics, and administration schedule on the intra tumoral conjugate concentration are investigated and the critical parameters that influence the localization and retention of the agent in the tumor are determined. We have incorporated the different cases of dosing intervals in the present model which makes our model more realistic than the model have been presented so far. Sensitivity analysis has been performed to validate the obtained analytical results. The finite difference technique has been used to solve the partial differential equations. Predictions made by the developed model can lead for the improvement of treatment protocols for two-step cancer chemotherapy.

ABSTRACT OF THE PAPERS FOR AMU PRIZE.

AMU-1: Generalized derivations and commutativity in near rings, Phool Miyan (Aligarh : *phoolmiyan83@gmail.com*).

Let N be a near ring. An additive mapping $f : N \longrightarrow N$ is said to be a right generalized (resp. left generalized) derivation with associated derivation d on N if $f(xy) = f(x)y + xd(y)$ (resp. $f(xy) = d(x)y + xf(y)$) for all $x, y \in N$. A mapping $f : N \longrightarrow N$ is said to be a generalized derivation with associated derivation d on N if f is both a right generalized and a left generalized derivation with associated derivation d on N. The purpose of the present paper is to prove some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a generalized derivation, thereby extending some known results proved by Bell and Mason [North-Holland Mathematical Studies, 137 (1987), 31-35] and Bell [Kluwer Academic Publ. Math. Appl. Dordr., 426 (1997), 191-197] on derivation to generalized derivation of a 3-prime near ring.

AMU-2: Characterization of symmetric biderivations on prime rings, Faiza Shujat (Aligarh : *faiza.shujat@gmail.com*).

The purpose of the present paper is to prove some results concerning symmetric biderivation on a one sided ideal of a ring, which are of independent interest. Moreover, we obtain a generalization of a result of Bresar [J. Algebra 172 (1995)]. A biadditive mapping $D : R \times R \to R$ is called a biderivation if for every $x \in R$, the map $y \mapsto D(x, y)$ as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is a derivation of R, i.e., $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$ and $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$. In [Canad Math Bull. 22(4) (1979)] Herstein determined the structure of a prime ring R admitting a nonzero derivation d such that the values of d commute, that is for which $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$. Perhaps even more natural might be the question of what can be said on a derivation when elements in a prime ring commute with all the values of a nonzero derivation. Herstein addressed this question by proving the following result. If d is a nonzero derivation of a prime ring R and $a \notin Z(R)$ is such that $[a, d(x)] = 0$ for all $x \in R$, then R has a characteristic 2, $a^2 \in Z(R)$ and $d(x) = [\lambda a, x]$, for all $x \in R$ and $\lambda \in C$, the extended centroid of R. In the mentioned paper, Bresar generalized above result of Herstein and gave a dscription of derivations d, g and h of a prime ring satisfying $d(x) = ag(x) + h(x)b$, $x \in R$, where a, b are some fixed elements in R. Then d, g, h has the following forms $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$ for all $x \in R$. Inspired by all these observations, our aim is to generalize above results for the case of biderivations on two sided ideals of prime rings. We prove the following. Let R be a prime ring of char$\neq 2$, I a nonzero ideal of R and $D,\ G,\ H : I \times I \to Q$ be biderivations of R with trace d, g, h, respectively. Suppose there exists $a, b \in R$ such that $D(x, x) = aG(x, x) + H(x, x)b$ for all $x \in I$. If $a, b \notin Z(R)$, then there exists $\lambda \in C$ such that $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$ for all $x \in I$.

ABSTRACT OF THE PAPERS FOR V. M. SHAH PRIZE.

VM Shah-1: Bi Unique Range Sets For Meromorphic Functions, Abhijit Banerjee (West Bengal :*abanerjee_kal@yahoo.co.in, abanerjeekal@gmail.com*).

Let f and g be two non-constant meromorphic functions and S be a set of distinct elements of $\mathbb{C}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) = a\}$, where each point is counted according to its multiplicity. Denote by $\overline{E}_f(S)$ the reduced form of $E_f(S)$. If $E_f(S) = E_g(S)$, we say that f and g share the set S CM. If $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. In this paper we introduce a new kind of pair of finite range sets in $\mathbb{C}$ such that two meromorphic functions share the sets with some relaxed sharing conditions become identical.

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VISCOSITY APPROXIMATION METHODS FOR MINIMIZATION AND FIXED POINT PROBLEMS - A RELOOK*

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Abstract. The present lecture is intended to be a survey of two broad themes in mathematics: "minimization problems" and "fixed point problems" with a focus on a common thread which links these two themes- the so-called "viscosity approximation methods". The talk will broadly consist of two parts. The first part will be a revisit to the viscosity solutions of minimization problems. The second part will review viscosity approximation methods for fixed points of non-expansive mappings.

1. Introduction

I feel deeply indebted and honored that the Indian Mathematical Society has invited me to deliver this lecture, which is the 11^{th} Ganesh Prasad Memorial Lecture. At the outset, I must thank Prof. Mrs. Geetha Rao, the President, Prof. N. K. Thakare, the General Secretary and Prof. Satya Deo Tripathi, the Academic Secretary, for doing me this honor. Dr. Ganesh Prasad (1876-1935) who is verily considered as the father of mathematical researches in India was a mentor to many of the illustrious mathematicians from India including the late Dr. A. N. Singh, Dr. Gorakh Prasad, Dr. R. S. Verma, Dr. B. N. Prasad, Dr. N. G. Shabde, Dr.R. D. Mishra and others. Dr. Prasad made many important and lasting contributions to mathematics which included over fifty research papers and eleven books of which *'A Treatise on Spherical Harmonics and the Functions of Bessel and Lame'* has become almost a classic. Aside from being a great scholar and a researcher, Dr. Prasad was also a great philanthropic individual, and above

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all a great human being. I salute his memory before turning to the technical parts of this talk.

The present lecture is intended to be a survey of two broad themes in Mathematics (rather, I should say Modern Applied Mathematics) where one would like to explore a common underlying thread- the so-called *viscosity approximation methods*. The first part (Section 2) is a revisit to the viscosity solutions of minimization problems. The second part (Section 3) is a survey of viscosity approximation methods for fixed points of non-expansive maps which is attracting some renewed attention recently. It is perhaps appropriate to begin with the following three interesting quotes. The second one is related to the famous problem of brachistochrone posed by Johann Bernoulli.

"Because the shape of the whole universe is most perfect, and, in fact designed by the wisest creator, nothing in all of the world will occur in which no maximum or minimum rule is shining forth".

Leonard Euler

"If one considers motions with the same initial and terminal points then, the shortest distance between them being a straight line, one might think that the motion along it needs least time. It turns out that this is not so".

Galileo Galilei

"The profound significance of well-posed problems for advancement of mathematical science is undeniable".

David Hilbert

The so-called viscosity approximation methods have been long used in diverse problems arising in variational analysis and optimization. These have various applications to different areas such as mathematical programming(cf.,e.g.,[11]) , variational problems including the classical minimal hypersurface problem (cf.[12]), plasticity theory(cf.[25]), phase transition (cf.[21]), PDE's (cf., e.g.,[3, 5, 17]), control theory (cf.,e.g.,[3, 9, 11, 16]). The main feature of these viscosity approximation methods is to be able to capture as a limit of solutions of a sequence of approximating problems, a particular solution of the underlying problem, the so-called *viscosity solution*, which is often a preferred solution.

The viscosity approximation methods for fixed points seem to have originated in the initial questions pertaining to the *fixed point property* in the metric fixed point theory for non-expansive mappings. One of the first seminal results in this direction is the one due to Browder [7]. A number of others have also contributed to this topic(cf.,e.g.,[14, 18, 22, 23, 24, 29]). More recently, there is some renewed interest in the minimum norm fixed points of non-expansive maps (cf.,e.g.,[30]), which are related to viscosity approximations. We will review a few of these results in Section 3.

2. Viscosity Solutions Of Minimization Problems

2.1. **Existence analysis in Optimization and some preliminaries.** Let us begin by recalling the so-called *Max-Min Theorem* from elementary calculus which asserts that a continuous function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ defined on a compact subset D of $\mathbb{R}^n$ is bounded and that it attains its (global) maximum and its (global) minimum at some points of D. Since the topology of $\mathbb{R}^n$ that we are using is metrizable, by Heine-Borel theorem, saying that D is compact is equivalent to saying that D is closed and bounded. More importantly, it is equivalent to saying that every sequence $\mathbf{x}^{(\mathbf{n})}$ in D has a convergent subsequence $\mathbf{x}^{(\mathbf{n_k})}$ converging in D.

We intend to begin by looking at a slightly more general result than the above stated result. For this purpose, let us observe that since $max_D f = -min_D(-f)$, the problem of maximizing f is equivalent to the problem of minimizing $-f$. Thus, without loss of generality, we may confine our attention to minimization problems in what is sometimes called *unilateral analysis*. We recall below some of the standard definitions.

Definition 2.1. Let X be a topological space. A function $f : X \to \mathbb{R} \cup \{\infty\}$ is said to be (i) *inf-compact* if for each $\alpha \in \mathbb{R}$, the sub-level set of f at height α:

$$lev_\alpha f = \{x \in X : f(x) \leq \alpha\}$$

is compact. It is said to be (ii) *lower semi-continuous* (lsc) if $lev_\alpha f$ is closed for each $\alpha \in \mathbb{R}$. Furthermore, in case X is a normed linear space, f is said to be (iii) *coercive* if $lim_{\|x\| \to +\infty} f(x) = +\infty$.

Remark 2.1. It is clear from the definitions that for a function $f : X \to \mathbb{R} \cup \{\infty\}$ defined on a normed space X, f is coercive if and only if f is *inf-bounded*, that is to say that the sublevel set $lev_\alpha f$ of f at height α is bounded for each $\alpha \in \mathbb{R}$. As a result, we see that if $X = \mathbb{R}^n$ with the usual topology, then for a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ which is lsc, f is coercive if and only if f is inf-compact.

Remark 2.2. It is easily seen that $f : X \to \mathbb{R} \cup \{\infty\}$ is lsc if and only if

$$\forall x \in X, f(x) \leq liminf_{y \to x} f(y) := sup_{N \in \mathcal{N}_x} inf_{y \in N} f(y).$$

Here $\mathcal{N}_x$ denotes the family of neighborhoods of x. Moreover, in case X is metrizable, then it is easily seen that a function $f : X \to \mathbb{R} \cup \{\infty\}$ is lsc if and only if it is sequentially lsc: whenever a sequence x_n in X converges to x_0, $liminf_n f(x_n) \geq f(x_0)$.

Let us recall that one main reason for bringing in extended real-valued functions in variational analysis and optimization is that these provide a flexible approach to modeling of minimization problems with constraints. Most minimization problems can be formulated as

$$min\{f_0(x) : x \in \Omega\}, \tag{2.1}$$

where $f_0 : X \to \mathbb{R}$ is a real-valued function, and $\Omega \subseteq X$ is the so-called *constraint set* or *feasible set*, X being some vector space, usually $\mathbb{R}^n$ in case of a mathematical programming problem. A natural way of dealing with such a problem is to apply a penalty to it. For example, introduce a distance d on X and for any positive real number λ, let us consider the minimization problem

$$min\{f_0(x) + \lambda dist(x, \Omega) : x \in X\}, \tag{2.2}$$

where,

$$dist(x, \Omega) = inf\{d(x, y) : y \in \Omega\} \tag{2.3}$$

is the distance function from x to Ω. Let us note that the penalty is equal to zero if $x \in \Omega$ (that is if the constraint is satisfied), and when $x \notin \Omega$ it takes larger and larger values with λ (when the constraint is violated). Notice that the approximated problem (2.2) can be written as

$$min\{f_\lambda(x) : x \in X\}, \tag{2.4}$$

where

$$f_\lambda(x) = f_0(x) + \lambda dist(x, \Omega), \tag{2.5}$$

is a real-valued function. Thus the approximated problems (2.2) are unconstrained problems. As $\lambda \to +\infty$, the (generalized) sequence of functions (2.5) increases to the function $f : X \to \mathbb{R} \cup \{\infty\}$, which is equal to

$$f(x) = \begin{cases} f_0(x), & \text{if } x \in \Omega, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.6}$$

Thus we are led to minimization of an extended real-valued function f:

$$min\{f(x) : x \in X\}$$

where f is given by (2.6). Let us note that if we introduce the *indicator function* δ_Ω of the set Ω:

$$\delta_\Omega(x) = \begin{cases} 0, & \text{if } x \in \Omega, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.7}$$

then we have the convenient expression $f = f_0 + \delta_\Omega$. We now come to the following generalization of the *Min-Max Theorem* popularly called the Weierstrass theorem.

The set of all the global minimizers of f on X is usually denoted by $argmin_X(f)$. The next theorem gives the conditions under which $argmin_X(f) \neq \emptyset$.

Theorem 2.1. *Let X be a topological space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function which is lower semicontinuous and inf-compact. Then $inf_X f > -\infty$, and there exists some $\hat{x} \in X$ which minimizes f on X:*

$$f(\hat{x}) \leq f(x) \quad \forall x \in X.$$

Proof. We need to prove that $argminf \neq \emptyset$. Indeed,

$$\begin{aligned} argminf &= \bigcap_{\alpha > inf_X f} lev_\alpha f \\ &= \bigcap_{\alpha_0 > \alpha > inf_X f} lev_\alpha f, \end{aligned}$$

where $\alpha_0 > inf_X f$ is taken arbitrary. The conclusion now follows from the compactness of the level set $lev_{\alpha_0} f$ and the finite intersection property of the closed sets $lev_\alpha f$ for $\alpha_0 > \alpha > inf_X f$. □

Remark 2.3. As a corollary of the preceding theorem, we have: Let X be a topological space and K be a compact subset of X. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, then $argmin_K(f) \neq \emptyset$.

Remark 2.4. A scrutiny of the proof of the preceding theorem reveals that the inf-compactness assumption on f can be slightly relaxed. In fact, the compactness of some lower level set $lev_{\alpha_0} f$ at a height $\alpha_0 > inf_X f$ is all that we need for ensuring the conclusion of this theorem.

Theorem 2.2. *Let X be a topological space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function which is lsc and such that for some $\alpha_0 > inf_X f, lev_{\alpha_0}$ is compact. Then $inf_X f > -\infty$ and*

$$argmin_X(f) \neq \emptyset.$$

To consider the difference between the last two theorems, let us take $X = \mathbb{R}^n$ and $f(\mathbf{x}) = \frac{\|\mathbf{x}\|}{1+\|\mathbf{x}\|}$. Then $lev_\alpha f$ is compact for each $\alpha < 1$, but $lev_1 f = \mathbb{R}^n$. We can apply the preceding theorem to conclude the existence of a minimizer which is $\mathbf{0}$ in this case.

Corollary 2.1. *Let $X = \mathbb{R}^n$ with the usual topology, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function which is lsc and coercive. Then*

$$argmin_X(f) \neq \emptyset.$$

A natural generalization of the above corollary to infinite dimensional spaces is the next well known theorem.

Theorem 2.3. *Let X be a reflexive Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex lsc and coercive function. Then*

$$argmin_X(f) \neq \emptyset.$$

Proof. Since f is convex lsc, it is weakly lsc. Also the sublevel sets of f are bounded in view of the coercivity of f. Hence f is weakly inf-compact in view of the reflexivity of X. An application of Theorem 2.1 taking the topology as the weak topology gives the desired conclusion. □

In many situations of practical interest, the objective function f to be minimized fails to be lower semicontinuous for a topology τ for which a minimizing sequence is τ-relatively compact. This necessitates the introduction of a τ-lower closure of f for consideration of the *relaxed problem* as follows

Definition 2.2. Given a topological space (X, τ) and a function $f : X \to \mathbb{R} \cup \{+\infty\}$, the τ-lower envelope of f is defined as

$$lsc_\tau f = sup\{g : X \to \mathbb{R} \cup \{+\infty\}, g \;\; \tau - lsc, g \leq f\}.$$

Remark 2.5. It is easily seen that for a function $f : X \to \mathbb{R} \cup \{+\infty\}$, for any $x \in X$,

$$lsc_\tau f(x) = liminf_{y \to x} f(y) = min\{liminf_\lambda f(x_\lambda) : \text{ net } x_\lambda \to x\}.$$

Also, it is clear that f is τ-lsc at $x \in X$ if and only if $f(x) = lsc_\tau f(x)$.

Example 2.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{R}, x \leq 0 \\ 0, & \text{if } x \in \mathbb{R}, x > 0 \end{cases}. \tag{2.8}$$

Clearly, the function f is not lsc, its lower closure is the function

$$lsc_\tau f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{R}, x < 0 \\ 0, & \text{if } x \in \mathbb{R}, x \geq 0 \end{cases}. \tag{2.9}$$

Remark 2.6. For a function $f : X \to \mathbb{R} \cup \{+\infty\}$, it is easily seen that

$$inf_X(f) = inf_X(lsc_\tau f). \tag{2.10}$$

Also,

$$argmin_X(f) \subset argmin_X(lsc_\tau f).$$

2.2. **Tikhonov well-posedness of minimization problems.** Let X be a convergence space: convergence of sequences $< x_n >$ (or nets $< x_\lambda >$ is defined in X satisfying the 'Kuratowski axioms'. Let

$$f : X \to (-\infty, +\infty](= \mathbb{R} \cup \{+\infty\})$$

be a *proper* function which is bounded below. Consider the minimization problem (X, f) :

$$\min\{f(x) : x \in X\}.$$

Let us denote

$$v_X(f) = \inf_X f := \inf\{f(x) : x \in X\}$$

the *optimal value function*. Let

$$argmin_X(f) := \{x \in X : f(x) = v_X(f)\},$$

denote the *set of optimal solutions of* (X, f), and for $\epsilon \geq 0$, let

$$\epsilon - argmin_X(f) := \{x \in X : f(x) \leq v_X(f) + \epsilon\}$$

denote the non-void set of ϵ-*approximate solutions* of (X, f). Apparently, the first notion of well-posedness for minimization problems is due to Tikhonov (1963), which demands

- Existence and uniqueness of a global minimizer: $\exists x_0 \in X$ such that $\{x_0\} = argmin_X(f)$.
- Whenever, one is able to compute approximately the optimal value $f(x_0) = v_X(f)$, one automatically approximates the optimal solution x_0.
- More precisely, if $< x_n > \subset X$ is a sequence such that $f(x_n) \longrightarrow v_X(f)$ (such a sequence is called a *minimizing sequence* for (X, f)), then $x_n \to x_0$ where $\{x_0\} = argmin_X(f)$.
- Put differently, problem (X, f) is *Tikhonov well-posed (T.W.p.)* if f has a unique global minimizer on X towards which every minimizing sequence converges.

The following example illustrates that uniqueness of a global minimizer of f need not entail Tikhonov well-posedness of problem (X, f).

Example 2.2. Let $X = \mathbb{R}$ and let

$$f(x) = -x, \text{ if } x < 0, |x - 1|, \text{ if } x \geq 0.$$

Here $argmin_X(f) = \{1\}, x_0 = 1, v_X(f) = 0$.
Let $x_n = -\frac{1}{n}$. Then $< x_n >$ is a minimizing sequence which fails to converge to x_0.

The next example is a classical example usually encountered in mathematical programming.

Example 2.3. Let $\Omega \subset \mathbb{R}^n$, and $f \in \mathcal{C}^{(2)}(\Omega), \mathbf{x_0} \in \Omega$. Suppose $\nabla f(\mathbf{x_0}) = \mathbf{0}$, and that the Hessian $H_f(\mathbf{x_0})$ is positive definite. Then $\mathbf{x_0}$ is a local minimizer of f. By Taylor's formula, there is a $\alpha > 0$ such that

$$f(\mathbf{x}) \geq f(\mathbf{x_0}) + \alpha \|\mathbf{x} - \mathbf{x_0}\|^2, \mathbf{x} \in B. \tag{2.11}$$

Here the problem (B, f) is T.w.p.

The example to follow shows that if problem (X, f) is not Tikhonov well-posed, then one may find a minimizing sequence which is not convergent.

Example 2.4. Let $E = \mathbb{R}^2$, equipped with the box norm $\|x\| = max\{|x_1|, |x_2|\}, x = (x_1, x_2) \in \mathbb{R}^2$. Let $X = \{(x_1, 0) : x_1 \in \mathbb{R}\}$. Let $f(x) = \|(0,1) - (x_1, 0)\|, x \in X$. Here $v_X(f) = 1, argmin_X f = \{(x_1, 0) : -1 \leq x_1 \leq 1\}$. Clearly, (X, f) is not Tikhonov well-posed. However, if we define $x_n = (-1, 0)$, if n is even and $x_n = (1, 0)$, if n is odd, then clearly $\{x_n\}$ is a minimizing sequence which is not convergent.

The next theorem lists some of the classical sufficient conditions for Tikhonov well-posedness of problem (X, f) (cf.,e.g., [11]).

Theorem 2.4. *Under any one of the following conditions, problem* (X, f) *is T.w.p.*

(i) $X = \mathbb{R}^n, f : X \to \mathbb{R}$ *is strictly convex, and coercive:*

$$f(x) \longrightarrow +\infty \text{ as } \|x\| \to \infty.$$

(ii) E *is a reflexive Banach space,*
$X \subset E$ *is a nonempty closed convex set,* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is proper, strictly convex, lsc, and coercive.*

(iii) $X = \mathbb{R}^k, f : X \to \mathbb{R}$ *is convex and lsc,* $argmin_X(f)$ *is a singleton.*

For geometric notions in Banach spaces from the theory of best approximations needed in the next two results one may refer to Chapter 8 of [20].

Theorem 2.5. *Let* X *be a nonempty closed and convex subset of a Banach space* E *which is in the class* $(Rf) \cap (R) \cap (A)$ *of Banach spaces that are reflexive, strictly convex and satisfying the Kadec norm property* (A)*: the weak convergence of a norm one sequence* $\{x_n\}$ *in* E *implies its strong convergence. Let* $f_u : X \to \mathbb{R}$ *be defined by:*

$$f_u(x) = \|u - x\|, \ \ x \in X.$$

Then (X, f_u) *is T.w.p. for each* $u \in E$.

Corollary 2.2. *If* X *is a nonempty closed and convex subset of a uniformly convex Banach space* E *and* f_u *is the function as defined in the previous theorem, then* (X, f_u) *is T.w.p.for each* $u \in E$.

Remark 2.7. Let X be a subset of a normed linear space E. Let us recall that X is said to be (i)*Chebyshev* if each element $u \in E$ has a unique best approximation (nearest element) from X and it is said to be (ii) *approximatively compact* if for each $u \in E$ each minimizing sequence $\{x_n\} \in X$, i.e., a sequence such that $\|u - x_n\| \to dist(u, X)$, has a convergent subsequence converging in X. If X is *Chebyshev and approximatively compact*, then it is easily seen that (X, f_u) is T.w.p. for each $u \in E$. Here f_u is the function as defined in the last theorem.

Definition 2.3. Recall that a function $c : T \to [0, +\infty)$ is called a *forcing function* (or a *firm function*)if

$$0 \in T \subset [0, +\infty), < t_n > \subset T, c(t_n) \to 0 \Rightarrow t_n \to 0.$$

Also, recall that if K is a nonempty convex subset of a normed space then a function $f : K \to \mathbb{R}$ is said to be *quasi-uniformly convex* if there exists a forcing function $c : [0, +\infty) \to [0, +\infty)$ such that

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\} - c(\|x-y\|), \forall x, y \in K \text{ and } \alpha \in (0, 1). \quad (2.12)$$

Theorem 2.6. *(*[11]*) Let K be a nonempty closed and convex subset of a Banach space X, and $f : K \to \mathbb{R}$ be lower semicontinuous, bounded below, and quasi-uniformly convex. Then problem (K, f) is T.w.p.*

2.3. **Tikhonov regularization of ill-posed problems.** Let us consider problem (K, f) where $K \subset X$, a Banach space and $f : X \to \mathbb{R}$ are such that (K, f) is *Tikhonov ill-posed*. Our aim here is to explore a strongly convergent minimizing sequence for (K, f) by approximately solving *appropriate perturbations* of (K, f) by adding to f a small regularizing term.This procedure seems fairly practical since only approximate knowledge of f is all that is required. Fix up sequences $\alpha_n > 0, \epsilon_n \geq 0$ such that $\alpha_n \to 0$ and $\epsilon_n \to 0$. For regularizing (K, f), we add to f a small non-negative *quasi-uniformly convex* term $\alpha_n g$ defined on the whole of X.

Theorem 2.7. *Let X be a Banach space,$f : X \to \mathbb{R}$ be $w-$sequentially l.s.c., $K \subset X$ be nonempty w-compact, and $g : X \to [0, +\infty)$ be l.s.c. and quasi-uniformly convex. Let $\alpha_n > 0, \epsilon_n \geq 0$ be given sequences of numbers such that $\alpha_n \to 0$ and $\epsilon_n \to 0$. Then the following conclusions hold:*

(i) If $u_n \in \epsilon_n - argmin_K(f + \alpha_n g), n \in \mathbb{N}$, then $< u_n >$ is a minimizing sequence for $(K, f) : f(u_n) \to v_K(f)$.
Also, if $\frac{\epsilon_n}{\alpha_n} \to 0$, then we have:

(ii) $\emptyset \neq limsup_n[\epsilon_n - argmin_K(f + \alpha_n g)] \subset argmin_{argmin_K(f)}(g)$.
Furthermore, if K and f are both convex, then we have:

(iii) $argmin_{argmin_K(f)}(g)$ is a singleton and denoting this set by $\{\tilde{u}\}$, we have $u_n \to \tilde{u}$ if $u_n \in \epsilon_n - argmin_K(f + \alpha_n g)$, $n \in \mathbb{N}$.

Remark 2.8. The above result which is a modified form of a result in [11], attributed to [19], already motivates the use of the terms 'viscosity function' g and the 'viscosity solutions' $argmin_{argmin_K(f)}(g)$ in what follows.

2.4. **Viscosity approximation of minimization problems.** Viscosity methods have been long employed for studying diverse problems arising in variational analysis and optimization. (cf.,e.g.,[3] for viscosity solutions of minimization problems, [12] for minimal hypersurface problem, [25] for plasticity, [27] for Tikhonov regularization, [9] for control theory: Hamilton-Jacobi equations, [11] for well-posedness and Tikhonov regularization). A characteristic feature of these methods is to capture as a limit of solutions of approximating problems, obtained by perturbing the objective function by a small multiple of a well behaved function, a particular solution of the underlying problem, the so-called *viscosity solution* which is often the preferred solution.

An abstract setting

Let X be an arbitrary set to be equipped with a suitable topology τ. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended real-valued function whose definition may be determined by some constraints. Let us consider the minimization problem

$$(P) \qquad \min\{f(x) : x \in X\}.$$

Given $g : X \to \mathbb{R}^+ \cup \{+\infty\}$ called the *viscosity function* and a sequence $\langle \epsilon_n \rangle \subset \mathbb{R}^+$, convergent to 0, let us consider the sequence of perturbed minimization problems:

$$(P_n) \qquad \min\{f(x) + \epsilon_n g(x) : x \in X\}.$$

It is assumed that for each $n \in \mathbb{N}$, there exists a solution u_n of (P_n),

Our main goal here is to investigate the convergence of the sequence $\{u_n\}$ and to characterize its limit. To this end, the notion of variational convergence called Γ-*convergence* given below, introduced by De Giogi and Franzoni[10] (also, called *epi-convergence* in Attouch [2, 3]), is very useful.

Definition 2.4. Given a topological space (X, τ) and functions $< f, f_n : X \to \mathbb{R} \cup \{+\infty\}, n \in \mathbb{N} >$, the sequence $< f_n >$ is said to Γ-*converge* to f, written $\tau - \Gamma - \lim_{n\to\infty} f_n = f$, if for each $x \in X$, we have:

(i) There exists $< x_n >$ which is τ-convergent to x for which

$$\lim sup_{n\to\infty} f_n(x_n) \leq f(x);$$

(i) Whenever $< x_n >$ is τ-convergent to x, we have

$$f(x) \leq \lim inf_{n\to\infty} f_n(x_n).$$

For simplicity, we simply write $f_n \xrightarrow{\Gamma} f$, whenever $\{f_n\}$ Γ-converges to f. In general, there is no connection between Γ-*convergence* and *pointwize convergence*.

However, it is known (cf.[3]) that in case of monotone sequences the two notions coincide upto lower closures.

(i) If

$$f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots,$$

then

$$\tau - \Gamma - \lim_{n\to\infty} f_n = sup_{n\in\mathbb{N}}(lsc_\tau f_n);$$

(ii) If

$$f_1 \geq f_2 \geq \ldots \geq f_n \geq \ldots,$$

then

$$\tau - \Gamma - \lim_{n\to\infty} f_n = lsc_\tau(inf_n f_n).$$

This explains in some sense the success of *monotone approximation schemes and viscosity methods* in variational analysis and optimization.

Remark 2.9. Let (X, τ) be a first countable topological space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function . If we take the sequence $< f_n >$ to be the constant sequence $f_n = f, n \in \mathbb{N}$ then it is easy to see that $f_n \xrightarrow{\Gamma} lsc_\tau f$. From this, one concludes that Γ-convergence is not topological.

The following result is mostly well known (cf., e.g., [4, p. 466]).

Theorem 2.8. *Let us be given a sequence* $< \epsilon_n > \subset \mathbb{R}^+$ *such that* $\epsilon_n \to 0$ *and a sequence of minimization problems*

$$(P_n) \qquad \min\{f_n(x) : x \in X\}.$$

Assume that there exists a topology τ *on* X *such that:*

(i) For every $n \in \mathbb{N}$*, there exists an* ϵ_n*-approximate solution* u_n *to* $(P_n), u_n \in \epsilon_n - argmin_X(f_n)$*,* $n \in \mathbb{N}$*, such that the sequence* $< u_n : n \in \mathbb{N} >$ *is* τ*-relatively compact;*

(ii) $f = \tau - \Gamma - \lim_{n\to\infty} f_n$.

Then

$$\lim_{n\to\infty} v_X(f_n) = v_X(f),$$

and every τ*-cluster point* $\hat{u}$ *of* $< u_n >$ *minimizes* f *on* X*, i.e.,* $\hat{u} \in argmin_X(f)$.

The following theorem is a modifed version of Theorem 2.6 in [3] for the relaxed problem .

Theorem 2.9. *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a given function which is proper and bounded below. Consider the associated minimization problem:*

$$(P) \qquad \min\{f(x); x \in X\}.$$

Let $<\epsilon_n>, <\alpha_n>, n \in \mathbb{N}$ *be given sequences in* $\mathbb{R}^+, \epsilon_n \neq 0$, *such that* $\epsilon_n \to 0, \alpha_n \to 0$, *and letting* $\beta_n := \frac{\alpha_n}{\epsilon_n}, \beta_n \to 0$.
Let us be given a function $g : X \to \mathbb{R}^+ \cup \{+\infty\}$ *(called the viscosity function),and for each* $n \in \mathbb{N}$, *consider the perturbed minimization problem*

$$(P_n) \qquad \min\{f(x) + \epsilon_n g(x) : x \in X\}.$$

Assume that there exists an α_n*-approximate solution of* $(P_n) : u_n \in \alpha_n - argmin_X (f_n), n \in \mathbb{N}$, *where* $f_n := f + \epsilon_n g$ *such that for some topology* τ *on* X, *we have:*

(i) The sequence $<u_n>_{n\in\mathbb{N}}$ *is* τ*-relatively compact;*
(ii) $lsc_\tau(f + \delta_{dom(g)}) = lsc_\tau(f)$, *and the function* g *is* τ*-l.s.c.*
(iii) $lsc_\tau(f + \epsilon_n g) = lsc_\tau f + \epsilon_n g \ \ \forall n \in \mathbb{N}$;
(iv) $dom(g) \cap argmin_X(lsc_\tau f) \neq \emptyset$.
Here $dom(g)$ *denotes the effective domain of* g.

Then every τ*-cluster point* $\hat{u}$ *of* $<u_n>$ *minimizes the function* $lsc_\tau f$ *on* X, $\lim_n f(u_n) = v_X(f)$, *and* $\hat{u}$ *satisfies for all* $v \in argmin_X(lsc_\tau f)$, *the so-called relaxed viscosity selection criterion*

$$\hat{u} \in argmin_X(lsc_\tau f), g(\hat{u}) \leq g(v) \Leftrightarrow \hat{u} \in argmin_{argmin_X(lsc_\tau f)}(g).$$

Moreover, the sequence $<u_n>$ *is a minimizing sequence of problem* (P) :

$$\lim_{n\to\infty} \frac{1}{\epsilon_n}[f(u_n) - v_X(f)] = 0$$

and also, $\lim_{n\to\infty} g(u_n) = v_{argmin_X(lsc_\tau f)}(g)$.

Corollary 2.3. *Under the assumptions as in the previous theorem, except that the viscosity function* $g : X \to \mathbb{R}^+$ *is assumed to be finite-valued, the conditions (iii) and (iv) are dropped, and in place of (iii) we assume (iii)': f and g are both* τ*-lsc, then the conclusions as stated in the theorem hold.*

The next theorem for viscosity solutions of convex minimization problems is a modifed version of a result in [3].

Theorem 2.10. *Let* X *be a reflexive Banach space (resp. the dual* E^* *of a separable normed space* E*),and* f *as above is a proper, convex, l.s.c. (resp.* w^**-l.s.c.) function which is bounded below and* $g : X \to \mathbb{R}^+$ *is coercive:*

$$lim_{\|x\|\to\infty} g(x) = \infty.$$

Then the sequence u_n *as in the previous theorem is bounded if and only if* $argmin_X(f) \neq \emptyset$. *In that case, every* $w-$*(resp.* $w*-$*)cluster point* $\hat{u}$ *of the sequence* $<u_n>$ *minimizes* f *on* X *and satisfies the following viscosity selection property:*

$$(VSP) \qquad \hat{u} \in argmin_{argmin_X(f)}(g).$$

Moreover, the sequence $< u_n >$ *is a minimizing sequence of problem* (P) :

$$\lim_{n\to\infty} \frac{1}{\epsilon_n}[f(u_n) - v_X(f)] = 0$$

and also, $\lim_{n\to\infty} g(u_n) = v_{argmin_X(f)}(g)$.

3. Viscosity Approximation of Fixed Points

3.1. **Introduction.** Let K be a nonempty, closed and convex subset of a (real) Banach space X. Let $T : K \to K$ be a non-expansive map:

$$\|Tx - Ty\| \leq \|x - y\| \ \ \forall x, y \in K.$$

We are mostly concerned here with the approximation of fixed points of T. Let $Fix(T)$ denote the set

$$\{x \in K : x = Tx\}$$

of fixed points of T. It is well known that, even if $Fix(T) \neq \emptyset$, the sequence of Picard iterates $x_n = T^n x_0, n \in \mathbb{N}$, of T may fail to converge. Also, $Fix(T)$ need not contain just one element.

3.2. **Fixed point property.**

Definition 3.1. A nonempty, bounded closed convex subset K of a Banach space X is said to satisfy the *fixed point property* (f.p.p.) if $Fix(T) \neq \emptyset$ for every non-expansive map $T : K \to K$.

In this connection, a *central question* which was being asked is the following: *What conditions on* K *or* X *ensure that* K *satisfies f.p.p.?*

The question, as stated above, arose in four interesting papers which appeared in 1965. The answers are given in the theorem given below.

Theorem 3.1. *Let* K *be a nonempty bounded closed convex subset of a Banach space* X*. Then* K *admits f.p.p. under any of the following conditions.*

(i) *(Browder* [6]*)*
X is a Hilbert space.

(ii) *(Browder*[7], *Göhde*[13]*)*
X is uniformly convex.

(iii) *(Kirk*[15]*)*
K is weakly compact, and it has normal structure:
Every nonempty bounded convex subset S *of* K *, containing at least two points, admits a nondiametral point* x*:*

$$r(x, S) := sup_{y\in S}\|x - y\| < diam(S) := sup\{\|y_1 - y_2\| : y_1, y_2 \in S\}.$$

A historical conjecture

For a while it was conjectured that

Any nonempty weakly compact convex subset of a Banach space must satisfy f.p.p.

This conjecture was settled in the negative by Alspach [1], who gave an interesting example of a weakly compact convex subset of $L^1[0,1]$ which admits an isometry lacking a fixed point.

3.3. **Viscosity approximation of fixed points.** Let us be given a non-expansive map $T : K \to K$, where K is a nonempty closed convex subset of a Banach space X. It is easily seen that $Fix(T)$ is closed, and that it is convex provided X is strictly convex. Let us be given a contraction $f : K \to K$:

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad \forall x, y \in K,$$

for some $\alpha \in (0,1)$. Assuming $Fix(T) \neq \emptyset$, for each $\epsilon \in (0,1)$, let us consider the perturbed map $T_\epsilon : K \to K$ given by

$$T_\epsilon x = \epsilon f(x) + (1-\epsilon) Tx, \quad x \in K. \tag{3.1}$$

Approximation of fixed points

Let us note that

- T_ϵ is a contraction on K. Hence,
- it has a unique fixed point x_ϵ in K:

$$x_\epsilon = \epsilon f(x_\epsilon) + (1-\epsilon) T(x_\epsilon). \tag{3.2}$$

Our aim here is to review some of the results concerning convergence of the net $< x_\epsilon >$, as given in (3.2), to a fixed point of T. Apparently, the first result in this direction is due to Browder[8], which is stated below.

Fix up $u \in K$, and consider the contraction P_ϵ on K defined by

$$P_\epsilon x = \epsilon u + (1-\epsilon) Tx, \quad x \in K. \tag{3.3}$$

Let $u_\epsilon \in K$ be the unique fixed point of P_ϵ:

$$u_\epsilon = \epsilon u + (1-\epsilon) T u_\epsilon. \tag{3.4}$$

First viscosity result for fixed points

In case X is a Hilbert space, using monotonicity and demi-closedness of the operator $I - P$, Browder [8] established the following result.

Theorem 3.2. *(Browder)*
In a Hilbert space X, the net $< u_\epsilon : \epsilon \in (0,1) >$, as defined in (3.4), converges to the unique fixed point of T that is nearest to u, in other words, to the best approximation of u onto $Fix(T)$.

In order to review some generalizations of this result to Banach spaces, we need to recall below some definitions.

Definition 3.2. Let$S(X)$ denote the unit sphere $\{x \in X : \|x\| = 1\}$ of the Banach space X. The norm $\|.\|$ in X is said to be *Gâteaux differentiable* if the limit

$$lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x, y \in S(X)$; X is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S(X)$, the above limit is uniform for $x \in S(X)$.

Definition 3.3. If $dim(X) \geq 2$, the *modulus of smoothness* of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(\theta) := sup\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \theta\}.$$

The space X is said to be

(i) *smooth*, if the norm $\|.\|$ is Gâteaux differentiable;

(ii) *uniformly smooth*, if

$$lim_{\theta\to 0}\frac{\rho_X(\theta)}{\theta} = 0.$$

The following theorem partially generalizes Theorem 3.2.

Theorem 3.3. *(Reich*[24]*, Morales and Jung*[22]*)*
Let K be a nonempty closed and convex subset of a Banach space X which has a uniformly Gâteaux differentiable norm, and let $T : K \to K$ be a non-expansive map with $Fix(T) \neq \emptyset$. Then for any choice of $u \in K$ the net $< u_\epsilon >$, as defined in (3.4), converges to a fixed point of T.

Explicit iteration

An *explicit* iterative process suggested by discretization of (3.2) is

$$x_{n+1} = \epsilon_n f(x_n) + (1-\epsilon_n)Tx_n, n \in \mathbb{N}, \tag{3.5}$$

where $\{\epsilon_n\}$ is a sequence in $(0,1)$.
A special case of (3.5) is the iterative process

$$u_{n+1} = \epsilon_n u + (1-\epsilon_n)Tu_n, \;\; n \in \mathbb{N}, \tag{3.6}$$

where u, u_0 in K are arbitrary and $\{\epsilon_n\} \subset (0,1)$.

This iteration was first introduced by Halpern[14] in the framework of Hilbert spaces. He proved the *weak* convergence of $\{u_n\}$ to a fixed point of T, where $\epsilon_n := n^{-a}, a \in (0,1)$. Lions[18] further improved the result of Halpern as follows.

Theorem 3.4. *In a Hilbert space X, the sequence $\{u_n\}$ as defined in (3.6) converges to the unique best approximation of u onto $Fix(T)$, provided $\{\epsilon_n\}$ satisfies the following:*

(L1) $\lim_{n\to\infty} \epsilon_n = 0$;
(L2) $\sum \epsilon_n = \infty$;
(L3) $\lim \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n^2} = 0$.

Whittman[28] showed that the above theorem holds with condition (L3) replaced by the condition

$$(W3) \quad \sum |\epsilon_{n+1} - \epsilon_n| < \infty,$$

the other conditions remaining the same.
For viscosity approximation, Moudafi [23] established the convergence in the norm of the implicit method (3.2) by taking $\epsilon = \epsilon_n = \frac{\delta_n}{1+\delta_n}$ as well as the strong convergence of the explicit method (3.5) by taking ϵ_n as above. In the latter case, the condition (L3) of Lions was replaced by

$$(M3) \quad \lim_{n\to\infty} |\frac{1}{\delta_n} - \frac{1}{\delta_{n-1}}| = 0.$$

In each case, the iterates converge to the unique fixed point of $P_{Fix(T)} \circ f$.

Both the results of Moudafi[23] were extended by Xu[29] as follows:

Theorem 3.5. *In a Hilbert space X, let the net $< x_\epsilon >$ be given by (3.2). Then we have:*

(i) $\lim_{\epsilon\to 0} x_\epsilon = \hat{x}$ *exists;*
(ii) $\hat{x} = P_{Fix(T)} f(\hat{x})$, *or equivalently, $\hat{x}$ is the unique solution in $Fix(T)$ to the variational inequality*

$$(VI)\ \langle (I-f)\hat{x},\, \hat{x} - x\rangle \le 0,\ x \in Fix(T), \tag{3.7}$$

where $P_{Fix(T)}$ denotes the metric projection onto $Fix(T)$

In the sequel, we denote by Λ_K the set of all contractions $f : K \to K$.

Theorem 3.6. *Let X be a Hilbert space, K be a nonempty closed convex subset of $X, T : K \to K$ be a non-expansive map with $Fix(T) \neq \emptyset$, and $f : K \to K$ be a contraction. Let the sequence $\{x_n\}$ be generated by (3.5). Then under the hypothesis (H1)-(H3), where (H1) = (L1), (H2) = (L2), and*

$$(H3) \quad \textit{either} \sum |\epsilon_{n+1} - \epsilon_n| < \infty \ \textit{or} \ \lim \frac{\epsilon_{n+1}}{\epsilon_n} = 1, \tag{3.8}$$

$x_n \to \hat{x}$, where $\hat{x}$ is the unique solution of the variational inequalities (3.7).

The next two theorems from [29] are Banach space versions of Theorems 3.5 and 3.6.

Theorem 3.7. *Let X be a uniformly smooth Banach space, K a nonempty closed convex subset of $X, T : K \to K$ a nonexpansive map with $Fix(T) \neq \emptyset$, and $f \in \Lambda_K$. Then the net $< x_\epsilon : \epsilon \in (0,1) >$ defined by (3.2) converges to an element of $Fix(T)$. If $Q : \Lambda_K \to Fix(T)$ is defined by*

$$Q(f) := \lim_{\epsilon \to 0} x_\epsilon, \quad f \in \Lambda_K, \tag{3.9}$$

then $Q(f)$ solves the variational inequalities

$$\langle (I-f)Q(f),\, J(Q(f)-x)\rangle \leq 0,\ f \in \Lambda_K,\ x \in Fix(T). \tag{3.10}$$

Here J denotes the duality map of X defined by

$$J(x) = \{x^* \in X^* : \langle x,\, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, x \in X.$$

Remark 3.1. In particular, if we take $f = u \in K$, a constant, then (3.10) is reduced to

$$\langle Qu - u,\, J(Qu - x)\rangle \leq 0, u \in K,\ x \in Fix(T), \tag{3.11}$$

which is a result of Reich [24].

Theorem 3.8. *Let X be a uniformly smooth Banach space, K a nonempty closed convex subset of $X, T : K \to K$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f \in \Lambda_K$. Then the sequence $\{x_n\}$ defined by (3.5), where $\{\epsilon_n\}$ satisfies (H1)-(H3) as in Theorem 3.6, converges to $Q(f)$, where $Q : \Lambda_K \to Fix(T)$ is defined by (3.6).*

3.4. **Minimum-norm fixed points.** For an ill-posed minimization problem posed over a Hilbert space X, if the viscosity function used for regularization is $g(x) = \frac{1}{2}\|x\|^2$, then one knows that the so-called *viscosity solution* is a minimum norm solution among all the solutions of the original problem. As a typical example, let us consider the *ill-posed constrained linear inverse problem*:

$$Tx = y, \quad x \in K, \tag{3.12}$$

where X, Y are Hilbert spaces, K is a nonempty closed convex subset of Y, $T : X \to Y$ is a bounded linear map and $y \in Y$. The problem is said to be *well-posed* if there is a unique solution which depends continuously on the *data y*; otherwise, it is said to be *ill-posed*. In case the problem is ill-posed, one looks for a *least residual norm (LRN) solution* which is the least squares solution of the convex quadratic problem

$$(P) \quad min\{\|Tx - y\|^2 : x \in K\}.$$

Let S_y denote the solution set of (P) which is closed and convex. It is well known that

$$S_y \neq \emptyset \Leftrightarrow P_{\overline{T(K)}}(y) \in T(K).$$

Problem (3.12) has a solution if this *feasibility condition* is satisfied.

Theorem 3.9. *The following statements are equivalent.*

(i) (feasibility condition)
$\hat{x}$ *is a solution of problem* $(P) : \hat{x} \in S_y$.

(ii) (Euler condition)
$\hat{x}$ *satisfies the variational inequalities*

$$\langle T^*(T\hat{x} - y), \hat{x} - x\rangle \leq 0, \quad x \in K, \tag{3.13}$$

where T^* *denotes the adjoint of* T.

(iii) (fixed point condition)
$\hat{x}$ *satisfies the fixed point equation*

$$\hat{x} = P_K(\hat{x} - \lambda T^*(T\hat{x} - y)), \tag{3.14}$$

for any scalar $\lambda > 0$.

It can be seen that for $0 < \lambda < \frac{2}{\|T\|^2}$, the map

$$x \longrightarrow P_K(x - \lambda T^*(Tx - y))$$

is non-expansive. Since the *viscosity solution* $\tilde{x}$ of the constrained linear inverse problem (3.12) is the minimum-norm element of the set S_y of solutions of the minimization problem (P), the problem of finding the viscosity solution of (3.12) is equivalent to the one of finding the minimum-norm fixed point of the non-expansive map

$$x \longrightarrow P_K(x - \lambda T^*(Tx - y)).$$

This motivates the consideration of the minimum-norm fixed point of a non-expansive map $T : K \to K$, given that $Fix(T) \neq \emptyset$,

$$\text{find } \tilde{x} \in Fix(T), \|\tilde{x}\|^2 = min\{\|x\|^2 : x \in Fix(T)\}. \tag{3.15}$$

In the framework of Hilbert spaces, let us consider the implicit scheme (3.2) of Browder as well as the explicit scheme (3.5) of Halpern under condition (3.8). If $0 \in K$, then by Theorem 3.5 and Theorem 3.6, it is clear that both these schemes converge to the unique minimum-norm fixed point of T. If $0 \notin K$, both these iterations fail. In this case, in Yao and Xu[30] the following alternative for Browder's scheme

$$x_\epsilon = P_K((1 - \epsilon)Tx_\epsilon)$$

and the following alternative

$$x_{n+1} = P_K((1 - \epsilon_n)Tx_n), \quad n \in \mathbb{N}$$

for the Halpern's scheme were considered. It was shown that both these schemes converge to the unique minimum-norm fixed point of T. Natural generalizations of Theorem 3.5 and Theorem 3.6 to minimum-norm fixed points of T are given in [30].

Finally, let us mention that except for a few results in the beginning, we have not attempted to give detailed proofs of any of the theorems given here, keeping in view the expected survey nature of this article.

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GENERALIZATIONS OF SOME GRAPH-THEORETIC RESULTS TO MATROIDS *

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Abstract. Subdivision of an edge, addition of an edge, the splitting operation (H. Fleischner [1990])and vertex splitting operation (P. Slater [1974]) are well known operations on graphs and have important applications. We consider generalizations of these operations and related graph results to binary matroids.

1. Introduction

First we take a review of some graph theoretic operations, related results and their applications. We then provide basic concepts in matroid theory and later on concentrate on generalizations of these graph theoretic results to matroids.

The connectivity $k(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. The graph G is n-connected if $k(G) \geq n$. Subdivision of an edge means replacing an edge of a graph by a path of length two. Addition of an edge means adding an edge between two non adjacent vertices of a graph.

Hedetniemi [8] classified 2-connected graphs in terms of the above two operations. He proved the following theorem.

Theorem 1.1 ([8])**.** *Let G be a nontrivial graph that is not a bridge. Then G is 2-connected if and only if G is a loop or can be obtained from a loop by a finite sequence of subdivisions and edge additions.*

Slater [23] specified the notion of n-vertex splitting operation on graphs in the following way: Let G be a graph and u be a vertex of G with deg $u \geq 2n-2$. Let H be the graph obtained from G by replacing u by two adjacent vertices u_1 and

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u_2, and if x is adjacent to u in G, written x adj u, then make x adj u_1 or x adj u_2 (but not both) such that deg $u_1 \geq n$ and deg $u_2 \geq n$. The transition from G to H is called an n-vertex splitting operation (see Figure 1).

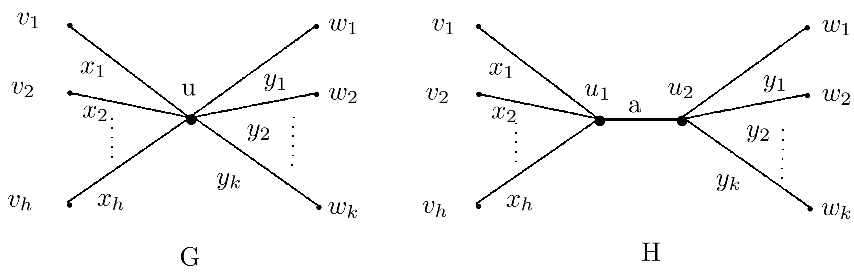


Figure 1

that if G is connected then the new graph H is connected. We have the following theorem.

Theorem 1.2 ([23])**.** *If G is n-connected and H arises from G by n-vertex splitting, then H is n-connected.*

The following theorems provide characterizations of classes of 1-connected and 2-connected graphs in terms of the 1-vertex, and 2-vertex splitting operations respectively.

Theorem 1.3 ([23])**.** *The class of 1-connected graphs is the class of graphs obtained from K_2 by finite sequences of edge addition and 1-vertex splitting.*

Theorem 1.4 ([23])**.** *The class of 2-connected graphs is the class of graphs obtained from K_3 by finite sequences of edge addition and 2-vertex splitting.*

Slater [24], proved that the class of n-connected graphs can be generated from K_{n+1} using n-vertex splitting operation together with some other operations. The sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 + G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set consists of $E_1 \cup E_2$ and all edges joining every vertex of V_1 to every vertex of V_2. For $n \geq 4$ the wheel W_n is defined to be the graph $K_1 + C_{n-1}$ where K_1 is the complete graph with one vertex and C_{n-1} is a cycle with $n - 1$ vertices.

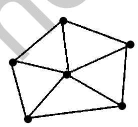


Figure 2 : W_6

Tutte [26] provided the following property of 3-connected graphs.

Theorem 1.5. *For a 3-connected simple graph the following are equivalent.*

(1) For every edge e, neither $G \setminus e$ nor G/e is both simple and 3-connected.
(2) G is a wheel with at least four vertices.

In the following theorem, Tutte gave the constructive characterization of 3-connected graphs.

Theorem 1.6 ([26]). *A graph is 3-connected if and only if G is a wheel or can be obtained from a wheel by a sequence of operations of edge addition and 3-vertex splitting.*

A walk in a graph G is a list $v_0, e_1, v_1, ..., e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has end vertices v_{i-1} and v_i. A trail is a walk with no repeated edge. v_0 and v_k are the end vertices of the walk. A walk or trail is closed if its end-vertices are the same.

A graph is Eulerian if it has a closed trail containing all edges. In other words, a graph is Eulerian if it contains a closed walk that traverses each edge exactly once. Eulerian graphs form an important class of graphs. One can find several applications of Eulerian graphs in literature (see [6]). For terminology concerning graphs, we refer to [6, 7].

Several characterizations of Eulerian graphs have been given by mathematicians. The following classic characterization of Eulerian graphs due to Euler is well known.

Theorem 1.7. *A connected graph is Eulerian if and only if all its vertices have even degree.*

A bond in a graph G is a minimal disconnecting set of edges of the graph. It is not difficult to show that every vertex of a graph has even degree if and only if every bond has even cardinality. Therefore the above theorem can be restated as follows.

Theorem 1.8. *A connected graph is Eulerian if and only if every bond has even cardinality.*

Shank [13] characterized Eulerian graphs in terms of the number of subsets of their spanning trees. In fact, he proved the following theorem.

Theorem 1.9 ([13]). *A connected graph G is Eulerian if and only if the number of subsets of $E(G)$ each of which is contained in a spanning tree of G is odd.*

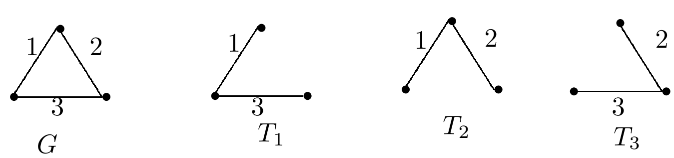


Figure 3: A graph G and its spanning trees

Let G be a complete graph on three vertices and $E(G) = \{1, 2, 3\}$. T_1, T_2 and T_3 are the spanning trees of G (see Figure 3). The subsets of $E(G)$ each of which is contained in a spanning tree of G are ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. Total number of sets is 7, an odd number.

Toida [25] characterized the class of Eulerian graphs in terms of the number of cycles of a graph containing each edge of the graph. Indeed, he proved the following result.

Theorem 1.10 ([25]). *A connected graph G is Eulerian if and only if every edge of G lies on an odd number of cycles.*

We observe that a connected graph is Eulerian if and only if its edge set can be partitioned into cycles. Bondy and Helberstan [2] provided the following characterization of Eulerian graphs.

Theorem 1.11 ([2]). *A graph is Eulerian if and only if the edge set has an odd number of cycle partitions.*

Fleischner [6] defined the splitting operation on graphs in the following way: Let G be a connected graph and let v be a vertex of degree at least three in G. If $x = vv_1$ and $y = vv_2$ are two edges incident at v, then splitting the pair x, y from v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2 . The transition from G to $G_{x,y}$ is called the splitting operation. For practical purpose, we also denote the new edges $v_{x,y}v_1$ and $v_{x,y}v_2$ in $G_{x,y}$ by x and y, respectively. Note the the edges x and y form a bond of $G_{x,y}$. The graph G can be retrieved from $G_{x,y}$ by identifying the vertices v and $v_{x,y}$. Figure 4 explicitly demonstrates this construction.

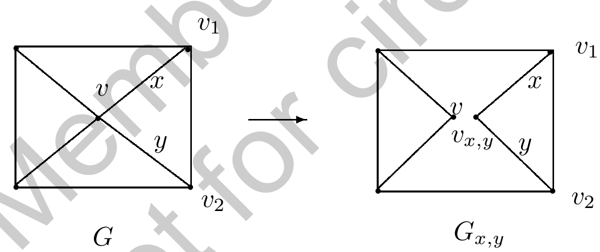


Figure 4: Graph and its splitting

Fleischner used splitting operation to characterize Eulerian graphs. In fact, he proved the following theorem.

Theorem 1.12 ([6]). *A graph G is Eulerian if and only if G can be transformed into a cycle C through repeated applications of the splitting procedure on vertices of degree exceeding two.*

In the following proposition, we characterize the cycles of the graph $G_{x,y}$ in terms of the cycles of the graph G.

Proposition 1.1 ([11]). *A subset C of edges of $G_{x,y}$ forms a cycle of $G_{x,y}$ if and only if C satisfies one of the following conditions:*
(1) C is a cycle of G containing both x and y or neither.
(2) $C = C_1 \cup C_2$ where C_1 and C_2 are edge disjoint cycles of G with $x \in C_1$, $y \in C_2$ such that $C_1 \cup C_2$ contains no cycle of Type (1).

In other words, the proposition characterizes the splitting operation in terms of the cycles of the graphs. This proposition will be useful for extending the splitting operation from graphs to binary matroids.

Remark 1.1. We observe that the graph $G_{x,y}$ has one additional vertex than the graph G viz. the vertex $v_{x,y}$; x and y are the only edges incident at this vertex. Consequently, the incidence matrix of $G_{x,y}$ has one extra row(corresponding to $v_{x,y}$) than the incidence matrix of G. The entries in this row are zero everywhere except in the columns corresponding to the edges x and y where it takes the value 1.

The splitting operation on a connected graph may not yield a connected graph. Fleischner provided a sufficient condition for the splitting operation to preserve the connectedness of the graph. Splitting Lemma is an useful tool in graph theory.

Splitting Lemma 1.1 ([6]). *Let H be a connected bridgeless graph. Suppose $v \in V(H)$ with $d(v) > 3$ and x, y, z are the edges incident at v. Form the graphs $H_{x,y}$ and $H_{x,z}$ by splitting away the pairs x, y and x, z respectively, and assume that x and z belong to different blocks if v is a cut vertex of H. Then either $H_{x,y}$ or $H_{x,z}$ is connected and bridgeless. In particular if v is a cut vertex, then $H_{x,z}$ has this property. Finally, if $B \subseteq H$ is a block and x, y, z are edges in B then both $H_{x,y}$ and $H_{x,z}$ are connected*

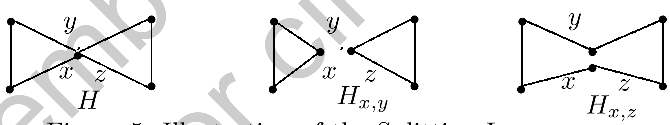


Figure 5: Illustration of the Splitting Lemma

The above graph theoretic results can be extended to the matroids. Matroids are common generalizations of graph theoretic properties and linear algebraic properties. They have basic links to lattices, codes and projective geometry, and are of fundamental importance in combinatorial optimization. Their applications extend into electrical engineering and statics. For terminology in matroid theory, we refer to [10].

2. Basics of Matroid Theory

Matroids can be defined in many different but equivalent ways. Here we consider the two approaches - the Independent sets approach and the circuits based approach.

2.1. **Independent sets and circuits.** A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying

(1) $\phi \in \mathcal{I}$

(2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$

(3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members of $\mathcal{I}$ are called the Independent sets of M. E is the ground set of M. A base of M is a maximal independent set. All bases of M have same number of elements. The number of elements in a basis of M is known as the rank of M and is denoted by $r(M)$. The dependent sets of M are subsets of E that are not independent. A circuit of M is a minimal dependent set. We denote by $\mathcal{I}(M)$, $\mathcal{B}(M)$ and $\mathcal{C}(M)$, respectively the set of independent sets, set of bases and the set of circuits of M. $E(M)$ will denote the ground set of M.

A matroid M is also defined as a pair $(E, \mathcal{C})$, where E is a finite set and $\mathcal{C}$ is a collection of nonempty subsets of E satisfying

(1) No member of $\mathcal{C}$ contains another member properly.

(2) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Members of $\mathcal{C}$ are called circuits of M.

If $\mathcal{I}(M)$ is known then $\mathcal{C}(M)$ can be determined easily viz. identify dependent sets and then choose minimal ones. On the other hand if $\mathcal{C}(M)$ is known then $\mathcal{I}(M)$ viz. those subsets of E each of which contains no member of $\mathcal{C}(M)$. If the set of bases of M is known then the set of independent sets can be determined viz. those subsets of $E(M)$ each of which is contained in some basis of M.

2.2. **Matroids associated to graphs and matrices.** We can associate a matroid to a graph. Let G be a graph and E be the set of edges of G. Let $\mathcal{C}$ be the set of edge sets of cycles of G. Then $\mathcal{C}$ is the set of circuits of a matroid on E. This matroid is called the cycle matroid of G and is denoted by $M(G)$.

Example 2.1. Consider the graph G as shown in Figure 6.

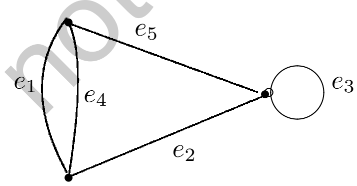


Figure 6 : Graph G

Then

$$E(M(G)) = \{e_1, e_2, e_3, e_4, e_5\}$$

and

$$C(M(G)) = \{\{e_3\}, \{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_2, e_4, e_5\}\}$$

A matroid can also be associated to a matrix. Let A be an $m \times n$ matrix over a field F, E be the set of column labels of A and $\mathcal{I}$ be the set of subsets X of E for which the set of columns labelled by X is linearly independent in the vector space $V(m, F)$. Then $(E, \mathcal{I})$ is a matroid. This matroid is called the vector matroid of A and it is denoted by $M[A]$.

Example 2.2. Let A be the matrix

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

over the field R of real numbers. Then

$$E(M[A]) = \{1, 2, 3, 4, 5\}$$

and

$$\mathcal{I} = \{\phi, \{1\}, \{2\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}.$$

The set of circuits of this matroid is

$$\mathcal{C} = \{\{3\}, \{1, 4\}, \{1, 2, 5\}, \{2, 4, 5\}.\}$$

Two matroids M_1 and M_2 are isomorphic, if there exists a bijection f from $E(M_1)$ to $E(M_2)$ such that for all $X \subseteq E(M_1)$, the set $f(X)$ is independent in M_2 if and only if X is independent in M_1. Such a bijection f is called an isomorphism from M_1 to M_2. M_1 is isomorphic to M_2 is written as $M_1 \cong M_2$. For example, the cycle matroid $M(G)$ in Example 2.1 is isomorphic to the vector matriod $M[A]$ of Example 2.2. The bijection from $E(M(G))$ to $E(M[A])$ is $f(e_i) = i$ for $i = 1, 2, \cdots, 5$.

A matroid M is said to be graphic if $M \cong M(G)$ for some graph G. For example, the cycle matroid of a graph is graphic. An example of a non-graphic matroid is as follows. Let $E = \{1, 2, 3, 4\}$ and $\mathcal{C}(M) = \{$X⊆E: $|X|$ =3$\}$. Then $M = (E, \mathcal{C})$ is a non-graphic matroid.

Let M be a matroid on E with set of bases $\mathcal{B}$. Then the set $\mathcal{B}' = \{E - B : B \in \mathcal{B}\}$ is a set of bases of a matroid on E. This matroid is called dual of M and is denoted by M^*. The circuits of M^* are called the cocircuits of M. If $M = M(G)$ then circuits of M^* are the bonds (or cut sets) of G. A matroid M is said to be cographic if its dual M^* is graphic, i. e. $M^* = M(G)$ for some graph G. For example, the matroid $M(K_5)$ is graphic but not cographic.

The Fano matroid, denoted by F_7, is the matroid defined on the set $E = \{1, 2, 3, 4, 5, 6, 7\}$, whose bases are all 3-element subsets of E except $\{1, 2, 4\}, \{1, 3, 6\}$,

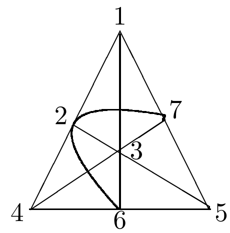


Figure 7

$\{2, 6, 7\}, \{4, 5, 6\}, \{1, 5, 7\}, \{3, 4, 7\}$, and $\{2, 3, 5\}$. In fact, these seven subsets of E and their complements form the set of circuits of this matroid. F_7 may be represented geometrically by the Figure (see Fig. 7). The elements of the ground set are shown by the points in the plane. If the three elements form a circuit then they are collinear. Note that the rank of F_7 is 3 and the matroid is neither graphic nor cographic.

If a matroid M is isomorphic to the vector matroid of a matrix A over a field F, then M is said to be representable over F and A is called a representation for M over F. A matroid is called binary if it is representable over $GF(2)$. Several characterizations of binary matroids are known (see [10]). Equivalently, a matroid M is binary if symmetric difference of any number of circuits of M is a union of disjoint circuits of M. Fano-matroid F_7 is reprentable over $GF(2)$. The following matrix gives a representation of F_7 over $GF(2)$.

$$A = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

2.3 Minors of Matroids

Let $M = (E, \mathcal{I})$ be a matroid and $X \subset E$. Then deletion of X from M is a matroid $M \setminus X = (E - X, \mathcal{I}')$ where $\mathcal{I}' = \{Y \subseteq E - X : Y \in \mathcal{I}\}$. Contraction of X in M is defined in the following way. Let X_0 be a maximal independent subset of X in M. Then a contraction of X in M is the matroid $M/X = (E - X, \mathcal{I}")$ where $\mathcal{I}" = \{Y \subseteq E - X : Y \cup X_0 \in \mathcal{I}\}$. Matroids obtained from a given matroid M by sequence of deletions and contractions of subsets of $E(M)$ are called as **minors** of M. Minors are useful in characterizing various classes of matroids.

2.4 Connectivity in matroids

A matroid M is said to be *connected* if for every pair x, y of elements of $E(M)$, there exists a circuit of M containing both x and y. The higher Connectivity of the matroids is defined in terms of the separations of the ground set of the matroid. Let k be a positive integer. Then, for a matroid M, a partition (X, Y) of E is a k-separation if

$$\min\{|X|, |Y|\} \geq k, \text{ and } r(X) + r(Y) - r(M) \leq k - 1.$$

If M has a k-separation, then M is called *k-separable*. If M is k-separable for some k, then the *connectivity* $\lambda(M)$ of M is $\min\{j : M \text{ is } j\text{-separable}\}$; otherwise we take $\lambda(M)$ to be ∞. If n is an integer exceeding one, we shall say that M is *n-connected* if $\lambda(M) \geq n$.

It is well known that a graph is Eulerian if and only if its edge set can be partitioned into cycles. Considering this property, Welsh [28] defined a matroid M to be Eulerian if its ground set is a disjoint union of circuits of M. It is known that a graph is bipartite if andonly if every cycle is of even length. We define a matroid M tobe bipartite if every circuit has even cardinality.

3. Generalizations of Graph results to matroids

In this section, we consider the generalizations of the graph theoretic results discussed in the first section to matroids. The operations of subdivision of an edge of a graph is extended to matroids in the following way. A matroid M is a series extension of a matroid N if M has a 2-element cocircuit $\{x, y\}$ such that the contraction M/x of x from M is N. Also the operation of addition of an edge to a graph is extended to matroids as follows: A matroid M is a single element extension of a matroid N if M has an element e such that the deletion of e from M is N. This extension is non-trivial if e is neither a loop nor a coloop of M.

Shikare and Waphare [21] generalized Hedetniemi's characterization of two connected graphs (Theorem 1.1) to matroids. In fact, they proved the fololwing theorem.

Theorem 3.1. *Let M be a nonempty matroid that is not a coloop. Then M is connected if and only if M is a loop, or can be obtained from a loop by a sequence of operation each consisting of non-trivial single element extension or a series extension.*

The term wheel is used for the graph $\mathcal{W}_r$ and its cycle matroid $M(\mathcal{W}_r)$. The whirl of rank r, $\mathcal{W}^r$ is a matroid whose ground set is $E(\mathcal{W}_r)$ and whose bases consists of the rim together with all edge sets of spanning trees of $\mathcal{W}_r$. Tutte [27] extended Theorem 1.5 to matroids. The corresponding result is stated by Theorem 3.2.

Theorem 3.2 ([27]). *Let M be a $3-$connected matroid with at least one element. Then following statements are equivalent.*

(1) For every element e of M , neither $M \setminus e$ nor M/e is 3-connected.
(2) M has rank at least three and is isomorphic to a wheel or a whirl.

Welsh [28] extended the classic characterization of Eulerian graphs viz. Theorem 1.7 to binary matroids. He proved the following theorem.

Theorem 3.3 ([28]). *A binary matroid is Eulerian if and only if the dual matroid is bipartite.*

The result is not true if the matroid is non binary.

Shikare and Raghunathan [20] generalized Shank's characterization of Eulerian graphs viz. Theorem 1.9 to binary matroids.

Theorem 3.4 ([18]). *A binary matroid M is Eulerian if and only if the number of independent sets of M is odd.*

Shikare [15] extended Theorems 1.10 and 1.11 to binary matroids. In fact, he proved the following theorems.

Theorem 3.5 ([15]). *A binary matroid M on a set E is Eulerian if and only if every element of E is contained in odd number of circuits of M.*

Theorem 3.6 ([15]). *A binary matroid M is Eulerian if and only if its ground set can be partitioned into circuits of M in an odd number of ways.*

We wish to extend results concerning splitting operation on graphs to binary matroids. In particular, we wish to generalize Theorem 1.12 and Splitting Lemma 1.1 to binary matroids. So the first step in this direction is to generalize notion of the splitting operation to binary matroids. Proposition 1.1 and Remark 1.1 play crucial role in this matter.

Raghunathan et. al. [11] defined splitting operation for binary matroids in the following way: Let $M = (E, \mathcal{C})$ be a binary matroid on a set E together with the set $\mathcal{C}$ of circuits and let $x, y \in E$.
Let $\mathcal{C}_0$=$\{C \in \mathcal{C}$: $x, y \in C$, or $x, y \notin C\}$, and $\mathcal{C}_1$=$\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \phi, x \in C_1, y \in C_2$, and $C_1 \cup C_2$ contains no member of $\mathcal{C}_0\}$.

Let $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1$. Then the pair $(E, \mathcal{C}')$ is a binary matroid. We denote this matroid by $M_{x,y}$ and say that $M_{x,y}$ has been obtained from M by splitting the pair of elements x, y. Moreover, the transition from M to $M_{x,y}$ is called a splitting operation. We note that an arbitrary circuit of $M_{x,y}$ contains either both x and y, or neither.

The splitting operation can also be defined in terms of the matrices representing the matroids over $GF(2)$. This provides more elegant apporach to the splitting operation.

Let M be a binary matroid on a set E, and let A be a matrix that represents M over $GF(2)$. Suppose that $x, y \in E$. Let $A_{x,y}$ be the matrix obtained from A by adjoining an extra row with zeros everywhere except in the columns of x and y where it takes value 1. Let $M_{x,y}$ be the vector matroid of the matrix $A_{x,y}$. The transition from M to $M_{x,y}$ is called a splitting operation. The matroid $M_{x,y}$ is referred as the splitting matroid.

Raghunathan et. al. [11] extended Fleischner's characterization of Eulerian graphs (Theorem 1.12) to binary matroids as follows.

Theorem 3.7 ([10]). *A binary matroid on a set E is Eulerian if and only if M can be transformed by repeated applications of the splitting operation into a matroid in which E is a circuit.*

It is shown that if M has rank k and $|E| = n$ then the minimum number of splitting operations required in the process is $n - (k+1)$.

Shikare and G. Azadi [18] characterized the bases of the splitting matroid $M_{x,y}$ in terms of the bases of the original matroid M. Indeed, they proved the following theorem.

Theorem 3.8. *. Let $M = (E, \mathcal{C})$ be a binary matroid and $x, y \in E$. Let $\mathcal{B}$ be the set of all bases of M and*
$\mathcal{B}_{x,y} = \{B \cup \{\alpha\} : B \in \mathcal{B}, \alpha \in E - B$, such that the unique circuit contained in $B \cup \{\alpha\}$, contains exactly one of x and $y\}$. Then $\mathcal{B}_{x,y}$ is the set of bases of $M_{x,y}$.

When specialized to graphs, we obtain the following result.

Corollary. Let G be a connected graph and x and y be adjacent edges of G with common vertex v. Assume that x and y belong to different blocks if v is a cut vertex of G. Let T be a spanning tree of G and $g \notin E(T)$. Then $T \cup g$ is a spanning tree of $G_{x,y}$ if and only if the fundamental cycle of G formed by g and T contains either x or y.

Allan Mills [9] described the cocircuits of the matroid $M_{x,y}$ in terms of the cocircuits of the original matroid M. His results are stated below.

Theorem 3.9. *. If $\{x\}$ is a cocircuit of M and y is not, then the cocircuit of $M_{x,y}$ consists of $\{x\}$, $\{y\}$ and the sets in the following collections.*

(1) $\{C^* - \{x\} : C^*$ is a cocircuit of M containing $\{x\}$ properly $\}$; and

(2) $\{C^* \in \mathcal{C}^*(M) : x \notin C^*$ and C^* is the only cocircuits of M contained in $C^* \cup \{x\}\}$.

Theorem 3.10. *. If $\{x, y\}$ does not contain a cocircuit of M, then $\{x, y\}$ and each non-empty set in the following collections is a cocircuit of $M_{x,y}$.*

(1) $\{C^* - \{x, y\} : C^* \in \mathcal{C}^*(M)$ and C^* contains $\{x, y\}$ properly $\}$

(2) $\{C^* : C^* \in \mathcal{C}^*(M)$ and C^* does not contain a Type (1) set $\}$

(3) $\{C^* \Delta \{x, y\} : C^* \in \mathcal{C}^*(M)$ and C^* contains exactly one of x and y and does not contain a Type (1) set $\}$.

The splitting operation on a connected binary matroid, in general, need not yield a connected binary matroid. The following theorems provide conditions for the splitting operation to preserve connectedness of the matroid M.

Theorem 3.11 ([16]). *Let M be a 4-connected binary matroid on a set E with at least nine elements and $x, y \in E(M)$. Then the matroid $M_{x,y}$ is connected.*

Corollary. The splitting operation on an n-connected ($n \geq 4$) binary matroid with at least nine elements yields a connected binary matroid.

Theorem 3.12 ([3]). *Let M be a 3-connected binary matroid and $x, y \in E(M)$. Suppose that every cocircuit Q of M containing $\{x, y\}$ has size at least 4, and Q contains no circuit of size 2. Then $M_{x,y}$ is connected.*

If M is a binary matroid and $x, y \in E(M)$ then the splitting of the dual matroid M^* with respect to $\{x, y\}$, in general, may not be same as the dual of the the splitting matroid $M_{x,y}$. Dhotre [4] provided conditions under which these two matroids will be same. Two elements x and y of a matroid M are in series in M if $\{x, y\}$ is a cocircuit of M.

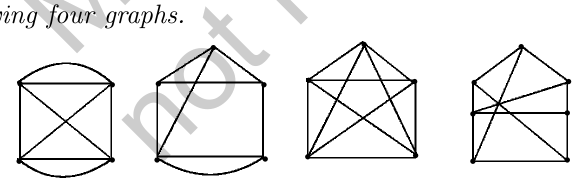
Theorem 3.13 ([5]). *Let M be a binary matroid and let $x, y \in E(M)$. Then ${M^*}_{x,y} = (M_{x,y})^*$ if and only if x and y are in series and $\{x, y\}$ is a circuit of M.*

Corollary. Let M be a binary connected matroid with at least 3 elements. Then ${M^*}_{x,y} \neq (M_{x,y})^*$ for every pair $\{x, y\}$ of $E(M)$.

The splitting operation on a graphic matroid may not yield a graphic matroid. Luis Goddyn, Simon Fraser University, Canada in 1996 raised the following Problem: Determine precisely those graphs G which have the property that the splitting operation on the cycle matroid $M(G)$ by every pair of edges yields a graphic matroid.

Shikare and Waphare [22] solved this problem by proving that there are exactly four minor-minimal graphs that do not have this property. In fact, they proved the following theorem.

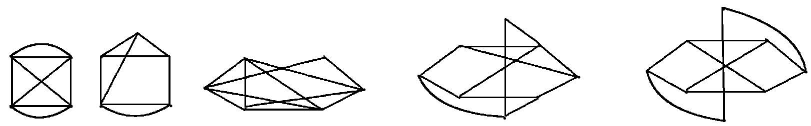
Theorem 3.14 ([22]). *The splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if the cycle matroid of the corresponding graph has no minor isomorphic to the cycle matroid of any of the following four graphs.*



The forbidden graphs for graphic matroids

The splitting operation on a cographic matroid, in general, may not yield a cographic matroid. Borse et. al. [4] provided a necessary and sufficient condition for the splitting operation to yield a cographic matroid from a cographic matroid.

Theorem 3.15 ([4])**.** *The splitting operation, by any pair of elements, on a co-graphic matroid yields a co-graphic matroid if and only if the cycle matroid of the corresponding graph has no minor isomorphic to any of the five matroids $M(H_i)$ for $i = 1, 2$ and $M^*(H_i)$ for $i = 3, 4, 5$ where Hi are the graphs as shown in the following Figure.*



H_1 H_2 H_3 H_4 H_5

Azadi [1] extended the notion of the n-vertex splitting operation on graphs to binary matroids. The corresponding operation on matroids is called element splitting operation. This operation is defined in terms of the matrices representing the matroids. Let M be a binary matroid on a set E and let A be a matrix over GF(2) that represents M. Suppose that T is a subset of E(M). Let A_T be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to the elements of T where it takes the value 1, and then adjoining an extra column (corresponding to a) with this column being zero everywhere except in the last row where it takes the value 1. Suppose M_T be the vector matroid of the matrix A_T. The transition from M to M_T is called the element splitting operation. Several properties of this operation have been explored in [1, 17].

We would like to mention that the splitting operation and the element splitting operation have been defined to binary matroids as extensions of the corresponding operations on graphs. It will be a great achievement if one can define these operations to arbitrary matroids and explore their properties.

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FROM FOURIER SERIES TO HARMONIC ANALYSIS ON LOCALLY COMPACT GROUPS*

AJAY KUMAR

Abstract. During the investigation of heat flow in 1807, Joseph Fourier came up with the idea that the graph of any function on a bounded interval can be obtained as a linear superposition of sines and cosines.

If f is a periodic function of one real variable of period 2π, then we can think f as a function on the circle group i.e. additive group of reals modulo integer multiples of 2π. Fourier series live on circle group T. The Fourier analysis of the real line (i.e., the Fourier transform) was introduced at about the same time as Fourier series. But it was not until the mid-twentieth century that Fourier analysis on $\mathbb{R}^N$ came to existence. Meanwhile, abstract harmonic analysis (i.e., the harmonic analysis of locally compact abelian groups) had developed a life of its own. And the theory of Lie group/ non abelian group representations provided a natural crucible for noncommutative harmonic analysis.

We discuss Fourier transform on Nilpotent Lie groups and representations of some classes of non-abelian locally compact groups. Some important results regarding Plancherel formula, Uncertainty Principles and Potential theory will be presented.

1. Fourier Series

There are several natural phenomena that are described by periodic functions. The position of a planet in its orbit around the sun is a periodic function of time; in Chemistry, the arrangement of molecules in crystals exhibits a periodic structure. The theory of Fourier series deals with periodic functions. We begin with the concept of Fourier Series.

Definition 1.1. A function $f(t)$ is said to have a period T or to be periodic with period T if for all t, $f(t+T) = f(t)$ where T is a positive constant. The least

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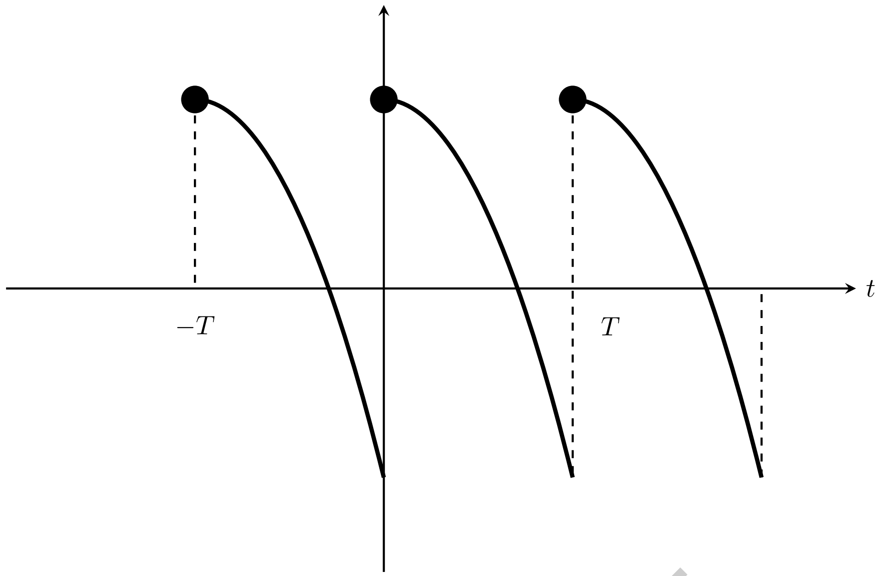


value of $T > 0$ is called the principal period or the fundamental period or simply the period of $f(t)$

Suppose $f(x)$ has the period 2π and

$$F(t) := f(\omega t) := f(\frac{2\pi}{T}t).$$

then F has the period T. Let a function f be declared on the interval $[0, T)$. The periodic expansion of f is defined by the formula

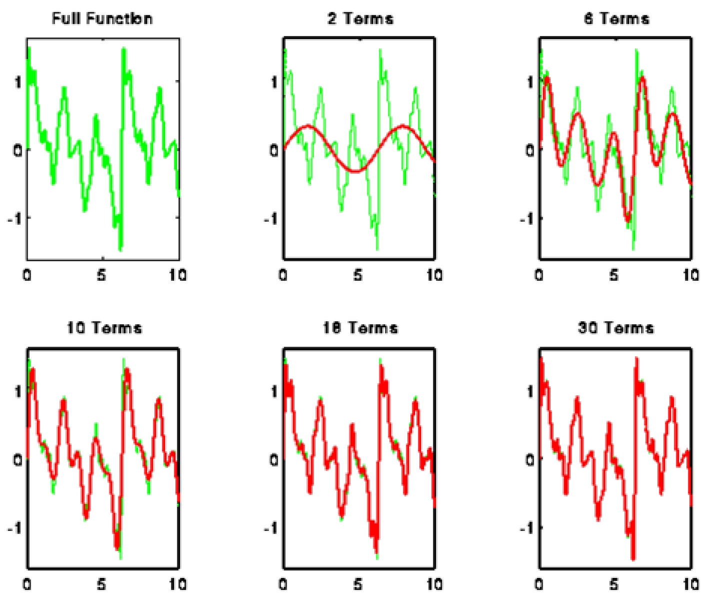
$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } 0 \leq t < T, \\ \tilde{f}(t-T), & \text{for all } t \in \mathbb{R}, \end{cases}$$

Let f be continuous on $I = [-\pi, \pi]$. Suppose that the series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$ converges uniformly to f for all $x \in I$. Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ cos\ ntdt \quad \text{for } n = 0, 1, 2, \ldots \quad (1)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ sin\ ntdt, \text{ for } n = 0, 1, 2, \ldots \quad (2).$$

The numbers a_n and b_n are called the Fourier coefficients of f. When a_n and b_n are given by (1) and (2), the trigonometric series is called the Fourier series of the function f. Below is an example of an arbitrary function which we approximate with Fourier series of various lengths. As you can see, the ability to mimic the behavior of the function increases with increasing series length, and the nature of the fit is that the "spikier" elements are fit better by the higher order function



In mathematics, the question of whether the Fourier series of a function converges to the given function is researched by a field known as classic harmonic analysis, a branch of pure mathematics. For most engineering uses of Fourier analysis, convergence is generally simply assumed without justification. However, convergence is not necessarily given in the general case, and there are criteria which need to be met in order for convergence to occur.

Example 1.1. (1) The Fourier series of the function $f(x) = x$, $-\pi \leq x \leq \pi$, is given by $f(x) \sim 2(sin(x) - \frac{sin(2x)}{2} + \dots)$.

(2) The Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \pi, & 0 \leq x \leq \pi, \end{cases}$$

is given by $f(x) \sim \frac{\pi}{2} + 2(sin(x) + \frac{sin(3x)}{3} + \dots)$.

(3) The Fourier series of the function

$$f(x) = \begin{cases} -\frac{\pi}{2}, & -\pi \leq x \leq 0, \\ \frac{\pi}{2}, & 0 \leq x \leq \pi, \end{cases}$$

is given by $f(x) \sim \frac{\pi}{2} + 2(sin(x) + \frac{sin(3x)}{3} + \dots)$.

Remark 1.1. We defined the Fourier series for functions which are 2π-periodic, one would wonder how to define a similar notion for functions which are L-periodic. Assume that $f(x)$ is defined and integrable on the interval $[-L, L]$. Set $F(x) = f(\frac{Lx}{\pi})$. The function F(x) is defined and integrable on $[-\pi, \pi]$. Consider the

Fourier series of $F(x)$

$$F(x) = f(\frac{L\pi}{\pi}) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx).$$

Using the substitution $t = \frac{Lx}{\pi}$, we obtain the following definition

Definition 1.2. Let $f(x)$ be a function defined and integrable on $[-L, L]$. The Fourier series of $f(x)$ is

$$f(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\frac{\pi t}{L} + b_n \sin n\frac{\pi t}{L}),$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\frac{Lx}{\pi}) \ \cos\ nxdx = \frac{1}{2L}\int_{-L}^{L} f(x)\ \cos\ (n\frac{\pi x}{L})dx,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{Lx}{\pi}) \ \cos\ nxdx = \frac{1}{L}\int_{-L}^{L} f(x)\ \cos\ (n\frac{\pi x}{L})dx,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{Lx}{\pi}) \ \sin\ nxdx = \frac{1}{L}\int_{-L}^{L} f(x)\ \sin\ (n\frac{\pi x}{L})dx,$$

for $n \geq 1$.

Example 1.2. The Fourier series of the function

$$f(x) = \begin{cases} 0, & -2 \leq x \leq 0, \\ x, & 0 \leq x \leq 2, \end{cases}$$

is given by $f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}[\frac{2}{n^2\pi^2}((-1)^n - 1)cosn\frac{\pi x}{2} + \frac{2}{n\pi}(-1)^{n+1}sinn\frac{\pi x}{2}]$.

Using Euler's identities $e^{i\theta} = \cos\theta + i\sin\theta$, the Fourier series of $f(x)$ can be written in complex form as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ and $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$, and $c_{-n} = \frac{1}{2}(a_n + ib_n)$, $n = 1, 2, \ldots$. In what sense does the series on the right converge, and if it does converge, in what sense is it equal to $f(x)$? These questions depend on the nature of the function $f(x)$. Considering functions $f(x)$ defined on $\mathbb{R}$ which satisfy a reasonable condition like $\int_{-\infty}^{\infty}|f(x)|dx < \infty$ or $\int_{-\infty}^{\infty}|f(x)|^2dx < \infty$. For an integrable function $x(t)$, define the Fourier transform by $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$ for every real number ω. The independent variable t represents time, the transform variable ω represents angular frequency. Other notations for this same function are $\widehat{x}(\omega)$ and $F(\omega)$. The function is complex-valued in general. If $X(\omega)$ is is defined as above, and $x(t)$ is sufficiently smooth, then it can be reconstructed by the inverse transform:

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega$$

for every real number t. Let's see how we compute a Fourier Transform: consider a particular function $f(x)$ defined as

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & \text{otherwise}, \end{cases}.$$

Its Fourier transform is:

$$\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi x u} dx \\ &= \int_{-1}^{1} x e^{-2\pi x u} dx \\ &= -\frac{1}{2\pi i u}(e^{2\pi u i} - e^{-2\pi u i}) \\ &= \frac{sin 2\pi u}{\pi u}. \end{aligned}$$

In this case $F(u)$ is purely real, which is a consequence of the original data being symmetric in x and $-x$.

2. Fourier Transform- Gaussian

The Fourier transform of a Gaussian function $f(x) = e^{-ax^2}$ is given by

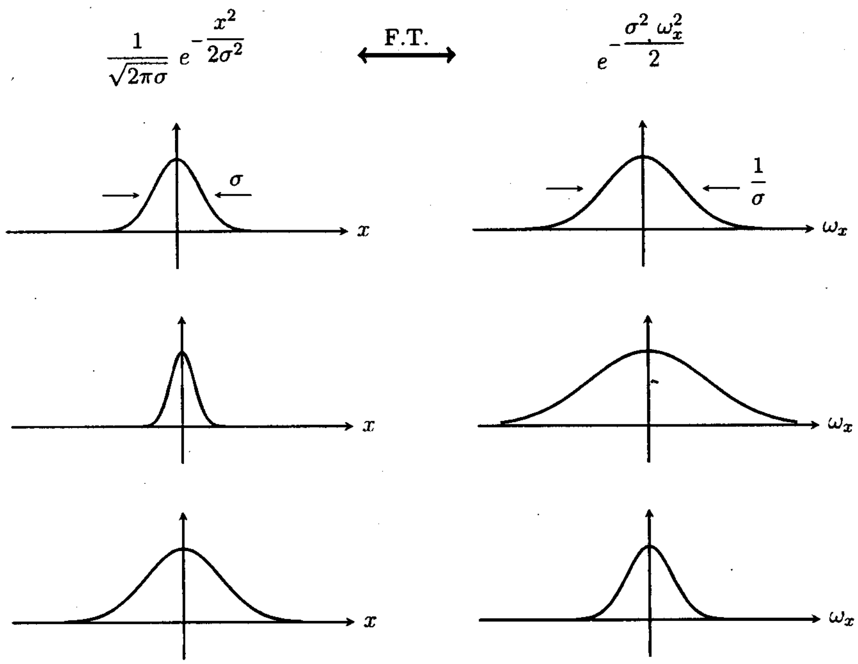
$$\begin{aligned} F_X[e^{-ax^2}](k) &= \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi x k} dx \\ &= \int_{-\infty}^{\infty} e^{-ax^2}[\cos(2\pi x k) - i\sin(2\pi x k)]dx \\ &= \int_{-\infty}^{\infty} e^{-ax^2}\cos(2\pi x k)dx - i\int_{-\infty}^{\infty} e^{-ax^2}\sin(2\pi x k)dx \end{aligned}$$

The second integrand is odd, so integration over a symmetrical range gives 0.The value of the first integral is given by Abramowitz and Stegun, so

$$F_X[e^{-ax^2}](k) = \sqrt{\frac{\pi}{a}} e^{-\pi^2 k^2/a},$$

a Gaussian transforms to another Gaussian. The Fourier transform of the Gaussian function is another Gaussian:

Note that the width sigma is oppositely positioned in the arguments of the exponentials. This means the narrower a Gaussian is in one domain, the broader it is in the other domain. The Fourier transform can also be extended to the space



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integrable functions defined on

$$F : L^1(\mathbb{R}^n) \to C(\mathbb{R}^n),$$

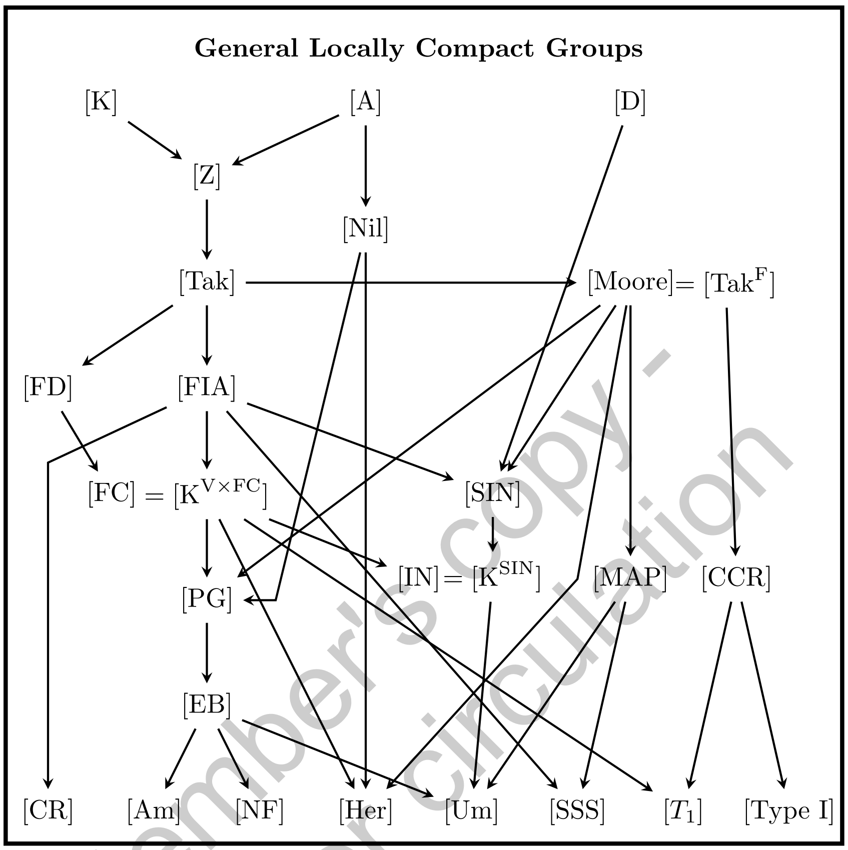
where $L^1(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} | \int_{\mathbb{R}^n} |f(x)|dx < \infty\}$ and $C(\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}^n$. In this case the definition usually appears as $\widehat{f}(w) = \int_{\mathbb{R}^n} f(x)e^{-i\omega x}dx$, where $\omega \in \mathbb{R}^n$ and $\omega.x$ is the inner product of the two vectors ω and x.

One may now use this to define the continuous Fourier transform for compactly supported smooth functions, which are dense in $L^2(\mathbb{R}^n)$. We now summarize the properties of $f \in L^1(\mathbb{R}^n)$.

(1) $\widehat{f}$ is continuous and $\|\widehat{f}\|_\infty \leq \|f\|_1$.
(2) If f is radial in the sense that there is a function F on $[0, \infty)$ such that $f(x) = F(|x|)$ then $\widehat{f}$ is radial.
(3) Let T_h denote the translation by $h \in \mathbb{R}^n$, *i.e.*, $T_h(f(x)) = f(x + h)$ then $\widehat{T_h f}(\xi) = e^{ih\xi}\widehat{f}(\xi)$ and for a nice function $\frac{1}{x_j}T_h f = T_h$.
(4) (Riemann Lebsegue Lemma) $\widehat{f} \in C_0(\mathbb{R}^n)$.
(5) (Plancheral) If f is in $L^2(\mathbb{R}^n)$ then $\widehat{f} \in L^2(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |f(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi$

These result were known about locally compact abelian groups and about compact groups till 1960. Thus it was natural to look for a common generalization of these two quite different groups abelian and compact.

Grosser, Moskowitz, Mosak, Kaniuth, Leptin , Ludwig and their co-workers looked at compactness conditions in topological groups.



$[A]$-Abelian groups
$[K]$-Compact groups
$[D]$-Discrete groups
$[Z]$-Central groups
$[Nil]$-Nilpotent groups
$[Tak]$-$[MAP]\bigcap\overline{[FD]}$
$[MAP]$-groups having enough finite dimensional irreducible representations to separate points.
$\overline{[FD]}$- groups having relatively compact commutator subgroup.
$\overline{[FC]}$-groups such that the closure of each conjugacy class is compact.

$\overline{[SIN]}$-groups such that every neighborhood of the identity contains a compact neighborhood which is invariant under all inner automorphisms.
$\overline{[IN]}$-groups such that closure of each conjugacy class is compact.
$\overline{[PG]}$-if for each compact neighborhood W of the identity there is an integer p such that $m(W^n) = O(n^p)$.
$\overline{[EB]}$- any compact neighborhood of the identity satisfies $m(W^n) = O(t^n)$ for all $t > 1$.
For other classes of non-abelian groups, the reader can refer [12]. Among several properties of locally compact group, we briefly mention three of these viz Plancheral Formula, Uncertainty Principle and Potential theory.

2.1. **Plancheral Formula.** The classical Plancheral theorem proved in 1910 by Michel Plancheral can be stated as follows:
Let $f \in L^2(\mathbb{R})$ and define $\phi_n : \mathbb{R} \to \mathbb{C}$ for $n \in \mathcal{N}$ by

$$\phi_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x)e^{iyx}dx.$$

The sequence ϕ_n is cauchy in $L^2(\mathbb{R})$ and we write $\phi = \lim_{n\to\infty} \phi_n$.

Define $\psi_n : \mathbb{R} \to \mathbb{C}$ by

$$\psi_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \phi(y)e^{-iyx}dy.$$

The sequence ψ_n is Cauchy in $L^2(\mathbb{R})$ and we write $\psi = \lim_{n\to\infty} \psi_n$. Then $\psi = f$ a.e, and $\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\phi(y)|^2 dy$. This theorem is true in various forms for any locally compact abelian groups.

Work on the Plancherel Formula for non-abelian groups began in earnest in the late 1940s. There were two distinct approaches. The first, for separable, locally compact, unimodular groups, was pursued by Mautner, Segal , and others. The second, for semisimple Lie groups, was followed by Gelfand-Naimark, and Harish-Chandra, along with others. Segal's paper and Mautner's paper led eventually to the following statement.

Let G be a separable unimodular, type I group, and let dx be a fixed Haar measure on G. There exists a positive measure μ on $\widehat{G}$ (determined uniquely up to a constant that depends only on dx) such that, for $f \in L^1(G) \bigcap L^2(G)$, $\pi(f)$ is a Hilbert-Schmidt operator for μ-almost all $\pi \in \widehat{G}$, and

$$\int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\pi(f)\|_{HS}^2 d\mu(x).$$

Here, of course, $\widehat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G. At about the same time, Harish-Chandra stated the follwing theorem in his paper Plancheral Formula for complex semisimple connected Lie groups. Let G be a connected, complex, semisimple Lie group. Then for $f \in C_c^\infty(G)$.

$$f(1) = \lim_{H \to 0} \prod_{\alpha \in P} D_\alpha \overline{D_\alpha}[e^{\rho(H)+\overline{\rho(H)}} \int_{K \times N} f(ue^H nu^{-1}) du dn].$$

Explanation of notation can be seen in [4]. Explicit Plancherel formulas can be obtained for groups like $SL(2,\mathbb{C})$, $SL(2,\mathbb{R})$, Reductive groups over Q_p, $GL(n;Q_p)$, semi direct product of groups, group extension, central groups.

2.2. **Uncertainty Principle.** The uncertainty principle is partly a description of a characteristic feature of quantum mechanical systems, partly a statement about the limitations of one's ability to perform measurements on a system without disturbing it, and partly a meta-theorem in harmonic analysis that can be summed up as follows. A non-zero function and its Fourier transform cannot both be sharply localized.

1. Heisenberg's inequality. Heisenberg's inequality gives the precise quantitative formulation of the uncertainty principle. Heisenberg's Inequality for the functions on $\mathbb{R}^n$. If $f \in L^2(\mathbb{R}^n)$ and $a, b \in \mathbb{R}^n$ then

$$\int |x-a|^2 |f(x)|^2 dx \int |\xi - b|^2 |\widehat{f}(\xi)|^2 d\xi \geq \frac{n^2 \|f\|_2^4}{16\pi^2}.$$

Heisenberg's inequality says that if f is highly localized, then $\widehat{f}$ cannot be concentrated near a single point. But it does not preclude f from being concentrated in a small neighborhood of two or more widely separated points. In fact, the latter phenomenon cannot occur either, and it is the object of local uncertainty inequalities to make this precise. Heisenberg Uncertainty Inequality has been proved for the spaces like

- Heisenberg Group
- Symmetric spaces of non-compact type (Price and Sitaram)
- Locally compact Abelian groups (Hogan)
- Infinite dimensional abstract Wiener space (Lee)

2. Benedick's Theorem Qualitative Uncertainty Principle If we think of concentration in terms of f living entirely on a set of finite measure, then we have the following beautiful result of Benedicks: Let f be a non-zero square integrable function on $\mathbb{R}$. Then the Lebesgue measures of the sets $\{x : f(x) \neq 0\}$ and $\{y, \widehat{f}(y) \neq 0\}$ cannot both be finite.

- J.A. Hogan (in 1988) proved that the QUP holds for a non-compact and non-discrete abelian locally compact group G with connected component of the identity G_0 if and only if G_0 is non-compact.
- J.A. Hogan (in 1993) proved that an infinite compact group satisfies the QUP if and only if it is connected.

- T. Matolcsi and J. Szucs (in 1973) proved that the weak QUP holds for every locally compact abelian group.
- Echterhoff, Kaniuth and Kumar (1991) proved that QUP holds for several lower dimensional nilpotent Lie groups and Weyl extension of Abelian groups.
- G. Kutyniok (in 2003) proved that a compact group G satisfies the weak QUP precisely when the quotient group $\frac{G}{G_0}$ is abelian.
- E. Kaniuth (in 2009) proved that a group G which is neither compact nor discrete and having finite dimensional irreducible representations satisfies the QUP if and only if G_0 is non-compact.

3. Hardy's Theorem The rate at which a function decay at infinity can also be considered a measure of concentration. The following elegant result of Hardy's states that both f and $\widehat{f}$ cannot be very rapidly decreasing: Suppose f is a measurable function on $\mathbb{R}$ such that

$$|f(x)| \leq Ae^{-\alpha\pi x^2}$$

and

$$|\widehat{f}(y)| \leq Be^{-\beta\pi y^2}$$

for some positive constants A, B, α, β then, if

(1) $a\beta > 1$,then f must necessarily be a zero function *a.e.*
(2) $a\beta < 1$, then there are infinitely many linearly independent functions
(3) $a\beta = 1$, then $f(x) = ce^{-apx2}$ for some constant c.

During the past fourteen years, there has been much effort to prove Hardy-like theorems for various classes of non-abelian connected Lie groups. Specifically, analogues of Hardy's theorem have been established for the following classes of groups

- The n-dimensional Euclidean motion group
- Cartan motion groups
- Abelian groups with noncompact connected component
- Non-compact connected semi-simple Lie groups G with finite centre
- SL(2;R)
- All semi-simple Lie groups G with finite center
- All non-compact semi-simple Lie groups
- The Heisenberg groups and for general simply connected nilpotent Lie groups.
- Connected nilpotent Lie groups with non- compact centre
- Exponential solvable Lie groups with non trivial centre

4. Cowling Price Theorem Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. Suppose for some k, $1 \leq k \leq n$.

$$(i) \int_{\mathbb{R}^n} e^{pa\pi x_k^2} |g(x_1, \ldots, \widehat{x}_k, \ldots, x_n)|^p |f(x_1, \ldots, x_n)|^p dx_1 dx_2 \ldots dx_n < \infty,$$

$$(ii) \int_{\mathbb{R}^n} e^{qb\pi y_k^2} |g(y_1, \ldots, \widehat{y}_k, \ldots, y_n)|^p |\hat{f}(y_1, \ldots, y_n)|^p dy_1 dy_2 \ldots dy_n < \infty,$$

where $a, b > 0$, $g, h : \mathbb{R}^{n-1} \to \mathbb{C}$ are measurable with $g \geq \alpha > 0$, $h \geq \beta > 0$, α, β are constants, $\frac{1}{g} \in L^p(\mathbb{R}^{n-1})$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{h} \in L^q(\mathbb{R}^{n-1})$, $\frac{1}{q} + \frac{1}{q'} = 1$. If $ab \geq 1$, then $f = 0$ almost everywhere.

Analogues of Cowling Price have been established for the following classes of groups.

- Heisenberg group
- Non-compact Riemannian symmetric spaces (Ray Sarkar)
- Non-compact connected semi-simple Lie groups
- Semi-simple Lie groups (Ebata, et. al.)
- Simply connected nilpotent Lie group
- Two-step nilpotent Lie groups (Ray)
- Threadlike nilpotent Lie Groups (Myself Bhatta)

Theorem 1 (Beurling' Theorem (1991))**.** *For $f \in L^1(\mathbb{R})$,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| |\widehat{f}(y)| e^{2\pi |xy|} dx dy < \infty,$$

then $f = 0$ almost everywhere.

Generalizations of these results have been obtained for a variety of locally compact groups like

- Heisenberg group
- Motion group
- Non-compact connected semi-simple Lie groups
- Connected nilpotent Lie groups
- Simply connected nilpotent Lie group

3. Potential theory on Stratified Lie Groups

The fundamental role of stratified Lie groups in analysis was envisaged by E. M. Stein [13] in his address at the Nice International Congress of Mathematicians in 1970. Since then, there has been a tremendous development in the analysis of stratified Lie groups also known as Carnot group. Indeed, Carnot groups appear as tangent spaces of sub-Riemannian manifolds and they find applications in many settings: mechanics, control theory, geometric theory of several complex variables, curvature problems for CR-manifolds, diffusion processes. The theory of sub-elliptic operators on stratified Lie groups is very interesting and has been attracting mathematicians. The Potential Theory related to sub-Laplacians $\Delta_{\mathbb{G}}$ has been investigated in the recent literature: see the paper by Trudinger & Wang [17] where several results of Potential Theory are established for a class of quasilinear subelliptic operators including sub-Laplacians; see also [14, 15, 16] where fully nonlinear Hessian operators are treated; see also Labutin [11] for the study of subharmonic functions related to this last class of operators; see the papers Bonfiglioli and Lanconelli [1, 2, 3] in which potential-theoretic results and applications are established in the very framework of stratified groups.

3.1. **Homogeneous Carnot groups and stratified groups.**

- A Lie group on $\mathbb{R}^n$ is a pair $\mathbb{G} = (\mathbb{R}^n, \circ)$ where $\circ$ is a binary operation on $\mathbb{R}^n$ such that the group operations are smooth. The notation $\mathcal{G}$ will be reserved for the Lie algebra of $\mathbb{G}$.
- **Examples :**
  (1) The usual addition structure of $\mathbb{R}^n$ turns it into a Lie group.
  (2) On $\mathbb{R}^3$, define the group operation by
  $$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1 + x_3 y_2, x_2 + y_2, x_3 + y_3).$$
  (3) On $\mathbb{R}^4$, define group operation by
$$(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1+y_1+x_4 y_2+\frac{1}{2}x_4^2 y_3, x_2+y_2+x_4 y_3, x_3+y_3, x_4+y_4).$$
- Let $(\mathbb{R}^n, \circ)$ be a Lie group equipped with a family $\{\delta_\lambda\}_{\lambda>0}$ of automorphisms of the form
  $$\delta_\lambda(x^{(1)}, x^{(2)}, \ldots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)})$$
  where $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \ldots, r$ and $N_1 + N_2 + \ldots + N_r = N$. For $1 \le i \le N_1$, let Z_i be vector field of $\mathcal{G}$ agreeing with $\frac{\partial}{\partial x_i}$ at the origin. If the Lie algebra generatted by $Z_1, \ldots, Z_{N_1}$ is whole of $\mathcal{G}$ then $\mathbb{G}$ is called a homogeneous Carnot group.
- A stratified group $\mathbb{H}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{h}$ admits a stratification $\mathfrak{h} = V_1 \oplus V_2 \oplus \ldots \oplus V_r$ such that $[V_1, V_{i-1}] = V_i$ for $2 \le i \le r$ and $[V_1, V_r] = 0$.
- The Lie algebra of a stratified group can admit more than one stratifications.
- Let $\mathbb{H}$ be a stratified group and $\mathfrak{V} = (V_1, \ldots, V_r)$ be a fixed stratification of the Lie algebra $\mathfrak{h}$. We say that a basis $\mathfrak{B}$ of $\mathfrak{h}$ is adapted to $\mathfrak{V}$ if
  $$\mathfrak{B} = (E_1^{(1)}, \ldots, E_{N_1}^{(1)}; \ldots; E_1^{(r)}, \ldots, E_{N_r}^{(r)})$$
  and $(E_1^{(i)}, \ldots, E_{N_i}^{(i)})$ is a basis for V_i.

- If $(V_1, \ldots, V_r)$ is a stratification of the algebra of $\mathbb{H}$ and dim $(V_1) = m$, we say that $\mathbb{H}$ as step r and has m generators.
- The following result shows that above definitions are well-posed
  Suppose that $(V_1, \ldots, V_r)$ and $(\tilde{V}_1, \ldots, \tilde{V}_{\tilde{r}})$ be any two stratifications of the algebra of $\mathbb{H}$ then $r = \tilde{r}$ and dim(V_i) =dim$(\tilde{V}_i)$ for every r. Moreover the algebra of $\mathbb{H}$ is a nilpotent Lie algebra of step r.
- The number $Q = \sum_{i=1}^{r} i\dim(V_i)$ is independent of the choice of stratification and is termed as the homogeneous dimension of $\mathbb{H}$.
- A homogeneous Carnot group is a stratified group and conversely given a stratified group $\mathbb{H}$, there exists a homogeneous Carnot group $\mathbb{H}^*$ which is isomorphic to $\mathbb{H}$.
- The Heisenberg Group
  On $\mathbb{R}^{2n+1}$, define the group operation as

$$(x, y, t).(u, v, s) = (x + u, y + v, t + s + 2(y.u - x.v)), x, y, u, v \in \mathbb{R}^n, s, t \in \mathbb{R}.$$

  With the above operation, $\mathbb{R}^{2n+1}$ becomes a Lie group denoted by $\mathbb{H}_n$. $\mathbb{H}_n$ is a homogeneous Carnot group with the following dilation structure. For $r > 0$ define δ_r on $\mathbb{H}_n$ as

$$\delta_r(x, y, t) = (rx, ry, r^2t).$$

- On $\mathbb{R}^4$ define $\circ$ as

$$x \circ y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(y_2x_1 - y_1x_2) \\ x_4 + y_4 + \frac{1}{2}(y_3x_1 - y_1x_3) + \frac{1}{12}(x_1 - y_1)(y_2x_1 - y_1x_2) \end{pmatrix}.$$

  $(\mathbb{R}^4, \circ)$ is a homogeneous Carnot group with the dilations

$$\delta_r(x_1, x_2, x_3, x_4) = (rx_1, rx_2, r^2x_3, r^3x_4).$$

3.2. **Sub-Laplacians of a Stratified Group.**

- Let $\mathbb{H}$ be a stratified group. The second order differential operator

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2$$

  is referred to as a sub-Laplacian on $\mathbb{H}$, if there exists a stratification $\mathfrak{h} = V_1 \oplus V_2 \oplus \ldots \oplus V_r$ such that $X_1, \ldots, X_m$ is a basis of V_1.
- The sub-Laplacian on the Heisenberg group $\mathbb{H}_1$ given by

$$\Delta_0 = (\partial_{x_1})^2 + (\partial_{x_2})^2 + 4(x_1^2 + x_2^2)(\partial_{x_3})^2 + 4x_2\partial_{x_1,x_3} - 4x_1\partial_{x_2,x_3}$$

  is known as the Kohn Laplacian.

- The sub-Laplacian is a hypoelliptic operator.(A differential operator D is called hypoelliptic if for any distribution u, $Du = f$ implies u is C^∞ whenever f is C^∞)

3.3. **The Fundamental Solution for a Sub-Laplacian.**

- Let D be a differential operator. A distribution u is said to be fundamental solution for D if $Du = \delta$ where δ is the Dirac distribution.
- For $N \geq 3$ the fundamental solution of the usual Laplacian in $\mathbb{R}^N$ is $c_N|x|^{2-N}$ while the fundamental solution in $\mathbb{R}^2$ is $log|x|$.
- One of the most striking analogies between $\mathcal{L}$ and the classical Laplacian is that for $Q \geq 3$, $\mathcal{L}$ possesses a fundamental solution Γ of the form $\Gamma = d^{2-Q}$ where d is a symmetric homogeneous norm on $\mathbb{G}$.

Homogeneous Norms

- Let $\mathbb{G}$ be a Carnot group. A function $d : \mathbb{G} \to [0, \infty)$ is a homogeneous norm on $\mathbb{G}$ if
  (1) $d(\delta_\lambda(x)) = \lambda d(x)$ for every $\lambda > 0$ and $x \in \mathbb{G}$;
  (2) $d(x) > 0$ iff $x \neq 0$
  Moreover, we say that d is symmetric if
  (3) $d(x^{-1}) = d(x)$ for every $x \in \mathbb{G}$.
- **Example:** Define

$$|x|_{\mathbb{G}} = \Big(\sum_{j=1}^{r} |x^{(j)}|^{\frac{2r!}{j}}\Big)^{\frac{1}{2r!}}$$

  Then $|x|_{\mathbb{G}}$ is a homogeneous norm on $\mathbb{G}$, smooth away from the origin.
- On any Carnot group $\mathbb{G}$ the map $x \mapsto |Log(x)|_{\mathbb{G}}$ is a symmetric homogeneous normon $\mathbb{G}$ smooth away from the origin.
- (Equivalence of the homogeneous norms) Let d be a homogeneous norm on $\mathbb{G}$ then there exists a constant $c > 0$ such that

$$c^{-1}|x|_{\mathbb{G}} \leq d(x) \leq c|x|_{\mathbb{G}} \ \forall \ x \in \mathbb{G}$$

Fundamental solution for Lie groups on $\mathbb{R}^N$

- Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}(Q > 2)$ then there exists a unique fundamental solution Γ for $\mathcal{L}$ satisfying the following:
  1. $\Gamma \in C^\infty(\mathbb{R}^N - \{0\})$;
  2. $\Gamma \in L^1_{loc}(\mathbb{R}^N)$ and vanishes at infinity.
  3. (Symmetry of Γ) The fundamental solution Γ of $\mathcal{L}$ satisfies

$$\Gamma(x^{-1}) = \Gamma(x) \ \ \forall x \in \mathbb{G} - \{0\}$$

  4.(δ_λ homogeneity of Γ) Γ is δ_λ-homogeneous of degree $2 - Q$ *i.e.*

$$\Gamma(\delta_\lambda(x)) = \lambda^{2-Q}\Gamma(x) \ \ \forall x \in \mathbb{G} - \{0\}.$$

5. $\Gamma(x) > 0 \ \ \forall x \in \mathbb{G} - \{0\}$.

4. Green Function

$\mathcal{L}$ Green Function

- Given a differential operator $\mathcal{L}$, if there exists, for each y a distribution $G(x,y)$ satisfying $\mathcal{L}G(x,y) = \delta(y-x)$ then it is called Green function for the operator $\mathcal{L}$. A solution to the differential equation
$$\mathcal{L}u = f \ \ for\ some\ suitable\ f$$
can be given by
$$u(x) = \int G(x,y)f(y)dy$$
- For $\mathcal{L} = d/dx$ on $\mathbb{R}$, the heaviside function given by
$$H(x,y) = \begin{cases} 1 & y \geq x \\ 0 & y < x \end{cases}$$
is a Green's function.

5. The Mean Value Formulas

The mean value formulas have got a wide application in the theory of Dirichlet Problem and in Integral Geometry. For the stratified Lie groups, mean value formulas were obtained by Lanconelli in 1990.

Surface Mean Value Formula

Let $\mathcal{L}$ be a sub-Laplacian and d be a gauge on a Stratified Lie group $\mathbb{G}$. For $x \in \mathbb{G}$ and $r > 0$ define the d ball as
$$B_d(x,r) = \{y \in \mathbb{G} : d(x^{-1}y) < r\}.$$

Theorem 2. *Let O be an open subset of $\mathbb{G}$ and u be harmonic on O. Then*
$$u(x) = \mathcal{M}_r(u)(x) = \frac{(Q-2)\gamma}{r^{Q-1}} \int_{\partial B_d} \mathcal{K}(x,z)u(z)dz,$$
where the kernel $\mathcal{K}$ depends upon the gradient of d and γ is volume of $\mathcal{K}$ in the second variable over the surface of B_d. Q denotes the homogeneous dimension of $\mathbb{G}$.

Solid Mean Value Formula

- $u(x) = M_r(u)(x)\frac{\nu}{r^Q}\int_{B_d(x,r)} \Psi(x,y)u(y)dy$, where the kernel function Ψ again depends on gradient of d. ν is average of Ψ.

Converse of Mean Value Theorems

Theorem 3. *Let $\mathcal{L}$ be a sub-Laplacian on a stratified Lie group $\mathbb{G}$ and d be an $\mathcal{L}$-gauge.Let u be a continuous function on an open subset O of $\mathbb{G}$. If one of the following is satisfied*

(i) $u(x) = \mathcal{M}_r(u)(x)$ *for every $x \in O$ and for all $r > 0$ such that ball of radius r around x is in O.*

(ii) $u(x) = M_r(u)(x)$ *for every $x \in O$ and for all $r > 0$ such that ball of radius r around x is in O.*

Then $u \in C^\infty(O)$ and $\mathcal{L}u = 0$.

Harnack Inequality

Obtained in various contexts. First in for the Heisenberg group in 1969 by Bony. Later by Franchi and Lanconelli (1982, 83), and Gutierrez and Lanconelli (2003).

Theorem 4 (Harnack Inequality)**.** *Let Ω be an open subset of $\mathbb{G}$ and $u : \Omega \to \mathbb{R}$ be a non-negative smooth harmonic function. Then*

$$\sup_{B_d(x_0,r)} u \leq \mathbf{c} \inf_{B_d(x_0,r)} u,$$

for each x_0 in Ω and suitably chosen $r > 0$. The constant $\mathbf{c}$ depends on u, r, x_0 and Ω.

The Dirichlet Problem

The Dirichlet problem asks to find a harmonic function whose continuous extension to the boundary of the domain coincides with a prescribed continuous function on the boundary. In case of domains in $\mathbb{C}$, the Perron method gives existence of the solution of the Dirichlet problem and hence well posedness of the problem. We will give here the descriprion of Perron family in the case of Stratified Lie group.

Subharmonic and Superharmonic Functions

Definition 5.1. Let Ω be an open subset of $\mathbb{G}$, $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function finite in a dense subset of Ω. u is called subharmonic if it satisfies

$$u(x) \leq \mathcal{M}_r(x) \quad \text{for all} \quad x \in \mathbb{G}, r > 0.$$

Definition 5.2. Let Ω be an open subset of $\mathbb{G}$, $u : \Omega \to (-\infty, \infty]$ be a lower semicontinuous function finite in a dense subset of Ω. u is called superharmonic if it satisfies

$$u(x) \geq \mathcal{M}_r(x) \, \text{for all} \, x \in \mathbb{G}, r > 0.$$

Perron Method As in the case of classical potential theory, the Perron family for an open subset Ω of $\mathbb{G}$ can be defined.

Theorem 5 (Fundamental theorem of Perron families)**.** *Let Ω be open in $\mathbb{G}$ and $\mathcal{F}$ be a Perron family then $u = \inf \mathcal{F}$ is harmonic in Ω.*

6. Examples

We now discuss some examples and illustrate some of the ideas discussed above with these examples.

6.1. **Some Simple Examples.**

- On $\mathbb{R}^4$ define $\circ$ as

$$x \circ y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + y_1 x_2 \\ x_4 + y_4 + y_1 x_2^2 + 2x_2 y_3 \end{bmatrix},$$

  and define dilations as

$$\delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4).$$

  $(\mathbb{R}^4, \circ)$ with the family δ_λ of dilations becomes a homogeneous Carnot group.
- On $\mathbb{R}^4$, define $*$ as

$$\xi * \eta = \begin{bmatrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \xi_3 + \eta_3 \\ \xi_4 + \eta_4 + 2\eta_1 \xi_2 - \eta_2 \xi_1 \end{bmatrix}.$$

  The decomposition of Lie algebra is

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 = span\{Z_1, Z_2, \partial_{\xi_3}\} \oplus span\{\partial_{\xi_4}\},$$

  where

$$Z_1 = \partial_{\xi_1} + 2\xi_2 \partial_{\xi_4},$$
$$Z_2 7 = \partial_{\xi_2} - 2\xi_1 \partial_{\xi_4}.$$

6.2. **The Heisenberg Group.** A Classic Example:

The Heisenberg Group I

- Consider the set of matrices of the form

$$\begin{pmatrix} 0 & x & t + 2x.y \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, t \in \mathbb{R}.$$

  The set with matrix multiplication forms a group called the Heisenberg group(of three dimensions). If x is replaced by the row vector x and y by the column vector y for $x, \mathbf{y} \in \mathbb{R}^{\mathbf{n}}$ then we get the $2n + 1$ dimensional Heisenberg group.

- The Heisenberg Group can also be defined on the set $\mathbb{C}^n \times \mathbb{R}$
  The multiplication being:
  $(z,t)\cdot(\zeta,s) = (z+\zeta, t+s+2\mathrm{Im}\ z\cdot\bar{\zeta}),\ \forall\ (z,t),(\zeta,s) \in \mathbb{C}^n \times \mathbb{R}$.
- with this operation and usual C^∞ structure on $\mathbb{C}^n \times \mathbb{R}$, this group becomes a Lie group denoted by $\mathbb{H}_n$.
- If $z = x+iy$ and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real coordinates on $\mathbb{H}_n$, we set

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}$$
$$Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$
$$T = \frac{\partial}{\partial t}$$

Then $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, T$ form a basis for the Lie algebra of $\mathbb{H}_n$.

The Heisenberg Group II

- The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ can be expressed as

$$\mathfrak{h}_n = \mathfrak{v}_n \oplus \mathfrak{z}_n,$$

where $\mathfrak{z}_n$ is span of T and $\mathfrak{v}_n$ is that generated by $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$. This gives a stratification of $\mathfrak{h}_n$.
- The canonical sub-Laplacian on $\mathbb{H}_n$ is given by

$$\mathcal{L}_0 = \sum_{i=1}^{n} (X_j^2 + Y_j^2).$$

- The fundamental solution for $\mathcal{L}_0$ with pole at the identity was given by Folland in the form

$$g_e([z,t]) = c\left(|z|^4 + t^2\right)^{-\frac{n}{2}},$$

c is a normalization constant.

The Kelvin Transform on $\mathbb{H}_n$

- On $\mathbb{H}_n - \{0\}$, define h as

$$h([z,t]) = \left[-\frac{z}{|z|^2 - it}, -\frac{t}{|z|^4 + t^2}\right]$$

- The Kelvin transform of f defined on $\mathbb{H}_n - \{0\}$ is given by

$$Kf = c^{-1} g_e . f \circ h$$

This transform, as in Euclidean case, sends harmonic functions to harmonic functions.

- Explicit expressions of Green's function and Poisson kernel for a class of polyradial fucnctions are known in case of the Heisenberg group.

6.3. **Another Interesting Class of Examples. H-type Groups I**

- H-type groups were first introduced by Kaplan in 1980 as a generalization of the Heisenberg group.
- Let $\mathfrak{n}$ be a real Lie algebra equipped with an inner product, which can be writetn as an orthogonal direct sum,

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z},$$

where $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{z}$ and $[\mathfrak{v}, \mathfrak{z}] = [\mathfrak{z}, \mathfrak{z}] = 0$. Define the linear mapping $J : \mathfrak{z} \to End(\mathfrak{v})$ by the formula

$$\langle J_Z X, X' \rangle = \langle Z, [X, X'] \rangle \quad \forall \quad Z \in \mathfrak{z},\ X, X' \in \mathfrak{v}.$$

$\mathfrak{n}$ is called a Lie algebra of H-type if

$$J_Z^2 = -|Z|^2 I.$$

- A Lie connected and simply connected Lie group is called H-type if its Lie algebra is of H-type.

H-type Groups II

- A Lie group of H-type is said to satisfy the J^2 condition if for every pair of orthogonal vectors $Z, Z' \in \mathfrak{z}$ and $X \in \mathfrak{v}$, there exists $Z'' \in \mathfrak{z}$ such that

$$J_Z J_{Z'} X = J_{Z''} X.$$

- There is a nice Kelvin transform defined on H-type groups which satisfy J^2 condition.

HM **Groups**

Definition 6.1. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{z}$ be its center. We say that $\mathfrak{g}$ is HM type if it admits a vector space decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $[\mathfrak{g}_1, \mathfrak{g}_2] \subseteq \mathfrak{g}_2$ and $\mathfrak{g}_2 \subseteq \mathfrak{z}$ with the additional property that for every $0 \neq \eta \in \mathfrak{g}_2^*$, the bilinear form $B_\eta(X, X') = \eta([X, X'])$ is non degenrate.

A Model Two Step Group

- Consider $(n+p)$ dimensional space $\mathbb{R}^n \times \mathbb{R}^p$ and an element of this will be written as $(x_1, \ldots, x_n, t_1 \ldots, t_p)$. We often use the pair (x, t) to denote an element of $\mathbb{R}^n \times \mathbb{R}^p$ where x will denote the tuple $(x_1, \ldots, x_n)$ and t will denote the tuple $(t_1, \ldots, t_p)$. On this space, we define a binary operation $*$ as

$$(x, t) * (x', t') = (x + x', t + t' + \sum_{j,k} a_{jk}^{\alpha} x'_j x_k),$$

where, for each α, (a^{α}_{jk}) is an antisymmetric matrix. Clearly, the operation $*$ turns $\mathbb{R}^n \times \mathbb{R}^p$ into a Lie group, denoted as N with the usual C^∞ structure on $\mathbb{R}^n \times \mathbb{R}^p$.

- Define vector fields X_j's as follows

$$X_j = \frac{\partial}{\partial x_j} + \sum_{\alpha=1}^{p} \sum_{k=1}^{n} a^{\alpha}_{jk} x_k \frac{\partial}{\partial t_\alpha}, \quad 1 \le j \le n$$

. The vector fields X_j's and $T_\alpha = \frac{\partial}{\partial t_\alpha}$ form a basis for the Lie algebra $\mathfrak{n}$ of N. It can be easily seen that the brackets of $\mathfrak{n}$ are given by

$$[X_j, X_k] = 2 \sum_{\alpha} a^{\alpha}_{jk} \frac{\partial}{\partial t_\alpha},$$

and all other brackets are trivial. It now follows that the group N is a two step nilpotent Lie group. If we assume that $p \le n(n-1)/2$ then the vector fields X_j's and their first brackets generate the Lie algebra $\mathfrak{n}$ of N.The operator

$$\Delta = \frac{1}{2} \sum_{j=1}^{n} X_j^2,$$

is the canonical sub-Laplacian on N. From [6], this operator is hypoelliptic. A fundamental solution for Δ with pole at identity is given by the integral

$$G(x,t) = \int_{\mathbb{R}^p} \frac{V}{f^q},$$

where V is a solution of a generalized transport equation and f is a solution of a generalized Hamilton-Jacobi equation. $q = \frac{n}{2} + p - 1$.

7. Open problems

Some open problems

- Determining analytic hypoellipticity for groups of higher order of nilpotency.
- To find a fundamental solution for the sub-Laplacian on a Filiform group where the operator is known to be not analytic hypoelliptic.(Let $\mathfrak{h}$ be a Lie algebra and $\mathfrak{h} \supset \mathfrak{h}_1 \supset \mathfrak{h}_2 \ldots$ be its lower central series. $\mathfrak{h}$ is called Filiform if $\dim \mathfrak{h}_k = \dim \mathfrak{h} - k$ for $3 \le k \le \dim \mathfrak{h}$)
- Finding Poisson Kernel on H-type groups without the J^2 condition and non polarizable groups.
- Generalizing Kelvin transform to groups beyond the class of H-type groups with J^2 condition.

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TOPOLOGICAL COMBINATORICS- THE KNESER CONJECTURE†

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ABSTRACT. The proof of Kneser Conjecture given by L. Lov*ász* is believed to have laid the foundations of Topological Combinatorics. In this expository paper, we present three distinct proofs of the Kneser Conjecture, each of which essentially uses the Bousruk-Ulam theorem of algebraic topology. We also discuss various generalizations of this conjecture along with the methods employed in their proofs.

1. Introduction

The Kneser conjecture, which we explain below, concerns with a simple problem of Combinatorics. We start with a finite set and consider all of its subsets of a given uniform size. Then we partition this collection of subsets into a fixed number of classes and ask whether any of those classes admits a pair of disjoint sets. More precisely, let X be a set having $2n+k$ elements, $n \geq 1, k \geq 0$. Let Σ_n be the collection of all n-sets of X, i.e., those subsets of X which have n-elements. Note that in Σ_n, we can obviously find two n-sets A and B such that $A \cap B = \phi$. Now here is a question :

Question: If we decompose Σ_n into two or more than two disjoint classes, then can we find a class K_i such that K_i has a pair of disjoint n-sets of X?. May or may be not!

To illustrate it by a simple example, let us consider the set $X = \{1, 2, 3, 4, 5\}$ and its 2-subsets Σ_2. Then

$$\Sigma_2 = \{\{1,2\}, \{1,3\}, \cdots \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \cdots, \{4,5\}\}.$$

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We can decompose Σ_2 into 3-classes as (Here $5 = 2.2 + 1$):
$K_1 = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\}$
$K_2 = \{\{2,3\},\{2,4\},\{2,5\}\}$
$K_3 = \{\{3,4\},\{3,5\},\{4,5\}\}$

Then note that none of the K_i's has a pair of disjoints sets. But if we decompose Σ_2 into two disjoint classes K_1 and K_2, then we can verify that at least one of them will have a pair of disjoints 2-sets. M. Kneser in the year 1955 made the following conjecture (see [9]):

Kneser Conjecture : Suppose $|X| = 2n + k$, and $\Sigma_n = \binom{2n+k}{n}$ denotes the set of all n-subsets of X. Decompose Σ_n into $k + 1$ disjoint classes. Then there is a class K_i which has a pair of disjoint sets.

Remark 1.1. The above conjecture reminds one the following version of the Borsuk-Ulam Theorem (Lusternik-Schnirelman Theorem): Decompose $\mathbb{S}^n$ as the union of $n + 1$ closed subsets $C_1, C_2 \cdots, C_{n+1}$. Then there is a set C_i which has a pair of antipodal points.

Remark 1.2. The Kneser Conjecture can obviously be stated as follows:
If Σ_n is decomposed into r disjoint classes where $r \leq k + 1$, then at least one of them will have a pair of disjoint subsets.

Remark 1.3. The Kneser Conjecture is not true if Σ_n is decomposed into $k + 2$ disjoint classes. An example is already given above, but for a general example, let $X = \{1, 2, \cdots, 2n + k\}$ and Σ_n be the set of all n-subsets of X. Define the class K_i = all n-subsets whose first member is $i, 1 \leq i \leq k + n + 1$. Then $K_1, K_2, \cdots, K_k, K_{k+1}$ and $K_{k+2} \cup K_{k+3} \cup \cdots \cup K_{k+n+1}$ are the $k + 2$ disjoint subsets of Σ_n and there is no K_i which contains a pair of disjoint n-sets !!

Proof. Note that for each $i = 1, 2, \cdots k+1$, any pair of n-sets in the class K_i always intersect viz. they have the index i in common. However, in $K_{k+2} \cup K_{k+3} \cup \cdots \cup K_{k+n+1}$, note that any two n-sets in any of the $K_{k+i}, 2 \leq i \leq n+1$ are not disjoint by definition. In case they are in different $K_{k+i_1}, K_{k+i_2}, i_1 \neq i_2$, then also they must have something in common, otherwise the total number of elements in X must be

$$k + 1 + n + n = 2n + k + 1 > 2n + k,$$

a contradiction. □

2. The Method of Lovász

Verifying the conjecture for small values of n and k indicates the possibility of the conjecture being true in general, but we need a proof for arbitrary values of n and k. No proof of this was available for a long period of time until in 1978 when

L. Lov*ász* gave a proof using the ideas of algebraic topology, in an unexpected manner. Here is the approach adopted by L.Lov*ász* in his seminal paper (see[11]).

Let us define a graph $GK_{n,k}$, called the Kneser graph associated to Σ_n, as follows :

- Vertices are n-subsets of X i.e., the elements of Σ_n
- Two vertices A and $B \in \Sigma_n$ will form an edge provided $A \cap B = \phi$.

Let us color the vertices of the above graph so that the two vertices of an edge have different colors. If we prove that the minimum number of colors required to color the graph $GK_{n,k}$ is $k+2$, then that will mean that if we color the vertices using only $k+1$ colors i.e., we divide Σ_n into $k+1$ disjoint sets, then there is a member of the partition which will have an edge (A, B) in it, i.e., $A \cap B = \phi$. In other words, the Kneser conjecture will be proved if we can show that the *Chromatic Number* $\chi(GK_{n,k})$ of the Kneser graph $GK_{n,k}$ is $k+2$. Thus, a proof of the conjecture was translated into a coloring problem of a graph. In fact, the Kneser conjecture for $X = \{1, 2, \cdots, 2n+k\}$ is equivalent to proving that $\chi(GK_{n,k})$ is $k+2$, i.e., $GK_{n,k}$ is not $k+1$ colorable.

In 1978, Lov*ász* proved the above result viz., that *the graph* $GK_{n,k}$ *is not* $(k+1)$*-colorable.* However, the proof uses the homotopical connectivity of the simplicial complex related to the Kneser graph $GK_{n,k}$ and an important generalized version of the Borsuk-Ulam Theorem of Algebraic Topology. This proof has given rise to a lot of good mathematics known as Topological Combinatorics. The subject matter of Topological Combinatorics usually arises in the following manner. Given a problem of combinatorics, we define a simplicial complex which converts the problem of combinatorics into a problem of algebraic topology. There are then powerful tools of algebraic topology like the homotopy theory, homology and cohomology theory, obstruction theory etc., which present themselves naturally in tackling that problem. Very often we are successful in solving these problems, which in turn solves the original problem of combinatorics. The combinatorial problems studied and resolved in this fashion constitute the subject matter of topological combinatorics. L. Lov*ász* was the first person to invent this method for proving the Kneser conjecture. By now a large number of problems of combinatorics have been successfully attacked using this method. We will deal with Lov*ász*'s method in the last Section of this paper because it is somewhat technical and difficult to immediately see its correctness. In section 3, we review the Borsuk-Ulam theorem itself in its various forms. In Section 4, we present the second proof (of 1978) and the third proof (of 2002) of the Kneser conjecture, which are short and use only the Borsuk-Ulam theorem and nothing else from algebraic topology.

3. Borsuk-Ulam Theorem

As we will see, the basic result for proving the Kneser conjecture, besides other results, is the Borsuk-Ulam theorem of algebraic topology. It is, therefore, very desirable that we understand this theorem and its several variants well before we embark upon different proofs of the Kneser conjecture. Let us recall that the two points x_1, x_2 of the standard $n-$ sphere S^n are said to be *antipodal points* if they are diametrically opposite to each other, i.e., $x_2 = -x_1$. The sphere S^n admits a homeomorphism $h : S^n \to S^n$ such that $h(x) = -x.$. This map defines a natural free action of the multiplicative group $G = \{-1, 1\}$ on S^n, which is called the antipodal action on S^n. A topological space X which admits a free $G-$ action, is called an antipodality space. A map $f : S^m \to S^n$ is called an antipodal map if it preserves the antipodal points, *i.e.*, $f(-x) = -f(x)\ \forall x \in S^m$.

Theorem 3.1. *The following statements are equivalent:*

(1) *There exists no continuous antipodal map $f : S^n \to S^{n-1}$ for any $n \geq 1$.*
(2) *(Borsuk-Ulam Theorem-continuous version) For any continuous map $f : S^n \to \mathbb{R}^n$, there is a pair of antipodal points $x, -x \in S^n$ such that $f(x) = f(-x)$.*
(3) *(Lusternik-Schnirelmann Theorem-closed version) Given any $n+1$ closed subsets $F_1, F_2, \cdots F_{n+1}$ which cover S^n, there exists some member F_i which has a pair of antipodal points.*
(4) *(Lusternik-Schnirelmann Theorem-open version) Given any $n+1$ open sets $U_1, U_2, \cdots U_{n+1}$ which cover S^n, there exists some member U_i which has a pair of antipodal points.*
(5) *(Lusternik-Schnirelmann Theorem-mixed version) Given any $n+1$ subsets $G_1, G_2, \cdots G_{n+1}$ which cover S^n and each of which is either open or closed in S^n, there exists some G_i which has a pair of antipodal points.*
(6) *(Antipodal on boundary version)There is no continuous map $f : \mathbb{D}^n \to S^{n-1}$ which is antipodal on the boundary of $\mathbb{D}^n$.*
(7) *For any antipodal map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = 0$.*

Most proofs of Borsuk-Ulam Theorem use tools of algebraic topology like degree of a map, the homotopy groups or homology-cohomology groups of a space and the product structure in cohomology (see for example [5], [15] or [8]). It must be remarked, however, that there is also a combinatorial proof of the Borsuk-Ulam theorem which uses Tucker's Lemma, and that Lemma is well-known to have a combinatorial proof (see [12]).

Proof. We will assume the Borsuk Ulam Theorem viz. (2) (see [5, p. 183]), and prove that the rest of the statements are all equivalent to the Borsuk-Ulam Theorem.

(1)⇒(2): If (2) is not true, then $\forall x \in S^n, f(x) \neq f(-x)$. Hence the map $g : S^n \to S^{n-1}$ defined by $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ is continuous and antipodal, a contradiction.

(2)⇒ (3): Let $F_1, F_2, \cdots F_{n+1}$ be a closed cover of S^n. For any $i, 1 \leq i \leq n$, the distance function $d(x, F_i) : S^n \to \mathbb{R}$ is a continuous function. Hence the map $f : S^n \to \mathbb{R}^n$ defined by $f(x) = (d(x, F_1), d(x, F_2), \cdots , d(x, F_n))$ is continuous. Hence by (2), there is an x_0 such that $f(x_0) = f(-x_0)$, i.e., $d(x_0, F_i) = d(-x_0, F_i)$, $\forall i \leq n$. Now if $x_0 \in F_i$ for some i, then since F_i is closed, $d(x_0, F_i) = 0 = d(-x_0, F_i)$, which means both $x_0, -x_0 \in F_i$ and (3) is proved. Otherwise $d(x_0, F_i) > 0$, $\forall i \leq n$, which means $x_0, -x_0 \notin F_i$. Since $F_1, F_2, \cdots F_{n+1}$ is a cover of S^n, we find that both $x_0, -x_0$ are in F_{n+1}. This proves (3).

(3)⇒ (1): If (1) is not true, then there exists a continuous antipodal map, say $f : S^n \to S^{n-1}$. Let us consider S^{n-1} as the boundary of an $n-$simples Δ^n so that origin is at the center of the simplex. Then observe that Δ^n has $n + 1$ faces and the radial projection from the origin will give us $n+1$ closed subsets, say $F_1, F_2, \cdots , F_{n+1}$ covering S^{n-1}. Clearly, none of these closed sets has a pair of antipodal points. Therefore, since f is antipodal, $f^{-1}(F_1), f^{-1}(F_2), \cdots , f^{-1}(F_{n+1})$ is a collection of $n + 1$ closed subsets of S^n such that none of these has a pair of antipodal points, *i.e.*, (3) is not true.

(3)⇒ (4): Let $U_1, U_2, \cdots U_{n+1}$ be an open cover of S^n. Suppose λ is a Lebesgue number for the above covering. This means $\forall x \in S^n$, the closed ball $\overline{B(x, \lambda)}$ around x will be contained in U_i for some i. Since S^n is compact, there is a finite number of such balls $\overline{B(x, \lambda)}, x \in S^n$ covering S^n. For each i, we choose all such balls which are contained in U_i. Let F_i denote the union of such balls. Then $F_i \subset U_i$ for each i, and $F_1, F_2, \cdots F_{n+1}$ cover S^n. Hence by (3). there exists an F_i which has a pair of antipodal points. But this means U_i has a pair of antipodal points.

(4)⇒ (5): Let $G_1, G_2, \cdots , G_{n+1}$ be a covering of S^n in which each G_i is either a closed set or an open set. To prove (5), we proceed by induction on the number k of closed subsets in the given covering $\{G_i\}$. If $k = 0$, then all of the G_i's are open and the result follows from (4). Suppose $0 < k < n + 1$ and assume that the result is true when there are $k - 1$ number of closed subsets in G_i. Fix a closed subset F in G_i and suppose F does not contain a pair of antipodal points. Hence its diameter will be less that 2, *i.e.*, $diam(F) = 2 - \epsilon$ for some $\epsilon > 0$. Define a set

$$U = \{x \in S^n | d(x, F) < \epsilon/2\}.$$

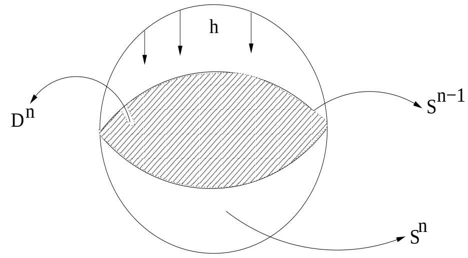

Figure 1

Then U is an open set and $\{G_1, ..., \hat{F}..., G_{n+1}\} \cup \{U\}$ is a cover of S^n with $n+1$ sets, of which exactly $k-1$ are closed and the remaining are open sets. Hence, by the induction hypothesis, some set in the above cover contains a pair of antipodal points. By construction, U cannot contain a pair of antipodal points and therefore some set in the original covering must contain a pair of antipodal points. This completes the proof.

(5)$\Rightarrow$ (3): Obvious.

(2)$\Rightarrow$ (7): By (2), there exists a pair of antipodal points $x, -x$ such that $f(x) = f(-x)$. Since f is antipodal, $f(-x) = -f(x)$ and so $2.f(x) = 0$, i.e., $f(x) = 0$.

(7)$\Rightarrow$ (1): If (1) is not true then there exists a continuous antipodal map $f : S^n \to S^{n-1}$. But this map is also an antipodal map $S^n \to \mathbb{R}^n$ having no zero, *i.e.*, (7) is not true.

(6)$\Rightarrow$ (1): If (1) is not true, then there is a continuous map $f : S^n \to S^{n-1}$ which is antipodal. Observe that we can consider S^n as the union of two hemispheres $H+$ and $H-$ which have S^{n-1} as the common boundary (See Fig 1.). Also, there is the projection homeomorphism $h : H+ \to \mathbb{D}^n$ defined by

$$h(x_1, ..., x_{n+1}) = (x_1, ..., x_n)$$

which is identity on the boundary S^{n-1} of $\mathbb{D}^n$. Hence the map $f0h^{-1}$ is continuous map from $\mathbb{D}^n \to S^{n-1}$ which is antipodal on the boundary S^{n-1}, contradicting (6).

(1)$\Rightarrow$ (6): Suppose (6) is not true. Then there is a continuous map $g : \mathbb{D}^n \to S^{n-1}$ which is antipodal on the boundary S^{n-1} of $\mathbb{D}^n$. We can now use the homeomorphism $h : H+ \to \mathbb{D}^n$ defined above (see Fig 1.) to construct a map $f : S^n \to S^{n-1}$ as follows:

$$f(x_1, ...x_{n+1}) = \begin{cases} g(h(x_1, ..., x_{n+1})) & \text{if } x_{n+1} \geq 0, \\ -g(h(x_1, ..., x_{n+1})) & \text{if } x_{n+1} \leq 0. \end{cases}$$

Then f is continuous because its restriction on the two closed hemispheres are continuous and they match on the common boundary. Clearly, f is also antipodal from $S^n \to S^{n-1}$, a contradiction to (1). □

C.T.Yang [17] has given an interesting generalization of the Borsuk-Ulam theorem using the index theory (see [6]). A special case of that theorem, which is analogous to (1), was utilized by Lov*á*sz in proving the Kneser conjecture.

Theorem 3.2. *(Antipodality version) Let X be a $(n-1)$-connected compact Hausdorff space admitting a free $\mathbb{Z}_2$-action. Then there is no continuous antipodal map $X \to S^{n-1}$.*

It is interesting to point out here that the Borsuk-Ulam theorem is related to the Brouwer's fixed point theorem [16]. We have

Theorem 3.3. *The Borsuk-Ulam theorem implies the Brouwer's fixed point theorem.*

Proof. Let $\mathbb{D}^n$ be an n-disk and $f : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous map. Suppose f has no fixed points. Then we can define a map $g : \mathbb{D}^n \to S^{n-1}$ as follows: For any $x \in \mathbb{D}^n$, draw the line segment joining $f(x)$ to x and produce the segment so that it meets the boundary S^{n-1} of $\mathbb{D}^n$ in a unique point, say $g(x)$. Then g is continuous and is identity on S^{n-1}. Since the identity map on S^{n-1} is antipodal, we can use part (6) to conclude that the Borsuk-Ulam is not true, a contradiction. □

A proof avoiding algebraic topology of the above theorem is given in [16]

Immediately after the proof of Kneser conjecture by Lov*á*sz in the year 1978, I. B*á*r*á*ny gave a short and simple proof of the Kneser conjecture, which appeared in the same volume of the same journal as that of Lov*á*sz [11]. The proof given by B*á*r*á*ny used a variant of the Borsuk-Ulam theorem and a combinatorial Lemma due to D.Gale, already proved in 1956. He did not make any use of algebraic topology other than the well-known Lusternik-Schnirelman theorem (open version, Theorem 3.1(4)) equivalent to the Borsuk-Ulam Theorem. For the following we refer to Fig 2.

Lemma 3.1 (David Gale(1956))**.** *Given $2n+k$ points, $n \geq 1, k \geq 0$ in a set S, we can place these points on the standard $k-$dimensional sphere S^k in such a manner that any open hemisphere around the point $x \in S^k$ contains at least n points of the set S.*

Since Gale's Lemma can be proved using elementary linear algebra, B*á*r*á*ny's proof of the Kneser conjecture confirmed the Lov*á*sz's proof of Kneser conjecture,

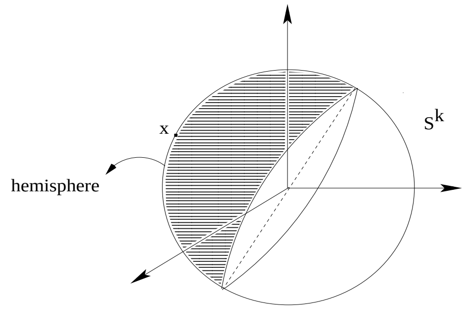

Figure 2

specially because the Lov*ász*'s proof was somewhat complicated and assumed familiarity with the concepts of Algebraic Topology rather well.

The Second Proof- due to I. B*á*r*á*ny :

Proof. Let us consider the sphere $\mathbb{S}^k$. Because of a Lemma due to Gale, we can place the points of $X = \{1, 2, \cdots, 2n+k\}$ on $\mathbb{S}^k$ such that any open hemisphere contains at least n-points of the set X. Let us now decompose the n-sets of X into $k+1$ disjoint classes say, $K_0, K_1, \cdots, K_k$. Now for each i, define the set

$$\theta_i = \{x \in \mathbb{S}^k \quad | \quad \text{the open hemisphere centered at } x \text{ contains at least one point from the class } K_i\}$$

If $x \in \theta_i$, it is easy to see that there is a small neighborhood of x such that all points of that neighborhood are in θ_i. Hence θ_i is an open set of $\mathbb{S}^k$ for $i = 0, \cdots, k$. If $x \in \mathbb{S}^k$, then there is a open hemisphere (by Gale's Lemma) around x (see Figure 2) which contains some n-set from X and that n-set is in some class K_i. Hence $x \in \theta_i$. This means $\{\theta_i \mid i = 0, \cdots, k\}$ is an open cover of $\mathbb{S}^k$. Hence by the Borsuk-Ulam Theorem (Lusternik-Schnirelman theorem-open version), some open set θ_i must contain a pair of antipodal points, say x and $-x$. Then, by definition, there is a n-set from the class K_i which lies in the hemisphere around x and also there is a n-set from the same class K_i which lies in the hemisphere around $-x$. Then clearly there is a pair of disjoint n-sets in the class K_i. □

Here is yet another proof of the Kneser conjecture given by an undergraduate student of USA which is as easy as the proof of Bárány, but it does not use the Gale's Lemma. It actually uses the well-known idea of a set of points in the Euclidean space being in *general position* and the *Borsuk-Ulam Theorem* (mixed version) only so that it is simpler.

The Third Proof- due to J. Green:

Proof. We start with the set X having $2n+k$ points and place them on the sphere S^{k+1} in general position. This can obviously be done. Now partition the set of n-subsets of X into $k+1$ disjoint classes, say $K_1, K_2, \cdots, K_{k+1}$. For each $i, 1 \leq i \leq k+1$, let U_i denote the set of all those points $x \in S^{k+1}$ such that the open hemisphere $H(x)$ centered at x, contains an n-subset from K_i (see Figure 2). This crucial definition of U_i is due to I. B*á*r*á*ny used in the second proof given earlier. Then, by a small perturbation of $H(x)$, one can easily see that U_i is an open set of S^{k+1}. Hence $F = S^{k+1} \setminus (U_1 \cup U_2 \cup ... \cup U_{k+1})$ is a closed set. Note that if $x \in F$, then $H(x)$ cannot contain any $(n+i)$-set for $i \geq 0$, otherwise, it will contain an n-set from X, and will be in some U_i. Thus, the collection $\{F, U_1, \cdots, U_{k+1}\}$ of $k+2$ sets is a covering of S^{k+1} by closed or open sets. Hence, by the mixed version of the Lusternik-Schnirelman Theorem proved earlier, one of the members of the above collection contains a pair of antipodal points, say $x_0, -x_0$. Now observe that F cannot contain such a pair because otherwise $H(x_0)$ and $H(-x_0)$, each will contain fewer than n points from the set of $2n+k$ points. This will mean the remaining points, whose number would be at least $k+2$, will lie on the great circle S^k of S^{k+1}. But this will mean that the set of $2n+k$ on S^{k+1} are not in general position, a contradiction. Hence both x_0 and $-x_0$ will lie in U_i for some i. Therefore $H(x_0)$ and $H(-x_0)$ both contain n-subset from the same class K_i and these are obviously disjoint. This completes the proof of the Kneser Conjecture. □

4. The First Proof- due to L.Lov*á*sz

In this Section we are now going to briefly present the first original proof of the Kneser Conjecture given by L.Lov*ász* in 1978. As pointed out earlier, though the proof is somewhat complicated and uses freely several concepts of algebraic topology (neither Gale's Lemma nor the idea of general position), yet it is quite instructive and has already proved its enormous power in tackling other problems of combinatorics. Let us go back to Section 2. Observe that we have already converted the Kneser Conjecture into a problem of Graph theory. We associated a graph, called Kneser Graph $G_{n,k}$, with the set of all n-sets of the set X, and then found that proving the Kneser conjecture is equivalent to showing that the Chromatic Number of this graph $\chi(G_{n,k})$ is exactly $k+2$. Lov*ász* accomplished this as follows:

He proved the Kneser conjecture in the following two steps:

(1) With any given graph G, Lov*ász* associated a simplicial complex $N(G)$, called the *nbd complex of* G, and proved that if $N(G)$ is $(k-1)$-connected in the sense of homotopy, then G cannot be $(k+1)$-colorable.

(2) He proved that the nbd complex $N(G_{n,k})$ of the Kneser graph $G_{n,k}$ is $(k-1)$-connected.

We now explain the topological ideas used by Lovasz. We begin with the following

Definition 4.1. For any graph $G = (V, E)$, let us define a simplicial complex $N(G)$, called *the neighborhood complex* (nbd complex) of G, whose vertices are the vertices of G and a set A of vertices forms a simplex if all the elements of A have a common neighbor, i.e., $\exists\, v \in V$ such that for all $a \in A$, (v, a) is an edge in G(see Fig 3.)

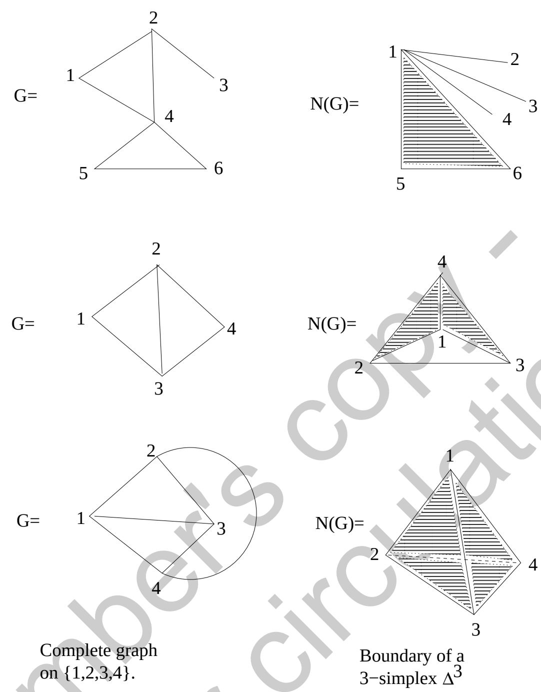

Figure 3

Examples: In the last example of a complete graph $G = K_4$ on 4 vertices, we easily see that the nbd complex $N(G)$ is the boundary of a 3-simplex, i.e., $N(G) \simeq S^2$.

Example 4.1. More generally, let us take a complete graph $G = K_{k+1}$ on $k+1$ vertices $\{0, 1, 2, \cdots, k\} = V$. Then every proper subset of V is a simplex in $N(G)$. In other words, subsets of V having all but one element, will be simplices of $N(G)$. Therefor the simplicial complex $N(G)$ can be identified with boundary complex of the simplex $\Delta^k =< 0, 1, 2, \cdots, k >$. Since $||\partial\Delta^k|| \simeq \mathbb{S}^{k-1}$, we find that a neighborhood complex need not be simply connected, even k-connected for any $k \geq 1$.

Theorem 4.1. *The neighborhood complex $N(KG_{n,k})$ is $(k-1)$-connected*

Proof. Consider the set of all n-subsets of S where $|S| = 2n+k$. Define a simplicial complex K as follows: (1) Vertices are all n-sets of S. (ii) The $(m+1)$-tuple $(A_0, \cdots, A_m)$ forms a simplex of K iff $|A_0 \cup \cdots \cup A_m| \leq n+k$. Then, clearly K is a simplicial complex, and what is interesting is that this simplicial complex K is exactly the neighborhood complex $N(GK_{n,k})$ of the Kneser graph $GK_{n,k}$. To see this observe that the vertices $A_0, A_1, \cdots, A_m$ of a m-simplex of K have a common neighborhood viz., the complement of the union $\cup_0^m A_i$ and hence forms a simplex in $N(GK_{n,k})$. The converse is also easily see to be true. Hence we find that $K = N(GK_{n,k})$.

Now we will prove that the space $|K|$ of the simplicial complex K is a $(k-1)$-connected. By the simplicial approximation theorem, it suffices to show that any simplicial map $f : \mathbb{S}^r \to |K|$ is null-homotopic, $\forall r \leq k-1$. In other words, we must show that $(k-1)$ -skeleton K_0 of K is null-homotopic in $|K|$. This will be proved by induction on numbers $|S|$, where $n+k \leq |S| \leq 2n+k$. When $|S| = n+k$, the simplicial complex K is indeed a simplex and the result is trivial.

Now assume that $|S| > n+k$. Let us introduce another simplicial complex K' as follows: (i) vertices are exactly the vertices of K (ii) the set $(A_0, \cdots, A_m)$ is a simplex of K' iff $|A_0 \cup \cdots \cup A_m| < n+k$. Then $|K'|$ is a closed subcomplex of $|K|$.

Claim 1. The simplicial complex $|K_0|$ can be deformed into $|K'|$ in $|K|$. To prove this we will define a simplicial map $\psi : |K_0| \to |K'|$ such that for any simplex $A = (A_0, \cdots, A_m)$ in K_0 $\psi(A)$ is contained in the simplex of K' spanned by all n-subsets of $A_0 \cup \cdots \cup A_m$.
This means there is a straight line homotopy in $|K|$ joining the points of $|A|$ with those $|\psi(A)|$. The definition of ψ is by induction on $\dim(A)$. If $\dim(A) = 0$, then A is vertex of K, we put $\psi(A) = A$, and $\psi(A) \in K'$. Assume now that $\dim(A) > 0$ and ψ is defined on the boundary $b(A)$ complex of the complex $Cl(A)$ such that $\psi(A) \in K'$. Now consider the subcomplex K'_A induces by K' on the set of vertices of A. Clearly, $\psi(A)$ lies in K'_A by induction and K'_A is $(k-1)-$ connected by the original induction hypothesis. Hence this map can be continuously extended to the whole of $Cl(A)$ and the required condition remains valid. This proves our claim that $|K_0|$ can be deformed into $|K'|$ in $|K|$.

Claim 2. The simplicial complex $|K'|$ is null homotopic in $|K|$.

Let $u, v \in S$. We define a map $\phi_{u,v} : K' \to K'$. If X is a vertex of K' we put

$$\phi_{uv}(X) = \begin{cases} (X - \{u\}) \cup \{v\} & \text{if } u \in X, v \notin X \\ X, & \text{otherwise.} \end{cases}$$

Then we observe that ϕ is a simplicial : If $A = (A_0, \cdots, A_m)$ is a simplex in K', then $\phi_{uv}(A) = (\phi_{uv}(A_0), \cdots, \phi_{uv}(A_m))$ is also a simplex in K'. This is so because if $u \notin A_0 \cup \cdots \cup A_m$ or $v \in A_0 \cup \cdots \cup A_m$, then

$$\phi_{uv}(A_0), \cdots, \phi_{uv}(A_m) \subseteq A_0 \cup \cdots \cup A_m,$$

and if $u \in A_0 \cup \cdots \cup A_m$ and $v \notin A_0 \cup \cdots \cup A_m$, then

$$\phi_{uv}(A_0), \cdots, \phi_{uv}(A_m) \subset (A_0 \cup \cdots \cup A_m - \{u\}) \cup \{v\}$$

In both cases

$$|\phi_{uv}(A_0), \cdots, \phi_{uv}(A_m)| \leq |A_0 \cup \cdots \cup A_m| \leq n + k - 1$$

Since $\bar{A} \cup \phi_{uv}(\bar{A})$ is contained in the simplex spanned by all n-sets of $A_0 \cup \cdots \cup A_m \cup \{v\}$, there is a straight line homotopy joining points of $\bar{A}$ to the points of $\phi_{uv}(\bar{A})$. Since S is a finite set, u, v can be allowed to vary over all pairs in S so that we have a finite composition

$$\phi_{u_p u_1} \phi_{u_p u_2} \cdots \phi_{u_p u_{p-1}} \phi_{u_{p-1} u_1} \cdots \phi_{u_2 u_1}$$

which maps every vertex X of K' into the vertex $\{u_1, \cdots u_n\}$ of K. This implies that the inclusion map $K' \to K$ is homotopic to a constant map i.e., $|K'|$ is null-homotopic.

Since $|K_0|$ can be deformed into $|K'|$ and $|K'|$ is null-homotopic in $|K|$, we find that $|K_0|$ is null-homotopic in K. Thus $K = N(GK_{n,k})$ is $(k-1)$-connected. □

The most fundamental result proved by Lovaz is the following:

Theorem 4.2. *Let G be any graph. If the neighborhood graph $N(G)$ of G is $(k-1)$-connected, then G is not $(k+1)$-colorable.*

Proof. Let $G = (V, E)$ be a finite graph. Let 2^V denote the collection of all subsets of V. Define a map $\nu : 2^V \to 2^V$ as follows:

$$\nu(A) = \text{the set of all common neighbours of all elements of } A$$

$$= \{v \in V | \, v(a) \in E(G), \forall a \in A\}$$

Let $N_1(G)$ denote the barycentric subdivision of the neighbourhood graph $N(G)$ of G. Then the above map induces a map $\nu : N_1(G) \to N_1(G)$ as follows: Any vertex of $N_1(G)$, which is a simplex of $N(G)$, is just an element $A = \{a_1, \cdots, a_p\}$ of 2^V. Then $\nu(A)$ is also in 2^V. Note that $\nu(A)$ will have at least one common neighborhood, say v_1. There may be more, say, $v_2, \cdots, v_p$. The $\{v_1, \cdots, v_p\}$ has a common neighbors, viz., every element of A. The $\{v_1, \cdots, v_p\}$ is a simplex of $N(G)$ and hence is a point of $N_1(G)$. Furthermore, ν reverses inclusions i.e., $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_p$ implies $\nu(A_1) \supseteq \nu(A_2) \supseteq \cdots \supseteq \nu(A_p)$ This means $\nu : N_1(G) \to N_1(G)$ is indeed a simplicial map. Note that $\nu(\nu(A)) \supseteq A$ and the containment may be proper, *e.g.*,

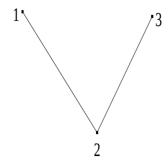

Figure 4

$$\begin{aligned} \nu(1) &= \{2\} \\ \nu(2) &= \{1,3\} \\ \nu^2(1) &\supsetneq (1) \end{aligned}$$

Now we select the set $L(G)$ of those points of $N_1(G)$ which are fixed under the map ν, *i.e.*,

$$L(G) = \{A \in N_1(G) \mid \nu^2(A) = A\}.$$

Then $L(G)$ is a subcomplex of $N_1(G)$. In the simple case of a complete graph on 3 vertices, let us see the complexes $N(G), N_1(G), L(G)$ (see Fig. 6, Fig. 7)

Let us take a complete graph G on 3 vertices.Then $G = N(G)$.(Fig. 5)

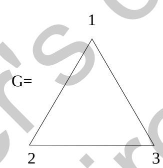

Figure 5

The barycentric subdivision $N_1(G)$ of $N(G)$ is given in Fig. 6.

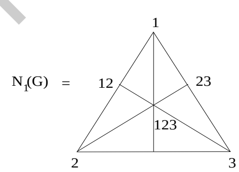

Figure 6

Hence $||L(G)|| \simeq \mathbb{S}^1$. In fact, this is true in general i.e., If $G = K_{k+1}$ is the complete graph on $k+1$ vertices, then $G = N(G)$, $||N_1(G)|| = ||N(G)||$ and $||L(G)|| \simeq \mathbb{S}^{k-1}$.

L(G)=

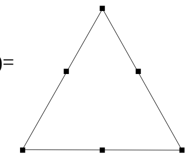

Figure 7

Also $\nu : L(G) \to L(G)$ is a free involution on $L(G)$ i.e., $\nu^2 =$ Identity on $L(G)$. Hence $(|L(G)|, \nu)$ is an antipodality space and we can talk of $\mathbb{Z}_2$-index $\mathrm{ind}_{\mathbb{Z}_2}|L(G)|$. We have

Proposition 4.1. *$|L(G)|$ is a deformation retract of $|N_1(G)| = |N(G)|$.* □

Thus our assumption that $N(G)$ is $(k-1)$-connected implies that $|L(G)|$ is $(k-1)$-connected. This means ([12, Prop. 5.3 (v) p. 96])

$$\mathrm{Ind}_{\mathbb{Z}_2}|L(G)| \geq k.$$

To prove Theorem 4.2, let us suppose that G admits a $(k+1)$-colouring. This means there is a graph homomorphism

$$f : G \to K_{k+1}$$

Here K_{k+1} denotes the complete graph on $k+1$ vertices. Then we can now show that f induces a $\mathbb{Z}_2$-map $|L(G)| \to |L(K_{k+1})|$. But $|L(K_{k+1})| \approx \mathbb{S}^{k-1}$ (see Example 4.1). This implies $\mathrm{Ind}_{\mathbb{Z}_2}|L(G)| \leq k-1$, a contradiction. Hence G can not be $k+1$ colorable. □.

Remark 4.1. As proved above, the assumption that G admits a $(k+1)$-colouring implies that we have a continuous map $|L(G)| \to \mathbb{S}^{k-1}$ where $|L(G)|$ is a $(k-1)$-connected antipodality space. This violates the Borsuk-Ulam Theorem 3.2 (antipodality version) and proves the Lov*á*sz's Theorem directly. We don't have to introduce the concept of $\mathbb{Z}_2$-index as in the above proof. We did this simply because the index Theorem can be generalized to p-groups for any prime p that is used in proving further generalizations of Kneser conjecture.

In the year 1973 Paul Erdös proposed the following generalization of the Kneser conjecture:

Let us take a set X having $(tn + (t-1)k)$ elements. When $t = 2$, the set has $2n + k$ elements, when $t = 3$, the set X has $3n + 2k$ elements, and so on. Then we can decompose the set X into $k+1$ disjoint classes such that at least one class has "t pairwise disjoint sets" instead of "a pairwise disjoint sets"? This generalized conjecture has also been proved by Alon, Frankl and Lov*á*sz in 1986 (see [1]) as follows :

Theorem 4.3. *Let $n, t \geq 1$ and $k \geq 0$. Suppose a set X has $(tn + (t-1)k)$ elements and the set Σ_n of its n-sets are decomposed into $k+1$ disjoint classes. Then, there is a class which contains "t pairwise disjoint n-sets".*

The above theorem was further generalized by K. S. Sarkaria by using the topological method of "deleted join" discovered by himself in [14]. Let us see what is this generalization. A collection K of n-subsets is said to be pairwise disjoint if the intersection of any two members of K is empty. More generally, suppose $j > 1$ is a positive integer. Then we say that K is j-wise disjoint if the intersection of any j members of K is empty, "pairwise" is now "2-wise". Now we have

Theorem 4.4. *(Sarkaria) Let X be a set having N elements such that $N(j-1)-1 \geq k(p-1)+p(n-1)$. If X is decomposed into $k+2$ disjoint classes of n-subsets of X, then there exists a class having p members whose elements are j-wise disjoint.*

When we take $j = 2$, we get the Erdös conjecture. When $j = p = 2$, we get the Kneser conjecture. Thus the above theorem of Sarkaria includes the above conjectures and much more!

It is remarkable to point out that the proofs of above theorems are all topological and analogous to the Lovász's proof of the Kneser conjecture. They can be found in the given references [1] and [14], where the basic ideas of proofs have been nicely formulated using equivariant topology.

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RECENT ADVANCES IN THE THEORY OF HARMONIC UNIVALENT MAPPINGS IN THE PLANE*

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Abstract. Planar harmonic univalent mappings have long been used in the representation of minimal surfaces. Such mappings and related functions have applications in the seemingly diverse fields of Engineering, Physics, Electronics, Medicine, Operational Research, Aerodynamics, and other branches of applied mathematical sciences. That is why; the theory of harmonic univalent mappings has become a very popular research topic in recent years. The aim of this expository article is to present recent advances in the theory of the planar harmonic univalent and related mappings with emphasis on recent results, conjectures and open problems and, in particular, to look at the harmonic analogues of the theory of analytic univalent functions in the unit disk.

1. Introduction

Harmonic mappings or harmonic functions, are critical components in the solutions of numerous physical problems, such as the flow of water through an underground aquifer, steady-state temperature distribution, electrostatic field intensity, the diffusion of, say, salt through a channel. As early as 1920s, the differential geometers used harmonic mappings in the representation of minimal surfaces. For example, the properties of minimal surfaces such as Gauss curvature can be studied through these harmonic mappings. Such mappings and related functions have also applications in the seemingly diverse fields of Engineering, Physics, Electronics, Medicine, Operational Research, Aerodynamics, and other branches of applied

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Key words and phrases: Harmonic mappings or functions, orientation or sense-preserving ha rmonic functions, open problems and conjectures, univalent functions, conformal mappings, geo metric function theory, harmonic Koebe mapping, harmonic starlike functions, harmonic convex functions, convex in one direction, convex in orthogonal directions, connections with Fourier se ries, connections with hypergeometric functions, biharmonic mappings, fully starlike and convex harmonic mappings.

mathematical sciences. For results and some of the related references, one may refer to [40], [48], [49], [53], [55].

Harmonic univalent mappings have attracted the serious attention of complex analysts only recently after the appearance of a basic paper by Clunie and Sheil-Small [32] in 1984. These researchers laid the foundation for the study of harmonic univalent mappings over the unit disk as a generalization of analytic univalent functions. Interestingly, almost at the same time, a famous Bieberbach conjecture posed in 1916 by L. Bieberbach [19] on the size of the moduli of the Taylor coefficients, was affirmatively settled by Louiz de Branges in 1985; see [23].

Although analogues of the classical growth and distortion theorems, covering theorems, and coefficient estimates are known for suitably normalized subclasses of harmonic univalent mappings, still many fundamental questions and conjectures remain unresolved in this area. There is a great expectation that the harmonic Koebe function will play the extremal role in many of these problems, much like the role played by the Koebe function in the classical theory of analytic univalent functions; see for example, [10], [36], [39].

Since there are several survey articles and books ([9], [20], [34], [35], [38]) on harmonic mappings and related areas, we shall make a selection of the results relevant to our precise objective. The purpose of this article is to survey some of the recent advances in the theory of harmonic mappings. We first give basic important definitions, results and classes of harmonic mappings in Section 2. We next give brief survey of the family S; that is, conformal mappings or the theory of analytic, univalent, and normalized functions in the open unit disk. In sections 4 and 5, we briefly survey subclasses of harmonic mappings and outstanding open problems, conjectures and some recent advances in new areas. Section 6 covers different methods of constructing new harmonic mappings. In Sections 7 to 9, we give some recent advances in areas of fully starlike and fully convex harmonic functions, and connections of harmonic mappings with the Fourier series and hypergeometric functions. In Section 10, we briefly survey biharmonic mappings. Since it is not possible to cover all areas of research in harmonic mappings in this short survey, we close this paper with some of the areas of research topics in the theory of harmonic mappings and related areas.

2. Preliminary: definitions and basic results

Let G be a domain of the complex plane $\mathbb{C}$. Twice continuously differentiable function $f(z) = u + iv, z = x + iy$ is called a *harmonic mapping* (or *harmonic function*) of G if it satisfies the Laplace's equation

$$\nabla f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{2.1}$$

Recall that $f = u + iv$ is a harmonic mapping in G if and only if u and v are real harmonic in G; but not necessarily conjugates in G. For example, the function $f(z) = u(x, y) + iv(x, y) = (x^2 - y^2) + i2xy$ is complex-valued harmonic because u and v satisfy Laplace's equation.The following two figures explain uses of harmonic mappings.

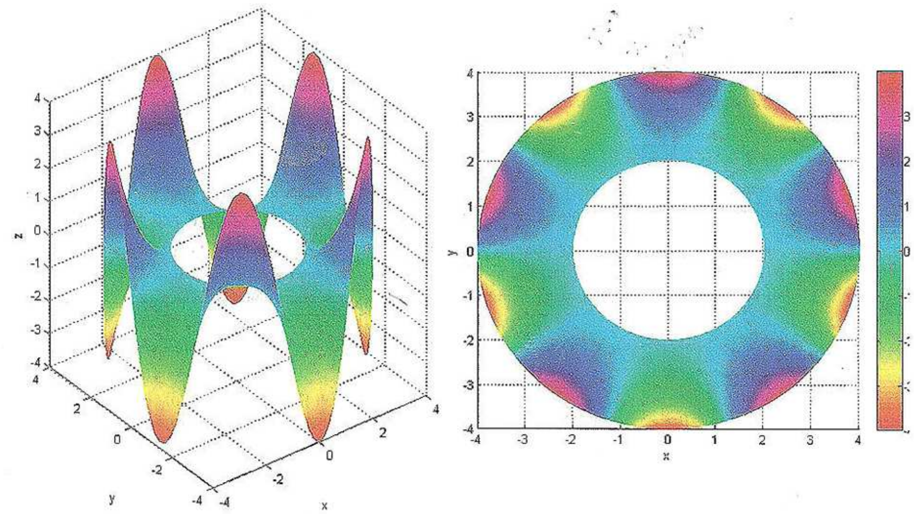

Figure 1. Harmonic map defined on an annulus
http://en.wikipedia.org/wiki/Harmonic _mapping.

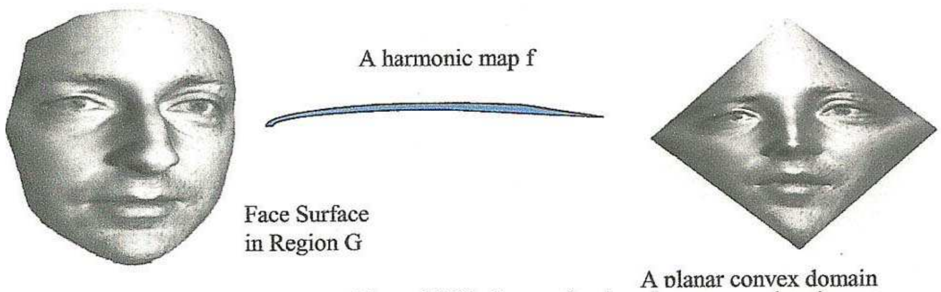

Figure 2 [40]. Face surface in a planar convex domain

Theorem 2.1 ([32], [34]). *If f is a harmonic mapping of a complex region G, then the following equations are equivalent representations of f*

$$\begin{aligned} &(a)\, f(z) = h(z) + \overline{g(z)}, \\ &(b)\, f(z) = Re\{h(z) + g(z)\} + iIm\{h(z) - g(z)\}, \qquad (2.2) \\ &(c)\, f(z) = \{h(z) - g(z)\} + 2Re(g(z)) \end{aligned}$$

where h and g are analytic functions in G.

Remark 2.1. Representations (a), (b) and (c) are not unique.

Remark 2.2. If G is a simply connected domain, then h and g are single-valued analytic functions. On the other hand, if G is not a simply connected domain, then h and g may be multiple-valued functions. Unless otherwise stated, G is a simply connected domain in this paper. In the former case, the function h is called the *analytic part* of f, and g is called the *co-analytic part* of f.

Example 2.1 ([32], [34]). A harmonic mapping

$$f(z) = z + \frac{1}{2}\bar{z}^2 = h(z) + \overline{g(z)}$$

can be written as $f(z) = Re\{h(z) + g(z)\} + iIm\{h(z) - g(z)\}$, where

$$h(z) = z \,\text{and}\, g(z) = \frac{1}{2}z^2.$$

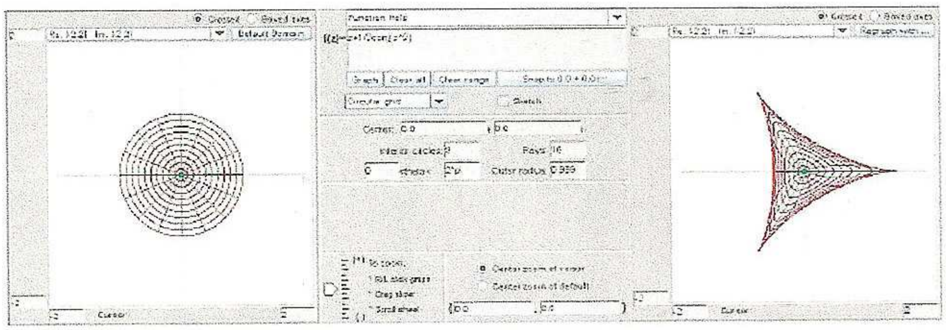

Figure 3: Image of the unit disk under the function in 2.1.

Example 2.2 ([32], [34]). A harmonic mapping

$$L(z) = h(z) + \overline{g(z)} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \frac{\frac{1}{2}\bar{z}^2}{(1-\bar{z})^2} = Re\{\frac{z}{1-z}\} + iIm\{\frac{z}{(1-z)^2}\} \quad (2.3)$$

is a right half-plane mapping; see Figure 4.

A subject of considerable importance in harmonic mappings is the *Jacobian* J_f of a function $f = u + iv$, defined by $J_f(z) = u_x(z)v_y(z) - u_y(z)v_x(z)$. Or, in terms of f_z and $f_{\bar{z}}$, we have

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2 \quad (2.4)$$

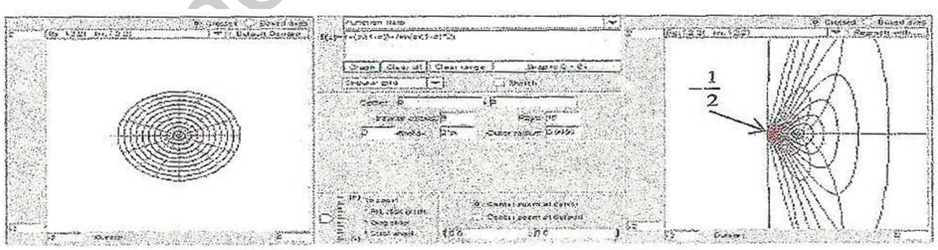

Figure 4: Image of the open unit disk under the function in Example 2.2

When J_f is positive in G, the harmonic mapping f is called *orientation-preserving* or *sense-preserving* in G. An analytic univalent function is a special case of an orientation-preserving harmonic univalent function. For analytic functions h, it is well-known that $J_h(z) \neq 0$ if and only if h is locally univalent at z. For harmonic function we have the following useful result due to Lewy [56].

Theorem 2.2. *A harmonic mapping f is locally univalent at any point z in G if and only if Jacobian $J_f(z)$is non-zero.*

In view of this Lewy's Theorem, harmonic univalent mappings are either *sense-preserving* (or *orientation preserving*) with $J_f(z) > 0$, or sense-reversing with $J_f(z) < 0$ for every point z in G. Clunie and Sheil-Small in 1984 [32] made the following important observation.

Theorem 2.3 ([32])**.** *A function f is locally univalent and sense-preserving in G if and only if $J_f(z) > 0$ in G; equivalently,*

$$|w(z)| = \left|\frac{g'(z)}{h'(z)}\right| < 1, \; or |g'(z)| < |h'(z)| \; in \; G. \tag{2.5}$$

Definition 2.1. The meromorphic function w given by $w(z) = g'(z)/h'(z)$ is called the *second dilatation*(or *dilatation*) of f.

Note that the harmonic mappings with dilatation $w(z) \equiv 0$ are precisely the conformal mappings. More generally, it is easy to see that harmonic mappings with constant dilatation $w(z) \equiv \alpha$ have the form $f = h + \overline{\alpha}\overline{h}$ for some analytic locally univalent function h.

Instead of taking any simply connected domain G in the plane, there is no essential loss of generality in taking the open unit disk $\triangle = \{z : |z| < 1\}$ as the domain of definition. From now onwards, unless it is otherwise stated, we assume that a function f is a harmonic mapping defined on $\triangle \subset \mathbb{C}$ onto a domain Ω, with $\triangle \neq \mathbb{C}$. If $g(0) = 0$, then the representations of f in (2.2) are unique and can be expanded in a series

$$f(re^{i\theta}) = \overline{g}(re^{i\theta}) + h(re^{i\theta}) = \sum_{-\infty}^{\infty} C_n(f) r^{|n|} e^{in\theta} \quad (0 \leq r \leq 1), \tag{2.6}$$

where g and h can be written as power series representations given by

$$g(z) = \sum_{n=-\infty}^{-1} \overline{C_n(f)} z^n, \quad h(z) = \sum_{n=0}^{\infty} C_n(f) z^n.$$

In order to streamline these representations, if we use normalization conditions $C_0(f) = 0$ and $C_1(f) = 1$ then we can write a harmonic mapping f defined on $\triangle$

as

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \tag{2.7}$$

where $a_n = C_n(f)(n \geq 2)$ and $b_n = \overline{C_{-n}(f)}(n \geq 1)$.

Let H be the family of all harmonic mappings of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \triangle, |b_1| < 1 \tag{2.8}$$

are analytic functions in $\triangle$. There are following three main subclasses of the family H

$$S_H = \{f \in H : f \text{ is univalent and sense-preserving in} \triangle\} \tag{2.9}$$

$$S_H^0 = \{f : f \in S_H \text{ and } f_{\overline{z}} = b_1 = 0\} \tag{2.10}$$

$$\begin{aligned} S &= \{f : f = h + \overline{g} \in S_H \,\text{and}\, g(z) = 0\} \\ &= \{h : h \text{ is analytic, univalent, } h(0) = 0, \text{ and} \, h'(0) = 1 \,\text{in}\, \Delta\} \end{aligned} \tag{2.11}$$

The class S is well-known for over 100 years. There are thousands of research articles, surveys, and monographs on family S and related to family S. It is also well-known that the family S_H is a natural generalization of the family S. Though the subclasses S_H^0 and S are compact, but S_H is not compact [32]. The first two classes were discovered by Clunie and Sheil-Small in 1984 [32]. Since $S \subset S_H^0 \subset S_H \subset H$, these inequalities raise the following research problems:

(a) What properties of S are true for the families S_H^0 and S_H?
(b) What concepts and results for S be extended to S_H^0 and S_H?
(c) What properties of the families S_H^0 and S_H don't hold in S?

Since a harmonic function may not be analytic, we get serious challenges, open problems, conjectures, and exciting results in the Theory of Harmonic Mappings.

3. Brief story of the family S (1907 – 1984)

The theory of univalent functions is largely related to family S. It is well-known that S is a compact subset of the locally convex linear topological space of all analytic normalized functions defined on Δ with respect to the topology of uniform convergence on compact subsets of Δ. The Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, z \in \Delta. \tag{3.1}$$

and its rotations are extremal for many problems in S. Note that $k(\Delta)$ is the entire complex plane minus the slit along the negative real axis from $-\infty$ to $-1/4$. For the family S, we have the following powerful and fascinating result which was discovered in 1907 by Koebe [54]:

Theorem 3.1. *There exists a positive constant c such that*

$$\{w : |w| \leq c\} \subset h(\Delta) \quad \forall h \in S.$$

But, this interesting result did not find many applications until Bieberbach [19] in 1916 proved that $c = 1/4$. More precisely, he proved that the open disk $|w| < 1/4$ is always covered by the map of Δ of any function $h \in S$. Interestingly, the one-quarter disk is the largest disk that is contained in $k(\Delta)$, where k is the Koebe function given by (3.1). In the same paper, Bieberbach also observed the following

Conjecture 3.1. **Bieberbach**. If $h \in S$ is any function given by (2.8) , then

$$|a_n| \leq n, \quad \forall n \geq 2. \tag{3.2}$$

furthermore, $|a_n| = n$ for all n for the Koebe function k defined by (3.1) and its rotations.

Note 3.1. Conjecture 3.1 may now be called as a special case of *de Branges Theorem* after the name of L. de Branges who settled this conjecture in 1984 and published paper in 1985 [23].

Failure to settle the Bieberbach conjecture until 1984 led to the introduction and investigation of several subclasses of S. An important subclass of S, denoted by S^*, consists of the functions that map Δ onto a domain *starshaped with respect to the origin*. Another important subclass of S is the family K of *convex functions* that map Δ onto a convex domain. Furthermore, a function h, analytic in Δ, is said to be *close-to-convex* in $\Delta, h \in C$, if $h(\Delta)$ is a close-to-convex domain; that is, if the complement of $h(\Delta)$ can be written as a union of non-crossing half-lines. It is well-known that $K \subset S^* \subset C \subset S$. We remark that various subclasses of these classes have been studied by many researchers including the author; see for example in [10], [36], [39].

Various attempts to prove or disprove the Bieberbach conjecture gave rise to eight major conjectures which are related to each other by a chain of implications; see for example, [10], [36], and [39]. Many powerful new methods were developed and a large number of related problems were generated in attempts to prove these conjectures, which were finally settled in mid 1984 by Louis de Branges [23]. For a historical development of the Bieberbach Conjecture and its implications on univalent function theory, one may refer to the survey by the author in [10], and [11].

4. Subclasses of harmonic univalent mappings

Analogous to well-known subclasses of the family S, there are several subclasses of the families S_H and S_H^0; see, for example, [9], [22], [32], [34], [35], [64], [70],

[75], [78]. A sense-preserving (or orientation-preserving) harmonic mapping $f \in S_H (f \in S_H^0)$ is in the class S_H^* (S_H^* respectively) if the range $f(\Delta)$ is starlike with respect to the origin. A function $f \in S_H^*$ (or $f \in S_H^{*0}$) is called a *harmonic starlike mapping* in Δ. Likewise a function f defined in Δ belongs to the class $K_H(K_H^0)$ if $f \in S_H$(or $f \in S_H^0$ respectively) and if $f(\Delta)$ is a *convex domain.* A function $f \in K_H$(or $f \in K_H^0$)is called a *harmonic convex* in Δ. See [9], [32], [35], and [36] for definitions, analytical conditions and results for these classes.

Similar to the subclass C of S, let C_H and C_H^0 denote the subsets, respectively, of S_H and S_H^0 such that for any $f \in C_H$ or $C_H^0, f(\Delta)$ is a *close-to-convex* domain. Recall that a domain G is close-to-convex if the complement of G can be written as a union of non-crossing half-lines. There are several papers on C_H and its subclasses in last few years; for example, see [22], [34], [64], [70], [75], and [78].

Comparable to the positive order $\alpha(0 \le \alpha < 1)$ in the subclasses S^* and K of S, Jahangiri [50] and [51] defined and denoted the corresponding subclasses of the functions which are *harmonic starlike of order* α and *harmonic convex of order* α, respectively, by $S_H^*(\alpha)$ and $K_H(\alpha)$. Note that $S_H^*(0) = S_H^*$ and $K_H(0) = K_H$. Also, note that whenever the co-analytic parts of each $f = h + \overline{g}$, that is g, is zero, then $S_H^*(\alpha) \equiv S^*(\alpha)$ and $K_H(\alpha) = K(\alpha)$, where $S^*(\alpha)$ and $K(\alpha)$ are the subclasses of the family S which consist of functions, respectively, of *starlike of order* α and *convex of order* α. See, for example, [10].

A domain G is *convex in the direction* θ if every line parallel to the line $z = te^{i\theta}$ has either connected or empty intersection with G. If $\theta = 0$, the domain is convex in the *horizontal direction or real axis*; denote it by CHD. On the other hand, if $\theta = \pi/2$, the domain is *convex in a vertical direction or imaginary axis*; denote it by VHD.

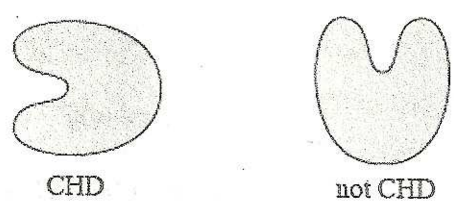

Figure 5: CHD and VHD

We next recall analogous concept of convolution for family S in case of harmonic mappings.

Definition 4.1. The convolution of two complex-valued harmonic functions

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{i_n} z^n + \overline{\sum_{n=1}^{\infty} b_{i_n} z^n}, |b_{i_n}| < 1, \quad i = 1, 2 \tag{4.1}$$

is given by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1_n} a_{2_n} z^n + \overline{\sum_{n=1}^{\infty} b_{1_n} b_{2_n} z^n}. \tag{4.2}$$

This convolution formula reduces to the famous Hadamard product defined for family S if the co-analytic parts of f_1 and f_2 are zero. Many researchers have recently introduced and studied several subclasses of the families of H; for example, see [4], [5], [6], [7], [8], [18], [22], [25], [41], [52], [64], [65], [66].

5. Old and new open problrms and conjectures

Before we survey some old and new research problems and conjectures, let us first recall analytic Koebe function and analogous harmonic Koebe mapping. Recall that the Koebe function given by (3.1) is univalent in Δ, its range is convex in the horizontal direction, $k \in S$, and it maps the unit disk onto the entire plane minus the real interval $(-\infty, -1/4]$. However, the *harmonic Koebe function* k_0 given by

$$\begin{aligned} k_0(z) &= Re\left(\frac{z + (1/3)\, z^3}{(1-z)^3}\right) + iIm\left(\frac{z}{(1-z)^2}\right) \\ &= \frac{z - (1/2)z^2 + (1/6)z^3}{(1-z)^3} + \overline{\frac{(1/2)z^2 + (1/6)z^3}{(1-z)^3}} \end{aligned} \tag{5.1}$$

is univalent and sense-preserving in Δ and, in fact, $k_0 \in S_H^0$. The function k_0 maps the unit disk onto the entire complex plane minus the real interval $(-\infty, -1/6]$. For detailed study of the harmonic Koebe mapping k_0, one may visit [35, Section 5.3] and [32].

Unlike for the family S, there is no overall positive lower bound for $|f(z)|$ depending on $|z|$ when $f \in S_H$. It is so because, for example, $z + \varepsilon\overline{z} \in S_H$ for all ε with $|\varepsilon| < 1$. However, using an extremal length method, Clunie and Sheil-Small [32] discovered the following interesting result analogous to the distortion property for functions in the family S.

Theorem 5.1. *If $f \in S_H^0$, then*

$$|f(z)| \geq \frac{|z|}{4(1+|z|)^2} \quad (z \in \Delta).$$

In particular,

$$\{w \in \mathbb{C} : |w| < \frac{1}{16}\} \subset f(\Delta) \quad \forall\, f \in S_H^0. \tag{5.2}$$

The result in Theorem 5.1 is non-sharp.

Harmonic Koebe function given by (5.1) suggests that the 1/16 radius in (5.2) may be improved to 1/6.

Conjecture 5.1. [32] $\{w \in \mathbb{C} : |w| < \frac{1}{6}\} \subset f(\Delta) \quad \forall\, f \in S_H^0$.

This conjecture is true for close-to-convex functions in C_H^0(see [21]). Clunie and Sheil-Small [32] posed the following *Harmonic Bieberbach* Conjecture for the family S_H^0:

Conjecture 5.2. If $f = h + \overline{g} \in S_H^0$ is given by (2.8), then

$$\begin{aligned} &||a_n| - |b_n|| \le n \\ &|a_n| \le \frac{(2n+1)(n+1)}{6},\ |b_n| \le \frac{(2n-1)(n-1)}{6}, \quad (n = 2, 3, \cdots). \end{aligned} \tag{5.3}$$

Equality occurs for $f = k_0$ given by (5.1).

For $f = h + \overline{g} \in S_H^0$, we have $|g'| \le |h'|(z \in \Delta)$. In particular, it follows that $|b_2| \le 1/2$. Conjecture 5.2 was proved for the functions in the class S_H^{*0}, and when $f(\Delta)$ is *convex in one direction* ([32], [75]). The results also hold if all the coefficients of f in S_H^{*0} are real [32]. It was proved in [78] that this conjecture is also true for $f \in C_H^0$. Later on, Sheil-Small [75] proposed the following harmonic analogue of the Bieberbach conjecture.

Conjecture 5.3. If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n + \overline{\sum\limits_{n=1}^{\infty} a_{-n} z^n} \in S_H$, then

$$|a_n| < \frac{2n^2+1}{3} \quad (|n| = 2, 3, \cdots). \tag{5.4}$$

Open Problem 5.1 [32]. For $f \in S_H^0$, find the best possible bound for $|a_2|$. It is conjectured that $|a_2| \le 5/2$.

In [32], it was discovered that $|a_2(f)| < 12,173, \forall\, f \in S_H$. This result was improved to $|a_2(f)| < 57.05, \forall \in S_H^0$ in [75]. These bounds were further improved to $|a_2(f)| < 49$ for S_H^0 in [35]. On the other hand, Conjecture 5.3 was proved for the class $\tilde{C}_H$. where $\tilde{C}_H$ denotes the closure of C_H [32]. The researchers in [78] improved Conjecture 5.2 and Conjecture 5.3 for $f \in C_H$ and $f \in C_H^0$. They also proposed to rewrite the following harmonic analogue of the Bieberbach conjecture.

Conjecture 5.4. If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n + \overline{\sum\limits_{n=1}^{\infty} b_n z^n} \in S_H$, then

(a) $\quad ||a_n| - |b_n|| \le (1 + |b_1|)n \quad (n = 2, 3, \cdots)$

(b) $\quad |a_n| \le \dfrac{(n+1)(2n+1)}{6} + |b_1| \le \dfrac{(n-1)(2n-1)}{6} \quad (n = 2, 3, \cdots)$

(c) $\quad |b_n| \le \dfrac{(n-1)(2n-1)}{6} + |b_1| \le \dfrac{(n+1)(2n+1)}{6} \quad (n = 2, 3, \cdots)$

Since $|b_1| < 1$ the above conjecture may be rewritten as

Conjecture 5.5. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \in S_H$, then

$$(a) \quad \|a_n| - |b_n\| < 2n \quad (n = 2, 3, \cdots)$$

$$(b) \quad |a_n| < \frac{2n^2+1}{3} \quad (n = 2, 3, \cdots)$$

$$(c) \quad |b_n| < \frac{2n^2+1}{3} \quad (n = 2, 3, \cdots)$$

Mocanu in [62] posed the following conjecture

Conjecture 5.6. If h and g are analytic functions on Δ, with $h'(0) \neq 0$, that satisfy

$$g'(z) = zh'(z) \tag{5.5}$$

that is, dilation $w = z$, and

$$Re\{1 + z\frac{h''(z)}{h'(z)}\} > -\frac{1}{2} \tag{5.6}$$

for all $z \in \Delta$, then the harmonic mapping $f = h + \overline{g}$ is univalent in Δ.

Recently, Bshouty and Lyzzaik [22] proved the conjecture by establishing the following stronger result.

Theorem 5.2. *If $f = h + \overline{g}$ is a harmonic mapping of Δ, with $h' \neq (0)$ that satisfies (5.5) and (5.6) for all $z \in \Delta$, then $f \in C_H$.*

In [22], it is shown by an example that the inequality (5.6) is best possible. These authors also offered the following
Open problem 5.2. Find sufficient conditions on h that would lead to the univalence of harmonic mappings $f = h + \overline{g}$ of Δ that satisfies the condition (5.5).

Next we note that a function

$$L(z) = h(z) + \overline{g(z)} = \frac{z - (1/2)\,z^2}{(1-z)^2} + \left(\overline{\frac{-(1/2)z^2}{(1-z)^2}}\right) = Re\left(\frac{z}{1-z}\right) + iIm\left(\frac{z}{(1-z)^2}\right) \tag{5.7}$$

is in K_H^0 and it maps Δ onto the half plane; see [35]. Moreover, parallel to a well-known coefficient bound theorem of family S, we have

Theorem 5.3. [32]. *If $f \in K_H^0$, then for $n = 1, 2, ...$ we have*

$$\|a_n| - |b_n\| \leq 1, |a_n| \leq \frac{(n+1)}{2}, |b_n| \leq \frac{(n-1)}{2},$$

The results are sharp for the function $f = L$ given by (5.7).

In view of the sharp coefficient bounds given for functions in K_H^0 in Theorem 5.3, we may take $f_1, f_2 \in K_H^0$ and define $f_1 * f_2$ by (4.2). Clunie and Sheil-Small [32] showed that if $\varphi \in K$ and $f = h + \overline{g} \in K_H$, then $f * (\varphi + a\overline{\varphi}) = h * \varphi + a\overline{g * \varphi}, |a| \leq 1,$

is a univalent mapping of Δ onto a close-to-convex domain. They raised the following

Open Problem 5.3 Which complex-valued harmonic functions φ have the property that $\varphi * f \in K_H \; \forall f \in K_H$?

In [41], the researcher constructed some examples in which the property of convexity is preserved for convolution of certain convex harmonic mappings. On the other hand, the researchers in [42] obtained integral means of extreme points of the closures of univalent harmonic mappings onto the right half plane $\{w : Re w > -1/2\}$ and onto the one-slit plane $\mathbb{C}\backslash(-\infty, a], a < 0$.

It is of interest to determine the largest disk $|z| < r$ in which all the members of one family possess properties of those in another. For example, all functions in K_H are convex in $|z| < \sqrt{2} - 1$; that is, the *radius of convexity* of f in K_H is $\sqrt{2} - 1$; see [71]. It is known [71] that $\{w : |w| < 1/2\} \subset f(\Delta) \; \forall f \in K_H^0$. It is also a known fact [75] that $f \in C_H$, then f is convex for $|z| < 3 - \sqrt{8}$. However, analogous to the radius problem for the family S and its subclasses, nothing much is known for S_H, S_H^0 and some of their subclasses. For example

Open Problem 5.4 [35]. Find the radius of starlikeness for starlike mappings in S_H.

Open Problem 5.5 [35]. Find the radius of convexity for harmonic isomorphism of Δ.

Conjecture 5.7. [73]. If $f = h + \overline{g} \in S_H$, then the radius of univalence of h is $1/\sqrt{3}$.

Sheil-Small [75] conjectured that for $f \in S_H$, the radius of convexity is $3 - \sqrt{8}$. It is known [32] that the radius of convexity for close-to-convex mappings in S_H is $3 - \sqrt{8}$; also see [71].

Michalski [58] introduced and settled coefficient conjectures and obtained other estimates for new subclasses of S_H and S_H^0 that are convex in two orthogonal directions. Motivated by this paper, the author [4] employed the shear construction method and connected S and S_H by using four different subclasses of harmonic mappings that are convex in any two orthogonal directions.

In another research area, harmonic mappings make contact with partial differential equations and the well-developed theory of *quasi-conformal mappings*. Some researchers have made efforts to find an appropriate harmonic analogue of the *Riemann's Mapping Theorem* related to the harmonic univalent mappings; for example, see [35], [44]. Though some forms of extensions of Riemann's mapping theorem do exist if dilatation is restricted or the term onto is interpreted in weaker sense. However, the question of uniqueness has not been fully settled.

Open Problem 5.6 What is the exact analogue of the Riemann Mapping Theorem for harmonic mappings?

Corresponding to the neighborhood problem and duality techniques for the family S, Nezhmetdinov [66] studied problems related to the family S^0_H. For several other recent problems and conjectures in planar harmonic mappings, one may refer to the article by Bshouty and Lyzzaik in [21].

6. Construction methods and advances in harmonic mappings

In this section, we first give four construction methods for constructing new harmonic mappings. First important technique is *'Shear Construction Method'* that depends on the following famous result.

Theorem 6.1. *(see [32]). If $f = h+\overline{g}$ is a harmonic and locally univalent function in Δ then*

(1) $F = h - g \in S$ and $F(\Delta)$ is convex in horizontal (or real) axis if and only if $f = h + \overline{g}$ is univalent and convex in the same direction,

(2) $F = h + g \in S$ and $F(\Delta)$ is convex in the imaginary axis if and only if $f = h + \overline{g}$ is univalent and convex in the same direction.

Observe that the domain is convex if and only if it is convex in *every* direction. Thus a harmonic mapping $f = h + \overline{g}$ has a convex range if and only if the range of every rotation $e^{i\theta}f$ is convex in horizontal direction for $0 \leq \theta < 2\pi$. Therefore, $e^{-i\theta}f$ and $e^{-i\theta}h - e^{i\theta}$ are convex in the direction of the real axis. It, therefore, follows that the function $h - e^{i2\theta}g$ is convex in the direction of the line $te^{i\theta},\ t \in \mathbb{R}$ In view of these observations; Theorem 6.1 gives the following generalized version of shear construction theorem.

Theorem 6.2. [4] [58]. *A harmonic and locally univalent function $f = h + \overline{g}$ is univalent mapping of Δ onto a domain convex in the direction of the line $z = te^{i\theta}, 0 \leq \theta < \pi, t \in \mathbb{R}$, if and only if $h - e^{i2\theta}g$ is a conformal univalent mapping of Δ onto a domain convex in the same direction.*

Theorem 6.1 provides "Shear construction method" to construct new harmonic mappings with prescribed dilatation. The word "shear" means to cut a function; similar to "shearing" a sheep to get its wool. This theorem gives the following steps to construct new harmonic univalent mappings.

Step 1 : Suppose F is an analytic univalent function convex in real direction.

Step 2 : Set $F = h - g$ and suppose w is analytic in Δ with $w = g'/h'$ and $|w| < 1$

Step 3 : Find h and g by solving partial differential equations

$$(i)\, h' - g' = F', \text{ and } (ii)\, wh' - g' = 0$$

with normalization $h(0) = F(0)$ and $g(0) = 0$.

Step 4 : Find harmonic shear function f using the relationship

$$f(z) = h(z) + \overline{g(z)} = h - g + 2Re(g(z)).$$

Then $f \in S_H^0$ by Theorem 6.1.

Remark 6.1. The requirement of a dilatation with $|w(z)| < 1$ in step 2 guarantees the local univalence of the harmonic mapping f constructed from a given analytic univalent mapping. Also, it is easy to show that the Jacaboian J_f of $f = h + \overline{g}$ is positive by the univalence of $h - g$; see [35].

Example 6.1. [32]. Set Koebe function

$$k(z) = \frac{z}{(1-z)^2} = h(z) - g(z) \,\text{and} w(z) = z = g'(z)/h'(z).$$

Recall that the Koebe function k(z) in the family S is convex in real axis. Solve the differential equations

$$\begin{aligned} h'(z) - g'(z) &= \left(\frac{z}{(1-z)^2}\right)' \quad (z \in \Delta) \\ g'(z) - zh'(z) &= 0 \end{aligned}$$

with the boundary conditions $h(0) = g(0) = 0$. Applying the shear construction method, we obtain the harmonic Koebe function k_0 given by (5.1). This function k_0 maps Δ univalently onto the entire complex plane minus the real slit $-\infty < t \leq -1/6$.

Another method of constructing new harmonic mappings was developed recently by Muir [59]. Motivated by this new method, the author, Ravichandran and Nagpal [13] have introduced the following operator

Definition 6.1. If f is a function in family S, then define

$$T_{\lambda p}[f] = \frac{D^\lambda f + pz(D^\lambda f)'}{1+p} + \overline{\frac{D^\lambda f - pz(D^\lambda f)'}{1+p}}, z \in \Delta, p > 0, \lambda \geq 0, \tag{6.1}$$

where

$$(D^\lambda f)(z) = \frac{z}{(1-z)^{\lambda+1}} * f, \quad \lambda \geq 0 \tag{6.2}$$

is the *Ruscheweyh derivative* of f.

The operator $D^\lambda f$ for $\lambda \geq 0$ was studied by the author and Silverman in [15] and [16]. The operator $T_{\lambda,p}$ defined in (6.1) generalizes the following well-known operators

$$T_{0,1}\left[\frac{z}{1-z}\right] = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) + \frac{1}{2}\left(\overline{\frac{z}{1-z} - \frac{z}{(1-z)^2}}\right), \tag{6.3}$$

$$T_{0,p}\left[\frac{z}{1-z}\right] = \frac{1}{p+1}\left(\frac{z}{1-z} + p\frac{z}{(1-z)^2}\right) + \frac{1}{p+1}\left(\overline{\frac{z}{1-z} - p\frac{z}{(1-z)^2}}\right), \tag{6.4}$$

$$T_{0,p}[f] = \frac{1}{p+1}(f + pzf') + \frac{1}{p+1}(\overline{f - pzf'}). \tag{6.5}$$

The operators (6.3), (6.4), (6.5) were studied, respectively, in [32], [60], and [58]. For $p > 0$ and $\lambda \geq 0$, the operator

$$T_p[V_\lambda] = \frac{V_\lambda + pzV'_\lambda}{p+1} + \overline{\frac{V_\lambda - pzV'_\lambda}{p+1}}. \tag{6.6}$$

where

$$V_\lambda(z) = \left(\frac{\lambda z}{\lambda+1}\right) {}_2F_1(1, 1-\lambda; 2+\lambda; -z) \tag{6.7}$$

introduced and studied in [73] appears to have the same structure as the operator $T_{\lambda,p}[f]$ defined by (6.1). In [58], the following problems are raised.

Open problem 6.1. Find a subset of family K so that $f \in K \Rightarrow T_{0,p}[F] \in S_H^*$.

Open problem 6.2. Find a subset of family $M \subset DCP$ so that $f \in M \Rightarrow T_{0,p}[f] \in K(a)$, where DCP is the set of *direction convexity* preserving functions.

We next describe the third method of constructing harmonic mappings that depends on the following result deduced from a theorem in [32].

Corollary 6.1. *If h is analytic convex and $w = g'/h'$is analytic with $|w| < 1$ in Δ, then $f = h + \overline{g} \in S_H$.*

Example 6.2. [34]. Suppose

$$h(z) = z - \frac{1}{4}z^2, w(z) = \frac{g'(z)}{h'(z)} = e^{(z+1)/(z-1)}.$$

It is easy to check that h is convex analytic in Δ. Then

$$g(z) = \int h'(z)w(z)dz = \int (1 - \frac{1}{2}z)e^{(z+1)/(z-1)}dz = \frac{1}{4}(z-1)^2 e^{(z+1)/(z-1)}.$$

Therefore, it follows that

$$f(z) = h(z) + \overline{g(z)} = z - \frac{1}{4}z^2 - \frac{1}{4}(\overline{z} - 1)^2 e^{(\bar{z}+1)/(\bar{z}-1)} \in S_H.$$

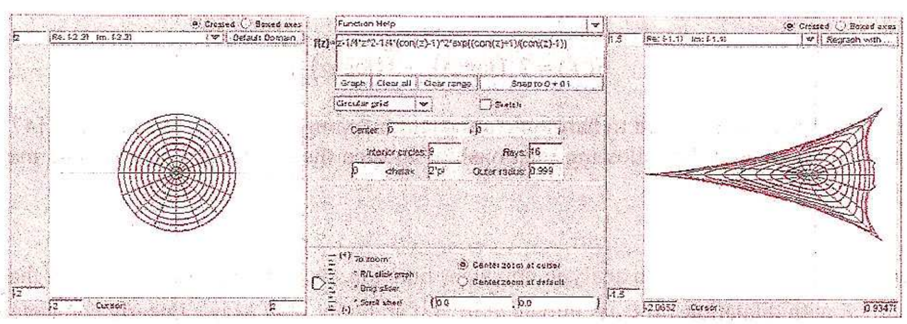

Figure 6: Image of open unit disk under function f in Example 6.2

We next give fourth method of constructing harmonic mappings. This method uses the following theorem by Pommerenke [67].

Lemma 6.1. *Let F be an analytic function in Δ with $F(0) = 0$ and $F'(0) = 1$ and suppose*

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta})(1 + ze^{-i\theta})},\ \theta \in \mathbb{R}. \tag{6.8}$$

If

$$Re\left(\frac{zF'(z)}{\varphi(z)}\right) > 0, \quad \forall\, z \in \Delta, \tag{6.9}$$

then F is convex in the direction of real axis.

Example 6.3 ([34]). Suppose

$$F(z) = h(z) - g(z) = \frac{z}{1-z} + \frac{1}{2}e^{(z+1)/(z-1)}$$

with dilatation $w(z) = e^{(z+1)/(z-1)}$. Then

$$F'(z) = h'(z) - g'(z) = \frac{z}{(1-z)^2}(1 - e^{(z+1)/(z-1)})$$

Using $\theta = \pi$ in (6.8) gives

$$\varphi(z) = \frac{z}{(1-z)^2}.$$

Then

$$Re\left[\frac{zF'(z)}{\varphi(z)}\right] = Re\left[1 - e^{(z+1)/(z-1)}\right] > 0.$$

By Lemma 6.1, $F(z) = h(z) - g(z)$ is convex in the direction of real axis. Shearing h-g with dilatation

$$w(z) = \frac{g'}{h'} = e^{\frac{z+1}{z-1}}$$

and normalizing, we get

$$h(z) = \int \frac{1}{(1-z)^2}dz = \frac{z}{1-z}, g(z) = -\frac{1}{2}e^{(z+1)/(z-1)}.$$

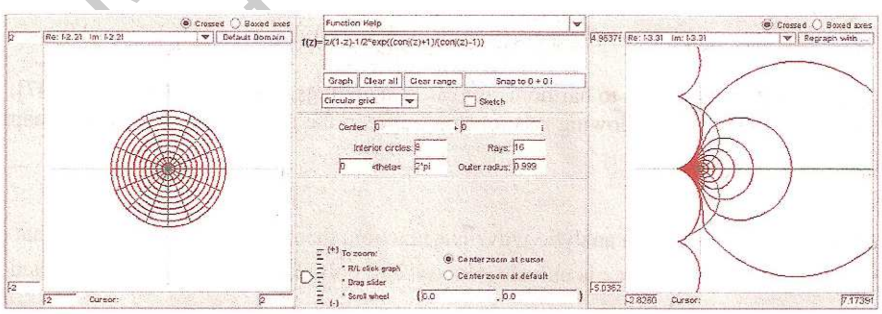

Figure 7: Image of open unit disk under function in Example 6.3

Therefore it follows that

$$f(z) = h(z) + \overline{g(z)} = \frac{z}{1-z} - \frac{1}{2}e^{\frac{\overline{z}+1}{\overline{z}-1}} \in S_H.$$

Note that shearing method does not always help to get closed form of h.

Another current research area of interest is to connect subclasses of harmonic mappings and minimal surfaces; for example, see [70], [79].

It is a common strategy to construct new functions with a given property to take the *linear combination* of two functions with the same property. But, there are examples in the literature to show that that the linear combination of two univalent functions may not be univalent. We first give the following result that determines whether a function f maps onto a domain convex in the direction of the imaginary axis.

Lemma 6.2 ([46]). *Suppose f is analytic and non-constant in Δ. Then*

$$Re\left((1-z^2)f'(z)\right) \geq 0 \ (z \in \Delta) \tag{6.10}$$

if and only if

(A) f is univalent in Δ;

(B) f is convex in the direction of the imaginary axis;

(C) there exists two points z_n' and z_n'' converging to $z = 1$, and $z = -1$, respectively, such that

$$\begin{aligned} \lim_{n\to\infty} Re(f(z_n')) &= \sup_{|z|<1} Re(f(z)), \\ \lim_{n\to\infty} Re(f(z_n'')) &= \sup_{|z|<1} Re(f(z)). \end{aligned} \tag{6.11}$$

Using Lemma 6.2, Dorff [34] proved the following sufficient condition for the linear combination of two harmonic univalent mappings in Δ.

Theorem 6.3. *Let $f_1 = h_1 + \overline{g_1}, f_2 = h_2 + \overline{g_2}$ be harmonic univalent mappings convex in the imaginary direction and $w_1 = w_2$. If f_1, f_2, satisfy the conditions given by (6.11), then $f_3 = tf_1 + (1-t)f_2\,(0 \leq t \leq 1)$ is univalent and convex in the direction of the imaginary axis.*

In [77], researchers derived several sufficient conditions of the linear combinations of harmonic univalent mappings that are univalent and convex in the direction of the real axis. Nagpal and Ravichandran [65] employed a different methodology to construct subclasses of S_H from some subfamilies of S.

There are several concepts and results for harmonic mappings that were recently extended from the corresponding known results for the classical family S; for example, *Schwarzian derivative* of an analytic tool in complex analysis, in general, and for family S, in particular; for example, see [29], [30], [68]. In [29], Chuaqui, Peter, and Osgood offered a definition of Schwarzian derivative that applies more generally to the theory of harmonic mappings. Their definition of the *Schwarzian derivative S(f) of a harmonic mapping f* is given in terms of the metric $ds = \rho|dz|$, for some positive function ρ of the associated minimal surface $K = -\rho^{-2}\nabla(\log\rho)$, where ∇ denotes the Laplacian operator defined in (2.1). These researchers defined and gave the geometric derivation and properties of the formula

$$S(f) = 2\{(\log\,\rho)_{zz} - ((\log\,\rho)_z)^2\} \tag{6.12}$$

For several other areas related to harmonic univalent mappings, one may refer to [45], [47], [59], [60]. We close this section by looking at the following relationship between the univalence of harmonic mapping $f = h + \overline{g}$ and its analytic factor h.

Theorem 6.4 ([31])**.** *If $h : \Delta \to \mathbb{C}$ is an analytic univalent function, then there exists $c > 0$ such that every harmonic mapping $f = h + \overline{g}$ with dilatation $|w| < c$ is univalent in Δ if and only if $h(\Delta)$is a linearly connected domain.*

For $c = 1$ and h convex, we have the following important result.

Corollary 6.2. *If h is analytic and convex in Δ, then every harmonic mapping of the form $f = h + \overline{g}$ with $|g'| < |h'|$ is univalent in Δ.*

In [32], the researchers found the conditions under which the harmonic mappings $F = h + e^{i\theta}\overline{g}$ remain univalent for all $\theta \in [0, 2\pi]$.

7. Fully starlike and fully convex harmonic mappings

Let$\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ for each $0 < r < 1$. If f is in family S and $f(\Delta)$ is a convex (starlike) domain, then for each $0 < r < 1, f(\Delta)$ is also convex (respectively, starlike) domain. But, the next example shows that the hereditary property of convex analytic mappings does not generalize to harmonic mappings.

Example 7.1 ([35]). The harmonic half-plane mapping L defined by (5.7) is a mapping of Δ onto the half-plane $Re\{w\} > -1/2$. But, $L(\Delta_r)$ is not convex for every r in the interval $(\sqrt{2} - 1, 1)$ Similarly, starlikeness is also not a hereditary property for harmonic mappings.

Motivated by Example 7.1, Chuaqui, Duren and Osgood [29] introduced the notions of *fully starlike harmonic* and *fully convex harmonic mappings* that do inherit the properties of starlikeness and convexity respectively. The positive order $\alpha(0 \le \alpha < 1)$ was studied by Nagpal and Ravichandran [63].

Definition 7.1 ([63])**.** A harmonic mapping f of Δ with f(0)=0 is called *fully starlike* of order $\alpha(0 \le \alpha < 1)$ if it maps every circle $|z| = r < 1$ in one-to-one manner onto a curve that bounds a domain starlike with respect to the origin satisfying

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) > \alpha,\ 0 \le \theta < 2\pi,\ 0 < r < 1. \tag{7.1}$$

If $\alpha = 0$, then f is fully *fully harmonic starlike*, see [29].

Definition 7.2 ([29]). A harmonic mapping f of Δ is called *full convexity* of order α $(0 \le \alpha < 1)$ if it maps every circle $|z| = r < 1$ in one-to-one manner onto a convex curve satisfying

$$\frac{\partial}{\partial\theta}(\arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)) > \alpha,\ 0 \le \theta < 2\pi,\ 0 < r < 1. \tag{7.2}$$

If $\alpha = 0$, then f is *fully harmonic convex*; see [29].

Remark 7.1. In particular, $f(z) \neq 0$ for $0 < |z| < 1$.

The following results give characterizations of fully harmonic convex and fully harmonic starlike functions in Δ.

Theorem 7.1 ([29]). *Let $f(z) = h(z) + \overline{g(z)}$ be a locally univalent orientation preserving harmonic mapping of Δ. Then f is fully convex if and only if*

$$|zh'(z)|^2 Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} \ge$$
$$|zg'(z)|^2 Re\left\{1 + \frac{zg''(z)}{g'(z)}\right\} + Re\{z^3[h''(z)g'(z) - h'(z)g''(z)]\}$$

for all z in Δ. If $f(0) = 0$, then f is fully starlike if and only if $f(z) \neq 0$ for $0 < |z| < 1$ and

$$|h(z)|^2 Re\left\{\frac{zh'(z)}{h(z)}\right\} \ge |g(z)|^2 Re\left\{\frac{zg'(z)}{g(z)}\right\} + Re\{z[h(z)g'(z) - h'(z)g(z)]\}$$

for all $z \in \Delta$.

The following theorem adapts an observation by the Sheil-Small [75] to the context of fully convex and fully starlike mappings.

Theorem 7.2 ([29]). *Let h, g, H, and G be analytic functions in Δ related by*

$$zH'(z) = h(z), zG'(z) = -g(z),\ z \in \Delta.$$

Then a harmonic mapping $f = h + \overline{g}$ is fully starlike if and only if $F = H + \overline{G}$ is fully convex.

According to the Rado'- Kneser-Choquet theorem (see, [35]), a fully convex harmonic mapping is necessarily univalent in Δ. However; the following example shows that a fully starlike harmonic mapping need not be univalent.

Example 7.2 ([29]). Using Theorem 7.2, it is straight forward to show that the harmonic half-plane mapping L defined by (5.7) has a Jacobian $J(z) > 0$ for $|z| < 2 - \sqrt{3}$ but $J(-2+\sqrt{3})$vanishes. Thus $f = h + \overline{g}$ is not univalent in any disk $|z| < r$ with $r > 2 - \sqrt{3}$. On the other hand, using Theorem 7.2 and function L, it is straightforward analysis to show (see, [29, page 140]) that fully starlike mapping need not be univalent in any disk $|z| < r$ of radius larger than

$$\frac{2-\sqrt{3}}{\sqrt{2}-1} = 0.646\cdots$$

Open Problem 7.1 If $f = h + \overline{g}$ is any fully harmonic starlike (convex) mapping defined in Δ, then verify or get different results for Conjecture 5.4 and Conjecture 5.5.

Nagpal and Ravichandran [63] determined the bounds for the radius of full starlikeness of order $\alpha(0 \le \alpha < 1)$ well as the radius of full convexity of order $\alpha(0 \le \alpha < 1)$ for certain families of univalent harmonic mappings.

Open Problem 7.2 Find the exact radius of starlikness of order $\alpha(0 \le \alpha < 1)$ for S_H^*, K_H and C_H

For $\alpha = 0$, Nagpal and Ravichandran [63] found the following

Theorem 7.3. *Suppose $f = h + \overline{g} \in S_H$. Then*

(1) $f \in K_H \Rightarrow f$ is fully starlike in atleast $|z| < 4\sqrt{2} - 5$
(2) $f \in C_H \Rightarrow f$ is fully starlike in atleast $|z| < 3 - \sqrt{8}$
(3) $f \in S_H^* \Rightarrow f$ is fully starlike in atleast $|z| < \sqrt{2} - 1$

These authors also found many other interesting bounds and radius problems. For example, they observed that Figure 8 is the image of the subdisk $|z| < \sqrt{(7\sqrt{7}-17)/2}$ under the mapping L given by (5.7).

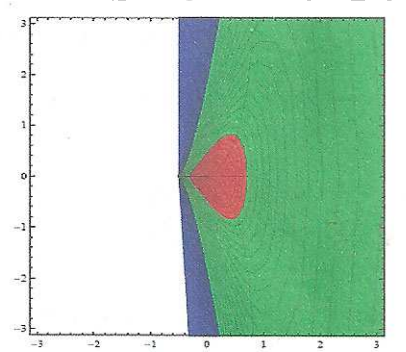

Figure 8: Image of subdisk under mapping L

Conjecture 7.1 ([63]). If $f \in K_H^0$, and $\alpha(0 \le \alpha < 1)$, then f is fully starlike of order $\alpha(0 \le \alpha < 1)$ in $|z| < r_1$ where $r_1 = r_1(\alpha)$ is given by

$$r_1(\alpha) = \begin{cases} \dfrac{\sqrt{1+8\alpha} - (1+2\alpha)}{2\alpha} & \text{if } 0 < \alpha < 1, \\ \sqrt{\dfrac{7\sqrt{7}-17}{2}} & \text{if } \alpha = 0. \end{cases}$$

Conjecture 7.2 ([63]). If $f = K_H^0$ then f is fully convex of order $\alpha (0 \leq \alpha \leq 1)$ in $|z| < r_2$ where $r_2 = r_2(\alpha)$ is the positive root of the equation $p(r, u_0) = 0$ in $(0, 1)$ with

$$p(r, u) = 1-6r^2+r^4+12r^2u^2-4r(1+r^2)u^3-\alpha[1+(2u^2-3)\{4u(1+r^2)-6r\}r+r^4]$$

and

$$u_0 = \frac{r(1+\alpha) - \sqrt{\alpha(1+2\alpha)(1+r^4)+(1+4\alpha+5\alpha^2)r^2}}{(1+r^2)(1+2\alpha)}$$

For $\alpha = 0$,this conjecture was confirmed (see [71]).

8. Connection with Fourier series

In this section, we briefly mention about recent interest in the interplay between Fourier series and harmonic mappings of the open unit disk, with particular emphasis on connections with the topology of curves. Without going in details, we outline steps that lead to step functions on the unit circle and harmonic extension of function f defined on the unit circle to the open disk Δ.

Suppose p and q are any two polynomials and let $f(z) = p(z) + \overline{q(z)}$. Then f is a harmonic function. We may write

$$f(z) = \overline{q(z)} + p(z) = \sum_{n=0}^{\infty} \overline{b_n} \overline{z}^n + \sum_{n=0}^{\infty} a_n z^n.$$

Note that $f(z) = p(z) + \overline{p(z)} = 2Re(p(z))$. Locally, an analytic function p(z) can be uniformly approximated by polynomials. Suppose $\overline{b_n} = c_{-n,} a_n = c_n, c_0 = \overline{b_0} + a_0, z = re^{it}$. Then

$$f(re^{it}) = \sum_{-\infty}^{-1} c_n r^n e^{-int} + \sum_{0}^{\infty} c_n r^n e^{int} = \sum_{-\infty<n<\infty} c_n r^{|n|} e^{int} \qquad (8.1)$$

In fact, (8.1) is what we noticed in (2.6). We assume that the coefficients vanish for large values of n. Take r = 1. Then

$$f(e^{it}) = \sum_{-\infty<n<\infty} c_n z^{int} \qquad (8.2)$$

Right side of (8.2) is the *Fourier series* of $f(e^{it})$ in the complex form. The right side can be expressed in the following Trigonometric polynomial

$$f(e^{it}) = \sum_{-\infty<n<\infty} c_n(\cos nt + i \sin nt) \qquad (8.3)$$

Let $T = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle on a complex plane $\mathbb{C}$. Consider a step function $f : T \to \mathbb{C}$.defined by

$$f(e^{it}) = c_k \qquad (t_{k-1} < t < t_k,\, 1 \leq k < n), \qquad t_0 < t_1 < t_2 < \cdots < t_n = t_0 + 2\pi.$$

The *harmonic extension* of f to Δ is defined by

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\left[Re\frac{1+ze^{-it}}{1-ze^{-it}}\right]f(e^{it})dt \tag{8.4}$$

and takes the form $= h(z) + \overline{g(z)}$. where h and g are analytic in Δ and $g(0) = 0$.

In [74], Sheil-Small explored topological properties of the harmonic extension f defined in (8.4). Connection of harmonic univalent mappings and Fourier series is almost a new research area.

9. Connections with hypergeometric functions

S. Ramanujan's Notebooks and his later work have motivated many researchers to make use of generalized hypergeometric functions in many areas of research. But, it was surprising to discover the use of hypergeometric functions in the proof of Bieberbach conjecture by L. de Branges [23]. This discovery has prompted renewed interests in these classes of functions [61], [72]. However, connections between the theory of harmonic univalent functions and hypergeometric functions are relatively new. But, we first recall some basic definitions and notations for hypergeometric functions.

Let a, b, c be complex numbers with $c \neq 0, -1, -2, -3, \cdots$ Then the Gauss hypergeometric function written as ${}_2F_1(a, b; c; z)$ or simply as $F(a, b; c; z)$ is defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \tag{9.1}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda) = \lambda(\lambda+1)\cdots(\lambda+n_1) \text{ for } n = 1, 2, 3, \cdots, \cdots \text{ and } (\lambda)_0 = 0 \tag{9.2}$$

Since the hypergeometric series in (9.1) converges absolutely in Δ, it follows that $F(a, b; c; z)$ defines a function which is analytic in Δ, provided that c is neither zero nor a negative integer. Recall that in terms of Gamma function, we are led to the well-known Gauss's summation theorem: If $Re(c - a - b) > 0$, then

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c \neq 0, -1, -2, \cdots \tag{9.3}$$

In particular, the incomplete beta function, related to the Gauss hypergeometric function,$\varphi(a, c; z)$, is defined by

$$\varphi(a, c; z) = zF(a, 1; c; z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1},\ z \in \Delta, c \neq 0, -1, -2, \cdots \tag{9.4}$$

It has an analytic continuation to the z-plane cut along the positive real axis from 1 to ∞. Note that $\varphi(a, 1, z) = z/(1-z)^a$. Moreover, $\varphi(2, 1, z) = z/(1-z)^2$ is the Koebe function given by (3.1). The hypergeometric series in (9.4) converges absolutely in Δ and thus $\varphi(a,c;z)$ is analytic function in Δ, provided that c is neither zero nor a negative integer. For further information about hypergeometric functions, one may refer to [76].

Driver and Duren in [37] used shears construction method to construct harmonic shears of regular polygons of order n in terms of hypergeometric functions. For any two normalized hypergeometric functions $zF(a_1,b_1;c_1;z)$ and $zF(a_2,b_2;c_2;z)$, the author in [7] defined Hohlov-type harmonic convolution operator

$$\wedge \equiv \wedge\begin{matrix} a_1,b_1,c_1 \\ a_1,b_1,c_1 \end{matrix} : H \to H$$

by

$$\begin{aligned}\wedge f &= f\tilde{*}\left(zF(a_1,b_1;c_1;z)+\overline{zF(a_2,b_2;c_2;z)}\right)\\ &= h * zF(a_1,b_1;c_1;z)+\overline{g * zF(a_2,b_2;c_2;z)}\end{aligned}$$

for any function $f = h+\overline{g}$ in H. We observe that

$$\wedge\begin{pmatrix} a_1,1,a_1 \\ a_2,1,a_2 \end{pmatrix} f(z) = f(z) = f(z)\tilde{*}\left(\frac{z}{1-z}+\left(\overline{\frac{z}{1-z}}\right)\right)$$

is the identity mapping and

$$\wedge\begin{pmatrix} -m,-m,c \\ -m,-m,c \end{pmatrix} f(z) = f(z)\tilde{*}\left(z+\sum_{n=2}^{m+1}\frac{\left\{(-m)_{n-1}\right\}^2}{(c)_{n-1}(1)_{n-1}}z^n+\sum_{n=2}^{m+1}\frac{\left\{(-m)_{n-1}\right\}^2}{(c)_{n-1}(1)_{n-1}}\overline{z^n}\right)$$

is a polynomial operator. In the same paper, the author also defines harmonic convolution incomplete beta operator $I : H \to H$ by

$$I\begin{pmatrix} b_1,c_1 \\ b_2,c_2 \end{pmatrix} f \equiv L(f) = f^{\tilde{*}}(\varphi_1+\varphi_2) = h*\varphi_1+\overline{g^*\varphi_2}$$

where φ_1 and φ_2 are incomplete beta functions given by

$$\varphi_1(z) = \varphi(b_1,c_1;z) = zF(1,b_1,c_1;z), \varphi_2(z) = \varphi(b_2,c_2;z) = zF(1,b_2,c_2;z).$$

Open Problem 9.1 Under what restrictions on the complex parameters, the harmonic convolution operators $\wedge : H \to H$: and $I : H \to H$ map various subclasses of S_H and S_H^0 respectively, into various subclasses of S_H and S_H^0.

In [7], [8], and [9], the author establishes some important connections between various subclasses of H, S_H^0 or S_H by applying the convolution hypergeometric operators $\wedge$ and I. We next define harmonic integral operator $\prod : H \to H$ by

$$\begin{aligned}\prod(f)(z) &= \prod\begin{bmatrix} a_1,b_1,c_1 \\ a_2,b_2,c_2 \end{bmatrix}(f)(z)\\ &= h(z) * \int_0^z F(a_1,b_1;c_1;t)dt + \overline{\overline{\sigma}g(z) * \int_0^z F(a_2,b_2;c_2;t)dt}\end{aligned}$$

Example 9.1. The operators

$$\prod \begin{bmatrix} a_1, 2, a_1 \\ a_2, 2, a_2 \end{bmatrix} (f)(z) = f(z) = f(z)\tilde{*}\left(\frac{z}{1-z} + \overline{\frac{z}{1-z}}\right)$$

and

$$\prod \begin{bmatrix} -m, -m, c \\ -m, -m, c \end{bmatrix} (f)(z)$$
$$= f(z) = f(z)\tilde{*}\left(z + \sum_{n=2}^{m+1} \frac{\{(-m)_{n-1}\}^2}{(c_1)_{n-1}(1)_n} z^n + \sum_{n=2}^{m+1} \frac{\{(-m)_{n-1}\}^2}{(c_1)_{n-1}(1)_n} \overline{z^n}\right)$$

are examples of harmonic integral operators. The author studied several properties of the operator $\prod$ in [6].

There are several published papers that connect (or use) hypergeometric and other special functions and harmonic univalent (or multivalent) mappings; for example, see [5], [12], [14], [17].

10. Landau's theorem, biharmonic and polyharmonic mappings

Biharmonic mappings have several applications in physical situations in Fluid Dynamics, Elasticity, Engineering and Biology; see [43], [53], and [55]. In Geometric Function Theory, use of biharmonic and polyharmonic mappings is new; there are only a few papers from 2005 to 2013. In [1], Abdulhadi, Abu Muhanna, and Khour introduced and studied biharmonic mappings; also, see [2], [3], [27], [33].

Definition 10.1. A four times continuously differentiable function $F = u + iv$ in domain D is said to be *biharmonic* if the Laplacian of F is harmonic in a domain D. Every biharmonic mapping F in a simply connected domain D has the representation

$$F(z) = |z|^2 G + K \tag{10.1}$$

where G and K are some complex-valued harmonic mappings in D.

Note that harmonic mappings G and K in (10.1) can be expressed as

$$G = g_1 + \overline{g_2}, \qquad K = k_1 + \overline{k_2} \tag{10.2}$$

where g_1, g_2, k_1, k_2 are analytic functions in D. The following example shows that if G is univalent, it does not follow that F is univalent.

Example 10.1 ([1]). Let $G(z) = ze^z + 0.4 + 0.2i$ and $F(z) = r^2 G(z)$ Then G is univalent in $|z| < 0.8$; but F is not univalent.

The researchers in [1] showed that a biharmonic mapping F given by (10.1) is locally univalent in Δ if G is starlike and K is orientation preserving. In [3] these authors extended Landau's Theorem to bounded biharmonic mappings of Δ. Also, see [2], [3], [27], [33].

Ponnusami and Qiao [69] studied polynomial approximation and fully harmonic functions for biharmonic mappings. In another paper [68], they explored the properties of the Schwarzian derivative, integral means, and the affine and linear invariant families of biharmonic mappings. Also, see papers in [27], [80]].

Definition 10.2 ([26])**.** . A $2\,p\,(p \geq 1)$ times continuously differentiable complex-valued function $F = u+iv$ in a domain D is p-harmonic if F satisfies the p-harmonic equation

$$\nabla^p F = \nabla(\nabla^{p-1})F = 0, \tag{10.3}$$

where ∇ represents the complex operator defined in (2.1).

It is also known (see [26], [57]) that a mapping F is *p-harmonic* or *polyharmonic* in a simply connected domain D if and only if F has the representation

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z), \tag{10.4}$$

where each G_{p-k+1} is harmonic, *i.e.*,$\nabla G_{p-k+1}(z) = 0$ for $k \in \{1, 2, \cdots, p\}$. These researchers got several properties of p-harmonic mappings; also, see [28] and [57].

11. Some old and new research areas in theory of harmonic mappings

The theory of harmonic univalent mappings is very vast and applicable branch of mathematics. It is not possible to survey all the recent developments in various research areas in this survey. That is why, many challenging open problems, conjectures, and emerging areas in the theory of harmonic univalent mappings and related functions could not be included in this survey; however, the following is a short list of some old and new research areas.

1. Harmonic mappings and minimal surfaces
2. Coefficient estimates and conjectures
3. Properties of special subclasses of harmonic univalent and related mappings
4. Growth, distortion, and covering theorems
5. Harmonic univalent mappings with non-positive coefficients
6. Meromorphic harmonic mappings
7. Multivalent harmonic mappings
8. Subordination problems in harmonic context
9. Harmonic Polynomials
10. Extremal problems in harmonic context
11. Special functions and harmonic mappings
12. Constructions of new harmonic mappings

13. Multiply connected domains
14. Inner mapping radius in harmonic context
15. Hardy space and integral means in harmonic context
16. Curvature problems and Boundary behavior in harmonic context
17. Inverse problems in harmonic context
18. Using harmonic mappings in Fourier Series or vice versa
19. Connections between harmonic and minimal surfaces
20. Harmonic mappings connected to quasi-conformal mappings
21. Riemann mapping theorem in harmonic context
22. Landau's theorem, Biharmonic and polyharmonic mappings
23. Connections with quasiconformal mappings
24. Connections with the Schwarzian derivatives

In 2007, European Science Foundation Research Program initiated a special project entitled, *"Harmonic and Complex Analysis and its Applications"* (HCAA); for example, see http://org.uib.no/hcaa/ . The main idea of this project is to establish a fruitful cooperation between two scientific communities: analysts with a broad background in Complex and Harmonic Analysis, and Mathematical Physics, and specialists in Physics and Applied Sciences. This project is a multidisciplinary program at the crossroads of mathematics and mathematical physics, mechanics and applications that proposes a set of coordinated actions for advancing in Harmonic and Complex Analysis and for increasing its applications to challenging scientific problems. Particular topics which will be considered by this Program include Conformal and Quasiconformal Mappings, Potential Theory, Banach Spaces of Analytic Functions and their applications to the problems of Fluid Mechanics, Conformal Field Theory, Hamiltonian and Lagrangian Mechanics, and Signal Processing.

12. Conclusion

In this article, we have made an attempt to present recent developments in the theory of harmonic mappings in the plane. We have been compelled to omit a number of related areas and interesting problems. However, we hope that this article may serve as a useful guide for new researchers in the theory of planar harmonic mappings and related areas. This article may also be useful to pure and applied mathematicians working in several diverse areas.

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GENERALIZED WALLIS-EULER PRODUCTS AND NEW INFINITE PRODUCTS FOR π

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Abstract. Wallis discovered in 1655 a marvelous infinite product for Pi. Around eighty years later, Euler developed the product representation for the sine function which generalized Wallis formula. Basing on these, Sondow and Huang derived few such products recently. Their paper motivated the author to discover three general formulae which generate infinite classes of such products and to derive a number of algebraic irrational-free infinite products for π. In the process, he also derives a new product expansion for $\sin \pi x$.

§ 1

Infinite products for Pi entered into mathematics with the following formula given in 1593 by the French mathematician Francois Viete (1540-1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}.\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}.\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots . \tag{1}$$

However, (1) is practically useless for evaluating Pi as it involves computation of increasingly complex algebraic irrationals.

John Wallis (1616-1703), England's most influential mathematician before Newton, discovered, sometime during the first half of 1655, his celebrated infinite product for Pi that appeared in his Arithmetica infinitorum [6]:

$$\frac{4}{\pi} = \frac{3.3}{2.4}.\frac{5.5}{4.6}.\frac{7.7}{6.8}... = \prod_{n=1}^{\infty} \frac{(2n+1)^2}{2n(2n+2)}. \tag{2}$$

Euler (1707-1783) came up with the product expansion of the sine function [1] which can be rewritten for $x \neq \pm n\pi, n \in \mathbb{N}$, as

$$\frac{x}{\sin x} = \frac{\pi^2}{(\pi^2 - x^2)}.\frac{4\pi^2}{(4\pi^2 - x^2)}.\frac{9\pi^2}{(9\pi^2 - x^2)}\cdots . \tag{3}$$

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(3) yields some important numerical product expansions for special values of x. Euler [2] gives three examples: $= \frac{\pi}{2}; \frac{\pi}{4}; \frac{\pi}{6}$. He obtains a product akin to (2)

$$\frac{\pi}{2} = \frac{2.2}{1.3}.\frac{4.4}{3.5}.\frac{6.6}{5.7}... = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}. \tag{4}$$

Euler pointed out at another place [3] that there is no difference, in whatever order the individual factors in this product are set out, provided none are left out. So by taking some number from the beginning, the remainder can be set out in order, just as

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2.4}{3.3}.\frac{4.6}{5.5}.\frac{6.8}{5.5}.\frac{8.10}{5.5}.etc$$
$$\frac{\pi}{2} = \frac{2.4}{1.3} \times \frac{2.6}{3.5}.\frac{4.8}{5.7}.\frac{6.10}{7.9}.\frac{8.12}{9.11}.etc$$
$$\frac{\pi}{2} = \frac{2}{3} \times \frac{2.4}{1.5}.\frac{4.6}{3.7}.\frac{6.8}{5.9}.\frac{8.10}{7.11}.etc$$
$$\frac{\pi}{2} = \frac{2.4}{3.5} \times \frac{2.6}{1.7}.\frac{4.8}{3.9}.\frac{6.10}{5.11}.\frac{8.12}{7.13}.etc$$

Surprisingly, Jonathan Sondow and Huang Yi [5] have given Euler's product for $x = \frac{\pi}{8}$ as their formula (8) and his product for $x = \frac{\pi}{3}$ as their formula (12). Further, what they prove as their general formula (13) is a straightforward case of $x = \frac{\pi}{k}$ deduced from (3). However, they have found a nice product for 2

$$2 = \prod_{n=1}^{\infty} \frac{(8n-6)(8n-4)(8n-4)(8n-2)}{(8n-7)(8n-5)(8n-3)(8n-1)}. \tag{5}$$

Using (5), they thus deduced from (3) a new product akin to (2)

$$\frac{\pi}{4} = \frac{2.6.8.8}{3.5.7.9}.\frac{10.14.16.16}{11.13.15.17}.\frac{18.22.24.24}{19.21.23.25}... = \prod_{n=1}^{\infty} \frac{(8n-6)(8n-2)(8n)(8n)}{(8n-5)(8n-3)(8n-1)(8n+1)}. \tag{6}$$

If we divide (6) by (5), we get

$$\frac{\pi}{8} = \prod_{n=1}^{\infty} \frac{(2n)^2(8n-7)}{(2n-1)^2(8n+1)}. \tag{7}$$

(7) suggests

General Formula I:

$$\frac{\pi}{m} = \prod_{n=1}^{\infty} \frac{(2n)^2(mn-m+1)}{(2n-1)^2(mn+1)}, \ \mathbf{m} \in \mathbb{N}. \tag{8}$$

General Formula I.1:

$$\frac{m\pi}{4} = \prod_{n=1}^{\infty} \frac{(2n)^2(mn+1)}{(2n+1)^2(mn-m+1)}, \ \mathbf{m} \in \mathbb{N}. \tag{8.1}$$

I present two proofs of (8) here. The first (more constructive) proof deals with even and odd m separately while the second (very brief) does it for all m at once. To evaluate the general class of infinite products, I follow the method (using the Gamma function) described in [7]. If k is a positive integer and $a_1+a_2+...+a_k = b_1+b_2+...+b_k$ and $n+b_j \neq 0$, then

$$\prod_{n=0}^{\infty} \frac{(n+a_1)\cdots(n+a_k)}{(n+b_1)\cdots(n+b_k)} = \prod_{m=1}^{k} \frac{\Gamma(b_m)}{\Gamma(a_m)}.$$

Further, $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$. Armed with these two results, now I take up the first proof.

Proof A. Let $m = 2k$. I now construct a product and compute its value.

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{(2kn-k)(2kn-k)}{(2kn-2k+1)(2kn-1)} &= \prod_{n=0}^{\infty} \frac{(2kn+k)(2kn+k)}{(2kn+1)(2kn+2k-1)} \\ &= \prod_{n=0}^{\infty} \frac{(n+\frac{1}{2})(n+\frac{1}{2})}{(n+\frac{1}{2k})(n+1-\frac{1}{2k})} = \frac{\Gamma(\frac{1}{2k})\Gamma(1-\frac{1}{2k})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \\ &= \frac{\pi/\sin\frac{\pi}{2k}}{\pi} = \frac{1}{\sin\frac{\pi}{2k}}. \end{aligned} \tag{9}$$

Putting $x = \pi/2k$ in (2) and using (9), we obtain

$$\begin{aligned}\frac{\pi}{2k} &= \sin\frac{\pi}{2k} \prod_{n=0}^{\infty} \frac{(2kn)^2}{(2kn-1)(2kn+1)} \\ &= \prod_{n=0}^{\infty} \frac{(2kn)^2}{(2kn-1)(2kn+1)} \Big/ \prod_{n=0}^{\infty} \frac{(2kn-k)^2}{(2kn-2k+1)(2kn-1)} \\ &= \prod_{n=1}^{\infty} \frac{(2kn)^2}{(2kn-k)^2} \frac{(2kn-2k+1)(2kn-1)}{(2kn-1)(2kn+1)} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)^2} \frac{(2kn-2k+1)}{(2kn+1)} \\ &= \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)^2} \frac{(mn-m+1)}{(mn+1)}. \end{aligned}$$

This gives the required result.

For odd m, we take an indirect route. Let m be odd and let

$$P = \prod_{n=1}^{\infty} \frac{(2n)^2(mn-m+1)}{(2n-1)^2(mn+1)}. \tag{a}$$

By the established even case, one gets

$$\frac{\pi}{2m} = \prod_{n=1}^{\infty} \frac{(2n)^2(2mn-2m+1)}{(2n-1)^2(2mn+1)}.$$

An application of the property $\Gamma(1+x)=x\Gamma(x)$ of the Gamma function gives

$$\frac{\pi}{2mP}=\prod_{n=1}^{\infty}\frac{(2n)^2(2mn-2m+1)}{(2n-1)^2(2mn+1)}/\prod_{n=1}^{\infty}\frac{(2n)^2(mn-m+1)}{(2n-1)^2(mn+1)}$$
$$=\prod_{n=1}^{\infty}\{\frac{(2mn-2m+1)(mn+1)}{(2mn+1)(mn-m+1)}\}=\prod_{n=0}^{\infty}\{\frac{(2mn+1)(mn+m+1)}{(2mn+2m+1)(mn+1)}\}$$
$$=\prod_{n=0}^{\infty}\frac{(n+\frac{1}{2m})(n+1+\frac{1}{m})}{(n+1+\frac{1}{2m})(n+\frac{1}{m})}=\frac{\Gamma(1+\frac{1}{2m})\Gamma(\frac{1}{m})}{\Gamma(\frac{1}{2m})\Gamma(1+\frac{1}{m})}=\frac{\frac{1}{2m}\Gamma(\frac{1}{2m})\Gamma(\frac{1}{m})}{\Gamma(\frac{1}{2m})\frac{1}{m}\Gamma(\frac{1}{m})}=\frac{1}{2}.$$

Therefore $P=\frac{\pi}{m}$, and our assertion for odd m is proved as well, completing the proof. □

Proof B: Let m be either even or odd and let

$$P=\prod_{n=1}^{\infty}\frac{(2n)^2(mn-m+1)}{(2n-1)^2(mn+1)}. \tag{b}$$

We now compute the quotient (4)/(b), *i.e.*,

$$\frac{\pi}{2P}=\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}/\prod_{n=1}^{\infty}\frac{(2n)^2(mn-m+1)}{(2n-1)^2(mn+1)}$$
$$=\prod_{n=1}^{\infty}\{\frac{(2n)^2(2n-1)^2(mn+1)}{(2n-1)(2n+1)(2n)^2(mn-m+1)}\}$$
$$=\prod_{n=1}^{\infty}\{\frac{(2n-1)(mn+1)}{(2n+1)(mn-m+1)}\}=\prod_{n=0}^{\infty}\{\frac{(2n+1)(mn+m+1)}{(2n+3)(mn+1)}$$
$$=\prod_{n=0}^{\infty}\frac{(n+\frac{1}{2})(n+1+\frac{1}{m})}{(n+\frac{3}{2})(n+\frac{1}{m})}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{m})}{\Gamma(\frac{1}{2})\Gamma(1+\frac{1}{m})}$$
$$=\frac{\frac{\sqrt{\pi}}{2}\Gamma(\frac{1}{m})}{\sqrt{\pi}\frac{1}{m}\Gamma(\frac{1}{m})}=\frac{m}{2}.$$
$$\therefore P=\prod_{n=1}^{\infty}\{\frac{(2n)^2(mn-m+1)}{(2n-1)^2(mn+1)}=\frac{\pi}{2}.\frac{2}{m}=\frac{\pi}{m}.$$

This completes the proof. □

Multiplication of (8) and (8.1) gives (4) squared proving (8.1). We deduce the following product using (2), (4) and (8.1).

General Formula I.2:

$$\frac{\pi}{2m}=\prod_{n=1}^{\infty}\frac{2n(2n+2)(mn-m+1)}{(2n-1)(2n+1)(mn+1)},\ \mathbf{m}\in\mathbb{N}. \tag{8.2}$$

§ 2

I shall now derive some new *a*lgebraic irrational-free infinite products for π.

On putting $x = \frac{\pi}{3}$ in (3), we get

$$\frac{2\pi}{3\sqrt{3}} = \prod_{n=1}^{\infty} \frac{(3n)^2}{(3n-1)(3n+1)}. \tag{10}$$

Letting $x = \frac{\pi}{6}$ in (3) yields

$$\frac{\pi}{3} = \prod_{n=1}^{\infty} \frac{(6n)^2}{(6n-1)(6n+1)}. \tag{11}$$

We thus obtain, on dividing (10) by (11),

$$\frac{2}{\sqrt{3}} = \prod_{n=1}^{\infty} \frac{(6n-1)(6n+1)}{(6n-2)(6n+2)}. \tag{12}$$

We have this product for $\sqrt{3}$

$$\sqrt{3} = \prod_{n=1}^{\infty} \frac{(6n-4)(6n-2)}{(6n-5)(6n-1)}. \tag{13}$$

One may easily verify (13)

$$\prod_{n=1}^{\infty} \frac{(6n-4)(6n-2)}{(6n-5)(6n-1)} = \prod_{n=0}^{\infty} \frac{(6n+2)(6n+4)}{(6n+1)(6n+5)} =$$
$$\prod_{n=0}^{\infty} \frac{(n+\frac{1}{3})(n+\frac{2}{3})}{(n+\frac{1}{6})(n+\frac{5}{6})} = \frac{\Gamma(\frac{1}{6})\Gamma(1+\frac{5}{6})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} = \frac{\sin\frac{\pi}{6}}{\sin\frac{\pi}{3}} = \sqrt{3}.$$

Hence (12) and (13) multiplied with (10) will respectively generate

$$\frac{4\pi}{9} = \frac{5.6.6.7}{4.4.8.8}.\frac{11.12.12.13}{10.10.14.14}.\frac{17.18.18.19}{16.16.20.20}... = \prod_{n=1}^{\infty} \frac{(6n-1)(6n)^2(6n+1)}{(6n-2)^2(6n+2)^2}. \tag{14}$$

$$\frac{2\pi}{3} = \frac{2.6.6}{1.5.8}.\frac{8.12.12}{7.11.14}.\frac{14.18.18}{13.17.20}... = \prod_{n=1}^{\infty} \frac{(6n-4)(6n)^2}{(6n-5)(6n-1)(6n+2)}. \tag{15}$$

Note that in the product to follow all terms after the first cancel and hence

$$\frac{1}{2} = \prod_{n=1}^{\infty} \frac{(8n-7)(8n+2)}{(8n-6)(8n+1)}. \tag{16}$$

Hence the product (5) $\times$ (16) gives

$$1 = \prod_{n=1}^{\infty} \frac{(8n-4)(8n-4)(8n-2)(8n+2)}{(8n-5)(8n-3)(8n-1)(8n+1)}. \tag{17}$$

I derived the next product through (7)/(17)

$$\frac{\pi}{8} = \prod_{n=1}^{\infty} \frac{(8n)^2(8n-7)(8n-5)(8n-3)(8n-1)}{(8n-4)^4(8n-2)(8n+2)}. \tag{18}$$

Let us now consider Euler's product for $x = \frac{\pi}{4}$

$$\frac{\pi}{2\sqrt{2}} = \frac{4.4}{3.5}.\frac{8.8}{7.9}.\frac{12.12}{11.13}... = \prod_{n=1}^{\infty} \frac{(4n)^2}{(4n-1)(4n+1)}. \tag{19}$$

We can transform it into

$$\frac{3\pi}{8\sqrt{2}} = \frac{4.8}{5.7}.\frac{8.12}{9.11}.\frac{12.16}{13.15}... = \prod_{n=1}^{\infty} \frac{4n(4n+4)}{(4n+1)(4n+3)}. \tag{20}$$

Now we have two products for $\sqrt{2}$. The first is due to Euler [2]

$$\sqrt{2} = \frac{2.2}{1.3}.\frac{6.6}{5.7}.\frac{10.10}{9.11}... = \prod_{n=1}^{\infty} \frac{(4n-2)^2}{(4n-3)(4n-1)}. \tag{21}$$

(21) put in (20) yields a product which approaches π from above

$$\frac{3\pi}{8} = \frac{2.2.4.8}{1.3.5.7}.\frac{6.6.8.12}{5.7.9.11}.\frac{10.10.12.16}{9.11.13.15}... = \prod_{n=1}^{\infty} \frac{(4n-2)^2 4n(4n+4)}{(4n-3)(4n-1)(4n+1)(4n+3)}. \tag{22}$$

The next product is obtained through division of (4) by (19)

$$\sqrt{2} = \frac{3.5}{2.6}.\frac{7.9}{6.10}.\frac{11.13}{10.14}... = \prod_{n=1}^{\infty} \frac{(4n-1)(4n+1)}{(4n-2)(4n+2)}. \tag{23}$$

(23) together with (20) gives a product which approaches π from below

$$\frac{3\pi}{8} = \frac{3.4.8}{2.6.7}.\frac{7.8.12}{6.10.11}.\frac{11.12.16}{10.14.15}... = \prod_{n=1}^{\infty} \frac{(4n-1)4n(4n+4)}{(4n-2)(4n+2)(4n+3)}. \tag{24a}$$

Note that 12 is a common factor in every numerator/denominator. Hence (24a) reduces to a formula where the numerator and the denominator are co-prime:

$$\frac{3\pi}{8} = \frac{8}{7}.\frac{56}{55}.\frac{176}{175}.\frac{400}{399}.\frac{760}{759}.\frac{1288}{1287}.\frac{2016}{2015}\cdots = \prod_{n=1}^{\infty} \frac{\{\frac{4}{3}(4n^3+3n^2-n)\}}{\{\frac{4}{3}(4n^3+3n^2-n)\}-1}. \tag{24b}$$

(24b) is a special case of

General Formula II:

$$(\frac{m-1}{m})\frac{\pi}{\sqrt{m}}\frac{1}{\sin\frac{\pi}{\sqrt{m}}} = \prod_{n=1}^{\infty} \frac{(mn-1)mn(n+1)}{(\sqrt{m}n-1)(\sqrt{m}n+1)(mn+m-1)}. \tag{25}$$

Proof.

$$\begin{aligned}
&\prod_{n=1}^{\infty} \frac{(mn-1)mn(n+1)}{(\sqrt{m}n-1)(\sqrt{m}n+1)(mn+m-1)} \\
&= \prod_{n=0}^{\infty} \frac{(mn+m-1)(mn+m)(n+2)}{(\sqrt{m}n+\sqrt{m}-1)(\sqrt{m}n+\sqrt{m}+1)(mn+2m-1)} \\
&= \prod_{n=0}^{\infty} \frac{(n+1-\frac{1}{m})(n+1)(n+2)}{(n+1-\frac{1}{\sqrt{m}})(n+1+\frac{1}{\sqrt{m}})(n+2-\frac{1}{m})}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(1-\frac{1}{\sqrt{m}})\Gamma(1+\frac{1}{\sqrt{m}})\Gamma(2-\frac{1}{m})}{\Gamma(1-\frac{1}{m})\Gamma(1)\Gamma(2)} \\
&= \frac{\Gamma(1-\frac{1}{\sqrt{m}})\frac{1}{\sqrt{m}}\Gamma(\frac{1}{\sqrt{m}})\Gamma(1+1-\frac{1}{m})}{\Gamma(1-\frac{1}{m})\Gamma(1)\Gamma(2)} \\
&= \frac{\Gamma(1-\frac{1}{\sqrt{m}})\frac{1}{\sqrt{m}}\Gamma(\frac{1}{\sqrt{m}})(1-\frac{1}{m})\Gamma(1-\frac{1}{m})}{\Gamma(1-\frac{1}{m})\Gamma(1)\Gamma(2)} \\
&= \frac{\Gamma(1-\frac{1}{\sqrt{m}})\frac{1}{\sqrt{m}}\Gamma(\frac{1}{\sqrt{m}})(1-\frac{1}{m})}{1.1} \\
&= \Gamma(1-\frac{1}{\sqrt{m}})\Gamma(\frac{1}{\sqrt{m}})\frac{1}{\sqrt{m}}(\frac{m-1}{m}) \\
&= \frac{\pi}{\sin\frac{\pi}{\sqrt{m}}}\frac{1}{\sqrt{m}}(\frac{m-1}{m}) \\
&= (\frac{m-1}{m})\frac{\pi}{\sqrt{m}}\frac{1}{\sin\frac{\pi}{\sqrt{m}}}.
\end{aligned}
$$

This completes the proof. □

Setting $x = \frac{\pi}{8}$ and $x = \frac{3\pi}{8}$ respectively in (3), we get Euler's products

$$\frac{\pi}{4\sqrt{2-\sqrt{2}}} = \frac{8.8}{7.9}.\frac{16.16}{15.17}.\frac{24.24}{23.25}\cdots. \tag{26}$$

$$\frac{3\pi}{4\sqrt{2-\sqrt{2}}} = \frac{8.8}{5.11}.\frac{16.16}{13.19}.\frac{24.24}{21.27}\cdots. \tag{27}$$

As $\sqrt{2} = \sqrt{2-\sqrt{2}} \times \sqrt{2+\sqrt{2}}$, Jonathan Sondow and Huang Yi [5] discovered a fine decomposition of Euler's product for

$$\sqrt{2} = \prod_{n=1}^{\infty} \frac{(8n-6)^2(8n-2)^2}{(8n-7)(8n-5)(8n-3)(8n-1)}$$

recorded in (21)

$$\sqrt{2-\sqrt{2}} = \frac{2.6}{3.5}.\frac{10.14}{11.13}\frac{18.22}{19.21}... = \prod_{n=1}^{\infty} \frac{(8n-6)(8n-2)}{(8n-5)(8n-3)}. \tag{28}$$

$$\sqrt{2+\sqrt{2}} = \frac{2.6}{1.7}.\frac{10.14}{9.15}.\frac{18.22}{17.23}... = \prod_{n=1}^{\infty} \frac{(8n-6)(8n-2)}{(8n-7)(8n-1)}. \tag{29}$$

Using (17) and their 6-factor product (10), I deduced this 3-factor product

$$\frac{\pi}{2\sqrt{2+\sqrt{2}}} = \frac{4.2.8}{5.3.5}.\frac{8.10.16}{9.11.13}.\frac{12.18.24}{13.19.21}... = \prod_{n=1}^{\infty} \frac{(4n)(8n-6)(8n)}{(4n+1)(8n-5)(8n-3)}. \tag{30}$$

Combining (29) and (30), I derived a 5-factor irrational-free product comparable to (4)

$$\frac{\pi}{2} = \frac{4.2.2.6.8}{5.1.3.5.7} \cdot \frac{8.10.10.14.16}{9.9.11.13.15} \cdot \frac{12.18.18.22.24}{13.17.19.21.23} \cdots$$
$$= \prod_{n=1}^{\infty} \frac{(4n)(8n-6)(8n-6)(8n-2)(8n)}{((4n+1)(8n-7)(8n-5)(8n-3)(8n-1)}. \tag{31}$$

§ **3**

The only other algebraic irrational-free formula generated by (25) is

$$\frac{35\pi}{108} = \frac{6.12.35}{5.7.71} \cdot \frac{12.18.71}{11.13.107} \cdot \frac{18.24.107}{17.19.143} \cdots$$
$$= \prod_{n=1}^{\infty} \frac{6n(6n+6)(36n-1)}{(6n-1)(6n+1)(36n+35)}. \tag{32}$$

Melnikov [4] gives alternative expansion of the sine function in his formula (12)

$$\sin x = \frac{2x}{\pi} \prod_{n=1}^{\infty} \left(1 + \frac{4x^2 - \pi^2}{(1-4n^2)\pi^2}\right). \tag{33}$$

By putting $x = \frac{\pi}{6}$ in (33), we get

$$\prod_{n=1}^{\infty} \frac{(6n-1)(6n+1)}{(6n-3)(6n+3)} = \frac{3}{2}. \tag{34}$$

Hence (32) × (34) yields

$$\frac{35\pi}{72} = \frac{2.4.35}{1.3.71} \cdot \frac{4.6.71}{3.5.107} \cdot \frac{6.8.107}{5.7.143} \cdots = \prod_{n=1}^{\infty} \frac{2n(2n+2)(36n-1)}{(2n-1)(2n+1)(36n+35)}. \tag{35}$$

Let us divide (35) by (4) to obtain

$$\frac{35}{36} = \prod_{n=1}^{\infty} \frac{(n+1)(36n-1)}{n(36n+35)}. \tag{36}$$

Multiply (35) by the general formula I (case $m = 1$), *i.e.*, $\pi = \prod_{n=1}^{\infty} \frac{(2n)^2.n}{(2n-1)^2.(n+1)}$

$$\frac{35\pi}{36} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)^2} \frac{(36n-1)}{(36n+35)}. \tag{37}$$

(37) gives a clue to

General Formula III:

$$\left(\frac{m-1}{m}\right)\pi = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)^2} \frac{(mn-1)}{(mn+m-1)}, \ \mathbf{m} \in \mathbb{N}. \tag{38}$$

Proof. If we divide (38) by (8), we get

$$(m-1) = \prod_{n=1}^{\infty} \frac{(mn-1)(mn+1)}{(mn-m+1)(mn+m-1)}. \tag{39}$$

If we establish (39), that would prove (38). Let us compute the R.H.S. of (39), *i.e.*,

$$\prod_{n=0}^{\infty}\frac{(mn+m-1)(mn+m+1)}{(mn+1)(mn+2m-1)}=\prod_{n=0}^{\infty}\frac{\left(n+1-\frac{1}{m}\right)\left(n+1+\frac{1}{m}\right)}{\left(n+\frac{1}{m}\right)\left(n+2-\frac{1}{m}\right)}$$
$$=\frac{\Gamma(\frac{1}{m})(1-\frac{1}{m})\Gamma(1-\frac{1}{m})}{\Gamma(1-\frac{1}{m})\frac{1}{m}\Gamma(\frac{1}{m})}$$
$$=\frac{(1-\frac{1}{m})}{\frac{1}{m}}=(m-1).$$

This completes the proof. □

We thus deduce from (25) and (38)

$$\sin\frac{\pi}{\sqrt{m}}=\frac{1}{\sqrt{m}}\prod_{n=1}^{\infty}\frac{4n(mn^2-1)}{m(n+1)(2n-1)^2}. \tag{40}$$

In fact, we can also deduce from Euler's product expansion for $\sin\pi x$ by employing the case $m=1$ of (8), the following expansion valid for all real x

$$\sin\pi x=x\prod_{n=1}^{\infty}\frac{4n(n^2-x^2)}{(n+1)(2n-1)^2}. \tag{41}$$

On setting $x=\frac{\pi}{10}$ in (3), we derive the following infinite product involving the numbers π and φ, which represents the golden ratio $\frac{1+\sqrt{5}}{2}$:

$$\pi\varphi=5\prod_{n=1}^{\infty}\frac{(10n)^2}{(10n-1)(10n+1)}. \tag{42}$$

Interestingly, $\pi\varphi$ represents the area of the annulus between two concentric circles - the unit circle and the outer circle having radius φ. Using the case $m=10$ in (8), dividing (42) by the resulting quantity and doing a little manipulation, I found a beautiful infinite product for Phi:

$$2\varphi=\prod_{n=1}^{\infty}\frac{(10n-5)^2}{(10n-9)(10n-1)}. \tag{43}$$

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A COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS VIA AN IMPLICIT RELATION IN AN INTUITIONISTIC FUZZY METRIC SPACE

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Abstract. In this paper, employing the common $(E.A)$ property, we prove a common fixed point theorem for weakly compatible mappings via an implicit relation in intuitionistic fuzzy metric space. Our results generalize the results of S. Kumar [11] and C. Alaca, D. Turkoglu and C. Yildiz [2].

1. Introduction

In 1986, Jungck [8] introduced the notion of compatible maps for a pair of self mappings. However, the study of common fixed points of non-compatible maps is also very interesting (see [16]). Aamri and El. Moutawakil [1] generalized the concept of non-compatibility by defining the notion of property $(E.A)$ and in 2005, Liu, Wu and Li [13] defined common $(E.A)$ property in metric spaces and proved common fixed point theorems under strict contractive conditions. Jungck and Rhoades [9] initiated the study of weakly compatible maps in metric spaces and showed that every pair of compatible maps is weakly compatible but reverse is not true. In the literature, many results have been proved for contraction maps satisfying property $(E.A)$ in different settings such as probabilistic metric spaces [5, 7]; fuzzy metric spaces [12, 15]; intuitionistic fuzzy metric spaces [11]. Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [19] and later there has been much progress in the study of intuitionistic fuzzy sets [4]. In 2004, Park [17] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [6]. Fixed point theory

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has important applications in diverse disciplines of mathematics, statistics, engineering, and economics in dealing with problems arising in: Approximation theory, potential theory, game theory, mathematical economics, etc. In this paper, employing the common $(E.A)$ property, we prove a common fixed point theorem for weakly compatible mappings via an implicit relation in intuitionistic fuzzy metric space. Our results generalize the results of S. Kumar [11] and C. Alaca, D. Turkoglu and C. Yildiz [2].

2. Preliminaries

The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) were originally introduced by Menger [14] in study of statistical metric spaces as follows.

Definition 2.1 ([18])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2 ([18])**.** A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions

(i) $\diamond$ is commutative and associative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

C. Alaca, D. Turkoglu and C. Yildiz [2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [10] as follows.

Definition 2.3 ([2])**.** A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all$x, y \in X$ and $t > 0$,
(ii) $M(x, y, 0) = 0$ for all $x, y \in X$,
(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$,
(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$,
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s,\ t > 0$,
(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(vii) $lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$,

(viii) $N(x,y,0) = 1$ for all $x, y \in X$,

(ix) $N(x,y,t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$,

(x) $N(x,y,t) = N(y,x,t)$ for all $x, y \in X$ and $t > 0$,

(xi) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$ for all $x, y, z \in X$ and $s,\, t > 0$,

(xii) for all $x, y \in X$, $N(x,y,.) : [0,\infty) \to [0,1]$ is right continuous,

(xiii) $lim_{t\to\infty} N(x,y,t) = 0$ for all $x, y \in X$.

We say (M,N) is an intuitionistic fuzzy metric space on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y w. r. t. t respectively.

Remark 2.1 ([2]). Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated as $x \diamond y = 1 - ((1-x) * (1-y))$ for all $x, y \in X$.

Remark 2.2 ([2]). In an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x,y,.)$ is non-decreasing and $N(x,y,.)$ is non-increasing for all $x, y \in X$.

Alaca, Turkoglu and Yildiz [2] introduced the following notions:

Definition 2.4 ([2])**.** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be

(i) convergent to a point $x \in X$ if, for all $t > 0, lim_{n\to\infty} M(x_n, x, t) = 1$ and $lim_{n\to\infty} N(x_n, x, t) = 0$,

(ii) Cauchy sequence if, for all $t > 0$ and $p > 0$, $lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ and $lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$.

Definition 2.5 ([2])**.** An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 2.1 ([2]). Let $X = \{\frac{1}{n} : n = 1, 2, 3, ...\} \cup \{0\}$ and let $*$ be the continuous t-norm and $\diamond$ be the continuous t-conorm defined by $a * b = ab$ and $a \diamond b = \min\{1, a+b\}$ respectively, for all $a, b \in [0,1]$. For each $t > 0$ and $x, y \in X$, define (M,N) by

$$M(x,y,t) = \frac{t}{t+|x-y|}, \qquad N(x,y,t) = \frac{|x-y|}{t+|x-y|}.$$

Clearly, $(X, M, N, *, \diamond)$ is a complete intuitionistic fuzzy metric space.

Definition 2.6 ([1])**.** A pair of self mappings (T, S) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to satisfy the property $(E.A)$ if there exist a sequence $\{x_n\}$ in X such that

$$lim_{n\to\infty} Tx_n = lim_{n\to\infty} Sx_n = z \text{ in } X$$

.

Example 2.2. Let $X = [0, \infty)$. Let $(X, M, N, *, \diamond)$ be the intuitionistic fuzzy metric space as in Example 2.8. Define $T, S : X \to X$ by $Tx = \frac{x}{5}$ and $Sx = \frac{2x}{5}$ for all $x \in X$. Clearly, for sequence $\{x_n\} = \{\frac{1}{n}\}$, T and S satisfies property $(E.A)$.

Definition 2.7 ([13]). Two pairs (A, S) and (B, T) of self mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ are said to satisfy the common $(E.A)$ property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = lim_{n\to\infty} By_n = lim_{n\to\infty} Ty_n = z$$

for some $z \in X$.

Example 2.3. Let $X = [-1, 1]$ and $(X, M, N, *, \diamond)$ be the intuitionistic fuzzy metric space as in Example 2.8. Define self mappings A, B, S and T on X as $Ax = \frac{x}{3}$, $Bx = \frac{-x}{3}$, $Sx = x$, and $Tx = -x$ for all $x \in X$. Then with sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{-1}{n}\}$ in X, one can easily verify that

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = lim_{n\to\infty} By_n = lim_{n\to\infty} Ty_n = 0.$$

Therefore, pairs (A, S) and (B, T) satisfy the common $(E.A.)$ property.

Definition 2.8 ([9]). A pair of self mappings (T, S) of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be weakly compatible if they commute at coincidence points, *i.e.*, if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

3. Main results

Implicit relations play important role in establishing of common fixed point results. Let M_6 be the set of all continuous functions $\phi : [0, 1]^6 \to R$ and $\psi : [0, 1]^6 \to R$ satisfying the following conditions

(A) $\phi(u, v, u, v, v, u) \geq 0$ imply $u \geq v$ for all $u, v \in [0, 1]$,
(B) $\phi(u, v, v, u, u, v) \geq 0$ imply $u \geq v$ for all $u, v \in [0, 1]$,
(C) $\phi(u, u, v, v, u, u) \geq 0$ imply $u \geq v$ for all $u, v \in [0, 1]$,
(D) $\psi(u, v, u, v, v, u) \leq 0$ imply $u \leq v$ for all $u, v \in [0, 1]$,
(E) $\psi(u, v, v, u, u, v) \leq 0$ imply $u \leq v$ for all $u, v \in [0, 1]$,
(F) $\psi(u, u, v, v, u, u) \leq 0$ imply $u \leq v$ for all $u, v \in [0, 1]$.

Example 3.1. Define $\phi, \psi : [0, 1]^6 \to R$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi_1(min\{t_2, t_3, t_4, t_5, t_6\})$$

where $\phi_1 : [0, 1] \to [0, 1]$ is an increasing and a continuous function such that $\phi_1(s) > s$ for all $s \in (0, 1)$, and

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi_1(max\{t_2, t_3, t_4, t_5, t_6\}),$$

where $\psi_1 : [0,1] \to [0,1]$ is increasing and a continuous function such that $\psi_1(k) < k$ for all $k \in (0,1)$. Clearly, ϕ and ψ satisfy all conditions (A), (B), (C), (D), (E) and (F).

We begin with following observation.

Lemma 3.1. *Let A, B, S and T be self mappings of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ satisfying the following*

(1) the pair (A, S) or (B, T) satisfies the property $(E.A.)$,

(2) for any $x, y \in X$, ϕ and ψ in M_6 and for all $t > 0$,

$$\phi\Big(\frac{M(Ax,By,t) + \min M(Sx,By,t), M(Ax,Ty,t)}{2},$$
$$M(Sx,Ty,t), M(Sx,Ax,t), M(Ty,By,t),$$
$$M(Sx,By,t), M(Ty,Ax,t)\Big) \geq 0$$

and

$$\psi\Big(\frac{N(Ax,By,t) + \max N(Sx,By,t), N(Ax,Ty,t)}{2},$$
$$N(Sx,Ty,t), N(Sx,Ax,t), N(Ty,By,t),$$
$$N(Sx,By,t), N(Ty,Ax,t)\Big) \leq 0.$$

(3) $A(X) \subseteq T(X)$ or $B(X) \subseteq S(X)$.

Then the pairs (A, S) and (B, T) share the common $(E.A.)$ property.

Proof. Suppose that the pair (A, S) satisfies property $(E.A.)$, then there exist a sequence $\{x_n\}$ in X such that $lim_n Ax_n = lim_n Sx_n = z$ for some $z \in X$. Since $A(X) \subseteq T(X)$, hence for each $\{x_n\}$, there exist $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore $lim_n Ax_n = lim_n Sx_n = lim_n Ty_n = z$. Now we claim that $lim_n By_n = z$. In fact, applying inequality (2) on $x = x_n, y = y_n$, we obtain

$$\phi\Big(\frac{M(Ax_n,By_n,t) + min\{M(Sx_n,By_n,t), M(Ax_n,Ty_n,t)\}}{2}, M(Sx_n,Ty_n,t),$$
$$M(Sx_n,Ax_n,t), M(Ty_n,By_n,t), M(Sx_n,By_n,t), M(Ty_n,Ax_n,t)\Big) \geq 0$$

and

$$\psi\Big(\frac{N(Ax_n,By_n,t) + max\{N(Sx_n,By_n,t), N(Ax_n,Ty_n,t)\}}{2}, N(Sx_n,Ty_n,t),$$
$$N(Sx_n,Ax_n,t), N(Ty_n,By_n,t), N(Sx_n,By_n,t), N(Ty_n,Ax_n,t)\Big) \leq 0$$

which on making $n \to \infty$ reduces to

$$\phi\Big(\frac{M(z, lim_n By_n, t) + min\{M(z, lim_n By_n, t), M(z, z, t)\}}{2}, M(z, z, t),$$
$$M(z, z, t), M(z, lim_n By_n, t), M(z, lim_n By_n, t), M(z, z, t)\Big) \geq 0,$$

$$\phi\Big(M(z, lim_n By_n, t), 1, 1, M(z, lim_n By_n, t), M(z, lim_n By_n, t), 1\Big) \geq 0;$$

and

$$\psi\Big(\frac{N(z, lim_n By_n, t) + max\{N(z, lim_n By_n, t), N(z, z, t)\}}{2}, N(z, z, t),$$
$$N(z, z, t), N(z, lim_n By_n, t), N(z, lim_n By_n, t), N(z, z, t)\Big) \leq 0,$$
$$\psi\Big(N(z, lim_n By_n, t), 0, 0, N(z, lim_n By_n, t), N(z, lim_n By_n, t), 0\Big) \leq 0.$$

By using (A) and (D), we have $M(z, lim_n By_n, t) \geq 1$ and $N(z, lim_n By_n, t) \leq 0$, and therefore, $lim_n By_n = z$. Hence,

$$lim_n Ax_n = lim_n Sx_n = lim_n By_n = lim_n Ty_n = z$$

for some $z \in X$. Hence, the pairs (A, S) and (B, T) share the common $(E.A.)$ property. □

Theorem 3.1. *Let A, B, S and T be self mappings of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ satisfying the conditions (2) and*

(4) the pair (A, S) and (B, T) share the common (E.A.) property,

(5) $S(X)$ and $T(X)$ are closed subsets of X.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. In view of (4), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$lim_n Ax_n = lim_n Sx_n = lim_n By_n = lim_n Ty_n = z$$

for some $z \in X$. As $S(X)$ is a closed subset of X, therefore, there exists a point $u \in X$ such that $z = Su$. We claim that $Au = z$. By (2), take $x = u, y = y_n$,

$$\phi\Big(\frac{M(Au, By_n, t) + min\{M(Su, By_n, t), M(Au, Ty_n, t)\}}{2}, M(Su, Ty_n, t),$$
$$M(Su, Au, t), M(Ty_n, By_n, t), M(Su, By_n, t), M(Ty_n, Au, t)\Big) \geq 0$$

and

$$\psi\Big(\frac{N(Au, By_n, t) + max\{N(Su, By_n, t), N(Au, Ty_n, t)\}}{2}, N(Su, Ty_n, t),$$
$$N(Su, Au, t), N(Ty_n, By_n, t), N(Su, By_n, t), N(Ty_n, Au, t)\Big) \leq 0$$

which on making $n \to \infty$ reduces to

$$\phi\Big(\frac{M(Au,z,t)+min\{M(z,z,t),M(Au,z,t)\}}{2}, M(z,z,t), M(z,Au,t),$$
$$M(z,z,t), M(z,z,t), M(z,Au,t)\Big) \geq 0,$$
$$\phi(M(Au,z,t),1,M(z,Au,t),1,1,M(z,Au,t)) \geq 0$$

and

$$\psi\Big(\frac{N(Au,z,t)+max\{N(z,z,t),N(Au,z,t)\}}{2}, N(z,z,t), N(z,Au,t),$$
$$N(z,z,t), N(z,z,t), N(z,Au,t)\Big) \leq 0,$$
$$\psi(N(Au,z,t),0,N(z,Au,t),0,0,N(z,Au,t)) \leq 0.$$

By using (A) and (D), we have $M(Au,z,t) \geq 1$ and $N(Au,z,t) \leq 0$. Therefore, $Au = z = Su$, which shows that u is a coincidence point of the pair (A,S). Since $T(X)$ is also a closed subset of X, therefore $lim_n Ty_n = z$ in $T(X)$ and hence there exists $v \in X$ such that $Tv = z = Au = Su$. Now, we show that $Bv = z$. By using inequality (2), take $x = u, y = v$, we have

$$\phi\Big(\frac{M(Au,Bv,t)+min\{M(Su,Bv,t),M(Au,Tv,t)}{2}, M(Su,Tv,t),$$
$$M(Su,Au,t), M(Tv,Bv,t), M(Su,Bv,t), M(Tv,Au,t)\Big) \geq 0,$$
$$\phi\Big(\frac{M(z,Bv,t)+min\{M(z,Bv,t),M(z,z,t)}{2}, M(z,z,t),$$
$$M(z,z,t), M(z,Bv,t), M(z,Bv,t), M(z,z,t)\Big) \geq 0,$$
$$\phi\Big(M(z,Bv,t),1,1,M(z,Bv,t),M(z,Bv,t),1) \geq 0;$$

and

$$\psi\Big(\frac{N(Au,Bv,t)+max\{N(Su,Bv,t),N(Au,Tv,t)}{2}, N(Su,Tv,t),$$
$$N(Su,Au,t), N(Tv,Bv,t), N(Su,Bv,t), N(Tv,Au,t)\Big) \leq 0,$$
$$\psi\Big(\frac{N(z,Bv,t)+max\{N(z,Bv,t),N(z,z,t)}{2}, N(z,z,t),$$
$$N(z,z,t), N(z,Bv,t), N(z,Bv,t), N(z,z,t)\Big) \leq 0,$$
$$\psi\Big(N(z,Bv,t),0,0,N(z,Bv,t),N(z,Bv,t),0\Big) \leq 0.$$

By using (B) and (E), we have

$$M(z,Bv,t) \geq 1, \quad N(z,Bv,t) \leq 0$$

which gives $Bv = z = Tv$ showing that v is a coincidence point of the pair (B,T). Since the pairs (A,S) and (B,T) are weakly compatible and $Au =$

$Su, Bv = Tv$, therefore, $Az = ASu = SAu = Sz, Bz = BTv = TBv = Tz$. Again, by using inequality (2), take $x = z, y = v$, we have

$$\phi\Big(\frac{M(Az,Bv,t) + min\{M(Sz,Bv,t), M(Az,Tv,t)}{2}, M(Sz,Tv,t),$$
$$M(Sz,Az,t), M(Tv,Bv,t), M(Sz,Bv,t), M(Tv,Az,t)\Big) \geq 0,$$
$$\phi\Big(\frac{M(Az,z,t) + min\{M(Az,z,t), M(Az,z,t)}{2}, M(Az,z,t),$$
$$M(Az,Az,t), M(z,z,t), M(Az,z,t), M(z,Az,t)\Big) \geq 0,$$
$$\phi(M(Az,z,t), M(Az,z,t), 1, 1, M(Az,z,t), M(Az,z,t)) \geq 0;$$

and

$$\psi\Big(\frac{N(Az,Bv,t) + max\{N(Sz,Bv,t), N(Az,Tv,t)}{2}, N(Sz,Tv,t),$$
$$N(Sz,Az,t), N(Tv,Bv,t), N(Sz,Bv,t), N(Tv,Az,t)\Big) \leq 0,$$
$$\psi\Big(\frac{M(Az,z,t) + max\{N(Az,z,t), N(Az,z,t)}{2}, N(Az,z,t),$$
$$N(Az,Az,t), N(z,z,t), N(Az,z,t), N(z,Az,t)\Big) \leq 0,$$
$$\psi(N(Az,z,t), N(Az,z,t), 0, 0, N(Az,z,t), N(Az,z,t)) \leq 0.$$

By using (C) and (F), we have

$$M(Az,z,t) \geq 1 \text{ and } N(Az,z,t) \leq 0.$$

Therefore, $Az = z = Sz$. Similarly, one can prove that $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz$, and z is common fixed point of A, B, S and T.

Uniqueness: Let z and w be two common fixed points of A, B, S and T, then by using inequality (2), we have

$$\phi\Big(\frac{M(Az,Bw,t) + min\{M(Sz,Bw,t), M(Az,Tw,t)}{2}, M(Sz,Tw,t),$$
$$M(Sz,Az,t), M(Tw,Bw,t), M(Sz,Bw,t), M(Tw,Az,t)\Big) \geq 0,$$
$$\phi\Big(\frac{M(z,w,t) + min\{M(z,w,t), M(z,w,t)}{2}, M(z,w,t),$$
$$M(z,z,t), M(w,w,t), M(z,w,t), M(w,z,t)\Big) \geq 0,$$
$$\phi(M(z,w,t), M(z,w,t), 1, 1, M(z,w,t), M(w,z,t)) \geq 0;$$

and

$$\psi\Big(\frac{N(Az,Bw,t) + max\{N(Sz,Bw,t), N(Az,Tw,t)}{2}, N(Sz,Tw,t),$$
$$N(Sz,Az,t), N(Tw,Bw,t), N(Sz,Bw,t), N(Tw,Az,t)\Big) \leq 0,$$

$$\psi\Big(\frac{N(z,w,t)+max\{N(z,w,t),N(z,w,t)}{2},N(z,w,t),N(z,z,t),$$
$$N(w,w,t),N(z,w,t),N(w,z,t)\Big)\leq 0,$$
$$\psi(N(z,w,t),N(z,w,t),0,0,N(z,w,t),N(w,z,t))\leq 0.$$

By using (C) and (F), we have $M(z,w,t)\geq 0$ and $N(z,w,t)\leq 0$. This gives $z=w$. $\square$

By choosing A,B,S and T suitably, one can derive corollaries involving two or more mappings. As a sample, we deduce the following natural result for a pair of self mappings by setting $B=A$ and $T=S$ in the above theorem.

Corollary 3.1. *Let A and S be self mappings of an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ satisfying the following:*

(6) the pair (A,S) satisfies the property (E.A.),

(7) for any $x,y\in X$, ϕ and ψ in M_6 and for all $t>0$,

$$\phi\Big(\frac{M(Ax,Ay,t)+min\{M(Sx,Ay,t),M(Ax,Sy,t)}{2},M(Sx,Sy,t),$$
$$M(Sx,Ax,t),M(Sy,Ay,t),M(Sx,Ay,t),M(Sy,Ax,t)\Big)\geq 0,$$

and

$$\psi\Big(\frac{N(Ax,Ay,t)+max\{N(Sx,Ay,t),N(Ax,Sy,t)}{2},N(Sx,Sy,t),$$
$$N(Sx,Ax,t),N(Sy,Ay,t),N(Sx,By,t),N(Sy,Ax,t)\Big)\leq 0,$$

(8) $S(X)$ is a closed subset of X.

Then, A and S have a point of coincidence each. Moreover, if the pair (A,S) is weakly compatible, then A and S have a unique common fixed point.

The following example illustrates Theorem 3.3.

Example 3.2. Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space where $X=[0,2)$ and define $\phi,\psi:[0,1]^6\to R$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-\phi_1(min\{t_2,t_3,t_4,t_5,t_6\})$$

where $\phi_1:[0,1]\to[0,1]$ is increasing and a continuous function such that $\phi_1(s)>s$ for all $s\in(0,1)$; and

$$\psi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-\psi_1(max\{t_2,t_3,t_4,t_5,t_6\})$$

where $\psi_1:[0,1]\to[0,1]$ is increasing and a continuous function such that $\psi_1(k)<k$ for all $k\in(0,1)$.

Clearly, ϕ and ψ satisfy all conditions (A), (B), (C), (D), (E) and (F). Define A, B, S and T by $Ax = Bx = 1$, $Sx = 1$ if $x \in Q$, $Sx = \frac{2}{3}$ otherwise; $Tx = 1$ if $x \in Q, Tx = \frac{1}{3}$ otherwise. Let $M(x,y,t) = \frac{t}{t+|x-y|}, N(x,y,t) = \frac{|x-y|}{t+|x-y|}$ for all $x, y \in X = [0,2)$ and $t > 0$. Then for the sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{-1}{n}\}$ in X, one observes that $lim_n Ax_n = lim_n Sx_n = lim_n By_n = lim_n Ty_n = 1 \in X$. This shows that pairs (A,S) and (B,T) share the common $(E.A.)$ property. Now, for verifying condition (2), take $x \in Q, y \in Q$; we have

$$\phi\Big(\frac{M(Ax,By,t) + min\{M(Sx,By,t), M(Ax,Ty,t)}{2}, M(Sx,Ty,t), M(Sx,Ax,t), M(Ty,By,t), M(Sx,By,t), M(Ty,Ax,t)\Big) \geq 0,$$

$$\phi\Big(\frac{M(1,1,t) + min\{M(1,1,t), M(1,1,t)}{2}, M(1,1,t), M(1,1,t), M(1,1,t), M(1,1,t), M(1,1,t)\Big) \geq 0,$$

$$\phi(\frac{1+1}{2}, 1, 1, 1, 1, 1) \geq 0, \quad \phi(1, 1, 1, 1, 1, 1) \geq 0, \quad 0 \geq 0,$$

$$1 - \phi_1(min\{1, 1, 1, 1, 1\}) \geq 0, \quad 1 - \phi_1(1) \geq 0, \quad 1 - \sqrt{1} \geq 0;$$

and

$$\psi\Big(\frac{N(Ax,By,t) + max\{N(Sx,By,t), N(Ax,Ty,t)}{2}, N(Sx,Ty,t), N(Sx,Ax,t), N(Ty,By,t), N(Sx,By,t), N(Ty,Ax,t)\Big) \leq 0,$$

$$\psi\Big(\frac{N(1,1,t) + max\{N(1,1,t), N(1,1,t)}{2}, N(1,1,t), N(1,1,t), N(1,1,t), N(1,1,t), N(1,1,t)\Big) \leq 0,$$

$$\psi(\frac{0+0}{2}, 0, 0, 0, 0, 0) \leq 0, \quad \psi(0, 0, 0, 0, 0, 0) \leq 0, \quad 0 \leq 0,$$

$$0 - \psi_1(max\{0, 0, 0, 0, 0\}) \leq 0, \quad 0 - \psi_1(0) \leq 0, \quad 0 - \sqrt{0} \leq 0,$$

which is true, hence condition (2) is verified for $x \in Q, y \in Q$. Similarly, we verify condition (2) for other cases. Thus, all the conditions of Theorem 3.3 are satisfied and $x = 1$ is the unique common fixed point of A, B, S and T.

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SOME PROPERTIES FOR GENERALIZED CAUCHY NUMBERS OF TWO KINDS

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Abstract. In this paper, we are concerned with generalized Cauchy numbers of two kinds. We first establish some identities for the above generalized Cauchy numbers. We further discuss the log-convexity or log-concavity of generalized Cauchy numbers under some conditions.

1. Introduction

Cauchy numbers of the first kind a_n and Cauchy numbers of the second kind b_n are defined by (see [5])

$$a_n = \int_0^1 (x)_n \mathrm{d}x, \quad b_n = \int_0^1 \langle x\rangle_n \mathrm{d}x,$$

where

$$(x)_n = \begin{cases} x(x-1)\cdots(x-n+1) & n \geq 1, \\ 1, & n = 0; \end{cases}$$

$$\langle x\rangle_n = \begin{cases} x(x+1)\cdots(x+n-1) & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Cauchy numbers of two kinds play important roles in many subjects such as approximate integrals, difference-differential equations, and combinatorics (see [2, 9, 10, 13]). For example, Cauchy numbers of two kinds are related to Stirling numbers, and Cauchy numbers appear in the Laplace summation formula (the Laplace summation formula is analogues to the Euler-Maclaurin formula). Hence the properties of Cauchy numbers deserve to be studied. In [13, 18 - 20], many properties of $\{a_n\}$ and $\{b_n\}$ are discussed. For example, several new identities relating Cauchy numbers with Stirling, Bernoulli and harmonic numbers are established in [13]. In this paper, we are interested in generalized

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Cauchy numbers of two kinds, and discuss their properties.
We first give the definitions of generalized Cauchy numbers of two kinds.

Definition 1.1. Let $f = (f_0, f_1, \cdots f_n, \cdots)$ be a sequence of nonnegative integers. Generalized Cauchy numbers of the first kind $c_f^{[1]}(n)$ and generalized Cauchy numbers of the second kind $c_f^{[2]}(n)$ are defined by

$$c_f^{[1]}(n) = \int_0^1 (x|f)_n \mathrm{d}x, \quad c_f^{[2]}(n) = \int_0^1 \langle x|f\rangle_n \mathrm{d}x,$$

where

$$(x|f)_n = \begin{cases} (x-f_0)(x-f_1)\cdots(x-f_{n-1}), & n \geq 1, \\ 1, & n = 0; \end{cases}$$

$$\langle x|f\rangle_n = \begin{cases} (x+f_0)(x+f_1)\cdots(x+f_{n-1}), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

It is clear that $c_f^{[1]}(n) = a_n$ and $c_f^{[2]}(n) = b_n$, when $f = (0, 1, 2, \cdots, n, \cdots)$. Some other definitions and notations will be used in this paper.

Definition 1.2. For a sequence of nonnegative integers $f = (f_0, f_1, \cdots, f_n, \cdots)$, the generalized Stirling numbers of the first kind $s_f(n, k)$ and the generalized Stirling numbers of the second kind $S_f(n, k)$ are defined by

$$\sum_{k=0}^{n} s_f(n, k)x^k = (x|f)_n,$$

$$\sum_{k=0}^{n} S_f(n, k)(x|f)_n = x^n.$$

When $f = (0, 1, 2, \cdots, n, \cdots)$, $s_f(n, k)$ and $S_f(n, k)$ become Stirling numbers of the first kind $s(n, k)$ and $S(n, k)$, respectively. For some properties of $s_f(n, k)$ and $S_f(n, k)$, see [16, 17].

Definition 1.3. Let $\{z_n\} \geq 0$ be a sequence of nonnegative numbers. $\{z_n\} \geq 0$ is called log-convex (or log-concave) if $z_n^2 \leq z_{n-1}z_{n+1}$(or $z_{n-1}z_{n+1} \leq z_n^2$) for all $n \geq 1$.

Definition 1.4. If $z_0 \leq z_1 \cdots \leq z_{m-1} \leq z_m \geq z_{m+1} \geq \cdots$ for some m, $\{z_n\}_{n\geq 0}$ is called unimodal, and m is called a mode of the sequence.

The object of this paper is to discuss some properties of $\{c_f^{[1]}(n)\}$ and $\{c_f^{[2]}(n)\}$. The main results of this paper can be summarized as follows: In

Section 2, we establish some identities for $c_f^{[1]}(n)(c_f^{[2]}(n))$ and derive generalized Stirling numbers of two kinds $s_f(n,k)(S_f(n,k))$. In Section 3, we mainly discuss the log-convexity (log-concavity) of generalized Cauchy numbers of two kinds under some conditions.

Throughout this paper,$\sigma_f^{[1]}(n) = (-1)^n c_f^{[1]}(n) \quad \text{for} \;\; n \geq 0.$

2. Some Identities for Generalized Cauchy Numbers of Two Kinds

In this section, we give some identities for generalized Cauchy numbers of two kinds. Cauchy numbers of two kinds $\{a_n\}$ and $\{b_n\}$ satisfy that (see [5])

$$a_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}(n)_k a_{n-k}}{k+1}, \quad b_n = \sum_{k=1}^{n} \frac{(n)_k b_{n-k}}{k+1}$$

Now we derive some recurrence relations for $\{c_f^{[1]}(n)\}$ and $\{c_f^{[2]}(n)\}$.

Theorem 2.1. *For $n \geq 1$, generalized Cauchy numbers of two kinds $\{c_f^{[1]}(n)\}$ and $\{c_f^{[2]}(n)\}$. satisfy that*

$$c_f^{[1]}(n+1) = \left(\frac{p_f(n)}{p_f(n-1)} - f_n\right) c_f^{[1]}(n) + \frac{f_{n-1}p_f(n)}{p_f(n-1)} c_f^{[1]}(n-1), \tag{2.1}$$

$$c_f^{[2]}(n+1) = \left(\frac{q_f(n)}{q_f(n-1)} + f_n\right) c_f^{[2]}(n) - \frac{f_{n-1}q_f(n)}{q_f(n-1)} c_f^{[2]}(n-1), \tag{2.2}$$

where

$$p_f(n) = \int_0^1 x(x|f)_n \, dx, \quad q_f(n) = \int_0^1 x\langle x|f\rangle_n \, dx,$$

Proof. For $n \geq 1$, one can verify that

$$c_f^{[1]}(n+1) = p_f(n) - f_n c_f^{[1]}(n),$$
$$c_f^{[2]}(n+1) = q_f(n) + f_n c_f^{[2]}(n).$$

Then we have

$$\frac{c_f^{[1]}(n+1) + f_n c_f^{[1]}(n)}{p_f(n)} = \frac{c_f^{[1]}(n) + f_{n-1} c_f^{[1]}(n-1)}{p_f(n-1)}$$

$$\frac{c_f^{[2]}(n+1) - f_n c_f^{[2]}(n)}{q_f(n)} = \frac{c_f^{[2]}(n) - f_{n-1} c_f^{[2]}(n-1)}{q_f(n-1)}$$

Hence (2.1)–(2.2) hold. □

Recall the following identities for Cauchy and Stirling numbers of two kinds (see [5]):

$$\sum_{k=0}^{n} \frac{s(n,k)}{k+1} = a_n, \tag{2.3}$$

$$\sum_{k=0}^{n} \frac{c(n,k)}{k+1} = b_n, \tag{2.4}$$

where $c(n,k) = (-1)^{n+k} s(n,k)$. Now we establish some identities for generalized Cauchy numbers and generalized Stirling numbers.

Theorem 2.2. *For generalized Cauchy numbers and generalized Stirling numbers, we have*

$$\sum_{k=0}^{n} s_f(n,k)\frac{1}{k+1} = c_f^{[1]}(n), \tag{2.5}$$

$$\sum_{k=0}^{n} s_f(n,k)\frac{(-1)^{n+k}}{k+1} = c_f^{[2]}(n), \tag{2.6}$$

$$\sum_{k=0}^{n} S_f(n,k)c_f^{[1]}(k) = \frac{1}{n+1}, \tag{2.7}$$

$$\sum_{k=0}^{n} S_f(n,k)(-1)^k c_f^{[2]}(k) = \frac{(-1)^n}{n+1}. \tag{2.8}$$

Proof. It follows from the definitions of generalized Cauchy numbers and generalized Stirling numbers that (2.5) – (2.8) hold. □

Remark 2.1. We note that (2.5) and (2.6) generalizes (2.3) and (2.4), respectively.

3. Log-Convexity (Log-Concavity) of Generalized Cauchy Numbers of two kinds

Log-concavity and log-convexity are important properties of combinatorial sequences and they are fertile sources of inequalities. The log-convexity or log-concavity of a sequence can help us obtaining its growth rate and asymptotic behavior. Unimodal and log-concave (log-convex) sequences often appear in many other subjects such as algebra, geometry, statistics, quantum physics, white noise theory, probability and mathematical biology. For some applications of the log-convexity or log-concavity of a sequence, see for instance [1, 3, 4, 7, 8, 11, 15]. In [20], the log-convexity of Cauchy numbers of the second kind $\{b_n\}$ is proved. In this section, we mainly discuss the log-convexity or log-concavity of generalized Cauchy numbers of two kinds according to the properties of $f = (f_0, f_1, \cdots, f_n, \cdots)$. Now we state and prove the main results of this section.

Lemma 3.1. *For generalized Cauchy numbers of the second kind $c_f^{[2]}(n)$ and its derived sequence $q_f(n)$, we have*

$$\frac{1}{2} c_f^{[2]}(n) \le q_f(n) \le c_f^{[2]}(n)$$

Proof. From the definition of generalized Stirling numbers of the first kind, we can get

$$\sum_{k=0}^{n} s_f(n,k)(-1)^{n+k} x^k = \langle x|f\rangle_n,$$

and $s_f(n,k)(-1)^{n+k} \ge 0$ for the definition of $\langle x|f\rangle_n$. Let $g_{n,k} = s_f(n,k)(-1)^{n+k}$,

$$c_f^{[2]}(n) = \int_0^1 \sum_{k=0}^{n} g_{n,k} x^k \,\mathrm{d}x = \sum_{k=0}^{n} \frac{g_{n.k}}{k+1}$$

and

$$q_f(n) = \sum_{k=0}^{n} \frac{g_{n,k}}{k+2}.$$

Clearly

$$\frac{1}{2(k+1)} \le \frac{1}{k+2} \le \frac{1}{k+1}, \text{for } 0 \le k \le n,$$

thus we get the desired inequality. □

Theorem 3.1. *For generalized Cauchy numbers of the second kind $\{c_f^{[2]}(n)\}$, $\{c_f^{[2]}(n)\}$ is log-convex if $f = (f_0, f_1, \cdots, f_n, \cdots)$ satisfies $f_n \ge \frac{1}{2} + f_{n-1}$ for $n \ge 1$.*

Proof. Applying the definition of log-convexity (log-concavity), we discuss the log-convexity (log-concavity) of $\{c_f^{[2]}(n)\}$. For $n \ge 1$, we can verify that

$$\left(c_f^{[2]}(n)\right)^2 - c_f^{[2]}(n-1)\, c_f^{[2]}(n+1) = \left(\int_0^1 \langle x|f\rangle_n \mathrm{d}x\right)^2 - \int_0^1 \langle x|f\rangle_{n-1} \mathrm{d}x \int_0^1 \langle x|f\rangle_{n+1} \mathrm{d}x.$$

As $f_n \le x + f_n \le 1 + f_n$ for $0 \le x \le 1$ and $n \ge 0$,

$$\begin{aligned}
&\left(c_f^{[2]}(n)\right)^2 - c_f^{[2]}(n-1) c_f^{[2]}(n+1) \\
&\le \int_0^1 (1 + f_{n-1})\langle x|f\rangle_n \mathrm{d}x \int_0^1 \langle x|f\rangle_{n-1} \mathrm{d}x - \int_0^1 \langle x|f\rangle_{n-1} \mathrm{d}x \int_0^1 (x + f_n)\langle x|f\rangle_n \mathrm{d}x \\
&= \int_0^1 \langle x|f\rangle_{n-1} \mathrm{d}x \int_0^1 \langle x|f\rangle_n (1 + f_{n-1} - x - f_n) \mathrm{d}x
\end{aligned}$$

$$= c_f^{[2]}(n-1)\Big((1+f_{n-1}-f_n)c_f^{[2]}(n) - q_f(n)\Big)$$

$$\leq c_f^{[2]}(n-1)(\frac{1}{2}+f_{n-1}-f_n)\,c_f^{[2]}(n),$$

and hence the proof. □

We note that $b_n = c_f^{[2]}(n)$, where $f = (0,1,2,\cdots,n,\cdots)$. From Theorem 3.1, we can immediately prove that the sequence of Cauchy numbers of the second kind $\{b_n\}$ is log-convex.

Now we discuss the log-convexity (log-concavity) of $\{c_f^{[2]}(n)\}$ when $f_n (n \geq 0)$ is a constant. We first recall a lemma.

Lemma 3.2. *(Davenport-Pólya Theorem)* [6, 12]. *If both $\{x_n\}$ and $\{y_n\}$ are log-convex, then so is their binomial convolution*

$$\omega_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}, \quad n = 0,1,2,\cdots.$$

Theorem 3.2. *For $n \geq 0, \ell$ is a nonnegative integer, assume that $f_n = \ell$, where ℓ is a constant. Then $\{c_f^{[2]}(n)\}$ is log-convex.*

Proof. For $\ell = 0, c_f^{[2]}(n) = 1/(n+1)$. Clearly, $\{c_f^{[2]}(n)\}$ is log-convex. For $\ell > 0$,

$$c_f^{[2]}(n) = \sum_{k=0}^{n} \binom{n}{k} \frac{\ell^{n-k}}{k+1}$$

It is easy to verify that $\{\ell^k\}$ and $\{1/(k+1)\}$ are log-convex. From the Davenport-Pólya Theorem, we know that $\{c_f^{[2]}(n)\}$ is log-convex. □

Theorem 3.3. *Suppose that $f = (f_0, f_1, \cdots, f_n, \cdots)$ satisfies $f_n \geq 1$ for $n \geq 0$. Then $\{\sigma_f^{[1]}(n)\}$ is log-convex if $f = (f_0, f_1, \cdots, f_n, \cdots)$ satisfies $f_n \geq 1 + f_{n-1}$ for $n \geq 1$.*

Proof. For $n \geq 1$, we can verify that

$$\begin{aligned}
&\Big(\sigma_f^{[1]}(n)\Big)^2 - \sigma_f^{[1]}(n-1)\,\sigma_f^{[1]}(n+1) \\
&= \left(\int_0^1 \langle -x|f\rangle_n \mathrm{d}x\right)^2 - \int_0^1 \langle -x|f\rangle_{n-1}\mathrm{d}x \int_0^1 \langle -x|f\rangle_{n+1}\mathrm{d}x \\
&\leq \left(\int_0^1 \langle -x|f\rangle_n \mathrm{d}x\right)^2 - (f_n-1)\int_0^1 \langle -x|f\rangle_{n-1}\mathrm{d}x \int_0^1 \langle -x|f\rangle_n \mathrm{d}x \\
&= \int_0^1 \langle -x|f\rangle_n \mathrm{d}x \int_0^1 \langle -x|f\rangle_{n-1}(-x+f_{n-1}-f_n+1)\mathrm{d}x.
\end{aligned}$$

$\square$

For the remainder of this section, we discuss the unimodality of $\{1/c_f^{[2]}(n)\}$ and $\{1/\sigma_f^{[1]}(n)\}$ under some conditions.

Theorem 3.4. *For $f = (f_0, f_1, \cdots, f_n, \cdots)$, there exists $m \geq 1$ such that $f_k = 0$ for $0 \leq k \leq m$ and $f_k \geq 1$ for $k \geq m+1$. Then $\{1/c_f^{[2]}(n)\}$ is unimodal, and its single peak is in $m + 1$.*

Proof. It is evident that $c_f^{[2]}(0) = 1$ and

$$c_f^{[2]}(k) = \int_0^1 x^k \mathrm{d}x = \frac{1}{k+1}$$

for $1 \leq k \leq m+1$. This indicates that $\{c_f^{[2]}(k)\}_{0 \leq k \leq m}$ is monotonic decreasing. For $k \geq m+1$,

$$c_f^{[2]}(k) - c_f^{[2]}(k+1) = \int_0^1 \langle x|f\rangle_k \,(1 - x - f_k)\,\mathrm{d}x < 0.$$

This means that $\{c_f^{[2]}(k)\}_{k \geq m+1}$ is monotonic increasing. Hence $\{1/c_f^{[2]}(n)\}$ is unimodal, and its single peak is in $m+1$. Then $\{1/c_f^{[2]}(n)\}$ is unimodal, and its single peak is in $m+1$. $\square$

Theorem 3.5. *For $f = (f_0, f_1, \cdots, f_n, \cdots)$, there exists $m \geq 1$ such that $f_k = 1$ for $0 \leq k \leq m$ and $f_k \geq 2$ for $k \geq m+1$. Then $\{1/\sigma_f^{[1]}(n)\}$ is unimodal, and its single peak is in $m+1$.*

The proof of Theorem 3.5 is similar to that of Theorem 3.4, and is omitted here.

4. Conclusions

We have established the connection between generalized Cauchy numbers and generalized Stirling numbers and have further discussed the log-convexity (log-concavity) of generalized Cauchy numbers. When $n \to \infty$, the asymptotic values of Cauchy numbers of two kinds are (see [5]):

$$\frac{a_n}{n!} \sim \frac{(-1)^{n+1}}{n(\mathrm{In}\, n)^2}, \quad \frac{b_n}{n!} \sim \frac{1}{\mathrm{In}\, n}, \quad n \to \infty.$$

The future work is to study the asymptotic approximation of generalized Cauchy numbers of two kinds.

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CONSTRUCTION OF A p-SYLOW SUBGROUP IN S_n

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Abstract. This article not only proves the existence of a p-sylow subgroup in symmetric group S_n but rather focus on to give an inductive method to construct it for any prime p and positive integer n.

1. Introduction

Most of the work done is inspired by the book [1]. In first section we introduce the terminologies and notations used throughout. We also prove a preliminary result as given in proposition 2.1 (see [1, page 102] and [2, page 84]). Section 2 deals with the construction of p-sylow subgroup in S_{p^k} (c.f. [1, 2.12.2]). In the last section we describe the general method for constructing a p-sylow subgroup in S_n for any positive integer n.

2. Notation and Preliminaries

Throughout this article, p stands for a prime and n for natural number. We denote the symmetric group of degree n by S_n. We will often refer reader to the following result of [1, page 102],

Proposition 2.1. *Let p be a prime and n be a positive integer. Then the power of p which exactly divides $n!$ is given by*

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$$

where $[x]$ is the greatest integer $\leq x$ for any real $x > 0$.

Proof. If $n < p$, then $p \nmid n!$. Suppose $n > p$. By division algorithm, there exist t, r such that $n = tp + r$ where $0 \leq r < p$. We write out $n!$ as

$$n! = 1 \cdot 2 \cdot \cdots \cdot p \cdot (p+1) \cdots (2p) \cdot (2p+1) \cdots (tp) \cdot (tp+1) \cdot (tp+2) \cdots (tp+r)$$

Notice that there are t factors that are divisible by p in $n!$ namely: $p, 2p, 3p, \cdots tp$. It follows that the power of p which exactly divides $n!$ is same as that of p

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which exactly divides $(tp)!$. Hence it suffices to prove the result for $n = tp$. We now consider the t factors

$$p \cdot 2p \cdot 3p \cdot 4p \cdots tp = p^t \cdot t!$$

Since $t < n$, induction on n will imply that the power of p which exactly divides $n!$ is

$$\begin{aligned} t + (\text{power of } p \text{which exactly divides } t!) &= t + \left(\left[\frac{t}{p}\right] + \left[\frac{t}{p^2}\right] + \left[\frac{t}{p^3}\right] + \cdots \right) \\ &= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots \end{aligned}$$

□

Let x_k denote power of p which exactly divides $p^k!$. The following corollary is an immediate consequence of proposition 2.1.

Corollary 2.1. *Let p be a prime and k be a positive integer. Then the power of p which exactly divides $p^k!$ is given by*

$$x_k = p^{k-1} + p^{k-2} + \cdots + p + 1$$

Definition 2.1. *Let G be a finite group of order n. For a prime p, a subgroup of order p^m in G is said to be a p-sylow subgroup if $p^m | n$ but $p^{m+1} \nmid n$.*

Remark 2.1. In view of what we have done above, we notice that order of a p-sylow subgroup

(1) in S_{p^k} is p^{x_k}, where $x_k = p^{k-1} + p^{k-2} + \cdots + p + 1$.
(2) in S_n is p^t, where $t = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$.

3. Construction of a p-sylow subgroup in S_{p^k}

In this section we inductively construct a p-sylow subgroup in group of permutations S_{p^k} on p^k-symbols as given in [1, 2.12.2]. We state this in the following theorem:

Theorem 3.1. *Construct a p-sylow subgroup in S_{p^k}.*

Proof. By the remark 2.1, the order of p-sylow subgroup in S_{p^k} is p^{x_k}. We apply induction on k. If $k = 1$ then $x_k = 1$. So we have a p-sylow subgroup namely the subgroup generated by the p-cycle $(1, 2, 3, \cdots, p)$. Suppose the result is true for $k - 1$. We now do it for k.

Let P_1 be a p-sylow subgroup of order $p^{x_{k-1}}$ that we have constructed in $S_{p^{k-1}}$. Consider a permutation $\tau \in S_{p^k}$ defined by

$$\begin{aligned} \tau &= \tau_1 \tau_2 \tau_3 \cdots \tau_{p^{k-1}}, \\ \text{where for each } j, \quad \tau_j &= \left(j, p^{k-1} + j, 2p^{k-1} + j, \cdots, (p-1)p^{k-1} + j \right) \end{aligned}$$

We see that τ is a product of p^{k-1} distinct p-cycles. This implies that $\circ(\tau) = p$. Since $S_{p^{k-1}}$ can be regarded as a subgroup of S_{p^k} by treating the elements of $S_{p^{k-1}}$ as permutations that fix all $j > p^{k-1}$, then P_1 can be thought as a subgroup in S_{p^k} of order $p^{x_{k-1}}$. Let us define

$$P_{i+1} = \tau^i P_1 \tau^{-i} \text{ for } i = 1, 2, 3, \cdots, (p-1).$$

Notice that $P_1, P_2, \cdots, P_p$ all are subgroup of S_{p^k} of same order $p^{x_{k-1}}$ such that the elements in P_i influence only the following set of p^{k-1} numbers:

$$\left\{(i-1)p^{k-1}+1,\ (i-1)p^{k-1}+2,\ (i-1)p^{k-1}+3, \cdots,\ ip^{k-1}\right\}.$$

This says that elements in distinct P_i commute and hence $T = P_1 P_2 \cdots P_p$ is a subgroup of S_{p^k}. Notice that $P_i \cap P_j = \{e\}$ if $0 \le i \ne j \le p$. We now claim that $\circ(T) = p^{p \cdot x_{k-1}}$.

proof of claim: We use inductive argument. Since $P_1 \cap P_2 = \{e\}$, it follows from [1, 2.5.1] that $\circ(P_1P_2) = p^{2 \cdot x_{k-1}}$. We know that any element in subgroup P_1P_2 influences only the following numbers:

$$\left\{1, 2, \cdots, p^{k-1}, p^{k-1}+1, p^{k-1}+2, \cdots,\ 2p^{k-1}\right\}$$

and any element in P_3 influences only the following numbers:

$$\left\{2p^{k-1}+1, 2p^{k-1}+2, \cdots,\ 3p^{k-1}\right\}.$$

This yields $P_1P_2 \cap P_3 = \{e\}$. By the use of [1, 2.5.1] again, we have $\circ(P_1P_2P_3) = p^{3 \cdot x_{k-1}}$. This inductive process will prove our claim.

From definition of p-cycle τ_j, we see that

$$\tau_j \notin P_i \qquad \text{for all } i = 1, 2, \cdots, p \ \ \& \ \ j = 1, 2 \cdots p^{k-1}$$

$$\Rightarrow \quad \text{any product of } \tau_j \quad \notin \quad P_i \quad \text{for all } \ i = 1, 2, \cdots, p$$

Let $t \in T = P_1 P_2 \cdots P_p$ be any element. Then t will be of the form:

$$t = t_1 t_2 \cdots t_p,$$

for some $t_1, t_2, \cdots, t_p$ with $t_i \in P_i$, so each t_i influences only the following numbers:

$$\left\{(i-1)p^{k-1}+1,\ (i-1)p^{k-1}+2,\ (i-1)p^{k-1}+3, \cdots,\ ip^{k-1}\right\}.$$

Therefore $\tau = \tau_1\tau_2\tau_3 \cdots \tau_{p^{k-1}}$ and t cannot have same cycle decomposition. Hence $\tau \notin T$. Now since $P_iP_j = P_jP_i$ for $0 \le i \ne j \le p$, we have

$$\tau T \tau^{-1} = \tau(P_1 P_2 \cdots P_p)\tau^{-1} = P_2 P_3 \cdots P_1 = T.$$

Let P be the subgroup of S_{p^k} generated by the element τ and T, that is

$$P = \{\tau^i t \ : \ t \in T \text{ and } 1 \le i \le p\}.$$

We determine the order of P. It is

$$\circ(P) = p^{px_{k-1}+1} = p^{p\cdot(p^{k-2}+p^{k-1}+\cdots+p+1)+1} = p^{p^{k-1}+p^{k-2}+\cdots+p+1} = p^{x_k}$$

Hence P is indeed a p-sylow subgroup in S_{p^k}. □

4. Construction of a p-sylow subgroup in S_n

In this section we generalize the inductive method adopted above for constructing p-sylow subgroup of the symmetric group S_n on n-symbols for any positive integer n. We state this in the following theorem:

Theorem 4.1. *Construct a p-sylow subgroup in S_n.*

Proof. (Construction of p-sylow subgroup in S_n)
Case-1: If $n < p$, there is no p-sylow subgroup in S_n.
Case-2: If $n = p$ then we have a p-sylow subgroup namely subgroup generated by the p-cycle $(1, 2, 3, \cdots, p)$.
Case-3: Now if $n > p$, then by division algorithm there exist integers t & r such that

$$n = tp + r \text{ where } 0 \leq r \leq p - 1.$$

For $0 < r < p-1$, this is not hard to see that the order of p-sylow subgroup in S_n is same as that of p-sylow subgroup in S_{pt}. Also since a p-sylow subgroup in S_{pt} can be regarded as a p-sylow subgroup in S_n, it is therefore sufficient to prove the result for $n = tp$.

The case when n is a multiple of p, we decompose n into the following form:

$$n = t_k p^k + t_{k-1} p^{k-1} + \cdots + t_2 p^2 + t_1 p \quad \text{for some } k,\ t_j$$

with $k > 0$ and $0 \leq t_j \leq p - 1$ for all $j = 0, 1, 2, \cdots, k$.
Subcase-1: We now construct a p-sylow subgroup in S_n when $n = tp^k$ with $0 < t < p$. Since

$$\left[\frac{tp^k}{p}\right] + \left[\frac{tp^k}{p^2}\right] + \left[\frac{tp^k}{p^3}\right] + \cdots + \left[\frac{tp^k}{p^k}\right] = t\left(p^{k-1} + p^{k-2} + \cdots + p + 1\right) = tx_k$$

then it follows that the order of p-sylow subgroup in S_n is p^{tx_k}.

If $k = 1$ then $n = tp$ then the order of p-sylow subgroup is p^t. Consider the following subgroups:

P_1 = subgroup generated by the p-cycle $(1, 2, 3, \cdots, p)$

P_2 = subgroup generated by the p-cycle $(p + 1, p + 2, p + 3, \cdots, 2p)$

$\vdots$

P_t = subgroup generated by the p-cycle $\Big((t - 1)p + 1, (t - 1)p + 2, \cdots, tp\Big)$

Clearly each P_i is a subgroup of order p in S_{tp} such that $P_iP_j = P_jP_i$ for all i, j. Therefore it follows that $P = P_1P_2\cdots P_t$ is a subgroup of order p^t in S_{tp}.

Now if $k > 1$ then $n = tp^k$ and the order of p-sylow subgroup is p^{tx_k}. Consider the following t-plies each with p^k elements:

$$\{1, 2, \cdots, p^k\}, \{p^k+1, p^k+2, \cdots, 2p^k\}, \cdots, \{(t-1)p^k+1, (t-1)p^k+2, \cdots, tp^k\}$$

We use theorem 3.1 to construct t number of p-sylow subgroups $P_1, P_2, \cdots, P_t$ in S_{p^k} with respect to each pile of p^k numbers. That is for $i = 1, 2, 3, \cdots, t$,

$$\begin{gathered} P_i = \text{a } p\text{-sylow subgroup in the symmetric group } S_{p^k} \text{ on numbers} \\ (i-1)p^k + 1,\ (i-1)p^k + 2,\ (i-1)p^k + 3,\ \cdots,\ ip^k. \end{gathered}$$

Notice that $\circ(P_i) = p^{x_k}$. Since all P_i can be treated as a subgroup of S_{tp^k} influencing only disjoint set of numbers, this yields that $P_i \cap P_j = \{e\}$ and $P_iP_j = P_jP_i$ for all $0 \leq i \neq j \leq p$. Hence it follows that $P = P_1P_2\cdots P_t$ is a subgroup in S_{tp^k} with order

$$\circ(P) = p^{x_k} \cdot p^{x_k} \cdots p^{x_k} \quad (t\text{-times}) = p^{tx_k}.$$

Hence P is the required p-sylow subgroup in S_{tp^k}.

Subcase-2: When $n = t_1p^k + t_2p^{k-1}$, $0 \leq t_1, t_2 < p$. We now construct a p-sylow subgroup in S_n. Let P_1 be a p-sylow subgroup in $S_{t_1p^k}$ that we have just constructed above. Notice that $\circ(P_1) = p^{t_1x_k}$. Consider the symmetric group $S_{t_2p^{k-1}}$ with t_2p^{k-1} symbols as

$$t_1p^k + 1,\ t_1p^k + 2,\ t_1p_k + 3,\ \cdots,\ t_1p^k + t_2p^{k-1}.$$

Let P_2 be the p-sylow subgroup in $S_{t_2p^{k-1}}$ that we can construct using above procedure as adopted for P_1. Also notice that $\circ(P_2) = p^{t_2x_{k-1}}$. We now treat P_1 and P_2 as a subgroups in S_n where $n = t_1p^k + t_2p^{k-1}$. Since the elements in P_1 and P_2 influence only disjoint subsets of numbers

$$\{1,\ 2,\ 3,\ \cdots, t_1p^k\}\ \&\ \{t_1p^k + 1,\ t_1p^k + 2,\ t_1p_k + 3,\ \cdots,\ t_1p^k + t_2p^{k-1}\}$$

respectively then $P = P_1P_2$ is a subgroup of S_n such that

$$\circ(P) = p^{t_1x_k} \cdot p^{t_2x_{k-1}} = p^{t_1x_k + t_2x_{k-1}}$$

which is the order of p-sylow subgroup in S_n.

In this way, we can construct a p-sylow subgroup in S_n where $n = t_1p^k + t_2p^{k-1} + t_3p^{k-3}$ by taking the product of a p-sylow subgroup in $S_{t_1p^k + t_2p^{k-1}}$ and a p-sylow subgroup in $S_{t_3p^{k-3}}$. Hence in general we can construct a p-sylow subgroup in any S_n with $n = t_1p^k + t_2p^{k-1} + \cdots + t_{k-1}p^2 + t_kp$. $\square$

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GENERAL FORMULAS FOR REMOVING THE SECOND AND THE THIRD TERM IN A POLYNOMIAL EQUATION OF ANY DEGREE

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ABSTRACT. General formulas are given to eliminate $(N-1)$th and $(N-2)$th power terms in a polynomial equation of degree N, where N is any integer greater than two. The formulas obviate the need to determine the two unknowns of quadratic Tschirnhaus transformation involving cumbersome algebraic manipulations, each time a polynomial equation of newer degree is encountered. These formulas are derived through mathematical induction, by studying the cases of cubic equation, quartic equation, quintic equation, and the sextic equation.

1. **Introduction**

In 1683, Ehrenfried Walther von Tschirnhaus introduced a polynomial transformation, which he claimed, will eliminate all intermediate terms in a polynomial equation of any degree, thereby reducing it into a binomial form from which roots can be easily extracted [1]. Thus the cubic and the quartic equations were reduced to binomial forms using quadratic and cubic Tschirnhaus transformations respectively [2, 3].

However in the case of quintic equation, several workers struggled without any success to reduce it to the binomial form. In 1786, Bring could reduce the quintic equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ to the form $y^5 + Ay + B = 0$, using a quartic Tschirnhaus transformation $y = x^4 + fx^3 + gx^2 + hx + j$. It seems Bring's work got lost in the archives of University of Lund. Unaware of Bring's work, Jerrard (1859) also arrived at the same form of quintic equation, which is now referred as Bring-Jerrard quintic equation [4].

Finally, it was left to Abel (1826) and later Galois (1832) to prove conclusively that the polynomial equations of degree five and above cannot be solved in radicals – the reason why the Tschirnhaus transformation could not eliminate all the intermediate terms in a quintic equation [5].

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It is well-known that for a polynomial equation of degree N in x, such as: $x^N + a_{N-1}x^{N-1} + a_{N-2}x^{N-2} + ... + a_1 x + a_0 = 0$, one can use linear transformation, $x = y - (a_{N-1}/N)$, to remove $(N-1)$th power term. However no such straightforward formula (or a set of formulas) is available to remove both $(N-1)$th and $(N-2)$th power terms from polynomial equations. Conventionally, a quadratic Tschirnhaus transformation is used to remove these terms. The method involves determination of two unknowns in the quadratic Tschirnhaus transformation, through tedious algebraic manipulations, each time a polynomial equation of newer degree is encountered.

In this paper a set of general formulas is derived for removing the $(N-1)$th and $(N-2)$th power terms from a polynomial equation of degree N (where $N > 2$), obviating the pain of going through cumbersome algebraic manipulations each time a polynomial equation of newer degree is encountered. We derive these formulas for the cases of cubic equation, quartic equation, quintic equation, and sextic equation, and then generalize these findings using mathematical induction.

2. Case 1: cubic equation

Consider the following cubic equation

$$x^3 + a_2 x^2 + a_1 x + a_0 = 0, \tag{2.1}$$

where a_0, a_1, and a_2 are coefficients in (2.1). Let us define a quadratic Tschirnhaus transformation by

$$x^2 = 2b_1 x + b_0 + y, \tag{2.2}$$

where b_0 and b_1 are two unknowns to be determined, and y is a new variable. Solving the quadratic equation (2.2), two solutions of x are obtained as

$$x = b_1 \pm \sqrt{b_1^2 + b_0 + y}.$$

We use one of the above two solutions, say

$$x = b_1 + \sqrt{b_1^2 + b_0 + y}, \tag{2.3}$$

in the given cubic equation (2.1) to eliminate x from it. This results in an expression in new variable y as follows

$$\begin{aligned}\left(b_1 + \sqrt{b_1^2 + b_0 + y}\right)^3 &+ a_2\left(b_1 + \sqrt{b_1^2 + b_0 + y}\right)^2 \\ &+ a_1\left(b_1 + \sqrt{b_1^2 + b_0 + y}\right) + a_0 = 0.\end{aligned} \tag{2.4}$$

Expanding above expression leads to

$$\begin{aligned} & b_1^3 + 3b_1^2\sqrt{b_1^2+b_0+y} + 3b_1(b_1^2+b_0+y) + (b_1^2+b_0+y)\sqrt{b_1^2+b_0+y} + \\ & \quad a_2\left(b_1^2 + 2b_1\sqrt{b_1^2+b_0+y} + b_1^2+b_0+y\right) + a_1\left(b_1 + \sqrt{b_1^2+b_0+y}\right) \\ & \qquad +a_0 = 0 \end{aligned} \tag{2.5}$$

and we rearrange it such that all square-root terms are kept on one side as shown below

$$\begin{aligned} & 4b_1^3 + 3b_0b_1 + a_2(2b_1^2+b_0) + a_1b_1 + a_0 + (3b_1+a_2)y \\ & \quad = -\sqrt{b_1^2+b_0+y}\left(4b_1^2 + 2a_2b_1 + b_0 + a_1 + y\right). \end{aligned} \tag{2.6}$$

Now squaring equation (2.6) and rearranging it in descending powers of y results in the transformed cubic equation in y as

$$y^3 + c_2y^2 + c_1y + c_0 = 0, \tag{2.7}$$

where the coefficients, c_0, c_1, and c_2, are given by

$$\begin{aligned} c_0 &= (b_1^2+b_0)(4b_1^2+2a_2b_1+b_0+a_1)^2 - [4b_1^3+3b_0b_1+a_2(2b_1^2+b_0)+a_1b_1+a_0]^2, \\ c_1 &= 4a_1b_1^2 + (2a_1a_2 - 6a_0 - 4a_2b_0)b_1 + 3b_0^2 + (4a_1 - 2a_2^2)b_0 + a_1^2 - 2a_0a_2 \\ c_2 &= 3b_0 - 2a_2b_1 + 2a_1 - a_2^2. \end{aligned} \tag{2.8}$$

Our aim is to force the coefficients c_1 and c_2 to vanish and then find expressions for the unknowns b_0 and b_1 in terms of coefficients of given cubic equation (2.1). Therefore equating c_2 to zero results in the following expression for b_0

$$b_0 = (2a_2b_1 - 2a_1 + a_2^2)/3. \tag{2.9}$$

Using (2.9) we eliminate b_0 from the expression for c_1 given in (2.8), and then equate c_1 to zero. This leads us to a quadratic equation in b_1 as shown below

$$\begin{aligned} & [4a_1 - (4a_2^2/3)]b_1^2 + [(14a_1a_2/3) - 6a_0 - (4a_2^3/3)]b_1 \\ & \quad +(4a_1a_2^2/3) - 2a_0a_2 - (a_1^2/3) - (a_2^4/3) = 0. \end{aligned} \tag{2.10}$$

Solving (2.10) we determine b_1, and from the relation (2.9) the remaining unknown b_0 is determined. With these values the y^2 and y terms in cubic equation (2.7) vanish, leading to the following binomial cubic in y

$$y^3 + c_0 = 0. \tag{2.11}$$

2.1. **Remarks.** Let $f(X) = X^3 + a_2X^2 + a_1X + a_0,$ where a_i's are indeterminates. Let L be the splitting field of f, $K = Q(w, a_0, a_1, a_2)$, where w denotes the non-trivial cube root of unity, and x — a root of f. Then L/K is a Galois extension of degree 6. If D is the discriminant of f, then $L = K[\sqrt{D},\ x]$. We have $L/K[\sqrt{D}]$ is a cyclic Galois extension of degree 3 and since L contains w, $L = K[\sqrt{D},\ y]$ where y^3 belongs to $K[\sqrt{D}]$. Since x satisfies a polynomial of degree 3, y is a polynomial in x of degree at most 2.

3. Case 2: quartic equation

Consider the following quartic equation

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{3.1}$$

where a_0, a_1, a_2, and a_3 are coefficients in (3.1). We use the solution (2.3) of the quadratic Tschirnhaus transformation (2.2) for eliminating x from the quadratic equation (3.1) as shown below

$$\begin{aligned} \left(b_1 + \sqrt{b_1^2 + b_0 + y}\right)^4 + a_3\left(b_1 + \sqrt{b_1^2 + b_0 + y}\right)^3 \\ + a_2\left(b_1 + \sqrt{b_1^2 + b_0 + y}\right)^2 + a_1\left(b_1 + \sqrt{b_1^2 + b_0 + y}\right) + a_0 = 0. \end{aligned} \tag{3.2}$$

Further expanding (3.2) leads to following expression

$$\begin{aligned} b_1^4 + 4b_1^3\sqrt{b_1^2 + b_0 + y} + 6b_1^2(b_1^2 + b_0 + y) \\ + 4b_1(b_1^2 + b_0 + y)\sqrt{b_1^2 + b_0 + y} + (b_1^2 + b_0 + y)^2 \\ + a_3[b_1^3 + 3b_1^2\sqrt{b_1^2 + b_0 + y} + 3b_1(b_1^2 + b_0 + y) + (b_1^2 + b_0 + y)\sqrt{b_1^2 + b_0 + y}\,] \\ + a_2(2b_1^2 + b_0 + y) + 2a_2b_1\sqrt{b_1^2 + b_0 + y} + a_1b_1 + a_1\sqrt{b_1^2 + b_0 + y} + a_0 = 0. \end{aligned} \tag{3.3}$$

As in the cubic case, we rearrange (3.3) by segregating all square-root terms on one side as follows

$$\begin{aligned} y^2 + (8b_1^2 + 2b_0 + 3a_3b_1 + a_2)y + \\ 8b_1^4 + 8b_0b_1^2 + b_0^2 + 4a_3b_1^3 + 3a_3b_0b_1 + 2a_2b_1^2 + a_2b_0 + a_1b_1 + a_0 = \\ -\sqrt{b_1^2 + b_0 + y}\left[(4b_1 + a_3)y + 8b_1^3 + 4b_0b_1 + 4a_3b_1^2 + 2a_2b_1 + a_3b_0 + a_1\right]. \end{aligned} \tag{3.4}$$

Squaring (3.4) and rearranging terms in descending powers of y results in a quartic equation in y

$$y^4 + c_3y^3 + c_2y^2 + c_1y + c_0 = 0, \tag{3.5}$$

where c_0, c_1, c_2, and c_3 are the coefficients in (2.15) given by

$$c_0 = (8b_1^4 + 8b_0b_1^2 + b_0^2 + 4a_3b_1^3 + 3a_3b_0b_1 + 2a_2b_1^2 + a_2b_0 + a_1b_1 + a_0)^2$$
$$-(b_1^2 + b_0)(8b_1^3 + 4b_0b_1 + 4a_3b_1^2 + 2a_2b_1 + a_3b_0 + a_1)^2,$$

$$c_1 = 2(8b_1^2 + 3a_3b_1 + 2b_0 + a_2)(8b_1^4 + 8b_0b_1^2 + b_0^2 + 4a_3b_1^3 + 3a_3b_0b_1 + 2a_2b_1^2$$
$$+a_2b_0 + a_1b_1 + a_0) - 2(4b_1 + a_3)(b_1^2 + b_0)(8b_1^3 + 4b_0b_1 + 4a_3b_1^2 + 2a_2b_1$$
$$+a_3b_0 + a_1) - (8b_1^3 + 4b_0b_1 + 4a_3b_1^2 + 2a_2b_1 + a_3b_0 + a_1)^2,$$

$$c_2 = 4a_2b_1^2 + (2a_2a_3 - 6a_1)b_1 + a_2^2 + 2a_0 - 2a_1a_3 + (6a_2 - 3a_3^2 - 6a_3b_1)b_0 + 6b_0^2$$

and

$$c_3 = 4b_0 + 2a_2 - a_3^2 - 2a_3b_1. \tag{3.6}$$

At present we are interested in the expressions for c_2 and c_3, since these coefficients are to be equated to zero. Equating c_3 to zero, we get an expression for b_0 as

$$b_0 = (2a_3b_1 - 2a_2 + a_3^2)/4. \tag{3.7}$$

We eliminate b_0 from the expression for c_2 given in (3.6) using (3.7) and then equate c_2 to zero. This results in a quadratic equation in b_1 as shown below

$$\begin{aligned}[4a_2 - (3a_3^2/2)]b_1^2 + [5a_2a_3 - 6a_1 - (3a_3^3/2)]b_1 + 2a_0 + (3a_2a_3^2/2)\\ -(a_2^2/2) - 2a_1a_3 - (3a_3^4/8) = 0.\end{aligned} \tag{3.8}$$

Notice that b_1 and b_0 get determined from (3.8) and (3.7) respectively for the condition $c_2 = c_3 = 0$. Thus the transformed quartic (3.5) in y has no y^2 and y^3 terms as is seen from

$$y^4 + c_1y + c_0 = 0. \tag{3.9}$$

4. **Case 3: quintic equation**

Let us consider the following quintic equation

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{4.1}$$

where a_0, a_1, a_2, a_3, and a_4 are its coefficients. As above, using (2.3), we eliminate x from the quintic (4.1) resulting in the following expression in y

$$(5b_1 + a_4)y^2 + (20b_1^3 + 10b_0b_1 + 8a_4b_1^2 + 2a_4b_0 + 3a_3b_1 + a_2)y + b_1^5$$
$$+10b_1^3(b_1^2 + b_0) + 5b_1(b_1^2 + b_0)^2 + a_4b_1^4 + 6a_4b_1^2(b_1^2 + b_0) + a_4(b_1^2 + b_0)^2$$
$$+a_3b_1^3 + 3a_3b_1(b_1^2 + b_0) + a_2b_1^2 + a_2(b_1^2 + b_0) + a_1b_1 + a_0 =$$
$$-\sqrt{b_1^2 + b_0 + y}\{y^2 + [10b_1^2 + 2(b_1^2 + b_0) + 4a_1b_1 + a_3]y + 5b_1^4 + 10b_1^2(b_1^2 + b_0)$$
$$+(b_1^2 + b_0)^2 + 4a_4b_1^3 + 4a_4b_1(b_1^2 + b_0) + 3a_3b_1^2 + a_3(b_1^2 + b_0) + 2a_2b_1 + a_1\} \tag{4.2}$$

Squaring equation (4.2) and rearranging terms in descending powers of y, we obtain a quintic equation in y as follows

$$y^5 + c_4y^4 + c_3y^3 + c_2y^2 + c_1y + c_0 = 0, \tag{4.3}$$

where c_0, c_1, c_2, c_3, and c_4 are coefficients of quintic (4.3). Since our aim is to make c_3 and c_4 vanish, we focus on the expressions for these coefficients. The coefficient c_4 is expressed as

$$c_4 = 5b_0 + 2a_3 - a_4^2 - 2a_4b_1. \tag{4.4}$$

Now equating c_4 to zero we get an expression for b_0 as

$$b_0 = (2a_4b_1 - 2a_3 + a_4^2)/5. \tag{4.5}$$

The expression for c_3 is obtained as

$$\begin{aligned} c_3 = 4a_3b_1^2 + (2a_3a_4 - 6a_2)b_1 + a_3^2 + 2a_1 - 2a_2a_4 \\ +(8a_3 - 4a_4^2 - 8a_4b_1)b_0 + 10b_0^2. \end{aligned} \tag{4.6}$$

Eliminating b_0 from (4.6), using (4.5), and then equating c_3 to zero leads to the following quadratic equation in b_1

$$\begin{aligned} [4a_3 - (8a_4^2/5)]b_1^2 + [(26a_3a_4/5) - 6a_2 - (8a_4^3/5)]b_1 \\ +2a_1 + (8a_3a_4^2/5) - (3a_3^2/5) - 2a_2a_4 - (2a_4^4/5) = 0. \end{aligned} \tag{4.7}$$

Note that b_1 determined from (4.7) and b_0 from (4.5) for the condition $c_3 = c_4 = 0$ renders the following quintic equation (4.3) in y to have no y^3 and y^4 terms

$$y^5 + c_2y^2 + c_1y + c_0 = 0. \tag{4.8}$$

5. Case 4: sextic equation

Now consider the following sextic equation

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{5.1}$$

where a_0, a_1, a_2, a_3, a_4, and a_5 are its coefficients. The technique as in the previous cases transforms a sextic equation (5.1) in x to the following sextic equation in y

$$y^6 + c_5y^5 + c_4y^4 + c_3y^3 + c_2y^2 + c_1y + c_0 = 0, \tag{5.2}$$

where c_0, c_1, c_2, c_3, c_4, and c_5 are the coefficients of sextic equation (5.2). We wish to make the coefficients c_4 and c_5 zero. The coefficient c_5 is expressed as

$$c_5 = 6b_0 + 2a_4 - a_5^2 - 2a_5b_1. \tag{5.3}$$

Equating c_5 to zero, we obtain an expression for b_0 as

$$b_0 = (2a_5b_1 - 2a_4 + a_5^2)/6. \tag{5.4}$$

The coefficient c_4 is obtained as

$$\begin{aligned} c_4 = 4a_4b_1^2 + (2a_4a_5 - 6a_3)b_1 + a_4^2 + 2a_2 - 2a_3a_5 \\ +(10a_4 - 5a_5^2 - 10a_5b_1)b_0 + 15b_0^2. \end{aligned} \tag{5.5}$$

Equating c_4 to zero and eliminating b_0 from (5.5) using (5.4), we obtain the following quadratic equation in b_1

$$\begin{aligned} [4a_4 - (5a_5^2/3)]b_1^2 + [(16a_4a_5/3) - 6a_3 - (5a_5^3/3)]b_1 \\ +2a_2 + (5a_4a_5^2/3) - (2a_4^2/3) - 2a_3a_5 - (5a_5^4/12) = 0. \end{aligned} \tag{5.6}$$

Since b_1 and b_0 are determined from (5.6) and (5.4) respectively with the condition $c_4 = c_5 = 0$, the sextic equation (5.2) in y has no y^4 and y^5 terms as shown in the following

$$y^6 + c_3y^3 + c_2y^2 + c_1y + c_0 = 0. \tag{5.7}$$

6. **Discussion**

Now we shall analyze the expressions so far obtained for b_0 and b_1, the coefficients of quadratic Tschirnhaus transformation (2.2), for the cases of cubic, quartic, quintic, and sextic equations. The expressions (2.9), (3.7), (4.5) and (5.4), for b_0 indicate that a general formula for b_0 for a polynomial equation of degree N (where $N > 2$) can be written as:

$$b_0 = \frac{1}{N}\left(2a_{N-1}b_1 + a_{N-1}^2 - 2a_{N-2}\right). \tag{6.1}$$

Observing the expressions (2.10), (3.8), (4.7) and (5.6) for quadratic equations in b_1, for the cases of cubic, quartic, quintic, and sextic equations, we note that following is the general formula for the quadratic equation in b_1 for a polynomial equation of degree N ($N > 2$)

$$\begin{aligned} \left[4a_{N-2} - \frac{(2N-2)a_{N-1}^2}{N}\right] b_1^2 + \\ \left[\frac{(6N-4)a_{N-2}a_{N-1}}{N} - 6a_{N-3} - \frac{(2N-2)a_{N-1}^3}{N}\right] b_1 + 2a_{N-4} \\ + \frac{(2N-2)a_{N-2}a_{N-1}^2}{N} - 2a_{N-3}a_{N-1} - \frac{(N-2)a_{N-2}^2}{N} - \frac{(N-1)a_{N-1}^4}{2N} = 0 \end{aligned} \tag{6.2}$$

Observe that for $N = 3$, we encounter a_{-1} in (6.2), implying it is coefficient of x^{-1}; as such this term is absent in a polynomial equation, hence $a_{-1} = 0$. The above general formulas [(6.1) and (6.2)] for b_0 and b_1 derived through mathematical induction will obviate the need to determine these unknowns each time a polynomial equation of newer degree is encountered. Let us solve one numerical example to illustrate the utility of these formulas.

7. A numerical example

Consider the quartic equation

$$x^4 + 5x^3 + 10x^2 + 10x + 4 = 0 \tag{7.1}$$

to be transformed to a quartic equation in y with no y^2 and y^3 terms, using quadratic Tschirnhaus transformation (2.2). The quadratic equation in b_1, obtained from (6.2) with $N = 4$, is given by

$$2.5b_1^2 + 2.5b_1 - 1.375 = 0.$$

Solving above quadratic equation, we obtain two values for b_1 as 0.394427191 and -1.394427191, and the corresponding two values of b_0 determined from (6.1) for $N = 4$ are 2.236067977 and -2.236067977. Using the expressions given in (3.6), we determine two values each of c_0 and c_1, corresponding to two values of b_0 and b_1. Accordingly we obtain two transformed quartic equations in y which have no y^2 and y^3 terms as shown below.

$$y^4 - 32.89968944y - 14.7531884 = 0 \tag{7.2}$$

corresponding to $b_1 = 0.394427191$, and

$$y^4 - 0.700310562y + 0.273188404 = 0 \tag{7.3}$$

corresponding to $b_1 = -1.394427191$. Now the issue is, which of the two quartics [(7.2) or (7.3)] is the true transformed version of the quartic (7.1). The only way is to decide through numerical calculations by finding the roots [of (7.2) or (7.3)], which are related to the roots of (7.1) through expression (2.3).

We determine one root of (7.2) as -0.4472135955; and using (2.3) corresponding value of x is determined as 1.788854382. However this is not the root of (7.1). Therefore the quartic (7.2) cannot be the transformed version of (7.1).

One root of (7.3), evaluated as 0.4472135955, gives a corresponding value of x as -1, using (2.3). This ($x = -1$) happens to be a root of (7.1). Remaining three roots of (7.3) also yield correct roots of (7.1) through the transformation (2.3). Hence, we conclude that (7.3) is the true transformed version of quartic equation (7.1).

Thus the quadratic Tschirnhaus transformation yields two transformed equations, of which one will be true transformed version; however this (true version) can be found out only by numerical methods [4].

8. Conclusions

The paper proves the existence of general formulas for the two unknowns of quadratic Tschirnhaus transformation, which are obtained through mathematical induction. These formulas can be used to eliminate the second and the third terms in a polynomial equation of any degree greater than two.

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ON $A^4 + nB^3 + C^2 = D^4$

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Abstract. In this paper, we give some parametric solutions of the Diophantine equation $A^4 + nB^3 + C^2 = D^4$ such that we can get infinitely many primitive and non-trivial integral solutions for (A, B, C, D). For the sake of completeness, we will include solutions in which A, B, C and D have a common factor.

1. Introduction

Dem'janenko [1] gave the parametric solution of

$$x^4 - y^4 = z^4 + t^2, \tag{1.1}$$

where x, y, z and t are integer variables. Basing on (1.1), Noam Elkies [3] has disproved Euler's conjecture [2] for fourth powers that at least four positive integral fourth powers are required to sum to an integral fourth power, except for the trivial case $y^4 = y^4$. But, the Diophantine equation

$$A^4 + nB^3 + C^2 = D^4, \tag{1.2}$$

where n is any non-zero integer has not been considered yet. Hence, in this paper, we want to give the parametric solutions of (1.2) such that we can get infinitely many primitive and non-trivial integral solutions for (A, B, C, D). To get the primitive solutions of (1.2), we assume that $gcd(A, B, C, D) = 1$. For the sake of completeness, we will include solutions in which A, B, C and D have a common factor.

2. Main Results

To find some parametric solutions of (1.2) we need the following lemma

Lemma 2.1. *For any pair of integers p and q, there is an algebraic identity*

$$\begin{aligned}&(2p^2 - q^2)^4 + 16pq(2p^2 + 2pq + q^2)^3 \\ &\quad + (4pq(2p^2 + 2pq + q^2))^2 = (2p^2 + 4pq + q^2)^4.\end{aligned} \tag{2.1}$$

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Proof. In a modified form, (2.1) is same as

$$\begin{aligned}&(2p^2+4pq+q^2)^4-(2p^2-q^2)^4\\&=16pq(2p^2+2pq+q^2)^3+(4pq(2p^2+2pq+q^2))^2,\end{aligned}\tag{2.2}$$

which can be easily verified. Thus, from (2.2) we get (2.1). □

Now, based on *Lemma* 2.1, we have the following theorem:

Theorem 2.1. *If m,n,t are non-zero integers and $gcd(2mn,t)=1$, the Diophantine equation $A^4+nB^3+C^2=D^4$ has infinitely many non-trivial and primitive solutions of the form* $(A,B,C,D)=\{(32n^2m^6-t^6), 4mt(32n^2m^6+8nm^3t^3+t^6), 16nm^3t^3(32n^2m^6+8nm^3t^3+t^6), (32n^2m^6+16nm^3t^3+t^6)\}$.

Proof. To prove *Theorem* 2.1 we have to establish that

$$\begin{aligned}&(32n^2m^6-t^6)^4+n(4mt(32n^2m^6+8nm^3t^3+t^6))^3\\&+(16nm^3t^3(32n^2m^6+8nm^3t^3+t^6))^2\\&=(32n^2m^6+16nm^3t^3+t^6)^4.\end{aligned}\tag{2.3}$$

Take $p=4nm^3, q=t^3$ in (2.1) to get (2.3). To ensure that the solutions for (A,B,C,D) are primitive, we take $gcd(2mn,t)=1$. For $gcd(2mn,t)\geq 2$, we will get solutions of (1.2) in which A,B,C and D have a common factor. □

Theorem 2.2. *If m,n,t are non-zero integers and $gcd(2m,nt)=1$, the Diophantine equation $A^4+nB^3+C^2=D^4$ has infinitely many non-trivial and primitive solutions of the form* $(A,B,C,D)=\{(32m^6-n^2t^6), 4mt(32m^6+8nm^3t^3+n^2t^6), 16nm^3t^3(32m^6+8nm^3t^3+n^2t^6), (32m^6+16nm^3t^3+n^2t^6)\}$.

Proof. The condition $gcd(2m,nt)=1$ will make both n and t to be odd integers. To prove *Theorem* 2.2 we need to prove that

$$\begin{aligned}&(32m^6-n^2t^6)^4+n(4mt(32m^6+8nm^3t^3+n^2t^6))^3\\&+(16nm^3t^3(32m^6+8nm^3t^3+n^2t^6))^2\\&=(32m^6+16nm^3t^3+n^2t^6)^4.\end{aligned}\tag{2.4}$$

Take $p=4m^3, q=nt^3$ in (2.1) to get (2.4). To ensure that the solutions for (A,B,C,D) are primitive, we take $gcd(2m,nt)=1$. For $gcd(2m,nt)\geq 2$, we will get solutions of (1.2) in which A,B,C and D have a common factor. □

The referee of this note found other three parametric solutions of (1.2) as

(*i*) $(A,B,C,D)=\{a^3n,-na^4,b^2,b\}$;

(*ii*) $(A,B,C,D)=\{a,-nb^2,n^2b^3,a\}$;

(*iii*) $(A,B,C,D)=\{(8n^4t^6-1),4nt^2,64n^6t^9,(8n^4t^6+1)\}$.

The present paper does not give a complete solution of the title equation. There may still exist other parametric solutions which the readers may wish to find out.

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ON KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study Kenmotsu manifolds admitting W_2-curvature tensor and quasi-conformal curvature tensor.

1. Introduction

In 1958, Boothby and Wong [2] studied odd dimensional manifolds with contact and almost contact structures from topological point of view. Sasaki and Hatakeyama [14] re-investigated them using tensor calculus in 1961. S. Tano [19] classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold M^n, the sectional curvature of plane sections containing ξ is a constant, say c. If $c > 0$, M^n is homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M^n is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M^n is warped product space $\mathbb{R} \times_f \mathbb{C}^n$. In 1972, K. Kenmotsu studied a class of contact Riemannian manifold and call them Kenmotsu manifold [9]. He proved that if Kenmotsu manifold satisfies the condition $R(X,Y).R = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type (1, 3) and R(X, Y) denotes the derivation of the tensor algebra at each point of the tangent space. Recently, Kenmotsu manifolds have been studied by several authors such as De [5], Sinha and Shrivastava [18], Jun, De and Pathak [8], De and Pathak [4], De, Yildiz and Yaliniz [6], Özgur and De [11], Chaubey and Ojha [3], Singh, Pandey and Pandey ([15], [16] [17]) and many others. In the present paper we have studied some curvature conditions on Kenmotsu manifolds. In section 2, some preliminary results regarding Kenmotsu manifold are recalled. Section 3 is devoted to study of W_2-recurrent Kenmotsu manifolds. In section 4, we have studied $W_2 - \phi-$recurrent Kenmotsu manifolds. Section 5 deals with example of $W_2 - \phi-$recurrent Kenmotsu manifolds. In the last section, we have studied quasi-conformally ϕ-recurrent Kenmotsu manifolds.

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2. Preliminaries

If on an odd dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifold [1] and the structure (ϕ, ξ, η, g) is called an almost contact metric structure to M^{2n+1}. In view of equations (2.1), (2.2) and (2.3), we have

$$\eta(\xi) = 1, \tag{2.4}$$

$$g(X, \xi) = \eta(X), \tag{2.5}$$

$$\phi(\xi) = 0. \tag{2.6}$$

An almost contact metric manifold is called Kenmotsu manifold [9] if

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.7}$$

$$(\nabla_X \xi) = X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

where ∇ is the Levi-Civita connection of g. Also, the following relations hold in Kenmotsu manifold [4], [6], [8]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(\xi, X)Y = -R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.11}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{2.12}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.13}$$

$$Q\xi = -2n\xi, \tag{2.14}$$

where Q is the Ricci operator, *i.e.* $g(QX, Y) = S(X, Y)$ and

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

for arbitrary vector fields X, Y, Z on M^{2n+1}.
A Kenmotsu manifold M^{2n+1} is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y), \tag{2.16}$$

for arbitrary vector fields X and Y, where λ_1 and λ_2 are smooth functions on M^{2n+1}.
The W_2-curvature tensor of type (0, 4) is defined by

$$'W_2(X,Y,Z,U) = {}'R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)], \tag{2.17}$$

where $'R$ is a Riemannian curvature tensor of type (0, 4) defined by $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and S is the Ricci tensor of type (0, 2).

For a $(2n+1)$-dimensional $(n > 1)$ almost contact metric manifold the W_2-curvature tensor is given by [12]

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[g(X,Z)QY - g(Y,Z)QX]. \tag{2.18}$$

The W_2-curvature tensor for a Kenmotsu manifold is given by

$$W_2(X,Y)\xi = [\eta(Y)X - \eta(X)Y] + \frac{1}{2n}[\eta(X)QY - \eta(Y)QX], \tag{2.19}$$

$$\eta(W_2(X,Y)\xi) = 0, \tag{2.20}$$

$$W_2(\xi,Y)Z = -W_2(Y,\xi)Z = \eta(Z)Y + \frac{1}{2n}\eta(Z)QY, \tag{2.21}$$

$$\eta(W_2(\xi,Y)Z) = -\eta(W_2(Y,\xi)Z) = 0, \tag{2.22}$$

and

$$\eta(W_2(X,Y)Z) = 0. \tag{2.23}$$

The notion of the quasi-conformal curvature tensor $\tilde{C}$ was introduced by Yano and Sawaki [20]. They defined the quasi- conformal curvature tensor by

$$\begin{aligned}\tilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] - \frac{r}{n}\{\frac{a}{n-1} + 2b\}[g(Y,Z)X - g(X,Z)Y],\end{aligned}$$

where a and b are constants such that $ab \neq 0$, R is the Riemannian curvature tensor, S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the manifold. If $a = 1$ and $b = -\frac{1}{n-2}$, then above equation takes the form

$$\begin{aligned}\tilde{C}(X,Y)Z =& R(X,Y)Z - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ =& C(X,Y)Z,\end{aligned}$$

where C is the conformal curvature tensor [20]. Thus the conformal curvature tensor C is a particular case of the quasi -conformal curvature tensor $\tilde{C}$.
For a $(2n+1)$-dimensional $(n>1)$ almost contact metric manifold the quasi-conformal curvature tensor is given by [20]

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] - \frac{r}{(2n+1)}\{\frac{a}{2n}+2b\}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{2.24}$$

Putting $X=\xi$ in equation (2.24) and using equations (2.11) and (2.13), we get

$$\begin{aligned}\tilde{C}(\xi,Y)Z = -\tilde{C}(Y,\xi)Z =& [a+2nb+\frac{r}{(2n+1)}(\frac{a}{2n}+2b)][\eta(Z)Y - g(Y,Z)\xi] \\ &+ b[S(Y,Z)\xi - \eta(Z)QY].\end{aligned} \tag{2.25}$$

Again, put $Z=\xi$ in equation (2.24) and using equations (2.10) and (2.13), we get

$$\begin{aligned}\tilde{C}(X,Y)\xi =& [a+2nb+\frac{r}{(2n+1)}(\frac{a}{2n}+2b)][\eta(X)Y - \eta(Y)X] \\ &+ b[\eta(Y)QX - \eta(X)QY].\end{aligned} \tag{2.26}$$

Now, taking the inner product of equations (2.24), (2.25) and (2.26) with ξ, we get

$$\begin{aligned}\eta(\tilde{C}(X,Y)Z) =& [a+2nb+\frac{r}{(2n+1)}(\frac{a}{2n}+2b)][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ &+ b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)],\end{aligned} \tag{2.27}$$

$$\begin{aligned}\eta(\tilde{C}(\xi,Y)Z) = -\eta(\tilde{C}(Y,\xi)Z) =& [a+2nb+\frac{r}{(2n+1)}(\frac{a}{2n}+2b)][\eta(Z)\eta(Y) - g(Y,Z)] \\ &+ b[S(Y,Z) + \eta(Y)\eta(Z)]\end{aligned} \tag{2.28}$$

and

$$\eta(\tilde{C}(X,Y)\xi) = 0 \tag{2.29}$$

respectively.

3. W_2-recurrent Kenmotsu manifolds

Definition 3.1. A non-flat Riemannian manifold M^{2n+1} is said to be W_2-recurrent if its W_2-curvature tensor satisfies the condition

$$\nabla W_2 = A \otimes W_2, \tag{3.1}$$

where A is a non-zero 1-form.

Definition 3.2. A $(2n+1)-$dimensional $(n>1)$ Kenmotsu manifold is said to be W_2-semisymmetric [7] if it satisfies $R(X,Y).W_2 = 0$, where R(X, Y) is the derivation of the tensor algebra at each point of the tangent space and W_2 is the W_2-curvature tensor of Kenmotsu manifold.

Theorem 3.1. *A W_2-recurrent Kenmotsu manifold is W_2-semisymmetric.*

Proof: We define a function $f^2 = g(W_2, W_2)$ on M^{2n+1}, where the metric g is extended to the inner product between the tensor fields. Then we have

$$f(Yf) = f^2 A(Y).$$

This can be written as

$$Yf = f(A(Y)), (f \neq 0). \tag{3.2}$$

From above equation, we have

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of above equation is identically zero and $f \neq 0$ on M^{2n+1}. Then

$$dA(X,Y) = 0, \tag{3.3}$$

i.e. 1-form A is closed.
Now from

$$(\nabla_Y W_2)(Z,U)V = A(Y)W_2(Z,U)V,$$

we have

$$(\nabla_X \nabla_Y W_2)(Z,U)V = \{XA(Y) + A(X)A(Y)\}W_2)(Z,U)V. \tag{3.4}$$

In view of equations (3.3) and (3.4) , we have

$$\begin{aligned} (R(X,Y).W_2)(Z,U)V &= [2dA(X,Y)]W_2)(Z,U)V \\ &= 0. \end{aligned} \tag{3.5}$$

This completes the proof.

Lemma 3.1 ([21]). *A W_2-semisymmetric Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(-1)$.*

In view of theorem (3.1) and lemma (3.2), we have a theorem as

Theorem 3.2. *A W_2-recurrent Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(-1)$.*

4. $W_2 - \phi$ -recurrent Kenmotsu manifolds

Definition 4.1. A Kenmotsu manifold M^{2n+1} is said to be locally $W_2 - \phi$-symmetric if the relation

$$\phi^2((\nabla_U W_2)(X,Y)Z) = 0, \tag{4.1}$$

holds for all vector fields X, Y, Z, U orthogonal to ξ.

Definition 4.2. A Kenmotsu manifold M^{2n+1} is said to be $W_2 - \phi$-recurrent if and only if there exists a 1-form A such that

$$\phi^2((\nabla_U W_2)(X,Y)Z) = A(U)W_2(X,Y)Z, \tag{4.2}$$

for all vector fields X, Y, Z, U. Here X, Y, Z, U are arbitrary vector fields not necessarily orthogonal to ξ.

Theorem 4.1. *A $W_2-\phi$-recurrent Kenmotsu manifold is an η-Einstein manifold.*

Proof: By virtue of equations (2.1) and (4.2), we have

$$-(\nabla_U W_2)(X,Y)Z + \eta((\nabla_U W_2)(X,Y)Z)\xi = A(U)W_2(X,Y)Z, \tag{4.3}$$

i.e.

$$-g((\nabla_U W_2)(X,Y)Z,V) + \eta((\nabla_U W_2)(X,Y)Z)\eta(V) = A(U)g(W_2(X,Y)Z,V). \tag{4.4}$$

From equation (2.18), we have

$$\begin{aligned} g((\nabla_U W_2)(X,Y)Z,V) = g((\nabla_U R)(X,Y)Z,V) + \frac{1}{2n}[g(X,Z)(\nabla_U S)(Y,V) \\ - g(Y,Z)(\nabla_U S)(X,V)]. \end{aligned} \tag{4.5}$$

Now, from equations (2.18), (4.4) and (4.5), we have

$$\begin{aligned} &- g((\nabla_U R)(X,Y)Z,V) + \eta((\nabla_U R)(X,Y)Z,V)\eta(V) - \frac{1}{2n}[g(X,Z)(\nabla_U S)(Y,V) \\ &- g(Y,Z)(\nabla_U S)(X,V)] + \frac{1}{2n}[g(X,Z)(\nabla_U S)(Y,\xi) - g(Y,Z)(\nabla_U S)(X,\xi)]\eta(V) \\ &= A(U)g(R(X,Y)Z,V) + \frac{1}{2n}A(U)[g(X,Z)S(Y,V) - g(Y,Z)S(X,V)]. \end{aligned} \tag{4.6}$$

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = V = e_i$ in above equation and summing over i, $1 \le i \le 2n+1$, we get

$$\begin{aligned} &-(\frac{2n+1}{2n})(\nabla_U S)(Y,Z) + \sum_{i=1}^{2n+1} \eta(\nabla_U R)(e_i,Y)Z)\eta(e_i) + \frac{dr(U)}{2n}g(Y,Z) \\ &= A(U)S(Y,Z) + \frac{1}{2n}A(U)[S(Y,Z) - rg(Y,Z)] + \eta(Z)[g(Y,U) - \eta(U)\eta(Y)]. \end{aligned} \tag{4.7}$$

Now putting $Z = \xi$ in the second term of above equation and denoting it by L which is of the form

$$L = g((\nabla_U R)(e_i, Y)\xi, \xi)g(e_i, \xi).$$

In this case L vanishes. Namely, we have

$$\begin{aligned} g((\nabla_U R)(e_i, Y)\xi, \xi) &= g(\nabla_U R(e_i, Y)\xi, \xi) - g(R(\nabla_U e_i, Y)\xi, \xi) \\ &\quad - g(R(e_i, \nabla_U Y)\xi, \xi) - g(R(e_i, Y)\nabla_U \xi, \xi) \end{aligned} \tag{4.8}$$

at p $\in$ M. In local co-ordinates $\nabla_U e_i = U^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore $\nabla_U e_i = 0$. Since R is skew-symmetric, we have

$$g(R(e_i, \nabla_U Y)\xi, \xi) = 0. \tag{4.9}$$

Using equation (4.9) and $\nabla_U e_i = 0$ in equation (4.8), we get

$$g((\nabla_U R)(e_i, Y)\xi, \xi) = g(\nabla_U R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_U \xi, \xi). \tag{4.10}$$

In view of $g(R(e_i, Y)\xi, \xi) = g(R(\xi, \xi)e_i, Y) = 0$ and $(\nabla_U g) = 0$, we have

$$g(\nabla_U R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_U \xi) = 0, \tag{4.11}$$

which implies

$$g((\nabla_U R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_U \xi) - g(R(e_i, Y)\nabla_U \xi, \xi).$$

Since R is skew-symmetric, we have

$$g((\nabla_U R)(e_i, Y)\xi, \xi) = 0. \tag{4.12}$$

Replacing $Z = \xi$ in equation (4.7) and using equations (2.4), (2.5), (2.13) and (4.12), we get

$$\begin{aligned} -(\frac{2n+1}{2n})(\nabla_U S)(Y, \xi) &= -(2n+1)A(U)\eta(Y) - \frac{r}{2n}A(U)\eta(Y) \\ &\quad - \frac{dr(U)}{2}\eta(Y) + g(Y, U) - \eta(Y)\eta(U). \end{aligned} \tag{4.13}$$

Now, we have

$$(\nabla_U S)(Y, \xi) = \nabla_U S(Y, \xi) - S(\nabla_U Y, \xi) - S(Y, \nabla_U \xi),$$

which on using equations (2.8) and (2.13) takes the form

$$(\nabla_U S)(Y, \xi) = -[S(Y, U) + 2ng(Y, U)]. \tag{4.14}$$

Form equations (4.13) and (4.14), we have

$$\begin{aligned} (\frac{2n+1}{2n})S(Y, U) + (2n+1)g(Y, U) &= -(2n+1)A(U)\eta(Y) \\ - \frac{r}{2n}A(U)\eta(Y) - \frac{dr(U)}{2}\eta(Y) &+ g(Y, U) - \eta(Y)\eta(U). \end{aligned} \tag{4.15}$$

Now replacing Y by ϕY and U by ϕU in above equation and using equations (2.2), (2.3) and (2.15), we get

$$S(X,U) = \lambda_1 g(X,U) + \lambda_2 \eta(X)\eta(U), \tag{4.16}$$

where $\lambda_1 = \frac{-4n^2}{4n+1}$ and $\lambda_2 = \frac{-2n^2}{2n+1}$ are constant. This shows that M^{2n+1} is an $\eta-$Einstein manifold. □

Lemma 4.1 ([9]). *If in an η-Einstein Kenmotsu manifold of the form*

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y),$$

either λ_1 or λ_2=constant, then the manifold reduces to an Einstein manifold.

Thus in view of theorem (4.1) and lemma (4.2), we have a corollary as

Corollary 4.1. *A $W_2 - \phi$-recurrent Kenmotsu manifold is an Einstein manifold.*

Theorem 4.2. *A locally $W_2 - \phi$-recurrent Kenmotsu manifold is an Einstein manifold.*

Proof: Consider a $W_2 - \phi$-recurrent Kenmotsu manifold. Then by virtue of equations (2.1) and (4.2), we have

$$-(\nabla_U W_2)(X,Y)Z + \eta((\nabla_U W_2)(X,Y)Z)\xi = A(U)W_2(X,Y)Z. \tag{4.17}$$

Putting $Z = \xi$ in above equation and using equation (2.20), we get

$$-(\nabla_U W_2)(X,Y)\xi = A(U)W_2(X,Y)\xi, \tag{4.18}$$

which gives

$$\begin{aligned} &-[(\nabla_U R)(X,Y)\xi + \frac{1}{2n}\{g(X,U)QY - g(Y,U)QX \\ &+ \eta(U)\eta(Y)QX - \eta(U)\eta(X)QY\}] = A(U)[R(X,Y)\xi \\ &+ \frac{1}{2n}\{\eta(X)QY - \eta(Y)QX\}]. \end{aligned} \tag{4.19}$$

Now, in view of equations (2.9) and (2.10), we have

$$(\nabla_U R)(X,Y)\xi = g(X,U)Y - g(Y,U)X - R(X,Y)U. \tag{4.20}$$

Using equation (4.20) in equation (4.19), we get

$$\begin{aligned} &-[g(X,U)Y - g(Y,U)X - R(X,Y)U + \frac{1}{2n}\{g(X,U)QY - g(Y,U)QX \\ &+ \eta(U)\eta(Y)QX - \eta(U)\eta(X)QY\}] = A(U)[R(X,Y)\xi \\ &+ \frac{1}{2n}\{\eta(X)QY - \eta(Y)QX\}]. \end{aligned} \tag{4.21}$$

If X, Y, and U are orthogonal to ξ, then we have

$$R(X,Y)U = g(X,U)Y - g(Y,U)X + \frac{1}{2n}\{g(X,U)QY - g(Y,U)QX\}. \tag{4.22}$$

Taking inner product of above equation with Z, we get

$$\begin{aligned} R(X,Y,U,Z) &= g(X,U)g(Y,Z) - g(Y,U)g(X,Z) \\ &\quad + \frac{1}{2n}\{g(X,U)S(Y,Z) - g(Y,U)S(X,Z)\}. \end{aligned} \tag{4.23}$$

Putting $Y = U = e_i$ and summing over i, $1 \leq i \leq 2n+1$, we get

$$S(X,Z) = -ng(X,Z), \tag{4.24}$$

which shows that M^{2n+1} is an Einstein manifold. This completes the proof. □

By virtue of equations (4.23) and (4.24), we have

$$R(X,Y,U,Z) = \frac{1}{2}[g(X,U)g(Y,Z) - g(Y,U)g(X,Z)]. \tag{4.25}$$

Thus in view of above equation, we have a corollary as

Corollary 4.2. *A locally $W_2 - \phi$-recurrent Kenmotsu manifold is of constant curvature.*

5. Example of W_2-ϕ - recurrent Kenmotsu manifolds

We consider 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3 \mid z \neq 0\}$, where (x,y,z) are standard coordinates of $\mathbb{R}^3$. The vector fields

$$e_1 = \frac{x}{z}\frac{\partial}{\partial x}, \; e_2 = \frac{y}{z}\frac{\partial}{\partial y}, \; e_3 = \frac{\partial}{\partial z} \tag{5.1}$$

are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1,e_1) = 1, \, g(e_1,e_2) = 0, \, g(e_1,e_3) = 0, \, g(e_2,e_2) = 1, \, g(e_2,e_3) = 0, \, g(e_3,e_3) = 1. \tag{5.2}$$

Let η the 1-form defined by $\eta(X) = g(X,\xi)$ for any vector field X. Let ϕ be the (1, 1)-tensor field defined by

$$\phi(e_1) = e_2, \; \phi(e_2) = -e_1, \; \phi(e_3) = 0. \tag{5.3}$$

Then by using the linearity of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$.

Further, $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for any vector fields X and Y. Hence for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact structure on M^3. Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$[e_1,e_2] = 0, \; [e_1,e_3] = \frac{1}{z}e_1, \; [e_2,e_3] = \frac{1}{z}e_2. \tag{5.4}$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned} \tag{5.5}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{aligned}
&\nabla_{e_1}e_1 = -\frac{1}{z}e_1, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = \frac{1}{z}e_1 + e_2 \\
&\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -\frac{1}{z}e_2, \quad \nabla_{e_2}e_3 = \frac{1}{z}e_2 - e_1 \\
&\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.
\end{aligned} \tag{5.6}$$

We observe that the manifold satisfies the condition $\nabla_X\xi = X - \eta(X)\xi$, for $e_3 = \xi$. Hence $M^3(\phi, \xi, \eta, g)$ is 3-dimensional Kenmotsu manifold. By using the above results, we can easily obtain the non-vanishing components of the curvature tensor as follows

$$\begin{aligned}
R(e_1, e_3)e_3 &= \frac{1}{z^2}e_1, \ R(e_3, e_1)e_3 = -\frac{1}{z^2}e_1, \\
R(e_2, e_3)e_3 &= \frac{1}{z^2}e_2, \ R(e_3, e_2)e_3 = -\frac{1}{z^2}e_2.
\end{aligned} \tag{5.7}$$

The definition of Ricci tensor in 3-dimensional manifold M^3 implies

$$S(X, Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i), \tag{5.8}$$

which on using equation (5.7), gives

$$\begin{aligned}
&S(e_1, e_1) = 0, \ S(e_1, e_2) = 0, \ S(e_1, e_3) = 0, \\
&S(e_2, e_2) = 0, \ S(e_2, e_3) = 0, \ S(e_3, e_3) = \frac{2}{z^2}.
\end{aligned} \tag{5.9}$$

In view of equations (2.18), (5.7) and (5.9), we have

$$\begin{aligned}
W_2(e_1, e_3)e_3 &= \frac{1}{z^2}e_1 - \frac{1}{2}Qe_1, W_2(e_3, e_1)e_3 = -\frac{1}{z^2}e_1 + \frac{1}{2}Qe_1, \\
W_2(e_2, e_3)e_3 &= \frac{1}{z^2}e_2 - \frac{1}{2}Qe_2, W_2(e_3, e_2)e_3 = -\frac{1}{z^2}e_2 + \frac{1}{2}Qe_2.
\end{aligned} \tag{5.10}$$

The vectors $\{e_1, e_2, e_3\}$ forms a basis of M^3 and so any vector X can be expressed as $X = a_1e_1 + a_2e_2 + a_3e_3$, where $a_i \in \mathbb{R}^+, i = 1, 2, 3$. By virtue of equations (5.6), (5.7) and (5.10), we have

$$(\nabla_X W_2)(e_1, e_3)e_3 = -\frac{a_1}{z^3}e_1 + \frac{a_1}{2z}Qe_1 \tag{5.11}$$

Applying ϕ^2 on both sides of above equation and using equations (5.3) and (5.10), we get

$$\phi^2((\nabla_X W_2)(e_1, e_3)e_3) = A(X)W_2(e_1, e_3)e_3, \tag{5.12}$$

where $A(X) = \frac{a_1}{z}$ is a non-vanishing 1-form. This shows that there exists a $W_2 - \phi-$recurrent Kenmotsu manifold of dimension 3.

6. Quasi-conformally ϕ - recurrent Kenmotsu manifolds

Definition 6.1. A Kenmotsu manifold M^{2n+1} is said to be locally quasi-conformally ϕ-symmetric if the relation

$$\phi^2((\nabla_U \tilde{C})(X,Y)Z) = 0, \tag{6.1}$$

holds for all vector fields X, Y, Z, U orthogonal to ξ.

Definition 6.2. A Kenmotsu manifold M^{2n+1} is said to be quasi-conformally ϕ-recurrent if and only if there exists a 1-form A such that

$$\phi^2((\nabla_U \tilde{C})(X,Y)Z) = A(U)\tilde{C}(X,Y)Z, \tag{6.2}$$

for all vector fields X, Y, Z, U. Here X, Y, Z, U are arbitrary vector fields not necessarily orthogonal to ξ.

Theorem 6.1. *A quasi-conformally $\phi-$recurrent Kenmotsu manifold M^{2n+1} is an $\eta-$Einstein manifold.*

Proof: By virtue of equations (2.1) and (6.2), we have

$$-((\nabla_U \tilde{C})(X,Y)Z) + \eta((\nabla_U \tilde{C})(X,Y)Z)\xi = A(U)\tilde{C}(X,Y)Z. \tag{6.3}$$

Taking inner product of above equation with V, we get

$$-g((\nabla_U \tilde{C})(X,Y)Z,V) + \eta((\nabla_U \tilde{C})(X,Y)Z)\eta(V) = A(U)g(\tilde{C}(X,Y)Z,V). \tag{6.4}$$

In view of equation (2.24), we have

$$\begin{aligned} &g((\nabla_U \tilde{C})(X,Y)Z,V) = ag((\nabla_U R)(X,Y)Z,V) + b\{(\nabla_U S)(Y,Z)g(X,V) \\ &- (\nabla_U S)(X,Z)g(Y,V) + (\nabla_U S)(X,V)g(Y,Z) - (\nabla_U S)(Y,V)g(X,Z)\} \\ &- \frac{dr(U)}{(2n+1)}(\frac{a}{2n} + 2b)[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)]. \end{aligned} \tag{6.5}$$

From equations (6.4) and (6.5), we have

$$\begin{aligned} &- ag((\nabla_U R)(X,Y)Z,V) + a\eta((\nabla_U R)(X,Y)Z)\eta(V) - b\{(\nabla_U S)(Y,Z)g(X,V) \\ &- (\nabla_U S)(X,Z)g(Y,V) + (\nabla_U S)(X,V)g(Y,Z) - (\nabla_U S)(Y,V)g(X,Z)\} \\ &+ b\{(\nabla_U S)(Y,Z)\eta(X) - (\nabla_U S)(X,Z)\eta(Y) + (\nabla_U S)(X,\xi)g(Y,Z) \\ &- (\nabla_U S)(Y,\xi)g(X,Z)\}\eta(V) + \frac{dr(U)}{(2n+1)}(\frac{a}{2n} + 2b)[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)] \\ &- \frac{dr(U)}{(2n+1)}(\frac{a}{2n} + 2b)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = aA(U)g(R(X,Y)Z,V) \\ &+ bA(U)\{S(Y,Z)g(X,V) - S(X,Z)g(Y,V) + g(Y,Z)S(X,V) - g(X,Z)S(Y,V)\} \\ &- \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)A(U)[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)]. \end{aligned} \tag{6.6}$$

Let $\{e_i\}, i = 1, 2, ..., 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = V = e_i$ in above equation and summing over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned}
&-(a+2b(n-1))(\nabla_U S)(Y,Z) + a\sum_{i=1}^{2n+1}\eta((\nabla_U R)(e_i,Y)Z)\eta(e_i) \\
&+ b[2n\{g(Z,U)\eta(Y) + g(Y,U)\eta(Z) - 2\eta(Y)\eta(Z)\eta(U)\} \\
&- g(Y,Z)dr(U)] + \frac{dr(U)}{(2n+1)}(\frac{a}{2n} + 2b)[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)] \\
&= (a+b(2n-1))S(Y,Z)A(U) + bA(U)rg(Y,Z) - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)2nA(U)g(Y,Z).
\end{aligned} \tag{6.7}$$

Replacing $Z = \xi$ in equation (6.7) and using equations (2.4), (2.5), (2.13) and (4.12), we get

$$\begin{aligned}
&-(a+2b(n-1))(\nabla_U S)(Y,\xi) + b[2n\{g(Y,U) - \eta(Y)eta(U)\} - \eta(Y)dr(U)] \\
&+ \frac{dr(U)}{(2n+1)}(\frac{a}{2n} + 2b)2n\eta(Y) = (a+b(2n-1))S(Y,\xi)A(U) \\
&+ bA(U)r\eta(Y) - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)2nA(U)\eta(Y).
\end{aligned} \tag{6.8}$$

Now, using equation (4.14) in above equation, we get

$$\begin{aligned}
&(a+2b(n-1))[S(Y,U) + 2ng(Y,U)] + 2nb\{g(Y,U) - \eta(Y)eta(U)\} - b\eta(Y)dr(U) \\
&+ \frac{dr(U)}{(2n+1)}(\frac{a}{2n} + 2b)2n\eta(Y) = -2n(a+b(2n-1))\eta(Y)A(U) \\
&+ bA(U)r\eta(Y) - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)2nA(U)\eta(Y).
\end{aligned} \tag{6.9}$$

Now replacing Y by ϕY and U by ϕU in above equation and using equations (2.2),(2.3) and (2.15) we get

$$S(Y,U) = \lambda_1 g(Y,U) + \lambda_2\eta(Y)\eta(U), \tag{6.10}$$

where $\lambda_1 = \frac{-2nb-2n(a+2b(n-1))}{a+2b(n-1)}$ and $\lambda_2 = \frac{2nb}{a+2b(n-1)}$ are constants. This shows that M^{2n+1} is an η-Einstein manifold. □

Now, in view of lemma (4.2) and theorem (5.1), we have a corollary as

Corollary 6.1. *A quasi-conformally $\phi-$recurrent Kenmotsu manifold M^{2n+1} is an Einstein manifold.*

Theorem 6.2. *A locally quasi-conformally $\phi-$recurrent Kenmotsu manifold M^{2n+1} is an Einstein manifold.*

Proof: By virtue of equations (2.1) and (6.1), we have

$$-(\nabla_U \tilde{C})(X,Y)Z + \eta((\nabla_U \tilde{C})(X,Y)Z)\xi = 0. \tag{6.11}$$

Replacing $Z = \xi$ in above equation, we get

$$-(\nabla_U \tilde{C})(X,Y)\xi + \eta((\nabla_U \tilde{C})(X,Y)\xi)\xi = 0. \tag{6.12}$$

Now, from equation (2.24), we have

$$\begin{aligned}(\nabla_U \tilde{C})(X,Y)\xi &= a(\nabla_U R)(X,Y)\xi + b[(\nabla_U S)(Y,\xi)X - (\nabla_U S)(X,\xi)Y \\ &+ \eta(Y)(\nabla_U Q)X - \eta(X)(\nabla_U Q)Y] - \frac{dr(U)}{(2n+1)}\{\frac{a}{2n} + 2b\}[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{6.13}$$

Also, in view of equation (4.20), we have

$$(\nabla_U R)(X,Y)\xi = g(X,U)Y - g(Y,U)X - R(X,Y)U.$$

Taking the inner product of above equation with ξ, we get

$$\eta((\nabla_U R)(X,Y)\xi) = 0. \tag{6.14}$$

From equations (6.13) and (6.14), we have

$$\eta((\nabla_U \tilde{C})(X,Y)\xi) = 0. \tag{6.15}$$

Now, using equation (6.15) in equation (6.12), we get

$$\begin{aligned}&a(\nabla_U R)(X,Y)\xi + b\{(\nabla_U S)(Y,\xi)X - (\nabla_U S)(X,\xi)Y + \eta(Y)(\nabla_U Q)(X) \\ &- \eta(X)(\nabla_U Q)(Y)\} - \frac{dr(U)}{(2n+1)}(\frac{a}{2n} + 2b)[\eta(Y)X - \eta(X)Y] = 0.\end{aligned} \tag{6.16}$$

If X, Y are orthogonal to ξ, then above equation takes the form

$$a(\nabla_U R)(X,Y)\xi + b\{(\nabla_U S)(Y,\xi)X - (\nabla_U S)(X,\xi)Y\} = 0, \tag{6.17}$$

which on using equations (4.14) and (4.20), gives

$$aR(X,Y)U = (a+2nb)\{g(X,U)Y - g(Y,U)X\} - b\{S(Y,U)X - S(X,U)Y\}. \tag{6.18}$$

Taking the inner product of above equation with V, we get

$$\begin{aligned}aR(X,Y,U,V) &= (a+2nb)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\} \\ &- b\{S(Y,U)g(X,V) - S(X,U)g(Y,V)\}.\end{aligned} \tag{6.19}$$

Putting $X = V = e_i$ in above equation and summing over i, $1 \leq i \leq 2n+1$, we get

$$S(Y,U) = -2ng(Y,U), \tag{6.20}$$

which shows that M^{2n+1} is an Einstein manifold. This completes the proof. □

Now, using equation (6.20) in equation (6.19), we get

$$R(X, Y, U, V) = -[g(Y, U)g(X, V) - g(X, U)g(Y, V)]. \tag{6.21}$$

Thus we have a corollary as

Corollary 6.2. *A locally quasi-conformally $\phi-$recurrent Kenmotsu manifold M^{2n+1} is a manifold of constant curvature -1, i.e. M^{2n+1} is locally a hyperbolic space.*

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A CHARACTERIZATION OF A PRIME P FROM THE BINOMIAL COEFFICIENT $\binom{n}{p}$

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Abstract. We complete a proof of a theorem that was inspired by an Indian Olympiad problem, which gives an interesting characterization of a prime number p with respect to the binomial coefficients $\binom{n}{p}$. We also derive a related result which generalizes the theorem in one direction.

1. Introduction and Motivation

Problem 1.1. 7 divides $\binom{n}{7} - \lfloor \frac{n}{7} \rfloor$, $\forall n \in \mathbb{N}$.

The above appeared as a problem in the Regional Mathematical Olympiad, India in 2003. Later in 2007, a similar type of problem was set in the undergraduate admission test of Chennai Mathematical Institute, a premier research institute of India where 7 was replaced by 3.

This became the basis of the following

Theorem 1.1 ([3], Saikia-Vogrinc)**.** *A natural number $p > 1$ is a prime if and only if $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p for every non-negative n, where $n > p + 1$ and the symbols have their usual meanings.*

Proof of Theorem 1.1

In [3], the above theorem is proved. The authors give three different proofs, however the third proof is incomplete. We present below a completed version of that proof.

Proof. First we assume that p is prime. Now we consider n as $n = ap + b$ where a is a non-negative integer and b an integer $0 \le b < p$. Obviously,

$$\left\lfloor \frac{n}{p} \right\rfloor = \lfloor \frac{ap+b}{p} \rfloor \equiv a \ (mod \ p). \tag{1.1}$$

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Now let us calculate $\binom{n}{p}$ $(mod\ p)$.

$$\begin{aligned}\binom{n}{p} &= \binom{ap+b}{p} \\ &= \frac{(ap+b)\cdot(ap+b-1)\cdots(ap+1)\cdot ap\cdot(ap-1)\cdots(ap+b-p+1)}{p\cdot(p-1)\cdots 2\cdot 1} \\ &= \frac{a\cdot(ap+b)\cdot(ap+b-1)\cdots(ap+1)\cdot(ap-1)\cdots(ap+b-p+1)}{(p-1)\cdot(p-2)\cdots 2\cdot 1} \\ &= \frac{aX}{(p-1)!}\end{aligned}$$

where $X = (ap+b)\cdot(ap+b-1)\cdots(ap+1)\cdot(ap-1)\cdots(ap+b-p+1)$.

We observe that there are $(p-1)$ terms in X and each of them has one of the following forms,
(a) $ap + r_1$, or
(b) $ap - r_2$
where $1 \leq r_1 \leq b$ and $1 \leq r_2 \leq (p-1-b)$.
Thus any two terms from either (a) or (b) differs by a number strictly less than p and hence not congruent modulo p. Similarly, if we take two numbers - one from (a) and the other from (b), it is easily seen that the difference between the two would be $r_1 + r_2$ which is at most $(p-1)$ (by the bounds for r_1 and r_2); thus in this case too we find that the two numbers are not congruent modulo p. Thus the terms in X forms a reduced residue system modulo p and so, we have,

$$X \equiv (p-1)!\ (mod\ p). \tag{1.2}$$

Thus, using (1.2) we obtain

$$\binom{n}{p} = a\frac{X}{(p-1)!} \equiv a(mod\ p). \tag{1.3}$$

So, (1.1) and (1.3) together gives

$$\left\lfloor \frac{n}{p} \right\rfloor \equiv \binom{n}{p}\ (mod\ p). \tag{1.4}$$

This proves forward implication.

To prove the reverse implication, we adopt a contrapositive argument meaning that if p were not prime (that is composite) then we must construct an n such that (1.4) does not hold. So, let q be a prime factor of p. We write p as $p = q^x k$, where $(q,k) = 1$. In other words, x is the largest power of q such that $q^x | p$ but $q^{x+1} \nmid p$ (in notation, $q^x || p$). By taking, $n = p + q = q^x k + q$, we have

$$\binom{p+q}{p} = \binom{p+q}{q} = \frac{(q^x k+q)(q^x k+q-1)\ldots(q^x k+1)}{q!}$$

which after simplifying the fraction equals $(q^{x-1}k+1)\frac{(q^xk+q-1)\dots(q^xk+1)}{(q-1)!}$. Clearly, $(q^xk+q-1)\dots(q^xk+1) \equiv (q-1)! \not\equiv 0 \ (mod\ q^x)$. Therefore,

$$\frac{(q^xk+q-1)\dots(q^xk+1)}{(q-1)!} \equiv 1 \ (mod\ q^x)$$

and

$$\binom{p+q}{p} \equiv q^{x-1}k+1 \ (mod\ q^x).$$

On the other hand, obviously

$$\left\lfloor \frac{p+q}{p} \right\rfloor = \left\lfloor \frac{q^xk+q}{q^xk} \right\rfloor \equiv 1 \ (mod\ q^x).$$

Now, since $(q,k)=1$, it follows that $q^{x-1}k+1 \not\equiv 1 \ (mod\ q^x)$. So we conclude,

$$\binom{p+q}{p} \not\equiv \left\lfloor \frac{p+q}{p} \right\rfloor \ (mod\ q^x). \tag{1.5}$$

So, $p \nmid (\binom{p+q}{p} - \lfloor \frac{p+q}{p} \rfloor)$, for if $p|(\binom{p+q}{p} - \lfloor \frac{p+q}{p} \rfloor)$, then since $q^x|p$, we would have $q^x|(\binom{p+q}{p} - \lfloor \frac{p+q}{p} \rfloor)$, a contradiction to (5). Thus, $\binom{p+q}{p} \not\equiv \lfloor \frac{p+q}{p} \rfloor \ (mod\ p)$. Hence we are through with the reverse implication too.

This completes the proof of Theorem 1.1. □

2. Another simple result

We state and prove the following simple result which generalizes one part of Theorem 1.1

Theorem 2.1. *For $n = ap + b = a_{(k)}p^k + b_{(k)}$, we have*

$$\binom{a_{(k)}p^k + b_{(k)}}{p^k} - \left\lfloor \frac{a_{(k)}p^k + b_{(k)}}{p^k} \right\rfloor \equiv 0 \ (\mathrm{mod}\ p)$$

with p a prime, $0 \le b_{(k)} \le p^k - 1$ and k a positive integer such that $1 \le k \le l$, where

$$n = a_0 + a_1p + \ldots + a_kp^k + a_{k+1}p^{k+1} + \ldots + a_lp^l$$

and for $k \ge 1$

$$a_{(k)} = a_k + a_{k+1}p + \ldots + a_lp^{l-k}$$

and

$$b_{(k)} = a_0 + a_1p + \ldots + a_{k-1}p^{k-1}.$$

.

The proof of this follows from the reasoning of the proof of Theorem 1.1 although there are some subtleties. In particular, we have

$$a = a_{(1)} = a_1 + a_2p + \ldots + a_lp^{l-1}$$

and

$$b = b_{(0)} = a_0.$$

For $k=0$, we set the convention that $a_{(0)} = n = a_0 + a_1 p + \ldots + a_l p^l$ and $b_{(0)} = 0$. Notice that Theorem 2.1 is obviously true for $k=0$. But the case $k=0$ doesn't correspond really to a power of p where p is a prime.

Proof: We have

$$\binom{n}{p^k} = \binom{a_{(k)}p^k + b_{(k)}}{p^k}$$

$$= \frac{(a_{(k)}p^k + b_{(k)}) \cdot (a_{(k)}p^k + b_{(k)} - 1) \cdots (a_{(k)}p^k + 1) \cdot a_{(k)}p^k \cdot (a_{(k)}p^k - 1) \cdots (a_{(k)}p^k + b_{(k)} - p^k + 1)}{p^k \cdot (p^k - 1) \cdots 2 \cdot 1}$$

$$= \frac{a_{(k)} \cdot (a_{(k)}p^k + b_{(k)}) \cdot (a_{(k)}p^k + b_{(k)} - 1) \cdots (a_{(k)}p^k + 1) \cdot (a_{(k)}p^k - 1) \cdots (a_{(k)}p^k + b_{(k)} - p^k + 1)}{(p^k - 1) \cdot (p^k - 2) \cdots 2 \cdot 1}.$$

Thus we obtain

$$(p^k - 1)! \binom{n}{p^k} = a_{(k)} \left(\prod_{r=1}^{b} (a_{(k)}p^k + r) \right) \left(\prod_{r=1}^{p^k - 1 - b} (a_{(k)}p^k - r) \right).$$

Or $a_{(k)}p^k + r \equiv r \pmod{p^k}$ and $a_{(k)}p^k - r \equiv -r \equiv p^k - r \pmod{p^k}$ with $0 < r < p^k$. It follows

$$\left(\prod_{r=1}^{b} (a_{(k)}p^k + r) \right) \left(\prod_{r=1}^{p^k - 1 - b} (a_{(k)}p^k - r) \right) \equiv \left(\prod_{r=1}^{b} r \right) \left(\prod_{r=1}^{p^k - 1 - b} (p^k - r) \right) \pmod{p^k}.$$

Since

$$\left(\prod_{r=1}^{b} r \right) \left(\prod_{r=1}^{p^k - 1 - b} (p^k - r) \right) = \left(\prod_{r=1}^{b} r \right) \left(\prod_{r=b+1}^{p^k - 1} r \right) = \prod_{r=1}^{p^k - 1} r = (p^k - 1)!$$

we have

$$\left(\prod_{r=1}^{b} (a_{(k)}p^k + r) \right) \left(\prod_{r=1}^{p^k - 1 - b} (a_{(k)}p^k - r) \right) \equiv (p^k - 1)! \pmod{p^k}.$$

We can notice that,

$$(p^k - 1)! = q(p-1)!\, p^{1+p+\ldots+p^{k-1}-k}$$

with $\gcd(p,q) = 1$ and because $\mathrm{ord}_p((p^k - 1)!) = 1 + p + \ldots + p^{k-1} - k$. Therefore we have

$$a_{(k)} c_{(k)} p^{k(p^k - 1)} + (p^k - 1)! \left\{ a_{(k)} - \binom{n}{p^k} \right\} = 0.$$

Equivalently

$$a_{(k)} c_{(k)} p^{k(p-1)(1+p+\ldots+p^{k-1})} + q(p-1)!\, p^{1+p+\ldots+p^{k-1}-k} \left\{ a_{(k)} - \binom{n}{p^k} \right\} = 0.$$

Dividing the above equation by $p^{1+p+\ldots+p^{k-1}-k}$ we have

$$q(p-1)! \left\{ a_{(k)} - \binom{n}{p^k} \right\} + a_{(k)} c_{(k)} p^{k+(k(p-1)-1)(1+p+\ldots+p^{k-1})} = 0.$$

Thus
$$q(p-1)!\left\{a_{(k)}-\binom{n}{p^k}\right\}\equiv 0 \pmod{p^k}.$$
Since if $m\equiv n \pmod{p^k}$ implies $m\equiv n \pmod{p}$ (the converse is not always true), we also have
$$q(p-1)!\left\{a_{(k)}-\binom{n}{p^k}\right\}\equiv 0 \pmod{p}.$$
As $q(p-1)!$ with $\gcd(p,q)=1$ and p are relatively prime, we get
$$\binom{n}{p^k}-a_{(k)}\equiv 0 \pmod{p}.$$

We finally have
$$\binom{n}{p^k}\equiv\left\lfloor\frac{n}{p^k}\right\rfloor \pmod{p}.$$
□

Theorem 2.2. *Let p be a prime number, let k be a natural number and let x be a positive integer such that*
$$x\equiv r \pmod{p^k}$$
with $0\leq r<p^k$. Denoting $q=\lfloor\frac{x}{p^k}\rfloor$ the quotient of the division of x by p^k, if there exists $s\in\mathbb{N}^\star$ for which
$$\lfloor\frac{x}{p^{ks}}\rfloor=q^s$$
then we have
$$x\equiv r \pmod{p^{ks}}.$$

Proof: Given p a prime number, let x be a positive integer such that $x\equiv r \pmod{p^k}$ with $0\leq r<p^k$. Denoting $q=\lfloor\frac{x}{p^k}\rfloor$, we assume that there exists $s\in\mathbb{N}^\star$ for which
$$\lfloor\frac{x}{p^{ks}}\rfloor=q^s.$$
If $k=0$, the result is obvious since for all integers x,r, we have $x\equiv r \pmod{1}$. In the following, we assume that $k\in\mathbb{N}^\star$.

Then, we have
$$x=qp^k+r$$
and
$$x=q^sp^{ks}+r'$$
with $0\leq r'<p^{ks}$. It comes that
$$qp^k+r=q^sp^{ks}+r'.$$
So ($s\in\mathbb{N}^\star$)
$$q^sp^{ks}-qp^k=r-r'\geq 0.$$

Since $0 \leq r < p^k$, we have

$$0 \leq r - r' < p^k.$$

Moreover, rewriting the equality $q^s p^{ks} - qp^k = r - r'$ as

$$qp^k(q^{s-1}p^{k(s-1)} - 1) = r - r'$$

we can notice that $p^k | r - r'$. Since $0 \leq r - r' < p^k$, it is only possible if $r - r' = 0$ and so

$$r = r'.$$

From the equality $x = q^s p^{ks} + r'$, we deduce that

$$x \equiv r \pmod{p^{ks}}.$$

□

A consequence of the Theorem 2.2 is that if an integer y is congruent to a positive integer x modulo p^k such that $x \equiv r \pmod{p^k}$, provided the conditions stated in the Theorem 2.2 are fulfilled, we have also $y \equiv r \pmod{p^{ks}}$.

It can be verified easily that the product $\left(\prod_{r=1}^{b}(a_{(k)}p^k + r)\right)$ $\left(\prod_{r=1}^{p^k-1-b}(a_{(k)}p^k - r)\right)$ contains the term $(a_{(k)}p^k)^{p^k-1} = a_{(k)}^{p^k-1}p^{k(p^k-1)}$. The term $(a_{(k)}p^k)^{p^k-1}$ is the only term in $p^{k(p^k-1)}$ which appears in the decomposition of this product into sum of linear combination of powers of p. Notice also that the number $k(p^k - 1)$ is the greatest exponent of p in this product when we decompose this product into sum of linear combination of powers of p (like a polynomial expression in variable p). Afterwards, we write $a_{(k)}^{p^k-1}$ as $c_{(k)}$ in order to simplify the notation. Thus, the quotient of the division of this product by $p^{k(p^k-1)}$ is $c_{(k)} = a_{(k)}^{p^k-1}$.

So, from the Theorem 2.2, we can now write

$$\left(\prod_{r=1}^{b}(a_{(k)}p^k + r)\right)\left(\prod_{r=1}^{p^k-1-b}(a_{(k)}p^k - r)\right) = c_{(k)}p^{k(p^k-1)} + (p^k - 1)!.$$

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ON THE VECTOR VALUED FUNCTIONS OF BOUNDED VARIATION IN TWO VARIABLES

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Abstract. Here, it is observed that $(\Lambda^1,\Lambda^2)^*BV(p(n),[a_1,b_1]\times[a_2,b_2],\mathbb{B})$, the class of functions of bounded $p(n)-(\Lambda^1,\Lambda^2)^*-$ variation from $[a_1,b_1]\times[a_2,b_2]$ into a commutative unital Banach algebra $\mathbb{B}$, is a commutative unital Banach algebra with respect to the pointwise operations and the generalized variation norm.

1. Introduction

The notion of bounded variation has undergone generalizations in several directions. It is known that, when equipped with certain natural norms involving the generalized variation, linear spaces of functions of generalized bounded variation become Banach spaces (see [2, 9, 10]). Recently, it is proved that $(\Lambda^1,...,\Lambda^N)^*BV^{(p)}(\Pi_{i=1}^N\sigma_i,\mathbb{B})$, the class of functions of bounded $p-(\Lambda^1,...,\Lambda^N)^*-$ variation from $\Pi_{i=1}^N\sigma_i$ into a commutative unital Banach algebra $\mathbb{B}$, is a Banach space, where σ_i are non-empty compact subsets of $\mathbb{R}$, for $i=1$ *to* N. In many cases, the norms are also submultiplicative, and so the function spaces carries the additional structure of commutative unital Banach algebras with respect to pointwise multiplication of the functions. Berkson and Gillespie [4] proved that $BV_H([a_1,b_1]\times[a_2,b_2],\mathbb{C})$, the class of complex valued functions of bounded variation (in the sense of Hardy) over $[a_1,b_1]\times[a_2,b_2]$, is a commutative unital Banach algebra with respect to the pointwise operations and the variation norm. Generalizing this result, here, it is proved that $(\Lambda^1,\Lambda^2)^*BV(p(n),[a_1,b_1]\times[a_2,b_2],\mathbb{B})$, the class of functions of bounded $p(n)-(\Lambda^1,\Lambda^2)^*-$variation from $[a_1,b_1]\times[a_2,b_2]$ into a commutative unital Banach algebra $\mathbb{B}$, is a commutative unital Banach algebra with respect to the pointwise operations and the generalized variation norm.

In the sequel, $\varphi(n)$ is a real sequence such that $\varphi(1)\geq 2$, $\varphi(n)\to\infty$ as $n\to\infty$ and $\mathbb{L}$ is the class of non-decreasing sequences $\Lambda=\{\lambda_i\}_{i=1}^\infty$ of positive numbers such that $\sum_i\frac{1}{\lambda_i}$ diverges.

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2. THE CLASS $\Lambda BV(p(n),[a,b],\mathbb{B})$

Definition 2.1. For given Banach algebra $\mathbb{B}$, $\Lambda=\{\lambda_i\}_{i=1}^{\infty}\in\mathbb{L}$ and $1\le$p(n)$\uparrow p$ as n$\to\infty$ $(1\le p\le\infty)$, we say that a function f from an interval $I=[a,b]$ into $\mathbb{B}$ is of bounded $p(n)-\Lambda-$variation relative to φ (that is, $f\in\Lambda BV(p(n),I,\mathbb{B})$) if

$$V_{\Lambda_{p(n)}}(f,I,\mathbb{B})=\begin{matrix} sup & sup \\ n\ge 1 & \{I_i\}\end{matrix}\left\{V_{\Lambda_{p(n)}}(f,\mathbb{B},\{I_i\}):\delta\{I_i\}\ge\frac{b-a}{\varphi(n)}\right\}<\infty,$$

where $\{I_i\}=\{[s_{i-1},s_i]\}$ is a finite collection of non-overlapping subintervals in I,

$$\delta\{I_i\}=\begin{matrix} inf \\ 1\le i\le m\end{matrix}\mid s_i-s_{i-1}\mid$$

and

$$V_{\Lambda_{p(n)}}(f,\mathbb{B},\{I_i\})=\left(\sum_{i=1}^{m}\frac{\|f(I_i)\|_{\mathbb{B}}^{p(n)}}{\lambda_i}\right)^{1/p(n)},$$

in which $f(I_i)=f(s_i)-f(s_{i-1})$.

In Definition 2.1, if λ_i =1, for all i, one gets the class $BV(p(n),I,\mathbb{B})$; if $p=\infty$, one gets the class $\Lambda BV(p(n)\uparrow\infty,I,\mathbb{B})$; and if $p(n)=p$, for all n, $(1\le p<\infty)$ one gets the class $\Lambda BV^{(p)}(I,\mathbb{B})$. For $\mathbb{B}=\mathbb{C}$, we omit writing $\mathbb{C}$, the class $\Lambda BV(p(n),I,\mathbb{B})$ reduces to the class $\Lambda BV(p(n),I)$ [8, Definition 1.1, p.215]

We prove the following theorem.

Theorem 2.1. *Let $\mathbb{B}$ be a commutative unital Banach algebra. The class $\Lambda BV(p(n),[a,b],\mathbb{B})$ is a commutative unital Banach algebra with respect to the pointwise operations and the variation norm*:

$$\|f\|=\|f\|_{\infty}+V_{\Lambda_{p(n)}}(f,[a,b],\mathbb{B}),\ f\in\Lambda BV(p(n),[a,b],\mathbb{B}).$$

For $p(n)=p$, for all n, Theorem 2.1 gives the result [10, Corollary 2.7, p.183] as a particular case.

Proof of Theorem 2.1. Let $\{f_k\}$ be any Cauchy sequence in $\Lambda BV(p(n),[a,b],\mathbb{B})$. Then it converges uniformly to some function say f on $I=[a,b]$. For any finite collection of non-overlapping subintervals $\{I_i\}$ in I, we get

$$\begin{aligned}V_{\Lambda_{p(n)}}(f_k,\mathbb{B},\{I_i\})&\le V_{\Lambda_{p(n)}}(f_k-f_l,\mathbb{B},\{I_i\})+V_{\Lambda_{p(n)}}(f_l,\mathbb{B},\{I_i\})\\&\le V_{\Lambda_{p(n)}}(f_k-f_l,I,\mathbb{B})+V_{\Lambda_{p(n)}}(f_l,I,\mathbb{B}).\end{aligned}$$

This implies,

$$V_{\Lambda_{p(n)}}(f_k,I,\mathbb{B})\le V_{\Lambda_{p(n)}}(f_k-f_l,I,\mathbb{B})+V_{\Lambda_{p(n)}}(f_l,I,\mathbb{B})$$

and

$$|V_{\Lambda_{p(n)}}(f_k,I,\mathbb{B})-V_{\Lambda_{p(n)}}(f_l,I,\mathbb{B})|\le V_{\Lambda_{p(n)}}(f_k-f_l,I,\mathbb{B})\to 0\ as\ k,l\to\infty.$$

Hence, $\{V_{\Lambda_{p(n)}}(f_k, I, \mathbb{B})\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore

$$\begin{aligned} V_{\Lambda_{p(n)}}(f, \mathbb{B}, \{I_i\}) &= \lim_{k\to\infty} V_{\Lambda_{p(n)}}(f_k, \mathbb{B}, \{I_i\}) \\ &\leq \lim_{k\to\infty} V_{\Lambda_{p(n)}}(f_k, I, \mathbb{B}) \leq M < \infty. \end{aligned}$$

Thus, $f \in \Lambda BV(p(n), I, \mathbb{B})$.

Since $\{f_k\}$ is a Cauchy sequence, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$V_{\Lambda_{p(n)}}(f_k - f_l, \mathbb{B}, \{I_i\}) < \epsilon,\ for\ all\ k, l \geq n_0.$$

Letting $l \to \infty$ and taking supremum on the both sides of the above inequality, we get $V_{\Lambda_{p(n)}}(f_k - f, I, \mathbb{B}) \leq \epsilon,\ for\ all\ k \geq n_0$.

Thus, $\|f_k - f\| \to 0\ as\ k \to \infty$.

Hence, $(\Lambda BV(p(n), I, \mathbb{B}), \|.\|)$ is a Banach space.

For any $f, g \in \Lambda BV(p(n), I, \mathbb{B})$,

$$\begin{aligned} &\sum_{i=1}^{m} \frac{\|fg(s_i) - fg(s_{i-1})\|_{\mathbb{B}}^{p(n)}}{\lambda_i} \\ &= \sum_{i=1}^{m} \frac{\|f(s_i)g(s_i) - f(s_{i-1})g(s_i) + f(s_{i-1})g(s_i) - f(s_{i-1})g(s_{i-1})\|_{\mathbb{B}}^{p(n)}}{\lambda_i} \\ &= \sum_{i=1}^{m} \frac{\|g(s_i)(f(s_i) - f(s_{i-1})) + f(s_{i-1})(g(s_i) - g(s_{i-1}))\|_{\mathbb{B}}^{p(n)}}{\lambda_i} \\ &\leq \sum_{i=1}^{m} \frac{(\|g\|_\infty \|f(s_i) - f(s_{i-1})\|_{\mathbb{B}} + \|f\|_\infty \|g(s_i) - g(s_{i-1})\|_{\mathbb{B}})^{p(n)}}{\lambda_i} \\ &= \sum_{i=1}^{m} \Big(\frac{\|g\|_\infty \|f(s_i) - f(s_{i-1})\|_{\mathbb{B}}}{\lambda_i^{1/p(n)}} + \frac{\|f\|_\infty \|g(s_i) - g(s_{i-1})\|_{\mathbb{B}}}{\lambda_i^{1/p(n)}}\Big)^{p(n)} \\ &\leq \Big[\Big(\sum_{i=1}^{m} \frac{\|g\|_\infty^{p(n)} \|f(s_i) - f(s_{i-1})\|_{\mathbb{B}}^{p(n)}}{\lambda_i}\Big)^{1/p(n)} \\ &\qquad + \Big(\sum_{i=1}^{m} \frac{\|f\|_\infty^{p(n)} \|g(s_i) - g(s_{i-1})\|_{\mathbb{B}}^{p(n)}}{\lambda_i}\Big)^{1/p(n)}\Big]^{p(n)} \\ &= \Big[\|g\|_\infty \Big(\sum_{i=1}^{m} \frac{\|f(s_i) - f(s_{i-1})\|_{\mathbb{B}}^{p(n)}}{\lambda_i}\Big)^{1/p(n)} \\ &\qquad + \|f\|_\infty \Big(\sum_{i=1}^{m} \frac{\|g(s_i) - g(s_{i-1})\|_{\mathbb{B}}^{p(n)}}{\lambda_i}\Big)^{1/p(n)}\Big]^{p(n)} \\ &\leq [\|g\|_\infty\ V_{\Lambda_{p(n)}}(f, I, \mathbb{B}) + \|f\|_\infty\ V_{\Lambda_{p(n)}}(g, I, \mathbb{B})]^{p(n)}. \end{aligned}$$

Thus,

$$\begin{aligned}\|fg\| &= \|fg\|_\infty \ + \ V_{\Lambda_{p(n)}}(fg, I, \mathbb{B}) \\ &\leq \|f\|_\infty \ \|g\|_\infty + \|g\|_\infty \ V_{\Lambda_{p(n)}}(f, I, \mathbb{B}) + \|f\|_\infty \ V_{\Lambda_{p(n)}}(g, I, \mathbb{B}) \\ &\leq (\|f\|_\infty + V_{\Lambda_{p(n)}}(f, I, \mathbb{B}))(\|g\|_\infty + V_{\Lambda_{p(n)}}(g, I, \mathbb{B})) \\ &= \|f\| \ \|g\|.\end{aligned}$$

This completes the proof of Theorem 2.1.

3. THE CLASS $(\Lambda^1, \Lambda^2)^* BV(p(n), [a_1, b_1] \times [a_2, b_2], \mathbb{B})$

Let f be a function from $\mathbf{R}^2 = I^1 \times I^2$ into a Banach algebra $\mathbb{B}$, where $I^1 = [a_1, b_1]$ and $I^2 = [a_2, b_2]$. Then

$$f(I^1 \times I^2) = f(I^1, b_2) - f(I^1, a_2) = f(b_1, b_2) - f(a_1, b_2) - f(b_1, a_2) + f(a_1, a_2).$$

Definition 3.1. For given Banach algebra $\mathbb{B}$, $\bigwedge = (\Lambda^1, \Lambda^2)$, where $\Lambda^k = \{\lambda_i^k\}_{i=1}^{\infty} \in \mathbb{L}$, for $k = 1,\ 2$, and $1 \leq$p(n)$\uparrow p$ as n$\to \infty$ $(1 \leq p \leq \infty)$, we say that a function f from $\mathbf{R}^2 = I^1 \times I^2 = [a_1, b_1] \times [a_2, b_2]$ into $\mathbb{B}$ is of bounded $p(n) - \bigwedge -$variation relative to φ (that is, $f \in \bigwedge BV(p(n), \mathbf{R}^2, \mathbb{B})$) if

$$\begin{aligned}&V_{\bigwedge_{p(n)}}(f, \mathbf{R}^2, \mathbb{B}) \\ &= \sup_{n \geq 1} \sup_{\{I_i^1 \times I_j^2\}} \left\{ V_{\bigwedge_{p(n)}}(f, \mathbb{B}, \{I_i^1 \times I_j^2\}) : \delta\{I_i^1 \times I_j^2\} \geq \frac{(b_1 - a_1)\ (b_2 - a_2)}{\varphi(n)} \right\} \\ &< \infty,\end{aligned}$$

where $\{I_i^1\} = \{[s_{i-1}, s_i]\}$ and $\{I_j^2\} = \{[t_{j-1}, t_j]\}$ are finite collections of non-overlapping subintervals in I^1 and I^2 respectively,

$$V_{\bigwedge_{p(n)}}(f, \mathbb{B}, \{I_i^1 \times I_j^2\}) = \left(\sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\|f(I_i^1 \times I_j^2)\|_{\mathbb{B}}^{p(n)}}{\lambda_i^1\ \lambda_j^2} \right)^{1/p(n)}$$

and

$$\delta\{I_i^1 \times I_j^2\} = \inf_{i,\ j} \mid (s_i - s_{i-1}) \times (t_j - t_{j-1}) \mid .$$

If $f \in \bigwedge BV(p(n), \mathbf{R}^2, \mathbb{B})$ is such that the marginal functions $f(a_1, .) \in \Lambda^2 BV(p(n), I^2, \mathbb{B})$ and $f(., a_2) \in \Lambda^1 BV(p(n), I^1, \mathbb{B})$ then f is said to be of bounded $p(n) - \bigwedge^* -$ variation over $\mathbf{R}^2$ relative to φ (that is, $f \in \bigwedge^* BV(p(n), \mathbf{R}^2, \mathbb{B})$).

If $f \in \bigwedge^* BV(p(n), \mathbf{R}^2, \mathbb{B})$ then f is bounded and each of the marginal functions $f(., t) \in \Lambda^1 BV(p(n), I^1, \mathbb{B})$ and $f(s, .) \in \Lambda^2 BV(p(n), I^2, \mathbb{B})$, where $t \in I^2$ and $s \in I^1$ are fixed.

Note that, if $p(n) = p$, for all n, $(1 \le p < \infty)$ the classes $\bigwedge BV(p(n), \mathbf{R}^2, \mathbb{B})$ and $\bigwedge^* BV(p(n), \mathbf{R}^2, \mathbb{B})$ reduce to the classes $\bigwedge BV^{(p)}(\mathbf{R}^2, \mathbb{B})$ and $\bigwedge^* BV^{(p)}(\mathbf{R}^2, \mathbb{B})$ respectively. For $\mathbb{B} = \mathbb{C}$, we omit writing $\mathbb{C}$, the classes $\bigwedge BV(p(n), \mathbf{R}^2, \mathbb{B})$ and $\bigwedge^* BV(p(n), \mathbf{R}^2, \mathbb{B})$ reduce to the classes $\bigwedge BV(p(n), \mathbf{R}^2)$ and $\bigwedge^* BV(p(n), \mathbf{R}^2)$ respectively.

We prove the following theorem.

Theorem 3.1. *Let $\mathbb{B}$ be a commutative unital Banach algebra. The class $\bigwedge^* BV(p(n), [a_1, b_1] \times [a_2, b_2], \mathbb{B})$ is a commutative unital Banach algebra with respect to the pointwise operations and the variation norm :*

$$\begin{aligned} \|f\| = \|f\|_\infty + V_{\bigwedge_{p(n)}}(f, [a_1, b_1] \times [a_2, b_2], \mathbb{B}) + V_{\Lambda^1_{p(n)}}(f(., a_2), [a_1, b_1], \mathbb{B}) \\ + V_{\Lambda^2_{p(n)}}(f(a_1, .), [a_2, b_2], \mathbb{B}), \;\; f \in \bigwedge{}^* BV(p(n), [a_1, b_1] \times [a_2, b_2], \mathbb{B}). \end{aligned} \tag{3.1}$$

For $p(n) = p$, for all n, Theorem 3.1 generalize the result [10, Corollary 3.7, p.185]

Proof of Theorem 3.1.
Let $\{f_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $\bigwedge^* BV(p(n), [a_1, b_1] \times [a_2, b_2], \mathbb{B})$. Then it converges uniformly to some function say f. From Theorem 2.1, we get

$$\lim_{k\to\infty} V_{\Lambda^1_{p(n)}}((f_k(., a_2) - f(., a_2)), I^1, \mathbb{B}) = 0 \tag{3.2}$$

and

$$\lim_{k\to\infty} V_{\Lambda^2_{p(n)}}((f_k(a_1, .) - f(a_1, .)), I^2, \mathbb{B}) = 0. \tag{3.3}$$

Now, for any $\{I^1_i \times I^2_j\}$, where $\{I^1_i\}$ and $\{I^2_j\}$ are finite collections of non-overlapping subintervals in I^1 and I^2 respectively,

$$\begin{aligned} V_{\bigwedge_{p(n)}}(f_k, \mathbb{B}, \{I^1_i \times I^2_j\}) &\le V_{\bigwedge_{p(n)}}(f_k - f_l, \mathbb{B}, \{I^1_i \times I^2_j\}) + V_{\bigwedge_{p(n)}}(f_l, \mathbb{B}, \{I^1_i \times I^2_j\}) \\ &\le V_{\bigwedge_{p(n)}}(f_k - f_l, \mathbf{R}^2, \mathbb{B}) + V_{\bigwedge_{p(n)}}(f_l, \mathbf{R}^2, \mathbb{B}). \end{aligned}$$

This implies,

$$V_{\bigwedge_{p(n)}}(f_k, \mathbf{R}^2, \mathbb{B}) \le V_{\bigwedge_{p(n)}}(f_k - f_l, \mathbf{R}^2, \mathbb{B}) + V_{\bigwedge_{p(n)}}(f_l, \mathbf{R}^2, \mathbb{B})$$

and

$$|V_{\bigwedge_{p(n)}}(f_k, \mathbf{R}^2, \mathbb{B}) - V_{\bigwedge_{p(n)}}(f_l, \mathbf{R}^2, \mathbb{B})| \le V_{\bigwedge_{p(n)}}(f_k - f_l, \mathbf{R}^2, \mathbb{B}) \to 0 \; as \; k, l \to \infty.$$

Hence, $\{V_{\bigwedge_{p(n)}}(f_k, \mathbf{R}^2, \mathbb{B})\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore

$$\begin{aligned} V_{\bigwedge_{p(n)}}(f, \mathbb{B}, \{I_i^1 \times I_j^2\}) &= \left(\sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\|f(I_i^1 \times I_j^2)\|_{\mathbb{B}}^{p(n)}}{\lambda_i^1 \ \lambda_j^2} \right)^{1/p(n)} \\ &= \lim_{k \to \infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\|f_k(I_i^1 \times I_j^2)\|_{\mathbb{B}}^{p(n)}}{\lambda_i^1 \ \lambda_j^2} \right)^{1/p(n)} \\ &\leq \lim_{k \to \infty} V_{\bigwedge_{p(n)}}(f_k, \mathbf{R}^2, \mathbb{B}) \leq M < \infty. \end{aligned}$$

This together with (3.2) and (3.3) imply $f \in \bigwedge^* BV(p(n), \mathbf{R}^2, \mathbb{B})$. Moreover,

$$\begin{aligned} V_{\bigwedge_{p(n)}}(f_k - f, \mathbb{B}, \{I_i^1 \times I_j^2\}) &= \left(\sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\|(f_k - f)(I_i^1 \times I_j^2)\|_{\mathbb{B}}^{p(n)}}{\lambda_i^1 \ \lambda_j^2} \right)^{1/p(n)} \\ &= \lim_{l \to \infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\|(f_k - f_l)(I_i^1 \times I_j^2)\|_{\mathbb{B}}^{p(n)}}{\lambda_i^1 \ \lambda_j^2} \right)^{1/p(n)} \\ &\leq \lim_{l \to \infty} V_{\bigwedge_{p(n)}}(f_k - f_l, \mathbf{R}^2, \mathbb{B}) \to 0 \ as \ k \to \infty. \end{aligned}$$

This together with (3.2) and (3.3) imply $(\bigwedge^* BV(p(n), \mathbf{R}^2, \mathbb{B}), \|.\|)$ is a Banach space.

For any $f, g \in \bigwedge^* BV(p(n), \mathbf{R}^2, \mathbb{B})$,

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\|(fg)(s_i, t_j) - (fg)(s_i, t_{j-1}) - (fg)(s_{i-1}, t_j) + (fg)(s_{i-1}, t_{j-1})\|_{\mathbb{B}}^{p(n)}}{\lambda_i^1 \ \lambda_j^2} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\|S_1^{ij} + S_2^{ij} + S_3^{ij} + S_4^{ij} + S_5^{ij} + S_6^{ij} + S_7^{ij} + S_8^{ij} + S_9^{ij} + S_{10}^{ij}\|_{\mathbb{B}}^{p(n)}}{\lambda_i^1 \ \lambda_j^2}, \end{aligned}$$

where

$$\begin{aligned} S_1^{ij} &= [f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1})] \ g(s_{i-1}, t_{j-1}), \\ S_2^{ij} &= [g(s_i, t_j) - g(s_i, t_{j-1}) - g(s_{i-1}, t_j) + g(s_{i-1}, t_{j-1})] \ f(s_i, t_j), \\ S_3^{ij} &= [f(a_1, t_j) - f(a_1, t_{j-1})] \ [g(s_i, a_2) - g(s_{i-1}, a_2)], \\ S_4^{ij} &= [f(a_1, t_j) - f(a_1, t_{j-1})] \ [g(s_{i-1}, a_2) + g(s_i, t_{j-1}) - g(s_i, a_2) - g(s_{i-1}, t_{j-1})], \\ S_5^{ij} &= [f(a_1, t_{j-1}) + f(s_i, t_j) - f(a_1, t_j) - f(s_i, t_{j-1})] \ [g(s_i, a_2) - g(s_{i-1}, a_2], \end{aligned}$$

$$
\begin{aligned}
S_6^{ij} &= [f(a_1,t_{j-1}) + f(s_i,t_j) - f(a_1,t_j) - f(s_i,t_{j-1})]\ [g(s_{i-1},a_2) + g(s_i,t_{j-1}) \\
&\qquad g(s_{i-1},t_{j-1}) - g(s_i,a_2)]\,, \\
S_7^{ij} &= [f(s_i,a_2) - f(s_{i-1},a_2)]\ [g(a_1,t_j) - g(a_1,t_{j-1})], \\
S_8^{ij} &= [f(s_i,a_2) - f(s_{i-1},a_2)]\ [g(a_1,t_{j-1}) + g(s_{i-1},t_j) - g(a_1,t_j) - g(s_{i-1},t_{j-1})], \\
S_9^{ij} &= [f(s_{i-1},a_2) + f(s_i,t_j) - f(s_{i-1},t_j) - f(s_i,a_2)]\ [g(a_1,t_j) - g(a_1,t_{j-1})]
\end{aligned}
$$

and

$$
\begin{aligned}
S_{10}^{ij} &= [f(s_{i-1},a_2) + f(s_i,t_j) - f(s_{i-1},t_j) - f(s_i,a_2)]\ [g(a_1,t_{j-1}) + g(s_{i-1},t_j) \\
&\qquad - g(a_1,t_j) - g(s_{i-1},t_{j-1})].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_1^{ij}+S_2^{ij}+S_3^{ij}+S_4^{ij}+S_5^{ij}+S_6^{ij}+S_7^{ij}+S_8^{ij}+S_9^{ij}+S_{10}^{ij}||_{\mathbb{B}}^{p(n)}}{\lambda_i^1\ \lambda_j^2} \\
&\le \sum_{i=1}^{m}\sum_{j=1}^{r}\Big(\frac{||S_1^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}} + \frac{||S_2^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}} + \frac{||S_3^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}}\frac{||S_4^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}} \\
&\quad + \frac{||S_5^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}} + \frac{||S_6^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}}\frac{||S_7^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}} + \frac{||S_8^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}} \\
&\quad + \frac{||S_9^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}}\frac{||S_{10}^{ij}||_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{1/p(n)}}\Big)^{p(n)} \\
&\le \Big[\Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_1^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_2^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} \\
&\quad + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_3^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_4^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} \\
&\quad + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_5^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_6^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} \\
&\quad + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_7^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_8^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} \\
&\quad + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_9^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)} + \Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{||S_{10}^{ij}||_{\mathbb{B}}^{p(n)}}{(\lambda_i^1\ \lambda_j^2)}\Big)^{1/p(n)}\Big]^{p(n)} \\
&\le [\|g\|\ V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B}) + \|f\|\ V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) \\
&\quad + V_{\Lambda^2_{p(n)}}(f(a_1,\cdot),I^2,\mathbb{B})V_{\Lambda^1_{p(n)}}(g(\cdot,a_2),I^1,\mathbb{B}) \\
&\quad + V_{\Lambda^2_{p(n)}}(f(a_1,\cdot),I^2,\mathbb{B})\ V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) + V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B})V_{\Lambda^1_{p(n)}}(g(\cdot,a_2),I^1,\mathbb{B})
\end{aligned}
$$

$$+ V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B})V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) + V_{\Lambda^1_{p(n)}}(f(\cdot,a_2),I^1,\mathbb{B})V_{\Lambda^2_{p(n)}}(g(a_1,\cdot),I^2,\mathbb{B})$$
$$+ V_{\Lambda^1_{p(n)}}(f(\cdot,a_2),I^1,\mathbb{B})V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) + V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B})V_{\Lambda^2_{p(n)}}(g(a_1,\cdot),I^2,\mathbb{B})$$
$$+ V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B})V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B})]^{p(n)}.$$

In view of the Theorem 2.1, we have

$$\begin{aligned}
&\|fg\| \\
&= \|fg\|_\infty + V_{\bigwedge_{p(n)}}(fg,\mathbf{R}^2,\mathbb{B}) + V_{\Lambda^1_{p(n)}}(fg(.,a_2),I^1,\mathbb{B}) + V_{\Lambda^2_{p(n)}}(fg(a_1,.),I^2,\mathbb{B}) \\
&\le \|f\|_\infty\ \|g\|_\infty + \|g\|_\infty\ V_{\Lambda^1_{p(n)}}(f(.,a_2),I^1,\mathbb{B}) + \|f\|_\infty\ V_{\Lambda^1_{p(n)}}(g(.,a_2),I^1,\mathbb{B}) \\
&\quad + \|g\|_\infty\ V_{\Lambda^2_{p(n)}}(f(a_1,.),I^2,\mathbb{B}) + \|f\|_\infty\ V_{\Lambda^2_{p(n)}}(g(a_1,.),I^2,\mathbb{B}) \\
&\quad + \|g\|\ V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B}) + \|f\|\ V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) \\
&\quad + V_{\Lambda^2_{p(n)}}(f(a_1,\cdot),I^2,\mathbb{B})V_{\Lambda^1_{p(n)}}(g(\cdot,a_2),I^1,\mathbb{B}) \\
&\quad + V_{\Lambda^2_{p(n)}}(f(a_1,\cdot),I^2,\mathbb{B})V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) \\
&\quad + V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B})\ V_{\Lambda^1_{p(n)}}(g(\cdot,a_2),I^1,\mathbb{B}) \\
&\quad + V_{\Lambda^1_{p(n)}}(f(\cdot,a_2),I^1,\mathbb{B})\ V_{\Lambda^2_{p(n)}}(g(a_1,\cdot),I^2,\mathbb{B}) \\
&\quad + V_{\Lambda^1_{p(n)}}(f(\cdot,a_2),I^1,\mathbb{B})V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) \\
&\quad + V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B})\ V_{\Lambda^2_{p(n)}}(g(a_1,\cdot),I^2,\mathbb{B}) + 2V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B})\ V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) \\
&\le 2(\|f\|_\infty + V_{\bigwedge_{p(n)}}(f,\mathbf{R}^2,\mathbb{B}) + V_{\Lambda^1_{p(n)}}(f(.,a_2),I^1,\mathbb{B}) + V_{\Lambda^2_{p(n)}}(f(a_1,.),I^2,\mathbb{B})) \\
&\quad (\|g\|_\infty + V_{\bigwedge_{p(n)}}(g,\mathbf{R}^2,\mathbb{B}) + V_{\Lambda^1_{p(n)}}(g(.,a_2),I^1,\mathbb{B}) + V_{\Lambda^2_{p(n)}}(g(a_1,.),I^2,\mathbb{B})) \\
&= 2\|f\|\ \|g\|.
\end{aligned}$$

This completes the proof of Theorem 3.1.

Remark 3.1. As it is known from the theory of Banach algebras, the norm (3.1) can always be replaced by an equivalent norm $|||\cdot|||$ on $\bigwedge^* BV(p(n),\mathbf{R}^2,\mathbb{B})$ such that

$$|||fg||| \le |||f|||\ |||g|||, \qquad f,g \in \bigwedge{}^* BV(p(n),\mathbf{R}^2,\mathbb{B}). \tag{3.4}$$

In fact, given $f \in \bigwedge^* BV(p(n),\mathbf{R}^2,\mathbb{B})$, consider the linear continuous operator T_f from $\bigwedge^* BV(p(n),\mathbf{R}^2,\mathbb{B})$ into itself defined by $T_f(g) = f\cdot g$ whenever $g \in \bigwedge^* BV(p(n),\mathbf{R}^2,\mathbb{B})$. Then the operator norm $|||f||| = \|T_f\| = \sup\{\|f\cdot g\|;\ \|g\| = 1\}$ is the desired norm on $\bigwedge^* BV(p(n),\mathbf{R}^2,\mathbb{B})$ satisfying (3.4) and $\|f\| \le |||f||| \le 2\|f\|$ for all $f \in \bigwedge^* BV(p(n),\mathbf{R}^2,\mathbb{B})$.

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BOOK-REVIEW

TITLE : CYCLIC MODULES AND STRUCTURE OF THE RINGS.
By S. K. Jain, Ashish K. Srivastava and Askar A. Tuganbaev;
Oxford University Press; 2012; Hard cover, x+220 pages;
ISBN No. 978 - 0 - 19 - 966451 - 1 ;
Library of Congress Control Number - 2012942048.
Reviewer : **SERGIO R. LÓPEZ-PERMOUTH.**

Cyclic modules are the stuff all other modules are made out of . Every module M over a unital ring R is the sum of all of its cyclic submodules. It is to be expected then that properties possessed by all right modules would have a strong influence on the structure of all other modules and consequently on those aspects of the structure of the ring that are determined by its category of modules. This is the philosophical backbone of much research on Rings and Modules going back to the 1960s. The match that lit this fire was Barbara Osofsky's tour-de-force theorem in her 1966 doctoral dissertation showing that if all cyclic right modules are injective then the ring is semisimple artinian.

While many times a module theorist may ask what happens when all modules share certain property, the problems grow in difficulty significantly when the requirements is made only of cyclic modules. Such considerations are the central topic of the monograph "Cyclic Modules and the Structure of Rings" by Jain, Srivastava and Tuganbaev. Modifications of this line of research can be obtained by asking for more modules (say, fur example, for all finitely generated modules) to satisfy the property of asking for fewer ones (say, proper cyclic or simple modules) to do so. Those various related thoughts intertwine throughout the text.

Research monographs are an important part of the mathematical literature, When a topic has been studied for a considerable length of time, the results may already be abundant and scattered throughout the literature; the need then arises for a central place where information about the specific questions around the topic may easily be found. The question would then remain about the way in which all of this aggregated information will be delivered. One possibility is to simply list results and include a few proofs in more or less the same way that they appear elsewhere in the literature. When this approach is taken, the authors of the monograph can idiosyncratically choose which proofs to include and they are also free to make decisions regarding the depth of the discussions they will highlight. A different approach would be to revisit the topic trying to give a self-contained (modulo some clearly stated prerequisites) logical exposition of the concepts and results about the subject in a way in which a person approaching the material for the first time can reasonably become acquainted with it in a progressively meaningful way. The book under review is of the monographic nature described at the beginning of the paragraph. I visited various spots in the text, looking for proofs that I distinctly remember as being particularly challenging to understand when I first met them in the original references, to see if there was any clarity gained by reading them in the context of the new book. I found very little gain from that perspective. On the other hand, from the monographic point of view, the book is very well-written and presents a sizable chunk of the ring and module theory developed since the late 1960s to the present under the umbrella of "rings characterized by the properties of their cyclic modules."

The book focuses on results that characterize the structure of rings over which all cyclic modules or all proper cyclic modules satisfy given finiteness conditions or homological properties. Cyclic right modules over unital rings are precisely the quotients of the right regular module (the ring itself thought of as a right module) by right ideals, It thus makes sense to set aside first some time to look at properties of rings in terms of the ring theoretic properties of their ring homomorphic images, that is to say, the quotients of the ring by two-sided ideals. Mostly, the defining conditions are imposed explicitly only on proper homomorphic images (that is on quotients modulo non-zero ideals) and only occasionally on the ring itself. The properties :onsidered in this first part of the book include rings all of whose proper factor rings are Artinian, rings all of whose proper factor rings are perfect, nonprime rings all of whose proper factor rings are von Neumann regular; commutative rings all of whose proper factor rings are self-injective, and rings with the restricted minimum condition.

The core of the book is devoted to module-theoretic properties of quotients of the right regular module. Sometimes, the net cast is widened by requiring only that proper cyclic modules share a given property. By proper cyclic in this context, sometimes the requirement is not only that the right ideal to be modded out be non-zero but actually that the quotient obtained upon modding out be not isomorphic to R itself. A highlight along this direction is the theory of PCI rings (proper cyclics are injective) which are not semisimple artinian; this is in clear contrast with the Osofsky mentioned above. This part of the book includes sections on rings over which every proper cyclic module is Artinian, rings over which every proper cyclic module is perfect, rings over which every cyclic module is injective rings over which every cyclic module has the property that complement submodules are direct submodules; rings over which every proper cyclic module is injective, etc. Sometimes, the range of the cyclics under consideration is reduced to only the simple modules; such is the case, for example, when V-rings (*i.e.* rings over which every simple module is injective) and their generalizations are considered. Other topics include rings for which the injective hull of every cyclic module is Sigma-injective, rings over which every cyclic module is quasi-injective (or continuous); rings over which every cyclic module is Sigma-injective, etc. The book closes with a topic that is only tangentially related to the central purpose of the monograph, namely, the last chapter deals with the structure certain rings as determined by properties of their one-sided ideals. That final section still manages to include several interesting results and is a valuable addition.

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REPORT ON IMS SPONSORED LECTURES

The details of the IMS Sponsored Lectures/Popular Talks held is as under :

1.	Speaker:	Mahan Mj. (Ram Krishna Mission Vivekanand University, Belur, Kolkata)
	Title of the Lecture:	Low dimensional projective groups.
	Day and Date:	Monday, Jan 27, 2014 at 4:00 pm.
	Venue :	H. R. I. Auditorium, Allahabad.
	Organizer:	Prof. Satya Deo. NASI Senior Scientist, Harish-Chandra Research Institute Chhatnag Road, Jhusi, Allahabad- 211 019, India.

Abstract : Fundamental groups of smooth projective varieties are called projective groups. We shall discuss (cohomological) conditions on dimension that force such a group to be the fundamental group of a Riemann surface.

2.	Speaker:	C. S. Bagewadi. (Department of Mathematics, Kuvempu University, Shimoga)
	Title of the Lecture:	Differential Geometry.
	Day and Date:	Monday, October 04, 2013 at 3:00 pm.
	Venue :	R. L. Science Institute, Belgaum. Karnataka.
	Organizer:	Dr. (Smt.) S. N. Banasode. P.G. Studies in Mathematics, R. L. Science Institute, Belgaum. Karnataka.

Abstract : Differential Geometry is a mathematical discipline that uses the technique of differential calculus and Integral Calculus as well as Linear Algebra and Multi-Linear Algebra to study problems in Geometry. Since the late 19 century Differential Geometry has grown into a field concerned more generally with geometric structures on Differential Manifolds. Instead of Calculus an axiomatic treatment of Differential Geometry is build via Sheaf Theory and Sheaf Co homology using Vector Sheafs in place of bundles based on arbitrary Topological Space.

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79^{th} IMS CONFERENCE : A BRIEF REPORT

The 79^{th} Annual Conference of the Indian Mathematical Society was held at the Rajagiri School of Engineering and Technology, Rajagiri Valley, Kakkanad, Cochin - 682 039 (Kerala), during December 28 - 31, 2014 under the Presidentship of Prof. Geetha S. Rao, Retd. Professor, Ramanujan Institute of Advanced Study in Mathematics, University of Madras, Chepauk, Chennai - 600 060 (TN), India. The conference was attended by more than 270 delegates from all over the country.

The inaugural function of the Conference, held in the morning of December 28, was presided over by Prof. Geetha S. Rao, President of the Society. The Conference was inaugurated by Prof. K. V. Thomas, Minister of Agriculture and Minister of State in the Ministry of Consumer Affairs, Food and Public Distribution, Govt. of India. Rev. Fr. Dr. Antony Kariyil CMI, Director, RSET, offered a warm welcome to the delegates. The General Secretary of IMS, Prof. N. K. Thakare spoke about the Indian Mathematical Society and on behalf of the Society expressed his sincere and profuse thanks to the host for organizing the Conference. He released IMS publications and also reported about (i) A. Narasinga Rao Memorial Prize and (ii) *P. L. Bhatnagar Memorial Prize* - the details of these Prizes are given later on. Prof. Satya Deo, Editor, JIMS, Prof. Ambat Vijayakumar, CUSAT and Dr Babu Paul, IAS also addressed the delegates.

Three renowned mathematicians of Kerala - Prof. K.S.S. Nambooripad, Prof. R. Sivaramakrishnan and Prof. T. Thrivikraman were honoured by the Rajagiri School of Engineering and Technology, during the inaugural ceremony.

Prof. Geetha S. Rao delivered her Presidential address (General) on *"The universal appeal of Mathematics"*. The inaugural function ended with a vote of thanks by the Local Organizing Secretary, Dr. Vinod Kumar P. B.

The Academic Sessions of the conference began with the Presidential Technical Address by Prof. Geetha S. Rao, the President of the Society, on *"A New Tool in Approximation Theory"* which was presided over by Prof. N. K. Thakare, the senior most past president of the Society present in the Conference.

One Plenary Lecture, **Five** Memorial Award Lectures and **Four** Invited Lectures were delivered by eminent mathematicians during the conference. Besides this, there were **Five** symposia on various topics of mathematics.

A Paper Presentation Competition for various IMS and other prizes was held (without parallel sessions) in which 14 papers were presented. The result of this Competition is declared in the Valedictory Function and appears later on in this report. In all 90 research papers were accepted for presentation during the Conference including 14 research papers for the Paper Presentation Competitions for various Prizes

A Cultural Programme was also arranged in the evening of December 29, 2013. The conference was supported by the NBHM, DST, CSIR, and the management of the Rajagiri School (through its UGC unassigned grant).

The Annual General Body Meeting of the Indian Mathematical Society was held at 12.15 p.m. on January 31, 2013 in the Seminar Hall-1 of the RSET, which was followed by the Valedictory Function in which the IMS and other prizes were awarded to the winners of the Paper Presentation Competition. It was also announced that Prof. S. G. Dani, (Department of Mathematics, IIT Bombay, Powai, Mumbai) will be the next President of the Society with effect from April 1, 2014 and that the next Annual Conference of the Society will be held at the Indian School of Mines, Dhanbad, during December 2014 with Prof. S. P. Tiwari, Head, Department of Applied Mathematics, Indian School of Mines, Dhanbad, as the Local Organizing Secretary.

The details of the academic programme of the Conference are as follows.

Details of the Plenary Lecture:

Prof. M. Ram Murty (Queen's Research Chair and Head, Department of Mathematics and Statistics, Queen's University, Ontario, Canada) delivered a Plenary Lecture on *"Ramanujan and the Zeta function"*. This Lecture was given in honour of Prof. V. M. Shah who served the IMS for nearly thirty years in various capacities.

Details of the Memorial Award Lectures:

1. The 27^{th} P. L. Bhatnagar Memorial Award Lecture was delivered by Dr. Indira Narayanaswamy (Aeronautical Development Agency, Bangalore) on *"Mathematics in Tejas Design: Configuration to Flight Testing"*.
2. The 24^{th} V. Ramaswami Aiyar Memorial Award Lecture was delivered by Prof. M. M. Shikare (University of Pune, Pune) on *"Generalizations of some graph-theoretic results to matroids"*.
3. The 24^{th} Srinivasa Ramanujan Memorial Award Lecture was delivered by Prof. D. P. Patil (IISc, Bangalore) on *" Burnside Algebras of Finite Groups"*.
4. The 24^{th} Hansraj Gupta Memorial Award Lecture was delivered by Prof. Ajit Iqbal Singh (ISI, New Delhi) on *"How permutations, partitions, projective representations and positive maps entangle well for Quantum Information Theory"*.
5. The 11^{th} Ganesh Prasad Memorial Award Lecture was delivered by Prof. D. V. Pai (IIT, Gandhinagar) on *"Viscosity approximation methods for minimization and fixed point problems - a relook"*.

1. A. Narasinga Rao Memorial Prize.

This prize for 2011 was not recommended by the Committee appointed for the purpose, and hence is not awarded to any one.

2. P. L. Bhatnagar Memorial Prize.

For the year 2013, this prize is presented jointly to Mr. Sagnik Saha (Kolkata) and Mr. Shubham Sinha (New Delhi), the top scorers from the Indian Team with 28 points each at the 54^{th} International Mathematical Olympiad held during July 18-26, 2013 at Santa Marta, Colombia.

3. Various prizes for the Paper Presentation Competition:

A total of 14 papers were received for Paper Presentation Competition for Six IMS Prizes, AMU Prize and VMS Prize. There were 11 entries for six IMS prizes, 02 entries for the AMU prize and 01 entry for the VMS prize.

Prof. Satya Deo (Chairperson), Prof. J. R. Patadia, Prof. B. N. Waphare, Prof. M. A. Sofi and Prof. P. K. Jain were the judges. Following is the result for the award of various prizes :

IMS Prize - Group-1:	07 Presentations. Prize is awarded to: Aparna Lakshmanan S. (Kerala).
IMS Prize - Group-2:	01 Presentation. Prize not awarded.
IMS Prize - Group-3 :	No presentation.
IMS Prize - Group-4:	Prize not awarded.
IMS Prize - Group-5:	No presentation.
IMS Prize - Group-6:	01 Presentation. Prize not awarded.
AMU Prize :	02 Presentations. Prize not awarded.
V M Shah Prize :	01 Presentation. Prize is awarded to : Abhijit Banerji, Kalyani University (W. B.).

Details of Invited Lectures delivered :

1. R. N. Mohapatra, Florida, USA on *" Riesz bases, Frames and Eraser".*
2. Gangadhar Hiremath, North Carolina, USA on *" Metrization, diagonal properties and invariance"*.
3. Om Ahuja, Ohio, USA on *"Recent Developments in the theory of Harmonic univalent mappings in the plane"*.
4. M. Wasadikar, Aurangabad (M.S.) on *"Graphs derived from rings and ordered structures".*

Details of the Symposia organized :

1. On **"Differential manifolds and related areas"**.
Convener: Manjusha Majumdar (Calcutta).

Speakers :

1. Manjusha Majumdar (Calcutta).
2. A. Sheikh (Burdwan University).
3. Bhattacharyya (Jadavpur University).
4. S. K. Srivastava (Gorakhpur University)

2. On **"Srinivasa Ramanujan's work"**
Convener: K. Srinivasa Rao (Retd. Prof. IMSC, Chennai).

Speakers :

1. Arun Verma (IIT Roorkee, Utharakhand).
2. Ahmad Ali (B. B. D. University, Lucknow).
3. A. K. Agarwal (Punjab University, Chandigarh).
4. K. Srinivasa Rao (Retd. Prof. IMSC, Chennai).

3. On **"Advances in Ordered Structures and Enumeration"**
Convener: S. K. Nimbhorkar, (Aurangabad).

Speakers :

1. V. V. Joshi (Pune),
2. Y. M. Borse (Pune),
3. S. K. Nimbhorkar (Aurangabad),
4. N. Bhavale (Pune).

4. On **"Topology and Applications".**
Convener: Satya Deo, HRI, Allahabad.

Speakers :

1. Kashyap Rajeevsarathy (IISER, Bhopal).
2. Mahender Singh (IISER, Mohali).
3. Satya Deo (HRI, Allahabad).

5. On **"Graph Theory and Applications".**
Convener: A. Vijayakumar (Cochin Uni. of Sc. and Tech.)

Speakers :

1. B. N. Waphare (Pune),
2. S. Arumugum (Kalaslingam),
3. Ms. A. Anuradha (NIT, Trichy),

4. Balaraman Ravindran (IITM, Chennai)
5. G. Sethuraman (Anna University, Chennai).

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I , N. K. Thakare, hereby declare that the particulars given above are true to the best of my knowledge and belief.

N. K. THAKARE
Signature of the Publisher

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