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Edited by

J. R. PATADIA

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(Dedicated to Prof. V. M. Shah in honour of his long and outstanding services to the Indian Mathematical Society)

THE MATHEMATICS STUDENT

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In keeping with the current periodical policy, THE MATHEMATICS STUDENT will seek to publish material of interest not just to mathematicians with specialized interest but to the postgraduate students and teachers of mathematics in India. With this in view, it will ordinarily publish material of the following type:

1. the texts of Presidential Addresses, the Plenary talks and the Award Lectures delivered at the Annual Conferences.
2. the abstracts of the research papers presented at the Annual Conferences.
3. the texts of the invited talks delivered at the Annual Conferences.
4. research papers, and
5. the Proceedings of the Society's Annual Conferences, Expository and Popular articles, Book Reviews, Reports of IMS Sponsored lectures, etc.

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Manuscripts (including bibliographies, tables, etc.) should be typed double spaced on A4 size white paper with 1 inch (2.5 cm.) margins on all sides with font size 11 pt., preferably in LaTeX . Sections should appear in the following order: Title Page, Abstract, Text, Notes and References. Comments or replies to previously published articles should also follow this format with the exception of abstracts. In LaTeX , the following preamble be used as is required by the Press:

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PROFESSOR V. M. SHAH - AN ACADEMICIAN WITH MULTIFACETED PERSONALITY

J. R. PATADIA

Prof. V. M. Shah voluntarily decided to take retirement from his services to the Indian Mathematical Society once his term got over on March 31, 2013 - in spite of whole hearted pursuing from all colleagues in the Council of the Indian Mathematical Society to continue to serve the Society for at least one more term. His decision was out of his *firm convictions:* **first**, that the task he had in his mind is more or less completed - the task of leading the Society in a specific direction for inclusive participation by the members of the entire mathematical community of the nation to meet the challenges India faces in mathematics teaching, learning and researches in the 21st Century and of preparing the appropriate platform / background in the Society for that; and **second**, that it is now *the right time*, under prevailing situation, for the visionary young generation of Indian mathematicians to take over the charge of leading this oldest mathematical society of the country for the fulfillment of its goals the founder fathers had set and the Indian Mathematical Community of the last hundred and five years has endorsed.

On this occasion of his retirement, in appreciation of his long and dedicated outstanding services of 30 years to the Society in his capacity as Treasurer, President, Editor of both the periodicals and finally as General Secretary, the IMS Council resolved at its meeting at Banaras Hindu University, Varanasi, during the 78th Annual Conference of the Society to dedicate the Volume 82, Nos. 1- 4 (2013) of The Mathematics Student to him in his honour.

Among the existing colleagues in the IMS Council, I am the one who has perhaps the longest association with Professor V. M. Shah - first as his M.Sc. Student during 1969-71 and as his research student during 1973-1978 in the Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University (The M. S. University) of Baroda, Vadodara - 390 002, Gujarat, India, where he worked throughout his career. Then I was his junior colleague in the same department since 1978; and lately, since 2004, also as his colleague in the IMS Council. During this long association with him, I have observed him as a teacher, as a researcher, as an administrator, as a counselor to his students and colleagues, as a guardian lending helping hand to his students and colleagues (at times even in their personal problems), as an organizer and surprisingly even as a social worker thriving to spread scientific temperament, awareness and attitude in the Society. Throughout I have observed that he is an academician with multifaceted personality and a man of firm conviction and with rare courage.

Researcher and Teacher :

Born in a noble family hailing from a small village "SARAR' in Baroda district of erstwhile State of Baroda, he had his primary education in the same village. His secondary and university education was in Baroda - staying in a boarding house, away from parents, since the age of ten plus. He was a merit scholar and the Fellow of the Faculty of Science of the M. S. University of Baroda as a student of M. S. University. With a brilliant academic career at the M. S. University, he joined the Department of Mathematics of the M. S. University of Baroda as an Assistant Lecturer in July 1952.

From the beginning itself he had a strong urge for doing researches and has realized its importance as well. However, during those days, there existed no facility for researches in the Department at Baroda (Prof. U. N. Singh joined the Department at M. S. University later). Under the circumstances, availing leave without pay and some scholarship granted by his University, he joined Prof. S. M. Shah for research work in "Complex Analysis" at Aligarh Muslim University - proudly the first initiative of its kind from the department then. In fact it was Prof. N. M. Shah (a wrangler from Cambridge and senior mathematician in the earstwhile Bombay University of the earstwhile Bombay State who was then a member of the Board of Studies of the Science Faculty of the M. S. University of Baroda) who helped V M Shah in arranging a meeting with Prof. S. M. Shah for research purpose. However, almost within a year of his joining Prof. S. M. Shah at Aligarh, Prof. S. M. Shah left for USA never to return to India. Prof. V. M. Shah returned to Baroda and then after completed his Ph. D. research work in "Fourier Analysis" under the supervision of Prof. U. N. Singh.

Sound analyst in general and Fourier analyst in particular, his Ph. D. research work as well as his joint work with his students, especially in the field of lacunary Fourier series, is of quite high order and has found place in some of the journals of International repute, such as the Proceedings of the American Mathematical Society, Journal of the London Mathematical Society, Journal of the Mathematical Analysis and Applications (USA), etc. As a research supervisor he took special care to see that his students were never under any tension, were always encouraged, they got full freedom and more importantly, worked as a counselor and friend whenever any one was under depression at times. All his research students found him very innovative, open to new ideas, considerate, liberal and loving. He also knows very well how close should one go to an individual.

He has a long and versatile experience of teaching not only the undergraduate and post graduate students of Science but also the graduate and post graduate students of engineering. Whenever I have met a person who happened to be his student in the past, he/she has always recalled him with high esteem and regarded him as an excellent teacher. He has delivered a lecture or a series of lectures at several universities and colleges all over country - always targeting common students and choosing interesting and innovative topics. Therefore he is always welcome there again, is very popular and there remains a lasting impact on students for the specific conceptual clarity. So popular he was as a teacher among students that just recently he was specially invited and felicitated / honoured by a group of his well placed engineering students of sixties at a special gathering in Vadodara.

Administrator and Counselor:

The appreciation, respect and faith which he earned as an administrator not only from the two former successful Vice-Chancellors, Prof. N. K. Thakare and Prof. Satya Deo, working with him as office bearers in IMS since last several years, but also from all the Council members I happened to be with since 2004 onwards as well as from his earlier IMS stalwart colleagues like late Professors J. N. Kapur, R. P. Agarwal, M. K. Singal and others is remarkable. Some of the important qualities for this success as administrator is his ability to listen to others and other views with open mind and great patience, his persuading power, his ability to work as Counselor, and of course his courage and conviction. So committed and thoughtfully dedicated he is to any task on hand that his administrative ability was recognized and appreciated right from the beginning of his teaching career, whether as a Warden to Halls of Residence or as a Head of the Department of Mathematics or as a Dean of the Science Faculty of his alma mater. I may recall that as a research student I was staying in one of the University Halls of residence where Prof. V M Shah was a Warden. I distinctly remember many students telling me "*Your Professor has a tremendous persuading power. Whenever we go to him for complaining about something, once we meet him and come back, we always get convinced that our complain was not well founded*". His sense of judgment, rational as well as diplomatic approach and ability to deal with students was so excellent that he proved himself as one of the best wardens; and it was with great reluctance that the university authorities agreed to his request to relieve him from the office of the warden when he was appointed as the Head .

As a Head of the Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Prof. V. M. Shah lead the department for a long time. He knew how to extract the best of every Faculty member for the development of the Department. Observing the rapid industrial growth of Baroda, he could foresee the need of introducing certain courses at the undergraduate and post graduate level so that students get ample opportunities for jobs, and was the first in the whole of Gujarat to introduce courses like Linear Programming, Operations Research, Numerical Analysis and Computer Programming at B. Sc. Level and Computer Programming even at the M. Sc. Level as early as in mid-seventies. He knew the potential and ability of each of his colleagues and was always very careful to keep the right person for the right work for the optimum output - at times taking academically and otherwise bold decisions. He was also instrumental in establishing a nice Computer Lab in the Department.

He encouraged young, energetic and sincere colleagues providing opportunities to learn and teach more, and was instrumental in instituting M. Phil. Degree in the Faculty of Science during his tenure as Dean, Faculty of Science (with more than 3500 students) from May 1985 till his retirement in October 1989. He was also the Director, University Computer Center since 1983 till his retirement. So much faith and esteem as a man of integrity he earned from the succession of authorities of the M. S. University as well as from all sections in the University (teachers, students, administrative members, Senate and Syndicate Members) for his bold and firm decisions, without fear or favour, may it be harsh, and at the same time remaining considerate and just to all concerned, that till as late as 2008, he had headed several University Committees for resolving controversial issues faced by the University authorities.

He is a Fellow of the Indian National Academy of Sciences. He is also a Founder Fellow of the Gujarat Science Academy and was also its Vice-President. He was a member of the UGC sponsored Curriculum Development Centre for Mathematics set up at the Ramanujan Institute of Advanced Study in Mathematics, Madras University. He was a member of the UGC Mathematics Panel and a member of its various Committees. He was entrusted in 1983 by the UGC to prepare a report by conducting a sample survey and case studies in Mathematics teaching at the

Under Graduate and Post Graduate levels in the universities and colleges in Western India. The survey was aimed at ascertaining the drawbacks and short falls in the system so that the same could be used to make specific recommendations for the improvement of teaching / research and quantifying the various inputs by the UGC.

An organizer and a Social worker :

It is now recognized, with reasonable appreciation from the large number of members of the Society in particular and many members of the mathematical Community of the nation in general, that it is during the tenure of Prof. V M Shah as General Secretary that the Council of the Society has given a turn to the Society in a specific direction for inclusive participation by the members of the entire mathematical community of the nation. As I have observed him for long, taking innovative decisions with changing times and making sincere efforts for successfully implementing it is one of his appreciable characteristics that is responsible for several such acts right from early days of his career. For example,

- When New Mathematics was introduced at school level in seventies in Gujarat State, he, as Head, took initiatives in several directions to offer his own contribution for the educational cause and service to the Society at large. On behalf of the State Education Department and with the support of the expertise from his own department, he organized four training courses for Higher Secondary Teachers of three districts of Gujarat, including Baroda, running over a period of two years. He organized several orientation / training programmes for School Teachers as well as for employees of industrial units like IPCL, GSFC, etc. From the platform of the Centre for Continuing and Adult Education of the University, he conducted several programmes, of specified weeks duration, titled "New Mathematics for Parents". As an outcome of this activity, he wrote a popular book entitled "New Mathematics for Parents and Pupils".
- He was a founder Editor (1972) of a monthly "Ganit Darshan" published with a view to bring home the concepts of New Mathematics to school teachers.
- He significantly contributed to the progress and services to the "Gujarat Ganit Mandal" and served the Mandal as its Secretary for four years and later as its President in 1979.
- When Computer Programming was introduced in seventies, he wrote an excellent introductory book "Computer Darshan" in Gujarati for the benefit of students and a common man of the Society. Written in a lucid

way for creating computer awareness. The book was widely popular and had undergone four editions.

- He is a Founder Secretary of voluntary organization "Gujarati Vigyan Parishad" engaged in Science Popularization activities. He is an invitee on the Governing Council of the "Community Science Center" which is engaged in popularizing Science among masses.
- With the co-operation of like minded people he also established in Baroda a "Primary Teachers Forum", with him as president. The whole lot of about 1500 schools were then covered under the programme.

On his retirement, his admirers and well wishers constituted a committee which created a "**V. M. Shah Felicitation Fund**" and a fairly good amount got generated then. This was distributed for various purposes; *e.g.*, for instituting *V. M. Shah Gold Medal* for mathematics at the M. S. University, for instituting prizes for Post Graduate mathematics students of the University, for instituting V. M. Shah Prize for best research paper presentation in Analysis at the Annual Conference of the IMS, towards endowment fund to Gujarat Ganit Mandal and for creating a public trust " *Prof. (Dr.) V. M. Shah Charitable Trust*" at Baroda for carrying out various socially useful academic activities . Just few years before in 2008, this trust donated Rupees Two Lakhs to the M. S. University of Baroda for conducting every year "*Professor V. M. Lecture Series*" in the Department of Mathematics from the interest amount. Prof. R. Balasubramaniam (Director, IMSc, Chennai) gave the first Lecture series in this.

He is convinced that our children must be trained for logical thinking and scientific temper. Such trained children will not only become better scientists but also rational citizen with scientific temper. For this purpose, he firmly believes in the dictum "*Catch them TOO young*" instead of "*Catch them young*". With profound confidence that Mathematics is the only tool available and handy at Primary level for inculcating logical thinking habit among young children, targeting them he has undertaken a task of preparing a series of FIVE books in Gujarati for the children above the eight years of age. The series is titled "**Majedar Ganit**" ("Joyful Mathematics" or "Mathematics one will enjoy"). The first three books have been published few years back and undergone Five editions already; and the first edition of the last two books is published in September 2013.

He feels that the Long term success of any organization is highly dependent upon the thoughtful strategy and its effective implementation. Setting of definite goals and clear descriptions of opportunities, resources and limitations provides significant guidance in the thoughtful strategy development and planning of the process of implementation.

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PROFESSOR V. M. SHAH : SOUL OF IMS

N. K. THAKARE

Professor Vadilal Maganlal Shah of Vadodara reclused himself from the onerous responsibilities of the General Secretaryship of the Indian Mathematical Society from, 1st April 2013. His action may remind the 'Santhara Vrat' some from the Jain community adopt. Professor Shah was adjoined with the activities of the IMS around 1981-82. Since, then he worked relentlessly for the IMS till 31st March 2013 in various capacities such as the Council member, the Treasurer, the President, the Editor of both the periodicals of IMS, namely, the Mathematics Student and the Journal of the IMS, and above all the General Secretary. He became synonym with the Indian Mathematical Society. He was in a way a soul of the society during the period of his association with the IMS. The condition of the IMS, during those days was perilous and precarious to say the least. The publication of periodicals was erratic. There were financial difficulties. There was almost a constitutional collapse of the IMS. However, Late Professors R. P. Agarwal , M. K. Singal, S. R. Sinha, D. N. Varma alongwith Professor Shah prepared a new constitution for the IMS that is in vogue now. Later many like Professor H. C. Khare also supported them. They saved the IMS from getting extinguished.

At later stages, there were attempts to hijack the IMS, the oldest scientific society of the country. Because of Professor Shah's sagacity and sincerity those attempts were nullified. His implicit capacity to take all the sections – regionwise and otherwise – alongwith him precluded some, self- professed intellectual cream of India the snatching away of the reigns of IMS from democratically elected members. He stood like a rock then. Even he seemed, sometimes, alone in his struggle to save the IMS. Thus he served not only the cause of mathematics, but a large number of researchers coming from the University system. He also nipped in the bud some attempts to convert the IMS into a family enterprise.

He served in Maharaja Sayajirao University of Baroda as Professor, Head in the Department of Mathematics and held a very important position of the Dean, Faculty of Science for many years. He was known as a trusted crisis – manager amongst students as well as teachers. He was, hence, accorded a huge felicitation by students, his colleagues and lovers of mathematics, some twenty four years ago.

Professor Shah is considered a pioneer and pillar for popularization of mathematics in Gujarat through his 5 – part books in vernacular Gujarathi titled "Mazedar Ganit". His books are so popular that some parts saw even five editions, a unique phenomena for books on mathematics and that too in a regional language. His man(a) (मन/मना) is always engulfed in rama (राम/रमा)of mathematics. Father of four daughters and grand – children, Professor Shah lives in his house "Anant (∞)" and is always ramamana (रममाण=रमा+मना) in his subject of choice.

The IMS awards every year V. M. Shah Prize to the best presentation of research paper in analysis during the Annual Conference of the IMS. As far as I remember Professor Shah never missed any Annual Conference till December 2010. In fact, Ramabhabhi also always graced the IMS conferences alongwith her husband.

Let me end this write-up on a very personal note. Professor Shah was born in a small village named 'Sarar', the population of which at that time was hardly 1300. He had his primary education in the same village that came under the jurisdiction of the princely State of Baroda. Incidently, Maharaja Sayajirao Gaikwad himself was born in a very small village named Kaulane near Malegaon in North Maharashtra, the population of this village is around 3000 today. The beginnings of Professor Shah almost parallel my own beginnings in a small village of North Maharashtra. In 1996, my daughter got married to a software engineer from Vadodara and since then Professor Shah became my "Mota Bhai"(elder brother) and helped me and members of the family of my daughter's sasural very earnestly. As the Father -In - Law of my daughter had expired long before her marriage in nineteen eighties, Professor Shah became her Father - In - Law as well as Father in absentia. His house became a place of many visits for us while in Vadodara. I wish Professor V. M. Shah very healthy and painless life. Even at the age of 84, he is very active and that emboldens me to predict that he will complete century of years of his life. I also wish Ramabhabhi good health and all the happiness that she desired. We also pray that Professor V. M. Shah shall remain connected with the IMS through his student

Professor J. R. Patadia who presently edits “The Mathematics Student”.

N. K. THAKARE (DHULE)
FOUNDER VICE CHANCELLOR, NORTH
MAHARASHTRA UNIVERSITY, JALGAON
AND EX-PROFESSOR OF MATHEMATICS
(LOKMANYA TILAK CHAIR) AND EX-HEAD,
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF PUNE, PUNE.

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PROFESSOR V. M. SHAH - IN HONOUR OF HIS RETIREMENT FROM THE INDIAN MATHEMATICAL SOCIETY

SATYA DEO

Professor V. M. Shah, former Head of the Mathematics Department and Dean, Faculty of Science, M. S. University of Baroda, Vadodara, took retirement from the position of General Secretary of the Indian Mathematical Society (IMS) w. e. f. April 01, 2013. He did so in spite of the fact that Prof N. K. Thakare, myself and most of the other Council Members of the IMS insisted that he should continue for yet another term. We were all disappointed when he stuck to his decision taken two years ago that he should now retire from IMS because of age factor, and that he should now give opportunity to younger generations of mathematicians to shoulder such responsibilities of IMS. He was generous to promise that he will keep advising us on all important matters even if he is not officially associated with IMS.

A Man of Conviction

I came in contact with Prof V.M.Shah mostly because of our assignments with the Indian Mathematical Society. When I became associated with the IMS as its Academic Secretary, Prof Shah was the General Secretary. Beginning as Editor, the Mathematics Student, President of IMS, as Editor of JIMS and with several other responsibilities, his long association with IMS has been really impressive. Prof Shah has been a past Sectional President (Mathematics including Statistics) of the Indian Science Congress Association also. His seniority as an established administrator and a matured mathematician stands out to impact all the offices that he has held in the past. I had the good luck of witnessing all of this within a short span of my companionship with him in IMS. Ever since he became the General Secretary of IMS, the Society had to face serious challenges concerning such things as the Registration Number of the Society, its Headquarters , its finances, its publications etc. But, with exemplary team work, he met all of them with firmness and conviction giving utmost consideration to the future of the Society. His clarity of objectives would always lead the considerations and decisions of the

Society. There was always a responsible patronship flowing from him while dealing with delicate and sensitive issues of IMS

A few Tests as General Secretary

During a transition period of the office bearers of IMS , there were suggestions by some of us, apparently in good faith, to increase the activities of the Society on the pattern of the American Mathematical Society. To realize this, some unofficial meetings were also held, and the academic programmes were chalked out and even announced on a particular website. Unfortunately, the people involved in doing so didn't take the IMS Council into confidence and thought that the Council will approve their decisions anyway. Some Council members of the IMS were visibly hurt and annoyed at this incident simply because the bylaws of the IMS were not followed. There were serious objections raised by them and it appeared that the IMS might be thrown into disarray. Prof Shah held the view that IMS is not only for a small group of people. The decisions taken by anyone ignoring the IMS bylaws cannot be accepted. The IMS has its own status and glorious history, and cannot follow the path and bylaws of the American Mathematical Society , even if it were otherwise sound. He stuck to the view that IMS is for all mathematicians of the country having a mass base, not meant for a select group of mathematicians. Using his remarkable experience and leadership, he acted and made sure that the main structure of the society is not broken. He took each of the Council Members into confidence, overturned all the decisions taken against the bylaws of the Society and gradually succeeded in meeting all the challenges with firm conviction. He would do exactly what was in the interest of an emerging Indian Mathematical Society as a whole. He made sure that the unity of people serving the Society should not be disturbed at any cost. This was a hard decision, but the Society came out of the above controversy with admirable success.

Simple and Straightforward

Prof Shah is nearing the age of 84 and has accomplished a lot as a mathematician. He stayed and served the M. S. University of Baroda at Vadodara (Gujarat) as Professor and Head of the Department of Mathematics, and also the Dean of the Faculty of Science. He never moved to any other place for a job. He always attracts people very easily who come in his contact. He never makes any attempt for that, it comes spontaneously because of his humility and the manner of talking to the people at large. In all the annual sessions that I have attended with him, most of the hosts, local faculty and delegates respected and talked to him as the seniormost officer of IMS which he was. He was at ease while talking to every unknown commoner as well as with mathematicians of highest possible stature. Even the media people followed him closely rather than the official president or the chairman of a particular annual session. This has not been embarrassing to the President or the Chairman except only on some rare occasion. He has given the best of what he had in himself to the Indian Mathematical Society. As I understand, he commands an enviable respect from the fraternity of the M. S. University of Baroda at Vadodara and its authorities, particularly from the faculty members of Mathematics Department even long after his retirement from that university. I mention this because such a nice tradition nowadays is vanishing very fast in our modern set up of educational atmosphere.

Very Methodical

Prof Shah tried to streamline everything in the IMS where he could as its General Secretary. Many people, especially the young ones, offer to host an annual session, but they don't know how to proceed. Also, there was nothing written in the IMS records as to what are the responsibilities of the hosts, how he/she will manage funds, who will select a chief guest,*etc*. Prof Shah, in consultation with the senior council members of IMS, drafted a "document" which has become a complete guide for the hosts. It has now been adopted by the IMS Council also. Now the hosts have to ask very little which is not mentioned in the document. This is just an example, but he has also streamlined several other functions of the Society like this one, and has been suggesting all of us to keep things streamlining as far as possible.

In every Council meeting or any other meeting, he would always carry a bundle of hand-written papers including the agenda of the meeting, and will discuss each one of them, carefully recalling the complete status of the matter. His memory is remarkably sharp even at this age, but still he didn't want to miss even a single item, and so they were all recorded beforehand in these hand-written papers. He has been always very patient and cool, and never lost his temper except on one occasion. In fact, it was an outburst against an attempt which might have damaged the very fabric of the Society. It will not be fair to recall that incident and so I must leave it here.

Author of "Majedar Ganit":

He has a wonderful art of narrating and explaining things, mathematical or otherwise. Since he likes to teach mathematics at all levels, he has written a series of books, called "Majedar Ganit" for young children. Unfortunately, I have not seen them, but as casually mentioned by Prof Shah himself, they are in Gujarati and are in great demand in that part of the country. He keeps them updating and gets regular applaud for those books from encouraged readers. No one wants to quit a talent which he has nourished and enjoyed throughout his life. These books are testimony to his love for teaching and enjoying mathematics.

I wish Prof V.M. Shah a long, healthy and cheerful life and that he keeps encouraging generations of mathematicians like all of us.

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IMPORTANCE OF MATHEMATICS AT SCHOOL LEVEL

HUZOOR H KHAN

Let me commence my address by stating that I am grateful to the members of the Indian Mathematical Society for electing me the President of the Indian Mathematical Society for 2012 - 13. It is a pleasure for me to be the President of the society for the year which is declared as the National Mathematical Year in recognition of Mr Srinivasa Ramanujan's (1987 - 1920) contribution to Mathematics on his 125th birth anniversary. He was one of the greatest Mathematicians of the twentieth century. Well known mathematicians, Professors G.H. Hardy and J.E. Littlewood, compared Mr. Ramanujan's mathematical abilities and natural genius with all-time great mathematicians like Leonhard Euler, Carl Friedrich Gauss and Karl Gustav Jacobi. The influence of Mr. Ramanujan on number theory is without parallel in mathematics. His papers, problems and letters would continue to captivate mathematicians in the future. He rediscovered a century of mathematics and made new discoveries. Different types of programmes in the discipline of Mathematics were organized in schools, colleges and Universities to celebrate the National Mathematical Year 2012.

Mathematics, the queen of Sciences, is so central for the development of Science and industry that it is impossible to think of modern life without Mathematics. Mathematical techniques, tools and mathematical approaches nourish, develop and promote the health of all the branches of Science, technology and humanities. Many scholars of the world paid rich compliments to Mathematics. Roger Bacon (1220 - 1292) stated that "Mathematics is the gate and key to the sciences" while De Morgan's (1806 - 1871) opinion is very different. "We know that mathematician care no more for logic than logician for mathematics. The two eyes of exact science are mathematics and logic; the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two". The three R's in education include

* The text of the Presidential Address (general) delivered at the 78th Annual Conference of the Indian Mathematical Society held at Banaras Hindu University, Varanasi - 221 005, UP, during the period January 22 - 25, 2013.

Arithmetic and this indicates the fundamental importance of Mathematics, as a component of culture and education. In fact, mathematics and language are the two essential tools with human beings which are used to explore and acquire new knowledge in emerging areas of science and technology. Mathematics develops the intelligence of children and helps in cultivating higher intellectual skills. It develops and enhances rationality among citizens. Thus, it has been recognized as a mental tool. It is a matter of general experience that school children who do well in mathematics, also do well in most of the other subjects. Mathematics is one of the easiest subjects, but all depends on how it is taught. The need of the hour is to regard Mathematics as the pre requisite of all programmes of social development. Even the Arts and Philosophy depend on Mathematics. Mathematics is also the language of Science and promoter of the development in the field of computers and information technology. This shows that Mathematics can be made easier and more accessible to the learners. The future of mankind is going to be governed by Mathematics.

Presently researches are going on in different parts of the world regarding the need of adaption of strategies which would provide opportunities for children to think rather than use the product of thoughts, to do mathematics instead of reading and listening mathematics. In our country there is great dissatisfaction among the mathematicians, parents and children about the way mathematics is being taught and tested at school level. Government policy regarding **No Examination** up to Secondary level will decrease the interest of the children in Mathematics as well as affect the quality of teaching and learning at school stage. This is a dangerous move with serious social implications. Examination occupies a prominent position in the entire educative process. After a course of study or a session of teaching, it is desirable to assess the degree to which the subject matter taught has been learnt by the students. Therefore, some kind of examination or evaluation system is essential at every grade level to ensure that a child being promoted to the next higher grade has acquired minimum knowledge and skills expected at the grade level which he has completed. To a common man the idea of having no examination up to secondary stage may sound very reasonable both psychologically and pedagogically. But beyond that level all educational and job-related opportunities are awarded on the basis of large-scale competitive examinations. Who will prepare students for those competitive examinations? In fact, the concept of formal examination germinated earlier in history than the concept of teaching. It was at some point of time during 12th Century that an emperor in China began to conduct examinations to select qualified and competent persons to help him manage the affairs related to governance. Thereafter, formal teaching classes (like present coaching classes) began for preparing individuals for that examination. Thus, teaching and examination are so closely associated that it is difficult to conceive of one without conceiving of the other. That is why we say that examination or evaluation is an integral part of teaching-learning process.

All human endeavors involve some kind of evaluation. Measurement and assessment are also partially or wholly associated with the overall process of evaluation. Education is an activity, which is aimed at modifying the behavior of children in desired direction. We assess, not only the extent and quality of learning, but also the degree to which the curriculum, methods of instruction, and instructional material used have been instrumental in achieving educational objectives. In this way, evaluation is an integral part of teaching-learning process. Examination, whether organized by individual teachers for assessing the effectiveness of their instructional approaches or by an outside agency for the purpose of certification, is an important instrument which is applied to ensure whether pre-determined instructional objectives have been achieved to a desired degree. There are two main approaches followed in schools by which student'learning is assessed - an internal formative evaluation conducted by the teachers who teach and an external summative evaluation conducted by a national or state board of education. The internal formative evaluation is a part of continuous and comprehensive evaluation (CCE) organized by teachers for the purpose of obtaining feedback from the students which might be used for improving curriculum and instructional strategies. On the other hand, external summative evaluation is organized for the purpose of issuing certificates of merit indicating the quantity and quality of learning acquired by the learners over a long period of time.

It is believed that examination generates anxiety and stress among children which result in an adverse effect on their learning. This belief is only partially true. Of course, heightened anxiety certainly has adverse impact on a child's learning and motivation, but a moderate level of anxiety is essential for a meaningful learning. This has been proved through research studies in education and psychology. Another argument in favour of not having any examination up to secondary stage is that children can learn better when they would be free from the worries related to examination. On the other hand, it is also argued that many children resort to drastic steps like committing suicide after failing in examination. If examinations are not held till the end of secondary school, no such thing will happen.

The belief that examination results throws a child into depression which culminates into suicide is also not wholly correct. It is not the examination that drags some children towards committing suicide; rather, it is the long lasting mental pressure exerted by their parents to do well in the examination. Every parent now wants that his child should become an IAS or an Engineer or a Doctor, without caring for his interest abilities and level of motivation which is so essential for learning of any kind. I am of the opinion that the decision taken by the Government in this regard will not help the students. The Government may claim statistically that literacy rate is improved but the students will pass their examination without the knowledge.

In the present system, we are witnessing a growing tendency among talented students to pursue studies in areas other than mathematics. How could we effectively tackle the challenge of attracting more talented students to Mathematics? I personally feel that the teachers as well as the students will not take the teaching seriously because they know that the student will be promoted without any examination. Mathematics is such a subject where the student has to understand the basic concepts. If the student is lenient in his studies especially in Mathematics due to this policy, he will lose interest and will be afraid of Mathematics in future. This will led to a talent crunch, seriously impeding the development of the future generation of Mathematician. I am of the opinion that in near future we will not be able to find students in Mathematics with good understanding. This growing backwardness in Mathematics will eventually retard its ability to be globally competitive and affect economic growth and social well-being of our country. To promote the attractiveness of Mathematics as subjects and professions, it is essential for the Mathematician to think about the problem which will occur in future.

I am also fortunate enough of presiding the conference at Banaras Hindu University, a great seat of learning who celebrated the 150th birth anniversary of the founder Mahamana Pandit Madan Mohan Malaviya (1861 - 1946), the great Indian leader, scholar and social reformer about a month ago. He has laid down three steps for conducting a programme:

- **"To sit together is a beginning;**
- **To think together is a progress; and**
- **To act together is a Success."**

I hope that the participants here will sit together, will think together and will interact/ discuss together to make this conference a complete success. One of the most important criterions for the success of a conference is the participation of a large number of young workers in the academic programmes of the conference. Thus it is the duty of the seniors to help and promote the calibre of the younger.

In the last, I suggest the followings:

Mathematics should be application oriented and mathematicians should be aware of scope, significance and limitations of the subject.

Teaching of core subjects should not be ignored while exploring new trends.

Promote excellence in the field of Mathematics and research and encourage those students/research scholars who are interested in Mathematics teaching /research.

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GROWTH AND POLYNOMIAL APPROXIMATION OF ENTIRE FUNCTION: A BRIEF SURVEY*

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Abstract. The main purpose of my today's talk is to provide a brief review of interrelations between the growth of entire functions and polynomial approximation and interpolation error.

1. Introduction

The origin of the general theory of entire functions, that is to say of functions which are regular in every finite region of the complex plane, found in the work of Weireastrass [41] and was developed by Picard, Borel, Poincare, Hadamard and others. Many new concepts were introduced in the beginning of twentieth century by various mathematicians and the theory of entire functions has been enriched by the works of Whittaker, Hayman, Boss, Shah, Clunie, Seremata, Juneja and others.

Due to the property of regularity in every finite region of the complex plane, the entire function $f(z)$ can be expanded in a Taylor series around any point $z = z_0$ of the plane. In particular we may assume $z_0 = 0$ and have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We write the maximum modules of $f(z)$ by $M(r) = \max_{|z|=r} |f(z)|$.

2. Measures of Growth

For classifying entire functions by their growth, the concept of order was introduced and given by the formula

$$\rho = \lim_{r\to\infty} \sup \frac{loglogM(r)}{log(r)} \tag{2.1}$$

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If $0 < \rho < \infty$, then the concept of type permits a subclassification by defining their type as

$$T = \lim_{r\to\infty} \sup \frac{logM(r)}{r^{\rho})}, \quad 0 \leq \tau \leq \infty \tag{2.2}$$

In 1954 Boas [2, pp. 9-11] obtained the following characterization

$$\lim_{r\to\infty} \sup \frac{loglogM(r)}{log(r)} = \rho = \lim_{n\to\infty} \sup \frac{nlogn}{log|a_n|^1} \tag{2.3}$$

$$\lim_{r\to\infty} \sup \frac{loglogM(r)}{r^\rho} = T = \lim_{n\to\infty} \sup \left(\frac{1}{e\rho} n|a_n|^{\rho/n}\right). \tag{2.4}$$

For the class of order $\rho = 0$ (slow growth) and $\rho = \infty$ (fast growth) many characterizations like (2.3) and (2.4) have been given. We call the entire function $f(z)$ of maximal, minimal and normal type as $T = \infty, T = 0$ or $0 < T < \infty$.

In 1933, Whittaker [42] introduced the concept of lower order for entire function as given by the formula

$$\lambda = \lim_{r\to\infty} \inf \frac{loglogM(r)}{log(r)}, \quad (0 < \lambda < \infty). \tag{2.5}$$

In analogy with the lower order, lower type is also defined as

$$T == \lim_{r\to\infty} \inf \frac{loglogM(r)}{r^\rho}, \quad (0 < t < \infty). \tag{2.6}$$

Since the sequence a_n determines the function $f(z)$ completely and uniquely, it should be possible to discover all the properties of the functions $f(z)$ by examining the behavior of its coefficients. Several workers such as Shah [34, 35, 36, 37], Rahman [29], Boas [2] and other have found the formulas relating the coefficients of Taylor series with order, lower order, type, lower type etc. Juneja [4] has obtained results which give formula for order *etc.* in terms of ratio of consecutive coefficients.

3. Concept of Index and Index-pair

For entire function $f(z)$ of class of order $\rho = 0$ or $\rho = \infty$, the usual definitions of type and lower type are not feasible and so growth of $f(z)$ can be precisely measured by confining to above concepts only. Various attempts, though in different directions have been made to study the growth of such functions.

In 1963 Sato [33] introduced the concept of 'index' of an entire function. Thus, if

$$\rho(q) = \lim_{r\to\infty} \sup \frac{log^{|q|} M(r)}{log(r)}, \quad q = 3, 4, \cdots \tag{3.1}$$

and

$$T(q) = \lim_{r\to\infty} \sup \frac{log^{|q-1|} M(r)}{r^{\rho(q)}}, \quad (0 < \rho(q) < \infty) \tag{3.2}$$

Where $log^{[0]}M(r) = M(r)$ and $log^{[q]}M(r) = log(log^{[q-1]}M(r))$, then $f(z)$ is said to be of index q if $\rho(q?1) = \infty$ and $\rho(q) < -infty$. Sato [33] obtained the coefficient characterization for $\rho(q)$ and $T(q)$.

The results of Sato [33] failed to compare the growth of entire functions of order $\rho = 0$ and $\rho(q?1) = \infty, \rho(q) = 0$.

In 1952, Shah and Ishaq [38] also studied growth in a more generalized manner than above and obtained the formulae

$$\rho^{*(k)} = \lim_{r\to\infty} \sup \frac{log^{(k)}M(r)}{log^{(k-1)r}}, \quad (0 \le \rho^{*(k)} \le \infty) \tag{3.3}$$

and

$$\rho'^{(k)} = \lim_{r\to\infty} \sup \frac{log^{(k)}M(r)}{log^{(k)}r}, \quad (0 \le \rho'^{(k)} \le \infty) \tag{3.4}$$

$k = 1, 2, 3, \cdots$ and obtained coefficients characterizations for $\rho^{*(k)}$ and $\rho'^{(k)}$. But they did not introduced the concept of type for functions with $\rho^{*(k)} = \rho'^{(k)}$. Thus this approach is not sufficient for functions of fast growth.

In 1976, Juneja et.al [6] introduced the concept of index-pair $(p, q), (p \ge q \ge 1)$ for an entire function $f(z)$ and defined the (p, q)-order ρ and lower (p, q)-order λ with index pair (p, q) by In 1977, Juneja et.al [7] defined (p, q)-type T and lower (p, q)-type t of an entire function with index-pair (p, q) by

$$\lim_{r\to\infty} \sup \frac{log^{(p)}M(r)}{log^{[q]}r} = \rho(p, q) \equiv \rho$$

$$\lim_{r\to\infty} \inf \frac{log^{(p)}M(r)}{log^{[q]}r} = \lambda(p, q) \equiv \lambda \tag{3.5}$$

In 1977, Juneja et.al [7] defined (p, q)- type T and lower (p, q)- type t of an entire function with index-pair (p, q) by

$$\lim_{r\to\infty} \sup \frac{log^{(p-1)}M(r)}{(log^{[q-1]}r)^{\rho}} = T(p \cdot q) \equiv T$$

$$\lim_{r\to\infty} \inf \frac{log^{(p-1)}M(r)}{(log^{[q-1]}r)^{\rho}} = t(p, q) \equiv t; \quad (0 \le t \le T \le \infty), \tag{3.6}$$

where $b < \rho < \infty, b = 0$ if $p > q$ and $b = 1$ if $p = q$.

Remark 3.1. The entire function with index-pair (1,1) are polynomials of degree > 1.

Remark 3.2. Transcendental entire functions of finite order are of index pair (2,1).

Remark 3.3. Entire functions having index-pair (2,2) and ρ, λ defined in (3.5) are called logarithmic order and lower logarithmic order introduced by Shah and Ishaq [38].

Remark 3.4. T, t defined by (3.6) are called logarithmic type and lower logarithmic type respectively for index-pair (2,2).

In [6, 7] Juneja et. al obtained coefficient characterization of entire functions for (p, q)-order, lower (p, q)-order, (p, q)-type and lower (p, q)-type.

4. Proximate Order and Generalized Types

The concepts discussed so far are inadequate to compare the growth of those functions which are of same order and of infinite type. Hence, for refinement of the above growth scale, the concept of proximate order was introduced by Levin [21] which is extended by Nandan et al [26] to entire functions with (p, q)-growth:

Definition 4.1. A real valued positive function $\rho(r)$ defined on $[r_0, \infty), r_0 > exp^{[q-1]}1$, is said to be of proximate order of an entire function with index-pair (p, q) if

(i) $\rho(r) \to \rho$ as $r \to \infty, b < \rho < \infty$

(ii) $\Delta_q(r)\rho'(r) \to 0$ as $r \to \infty; \rho'(r)$ denotes the derivatives of $\rho(r)$ and $\Delta_q = \prod_{I=0}^{q} log^{(i)}(r)$. Also

$$\lim_{r\to\infty} \sup \frac{log^{(p-1)}M(r)}{(log^{[q-1]}r)^{\rho}} = 1$$

Definition 4.2. Let $f(z)$ be an entire function of (p, q)-order $\rho(p, q)(b < \rho(p, q) < \infty)$ such that

$$\lim_{r\to\infty} \sup \frac{log^{(p-1)}M(r)}{(log^{[q-1]}r)^{\rho(r)}} = T*(p, q) \equiv T*$$

$$(0 \leq t* \leq T* \leq \infty).$$

$$\lim_{r\to\infty} inf \frac{log^{(p-1)}M(r)}{(log^{[q-1]}r)} = t*(p, q) \equiv t*.$$

If $0 < t* \leq T* < \infty$, then $\rho(r)$ is said to be proximate order and $t*$ and $T*$t as its generalized lower type and generalized type respectively.

In 1980 Nandan et.al [28] also defined lower proximate order and obtained coefficients characterization of generalized (p, q)-type $T*$ and generalized lower (p, q)-type $t*$. Their results include some results of Levin [22], Boas [2], Shah [34] *etc.*

Remark 4.1. The formula for (p, q)-order, lower (p, q)-order, (p, q)-type, lower (p, q)-type, generalized type and generalized lower type are not in general applicable to an entire function $f(z) = \sum_{n=1}^{\infty} a_n z^n$, if the coefficients $a_n' s$, vanish for infinity many values of n. Hence in studying the growth of such functions it is preferable to consider the gap power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}, \tag{4.1}$$

where $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1}$ as $n \to \infty$. No element of the sequence $\{a_n\}$ is zero.

Juneja et.al [6, 7] obtained coefficients characterization of $f(z)$ given by (4.1) with respect to index-pair (p, q).

Kasana et.al [11] characterized the lower (p, q)-type with respect to proximate order of an entire function in terms of coefficients and exponents of its gap power series expansion. They obtained the following results.

Theorem 4.1. *Let $f(z)$ be an entire function such that $\phi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ forms a nondecreasing function of n for $n > n_0$. Then*

$$t* = \lim_{n\to\infty} \ inf \left[\frac{\phi(log^{(p-2)}\lambda_{n-1})}{log^{(q-1)}|a_n|^{-1/\lambda_n}}\right]^{\rho}, \quad p \geq 3 \tag{4.2}$$

The relation (4.2) holds for $p = 2$ also, if $\lambda_{n?1} \cong \lambda_n$ as $\to \infty$, we have

$$\frac{t*}{M} = \lim_{n\to\infty} \ inf \left[\frac{\phi(\lambda_{n-1})}{log^{(A)}|a_n|^{-1/\lambda_n}}\right]^{\rho-A}, \tag{4.3}$$

where $A = 1$ if $q = 2, A = 0$ if $q \neq 2$, and

$$M = \begin{cases} (\rho-1)^{p-1}/\rho^p & if, (p-q) = (2,2) \\ 1/erho & if, (p-q) = (2,1) \\ 1 & otherwise \end{cases}$$

here $\phi(x)$ is the unique solution of the equation

$$x = (log^{(q-1)}r)^{\rho(r)-A} \leftrightarrow \phi(x) = log^{(q-1)}r$$

If we omit the condition on $\phi(x)$ in Theorem 4.1, then we have

Theorem 4.2. *For an entire function $f(z)$, we have*

$$t* = \max_{(m_k)} \{\lim_{k\to\infty} \ inf \left[\frac{\phi(log^{(p-2)}\lambda_{m_{k-1}})}{log^{(q-1)}|a_{m_k}|^{-1/\lambda_{m_k}}}\right]^{\rho}\}, \quad for\, p \geq 3.$$

Further, if $\{\lambda_{m_k}\}$ be the sequence of principal indices such that $\lambda_{n_{k-1}} \cong \lambda_{n_k}$ as $k \to \infty$, then for $p = 2$,

$$\frac{t*}{M} = \max_{(m_k)} \{\lim_{k\to\infty} \ inf \left[\frac{\phi(\lambda_{m_{k-1}})}{log^{(A)}|a_{m_k}|^{-1/\lambda_{m_k}}}\right]^{\rho-A}\}$$

The maximum is taken over all increasing sequences of positive integers $\{m_k\}$.

In a subsequent paper Kasana et al. [11] proved a decomposition on theorem, some results involving coefficients, ratio of coefficients and exponents of entire gap power series.

5. Polynomial Approxmiation and Interpolation of Entire Functions

The theory of approximation of functions originated in the classical approximation theorem of Weierstrass. Chebyshev's concepts of best approximation and the converse theorem of Berstein on the existence of a function with a given sequence of best approximants gave it a broad base and it is known that by Runge's theorem. "If E is a connected region in the complex plane then an analytic function can be approximated on arbitrary compact subsets of E, by rational functions". This theorem aroused interest in the approximation of entire functions by polynomials and rational functions on subsets of complex plane. The degree of approximation is seen to depend not only on the region of convergence of the function in question but also on its growth.

Thus, we assume that E be a compact set in complex plane and $\xi^{(n)} = \{\xi_{n_0}, \xi_{n_1}, \cdots, \xi_{n_k}\}$ be a system of $(n+1)$ points of the set E such that

$$V(\xi^{(n)}) = \prod_{0 \le j \le k \le n} |\xi_{n_j} - \xi_{n_k}|$$

and

$$\Delta^{(j)}(\xi^{(n)}) = \prod_{\substack{k=0 \\ k \ne j}}^{n} |\xi_{n_j} - \xi_{n_k}|, \quad j = 0, 1, \cdots, n$$

Again, let $\eta^{(n)} = \eta_{n_1}, \eta_{n_2}, \cdots, \eta_{n_n}$, be the system of $(n+1)$ points in E such that

$$\begin{aligned} V_n &\equiv V(\eta^{(n)}) \\ &= \sup_{\xi_{(n)} \subset E} V(\eta^{(n)}) \end{aligned}$$

and

$$\Delta^0(\eta^{(n)}) \le \Delta^{(j)}(\eta^{(n)}) \quad \text{for } j = 1, 2, \cdots, n.$$

Such a system always exists and is called the n^{th} extremal system of E. The polynomials

$$L^{(j)}(z, \eta^{(n)}) = \prod_{\substack{k=0 \\ k \ne j}} \left(\frac{z - \eta_{n_k}}{\eta_{n_j} - \eta_{n_L}} \right). \quad j = 0, 1, \cdots, n,$$

are called Lagrange extremal polynomials and the limit $d \equiv d(E) = \lim_{n \to \infty} V_n^{2/n(n+1)}$ is called the transfinite diameter of E.

Let $C(E)$ denote the algebra of analytic function on the set E. Define approximation errors as

$$e_{n,1}(f) = \inf_{g \in \pi_n} ||f - g||,$$
$$e_{n,2}(f) = ||L_n - L_{n-1}||. \quad n \geq 2$$
$$e_{n,3}(f) = ||L_n - f||, \quad n \geq 0$$

where $||.||$ is the sup norm and π_n denotes the set of all polynomials of degree $\leq n$.

$$L_n(z) = \sum_{j=1}^{n} L^{(j)}\left(z, \eta^{(n)} f(\eta_{n_j})\right), \quad n \in N$$

is the Lagrange interpolating polynomial.

Bernstein [1] proved that if $f(z) : [?1, 1] \to R$ is continuous then

$$\lim_{n\to\infty} e_{n,1}^{1/n}(f) = 0$$

if and only if $f(x)$ has an entire function extension $f(z)$.

Reddy [30, 31] connected classical order and type with polynomial approximation error of an entire function which is an extension of a continuous function on $[-1, 1]$. Juneja [3] extended these results for lower order and Massa [23] studied for the lower type. Contemporarily, Rice [32] and Winiarski [43] studied order and type for different approximation errors of a continuous functions on the arbitrary domains. Kasana and Kumar [9, 10] characterized the (p, q)-order, lower (p, q)-order, (p, q)-type and generalized lower (p, q) type in terms of approximation errors $e_{n,1}(f), e_{n,2}(f), e_{n,3}(f)$.

6. Approximation of Generalized Axisymmetric/ Biaxisymmetric Potentials

Mccoy [24, 25, 26], Kasana and Kumar [18] and Srivastava [39, 40] studied polynomial approximation and growth of generalized axisymmetric/biaxisymmetric potentials. The regular solutions of the generalized axisymmetric/biaxisymmetric potential equation were related to an analytic function by the use of transform operators. The generalized biaxisymmetric potentials are the solution of the potential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + \frac{2\beta+1}{y}\frac{\partial}{\partial y}\right) F^{(\alpha,\beta)} = 0, \quad \alpha > \beta > -1/2.$$

(α, β) fixed in a neighborhood of the origin where the analytic Cauchy data, $F_x^{(\alpha,\beta)}(0, y) = F_y^{(\alpha,\beta)}(x, 0) = 0$ is satisfied along the singular lines.

Considering the L^p-approximation of the entire GBASP function $F^{(\alpha,\beta)}$ through the set of all real harmonic polynomials of degree at most $n, H_n^{(\alpha,\beta)}$, and defining the L^p-approximation error as

$$E_n^p(F^{(\alpha,\beta)}) = inf\{||F^{(\alpha,\beta)} - P^{(\alpha,\beta)}||_p, \quad P^{(\alpha,\beta)} \in H_n^{(\alpha,\beta)}\}, \quad n = 1, 2, \cdots$$

Khan et.al [13] obtained the characterization of lower order and lower type in terms of L^p-approximation $E_n^p(F^{(\alpha,\beta)})$. In the subsequent paper Khan et.al [14] the Chebyshev polynomial approximation of an entire solution of generalized axisymmetric Halmholtz equation (LASHE)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2v}{y}\frac{\partial u}{\partial y} + k^2 u = 0, \quad v > 0,$$

in Banach spaces have been studied. Some bounds on generalized order of GASHE functions of slow growth have been obtained in terms of the Bessel- Gegenbauer coefficients and approximation errors using function theoretic methods. In [19, 20] Kumar obtained some inequalities on growth parameters of entire function GASHE in terms of approximation errors.

7. Growth and Approximation over a certain Lemniscate

A lemniscate is a locus of the form

$$\Lambda : |p(z)| = \mu \to 0,\ p(z) \equiv (z?\beta_1)(z?\beta_2)\cdots(z?\beta_\lambda \tag{7.1}$$

where μ and the β_k are fixed, and z varies in accordance with (7.1); the β_k are not necessarily all distinct. If β_k are fixed and μ varies, equation (7.1) represents a family of lemniscates. The lemniscate (7.1) is the image of the circle $|\varpi(z)| = \mu$ under the transformation $\varpi = p(z)$.

The partial sum of Jacobi series of an entire function

$$f(z) = \sum_{k=0}^{\infty} qk(z)[\gamma(z)]^{k-1} \tag{7.2}$$

where $\gamma(z)$ is a polynomial of degree ξ and $qk(z)$ is a uniquely determined polynomial of degree $\xi - 1$ or less, interpolate $f(z)$ at the zeros of $\gamma(z)$.

Rice [32] has extended the results (2.3) and (2.4) by considering the polynomial expansion of $f(z)$ of the form (7.2). If Γ_R be the lemniscates $\Gamma_R = \{z : |\gamma(z)| = R\}, ||\Gamma_R||$ be the length of Γ_R and $M(\Gamma_R, f) = ||f||_{\Gamma_R} = \max_{z\in\Gamma_R} |(z)|$. Using the estimate

$$||\Gamma_R|| = 2\pi R^{\frac{1}{\xi}}(1 - 0(1)) \text{ as } R \to \infty,$$

showed that $f(z)$ given by (7.2) is an entire function of order ρ, if and only if,

$$\lim_{R\to\infty} \sup \frac{loglogM(\Gamma_R, f)}{logR} = \frac{\rho}{\xi}$$

and that $f(z)$ is of order $\rho > 0$ and type $T(0 < T < \infty)$, if and only if

$$\lim_{R\to\infty} \sup \frac{logM(\Gamma_R, f)}{R^{\rho/\xi}} = T,$$

The generalizations of (2.3) and (2.4) read as follows:

Let α' be fixed. Then $f(z)$, given by (7.2), is an entire function of order $\rho > 0$, if and only if,

$$\rho = \xi \lim_{k\to\infty} \sup \frac{klogk}{log||qk(z)||^{-1}_{\Gamma_\alpha}}$$

and that it is of order $\rho > 0$ and type $T(0 < T < \infty)$, if and only if,

$$E\rho T = \xi \lim_{k\to\infty} \sup\{k(||qk(z)||_{\Gamma_\alpha})^{\rho/\xi^k}\}.$$

Consider the function

$$H_\alpha(\varpi) = \sum_{k=1}^{\infty} ||qk(z)||\Gamma_\alpha \varpi^k, \quad \alpha < R,$$

Where $||qk(z)||\Gamma_\alpha = \max_{z\in\Gamma_\alpha}\{|qk(z)|\}$ as $k \to \infty$. It is known [32, Lemma 2] that if $f(z)$ is analytic in Γ_R, then there exists a polynomial $Q(z)$ of degree $\xi - 1$ independent of k and R such that for $\alpha' < R$ and $k = 1, 2, \cdots$,

$$||qk(z)||\Gamma_\alpha \leq \frac{||\Gamma_R||M(\Gamma_R, f)}{2\pi R^k} ||Q(z)||\Gamma_R \tag{7.3}$$

Using (7.3) we can easily seen that $H_\alpha(\varpi)$ is entire if and only if

$$[||qk(z)||\Gamma_\alpha]^{1/R} = 0.$$

Moreover, $H_\alpha(\varpi) = \sum_{k=1}^{\infty} ||qk(z)||\Gamma_\alpha varpi^k$ holds in the whole complex plane.

Let B be a Caratheodory domain (a domain bounded by Jordan curve). For $1 \leq p \leq \infty$ let $L^p(B)$ be the class of all function f holomorphic in B such that

$$||f||_{B,p} = \left[\frac{1}{A}\int\int_B |f(z)|^p \mathrm{d}x\mathrm{d}y\right]^{1/p} < \infty,$$

where A is the area of B. For $f \in L^p(B)$, set

$$E^p_\xi(f) = \inf_{t\in\pi_\xi} ||f - t||_{B,p}, \tag{7.4}$$

π_ξ consists of all polynomials of degree at most $\xi = nk$.

Kapoor and Nautiyal [8] have characterized generalized growth parameters for entire functions of fast growth in terms of $||qk(z)||\Gamma_\alpha$ in sup norm. Kumar [21] studied the generalized growth parameters in terms of approximation errors defined by (7.4). Khan et.al [16] generalized some results of Rahman [29] by considering the maximum of $|f(z)|$ over a certain lemniscates instead of considering the maximum of $|f(z)|$, for $|z| = r$ and obtained the analogous results for the entire function $f(z)$ given by (7.2). Moreover, we have obtained some inequalities on the lower order, type and lower type in terms of $||qk(z)||\Gamma_\alpha$.

In a subsequent paper Khan et.al [15] obtained the formulae for (p, q)- order and generalized (p, q)-type of $f(z)$ in terms of $||qk(z)||\Gamma_\alpha$ and polynomial approximation error $E^p_\xi(f)$. Their results include some results of Sato [33], Rice [32], Juneja [4], and Juneja and Kapoor [5]. Khan et.al [15] proved the following theorems:

Theorem 7.1. *If $f \in L^p(B), 1 \leq p\infty$, can be extended to entire function with index-pair $(p,q), (p,q)$-order$\rho(p,q,f)(b < \rho(p,q,f) < \infty)$ and generalized (p,q)-type $T*(p,q,f)$, then for every $||qk(z)||\Gamma_\alpha$, there exists an entire function*

$$H_\alpha(\varpi) = \sum_{k=1}^{\infty} ||qk(z)||_{\Gamma_\alpha} \varpi^k$$

such that

$$\rho(p,q,f)/\xi = \rho(p,q,H_\alpha)$$

and

$$T*(p,q,f) = T*(p,q,H_\alpha)$$

Theorem 7.2. *If $f \in L^p(B)$, can be extended to an entire function with index-pair$(p,q), (p,q)$-order$\rho(p,qf)(b < \rho(p,q,f) < \infty)$ and generalized (p,q)-type $T*(p,q,f)$, then for every $E_\xi^p(f)$, there exists an entire function $\tilde{H}(t) = \sum_{k=1}^{\infty} E_\xi^p(f)t^k$ such that*

$$\rho(p,q,f)/\xi = \rho(p,q,\tilde{H})$$

and

$$T*(p,q,f) = T*(p,q,\tilde{H}).$$

Theorem 7.3. *Let $f(z) \in L^p(B)$. Then $f(z)$ can be extended to an entire function of (p,q)-order$\rho(p,q,f)(b < \rho(p,q,f) < \infty)$ if and only if*

$$\rho(p,q,f)/\xi = P(L(p,q,H_\alpha)) = P(L(p,q,\tilde{H}))$$

where

$$L(p,q,H_\alpha) = \lim_{k\to infty} \sup \frac{log^{[p-1]}k}{log^{[q-1]}\left(||qk(z)||_{\Gamma_\alpha}\right)^{-1/k}}$$

and

$$L(p,q,\tilde{H} = \lim_{k\to infty} \sup \frac{log^{[p-1]}k}{log^{[q-1]}\left(E_\xi^p(f)\right)^{-1/k}}, \quad \xi = nk$$

Theorem 7.4. *Let $f \in L^p(B)$. Then $f(z)$ can be extended to an entire function of (p,q)-order $\rho(p,q,f)(b < \rho(p,q,f) < \infty)$ and generalized (p,q)-type $T*(p,q,f)(0 < T*(p,q,f) < \infty)$ if and only if*

$$\frac{T*(p,q,f)}{M(p,q)} = \lim_{k\to\infty} \sup \left[\frac{\phi\left(log^{[p-2]}k\right)}{log^{[q-1]}\left[||qk(z)||_{\Gamma_\alpha}\right]^{-1/k}}\right]^{\frac{\rho(p,q,f)}{\xi} - A}$$

$$= \lim_{k\to\infty} \sup \left[\frac{\phi\left(log^{[p-2]}k\right)}{log^{[q-1]}\left(E_{\xi}^{p}(f)\right)^{-1/k}} \right]^{\frac{\rho(p,q,f)}{\xi}-A} . \quad \xi = nk$$

where

$$M(p,q) = \begin{cases} (\zeta-1)^{(\zeta-1)}/\zeta^{\zeta} & \text{if } (p,q)=(2,2), \quad \zeta = \frac{\rho(2,2)}{\xi}, \\ \frac{\xi}{e\rho(2,1)} & \text{if } (p,q)=(2,1), \\ 1. & \text{otherwise} \end{cases}$$

Theorem 7.5. *If $f(z) \in L^{p}(B)$ can be extended to an entire function of (p,q)-order$\rho(p,q,f)$ such that $\rho(p,q,f)=b$, then for every $\delta > 0$*

$$\lim_{k\to\infty} \sup \frac{\left(log^{[p-2]}k\right)^{\delta})}{log^{[q-1]}\left(||qk(z)||_{\Gamma_{\alpha}}^{-1/k}\right)} = 0$$

Further, if $\rho(p,q,f) > b$ and $f(z)$ is of minimal generalized (p,q)-type, then

$$\lim_{k\to\infty} \sup \frac{\phi\left(log^{[p-2]}k\right)}{log^{[q-1]}\left(||qk(z)||_{\Gamma_{\alpha}}^{-1/k}\right)} = 0$$

Theorem 7.6. *If $f(z) \in L^{p}(B)$ can be extended to an entire function of (p,q)-order $\rho(p,q,f)$ such that $\rho(p,q,f)=b$, then for every $\delta > 0$,*

$$\lim_{k\to\infty} \sup \frac{\left(log^{[p-2]}k\right)^{\delta}}{log^{[q-1]}\left(E_{\xi}^{p}(f)\right)^{-1/k}} = 0$$

Further, if $\rho(p,q,f) > b$ and $f(z)$ is of minimal generalized (p,q)-type, then

$$\lim_{k\to\infty} \sup \frac{\phi\left(log^{[p-2]}k\right)}{log^{[q-1]}\left(E_{\xi}^{p}(f)\right)^{-1/k}} = 0$$

Recently, Khan et.al [17], characterized the lower (p,q)-order and generalized lower (p,q)-type with respect to proximate order in terms of polynomial coefficients and approximation errors.

References

[1] S. N. Bernstein,*Lecons sur Les properties extremales et la meilleure approximation des functions analytiques d'une variable reelle,* Gauthier - Villars, Paris, (1926).

[2] R. P. Boas,*Entire Functions,* Academic Press, New York, (1954).

[3] O. P. Juneja,*Approximation of an entire function,* J. Approx.Theory 4(1974), no.11, 343–349.

[4] O. P. Juneja,*On the coefficients of entire series,* J. Analyse Math. 24 (1971), 395-401.

[5] O. P. Juneja and G. P. Kapoor,*Polynomials coefficients of entire series,*Yokohama Math. J. 22 (1974), 125-133.

[6] O. P. Juneja, G. P. Kapoor and S. K. Bajpai,*On the (p, q)-order and lower order (p, q)-order of an entire function,* J. Reine Angew Math. 282 (1976), 53-67.

[7] O. P. Juneja, G. P. Kapoor and S. K. Bajpai,*On the (p, q) type and lower (p, q)-type of an entire function,* J. Reine Angew Math. 290 (1977), 180-190.

[8] G. P. Kapoor and A. Nautiyal,*Polynomial coefficients and generalized orders of an entire function,* Math. Japonica 30 (1985), no.1, 1-13.

[9] H. S. Kasana and D. Kumar,*On approximation and interpolation of entire functions with index-pair (p, q),*Publicacions Matematiques, 38 (1994), 255-267.

[10] H. S. Kasana and D. Kumar,*On polynomial approximation of entire functions with index-pair (p, q),* Rend. Instit. Mat. Univ. Treiste, 35 (2003), 49-59.

[11] H. S. Kasana, D. Kumar and G. S. Srivastava,*On the growth of coefficients of entire function,* Rend. Acad. Naz. delle Sci. delta dei X L Mem. Mate., 16 (1992), fasc. 1, 1-19.

[12] H. S. Kasana, D. Kumar and G. S. Srivastava,*On the generalized lower type of entire gap power series,* U.U.D.M. Report, 16 (1990), 1-12.

[13] Huzoor H. Khan and Rifaqat Ali,*On the best polynomial approximation of generalized bi-axisymmetric potentials in Lp-norm, p 1,*T.J.M.M. 3 (2011) no.2, 103-110.

[14] Huzoor H. Khan and Rifaqat Ali,*Slow growth and approximation of entire solution of generalized axially symmetric helmholtz equation,* Asian J. Math. Stat. 5 no.4 (2012), 104-120.

[15] Huzoor H. Khan and Rifaqat Ali,*Polynomial coefficients and approximation errors of entire series,* T.J.M.M. 4 (2012), no. 1, 25-35.

[16] Huzoor H. Khan and Rifaqat Ali,*On the growth and polynomial coefficients of entire series,* Applied Math. 2 (2011) no. 2, 1124-1128.

[17] Huzoor H. Khan and Rifaqat Ali,*Growth and approximation errors of entire series having index-pair (p, q),* Fasc.Math. 49 (2012), 61–73.

[18] D. Kumar and H. S. Kasana,*Approximation and interpolation of generalized biaxisymmetric potentials,* Pan AmericanMath. J. 9(1) (1999), 55-62.

[19] D. Kumar,*Growth and Chebyshev approximation of entire funciton solution of helmholtz equation in R^2,* European J. Pure and Appl. Math. 3 (2010), no. 6, 1062-1069.

[20] D. Kumar,*On the (p, q)-growth of entire function solutions of helmholtz equation,* J. Nonlinear Sci. Appl. 4 (2011), no. 1, 5-14.

[21] D. Kumar,*Approximation error and generalized orders of an entire function,* Tamsui Oxford J. Math. Sci., 25 (2) (2009), 225-235.

[22] B. Ja Levin,*Distribution of zeros of entire function, Monographs,* Amer. Math. Soc. Translation 5 (1968), Revised (1980).

[23] S. Massa,*Remarks on the growth of an entire function and degree of approximation,* Riev. Mat. Univ. Parma (4) 7 (1981), 1-6.

[24] P. A. McCoy,*Polynomial approximation and growth of generalized axisymmetric potentials,* Cand. J. Math. 31 (1979), no. 1, 49-59.

[25] P. A. McCoy,*Polynomial approximation of generalized biaxisymmetric potentials,* J. Approx. Theory, 25 (1979), 153-168.

[26] P.A. McCoy,*Best Lp-approximation of generalized biaxisymmetric potentials,* Proc. Amer. Math. Soc. 79 (1980), 435-440.

[27] K. Nandan, R. P. Doherey and R. S. L. Srivastava,*Proximate order of an entire function with index-pair (p, q),* Indian J. Pure Appl. Math. 11 (1963), 33-39.

[28] K. Nandan, R. P. Doherey and R. S. L. Srivastava,*On the generalized type and generalized lower type of an entire function with index-pair (p, q),* Indian J. Pure Appl. Math. 11 (1980), 1422-1433.

[29] Q. I. Rahman,*On the coefficients of an entire series of finite order,* Math. Student 25 (1957), 113-121.

[30] A. R. Reddy,*Approximation of an entire function,* J. Approx. Theory, 3 (1970), 128-137.

[31] A.R. Reddy,*Best polynomial approximation of certain entire functions,* J. Approx. Theory, 5 (1972), 97-112.

[32] J.R. Rice,*The degree of convergence of entire function,* Duke Math. J. 38 (1971), 429-440.

[33] D. Sato,*On the rate of growth of entire function of fast growth,* Bull. Amer. Math. Soc., 69 (1963), 411-414.

[34] S. M. Shah,*On proximate order of integral functions,* Bull. Amer. Math. Soc., 52 (1946), 326-328.

[35] S. M. Shah,*A note on lower proximate orders,* J. Indian Math. Soc., 12 (1948), 31-32.

[36] S. M. Shah,*On the coefficients of entire series of finite orders,* J. London Math. Soc., 26 (1951), 45-46.

[37] S. M. Shah,*Polynomial approximation of an entire function and generalized orders,* J. Approx. Theory, 19 (1977), 315-324.

[38] S. M. Shah and M. Ishaq,*On the maximum modulus and coefficients of an entire series,* J. Indian Math. Soc., 16 (1952), 177-182.

[39] G.S. Srivastava,*Approximation and growth of generalized axisymmetric potentials,* Approx. Theory Appl., 12 (4) (1996), 96-104.

[40] G. S. Srivastava,*On the growth and approximation of generalized axisymmetric potentials,* Functiones Et. Approx., 24 (1996), 113-123.

[41] K. Weirestrass,*Zur theorie des eindentiqu analytichen functionen werks,* Ganita 2, 77-124.

[42] J. M. Whittaker,*The lower order of integral functions,* J. London Math. Soc., 8 (1933), 20-27.

[43] T. N. Winiarski,*Approximation and interpolation of entire functions,* Ann. Polon. Math. 23 (1970), 259-273.

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ABSTRACTS OF THE PAPERS SUBMITTED FOR PRESENTATION AT THE 78TH ANNUAL CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY, HELD AT BANARAS HINDU UNIVERSITY, VARANASI - 221 005, DURING JANUARY 22 - 25, 2013

A: Combinatorics, Graph Theory and Discrete Mathematics :

A-1: The total graph of idealization of a finite commutative ring, *Moytri Sarmah and Kuntala Patra ,Guwahati: moytrisarmah@gmail.com, kuntalapatra@gmail.com*

Let R be a finite commutative ring with nonzero identity and M be an R-module. Then the set $R(+)M = \{(r,m) : r \in R, m \in M\}$ is a commutative ring with nonzero unity under the binary operations (+) and (:) defined as $(x,m) + (y,n) = (x+y, m+n)$ and $(x,m)(y,n) = (xy, xn+ym)$, for $(x,m), (x,m), (y,n) \in R(+)M$. The ring is called the Idealization of M in R. The total graph of idealization of M in R, denoted by $T(\Gamma(R(+)M))$ is the graph whose vertex set consists of all the elements of $R(+)M$ and any two vertices (x,m) and (y,n) are adjacent if and only if $(x,m)+(y,n) \in Z(R(+)M)$, where $Z(R(+)M)$ is the set of all zero divisors of $R(+)M$. In this paper we have studied the structure of $T(\Gamma(R(+)M))$, considering $Z(R(+)M)$ as in ideal of $R(+)M$. we have also studied some graphical parameters such as cycle basis, cycle rank, independent number etc. for this graph $T(\Gamma(R(+)M))$.

A-2: Prime graph of product of rings, *Kuntala Patra and Sanjoy Kalita, Guwahati: Kuntalapatra@gmail.com, sanjoykalita1@gmail.com.*

Let R be a commutative ring. The prime graph of the ring R is defined as a graph whose vertex set consists of all elements of R and any two distinct vertices x and y are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by PG(R). In this paper we obtain some results on the chromatic number of prime graph of the ring $\mathbb{Z}_m \times \mathbb{Z}_n$.We also investigate some relations between the chromatic number of prime graph of finite product of commutative rings and the chromatic number of prime graph of these rings.

A-3: Zero-divisor graphs of matrices over noncommutative rings, *Kuntala Patra and Priyanka Pratim Baruah* (*Guwahati: kuntalapatra@gmail.com, pratimbaruah6@rediffmail.com*).

Let R be a non-commutative ring and $M_n(R)$ denote the ring of $n \times n$ matrices over R. Let Γ and $\Gamma(M_n(R))$ be the (directed) zero-divisor graphs of R and $M_n(R)$ respectively. In this paper we investigate the properties of zero-divisor graphs of matrix rings and also discuss the relation between the diameter of $\Gamma(R)$ and that of $\Gamma(M_n(R))$.See.

A-4: Upper bounds for the largest eigenvalue of a bipartite graph, *Jorma K. Merikoski, Ravindra Kumar and Ram Asrey Rajput* (*Agra : jorma.merikoski@uta.fi, ravindra_dei@yahoo.com, ramasreyrajput@yahoo.co.in*).

New upper bounds for the largest eigenvalue of a finite, simple, undirected and bipartite graph G are presented. To obtain our bounds we exploit the special structure of its adjacency matrix A and properties of bipartite graphs. These result are compared with certain upper bounds that work more generally.

A-5: A characterization of the flats of the es-splitting binary matroids, *S.B. Dhotre and M. M. Shikare* (*Pune : dsantosh2@yahoo.co.in, mms@math.unipune.ac.in*).

In this paper, we characterize the flats and the rank function of the es-splitting matroid M_X in terms of the flats and the rank function, respectively of the original matroid M.

B: Algebra, Number Theory and Lattice Theory:

B-1: Generalized $(\theta, \varphi)-$ derivations as homomorphisms or as anti-homomorphisms on σ–prime rings, *Asma Ali and Khalid Ali Hamdin* (*Aligarh : asma_ali2@rediffmail.com, hamdinkh@yahoo.com*).

Let R be a 2-torsion free σ–prime ring and F be a generalized (θ, φ)–derivation on R with associated (θ, φ)–derivation d where σ–commutes with d; θ and φ. Let U be a nonzero square closed σ–Lie ideal of R. In the present paper, we prove that if F acts as a homomorphism or as an antihomomorphism on U, then either $d = 0$ or $U \subseteq Z(R)$.

B-2: Cocentralizing derivations on prime rings, *Asma Ali and Shahoor Khan* (*Aligarh: asma_ali2@rediffmail.com, shahoor.khan @rediffmail.com*).

Let R be a ring with center Z(R) and L be a Lie ideal of R. In the present paper, it is shown that if f and g are derivations of R and L is a nonzero square closed Lie ideal of R such that $f(x)y + yg(x) = 0$ for all $x, y \in L$; then $f(u)[x, y] = 0 = [x, y]g(u)$ for all $x, y, u \in L$ and f, g are central. This generalizes the results of Bresar [Theorem 4.1, Journal of Algebra, 156 (1993),385–394], Lee and Wong [Theorem, Bull. Inst. Math. Acad. Sinica 23 (1997),1–5].

B-3: Structure of certain conditioned rings, *Rekha Rani* (*Khurja : rekharani@rediffmail.com*).

Using commutativity of rings satisfying $(xy)^{n(x,y)} = xy$ proved by Searcoid and MacHale , Ligh and Luh gave a direct sum decomposition for rings with the mentioned condition. Further, Bell and Ligh sharpened the result and obtained a decomposition theorem for rings with the property $xy = (xy)^2 f(x, y)$ where $f(X, Y) \in Z < X, Y >$, the ring of polynomials in two noncommuting indeterminates over the ring $\mathbb{Z}$ of integers. In the present paper, we continue the study and investigate structure of certain rings and near rings satisfying the following condition: $xy = p(x, y)$ with $p(x, y)$ an admissible polynomial in $Z < X, Y >$. This condition is naturally more general than the above mentioned conditions. Moreover, we deduce the commutativity of such rings.

B-4: Regular f–derivations on TM–algebras, *T. Ganeshkumar and M. Chandramouleeswaran* (*Madurai : ganeshkumar.wbc@gmail.com*).

It is well known that BCK and BCI–algebras are two classes of algebras of logic. They were introduced by Imai and Iseki and have been extensively investigated by many researchers. It is known that the class of BCK − algebras is a proper sub class of the BCI − algebras. J.Neggers and H.S.Kim introduced the notion of d–algebras which is another generalization of BCK-algebras. Recently another algebra based on proportional calculi was introduced by Tamilarasi and Mekalai in the year 2010 known as TM-algebras. The notion of derivations on rings is quite old. However it got significance only after Ponser's work in 1957. After this many researchers started working in this direction. Recently, Kandaraj and Chandramouleeswaran introduced the notion of derivation on d–algebras, another generalization of BCK–algebras. In this paper, we introduce the notion of f–derivation on TM–algebras. We study the properties of regular f–derivations on X and prove that the set of all f–derivations on a TM–algebra X forms a semigroup under a suitable binary composition.

B-5: New methods of factorization, *Aswatha Narayan Muthkur* (*Bangalore : muthkur.marc@gmail.com*).

Factorization of big numbers with large number of digits required in public key cryptosystem is looked into by appealing to the very fundamental and a new approach given to the subject.

B-6: A generalization of quasi-ideals of a P-regular near-ring, *Manoj Kumar Manoranjan* (*Bihar : manojmanoranjan.kumar @gmail.com*).

A quasi regularity of an element in a ring can be visualized as a generalization of the concept of invertibility so formulated as to be meaningful for rings without unity element. In the case of a near-ring. the concept of radical and quasi-regularity have the same formulation in the wake of truncated distributive law and absence of commutative law of addition. In this paper we prove some characteristics of quasi-ideals of a regular near-ring. Particularly we consider the representations of elements of a P-regular near-ring related to the right ideals P. As a result it is proved that every elements of a quasi-ideal Q of a P-regular near-ring can be represented as the sum of two elements P and Q .Equivalent conditions are obtained for a near- ring to be P-Regular near-ring.

B-7: Characterization of strongly finite dimensional modules, *Jituparna Goswami and Helen K. Saikia* (*Guwahati : jituparnagoswami18@gmail.com, hsaikia@yahoo.com*).

Let R be an associative ring with identity. A unital right R..module M is called "strongly finite dimensional" if Sup $\{G.dim(M/N)|N \leq M\} = +\infty$, where G. dim denotes the Goldie dimension of a module. In this paper we attempt to provide a new insight to characterize strongly finite dimensional module. Some equivalent condition are obtained regarding finite Goldie dimension and Krull dimension. It is proved that if M is a serial module having no 0-critical submodule, then M has finite strong Goldie dimension if and only if M has krull dimension. Using the concepts of strong Goldie dimension and Krull dimension, extending modules are characterized.

B-8: Quadratic fields with class number divisible by 3, *Helen K. Saikai and Azizul Hoque* (*Guwahati : hsaikia@yahoo.com, ahoque.ms @gmail.com*).

The authors establish that the number of real quadratic fields whose absolute discriminant is $\leq x$ and whose class number is divisible by 3 is $\gg x^{\frac{15}{16}}$ improving the best known lower bound $\gg x^{\frac{7}{8}}$ of D. Byeon and E.Koh. Certain result on the divisibility by 3 of the class number of the imaginary quadratic fields $Q(\sqrt{-q})$ for different forms of q are presented in this paper.

B-9: Essential pseudo p-injective module, *Helen K. Saikai and Himashree Kalita* (*Guwahati : hsaikia@yahoo.com, himashree.kalita28 @gmail.com*).

A module M is said to be essential pseudo p-injective module if every monomorphism from its cyclic submodules to M extends to an endomorphism of M. We establish several equivalent conditions for a module to be pseudo p-injective. If a module M has no proper essential sub modules, then M is an essential Pseudo p-injective module. Essential pseudo p- injective module M has no proper essential submodules isomorphic to a direct summand of M. If K is an essential submodule of an essential pseudo p-injective module M then (i) K itself is essential pseudo p-injective if K is stable under mono endomorphism of M and (ii) K is essential pseudo stable. Moreover, if M is an essential pseudo p-injective module than every essential pseudo stable submodule of M is also essential pseudo p-injective and intersection of two invariant submodules of M is an essential pseudo stable submodule of M.

B-10: Notes on *N*-ideal of a *BF*-algebra, *P. Muralikrishna and M. Chandramouleeswaran* (*Department of Mathematics, VIT University, Vellore, Tamilnadu*).

This paper introduces the notion of *N*–Structured ideal of a *BF*–algebra by using the idea of an *N*–function.

B-11: A conjecture on integer powers and beyond, *Susil Kumar Jena* (*Odisha : susil_kumar@yahoo.co.uk*).

The 'Conjecture on integer powers' states that for any positive integer $n \geq 2$, the n^{th} power of any arbitrary integer including zero can be expressed, primitively and non-trivially, in infinitely many different ways as the sum or difference of $(n+1)$ number of other non-zero, but not necessarily distinct, integral n^{th} powers. We verify this inference for integral squares, fourth power and partly,for cubic powers. For other integral powers ≥ 5, the conjecture is open.

B-12: A note on subtractive extensions of ideals in semirings, *J.N. Chaudhari and K.J. Ingale* (*Jalgaon : jnchaudhari@rediffmail.com*).

The concept of subtractive extension of ideals is recently introduced by Chaudhari and Bonde. In this paper, we introduce the concept of closure of an ideal A of a semiring R with respect to an ideal I of R and hence extend some results of k-closure of ideals in semirings.

B-13: Some remarks on subtractive extensions of subsemimodules, *J.N. Chaudhari and D.R. Bonde* (*Jalgaon : drbonde@rediffmail.com*).

In this paper, we introduce the notion of subtractive extension of a subsemimodules which is a generalization of subtractive subsemimodules and prove some equivalent conditions for subtractive extensions of a subsemimodule.

B-14: Some remarks on a theorem of Huneke- Wiegand, *Ganesh S. Kadu* (*Pune :gsk@math.unipune.ac.in*).

We give one dimensional examples to show that rank assumption on the modules cannot be dropped from the two theorems of Miller. Miller doesn't mention the assumption concerning the rank of modules in her simplification of the proof of a theorem of Huneke and Weigand. Our example shows that the rank assumption in a theorem of Huneke and Weigand cannot be dropped.

B-15: On permuting 3-derivations and commutativity in prime near rings, *Asma Ali and Phool Miyan* (*Aligarh : asma_ali2@rediffmail.com, phoolmiyan83@gmail.com*).

In this note, we introduce a permuting 3-derivation on semigroup ideals of near rings and investigate some conditions for a near ring to be a commutative ring.

B-16: Products of generalized derivations on prime near rings, *Asma Ali, Howard E. Bell and Phool Miyan* (*Aligarh: asma_ali2@rediffmail.com, hbell@brocku.ca, phoolmiyan83@gmail.com*).

Let N be a near ring. An additive mapping $f : N \to N$ is said to be a right generalized (resp. left generalized) derivation with associated derivation d on N if $f(xy) = f(x)y + xd(y)$(resp. $f(xy) = d(x)y + xf(y)$) for all $x, y \in N$. A mapping $f : N \to N$ is said to be a generalized derivation with associated derivation d on N if f is both a right generalized and a left generalized derivation with associated derivation d on N. The purpose of the present paper is to prove some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a generalized derivation, thereby extending some known result on derivations.

B-17: On semiderivations of near rings, *Asma Ali, Howard E. Bell, Rekha Rani and Phool Miyan*(*Aligarh : asma_ali2 @rediffmail.com, hbell@brocku.ca, linerekha2@yahoo.co.in*).

Let N be a near ring. An additive mapping $f : N \to N$ is said to be a semiderivation on N if there exists a mapping $g : N \to N$ such that (i)

$f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and (ii) $f(g(x)) = g(f(x))$ for all $x, y \in N$. In this paper we prove some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a nonzero semiderivation, thereby extending some known result on derivations.

B-18: On generalized derivations and commutativity of prime and semiprime rings, *Asma Ali, Deepak Kumar and Phool Miyan* (*Aligarh : asma_ali2@rediffmail.com, deep math1@yahoo.com, phoolmiyan83@gmail.com*).

Let R be a prime ring and θ, ϕ endomorphisms of R. An additive mapping $F : R \to R$ is called a generalized (θ, ϕ) -derivation on R if there exists a (θ, ϕ)-derivation $d : R \to R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$. Let S be a nonempty subset of R. In the present paper for various choices of S we study the commutativity of a semiprime (prime) ring R admitting a generalized (θ, ϕ)-derivation F satisfying any one of the properties: (i) $F(x)F(y) - xy \in Z(R)$, (ii) $F(x)F(y) + xy \in Z(R)$, (iii) $F(x)F(y) - yx \in Z(R)$, (iv) $F(x)F(y) + yx \in Z(R)$, (v) $F[x, y] - [x; y] \in Z(R)$, (vi) $F[x, y] + [x, y] - inZ(R)$, (vii) $F(xoy) - xy \in Z(R)$, and (viii) $F(xoy) + xoy \in Z(R)$, for all $x, y \in S$.

B-19: Derivations and commutativity of prime near rings, *Asma Ali and Phool Miyan* (*Aligarh : asma ali2@rediffmail.com*).

As a generalization of derivation the notion of a semiderivation on a ring was introduced by Bergen. Let N be a near ring. We define an additive mapping $N \to N$ is said to be a semiderivation on N if there exists a function $g : N \to N$ such that (i) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and (ii) $f(g(x)) = g(f(x))$ hold for all $x, y \in N$. In case g is the identity map on N; f is of course just a derivation on N. In this paper we prove some commutativity results in 3-prime near ring admitting a semiderivation. In fact the Theorems obtained are generalizations of certain earlier results.

B-20: Generalized jordan (θ, ϕ) derivations in rings, *Asma Ali and Phool Miyan* (*Aligarh : asma_ali2@rediffmail.com, phool miyan @rediffmail.com*).

Let R; S be any rings and U a Lie ideal of S such that $u^2 \in U$. Let (θ, ϕ) be homomorphisms of R into S and M be a 2-torsion free S-bimodule such that $mRx = (0)$ with $m \in M; x \in R$ implies that either $m = 0$ or $x = 0$. An additive mapping $F : R \to M$ is called a generalized (θ, ϕ) -derivation (resp. generalized Jordon (θ, ϕ)- derivation) on U if there exists a (θ, ϕ)-derivation $d : S \to M$ such that $F(uv) = F(u)\theta(v) + \phi(u)d(v)$ (resp. $F(u^2) = F(u)\theta(u) + \phi(u)d(u)$), holds for all $u; v \in U$. In the present paper, it is shown that if θ is injective on R, then every generalized Jordan (θ, ϕ)- derivation F on U is a generalized (θ, ϕ)-derivation on

U. In fact our result is a wide generalization of certain results proved earlier.

B-21: Extensions of some results for rings with unity to s-unital rings, *Achlesh Kumari* (*Department of Mathematics, S.V. College, Aligarh*).

There are sufficient examples to show that many results which are true for rings with unity fail to hold for rings without unity. In an attempt to generalize the results for rings with unity to other wider classes of rings, algebraists like Tominaga and Hirano introduced the concepts of s-unital and one sided s-unital (left or right s-unital) rings. In this paper, we extend some theorems which are valid for rings with unity to these wider classes of rings. Also the examples are provided to demonstrate that these results can not be generalized for all rings.

B-22: Certain new modular identities for Ramanujan's cubic continued fraction, *M.S. Mahadeva Naika, S. Chandankumar and N. P. Suman* (*Bangalore : msmnaika@rediffmail.com, chandan.s17 @gmail.com, npsuman. 13@gmail.com*).

In this paper, first we establish some new relations for ratios of Ramanujan's theta functions.We establish some new general formulas for explicit evaluations of Ramanujan's theta functions.We also establish new relations connecting Ramanujan's Cubic continued fraction $V(q)$ with other four continued fractions $V(q^{15}); V(q^{5/3}); V(q^{21})$ and $V(q^{7/3})$.

B-23: A note on some characterizations of a normal subgroup of a group, *Vipul Kakkar* (*CSIR : vplkakkar@gmail.com*).

Let G be a group and H be a subgroup of G which is either finite or of finite index in G. In this note, we give some characterizations for normality of H in G. As a consequence we get a very short and elementary proof of a certain known theorem which avoids the use of the classification of finite simple groups.

B-24: Factorisation of groups through p-maps, *Swapnil Srivastava and Punish Kumar* (*Department of Mathematics, SHIATS, Allahabad*).

In this paper we have defined the concept of p-map. By using this map we have factorized the group G and also shown that p(G) is a subgroup of G and the condition under which it is normal. $S = \{x : p(x) = e\}$ is a transversal of p(G) which becomes group by using p-map and some more conditions.

C: Real and Complex Analysis(Including Special Functions, Summability and Transforms):

C-1: On zeros and zero free regions of bicomplex Riemann zeta function, *Rajiv K. Srivastava and Jogendra Kumar* (*Agra : jogendra_j @rediffmail.com*).

In this paper, we have studied the Bicomplex version of complex Riemann Zeta Function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. We categorized the non-trivial zeros of the Bicomplex Riemann Zeta function. By using the established fact that the non-trivial zeros of the Complex Riemann Zeta Function lie in the critical strip $0 < Re(s) < 1$ we have obtained the region in C_2 to which the non-trivial zeros of the Bicomplex Rieman Zeta function belong. Also we have given a geometrical location of the non-trivial zeros of Bicomplex Riemann Zeta function in some sense. We have given three types of Conjugation of Bicomplex Rieman Zeta function. We found some regions in which the Bicomplex Riemann Zeta function cannot vanish.

C-2: Some inequalities on the relative defects corresponding to the common roots of several meromorphic functions, *Sanjib Kumar Datta and Sudipta Kumar Pal* (*West Bengal: sanjib_kr_datta@yahoo.co.in, sudipta.pal07@yahoo.co.in*).

The purpose of this paper is to consider several meromorphic functions having common roots and find some new relations involving their relative defects.

C-3: Maximun terms of composite entire functions and their comparative growth rates, *Sanjib Kumar Datta, Arup Rate Das and Samten Tamang* (*West Bangal : sanjib_kr_datta@yahoo.co.in, amtentamang2012@gmail.com, das arupratan@yahoo.in*).

In the paper we investigate some comparative growth properties related to the maximun terms of composite entire functions.

C-4: Slow growth and optimal approximation of pseudoanalytic functions on the disk, *Devendra Kumar* (*Ghaziabad : d_kumar001 @rediffmail.com*).

Pseudoanalytic functions (PAF) are constructed as complex combination of real-valued analytic solutions to the Stokes-Betrami System. These solutions include the generalized biaxisymmetric potentials. McCoy considered the approximation of pseudoanalytic functions on the disk. Kumar et al. studied the generalized order and generalized type PAF in terms of the Fourier coefficients occurring in its local expansion and optimal approximation errors in Bernstein sense on the disk. The aim of this paper in to improve the results of McCoy and Kumar et al. Our results apply satisfactorily for slow growth.

C-5: Irreducible representation of $SL(2;\mathbb{C})$ and generating relations for the generalized hypergeometric functions, *V. Srinivasa Bhagavan* (*Vaddeswaram : drvsb002 @kluniversity.in*).

The aim of present paper is to obtain a new class of generating relations for the generalized hypergeometric functions $\psi_{\alpha,\beta,\gamma,m}(x)$ by constructing a realization of the algebraic representation $D(u, m_0)$ of $SL(2, \mathbb{C})$ (Complex special linear group). Here the suitable interpretation to various parameters have been given simultaneously. These generating functions, in turn yield, as special cases, a number of linear generating functions to various important classical orthogonal polynomials of mathematical physics and statistics; namely the Laguerre, Hermite (even and odd), Meixner, Gottlied and Krawtchouk polynomials. Many results obtained as special cases are known but some of them are believed to be new in the theory of special functions.

C-6: On the strong (j, p_n) summability of Fourier series, *Satish Chandra and Monika Varshney* (*Chandausi : satishchandra111960 @gmail.com, varsmonika92@gmail.com*).

In this paper we have proved a theorem on the strong (J, p_n) summability of Fourier series, which generalizes various knows results.

C-7: On the convergence of Jacobi series at poles, *Sajroo Prasad Yadav, Rakesh Kumar Yadav and Dinesh Kumar yadav* (*Satna : ydsrpa@yahoo.co.in*).

A Banach space X of signals defined on $[-1, +1]$ is considered. Signals are as functions which are p-power $(1 \leq p \leq \infty)$ Lebesgue integrable with weight

$$\omega(x) = (1-x)^{\alpha}(1+x)^{\beta}, (\alpha, \beta > -1)$$

on the interval $[-1, 1]$. We recognize some of the subspaces of X by the convergence behavior of Fourier-Jacobi expansions associated with the signals in X. These are new subspaces not known earlier.

C-8: Certain quantum calculus operators associated with the basic analogue of for-wright hypergeometric function, *D.K.Jain, Renu Jain and Farooq Ahmad* (*Gwalior : dkj@yahoo.co.in, renujain3 @rediffmail.com, sheikh-farooq85@gmail.com*).

The objective of the paper is to derive the relationship that exists between the basic -analogue of For-Wright hypergeometric function ${}_p\Psi_q(z; q)$ and the quantum calculus operators, in particular Riemann-Louville q-integral and q-differential operators. Some spacial cases been also discussed.

C-9: On approximation of functions of generalized Lipschitz class by $(E, q)(N, p_n)$ summability method of Fourier series, *Shyam lal* (*Varanasi : shyam_lal@rediffmail.com*).

In this paper, four new theorems on the degree of approximation of function $f \in Lip\alpha$ and $W(L^r, \xi)$ have been proved.

C-10: Representation of five dimensional Lie algebra and generating relations for the generalized hypergeometric functions, *V. S. Bhagavan and O.V.N.L. Umadevi* (*Guntur : drvsb002@kluniversity.in*).

The aim of present paper is to derive the generating functions for the generalized hypergeometric functions $\psi_{\alpha,\beta,\gamma,m}(x)$ by constructing a five dimensional Lie algebra, with the help of Weisner's method. Here, the suitable interpretation to various parameters have been given simultaneously. These generating functions, in turn yield, as special cases, a number of linear generating functions to various classical orthogonal polynomials; namely the Laguerre, Hermite (even and odd), Meixner, Gottlieb and Krawtchouk polynomials. Many results obtained as special cases are known but some of them are believed to be new in the theory of special functions.

C-11: Derivatives of divided differences, *Subhasis Ray* (*West Bengal : drvsb002@kluniversity.in*).

Let $V \subset \Re$ be a finite set and $f : \Re \to \Re$. $f(V)$ is divided difference of f at the points of V . The m-th order Peano derivative of $f(x \cap V)$ is defined and is denoted by $f_m(x; V)$. The properties of this derivative are studied.

D: Functional Analysis:

D-1: Fixed point theorems in a generalized metric space, *D.P. Shukla* (*Rewa : shukladpmp@gmail.com*).

In this paper, we have established fixed point theorems in generalized metric space using self mapping and contraction mappings and the obtained results can be considered as an extension and generalization of some well results of fixed point theorems in generalized metric spaces.

D-2: Fixed point and common fixed point theorems in metric space, *D.P. Shukla and Shiv Kant Tiwari* (*Rewa : shukladpmp @gmail.com*).

In this paper we have introduced fixed point theorems and common fixed point theorems in complete metric spaces.

D-3: Regularity of operator space projective tensor product of c*-algebras, *Ajay Kumar and Vandana Rajpal* (*Delhi : akumar@maths .du.ac.in, vandanarajpal.math@mail.com*).

The Banach *-algebra $A\widehat{\otimes}B$, the operator space projective tensor product of C*-algebras A and B, is shown to be *-regular if Tomiyama's property (F) holds for $A \otimes_{min} B$ and $A \otimes_{min} B = A \otimes_{max} B$ where $\otimes_{min}$ and $\otimes_{max}$ are the injective and projective C*-cross norm, respectively. However, $A\widehat{\otimes}B$ has a unique C*-norm if and only if $A \otimes B$ has. We also discuss the property (F) of $A\widehat{\otimes}B$ and $A\otimes_h B$, the Haagrerup tensor product of A and B.

D-4: On a new system of generalized variational inclusions with H(. , .)-cocoercive operator in Hilbert spaces, *Sanjeev Gupta and Shamshad Husain* (*Aligarh : guptasanmp@gmail.com*).

In this paper, we introduce and study a new system of generalized variational inclusions with H(. , .)-cocoercive operators which contains variational inequalities, variational inclusions, systems of variational inequalities and systems of variational inclusions in the literature as special cases. By using the resolvent technique for the H(. , .)-cocoercive operators, we prove the existence of solutions and the convergence of a new iterative algorithm for this system of variational inclusions in Hilbert spaces. The results in this paper unify, extend and improve some known results from the literature.

D-5: Some approximation properties of q-Baskakov- Beta- Stancu type operators, *Kejal Khatri* (*Department of Mathematics, SVNIT Surat*).

This paper deals with new type q-Baskakov-Beta-Stancu operators. First, we have used the properties of q- integral to establish the moments of these operators. We also obtain some approximation properties and asymptotic formulae for these operators. In the end we have also presented better error estimations for the q- operators.

D-6: On the durrmeyer type modification of the q-Baskakov-Stancu type operators, *Prashantkumar G. patel* (*Gujarat : prashant255 @gmail.com*).

In this paper, we are dealing with q analogue of Durrmeyer type modied Baskakov- Stancu operators, which introduces a new sequence of positive linear q-integral operators. We have shown that this sequence is an approximation process in the polynomial weighted space of continuous function defined on the interval $[0, \infty)$. We have studied, moments of operators, the rate of convergence, weighted approximation properties, asymptotic formula and better error estimation for these operators.

E: Differential Equations, Integral Equations and Functional Equations:

E-1: Extremal solution for random differential equations, *R.G. Metkar* (*Nanded : rammetkarmath@rediffmail.com*).

In this paper, an existence of extremal random solution is proved for a periodic boundary value problem of second order ordinary random differential equations using an algebraic random fixed point theorem.

E-2: Numerical range of certain composition operators on ℓ^2, *Kamalesh Kumar Rai* (*Itanagar : kkraib@rediffmail.com*).

Let ℓ^2 be the Hilbert space of all square summable sequences of complex numbers under the standard inner product on it. Let C_φ be a nonsurjective composition operator on ℓ^2 induced by a function φ on N into itself. Numerical range of C_φ is denoted by $W(C_\varphi)$ and is defined by

$$W(C_\varphi) = \{< C_\varphi(f), f >: |f| = 1\}.$$

In this paper numerical range of positive composition operator, self adjoint composition operator and certain other composition operator are characterized. In some cases these operators are polynominally compact while in others are not polynominally compact.

E-3: A simulation approach: security force versus insurgents combat model under decapitation warfare, *Lambodara Sahu* (*Pune : lsahucme@gmail.com*).

Besides regular combat, security forces, as a part of their strategy engage in decapitation warfare like killing or arresting the key leaders of insurgents, capturing their central base, etc. Insurgents are also involved in the decapitation warfare like assassination or abduction of key Govt.officials, ministers and high profile politicians. Concepts derived from Lanchester's and Lanchester-type combat models are used to project the effectiveness of forces of regular combat under decapitation warfare. A conceptual model dealing with certain operational factors like robustness of forces, undermining effects, decapitation combat status, break-points, attrition rates is being simulated to describe the effects of decapitation strategy on the regular battlefield. A discrete strength simulation approach is adopted to find the casualty of security force in the regular combat at various break points of insurgents. An illustrative example is provided to demonstrate the effect of decapitation in the regular combat.

E-4: Controllability results for second order nonlinear mixed neutral functional integrodifferential equations in abstract spaces, *Kishor D. Kucche, M.B Dhakne* (*Maharashtra : kdkucche @gmail.com, mbdhakne@yahoo.com*).

In this paper we establish the controllability results for second order mixed neutral functional integrodifferential equations in Banach spaces by applying Sadoviskii fixed point theorem and using the theory of strongly continuous cosine family of operator. An example is provided to illustrate our result.

E-5: Perturbation of AC - mixed type functional equation: a direct method, *M. Arunkumar and S.Ramamoorthi* (*Tiruvannamalai : anarun2002@yahoo.co.in, ram aishu9@yahoo.co.in*).

In this paper, we obtain the general solution and generalized Ulam - Hyers stability of AC- mixed type functional equation

$$f(2x + y) - f(2x - y) = 4[f(x + y) - f(x - y)] - 6f(y)$$

by direct method.

E-6: Stability of a n-dimensional additive functional in random normed space: a fixed point approach, *M. Arunkumar and T. Namachivayam*(*Tiruvannamalai : anarun2002@yahoo.co.in, namachi.siva@rediffmail.com*).

In this paper, the authors have established the general solution in vector space and generalized Ulam-Hyers stability of a n- dimensional additive functional equation of the form

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{i} x_j\right) = \sum_{i=1}^{n} (n-i+1)f(x_i),$$

where $n \geq 2$ in RN-Space using fixed point method.

E-7: Additive - quadratic functional equation are stable in Banach space, *M. Arunkumar and P. Agilan* (*Tiruvannamalai : anarun2002 @yahoo.co.in, agilram@gmail.com*).

In this paper, the authors have established the general solution and generalized Ulam-Hyers stability of a mixed type Additive Quadratic (AQ)-functional equation

$$f(x+2y+3z)+f(x-2y+3z)+f(x+2y-3z)+f(x-2y-3z)$$
$$= 4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)]$$

in Banach spaces.

E-8: HUR, JMR stabilities of an additive functional equation in generalized 2-normed spaces, *M. Arunkumar and N. Maheshkumar* (*Tiruvanna-malai : anarun2002@yahoo.co.in, mrnmahesh@yahoo.com*).

In this paper, the authors have established the solution and Hyers-Ulam- Rassias (HUR) and JMRassias (JMR) stability of the Additive function equation

$$g(x)+g(y+z) = g(x+y)+g(z)$$

in generalized 2-normed spaces.

E-9: Stability of n - dimensional cubic functional equation in fuzzy normed spaces, *M. Arunkumar and S. Karthikeyan* (*Tiruvannamalai : anarun2002@yahoo.co.in, sma204@yahoo.com*).

In this paper, the authors investigate the generalized Hyers-Ulam- stability of n- dimensional cubic functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{i,j}\right) = (n-6)f\left(\sum_{j=1}^{n} x_j\right) + 4\sum_{1\leq i<j\leq n} f(x_i+x_j)$$
$$- \frac{(n-2)}{2}\sum_{j=1}^{n} f(2x_j)$$

where

$$x_{ij} = \begin{cases} -x_j & \text{if } i = j \\ x_j & \text{if } i \neq j \end{cases}$$

in fuzzy normed space with $n \geq 3$.

E-10: AQ - functional equations are stable in RN - space, *M. Arunkumar, G. Ganapathy and A. Murthy* (*Tiruvannamalai : anarun 2002@yahoo.co.in, ganagandhi@yahoo.com*).

In this paper, the authors obtained the general solution and generalized Ulam- Hyers stability of AQ- functional equation

$$(2x \pm y \pm z) = 2f(-x \mp y \mp z) - 2f(\pm y \pm z) + f(\pm y \pm z) + 3f(x) + f(-x)$$

in Random Normed Spaces setting.

E-11: Convergence of β-operators, *T.A.K. Sinha and Manoj Kumar* (*Patna : thakurashok1212@gmail.com*).

In this paper we show that β- Operators defined as

$$\beta_n(f,x) = \int_0^\infty \frac{x^n}{B(n.n)} \frac{t^{n-1}}{(x+t)^{2n}} f(t)\mathrm{d}t$$

where $B(n,n)$ is beta function and $f \in L^p[0,\infty)$ approximate $f(.)$over compact subsets of $[0,\infty)$.

E-12: On a special type of operator named λ- jection, *Dr. Prabhat Chandra and Dr. L. B. Singh* (*Patna : lbsdmjpu@gmail.com*).

We investigate the case when a general expression of form $\alpha I + \alpha E + bE^2$ is a λ- jection if E is a λ- jection.

E-13: Growth properties of differential monomials and differential polynomials from the view point of slowly changing functions, *Sanjib Kumar Datta, Tanmay Biswas and Golok Kumar Mondal* (*West Bengal : sanjib_kr_datta@yahoo.co.in, tanmaybiswas_math@rediffmail.com, golok.mondal13@rediffmail.com*).

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using L^8-order and L^*-type and differential monomials, differential polynomials generated by one of the factors.

E-14: On existence of locally attractive solutions of fractional order nonlinear random integral equation in Banach algebras, *B.D. Karande* (*bdkarande@rediffmail.com*).

In this Paper, we prove an existence theorem for fractional order nonlinear random integral equation in Banach algebras under Lipchitz and Caratheodory conditions.The existence of locally attractive solutions is also proved.

E-15: Study of first order discontinuous ordinary differential equations, *Rakesh Kumar Singh* (*Department of Mathematics, J.P. University, Chappra*).

The existence and uniqueness theorems for the first order discontinuous differential equations have been obtained through Ascoli-Arzela theorem and Banach fixed point principle under the generalized measurability and Lipschitzian conditions. An existence theorem and the maximum interval for the existence of the solution are also obtained without using the generalized Lipchitz condition and is a nonlinear alternative of Granas et. al.

E-16: An analysis of periodic behaviors of a continuous system, *Nabajyoti Das* (*Boko : dnabajyoti09@gmail.com*).

In this paper, the nonlinear differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a - (b+1)x + x^2y, \ \frac{\mathrm{d}y}{\mathrm{d}t} = bx - x^2y,$$

where a and b are adjustable parameters, are considered to discuss the development of a general theory for determining periodic and cyclic behaviors in any nonlinear differential equations, and to analyze the stability of fixed points of the concerned model with graphical representations.

E-17: Numerical solutions of non-linear elliptic partial differential equations using orthogonal and biorothogonal wavelet based algebraic multigrid method, *S.C. Shiralashetti, M.H. Kantli, Sharada S. Naregal, B. Veeresh* (*Dharwad : shiralashettisc@yahoo.com, mkantli @gmail.com*).

In this paper we describe how biorthogonal wavelets may be used to solve non-linear elliptic partial differential equations. These problems are currently solved by techniques such as , finite elements and classical algebraic multigrid method. The wavelet method, however, offers several advantages over traditional methods. Wavelets have the ability to represent functions at different levels of resolution, thereby providing a logical means of developing a hierarchy of solutions. Construction of hierarchy of matrices in Algebraic Multigrid context is based on low-pass filter version of Fast Wavelet Transform (FWT). FWT based on biorthogonal wavelets is fast compared to orthogonal wavelets, because FWT depends upon the length of filters used. In order to demonstrate the Biorthogonal Wavelet based Algebraic Multigrid method, we consider the two non-linear Elliptic PDEs for their solutions. By comparison with a orthogonal Wavelet based Algebraic Multigrid method and classical Algebraic Multigrid method solutions of these problems with boundary conditions, we show how a Biorthogonal wavelet based algebraic multigrid technique may be efficiently developed. The convergence of the wavelet based Algebraic Multigrid solutions are examined and they are found to compare extremely favorably to the classical Algebraic Multigrid solutions. Preliminary investigations also indicate that the Biorthogonal wavelet based algebraic multigrid method is a strong contender to the orthogonal wavelet and classical Algebraic Multigrid method, at least for problems with simple geometries but equally well applicable for finite difference discretization.

E-18: Existence theorem for abstract measure integro-differential equations, *S.S. Bellale and B.C. Dhage* (*Latur : sidhesh.bellale@gmail.com, bcdhage@yahoo.co.in*).

In this paper, an existence theorem for a abstract measure integro- differential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for extremal solutions for Caratheodory as well as discontinuous cases of the nonlinearities involved in the equations.

F: Geometry:

F-1: Study of the geometric spectral asymptotics, *Ashwani Kumar*

Sinha(*Ara : mathashwani@yahoo.co.in*).

Those who do not know what "geometric spectral asymptotics" is should think in terms of this elementary example: A vibrating string of length L has normal modes (eigen functions) $\phi_n(x) = \sin(n\pi x/L)$ with freequencing $\varpi_n \equiv \sqrt{\lambda n} = n\pi, L$(in units where the speed of sound is 1) where n runs through all the positive integers. Note that

$$Z(s) \equiv \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

is essentially the famous Riemann zeta function. The eigenfunctions satisfy the equation and boundary condition.

$$\phi_n = -\lambda_n \phi_n, \qquad \phi_n(0) = \phi_n(L) = 0.$$

F-2: On Einstein-Sasakian conharmonic recurrent spaces, *A.K. Singh* (*Department of Mathematics, H.N.B. Garhwal University, Srinagar*).

In the present paper, we have defined and studied Einstein-Sasakian spaces with recurrent Bochner curvature tensor and several theorems have been investigated. The necessary and sufficient condition for an Einstein-Sasakian Conharmonic recurrent space to be Sasakian recurrent has been investigated.

F-3: Warped product submanifolds of LP-Sasakian manifolds, *Shyamal Kumar Hui* (*West Bengal : shyamal hui@yahoo.co.in*).

We study warped product submanifolds, especially warped product hemi- slant submanifolds of LP-Sasakian manifolds. We obtain the result on the non-existence or existence of warped product hemi-slant submanifolds and give some examples of LP- Sasakian manifolds. The existence of warped product hemi-slant submanifolds of an LP- Sasakian manifold is also ensured by an interesting example.

F-4: On a product semi-symmetric non-metric connection in a hyperbolic Kähler manifold, *R.N. Singh and Giteshwari Pandey* (*Rewa : rnsinghmp@rediffmail.com, math.giteshwari@gmail.com*).

The object of the present paper is to characterize a type of product semi-symmetric non-metric connection in a hyperbolic Kähler manifold and to study some of its curvature properties.

F-5: On the C-Bochner curvature tensor of generalized Sasakian-space-forms, *R.N. Singh and Shravan K. Pandey* (*Rewa : rnsinghmp @rediffmail.com, shravan.math@gmail.com*).

The object of the present paper is to characterize generalized Sasakian-space-forms satisfying certain curvature conditions on C-Bochner curvature tensor.

F-6: On LP-Sasakian manifold admitting a quarter symetric non metric ϕ-connection, *Jay Prakesh Singh* (*Mizoram : jpsmaths@gmail.com*).

In the present paper, we have introduced a Quarter symmetric non metric ϕ- connection in a Lorentzian Para-Sasakian manifold. We have studied some geometrical properties of this connection. It is proved that in a Lorentzian Para-Sasakian manifold if the curvature tensor with respect to Quarter symmetric non metric ϕ-connection ∇ vanishes, then it is conformally flat with respect to the same.

F-7: h-Rander's conformal change of Finsler metric, *H.S. Shukla and Arunima Mishra* (*Gorakhpur : profhsshuklagkp@rediffmail.com, arunima16oct@hotmail.com*).

A change of Finsler metric $L(x,y) \to \bar{L}(x,y)$ is called Randers change of L if $\overline{L}(x,y) = L(x,y) + \beta(x,y)$, where $\beta(= b_i(x)y^i)$ is a one-form on a smooth manifold M^n. A H-vector on a Finsler space has been defined by Izumi in 1980 which satisfies $\partial_j b_i = L^{-1}\rho h_{ij}$, where ρ is some scalar function and h_{ij} are components of angular metric tensor. In the present paper we define h-Randers conformal change as

$$L(x,y) \to L^*(x,y) = e^{\sigma x} L(x,y) + \beta(x,y)$$

where $\sigma(x)$ is a function of x and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M^n and satisfy the condition of h-vector.

In the present paper we have found out the relation between Cartan's connections of two Finsler spaces (M^n, L) and $(M^n, \overline{L})$. We have also obtained the conditions under which this change is projective and the condition under which Douglas space or a Landsberg space or a weakly Berwald space is invariant under h-Rander's conformal change of a Finsler metric.

G: Topology:

G-1: Characterization of rarely gI-continuous multifunctions, *M. Rameshkumar* (*P.A. College of Engineering and Technology, Pollachi, Tamil Nadu*).

This paper aims to introduce a new class of functions called rarely generalized ideal gI-continuous multifunctions. This paper is devoted to the study of upper (and lower) rarely gI-continuous multifunctions.

G-2: $g * b$ - closure operator and $g * b$ -open sets in topological paces, *M.Thirumalaiswamy and S. Saranya* (*Tamil Nadu : saranmath89@gmail.com*).

In this paper, we extend and study the closure operator of a generalized b-closed sets called, $g * b$- closure operator with the aid of $g * b$- closed sets in topological spaces. Also we define $g * b$-open sets in topological spaces.

G-3: Decompositions of weaker forms of continuity via idealization, *K. Viswanathan, J. Jayasudha* (*Tamil Nadu : visungm@yahoo.com*).

In this paper, we introduce and investigate the notions of $\omega\alpha - \mathcal{I}$- closed sets and $\omega\alpha - \mathcal{I}$-continuous maps in ideal topological spaces. Also we introduce the notions of $\eta^* - \mathcal{I}$- sets and $\gamma - \mathcal{I}$- sets to obtain decompositions of $\alpha - \mathcal{I}$- continuity and $\beta - \mathcal{I}$- continuity.

G-4: On regular spaces and normal spaces in generalized form, *Suwarnlatha N. Banasode* (*Karnataka : suwarn_nam@yahoo.co.in*).

In this paper the notions of g-minimal regular spaces and g-minimal normal spades are introduced and studied in topological spaces. A topological space (X, τ) is said to be generalized minimal regular (briefly $g - m_i$ regular) space if for every $g - m_i$ closed set F of X and each point $x \in F^c$ there exists disjoint open sets U and V of X such that $x \in U$ and $F \subset V$. A topological space (X, τ) is said to be generalized minimal normal (briefly $g - m_i$ normal) space if for any pair of disjoint $g - m_i$ closed sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Some basic properties of such spaces are obtained.

G-5: Bisotopological and bisogeometric aspects of intense BIS processes and resulting Cesaro-Orlicz spaces, *Ashwani Kumar Sinha, M.M. Bajaj and Rashmi Sinha* (*Ara : mathashwani@yahoo.co.in*).

The present paper deals with the bisotopological and bisogeometric properties of intense BIS processes taking place throughout the world. We have considered the associated Cesaro-Orlicz spaces. For the Luxemburg norm, Cesaro-Orlicz spaces have the Fatou property. The subspace of order continuous elements in Ces can be defined in two ways. Finally, we study criteria for strict monotonicity, uniform monotonicity and roundity (= strict convexity).

G-6: Sobriety in ivif-top, *Rana Noor and Arun K. Srivastava* (*Varanasi : rananoor4@gmail.com, arunksrivastava@gmail.com*).

We introduce the notion of sober spaces, within the class of interval-valued intuitionistic fuzzy topological spaces.We show that the resulting subcategory is the epireflective hull of a Sierpinski object in the category of To interval-valued intuitionistic fuzzy topological spaces.

G-7: On generalizations of fuzzy closed sets using various complement functions, *P. Thangavelu* (*Coimbatore : thangavelu@karunya.edu*).

The Fuzzy topology is a generalization of general topology. The connection between the fuzzy open sets and the fuzzy closed sets is given by the standard complement λ' of λ defined by $\lambda'(x) = 1 - \lambda(x)$. Several Fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complement in the fuzzy literature. The standard complement is obtained by using the functional$\varphi : [0,1] \to [0,1], \varphi(x) = 1 - x$, for all $x \in [0,1]$. In this paper, the concept of fuzzy φ -closed subsets of a fuzzy topological space is introduced where $\varphi : [0,1] \to [0,1]$ is a function. In fact the notions of fuzzy closed sets and fuzzy closure with respect to the standard complement are extended to the analog concepts with respect to an arbitrary complement function $\varphi : [0,1] \to [0,1]$. Using this extended fuzzy closure operator, some sets that are near to fuzzy open sets and fuzzy open sets and fuzzy closed sets are studied. Fuzzy continuous functions and fuzzy irresolute functions are also extended in this sense.

G-8: Full rationality of fuzzy choice functions on base domains through indicators, *Santosh Desai* (*Kolhapur : santosh_desai21 @rediffmail.com*).

The present paper introduces indicators of the weak fuzzy T-congruence axiom and fuzzy Chernoff axiom. These indicators measure the degree to which the fuzzy choice functions satisfy the weak fuzzy T-congruence axiom and the fuzzy Chernoff axiom. The indicator of the full rationality is expressed in terms of the indicator of the weak fuzzy T-congruence axiom and the fuzzy Chernoff axiom.

G-9: On pairwise minimal open sets and maps in bitopological spaces, *S.S. Benchalli and Basavaraj M. Ittanagi* (*Karnataka : dr.basavarajsit@gmail.com*).

The aim of this paper is to define and study a new class of sets called pairwise minimal open sets and pairwise maximal open sets in bitopological spaces. We also introduce the concepts of pairwise minimal continuous, pairwise maximal continuous, pairwise minimal irresolute and pairwise maximal irresolute functions in bitopological spaces and investigate the relations between these kinds of continuity.

G-10: On pairwise rw-closed sets in bitopological spaces, *R.S. Wali* (*Guledagudd : rswali@rediffmail.com*).

The aim of this paper is to define and study a new class of sets called pairwise-$\Upsilon\omega$-closed sets in bitopological spaces. A subset A of a bitopological space (X, τ_1, τ_2) is said to be pairwise-$r\omega$-closed set if $\tau_j - cl(A) \subset G$ and $G \subset RSO(X, \tau_i)$, where $\{i, j\} \in\in \{1, 2\}$ are fixed integers. This new class of sets properly lies between the class of pairwise-ω-closed sets and the class of pairwise-rg-closed sets. Some of their properties have been investigated.

G-11: A fuzzy analysis of causes and consequences of farmers suicide in selected talukas of Yeotmal district, *Nilima Puranik* (*Yavatmal : nilunarayan@yahoo.co.in*).

This paper aims at analyzing two major issues, namely causes and consequence of farmers suicide in selected talukas of Yeotmal district. The analysis should help planners and social organizations to arrive at strategies to overcome the deadly menace. Survey data has been used to illustrate how fuzzification can help in formulating problems when many aspects do not fall under the traditional two-valued logical domain.

The study is based on both, the primary data generated from a survey, of 26 selected farmers and the secondary data , obtained from various revenue departments.

G-12: Some non-unique common fixed point theorems in symmetric spaces through clr(s, t) property, *D.K. Patel, E. Karapinar, M. Imdad and D. Gopal* (*Surat : deepesh456@gmail.com*).

In this paper, we introduce a new class of mappings which enjoys the common limit in the range property and utilize the same to establish common fixed point theorems for such mappings in symmetric spaces. Our results improve the corresponding results contained in Imdad et al. [Appl. Math. Letters 23 (2010), 351-355], Soliman et al. [Commun. Korean Math. Soc. 25(4) (2010), 629-645], Gopal et al. [Commun. Korean Math. Soc. 27(2) (2012), 385-397] and some others. Some illustrative examples to highlight the realized improvements are also discussed.

G-13: Some set theoretic properties of continuous linear operators satisfying some generalised non-expansive type conditions, *Kalishankar Tiwary* (*Kolkata : tiwarykalishankar@yahoo.com*).

In this paper we investigate certain set theoretic properties in the set of all continuous linear operators mapping a Banach space into itself, of the set of continuous function.

G-14: A generalised common fixed point theorem of A-compatible and S-compatible mappings, *Yumnam Rohen and M. Popeshwar Singh* (*Manipur : ymnehor2008@yahoo.com*).

In this paper we prove a common fixed point theorem of four self mappings satisfying a generalized inequality using the concept of A-compatible and S-compatible mappings. Our result generalizes many earlier related results in the literature.

G-15: A characterization of the category Q-TOP, *Sheo Kumar Singh and Arun K. Srivastava* (*Varanasi : sheomathbhu@gmail.com, arunksrivastava@gmail.com*).

E.G. Manes [Compact Hausdorff objects, Gen. Top. and Appl., 4 (1974), 341-360], gave a characterization of the category TOP of topological spaces, in which 2_S (the Sierpinski object in TOP) played a crucial role. A.K. Srivastava [Fuzzy Sierpinski space, J. Math. Anal. Appl.,103 (1984), 103-105], gave a characterization of the category $[0,1]$-TOP of $[0,1]$-topological spaces (known more commonly as fuzzy topological spaces) in terms of a Sierpinski object I_S in $[0,1]$-TOP on the lines of Manes. In this note, we give an analogous characterization of Q-TOP, the category of the recently introduced Q-topological spaces (Q being a fixed member of a fixed variety of Ω-algebras) by S.A. Solovyov [Sobriety and spatiality in variety of algebras, Fuzzy Sets and Systems,159 (2008), 2567-2585] (which is a major generalization of fuzzy topological spaces), as a Sierpinski object in Q-TOP has already been obtained by Solovyov. This characterization clearly has the potential of becoming applicable to many instances of Q-TOP, of which the characterization of $[0,1]$-TOP, as given in Srivastava's paper, is an example.

H: Measure Theory, Probability Theory and Stochastic Processes, and Information Theory:

H-1: A note on Karl Pearson's coefficient of correlation and its application to mathematics, *M.S. Bhalachandra* (*Nashik : msbhalachandra@yahoo.co.in*)

It is well known that "$Y = -Z \Rightarrow \rho(X,Y) = -\rho(X,Z)$" where ρ is Karl Pearson's coefficient of correlation between X and Y. The first section of the paper discusses the truth of the converse of the above result with the help of trigonometric functions. In the second section of the paper, regression models are developed for estimating the values of Sine and Cosine functions.

I: Numerical Analysis, Approximation Theory and Computer Science:

I-1: On certain combinations of modified Beta operators, *Prerna Maheshwari and Diwaker Sharma* (*Ghaziabad : mprerna anand@yahoo.com*).

The present paper deals with the study on modified Beta operator proposed earlier by Gupta and Ahmad. We estimate simultaneous approximation properties of these operators and a local direct result in terms of modulus of continuity. Here we observe that the rate of convergence is $O(1/n)$ even in simultaneous approximation. In order to improve the order of approximation for these operators we take iterative combination. We also establish an asymptotic formula and error estimate in term of higher order modulus of continuity in simultaneous approximation for the iterative combinations of these operators.

I-2: Simultaneous approximation by Lupas-Szász type operators, *Bipin Bihari and T.A.K. Sonha* (*Department of Mathematics, Magadh University, S.M.D. College Punpun, Patna*).

In this paper we have shown that derivatives of operators do converge to the derivative of function of corresponding order. The Lupas-Sz asz type operators are defined as

$$L_n(f,x) = \int_0^\infty W(n,x,t)f(t)\mathrm{d}t,$$

where

$$W(n,x,t) = n\sum_0^\infty \rho_{nv}(x)s_{n,v}(t), p_{n,v}(x) = n + v - 1_{cv}\frac{x^v}{(1+x)^{n+v}},$$

$$S_{n,v(x)} = e^{-nx}\frac{(nx)^v}{v!}.$$

These operators are linear, positive and approximating sequence in $L^p[0,a]$. We explore relation between moments and find expression for derivatives of kernel function.We finally show that $L^{(\Upsilon)}{}_n(f,x)$ converges to $f^{(\Upsilon)}(x)$.

I-3: Numerical solution of fractional bio heat equation by implicit finite difference method, *Ramesh S. Damor, A. K. Shukla and Sushila Kumar* (*Surat :rameshsvnit2010@gmail.com, ajayshukla@rediffmail.com, skumariitr@gmail.com*).

In present paper, we deal with the anomalous diffusion in space fractional bio heat equation using implicit finite difference method. We consider Riemann-Liouville fractional derivative as space fractional derivative and studied the temperature distribution in the tissue.

I-4: Hybrid anti-synchronization of fractional order chaotic systems using active control method, *Mayank Srivastava* (*Varanasi : mayanksrivastava1983@ gmail.com*).

In this article, the active control method is used for hybrid anti-synchronization between two fractional order chaotic system Chen chaotic system as the master system and the other fractional order chaotic system is Lorenz system as slave system separately. The fractional derivative is described in Caputo sense. Numerical simulation results which are carried out using Adams-Bashforth-Moulton method show that the method is easy to implement and reliable for hybrid anti-synchronizing the two nonlinear fractional order chaotic systems while it also allows both the systems to remain in chaotic states.

J: Operations Research:

J-1: Solving multi objective transportation problem using fuzzy logic, *M.R. Fegade and V.A. Jadhav* (*Nanded : mrfegade@gmail.com*).

In this paper, we have proposed the method to solve the multi objective transportation problem. In this study we have proposed the algorithm for finding minimum cost and maximize the profit. Here we have proposed the different membership function for checking feasibility of the proposed method with a numerical example.

J-2: Backward bifurcation in smoking cessation model with media campaigns, *Anupama Sharma and A.K. Mishra* (*Varanasi : anupama bhu@yahoo.com*).

Nowadays most of the public policies to manage the social issues are focusing on motivating people through media campaigns. In view of this, we propose and analyze a nonlinear mathematical model to study the effect of media campaigns on the smoking cessation. The equilibria of the model have been obtained and their stability discussed. Using center manifold reduction theory, we reduce the proposed model to a system of lower dimension. The reduced system contains all the necessary information regarding the asymptotic behavior of small solutions of the original system.

The analysis shows that changing one parameter of the system (reproduction number, R, which depends on various other parameters), two different manifolds of fixed points cross each other and transcritical bifurcation occurs. Further, for large value of relapse rate the bifurcation is subcritical (backward). This shows that requirement $R < 1$ is only necessary, but not sufficient, for smoking cessation. Numerical simulation also supports the analytically obtained results.

K: Solid Mechanics, Fluid Mechanics, Geophysics and Relativity:

K-1: Stability of viscous flow in a narrow-gap annulus with radial temperature gradient and constant heat flux at inner rotating, *Rajendra Prasad, A.K. Singh and Sadhana Pandey* (*Varanasi : rajendrabhu108@gmail.com*).

This paper presents the stability of Couette flow of a viscous incompressible fluid between two concentric rotating cylinders in the presence of a radial temperature gradient due to constant heat flux at the inner cylinder.A trigonometric series method is used to solve the eigen value problem when the gap between the cylinders is narrow. The numerical value of the critical wave number and the critical Taylor number are computed from the obtained analytical expressions. It is found that the numerical value of the critical Taylor number obtained by third approximation agree very well with the earlier results obtained by a numerical method. Also the amplitude of the radial velocity and the cell-pattern are shown on graphs for different values of the ratio of the angular velocities.The stabilizing effect is more promising when the cylinders are counter-rotating.

K-2: Thermal instability in a porous medium layer of Walter's (Model-B) rotating fluid in the presence of suspended particles: Brinkman model, *S. K. Kango* (*Kullu : skkango72@gmail.com*).

The onset of thermal instability in a porous medium layer of Walters' (Model B') rotating fluid in the presence of suspended particles is studied analytically. For the porous medium, the Brinkman model is employed. By applying normal mode analysis methods, the dispersion relation has been derived and solved analyticity. It is observed that the rotation, suspended particles, gravity field and viscoelasticity introduce oscillatory modes. For stationary convection, it is observed that the rotation has stabilizing effect and suspended particles have destabilizing effect on the system whereas Darcy number and medium permeability have stabilizing/destabilizing effects under certain conditions.The result have been shown graphically.

K-3: Response of longitudinal wave from a plane free boundary of fibre-reinforced thermoelastic half-space, *S.S. Singh and C.Zorammuana* (*Mizoram : saratcha32@yahoo.co.uk*).

The problem of the reflection of plane waves due to incident longitudinal wave at a free fibrereinforced thermoelastic half-space has been investigated. There exist three types of couple plane waves which are of longitudinal waves, transverse waves and thermal waves in the thermoelastic medium. The analytical expressions of their velocities are obtained and it is observed that they depend on the angle of propagation. Using appropriate boundary conditions, the amplitude and energy ratios for the reflected waves are derived and computed numerically.

K-4: Mathematical analysis and concepts for industrial design of vibration & shock isolation systems for power plants and aerospace applications, *S.N.Bagchi* (*Noida : design@resisto ex.in*).

This is an industry academia oriented presentation on the correlation of mathematical concepts and engineering fabrication technology used for designing the Vibration and Shock isolation system related to railways, automobiles, buses and aircrafts for a reliable and comfortable journey. The power plant machineries and aerospace applications are focused. The mathematical modeling and modalities have an engineering-physics orientation. In view of the inter-disciplinary nature of the subject and wide spectrum of coverage from Astrophysics to Mechanical Engineering the presentation may contain more pictorial slides from real life examples and installations from industrial sector. The various types of isolators and the related system eigenfrequency calculations are detailed. The mathematical designs for the isolation of micro-seismic vibration from ground for high power microscopy, laser holography and nano- metrology using Air spring isolated worktables are also covered. The vibration isolation efficiency calculations will be detailed for a mechanical system. The mathematical results are validated by actual measurement after installation and are found to be well within an acceptable level of tolerance of $+/-5\%$. Case studies will be presented.

K-5: Some exact solutions of string cosmological models in Bianchi type-ii space-time, *J.K. Singh* (*New Delhi : saratcha32@yahoo.co.uk*).

In this paper we obtain some new exact solutions of the string cosmology in the presence and absence of the magnetic field in the context of Bianchi type-II space- times. We examine the energy conditions for a cloud of strings coupled to the Einstein equations.

K-6: Singular surface and steepening of waves in gravitational collapse of interstellar gas clouds, *J. Jena* (*New Delhi : saratcha32 @yahoo.co.uk*).

In this paper, the singular surface theory has been used to study the different modes of wave propagation and its culmination into discontinuities at the wave fronts in gravitational collapse of a spherically symmetric interstellar gas cloud. The transport equation for the jump discontinuity has been obtained. The effect of cooling - heating function has been studied in detail.

K-7: Certain solutions of shock-waves in non-ideal gases, *Kanti Pandey and Kiran Singh* (*Department of Mathematics, Lucknow University,Lucknow*).

In present paper non similar solutions for plane, cylindrical and spherical unsteady flows of nonideal gas behind shock wave of arbitrary strength initiated by the instantaneous release of finite energy and propagating in a non-ideal gas is investigated. Asymptotic analysis is applied to obtain a solution up to second order. Solution for numerical calculation Runga-Kutta method of fourth order is applied and is concluded that for non-ideal case there is a decrease in velocity, pressure and density for 0^{th} and II^{nd} order in comparison to ideal gas but a increasing tendency in velocity, pressure and density for Ist order in comparison to ideal gas. The energy of explosion J_0 for ideal gas is greater in comparison to non-ideal gas for plane, cylindrical and spherical waves.

K-8: Effect of reforestation on the control of atmospheric carbon dioxide: a delay mathematical model, *A.K. Mishra and Maitri Verma* (*Department of Mathematics, Banaras Hindu University, Varanasi*).

Carbon dioxide (CO_2) is one of the major greenhouse gases responsible for the burgeoning threat of global warming. Forest biomass plays an important role in sequestration of carbon dioxide from the atmosphere but the global forest biomass is declining with an alarming rate due to human activities. In this paper, we propose a non-linear mathematical model to study the effect of reforestation on the dynamics of atmospheric CO_2 by incorporating time delay in implementation of reforestation efforts. In modeling process, it is assumed that the reforestation efforts, which are applied to increase the forest biomass, depend on the difference between carrying capacity of forest biomass and its value measured some time earlier. The model is analyzed by using stability theory of delay differential equations. The model analysis shows that the atmospheric concentration of CO_2 decreases due to reforestation but longer delay in applying reforestation efforts has destabilizing effect on the dynamics of the system. The critical value of this time delay is found analytically. The Hopf- bifurcation analysis is performed by taking time delay as bifurcation parameter. The stability and direction of bifurcating periodic solutions arising through Hopf- bifurcation are also discussed.Numerical simulation is performed to support the analytical findings.

K-9: Wavelet-Galerkin method for solving boundary value problems of Bratu-type, *S.C. Shiralashetti, M.H. Kantli, P.B. Mutalik Desai, N.M. Bujurke* (*Dharwad : mkantli@gmail.com, shiralashettisc @yahoo.com*).

In this paper, we present a framework to determine a Wavelet -Galerkin solutions of Bratu-type equations that are widely applicable in fuel ignition of the combustion theory and heat transfer. The Daubechies family of wavelets will be considered due to their useful properties.Since the contribution of compactly supported wavelet by Daubechies and multiresolution analysis bases on Fast Fourier Transform (FWT) algorithm by Beylkin, wavelet based solution of ordinary differential equations gained momentum in attractive way. Advantages of Wavelet-Galerkin Method over finite difference or element method have led to tremendous application in science and engineering. In the present work Daubechies families of wavelets have been applied to solve Boundary Value Problems of Bratu-type equations. Solution obtained may the Daubechies-6 coefficients has been compared with exact solution. The good agreement of mathematical results, with the exact solution proves the accuracy and efficiency of Wavelet-Galerkin Method for Bratu-type models.

K-10: The onset of double difusive convection in a Maxwell fluid, *Bharati S. Biradar* (*Gulbarga : bharatibiradar@gmail.com*).

The onset of double diffusive convection in a Maxwell fluid saturated porous layer is studied using a linear stability analysis. The onset criterion for stationary and oscillatory convection is derived analytically. The effect of stress relaxation parameter, Lewis number, solute Rayleigh number,Darcy-Prandtl number and normalized porosity parameter on the stationary and oscillatory convection is shown graphically.

L: Electromagnetic Theory, Magneto-Hydrodynamics, Astronomy and Astrophysics:

L-1: Study of couple stresses on MHD poiseuille flow through a porous medium past an accelerated plate, *B.G. Prasad and Raj Shekhar Prasad* (*Patna : profbgprasad@gmail.com, shekharraj.2010 @gmail.com*).

In this paper a mathematical analysis of effect of couple stresses on magneto hydrodynamic (MHD)flow of viscous, incompressible and electrically conducting fluid in the presence of porous medium past an accelerated plate has been studied. An exact solution of the dimensionless governing equations has been obtained by using method of perturbation technique. The effects on velocity, discharge between the plates and well shear are studied for different parameters like Hartmann number, Reynolds number, pulsation, time, porosity parameter, pulsation frequency and reported graphically to show the interesting aspect of the solution.

L-2: Role of primary third order resonances on Mimas-tethys system, *Z.A. Taqvi, Govind Kumar Jha and L.M. Saha* (*New Delhi : jhagovi@gmail.com, lmsaha@sify.com*).

We have derived the equations of motion from the averaged Hamiltonian comprising with oblateness of Saturn, in the Mimas-tethys system. This includes the Keplerian Hamiltonian, secular Hamiltonian up to the second order and resonant Hamiltonian up to the third order of resonances. We have studied numerically the time series graph of Hamiltonian at epoch 2008 January and the results have been compared with the result of Champenois and Vienne(1999).

L-3: Role of resonances in chaotic evolution of satellites in the solar system, *Sarita Jha and Suresh Kumar Agrawal* (*Hazaribag : saritajhakbw.vbu@gmail.com*).

Theoretical efforts to determine the effects of Planetary interactions led to the development of perturbation theory and averaging methods, and to many other developments in mathematics. In recent years, Celestial Mechanics has evolved from its traditional habitat in the study of orbits of Solar system objects. The greatest difficulties in these studies – and also the most interesting dynamical behaviours – are associated with orbital resonances. Orbital resonances are ubiquitous in the Solar system. They play a decisive role in the long term dynamics, and in cases of some physical evolution, of the planets and of their natural satellites. Weak dissipative effects - also the role of resonances in chaotic evolution of satellites in the Solar system.

L-4: Dufour effect on a transient MHD flow past a uniformly moving porous plate with heat sink, *N. Ahmed, S. Sinha, S. Talukdar* (*Guwahati : saheel nazib@yahoo.com, mathssujangu@gmail.com, talukdar sujit@yahoo.com*).

An attempt has been made to study the Dufour effect on a transient MHD convective flow past a uniformly moving porous plate in presence of a heat source when suction velocity as well as free stream velocity varies periodically with time. A uniform magnetic field is applied normal to the plate. The plate is assumed to move with a constant velocity in the direction of the fluid flow. The equations governing the flow, heat and mass transfer characteristics are solved analytically using perturbation technique. The profiles of the dimensionless velocity, temperature, concentration as well as skin friction, Nusselt number and Sherwood number at the plate are demonstrated graphically for various values of the parameters involved in the problem and the results are discussed.

L-5: Soret effect on an MHD free convective mass transfer flow past an accelerated vertical plate with chemical reaction, *N. Ahmed and K. Kr. Das (Guwahati : saheel_nazib@yahoo.com, kishoredas969 @yahoo.in).*

An exact solution to the problem of an MHD heat and mass transfer flow with Soret effect past an accelerated infinite vertical plate embedded in a porous medium in presence of chemical reaction is presented. A magnetic field of uniform strength is applied normal to the plate directed into the fluid region. The governing equations are solved by Laplace-Transform technique in closed form.The expressions for the velocity field, the temperature field, the concentration field, the coefficient of skin friction at the plate in the direction of flow and the coefficient of heat and mass transfer in terms of Nusselt number and Sherwood number at the plate are obtained and their nature are demonstrated graphically for different values of the parameters involved in the problem. It is found that the Soret effect or thermal diffusion effect has a significant role in controlling the flow and heat and mass transfer characteristics.

L-6: Damped motion of electron in electric conductor due to presence of magnetic and alternative electric fields at right angles and closed forivi solutions, *Soumen Maitra (Pune : soumen wmaitra@yahoo.co.in).*

Flow of current in an electric conductor is attributed to movement of electrons in the opposite directions. ln this paper a second-order differential equation of an electron is formed taking into account a clamping force owing to collisions between electrons and two more applied forces, magnetic field and alternative electric field in perpendicular directions and has been completely solved in closed form subject to the prescribed initial conditions. The governing differential equation of motion of the electrons is separated into two tractable simultaneous equations by use of complex number $\iota = \sqrt{-1}$) It is proved that the effect of damping partially dies away after sometime vis-à-vis theoretically after a longtime, say after infinite time. The acceleration, velocity and distance described by the electron at any instant of time are determined. Thereafter their maximum and minimum values are found out.

M: Bio-Mathematics:

M-1: Numerical solution of magnetic fluid hyperthermia, *Sonalika Singh and Sushil Kumar* (*Surat: singhsonalika01 @gmail.com, sushilk @ashd.svnit @ashd.svnit.ac.in*).

Magnetic fluid hyperthermia is a cancer treatment in which body tissues are exposed to high temperatures to damage and kill cancer cells, usually with minimal injury to normal tissues. In this paper, the methods are described on physical problem. Here mathematical model of temperature distribution in bi-layered spherical tissue during magnetic fluid hyperthermia treatment of tumour has been discussed. The numerical method is applied to solve Pennes bio- heat transfer problem.

M-2: Mining quasi cliques in PPI network: an application to protein complex detection, *Sumanta Ray* (*Department of Mathematics University of Kalyani, Kalyani*).

Considering the large number of proteins and the high complexity of the protein-protein interaction (PPI) network, decomposing it into smaller and manageable modules is a valuable step toward the analysis of biological processes and pathways. There are different types of graph clustering approaches available for modularization of PPI networks. Most of them seek to identify groups or clusters of proteins on the basis of the local density of the interaction graph in the network. In this article we present a novel Multiobjective clustering in PPI network- to find quasi cliques in PPI network. Quasi cliques (almost clique) generally represent denser region in protein protein interaction graph and are expected to form protein complex. We consider graphical properties of PPI network based on density and some centrality measures as objective functions. The proposed technique is demonstrated on yeast PPI data and the results are compared with that of some existing algorithms.

N: History and teaching of mathematics:

N-1: Teaching of mathematics, *Ashwani Kumar Sinha and M.M. Bajaj* (*Ara : mathashwani@yahoo.co.in*).

The main goal of mathematics education in schools is the mathematics of child's thinking. Clarity of thought and pursuing assumption to logical conclusion is central to the mathematical enterprize. There are many ways of thinking, and the kind of thinking one learns in mathematics is an ability to handle abstractions and an approach to problem solving.

Universalization of schooling has important implications for mathematics curriculum. Mathematics being a compulsory subject of study, access to quality mathematics education is every child's right. We want mathematics education that is affordable to every child, and at the same time, enjoyable. With many children exiting the system after Class VIII, mathematics education at the elementary stage should help children,prepare for the challenges they face further in life.

N-2: New model on the development of mathematical reasoning, *Shaligram Shukla* (*West Bengal : mathashwani@yahoo.co.in*).

Mathematics is a way to settle in the mind a habit of reasoning where the results are developed through a process of reasoning and the conclusion are drawn from the given facts. The reasoning in mathematics has the following characteristics such as simplicity, accuracy, certainty of results, originality, similarity to the reasoning of life, and verifications. Mathematics in the making is intuitive, experimental and inductive, where mathematicians generalize their theories from particular examples. Deductive reasoning proceeds from abstract rules to concrete ideas. Mathematics in its widest sense is the development of all types of deductive reasoning which requires undefined terms, definitions, postulates, axioms and essentials. Study of mathematics disciplines the mind and develops the reasoning power of our different faculties. It gives us knowledge without taxing the memory in the least - resulting in the development of power rather then the acquisition of knowledge. In short, model leads a student of mathematics to an intelligent and judicious mode of study in general and enables him to acquire mental self-reliance habits and independence in out-look.

N-3: History of mathematics education with special reference to Ramanujan's contributions in mathematics, *R.B. Dash and H.K.*

Sharma (*Cuttak : rajaniballavdash@gmail.com, harikishoresharma9 @gmail.com*).

A brief account of his life, career and remarkable mathematical contributions is given to describe the gifted talent of Srinivasa Ramanujan.

N-4: Existence of zero and infinity in Vedas, *Omkar Lal Shrivastava and Sumita Shrivastava* (*Rajnandgaon : sumitashrivastava9@gmail.com, omkarlal@gmail.com*).

The Vedas are ancient religious and philosophical scriptures that originated in India . The term Veda is mostly used for four canonical Vedas : 1. Rigveda(RV) 2. Samaveda(SV) 3. Yajurveda (YV)and 4. Atharvaveda(AV) . To be as conservative as possible within the parameters of the new archaeological and astronomical evidence, we think it prudent to consider 2000 BC as the divide between the early Vedic and the later Vedic literature. The Vedas are composed in Sanskrit in the form of hymns to various deities and also hymns praising all forms of knowledge. Although the Vedas are not exclusively dealing with mathematics but many mathematical data and mathematical methods are contained in Vedas. Sharma (2009) remarks that "it is well recognized that from Sanskrit resulted Greek and Iranian languages in 500 BC Latin in 300 BC Goethic in 400 AD etc". But we will not go into linguistic detail. We have found contrary to the belief of Prof Witzel (1998, 2001) that "one must keep in mind that the concept of zero did not exist at that time and also there was no script". We have found some references in Vedas about six basic mathematical operations. In our paper we will discuss in detail about the existence of zero and infinity in Vedas in mathematical sense.

N-5: Contributions of Brahmgupta in algebra, *Daya Shankar Gupta* (*Kabir Sant Nagar, U.P.*).

In this paper we discuss some important items of time, place of work, extant text books etc. relating to the work of Brahmagupta's contributions in algebra.

ABSTRACT OF THE PAPERS FOR IMS PRIZES:

GROUP -1:

IMS-1: Induced complements of a graph, *Prameela Kolake* (*Surathkal : prameela.kolake@gmail.com*).

In this paper, we define a new generalization of the complement of a given graph. Let $G = (V, E)$ be a graph and $S \subseteq V$. The induced complement of the graph G with respect to the set S, denoted by $\overline{G_s}$, is the graph obtained from the graph G by removing the edges of $\langle S \rangle$ in G and adding the edges which are not in $\langle S \rangle$ in G. Given a set $S \subseteq V$ of vertices of a graph G, the graph G is said to be S-self complementary if $\overline{G_s} \cong G$. A graph G is said to be k-induced self complementary $(k - i.s.c.)$ if for every subset $S \subseteq V$ with $|S| = k, \overline{G_s} \cong G$. Given a set $S \subseteq V$, the graph G is said to be S-cocomplementary if $\overline{G_s} \cong \overline{G}$. A graph G is said to be k-co-induced complementary $(k - co - i.c.)$ if for every subset $S \subseteq V$ with $|S| = k, \overline{G_s} \cong \overline{G}$. This paper presents the study of the properties of the induced complements of a given graph.

IMS-2: Rogers-Ramanujan identities for Frobenius partitions, *Garima Sood* (*Chandigarh : garimasood18@gmail.com*).

We give new combinatorial interpretations of five basic series of Rogers in terms of generalized Frobenius partitions. This extends the recent work of Goyal and Agarwal and yields five new 3-way combinatorial identities.

IMS-3: Baer ideals in 0-distributive posets, *Nilesh D. Mundlik* (*Pune : mundliknilesh@gmail.com*).

In this paper, Baer ideals in posets are defined and some characterizations of baer ideals in 0-distributive posets are obtained. Also, sufficient conditions are obtained for an ideal to be a Baer ideal. Baer ideals in SSC and weakly SSC posets are studied. A weakly SSC poset is represented in terms of minimal prime ideals. In fact it is proved that a 0-distributive poset is weakly SSC if and only it the map $f : Q \to 2^{Min(Q)}$ is an isomorphism on to $f(Q)$, where $f(x) = \{M \in Min(Q) | x \notin M\}$. The concept of quasicomplemented lattices is extended to posets and its properties in terms of Bear ideals are studied. It can be noted that there are no implications between 0-distributive posets and quasicomplemented posets. Finally the topological properties of the space $Spec_B(Q)$ of prime Baer ideals of a join semilattice are given. In fact it is proved that $Spec_B(Q)$ is compact if and only if Q has a dense element.

IMS-4: Refined estimates on conjecture of Minkowski of $n = 10, 11$ and 12, *Leetika Kathuria* (*Chandigarh : kathurialeetika@gmail.com*).

Let $L_i = a_{i1}x_1 + \cdots + a_{in}x_n, 1 \leq i \leq n$, be n real linear forms of determinant $\Delta = det(a_{ij}) \neq 0$. The classical conjecture of Minkowski in the geometry of numbers asserts that for given any real numbers $c_1, \ldots, c_n$ there exist integers $u_1, \ldots, u_n$ such that $\prod_{i=1}^{n} |(L_1(u_1, \ldots, u_n) + c_i)| \leq 1/2^n|\Delta|$. This conjecture is known to be true for $n \leq 9$ only. Weaker results are known for higher n. Several authors such as Čhebotarev (1940), Davenport(1946), Woods(1958), Mordell(1960), Il'in(1991), Hans-Gill et al.(2010, 2011) have obtained the estimates on Minkowski's conjecture. In this paper we shall obtain improvements on the estimates for Minkowski's conjecture for $n = 10, 11$ and 12. To obtain these, we shall first obtain improvements on the estimates to Woods' conjecture for these values.

GROUP -2:

IMS-5: Clustering of bicomplex nets in idempotent order topology, *Sukhdev Singh* (*Agra : ssukhdev1209@yahoo.co.in*).

This paper initiates the study of clustering of bicomplex nets. Clustering on different types of zones in the bicomplex space have been defined. Clustering in idempotent order topology and idempotent product topology have been compared. Relation between clustering of bicomplex nets and the clustering of its component nets have been defined. Finally, investigations have been made connecting clustering of a bicomplex net and confinement of its subnets.

GROUP - 3:

{No paper}

GROUP - 4:

IMS-6: Second order random differential equations, *D.S. Palimkar (Nanded : dspalimkar@rediffmail.com).*

In this paper, an existence of extremal random solution for nonlinear second order ordinary random differential equations is proved under a Caratheodory condition using an application of the random version of the Leray-Schauder principle and an algebraic random fixed point Theorem of Dhage.

IMS-7: Heteroclinic solutions of singular Φ- Laplacian boundary value problems on infinite time scales, *Murali Penugurthi (Visakhapatnam : penugurthi.murali@gmail.com).*

In this paper, we derive sufficient conditions for the existence of heteroclinic solutions to the singular Φ-Laplacian boundary value problem,

$$\left[\Phi(y^{\Delta}(t))\right]^{\Delta} = f(t, y(t), y^{\Delta}(t)), t \in \mathbb{T}$$
$$y(-\infty) = -1, y(+\infty) = +1,$$

on infinite time scales by using the Brouwer invariance domain theorem. As an application we demonstrate our result with an example.

IMS-8: Eventual stability of impulsive functional differential equations, *Dilbaj Singh (Punjab : dilbaj.singh@lpu.co.in).*

In this paper, by using piecewise continuous Lyapunov functions, criteria of eventual stability for system of impulsive functional differential equations is investigated. The sufficient conditions that are obtained significantly depend on the moments of impulses. An example is provided to illustrate the effectiveness and the advantages of the results obtained.

IMS-9: Exact controllability of the hyperbolic system with bilinear controls, *Sonawane Ramdas baburao (Nashik : sonawaneramdas@gmail.com).*

We consider the problem of exact controllability of hyperbolic system with Dirichlet boundary conditions. A bilinear controls are coefficient like vy, uy_t. Controls are interpreted as the axial load, and the gain of the viscous (motion-activated) damping. Exact controllability result is stated and proved for some particular targets.

IMS-10: Nonlinear integral inequalities and its applications, *S.G. Latpate* (*Pune : sglatpate@gmail.com*).

In this paper, we have established some new non-liner integral inequalities and obtained bound on unknown function. These inequalities can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of some nonlinear differential and integral equations.

IMS-11: Optimal error estimates of a semidiscrete finite element method for a magnetic field diffusion equation, *Nisha Sharma* (*Chandigarh : ns01838@gmail.com*).

The purpose of the paper is to analyze semidiscrete Galerkin finite element method for a time dependent parabolic integro-differential equation in higher spatial dimensions which arise in the mathematical modeling of the penetration of a magnetic field into a substance. Optimal L^2 and H^1 error estimates for the continuous Galerkin approximation are obtained. θ method is used to obtain a numerical scheme in a vector matrix form. Further, a numerical experiment is presented to validate the theoretical results.

GROUP - 5:

IMS-12: Study of a problem of functionally graded hollowsphere under dual phase-lag thermoelasticity by finite element method, *Shweta Kothari* (*shweta.kothari5 @gmail.com, skothari.rs.apm@itbhu.ac.in*).

The present paper investigates the thermoelastic interactions inside a functionally graded hollow sphere under dual phase-lag thermoelasticity theory. The material of the sphere is assumed to be graded through its thickness in the radial direction such that the profiles of all material properties, except the phase-lags, are assumed to follow a volume-fraction based rule with a power law. The inner and outer boundaries of the sphere are subjected to different thermal and mechanical boundary conditions. Galerkin type finite element method is used to solve the coupled equations arising out of the present model in Laplace transform domain. Solution in space time domain is obtained by employing a numerical inversion method. The effects of non homogeneity index, phase-lags and time upon the different fields like displacement, temperature, radial stress and hoop stress are analyzed by various graphical plots of our numerical results.

GROUP - 6:

IMS-13: Nonsmooth multiobjective optimization via generalized approximate convexity and quasi efficiency using limiting subdifferentials, *B.B. Upadhyay* (*Varanasi : bhooshanbhu@gmail.com*).

In this paper, we consider a class of nonsmooth multiobjective optimization problem and establish a necessary optimality condition by using the concept of quasi efficient solutions and the tools of limiting subdifferentials. We define four new generalizations of the concept of approximate convexity for locally Lipschitz functions and obtain the relationship between these generalizations of the approximate convex functions and the classical notions of pseudoconvexity and quasiconvexity by using the properties of limiting subdifferentials. Suitable examples are given to illustrate the significance of these relationships. These generalizations are then used to establish certain sufficient optimality conditions for a nonsmooth multiobjective optimization problem.

IMS-14: On epsilon convexity and minty variational principle for differentiable vector optimization problems, *Vivek Laha* (*Varanasi : vivek333@gmail.com*).

This paper deals with the vector optimization problems involving the differentiable epsilon convex functions and the formulation of the epsilon vector variational inequalities of Stampacchia and Minty type to solve the vector optimization problems, and is devoted to study the relationships among the solutions of the epsilon vector variational inequalities of Stampacchia and Minty type and also the epsilon quasi efficient solutions of the differentiable vector optimization problems. Moreover, the perturbed epsilon vector variational inequalities are formulated and some relations with the epsilon Minty vector variational inequalities are established. The corresponding strong and weak versions of the epsilon vector variational inequalities are also formulated and various result for the weak and strong efficient solutions are established. The results in this paper extend and generalize various results present in the literature.

ABSTRACT OF THE PAPERS FOR AMU PRIZE:

AMU-1: Guass newton methods of non-linear ill-posed Hammerstein type equations, *Monnanda Erappa Shobha* (*Karnataka : shobha.me @gmail.com*).

A combination of Newton iterative method and Tikhonov regularization is proposed and applied to nonlinear ill-posed Hammerstein type operator equations $KF(x) = y$, where $K : X \to Y$ is a bounded linear operator and $F : X \to X$ is a non-linear operator. It is assumed that the Fréchet derivative of F is invertible in a neighbourhood which includes the initial guess x_0 and the solution $\hat{x}$. Local cubic convergence is established and order optimal error bounds are obtained by choosing the regularization parameter according to the the balancing principle of Pereverzev and Schock (2005). We also present the results of computational experiments giving the evidence of the reliability of our approach.

AMU-2: m^{th}-root Rander's change of a Finsler metric, *V.K. Chaubey* (*Gorakhpur : vkcoct@gmail.com*).

In the present paper we introduce a m^{th}-root Rander's changed Finsler metric as

$$\overline{L}(x, y) = L(x, y) + \beta(x, y)$$

where $L = \{a_{i_1,i_2,\ldots,i_m}(x)y^{i1}y^{i2 \ldots y^{im}}\}^{1/m}$ is a m^{th}-root metric and β is one form. Further we obtained the relation between the v- and hv- curvature tensor of m^{th}-root Finsler space and its m^{th}-root Rander's changes Finsler space and obtained some theorems for its $S3$ and $S4$-likeness of Finsler spaces and when this changed Finsler space will be Berwald space (resp. Landsberg space). Also we obtain T-tensor for the m^{th}-root Rander's changed Finsler space $\overline{F}^n$.

AMU-3: Semigroup ideals and semiderivations of prime near rings, *Phool Miyan* (*Aligarh : phoolmiyan83@gmail.com*).

Let N be a near ring. An additive mapping $f : N \to N$ is said to be a semiderivation on N if there exists a mapping $g : N \to N$ such that

(i) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and
(ii) $f(g(x)) = g(f(x))$ for all $x, y \in N$.

In this paper we prove some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a nonzero semiderivation, thereby extending some knome results on derivations.

AMU-4: Additive maps acting as a homomorphism or antihomomorphism in semiprime rings, *Basudeb Dhara* (*West Bengal : basu dhara@yahoo.com*).

R is a semiprime ring with center $Z(R)$. Let $F, d : R \to R$ be any two mappings such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper, our main object is to study the situations:
(1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$,(3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) + F(y)F(x) \in Z(R)$; for all x, y in some suitable subset of R.

AMU-5: Characterizations of generalized derivation in rings, *Faiza Shujat* (*Aligarh : faiza.shujat@gmail.com*).

Let R be a ring. An additive mapping $F : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In [Canad Math Bull. 22(4) (1979), 509-511] Herstein determined the structure of a prime ring R admitting a nonzero derivation d such that the values of d commute, that is $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$.

Perhaps even more natural might be the question of what can be said of a derivation when elements in a prime ring commute with all the values of a nonzero derivation. Herstein addressed this question by proving the following result: If d is a nonzero derivation of a prime ring R and $a = 2Z(R)$ is such that $[a, d(x)] = 0$ for all $x \in R$, then R has characteristic 2, $a^2 \in Z(R)$ and $d(x) = [\lambda a, x]$ for all $x \in R$ and $\lambda \in C$, the extended centroid of R. Recently Albas and Nurcan [Algebra Colloq, 11 (3) (2004), 399-410] proved that if R is a noncommutative prime ring and d is a generalized derivation with associated derivation α of R and for all $x \in R, [a, d(x)] = 0$, then either $a \in C$ or there exist $\lambda, \eta \in C$ such that $d(x) = \eta x + \lambda(ax + xa)$ for all $x \in R$: In the present note we characterize the generalized derivation F of a prime (semiprime) ring satisfying the above mentioned condition. Our theorems extend result of Bresar.[J. Algebra 156(1993), 385-394, Lemma 2.2], Aydin [Intern J. Algebra 5(1) (2011), 17-23, Theorem 2.3], Nurcan [Algebra Colloq, 13 (3) (2006), 371-380, Lemma 4.2] and a result of Herstein [Canad Math Bull. 22(4) (1979), 509-511, Theorem].

ABSTRACT OF THE PAPERS FOR V.M. SHAH PRIZE:

VMS-1: L_p -approximation of signals (functions) belonging to weighted $W(L_r;\xi(t))$- classes using product C^1, N_p summability method of conjugate series of its fourier series, *Vishnu Narayan Mishra* (*Surat: vishnunarayanmishra@gmail.com*).

Recently, Lal has determined the degree of approximation of function belonging to Lip α and weighted $W(L_r, \xi(t))$- classes using Product C^1, N_p summability with non-increasing weights $\{p_n\}$. In this paper, we determine the degree of approximation of function $\tilde{f}$, conjugate to a 2π- periodic function f belonging to weighted $W(L_r, \xi(t))$- class using semi-monotonicity on the generating sequence $\{p_n\}$ with proper set of conditions.

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WACHSPREES, MEAN VALUE AND GENERAL BARYCENTRIC COORDINATES AND THEIR MAPPING PROPERTIES

H. P. DIKSHIT

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Abstract. E. L. Wachspress introduced in 1975, certain rational polynomial basis functions, in particular, over closed convex polygons in $\mathbb{R}^2$ which had the remarkable property of reducing to polynomials along the edges of the polygon besides many other useful properties like partition of unity and linear precision. One of the important outcomes of these was a system of Wachspress coordinates for a unique representation of any point within the polygon and its boundary depending on the vertices of the polygon. We use the Wachspress rational function two folds. One is to employ complex analogue of Wachspress rational polynomial to define rational complex planar splines and discover their quasiconformal property. The other is to employ corresponding barycentric coordinates in developing Computer Aided Geometric Design (CAGD) applications. Motivated by the Mean Value Theorem for harmonic functions, Michael S. Floater introduced mean value coordinates to represent any point interior to the polygon. We will compare properties of these two types of barycentric coordinates and discuss injectivity issues related to them. We then present generalized barycentric coordinates, which give bounds for these coordinates and also show that there is a natural way to extend mean value coordinates to the boundary of the polygon.

1. INTRODUCTION

Motivated by Ceva's Theorem [4] that for any point v inside a planar triangle $[v_1, v_2, v_3]$ with vertices v_1, v_2, v_3 there exist three masses ω_1, ω_2 and ω_3, such that, if placed at the corresponding vertices of the triangle their centre of mass (or barycentre) will coincide with v, that is

$$v = (v_1\omega_1 + v_2\omega_2 + v_3\omega_3)/(\omega_1 + \omega_2 + \omega_3). \tag{1.1}$$

* This paper is dedicated in honour of Prof. V. M. Shah.

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Möbius [20] was the first to study such mass points and he defined $(\omega_1, \omega_2, \omega_3)$ as the barycentric coordinates of v. These barycentric coordinates are only unique up to multiplication by a common non-zero scalar and they are usually normalized so that their sum: $\omega_1 + \omega_2 + \omega_3 = 1$. These normalized triangular barycentric coordinates are linear in v and have the useful property that, the i-th coordinate has value 1 at v_i and 0 at any other v_j. This is why they are commonly used to linearly interpolate values given at the vertices of a triangle and have applications in computer graphics, triangular Bézier patches, splines over triangulations, geometric modelling and many other fields including the finite element method and terrain modelling.

Generalization of barycentric coordinates to convex polygons appears in the pioneering work of Wachspress [23] on finite element method. These Wachspress coordinates are rational polynomials and were later generalized to convex polytopes by Warren [24] who also showed that they have minimal degree [25]. They can be computed with simple and local formulas in the plane (see Meyer et al. [19]) as well as in higher dimensions (see Warren et al. [26]) and have many other nice properties like affine invariance.

We first got interested in Wachspress rational functions with a somewhat different objective (see, Dikshit et al. [9], [12], [13]). Motivated by results of Opfer et al. [21], [22] and our results in Dikshit et al. [9] in establishing certain important mapping properties of complex polynomial planar splines, which required proofs of injectivity and orientation preserving properties, we ventured to look for rational polynomial pieces which could enable us to ensure both injectivity and orientation preserving properties. We thus considered complex analogue in $z = x + iy$ and $\bar{z} = x - iy$ of Wachspress rational basis functions of degree (2,1) to define Wachspress rational complex planar splines and studied their mapping properties.

We later got interested in Wachspress coordinates over an admissible quadrilateral, that is, quadrilaterals with interescting opposite sides, to develop CAGD methods for designing surfaces over quadrilaterals. We (Dahmen et al. [5], Dikshit et al. [10], [11]) also showed that they inherit almost all the important properties of the well known Bézier-Bernstein methods for triangles.

The object of this survey article is to highlight some of our works with A. Ojha, A. Sharma and R.A. Zalik on quasiconformality of complex planar splines and Wachspress rational complex planar splines and later works with W. Dahmen and A. Ojha on Computer Aided Geometric Design (CAGD) applications using Wachspress barycentric coordinates over quadrilaterals.

There are many generalizations of barycentric coordinates over triangles to general polygons but we will focus on Wachspress and mean value coordinates which have some unique and distinctive properties. We will also update on some further interesting developments especially due to M. S. Floater and his coworkers.

2. QUASICONFORMAL WACHSPRESS RATIONAL COMPLEX PLANAR SPLINES

The main motivation for studies of piecewise real polynomial functions or real splines has been their capability to provide greater flexibility and rich mathematical properties while ensuring computational convenience. Thus such functions have found extensive applications in a variety of areas. For studies of their complex counterparts, there are two main approaches. The approach introduced by Ahlberg, et al. [1] consists in defining a piecewise complex polynomial function along the boundary of a closed region in the complex plane and then extending such a function to the interior by the Cauchy's integral formula. A second approach due to Opfer et al. [21] consists in starting with a partition of a closed region in the complex plane, and introducing complex planar splines in terms of pieces, which are nonanalytic functions. For, if we use analytic functions then even the continuity condition across the edges implies that all pieces are the same by an application of the Identity Theorem. Thus, Opfer et al. [21]-[22], used polynomials in z and $\bar{z}$, which are nonanalytic, as pieces, to define complex planar splines. Motivated by Wachspress' development of a rational finite element basis [23], Dikshit et al. [12] studied the properties of rational complex planar spline interpolants of degree (2,1) and showed that under certain conditions these interpolants have some nice shape preserving properties like quasiregularity and quasiconformality.

Let us consider a quadrilateral $P_0 = P_0(t)$ in complex plane with vertices $0, t, it, 1+i$, such that $0 < t < 1$. Then as observed in Dikshit et al. ([13, p. 237]), we can find a partition into quadrilaterals which can be obtained by translation and rotation of P_0. Thus we fix the Wachspress rational polynomial of degree (2,1) on P_0 and recover the same for any other element through affine functions. For this, we notice that the exterior diagonal of P_0 (which lies outside P_0) is represented by the equation

$$q(z) = (1-i)z + (1-i)\bar{z} + 2t/(1-t) = 0, 0 < t < 1. \tag{2.1}$$

Now employing the definition of Wachspress rational finite elements of degree (2,1), we set the polynomial piece over P_0 as

$$\rho^{2,1}(z) = a + bz + c\bar{z} + d(z^2 - \bar{z}^2)/q(z), z \in P_0. \tag{2.2}$$

The affine property is then used to ensure continuity along the edges of degree (2,1) rational complex planar splines. We observe that the affine property along the edges of the rational interpolants taking prescribed values at the vertices is a more stringent condition than the continuity. Thus to construct continuous rational complex planar splines, which is our objective, it is natural to expect from the foregoing observations that the requirement of continuity instead of affine property may provide more degrees of freedom. These considerations enable us to construct continuous rational complex planar splines of degree (3,1), with polynomial piece on P_0 defined by

$$\rho_f^{3,1}(z) = a + bz + c\bar{z} + (lz + m\bar{z} + n)(z^2 - \bar{z}^2)/q(z), z \in P_0. \tag{2.3}$$

We have shown in Dikshit et al. [13] that there exists a unique continuous rational complex planar spline $\rho_f^{3,1}(z)$ of degree (3,1) interpolating a given $f(z) = u(x, y) + iv(x, y), z = x + iy$ at all the vertices and midpoints of edges not parallel to any of the axes of the underlying quadrilaterals, which has nice convergence properties. Setting,

$$R(\rho_f^{j,1}, \epsilon) = \{z : |\rho_f^{j,1}(z) - f(z)| < \epsilon\}, j = 2, 3,$$

we gave numerical examples in ([13, p. 246]) [1] for which the interpolant $\rho_f^{3,1}$ does better than the corresponding rational complex planar spline interpolant $\rho_f^{2,1}$ in the sense that,

$$R(\rho_f^{2,1}, 0.01) \subset R(\rho_f^{3,1}, 0.01), \text{ for } f(z) = z^2; \sin z; \log(1 + z) \text{ and } \exp z.$$

In order to study orientation preserving property of $\rho_f^{3,1}$ we recall that for a given mapping $f(z) = u(x, y) + iv(x, y), z = x + iy$ such that the first partial derivatives of u and v exist, the Jacobian $J_f(z) = u_x v_y - u_y v_x$ is positive for orientation preserving property and is negative for orientation reversing property. We thus encountered the difficult task of close identification of the orientation preserving region $H^+_{\rho_f^{3,1}}$ in which $J_{\rho_f^{3,1}}(z) > 0$ or orientation reversing region $H^-_{\rho_f^{3,1}}$ in which $J_{\rho_f^{3,1}}(z) < 0$. This required us to find precise conditions on $\rho_f^{3,1}$ to ensure that the underlying quadrilateral lies within $H^+_{\rho_f^{3,1}}$. In support of this, we gave example of the following degree (3,1) polynomial interpolant $S_f^{3,1}$ over the quadrilateral $P_0 = P_0(1/2)$, with vertices as $0, 1/2, 1 + i, i/2$,

$$S_f^{3,1}(z) = (1.1)z - i\bar{z} + ((1 - i)z + (1 + i)\bar{z})/q(z), z \in P_0(1/2),$$

[1] For these studies of rational complex planar splines of degree (3,1), we (Dikshit et al. [13]) carried out the work with the aid of MACSYMA, a large symbolic manipulation program developed at the MIT Laboratory for Computer Science and supported from 1975 to 1983 by the National Aeronautics and Space Administration under grant NSG 1323, by the Office of the Naval Research under grant N00014-77C-0641, by the U.S. Department of Energy under grant F49620-79-C-020, and since 1982 by Symbolics Inc. of Bulington, MA) and presented some numerical examples that suggested that the interpolant studied here is a better approximation than the corresponding degree (2,1) rational complex planar spline interpolant.

which satisfies the conditions of Theorem 4.1 in [13] and is orientation preserving, since $P_0(1/2) \subset J^+_{S_f^{3,1}}$. A closely drawn graph of the region $J^+_{S_f^{3,1}}$ is also given in ([13, p.247]).

Further, we gave example of the following type (2.3) polynomial interpolant $r_f^{3,1}$ over the quadrilateral $P_0 = P_0(1/2)$.

$$r_f^{3,1}(z) = (1.1)z - i\bar{z} + ((i-1)z - (1+i)\bar{z})/q(z), z \in P_0(1/2),$$

which violates the conditions of Theorem 4.1 in [13] and is not orientation preserving, since $P_0 \not\subset J^+_{r_f^{3,1}(z)}$. It is not even orientation reversing as $P_0 \not\subset J^-_{r_f^{3,1}(z)}$. A closely drawn graph of the region $J^+_{r_f^{3,1}}$ is also given in ([13, p.248]), which shows that $J^+_{r_f^{3,1}}$ does not cover P_0.

We finally showed in Theorem 4.1 of [13] that under certain conditions $\rho_f^{3,1}$ is an orientation preserving homeomorphism.

For studying quasiregularity and quasiconformality of the degree (3,1) interpolant, we recall that the complex dilatation μ_f of f is defined by

$$\mu_f = f_{\bar{z}}/f_z,$$

and f is conformal at z_0, if and only if, $\mu_f(z_0) = 0$.

By construction degree (3,1) interpolant cannot be conformal, so it will be of interest to determine, how close the complex dilatation of the interpolant is to the complex dilatation of f. In Theorem 4.2 of [13], we show that

$$\left| ||\mu_f|| - \left|\left|\mu_{\rho_f^{3,1}}\right|\right| \right| \le \left|\left|\mu_f - \mu_{\rho_f^{3,1}}\right|\right| \le K\delta,$$

where K is some finite positive constant and δ is a sufficiently small number depending on the diameter of the underlying partition into quadrilaterals.

Another, measure of conformality is the dilatation D_f, defined by

$$D_f = (|f_z| + |f_{\bar{z}}|)/(|f_z| - |f_{\bar{z}}|).$$

f is said to be quasiconformal, if D_f is bounded. It is conformal at z_0, if $D_f(z_0) = 1$ (Ahlfors [2, p. 5]). We finally conclude the quasiconformal property of $\rho_f^{3,1}$ under certain conditions (see Theorems 4.2 and Theorem 4.3 in Dikshit et al. [13]).

3. WACHPRESS RATIONAL BASIS FUNCTIONS AND BARYCENTRIC COORDINATES FOR QUADRILATERALS

We recall that $(\lambda_1, \lambda_2, \lambda_3)$ the barycentric coordinates for any point v in a given triangle $[v_1, v_2, v_3]$ with vertices as points $v_1, v_2, v_3 \in \mathbb{R}^2$ are such that for any point v in the triangle

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3; \lambda_1 + \lambda_2 + \lambda_3 = 1; 0 \le \lambda_i \le 1,\ i = 1, 2, 3.$$

We begin defining Wachspress coordinates over a quadrilateral $Q \subset \mathbb{R}^2$ with vertices $v_1 = (0,0), v_2 = (0,t), v_3 = (1,1), v_4 = (t,0)$ such that $0 < t < 1$. Of course here $t \neq 0$. Also $t \neq 1$, for otherwise Q is a square which is not admissible as opposite sides do not intersect. For $j \in \mathbb{Z}_4$, let $e_j(u) = 0$ be the linear form representing the edge e_j joining the vertices v_j and v_{j+1} of Q. Clearly each of the pair of opposite edges e_1 and $e_3; e_2$ and e_4 ; of Q intersect, since, $0 < t < 1$. Thus, it is easily seen that the exterior diagonal of Q is represented by the linear polynomial:

$$r(u) = r(x,y) = x + y + t/(1-t) = 0; 0 < t < 1.$$

Clearly, $r(u) \neq 0, u \in Q$. The real Wachspress rational basis functions of degree (2,1) are then defined by

$$W_j(u) = K_j e_{j+1}(u) e_{j+2}(u)/r(u); j \in \mathbb{Z}_4; u \in Q \subset \mathbb{R}^2, \tag{3.1}$$

where the constant K_j is defined as

$$K_j = r(v_j)/\{e_{j+1}(v_j).e_{j+2}(v_j)\}, \quad j \in \mathbb{Z}_4,$$

so that $W_j(v_j) = 1$. Now observing that v_{j+1} and v_{j+2} lie on the linear form $e_{j+1}(u) = 0$, while v_{j+3} lies on the linear form $e_{j+2}(u) = 0$ we see from (3.1) that $W_i(v_j) = 0$ for $i = j+1, j+2, j+3$. We thus conclude that $W_i(v_j) = \delta_{ij}$.

For $j \in \mathbb{Z}_4$, we also see from (3.1) that $W_j(u) = 0$, along the edges e_{j+1} and e_{j+2}. As we shall see $W_j(u)$ reduces to linear polynomials along the edges e_j and e_{j+3}. More generally, Wachspress basis rational polynomial functions have the interesting property that the rational polynomials reduce to polynomials along the edges of the underlying convex polygon. This follows from the following consequence of Bozut's Theorem.

Lemma 3.1. *Let us denote by $p_r(x)$ any polynomial of degree $r \geq 0$, and let p_r denote the curve represented by $p_r(x) = 0$. Given polynomials $p_k(x), p_\ell(x), p_m(x)$ with $0 < k \leq \ell \leq m$, such that the three polynomial curves have j distinct common points of intersection with $1 \leq j \leq k$, that is,*

$$p_m(x_i) = p_\ell(x_i) = p_k(x_i);\ i = 1, \ldots, j.$$

Then the rational polynomial $p_m(x)/p_\ell(x)$, along p_k is a rational polynomial of degree $p_{m-j}(x)/p_{\ell-j}(x)$.

In particular, if $1 = j = k = \ell = m$, Lemma 3.1 implies that the three linear forms $p_r(x) = 0, r = k, \ell, m$, which intersect at x_1 are such that the ratio of any two of them is a constant along the third linear form. That is why we needed the opposite sides of the quadrilateral Q to be intersecting and the linear form in the denominator to be the exterior diagonal. Thus, Wachspress rational basis functions on Q reduce to linear polynomials along the edges of Q. We also have the following properties of $W_j(u)$.

$$W_j(u) \geq 0, \sum_{j=1}^{4} W_j(u) = 1; u \in Q; W_j(v_k) = \delta_{j,k}. \tag{3.2}$$

These Wachspress functions have linear precision property, that is, if $p(u)$ is any linear polynomial, then

$$p(u) = \sum_{j=1}^{4} p(v_j) W_j(u).$$

Thus, taking in particular $p(u) = u$, and using (3.2), we have

$$u = \sum_{j=1}^{4} v_j W_j(u); W_j(u) \geq 0, \sum_{j=1}^{4} W_j(u) = 1; u \in Q,$$

and the weights $(W_1(u), W_2(u), W_3(u), W_4(u))$ could be used to define barycentric coordinates for any point $u \in Q$. Thus, we developed in Dahmen et al. [5] a new method for construction of surfaces over quadrilaterals, similar to the well known Bėzier- Bernstein method for construction of surfaces over triangles. In the foregoing definition it was assumed that the opposite sides of Q intersect so that the exterior diagonal could be determined. However, in Dikshit [8], we have observed that if only two opposite sides are parallel and the other two intersect, then one can use the line passing through the point of intersection and parallel to the parallel edges of Q in place of exterior diagonal. In case of rectangles we have bilinear form.

We used these coordinates for CAGD applications in Dahmen et al. [5], Dikshit et al. ([10], [11]). In the next section we shall study Wachspress and mean value coordinates over general convex polygons.

4. WACHSPRESS AND MEANVALUE COORDINATES FOR POLYGONS

In earlier sections, we defined Wachspress rational functions of degree (2,1) over quadrilaterals and used them to extend the definition of barycentric coordinates over triangles to convex quadrilaterals, for representing any point in the underlying quadrilateral in terms of its vertices and their CAGD applications. We also used complex analogues of Wachspress rational functions of degree (2,1) and degree (3,1) over quadrilaterals to study quasiconformality of complex rational polynomial splines. As required for the foregoing works, we explicitly determined these rational polynomials.

We now introduce barycentric coordinates for a general polygon. Let $v, v_1, \ldots,$ v_n be given points in the plane with $v_1, \ldots, v_n$ arranged in an anticlockwise ordering around v. We look for weights, $\lambda_1, \ldots, \lambda_n \geq 0$ such that

$$v = \sum_{i=1}^{n} \lambda_i v_i; \text{ with } \sum_{i=1}^{n} \lambda_i = 1. \tag{4.1}$$

Assuming that the polygon $v_1, \ldots, v_n$ is convex, Wachspress [23] found a solution in which the coordinates of an interior point v can be expressed in terms of rational polynomials:

$$\lambda_i = \frac{w_i}{\sum\limits_{j=1}^{n} w_j}, \qquad w_i = \frac{A(v_{i-1}, v_i, v_{i+1})}{A(v_{i-1}, v_i, v)A(v_i, v_{i+1}, v)}, \tag{4.2}$$

where $A(a;b;c)$ is the signed area of triangle $[a;b;c]$. These Wachspress coordinates were later generalized to convex polytopes by Warren [24] who also showed that they have minimal degree (Warren [25]). They can be computed with simple and local formulas in the plane (see Meyer et al. [19]) and have many other nice properties like affine invariance.

There is another formulation of these coordinates in terms of the angles of the triangles $T_i(v) = [v, v_i, v_{i+1}] \equiv T_i$ formed by joining the point v to the vertices v_i, v_{i+1}. Let $\alpha_i, \beta_i, \gamma_i$ respectively, denote the angles $\angle v_i v v_{i+1}; \angle v v_i v_{i+1}$ and $\angle v_i v_{i+1} v$ of T_i, then the following alternative form for Wachspress coordinates is due to M. Meyer et al. [19].

$$\lambda_i = \frac{w_i}{\sum\limits_{j=1}^{n} w_j}, \qquad w_i = \frac{\cot \gamma_{i-1} + \cot \beta_i}{||v_i - v||^2}. \tag{4.3}$$

Wachspress coordinates are positive in the interior of the convex polygon and are C^∞ smooth. They are rational polynomials in the coordinates x and y of point $u = (x, y)$ with degree at most $n - 2$, which is the least possible degree as shown by J. Warren [25].

We now describe mean value coordinates introduced by Floater [14], which satisfy all the expected properties of barycentric coordinates. The motivation behind these coordinates was an attempt to approximate harmonic maps by piecewise linear maps over triangulations, in such a way that injectivity is preserved. Assuming that v is an interior point of the polygon $v_1, \ldots, v_n$, let $0 < \alpha_i < \pi$, be the angle at v in the triangle $[v, v_i, v_{i+1}]$, defined cyclically. Floater [14] proved the following proposition.

Proposition 4.1. *The weights*

$$\lambda_i = w_i / \sum_{j=1}^{n} w_j; \qquad w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{||v_i - v||}; \tag{4.4}$$

are the mean value coordinates for v with respect to $v_1, \ldots, v_n$.

We would like to sketch the proof of the foregoing proposition as given in [14]. Since, $0 < \alpha_i < \pi$, $\tan(\alpha_i/2)$ is defined and positive. Thus λ_i in (4.4) is well-defined and positive for $i = 1, 2, \ldots, n$ and by definition the sum $\sum_{i=1}^{n} \lambda_i = 1$. It remains to prove that

$$\sum_{i=1}^{n} w_i v_i = v, \text{ equivalently, } \sum_{i=1}^{n} w_i(v_i - v) = 0. \tag{4.5}$$

Putting the polar coordinates: $v_i = v + r(\cos\theta_i, \sin\theta_i)$ centred at v, we see that

$$(v_1 - v)/||v_1 - v|| = (\cos\theta_i, \sin\theta_i) \text{ and } \alpha_i = \theta_{i+1} - \theta_i.$$

Therefore, the second sum in (4.5) becomes

$$\sum_{i=1}^{n} (\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2))(\cos\theta_i, \sin\theta_i) = 0,$$

which is equivalent to

$$\sum_{i=1}^{n} (\tan(\alpha_i/2)((\cos\theta_i, \sin\theta_i) + (\cos\theta_{i+1}, \sin\theta_{i+1})) = 0, \tag{4.6}$$

by noticing that α_i is defined cyclically and effecting suitable change in summation index in the second sum and combining the two sums. The identity (4.6) is then proved to complete the proof of (4.4) and the Proposition 4.1.

The mean value coordinates are not only positive, but we can bound them away from zero [14]. For, if $L^* = \max_i ||v_i - v||$, $L_* = \min_i ||v_i - v||$, $\alpha^* = \max_i \alpha_i, \alpha_* = \min_i \alpha_i$, then from (4.4) we have

$$2\tan(\alpha_*/2)/L^* \leq w_i \leq 2\tan(\alpha^*/2)/L_*$$

and

$$\frac{1}{nC} \leq \lambda_i \leq \frac{C}{n}, \tag{4.7}$$

where

$$C = L^* \tan(\alpha^*/2)/L_*, \tan(\alpha_*/2) \geq 1.$$

If $v_1, \ldots, v_n$ is a regular polygon and v is its centre then $C = 1$ and we have $\lambda_i = \frac{1}{n}$, for all i.

The foregoing weights can be derived from an application of the mean value theorem for harmonic functions, which suggests calling them mean value coordinates. They obviously depend smoothly on the points $v, v_1, \ldots, v_n$.

Constructing a function that interpolates a set of values defined at vertices of a mesh is a fundamental operation in computer graphics. Such an interpolant has many uses in applications such as shading, parameterization and deformation. Wachspress and mean value coordinates both support construction of this type of interpolants and they have many other interesting properties, some of which are given below.

- When $n = 3$ both the mean value coordinates and Wachspress coordinates, are equal to the three barycentric coordinates.
- When $n = 4$ the mean value coordinates and Wachspress coordinates are different in general. For example, when the points v_1, v_2, v_3, v_4 form a rectangle, the Wachspress coordinates are bilinear, while the mean value coordinates are not.
- Both Wachspress and mean value coordinates are linear along the edges and have the Lagrange property: $\lambda_i(v_j) = \delta_{ij}$ at the vertices, that simplifies computation involving interpolation at vertices and the interpolants can be computed directly and efficiently without solving a linear system.
- Wachspress coordinates are affine invariant, that is, they are unchanged when any affine transformation $T = \mathbb{R}^2 \to \mathbb{R}^2$ is applied simultaneously to the points $v_1, v_2, \ldots, v_n$ and v. Specifically, if we express λ_i as $\lambda_i(v, v_1, v_2, \ldots, v_n)$ to indicate the dependency on the vertices, we have

$$\lambda_i(Tv, Tv_1, Tv_2, \ldots, Tv_n) = \lambda_i(v, v_1, v_2, \ldots, v_n). \tag{4.8}$$

  Comparing with other coordinates, Wachspress coordinates are special in the sense that they have full affine invariance, that is, they are in addition invariant to non-uniform scalings (*e.g.* a scaling in just the x coordinate).
- Though the mean value coordinates are not affine invariant, they have the invariance property (4.8) whenever $T = \mathbb{R}^2 \to \mathbb{R}^2$ is a similarity, that is, a combination of translation, rotation, or uniform scaling.
- The mean value coordinates can be used in the non-convex setting and even more generally for sets of arbitrary planar polygons without self-intersections, whereas the Wachspress coordinates are defined only for convex polygons.
- Wachspress and the mean value coordinates are positive over the interior of any convex polygon. They are the only positive coordinates with uniform scaling invariance that can be computed with a local three-point-formula.
- The Wachspress coordinateas are C^∞ smooth.
- The mean value coordinates are C^∞ smooth except at the vertices of the polygon where they are only C^0-continuous.
- Wachspress coordinates (4.2) are defined for all points v inside the convex polygon and its boundary because by multiplying by the product of all

areas$A(v_1, v_{i+1}, v)$, they have the well-known equivalent expression

$$\lambda_i = \frac{w_i}{\sum\limits_{j=1}^{n} w_j}, w_i = A(v_{i-1}, v_i, v_{i+1}) \prod_{j \neq i-1,i} A(v, v_j, v_{j+1}).$$

- If the points $v_1, \dots, v_n$ form a convex polygon, the mean value coordinates are defined for all points v inside the polygon, but due to the use of the angles α_i in formula (4.4), it is not obvious whether the coordinates can be extended to the polygon itself.

It was proved later that the mean value coordinates can also be continuously extended to the polygon itself. We will discuss this issue in the next section.

In Ju et al. [18] mean value coordinates have been generalized from closed $2D$ polygons to closed triangular meshes. Given such a mesh Δ, it has been shown in [18] that these coordinates are continuous everywhere and smooth on the interior of Δ. They are linear on the triangles of Δ and can reproduce linear functions on the interior of Δ. Several interesting applications including constructing volumetric textures and surface deformation have also been presented in [18].

5. INJECTIVITY OF WACHSPRESS AND MEAN VALUE MAPPINGS

In recent years generalized barycentric coordinates have been used to deform shapes, when modelling and processing geometry: shapes defined either by curves or surfaces. Most of these methods focus on Wachspress and mean value coordinates. In this section we shall examine the issue of injectivity of Wachspress and mean value mappings between convex polygons.

In this section, we shall for convenience use vector representations for points and functions. Let $P, Q \subset \mathbb{R}^2$ be strictly convex polygons with vertices $\mathbf{p}_1, \mathbf{p}_2, \dots,$ $\mathbf{p}_n$, and $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$, respectively, ordered anticlockwise, with $n \geq 3$. By strictly convex we mean that no three vertices in the polygon are collinear. We view both P, Q as open sets in $\mathbb{R}^2$ and $\bar{P}, \bar{Q}$ denote their closures, whereas ∂P and ∂Q denote their boundaries. We now use the barycentric coordinates to define a smooth mapping $\mathbf{f} : \bar{P} \to \bar{Q}$. Consider a sequence of nonnegative functions: $\lambda_1, \dots, \lambda_n : \bar{P} \to \mathbb{R}$, such that (see Floater et al. [16])

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x}) = 1, \quad \mathbf{x} \in \bar{P}, \tag{5.1}$$

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x}) \mathbf{p}_i = \mathbf{x}, \quad \mathbf{x} \in \bar{P}. \tag{5.2}$$

Using (5.1)-(5.2), it can be shown (see [15]) that if

$$\mathbf{x} = (1 - \mu)\mathbf{p}_\ell + \mu \mathbf{p}_{\ell+1}, \tag{5.3}$$

for some ℓ and some $\mu \in [0,1]$, with indexes treated cyclically, then

$$\lambda_\ell(\mathbf{x}) = 1-\mu,\ \lambda_{\ell+1}(\mathbf{x}) = \mu,\ \text{and } \lambda_i(\mathbf{x}) = 0,\ \text{for } i \neq \ell, \ell+1. \tag{5.4}$$

We define the barycentric mapping $\mathbf{f} : \bar{P} \to \mathbb{R}^2$,

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i(\mathbf{x})\mathbf{q}_i, \quad \mathbf{x} \in \bar{P}. \tag{5.5}$$

Then (5.1) implies that $\mathbf{f}(\mathbf{x})$ is a convex combination of the points $\mathbf{q}_i$ and so $\mathbf{f}(\bar{P}) \subset \bar{Q}$. Further, by (5.4), if $\mathbf{x}$ is the boundary point as in (5.3) then

$$\mathbf{f}(\mathbf{x}) = (1-\mu)\mathbf{q}_\ell + \mu\mathbf{q}_{\ell+1}.$$

Thus, $\mathbf{f}$ maps ∂P to ∂Q in a piecewise linear fashion, mapping vertices and edges of ∂P to corresponding vertices and edges of ∂Q. If $\mathbf{f}(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$ and ∂_1 and ∂_2 respectively, denote $\partial h(\mathbf{x})/\partial x$ and $\partial h(\mathbf{x})/\partial x^2$, then $J(\mathbf{f})$, the Jacobian of $\mathbf{f}$ is given by the determinant,

$$J(\mathbf{f}) = \begin{vmatrix} \partial_1 f & \partial_1 fg \\ \partial_2 f & \partial_2 g \end{vmatrix}.$$

Writing,

$$\mathcal{D}(a,b,c) = \begin{vmatrix} a & b & c \\ \partial_1 a & \partial_1 b & \partial_1 c \\ \partial_2 a & \partial_2 b & \partial_2 c \end{vmatrix}, \tag{5.6}$$

we have the following due to Floater et al. ([16, Lemma 1]).

Lemma 5.1. *For any set of differentiable functions $\lambda_1, \ldots, \lambda_n$ satisfying (5.1), the Jacobian*

$$J(\mathbf{f})(\mathbf{x}) = 2 \sum_{1 \le i < j < k \le n} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k), \tag{5.7}$$

where $\mathbf{A}(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)$ is the signed area of the triangle $[\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k]$.

We observe that on differentiation of (5.1) and (5.5), we get the matrix identity,

$$\begin{pmatrix} 1 & f & g \\ 0 & \partial_1 f & \partial_1 g \\ 0 & \partial_2 f & \partial_2 g \end{pmatrix} = \begin{pmatrix} \lambda_1 & \ldots & \lambda_n \\ \partial_1 \lambda_1 & \ldots & \partial_1 \lambda_n \\ \partial_2 \lambda_1 & \ldots & \partial_2 \lambda_n \end{pmatrix} = \begin{pmatrix} 1 & q_1^1 & q_1^2 \\ \vdots & \vdots & \vdots \\ 1 & q_n^1 & q_n^2 \end{pmatrix}$$

which on application of the Cauchy-Binet theorem (see Ando [3], formula (1.23)) proves (5.7), since the determinant of the matrix on the left equals $J(\mathbf{f})$

Using strictly convex property of Q and Lemma 5.1 we obtain the following sufficient condition for injectivity.

Theorem 5.1. *If $\lambda_1, \ldots, \lambda_n$ are differentiable barycentric coordinates such that for all $\mathbf{x} \in \bar{P}, \mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) \ge 0$ for all i, j, k with $1 \le i < j < k \le n$ and $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) > 0$ for some i, j, k with $1 \le i < j < k \le n$ then $\mathbf{f}$ is injective.*

The following theorem gives necessary condition for injectivity [16].

Theorem 5.2. *If If $\lambda_1, \ldots, \lambda_n$ are differentiable barycentric coordinates and* $\mathbf{f}$ *is injective then for all r, s, t with $1 \le r < s < t \le n$*

$$\sum_{1 \le i < s \le j < t \le k < n+r} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) \ge 0 \ \textit{in } P. \tag{5.8}$$

In order to prove Theorem 5.2, consider the polygon Q with $\mathbf{q}_i = \cdots = \mathbf{q}_{r-1} = (0,0)$, $\mathbf{q}_r = \cdots = \mathbf{q}_{s-1} = (1,0)$, $\mathbf{q}_s = \cdots = \mathbf{q}_{t-1} = (0,1)$, and $\mathbf{q}_t = \cdots = \mathbf{q}_n = (0,0)$. Clearly, then $1 \le i < j < r$ implies that $\mathbf{q}_i = \mathbf{q}_j = (0,0)$ which in its turn implies that $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) = 0$ for any $k \ge r$ Similarly, $r \le i < j < s$ implies that $\mathbf{q}_i = \mathbf{q}_j = (1,0)$ and $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) = 0$ for any $k \ge s < j < i \ge r$ or $1 \le k < s$ Analyzing this way, we see that $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) = 0$ unless

$$1 \le i < r \le j < s \le k < t \ \text{ or } \ r \le i < s \le j < t \le k \le n. \tag{5.9}$$

Also, we see that under the condition (5.9), $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)$ equals half of the area of the unit square, so that $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) = 1/2$. Since $\mathbf{f}$ is injective $J(\mathbf{f}) \ge 0$ therefore on substituting the foregoing values into (5.7), we see that

$$\sum_{1 \le i < r \le j < s \le k < t} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) + \sum_{r \le i < s \le j < t \le k \le n} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) \ge 0. \tag{5.10}$$

Replacing i, j, k by k, i, j in the first sum and using the fact that $\mathcal{D}(\lambda_k, \lambda_i, \lambda_j) = \mathcal{D}(\lambda_i, \lambda_j, \lambda_k)$, the first sum in (5.10) can be written as

$$\sum_{1 \le k < r \le i < s \le j < t} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) = \sum_{\substack{r \le i < s \le j < t \\ n+1 \le k < n+r}} \mathcal{D}(\lambda_i, \lambda_j, \lambda_{k-n}). \tag{5.11}$$

Finally replacing λ_{k-n} by λ_k, combining the expression for the above on the right of (5.11) with the second sum in (5.10), we get (5.8).

We have the following important consequence of Theorem 5.1 due to Floater et al. [16].

Theorem 5.3. *If $\lambda_1, \ldots, \lambda_n$ are Wachspress coordinates, then* $\mathbf{f}$ *is injective.*

Considering $\mathbf{x} \in P$ and $\mathbf{x} \in \partial P$, separately the above Theorem 5.3 is proved by an application of Theorem 5.1.

We now sketch the proof of Theorem 5.3, when $\mathbf{x} \in P$ but $\mathbf{x} \notin \partial P$. Since $\mathbf{x} \notin \partial P$ we can define $\lambda_i(\mathbf{x})$ in (5.2), using the rational weight function w_i of (4.2), *i.e.*, $\lambda_i = \mu w_i$, where $1/\mu = \sum_j w_j > 0$. Choosing i, j, k such that $1 \le i < j < k \le n$, it is required to be shown that

$$\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) = \mathcal{D}(\mu w_i, \mu w_j, \mu w_k)(\mathbf{x}) > 0. \tag{5.12}$$

We now first expand the second and third rows of the determinant $\mathcal{D}(\mu w_i, \mu w_j, \mu w_k)$ $(\mathbf{x})$ by using the product rule. Then, subtract $\partial_1 \mu / \mu$ times the first row from the second row and $\partial_1\mu/\mu$ times the first row from the third row, we see that (5.12) is equivalent to

$$\mathcal{D}(\mu w_i, \mu w_j, \mu w_k)(\mathbf{x}) = \mu^3 \mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) > 0,$$

or equivalently that $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) > 0$,, since $\mu > 0$. For detailed proof of this and the proof for the case when $\mathbf{x} \in \partial P$ we refer to Floater et al. [16].

On the basis of numerous numerical tests, it is conjectured in Floater et al. [16] that mean value mappings are always injective when $n = 4$. However, for $n = 5$ we have the following counterexample. Let us suppose that P is given by

$$\mathbf{p}_1 = (0,0); \mathbf{p}_2 = (0.7, 0.1), \mathbf{p}_3 = (2, 0.5), \mathbf{p}_4 = (0.7, 0.9), \mathbf{p}_5 = (0, 1).$$

Moreover, let $\mathbf{x} = (1.2, 0.5) \in P$. By evaluating $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x})$ for the mean value coordinates of P one obtains that

$$\begin{aligned}\mathcal{D}(\lambda_1, \lambda_2, \lambda_5)(\mathbf{x}) &= -0.00011838,\\ \mathcal{D}(\lambda_1, \lambda_3, \lambda_5)(\mathbf{x}) &= -0.00040841,\\ \mathcal{D}(\lambda_1, \lambda_4, \lambda_5)(\mathbf{x}) &= -0.00011838.\end{aligned}$$

Therefore,

$$\mathcal{D}(\lambda_1, \lambda_2, \lambda_5)(\mathbf{x}) + \mathcal{D}(\lambda_1, \lambda_3, \lambda_5)(\mathbf{x}) + \mathcal{D}(\lambda_1, \lambda_4, \lambda_5)(\mathbf{x}) = -0.00064517.$$

As this value is negative, it violates the condition (5.8) of Theorem 5.2 for $n = 5, r = 1, s = 2$ and $t = 5$. Consequently, by choosing Q to be, for example, given by

$$\mathbf{q}_1 = (2.5, 0),\ \mathbf{q}_2 = (3.4999, 0.4999),\ \mathbf{q}_3 = (3.5, 0.5),$$
$$\mathbf{q}_4 = (3.4999, 0.5001), \mathbf{q}_5 = (2.5, 1),$$

we arrive at an example of a mean value mapping over strictly convex polygons where the Jacobian is negative at some interior points. Indeed, $J(\mathbf{f})(\mathbf{q}) = -0.0006069$.

A second counterexample (see [16]) to show that mean value coordinates are not injective, is provided by choosing the vertices of P as

$$\mathbf{p}_1 = (0,0),\ \mathbf{p}_2 = (0.7, 0),\ \mathbf{p}_3 = (10, 0.5),\ \mathbf{p}_4 = (0.7, 1),\ \mathbf{p}_5 = (0, 1),$$

and that of Q as

$$\mathbf{q}_1 = (13, 0), \mathbf{q}_2 = (14.999, 4.99), \mathbf{q}_3 = (15, 5), \mathbf{q}_4 = (14.999, 5.01), \mathbf{q}_5 = (13, 10).$$

6. GENERALIZED BARYCENTRIC COORDINATES

In this section we will present generalized barycentric coordinates for convex polygons (see Floater et al. [15], Hormann et al. [17]). Let Ω be a convex polygon in the plane, regarded as a closed set, with vertices $v_1, v_2, \ldots, v_n, n \geq 3$, in an anticlockwise ordering. We will call any set of functions $\lambda_i : \Omega \to \mathbb{R},\ i = 1, \ldots, n$, barycentric coordinates if they satisfy, for all $v \in \Omega$ the three properties:

$$\lambda_i(v) \geq 0, \qquad i = 1, 2, \ldots, n. \tag{6.1}$$

$$\sum_{i=1}^{n} \lambda_i(v) = 1. \tag{6.2}$$

$$\sum_{i=1}^{n} \lambda_i(v) v_i = v. \tag{6.3}$$

The mean value coordinates are well defined for all $v \in \operatorname{Int}\Omega$ but, as we will see, these coordinates can be extended in a natural way to the boundary $\partial\Omega$ as well [15].

We now begin by deriving sharp upper and lower bounds on each coordinate function λ_i and use them to show that all barycentric coordinates which are continuous in the interior $\operatorname{Int}\Omega$, such as the mean value coordinates, extend continuously to the boundary $\partial\Omega$.

For each $i \in \{1, \ldots, n\}$, let $L_i : \Omega \to \mathbb{R}$ be the continuous piecewise linear function, which is linear on each triangle of the form $[v_j, v_{j+1}, v_i]$, $j \neq i-1, i$, and has the values $L_i(v_j) = \delta_{i,j}$ at the vertices of Ω.

Conversely, let $\ell_i : \Omega \to \mathbb{R}$ be the function which is linear on the triangle $[v_{i-1}, v_i, v_{i+1}]$ and zero on $\Omega \backslash [v_{i-1}, v_i, v_{i+1}]$ with $\ell_i(v_j) = \delta_{i,j}$ at the vertices. Then we have the following due to Floater et al. [15].

Proposition 6.1. *Let D be any subset of Ω and suppose that the functions $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$ satisfy the three defining properties of barycentric coordinates, (6.1), (6.2) and (6.3), for $v \in D$. Then, for $i = 1, \ldots, n$,*

$$0 \leq \ell_i(v) \leq \lambda_i(v) \leq L_i(v) \leq 1, \qquad v \in D. \tag{6.4}$$

We sketch the proof of the above for completeness [15]. We observe that the properties (6.2) and (6.3) ensure linear precision property, that is, for any linear polynomial $p(v)$,

$$p(v) = \sum_{k=1}^{n} \lambda_k(v) p(v_k). \tag{6.5}$$

We now notice that any point $v \in D \subset \Omega$, belongs to at least one triangle $[v_i, v_j, v_{j+1}]$ $j \neq i-1, i$ and $A(v_i, v_j, v_{j+1})$ is linear in v so by an application of (6.5), we have

$$A(v, v_j, v_{j+1}) = \sum_{k=1}^{n} \lambda_k(v) A(v_k, v_j, v_{j+1}) \geq \lambda_i(v) A(v_i, v_j, v_{j+1}), \tag{6.6}$$

the latter inequality is due to (6.1). We also see directly that over the triangle $[v_i, v_j, v_{j+1}]$, the linear polynomial: $A(v, v_j, v_{j+1})/A(v_i, v_j, v_{j+1})$, satisfies the definition of $L_i(v)$ and therefore it equals $L_i(v)$. Therefore, from (6.6),

$$\lambda_i(v) \leq A(v, v_j, v_{j+1})/A(v_i, v_j, v_{j+1}) = L_i(v).$$

The proof of the other inequality proceeds on somewhat similar lines when we observe that the linear polynomial:

$$A(v, v_{i+1}, v_{i-1})/A(v_i, v_{i+1}, v_{i-1}) = \ell_i(v).$$

Letting $D = \Omega$ and noting that the linear polynomials L_i and ℓ_i agree on the boundary $\partial\Omega$, we have the following consequence of the above Proposition.

Corollary 6.1. *If $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$ is any set of barycentric coordinates then, for $i = 1, \ldots, n$,*

$$\ell_i(v) \leq \lambda_i(v) \leq L_i(v), \qquad v \in \Omega, \tag{6.7}$$

and

$$\ell_i(v) = \lambda_i(v) = L_i(v), \qquad v \in \partial\Omega. \tag{6.8}$$

Therefore, λ_i is linear on each edge $[v_i, v_{i+1}]$ and satisfies $\lambda_i(v_j) = \delta_{i,j}$.

A further consequence is that it is sufficient to define barycentric coordinates only in the interior of Ω.

Corollary 6.2. *Suppose, $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$ are continuous and satisfy the three properties (6.1), (6.2) and (6.3) on* Int (Ω)*, then they are continuous barycentric coordinates in the sense that they have a unique continuous extension to $\partial\Omega$ and therefore satisfy (6.1), (6.2) and (6.3) on the whole of Ω.*

Proof of Corollary 6.2. As in [15], if w is any point in $\partial\Omega$ and $w_1, w_2, \ldots$, is any sequence of points in Int Ω such that $\lim_{k\to\infty} w_k = w$, then by (6.7)

$$\ell_i(w_k) \leq \lambda_i(w_k) \leq L_i(w_k),$$

and

$$\lim_{k\to\infty} \ell_i(w_k) = \lim_{k\to\infty} L_i(w_k) = \ell_i(w).$$

Therefore, we have

$$\lim_{k\to\infty} \lambda_i(w_k) = \ell_i(w) = L_i(w).$$

Thus, if we construct continuous functions $\lambda_1, \ldots, \lambda_n$ satisfying the three basic properties (6.1), (6.2) and (6.3) in Int Ω, as in the case of the mean value coordinates, we are justified in viewing them as continuous barycentric coordinates on the whole of Ω.

We can show that the functions $L_1, L_2, \ldots, L_n$ and $\ell_1, \ell_2, \ldots, \ell_n$, themselves do not form sets of barycentric coordinates, for $n > 3$. For example with $n = 4$ - a square, and v its centre, we have $L_i(v) = 1/2$ and $\ell_i(v) = 0$ for all $i = 1, 2, 3, 4$ and so

$$\sum_{i=1}^{4} L_i(v) = 1/2 \text{ and } \sum_{i=1}^{4} \ell_i(v) = 0 < 1,$$

which violate condition (6.2) and therefore $L_1, L_2, \ldots, L_n$ and $\ell_1, \ell_2, \ldots, \ell_n$ do not define barycentric coordinates.

In order to highlight the special significance of functions L_i and ℓ_i, we call any function $\lambda_i : \Omega \to \mathbb{R}$ an i-th barycentric coordinate if it is the i-th member of some set of barycentric coordinates $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$.

It follows that L_i and ℓ_i are the least upper bound and the greatest lower bound on λ_i in (6.7). It can be shown ([15, Proposition 3]) that the functions L_i and ℓ_i are the i-th barycentric coordinates $\Omega \to \mathbb{R}$ and that a function $\lambda_i : \Omega \to \mathbb{R}$ is an i-th barycentric coordinate if and only if,

$$\ell_i(v) \le \lambda_i(v) \le L_i(v), \qquad v \in \Omega. \tag{6.9}$$

This leads to another interesting property of barycentric coordinates. Since the function ℓ_i is clearly convex as well as satisfying $\ell_i(v_j) = \delta_{ij}$ a result of Dahmen and Micchelli [6] shows that ℓ_i is the point wise maximum of all convex functions f on Ω which satisfy $f(v_j) = \delta_{ij}$. Conversely, ℓ_i is concave and the point wise minimum of all concave functions f on Ω such that $f(v_j) = \delta_{ij}$. Therefore, we have the following [15].

Proposition 6.2. *Suppose, $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$ are barycentric coordinates. For each i if λ_i is convex then $\lambda_i = \ell_i$ and if λ_i is concave then $\lambda_i = L_i$. So if λ_i is differentiable in* Int (Ω) *then it is neither convex nor concave.*

Generalized barycentric coordinates have been extensively employed in many applications. Due to shortage of space we just indicate one such area of recent interest s mesh merging which is to construct a new mesh based on a mesh (identified target), while the new one is required to own the features of the other mesh (identified target) too. That is, the source is cloned to the target. Very recently in June 2013, appeared an important work by Deng, Z. et al. [7], who introduced a new model merging method with mean value coordinates. This method not only merges meshes, but also merges the textures. It firstly computes the mean value coordinates of the sources' vertices with respect to its boundary. After the target is specified, a function is constructed to present the vertex position differences between the boundaries of the source and the target. While the source boundaries can be merged through the boundary differences directly, the differences of the inside vertices are interpolated based on the boundary differences through mean value coordinates. With the differences, the source is merged with the target to construct a new mesh. In order to smoothen the merging boundary, the differences of the tangent vectors on the boundary are interpolated through mean value coordinates too like the position differences. The new mesh is textured based on the two original textures, while the source texture is adjusted by the colour differences, which are obtained by interpolating the boundaries colour differences through mean value coordinates. This method does not need to parameterize meshes, or to solve a large linear system. It merges the models only through computing close-form expressions based on the mean value coordinates and some kinds of differences.

References

[1] Ahlberg, J. H., Nilson, E. N. and Walsh, J. L., *Complex cubic splines,* Trans. Amer. Math. Soc., 129 (1967) 391–413.

[2] Ahlfors, L. V., *Lectures on Quasiconformal Mappings,* Van Nostrand Mathematical Studies, 10 (1966).

[3] Ando ,T., *Totally positive matrices,* Linear Algebra Appl., 90 (1987) 165 -219.

[4] Ceva, G.,De lineis, *De lineis rectis se invicem secantibus,* statica constructio. Ludovici Montiae, Mediolanum 1678.

[5] Dahmen, W., Dikshit, H.P. and Ojha, A., *Wachspress functions over convex quadrilaterals,* Computer Aided Geometric Design, 17 (2000) 879–890.

[6] Dahmen, W. and Micchelli, C., *Convexity of multivariate Bernstein polynomials and box spline surfaces.* Studia Scientiarum Mathematicarum Hungarica, 23 (1988) 265–287.

[7] Deng, Z., Chen, G. Wang, F., Zhou, F. and Liu, X., *Model Merging Through Mean Value Coordinates,* Journal of Information & Computational Science, 10 (2013) 2551-2560.

[8] Dikshit, H.P., *Geometric modelling and shape preserving properties of Wachspress functions,* Dr. Zakir Hussain Award Lecture, in Proceedings of an International Conference on Modelling and Engineering and Technological Problems, A.H.Siddiqi, A.K.Gupta and M. Brokate (eds.), 14–16,January 2009, American Institute of Physics Conference Proceedings, 1146, Melville, New York, 2009, pp.18–38.

[9] Dikshit, H. P. and Ojha, A., *On convergence and quasiconformality of complex planar spline interpolants,* Math. Proc. Cambridge Phil. Soc., 99 (1986) 347–356.

[10] Dikshit, H. P. and Ojha, A., *On C^1 continuity of Wachspress quadrilateral patches, for computer aided geometric design,* Computer Aided Geometric Design, 19 (2002) 207–222.

[11] Dikshit, H. P. and Ojha, A., *Subdivision formula for Wachspress quadrilateral patches,* Computer Aided Geometric Design, 20 (2003) 395–399.

[12] Dikshit, H. P., Ojha, A. and Sharma, A., *Certain mapping properties of rational complex planar splines,* Math. Proc. Cambridge Phil. Soc., 101 (1987) 141–149.

[13] Dikshit, H. P., Ojha, A. and Zalik, R. A., *Wachspress type rational complex planar splines of degree (3,1),* Adv. Comput. Math., 2 (1994) 235–249.

[14] Floater, M. S., *Mean value coordinates,* Computer Aided Geometric Design, 20 (2003) 19–27.

[15] Floater, M. S., Hormann, K. and Kós, G, *A general construction of barycentric coordinates over convex polygons,* Adv. Comput. Math., 24 (2006) 311–331.

[16] Floater, M. S. and Kosinka, J., *On the injectivity of Wachspress and mean value mappings between convex polygons,* Adv. Comput. Math., 32 (2010) 163–174.

[17] Horman, K. and Floater, M. S., *Mean value coordinates for arbitrary planar polygons,* ACM Trans. Graphics, 25 (2006) 1424–1441.

[18] Ju,T., Schaefer, S. and Warren, J., *Mean value coordinates for closed triangular meshes,* ACM Trans. on Graphics, 24 (2005) 561-566.

[19] Meyer, M., Barr, A., Lee, H. and Desbrun, M., *Generalized barycentric coordinates on irregular polygons,* J. Graphics Tools, 7 (2002) 13-22.

[20] Möbius, A. F., *Der barycentrische Calcul.* Johann Ambrosius Barth, Leipzig, 1827.

[21] Opfer, G. and Puri, M.L., *Complex planar splines,* J. Approx. Theory, 31 (1981) 383–402.

[22] Opfer, G. and Schober, G., *On convergence and quasiregularity of interpolating complex planar splines,* Math. Z., 180 (1982) 469–481.

[23] Wachspress, E. L., *A Rational Finite Element Basis,* Academic Press, New York, 1975.

[24] Warren, J., *Barycentric coordinates for convex polytopes,* Adv. in Comput. Math., 6 (1996) 97-108.

[25] Warren, J., *On the uniqueness of barycentric coordinates,* In Topics in Algebraic Geometry and Geometric Modeling, Goldman, R. and Krasauskas, R. Eds. Contemporary Mathematics, 334 (2003) 93-99, American Mathematical Society.

[26] Warren, J., Schaefer, S., Hirani, A. and Desbrun, M., *Barycentric coordinates for convex sets.* Adv. ComputMath., **27** (2007) 319–338.

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TRIANGULATIONS OF PROJECTIVE SPACES

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ABSTRACT. Projective spaces over real numbers and complex numbers are topological manifolds and interesting objects from algebraic and geometric point of view. Here, we discuss triangulations of these spaces.

1. PRELIMINARIES: MANIFOLDS

Euclidean Spaces: The space $\{(x_1, \dots, x_m) : x_i$ is a real number, for $1 \leq i \leq m\}$ with usual topology (*i.e.*, U is *open* if and only if U can be expressed as the union of open balls) is called the m-dimensional Euclidean space and is denoted by $\mathbb{R}^m$.

Complex n-Space: Similarly, $\mathbb{C}^n$ denotes the space $\{(z_1, \dots, z_n) : z_j$ is a complex number, for $1 \leq j \leq n\}$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the identification $(x_1 + y_1 i, \dots, x_n + y_n i) \equiv (x_1, y_1, \dots, x_n, y_n)$. This gives the standard topology on $\mathbb{C}^n$.

Topological Manifolds: A Hausdorff topological space X is called an n-dimensional topological manifold (in short, n-manifold) if for each $x \in X$, there exists an open set U in X such that $x \in U$ and U is homeomorphic to an open subset V of $\mathbb{R}^n$.

Thus, a Hausdorff topological space X is an n-manifold if and only if X can be express as the union $X = \cup_{\alpha \in A} U_\alpha$, where each U_α is an open subset of X and is homeomorphic to an open subset V_α of $\mathbb{R}^n$.

Examples of Manifolds:

(1) The Euclidean space $\mathbb{R}^n$ is an n-manifold.
(2) The space $\mathbb{C}^n$ is a $2n$-manifold.
(3) If X is a discrete topological space then X is 0-dimensional manifold.
(4) If M is an n-manifold and N is an open subset of M then N is an n-manifold.

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(5) If X is an m-manifold and Y homeomorphic to X then Y is also an m-manifold.
(6) If X is an m-manifold and Y is an n-manifold then $X \times Y$ is an $(m+n)$-manifold.
(7) If X and Y are two disjoint m-manifolds then the disjoint union $X \sqcup Y$ is also an m-manifold.
(8) The space $M_{m\times n}(\mathbb{R})$ of $m \times n$ real matrices can be identified with $\mathbb{R}^{mn}$. Hence $M_{m\times n}(\mathbb{R})$ is an mn-dimensional manifold.
(9) The space $GL_n(\mathbb{R})$ of $n \times n$ non-singular matrices is an open subset of $M_{n\times n}(\mathbb{R})$ and hence $GL_n(\mathbb{R})$ is an n^2-manifold.
(10) At a given time, our universe is a 3-dimensional manifold!

Unit Spheres: For $n \geq 0$, S^n denotes the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. So, $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$.

With the identification $\mathbb{R}^2 \equiv \mathbb{C}, S^1$ can be identified with the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

For $1 \leq j \leq n+1$, let $D^n_{j,+} = \{(x_1, \dots, x_{n+1}) \in S^n : x_j > 0\}$ and $D^n_{j,-} = \{(x_1, \dots, x_{n+1}) \in S^n : x_j < 0\}$ be the open hemispheres. Then $D^n_{j,\pm}$ is homeomorphic to the open set (n-ball) $D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\}$ via the map $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1})$. Since $S^n = (\cup_{j=1}^{n+1} D^n_{j,+}) \cup (\cup_{j=1}^{n+1} D^n_{j,-})$, it follows that S^n is an n-manifold.

Real Projective Spaces: For $n \geq 1$, the real projective n-space $\mathbb{RP}^n$ is defined as the set of all 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. We know that two non-zero vectors α and β define the same 1-dimensional subspace if and only if $\beta = a\alpha$ for some $a \in \mathbb{R} \setminus \{0\}$. Therefore, $\mathbb{RP}^n$ can be identified as the quotient space of $\mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\}$ defined by the relation: $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$ if $(x_1, \dots, x_{n+1}) = a(y_1, \dots, y_{n+1})$ for some $a \in \mathbb{R} \setminus \{0\}$. The topology on $\mathbb{RP}^n$ is the quotient topology.

The $\sim$ - class containing $(x_1, \dots, x_{n+1})$ is denoted by $[x_1 : \cdots : x_{n+1}]$. Thus, the points of $\mathbb{RP}^n$ are represented by homogeneous real coordinates $[x_1 : \cdots : x_{n+1}]$ with the condition that at least one $x_i \neq 0$ and $[x_1 : \cdots : x_{n+1}] = [ax_1 : \cdots : ax_{n+1}]$ for all $a \neq 0$. Let $\pi \colon \mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\} \to \mathbb{RP}^n$ be the quotient map $(x_1, \dots, x_{n+1}) \mapsto [x_1 : \cdots : x_{n+1}]$.

Since each 1-dimensional subspace of $\mathbb{R}^{n+1}$ intersects the unit sphere S^n in two points, $\mathbb{RP}^n$ is also obtained from S^n by identifying x with $-x$.

Equivalently, $\mathbb{RP}^n$ is homeomorphic to the orbit space $S^n/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2 = \{1, -1\}$ on S^n is given by: $(-1)(x_1, \dots, x_{n+1}) = (-x_1, \dots, -x_{n+1})$. Let η be the quotient map $\pi|_{S^n} \colon S^n \to \mathbb{RP}^n$. For $1 \leq j \leq n+1$, let $U_j = \{[x_1 : \cdots : x_{n+1}] \in \mathbb{RP}^n : x_j \neq 0\}$. Then $\varphi_j \colon U_j \to \mathbb{R}^n$ given by $\varphi_j([x_1 : \cdots : x_{n+1}]) \mapsto (\frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{n+1}}{x_j})$ is a homeomorphism (with $\varphi_j^{-1}((y_1, \dots, y_n)) = [y_1 :$

$\cdots : y_{j-1} : 1 : y_j : \cdots : y_n]$).

Since $\mathbb{RP}^n = \cup_{j=1}^{n+1} U_j$, it follows that $\mathbb{RP}^n$ is an n-manifold.

Complex Projective Spaces: For $n \geq 1$, the complex projective n-space $\mathbb{CP}^n$ is defined as the 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$ and hence can be identified as the quotient space of $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ defined by the relation: $(z_1, \dots, z_{n+1}) \sim (w_1, \dots, w_{n+1})$ if $(z_1, \dots, z_{n+1}) = a(w_1, \dots, w_{n+1})$ for some $a \in \mathbb{C} \setminus \{0\}$.

The $\sim$-class containing $(z_1, \dots, z_{n+1})$ is denoted by $[z_1 : \cdots : z_{n+1}]$.

Consider the unit sphere $S^{2n+1} = \{(z_1, \dots, z_{2n+1}) : |z_1|^2 + \cdots + |z_{n+1}|^2 = 1\} \subseteq \mathbb{C}^{n+1}$. Clearly, $\alpha, \beta \in S^{2n+1}$ give same point of $\mathbb{CP}^n$ if and only if $\alpha = t\beta$ for some $t \in S^1 \subseteq \mathbb{C}$.

Thus, the space $\mathbb{CP}^n$ can be identified with the orbit space S^{2n+1}/S^1, where the action of $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ on S^{2n+1} is given by: $z(z_1, \dots, z_{n+1}) = (zz_1, \dots, zz_{n+1})$. Let $\eta \colon S^{2n+1} \to \mathbb{CP}^n$ be the quotient map $(z_1, \dots, z_{n+1}) \mapsto [z_1 : \cdots : z_{n+1}]$. Then, for $[z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n$, $\eta^{-1}([z_1 : \cdots : z_{n+1}]) = \{z(z_1, \dots, z_{n+1}) : z \in S^1\}$.

For $1 \leq j \leq n+1$, let $V_j = \{[z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n : z_j \neq 0\}$. Then $\psi_j \colon V_j \to \mathbb{C}^n$ given by $\psi_j([z_1 : \cdots : z_{n+1}]) \mapsto (\frac{z_1}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_{n+1}}{z_j})$ is a homeomorphism.

Since $\mathbb{CP}^n = \cup_{j=1}^{n+1} V_j$, it follows that $\mathbb{CP}^n$ is a $2n$-manifold.

Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the one point compactification of $\mathbb{C}$. Then $\widehat{\mathbb{C}}$ is homeomorphic to the unit 2-sphere S^2 and the map $\mathbb{CP}^1 \to \widehat{\mathbb{C}}$, given by $[z_1 : z_2] \mapsto z_1/z_2$ if $z_2 \neq 0$ and $[z_1 : 0] \mapsto \infty$, is a homeomorphism. Thus, $\mathbb{CP}^1$ is homeomorphic to the unit 2-sphere S^2. If we identify $\mathbb{CP}^1$ with S^2 (via these homeomorphisms) then we get a natural map $\eta \colon S^3 \to S^2$. This map is called the *Hopf map*.

2. Preliminaries: Simplicial Complexes

Convex Sets: A subset $C \subseteq \mathbb{R}^m$ is called convex if for each pair of points $a, b \in C$, the line segment $\{sa + tb : s, t \in [0, 1]$ and $s + t = 1\}$ is in C.

For a set $A \subseteq \mathbb{R}^m$, let $\langle A \rangle := \{\sum_{i=1}^r t_i a_i : a_1, \dots, a_r \in A, t_1, \dots, t_r \in [0, 1]$ with $\sum_{i=1}^r t_i = 1$, r positive integer$\}$. Then $\langle A \rangle$ is the smallest convex set containing A, and is called the convex hull of A.

Simplex, Face of a Simplex: A set of points $a_1, \dots, a_k$ in $\mathbb{R}^m$ is called affinely independent if $\{a_2 - a_1, \dots, a_k - a_1\}$ is a linearly independent set.

A geometric n-simplex (or simplex of dimension n) σ in $\mathbb{R}^m$ is the affine hull of a set of $n+1$ affinely independent points.

If $\sigma = \langle \{a_0, \dots, a_n\} \rangle$ then σ is also denoted by $a_0 a_1 \cdots a_n$. The points $a_0, \dots, a_n$ are called the vertices of σ. If $B \subseteq \{a_0, \dots a_n\}$ then $\langle B \rangle$ is called a face of σ. The point $\hat{\sigma} := \frac{1}{n+1}(a_0 + \cdots + a_n)$ is called the barycenter of σ. By definition, the empty set is a geometric simplex of dimension -1 and is a face of every simplex.

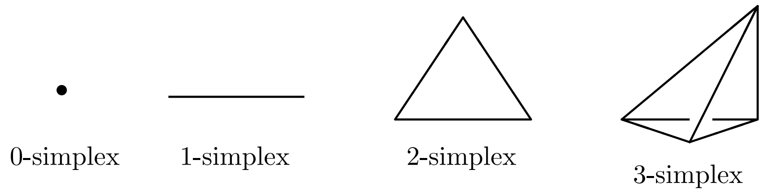

Figure 1 : Geometric Simplices

Geometric Simplicial Complex : A finite collection K of geometric simplices in some $\mathbb{R}^n$ is called a geometric simplicial complex if (i) $\alpha \in K$, β is a face of α imply $\beta \in K$ and (ii) σ, $\gamma \in K$ imply $\sigma \cap \gamma$ is a face of both σ and γ.

For $i = 0, 1$, the i-simplices in a geometric simplicial complex K are also called the vertices and edges of K, respectively.

The set of vertices is called the vertex set of K and is denoted by $V(K)$.

The set $|K| := \cup_{\sigma \in K}\sigma$ with subspace topology is called the geometric carrier of K.

For a geometric simplicial complex K, let $\widetilde{K} = \{A \subseteq V(K) : \langle A \rangle \in K\}$. Then

(i) $A \in \widetilde{K}$ and $B \subseteq A$ imply $B \in \widetilde{K}$ and
(ii) $x \in V(K)$ implies $\{x\} \in \widetilde{K}$.

We will see that we can construct back K from $\widetilde{K}$.

The collection $\widetilde{K}$ is called the abstract scheme of K.

This motivates us to consider the following.

Simplicial Complex : An (abstract) simplicial complex X is a collection of subsets of a finite set V such that

(i) $\alpha \in X$ and $\beta \subseteq \alpha$ imply $\beta \in X$ and
(ii) $x \in V$ implies $\{x\} \in X$.

The set V is called the vertex-set of X and denoted by $V(X)$.

For $i \geq 0$, a member $\alpha \in X$ with $i+1$ elements is called a simplex of dimension i (in short, an i-simplex or i-face) of X.

The empty set is a face (of each simplicial complex) of dimension -1.

The maximum of k such that X has a k-simplex is called the dimension of X.

For a simplex α in X if there is no simplex $\beta \neq \alpha$ such that $\alpha \subset \beta$ then α is said to be a maximal simplex. Clearly, a simplicial complex is uniquely determined by its maximal simplices. We sometime identify a simplicial complex with the set of maximal simplices of it.

A d-dimensional simplicial complex X is called pure if all the maximal simplices are of dimension d.

So, the abstract scheme of a geometric simplicial complex is a simplicial complex.

Isomorphisms and Automorphisms : An isomorphism between two simplicial complexes X and Y is bijection $\varphi\colon V(X) \to V(Y)$ such that $A \in X$ if and only if $\varphi(A) \in Y$. Two simplicial complexes are called isomorphic if such an isomorphism exists. We identify two isomorphic simplicial complexes.

An isomorphism from a simplicial complex X to itself is called an automorphism. All the automorphisms of X form a group (denoted by $\mathrm{Aut}(X)$) under composition.

Geometric Realization and Geometric Carrier : Let X be a simplicial complex. A geometric simplicial complex K is said to be a geometric realization of X if X is isomorphic to the abstract scheme of K. We identify a simplicial complex with a geometric realization of it.

If K and L are two geometric realizations of a simplicial complex X then it is not difficult to see that $|K|$ and $|L|$ are homeomorphic. Therefore, we can define the geometric carrier of a simplicial complex X (denoted by $|X|$) is the geometric carrier of a geometric realization of X.

Let K be a geometric realization of a simplicial complex X and $\sigma \in X$. The geometric simplex in K corresponding to σ is said to be the geometric carrier of σ and is denoted by $|\sigma|$ (or $\langle\sigma\rangle$). Thus, $|X| = \cup_{\sigma\in X}|\sigma|$.

Triangulation : If a topological space Y is homeomorphic to the geometric carrier $|X|$ of a simplicial complex X then we say X is a triangulation of Y or X triangulates Y.

Existence of Geometric Realizations : Consider the moment curve $C_m := \{(t, t^2, \dots, t^m) : t \in \mathbb{R}\}$ in $\mathbb{R}^m$.

For $i \leq m$, if A is a set of $i+1$ distinct points on C_m then $\langle A\rangle$ is a geometric i-simplex.

For a d-dimensional simplicial complex X, we take its vertices on the curve $C_{2d+1} \subseteq \mathbb{R}^{2d+1}$.

Then, for any two disjoint simplices α, β in X, $\langle\alpha\cup\beta\rangle$ is a geometric simplex whose vertex set is $\alpha\cup\beta$. Thus, $\langle\alpha\rangle\cap\langle\beta\rangle = \emptyset$.

Therefore, for any two simplices σ and τ of X, $\langle\sigma\rangle\cap\langle\tau\rangle = \langle\sigma\cap\tau\rangle$.

This implies that the collection $X_{gr} := \{\langle A\rangle : A \in X\}$ of geometric simplices in $\mathbb{R}^{2d+1}$ is a geometric simplicial complex.

Clearly, the abstract scheme of X_{gr} is X. So, X_{gr} is a geometric realization of X.

Subcomplex and Edge-Graph : If X and Y are simplicial complexes and

$Y \subseteq X$ then Y is called a subcomplex of X.

For a simplicial complex X, let $\mathrm{EG}(X)$ be the subcomplex of X consists of the vertices and the edges of X. This subcomplex is called the edge-graph of X. For $u, v \in V(X)$, let $d_X(u, v)$ denote the length of a shortest path from u to v in the edge graph of X. Then d_X is a metric on $V(X)$.

Join : If X and Y are two simplicial complexes with disjoint vertex-sets then their join $X * Y$ is the simplicial complex $\{\alpha \sqcup \beta : \alpha \in X, \beta \in Y\}$.

Quotient Complex and Proper Action : Let X be a simplicial complex and G be a subgroup of $\mathrm{Aut}(X)$. Let $\eta : V(X) \to V(X)/G$ be the natural projection. The simplicial complex $X/G := \{\eta(A) : A \in X\}$ is called the quotient complex.

Let G be a group of automorphism of X. It is easy to see that if X is a pure d-dimensional simplicial complex and $d_X(u, g(u)) \geq 2$ (*i.e.*, $ug(u)$ is a non-edge) for all $u \in V(X)$ and $1 \neq g \in G$ then X/G is also a pure d-dimensional simplicial complex. In this case, G induces an action on $|X|$.

We say that G acts properly on X if $d_X(u, g(u)) \geq 3$ for all $u \in V(X)$ and $1 \neq g \in G$.

We say that the action of G on X is good if

(i) $d_X(u, g(u)) \geq 2$ for all $u \in V(X)$ and $1 \neq g \in G$ and

(ii) $u \neq v \in V(X)$ are in a common G-orbit $\Rightarrow$ there is $g \in G$ such that $g(u) = v$ and g fixes all the vertices in $\{w \in V(X) : uw, vw \in X\}$.

Trivially, a proper action is good.

If action of G on X is good then (see [5]) X/G triangulates $|X|/G$.

Barycentric Subdivision : Let X be a simplicial complex. Consider the simplicial complex $X^{(1)}$ whose vertices are the nonempty simplices of X. A set $\{\alpha_1, \dots, \alpha_k\}$ is a simplex of $X^{(1)}$ if $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_k$.

Let K be a geometric realization of X. For each nonempty simplex $\alpha \in X$, choose a point $\hat{\alpha}$ in the interior of $|\alpha|$. Consider a simplex $\sigma \neq \emptyset$ of X. Let $C(\sigma) := \{\alpha_1 \cdots \alpha_i : \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_i \subseteq \sigma\}$. Then for each $A = \alpha_1 \cdots \alpha_i \in C(\sigma)$, we can take $|A| = \langle\{\hat{\alpha}_1, \dots, \hat{\alpha}_i\}\rangle \subseteq |\sigma|$. Then we get a geometric realization $K' := \{|A| : A \in C(\sigma), \sigma \in X\}$ of $X^{(1)}$ with $|K'| \subseteq |K|$. Since $|\sigma| = \cup_{A \in C(\sigma)} |A|$, it follows that $|K'| = |K|$. This implies that $|X^{(1)}| = |X|$.

The simplicial complex $X^{(1)}$ is called the (first) barycentric subdivision of X.

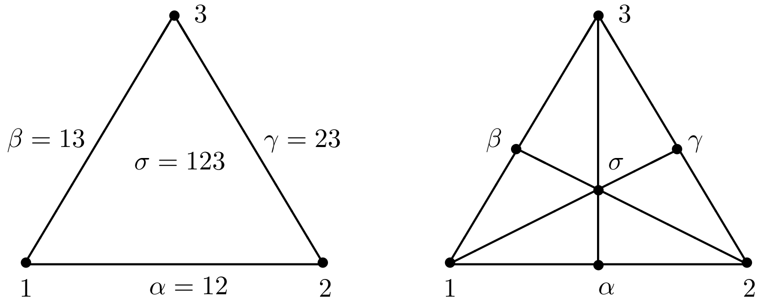

Figure 2: Barycentric Subdivision of a 2-simplex

Triangulated Manifolds: By a triangulated d-manifold, we mean a simplicial complex whose geometric carrier is a topological d-manifold.

Similarly, by a triangulated sphere/ball, we mean a simplicial complex whose geometric carrier is a topological sphere/ball.

Standard Balls and Spheres: A standard $(d+1)$-ball is a pure $(d+1)$-dimensional simplicial complex with one maximal simplex (of dimension $d+1$). The standard ball with the $(d+1)$-simplex σ is denoted by $\overline{\sigma}$. Clearly, $\overline{\sigma}$ is a triangulated ball.

Consider the subcomplex $\partial\sigma := \overline{\sigma} \setminus \{\sigma\}$ of $\overline{\sigma}$. It is easy to see that $\partial\sigma$ triangulates the d-sphere S^d and hence a triangulated d-manifold (and sphere). The simplicial complex $\partial\sigma$ is called the standard d-sphere, and is also denoted by S^d_{d+2}.

Examples of Triangulated 2-Manifolds: Here we present examples of triangulations of the 2-spheres, the torus and the Klein bottle.

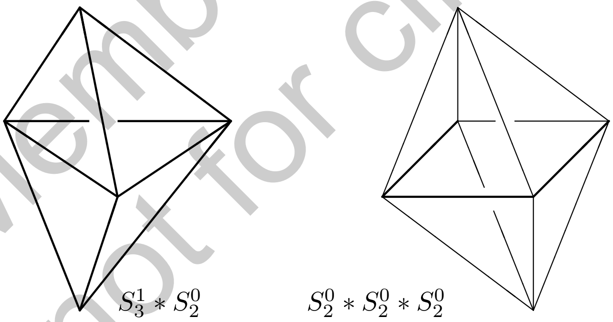

Figure 3: Triangulations of S^2 with 5 and 6 vertices

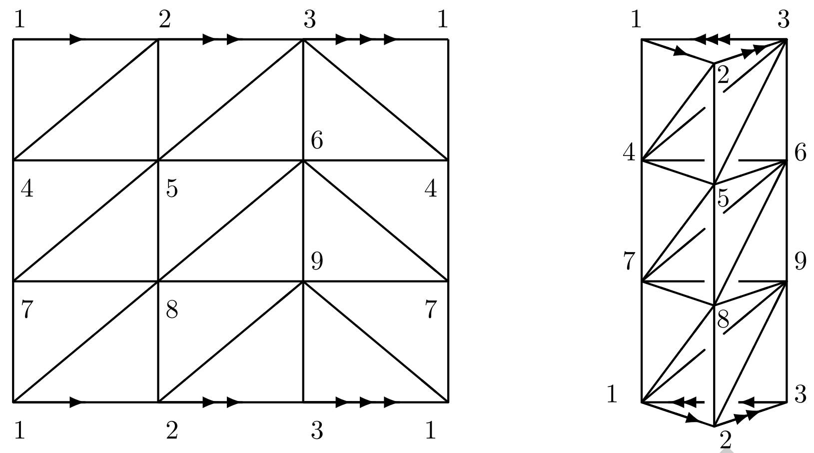

Figure 4 : A 9-vertex torus

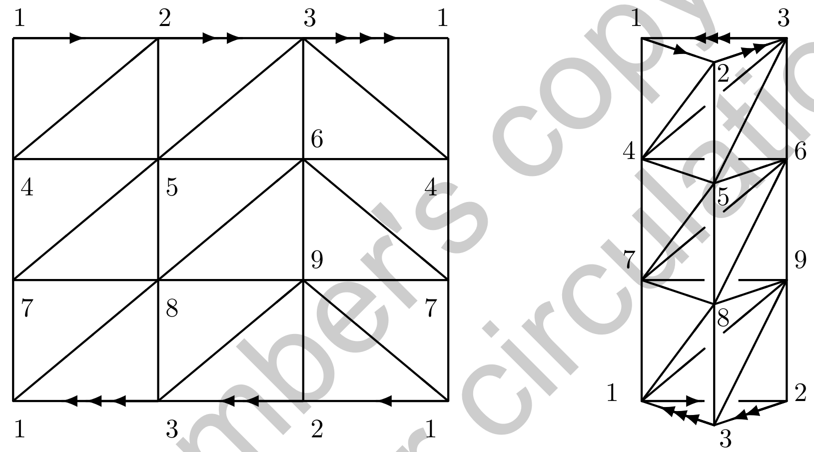

Figure 5 : A 9-vertex Klein bottle

3. Triangulations of Real Projective Spaces

Icosahedron and the 6-Vertex $\mathbb{RP}^2$: Consider the 12-vertex platonic solid icosahedron

$$\mathcal{I} = \langle \{ \frac{1}{\sqrt{\tau+2}}(0, \pm\tau, \pm 1), \frac{1}{\sqrt{\tau+2}}(\pm 1, 0, \pm\tau), \frac{1}{\sqrt{\tau+2}}(\pm\tau, \pm 1, 0) \} \rangle,$$

where $\tau = \frac{\sqrt{5}+1}{2}$. Let $\partial\mathcal{I}$ denote the simplicial complex corresponding to the boundary complex of $\mathcal{I}$. Then $\partial\mathcal{I}$ triangulates the 2-sphere S^2. The antipodal map on the boundary of $\mathcal{I}$ induces a proper $\mathbb{Z}_2$ action on $\partial\mathcal{I}$. Hence $\partial\mathcal{I}/\mathbb{Z}_2$ is a triangulated 2-manifold (with 6 vertices) and triangulates the real projective plane $\mathbb{RP}^2$.

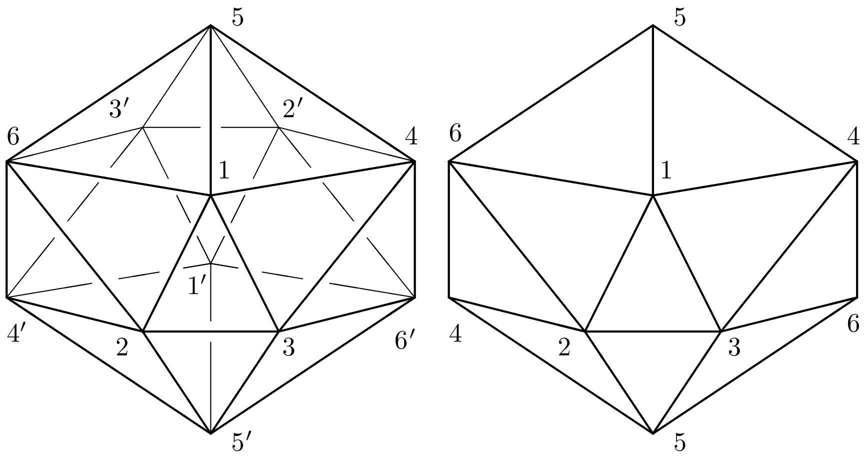

Figure 6 : Icosahedron and the 6-vertex $\mathbb{RP}^2$

Barycentric Subdivision of S^2_4 and a 7-Vertex $\mathbb{RP}^2$: Consider the regular tetrahedron T whose vertices are $v_1 = \frac{1}{\sqrt{3}}(-1,1,1)$, $v_2 = \frac{1}{\sqrt{3}}(1,-1,1)$, $v_3 = \frac{1}{\sqrt{3}}(1,1,-1)$, $v_4 = -\frac{1}{\sqrt{3}}(1,1,1)$ and take S^2_4 as the boundary complex of T. The vertices of the barycentric subdivision $(S^2_4)^{(1)}$ are the faces of S^2_4. For a geometric realization we take vertices corresponding to v_1v_2, v_1v_3, v_1v_4, v_3v_4, v_2v_4, v_2v_3, $v_2v_3v_4$, $v_1v_3v_4$, $v_1v_2v_4$ and $v_1v_2v_3$ are $v_5 = (0,0,1)$, $v_6 = (0,1,0)$, $v_7 = (-1,0,0)$, $u_5 = (0,0,-1)$, $u_6 = (0,-1,0)$, $u_7 = (1,0,0)$, $u_1 = -v_1$, $u_2 = -v_2$, $u_3 = -v_3$ and $u_4 = -v_4$ respectively.

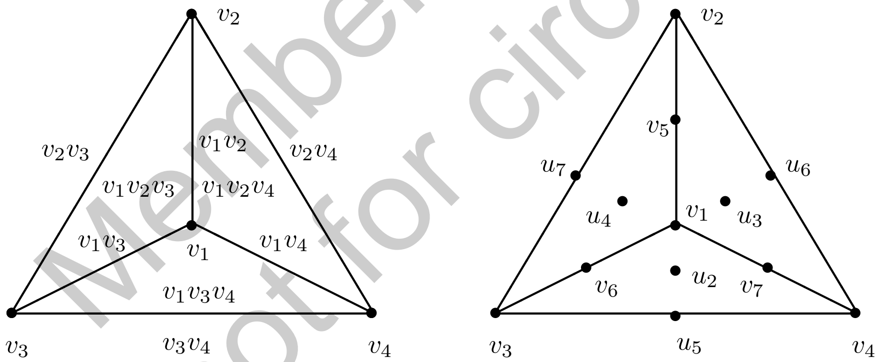

Figure 7 : Barycentric Subdivision $(S^2_4)^{(1)}$ of S^2_4

Consider the action of $\mathbb{Z}_2 = \{1,-1\}$ on $(S^2_4)^{(1)}$ given by $(-1)\cdot v_j = u_j$, $1 \leq j \leq 7$. This action is proper and hence $(S^2_4)^{(1)}/\mathbb{Z}_2$ is a (7-vertex) triangulated manifold and triangulates $|(S^2_4)^{(1)}|/\mathbb{Z}_2$. Since the above action of $\mathbb{Z}_2$ induces the antipodal action on $|(S^2_4)^{(1)}|$, $|(S^2_4)^{(1)}|/\mathbb{Z}_2$ is homeomorphic to $\mathbb{RP}^2$. Thus, $(S^2_4)^{(1)}/\mathbb{Z}_2$ triangulates $\mathbb{RP}^2$.

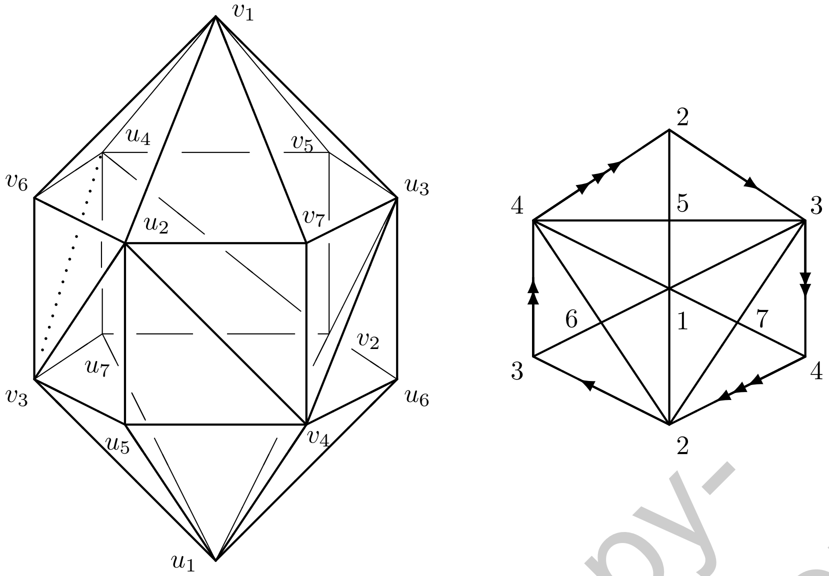

Figure 8: Geometric realizations of $(S^2_4)^{(1)}$ and $(S^2_4)^{(1)}/\mathbb{Z}_2$

Barycentric Subdivision of S^d_{d+2} and a Triangulation of $\mathbb{RP}^d$: For $d \geq 3$, let V be the vertex set of S^d_{d+2}. Then the vertex set of $(S^d_{d+2})^{(1)}$ is $P(V) \setminus \{\emptyset, V\} = \{A \subseteq V : \emptyset \neq A \neq V\}$. Consider the action of $\mathbb{Z}_2 = \{1, -1\}$ on $P(V) \setminus \{\emptyset, V\}$ given by $(-1)A = V \setminus A$. This induces a proper action of $\mathbb{Z}_2$ on $(S^d_{d+2})^{(1)}$. (In fact, $d_{(S^d_{d+2})^{(1)}}(A, V \setminus A) = 3$ for all $A \in P(V) \setminus \{\emptyset, V\}$.) Hence $R_d := (S^d_{d+2})^{(1)}/\mathbb{Z}_2$ is a triangulated d-manifold and triangulates $|S^d_{d+2})^{(1)}|/\mathbb{Z}_2$. Since the number of vertices in $(S^d_{d+2})^{(1)}$ is $2^{d+2} - 2$, the number of vertices in R_d is $2^{d+1} - 1$.

Let E be the $(d+1)$-dimensional subspace $\{(x_1, \dots, s_{d+2}) \in \mathbb{R}^{d+2} : x_1 + \cdots + x_{d+2} = 0\}$ of $\mathbb{R}^{d+2}$ and let S be the topological d-sphere $S^{d+1} \cap E$ in $\mathbb{R}^{d+2}$. Let P be the regular $(d+1)$-simplex in E whose vertices are $v_j = \frac{1}{\sqrt{d^2+3d+2}}((d+2)e_j - \sum_{i=1}^{d+2} e_i)$, $1 \leq j \leq d+2$, where $\{e_1, \dots, e_{d+2}\}$ is the standard basis of $\mathbb{R}^{d+2}$.

As in the 2-dimensional case, for a geometric realization of $(S^d_{d+2})^{(1)}$, we take S^d_{d+2} as the boundary complex of P. If σ is a k-simplex of S^d_{d+2}, then the barycenter c_σ of $|\sigma|$ is $\frac{1}{k+1}\sum_{v_i \in \sigma} v_i$. We choose the point $u_\sigma := \frac{c_\sigma}{\|c_\sigma\|}$ for the vertex σ of $(S^d_{d+2})^{(1)}$. So, the vertex set of our geometric realization of $(S^d_{d+2})^{(1)}$ is $\{u_\sigma : \sigma$ is a non-empty simplex of $S^d_{d+2}\}$. Observe that $u_{V \setminus \sigma} = -u_\sigma$. Thus, the above action of $\mathbb{Z}_2$ induces the antipodal action on $|(S^d_{d+2})^{(1)}|$ and hence $|(S^d_{d+2})^{(1)}|/\mathbb{Z}_2$ is homeomorphic to $\mathbb{RP}^d$. Therefore, R_d triangulates $\mathbb{RP}^d$.

Minimal Triangulations of $\mathbb{RP}^d$: Since only triangulated 2-manifolds with at most 5 vertices are S^2_4 and $S^1_3 * S^0_2$, any triangulation of $\mathbb{RP}^2$ needs at least 6 vertices and we have one such triangulation, namely $\mathbb{RP}^2_6$ defined above. In [1], Arnoux and Marin proved the following :

Theorem 3.1. *Let M be a triangulation of $\mathbb{RP}^d$. If M has n vertices then $n \geq \binom{d+2}{2}$. Moreover, equality is possible only for $d = 1$ and $d = 2$.*

So, any triangulation of $\mathbb{RP}^3$ needs at least 11 vertices. In 1970, Walkup [20] constructed one 11-vertex triangulation of $\mathbb{RP}^3$. In [19], Sulanke and Lutz have shown (using computer) that there exist exactly thirty 11-vertex triangulations of $\mathbb{RP}^3$.

By the above theorem, any triangulation of $\mathbb{RP}^4$ needs at least 16 vertices. In 1999, Lutz [12] has constructed one 16-vertex triangulation of $\mathbb{RP}^4$. We believe that this 16-vertex triangulation is the unique triangulation of $\mathbb{RP}^4$ with 16 vertices.

In 2005, Lutz [14] found one 24-vertex triangulation of $\mathbb{RP}^5$. No triangulation of $\mathbb{RP}^5$ is known with 22 or 23 vertices.

In [7], Basak and Sarkar have shown that $\mathbb{RP}^d$ has a triangulation with $2^d + d + 1$ vertices for all $d \geq 2$.

Recently, Sarkar [17] has constructed a $d(d+1)$-vertex triangulation and a $(d+1)^2$-vertex triangulation of $\mathbb{RP}^d$ for each $d \geq 2$.

4. Triangulations of Complex Projective Spaces

Triangulation of $\mathbb{CP}^1$ with 4 Vertices : Since $\mathbb{CP}^1$ is (homeomorphic to)S^2, it has a 4-vertex triangulation, namely, S^2_4. Clearly, it is vertex-minimal.

Triangulation of $\mathbb{CP}^2$ with 9 Vertices : Let $\mathbb{F}_3$ be field with three elements. Let $\mathcal{P}$ be the affine plane over $\mathbb{F}_3$. (The points of the incidence system $\mathcal{P}$ are the points of the 2-dimensional vector space $\mathbb{F}_3^2$ and the lines are $L + x$, where L is a 1-dimensional subspace of $\mathbb{F}_3^2$ and $x \in \mathbb{F}_3^2$.) There are 12 lines in $\mathcal{P}$ which form 4 parallel classes and each class is determined by a 1-dimensional subspace of $\mathbb{F}_3^2$.

Consider the 9-vertex simplicial complex $\mathcal{X}$ as follows. The vertices of $\mathcal{X}$ are the points of the affine plane $\mathcal{P}$. Let $J = \{j_1, j_2, j_3\}$, $K = \{k_1, k_2, k_3\}$, $L = \{l_1, l_2, l_3\}$ and $M = \{m_1, m_2, m_3\}$ be the 4 parallel classes of lines in $\mathcal{P}$ with fixed orderings.
Then

$$\mathcal{X} = (\bigcup_{i=1}^{3}(j_i * \partial j_{i+1})) \cup (\bigcup_{i=1}^{3}(k_i * \partial k_{i+1})) \cup (\bigcup_{i=1}^{3}(l_i * \partial l_{i+1})) \cup (\bigcup_{i=1}^{3}(m_i * \partial m_{i+1})).$$

(Addition in the suffix is modulo 3.) This gives $4 \times 9 = 36$ facets of $\mathcal{X}$. The simplicial complex $\mathcal{X}$ triangulates the complex projective plane $\mathbb{CP}^2$ and is denoted by $\mathbb{CP}^2_9$. This was first constructed by Kühnel and Banchoff in 1983 [10].

Several proofs of the uniqueness of $\mathbb{CP}^2_9$ are now known. The first was the computer-aided proof of Kühnel and Laßmann [11]. The second proof, due to Arnoux and Marin [1], uses cohomology theory with $\mathbb{Z}_2$ coefficients. The third and the fourth are combinatorial proofs, due to us [2, 3].

Morin and Yoshida gave another proof of the fact that the topological space triangulated by $\mathbb{CP}^2_9$ is $\mathbb{CP}^2$. One more proof of the last fact has been found by Madahar and Sarkaria in 2000 [15]. They constructed a 17-vertex triangulated 4-ball D^4_{17} whose boundary is a 12-vertex triangulated 3-sphere S^3_{12} and defined a combinatorial analogue $h\colon S^3_{12} \to S^2_4$ of the Hopf map so that the simplicial complex $S^2_4 \cup_h D^4_{17}$ is precisely $\mathbb{CP}^2_9$.

Minimal Triangulations of $\mathbb{CP}^d$: In 1991, Arnoux and Marin [1] proved the following:

Theorem 4.1. *Let M be a triangulation of $\mathbb{CP}^d$. If M has n vertices then $n \geq (d+1)^2$. Moreover, equality is possible only for $d = 1$ and $d = 2$.*

Triangulations of $\mathbb{CP}^d$ as Symmetric Product of S^2: The symmetric group $\mathrm{Sym}(d)$ acts on the Cartesian product $(S^2)^d$ by coordinate permutation, and the quotient space $(S^2)^d/\mathrm{Sym}(d)$ is homeomorphic to the complex projective space $\mathbb{CP}^d$.

In 2011, we [4] used the case $d = 2$ of this fact to construct a 10-vertex triangulation $\mathbb{CP}^2_{10}$ of $\mathbb{CP}^2$ which is a branched quotient of a 16-vertex triangulation of $S^2 \times S^2$. We have shown that the geometric carrier of $\mathbb{CP}^2_{10}$ is same as that of $\mathbb{CP}^2_9$. Since $\mathbb{CP}^2_{10}$ triangulates $\mathbb{CP}^2$, this gives one more proof of the fact that $\mathbb{CP}^2_9$ triangulates $\mathbb{CP}^2$.

In 2012, we [5] have constructed a 124-vertex simplicial subdivision $(S^2)^3_{124}$ of the 64-vertex standard cellulation $(S^2_4)^3$ of $(S^2)^3$, such that the Sym(3)-action on this cellulation naturally extends to an action on $(S^2)^3_{124}$. Further, the Sym(3)-action on $(S^2)^3_{124}$ is good, so that the quotient simplicial complex $\mathbb{CP}^3_{30} := (S^2)^3_{124}/\mathrm{Sym}(3)$ is a (30-vertex) triangulation of $\mathbb{CP}^3$. In other words, we have constructed a simplicial realization $(S^2)^3_{124} \to \mathbb{CP}^3_{30}$ of the branched covering $(S^2)^3 \to \mathbb{CP}^3$.

We ran the BISTELLAR program on the input $\mathbb{CP}^3_{30}$, in order to reduce the number of vertices. The end result was a 18-vertex triangulation $\mathbb{CP}^3_{18}$ of $\mathbb{CP}^3$. Thus, we have a 18-vertex triangulation of $\mathbb{CP}^3$. We suspect that there may not be any 17-vertex triangulation of $\mathbb{CP}^3$.

Another 10-Vertex Triangulation of $\mathbb{CP}^2$: Consider the 7-vertex simplicial complex $\mathbb{T}^2$ given in Figure 9. It triangulates the 2-dimensional torus $S^1 \times S^1$.

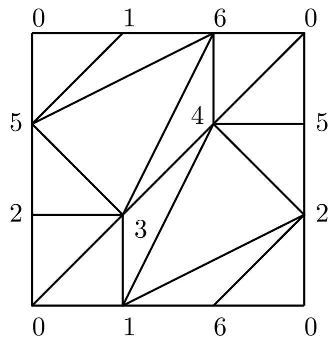

Figure 9 : 7-vertex torus $\mathbb{T}^2$

There are exactly three 7-vertex triangulations of the solid torus $D^2 \times S^1$ whose boundaries are $\mathbb{T}^2$. These are the following :

$$\begin{aligned} T_1 &= \{0123, 1234, 2345, 3456, 0456, 0156, 0126\}, \\ T_2 &= \{0246, 1246, 1346, 1356, 0135, 0235, 0245\}, \\ T_3 &= \{0145, 1245, 1256, 2356, 0236, 0346, 0134\}. \end{aligned}$$

Observe that $T_1 \cap T_2 = T_1 \cap T_3 = T_2 \cap T_3 = \mathbb{T}^2 = \partial T_1 = \partial T_2 = \partial T_3$. The triangulated 2-manifold $\mathbb{T}^2$ has fourteen 2-simplices. Among the remaining $\binom{7}{3} - 14 = 21$ possible 2-simplices, 7 are in T_1, 7 are in T_2 and 7 are in T_3. For $1 \leq i < j \leq 3$, let $S_{ij} := T_i \cup T_j$. Then, S_{ij} is a triangulated 3-manifold and triangulates the 3-sphere S^3.

Consider the pure 4-dimensional simplicial complex $\mathcal{Y}$ on the vertex-set $\{0, 1, \ldots, 6, x, y, z\}$ as follows :

$$\mathcal{Y} = (\{x\} * S_{12}) \cup (\{y\} * S_{13}) \cup (\{z\} * S_{23}).$$

The simplicial complex $\mathcal{Y}$ is a triangulated 4-manifold and triangulates $\mathbb{CP}^2$. This was constructed by Banchoff and Kühnel in 1992 [6].

Triangulations of $\mathbb{CP}^d$ for Higher d : Recently, Soumen Sarkar [18] has generalized the above construction of Banchoff and Kühnel and has constructed a $\frac{1}{4}(5^{d+1} - 1)$-vertex triangulation of $\mathbb{CP}^d$ for each $d \geq 2$. This gives first explicit triangulations of $\mathbb{CP}^d$ for $d \geq 4$.

References

[1] P. Arnoux and A. Marin, *The Kühnel triangulation of the complex projective plane from the view-point of complex crystallography (part II),* Memoirs of Fac. Sc., Kyushu Univ. Ser. A **45** (1991), 167–244.

[2] B. Bagchi and B. Datta, *On Kühnel's 9-vertex complex projective plane,* Geom. Dedicata **50** (1994), 1–13.

[3] B. Bagchi and B. Datta, *A short proof of the uniqueness of Kühnel's 9-vertex complex projective plane,* Advances in Geometry **1** (2001), 157–163.

[4] B. Bagchi and B. Datta, *From the icosahedron to natural triangulations of* $\mathbb{CP}^2$ *and* $S^2 \times S^2$*,* Discrete Comput Geom. **46** (2011), 542–560.

[5] B. Bagchi and B. Datta, *A triangulation of* $\mathbb{CP}^3$ *as symmetric cube of* S^2*,* Discrete Comput Geom. **48** (2012), 310–329.

[6] T. F. Banchoff and W. Kühnel, *Equilibrium triangulations of the complex projective plane,* Geom. Dedicata **44** (1992), 413–433.

[7] B. Basak and S. Sarkar, *Equivariant and equilibrium triangulations of some small covers with the minimum vertices,* arXiv:1306.1568v1, 2013, 19 pages.

[8] U. Brehm and W. Kühnel, *Combinatorial manifolds with few vertices,* Topology **26** (1987), 465–473.

[9] B. Datta, *Minimal triangulations of manifolds,* J. IISc **87** (2007), 429–449.

[10] W. Kühnel and T. F. Banchoff, *The 9-vertex complex projective plane,* Math. Intelligencer **5** (3) (1983), 11–22.

[11] W. Kühnel and G. Laßmann, *The unique 3-neighbourly 4-manifold with few vertices,* J. Combin. Theory, Ser. A **35** (1983), 173–184.

[12] F. H. Lutz, *Triangulated Manifolds with Few Vertices and Vertex-Transitive Group Actions,* Thesis (TU, Berlin), Shaker Verlag, Aachen, 1999.

[13] F. H. Lutz, *BISTELLAR, Version Nov/2003.* http://www.math.TU-Berlin.de/diskregeom/stellar/BISTELLAR.

[14] F. H. Lutz, *Triangulated manifolds with few vertices: Combinatorial manifolds.* arXiv:math/0506372v1, 2005, 37 pages.

[15] K. V. Madahar and K. S. Sarkaria, *A minimal triangulation of the Hopf map and its application,* Geom Dedicata **82** (2000), 105–114.

[16] B. Morin and M. Yoshida, *The Kühnel triangulation of the complex projective plane from the view-point of complex crystallography (part I),* Memoirs of Fac. Sc., Kyushu Univ. Ser. A **45** (1991), 55–142.

[17] S. Sarkar, *On* $\mathbb{Z}_2$*-equivariant triangulations of* $\mathbb{RP}^n$, arXiv:1306.2851v1, 2013, 12 pages.

[18] S. Sarkar, *Equilibrium triangulations of quasitoric manifolds with standard characteristic vectors,* (in preparation).

[19] T. Sulanke and F. H. Lutz, *Isomorphism free lexicographic enumeration of triangulated surfaces and 3-manifolds,* Euro. J. Combin. **30** (2009), 1965–1979.

[20] D. W. Walkup, *The lower bound conjecture for 3- and 4-manifolds,* Acta Math. **125** (1970), 75–107.

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DISMANTLABILITY AND REDUCIBILITY IN DISCRETE STRUCTURES*

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CONTENTS

1. Introduction

An important outgrowth of the discrete structures such as ordered sets, lattices, graphs, networks, etc. is not only due to applications in computer systems development and analysis but also due to the emergence of several combinatorial problems which, while simple in formulation, remain unsolved. These problems have an intrinsic appeal and often highlight concepts and techniques of wider applicability. Two of these problems are characterizing ordered sets which satisfy the fixed point property and the other is characterizing those undirected graphs which are "orientable" so as to produce the diagram of an ordered set. We highlight the known results of such open problems and emerged concepts and techniques. We begin with the basic concepts.

Definition 1.1. A **partially ordered set**(poset) is a nonempty set together with a binary relation which is reflexive, antisymmetric and transitive on it.

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Definition 1.2. A **graph** is a pair (V, E), where V is a finite nonempty set and E is a symmetric binary relation on V. The elements of V are called vertices and that of E are called edges.

Definition 1.3. A poset L is called a **lattice** if $inf\{a, b\}$ and $sup\{a, b\}$ exist, $\forall$ $a, b \in L$.

Definition 1.4. An ordered set is said to have **Fixed Point Property (FPP)** if every order-preserving map of the ordered set to itself fixes a point.

Open Problem 1.1 (D. Duffus and I. Rival[9]). *Characterize posets having FPP.*

Perhaps one of the most challenging unsolved problems in the theory of ordered sets is to characterize posets with FPP. The problem is completely settled for lattices.

Theorem 1.1 (A. Tarski[25]). *Every complete lattice has the fixed point property.*

Proof: If L is a complete lattice and f is an order-preserving map of L to itself, we set $A = \{x \in L | x \leq f(x)\}$ and $a = sup_L A$. Then, for each $x \in A$, $x \leq f(x) \leq f(a)$, hence, $a \leq f(a)$. From this it follows that $f(a) \leq f(f(a))$, whence $f(a) \leq a$, that is, $a = f(a)$ is a fixed point of f. ■

Theorem 1.2 (A. Davis[8]). *Every lattice with the fixed point property is complete.*

The above Theorem is proved using the fact that FPP is preserved under the formation of retract.

Definition 1.5. An order-preserving map $g : P \to Q$ is a **retraction** of P onto $Q \subseteq P$ provided that $g(x) = x$ *for each* $x \in Q$. If there is a retraction of P onto Q, then Q is a **retract** of P.

Proposition 1.1. *Every retract of a poset with the fixed point property has the fixed point property.*

The problem is largely open for general posets. In Section 2, we will focus on the developments with respect to FPP in finite posets.

Definition 1.6. The **covering relation** of a partially ordered set P is the binary relation which holds between comparable elements that are immediate neighbors. The graph on P with edges as covering relations is called a **cover graph**, denoted by $C(P)$.

Definition 1.7. A graph G is said to be **orientable** as an ordered set (poset) P if G and $C(P)$ are isomorphic as graphs.

Open Problem 1.2 (O. Ore [22]). *Characterize graphs which are cover graphs.*

The problem is still open. One of the well known partial solutions is in terms of girth and chromatic number of a graph.

Definition 1.8. The **girth** $g(G)$ of a graph G is the length of a shortest cycle contained in the graph.

Definition 1.9. The **chromatic number** of a graph G is the smallest number of colors $\chi(G)$ needed to color the vertices of G so that no two adjacent vertices share the same color.

Theorem 1.3 (D. Duffus and I. Rival [11]). *If $\chi(G) < g(G)$ then the graph G is orientable.*

In Section 3, we will see some recently obtained partial solutions.

The present development with respect to the two open problems mentioned above depends upon the study of the structures of ordered sets via retract and reducibility. G. Bordalo and B. Monjardet [3] introduced the concept of reducible classes of discrete structures.

Definition 1.10. An element x of a finite lattice/poset/graph L satisfying certain properties is **deletable** if $L - \{x\}$ is respectively a lattice/poset/graph satisfying the same properties.

Definition 1.11. A class is **reducible** if each non-trivial member of this class admits (at least) one deletable element (equivalently, if one can go from any member in this class to the trivial one by a sequence of members of the class obtained by deleting one element in each step).

G. Bordalo and B. Monjardet[3] proved that the classes of semimodular lattices, pseudocomplemented lattices and locally distributive lattices are reducible, however, the classes of modular lattices and distributive lattices are not reducible. There are two general problems in respect of reducibility.

Open Problem 1.3. *Identify classes of posets which are reducible and characterize the deletable elements.*

Whenever a class is not reducible, then there is a member of the class of which no singleton is deletable. By taking into consideration a smallest deletable set S, Kharat, Waphare and Thakare [18] introduced the following concepts.

Definition 1.12. Let $\mathscr{P}$ be a class of posets and $P \in \mathscr{P}$. We say that a non empty subset S of P is **$\mathscr{P}$- deletable** if $P - S \in \mathscr{P}$.

Definition 1.13 ([18])**.** The smallest positive number r is called **$\mathscr{P}$-reducibility number** of $P \in \mathscr{P}$, denoted by $red(P, \mathscr{P})$, such that there exists a $\mathscr{P}$-deletable subset S of P with $|S| = r$.

Open Problem 1.4. *Find the reducibility number of a structure with respect to a class and characterize the structures in terms of the reducibility number.*

In Section 4, we highlight some of the known reducible classes of discrete structures in view of Open Problem 1.3 and Section 5 is devoted to determine reducibility numbers.

2. Fixed Point Property in Posets

We consider some of the known results in terms of the fixed point property.

Definition 2.1 (I. Rival [24])**.** For even $n \geq 4$, a subset $C = \{c_1, c_2, ..., c_n\}$ of P is a **crown** provided that $c_1 < c_n$, and $c_1 < c_2, c_2 > c_3, ..., c_{n-2} > c_{n-1}, c_{n-1} < c_n$ are the only comparability relations that hold in C and, in the case $n = 4$, there is no $a \in P$ such that $c_1 < a < c_2, c_3 < a < c_4$.

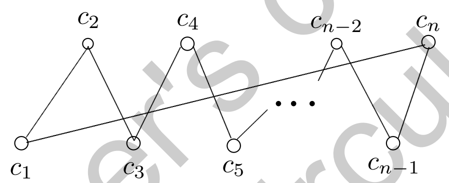


Fig. 1 (Crown)

Definition 2.2 (Richard A. Brualdi [5])**.** A poset is called a **generalized crown** if it is connected, of length 1, and all vertices of which have same degree.

Definition 2.3. An element a of a poset P is **irreducible** in P if a has precisely one upper cover or precisely one lower cover in P.

Definition 2.4 (Benoit Larose , Lfiszl Zadori [20])**.** An n-element **partially ordered set** P is **dismantlable by irreducibles** if the elements of P can be labelled $a_1, a_2, ..., a_n$ so that $a_i \in I(P - \{a_1, a_2, ..., a_{i-1}\})$ for each $i = 1, 2, ..., n - 1$, where $I(P)$ denote the set of all elements irreducible in P.

The following result shows that deletion of an irreducible element maintains FPP.

Proposition 2.1 (I. Rival [23])**.** *Let P be a partially ordered set in which every chain is finite and let $a \in I(P)$. Then P is fixed point free if and only if $P - \{a\}$ is fixed point free.*

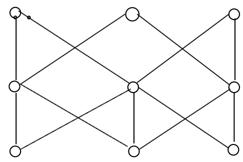


Fig. 2 (Poset with FPP but not dismantlable by irreducibles)

Proof Let $a \in I(P)$ where a has a unique lower cover b, say. If $f : P \to P$ is order-preserving and without fixed points then, since P contains no infinite chains, $f(b) \neq a$. The map $f'I : P - \{a\} \to P - \{a\}$ defined by

$$f'(x) = \begin{cases} b & \text{if } x \in f^{-1}(\{a\}); \\ f(x) & \text{otherwise.} \end{cases}$$

is order-preserving and without fixed points.
For example, if $x, y \in P - a$, $x \geq y, x \in f^{-1}(\{a\})$, and $y \notin f^{-1}(\{a\})$ then $f(x) = a > b \geq f(y)$ so that $f'(x) \geq f'(y)$. On the other hand, if $g : P - \{a\} \to P - \{a\}$ is order-preserving and without fixed points then it is easily verified that $g' : P \to P$ defined by

$$g'(x) = \begin{cases} g(b) & \text{if } x = a \\ g(x) & \text{otherwise} \end{cases}$$

is again order-preserving and without fixed points. ■

As a consequence of the above result we get

Theorem 2.1 (I.Rival [23])**.** *Every poset dismantlable by irreducibles has Fixed Point Property.*

The converse need not be true. A poset having Fixed Point Property need not be dismantlable. See the poset depicted in Fig. 2 given by I. Rival [23].

However, the converse is true for the posets of length 1.

Theorem 2.2 (Duffus and Rival [8])**.** *Let P be a finite, connected, partially ordered set of length* 1*. The following conditions are equivalent.*

(1) P has the fixed point property;
(2) P is dismantlable by irreducibles;
(3) P does not contain a crown.

In general posets, FPP is characterized using generalized crown. Recall that any retract of a dismantlable poset is dismantlable.

Theorem 2.3 (Richard A. Brualdi [5])**.** *Let P be a finite poset. Then P has the fixed point property if and only if a generalized crown is not a retract of P.*

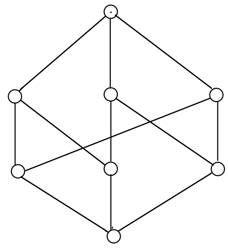


Fig. 3 (Smallest lattice containing a crown)

Dismantlability is also preserved under the formation of direct product.

Lemma 2.1 (Rival et al. [12]). *Let P and Q be dismantlable ordered sets. Then $P \times Q$ is dismantlable.*

From the above results it is clear that dismantlability provides a technique to achieve posets with FPP. It is therefore important to look into the structure and cover graph of dismantlable posets in the sense of orientability.

3. Orientability

Much of the combinatorial interests in ordered sets is inextricably linked to the combinatorial features of the diagrams associated with them. Still, there is little known about those undirected graphs which are orientable.

Theorem 3.1 (D.Duffus, I.Rival [11]). *A finite graph G is the covering graph of a Distributive Lattice of length n if and only if G is retract of $C(Q_n)$, and $diam(G) = n$.*

Similar type of characterizations are obtained for modular lattices by J.Jakubik [14] and geometric lattices by Duffus and Rival [11]. H. Grotzsch [13] shown that triangle-free planar graphs are 3-chromatic; consequently, they are orientable. Recently we have obtained characterization for the covering graphs of dismantlable lattices.

Definition 3.1 (Rival [17]). A finite lattice L of order n is called **dismantlable lattice** if there exists a chain $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n (= L)$ of sublattices of L such that $|L_i| = i$; for all i.

Note that if a lattice is dismantlable by irreducibles does not mean that it is a dismantlable lattice. The lattice in Fig. 3 is dismantlable by irreducibles, but it is not a dismantlable lattice as it contains a crown.

Theorem 3.2 (Rival[23]). *A finite lattice is a dismantlable lattice if and only if it contains no crown.*

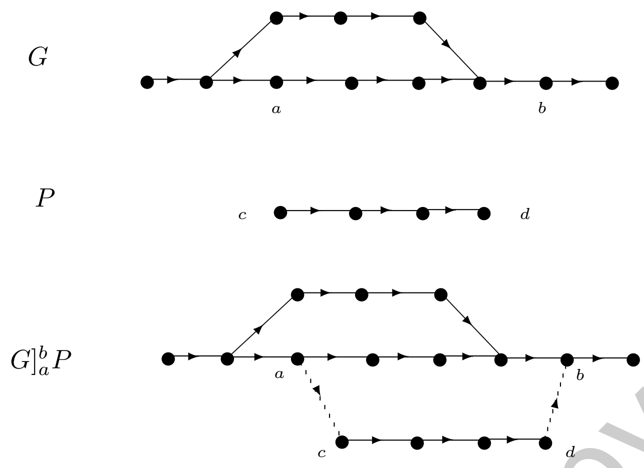


Fig. 4 (Adjunct of an ear)

In fact, dismantlable lattices are lattices which are posets dismantlable by doubly irreducible elements.

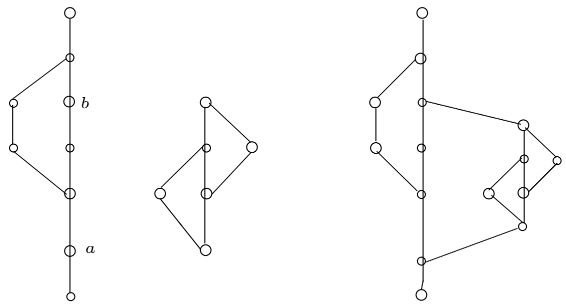
Theorem 3.3 (Brucker and Gely [6]). *A lattice L is a dismantlable lattice if and only if there exists $L_1 \subset L_2 \subset \cdots \subset L_n = L$ such that L_1 is a trival lattice and $L_{i-1} = L_i \setminus \{x\}$ where x is a doubly irreducible element of L_i.*

We gave a partial solution to the open problem of orientability by characterizing graphs of dismantlable lattices.

Definition 3.2. An **ear** of a graph G is a maximal non empty path having all internal vertices of degree 2 in G.

Definition 3.3 (Waphare and Bhavale[2013]). Let G be any directed graph and P be any directed path from c to d with $V(G) \cap V(P) = \emptyset$. Let (a, b) be a pair of vertices in G such that there is a directed path from a to b in G of length at least 2. We define **adjunct of directed ear** P to G at (a, b) to be a directed graph denoted by $G\ ]_a^b\ P$ having vertex set $V(G) \cup V(P)$ and arc set $A(G) \cup A(P) \cup \{(a, c), (d, b)\}$.

We say that a directed graph G is adjunct of directed ears if it can be obtained by adjunction of directed ears starting with a directed path. An underlying graph of a directed graph which is adjunct of directed ears is called simply "adjunct of ears".



L1 L2 $L1\ ]_a^b\ L2$

Fig. 5

Theorem 3.4 (Waphare and Bhavale[2013]). *A graph G is orientable as a dismantlable lattice if and only if G is an adjunct of ears.*

The above result is a consequence of the structure theorem (see Theorem 3.5) for dismantlable lattices. The structure theorem is based on an innovative operation called adjunct of lattices.

Definition 3.4 (Thakare, Pawar and Waphare [26]). Let L_1 and L_2 be two disjoint lattices and (a, b) be a pair of elements in L_1 such that $a < b$ and $a \not\prec b$. We define the partial order $\leq$ on $L = L_1 \cup L_2$ with respect to the pair (a, b) as follows:
$x \leq y$ in L if $x, y \in L_1$ and $x \leq y$ in L_1 or $x, y \in L_2$ and $x \leq y$ in L_2 or $x \in L_1, y \in l_2$ and $x \leq a$ in L_1, or $x \in L_2, y \in L_1$ and $b \leq y$ in L_1.
The procedure for obtaining L in this way is called **adjunct operation** of L_1 with L_2. We call a lattice L as adjunct of lattices L_1, L_2 with respect to (a, b) and denote it by $L = L_1]_a^b L_2$.

Theorem 3.5. *(***Structure Theorem***)(Thakare, Pawar and Waphare* [26]*) A finite lattice is dismantlable lattice if and only if it is an adjunct of chains, i.e.,* $L = (...((C_1]_{a_1}^{b_1} C_2)]_{a_2}^{b_2} C_3)...)]_{a_n-1}^{b_n-1} C_n$.

In the paper of Thakare, Pawar and Waphare [26], the Structure Theorem for dismantlable lattices is also used for finding exact formulae in terms of partition of n, for the number of non-isomorphic lattices of certain types having n elements. It is clear that the class of dismantlable lattices is reducible and a doubly irreducible element is always deletable. This intrinsic property of reducibility can be carried over to other structures.

4. Reducible Classes

The classes of dismantlable lattices, posets dismantlable by irreducibles are reducible. Actually, the concept of dismantlable graphs is motivated from the concept of dismantlable posets by irreducibles by taking into account corresponding comparability graph. The concept of dismantlability can be carried to dismantlable graphs.

Definition 4.1. We say that K_1 is a **dismantlable graph**. A graph G (with at least 2 vertices) is dismantlable if and only if there exist vertices y, z such that $N[z] \subseteq N[y]$ (y dominates z) and $G(V - z)$ is dismantlable, where $N[y] = \{x|(x, y) \in E(G)\} \cup \{y\}$.

The class of dismantlable graphs is also reducible. Further the classes of trees, connected graphs, complete graphs are reducible.

Definition 4.2. Let I be an ideal(semi-ideal) of a poset P. The set $Z_I(P)$ is called the set of **generalized zero divisors** with respect to I of P and is given by $Z_I(P) = \{x \in P \setminus I | \{x_1, y_1\}^\ell \subseteq I$, for some $x_1 \in (x] \setminus I$ and for some $y_1 \notin I\}$, where $A^\ell = \{x|x \leq a,\ \forall\ a \in A\}$ and $(x] = \{a|a \leq x\}$.

Definition 4.3 (V.V. Joshi, B. N. Waphare, H. Y. Pourali [15])**.** Let I be a non-prime ideal(semi-ideal) of a poset P. We associate an undirected graph called **generalized zero divisor graph** of P with respect to I, denoted by $\widehat{G_I(P)}$, in which the set of vertices $V(\widehat{G_I(P)}) = Z_I(P)$ and two distinct vertices x and y are adjacent if and only if $\{x_1, y_1\}^\ell \subseteq I$, for some $x_1 \in (x] \setminus I$ and for some $y_1 \in (y] \setminus I\}$.

Theorem 4.1 (V.V. Joshi, B. N. Waphare and H. Y. Pourali [15])**.** *The following conditions on a simple graph G are equivalent.*

(1) G is the generalized zero divisor graph of a poset with respect to some non-prime semi-ideal.
(2) G is the complete r-partite graph for some $r > 1$.
(3) The complement of G is disjoint union of cliques.

An easy consequence of the above representation theorem for generalized zero divisor graphs is the following result.

Corollary 4.1. *The class of generalized zero divisor graphs is reducible.*

The classes of regular graphs, Hamiltonian graphs & Eulerian graphs are not reducible.

5. Reducibility Number

It is easy to see that a class $\mathscr{P}$ is a reducible class of posets if and only if for any non-trivial $P \in \mathscr{P}$, $red(P, \mathscr{P}) = 1$. Further, if $\mathscr{P}$ is a class of lattices closed for sublattices and if $P \in \mathscr{P}$ is a dismantlable lattice then $red(P, \mathscr{P}) = 1$.

Some Notations:

(1) $\mathscr{D}$: Distributive Lattices.
(2) $\mathscr{M}$: Modular Lattices.
(3) $\mathscr{B}$: Boolean Lattices.
(4) $\mathscr{L}(G)$: Lattice of all subgroups of a group G.
(5) $\mathscr{P}$: Class of pseudo-complemented u-posets.

Theorem 5.1 (V. S. Kharat, B. N. Waphare and N. K. Thakare [19]). *Let $L = 2^n$, $n \geq 2$ then $red(L, \mathscr{D}) = 2^{n-2} = red(L, \mathscr{M})$.*

In general, they proved the following result.

Theorem 5.2. *Let $L = C_1 \times C_2 \times ... \times C_k$ be a lattice where each $C_i, (i = 1, 2, ..., k)$ is a chain of length e_i. Without loss of generality, assume that $e_1 \geq e_2 \geq ... \geq e_k$. Then $red(L, \mathscr{D}) = (e_3 + 1)(e_4 + 1)...(e_k + 1) = red(L, \mathscr{M})$.*

The above theorem can be used to obtain properties of groups.

Theorem 5.3 (V. S. Kharat, B. N. Waphare and N. K. Thakare [19]). *Let G be a finite group such that the lattice of all subgroups of G, i.e., $\mathscr{L}(G)$, is distributive. Then*

(1) $red(\mathscr{L}(G), \mathscr{D}) = 1$ if and only if $O(G)$ has at most two distinct prime divisors, where, $O(G)$ is order of the group G,
(2) if $red(\mathscr{L}(G), \mathscr{D})$ is a prime number p then $O(G)$ is divisible by exactly three distinct primes.

It is proved that the class of pseudo-complemented posets is not reducible. In fact, no element of the poset depicted in Fig. 6 is deletable.
The following problem is discussed in [18].

Open Problem 5.1. *Whether there are subclasses of the family of pseudo-complemented posets that are reducible?*

Definition 5.1. A poset P is said to be u**-poset** if for every set of atoms, $p_1, ..., p_k$, there exists an element w such that $p_i \leq w$, for $i = 1, 2, ..., k$ and there is no atom $q \leq w$, with $q \neq p_i$, for all $i = 1, 2, ..., k$.

Theorem 5.4 (V. S. Kharat and B. N. Waphare [18]). *If P is a pseudo-complemented u-poset with n atoms, then there are at least n elements which are $\mathscr{P}$ - deletable.*

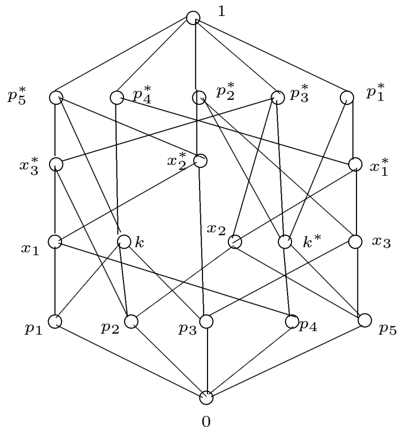


Fig. 6 (No element is $\mathscr{PC}$ - deletable)

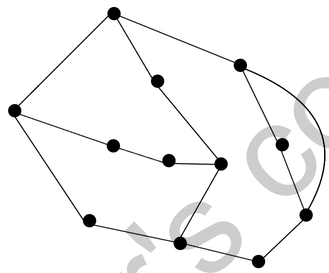


Fig. 7 (2-connected graph)

Theorem 5.5 (V. S. Kharat and B. N. Waphare [18]). *The class of pseudo-complemented u-posets is reducible.*

Y.M.Borse and B.N.Waphare [4] studied the reducibility in graphs.

Definition 5.2. An **ear decomposition** of an undirected graph G is a partition of its set of edges into a sequence of ears starting with a cycle, such that the one or two endpoints of each ear belong to earlier ears in the sequence and such that the internal vertices of each ear do not belong to any earlier ear.

Definition 5.3. An **open ear decomposition** or a proper ear decomposition is an ear decomposition in which the two endpoints of each ear after the first are distinct from each other.

Theorem 5.6 (H. Whitney [29]). *A graph $G = (V, E)$ with $|V| \geq 2$ is 2-vertex-connected if and only if it has an open ear decomposition.*

The class of two-(vertex) connected graphs is not reducible.For example, no vertex is deletable from a cycle.

Theorem 5.7. *(Y. M. Borse and B. N. Waphare* [4]*) Let G be a 2-connected graph and let $U \subset V(G)$ with $|U| \geq 2$. Then U is a minimal deletable set of G if and only if U is the set of internal vertices of a deletable ear of G.*

Theorem 5.8. *(Y. M. Borse and B. N. Waphare* [4]*) If G is critically 2-connected then $red(G) = k$, where k is the length of a smallest deletable ear of G.*

Definition 5.4. A graph is **series-parallel** if it can be created from K_2 by repeatedly duplicating and subdividing edges.

Series-parallel graphs are very well studied in connection with algorithmic graph theory. The series parallel graphs which are 2-connected can be obtained by nesting ears as given in the following lemma.

Lemma 5.1. *Every 2-connected series-parallel graph G has an ear decomposition $E_0, E_1, \cdots, E_n$ satisfying*
(i) for $i > 1$, there is some $j < i$ such that E_i is nested in E_j;
(ii) if two ears E_i and $E_{i'}$ are both nested in the same E_j, then either the nest interval of E_i contains that of $E_{i'}$, or vice versa, or the two nest intervals are disjoint; i.e., no two nest intervals in each ear E_j cross each other.

We investigated properties of a special class of critically 2-connected series-parallel graphs.

Definition 5.5. Let $\mathcal{C}$ be the class of series-parallel graphs each of which has a nested ear decomposition $E_0, E_1, \cdots, E_n$ satisfying
(i) $|E_0| \geq 4$,
(ii) $|E_i| \neq 2$ for each i and
(iii) the length of each nested interval is different from 2.

Theorem 5.9. *(Y. M. Borse and B. N. Waphare* [4]*) If $G \in \mathcal{C}$, then G is critically 2-connected.*

We close our discussion by recent work regarding reducibility of regular graphs.

Theorem 5.10 (A. B. Attar[2])**.** *Let R be an r-regular graph with p vertices. Suppose $U = \{u_1, u_2, \ldots, u_k\}$ is a deletable set of vertices with respect to the class of regular graphs. Then the following statements are true.*

(1) $r - d_{[U]}(u_i) \leq p - k, i = 1, 2, \ldots, k$, where $[U]$ denotes the vertex induced subgraph induced by U.
(2) $\frac{rk-2m}{p-k} = r - j$ where $m = |E([U])|$ and the $j =$ the degree of every vertex in $R - U$. In particular, $p - k$ divides $rk - 2m$ for some $0 \leq m \leq \frac{rk}{2}$.
(3) $r - j \leq k$ where j is the degree of every vertex in $R - U$.

Corollary 5.1 (A. B. Attar [2]). *Let R be an r-regular graph with p vertices. If a subset $U \subset V(R)$ with $|E([U])| = m$ and $|U| = k$ is deletable then $p - k$ is a divisor of $rk - 2m$*

Using these two results, all the graphs having reducibility number 1,2,3,4 are completely characterized as follows.

Theorem 5.11. *(A. B. Attar [2]) Let $R \in \mathcal{R}$ be a non-trivial graph. Then $v - red_{\mathcal{R}}(R) = 1$ if and only if R is a complete graph or a null graph.*

Theorem 5.12. *(A. B. Attar [2]) Let $R \in \mathcal{R}$ with $p > 2$ vertices and let r be the degree of every vertex. Then $v - red_{\mathcal{R}}(R) = 2$ if and only if p is even and one of the following statement holds.*

(1) $r = 1$.
(2) $r = \frac{p-2}{2}$ *and there exist vertices u, v in R such that $d(u, v) > 2$.*
(3) $r = \frac{p}{2}$ *and there is an edge which is not in a triangle.*
(4) $r = p - 2$.

Theorem 5.13 (S. Mane [21]). *For $n = 2^k - 1$, $v - red_{\mathcal{R}}(Q_n) = 2^{n-k}$. Further if D is the kernel of the linear transformation $H : Q_n \to Q_k$ where H is a matrix having columns the non-zero vectors of Q_k then D is the smallest deletable set with respect to $\mathcal{R}$, also $Q_n - D$ is bipancyclic i.e. having cycle of length m for every even integer $4 \leq m \leq |V(Q_n - D)|$.*

In general, it is proved that

Theorem 5.14 (S. Mane [21]). *For $2^k - 1 \leq n < 2^{k+1} - 1$ there exists $\mathcal{R}$-deletable set D of cardinality 2^{n-k} such that $Q_n - D$ is $(n-1)$-regular, $(n-1)$-connected and bipancyclic.*

References

[1] A. B. Attar, *Characterization the deletable set of vertices in the (p-3)-regular graphs*, J. Math. Computer Sci., **Vol.3 No.2**, (2011), 156–164.

[2] A. B. Attar, *Reducibility for certain classes of graphs and digraphs*, Ph. D. Thesis, Pune University, (2004).

[3] G. Bordalo, and B. Monjardet, *Reducible classes of finite lattices*, Order, **13, No. 4**, (1996), 379–390.

[4] Y. M. Borse and B. N. Waphare, *On determination and construction of critically 2-connected graphs*, AKCE. J. Graphs. Combin., **7, No.1**, (2010), 73–83.

[5] R. A. Brualdi, J. A. Dias da Silva, *A retract characterization of posets with the fixed-point property*, Discrete Math. **169(1-3)**,(1997),199–203

[6] F. Brucker and A. Gély, *Crown-free lattices and their related graphs*, Order, **28, No. 3**, (2011), 443–454

[7] A. Davis., *A characterization of complete lattices*, Pacific J. Math., **5**(1955), 311–319.

[8] D. Duffus and I. Rival, *Crowns in dismantlable partially ordered sets*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely), **Vol. I**, (1976), 271–292.

[9] D. Duffus , M. Pouze and I. Rival, *Complete ordered sets with no infinite antichains*, Discrete Math., **35**(1981), 39–52.

[10] D. Duffus and I. Rival, *A structure theory for ordered sets*, Discrete Math., **35**, (1981), 53–118.

[11] D. Duffus and I. Rival, *Graphs orientable as distributive lattices*, Proc. Amer. Math. Soc., **88, No. 2**, (1983), 197–200.

[12] T. Fofanova, I. Rival and A. Rutkowski, *Dimension two, fixed points and dismantlable ordered sets*, Order, **13, No. 3**, (1996), 245–253.

[13] H. Grotzsch, *Ein Dreifarbensatz fiir dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin Luther Univ., Halle-Wittenberg, Math. Naturwiss. Reihe, **8**, (1958), 109-120.

[14] J. Jakubík, *Modular lattices of locally finite length*, Acta Sci. Math. (Szeged), **37**, (1975), 79–82.

[15] V. V. Joshi, B. N. Waphare, H. Y. Pourali, *On generalized zero divisor graph of a poset*, Discrete Applied Mathematics, **161**,(2013), 1490–1495.

[16] D. Kelly and I. Rival, *Crowns, fences, and dismantlable lattices*, Canad. J. Math., **26**, (1974),1257–1271.

[17] D. Kelly and I. Rival , *Planar lattices*, Canad. J. Math. **27**, (1975), 636–665.

[18] V. S. Kharat and B. N. Waphare, *Reducibility in finite posets*, European J. Combin. **22, No. 2**, (2001), 197–205.

[19] V. S. Kharat, B. N. Waphare and N. K. Thakare, *Reducibility number, Discrete Applied Mathematics*, **155, No. 16**, (2007), 2069–2076.

[20] B. Larose and L. Zadori, *Algebraic properties and dismantlability of finite posets*, Discrete Math., **163**, (1997), 89–99.

[21] S. Mane, *Domination parameters and fault tolerance of hypercubes*, Ph. D. Thesis, Pune University, (2012).

[22] O. Ore, *Theory of Graphs*, Amer. Math. Soc., Providence, RI, 1962.

[23] I. Rival, *Lattices with doubly irreducible elements*, Canad. Math. Bull., **17**, (1974), 91–95.

[24] I. Rival, *A fixed point theorem for finite partially ordered sels*, J. Combinatorial Theory, Ser.A **21, No. 3**, (1976), 309–318.

[25] A. Tarski, *A lattice-theoretical fixed point theorem and its applications*, Pacific J. Math., **5**, (1955), 285–309.

[26] N. K. Thakare, M. M. Pawar and B. N. Waphare, *A structure theorem for dismantlable lattices and enumeration*, Period. Math. Hungar. **45, No. 1-2**,(2002), 147–160.

[27] W. T. Trotter and J.I. Moore, *Characterization problems for graphs, partially ordered sets, lattices, and families of sets*, Discrete Math., **16**, (1976), 361–381.

[28] B. N. Waphare and A. Bhavale, *Orientable graphs of dismantlable lattices*, 2013 (preprint).

[29] H. Whitney, *Non-separable and planar graphs*, Trans. Amer. Math. Soc., **34** (1932), 339–362.

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R. P. STANLEY'S SOLUTION OF ANAND-DUMIR-GUPTA CONJECTURES ABOUT MAGIC SQUARES*

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Abstract. In 1966 Anand-Dumir-Gupta formulated a number of conjectures concerning enumeration of doubly stochastic matrices with given line sum. In 1973 Stanley solved most of these conjectures using techniques from commutative algebra. We sketch Stanley's solution after recalling necessary results from commutative algebra.

1. THE ANAND-DUMIR-GUPTA CONJECTURES

The objective of this exposition is to outline Stanley's solution of the Anand-Dumir-Gupta (ADG) conjectures. The ADG conjectures concern enumeration of doubly stochastic matrices or magic squares. Let $\mathbb{N}$ denote the set of nonnegative integers and let $\mathbb{P}$ denote the set of positive integers. An $n \times n$ matrix M is called a *magic square* if its entries are in $\mathbb{N}$ and the sum of entries in any row or column is a given integer r. The number r is called the line sum of M. Let $H_n(r)$ denote the number of $n \times n$ magic squares with line sum r. It is clear that

$$H_1(r) = 1 \text{ and } H_2(r) = r + 1.$$

MacMahon [5] and independently Anand-Dumir-Gupta [1] proved that

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

Inspired by these formulas H. Anand, V. C. Dumir and Hansraj Gupta proposed the following conjectures in [1]:

Conjecture 1.1 (**Anand-Dumir-Gupta (1966)**). Fix $n \geq 1$. Then

(1) $H_n(r) \in \mathbb{C}[r]$.
(2) $\deg H_n(r) = (n-1)^2$.

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(3) $H_n(i) = 0$ for $i = -1, -2, \ldots, -(n-1)$.
(4) $H_n(-n-r) = (-1)^{n-1} H_n(r)$.

Quite often integer valued sequences are studied effectively by understanding the properties of their generating functions. Stanley translated the ADG conjectures in terms the generating function of $H_n(r)$. He showed that the four assertions about $H_n(r)$ are equivalent to the following:

$$\sum_{r=0}^{\infty} H_n(r)\lambda^r = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^{(n-1)^2+1}},$$

where $h_0, h_1, \ldots, h_d$ are integers, $d = (n-1)^2 + 1 - n$, $h_0 + h_1 + \cdots + h_d \neq 0$ and $h_{d-i} = h_i$ for $i = 0, 1, \ldots, d$. Stanley made the additional conjectures that
(5) $h_i \geq 0$ for all i and
(6) $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$.

Stanley settled (1)-(5) in 1973 [7]. A geometric proof based on Ehrhart polynomials of integral polytopes appears in Stanley's Green Book [8]. The conjecture (6) has been proved by Athanasiadis [2].

Stanley's solution used several techniques from commutative algebra to solve the ADG conjectures such as Hilbert series of graded modules, Cohen-Macaulay and Gorenstein rings, invariant theory, local cohomology modules and canonical modules of graded Cohen-Maculay rings. We will describe these techniques first and then return to a sketch of Stanley's solution. We have tried to simplify the proof by using the Grothendieck-Serre formula for the difference of Hilbert function and the Hilbert polynomial of a finitely generated graded module over a graded Noetherian ring. We refer the reader to [11] and an excellent survey article by W. Bruns [3] on Stanley's solution in which the techniques required from commutative algebra have been presented in full details. We also refer the reader to an expository account, by Anurag Singh [12], of a solution of the parts (1) and (2) of the ADG conjectures.

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2. HILBERT SERIES OF GRADED ALGEBRAS

Let k be a field. An **$\mathbb{N}$-graded k-algebra** R is a commutative ring containing a sequence of k-vector spaces $R_0 = k, R_1, R_2, \ldots$, such that $R = \oplus_{i=0}^{\infty} R_i$ and the multiplication in R obeys the rule $R_i R_j \subseteq R_{i+j}$ for all i, j. The elments of R_i are called homogeneous of degree i. If R is Noetherian then there exist homogeneous elements $a_1, a_2, \ldots, a_s$ with degree $a_i = e_i$, for certain positive integers $e_1, e_2, \ldots, e_s$ and $R = k[a_1, a_2, \ldots, a_s]$. Thus we may realize R as a quotient of a polynomial ring $A = k[y_1, y_2, \ldots, y_s]$ where $\deg y_i = e_i$ for $i = 1, 2, \ldots, s$. The algebra R is called **standard** if finitely many elements of R_1 generate R as a k-algebra.

A $\mathbb{Z}$ -graded R-module M is an R-module having a direct sum decomposition $M = \oplus_{i=0}^{\infty} M_i$ where each M_i is a k-vector space and $R_i M_j \subseteq M_{i+j}$ for all i, j. A map $f : M \longrightarrow N$ of graded R-modules M and N is called graded if $f(M_i) \subseteq N_i$ for all i. A submodule N of M is called homogeneous if it is generated by homogeneous elements. If N is homogeneous submodule of M, then $N = \oplus_{n \in \mathbb{Z}} N \cap [M]_n$. Here $[M]_n$ denotes the R_0-submodule of M generated by elements of degree n in M. If N is homogeneous submodule of M, then the quotient module M/N is a graded R-module with $[M/N]_n = M_n/N_n$ for all n.

It is clear that $\dim M_i < \infty$ for all i for a Noetherian $\mathbb{Z}$-graded R-module. The **Hilbert series** of M is the generating function

$$F(M, \lambda) = \sum_{n=0}^{\infty} \dim_k M_n \lambda^n.$$

The function $H(M, n) = \dim_k M_n$ is called the **Hilbert function** of M. Recall that the **dimension** of R, $\dim R$, is the maximum number of algebraically independent elements in R over k. Recall that the degree of a rational function $p(x)/q(x)$ is defined to be $\deg p(x) - \deg q(x)$. Finally the dimension of an R-module M is defined by $\dim M = \dim R/\operatorname{ann}(M)$, where $\operatorname{ann}(M) = \{r \in R : rM = 0\}$.

Theorem 2.1 (**Hilbert-Serre Theorem**). *Let R be a Noetherian graded k-algebra generated by elements of positive degrees $e_1, e_2, \ldots, e_s$ and M, a Noetherian graded R-module. Then*

1. *There exists $h(\lambda, \lambda^{-1}) \in \mathbb{Z}[\lambda, \lambda^{-1}]$ such that $F(M, \lambda) = \frac{h(\lambda, \lambda^{-1})}{(1-\lambda^{e_1}) \cdots (1-\lambda^{e_s})}$.*
2. *The order of the pole at $\lambda = 1$ of $F(M, \lambda) = \dim M$.*
3. *If R is standard then $H(M, n)$ is given by a polynomial $P(M, n)$ with rational coefficients for large n. Moreover $\deg P(M, n) = \dim M - 1$.*
4. *$\max\{n : H(M, n) \neq P(M, n)\} = \deg F(M, \lambda)$.*

3. COHEN-MACAULAY GRADED RINGS AND MODULES

In this section, we review the basics of Cohen-Macaulay rings and modules for Noetherian graded algebras over a field. Throughout this section R will denote an $\mathbb{N}$-graded Noetherian k-algebra where k is a field and M will be a graded $\mathbb{Z}$-module over R.

Definition 3.1. A partial homogeneous system of parameters (hsop) for M is a sequence of homogenous elements $\mathbf{a} = a_1, a_2, \ldots, a_r$ of positive degree in R such that $\dim M/\mathbf{a}M = \dim M - r$. We say $\mathbf{a}$ is an hsop for M if $r = \dim M$.

Lemma 3.1. *Let $\dim M = d$. A sequence of elements $\mathbf{a} = a_1, a_2, \ldots, a_d$ is an hsop for M if and only if M is a finitely generated module over $k[a_1, a_2, \ldots, a_d]$.*

Homogeneous systems of parameters for M exist and if k is infinite, we can choose an hsop for M from R_1.

Definition 3.2. A sequence $\mathbf{a} = a_1, a_2, \ldots, a_g$ of homogeneous elements of R of positive degree is called an M-sequence if for $i = 0, 1, \ldots, g-1$, a_{i+1} is a nonzerodivisor on $M/(a_1, a_2, \ldots, a_i)M$.

Theorem 3.1. *(1) All maximal M-sequences in ideal I with $IM \neq M$ have equal length.*

(2) The sequence $a_1, a_2, \ldots, a_g$ is an M-sequence if and only if $a_1, a_2, \ldots, a_g$ are algebraically independent over k and M is a free $k[a_1, a_2, \ldots, a_g]$-module.

(3) The sequence $\mathbf{a} = a_1, a_2, \ldots, a_g$ is an M-sequence if and only if

$$F(M, \lambda) = \frac{F(M/\mathbf{a}M), \lambda)}{\prod_{1=1}^{r}(1 - \lambda^{\deg a_i})}.$$

(4) $\operatorname{depth} M \leq \dim M$.

Definition 3.3. The length of longest M-sequence in an ideal I, (resp. the maximal homogeneous ideal of R) denoted by $\operatorname{depth}_I M$, (resp. $\operatorname{depth} M$) is called the $I-\operatorname{depth}$ of M. The module M is called **Cohen-Macaulay** R-module if $\operatorname{depth} M = \dim M$.

Definition 3.4. Let M be a finitely generated graded module over a graded Noetherian ring R. By projective dimension $\operatorname{pd}(M)$ of M over R, we mean the smallest nonnegative integer p so that there is an exact sequence of free R-modules of finite rank

$$0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

so that coker $\phi_1 = M$.

Theorem 3.2 (**Auslander-Buchsbaum Formula**). *Let M be a finitely generated graded module of finite projective dimension over a graded k-algebra. Then*

$$\operatorname{depth} M + \operatorname{pd} M = \operatorname{depth} R.$$

Theorem 3.3. *The following are equivalent:*

(1) M is Cohen-Macaulay.

(2) Every hsop for M is an M-sequence.

(3) M is a finitely generated free module over $k[\mathbf{a}]$ for some (hence every) hsop $\mathbf{a}$.

Theorem 3.4. *Let I be a proper ideal of a Cohen-Macaulay graded algebra R. Then*

$$\dim(R/I) + \operatorname{ht}(I) = \dim(R).$$

4. MACAULAY'S THEOREM ABOUT GORENSTEIN GRADED RINGS

The purpose of this section is to recall the basic definitions and facts about Gorenstein graded rings and provide a proof of Macaulay's theorem concerning Hilbert series of Gorenstein graded rings. We begin by describing

The category of graded modules

Let R be an $\mathbb{N}$-graded ring. Let $\mathcal{M}$ be the category of $\mathbb{Z}$-graded R-modules. Consider the $\mathbb{Z}$-graded modules

$$M = \bigoplus_{n\in\mathbb{Z}} M_n \text{ and } N = \bigoplus_{n\in\mathbb{Z}} N_n \in \mathcal{M}.$$

An R-linear map $f : M \longrightarrow N$ is a morphism in $\mathcal{M}$ if $f(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. By $M(n)$ we mean the module M with grading defined by $[M(n)]_m = M_{m+n}$ for all $m \in \mathbb{Z}$. Put

$${}^*\operatorname{Hom}(M,N)_n = \{f : M \longrightarrow N(n) \mid f \text{ is degree preserving }\}.$$

and ${}^*\operatorname{Hom}(M,N) = \bigoplus_{n\in\mathbb{Z}} {}^*\operatorname{Hom}(M,N)_n$. It is easy to check that if M is finitely generated then ${}^*\operatorname{Hom}(M,N) = \operatorname{Hom}(M,N)$. Let M be a graded R-module. Consider a projective resolution of M in the category $\mathcal{M}$:

$$\mathcal{P} : \cdots \longrightarrow P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} M \longrightarrow 0.$$

Apply the contravariant functor ${}^*\operatorname{Hom}(-,N)$ to the above resolution to get the complex

$$0 \longrightarrow {}^*\operatorname{Hom}(P_0,N) \xrightarrow{\phi_1\otimes id} {}^*\operatorname{Hom}(P_1,N) \xrightarrow{\phi_2\otimes id} {}^*\operatorname{Hom}(P_2,N) \longrightarrow \cdots.$$

Then ${}^*\operatorname{Ext}^i(M,N) = \operatorname{Kernel}(\phi_{i+1}\otimes 1)/\operatorname{Image}(\phi_i \otimes 1)$.

Proposition 4.1. *Let $A = k[x_1, x_2, \ldots, x_s]$ be polynomial ring over a field k. Let I be a homogeneous ideal of A. Let A/I be Cohen-Macaulay. Then*

$$\operatorname{Ext}^i(A/I, A) \neq 0 \iff i = h = ht(I).$$

Proof By Auslander-Buchsbaum formula

$$\operatorname{pd}(A/I) = \operatorname{depth} A - \dim A/I = s - (s-h) = h.$$

Write a graded minimal resolution of A/I as an A-module:

$$0 \longrightarrow A^{\beta_h} \longrightarrow A^{\beta_{h-1}} \longrightarrow \cdots \longrightarrow A^{\beta_1} \longrightarrow A \longrightarrow 0.$$

Thus $\operatorname{Ext}^i(A/I, A) = 0$ for $i > h$. Since A is Cohen-Macaulay ,

$$\operatorname{depth}_I(A) = \min\{i : \operatorname{Ext}^i(A/I, A) \neq 0\} = h.$$

■

Definition 4.1. The A-module $K_{A/I} = \operatorname{Ext}^h(A/I, A)$ is called the **canonical module** of A/I. The ring A/I is called **Gorenstein** if $K_{A/I} \simeq A/I(a)$. for some $a \in \mathbb{Z}$. The integer a is called the a-invariant of A/I.

Macaulay's theorem for Hilbert series of Gorenstein graded rings

Theorem 4.1. *Put $R = A/I$ and $d = \dim(R)$. Suppose $\deg x_i = e_i \in \mathbb{P}$ for $i = 1, 2, \ldots, s$. Then as rational functions of λ*

$$F(K_R, \lambda) = (-1)^d \; F(R, 1/\lambda) \; \lambda^{-\sum_{i=1}^{s} e_i}.$$

Proof (Stanley) Write a minimal free resolution of R as an A-module:

$$0 \longrightarrow M_h \xrightarrow{\phi_h} M_{h-1} \xrightarrow{\phi_{h-1}} \cdots \longrightarrow M_1 \xrightarrow{\phi_1} M_0 \longrightarrow 0.$$

Apply $\operatorname{Hom}(-, A)$ to the above resolution to get the complex:

$$0 \longrightarrow \operatorname{Hom}(M_0, A) \xrightarrow{\phi_0^*} \cdots \xrightarrow{\phi_h^*} \operatorname{Hom}(M_h, A) \longrightarrow 0.$$

Thus $K_R \simeq \operatorname{Hom}(M_h, A)/\operatorname{Im}(\phi_h^*)$. Hence we have the following minimal free resolution for K_R as an A-module:

$$0 \longrightarrow \operatorname{Hom}(M_0, A) \xrightarrow{\phi_0^*} \cdots \xrightarrow{\phi_h^*} \operatorname{Hom}(M_h, A) \longrightarrow K_R \longrightarrow 0.$$

It is easy to see that for integers m and n,

$$F(M(n), \lambda) = \lambda^{-n} F(M, \lambda) \text{ and } \operatorname{Hom}(A(m), A) \simeq A(-m).$$

Let $\operatorname{rank}(M_i) = \beta_i$, and $M_i = \oplus_{j=1}^{\beta_i} A(-g_{ij})$ for $i = 0, 1, \ldots, h$. Put $D(\lambda) = \prod_{p=1}^{s}(1 - \lambda^{e_p})$ and $N_i(\lambda) = \sum_{j=1}^{\beta_i} \lambda^{g_{ij}}$. Now we calculate the Hilbert series of R and K_R from their minimal free resolutions written above.

$$F(M_i, \lambda) = \sum_{j=1}^{\beta_i} F(A(-g_{ij}), \lambda) = \frac{\sum_{j=1}^{\beta_i} \lambda^{g_{ij}}}{\prod_{p=1}^{s}(1 - \lambda^{e_p})} = \frac{N_i(\lambda)}{D(\lambda)}.$$

Hence $F(R, \lambda) = \sum_{i=0}^{h} N_i(\lambda)/D(\lambda)(-1)^i$. To find $F(K_R, \lambda)$, note that

$$F(K_R, \lambda) = \sum_{i=0}^{h} (-1)^{i+h} F(M_i^*, \lambda) = \sum_{i=0}^{h} (-1)^{i+h} N_i(\lambda^{-1})/D(\lambda).$$

Since $D(\lambda^{-1}) = (-1)^s D(\lambda) \lambda^{-\sum_{p=1}^{s} e_p}$, we get

$$\begin{aligned} F(R, 1/\lambda) &= \sum_{i=0}^{h} (-1)^i \frac{N_i(\lambda^{-1})}{D(\lambda^{-1})} \\ &= (-1)^{s-h} \lambda^{\sum_{i=1}^{s} e_i} F(K_R, \lambda) \\ &= (-1)^d \lambda^{\sum e_i} F(K_R, \lambda). \end{aligned}$$

■

Corollary 4.1 (Macaulay's Theorem). *If the ring $R = A/I$ is Gorenstein of dimension d then for some $\sigma \in \mathbb{Z}$,*

$$F(R, 1/\lambda) = (-1)^d \lambda^\sigma F(R, \lambda).$$

If R is standard Gorenstein with $F(R,\lambda) = \frac{h_0 + h_1\lambda + \cdots + h_g\lambda^g}{(1-\lambda)^d}$, and $h_g \neq 0$, then

(1) $h_i = h_{g-i}$, for all $i = 0, 1, \ldots, g$.
(2) $\sigma = d - g$.
(3) If $\sigma \geq 1$, then $H(n) = \dim R_n$ is a polynomial for all n,
  (a) $H(-i) = 0$ for all $i = 1, 2, \ldots, (\sigma - 1)$, and
  (b) $H(n) = (-1)^{d-1} H(-\sigma - n)$ for all $n \in \mathbb{Z}$.

Proof (1) and (2): Put $e = \sum_{i=1}^{s} e_i$. Suppose R is Gorenstein. Then $K_R \simeq R(a)$, for some $a \in \mathbb{Z}$. Hence

$$F(K_R, \lambda) = \lambda^{-a} F(R, \lambda) = (-1)^d \lambda^{-e} F(R, 1/\lambda).$$

Hence $F(R, \lambda) = \lambda^{a-e}(-1)^d F(R, 1/\lambda)$. Now let R be standard Gorenstein. Write

$$F(R, \lambda) = (h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_g\lambda^g)/(1-\lambda)^d$$

where $h_g \neq 0$. Then

$$F(R, 1/\lambda) = (-1)^d \lambda^{d-g}(h_0\lambda^g + h_1\lambda^{g-1} + \cdots + h_g)/(1-\lambda)^d = \lambda^{e-a}(-1)^d F(R, \lambda).$$

Hence $d - g = e - a = \sigma$ and $h_i = h_{g-i}$ for all $i = 0, 1, \ldots, g$.
(3) By theorem 2.1, it is clear that if $\sigma \geq 1$, then $H(n)$ is a polynomial for all $n \in \mathbb{Z}$ and as $\dim R_n = 0$ for all $n < 0$, $H(n) = 0$ for all $n = -1, -2, \ldots, -(\sigma - 1)$, and $H(-\sigma) \neq 0$. We have for all $n \geq -(\sigma - 1)$,

$$H(n) = h_0\binom{n+d-1}{d-1} + h_1\binom{n+d-2}{d-1} + \cdots + h_g\binom{d-1+n-g}{d-1}.$$

Now use the fact that $h_i = h_{g-i}$ for all $i = 1, 2, \ldots, g$, and $\binom{n}{p} = (-1)^p\binom{p-n-1}{p}$, as polynomials,

$$\begin{aligned} H(n) &= \sum_{i=0}^{g} h_i \binom{d-1+n-i}{d-1} \\ &= \sum_{i=0}^{g} h_{g-i} \binom{d-1-(d-1+n-i)-1}{d-1}(-1)^{d-1} \\ &= \sum_{i=0}^{g} h_i \binom{g-i-n-1}{d-1}(-1)^{d-1} \\ &= \sum_{i=0}^{g} h_i \binom{d-\sigma-i-n-1}{d-1}(-1)^{d-1} \\ &= (-1)^{d-1}H(-\sigma-n). \quad \blacksquare \end{aligned}$$

Definition 4.2. The vector $(h_0, h_1, \ldots, h_g)$ is called the **h-vector** of the standard graded algebra R. If the condition $h_i = h_{g-i}$ is satisfied for all $i = 0, i, \ldots, g$ then we say that the h-vector of R is **symmetric.**

Example 4.1. The symmetry of the h-vector of a standard graded Cohen-Macaulay algebra R does not imply that R is Gorenstein. We construct an example. Consider the ideal $I = (xyz, xw, zw)$ of the polynomial ring $A = k[x, y, z, w]$. The ideal I is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} 0 & -z & x \\ w & -yz & 0 \end{bmatrix}$$

A resolution of $R = A/I$ as an A-module is:

$$0 \longrightarrow A(-3) \oplus A(-4) \xrightarrow{\ f\ } A(-3) \oplus A(-2)^2 \xrightarrow{\ g\ } A \longrightarrow 0,$$

where the maps f and g are defined as

$$f([r, s]) = [r, s] \begin{bmatrix} 0 & -z & x \\ w & -yz & 0 \end{bmatrix} \text{ and } g([r, s, t]) = rxyz + sxw + tzw.$$

It can be shown easily that the above sequence is a minimal resolution of R. Hence by Auslander-Buchsbaum formula, $\operatorname{depth} R = \operatorname{depth} A - \operatorname{pd} R = 4-2 = 2 = \dim R$. Hence R is Cohen-Macaulay. However it is not Gorenstein as the above resolution shows that $\operatorname{rank} K_R = 2$. The Hilbert series of R can be found from the resolution and it turns out to be $(1+2\lambda+\lambda^2)/(1-\lambda)^2$. Hence the h-vector of R is symmetric although it is not Gorenstein.

Remark 4.1. The principal result of [9] shows that the symmetry of the h-vector implies Gorenstein property provided R is a Cohen-Macaulay domain.

5. LOCAL COHOMOLOGY OF GRADED MODULES

We will need an alternative characterization of canonical modules of Cohen-Macaulay graded rings over fields in terms of local cohomology modules. Therefore we will introduce these modules and summarise their fundamental properties. For detailed proofs we refer the reader to [4]. We will also derive a formula for the difference between Hilbert function and Hilbert polynomial of a graded algebra over a field. This formula plays a crucial role in the solution.

Let R be a standard graded ring over a field k. Let $\mathfrak{m}$ be the maximal homogeneous ideal of R. Let $x = x_1, x_2, \ldots, x_r$ be a sequence of homogeneous elements of R of positive degree. Let R_f denote the localization of R at the multiplicatively closed set generated by f. For a $\mathbb{Z}$-graded R-module M, Let $\mathcal{K}(x^\infty, M)$ denote the complex

$$\bigotimes_{i=1}^{r}(0 \longrightarrow R \longrightarrow R_{x_i} \longrightarrow 0) \otimes M.$$

This is called the **localisation complex** of M. The components of this complex are displayed in the sequence:

$$0 \xrightarrow{\delta_0} M \xrightarrow{\delta_1} \prod_{i=1}^{r} M_{x_i} \xrightarrow{\delta_2} \prod_{i<j} M_{x_i x_j} \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_r} M_{x_1 x_2 \ldots x_r} \longrightarrow 0.$$

The map δ_{j+1} is defined as follows: Let $m \in N = M_{x_{i_1} x_{i_2} \ldots x_{i_j}}$. Put

$$[r] \setminus \{i_1, i_2, \ldots, i_j\} = \{l_1, l_2, \ldots, l_{r-j}\}$$

with $l_1 < l_2 < \cdots < l_{r-j}$. Let $\phi_{l_s} : N \longrightarrow N_{x_{l_s}}$ be the natural map for $1 \leq s \leq r-j$. Then

$$\delta_{j+1}(m) = \phi_{l_1}(m) - \phi_{l_2}(m) + \cdots + (-1)^{r-j}\phi_{l_{r-j}}(m).$$

The i^{th} local cohomology module $H_x^i(M)$ is the i^{th} cohomology module of the complex $\mathcal{K}(x^\infty, M)$. Hence

$$H_x^i(M) = H^i(\mathcal{K}(x^\infty, M)) = \ker\delta_{i+1}/\mathrm{im}\delta_i.$$

The local cohomology modules are $\mathbb{Z}$-graded modules. These modules are depth sensitive.

Lemma 5.1. *If the sequence of finite graded R-modules*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0,$$

is exact, then we have the long exact sequences of local cohomology modules

$$\begin{aligned} 0 \longrightarrow H_x^0(M_1) &\longrightarrow H_x^0(M_2) \longrightarrow H_x^0(M_2) \\ \longrightarrow H_x^1(M_1) &\longrightarrow H_x^1(M_2) \longrightarrow H_x^1(M_2) \longrightarrow \cdots \end{aligned}$$

Theorem 5.1. *Let R be a standard graded ring with maximal graded ideal $\mathfrak{m}$ generated by homogeneous elements $x_1, x_2, \ldots, x_r$. Let M be a finitely generated graded R-module. Then*

$$\operatorname{depth}(M) = \min\{i : H_x^i(M) \neq 0\}.$$

Corollary 5.1. *The module M is Cohen-Macaulay if and only if $H_x^i(M) = 0$ for all $i < \dim M$.*

Let $\ell(N)$ denote length of a module N. Let $R = \oplus_{n=0}^{\infty} R_n$ be a standard Noetherian graded ring over an artinian local ring R_0. Let R_+ denote the ideal of R generated by elements of positive degrees. Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded R-module. Let $H_M(n) = \ell(M_n)$ be the Hilbert function of M and $P_M(n)$ be the corresponding Hilbert polynomial. The Hilbert series of M is the formal power series $F(M, \lambda) = \sum_{n \in \mathbb{Z}} \ell(M_n)\lambda^n$. We know that $H_M(n) = P_M(n)$ for all $n > \deg F(M, \lambda)$. The graded local cohomology modules $H_{R_+}^i(M)$ can be used to find the difference $H_M(n) - P_M(n)$. Define the function $\chi_M : \mathbb{Z} \longrightarrow \mathbb{Z}$ by

$$\chi_M(n) = \sum_{i=0}^{\infty} (-1)^i \ell(H_{R_+}^i(M)_n)$$

We have the following more precise result of Grothendieck and Serre.

Theorem 5.2. **[Grothendieck-Serre]** *Let R and M be as above. Then for all $n \in \mathbb{Z}$,*

$$H_M(n) - P_M(n) = \chi_M(n).$$

Proof: First observe that if

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is an exact sequence of $\mathbb{Z}$-graded R-modules, then by using the long exact sequence of local cohomology modules and additivity of length, we get $\chi_N = \chi_M + \chi_P$. We apply induction on $d = \dim M$. If $d = 0$, then $M = H_{R_+}^0(M)$, and $P_M(n) = 0$. Hence the formula is true. Now let $d > 0$. We may replace M by $M/H_{R_+}^0(M)$ without any harm. Thus we have $H_{R_+}^0(M) = 0$. Hence there is an M-regular element a of degree one in R. Consider the exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{\mu_a} M \longrightarrow M/aM \longrightarrow 0.$$

Hence for all $n \in \mathbb{Z}$,

$$\chi_{M/aM}(n) = \chi_M(n) - \chi_M(n-1) \text{ and } H_{M/aM}(n) = H_M(n) - H_M(n-1)$$

Thus $P_{M/aM}(n) = P_M(n) - P_M(n-1)$. Put $f(n) = H_M(n) - P_M(n)$. Then these equations show that the first difference $\Delta f(n) = f(n) - f(n-1)$ coincides with $\Delta(\chi_M)(n)$. Since $f(n) = \chi_M(n) = 0$, for all large n, we have $f(n) = \chi_M(n)$ for all $n \in \mathbb{Z}$. ∎

6. LINEAR HOMOGENEOUS DIOPHANTINE EQUATIONS

Let $x_{ij};\ i,j = 1,2,\ldots,n$ be indeterminates. The entries of an $n \times n$ magic square are solutions to the following system of linear Diophantine equations:

$$x_{11} + x_{12} + \cdots + x_{1n} = \sum_{j=1}^{n} x_{ij} \text{ for } i = 2,3\ldots,n. \tag{6.1}$$

$$x_{11} + x_{12} + \cdots + x_{1n} = \sum_{i=1}^{n} x_{ij} \text{ for } j = 2,3\ldots,n.$$

Thus the problem of counting magic squares is a special case of counting nonnegative integer solutions of a system of linear Diophantine equations. Let Φ be an $r \times n$ $\mathbb{Z}$-matrix. Let $x_1, x_2, \ldots, x_n$ be indeterminates. Let X denote the column vector $(x_1, x_2, \ldots, x_n)^t$. We are interested in the $\mathbb{N}$-solutions to the system $\Phi X = 0$. We gather all the solutions in the monoid

$$E_\Phi = \{\beta \in \mathbb{N}^n : \Phi\beta = 0\}.$$

Let k be any field. Put $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$. With E_Φ we can associate the monoid algebra

$$R_\Phi = k[x^\beta : \beta \in E_\Phi].$$

Stanley studied the monoid algebra R_μ where μ is the $(2n-2) \times n^2$ coefficient matrix of the system (6.1). In particular he showed that the ring R_μ is a Gorenstein and calculated its canonical module and thus its a-invariant. We shall see that these observations are enough to settle the conjectures (1)-(5). Let us begin by observing the

Theorem 6.1. *The monoid algebra R_Φ is a finitely generated k-algebra.*

Proof: Let I denote the ideal in $R = k[x_1, x_2, \ldots, x_n]$ generated by the set

$$P = \{x^\beta : 0 \neq \beta \in E_\Phi\}.$$

Since R is Noetherian, I is generated by a finite subset $G = \{x^{\delta_1}, x^{\delta_2}, \ldots, x^{\delta_t}\}$ of P. We claim that

$$R_\Phi = k[x^\delta : x^\delta \in G].$$

Indeed, any $x^\beta \in R_\Phi$ can be written as $x^\beta = x^{\delta_i} x^\gamma$ for some i and $x^\gamma \in R$. Thus $\gamma = \beta - \delta_i \in E_\Phi$. The argument can be repeated for x^γ, eventually yielding an expression for x^β in terms of x^{δ_i} for $i = 1,2,\ldots,t$. ■

As far as the structure of R_μ is concerned, we have more precise information due to

Theorem 6.2 (Birkhoff-von Neumann Theorem). *Every $n \times n$ magic square is an $\mathbb{N}$-linear combination of the $n \times n$ permutation matrices.*

Thus R_μ is generated by $n!$ degree n monomials. Let $[R_\mu]_r$ denote the k-subspace of R_μ generated by the monomials of degree nr. These monomials are in one-to-one correspondence with magic squares of line sum r. Moreover, $R_\mu = \oplus_{r=0}^{\infty}[R_\mu]_r$. Thus

$$H(R_\mu, r) = \dim_k [R_\mu]_r = H_n(r).$$

This observation will eventually lead to the conclusion that $H_n(r)$ is a polynomial in r for all r. But for the time being we can see that it is so for all large values of r in view of the Hilbert-Serre theorem.

Lemma 6.1. *If $\Phi X = 0$ has a positive solution, then* $\dim R_\Phi = n - \operatorname{rank} \Phi$.

Proof: We show that the vectors $\beta_1, \beta_2, \ldots, \beta_d \in E_\Phi$ are $\mathbb{Q}$-linearly independent if and only if $x^{\beta_1}, x^{\beta_2}, \ldots, x^{\beta_d}$ are algebraically independent over k. Suppose that $\beta_1, \beta_2, \ldots, \beta_d \in E_\Phi$ are linearly independent over $\mathbb{Q}$. Let

$$\sum_\alpha a_\alpha (x^{\beta_1})^{\alpha_1}(x^{\beta_2})^{\alpha_2} \ldots (x^{\beta_d})^{\alpha_d} = 0,$$

for certain $a_\alpha \in k$ and distinct vectors $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. As $\beta_1, \beta_2, \ldots, \beta_d$ are linearly independent over $\mathbb{Q}$, the vectors $\alpha_1\beta_1 + \cdots + \alpha_d\beta_d$ are distinct. Hence $a_\alpha = 0$ for all α.

Conversely let $x^{\beta_1}, x^{\beta_2}, \ldots, x^{\beta_d}$ be algebraically independent over k. Suppose that $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}$ such that $\alpha_1\beta_1 + \cdots + \alpha_d\beta_d = 0$. Without loss of generality we may assume that $\alpha_1, \alpha_2, \ldots, \alpha_p > 0$ and $\alpha_{p+1}, \ldots, \alpha_d < 0$. Then

$$\alpha_1\beta_1 + \cdots + \alpha_p\beta_p = \alpha_{p+1}\beta_{p+1} + \cdots + \alpha_d\beta_d.$$

This yields the relation $x^{\alpha_1\beta_1} \cdots x^{\alpha_p\beta_p} = x^{\alpha_{p+1}\beta_{p+1}} \cdots x^{\alpha_d\beta_d}$.

Let $\alpha \in \mathbb{P}^n \cap E_\Phi$. Let $d = n - \operatorname{rank} \Phi$. Pick linearly independent solutions $\beta_1, \beta_2, \ldots, \beta_d \in \mathbb{Z}^n$ of $\Phi X = 0$. Let $t \in \mathbb{P}$. If $\alpha - t\beta_1, \alpha - t\beta_2, \ldots, \alpha - t\beta_d$ are linearly dependent over $\mathbb{Q}$, there exist $a_1, a_2, \ldots, a_d \in \mathbb{Z}$, not all zero such that

$$a_1(\alpha - t\beta_1) + a_2(\alpha - t\beta_2) + \cdots + a_d(\alpha - t\beta_d) = 0.$$

We have $b_1, b_2, \ldots, b_d \in \mathbb{Q}$ such that $\alpha = b_1\beta_1 + b_2\beta_2 + \cdots + b_d\beta_d$. Hence

$$a_1(b_1 - t)\beta_1 + a_2(b_2 - t)\beta_2 + \cdots + a_d(b_d - t)\beta_d = 0.$$

Hence by selecting $t \in \mathbb{P}$ sufficiently small, we get $a_i = 0$, for all $i = 1, 2 \ldots, d$. This contradiction proves that $\alpha - t\beta_1, \alpha - t\beta_2, \ldots, \alpha - t\beta_d$ are linearly independent solutions in $\mathbb{P}^n$. ■

7. STANLEY'S SOLUTION OF THE ADG CONJECTURES

Theorem 7.1. *$H_n(r)$ is a polynomial of degree $(n-1)^2$ for large r.*

Proof: The ring R_μ is a standard graded k-algebra. The r^{th} graded component of it is generated by monomials of degree rn corresponding to magic squares of line sum r. Hence by Theorem 2.1, $\deg H_n(r) = \dim R_\mu - 1$ and $H_n(r)$ is a polynomial for large r. We show that

$$\dim R_\mu = (n-1)^2 + 1.$$

By the above lemma, $\dim R_\mu = \text{nullity}\,\Phi$. Note that to construct a magic square, we may assign any nonnegative values to the variables x_{ij}, for $i, j = 1, 2, \ldots, (n-1)$ and a value for x_{1n} will determine the rest of the entries of it. Thus $\text{nullity}\,\Phi = (n-1)^2 + 1$. ■

The next important step in the solution is the observation that the graded ring R_Φ is Cohen-Macaulay.

This is proved by showing that it is a ring of invariants of an algebraic torus acting linearly on a polynomial ring. A well-known theorem of Hochster then implies that it is Cohen-Macaulay.

Let k^* denote the multiplicative group of k. Consider the algebraic torus

$$T = \{\text{diag}(u^{\gamma_1}, u^{\gamma_2}, \ldots, u^{\gamma_n}) : u = (u_1, u_2, \ldots, u_r) \in (k^*)^r\}.$$

Let us write the $r \times n$ matrix $\Phi = [\gamma_1, \gamma_2, \ldots, \gamma_n]$ where γ_i is the i^{th} column vector of Φ. T acts on $R = k[x_1, x_2, \ldots, x_n]$ via the automorphisms

$$\tau_u : x_i \longrightarrow u^{\gamma_i} x_i, \ \ i = 1, 2, \ldots, n.$$

Let $\beta \in \mathbb{N}^n$. Then

$$\begin{aligned}\tau_u(x^\beta) &= (u^{\gamma_1}x_1)^{\beta_1}(u^{\gamma_2}x_2)^{\beta_2}\cdots(u^{\gamma_n}x_n)^{\beta_n}\\ &= u^{\beta_1\gamma_1+\beta_2\gamma_2+\cdots\beta_n\gamma_n}x^\beta\end{aligned}$$

Hence $\tau_u(x^\beta) = x^\beta$ if and only if $\beta \in E_\Phi$. Hence R_Φ is the ring of invariants of the torus T acting linearly on R. By Hochster's theorem [6] R_Φ is Cohen-Macaulay.

We can now dispose the conjecture (5) of Stanley. Since R_μ is Cohen-Macaulay homogenous ring of dimension $d = (n-1)^2 + 1$, there exists an hsop $\mathbf{a}$ for R_μ of elements of degree one. Hence

$$F(R_\mu, \lambda) = \frac{F(R_\mu/(\mathbf{a}), \lambda)}{(1-\lambda)^d}.$$

Hence the numerator of the above Hilbert series is a polynomial with positive coefficients.

Finally we sketch the solutions of the parts (1)-(4) of the ADG-Conjecture By Corollary 4.1, we need to show that the degree of the Hilbert series of R_μ is $-n$ and it is Gorenstein. This is done by computing its canonical module. Stanley showed that the canonical module of R_Φ is precisely $k[x^\beta : \beta \in \mathbb{P}^n \cap E_\Phi]$. Thus if $\gamma = (1, 1, \ldots, 1) \in E_\Phi$, then

$$K_{R_\Phi} = x^\gamma R_\Phi.$$

Hence in this case, R_Φ is Gorenstein. For the case of magic squares, the $n \times n$ magic square whose each entry is 1 is the smallest positive solution. Thus $K_{R_\mu} = \prod_{1\le i,j\le n} x_{ij} R_\mu$. Since the degree of $\prod_{1\le i,j\le n} x_{ij}$ as an element of R_μ is n, the degree of its Hilbert series is $-n$. It proves that $H_n(r)$ is a polynomial for all $r > -n$. Moreover $H_n(-n) \neq 0$ and

$$H_n(-1) = H_n(-2) = \cdots = H_n(-(n-1)) = 0.$$

Corollary 4.1, we conclude that

$$H_n(r) = (-1)^{(n-1)^2} H_n(-r-n),$$

for all n. But $(n-1)^2$ and $(n-1)$ have same parity. Thus for all r

$$H_n(r) = (-1)^{n-1} H_n(-r-n),$$

References

[1] H. Anand, V. C. Dumir and H. Gupta, *A combinatorial distribution problem,* Duke Math. J., **33**(1966), 757–769.

[2] Ch. A. Athanasiadis, *Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley,* J. Reine Angew. Math. 583 (2005), 163–174.

[3] W. Bruns, *Commutative algebra arising out of Anand-Dumir-Gupta conjectures,* Proceedings of International Conference on Commutative Algebra and Combinatorics, Editor W. Bruns, Ramanujan Mathematical Society (2007), 1–38.

[4] W. Bruns and J. Herzog, *Cohen-Macaulay Rings, revised edition,* Cambridge University Press, 1998.

[5] P. A. MacMahon,*Combinatory Analysis*, vols 1 and 2, Cambridge, 1916; reprinted by Chelsea, New York, 1960.

[6] M. Hochster, *Rings of invariants of Tori, Cohen-Macaulay rings generated by monomials and polytopes,* Annals of Math,**96** (1972), 318–337.

[7] R. P. Stanley,*Linear homogeneous Diophatine equations and magic labelling of graphs,* Duke Math. J., **40**(1973), 607–632.

[8] R. P. Stanley, *Enumerative Combinatorics*, Vol I, Wardsworth and Brooks/Cole, Pacific Grove, CA, 1986.

[9] R. P. Stanley, *Hilbert functions of graded algebras,* Advances in Math, **28**(1978), 57–83.

[10] R. P. Stanley, *Linear Diophantine equations and local cohomology,* Inv. Math. **68**(1982), 175–193.

[11] R. P. Stanley, *Combinatorics and Commutative Algebra,* Second Edition, Birkhaüser, Boston, 1996.

[12] A. K. Singh, *Tight closure: applications and questions,* Proceedings of the 4^{th} Japan-Vietnam Joint Seminar on Commutative Algebra (2009), 1–18.

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CONVOLUTION FUNCTION OF SEVERAL VARIABLES WITH p(n)−BOUNDED VARIATION

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Abstract. Let L^1 be the class of all complex functions, with period 2π in each variable, in the space $L^1(\overline{\mathbb{T}}^N)$, where $\mathbb{T} = [0, 2\pi)$ is one dimensional torus. Here, it is observed that $L^1 * E \subset E$ for $E = BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N)$.

1. Introduction.

The convolution product is known for inheriting the best smoothness property as well as the best variation property of each parent function[9]. Because of these properties convolution product is used in digital image-processing for average filtering, for edge detection and for smoothing an edge of an image.

For any complex valued functions $f, g \in L^1(\overline{\mathbb{T}}^N)$, where f and g both are $2\pi-$periodic functions in each variable, the convolution of f and g (that is $f * g$) is defined as

$$(f * g)(\mathbf{x}) = \frac{1}{(2\pi)^N}\int_{\overline{\mathbb{T}}^N} f(\mathbf{y})g(\mathbf{x}\text{-}\mathbf{y})\, d\mathbf{y},\ \forall \mathbf{x} \in \overline{\mathbb{T}}^N.$$

For functions of several variables it is also well know that $L^1(\overline{\mathbb{T}}^N) * L^p(\overline{\mathbb{T}}^N) \subset L^p(\overline{\mathbb{T}}^N)$. Recently, it is observed that [8]

$$L^1(\overline{\mathbb{T}}^N) * E = \{f * g : f \in L^1(\overline{\mathbb{T}}^N) \text{ and } g \in E \text{ over } \overline{\mathbb{T}}^N\} \subset E$$

for $E = Lip(p; \alpha_1, \alpha_2, ..., \alpha_N)$ over $\overline{\mathbb{T}}^N$, for $E = AC(\overline{\mathbb{T}}^N)$, for $E = r - BV(\overline{\mathbb{T}}^N)$ and for $E = \Lambda BV^{(p)}(\overline{\mathbb{T}}^N)$.

Here, in Section 2 the concept of $p(n)-$bounded variation is developed for a function of two variables. In Section 3 properties of functions of $p(n)-$bounded variation are stated and proved. In Section 4 we prove our main result for $N = 2$ and finally in Section 5 we explain how the general case can be obtained.

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2. Notations and definitions.

Let f be a complex valued function on a rectangle $R^2 = [a_1, b_1] \times [a_2, b_2]$ in 2-dimensional Euclidean space $\mathbb{R}^2$. Let $P = \{R_1, R_2, \cdots, R_m\}$ be a (finite) partition of R^2, where R_i are pair wise disjoint (no two have common interior) rectangles of R^2 having their sides parallel to the standard coordinate axis and whose union is R^2. Consider a sub-rectangle $R' = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ of R^2 with $a_i \leq \alpha_i < \beta_i \leq b_i$ for all $i = 1, 2$. Define $\Delta f(R')$, ΔR_i and $\delta(P)$ as follow.

$$\begin{aligned} \Delta f(R') &= \Delta f(\ [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\ ) \\ &= f(\beta_1, \beta_2) - f(\beta_1, \alpha_2) - f(\alpha_1, \beta_2) + f(\alpha_1, \alpha_2), \\ \Delta R' &= (\beta_1 - \alpha_1) \times (\beta_2 - \alpha_2) \text{ and } \delta(P) = \min_{1 \leq i \leq m} \{\Delta R_i\}. \end{aligned}$$

Definition 2.1. Let $\{\varphi(n)\}_{n=1}^{\infty}$ be a real sequence such that $\varphi(1) \geq 2$ and $\lim n \to \infty\ \varphi(n) = \infty$. For $1 \leq p(n) \uparrow p\ as\ n \to \infty$, where $1 \leq p \leq \infty$, we say that $f \in BV_V(p(n) \uparrow p, \varphi, R^2)$(that is, f is said to be of $p(n)$ bounded variation) if

$$V(f, p(n), \varphi, R^2) = \sup_{n \geq 1} \sup_{P} \left\{ \left(\sum_{i=1}^{m} |\Delta f(R_i)|^{p(n)} \right)^{1/p(n)} : \delta(P) \geq \frac{\triangle R^2}{\varphi(n)} \right\} < \infty,$$

where P is a partition of R^2.

Remark 2.1. In the above definition, for $p(n) = 1$, $\forall\ n$, one gets the class of functions of BV in the sense of Vitali (see, for example, [6]) and for $p(n) = p$, $\forall\ n$, one gets the class $BV_V^{(p)}(R^2)$.

Note that, a function $f \in BV_V(p(n) \uparrow p, \varphi, R^2)$ is not necessarily Lebesgue integrable. This can be easily observed from the following example.

Example 2.1. [5, p.23]**:** Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x} + \frac{1}{y}, & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \frac{1}{x}, & \text{if } x \neq 0 \text{ and } y = 0, \\ \frac{1}{y}, & \text{if } x = 0 \text{ and } y \neq 0, \\ 0, & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Then $V(f, p(n), \varphi, [0,1]^2) = 0 < \infty$, however $f \notin B([0,1]^2)$.

Definition 2.2. We say that a function $f = f(x_1, x_2) \in BV_H(p(n) \uparrow p, \varphi, R^2)$, if $f \in BV_V(p(n) \uparrow p, \varphi, R^2)$ and if the marginal functions $f(., a_2)$ and $f(a_1, .)$ are of $p(n)$-bounded variation on the intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ respectively (refer [1, Definition 1,p.401] for the definition of $p(n)$-bounded variation over $[a, b]$).

Remark 2.2. Note that, for $p(n) = 1$, for all n, the class $BV_H(p(n) \uparrow p, \varphi, R^2)$ reduces to the class $BV_H(R^2)$ of functions bounded variation in the sense of Hardy [4].In the above definition 2.2, for $p(n) = p$, for all n , one gets the class $BV_H^{(p)}(R^2)$.

Both these generalized Wiener classes possess many interesting properties. Here, we prove the following properties.

3. Properties of functions of $p(n)-$bounded variation.

Lemma 3.1. *If $f \in BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2)$, then f is bounded over $\overline{\mathbb{T}}^2$, that is $BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2) \subseteq B(\overline{\mathbb{T}}^2)$.*

Proof. For any $(x, y) \in \overline{\mathbb{T}}^2$.

$$\begin{aligned}
&|f(x,y)| \\
&= |f(x,y) - f(0,y) - f(x,0) + f(0,0) + f(0,y) \\
&\quad - f(0,0) + f(x,0) - f(0,0) + f(0,0)| \\
&\quad \le |f(x,y) - f(0,y) - f(x,0) + f(0,0)| \\
&\quad + |f(0,y) - f(0,0)| + |f(x,0) - f(0,0)| + |f(0,0)| \\
&= \Big(|f(x,y) - f(0,y) - f(x,0) + f(0,0)|^{p(n)}\Big)^{1/p(n)} \\
&\quad + \Big(|f(0,y) - f(0,0)|^{p(n)}\Big)^{1/p(n)} \\
&\quad + \Big(|f(x,0) - f(0,0)|^{p(n)}\Big)^{1/p(n)} + |f(0,0)| \\
&\quad \le V(f, p(n), \varphi, [0,2\pi]^2) + V(f(0,.), p(n), \varphi, [0,2\pi]) \\
&\quad + V(f(.,0), p(n), \varphi, [0,2\pi]) + |f(0,0)|.
\end{aligned}$$

This completes the proof.

Lemma 3.2. *The class $BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2) = B(\overline{\mathbb{T}}^2)$, if and only if there is a positive constant M such that*

$$(\varphi(n))^{1/p(n)} \le M, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Proof. First, assume that the condition (3.1) is satisfied. Consider any partition P=$\{R_1, R_2, \cdot, \cdot, \cdot, R_m\}$ of $\overline{\mathbb{T}}^2$, where $\delta(R_i) \ge \frac{(2\pi)^2}{\varphi(n)}$, $\forall$ i=1 to m. Therefore, $m \le \varphi(n)$. Then for any bounded function $f(x, y)$ over $\overline{\mathbb{T}}^2$, we get

$$\begin{aligned}
&\{\sum_{i=1}^{m} |\triangle f(R_i)|^{p(n)}\}^{1/p(n)} \le \{\sum_{i=1}^{m} (4||f||_\infty)^{p(n)}\}^{1/p(n)} \\
&\quad = 4||f||_\infty \, m^{1/p(n)} \le 4||f||_\infty \, (\varphi(n))^{1/p(n)} \le 4||f||_\infty \, M.
\end{aligned}$$

Hence, from Lemma 3.1, we get $BV_H(p(n), \varphi, \mathbb{T}^2) = B(\mathbb{T}^2)$.

Converse: Suppose that the condition (3.1) is not carried out, *i.e.* there exist a sequence $\{n_k\}$ of natural numbers such that

$$\lim_{k\to\infty} (\varphi(n_k))^{1/p(n_k)} = \infty. \tag{3.2}$$

Let E be the set of those points of the segment [0,2π] which are represented as $\frac{m\pi}{\sqrt{\varphi(n)}}$, where m and n takes all possible values from the set of nonnegative integers. Then E is countable dense subset of [0,2π] implies its compliment F=$[0, 2\pi]\setminus E$ is also dense in [0,2π]. Define $f : \overline{\mathbb{T}}^2 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} 1, & \text{if } x = y \text{ and } x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

For all natural k such that $\varphi(k) \geq 36$, let us consider

$$t_{2i} = \frac{3\pi i}{\sqrt{\varphi(n_k)}} \text{ and } t_{2i+1} \in F(i = 1, 2, 3, ..., [\frac{\sqrt{\varphi(n_k)}}{3}])$$

such that

$$t_{2i+1} - t_{2i} \geq \frac{2\pi}{\sqrt{\varphi(n_k)}}.$$

Then, for

$$R_i = [t_i, t_{i+1}] \times [t_i, t_{i+1}]$$

we get

$$\begin{aligned} \{\sum_{i=1}^{[\sqrt{\varphi(n_k)}/3]^2} |\triangle(R_i)|^{p(n_k)}\}^{1/p(n_k)} &= ([\frac{\sqrt{\varphi(n_k)}}{3}]^2)^{1/p(n_k)} \\ &\geq (\frac{1}{36}\varphi(n_k))^{1/p(n_k)}. \end{aligned}$$

This together with (3.2) implies bounded function f$\notin BV(p(n), \varphi, \overline{\mathbb{T}}^2)$ - a contradiction.

This completes the proof.

Lemma 3.3. *We have*

$$B(\overline{\mathbb{T}}^2) \cap BV_V^{(p)}(\overline{\mathbb{T}}^2) \subseteq B(\overline{\mathbb{T}}^2) \cap BV_V(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$$

and

$$BV_H^{(p)}(\overline{\mathbb{T}}^2) \subseteq BV_H(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2).$$

For $1 \le p < \infty$, *we have*

$$B(\overline{\mathbb{T}}^2) \cap \bigcup_{1 \le q < p} BV_V^{(q)}(\overline{\mathbb{T}}^2) \subseteq B(\overline{\mathbb{T}}^2) \cap BV_V(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2) \subseteq B(\overline{\mathbb{T}}^2) \cap BV_V^{(p)}(\overline{\mathbb{T}}^2)$$

and

$$\bigcup_{1 \le q < p} BV_H^{(q)}(\overline{\mathbb{T}}^2) \subseteq BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2) \subseteq BV_H^{(p)}(\overline{\mathbb{T}}^2).$$

Proof. Let $f \in BV_H^{(p)}(\overline{\mathbb{T}}^2)$. Then from Lemma 3.1, we get $f \in B(\overline{\mathbb{T}}^2)$. Since $1 \le p(n) \uparrow \infty$ as $n \to \infty$, for a given p there exists some $n_0 \in \mathbb{N}$ such that $p < p(n) \le p(n+1) \le \cdots$ for all $n \ge n_0$. Let $P = \{R_1, R_2, \cdots, R_m\}$ be any finite partition of $\overline{\mathbb{T}}^2$ such that

$$\delta(P) \ge \frac{4\pi^2}{\varphi(n)},$$

where $\delta(P) = \min_{1 \le i \le m} \{\Delta R_i\}$.

When $1 \le n \le n_0$, we get

$$\begin{aligned}\{\textstyle\sum_{i=1}^m |\triangle f(R_i)|^{p(n)}\}^{1/p(n)} &\le \{\textstyle\sum_{i=1}^m (4||f||_\infty)^{p(n)}\}^{1/p(n)} \\ &= 4||f||_\infty\, m^{1/p(n)} \le 4||f||_\infty\, (\varphi(n))^{1/p(n)} \le 4||f||_\infty\, \varphi(n_0) < \infty.\end{aligned}$$

When $n > n_0$, we have

$$(\sum_{i=1}^m |\triangle f(R_i)|^{p(n)})^{1/p(n)} \le (\sum_{i=1}^m |\triangle f(R_i)|^p)^{1/p} \le V(f, p, \overline{\mathbb{T}}^2) < \infty.$$

Therefore $f \in B(\overline{\mathbb{T}}^2) \cap BV_V(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$ implies $B(\overline{\mathbb{T}}^2) \cap BV_V^{(p)}(\overline{\mathbb{T}}^2) \subseteq B(\overline{\mathbb{T}}^2) \cap BV_V(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$. Hence, $BV_H^{(p)}(\overline{\mathbb{T}}^2) \subseteq BV_H(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$ follows from Lemma 3.1 and the result [7, Lemma 2.7].

Let $1 < p < \infty$ and $1 \le p(n) \uparrow p$. Since $1 \le p(n) \le p$, it follows that

$$(\sum_{i=1}^m |\triangle f(R_i)|^p)^{1/p} \le (\sum_{i=1}^m |\triangle f(R_i)|^{p(n)})^{1/p(n)}.$$

Therefore, we get $B(\overline{\mathbb{T}}^2) \cap BV_V(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2) \subseteq B(\overline{\mathbb{T}}^2) \cap BV_V^{(p)}(\overline{\mathbb{T}}^2)$. Thus, $BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2) \subseteq BV_H^{(p)}(\overline{\mathbb{T}}^2)$ follows from Lemma 3.1 and the result [7, Lemma 2.7]. Let $1 \le q < p$. Then there is an integer n_0 such that $q < p(n) \le p(n+1) \le \cdots \le p$ for all $n \ge n_0$.

The remaining proof is similar to the proof of $B(\overline{\mathbb{T}}^2) \cap BV_V^{(p)}(\overline{\mathbb{T}}^2) \subseteq B(\overline{\mathbb{T}}^2) \cap BV_V(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$ and $BV_H^{(p)}(\overline{\mathbb{T}}^2) \subseteq BV_H(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$.

4. Main result for functions of two variables.

Theorem 4.1. *If $f \in L^1(\overline{\mathbb{T}}^2)$ and $g \in BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2)$ then $f * g \in BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2)$.*

Proof. If $f = 0$ then $f * g = 0$ and hence $f * g \in BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2)$. We may therefore assume without the loss of generality that $||f||_1 = 1$. Let $\{R_k := [a_k, b_k] \times [c_k, d_k]\}_{k=1}^m$ be any finite collection of pairwise non-overlapping sub-rectangles of $\overline{\mathbb{T}}^2$. Then by Holder's inequality, for any $k \in \{1, 2, ..., m\}$ one gets

$$|f * g(a_k, c_k) - f * g(b_k, c_k) - f * g(a_k, d_k) + f * g(b_k, d_k)|^{p(n)}$$
$$\leq (\int\int_{\overline{\mathbb{T}}^2} (|f(x, y)|^{1/q(n)})(|f(x, y)|^{1/p(n)})|g(a_k - x, c_k - y) - g(b_k - x, c_k - y)$$
$$-g(a_k - x, d_k - y) + g(b_k - x, d_k - y)|dx\ dy)^{p(n)}$$
$$\leq \int\int_{\overline{\mathbb{T}}^2} |f(x, y)||g(a_k - x, c_k - y) - g(b_k - x, c_k - y) - g(a_k - x, d_k - y)$$
$$+g(b_k - x, d_k - y)|^{p(n)} dx\ dy$$

where in $q(n)$ is the index conjugate to $p(n)$ for each n.
Summing both the sides of the above inequality over $k = 1$ to m, we have

$$\sum_{k=1}^m |f * g(a_k, c_k) - f * g(b_k, c_k) - f * g(a_k, d_k) + f * g(b_k, d_k)|^{p(n)}$$
$$\leq \int\int_{\overline{\mathbb{T}}^2} |f(x, y)| \mid \sum_{k=1}^m |g(a_k - x, c_k - y) - g(b_k - x, c_k - y)$$
$$-g(a_k - x, d_k - y) + g(b_k - x, d_k - y)|^{p(n)} dx\ dy.$$

Thus,
$V(f * g, p(n), \varphi, \overline{\mathbb{T}}^2) \leq V(g, p(n), \varphi, \overline{\mathbb{T}}^2)$.
Similarly, it is easy to prove that the marginal functions satisfy
$V(f * g(0, .), p(n), \varphi, [0, 2\pi]) \leq V(g(0, .), p(n), \varphi, [0, 2\pi])$ and
$V(f * g(., 0), p(n), \varphi, [0, 2\pi]) \leq V(g(., 0), p(n), \varphi, [0, 2\pi])$.

This proves the theorem.

5. Extension of the results for functions of several variables.

Let f be a complex valued function on a cube $R^N = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N]$ in N-dimensional Euclidean space $\mathbb{R}^N$. Let $P = \{R_1, R_2, \cdots, R_m\}$ be a (finite) partition of R^N, where R_i are pair wise disjoint (no two have common interior) parallelepiped of R^N having their sides (faces) parallel to the standard coordinate hyperplanes and whose union is R^N. Consider a sub-parallelepiped $R' = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_N, \beta_N]$ of R^N with $a_i \leq \alpha_i < \beta_i \leq b_i$ for all $i = 1, 2, 3, \cdots N$. Define $\Delta f(R')$, ΔR_i and $\delta(P)$ as follow.

For N=2,

$$\Delta f(R') = \Delta f(\ [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\ ) = f(\beta_1, \beta_2) - f(\beta_1, \alpha_2) - f(\alpha_1, \beta_2) + f(\alpha_1, \alpha_2)$$

and $\Delta R' = (\beta_1 - \alpha_1) \times (\beta_2 - \alpha_2)$.

For N=3,
$\Delta f(R') = \Delta f([\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times[\alpha_3,\beta_3])$

$$= [f(\beta_1,\beta_2,\beta_3) - f(\beta_1,\alpha_2,\beta_3) - f(\alpha_1,\beta_2,\beta_3) + f(\alpha_1,\alpha_2,\beta_3)]$$
$$-[f(\beta_1,\beta_2,\alpha_3) - f(\beta_1,\alpha_2,\alpha_3) - f(\alpha_1,\beta_2,\alpha_3) + f(\alpha_1,\alpha_2,\alpha_3)]$$
$$= \Delta_{[\alpha_3,\beta_3]}\Delta f([\alpha_1,\beta_1]\times[\alpha_2,\beta_2])$$

and $\Delta R' = (\beta_1-\alpha_1)\times(\beta_2-\alpha_2)\times(\beta_3-\alpha_3)$.

Successively, for any $N \geq 3$ we have

$$\Delta f(R') = \Delta f([\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times\cdots\times[\alpha_N,\beta_N])$$
$$= \Delta_{[\alpha_N,\beta_N]}\Delta f([\alpha_1,\beta_1]\times\cdot\cdot\times[\alpha_{N-1},\beta_{N-1}])$$

and $\Delta R' = \Pi_{i=1}^{N}(\beta_i - \alpha_i)$.

For a given partition P of R^N define

$$\delta(P) = \min_{1\leq i\leq m}\{\Delta R_i\}.$$

Definition 5.1. Let $\{\varphi(n)\}_{n=1}^{\infty}$ be a real sequence such that $\varphi(1) \geq 2$ and $\lim_{n\to\infty} \varphi(n) = \infty$. For $1 \leq p(n) \uparrow p \; as \; n \to \infty$, where $1 \leq p \leq \infty$, we say that $f \in BV_V(p(n)\uparrow p, \varphi, R^N)$ if

$$V(f,p(n),\varphi,R^N) = \underset{n\geq 1}{sup}\;\underset{P}{sup}\{(\sum_{i=1}^{m}|\Delta f(R_i)|^{p(n)})^{1/p(n)} : \delta(P) \geq \frac{\triangle R^N}{\varphi(n)}\} \;<\; \infty,$$

where P is a partition of R^N.

Remark 5.1. In the above definition, for $p(n) = 1$, $\forall\; n$, one gets the class of functions of BV in the sense of Vitali (see, for example, [6]) and for $p(n) = p$, $\forall\; n$, one gets the class $BV_V^{(p)}(R^N)$.

Note that, when N >1, a function $f \in BV_V(p(n)\uparrow p,\varphi,R^N)$ is not necessarily Lebesgue integrable.

In case of $N = 2$, we have already defined the class $BV_H(p(n) \uparrow p, \varphi, R^2)$ in Section 2. In case of $N \geq 3$, the notion of $p(n)$ bounded variation (in the sense of Hardy) over parallelepiped R^N can naturally be defined by the following recurrence:
$f \in BV_H(p(n)\uparrow p,\varphi,R^N)$ if $f \in BV_V(p(n)\uparrow p,\varphi,R^N)$ and each of the marginal functions $f(x_1,\cdots,a_k,\cdots,x_N)$ is in the class $f \in BV_H(p(n)\uparrow p,\varphi,R(a_k))$ where $k = 1,2,\cdots,N$ and

$R(a_k) = \{(x_1,\cdots,x_{k-1},x_{k+1},\cdots,x_N) : x_j \in [a_j,b_j], j = 1,2,\cdots,k-1,k+1,\cdots,N\}$.

This definition can be equivalently reformulated as follows: $f \in BV_H(p(n) \uparrow p, \varphi, R^N)$ if and only if $f \in BV_V(p(n) \uparrow p, \varphi, R^N)$ and for any choice of $(1 \le) j_1, < \cdots < j_m (\le N)$, $1 \le m < N$, the function $f(x_1, \cdots, a_{j_1}, \cdots, a_{j_m}, \cdots, x_N)$ is in the class $f \in BV_H(p(n) \uparrow p, \varphi, R(a_{j_1}, \cdots, a_{j_m}))$, where

$$R(a_{j_1}, \cdots, a_{j_m}) = \{(x_{l_1}, \cdots, x_{l_{N-m}}) \in R^{N-m} : a_j \le x_j \le b_j,$$
$$for \quad j = l_1, l_2, \cdots, l_{N-m}\}$$

and $\{l_1, \cdots, l_{N-m}\}$ is the complementary set of $\{j_1, \cdots, j_m\}$ with respect to $\{1, 2, \cdots, N\}$.

Remark 5.2. Note that, for $p(n) = 1$, for all n, the class $BV_H(p(n) \uparrow p, \varphi, R^N)$ reduces to the class $BV_H(R^N)$ [4].

Following are the extensions of the results of Section 3 and Section 4 in the general case N.

Lemma 5.1. *If $f \in BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N)$, then f is bounded over $\overline{\mathbb{T}}^N$, that is $BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N) \subseteq B(\overline{\mathbb{T}}^N)$.*

Lemma 5.2. *The class $BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N) = B(\overline{\mathbb{T}}^N)$, if and only if there is a positive constant M such that*

$$(\varphi(n))^{1/p(n)} \le M, \quad \forall n \in \mathbb{N}. \tag{5.1}$$

Lemma 5.3. *We have*

$$B(\overline{\mathbb{T}}^N) \cap BV_V^{(p)}(\overline{\mathbb{T}}^N) \subseteq B(\overline{\mathbb{T}}^N) \cap BV_V(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^N)$$

and
$BV_H^{(p)}(\overline{\mathbb{T}}^N) \subseteq BV_H(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^N)$.
For $1 \le p < \infty$, we have

$$B(\overline{\mathbb{T}}^N) \cap \bigcup_{1 \le q < p} BV_V^{(q)}(\overline{\mathbb{T}}^N) \subseteq B(\overline{\mathbb{T}}^N) \cap BV_V(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N) \subseteq B(\overline{\mathbb{T}}^N) \cap BV_V^{(p)}(\overline{\mathbb{T}}^N)$$

and

$$\bigcup_{1 \le q < p} BV_H^{(q)}(\overline{\mathbb{T}}^N) \subseteq BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N) \subseteq BV_H^{(p)}(\overline{\mathbb{T}}^N).$$

Theorem 5.1. *If $f \in L^1(\overline{\mathbb{T}}^N)$ and $g \in BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N)$ then $f * g \in BV_H(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N)$.*

All these extended results can be easily proved by induction of the arguments with respect to N.

References

[1] T. Akhobadze, *Functions of generalized Wiener classes* $BV(p(n)\uparrow\infty,\varphi)$ *and their Fourier coefficients,* Georgian Math. J., V.7, 3(2000), 401–416.

[2] R. E. Edwards, *Fourier series. A modern Introduction. Vol.1,* 2^{nd} *Ed.,* Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[3] V. Fulop and F. Moricz, *Order of magnitude of multiple Fourier coefficients of functions of bounded variation,* Acta Math. Hungar., 104 (12) (2004), 95–104.

[4] G. H. Hardy, *On double Fourier series,* Quart. J. Math., 37 (1905), 53–79.

[5] S. R. Ghorpade and B. V. Limaye, *A Course in Multivariable Calculus and Analysis,* Springer, 2009.

[6] F. Moricz and A. Veres, *Absolute convergence of multiple Fourier series revisited,* Analysis Mathematica, V.43, (2008), 145–162.

[7] R. G. Vyas, *Small gaps Fourier series and generalized variations,* Adv. Pure Appl. Math., V.3, 2(2012), 223–230.

[8] R. G. Vyas, *Convolution function of several variables with generalized bounded variation,* Analysis Mathematica, V.39, 2(2013), 153–161.

[9] R. G. Vyas, *On the convolution of functions of generalized bounded variations,* Georgian Math. J., V.13, 1(2006), 193–197.

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FIXED POINTS OF EXPANSION MAPPINGS IN MENGER SPACES USING WEAK COMPATIBILITY

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Abstract. In the present paper, we prove some common fixed point theorems for non-surjective expansion mappings in Menger spaces using weak compatibility. Our results improve, extend and generalize the results of Kumar and Pant [A common fixed point theorem for expansion mappings in probabilistic metric spaces, Ganita, 57(1) (2006), 89–95], Dimri et al. [Fixed point of a pair of non-surjective expansion mappings in Menger spaces, Stud. Cerc. St. Ser. Matematica Universitatea Bacău, 18 (2008), 55–62], Kumar et al. [Fixed point theorems for expansion mappings theorems in various spaces, Stud. Cerc. St. Ser. Matematica Universitatea Bacău, 16 (2006), 47–68] and Gujetiya et al. [Common fixed point theorem for expansive maps in Menger spaces through compatibility, Int. Math. Forum, 5(63) (2010), 3147–3158].

1. Introduction

In 1906, Fréchet [3] introduced the concept of metric space; which is a natural setting for many problems in which notion of distance appears. An essential feature is the fact that, for any two points in the space, there is defined a positive number called the distance between the two points. However, in practice we find very often that this association of a single number for each pair is, strictly speaking, an over-idealization. Therefore, Menger [9] introduced the concept of probabilistic metric space (briefly, PM-space) as a generalization of metric space. It has fundamental importance in probabilistic functional analysis, nonlinear analysis and applications.

Banach contraction principle [1] is an important tool in the theory of metric spaces. Probabilistic contraction was first defined and studied by Sehgal [13]. Banach contraction principle [1] also yields a fixed point theorem for a diametrically opposite class of mappings, viz. expansion mappings. The study of fixed point of single expansion mapping in a metric space is initiated by Wang et al. [17].

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In 1987, Pant et al. [11] initiated the study of fixed point theorems for expansion mappings on probabilistic metric spaces as follows:

Let $(X, \mathcal{F}, \triangle)$ be a Menger space. A mapping $T : X \to X$ will be called an expansion mapping iff for a constant $k > 1$,

$$F_{Tx,Ty}(kt) \leq F_{x,y}(t) \tag{1}$$

holds for all $x, y \in X$ and all $t > 0$. The interpretation of inequality (1) is as follows:

"The probability that the distance between the image points of Tx and Ty is less than kt is never greater than the probability that the distance between x and y is less than t".

Subsequently, Vasuki [16], Kumar et al. [7], Dimri et al. [2] and Kumar [6] also proved some fixed point theorems for expansion mappings in Menger spaces. In 2006, Kumar and Pant [8] proved a common fixed point theorem for surjective, continuous, compatible mappings using expansion type condition in a complete Menger space. Recently, Gujetiya et al. [4] proved a common fixed point theorem for expansive mappings in Menger space using the notion of sub-compatibility.

In this paper, our objective is to prove some common fixed point theorems for four non-surjective expansion mappings without assumption of continuity using weak compatibility in Menger spaces.

2. Preliminaries

Definition 2.1. [12] A mapping $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied: for all $a, b, c, d \in [0,1]$

(1) $\triangle(a, 1) = a$;
(2) $\triangle(a, b) = \triangle(b, a)$;
(3) $\triangle(a, b) \leq \triangle(c, d)$ for $a \leq c$, $b \leq d$;
(4) $\triangle(\triangle(a, b), c) = \triangle(a, \triangle(b, c))$.

Examples of basic t-norms are: $\triangle(a, b) = \min\{a, b\}$, $\triangle(a, b) = ab$ and $\triangle(a, b) = \max\{a + b - 1, 0\}$.

Definition 2.2. [12] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on

X and the value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3. [12] The ordered pair $(X, \mathcal{F})$ is called a PM-space if X is a non-empty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$

(1) $F_{x,y}(t) = H(t) \Leftrightarrow x = y$;
(2) $F_{x,y}(t) = F_{y,x}(t)$;
(3) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$.

Definition 2.4. [12] A Menger space is a triplet $(X, \mathcal{F}, \triangle)$ where $(X, \mathcal{F})$ is a PM-space and t-norm $\triangle$ is such that the inequality

$$F_{x,z}(t + s) \geq \triangle\left(F_{x,y}(t), F_{y,z}(s)\right),$$

holds for all $x, y, z \in X$ and all $t, s > 0$.

Every metric space (X, d) can be realized as a PM-space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.5. [12] Let $(X, \mathcal{F}, \triangle)$ be a Menger space with continuous t-norm $\triangle$. A sequence $\{x_n\}$ in X is said to be

(1) convergent to a point x in X if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $\mathbb{N}(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq \mathbb{N}(\epsilon, \lambda)$.
(2) Cauchy if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $\mathbb{N}(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq \mathbb{N}(\epsilon, \lambda)$.

A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.6. [10] Two self mappings A and B of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be compatible if and only if $F_{ABx_n,BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some $z \in X$ as $n \to \infty$.

Definition 2.7. [5] Two self mappings A and B of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Bz$ some $z \in X$, then $ABz = BAz$.

Two compatible self mappings are weakly compatible but the converse is not true (see [14, Example 1]). Therefore the concept of weak compatibility is more general than that of compatibility.

Lemma 2.1. [15] *Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that for $t > 0$ and $n \in \mathbb{N}$*

$$F_{x_n,x_{n+1}}(kt) \geq F_{x_{n-1},x_n}(t),$$

then $\{x_n\}$ *be a Cauchy sequence in* X.

Lemma 2.2. [10] *Let* $(X, \mathcal{F}, \triangle)$ *be a Menger space. If there exists a constant* $k \in (0,1)$ *such that*

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ *and* $t > 0$ *then* $x = y$.

3. Results

In [8], Kumar and Pant proved the following result:

Theorem 1. *Let* $(X, \mathcal{F}, \triangle)$ *be a complete Menger space, where* $\triangle(a,b) = \min\{a,b\}$ *for all* $a, b \in [0,1]$. *Further, let* A, B, S *and* T *be self mappings of* X *satisfying the following conditions:*

(1) A *and* B *are surjective;*
(2) One of A, B, S *or* T *is continuous;*
(3) The pairs (A, S) *and* (B, T) *are compatible;*
(4) There exists a constant $k > 1$ *such that*

$$F_{Ax,By}(kt) \leq F_{Sx,Ty}(t), \tag{2}$$

for all $x, y \in X$ *and* $t > 0$. *Then* A, B, S *and* T *have a unique common fixed point in* X.

Now we prove our main results:

Theorem 2. *Let* $(X, F, \triangle)$ *be a Menger space, where* $\triangle(a,b) = \min\{a,b\}$ *for all* $a, b \in [0,1]$. *Further, let* (A, S) *and* (B, T) *are weakly compatible pairs of self mapping of* X *satisfying inequality (2) of Theorem 1. Suppose that*

(1) $T(X) \subseteq A(X)$ *and* $S(X) \subseteq B(X)$;
(2) If one of $A(X)$, $B(X)$, $S(X)$ *or* $T(X)$ *is a complete subspace of* X.

Then A, B, S *and* T *have a unique common fixed point in* X.

Proof: Let $x_0 \in X$. By (1), we define the sequence $\{x_n\}$ in X such that

$$Ax_{2n+1} = Tx_{2n} = y_{2n} \tag{3}$$

and

$$Bx_{2n+2} = Sx_{2n+1} = y_{2n+1}, \tag{4}$$

for all $n = 0, 1, \ldots$. By inequality (2) and conditions (3), (4), for $k > 1$, $t > 0$, we have

$$F_{Ax_{2n+1},Bx_{2n+2}}(kt) \leq F_{Sx_{2n+1},Tx_{2n+2}}(t),$$

and so

$$F_{y_{2n},y_{2n+1}}(kt) \leq F_{y_{2n+1},y_{2n+2}}(t).$$

Similarly, by inequality (2),

$$F_{y_{2n+2},y_{2n+1}}(kt) \leq F_{y_{2n+3},y_{2n+2}}(t).$$

In general, for any n, $k > 1$ and $t > 0$, by inequality (2), we have

$$F_{y_n,y_{n+1}}(kt) \leq F_{y_{n+1},y_{n+2}}(t).$$

By Lemma 2.1, $\{y_n\}$ is a Cauchy sequence. Since $A(X)$ is a complete subspace of X, $\{y_n\}$ has a limit in $A(X)$, call it z. Hence there exists a point u in X such that $Au = z$. Consequently, the subsequences $\{Ax_{2n+1}\}$, $\{Bx_{2n}\}$, $\{Sx_{2n+1}\}$, $\{Tx_{2n}\}$ also converge to z. By inequality (2), we have

$$F_{Au,Bx_{2n}}(kt) \leq F_{Su,Tx_{2n}}(t).$$

Letting $n \to \infty$, we get

$$F_{z,z}(kt) \leq F_{Su,z}(t).$$

By Lemma 2.2, we have $Su = z$. Hence, $Au = Su = z$. Since $S(X) \subseteq B(X)$, so there exists $v \in X$ such that $Bv = Su (= Au = z)$. By inequality (2), we have

$$F_{Au,Bv}(kt) \leq F_{Su,Tv}(t),$$

and so

$$F_{z,z}(kt) \leq F_{z,Tv}(t),$$

In view of Lemma 2.2, we get $Tv = z$. Therefore, $Au = Su = Bv = Tv = z$. Since A and S are weakly compatible, $Az = ASu = SAu = Sz$.

Now, we show that z is a fixed point of A. By inequality (2), for $t > 0$ and $k > 1$, we get

$$F_{Az,Bv}(kt) \leq F_{Sz,Tv}(t),$$

and so

$$F_{Az,z}(kt) \leq F_{Az,z}(t).$$

By Lemma 2.2, we have $Az = z$. Hence $Az = Sz = z$. Also, the mappings B and T are weakly compatible, then $Bz = BTv = TBv = Tz$. Similarly, by inequality (2), it can be proved that $Bz = Tz = z$. Hence $Az = Sz = Bz = Tz = z$. Thus z is a common fixed point of the mappings A, B, S and T.

Finally in order to prove the uniqueness of z, suppose that $w (\neq z)$ be another common fixed point of A, B, S and T. Then by inequality (2), we have

$$F_{Az,Bw}(kt) \leq F_{Sz,Tw}(t),$$

and so

$$F_{z,w}(kt) \leq F_{z,w}(t).$$

By Lemma 2.2, it follows that $z = w$. This completes the proof of the theorem.

A similar argument holds by taking one of subspaces $B(X)$, $S(X)$ or $T(X)$ complete. ■

Theorem 3. *Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Further, let (A, S) and (B, T) are weakly compatible pairs of self mappings of X satisfying conditions (1) and (2) of Theorems 2 then there exists a constant $k > 1$ such that*

$$F_{Ax,By}(kt) \leq \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\}, \tag{5}$$

for all $x, y \in X$ and $t > 0$. Then A, B, S and T have a common fixed point in X.

Proof: The proof may be completed on the lines of Theorem 2. ■

Theorem 4. *Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $\triangle(a, b) = min(a, b)$ for all $a, b \in [0, 1]$. Further, let (A, S) and (B, T) are weakly compatible pairs of self mapping of X satisfying conditions (1) and (2) of Theorems 2 then there exists a constant $k > 1$ such that*

$$(F_{Ax,By}(kt))^2 \leq F_{Ax,Sx}(t)F_{By,Ty}(t) \tag{6}$$

for all $x, y \in X$ and $t > 0$. Then A, B, S and T have a unique common fixed point in X.

Proof: By (3), (4) and inequality (6), for $k > 1$ and $t > 0$ we have

$$(F_{Ax_{2n+1},Bx_{2n+2}}(kt))^2 \leq F_{Ax_{2n+1},Sx_{2n+1}}(t)F_{Bx_{2n+2},Tx_{2n+2}}(t),$$

and so

$$(F_{y_{2n},y_{2n+1}}(kt))^2 \leq F_{y_{2n},y_{2n+1}}(t)F_{y_{2n+1},y_{2n+2}}(t).$$

or, equivalently,

$$F_{y_{2n},y_{2n+1}}(kt) \leq F_{y_{2n+1},y_{2n+2}}(t).$$

Similarly, by inequality (6),

$$F_{y_{2n+2},y_{2n+1}}(kt) \leq F_{y_{2n+3},y_{2n+2}}(t).$$

In general, for any n, $k > 1$ and $t > 0$, by inequality (6), we have

$$F_{y_n,y_{n+1}}(kt) \leq F_{y_{n+1},y_{n+2}}(t).$$

By Lemma 2.1, $\{y_n\}$ ia Cauchy sequence. The rest of the proof runs on the lines of the proof of Theorem 2, therefore details are omitted. ■

Conclusion

Theorem 2 is a generalization of Theorem 1 in the sense it has been proved for non-surjective mappings and the pairs (A, S) and (B, T) are taken weakly compatible which are more general than that of compatibility. Theorem 2 extends the result of Kumar et al. [7, Theorem 3.2]. Theorem 3 improve and generalize the results of Gujetiya et al. [4, Theorem 3.1] without any requirement of completeness of the underlying space and continuity of the involved mappings. Theorem 4 is an extension of the result of Dimri et al. [2, Theorem 3.2] to four mappings.

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References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181.

[2] R. C. Dimri, B. D. Pant and S. Kumar, *Fixed point of a pair of non-surjective expansion mappings in Menger spaces*, Stud. Cerc. St. Ser. Matematica Universitatea Bacău, **18** (2008), 55–62.

[3] M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendic. Circ. Mat. Palermo, **22** (1906), 1–74.

[4] R. K. Gujetiya, V. K. Gupta, M. S. Chauhan and O. Sikhwal, *Common fixed point theorem for expansive maps in Menger spaces through compatibility*, Int. Math. Forum, **5**(63) (2010), 3147–3158.

[5] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., **9**(4) (1986), 771–779.

[6] S. Kumar, *Some fixed point theorems in Menger spaces and applications*, Ph. D. Dissertation, Kumaun Univ., Nainital (India), (2008).

[7] S. Kumar, R. Chugh and R. K. Vats, *Fixed point theorems for expansion mappings theorems in various spaces*, Stud. Cerc. St. Ser. Matematica Universitatea Bacău, **16** (2006), 47–68.

[8] S. Kumar and B. D. Pant, *A common fixed point theorem for expansion mappings in probabilistic metric spaces*, Ganita, **57**(1) (2006), 89–95.

[9] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. U.S.A., **28** (1942), 535–537.

[10] S. N. Mishra, *Common fixed points of compatible mappings in PM-spaces*, Math. Japon., **36** (1991), 283–289.

[11] B. D. Pant, R. C. Dimri and S. L. Singh, *Fixed point theorems for expansion mapping on probabilistic metric spaces*, Honam Math. J., **9**(1) (1987), 77–81.

[12] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics. North-Holland Publishing Co., New York, (1983).

[13] V. M. Sehgal, *Some fixed point theorems in functional analysis and probability*, Ph. D. Dissertation, Wayne State Univ., (1966).

[14] B. Singh and S. Jain, *A fixed point theorem in Menger Space through weak compatibility*, J. Math. Anal. Appl., **301** (2005), 439–448.

[15] S. L. Singh and B. D. Pant, *Common fixed point theorems in probabilistic metric spaces and extension to uniform spaces*, Honam Math. J., **6** (1984), 1–12.

[16] R. Vasuki, *Fixed point and common fixed point theorems for expansive maps in Menger spaces*, Bull. Cal. Math. Soc., **83** (1991), 565–570.

[17] S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iseki, *Some fixed point theorems on expansion mappings*, Math. Japon., **29** (1984), 631–636.

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AN EASY TO REMEMBER, ECONOMIC APPROACH TO THE RICCATI DIFFERENTIAL EQUATION

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ABSTRACT. We investigate particular instances of a differential equation derived from the Riccati equation $y' + ay = by^2 + c$ by an affine transformation $z = \alpha\, y + \beta$, where α and β are continuously differentiable functions such that $\alpha(x) \neq 0$ for all possible x.

In particular, we obtain the solutions of the linear equation $y' + ay = c$ in a new and quite natural way. Moreover, we obtain the classical methods of reducing the Riccati equation to a Bernoulli equation $z' + pz = qz^2$ and a canonical form $z' = z^2 + f$ in a unified, economic way.

1. Notation, Terminology, and Motivation

Let I be an open interval on the real line, and denote by $\mathcal{C}(I)\big(\text{resp. } \mathcal{C}_k(I)\big)$ the family of all continuous (resp. k-times continuously differentiable) complex valued functions on I.

Assume that $a, b, c \in \mathcal{C}(I)$, and consider the Riccati equation

$$y' + ay = by^2 + c. \tag{1.1}$$

If $b = 0$, then (1.1) reduces to the linear equation

$$y' + ay = c. \tag{1.2}$$

While, if $c = 0$, then (1.1) reduces to the Bernoulli equation

$$y' + ay = by^2. \tag{1.3}$$

If y is a solution of (1.3) on I such that $y(x) \neq 0$ for all $x \in I$, then we have

$$\frac{y'}{y^2} + a\,\frac{1}{y} = b.$$

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Hence, by noticing that $y'/y^2 = (-1/y)'$, we can naturally arrive at the substitution

$$z = -\frac{1}{y}. \tag{1.4}$$

This leads us to the linear equation

$$z' - az = b. \tag{1.5}$$

Therefore, the solutions of the Bernoulli equation (1.3) can be easily derived from those of the linear one (1.5). Namely, if y is a solution of (1.3) such that $y(\xi) \neq 0$ for some $\xi \in I$, then because of the continuity of y there exists an open subinterval J of I, containing ξ, such that $y(x) \neq 0$ for all $x \in J$. Therefore, it is not a severe restriction to assume that $J = I$.

By using the methods of 'separation of variables' and 'variation of constants' [4, p. 28], one can easily find a solution y of (1.2) on I in the form

$$y = \frac{v + \eta}{u}, \tag{1.6}$$

where $\eta = y(\xi)$ for some $\xi \in I$ and

$$u(x) = \exp\left(\int_\xi^x a(t)\,dt\right) \quad \text{and} \quad v(x) = \int_\xi^x u(t)\,c(t)\,dt \tag{1.7}$$

for all $x \in I$.

Hence, we can observe that

$$u\,y - v = \eta. \tag{1.8}$$

Therefore, to prove that every solution of (1.2) on I is of the form (1.6), it is enough to verify only that if y is a solution of (1.2), then $u\,y - v$ is a constant function. This suggests that to determine the possible forms of an assumed solution y of the more general equation (1.1) it may also be useful to consider some affine transform $z = \alpha y + \beta$ of y with $\alpha, \beta \in \mathcal{C}_1(I)$.

2. An Affine Transformation of Equation (1.1)

Suppose that y is a solution of the equation

$$y' + a\,y = b\,y^2 + c. \tag{2.1}$$

on I, and moreover $\alpha, \beta \in \mathcal{C}_1(I)$. Define

$$z = \alpha\,y + \beta. \tag{2.2}$$

Then, by equation (2.1), we have

$$\alpha\,y' + \alpha a\,y = \alpha\,b\,y^2 + \alpha\,c.$$

Moreover, if $\alpha(x) \neq 0$ for all $x \in I$, then we also have

$$\alpha\,y' = (\alpha\,y)' - \alpha' y = (\alpha\,y)' - \frac{\alpha'}{\alpha}\,\alpha\,y = (z - \beta)' - \frac{\alpha'}{\alpha}\,(z - \beta),$$

and moreover

$$\alpha\, a\, y = a\, \alpha\, y = a\,(z-\beta) \qquad \text{and} \qquad \alpha\, b\, y^2 = \frac{b}{\alpha}\,(\alpha\, y)^2 = \frac{b}{\alpha}(z-\beta)^2.$$

Therefore,

$$(z-\beta)' + \Big(a - \frac{\alpha'}{\alpha}\Big)(z-\beta) = \frac{b}{\alpha}\,(z-\beta)^2 + \alpha\, c.$$

Hence, by using that

$$(z-\beta)' = z' - \beta' \qquad \text{and} \qquad (z-\beta)^2 = z^2 - 2\,\beta\, z + \beta^2,$$

we can already infer that

$$z' + \Big(a - \frac{\alpha'}{\alpha} + 2\frac{b}{\alpha}\beta\Big) z = \frac{b}{\alpha}\, z^2 + \beta' + \Big(a - \frac{\alpha'}{\alpha}\Big)\beta + \frac{b}{\alpha}\,\beta^2 + \alpha\, c. \tag{2.3}$$

Thus, z is a solution of a more complicated Riccati equation (2.3). However, this equation can be greatly simplified by making some appropriate choices of the functions α and β.

3. **A First Simplification of Equation (2.3)**

If $b = 0$, then equations (2.1) and (2.3) reduce to the equations

$$y' + a\, y = c. \tag{3.1}$$

and

$$z' + \Big(a - \frac{\alpha'}{\alpha}\Big) z = \beta' + \Big(a - \frac{\alpha'}{\alpha}\Big)\beta + \alpha\, c. \tag{3.2}$$

Now, in accordance with the observations of Section 1, we may naturally take

$$\alpha = u \qquad \text{and} \qquad \beta = -v, \tag{3.3}$$

where

$$u\,(x) = \exp\Big(\int_\xi^x a(t)\, dt\Big) \quad \text{and} \quad v\,(x) = \int_\xi^x u\,(t)\, c(t) dt \tag{3.4}$$

for some $\xi \in I$ and all $x \in I$. Namely, in this case we have

$$\alpha' = u' = a\, u = a\, \alpha \qquad \text{and} \qquad \beta' = -v' = -u\, c = -\alpha\, c.$$

Therefore, equation (3.2) reduces to

$$z' = 0. \tag{3.5}$$

Hence, by taking $\eta = y\,(\xi)$ and observing that

$$z\,(\xi) = \alpha\,(\xi)\, y\,(\xi) + \beta\,(\xi) = u\,(\xi)\, y\,(\xi) - v\,(\xi) = y\,(\xi) = \eta, \tag{3.6}$$

we can infer that $z\,(x) = \eta$ for all $x \in I$. Therefore,

$$y = \frac{z-\beta}{\alpha} = \frac{v+\eta}{u}. \tag{3.7}$$

Thus, we have determined the possible form of an assumed solution y of the linear equation (3.1) on I in a new and quite natural way. Moreover, if y is of the form

$$y = \frac{v + \eta}{u}$$

for some complex number η, then $y\,u = v + \eta$.Hence, by using that

$$(y\,u)' = y'u + y\,u' = y'u + y\,a\,u \qquad \text{and} \qquad (v + \eta)' = v' = u\,c,$$

we can infer that $y'u + y\,a\,u = u\,c$, and thus $y' + a\,y = c$. Therefore, y is a solution of (3.1) on I.

4. A Second Simplification of Equation (2.3)

Suppose now that $\alpha(x) = 1$ for all $x \in I$. Then, since $\alpha' = 0$, equation (2.3) reduces to

$$z' + \big(a + 2\,b\,\beta\big)\,z = b\,z^2 + \beta' + a\,\beta + b\,\beta^2 + c. \tag{4.1}$$

Suppose next that $\beta = -y_o$, where y_o is a solution of (2.1) on I. Then, since

$$\beta' + a\,\beta = (-y_o)' + a\,(-y_o) = -(y_o' + a\,y_o) = -(b\,y_o^2 + c) = -(b\,\beta^2 + c),$$

equation (4.1) reduces to the Bernoulli equation

$$z' + \big(a - 2\,b\,y_o\big)\,z = b\,z^2. \tag{4.2}$$

If $z(x) \neq 0$, *i.e.*, $\alpha\,y(x) + \beta(x) \neq 0$, and thus

$$y(x) \neq -\frac{1}{\alpha(x)}\beta(x) = y_o(x) \tag{4.3}$$

for all $x \in I$, then by using the substitution

$$w = -\frac{1}{z} \tag{4.4}$$

we can arrive at the linear equation

$$w' - \big(a - 2\,b\,y_o\big)\,w = b. \tag{4.5}$$

Hence, by using the solution formula (3.7) of the linear equation (3.1), we can infer that w is of the form

$$w = \frac{\Psi + \theta}{\Phi}, \tag{4.6}$$

where $\theta = w(\xi)$ for some $\xi \in I$ and

$$\Phi(x) = \exp\Big(-\int_\xi^x \big(a - 2\,b\,y_o(t)\big)\,dt\Big)$$

$$\text{and} \qquad \Psi(x) = \int_\xi^x \Phi(t)\,b(t)\,dt \tag{4.7}$$

for all $x \in I$. Hence, it is clear that

$$y = \frac{z - \beta}{\alpha} = y_o + z = y_o - \frac{1}{w} = y_o - \frac{\Phi}{\Psi + \theta} \tag{4.8}$$

and

$$\theta = w\,(\xi) = -\frac{1}{z\,(\xi)} = -\frac{1}{\alpha\,(\xi)\,y\,(\xi) + \beta\,(\xi)} = \frac{1}{y(\xi) - y_o(\xi)}. \tag{4.9}$$

Moreover, if y is of the form

$$y = y_o - \frac{\Phi}{\Psi + \theta}.$$

for some complex number $\theta \notin -\Psi\,(I)$, then under the notation

$$w = \frac{\Psi + \theta}{\Phi},$$

we have $y = y_o - 1/w$, and thus

$$y\,w = y_o w - 1 \qquad \text{and} \qquad w = \frac{1}{y_o - y}.$$

Hence, by using that

$$\big(y\,w\big)' = y'w + y\,w' = y'w + y\,\big((a - 2\,b\,y_o)\,w + b\big) = \frac{y' + a\,y - 2\,b\,y_o y}{y_o - y} + b\,y,$$

and

$$\big(y_o\,w - 1\big)' = \big(y_o\,w\big)' = \frac{y_o' + a\,y_o - 2\,b\,y_o^2}{y_o - y} + b\,y_o = \frac{c - b\,y_o^2}{y_o - y} + b\,y_o,$$

we can infer that

$$\frac{y' + a\,y - 2\,b\,y_o y}{y_o - y} + b\,y = \frac{c - b\,y_o^2}{y_o - y} + b\,y_o.$$

This implies that

$$y' + a\,y - 2\,b\,y_o y + b\,y_o y - b\,y^2 = c - b\,y_o^2 + b\,y_o^2 - b\,y_o y.$$

and thus $y' + a\,y = b\,y^2 + c$. Therefore, y is a solution of (2.1) on I.

5. A Third Simplification of Equation (2.3)

If $b \in \mathcal{C}_1(I)$ such that $b\,(x) \neq 0$ for all $x \in I$, then we can take $\alpha = b$. Thus, equation (2.3) reduces to

$$z' + \Big(a - \frac{b'}{b} + 2\,\beta\Big)z = z^2 + \beta' + \Big(a - \frac{b'}{b}\Big)\,\beta + \beta^2 + b\,c. \tag{5.1}$$

Moreover, if $a \in \mathcal{C}_1(I)$ and $b \in \mathcal{C}_2(I)$, then we can take

$$\beta = -\frac{1}{2}\Big(a - \frac{b'}{b}\Big).$$

Thus, equation (5.1) reduces to

$$z' = z^2 + f, \tag{5.2}$$

where
$$f=-\frac{1}{2}\Big(a-\frac{b'}{b}\Big)'-\frac{1}{4}\Big(a-\frac{b'}{b}\Big)^2+b\,c. \tag{5.3}$$
Now, suppose that z_o is a solution of (5.2) on I and $z(x)\neq z_o(x)$. That is, $\alpha(x)\,y(x)+\beta(x)\neq z_o(x)$, and thus
$$y(x)\neq\frac{z_o(x)-\beta(x)}{\alpha(x)}=\frac{1}{b(x)}\left(z_o(x)+\frac{1}{2}\left(a(x)-\frac{b'(x)}{b(x)}\right)\right) \tag{5.4}$$
for all $x\in I$. Then, by using the solution formula (4.8) of the Riccati equation (2.1), we can infer that z is of the form
$$z=z_o-\frac{\varphi}{\psi+\vartheta}, \tag{5.5}$$
where $\vartheta=z(\xi)$ for some $\xi\in I$ and
$$\varphi(x)=\exp\left(\int_\xi^x 2\,z_o(t)\,dt\right)\qquad\text{and}\qquad\psi(x)=\int_\xi^x\varphi(t)\,dt \tag{5.6}$$
for all $x\in I$. Hence, it is clear that
$$y=\frac{z-\beta}{\alpha}=\frac{1}{b}\left(z+\frac{1}{2}\left(a-\frac{b'}{b}\right)\right)$$
$$=\frac{1}{b}\left(z_o-\frac{\varphi}{\psi+\vartheta}+\frac{1}{2}\left(a-\frac{b'}{b}\right)\right) \tag{5.7}$$
and
$$\vartheta=z(\xi)=\alpha(\xi)\,y(\xi)+\beta(\xi)=b(\xi)\,y(\xi)-\frac{1}{2}\left(a(\xi)-\frac{b'(\xi)}{b(\xi)}\right). \tag{5.8}$$
Moreover, if y is of the form
$$y=\frac{1}{b}\left(z_o-\frac{\varphi}{\psi+\vartheta}+\frac{1}{2}\Big(a-\frac{b'}{b}\Big)\right)$$
for some complex number $\vartheta\notin-\psi(I)$, then under the notation
$$z=z_o-\frac{\varphi}{\psi+\vartheta},$$
we have
$$b\,y=z+\frac{1}{2}\left(a-\frac{b'}{b}\right).$$
Hence, by using that $(b\,y)'=b'y+b\,y'$ and
$$\left(z+\frac{1}{2}\left(a-\frac{b'}{b}\right)\right)'=z'+\frac{1}{2}\left(a-\frac{b'}{b}\right)'=z^2+f\,+\,\frac{1}{2}\left(a-\frac{b'}{b}\right)'$$
$$=\left(b\,y-\frac{1}{2}\left(a-\frac{b'}{b}\right)\right)^2-\frac{1}{2}\Big(a-\frac{b'}{b}\Big)'-\frac{1}{4}\left(a-\frac{b'}{b}\right)^2+b\,c\,+\,\frac{1}{2}\left(a-\frac{b'}{b}\right)'$$
$$=b^2y^2-b\,y\Big(a-\frac{b'}{b}\Big)+b\,c=b^2y^2-a\,b\,y+b'y+b\,c,$$
we can infer that
$$b'y+b\,y'=b^2y^2-a\,b\,y+b'y+b\,c,$$

and thus $y' + a\,y = b\,y^2 + c$. Therefore, y is a solution of (2.1) on I.

From the above computation, we can also see that

$$y = \frac{1}{b}\left(z_o + \frac{1}{2}\left(a - \frac{b'}{b}\right)\right)$$

is also a solution of (2.1) on I.

6. Finding a Non-Vanishing Solution of Equation (5.2)

Suppose that z is a solution of the equation

$$z' = z^2 + f, \tag{6.1}$$

on I. Take $\xi \in I$ and define

$$w\,(x) = \exp\left(-\int_\xi^x z\,(t)\,dt\right) \tag{6.2}$$

for all $x \in I$.

Then, we have

$$w' + z\,w = 0, \qquad \text{and thus} \qquad w'' + (z\,w)' = 0.$$

Hence, by noticing that

$$(z\,w)' = z'w + z\,w' = \left(z^2 + f\right)w + z\,(-z\,w) = f\,w,$$

we can arrive at the linear equation

$$w'' + f\,w = 0. \tag{6.3}$$

Now, if w_o is a solution of (6.3) on I such that $w_o(x) \neq 0$ for all $x \in I$ and

$$z_o = -\frac{w_o'}{w_o},$$

then by using that $w_o' = -z_o w_o$ and

$$w_o'' = -f\,w_o \qquad \text{and} \qquad (-z_o w_o)' = -z_o' w_o - z_o w_o' = -z_o' w_o + z_o^2 w_o$$

we can see that $-f\,w_o = -z_o' w_o + z_o^2 w_o$, and thus $z_o' = z_o^2 + f$. Therefore, z_o is a solution of (6.1) on I.

Thus, the results of Section 5 can be applied. Moreover, we can note that if $w_o(x) \notin (-\infty, 0]$ for all $x \in I$, then

$$\begin{aligned}\varphi\,(x) &= \exp\left(\int_\xi^x 2\,z_o(t)\,dt\right) = \exp\left(-2\int_\xi^x \frac{w_o'(t)}{w_o(t)}\,dt\right)\\ &= \exp\left(-2\,\log\left(w_o(x)\right) + 2\,\log\left(w_o(\xi)\right)\right)\\ &= \exp\left(-2\,\log\left(w_o(x)\right)\right)\exp\left(2\,\log\left(w_o(\xi)\right)\right) = \frac{w_o(\xi)^2}{w_o(x)^2}\end{aligned}$$

for all $x \in I$.

7. Finding a Non-Vanishing Solution of Equation (6.3)

If w is a solution of the equation

$$w'' + f\,w = 0. \tag{7.1}$$

on I, then by taking $\xi \in I$ we can see that

$$\begin{aligned} w\,(x) &= \int_{\xi}^{x} w'(t)\,dt + w\,(\xi) \\ &= \int_{\xi}^{x} \left(\int_{\xi}^{t} w''(s)\,ds + w'(\xi) \right) dt + w\,(\xi) \\ &= -\int_{\xi}^{x} \left(\int_{\xi}^{t} f\,(s)\,w\,(s)\,ds \right) dt + w'(\xi)\,(x-\xi) + w\,(\xi) \end{aligned}$$

for all $x \in I$. Therefore, a non-vanishing solution of (6.3) can, in principle, be found by successive approximation.

Moreover, if f is, in particular, a polynomial or an analytical function, then a non-vanishing solution of (6.3) can, in principle, be also found in a polynomial or power series form. Note that, both for the series and approximation methods, equation (7.1) is somewhat more convenient than (6.1) since in its case we do not need to compute squares of difficult sums.

However, the best case is when

$$f = -\frac{1}{2}\Big(a - \frac{b'}{b}\Big)' - \frac{1}{4}\left(a - \frac{b'}{b}\right)^2 + b\,c$$

is a constant function on I. That is, equation (7.1) is of the form

$$w'' + \mu\,w = 0 \tag{7.2}$$

for some complex number μ.

Namely, if $\mu = 0$, then by taking $w_o(x) = 1$ for all $x \in I$ we can immediately get a non-vanishing solution w_o of (7.2) on I. While, if $\mu \neq 0$, then by taking

$$\lambda = \exp\big(\log\big(-\mu)/2\,\big) \qquad \text{and} \qquad w_o(x) = \exp\,(\lambda\,x\,)$$

for all $\in I$, we can at once see that

$$w_o'' = \lambda^2 w_o = -\mu\,w_o.$$

Thus, we again have a non-vanishing solution w_o of (7.2) on I. Note that if in particular μ is a real number, then both the real and imaginary parts of w_o are also solutions of (7.2).

8. An Illustrating Particular Case

If in addition to the assumptions of Section 5, we have

$$a = \frac{b'}{b} \qquad \text{and} \qquad c = \frac{1}{b},$$

then equations (2.1), (6.1) and (6.3) take the forms

$$y' + \frac{b'}{b}\, y = b\, y^2 + \frac{1}{b}, \tag{8.1}$$

$$z' = z^2 + 1, \tag{8.2}$$

$$w'' + w = 0. \tag{8.3}$$

If in particular $I = [-r,\, r]$ for some $0 < r < \pi/2$, then by the corresponding observation of Section 7 it is clear that the function w_o, given by

$$w_o(x) = \cos(x) = \operatorname{Re}\big(\exp(i\, x)\big) \tag{8.4}$$

for all $x \in I$, is a solution of (8.3) on I such that $w_o(x) \neq 0$ for all $x \in I$. Moreover, by the corresponding observation of Section 6 the function z_o, given by

$$z_o(x) = -\frac{w_o'(x)}{w_o(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x) \tag{8.5}$$

for all $x \in I$, is a solution of the equation (8.2) on I.

Hence, by the corresponding results of Sections 5 and 6, we can state that if y is a solution of (8.1) on I such that

$$y(x) \neq \frac{1}{b(x)} \left(z_o(x) + \frac{1}{2}\Big(a(x) - \frac{b'(x)}{b(x)} \Big) \right) = \frac{1}{b(x)} \tan(x) \tag{8.6}$$

for all $x \in I$, then

$$y = \frac{1}{b} \left(z_o - \frac{\varphi}{\psi + \vartheta} + \frac{1}{2}\Big(a - \frac{b'}{b} \Big) \right) = \frac{1}{b} \left(\tan - \frac{\varphi}{\psi + \vartheta} \right), \tag{8.7}$$

where

$$\vartheta = b(0)\, y(0) - \frac{1}{2}\Big(a(0) - \frac{b'(0)}{b(0)} \Big) = b(0)\, y(0), \tag{8.8}$$

and moreover

$$\varphi(x) = \exp\left(\int_0^x 2\, z_o(t)\, dt \right) = \frac{w_o(0)^2}{w_o(x)^2} = \frac{1}{\cos(x)^2} = \tan(x)^2 + 1 \tag{8.9}$$

and

$$\psi(x) = \int_0^x \varphi(t)\, dt = \int_0^x \big(\tan(t)^2 + 1 \big)\, dt = \tan(x) \tag{8.10}$$

for all $x \in I$. Therefore, we actually have

$$y = \frac{1}{b}\left(\tan - \frac{\tan^2 + 1}{\tan + \vartheta}\right) = \frac{1}{b}\,\frac{\vartheta\,\tan - 1}{\tan + \vartheta}. \tag{8.11}$$

Moreover, if ϑ is a complex number such that $\vartheta \notin [-\tan(r), \tan(r)]$ and

$$y = \frac{1}{b}\,\frac{\vartheta\,\tan - 1}{\tan + \vartheta},$$

then from the corresponding result of Section 5 we know that y is a solution of (8.1) on I. Furthermore, we also know that

$$y = \frac{1}{b}\,\tan$$

is also a solution of (8.1) on I.

9. Historical Notes

According to Watson [5, p. 1], the first Riccati's type equation

$$x^2\,d\,x + y^2\,d\,x = a^2 d\,y, \qquad i.e., \qquad x^2 + y^2(x) = a^2 y'(x) \tag{9.1}$$

occurred in a paper by John Bernoulli in 1694 without solutions.

In this respect, it is noteworthy that the some form of the equation

$$y' = P\,y + Q\,y^{\alpha}, \tag{9.2}$$

considered first by Jacob Bernoulli in 1695, was already solved by John Bernoulli in 1697.

A few years later, the particular equation

$$d\,y = y\,y\,d\,x + x\,x\,d\,x, \qquad i.e., \qquad y'(x) = y\,(x)^2 + x^2 \tag{9.3}$$

was reduced to the linear equation

$$d\,d\,y : y = -x^2\,d\,x^2, \qquad i.e., \qquad y''(x)\,/\,y\,(x) = -x^2 \tag{9.4}$$

by James Bernoulli in a letter written to Leibnitz in 1702.

Moreover, James Bernoulli tried to solve equations (9.4) and (9.3) in terms of power series. However, his work was not fully appreciated, since mathematicians of that time were mainly interested in solutions in finite forms containing only classical elementary functions.

Some more difficult equations of this type

$$x^m dx^n = du + uudx : x^n, \quad i.e., \quad nx^{m+n-1} = u'(x) + u\,(x)/x^n \tag{9.5}$$

and

$$nxxdx - nyydx + xxdy = xydx,$$

$$i.e., \qquad nx^2 - ny\,(x)^2 + x^2 y'(x) = xy\,(x) \tag{9.6}$$

were studied by Riccati in 1724 and Manfredius in 1702, respectively.

Solutions of the equation

$$a\,x^n\,dx + u\,u\,dx = b\,du, \qquad i.e., \qquad a\,x^n + u\,(x)^2 = b\,u'(x) \tag{9.7}$$

were given by Daniel Bernoulli in 1724 for some values $n = -4m/(2m \pm 1)$.

Stimulated by these early researches, several mathematicians turned with much interest to further study of these Riccati's type equations. In particular, the term 'Riccati's equation' was already used by D' Alembert in 1763, despite that Riccati does not contributed essentially to the subject.

Some form of the general Riccati equation

$$y' = P + Q\,y + R\,y^2 \tag{9.8}$$

was first studied by Euler in 1769.

Among others, Euler observed that if a particular solution y_1 of (9.8) is known, then by taking the substitution $y = y_1 + 1/u$ equation (9.8) can be reduced to a linear differential equation of the first order. Thus, it can be solved by two quadratures.

An obvious generalization of (9.8) is the equation

$$y' = \sum_{k=0}^{n} p_k\,y^k. \tag{9.9}$$

The $n = 3$ particular case is called the Abel equation [1, p. 74].

Namely, Abel's original equation

$$(y + s)\,y' + p + q\,y + r\,y^2 = 0 \tag{9.10}$$

can be converted into the above form by the transformation $y + s = 1/z$.

Moreover, Riccati's equations can also be generalized to higher orders by the elimination of the parameters λ_k in the fraction

$$y = \frac{\sum_{k=1}^{n} \lambda_k\,u_k}{\sum_{k=1}^{n} \lambda_k\,v_k}. \tag{9.11}$$

According to Davis [1, p. 76], this generalization was already introduced by E. Vessiot in 1895 and G. Wallenberg in 1899.

However, recently the matrix valued generalizations of the Riccati equations have been the subject of most studies. According to Reid [3], these comprise a highly significant class of nonlinear differential equations due to their frequent appearance in applications. Only on the MathSciNet one can find more than four thousands matches for the term 'Riccati equations'.

Added in Proof. Meantime, we observed that the paper [P. R. P. Rao and V. H. Ukidave, Some separable forms of the Riccati equation, Amer. Math. Monthly

75 (1968), 38-39] , and its three predecessors mentioned in its References, contain ideas similar to ours.

References

[1] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover Publications, New York, 1962.

[2] D. Dimitrovski, L. Stefanovska and M. Kujumdzieva-Nikolska, *Every Riccati equation can be solved by quadratures in a wider sense*, Math. Balkanica **11** (1997), 221–228.

[3] W. T. Reid, *Riccati Differential Equations*, Academic Press, New York, 1972.

[4] W. W. Stepanow, *Lehrbuch der Differentialgleichungen*, Deutscher Verlag der Wissenschaften, Berlin, 1956.

[5] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1962.

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EULER'S CONSTANT, HARMONIC SERIES AND RELATED INFINITE SUMS

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Abstract. This paper introduces general harmonic series for $log_e n$, extends Euler's constant γ and evaluates sums of the series $\sum_{n=1}^{\infty}\{(n-1)k\}!/(nk)!, k$ integer > 1. The author also derives an elegant infinite series for π.

I.

I begin with a result found in §.8 of Euler's 1734 paper titled *De Progressionibus Harmonicis Observationes*[1]:

$$\ln n = \left\{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-1\right\}+\left\{\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)-\frac{1}{2}\right\}$$
$$+\left\{\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\cdots+\frac{1}{3n}\right)-\frac{1}{3}\right\}+\ldots$$
$$=\sum_{m=1}^{\infty}\left[\left\{\sum_{r=(m-1)n+1}^{mn}\frac{1}{r}\right\}-\frac{1}{m}\right] \tag{1}$$

Euler introduces a new constant with notation C in §.11 of the same paper and records an incorrect (last digit) value C=0.57721*8* as against the actual value $\gamma = .577215\ldots$.

It is easy to establish (1) using definition $\gamma = \lim_{n\to\infty}\gamma_n$, where $\gamma_n = H_n - lnn$ and $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

$\sum_{m=1}^{\infty}\left[\left\{\sum_{r=(m-1)n+1}^{mn}\frac{1}{r}\right\}-\frac{1}{m}\right] = \sum_{m\to\infty}\left[\sum_{u=1}^{mn}\frac{1}{u}-\sum_{p-1}^{n}\frac{1}{\nu}\right] = \lim_{n\to\infty}[(In\, mn)+\gamma_{mn}-(In\, mn)-\gamma_n]$

$= \lim_{n\to\infty}[In(\frac{mn}{n})] + \lim_{n\to\infty}[\gamma_{mn}-\gamma_n] = In\, m.$

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corollary

$$
\begin{aligned}
\mathbf{In\,n} &= \sum_{m=1}^{\infty}\left[\frac{1}{mn-(m-1)}-\frac{1}{mn}\right]+\sum_{m=1}^{\infty}\left[\frac{1}{mn-(m-2)}-\frac{1}{mn}\right] \\
&\quad +\sum_{m=1}^{\infty}\left[\frac{1}{mn-(m-3)}-\frac{1}{mn}\right]+\cdots+\sum_{m=1}^{\infty}\left[\frac{1}{mn-1}-\frac{1}{mn}\right] \\
&= \sum_{m=1}^{\infty}\left[\left\{\sum_{r=1}^{n-1}\frac{1}{mn-(m-r)}\right\}-\frac{1}{mn}\right] \qquad (2)
\end{aligned}
$$

It may be noted that the harmonic series used above has common difference $d = 1$. I have found that any positive integer $d \geq 1$ can be employed as common difference for computing $log_e n$. We now state and prove the result for $d = 2$.

Theorem 1.

$$
In\,n = \sum_{i=0}^{\infty}\left\{\sum_{j=0}^{n^2}\frac{1}{2in^2+(2j-1)}-\frac{1}{2i+1}\right\} \qquad (3)
$$

Proof: The result is obviously true for $n = 1$. We shall now use induction to prove (3).
Assuming it true for $n = k$, we have

$$
\begin{aligned}
\text{In } k &= \left\{\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k^2-1}\right)-1\right\}+\left\{\left(\frac{1}{2k^2+1}+\frac{1}{2k^2+3}+\cdots+\frac{1}{4k^2-1}\right)-\frac{1}{3}\right\} \\
&\quad +\left\{\left(\frac{1}{4k^2+1}+\frac{1}{4k^2+3}+\cdots+\frac{1}{6k^2-1}\right)-\frac{1}{5}\right\}+\cdots
\end{aligned}
$$

Let us add and subtract certain terms to the RHS for bringing it to $n = k + 1$. The difference of terms in two would be $(k+1)^2 - k^2 = 2k + 1$. At the end of the first braces, we would have $2k + 1$ number of negative terms which get cancelled by the first $2k + 1$ positive terms of the second braces. But then we would have $2(2k+1)$ number of negative terms at the end of the second braces. These in turn will get canceled by equal number of positive terms leaving $3(2k + 1)$ negative terms at the end of the third braces.

Proceeding in this manner we would have $n(2k+1)$ negative terms at the end of the nth braces, namely, $-\left[\{\frac{1}{2nk^2+1}\}+\{\frac{1}{2nk^2+3}\}+\{\frac{1}{2nk^2+5}\}+\cdots+\{\frac{1}{2n(k+1)^2-1}\}\right] = -[S]$,say. To evaluate $-[S]$ we add and subtract terms with even denominators:

$$
\begin{aligned}
-[S] &= \left[\left\{\frac{1}{2nk^2+1}\right\}+\left\{\frac{1}{2nk^2+2}\right\}+\left\{\frac{1}{2nk^2+3}\right\}+\left\{\frac{1}{2nk^2+4}\right\}+\left\{\frac{1}{2nk^2+5}\right\}\right. \\
&\quad \left.+\left\{\frac{1}{2nk^2+6}\right\}+\cdots+\left\{\frac{1}{2n(k+1)^2-1}\right\}+\right]-\left[\left\{\frac{1}{2nk^2+2}\right\}+\left\{\frac{1}{2nk^2+4}\right\}\right. \\
&\quad \left.+\left\{\frac{1}{2nk^2+6}\right\}+\cdots+\left\{\frac{1}{2n(k+1)^2}\right\}+\right]
\end{aligned}
$$

$$= \left[\left(\sum_{i=1}^{2n(k+1)^2} \frac{1}{i}\right) - \left(\sum_{j=1}^{2nk^2} \frac{1}{j}\right)\right] - \frac{1}{2}\left[\left\{\frac{1}{nk^2+1}\right\} + \left\{\frac{1}{nk^2+2}\right\}\right.$$
$$\left. + \left\{\frac{1}{nk^2+3}\right\} + \cdots + \left\{\frac{1}{n(k+1)^2}\right\}\right]$$
$$= \left[\left(\sum_{i=1}^{2n(k+1)^2} \frac{1}{i}\right) - \left(\sum_{j=1}^{2nk^2} \frac{1}{j}\right)\right] - \frac{1}{2}\left[\left(\sum_{s=1}^{n(k+1)^2} \frac{1}{s}\right) - \left(\sum_{t=1}^{nk^2} \frac{1}{t}\right)\right]$$
$$= \left[In\ 2n(k+1)^2 + \gamma_{2n(k+1)^2}\right] - \left[In\ 2nk^2 + \gamma_{2nk^2}\right]$$
$$- \left[In\ n(k+1)^2 + \gamma_{n(k+1)^2}\right] + \left[In\ nk^2 + \gamma_{nk^2}\right]$$
$$= \left[In\ 2n(k+1)^2/2nk^2\right] + \left[\gamma_{2n(k+1)^2} - \gamma_{2nk^2}\right]$$
$$- \frac{1}{2}\left[In\ n(k+1)^2/nk^2\right] - \frac{1}{2}\left[\gamma_{n(k+1)^2} - \gamma_{nk^2}\right]$$
$$= [2In\{(k+1)/k\}] + \left[\gamma_{2n(k+1)^2} - \gamma_{2nk^2}\right] - [In\ \{(k+1)/k\}] - \frac{1}{2}\left[\gamma_{n(k+1)^2} - \gamma_{nk^2}\right]$$
$$= [In\{(k+1)/k\}] + \left[\gamma_{2n(k+1)^2} - \gamma_{2nk^2}\right] - \frac{1}{2}\left[\gamma_{nk^2} - \gamma_{nk^2}\right]$$

Evaluating this quantity as n tends to infinity, we have $-[S] = ln\,\{(k+1)/k\}$. Now shift $-[S]$ to LHS. Then, LHS $= ln\,k + ln\,\{(k+1)/k\} = ln\,(k+1)$. Thus the result is true for $n = k+1$ if assumed true for $n = k$. Hence, true for all n.□

In fact, we have a general result which can be proved in the above manner:
General theorem : Let d and n be positive integers. Then,

$$ln\,n = \sum_{m=1}^{\infty}\left[\left\{1 + \frac{1}{1+(m-1)dn^d} + \frac{1}{1+(m-1)dn^d+d}\right.\right.$$
$$\left.\left. + \frac{1}{1+(m-1)dn^d+2d} + \dots \frac{1}{1+(m-1)dn^d+(n^d-1)d} - \frac{1}{1+(m-1)d}\right\}\right]$$
$$= \sum_{m=1}^{\infty}\left[\sum_{k=1}^{n^d-1} \frac{1}{1+\{(m-1)dn^d\}+kd} - \frac{1}{1+\{(m-1)d\}}\right] \tag{4}$$

We have n^d number of positive terms always followed by one negative term beginning with 1.

We may now generalize Euler's constant. Let d be the positive integer which is the common difference between two consecutive denominators of the series for $ln\,n$. Then, I define the generalized constant as:

$$\gamma(d) = \lim_{n\to\infty}\left[\left\{1 + \frac{1}{1+d} + \frac{1}{1+2d} + \frac{1}{1+3d} + \cdots + \frac{1}{1+(n^d-1)d}\right\} - In\,n\right] \tag{5}$$

We may treat$\gamma(d)$ as an arithmetical function. I found that

$$\begin{aligned}\gamma(d) &= 1+\frac{\gamma}{d}-\frac{1}{d}\left[\sum_{n=1}^{\infty}\ \frac{1}{n(nd+1)}\right]\\ &= 1+\frac{\gamma}{d}+\frac{1}{d}\int_0^1\ In(1-x^d)\mathrm{d}x\end{aligned} \tag{6}$$

Further, evaluating the general integral, I discovered these general formulae:

$$\begin{aligned}\gamma(2n+1) = &\left[\frac{\gamma}{(2n+1)}\right]+\left[\frac{In(2n+1)}{(2n+1)}\right]+\left[\frac{In2}{2(2n+1)}\right]\\ &+\left[\frac{\pi}{(2n+1)^2}\right]\left[\sum_{k=1}^{n}\{2(n-k)+1\}\sin\frac{2k\pi}{2n+1}\right]\\ &-\left\{\frac{1}{(2n+1)}\right\}\left[\sum_{k=1}^{n}\ \left\{\cos\frac{2k\pi}{(2n+1)}\right\}In\left\{1-\cos\frac{2k\pi}{2n+1}\right\}\right]\end{aligned} \tag{7a}$$

$$\begin{aligned}\gamma(2n) = &\left[\frac{\gamma}{(2n)}\right]+\left[\frac{In4n}{(2n)}\right]+\left(\frac{\pi}{(2n)}\right)\left[\sum_{k=1}^{n-1}\left\{\frac{(n-k)}{n}\right\}\sin\frac{2k\pi}{n}\right]\\ &-\left(\frac{1}{2n}\right)\left[\sum_{k=1}^{n-1}\ \left(\cos\frac{k\pi}{n}\right)In\left\{1-\left(\cos\frac{k\pi}{n}\right)\right\}\right]\end{aligned} \tag{7b}$$

I computed $\gamma(d)$ for various d up to 24 and found that it assumes the greatest value at $d=5$.
Let me record first few values:

$\gamma(1)=\gamma=.5772156649...$

$\gamma(2)=ln2+\dfrac{\gamma}{2}=.98175501301...$

$\gamma(3)=\dfrac{In3}{2}+\dfrac{\pi}{6\sqrt{3}}+\dfrac{\gamma}{3}=1.044011...$

$\gamma(4)=\dfrac{3In2}{4}+\dfrac{\pi}{8}+\dfrac{\gamma}{4}=1.0568633...$

$\gamma(5)=\dfrac{In5}{4}+\dfrac{In(2+\sqrt{5})}{6\sqrt{5}}+\dfrac{\pi}{50\sqrt{2}}\left\{3\left(\sqrt{5+\sqrt{5}}\right)+\left(\sqrt{5+\sqrt{5}}\right)\right\}+\dfrac{\gamma}{5}$
$=1.057807979...$

$\gamma(6)=+\dfrac{In2}{3}+\dfrac{In3}{4}+\dfrac{\pi}{4\sqrt{3}}+\dfrac{\gamma}{6}=1.05535458...$

I discerned an interesting relation $\pi=\gamma(1)-6\gamma(2)+8\gamma(4)$ which yields an elegant series:

$$\pi=\sum_{n=0}^{\infty}\frac{3}{(n+1)(2n+1)(4n+1)} \tag{8}$$

Though the infinite series (8) looks good, it not useful for computational purpose because of its slow rate of convergence. Incidentally, I have discovered a few more series for π and would like to submit here one that involves the convergent $\frac{22}{7}$

$$\pi = \frac{22}{7} - \sum_{n=2}^{\infty} \frac{60}{(4n^2-1)(16n^2-1)(16n^2-9)}. \tag{9}$$

II

I shall now take up the series $\sum_{n=1}^{\infty} [\{(n-1)k\}!/(nk)!] = \sum_{n=1}^{\infty} \frac{1}{\prod_{m=1}^{k}(nk-(k-m))}$,where k is an integer > 1. We can decompose the n^{th} term in terms of the series mentioned in the corollary (2)using the following identity:

Identity: $$\frac{1}{\prod_{m=1}^{k}(nk-(k-m))} = \sum_{m=1}^{k} \frac{(-1)^{m-1}}{(m-1)!(k-1)!}\left[\left\{\frac{1}{nk-(k-m)}\right\} - \frac{1}{nk}\right] \tag{10}$$

it is easy to see [2] that

$$\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = In2 \tag{11}$$

Dr. Knopp [3] mentions that

$$\frac{1}{1.2.3} + \frac{1}{4.5.6} + \frac{1}{7.8.9} + \ldots = \frac{1}{4}\left[\frac{\pi\sqrt{3}}{3} - In3\right] \tag{12}$$

$$\frac{1}{1.2.3.4} + \frac{1}{5.6.7.8} + \frac{1}{9.10.11.12} + \ldots = \frac{1}{4}\left[In2 - \frac{\pi}{6}\right] \tag{13}$$

I could not find anywhere the sums for $k \geq 5$ factors in the denominator. I have evaluated the sum for $k = 5$ with the help of the corollary (2), the identity (10), the integral $\int_0^1 \frac{dx}{(1+x^k)} = \sum_{n=1}^{\infty}\left[\left\{\frac{1}{2nk-(2k-1)}\right\} - \frac{1}{2nk}\right]$, the value of $\gamma(5)$ and the expansion $\frac{\pi}{y}\cot\frac{\pi x}{y} = \frac{1}{x} + \frac{1}{x-y} + \frac{1}{x+y} + \frac{1}{x-2y} + \frac{1}{x+2y} + \frac{1}{x-3y} + \frac{1}{x+3y} + \ldots$, when $\frac{x}{y}$ is rational:

$$\frac{1}{1.2.3.4.5} + \frac{1}{6.7.8.9.10} + \frac{1}{11.12.13.14.15} + \ldots$$
$$= \frac{1}{96}\left[\left\{2\pi\sqrt{5-2\sqrt{5}}\right\} - In5 - \frac{2\sqrt{5}}{3}In(2+\sqrt{5})\right] \tag{14}$$

The sum for $k = 6$ and 8 are not that difficult to compute:

$$\frac{1}{1.2.3.4.5.6} + \frac{1}{7.8.9.10.11.12} + \frac{1}{13.14.15.16.17.18} + \cdots = \frac{1}{5}\left[\frac{2In2}{9} - \frac{3In3}{32} - \frac{7\sqrt{3}\pi}{864}\right] \tag{15}$$

$$\frac{1}{1.2.3.4.5.6.7.8} + \frac{1}{9.10.11.12.13.14.15.16} + \frac{1}{17.18.19.20.21.22.23.24} \cdots$$
$$= \frac{1}{840}\left[\frac{17In2}{8} - \sqrt{2}In(1+\sqrt{2}) - \frac{(10\sqrt{2}-11)\pi}{48}\right] \tag{16}$$

The following sum for $k = 10$ is again extremely difficult to compute:

$$\frac{1}{1.2.3.4.5.6.7.8.9.10} + \frac{1}{11.12.13.14.15.16.17.18.19.20} + \frac{1}{21.22.23.24.25.26.27.28.29.30} + \ldots$$
$$= \frac{1}{567}\left[\frac{2In2}{25} - \frac{25In5}{1024} - \frac{11\sqrt{5}In(2+\sqrt{5})}{2560}\right.$$
$$+ \frac{\pi}{12800}\left\{\frac{11\sqrt{2}}{2}(15-8\sqrt{5})\sqrt{5+\sqrt{5}} + \frac{11\sqrt{2}}{2}\left(5\sqrt{5-\sqrt{5}}\right)\right.$$
$$\left.\left. - \frac{2}{5}(124\sqrt{5}-255)\sqrt{5+2\sqrt{5}} + \frac{8}{5}(19\sqrt{5}-30)\sqrt{5-2\sqrt{5}}\right\}\right] \tag{17}$$

The sum for $k = 12$ is less difficult to evaluate:

$$\frac{1}{1.2.3.4.5.6.7.8.9.10.11.12} + \frac{1}{13.14.15.16.17.18.19.20.21.22.23.24}$$
$$+ \frac{1}{25.26.27.28.29.30.31.32.33.34.35.36} + \ldots$$
$$= \frac{1}{8800}\left[\frac{In2}{27} - \frac{3In3}{448} - \frac{65\sqrt{3}In(2+\sqrt{3})}{4536} - \frac{\pi(394-221\sqrt{3})}{108864}\right] \tag{18}$$

I find it impossible to compute the sum for any other value of k.

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References

[1] L. Euler, *De Progressionibus Harmonicis Observationes,* Commentarii academiae scientiarum Petropolitanae 7, 1740, 150–161. http://www.math.dartmouth.edu/ euler/pages/E043.

[2] W.L. Ferrar, *A Text-Book of Convergence,* Oxford University Press, 1947, p.74.

[3] Konrad Knopp, *Theory and Application of Infinite Series,* Dover Publications, New York 1990 reprint of Blackie & Son, London, 2nd ed. 1951, p.268, Ex.107(g)and(h).

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THE RANGE OF BERNOULLI PROBABILITIES

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Abstract. What are all the sets of numbers which can arise as probabilities of events in some finite repetition of Bernoulli trials? In this note, we give a partial answer to the problem of characterizing all such sets. It is seen that a certain special case; namely, when all the numbers or at least their pairwise ratios are rational, and their maximum at most $1/e$, conforms to this situation. A more general result depends on the clique number of a certain regular graph. It is also seen that starting with a fixed probability of success, the range is dense in [0,1] if all finite numbers of trials are considered and the whole of [0,1] if infinitely many trials are allowed.

1. Introduction

One of the commonest calculations in an introductory probability course concerns independent repititions of Bernoulli trials with the probability of successs in each being fixed at p for some $0 < p < 1$. One often models this experiment through a coin with probability of head p being repeatedly tossed. For several examples and illuminating discussions, the classic book by Feller ([2], chapters VI and VIII) remains perhaps the best source. Some more recent works containing extensive treatments include, for instance, Pitman [7] and Grinstead and Snell [4]. Just one among legions of real-life examples of modeling by Bernoulli trials is provided by Lou [6]. Their central role in many notable problems in probability theory of historical importance is brought out in Gorroochurn [3].

Write $\Omega = \{S,F\}$ and for $n \geq 1$, denote by $P_{p,n}$ the probability measure on all subsets of Ω^n obtained as follows: to every elementary outcome or sequence with k Ss and $(n-k)$ Fs assign the probability $p^k(1-p)^{n-k}$.

The various ways in which these probabilities of elementary outcomes can be added gives one the possible numbers that can arise as probabilities of some event in such an experiment. Thus, if we denote the range of $P_{p,n}$ as $R_{p,n}$ then

$$R_{p,n} = \left\{ \sum_{k=0}^{n} a_k p^k (1-p)^{n-k} : a_0, a_1, \ldots, a_n \in \mathbb{Z}^+, a_k \leq \binom{n}{k}, \ k = 1, 2, \ldots, n. \right\}$$

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is the set of all probabilities of various events in this experiment. Clearly the sets $R_{p,n}$ increase with n; we denote their union over n by R_p.

Given a finite subset of the unit interval, determining whether it is contained in R_p for some p is, though not apparently so, a markedly more nontrivial problem. We provide a few examples of such sets but a complete answer remains elusive.

Given $m \geq 1$, numbers $\alpha_1, \alpha_2, \ldots, \alpha_m \in (0,1)$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_m$ and $p \in (0,1)$, we say that $(\alpha_1, \alpha_2 \ldots, \alpha_m)$ is *p-achievable* if the set $A := \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq R_p$. By the finiteness of A, this is clearly equivalent to $A \subseteq R_{p,n}$ for some n. The vector $(\alpha_1, \alpha_2 \ldots, \alpha_m)$ will be called achievable if it is p-achievable for some $p \in (0,1)$. We shall alternatively call the set A, respectively p-achievable and achievable.

Thus, achievability really means that $\alpha_1, \alpha_2, \ldots, \alpha_m$ can be obtained as probabilities of m events in a certain number n of independent repetitions of the same trial; that is, there exists a $p \in [0,1]$, an $n \in \mathbb{N}$ and events $E_1, E_2, \ldots, E_m \subseteq \Omega^n$ such that $P_{p,n}(E_j) = \alpha_j$ for all $j = 1, 2, \ldots, m$.

For example, any finite subset of the set of dyadic rationals in (0,1) can be easily seen to be $\frac{1}{2}$-achievable.

Notice that if $\{a, b\}$ is p-achievable, any subset chosen from the following set is also p-achievable (possibly using different values of n):

$$\{a,\, b,\, a^2,\, 1-a,\, a(1-a),\, 2a(1-a),\, ab,\, a+b-ab,\, \ldots\}$$

In fact for all of them $2n$ trials would suffice. As indicated, the set could be extended by using the same operations on any two members of the above list and more elements added.

Convention: To check achievability of $(\alpha_1, \alpha_2 \ldots, \alpha_m)$, we shall throughout assume that $\alpha_m \leq \frac{1}{2}$. Since every other case is either trivially achievable or can easily be reduced to this, if necessary by subtracting entries larger than $\frac{1}{2}$ from 1, there is no loss of generality.

Notation: Throughout, we use $\binom{m}{m_1, m_2, \ldots, m_k}$ to denote the multinomial coefficient $\frac{m!}{m_1! m_2! \cdots m_k! \{m - (m_1 + m_2 + \cdots m_k)\}!}$ for non-negative integers m, m_1, m_2, $\ldots$, m_k with $m_1 + m_2 + \cdots m_k \leq m$.

2. A class of achievable sets

In this section, we give a complete description of a certain special case.

Theorem 1. *If all the pairwise ratios of $\alpha_1, \alpha_2, \ldots, \alpha_m$ are rational and $\alpha_m \leq e^{-1}$, then $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is achievable.*

Corollary 2.1. *If all among $\alpha_1, \alpha_2, \ldots, \alpha_m$ are rational and $\alpha_m < e^{-1}$, then $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is achievable.*

We use the following trivial facts:

Fact 1. For every $k, l \in \mathbb{N}$,

$$\max_{0\leq p\leq 1} p^k(1-p)^l = \frac{k^k l^l}{(k+l)^{k+l}}$$

achieved at $p = \frac{k}{k+l}$.

Fact 2. For every $n \in \mathbb{N}$,

$$\left(\frac{n+1}{n}\right)^n < e < \left(\frac{n+1}{n}\right)^{n+1}.$$

Proof of Theorem 2.1 The hypothesis implies that there exists $\gamma \in (0, e^{-1}]$ and positive integers $k_1, k_2, \ldots, k_m$ such that $\alpha_j = k_j\gamma$, $j = 1, 2, \ldots, m$. To see why, we may for instance write $\alpha_j = \frac{p_j}{q_j}\alpha_1$ for $1 \leq j \leq m$ and take $\gamma := \frac{\alpha_1}{q_1 q_2 \cdots q_m}$. Clearly, $k_m > k_{m-1} > \ldots > k_1 \geq 1$. Now

$$k_m\gamma = \alpha_m \leq e^{-1} < \left(\frac{k_m-1}{k_m}\right)^{k_m-1},$$

which implies $\gamma \leq \frac{(k_m-1)^{k_m-1}}{k_m^{k_m}} = \max_{0\leq p\leq 1} p^{k_m-1}(1-p)$. But obviously, $\min_{0\leq p\leq 1} p^{k_m-1}(1-p) = 0$. Hence by the intermediate value theorem, $\exists\, p \in (0,1)$ such that $p^{k_m-1}(1-p) = \gamma$. With this p and $n = k_m$, we denote by $E_1, E_2, \ldots, E_n$ the n events of $n-1$ Ss and 1 F in n trials. Putting $E^{(j)} = \cup_{t=1}^{k_j} E_t$, we note that

$$P_{p,n}(E^{(j)}) = k_j p^{k_m-1}(1-p) = k_j\gamma = \alpha_j, \quad j = 1, 2, \ldots, m,$$

completing the proof. ∎

As illustration, with $\alpha_1 = \frac{1}{5}$ and $\alpha_2 = \frac{1}{3}$, find p solving $p^4(1-p) = \frac{1}{15}$ and carry out 5 trials with this p. The events are $E = 4$ successes and a failure, and $F = \{$ SSSSF, SSSFS, SSFSS $\}$, for example.

Remark 2.1. Even if $\alpha_m \in (\frac{1}{e}, \frac{1}{2})$, one may try to achieve any one of the m-tuples $(\alpha_{1,n_1}, \alpha_{2,n_2}, \ldots \alpha_{m,n_m})$ where $\alpha_{j,0} = \alpha_j$, $\alpha_{j,1}$, $\alpha_{j,2}\ldots$ is the strictly decreasing sequence of numbers successively defined as

$$\begin{aligned} \alpha_{j,n-1} &= 2\alpha_{j,n}(1-\alpha_{j,n}) \\ \Leftrightarrow \quad \alpha_{j,n} &= \frac{1}{2}\left[1 - \sqrt{1-2\alpha_{j,n-1}}\right] \\ \Leftrightarrow \quad \alpha_{j,n} &= \frac{1}{2}\left[1 - \sqrt[2^n]{1-2\alpha_j}\right]. \end{aligned}$$

Clearly, $\alpha_{j,n} \downarrow 0$ strictly; hence for all large n (larger than $\log_2\left[\frac{\log(1-2\alpha_j)}{\log(1-2e^{-1})}\right]$), $\alpha_{j,n} < e^{-1}$.

If for some such $n_1, n_2, \ldots, n_m$, $\{\alpha_{1,n_1}, \alpha_{2,n_2}, \ldots \alpha_{m,n_m}\}$ have pairwise rational ratios, those terms can be used to achieve $(\alpha_1, \alpha_2, \ldots, \alpha_m)$. For instance, if $\alpha_1 < e^{-1}$ is rational and $\alpha_2 = \frac{3}{8}$, $\alpha_{2,1} = \frac{1}{4}$ works with α_1.

However, if $\alpha_m = \frac{1}{2}$, this method is of no use since each $\alpha_{m,n}$, defined in the above way, equals $\frac{1}{2}$ as well.

Remark 2.2. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are all rational and

$$\alpha_1 < \alpha_2 < \ldots < \alpha_t < e^{-1} < \alpha_{t+1} < \ldots < \alpha_m,$$

$(\alpha_1, \alpha_2, \ldots, \alpha_l)$ is achievable as long as $\alpha_t > \frac{e\alpha_l - 1}{e-1}$. For then, for $s = t+1, \ldots, l$, we have $x_s := \frac{\alpha_s - \alpha_t}{1-\alpha_t} < e^{-1}$ so that $\{\alpha_1, \alpha_2, \ldots, \alpha_t, x_{t+1}, \ldots, x_l\}$ is achievable (this sequence may not be in increasing order), all being rational. Consequently, $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ is achievable too, since for each $s = t+1, \ldots, l$, we have $\alpha_s = \alpha_t + x_s - \alpha_t x_s$.

In particular, if there exists a j such that α_j belongs to the interval $(\frac{e-2}{2(e-1)}, e^{-1})$ then $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is achievable since in that case $\alpha_t \geq \alpha_j > \frac{e\frac{1}{2}-1}{e-1} \geq \frac{e\alpha_m - 1}{e-1}$. This allows us to achieve, for instance, $(\alpha_1, \alpha_2) = (\frac{1}{3}, \frac{1}{2})$ through $(\alpha_1, x_2) = (\frac{1}{3}, \frac{1}{4})$, a feat that was not directly possible through either of the methods described earlier.

3. A graph-theoretical formulation

To tackle the general case where we wish to remove the restriction on the largest of the numbers, our first aim is to find the maximum number of pairwise disjoint events each with at least k successes and l failures in n trials, $n \geq 2k + l$. Here it has been assumed without loss of generality that $k \geq l$.

These events can be identified by fixing k positions for Ss and l for Fs in binary strings (composed of S and F) of length n; the remaining positions being arbitrary. Then, two such events are disjoint if and only if in at least one position among n, one of the strings has S while the other has F.

Consider the following regular graph: the $\binom{n}{k,l}$ events as above are the vertices. Join a pair if they represent disjoint events. The degree of each vertex can be checked to be

$$\begin{aligned} & \binom{n}{k,l} - \sum_{i=0}^{l} \binom{l}{i} \binom{n-k-l}{l-i} \binom{n-2l+i}{k} \\ = \quad & \binom{n}{k,l} - \sum_{j=0}^{k} \binom{k}{j} \binom{n-k-l}{k-j} \binom{n-2k+j}{l}. \end{aligned}$$

Some alternative expressions are also possible.

For justifying the first line, represent each of the events by a string of k Ss, l Fs and $(n-k-l)$ Us (for unrestricted) and consider the number of strings that represent events *not* disjoint to a given one. For a given pair of non-adjacent vertices, in all the k (respectively, l) positions of the first string containing S (respectively, F), the second string must contain either S or U (respectively, either F or U). Denote by i the number of positions out of the l of the first string containing F where the second string also has F. These positions can be chosen in $\binom{l}{i}$ ways. Having chosen these, there are $(n-k-i)$ positions to choose for the remaining $(l-i)$ Fs, and subsequently, $(n-2l+i)$ positions to choose from for the k Ss. Subtracting the resulting number from the size of the graph gives the degree. The second line is symmetric.

Then the maximal number of pairwise disjoint events in the collection clearly equals the clique number (size of a maximal complete subgraph) of the above graph.

Denote this clique number by $c(k, l, N)$. While Punnim [9] derives some bounds for the clique number for a regualr graph in terms of the degree, they are not useful for the present purpose, and no exact formula for obtaining it seems available.

Theorem 2. *If all the pairwise ratios are rational and*

$$\alpha_m < \inf_k \limsup_{r\to\infty} \sup_{l,N} c(kr, l, N) \frac{(kr)^{kr} l^l}{(kr+l)^{kr+l}},$$

then $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ *is achievable.*

Proof of the theorem is straightforward following the lines of section 1: first we find $\gamma > 0$ and positive integers $k_1 < k_2 < \ldots < k_m$ such that $\alpha_j = k_j \gamma$ for each j.
Now we find suitable r, l, N such that

$$\alpha_m < c(k_m r, l, N) \frac{(k_m r)^{k_m r} l^l}{(k_m r + l)^{k_m r + l}}.$$

Then, we have

$$\gamma = \frac{\alpha_m}{k_m} < \frac{c(k_m r, l, N)}{k_m} \frac{(k_m + r)^{k_m + r} l^l}{(k_m r + l)^{k_m r + l}}.$$

Notice that since $c(k_m r, l, N) \uparrow \infty$ as $r \to \infty$, so r can be,and we assume has been, chosen so large that

$$\gamma < \left\lfloor \frac{c(k_m r, l, N)}{k_m} \right\rfloor \frac{(k_m + r)^{k_m + r} l^l}{(k_m r + l)^{k_m r + l}} = c \frac{(k_m + r)^{k_m + r} l^l}{(k_m r + l)^{k_m r + l}} \text{ say.}$$

Hence by arguments as in section 1, there exists a $p \in (0, 1)$ such that

$$\gamma = c p^{k_m + r} (1-p)^l.$$

But then for each $1 \le j \le m$,

$$\alpha_j = k_j \gamma = k_j c p^{k_m r} (1-p)^l = c_j p^{k_m r} (1-p)^l, \text{ say,}$$

where each c_j is an integer less than or equal to $c(k_m r, l, N)$. This means that in N trials with the p of the above equation as probability of success, out of the $c(k_m r, l, N)$ pairwise disjoint events of $k_m r$ successes and l failures, the probability of the union of any c_j is exactly α_j. ∎

Remark 3.1. In the special case when l is always fixed at 1, we get back our above work in section 1.

Remark 3.2. While the condition of rational ratios is obviously restrictive, one necessary condition for p-achievability can easily be seen to be that each member belong to $\mathbb{Q}(p)$.
In particular, if any is algebraic, then so is p and hence, so are all.

4. Density and an alternative to nonatomicity

Here, we first check that for every $p \in (0,1)$, R_p is dense in $[0,1]$. The proof is simple and depends only on the fact that $\forall\ n \geq 1$,

$$\delta_n := \max_{0 \leq j \leq n} p^j (1-p)^{n-j} = \max\{p^n, (1-p)^n\} \to 0 \text{ as } n \to \infty.$$

Since if we order the elements of $R_{p,n}$ in increasing fashion, the difference between two successive terms is $p^j(1-p)^{n-j}$ for some j, it follows that for any interval I of length $\epsilon > 0$, one of the terms must lie inside I if $\delta_n < \epsilon$. Denseness follows.

Finally, we show that if $\tilde{\Omega} = \Omega^\infty$ and P_p is the product probability on $\tilde{\Omega}$, then the range $\tilde{R_p}$ of P_p is all of $[0,1]$. This also follows from the non-atomicity of P_p, using Exercise 2.19, page 35 of Billingsley [1] but we give a direct proof.

Let $\alpha \in (0,1)$. If $\alpha \in \tilde{R_p}$ there is nothing to prove; if not, we use the following facts:

(a) R_p is dense in [0,1];

(b) R_p is closed under products, as already observed earlier; and

(c) $R_p \subseteq \tilde{R_p}$.

We choose numbers $\{\beta_k, \alpha_k : k \geq 1\}$ from R_p recursively as follows: choose $\beta_1 = 1$ and $\alpha_1 \in (\alpha, 1) \cap R_p$. For $k \geq 2$, having defined β_{k-1} and α_{k-1}, put $\beta_k := \beta_{k-1}\, \alpha_{k-1}$; and choose

$$\alpha_k \in R_p \cap \left(\frac{\alpha}{\beta_k}, \frac{k}{k-1}\ \frac{\alpha}{\beta_k} \wedge 1 \right).$$

Then $\beta_{k+1} = \beta_k \alpha_k \in (\alpha,\ \frac{k}{k-1}\, \alpha)$ for $k \geq 2$, whence $\beta_k \downarrow \alpha$ as $k \to \infty$.

But observe that the sequence $\beta_k = \alpha_1 \alpha_2 \cdots \alpha_{k-1}$, $k \geq 2$, can be obtained as respective probabilities under P_p of a sequence of decreasing events in $\tilde{\Omega}$. It follows by the property of continuity from above of probabilities (see Billingsley [1], Theorem 2.1 *(ii)*) that the intersection of these events has probability α under P_p. We have thus proved

Theorem 3. *If $p \in (0,1)$, then R_p is dense in $[0,1]$ and $\tilde{R}_p = [0,1]$.*

As an aside, it is an instructive exercise to check that for a purely atomic probability with weights $\{p_n : n \geq 1\}$, a necessary and sufficient condition for the range to be all of $[0,1]$ is that $\forall\ n,\ p_n \leq \sum_{j|p_j<p_n} p_j$. This follows from a result first proved by S. Kakeya, described in Pólya and Szegö [8, page 29, no. **131**]. Kesava Menon [5] contains a very detailed analysis.

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References

[1] Billingsley, P. (1995). *Probability and Measure*, 3rd Ed. John Wiley & Sons.

[2] Feller, W. (2000). *An Introduction to Probability Theory and Its Applications, Vol. 1*, 3rd Ed (Revised Printing). John Wiley & Sons.

[3] Gorroochurn, P. (2012). *Classic Problems of Probability.* John Wiley & Sons.

[4] Grinstead, C. M. and Snell, J. L. (1997). *Introduction to Probability.* American Mathematical Society.

[5] Kesava Menon, P. (1948). On a class of perfect sets. *Bull. Amer. Math. Soc.* **54**, 706–711.

[6] Lou, W. Y. Wendy (2003). The exact distribution of the k-tuple statistic for sequence homology. *Statistics & Probability Letters.* **61**, issue *1*, 51–59.

[7] Pitman, J. (1993). *Probability.* Springer.

[8] Pólya, G and Szegö, G. (1972). *Problems and Theorems in Analysis, Vol. 1.* Springer-Verlag.

[9] Punnim, N. (2002). The clique numbers of regular graphs. *Graphs and Combinatorics.* **18**, no. *4*, 781–785.

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BIORTHOGANAL WAVELETS BASED PRECONDITIONERS FOR SPARSE LINEAR SYSTEMS

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Abstract. We present a class of efficient preconditioners, based on biorthogonal family of Cohen Daubechies Feauveau (CDF) wavelets for sparse matrices arising in numerical solution of partial differential equations. Theory developed (Lemma 3.1 and Theorem 3.1) and algorithm proposed enables in designing these efficient wavelet based preconditioners, which are applicable to one of the Krylov subspace iterative methods namely Generalised Minimum Residual (GMRES). For illustration representative matrices from Tim Davis collection of sparse matrices is considered. Preconditioners based on DWT, DWTPer and DWTPerMod with biorthogonal family of wavelets are found to be robust, fast converging and requiring less CPU time, compared with that of Daubechies wavelets. More natural scheme of handling boundary is considered using symmetric extension for biorthogonal wavelets.

1. Introduction

The linear system of algebraic equations

$$Ax = b \tag{1.1}$$

where A is $n \times n$ non-singular matrix and b is vector of size n arise while discretising differential equations using finite difference or finite element schemes. For the solution of (1.1), we have a choice between direct and iterative methods. Gaussian elimination solves linear systems directly; it is an important technique in computational science and engineering. Gaussian elimination is expensive computationally and requires $o(n^3)$ arithmetic operations. Direct methods are time consuming (CPU time and hence memory) and for sparse systems these become inefficient for large unstructured problems, because of excessive amount of fill-in that occurs during the solution. One way of attempting to reduce the number of arithmetic operations (complexity) and computer time to obtain the solution is to choose iterative methods such as relaxation schemes or Krylov subspace iterative methods [9, 16]. Each iterative method is aimed at providing a way of solving

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a particular class of linear system, but none can be relied upon to perform well for all problems, and usually it is necessary to make use of a preconditioner to accelerate the convergence of iteration. This means that the original system of equations is transformed into equivalent system, which is more amenable to solution by a chosen iterative method (Benzi [1]). The combination of preconditioning and Krylov subspace iterations could provide efficient and simple general-purpose procedures that compete with direct solvers (Saad [16]).

A preconditioner is a nonsingular matrix $M(\approx A)$ with $M^{-1}A \approx I$.The philosophy is that the preconditioner has to be close to A and still allow fast converging iteration. The broad class of these preconditioners are mainly obtained by splitting A, say, in the form $A = D + C$ and writing (1) in the form

$$(I + D^{-1}C)x = D^{-1}b \tag{1.2}$$

If $M$is the diagonal of A then the iteration scheme is due to Jacobi($M$is not close to$A$)and the other extreme is $M = A$(too close to A). In between these two extremities, we look for M as band matrix constructed by setting to zero all entries of the matrix outside a chosen diagonal bandwidth [1, 4, 8, 15].

The most efficient and trusted iterative methods for the large matrices are Krylov subspace iterative methods such as Conjuagate Gradient (CG), Generalised Minimum Residual (GMRES) and their variants etc. Krylov subspace methods, in a broad sense, have the combining characteristics of direct as well as classical relaxation (stationary) methods. Like direct methods they are reliable in obtaining accurate solution at a predictable maximum cost and like stationary iterative methods the actual cost depends on the number of iterations required for convergence to a specified accuracy. For the symmetric positive definite matrices CG and some of its variants, with appropriate preconditioners, work very well. For the unsymmetric matrices most often GMRES and its various modifications (restarted GMRES, etc) are the methods of choice [16]. Again, it is of interest to note that there are large class of problems where in the Krylov subspace methods or their preconditioned forms are inadequate requiring quite a large number of iterations for convergence. Wavelet based preconditioners are found to be more reliable and robust for these iteration schemes [4].

Chen [4] has exhaustively listed various orthogonal wavelet based schemes obtained by matrix/operator splitting with Discrete Wavelet Transform (DWT) to construct new and improved preconditioners for dense matrices. Chen [3] and Ford [8] have developed this category of preconditioners for dense matrices where as Kumar and Mehra [15] confine to large sparse matrices having high condition number. These authors have used Daubechies (orthogonal) wavelets with periodic extension and zero padding respectively. Zero padding [15] and periodic extension [3, 8] introduce discontinuities at the boundaries of the original function. These discontinuities cause local increase in wavelet coefficients and complicate compression of the function. The symmetric extension of the function at the boundary avoids these drawbacks and preserves its structure. As stressed forcefully by Unser [18], with symmetric extension for biorthogonal wavelets, the wavelet transform reduces boundary artifacts and these advantages cannot be achieved by other non-linear phase wavelets such as Daubechies wavelets. In this class of schemes DWT with Permutations (DWTPer) and DWTPerMod [8] are attractive and found to work efficiently compared with other classical methods. More about these will be emphasized in subsequent sections.

In 1992, Cohen, Feauveau and Daubechies[5] constructed the compactly supported biorthogonal wavelets, which are preferred over the orthonormal basis functions because of their efficiency in diverse fields - such as numerical analysis signal processing, medical imaging etc. The success story of digitizing fingerprints, by Federal Bureau of Investigation (FBI), using biorthogonal spline wavelets and the manufacturing of chips for Joint Photographic Expert Group (JPEG 2000), finer structure analysis of Electrocardiography (ECG), Electroencephalography (EEG) using these wavelet families, etc. are the testimony of the attractive activities in the field [13, 17, 18, 21].

For the best possible results, the wavelets should have several vanishing moments and a compact support as small as possible (central to the success of wavelet compression). Keinert [10] has shown that for some operators (six standard examples from Beylkin et al [2] paper), the biorthogonal wavelet exhibit both higher compression and faster execution than the corresponding orthogonal wavelets (Daubechies wavelets) at comparable accuracy. Also, numerical stability of broad class of biorthogonal wavelets is assured [11]. The most popular representatives of the Cohen Daubechies Feauveau (CDF) [5] class of biorthogonal wavelets are splines as well. As such they are easy to use and one can achieve same quality of approximation with representation at half the resolution compared with Daubechies wavelets for smooth data [19].

DWT is a big boon especially in signal and Image processing where in smooth data can be compressed sufficiently without loosing primary features of the data. Compressed data either using thresholding or cutting is of much use if we are not using it in further process such as using it as preconditioner, which causes large fill-in in its routine applications. However, the novel scheme, DWT with permutation, explained in [3], avoids these drawbacks especially creation of finger pattern matrices etc, and enables in selecting efficient preconditioners using wavelets. Non-smooth data may be of two types-diagonal band and horizontal / vertical bands. Suitable matrix-splitting enables identifying the non-smooth part of the matrix which could be a diagonal band or a portion bounded by some horizontal/vertical column. Standard DWT results in matrices with finger pattern. If DWT is followed by the permutation of the rows and columns of the matrix then it centres/brings the finger pattern about the leading diagonal. This strategy is termed as DWTPer. Elegant analysis presented by Chen enables in predicting the width of band of entries of matrix which have larger absolute values. Later, an approximate form of this can be formed and taken as preconditioner, which controls fill-in whenever schemes like decomposition is used. Similar criteria is adopted in transforming non-smooth parts which are horizontal/vertical bands and are shifted to the bottom or right hand side of the matrix after DWTPer is applied. This procedure is termed as DWTPerMod and takes care of other cases where non-smooth parts are located at the top or in the middle of the matrix [8].

Later, Ford [8] proposed similar ideas more effectively incorporating the missing finer details such as fixing of precise band width and automatic selection of optimal choice of transform level under DWTPerMod algorithm and presented its salient features by applying it to standard dense matrices arising in various disciplines/fields of interest. Both Chen and Ford [4, 8] confine to a periodised Daubechies Wavelets. Kumar and Mehra [15] use iDWTPer(Incomplete Discrete Wavelet Transform with permutation) and propose effective algorithms for selecting preconditioners for large sparse matrices but their work is silent about the selection of transform level.

The structure of the paper is organized as follows: In section 2 we explain briefly about orthogonal and biorthogonal wavelets. Section 3 contains practical computation details of transforms required. Here, we present the main results in the form of lemma, theorem and algorithm and describe the design of the CDF wavelet based preconditioners. In section 4 we illustrate with typical test problems, selected from Tim Davis collection, to emphasize the potential usefulness of preconditioners designed in this article. In section 5 brief discussion is given finally in section 6 we summarize our conclusion.

2. Wavelet Preliminaries

The effective introduction to wavelets via multiresolution analysis is broad based compared with conventional/classical way through scaling function concept and as such the following basics in compact form are given below.

2.1. **Multiresolution Analysis (MRA),(Mallat** [13]**).** A multiresolution analysis on $\Re$ is a sequence of closed subspaces $\{V_j\}_{j\in Z}$ of $L^2(\Re)$ satisfying the following conditions

i) nestedness: $V_j \subset V_{j+1}$
ii) completeness: closure of $\cup V_j = L^2(\Re)$, nullity: $\cap V_j = \{0\}$

iii) scale invariance: $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$

iv) shift invariance: $f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0, k \in Z$

v) shift invariant basis: there exists a scaling function $\phi \in V_0$ such that $\{\phi(x-k)\}_{k\in Z}$ form a stable (Riesz) basis for V_0.

Let W_0 be complement of V_0 in V_1,*i.e.* $V_1 = V_0 \bigoplus W_0$. This implies that there exists a function $\psi \in W_0$ such that its translates $\{\psi(x-k)\}_{k\in Z}$ form a Riesz basis for W_0. For each $j, k \in Z$, define $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ then these form a Riesz bases for V_j and W_j respectively. Biorthognal wavelets are defined by two hierarchies of approximation spaces-MRA's on $\Re$, denoted by $\{V_j\}_{j\in Z}$ and $\{\tilde{V}_j\}_{j\in Z}$,*i.e.* two scaling functions ϕ, $\tilde{\phi}$ two wavelet functions ψ, $\tilde{\psi}$ such that $V_j = \overline{\langle \phi_{j,k}\rangle}, \tilde{V}_j = \overline{\langle \tilde{\phi_{j,k}}\rangle}, W_j = \overline{\langle \psi_{j,k}\rangle}, \tilde{W}_j = \overline{\langle \tilde{\psi_{j,k}}\rangle}$ and satisfies the following biorthogonality conditions [12, 17, 21]
$\langle \phi_{n,j}, \tilde{\phi_{n,k}}\rangle = \delta_{j-k}$, $\langle \psi_{n,j}, \tilde{\psi_{n,k}}\rangle = \delta_{j-k}$, $\langle \phi_{n,j}, \tilde{\psi_{n,k}}\rangle = \langle \psi_{n,j}, \tilde{\phi_{n,k}}\rangle = 0$. where δ is the Kronecker delta function defined by

$$\delta_{j-k} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

Here,

$$V_{j+1} = V_j + W_j, \tilde{V}_{j+1} = \tilde{V}_j + \tilde{W}_j, V_j \perp \tilde{W}_j, \tilde{V}_j \perp W_j \tag{2.1}$$

Every space W_j will be a complement to V_j in V_{j+1} but not an orthogonal complement as above(in orthogonal wavelet). Since V_0 and W_0 are contained in V_1, we have

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k), \psi(x) = \sqrt{2}\sum_k g_k \phi(2x-k)$$
$$\tilde{\phi}(x) = \sqrt{2}\sum_k \tilde{h}_k \tilde{\phi}(2x-k), \tilde{\psi}(x) = \sqrt{2}\sum_k \tilde{g}_k \tilde{\phi}(2x-k)$$

for some $\{h_k\}, \{\tilde{h}_k\}, \{g_k\}, \{\tilde{g}_k\}$ in $l^2(\Re)$. These filter coefficients can be generated using factorization scheme of Daubechies polynomials and are related by $g_n = (-1)^n \tilde{h}_{1-n}, \tilde{g}_n = (-1)^n h_{1-n}$ [5, 17, 21]. Here $\tilde{\phi}, \tilde{\psi}$ are called dual of ϕ, ψ respectively. We define the projections $\tilde{P}_j, \tilde{Q}_j, P_j, Q_j$ from $L^2(\Re)$ onto $V_j, W_j, \tilde{V}_j, \tilde{W}_j$ by

$$P_j f = \sum_{k\in Z}\langle f, \phi_{j,k}\rangle \tilde{\phi}_{j,k}, \tilde{P}_j f = \sum_{k\in Z}\langle f, \tilde{\phi}_{j,k}\rangle \phi_{j,k} \tag{2.2}$$

$$Q_j f = \sum_{k\in Z}\langle f, \psi_{j,k}\rangle \tilde{\psi}_{j,k}, \tilde{Q}_j f = \sum_{k\in Z}\langle f, \tilde{\psi}_{j,k}\rangle \psi_{j,k} \tag{2.3}$$

$P_j, \tilde{P}_j$ and $Q_j, \tilde{Q}_j$ are also called approximation and detail operators on f.If $\phi = \tilde{\phi}$, this reduces to orthogonal MRA. The projectionP_j reduces to orthogonal projection.We are dealing only with Daubechies orthogonal MRA and biorthogonal MRA leading to CDF [5]. Our goal is to derive biorthogonal counterparts to Daubechies wavelet based preconditioners for linear system and give comparison of results obtained with existing methods. Motivated by the work of Chen [4], Ford [8] and Kumar and Mehra[15], we have developed biorthogonal wavelet based preconditioners for *sparse unsymmetric linear systems*.

3. Practical Computations

Suppose that a vector (signal)s from vector space $\Re^n$ is given. One may construct it as an infinite sequence by extending the signal as symmetric one at both ends and hence by periodic form/extension [12, 17, 21] and use this extended signal as a sequence of scaling coefficients for some underlying function f_L in some fixed space $\tilde{V}_L$ of L^2(from(2.2))as

$$f_L(x) = \sum_k s_{L,k}\tilde{\phi}_{L,k}(x)$$

From (2.1),$\tilde{V}_L = \tilde{V}_r + \tilde{W}_r + \tilde{W}_{r+1} + ... + \tilde{W}_{L-1}$, this implies that

$$f_L(x) = \sum_k s_{r,k}\tilde{\phi}_{r,k}(x) + \sum_{t=r}^{L-1}\sum_i d_{t,i}\tilde{\psi}_{t,i}(x)$$

s, d coefficients are called smooth (filtered by lowpass) and detail/difference (filtered by highpass) parts of f respectively, where

$$s_{j-1,k} = \sum_m h_{m-2k}s_{j,m} \text{ and } d_{j-1,k} = \sum_m g_{m-2k}s_{j,m} \tag{3.1}$$

$$s_{j+1,n} = \sum_k \tilde{h}_{n-2k}s_{j,k} + \sum_k \tilde{g}_{n-2k}d_{j,k} \tag{3.2}$$

Here$\{h, g\}$ are called analysis (decomposition) filters and $\{\tilde{h}, \tilde{g}\}$ are called synthesis (reconstruction) filters. The process of obtaining (3.1) for various j's is termed as Discrete Wavelet Transform and (3.2) is inverse of (3.1) (Mallat Algorithm [13]). DWT transforms the vector $s \in \Re^n$ to

$$w = [s_r^T, d_r^T, d_{r+1}^T, ..., d_{L-1}^T]^T \tag{3.3}$$

The goal of wavelet transform is to make the transformed vector to be nearly sparse. This can be achieved by increasing the number of vanishing moments of analysis wavelet *i.e.* $\int \psi(x)x^t dx = 0$ for $t = 0, 1, ..., \tilde{p} - 1$, which is equivalent to $G(z) = \sum g_k z^{-k}$ having $z = 1$ as zero of order $\tilde{p}, G(z)$ is $Z-$transform of $\{g_k\}$. Similarly p is defined for $\tilde{G}(z)$ [17].For a given $p, \tilde{p}$, such that $p + \tilde{p}$ is even, there exists compactly supported Biorthogonal Spline wavelets of order $p, \tilde{p}$ called Cohen Daubechies Feauveau wavelets (CDF$(p, \tilde{p})$). Length of $\{\tilde{h}\}$ is $\tilde{p}+1$ and that of $\{h\}$ is $2p + \tilde{p} - 1$ [5, 6]. For example with $p + \tilde{p} = 4$ [using Mathematica], we have the corresponding filter coefficients given in Table1.

Table 1 Filter Coefficients of CDF

$(p,\tilde{p})$	Scaling filter coefficients $\{h\}$	Dual Scaling filter coefficints$\{\tilde{h}\}$
CDF(3,1)	$\{\frac{-1}{8\sqrt{2}},\frac{1}{8\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{8\sqrt{2}},\frac{-1}{8\sqrt{2}}\},-2(=k)$	$\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\},0$
CDF(1,3)	$\{\frac{-1}{2\sqrt{2}},\frac{3}{2\sqrt{2}},\frac{3}{2\sqrt{2}},\frac{-1}{2\sqrt{2}}\},-1$	$\{\frac{1}{4\sqrt{2}},\frac{3}{4\sqrt{2}},\frac{3}{4\sqrt{2}},\frac{1}{4\sqrt{2}}\},-1$
CDF(2,2)	$\{\frac{-1}{4\sqrt{2}},\frac{1}{2\sqrt{2}},\frac{3}{2\sqrt{2}},\frac{1}{2\sqrt{2}},\frac{-1}{4\sqrt{2}}\},-2$	$\{\frac{1}{2\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}}\},-1$

In the above table $-2(=k)$ represents the starting index of filter coefficient. CDF$(1,1)$ is Haar orthogonal Wavelet. Daubechies of order 4(Daub-4) lowpass filter coefiicients are $h_0=\frac{1+\sqrt{3}}{4\sqrt{2}},h_1=\frac{3+\sqrt{3}}{4\sqrt{2}},h_2=\frac{3-\sqrt{3}}{4\sqrt{2}},h_3=\frac{1-\sqrt{3}}{4\sqrt{2}}$ and its highpass filtered coefficients are given by$g_n=(-1)^n h_{3-n}$. Here $h=\tilde{h},g=\tilde{g}$. We can show the importance of biorthogonal wavelet transform on Calderon-Zygmund matrix. Consider the Calderon-Zygmund(CZ) type matrix $A=(a_{ij})$, where

$$a_{i,j}=\begin{cases}3 & i=j\\ \frac{1}{|i-j|} & i\neq j\end{cases}$$

Table 2 gives the number of nonzeros after compression of $\tilde{A}$(after applying DWT) for level 3 with thresholding value 0.001,where the number of operations and number of vanishing moments are same for both wavelet families.

Table 2. Number of nonzeros after compression

Order of CZ	Daub-4	CDF(2,2)
512	11026	10514
1024	30190	29226
2048	92998	91346
4096	316676	313810

3.1. **Wavelet-Based Preconditioners.** For the brief description of DWT, DWTPer and DWTPerMod we have their matrix representations in the following forms. Assume $n=2^L$, for some positive integer L and r an integer such that $2^r<D$ and $2^{r+1}\geq D$(for biorthogonal wavelets $D=p+\tilde{p}$), where D is the order of Daubechies wavelet, $r=0$, for $D=2$ (Haar/CDF(1,1) wavelets) and $r=1$ for $D=4$ [Daubechies wavelet of order 4, CDF(2,2),CDF(1,3),CDF(3,1)]. The matrix form of (3.3) is

$$w=P_kW_kP_{k-1}W_{k-1}...P_2W_2P_1W_1 \tag{3.4}$$

$$w = Ws$$

where,

$$P_i = \begin{pmatrix} \overline{P_i} & \\ & J_i \end{pmatrix} \tag{3.5}$$

with $\bar{P}_i$ is a permutation matrix of size $\frac{n}{2^{i-1}}$, $\bar{P}_i = I(1, 3, 5, ..., \frac{n}{2^{i-1}} - 1, 2, 4, ..., \frac{n}{2^{i-1}})$, J_i is identity matrix of size $n - \frac{n}{2^{i-1}}$ and

$$W_i = \begin{pmatrix} \bar{W}_i & \\ & J_i \end{pmatrix}$$

where $\bar{W}_i$ defined by the following matrices.

Let W and $\hat{W}$ denote DWT and DWTPer analysis matrices respectively either for orthogonal or biorthogonal wavelets depending on the context of the analysis.

1)For Daubechies order four($D = 4$)with 2 vanishing moments, it is

$$\bar{W}_i = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & & & & \\ g_0 & g_1 & g_2 & g_3 & & & & \\ & & h_0 & h_1 & h_2 & h_3 & & \\ & & g_0 & g_1 & g_2 & g_3 & & \\ & & & \vdots & & \ddots & \ddots & \vdots \\ & & & & h_0 & h_1 & h_2 & h_3 \\ & & & & g_0 & g_1 & g_2 & g_3 \\ h_2 & h_3 & & & & & h_0 & h_1 \\ g_2 & g_3 & & & & & g_0 & g_1 \end{bmatrix}_{\frac{n}{2^{i-1}} \times \frac{n}{2^{i-1}}} \tag{3.6}$$

2)For CDF(2,2) with 2 vanishing moments, we have

$$\bar{W}_i = \begin{bmatrix} h_0 & 2h_1 & 2h_2 & & & & & & & \\ g_0 & g_1 & g_2 & & & & & & & \\ h_{-2} & h_{-1} & h_0 & h_1 & h_2 & & & & & \\ & & g_0 & g_1 & g_2 & & & & & \\ & & & \vdots & & \ddots & \ddots & \vdots & & \\ & & & & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & \\ & & & & & & g_0 & g_1 & g_2 & \\ & & & & & & h_{-2} & h_{-1} & h_0 + h_2 & h_1 \\ & & & & & & & & g_0 + g_2 & g_1 \end{bmatrix}_{\frac{n}{2^{i-1}} \times \frac{n}{2^{i-1}}} \tag{3.7}$$

3)and for CDF(3,1) with one vanishing moment

$$\bar{W}_i = \begin{bmatrix} h_0+h_{-1} & h_1+h_{-2} & h_2 & h_3 & & & & & & \\ g_0 & g_1 & & & & & & & & \\ h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 & & & & \\ & & g_0 & g_1 & & & & & & \\ & & & \vdots & & \ddots \ \ddots & \vdots & & & \\ & & & & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 \\ & & & & & & g_0 & g_1 & & \\ & & & & & & h_{-2} & h_{-1} & h_0+h_3 & h_1+h_2 \\ & & & & & & & & g_0 & g_1 \end{bmatrix}_{\frac{n}{2^{i-1}} \times \frac{n}{2^{i-1}}} \tag{3.8}$$

Chen[3] has defined one level DWTPer matrix for Daubechies wavelet of order D and it is given by

$$\hat{W}_i = \begin{bmatrix} h_0 & \varphi & h_1 & \varphi & h_2 & \varphi\dots & h_{D-1} & & \\ \varphi & I & \varphi & \varphi & \varphi & \varphi\dots & \varphi & & \\ g_0 & \varphi & g_1 & \varphi & g_2 & \varphi\dots & g_{D-1} & & \\ \varphi & \varphi & \varphi & I & \varphi & \varphi & \dots & \varphi & \\ & & & & h_0 & \varphi & \ddots & \ddots & \\ & & & & \varphi & I & \ddots & \ddots & \\ & & & \vdots & & \ddots \ \ddots & \vdots & & \\ h_2 & \varphi & \dots & h_{D-1} & & h_0 & \varphi & h_1 & \varphi \\ \varphi & \varphi & \dots & \varphi & & \varphi & I & \varphi & \varphi \\ g_2 & \varphi & \dots & g_{D-1} & & g_0 & \varphi & g_1 & \varphi \\ \varphi & \varphi & \dots & \varphi & & \varphi & \varphi & \varphi & I \end{bmatrix}_{n\times n}$$

Here I is an identity matrix of size $2^{i-1}-1$ and φ's block zero matrices. For $i=1$,both I and φ are of size 0, i.e. $\hat{W}_1 = \bar{W}_1 = W_1$. Similarly we define one level DWTPer matrix for CDF$(p,\tilde{p})$. Here we define for even p and $\tilde{p}$(same thing hold for odd p and $\tilde{p}$ [15, 12]) for this let $P_e = \frac{2p+\tilde{p}-2}{2}, q = \frac{\tilde{p}}{2}$ then one level DWTPer matrix for CDF$(p,\tilde{p})$is given by,

$$\hat{W}_i = \begin{bmatrix} h_0 & \varphi & 2h_1 & \varphi & 2h_2 & \varphi & \dots & & 2h_{P_e} & \\ \varphi & I & \varphi & \varphi & \varphi & \varphi\dots & \varphi & & & \\ g_0 & \varphi & g_1+g_{-1} & \varphi & g_2+g_{-2} & \varphi & \dots & g_{q+1} & & \\ \varphi & \varphi & \varphi & I & \varphi & \varphi & \dots & \varphi & & \\ & \ddots & & & \vdots & \ddots & \vdots & & & \\ & \varphi & h_{-1} & \varphi & h_0 & \varphi & h_1 & \dots & & \\ & & & & \varphi & I & \varphi & & & \\ & \varphi & g_{-1} & \varphi & g_0 & \varphi & g_1 & \dots & & \\ \vdots & & \vdots & & \ddots \ \vdots & \vdots & & & & \\ & & h_{-P_e}\dots & \varphi & h_{-1}+h_3 & \varphi & h_0+h_2 & \varphi & h_1 & \varphi \\ & & \varphi & \dots & \varphi & & \varphi & I & \varphi & \varphi \\ & & \varphi & & 2g_{1-q}\dots & \varphi & 2g_0 & \varphi & g_1 & \varphi \\ & & \varphi & \dots & \varphi & & \varphi & \varphi & \varphi & I \end{bmatrix}_{n\times n} \tag{3.9}$$

Here I is an identity matrix of size$2^{i-1}-1$ and φ's block zero matrices. For $i=1$, both I and φ's are of size 0, *i.e.*, $\hat{W}_1=\bar{W}_1=W_1$. In the above matrix h_0, I and g_1 lie on main diagonal, for CDF(2,2), it reduces to

$$\hat{W}_i=\begin{bmatrix} h_0 & \varphi & 2h_1 & \varphi & 2h_2 & & & & \\ \varphi & I & \varphi & \varphi & \varphi & & & & \\ g_0 & \varphi & g_1 & \varphi & g_2 & & & & \\ \varphi & \varphi & \varphi & I & \varphi & \varphi & \dots & \varphi & \\ h_{-2} & \varphi & h_{-1} & \varphi & h_0 & \varphi & h_1 & \varphi & h_2 \\ \varphi & \varphi & \varphi & \varphi & \varphi & I & \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi & \varphi & g_0 & \varphi & g_1 & \varphi & g_2 \\ \vdots & & \vdots & & \ddots\ \ddots & & \vdots & & \\ & & h_{-2} & \varphi h_{-1} & \varphi & h_0+h_2 & \varphi & h_1 & \varphi \\ & & \varphi & \varphi & \varphi & \varphi & I & \varphi & \varphi \\ & & & & \varphi & g_0+g_2 & \varphi & g_1 & \varphi \\ & & \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & I \end{bmatrix}_{n\times n} \tag{3.10}$$

The above wavelets were defined on the real line, i.e. on a one-dimensional domain. To create wavelets on higher dimensional domains, one of the approaches is to perform the wavelet transform independently for each dimension [20]. For two dimensional case, let A be a $n\times n$ matrix then its wavelet transform is $\tilde{A}=WAW^T$.

Now, DWTPer for a vector $s\in\Re^n$ can be defined by$\hat{w}=\hat{W}s$, with $\hat{W}=\hat{W}_k\hat{W}_{k-1}...\hat{W}_2\hat{W}_1$ for k levels. For a matrix A of order $n\times n$, the DWTPer would give $\hat{A}=\hat{W}A\hat{W}^T$. To relate $\hat{A}$ to $\tilde{A}$ from a standard DWT or relate $\hat{W} to W$, in [3], [8] it is proved that $\hat{A}=P\tilde{A}P^T$, where $P=P_1^TP_2^T...P_k^T$, each P_i is a permutation as defined in (3.5). Similar result holds for biorthogonal spline wavelets also. Here number of operations required for obtaining one smooth coefficient and one detail coefficient is $2D=2(p+\tilde{p})$. Daubechies wavelet has D number of lowpass filter coefficients and D number of highpass filter coefficients, where as for CDF$(p,\tilde{p})$ wavelet has $2p+\tilde{p}-1$ number of lowpass filter coefficients and $\tilde{P}+1$ number of highpass filter coefficients. Similar argument holds for the number of operations required for obtaining $\tilde{A}(=WAW^T)$ which are same for both families of wavelets (Neilsen [14], Uytterhoeven [20]).

3.2. **New algorithm for CDF wavelet based preconditioners.** A band (α,β), is a matrix whose lower bandwidth is α and upper bandwidth is β and we use the band (band$(\alpha_1,\beta_1)\times$ band(α_2,β_2)) = band$(\alpha_1+\alpha_2,\beta_1+\beta_2)$[9]. Now we shall discuss about the band width of $\hat{W}$ for biorthogonal wavelets which enables to select suitable preconditioners.

Lemma 3.1 For level k, the bandwidth of biorthogonal spline wavelet (CDF$(p,\tilde{p})$) analysis matrix $\hat{W}$ is given by

i) $((2^k-1)\frac{2p+\tilde{p}-2}{2},(2^k-1)\frac{2p+\tilde{p}-2}{2})$, for even $p,\tilde{p}$

ii) upper bandwidth is $(2^k-1)\frac{2p+\tilde{p}-1}{2}$ and lower bandwidth is

$$=\begin{cases}(2^k-1)(\frac{\tilde{p}+1}{2}) & \text{if } p=1\\ (2^k-1)\frac{2p+\tilde{p}-3}{2} & \text{otherwise}\end{cases}$$

for odd $p,\tilde{p}$

Proof: Let $p,\tilde{p}\in N$ such that their sum is even. Then the total number of filter coefficients is $2(p+\tilde{p})$, among them length of highpass filter coefficients ($\{g_k\}$) is $\tilde{p}+1$ and that of lowpass filter coefficients ($\{h_k\}$) is $2p+\tilde{p}-1$.

We prove this result for even p and $\tilde{p}$. Similar proof follows for odd p and $\tilde{p}$. Since p and $\tilde{p}$ are even, this implies that $h_k=h_{-k}$ and $g_{2-k}=g_k$ [21]. A typical strip in each matrix $\hat{W}_i$ (3.9) is given by

$$\begin{matrix} h_{-P_e}\dots & \varphi & h_{-1} & \varphi & h_0 & \varphi & h_1\dots & h_{P_e}\\ \varphi & \dots & \varphi & \varphi & \varphi & I & \varphi & \varphi\\ \varphi & & g_{1-q}\dots & \varphi & g_0 & \varphi & g_1\dots & g_{1+q}\end{matrix} \tag{3.11}$$

where $P_e=\frac{2p+\tilde{p}-2}{2}, q=\frac{\tilde{p}}{2}$. Here h_0, I and g_1 lie on main diagonal of $\hat{W}_i$, the size of zero vector φ(which lies between two nonzero filters) is $2^{i-1}-1$ and this implies that upper bandwidth of $\hat{W}_i$ is $P_e+(2^{i-1}-1)P_e=2^{i-1}P_e$. Since $\hat{W}=\hat{W}_k\hat{W}_{k-1}...\hat{W}_2\hat{W}_1$, its upper bandwidth is $P_e(2^{k-1}+2^{k-2}+...+1)=(2^k-1)P_e$. Since $max\{P_e,q\}=P_e=\frac{2p+\tilde{p}-2}{2}$ its lower band width is also same. In the above strip (3.11), φ is of different size and it depends on where it appears [3].

Theorem 3.1. *Let A be a band(α,β) matrix. Then the DWTPer of k levels, based on CDF$(p,\tilde{p})$, transforms A into $\hat{A}$ which is a band(λ_1,λ_2) matrix with,*

$$\begin{cases}\lambda_1-\alpha=\lambda_2-\beta=(p+\tilde{p})(2^k-1) & \textit{if } p{=}1\\ \lambda_1-\alpha=\lambda_2-\beta=(2p+\tilde{p}-2)(2^k-1) & \textit{otherwise}\end{cases}$$

Proof: Proof follows from Lemma 3.1 and the fact that band (band$(\alpha_1,\beta_1)\times$ band(α_2,β_2)) = band$(\alpha_1+\alpha_2,\beta_1+\beta_2)$.

To solve $Ax=b$, by using biorthogonal wavelet based preconditioner, we propose the following algorithm

Algorithm 3.1:

1) Apply DWTPer (CDF$(p,\tilde{p})$) to $Ax=b$ to obtain $\hat{A}u=z$
2) Select a suitable $\hat{M}$ band form of $\hat{A}$ (Theorem 3.1)
3) Use $\hat{M}$ as a preconditioner to solve $\hat{A}u=z$ iteratively
4) Apply Inverse Wavelet Transform on u to get required solution.

To apply algorithm discussed above, we must have an idea about the level of transform to be applied on A . To overcome this Ford [8] developed under DWTPerMod algorithm and the speciality of the algorithm is that it fixes the level automatically by knowing the size of A and the order of the wavelet $D(= p + \tilde{p})$ transform used. Further she sharpened the bounds for band(λ_1, λ_2) in Theorem 3.1 corresponding to Daubechies orthogonal wavelets. For the purpose of implementation n (order of matrix) need not be a power of two [4].

4. Numerical Results

To test the efficiency of biorthogonal wavelet based preconditioners and compare results with orthogonal wavelet based preconditioners we have considered the following examples from Tim Davis collection of matrices. The right hand side of linear system was computed from the solution vector x of all ones (except for Sherman4 matrix). We have implemented the proposed algorithms using Matlab-7.5 and Mathematica-7. The initial guess is always $x_0 = 0$ and stopping criteria is relative residual is less than or equal to10^{-6} $(i.e., \|b - Ax\|_2 \leq \|b\|_2 10^{-6})$ and the Krylov Subspace iterative method used is GMRES (25) [16]. The symbol Ω stands for slow convergence/no convergence in the following tables. In these tables number of iterations required for convergence and the corresponding CPU time in seconds are given in square brackets. Also n, nnz and $k(A)$, represent the size, number of nonzeros and condition number of the matrix A respectively. Here $k(A) = condest(A)$ [Matlab function for finding condition number of given sparse matrix]. To fix the level of wavelet transform, algorithm of [8] is used with $\alpha = \beta = 2$ for all examples except for gre-512 matrix.

Tols90: tols90 matrix arises in the stability analysis of a model of an airplane (aeroelasticity). For this matrix,$n = 90, nnz = 1746$ and $k(A) = 2.4913 \times 10^4$.Convergence details are presented in Table **3**.

Table 3(*Tols90*)

Wavelet family	DWT	DWTPer	DWTPerMod
CDF(3,1)	21[0.0624]	Ω	8[0.0156]
Haar/CDF(1,1)	20[0.0624]	Ω	Ω
Daub-4	48[0.0728]	Ω	Ω

Gre-216a: The matrix gre-216a arises in simulation studies in computer systems. Here $n = 216, nnz = 812$ and $k(A) = 3.0499 \times 10^2$. Convergence details are presented in Table **4**.

Table 4(*gre216a*)

Wavelet family	DWT	DWTPer	DWTPerMod
CDF(3,1)	Ω	10[0.0312]	5[0.0312]
Haar/CDF(1,1)	Ω	Ω	525[0.7644]
Daub-4	Ω	125[0.3276]	51[0.1404]

Gre-512: The matrix gre-512 arises in simulation studies in computer systems. Here $n = 512$,$nnz = 1976$ and $k(A) = 3.8478 \times 10^2$.With $\alpha = \beta \leq 24$ preconditioners developed for Haar and Daub-4 are not useful for GMRES(25). But for $\alpha = 24, \beta = 0$,convergence details are presented in Table **5**,Figures **1**and **2**.

Table 5(*gre-512*)

Wavelet family	DWT	DWTPer	DWTPerMod
CDF(3,1)	Ω	Ω	7[0.1248]
Haar/CDF(1,1)	Ω	Ω	Ω
Daub-4	Ω	Ω	Ω

Sherman4: this matrix arises in oil reservoir modeling. Here right hand side vector is available. Here $n = 1104, nnz = 3786$ and $k(A) = 7.1579 \times 10^3$.Convergence history presented given in Table **6**.

Table 6(*Sherman4*)

Wavelet family	DWT	DWTPer	DWTPerMod
CDF(2,2)	83[0.3120]	68[0.3588]	73[12.3865]
Daub-4	98[0.3120]	150[0.7800]	1016[18.4705]

Epb0: this matrix arises from plate-fin heat exchanger problem. For this matrix $n = 1794$,$nnz = 7764$ and $k(A) = 1.4719 \times 10^5$.Convergence history is given in Table **7**.

Table 7(*epb0*)

Wavelet family	DWT	DWTPer	DWTPerMod
CDF(3,1)	Ω	769[5.6316]	150[6.8172]
Daub-4	Ω	843[4.2432]	214[7.6440]

Rdb2048: This matrix arises in Chemical engineering discipline. For this matrix $n = 2408$,$nnz = 12032$ and $k(A) = 1.9081 \times 10^3$. Convergence history is provided in Table**8**.

Table 8(*rdb2048*)

Wavelet family	DWT	DWTPer	DWTPerMod
CDF(3,1)	249[2.1996]	94[4.1808]	5[0.7020]
Daub-4	526[4.7736]	435[2.4336]	134[1.8096]

Dw8192:this matrix arises from electromagnetic problem (square dielectric wave guide). For this matrix $n = 8192, nnz = 41746$ and $k(A) = 1.5001 \times 10^7$. Convergence history is given in Table **9**, Figures **3** and **4**.

Table 9(*dw8192*)

Wavelet family	DWT	DWTPer	DWTPerMod
CDF(3,1)	Ω	16[4.8828]	15[22.7605]
Daub-4	Ω	Ω	71[56.7688]

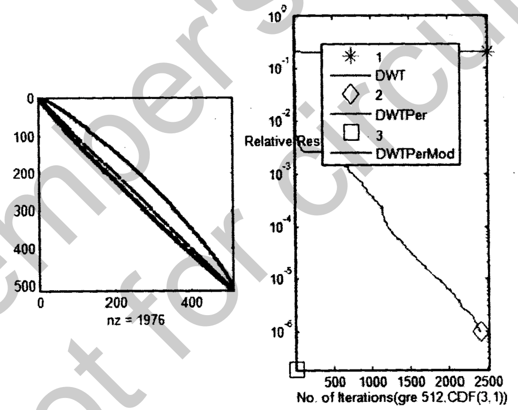

Figure 1

Left:Sparsity pattern of Gre-512 matrix, right: Graphical representation of convergence by using CDF(3,1).

Figure 1
Left:Sparsity pattern of Gre-512 matrix, right: Graphical representation of convergence by using CDF(3,1).

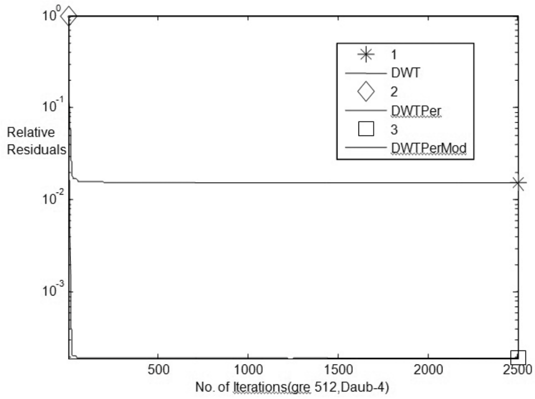

Figure 2
Graphical representation of convergence by using Daub-4.

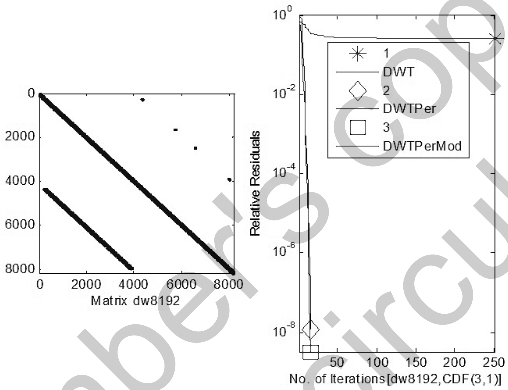

Figure 3
Left: Sparsity pattern of dw8192 matrix, right: Graphical representation of convergence by using CDF(3,1).

5. Discussion

The performance of Standard DWT and DWTPer band preconditioners depend on the "level" of wavelet transform used, which must be decided in advance for implementation. Our aim is to emphasize on the superiority of DWTPerMod algorithm in selecting preconditioner because it has built in scheme to select wavelet transform level automatically. In all cases DWTPerMod(biorthogonal) based preconditioner requires less number of iterations and also takes least CPU time compared with that of orthogonal wavelets, which are given in last column of Tables **3-9**.

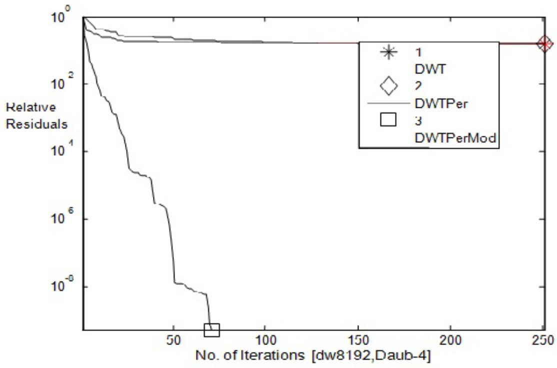

Figure 4

Graphical representation of convergence by using Daub-4.

Kumar and Mehra[15] have concluded that Haar transform is relatively more effective in most test problems. The same result can be achieved by using CDF$(p, 1)$ where p is odd natural number [5]. In this article we provide some sparse matrices for which Haar transform is inefficient/has slow convergence for iterative methods. For these test problems we have chosen CDF(3,1) for constructing band preconditioner and compared with Daub-4 wavelets. If we increase the order of wavelets (CDF$(p, 1)$ or Daubechies), which results in the more complexity for the test matrices discussed here, which is redundant.

Biorthogonal wavelet based preconditioners developed here are for *sparse unsymmetric linear systems*. The popular Krylov subspace iterative method is restarted GMRES. Restarted GMRES-GMRES(25) is chosen as our computations show slight improvements(number of iterations required/CPU time taken) compared with GMRES(20) and also no noticeable changes are found beyond GMRES(25). We have given the x-y plots for the number of iteration v/s related residuals for some tested representative matrices (Figures 1-4).

6. Conclusion

Biorthogonal wavelet based theory and algorithms are presented for the design of efficient preconditioners in dealing with unsymmetric sparse matrices from Tim Davis collection of matrices. More natural scheme of handling boundary is considered using symmetric extension. From our numerical experiments, in section **4**, it is noteworthy and of significance to find that the preconditioners developed, based on biorthogonal wavelets are found to be robust, efficient in accelerating the convergence of GMRES (25) and require less CPU time, compared with that of orthogonal wavelets. Also, these CDF based preconditioners are attractive in finding optimal compression of a given matrix. CDF biorthogonal wavelets have

a minimum support of size $p + \tilde{p} - 1$, for given p and $\tilde{p}$ among all biorthogonal wavelets, which is of great importance in numerical analysis.
For comparison, we consider Daubechies orthogonal wavelets of order D and CDF$(p, \tilde{p})$. Here for $D = 2$ and $p = \tilde{p} = 1(p + \tilde{p} = 2)$, they represent the same

family of wavelets, namely Haar. For $D = 4(p+\tilde{p} = 4)$, CDF$(2, 2)$ and Daub-4 have two vanishing moments, their performance is almost same. CDF (1,3) does not form Riesz bases for $L^2(\Re)$[5] because of this reason we can not give comparison of these results. We have compared the results using CDF$(3, 1)$ and Daub-4. As we increase the order$(Daub - D)$ of wavelet computational complexity increases [4].

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References

[1] M. Benzi,*Preconditioning techniques for large linear systems: A Survey,* J. Comput. Phy.,**182**(2002), 418–477.

[2] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms,* I. Comm. Pure Appl. Math., **44**(1991), No. 2, 141–183.

[3] K. Chen, *Discrete wavelet transforms accelerated sparse preconditioners for dense boundary element systems,* Elec. Trans. Numer. Anal., **8**(1999), 138–153.

[4] Ke Chen, *Matrix Preconditioning Techniques and Applications,* Cambridge University Press , 2005.

[5] A. Cohen, I. Daubechies and J. C. Feauveau, *Biorthogonal bases of compactly supported wavelets,* Comm. Pure Appl. Math.,**45**(1992), 485–560.

[6] I. Daubechies, *Ten Lectures on Wavelets,* SIAM, Philadelphia, 1992.

[7] T. Davis, *University of Florida Sparse Matrix Collection,* NA Digest, **97**, 1997.[Available from: http://www.cise.ufl.edu/research/sparse/matrices/].

[8] J. M. Ford, *An improved discrete wavelet transform preconditioner for dense matrix problems,* SIAM J. Matrix. Anal. Appl., **25**(2003), No.3, 642–661.

[9] G. H. Goulab and C. F. Van Loan, *Matrix Computations,* Hindustan Book Agency, India, 2007.

[10] F. Keinert, *Biorthogonal wavelets for fast matrix computations,* Appl. Comput. Harmon. Anal., **1**(1994), 147–156.

[11] F. Keinert, *Numerical stability of biorthogonal wavelet transforms,* Advances in Copm. Math. **4**(1995), No.1-2, 1–26.

[12] F. Keinert, *Wavelets and Multiwavelets,* Champan and Hall/CRC Press, Florida, USA, 2004.

[13] S. Mallat, *A Wavelet Tour of Signal Processing,* Academic Press, Elsevier, California,USA, 1999.

[14] O. M. Nielsen, *Wavelets in scientific computing,* Ph.D thesis,Technical University of Denmark, Denmark, 1998.

[15] B.V. Rathish Kumar, M. Mehra, *Wavelet based preconditioners for sparse linear systems,* Appl.Math.Comput. **171**(2005), 203–224.

[16] Y.Saad, *Iterative Methods for Sparse Linear Systems,* SIAM, Philadelphia, 2003.

[17] G. Strang and T. Nguyen, *Wavelets and Filter Banks,* Wellesley-Cambridge Press, Wellesley, MA, 1997.

[18] M. Unser, *Ten good reasons for using spline wavelets,* Pro. SPIE Vol.3169, Wavelets Applications in signal and Image Processing **V** (1997), 422–431.

[19] M. Unser, T. Blu, *Comparison of wavelets from the point of view of approximation Error.* Proc. Of SPIE Vol. **3458**, Wavelets Applications in Signal and Image Processing VI, ed. A.F Laine, A Aldroubi,14-21(1998).

[20] G. Uytterhoeven, *Wavelets: Software and Applications,* Ph.D thesis, K. U. Leuven, Belgium, 1999.

[21] D. F. Walnut, *An Introduction to Wavelet Analysis,* Birkhauser, 2002.

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COMPARATIVE GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS IN THE LIGHT OF THEIR RELATIVE ORDERS

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Abstract. In this paper we study some comparative growth properties of composite entire functions on the basis of their relative order (relative lower order) with respect to another entire function.

1. Introduction, Definitions and Notations.

Let f and g be two entire functions defined in the open complex plane $\mathbb{C}$ and $F(r) = \max\{|f(z)| : |z| = r\}$, $G(r) = \max\{|g(z)| : |z| = r\}$. The order and lower order of an entire function f are defined in the following way:

The order ρ_f and the lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} F(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} F(r)}{\log r},$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$.
If f is non-constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} F^{-1}(s) = \infty$.
Bernal [1] introduced the definition of relative order of f with respect to g , denoted by $\rho_g(f)$ as follows :

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\}$$
$$= \limsup_{r\to\infty} \frac{\log G^{-1} F(r)}{\log r}.$$

The definition coincides with the classical one [6] if $g(z) = \exp z$.
Similarly one can define the relative lower order of f with respect to g denoted by

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$\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

In this paper we wish to establish some results relating to the growth rates of composite entire functions on the basis of relative order (relative lower order). We do not explain the standard definitions and notations in the theory of entire functions as those are available in [7].

2. Lemmas.

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [2] *If f and g are two entire functions then for all sufficiently large values of r,*

$$F\left(\frac{1}{8}G\left(\frac{r}{2}\right) - |g(0)|\right) \le F\circ G(r) \le F(G(r)).$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let f and h be any two entire functions such that $0 < \lambda_h(f) \le \rho_h(f) < \infty$ and g be an entire function with finite lower order. Then for every positive constant μ and every real number α*

$$\lim_{r\to\infty} \frac{\log H^{-1}F\circ G(r)}{\{\log H^{-1}F(r^\mu)\}^{1+\alpha}} = \infty .$$

Proof: If α be such that $1+\alpha \le 0$ then the theorem is trivial. So we suppose that $1+\alpha > 0$. Since H^{-1} is an increasing function of r, it follows from Lemma 2.1 for all sufficiently large values of r that

$$\log H^{-1}F\circ G(r) \ge (\lambda_h(f) - \varepsilon)\frac{1}{16} + (\lambda_h(f) - \varepsilon)(\frac{r}{2})^{\lambda_g - \varepsilon} \tag{3.1}$$

where we choose $0 < \varepsilon < \min(\lambda_h(f), \lambda_g)$.

Again from the definition of $\rho_h(f)$ it follows for all sufficiently large values of r

$$\left\{\log H^{-1}F(r^\mu)\right\}^{1+\alpha} \le (\rho_h(f) + \varepsilon)^{1+\alpha} \mu^{1+\alpha} (\log r)^{1+\alpha}. \tag{3.2}$$

Now from (3.1) and (3.2) it follows for all sufficiently large values of r ,

$$\frac{\log H^{-1}F\circ G(r)}{\{\log H^{-1}F(r^\mu)\}^{1+\alpha}} \geqslant \frac{(\lambda_h(f) - \varepsilon)\frac{1}{16} + (\lambda_h(f) - \varepsilon)(\frac{r}{2})^{\lambda_g - \varepsilon}}{(\rho_h(f) + \varepsilon)^{1+\alpha} \mu^{1+\alpha} (\log r)^{1+\alpha}}.$$

Since $\frac{r^{\lambda_g - \varepsilon}}{(\log r)^{1+\alpha}} \to \infty$ as $r \to \infty$, the theorem follows from above. ■

Remark 3.1. Theorem 3.1 is still valid with "limit superior" instead of " limit " if we replace the condition " $0 < \lambda_h(f) \le \rho_h(f) < \infty$ " by " $0 < \lambda_h(f) < \infty$". Also for $h = \exp z$, $\alpha = 0$ and $\mu = 1$, Theorem 3.1 reduces to a theorem of Song and Yang {cf. [5] .}

In the line of Theorem 3.1 one may state the following theorem without proof:

Theorem 3.2. *Let f, g and h be any three entire functions where g is of finite lower order and $\lambda_h(f) > 0$, $\rho_h(g) < \infty$. Then for every positive constant μ and every real number α,*

$$\lim_{r\to\infty} \frac{\log H^{-1}F \circ G(r)}{\{\log H^{-1}G(r^{\mu})\}^{1+\alpha}} = \infty \ .$$

Remark 3.2. In Theorem 3.2 if we take the condition $\lambda_h(g) < \infty$ instead of $\rho_h(g) < \infty$, then also Theorem 3.2 remains true with "limit superior" in place of " limit ". Also for $h = \exp z$ and $\alpha = 0$, Theorem 3.2 reduces to a theorem of Singh and Baloria { cf. [4]} .

Theorem 3.3. *Let f and h be any two entire functions with $0 < \lambda_h(f) \le \rho_h(f) < \infty$ and g be an entire function with finite order. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\{\log H^{-1}F \circ G(r)\}^{1+\alpha}}{\log H^{-1}F(\exp r^{\mu})} = 0 \quad \text{if } \mu > (1+\alpha)\rho_g \ .$$

Proof: If $1+\alpha \le 0$, then the theorem is obvious. We consider $1+\alpha > 0$. Since H^{-1} is an increasing function of r, it follows from the second part of Lemma 2.1 that for all sufficiently large values of r

$$\log H^{-1}F \circ G(r) \le (\rho_h(f) + \varepsilon)\, r^{\rho_g+\varepsilon} \ . \tag{3.3}$$

Again for all sufficiently large values of r we get that

$$\log H^{-1}F(\exp r^{\mu}) \ge (\lambda_h(f) - \varepsilon)\, r^{\mu} \ . \tag{3.4}$$

Hence for all sufficiently large values of r we obtain from (3.3) and (3.4) that

$$\frac{\{\log H^{-1}F \circ G(r)\}^{1+\alpha}}{\log H^{-1}F(\exp r^{\mu})} \le \frac{(\rho_h(f) + \varepsilon)^{1+\alpha}\, r^{(\rho_g+\varepsilon)(1+\alpha)}}{(\lambda_h(f) - \varepsilon)\, r^{\mu}} \ , \tag{3.5}$$

where we choose $0 < \varepsilon < \min\left\{\lambda_h(f), \frac{\mu}{1+\alpha} - \rho_g\right\}$. So from (3.5) we obtain that

$$\lim_{r\to\infty} \frac{\{\log H^{-1}F \circ G(r)\}^{1+\alpha}}{\log H^{-1}F(\exp r^{\mu})} = 0 \ .$$

This proves the theorem. ∎

Remark 3.3. In Theorem 3.3 if we take the condition $0 < \rho_h(f) < \infty$ instead of $0 < \lambda_h(f) \le \rho_h(f) < \infty$, the theorem remains true with " limit inferior" in place of "limit " and in that case it improves Theorem 2 of Singh and Baloria { cf. [4]} .

In view of Theorem 3.3 the following theorem can be carried out:

Theorem 3.4. *Let f, g and h be any three entire functions where g is with finite order and $\lambda_h(g) > 0$, $\rho_h(f) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log H^{-1}F \circ G(r)\right\}^{1+\alpha}}{\log H^{-1}G(\exp r^{\mu})} = 0 \quad \text{if } \mu > (1+\alpha)\rho_g \ .$$

The proof is omitted.

Remark 3.4. In Theorem 3.4 if we take the condition $\rho_h(g) > 0$ instead of $\lambda_h(g) > 0$, the theorem remains true with " limit " replaced by " limit inferior".

Theorem 3.5. *Let f and h be any two entire functions such that $0 < \lambda_h(f) \le \rho_h(f) < \infty$. Suppose g be an entire function with $0 < \mu < \rho_g \le \infty$. Then for a sequence of values of r tending to infinity,*

$$H^{-1}F \circ G(r) > H^{-1}F(\exp r^{\mu}) \ .$$

Proof: Since H^{-1} is an increasing function of r, in view of Lemma 2.1 we get for a sequence of values of r tending to infinity,

$$\log H^{-1}F \circ G(r) \ge (\lambda_h(f) - \varepsilon)\frac{1}{16} + (\lambda_h(f) - \varepsilon)\left(\frac{r}{2}\right)^{\rho_g - \varepsilon} . \tag{3.6}$$

Again from the definition of $\rho_h(f)$ we obtain for all sufficiently large values of r,

$$\log H^{-1}F(\exp r^{\mu}) \le (\rho_h(f) + \varepsilon)\, r^{\mu} \ . \tag{3.7}$$

Now from (3.6) and (3.7), it follows for a sequence of values of r tending to infinity that

$$\frac{\log H^{-1}F \circ G(r)}{\log H^{-1}F(\exp r^{\mu})} \ge \frac{(\lambda_h(f) - \varepsilon)\frac{1}{16} + (\lambda_h(f) - \varepsilon)(\frac{r}{2})^{\rho_g - \varepsilon}}{(\rho_h(f) + \varepsilon)\, r^{\mu}} \ . \tag{3.8}$$

As $\mu < \rho_g$ we can choose $\varepsilon (> 0)$ in such a way that

$$\mu < \rho_g - \varepsilon \ . \tag{3.9}$$

Thus, from (3.8) and (3.9), we get that

$$\limsup_{r\to\infty} \frac{\log H^{-1}F \circ G(r)}{\log H^{-1}F(\exp r^{\mu})} = \infty \ . \tag{3.10}$$

From (3.10), we obtain for a sequence of values of r tending to infinity that

$$H^{-1}F \circ G(r) > H^{-1}F(\exp r^{\mu}) \ .$$

This proves the theorem. ■

Theorem 3.6. *Let f and h be any two entire functions with $0 < \lambda_h(f) \le \rho_h(f) < \infty$. Suppose g be an entire function with finite relative order with respect to h and $0 < \mu < \rho_g$. Then for a sequence of values of r tending to infinity,*

$$H^{-1}F \circ G(r) > H^{-1}G(\exp(r^{\mu})) \ .$$

Proof: Let $0 < \mu < \mu_0 < \rho_g$.Then in view of Theorem 3.5, for a sequence of values of r tending to infinity, we get that

$$\log H^{-1}F \circ G(r) > (\lambda_h(f) - \varepsilon)\, r^{\mu_0}. \tag{3.11}$$

Again from the definition of $\rho_h(g)$, we obtain for all sufficiently large values of r,

$$\log H^{-1}G(\exp r^{\mu}) \leq (\rho_h(g) + \varepsilon)\, r^{\mu}\ . \tag{3.12}$$

So combining (3.11) and (3.12), we obtain for a sequence of values of r tending to infinity,

$$\frac{\log H^{-1}F \circ G(r)}{\log H^{-1}G(\exp r^{\mu})} \geqslant \frac{(\lambda_h(f) - \varepsilon)\, r^{\mu_0}}{(\rho_h(g) + \varepsilon)\, r^{\mu}}. \tag{3.13}$$

Since $\mu_0 > \mu$, from (3.13), it follows that

$$\limsup_{r\to\infty} \frac{\log H^{-1}F \circ G(r)}{\log H^{-1}G(\exp r^{\mu})} = \infty\ . \tag{3.14}$$

Thus the theorem follows from (3.14). ■

Theorem 3.7. *Let f, g and h be any three entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$H^{-1}F \circ G(r) < H^{-1}F(\exp r^{\mu})\ .$$

Proof: Since H^{-1} is an increasing function of r, it follows from the second part of Lemma 2.1 that for a sequence of values of r tending to infinity,

$$H^{-1}F \circ G(r) \leq H^{-1}F(G(r))$$

$$i.e.,\ \log H^{-1}F \circ G(r) \leq (\rho_h(f) + \varepsilon)\, r^{\lambda_g + \varepsilon}\ . \tag{3.15}$$

Now, from (3.4) and (3.15), it follows for a sequence of values of r tending to infinity,

$$\frac{\log H^{-1}F(\exp r^{\mu})}{\log H^{-1}F \circ G(r)} \geq \frac{(\lambda_h(f) - \varepsilon)\, r^{\mu}}{(\rho_h(f) + \varepsilon)\, r^{\lambda_g + \varepsilon}}\ . \tag{3.16}$$

As $\lambda_g < \mu$ we can choose $\varepsilon\,(> 0)$ in such a way that

$$\lambda_g + \varepsilon < \mu < \rho_g\ . \tag{3.17}$$

Thus, from (3.16) and (3.17), we obtain that

$$\limsup_{r\to\infty} \frac{\log H^{-1}F(\exp r^{\mu})}{\log H^{-1}F \circ G(r)} = \infty\ . \tag{3.18}$$

From (3.18), we obtain for a sequence of values of r tending to infinity that

$$H^{-1}F(\exp r^{\mu}) > H^{-1}F \circ G(r)\ .$$

Thus the theorem follows. ■

On the line of Theorem 3.7, we may state the following theorem without proof:

Theorem 3.8. *Let g and h be any two entire functions with $\lambda_h(g) > 0$ and f be an entire function with finite relative order with respect to h. Also suppose that $\lambda_g < \mu < \infty$, then for a sequence of values of r tending to infinity,*

$$H^{-1}F \circ G(r) < H^{-1}G(\exp r^{\mu}) \ .$$

As an application of Theorem 3.5 and Theorem 3.7, we may state the following theorem:

Theorem 3.9. *Let f, g and h be any three entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\lambda_g < \mu < \rho_g$. Then*

$$\liminf_{r\to\infty} \frac{H^{-1}F \circ G(r)}{H^{-1}F(\exp r^{\mu})} \leq 1 \leq \limsup_{r\to\infty} \frac{H^{-1}F \circ G(r)}{H^{-1}F(\exp r^{\mu})} \ .$$

The proof is omitted.

In view of Theorem 3.6 and Theorem 3.8, the following theorem can be carried out:

Theorem 3.10. *Let f, g and h be any three entire functions with $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_h(g) \leq \rho_h(g) < \infty$ and $0 < \lambda_g < \mu < \rho_g < \infty$. Then*

$$\liminf_{r\to\infty} \frac{H^{-1}F \circ G(r)}{H^{-1}G(\exp r^{\mu})} \leq 1 \leq \limsup_{r\to\infty} \frac{H^{-1}F \circ G(r)}{H^{-1}G(\exp r^{\mu})} \ .$$

The proof is omitted.

Theorem 3.11. *Let f, g and h be any three entire functions such that $\rho_h(f) < \infty$ and $\lambda_h(f \circ g) = \infty$. Then for every $\mu\,(>0)$,*

$$\lim_{r\to\infty} \frac{\log H^{-1}F \circ G(r)}{\log H^{-1}F(r^{\mu})} = \infty \ .$$

The proof is omitted because it can be carried out using the same technique involved in Theorem 4 of [3].

Remark 3.5. Theorem 3.11 is also valid with "limit superior" instead of "limit" if $\lambda_h(f \circ g) = \infty$ is replaced by $\rho_h(f \circ g) = \infty$ and the other conditions remain the same.

Corollary 3.1. *Under the assumptions of Theorem 3.11 and Remark 3.5,*

$$\lim_{r\to\infty} \frac{H^{-1}F \circ G(r)}{H^{-1}F(r^{\mu})} = \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{H^{-1}F \circ G(r)}{H^{-1}F(r^{\mu})} = \infty$$

respectively holds.

The proof is omitted.

Analogously one may also state the following theorem and corollaries without proofs as they may be carried out in the line of Remark 3.5, Theorem 3.11 and Corollary 3.1, respectively.

Theorem 3.12. *If f, g and h be any three entire functions with $\rho_h(g) < \infty$ and $\rho_h(f \circ g) = \infty$ then for every $\mu\,(> 0)$*

$$\limsup_{r\to\infty} \frac{\log H^{-1} F \circ G(r)}{\log H^{-1} G(r^{\mu})} = \infty \ .$$

Corollary 3.2. *Theorem 3.12 is also valid with "limit " instead of "limit superior" if $\rho_h(f \circ g) = \infty$ is replaced by $\lambda_h(f \circ g) = \infty$ and the other conditions remain the same.*

Corollary 3.3. *Under the assumptions of Theorem 3.12 and Corollary 3.2,*

$$\limsup_{r\to\infty} \frac{H^{-1} F \circ G(r)}{H^{-1} G(r^{\mu})} = \infty \quad \textit{and} \quad \lim_{r\to\infty} \frac{H^{-1} F \circ G(r)}{H^{-1} G(r^{\mu})} = \infty$$

respectively holds.

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References

[1] L. Bernal, *Orden relative de crecimiento de funciones enteras,* Collect. Math., 39(1988), 209–229.

[2] J. Clunie , *The composition of entire and meromorphic functions,* Mathematical essays dedicated to A.J.Macintyre, Ohio University Press, (1970), 75–92.

[3] S. K. Datta, *Growth proerties of composite entire and meromorphic functions,* Bull. Cal. Math. Soc., 95(2003)(5),.429–436.

[4] A. P. Singh and M. S. Baloria, *On the maximum modulus and maximum term of composition of entire functions,* Indian J. Pure Appl. Math., 22 (1991), 1019–1026.

[5] G. D. Song and C. C. Yang, *Further growth properties of composition of entire and meromorphic functions,* Indian J. Pure Appl. Math., 15(1984), 67–82.

[6] E. C. Titchmarsh, *The theory of functions, 2nd edition,* Oxford University Press, Oxford, 1939.

[7] G. Valiron, *Lectures on the General Theory of Integral Functions,* Chelsea Publishing Company, 1949.

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ON SOME INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study invariant submanifolds of Kenmotsu manifolds. We consider the semiparallel, 2- semiparallel, pseudo-parallel and generalized Ricci pseudo-parallel invariant submanifolds of Kenmotsu manifolds. Also we search for the conditions $\mathcal{Z}(X,Y).\alpha = 0$ and $\mathcal{Z}(X,Y).\bar{\nabla}\alpha = 0$ on an invariant submanifold of Kenmotsu manifold, where $\mathcal{Z}$ is the concircular curvature tensor.

1. Introduction

Among Riemannian manifolds, the most interesting and most important for applications are the symmetric ones. From the local point of view they were introduced independently by Shirokov [25] and Levy [18] as a Riemannian manifold with covariant constant curvature tensor R, *i.e.*, with

$$\nabla R = 0,$$

where ∇ is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was worked out by Cartan in 1927.

Later, a similar development took place in the geometry of submanifolds in the space forms, where a fundamental role is played by metric tensor g (as the induced Riemannian metric) and the second fundamental form α. Besides the Levi-Civita connection ∇, with $\nabla g = 0$, a normal connection $\nabla^{\perp}$ is also defined. The submanifolds with parallel second fundamental form, *i.e.*, with

$$\bar{\nabla}\alpha = 0,$$

where $\bar{\nabla}$ is the pair of ∇ and $\nabla^{\perp}$, deserve special attention. The theory of parallel submanifolds is concisely treated in recent monograph by Lumiste [19].

As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric

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if the curvature tensor satisfies $R(X,Y).R = 0$, where $R(X,Y)$ is considered as a field of linear operators, actng on R.

Semisymmetric manifolds were classified by Szabó, locally in [26]. The classification results of Szabó were presented in the book [7].

Parallel submanifolds were likewise later placed in a more general class of submanifolds generalizing the notion of parallel submanifolds.

Let M and $\tilde{M}$ be two Riemannian or semi-Riemannian manifolds, $f : M \longrightarrow \tilde{M}$ an isometric immersion, α the second fundamental form and $\bar{\nabla}$ the van der Waerden-Bortolotti connection of M. An immersion is said to be *semiparallel* if

$$\bar{R}(X,Y).\alpha = (\bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]})\alpha = 0 \tag{1.1}$$

holds for all vector fields tangent to M [9], where $\bar{R}$ denots the curvature tensor of the conncetion $\bar{\nabla}$. Semiparallel immersion have been studied by many authors, see, for example, ([10], [13], [14], [17] and [19]).
In [2] Arslan et al defined and studied submanifolds satisfying the condition

$$\bar{R}(X,Y).\bar{\nabla}\alpha = 0 \tag{1.2}$$

for all vector fields X, Y tangent to M. Submanifolds satisfying (1.2) are called 2-*semiparallel*.

An immersion is said to be pseudo-parallel ([3], [4]) if

$$\bar{R}(X,Y).\alpha = fQ(g,\alpha) \tag{1.3}$$

holds for all vector fields X, Y tangent to M, where f denotes real valued fuction on M and for a $(0,k)$, $k \geq 1$ tensor T and a $(0,2)$ tensor E, $Q(E,T)$ is defined by [27]

$$\begin{aligned} Q(E,T)(X_1, X_2, ..., X_k; X, Y) = & -T((X \wedge_E Y)X_1, X_2, ..., X_k) \\ & - T(X_1, (X \wedge_E Y)X_2, ..., X_k) \\ & - ... - T(X_1, X_2, ..., X_{k-1}, (X \wedge_E Y)X_k), \end{aligned} \tag{1.4}$$

where $X \wedge_E Y$ is defined by

$$(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y. \tag{1.5}$$

Therefore we can take the following equation instead of the equation (1.1)

$$\begin{aligned} R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) \\ = -f[\alpha(X \wedge_g Y)U,V) + \alpha(U,(X \wedge_g Y)V)] \end{aligned} \tag{1.6}$$

Also Murathan, Arsalan and Ezentas [20] defined submanifolds satisfying the condition

$$\bar{R}(X,Y).\alpha = fQ(S,\alpha). \tag{1.7}$$

The equation (1.7) can be written as

$$\begin{aligned} R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) \\ = -f[\alpha(X\wedge_S Y)U,V) + \alpha(U,(X\wedge_S Y)V)]. \end{aligned} \tag{1.8}$$

This kind of submanifolds are called generalized Ricci-pseudo parallel.

In a recent paper [22], Özgür and Murathan studied semiparallel and 2-semiparallel invariant submanifolds of Lorentzin para-Sasakian manifolds. Also in [21] V. Mangione studied invariant submanifolds of Kenmotsu manifolds and some conditions that these submanifolds are totally geodesic are given.

Motivated by these studies of the above authors ([21], [22]), in this present paper we consider invariant submanifolds of Kenmotsu manifolds. We consider the semiparallel, 2- semiparallel, pseudo-parallel and generalized Ricci pseudo-parallel invariant submanifolds of Kenmotsu manifolds. Also we search for the conditions $\mathcal{Z}(X,Y).\alpha = 0$ and $\mathcal{Z}(X,Y).\bar{\nabla}\alpha = 0$ on an invariant submanifolds of Kenmotsu manifolds, where $\mathcal{Z}$ is the concircular curvature tensor.

The paper is organized as follows:

In section 2, we give necessary details about submanifolds and the concircular curvature tensor. In section 3, we give a brief account of Kenmotsu manifolds and their submanifolds. In section $4-7$, we study semiparallel, 2- semiparallel, pseudo-parallel and generalized Ricci pseudo-parallel invariant submanifolds of Kenmotsu manifolds respectively and we show that these type of submanifolds are totally geodesic. Section 8 is devoted to study of invariant submanifolds satisfying the condition $\mathcal{Z}(X,Y).\alpha = 0$. Finally in section 9 we study invariant submanifolds satisfying $\mathcal{Z}(X,Y).\bar{\nabla}\alpha = 0$. As a consequence of the above results we obtain some important Corollaries.

2. Basic Concepts

Let (M,g) be an n-dimensional Riemannian submanifold of an $(n+d)$-dimensional Riemannian manifold $(\tilde{M},\tilde{g})$. We denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connection of $\tilde{M}$ and M, respectively. Then we have the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y), \tag{2.1}$$

and

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

where X, Y are the vector fields tangent to M and N is a normal vector field on M, respectively. $\nabla^{\perp}$ is called the *normal connection* of M. We call α the *second fundamental form* of submanifolds M. If $\alpha = 0$ then the manifold is said to be *totally geodesic*. For the second fundamental form α, the covariant derivative of α is defined by

$$(\bar{\nabla}_X \alpha)(Y, Z) = \nabla^{\perp}_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) \tag{2.2}$$

for any vector fields X, Y, Z tangent M. Then $\bar{\nabla}\alpha$ is a normal bundle valued tensor of type $(0, 3)$ and is called the *third fundamental form* of M. $\bar{\nabla}$ is called the *van der waerden-Bortolotti connection* of M, *i.e.*, $\bar{\nabla}$ is the connection in $TM \oplus T^{\perp}M$ built with ∇ and $\nabla^{\perp}$. If $\bar{\nabla}\alpha = 0$, then M is said to have *parallel second fundamental* form [8]. From the Gauss and Weingarten formulas we obtain

$$(\tilde{R}(X, Y)Z)^T = R(X, Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X. \tag{2.3}$$

Therefore from (1.1) we have

$$(\bar{R}(X, Y).\alpha)(U, V) = R^{\perp}(X, Y)\alpha(U, V) - \alpha(R(X, Y)U, V) - \alpha(U, R(X, Y)V) \tag{2.4}$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X, Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]} \tag{2.5}$$

and $\bar{R}$ denotes the curvature tensors of $\bar{\nabla}$. Similarly

$$\begin{aligned}(\bar{R}(X, Y).\nabla\alpha)(U, V, W) = R^{\perp}(X, Y)(\bar{\nabla}\alpha)(U, V, W)\\ - (\bar{\nabla}\alpha)(R(X, Y)U, V, W)\\ - (\bar{\nabla}\alpha)(U, R(X, Y)V, W)\\ - (\bar{\nabla}\alpha)(U, V, R(X, Y)W)\end{aligned} \tag{2.6}$$

for all vector fields X, Y, U, V and W tangent to M, where $(\bar{\nabla}\alpha)(U, V, W) = (\bar{\nabla}_U \alpha)(V, W)$ [2]. For an n-dimensional ($n \geq 3$) Riemannian manifold (M, g), the *concircular curvature tensor* $\mathcal{Z}$ is defined by [28]

$$\mathcal{Z}(X, Y)V = R(X, Y)V - \frac{r}{n(n-1)}\{g(Y, V)X - g(X, V)Y\} \tag{2.7}$$

for all vector fields X, Y and V on M^n, where r is the scalar curvature of M^n. We observe immediately from the form of the concircular curvature tensor that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus one can think of the concircular tensor as a measure of failure of the Riemannian manifold to be of constant curvature.
Similarly to (2.6) and (2.7) the tensor $\mathcal{Z}(X, Y).\alpha$ and $\mathcal{Z}(X, Y).\bar{\nabla}\alpha$ are defined by

$$(\mathcal{Z}(X, Y).\alpha)(U, V) = R^{\perp}(X, Y)\alpha(U, V) - \alpha(\mathcal{Z}(X, Y)U, V) - \alpha(U, \mathcal{Z}(X, Y)V) \tag{2.8}$$

and

$$\begin{aligned}(\mathcal{Z}(X,Y).\bar{\nabla}\alpha)(U,V,W) = R^{\perp}(X,Y)(\bar{\nabla}\alpha)(U,V,W)\\ -(\bar{\nabla}\alpha)(\mathcal{Z}(X,Y)U,V,W)\\ -(\bar{\nabla}\alpha)(U,\mathcal{Z}(X,Y)V,W)\\ -(\bar{\nabla}\alpha)(U,V,\mathcal{Z}(X,Y)W),\end{aligned} \tag{2.9}$$

respectively.

3. Kenmotsu manifolds

An odd dimensional smooth manifold M^n $(n \geq 3)$ is said to admit an *almost contact structure*, sometimes called a (ϕ,ξ,η)-*structure*, if it admites a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([5],[6])

$$\phi^2 = -I + \eta\otimes\xi,\ \ \eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \eta\circ\phi = 0. \tag{3.1}$$

The first and one of the remaining three relations in (3.1) imply the other two relations in (3.1). An almost contact structure is said to be *normal* if the induced almost complex structure J on $M^n\times\mathbb{R}$ defind by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n\times\mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ,ξ,η), that is,

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3.2}$$

or equivalently,

$$g(X,\phi Y) = -g(\phi X,Y)\ \ and\ \ g(X,\xi) = \eta(X)$$

for all vector fields X, Y tangent to M. Then M becomes an *almost contact metric manifold* equipped with an almost contact metric structure (ϕ,ξ,η,g). An almost contact metric structure becomes an *contact metric structure* if

$$g(X,\phi Y) = d\eta(X,Y), \tag{3.3}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field. Moreover if

$$(\nabla_X\phi)Y = -g(X,\phi Y) - \eta(Y)\phi X, \tag{3.4}$$

$$\nabla_X\xi = X - \eta(X)\xi, \tag{3.5}$$

then the almost contact manifold is called an almost *Kenmotsu* manifold [16]. Also in a Kenmotsu manifold the following relations holds:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.6}$$

$$(\nabla_X\eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{3.7}$$

$$S(X,\xi) = -(n-1)\eta(X). \tag{3.8}$$

Kenmotsu manifolds have been studied by several authors such as ([11], [12], [15], [23], [28], [29]) and many others.
A submanifold M of a Kenmotsu manifold $\tilde{M}$ is called an invariant submanifold of $\tilde{M}$ if $\phi(TM) \subset (TM)$. In an invariant submanifold of a Kenmotsu manifold $\alpha(X,\xi) = 0$ for any vector field X tangent to M.

Proposition 3.1. [21] *Let M^n be a invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then the following equalities holds on M^n*

$$\nabla_X \xi = X - \eta(X)\xi, \tag{3.9}$$

$$\alpha(X,\xi) = 0, \tag{3.10}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{3.11}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{3.12}$$

$$(\nabla_X \phi)Y = -g(X,\phi Y) - \eta(Y)\phi X, \tag{3.13}$$

$$\alpha(X,\phi Y) = \phi\alpha(X,Y), \tag{3.14}$$

for all vector fields X, Y tangent to M.

So we can state the following:

Theorem 3.1. [21] *An invariant submanifold M^n of a Kenmotsu manifold $\tilde{M}$ is a Kenmotsu manifold.*

Proposition 3.2. *If a non-flat Riemannian manifold has a recurrent second fundamental form, then it is semiparallel.*

Proof The second fundamental form α is said to be recurrent if

$$\nabla\alpha = A \otimes \alpha,$$

where A is an every where non-zero 1-form. We now define a function f on M by

$$f^2 = g(\alpha,\alpha),$$

where the metric g is extended to the inner product between the tensor fields in the standard fashion. Then we know that $f(Yf) = f^2 A(Y)$. So from this we have $Yf = fA(Y)$, since f is non zero. This implies that

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XA(Y))f.$$

$$\text{Hence, } X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f.$$

Therefore, we get

$$(\bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]})f = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and f is non-zero on M by our assumption, we obtain

$$dA(X,Y) = 0, \tag{3.15}$$

i.e., the 1-form A is closed.

Now from

$$(\nabla_X \alpha)(U,V) = A(X)\alpha(U,V),$$

we get

$$(\bar{\nabla}_U \bar{\nabla}_V \alpha)(X,Y) - (\bar{\nabla}_{\bar{\nabla}_U V} \alpha)(X,Y) = [(\bar{\nabla}_U A)V + A(U)A(V)]\alpha(X,Y).$$

Hence using (3.15), We get

$$(\bar{R}(X,Y).\alpha)(U,V) = [2dA(X,Y)]\alpha(X,Y) = 0.$$

Therefore, we have

$$\bar{R}(X,Y).\alpha = 0. \tag{3.16}$$

for any X, Y. If $f = 0$, then from $f^2 = g(\alpha,\alpha)$ we get $\alpha = 0$ and hence obviously $R(X,Y).\alpha = 0$. Hence the result.

4. Semiparallel invariant submanifolds of Kenmotsu manifolds

In this section we consider semiparalel invariant submanifolds M of a Kenmotsu manifolds $\tilde{M}$. Therefore we have $\bar{R}.\alpha = 0$. Then from (2.4) we have

$$R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) = 0 \tag{4.1}$$

for all vector fields X, Y, U and V tangent to M. Putting $X = V = \xi$ in (4.1) we obtain

$$R^{\perp}(\xi,Y)\alpha(U,\xi) - \alpha(R(\xi,Y)U,\xi) - \alpha(U,R(\xi,Y)\xi) = 0. \tag{4.2}$$

Therefore (4.2) and (3.10) yields

$$\alpha(U,R(\xi,Y)\xi) = 0. \tag{4.3}$$

Then by use of (3.11) we have

$$\alpha(Y,U) = 0, \tag{4.4}$$

which gives us M^n is totally geodesic. On the other hand it is easy to see that if M^n is totally geodesic, then it is semiparallel. Thus we can state the following:

Theorem 4.1. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M^n is semiparallel if and only if M^n is totally geodesic.*

Now, if M^n has parallel second fundamental form, then it is semiparallel. Therefore we can state the following:

Corollary 4.1. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M^n has parallel second fundamental form if and only if M^n is totally geodesic.*

By virtue of the Theorem 4.1 and Prop. 3.2 we can state the following:

Corollary 4.2. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M^n has recurrent second fundamental form if and only if M^n is totally geodesic.*

The second fundamental form α is birecurrent [24] if there exists a non-zero covariant tensor field B such that $(\nabla_X\nabla_W\alpha - \nabla_{\nabla_X W}\alpha)(Y,Z) = B(X,W)\alpha(Y,Z)$. In a recent paper [1] Aikawa and Matsuyama proved that if a tensor field T is birecurrent, then $R(X,Y).T = 0$. Therefore by virtue of the Theorem 4.1 we can state the following:

Corollary 4.3. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M^n has birecurrent second fundamental form if and only if M^n is totally geodesic.*

5. 2-semiparallel invariant submanifolds of Kenmotsu manifolds

In this section we consider 2-semiparalel invariant submanifolds M of a Kenmotsu manifold $\tilde{M}$. Therefore we have $\bar{R}.\nabla\alpha = 0$. Then from (2.6) we have

$$\begin{aligned} &R^{\perp}(X,Y)(\bar{\nabla}\alpha)(U,V,W) - (\bar{\nabla}\alpha)(R(X,Y)U,V,W) \\ &-(\bar{\nabla}\alpha)(U,R(X,Y)V,W) - (\bar{\nabla}\alpha)(U,V,R(X,Y)W) = 0 \end{aligned} \tag{5.1}$$

for all vector fields X, Y, U, V and W tangent to M. Taking $X = V = \xi$ in (5.1) we obtain

$$\begin{aligned} &R^{\perp}(\xi,Y)(\bar{\nabla}\alpha)(U,\xi,W) - (\bar{\nabla}\alpha)(R(\xi,Y)U,\xi,W) \\ &-(\bar{\nabla}\alpha)(U,R(\xi,Y)\xi,W) - (\bar{\nabla}\alpha)(U,\xi,R(\xi,Y)W) = 0. \end{aligned} \tag{5.2}$$

Therefore in the view of (3.10), (3.11) and (3.14) we have the following equalities:

$$\begin{aligned} (\bar{\nabla}\alpha)(U,\xi,W) &= (\bar{\nabla}_U\alpha)(\xi,W) \\ &= \nabla^{\perp}_U(\alpha(\xi,W)) - \alpha(\nabla_U\xi,W) - \alpha(\xi,\nabla_U W) \\ &= -\alpha(U - \eta(U)\xi,W) \\ &= -\alpha(\phi U,W). \end{aligned} \tag{5.3}$$

$$
\begin{aligned}
(\bar{\nabla}\alpha)(R(\xi,Y)U,\xi,W) &= (\bar{\nabla}_{R(\xi,Y)U}\alpha)(\xi,W) \\
&= \nabla^{\perp}_{R(\xi,Y)U}(\alpha(\xi,W)) - \alpha(\nabla_{R(\xi,Y)U}\xi,W) \\
&\quad - \alpha(\xi,\nabla_{R(\xi,Y)U}W) \\
&= -\alpha(\nabla_{R(\xi,Y)U}\xi,W) \\
&= -\alpha(R(\xi,Y)U - \eta(R(\xi,Y)U)\xi,W)
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
(\bar{\nabla}\alpha)(U,R(\xi,Y)\xi,W) &= (\bar{\nabla}_U\alpha)(R(\xi,Y)\xi,W) \\
&= \nabla^{\perp}_U\alpha(R(\xi,Y)\xi,,W) - \alpha(\nabla_U R(\xi,Y)\xi,W) \\
&\quad - \alpha(R(\xi,Y)\xi,\nabla_U W) \\
&= \nabla^{\perp}_U\alpha(Y,W) - \alpha(\nabla_U(Y-\eta(Y)\xi),W) \\
&\quad - \alpha(Y,\nabla_U W)
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
(\bar{\nabla}\alpha)(U,\xi,R(\xi,Y)W) &= (\nabla^{\perp}_U\alpha)(\xi,R(\xi,Y)W) \\
&= \nabla^{\perp}_U\alpha(\xi,R(\xi,Y)W) - \alpha(\nabla_U\xi,R(\xi,Y)W) \\
&\quad - \alpha(\xi,\nabla_U R(\xi,Y)W) \\
&= -\alpha(U-\eta(U)\xi,R(\xi,Y)W).
\end{aligned} \tag{5.6}
$$

Using (5.3) – (5.6) in (5.2), we obtain

$$
\begin{aligned}
&- R^{\perp}(\xi,Y)\alpha(U,W) + \alpha(\phi R(\xi,Y)U,W) - \nabla^{\perp}_U\alpha(Y,W) \\
&+ \alpha(\nabla_U(Y-\eta(Y)\xi),W) + \alpha(Y,\nabla_U W) - \alpha(U-\eta(U)\xi,R(\xi,Y)W) = 0.
\end{aligned} \tag{5.7}
$$

Putting $W=\xi$ in (5.7) we have

$$\alpha(Y,U) + \alpha(Y,U) = 0. \tag{5.8}$$

It follows that

$$2\alpha(Y,U) = 0. \tag{5.9}$$

Hence

$$\alpha(Y,U) = 0. \tag{5.10}$$

Therefore M^n is totally geodesic. Conversely, if M^n is totally geodesic ($\alpha = 0$), then $\bar{R}.\bar{\nabla}\alpha = 0$ trivially. Thus we can state the folowing:

Theorem 5.1. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M^n is 2-semiparallel if and only if M^n is totally geodesic.*

6. PSEUDO-PARALLEL INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

In this section we study pseudo-parallel invariant submanifolds of Kenmotsu manifolds. Therefore we have

$$\bar{R}(X,Y).\alpha = fQ(g,\alpha) \tag{6.1}$$

holds for all vector fields X, Y tangent to M, where f denotes real valued function on M^n. The equation (6.1) can be writen as

$$\begin{aligned} R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) \\ = -f[\alpha(X\wedge_g Y)U,V) + \alpha(U,(X\wedge_g Y)V)], \end{aligned} \tag{6.2}$$

where $(X \wedge_g Y)$ defined by

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{6.3}$$

Using (6.3) in (6.2) we have

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) \\ &= -f[g(Y,U)\alpha(X,V) - g(X,U)\alpha(Y,V) + g(Y,V)\alpha(U,X) - g(X,V)\alpha(U,Y)]. \end{aligned} \tag{6.4}$$

Putting $X = V = \xi$ in (6.4) we get

$$\begin{aligned} &R^{\perp}(\xi,Y)\alpha(U,\xi) - \alpha(R(\xi,Y)U,\xi) - \alpha(U,R(\xi,Y)\xi) \\ &= -f[g(Y,U)\alpha(\xi,\xi) - g(\xi,U)\alpha(Y,\xi) + g(Y,\xi)\alpha(U,\xi) - g(\xi,\xi)\alpha(U,Y)]. \end{aligned} \tag{6.5}$$

Thus (3.5), (3.10) and (3.11) yields

$$-\alpha(U,Y-\eta(Y)\xi) = -f[-\alpha(U,Y)]. \tag{6.6}$$

It follows that

$$(f+1)\alpha(U,Y) = 0 \tag{6.7}$$

i.e., $\alpha(U,Y) = 0$ which gives M^n it totally geodesic provided $f \neq -1$.
Conversely, let M^n is totally geodesic. Clearly from (6.4) it follows that M^n is totally geodesic. Thus we can state the following:

Theorem 6.1. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M^n is pseudo-parallel if and only if M^n is totally geodesic provided $f \neq -1$.*

7. Generalized Ricci pseudo-parallel invariant submanifolds of Kenmotsu manifolds

In this section we study generalized Ricci pseudo-parallel invariant submanifolds of Kenmotsu manifolds. Therefore we have

$$\bar{R}(X,Y).\alpha = fQ(S,\alpha) \tag{7.1}$$

holds for all vector fields X, Y tangent to M, where f denotes real valued function on M^n. The equation (7.1) can be written as

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) \\ &= -f[\alpha(X \wedge_S Y)U,V) + \alpha(U,(X \wedge_S Y)V)], \end{aligned} \tag{7.2}$$

where $(X \wedge_g Y)$ defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{7.3}$$

Using (7.3) in (7.2) we have

$$\begin{aligned} &R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) \\ &= -f[S(Y,U)\alpha(X,V) - S(X,U)\alpha(Y,V) + S(Y,V)\alpha(U,X) - S(X,V)\alpha(U,Y)]. \end{aligned} \tag{7.4}$$

Putting $X = V = \xi$ in (7.4) we get

$$\begin{aligned} &R^{\perp}(\xi,Y)\alpha(U,\xi) - \alpha(R(\xi,Y)U,\xi) - \alpha(U,R(\xi,Y)\xi) \\ &= -f[S(Y,U)\alpha(\xi,\xi) - S(\xi,U)\alpha(Y,\xi) + S(Y,\xi)\alpha(U,\xi) - S(\xi,\xi)\alpha(U,Y)]. \end{aligned} \tag{7.5}$$

Using (3.10) in (7.5), we have

$$-\alpha(U,R(\xi,Y)\xi) = -f[-S(\xi,\xi)\alpha(U,Y)]. \tag{7.6}$$

It follows that from (3.11) and (3.12)

$$-\alpha(U,Y-\eta(Y)\xi) = -f[(n-1)\alpha(U,Y)], \tag{7.7}$$

which gives us

$$[(n-1)f - 1]\alpha(U,Y) = 0 \tag{7.8}$$

i.e., $\alpha(U,Y) = 0$ which gives M^n it totally geodesic provided $f \neq \frac{1}{n-1}$.
Conversely, let M^n is totally geodesic. Clearly from (7.4) it follows that M^n is totally geodesic. Thus we can state the following:

Theorem 7.1. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M^n is generalized pseudo-parallel if and only if M^n is totally geodesic provided $f \neq \frac{1}{n-1}$.*

8. Invariant submanifolds of Kenmotsu manifolds satisfying $\mathcal{Z}(X,Y).\alpha = 0$

In this section we study invariant submanifolds of Kenmotsu manifolds satisfying $\mathcal{Z}(X,Y).\alpha = 0$. Therefore we have

$$R^{\perp}(X,Y)\alpha(U,V) - \alpha(\mathcal{Z}(X,Y)U,V) - \alpha(U,\mathcal{Z}(X,Y)V) = 0. \tag{8.1}$$

Putting $X = V = \xi$ in (8.1) we have

$$R^{\perp}(\xi,Y)\alpha(U,\xi) - \alpha(\mathcal{Z}(\xi,Y)U,\xi) - \alpha(U,\mathcal{Z}(\xi,Y)\xi) = 0. \tag{8.2}$$

It follows that from (2.7)

$$\alpha[U, R(\xi,Y)\xi - \frac{r}{n(n-1)}\{g(Y,\xi)\xi - g(\xi,\xi)Y\}] = 0, \tag{8.3}$$

which gives us

$$(1 + \frac{r}{n(n-1)})\alpha(U,Y) = 0. \tag{8.4}$$

Hence $\alpha(U,Y) = 0$ which gives M^n it totally geodesic provided $r \neq -n(n-1)$. Conversely, let M^n is totally geodesic. Clearly from (8.1) it follows that M^n is totally geodesic. Thus we can state the following:

Theorem 8.1. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $r \neq -n(n-1)$. The condition $\mathcal{Z}(X,Y).\alpha = 0$ holds on M^n if and only if M^n is totally geodesic.*

9. Invariant submanifolds of Kenmotsu manifolds satisfying $\mathcal{Z}(X,Y).\bar{\nabla}\alpha = 0$

In this section we study invariant submanifolds of Kenmotsu manifolds satisfying $\mathcal{Z}(X,Y).\bar{\nabla}\alpha = 0$. Therefore we have

$$\begin{aligned} &R^{\perp}(X,Y)(\bar{\nabla}\alpha)(U,V,W) - (\bar{\nabla}\alpha)(\mathcal{Z}(X,Y)U,V,W) \\ &- (\bar{\nabla}\alpha)(U,\mathcal{Z}(X,Y)V,W) - (\bar{\nabla}\alpha)(U,V,\mathcal{Z}(X,Y)W) = 0. \end{aligned} \tag{9.1}$$

Putting $X = V = \xi$ in (9.1) yields

$$\begin{aligned} &R^{\perp}(\xi,Y)(\bar{\nabla}\alpha)(U,\xi,W) - (\bar{\nabla}\alpha)(\mathcal{Z}(\xi,Y)U,\xi,W) \\ &- (\bar{\nabla}\alpha)(U,\mathcal{Z}(\xi,Y)\xi,W) - (\bar{\nabla}\alpha)(U,\xi,\mathcal{Z}(\xi,Y)W) = 0. \end{aligned} \tag{9.2}$$

In the view of (3.10), (3.12) and (3.14) we have the following

$$\begin{aligned} (\bar{\nabla}\alpha)(U,\xi,W) &= (\bar{\nabla}_U\alpha)(\xi,W) \\ &= \nabla^{\perp}_U(\alpha(\xi,W)) - \alpha(\nabla_U\xi,W) - \alpha(\xi,\nabla_U W) \end{aligned} \tag{9.3}$$

$$
\begin{aligned}
&= -\alpha(U - \eta(U)\xi, W) \\
&= -\alpha(U, W) \\
(\bar{\nabla}\alpha)(\mathcal{Z}(\xi, Y)U, \xi, W) &= (\bar{\nabla}_{\mathcal{Z}(\xi,Y)U}\alpha)(\xi, W) \qquad (9.4) \\
&= \nabla^{\perp}_{\mathcal{Z}(\xi,Y)U}\alpha(\xi, W) - \alpha(\nabla_{\mathcal{Z}(\xi,Y)U}\xi, W) \\
&\quad - \alpha(\xi, \nabla_{\mathcal{Z}(\xi,Y)U}W). \\
&= -\alpha(\nabla_{\mathcal{Z}(\xi,Y)U}, W) \\
(\bar{\nabla}\alpha)(U, \mathcal{Z}(\xi, Y)\xi, W) &= (\bar{\nabla}_U\alpha)(\mathcal{Z}(\xi, Y)\xi, W) \qquad (9.5) \\
&= \nabla^{\perp}_U(\alpha(\mathcal{Z}(\xi, Y)\xi, W)) - \alpha(\nabla_U\mathcal{Z}(\xi, Y)\xi, W) \\
&\quad - \alpha(\mathcal{Z}(\xi, Y)\xi, \nabla_U W)
\end{aligned}
$$

and

$$
\begin{aligned}
(\bar{\nabla}\alpha)(U, \xi, \mathcal{Z}(\xi, Y)W) &= (\bar{\nabla}_U\alpha)(\xi, \mathcal{Z}(\xi, Y)W) \qquad (9.6) \\
&= \nabla^{\perp}_U\alpha(\xi, \mathcal{Z}(\xi, Y)W) - \alpha(\nabla_U\xi, \mathcal{Z}(\xi, Y)W) \\
&\quad - \alpha(\xi, \nabla_U\mathcal{Z}(\xi, Y)W) \\
&= -\alpha(U - \eta(U)\xi, \mathcal{Z}(\xi, Y)W).
\end{aligned}
$$

Substituting (9.3)-(9.6) in (9.2) we get

$$
\begin{aligned}
&- R^{\perp}(\xi, Y)\alpha(\phi U, W) + \alpha(\nabla_{\mathcal{Z}(\xi,Y)U}, W) \\
&- \nabla^{\perp}_U(\alpha(\mathcal{Z}(\xi, Y)\xi, W)) + \alpha(\nabla_U\mathcal{Z}(\xi, Y)\xi, W) \qquad (9.7) \\
&+ \alpha(\mathcal{Z}(\xi, Y)\xi, \nabla_U W) + \alpha(U - \eta(U)\xi, \mathcal{Z}(\xi, Y)W) = 0.
\end{aligned}
$$

Putting $W = \xi$ in (9.7) and using (3.9) and (3.10) yields

$$2\alpha(U - \eta(U)\xi, \mathcal{Z}(\xi, Y)\xi) = 0. \qquad (9.8)$$

It follows that from (2.7)

$$\alpha[U - \eta(U)\xi, R(\xi, Y)\xi - \frac{r}{n(n-1)}\{g(Y, \xi)\xi - g(\xi, \xi)Y\}] = 0, \qquad (9.9)$$

which gives us

$$(1 + \frac{r}{n(n-1)})\alpha(U, Y) = 0. \qquad (9.10)$$

Hence $\alpha(U, Y) = 0$ which gives M^n it totally geodesic provided $r \neq -n(n-1)$.

Conversely, let M^n is totally geodesic. Clearly from (9.1) it follows that M^n is totally geodesic. Thus we can state the following:

Theorem 9.1. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $r \neq -n(n-1)$. The condition $\mathcal{Z}(X,Y).\bar{\nabla}\alpha = 0$ holds on M^n if and only if M^n is totally geodesic.*

In view of the Theorem 4.1-9.1, we can state the following:

Theorem 9.2. *Let M^n be an invariant submanifold of a Kenmotsu manifold $\tilde{M}$. Then the following statements are equivalent:*

(1) M^n is semiparallel;
(2) M^n is 2- semiparallel;
(3) M^n is pseudo-parallel provided $f \neq -1$;
(4) M^n is generalized Ricci pseudo-parallel provided $f \neq \frac{1}{n-1}$;
(5) M^n satisfies the condition $\mathcal{Z}(X,Y).\alpha = 0$ with $r \neq -n(n-1)$;
(6) M^n satisfies the condition $\mathcal{Z}(X,Y).\bar{\nabla}\alpha = 0$ with $r \neq -n(n-1)$;
(7) M^n is totally geodesic .

References

[1] Aikawa, R. and Matsuyama, Y., *On the local symmetry of Kaehler hypersurfaces,* Yokohoma mathematical J. 51(2005).

[2] Arslan, K., Lumiste., Murathn, C. and Özgür, C., 2-*semiparallel surfaces in space forms. I: Two particular cases,* Proc. Est. Acad. Sci. Phys. Math. 49(3)(2000), 139–148.

[3] Asperti, A. C., Lobos, G. A. and Mercuri, F., *pseudo-parallel immersions in space forms,* Mat. Contemp. 17(1999), 59–70.

[4] Asperti, A. C., Lobos, G. A. and Mercuri, F., *pseudo-parallel submanifolds of a space forms,* Adv. Geom. 2(2002), 57–71.

[5] Blair, D. E., *Contact manifold in Riemannian geometry,* Lecture notes on mathematics, 509, Springer-Verlag, Berlin, 1976.

[6] Blair, D. E., *Riemannian geometry on contact and symplectic manifolds,* Progress in Math., Vol. 203, Berlin.

[7] Boeckx, E., Kowalski, O., and Vanhecke, L., *Riemannian manifolds of conullity two,* Singapore World Sci. Publishing, 1996.

[8] Chen, B. Y., *Geometry of submanifolds,* Pure and Appl. Math. 22, Marcel Dekker, Inc., New York, 1973.

[9] Deprez, J., *Semiparallel surfaces in Euclidean space,* J. Geom. 25(1985), 192–200.

[10] Deprez, J., *Semiparallel hypersurfaces,* Rend. sem. Mat. Univ. Politechn. Torino, 45(1986), 303-316.

[11] De, U. C. and Pathak, G., *On 3-dimensional Kenmotsu manifolds,* Indian J. Pure Applied Math., 35(2004), 159–165.

[12] De, U. C. and De, Krishnendu., *On $\phi-$ Concircurly symmetric Kenmotsu manifolds,* Thai J. Math. 10(2012)(1), 1–11.

[13] Dillen, F., *Semiparallel hypersurfaces of a real space form,* Israel J. Math. 75(1991), 193–202.

[14] Endo, H., *Invariant submanifolds in contact metric manifolds,* Tensor(N.S) 43(1)(1986), 83–87.

[15] Jun, J-B., De, U. C. and Pathak, G., *On Kenmotsu manifolds,* J. Korean Math. Soc., 42(2005), 435–445.

[16] Kenmotsu, K., *A class of contact Riemannian manifolds,* Tohoku Math. J., 24(1972), 93–103.

[17] Kon, M., *Invariant submanifolds of Normal contact metric manifolds,* Kodai Math. Sem. Rep. 25(1973), 330–336.

[18] Levy, H., *Symmetric tensors of the second order whose covariant derivatives vanish,* Ann. Math. 27(2)(1925), 91–98.

[19] Lumiste, Ü., *Semisymmetric submanifolds as second order envelope of a symmetric submanifolds,* Proc. Estonian Acad. Sci. Phys. Math. 39(1990), 1–8.

[20] Murathan, C., Arslan K. and Ezentas, E., *Ricci generalized pseudo-symmetric immersions,* Diff. Geom. and Appl. Matfypress Prague, (2005), 99–108.

[21] Mangione, V., *Totally geodesic submanifolds of a Kenmotsu space form,* Math. Reports 7(57),4(2005), 315–324.

[22] Özgür, C. and Murathan C., *On invariant submanifolds of Lorentzian Para-Sasakian manifolds,* The Arab. J. for Sci. and Eng. 34(2008), 177–185.

[23] Özgür, C. and De, U. C., *On the quasi-conformal curvature tensor of a Kenmotsu manifold,* Mathematica Pannonica, 17/2, (2006), 221–228.

[24] Roter W., *On conformally recurrent Ricci-recurrent manifolds,* Colloq. Math. (46)(1982), 45–57.

[25] Shirokov, P. A., *Constant vector fields and tensor fields of second order in Riemannian spaces,* Izv. Kazan Fiz.-Mat. Obshchestva Ser. 25(2)(1925), 86–114 (in Russian).

[26] Szabó, Z. I., *Structure theorems on Riemannian spaces satisfying* $R(X,Y).R = 0$, *the local version,* J. Diff. Geom., 17(1982), 531–582.

[27] Verstraelen, L., *Comments on pseudosymmetry in the sense of Ryszard Deszcz,* In: Geometry and Topology of submanifolds, VI. River Edge, NJ: World Sci. Publishing, 1994, 199–209.

[28] Yano, K. and Kon, M., *Structures on manifolds, Series in Pure Math,* 3. World Scientific Publ. Co., Singapore, 1984.

[29] Yildiz, A., De, U. C. and Acet, B. E., *On Kenmotsu manifolds satisfying certain curvature conditions,* SUT J. Math. 45(2)(2009), 89–101.

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RADIO NUMBER FOR LINEAR CACTI

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Abstract. A radio labeling of a graph G is a function f from the vertex set V (G) to the set of non-negative integers such that $|f(u)-f(v)| \geq diam(G)+1-d(u,v)$, where diam(G) and $d(u,v)$ are respectively the diameter and distance between u and v in graph G. The radio number $rn(G)$ is the smallest number k such that G has radio labeling with $\max\{f(v) : v \in V(G)\} = k$. We investigate radio number for linear cacti.

1. Introduction

A radio labeling of a graph G is an extension of distance two labeling which is motivated by the channel assignment problem modeled by Hale [4] in 1980. For a given set of transmitters, the important task is to assign channels to the transmitters such that network is free of interference. This problem is known as channel assignment problem. In network of transmitters, it is observed that the interference is closely related to the geographical location of transmitters. Closer the transmitters are, higher the interference is. If we consider two level interference, namely major and minor, then two transmitters are very close if the level of interference between them is major while close if the level of interference between them is minor. Roberts [10] proposed a variation of the channel assignment problem in which close transmitters must receive different channels and very close transmitters must receive channels that are at leat two apart.

In a graph model of this problem, the transmitters are represented by the vertices of a graph; two vertices are very close if they are adjacent and close if they are at distance two apart in the graph. Motivated through this problem Griggs and Yeh [3] introduced the concept of distance two labeling which is defined as follows:

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Definition 1.1. A distance two labeling (or L(2, 1)-labeling) of a graph $G = (V(G), E(G))$ is a function f from the vertex set V (G) to the set of nonnegative integers such that the following conditions are satisfied:

(1) $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$
(2) $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$

The difference between the largest and the smallest label assigned by f is called the span of f and the minimum span over all L(2, 1)-labeling of G is called the λ - number of G, denoted by $\lambda(G)$. The L(2, 1)-labeling has been explored in past two decades by many researchers like Chang and Kuo [1], Griggs and Yeh [3], Sakai [9], Vaidya and Bantva [11, 12] and Vaidya et al. [14].

But as time passed, practically it has been observed that the interference among transmitters might go beyond two levels. Radio labeling extends the level of interference considered in L(2, 1)-labeling from two to the largest possible - the diameter of G. The diameter of G is denoted by $diam(G)$ or simply by d is the maximum distance among all pairs of vertices in G. Motivated through the problem of channel assignment of FM radio stations Chartrand et al. [2] introduced the concept of radio labeling of graph as follows.

Definition 1.2. A radio labeling of a graph G is an injective function $f : V(G) \rightarrow \{0, 1, 2, \dots\}$ such that

$$|f(u) - f(v)| \geq diam(G) + 1 - d(u, v), \text{ for all } u, v \in V(G).$$

The radio number denoted by $rn(G)$ is the minimum span of a radio labeling for G. Note that when diam(G) is two then radio labeling and distance two labeling are identical. The radio labeling is studied in the past decade by many researchers like Liu [5], Liu and Xie [6, 7], Liu and Zhu [8], Vaidya and Bantva [13] and Vaidya and Vihol [15].

In present work, a graph $G = (V(G), E(G))$ is finite, connected and undirected without loops and multiple edges. The distance from u to v, denoted as $d(u, v)$, is the least length of a u, v-path. The eccentricity of the vertex u, denoted as $\epsilon(u)$, is $\max\{d(u, v) : v \in V(G)\}$. The center of a graph G, denoted by C(G), is the subgraph induced by the vertices of minimum eccentricity. A vertex v of a graph G is called a cut vertex if its deletion leaves a graph disconnected. A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. If H is a block of graph G then H has no cut vertex but H may contain vertices that are cut vertices of G and two blocks in a graph share at most one vertex. The block cutpoint graph G is a bipartite graph H in which one partite set consists of the cut vertices of G and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of H if and only if $v \in B_i$. Moreover terms not defined here are used in the sense of West [16].

In this paper, we completely determine the radio number for linear cacti $P_m(K_n)$. Through out this discussion we consider $m \geq 3$ as for $m = 1$ and $m = 2, P_m(K_n)$ is complete graph K_n and one point union of two K_n is diameter two graph for which $L(2,1)$-labeling and radio labeling coincide. We also take $n > 2$ as for $n = 2$, the linear cactus is a path P_{m+1} and radio number is completely determined by Liu and Zhu [8].

2. Main Results

In this section, first we define linear cacti.

Definition 2.1. A linear cactus $P_m(K_n)$ is a connected graph whose all the blocks are isomorphic to complete graph K_n and block cutpoint graph is a path P_{2m-1}.

From the definition of linear cactus $P_m(K_n)$, it is clear that the linear cactus has m blocks and $m-1$ cut vertices and let $V(P_m(K_n)) = \{v_i^j, v_{m+1}^1 : 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and $E(P_m(K_n)) = \{v_i^j v_{i+1}^1, v_i^j v_i^k : 1 \leq i \leq m, 1 \leq j, k \leq n-1, j \neq k\}$. Then $v_i^1, i = 2, 3, \ldots, m$ be the cut vertices say $c_i, i = 1, 2, \ldots, m-1$ and the subgraphs induced by $\{v_i^j, v_{i+1}^1 : 1 \leq j \leq n-1\}$ for each $i = 1, 2, \ldots, m$ are blocks say $B_i, i = 1, 2, \ldots, m$ of $P_m(K_n)$ such that for each $i = 1, 2, \ldots, m-1, B_i \cap B_{i+1} = c_i$. From the definition of linear cactus it is clear that $p = |V(P_m(K_n))| = m(n-1)+1$ and $diam(P_m(K_n)) = d = m$.

Example 2.1. A linear cactus $P_5(K_6)$ is shown in Figure 1 while the corresponding block-cut-point graph is shown in Figure 2.

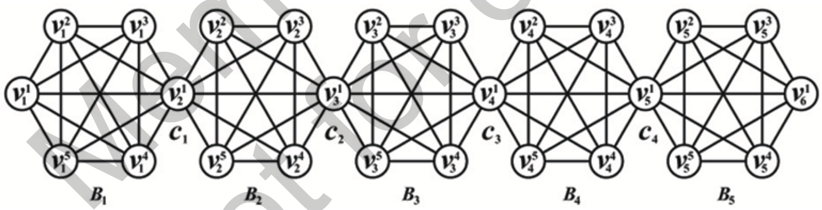

Figure 1: $P_5(K_6)$

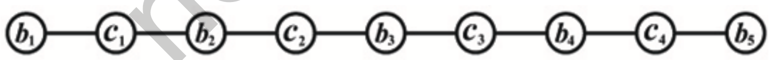

Figure 2: Block-cut-point graph of $P_5(K_6)$

Lemma 2.1. *For* $P_m(K_n)$,

$$C(P_m(K_n)) = \begin{cases} K_n, & if m \text{ is odd} \\ K_1, & if m \text{ is even} \end{cases}$$

Proof. Let $P_m(K_n)$ be the linear cactus of m blocks and $k = \lfloor \frac{m}{2} \rfloor$. Let S be the set of all vertices with minimum eccentricity in linear cactus $P_m(K_n)$. In $P_m(K_n)$, the eccentricity of each vertex v_i^j and $v_{m+1}^1, 1 \leq i \leq m, 1 \leq j \leq n-1$ is given as follows.

Case - 1: m is odd.
If m is odd then $m = 2k+1$ and in this case

$$\epsilon(v_i^j) = \begin{cases} m+1-i, & 1 \leq i \leq k+1, 1 \leq j \leq n-1 \\ i-1, & k+2 \leq i \leq m+1, j=1 \\ i, & k+2 \leq i \leq m, 2 \leq j \leq n-1 \end{cases}$$

$S = \min\{\epsilon(v_i^j) : 1 \leq i \leq m+1, 1 \leq j \leq n-1\}$
$= \{v_{k+1}^j, v_{k+2}^1 : 1 \leq j \leq n-1\}$
The graph induced by S is isomorphic to the complete graph K_n and hence $C(P_m(K_n)) = K_n$.

Case - 2: m is even.
If m is even then $m = 2k$ and in this case

$$\epsilon(v_i^j) = \begin{cases} m+1-i, & 1 \leq i \leq k, 1 \leq j \leq n-1 \\ i-1, & k+1 \leq i \leq m+1, j=1 \\ i, & k+1 \leq i \leq m, 2 \leq j \leq n-1 \end{cases}$$

$S = \min\{\epsilon(v_i^j) : 1 \leq i \leq m+1, 1 \leq j \leq n-1\}$
$= \{v_{k+1}^1\}$
The graph induced by S is isomorphic to the complete graph K_1 and hence $C(P_m(K_n)) = K_1$.
Thus, from Case - 1 and Case - 2, we have

$$C(P_m(K_n)) = \begin{cases} K_n, & \text{if } m \text{ is odd} \\ K_1, & \text{if } m \text{ is even} \end{cases}$$

■

We define left and right hand side vertices of linear cactus according to its center. For that we rename the vertices of $P_m(K_n)$ as follows.
For $m = 2k+1$, rename the vertices $v_i^j, 1 \leq i \leq m, 1 \leq j \leq n-1$ and v_{m+1}^1 by

$$v_i^j = \begin{cases} v_{L(k+1-i)}^j, & 1 \leq i \leq k, 1 \leq j \leq n-1 \\ v_c^j, & i = k+1, 1 \leq j \leq n-1 \\ v_c^j, & i = k+2, j = 1 \\ v_{R(i-(k+1))}^j, & k+2 \leq i \leq 2k+1, 2 \leq j \leq n-1 \\ v_{R(i-(k+2))}^1, & k+3 \leq i \leq 2k+2, j = 1 \end{cases}$$

For $m = 2k$, rename the vertices $v_i^j, 1 \le i \le m, 1 \le j \le n-1$ and v_{m+1}^1 by

$$v_i^j = \begin{cases} v_{L(k+1-i)}^j, & 1 \le i \le k, 1 \le j \le n-1 \\ v_c^1, & i = k+1, j = 1 \\ v_{R(i-k)}^j, & k+1 \le i \le 2k, 2 \le j \le n-1 \\ v_{R(i-(k+1))}^1, & k+2 \le i \le 2k+1, j = 1 \end{cases}$$

Example 2.2. In Figure 3 and Figure 4, the linear cacti $P_5(K_6)$ and $P_6(K_6)$ are shown with new notation of vertices.

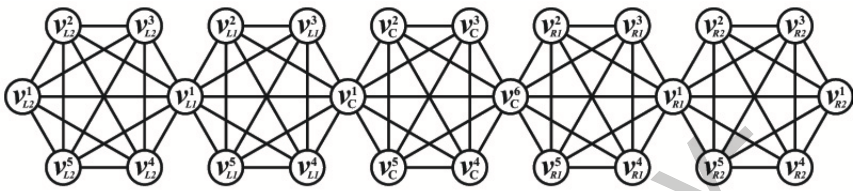

Figure 3: $P_5(K_6)$

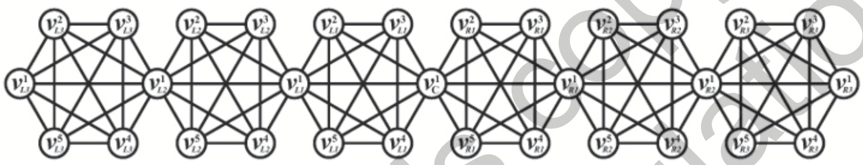

Figure 4: $P_6(K_6)$

In a $P_m(K_n)$, two vertices u and v are called vertices of opposite side if $u = v_{Li}^j$ and $v = v_{Ri}^j$ and vice versa, for some v_{Li}^j and v_{Ri}^j except one of them is vertex of $C(P_m(K_n))$.

We define the level function on $V(P_m(K_n))$ to the set of whole numbers W by $L(u) = \min\{d(u,w) : w \in V(C(P_m(K_n)))\}$, for any $u \in V(P_m(K_n))$

In $P_m(K_n)$, the maximum level is $\lfloor \frac{m}{2} \rfloor$ and notice that the level of each v_{Li}^j and $v_{Ri}^j, 1 \le i \le k = \lfloor \frac{m}{2} \rfloor, 1 \le j \le n-1$ is i.
The total level of graph G, denoted by L(G), is defined as

$$L(G) = \sum_{u \in V(G)} L(u)$$

Remark 2.1. For $G = P_m(K_n)$,

$$(a) \qquad d(u,v) \le \begin{cases} L(u) + L(v) + 1 & \text{if } m \text{ is odd} \\ L(u) + L(v) & \text{if } m \textit{ iseven} \end{cases}$$

Moreover, equality hold if u and v are on opposite side.

$$(b) \qquad L(G) = \begin{cases} \frac{1}{4}(m^2-1)(n-1) & \text{if } m \text{ is odd} \\ \frac{1}{4}m(m+3)(n-1) & \text{if } m \text{ is even} \end{cases}$$

Lemma 2.2. *Let f be an assignment of non negative integers to $V(P_m(K_n))$ and $(u_1, u_2, u_3, \dots, u_p)$ be the ordering of $V(P_m(K_n))$ such that $f(u_i) < f(u_{i+1})$ defined by $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$ and $d(u_i, u_{i+1}) \leq k + 1$, where $k = \lfloor \frac{m}{2} \rfloor$. Then f is radio labeling.*

Proof. Let $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$ for $1 \leq i \leq p - 1$ and $d(u_i, u_{i+1}) \leq k + 1$, where $k = \lfloor \frac{m}{2} \rfloor$.

For each $i = 1, 2, \dots, p - 1$, let $f_i = f(u_{i+1}) - f(u_i)$. Now, we want to prove that f is a radio labeling. *i.e.* for any $i \neq j, |f(u_j) - f(u_i)| \geq d + 1 - d(u_i, u_j)$ Without loss of generality, let $j \geq i + 2$ then

$$f(u_j) - f(u_i) = f_i + f_{i+1} + \cdots + f_{j-1}$$
$$= (j - i)(d + 1) - d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2}) - \cdots - d(u_{j-1}, u_j)$$

Case - 1: m is odd.

If $m = 2k + 1$ then $d = 2k + 1$ and hence

$$\begin{aligned} f(u_j) - f(u_i) &\geq (j - i)(d + 1) - (j - i)(k + 1) \text{ as } d(u_i, u_{i+1}) \leq k + 1 \\ &= (j - i)(2k + 2) - (j - i)(k + 1) \\ &= (j - i)(2k + 2 - k - 1) \\ &= (j - i)(k + 1) \\ &\geq 2(k + 1) \\ &\geq d + 1 - d(u_i, u_j) \end{aligned}$$

Case - 2: m is even.

If $m = 2k$ then $d = 2k$ and hence

$$\begin{aligned} f(u_j) - f(u_i) &\geq (j - i)(d + 1) - (j - i)(k + 1) \text{ as } d(u_i, u_{i+1}) \leq k + 1 \\ &= (j - i)(2k + 1) - (j - i)(k + 1) \\ &= (j - i)(2k + 1 - k - 1) \\ &= (j - i)(k) \\ &\geq 2k = 2k + 1 - 1 \\ &\geq d + 1 - d(u_i, u_j) \end{aligned}$$

Thus, in both cases, f is a radio labeling. ■

Theorem 2.1. *For $G = P_m(K_n)$,*

$$rn(P_m(K_n)) \geq \begin{cases} \frac{1}{2}(m^2 + 1)(n - 1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m - 1)(n - 1) + 1, & \text{if } m \text{ is even} \end{cases}$$

Proof. Let f be an optimal radio labeling for $P_m(K_n)$ and $(u_1, u_2, u_3, \ldots, u_p)$ be the ordering of $V(P_m(K_n))$ such that $0 = f(u_1) < f(u_2) < f(u_3) < \cdots < f(u_p)$. Then $f(u_{i+1}) - f(u_i) \geq d + 1 - d(u_i, u_{i+1})$, for all $1 \leq i \leq p-1$. Summing these $p-1$ inequalities we get

$$rn(P_m(K_n)) = f(u_p) \geq (p-1)(d+1) - \sum_{i=1}^{p-1} d(u_i, u_{i+1}) \tag{2.1}$$

Case - 1: m is odd.
For $P_m(K_n)$, we have

$$\begin{aligned}\sum_{i=1}^{p-1} d(u_i, u_{i+1}) &\leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1}) + 1] \\ &= 2\sum_{u \in V(G)} L(u) - L(u_1) - L(u_p) + (p-1) \\ &= 2L(G) + (p-1) \text{ (Choosing } u_1, u_p \in V(C(G))) \end{aligned} \tag{2.2}$$

Substituting (2.2) in (2.1), we get

$$\begin{aligned} rn(P_m(K_n)) &= f(u_p) \geq (p-1)(d+1) - 2L(G) - (p-1) \\ &= (p-1)d - 2L(G) \\ &= m(n-1)(m) - 2\frac{1}{4}(m^2-1)(n-1) \\ &= \frac{1}{2}(2m^2 - m^2 + 1)(n-1) \\ &= \frac{1}{2}(m^2+1)(n-1) \end{aligned}$$

Case - 2: m is even.
For $P_m(K_n)$, we have

$$\begin{aligned}\sum_{i=1}^{p-1} d(u_i, u_{i+1}) &\leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1})] \\ &= 2\sum_{u \in V(G)} L(u) - L(u_1) - L(u_p) \\ &= 2L(G) - 1 \text{ (Choosing } u_1 \in V(C(G)) \text{ and } u_p \in N(u_1)) \end{aligned} \tag{2.3}$$

Substituting (2.3) in (2.1), we get

$$\begin{aligned} rn(P_m(K_n)) &= f(u_p) \geq (p-1)(d+1) - [2L(G) - 1] \\ &= (p-1)(d+1) - 2L(G) + 1 \\ &= m(n-1)(m+1) - 2\frac{1}{4}m(m+3)(n-1) + 1 \\ &= \frac{1}{2}(m^2 - m)(n-1) + 1 \\ &= \frac{1}{2}m(m-1)(n-1) + 1 \end{aligned}$$

Thus, from Case - 1 and Case - 2, we have

$$rn(P_m(K_n)) \geq \begin{cases} \frac{1}{2}(m^2+1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even} \end{cases}$$

■

Theorem 2.2. *Let $G = P_m(K_n)$ be a linear cactus and $k = \lfloor \frac{m}{2} \rfloor$ then*

$$rn(P_m(K_n)) \leq \begin{cases} \frac{1}{2}(m^2+1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even} \end{cases}$$

Proof. Here we consider following two cases.
Case - 1: m is odd.

For $G = P_m(K_n)$, define $f : V(G) \to \{0, 1, 2, \ldots, \}$ by $f(u_1) = 0, f(u_{i+1}) = f(u_i) + d - L(u_i) - L(u_{i+1}), 1 \leq i \leq p-1$ as per following ordering of vertices.

v_C^2	$\to v_{Rk}^2$	$\to v_{Lk}^2$	$\to v_C^3$	$\to v_{Rk}^3$	$\to v_{Lk}^3$	$\ldots$
v_C^{n-1}	$\to v_{Rk}^{n-1}$	$\to v_{Lk}^{n-1}$	$\to v_C^n$	$\to v_{Lk}^1$	$\to v_{R(k-1)}^1$	
v_{L1}^2	$\to v_{R(k-1)}^2$	$\to v_{L1}^3$	$\to v_{R(k-1)}^3$	$\to v_{L1}^4$	$\to v_{R(k-1)}^4$	$\ldots$
v_{L1}^{n-1}	$\to v_{R(k-1)}^{n-1}$	$\to v_{L1}^1$	$\to v_{R(k-2)}^1$			
v_{L2}^2	$\to v_{R(k-2)}^2$	$\to v_{L2}^3$	$\to v_{R(k-2)}^3$	$\to v_{L2}^4$	$\to v_{R(k-2)}^4$	$\ldots$
v_{L2}^{n-1}	$\to v_{R(k-2)}^{n-1}$	$\to v_{L2}^1$	$\to v_{R(k-3)}^1$			
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	
$v_{L(k-1)}^2$	$\to v_{R2}^2$	$\to v_{L(k-1)}^3$	$\to v_{R2}^3$	$\to v_{L(k-1)}^4$	$\to v_{R2}^4$	$\ldots$
$v_{L(k-1)}^{n-1}$	$\to v_{R2}^{n-1}$	$\to v_{L(k-1)}^1$		$\to v_{Rk}^1$	$\to v_C^1$	

Case - 2: m is even.

For $G = P_m(K_n)$, define $f : V(G) \to \{0, 1, 2, \dots,\}$ by $f(u_1) = 0, f(u_{i+1}) = f(u_i) + d + 1 - L(u_i) - L(u_{i+1}), 1 \le i \le p - 1$ as per following ordering of vertices.

$$
\begin{array}{llllllll}
v^2_{L1} & \to v^2_{RK} & \to v^3_{L1} & \dots & \to v^{n-1}_{L1} & \to v^{n-1}_{Rk} & \to v^1_{L1} & \to v^1_{Rk} \\
v^2_{L2} & \to v^2_{R(k-1)} & \to v^3_{L2} & \dots & \to v^{n-1}_{L2} & \to v^{n-1}_{R(k-1)} & \to v^1_{L2} & \to v^1_{R(k-1)} \\
\dots & \dots & \dots & \dots\ \dots & & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots\ \dots & & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots\ \dots & & \dots & \dots & \dots \\
v^2_{Lk} & \to v^2_{R1} & \to v^3_{Lk} & \dots & \to v^{n-1}_{Lk} & \to v^{n-1}_{R1} & \to v^1_{Lk} & \to v^1_{Rk} \\
v^1_C & & & & & & &
\end{array}
$$

Thus, it is possible to assign labeling to the vertices of $P_m(K_n)$ with span equal to the lower bound satisfying the condition of Lemma 2.2 and hence f is a radio labeling.

Thus, we have,

$$rn(P_m(K_n)) \le \begin{cases} \frac{1}{2}(m^2+1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even} \end{cases}$$

■

Theorem 2.3. *For $G = P_m(K_n)$,*

$$rn(P_m(K_n)) = \begin{cases} \frac{1}{2}(m^2+1)(n-1), & \textit{if } m \textit{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \textit{if } m \textit{ is even} \end{cases}$$

Proof. Proof follows from Theorem 2.1 and Theorem 2.2. ■

Corollary 2.1. *For $G = P_{an}(K_n)$, where $a \ge 1$*

$$rn(P_{an}(K_n)) = \begin{cases} \frac{1}{2}(n-1)(a^2n^2+1), & \textit{if } an \textit{ is odd} \\ \frac{1}{2}an(n-1)(an-1)+1, & \textit{if } an \textit{ is even} \end{cases}$$

Proof. Let $G = P_m(K_n)$ be a linear cactus then by Theorem 2.3,

$$rn(P_m(K_n)) = \begin{cases} \frac{1}{2}(m^2+1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even} \end{cases}$$

We take $m = an$ and consider the following two cases

Case - 1: an is odd.

$$\begin{aligned} rn(P_{an}(K_n)) &= (p-1)d - 2L(G) \\ &= an(n-1)an - \frac{1}{2}(an-1)(an+1)(n-1) \\ &= \frac{1}{2}(n-1)(a^2n^2+1) \end{aligned}$$

Case - 2: an is even.

$$\begin{aligned} rn(P_{an}(K_n)) &= (p-1)(d+1) - 2L(G) + 1 \\ &= an(n-1)(an+1) - \frac{1}{2}an(an+3)(n-1) + 1 \\ &= \frac{1}{2}an(n-1)(an-1) + 1 \end{aligned}$$

Thus, from Case - 1 and Case - 2, we have

$$rn(P_{an}(K_n)) = \begin{cases} \frac{1}{2}(n-1)(a^2n^2+1), & \text{if } an \text{ is odd} \\ \frac{1}{2}an(n-1)(an-1)+1, & \text{if } an \text{ is even} \end{cases}$$

■

Corollary 2.2. *For $G = P_m(K_{am})$, where $a \geq 1$*

$$rn(P_m(K_{am})) = \begin{cases} \frac{1}{2}(am-1)(n^2+1), & if\, m \; is \; odd \\ \frac{1}{2}m(m-1)(am-1)+1, & if\, m \; is \; even \end{cases}$$

Proof. Let $G = P_m(K_n)$ be a linear cactus then by Theorem 2.3,

$$rn(P_m(K_n)) = \begin{cases} \frac{1}{2}(m^2+1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even} \end{cases}$$

We take $n = am$ and consider the following two cases

Case - 1: m is odd.

$$\begin{aligned} rn(P_m(K_{am})) &= (p-1)d - 2L(G) \\ &= m(am-1)m - \frac{1}{2}(m-1)(m+1)(am-1) \\ &= \frac{1}{2}(am-1)(m^2+1) \end{aligned}$$

Case - 2: m is even.

$$\begin{aligned} rn(P_m(K_{am})) &= (p-1)(d+1) - 2L(G) + 1 \\ &= m(am-1)(m+1) - \frac{1}{2}m(m+3)(am-1) + 1 \\ &= \frac{1}{2}m(m-1)(am-1) + 1 \end{aligned}$$

Thus, from Case - 1 and Case - 2, we have

$$rn(P_m(K_{am})) = \begin{cases} \frac{1}{2}(am-1)(n^2+1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(am-1)+1, & \text{if } m \text{ is even} \end{cases}$$

■

In particular, if $m = n$ then we have the following useful result in terms of n only.

Corollary 2.3. *For $G = P_n(K_n)$,*

$$rn(P_n(K_n)) = \begin{cases} \frac{1}{2}(n-1)(n^2+1), & \text{if } n \text{ is odd} \\ \frac{1}{2}n(n-1)^2+1, & \text{if } n \text{ is even} \end{cases}$$

Example 2.3. In Figure 5, the ordering of the vertices and optimal radio labeling for $P_5(K_6)$ is shown.

$$\begin{array}{lllll} v_C^2 & \to v_{R2}^2 & \to v_{L2}^2 & \to v_C^3 & \to v_{R2}^3 \\ v_{L2}^3 & \to v_C^4 & \to v_{R2}^4 & \to v_{L2}^4 & \to v_C^5 \\ v_{R2}^5 & \to v_{L2}^5 & \to v_C^6 & \to v_{L2}^1 & \to v_{R1}^1 \\ v_{L1}^2 & \to v_{R1}^2 & \to v_{L1}^3 & \to v_{R1}^3 & \to v_{L1}^4 \\ v_{R1}^4 & \to v_{L1}^5 & \to v_{R1}^5 & \to v_{L1}^1 & \to v_{R2}^1 \\ v_C^1 & & & & \end{array}$$

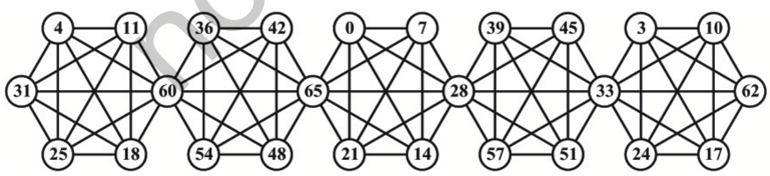

Figure 5: $\lambda(P_5(K_6)) = 65$

Example 2.4. In Figure 6, the ordering of the vertices and optimal radio labeling for $P_6(K_6)$ is shown.

$$
\begin{array}{lllll}
v^2_{L1} & \to v^2_{R3} & \to v^3_{L1} & \to v^3_{R3} & \to v^4_{L1} \\
v^4_{R3} & \to v^5_{L1} & \to v^5_{R3} & \to v^1_{L1} & \to v^1_{R3} \\
v^2_{L2} & \to v^2_{R2} & \to v^3_{L2} & \to v^3_{R2} & \to v^4_{L2} \\
v^4_{R2} & \to v^5_{L2} & \to v^5_{R2} & \to v^1_{L2} & \to v^1_{R2} \\
v^2_{L3} & \to v^2_{R1} & \to v^3_{L3} & \to v^3_{R1} & \to v^4_{L3} \\
v^4_{R1} & \to v^5_{L3} & \to v^5_{R1} & \to v^1_{L3} & \to v^1_{R3} \\
v^1_{C} & & & &
\end{array}
$$

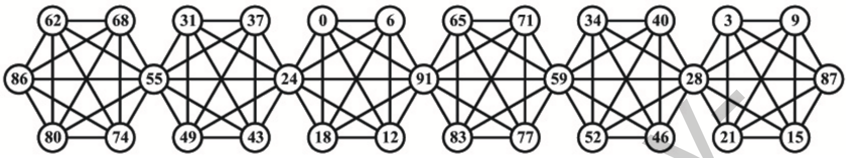

Figure 6: $\lambda(P_6(K_6)) = 91$

3. Concluding Remarks

The channel assignment problem is well explored and important as well for the establishment of radio transmitters network. The $L(2,1)$-labeling and its extension - radio labeling are greatly studied in this context. As exact radio number is known only for handful of graph families. It is very interesting to investigate bounds or exact radio number for some graphs. We consider a complex network - linear cacti and achieve exact radio number for the same.

References

[1] G. J. Chang and D. Kuo, *The $L(2,1)$-labeling problem on graphs,* SIAM Journal on Discrete Mathematics, 9(2) (1996) 309–316.

[2] G. Chartrand, D. Erwin, F. Harary and P. Zhang, *Radio labeling of graphs,* Bulletin of the Institute of Combinatorics Appllications, 33 (2001) 77–85.

[3] J. R. Griggs and R. K. Yeh, *Labeling graphs with a condition at diameter 2,* SIAM Journal on Discrete Mathematics, 5 (1992) 586–595.

[4] W. K. Hale, *Frequency assignment:Theory and applications,* Proceedings of the IEEE 68(12) (1980) 1497–1514.

[5] D. Liu, *Radio number for trees,* Discrete Mathematics, 308 (2008) 1153–1164.

[6] D. Liu and M. Xie, *Radio number of square cycles,* Congressus Numerantium, 169 (2004) 105–125.

[7] D. Liu and M. Xie, *Radio number of square paths,* Ars Combinatoria, 90 (2009) 307–319.

[8] D. Liu and X. Zhu, *Multilevel distance labelings for paths and cycles,* SIAM Journal on Discrete Mathematics, 19(3) (2005) 610–621.

[9] D. Sakai, *Labeling Chordal Graphs: Distance Two Condition,* SIAM Journal on Discrete Mathematics, 7(1) (1994) 133–140.

[10] F. S. Roberts, *T-coloring of graphs: Recent results and open problems,* Discrete Mathematics, 93 (1991), 229–245.

[11] S. K. Vaidya and D. D. Bantva, *Distance two labeling of generalized cacti,* International Journal of Mathematical Combinatorics, 2 (2012) 103–109.

[12] S. K. Vaidya and D. D. Bantva, *Labeling cacti with a condition at distance two,* Le Matematiche, 66 (2011) 29–36.

[13] S. K. Vaidya and D. D. Bantva, *Radio number for total graph of paths,* ISRN Combinatorics, **2013**(2013), 1–5.

[14] S. K. Vaidya, P. L. Vihol, N. A. Dani and D. D. Bantva, *$L(2,1)$-labeling in the context of some graph operations,* Journal of Mathematics Research, 2(3) (2010) 109–119.

[15] S. K. Vaidya and P. L. Vihol, *Radio labeling for some cycle related graphs,* International Journal of Mathematics and Soft Computing, 2(2) (2012) 11–24.

[16] D. B. West, *Introduction to Graph theory,* Prentice - Hall of India, 2001.

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ON ABEL PRIZE 2011 TO JOHN WILLARD MILNOR:

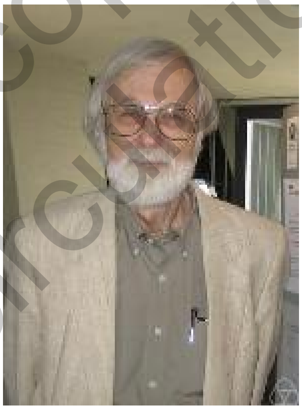
A BRIEF DESCRIPTION OF HIS SIGNIFICANT WORK.

S. S. KHARE

(Received : 06-06-2012 ; Revised : 20 - 02 - 2013)

1. Introduction

Abel Prize is one of the most prestigious international prizes in mathematics given every year since 2003. It is awarded annually by the Norwegian Academy of Science and Letters and carries a prize money of around 1 million. So far, Abel prize winners have been J.P. Serre (2003), Sir M. F. Atiyah & I. M.Singer (2004), P. T. Lax (2005), Lennart Car-leson (2006), S. R. Srinivasa Varadhan (2007),John G. Thompson & J. Tits (2008), M. L. Gromov (2009), J. T. Tate (2010).



John Willard Milnor

The Abel Prize 2011 went to Prof. John Willard Milnor, a distinguished Professor and Co-director of the Institute of Mathematical Sciences at Stony Brook University, New York.

He was born in 1931 in New Jersey, U.S.A. He completed his Ph.D at the age of 23. He wrote his first research paper "On the total curvature of knots" at the age of 18 which was published in Annals of Math. in 1950. He started his

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teaching career in Princeton University and was a full professor at the age of 29. He joined the Institute of Advance Study at Princeton in 1970. He was the first Director of the Institute for Mathematical sciences at Stony Brook University.

John Milnor has played a major role in American Mathematical Society and served as Vice-President of A.M.S. in 1975-76. His profound ideas and fundamental discoveries have largely shaped mathematical landscape of the second half of the 20^{th} century. He received Abel Prize 2011 for his pioneering discoveries in topology, geometry and algebra. His work display features of deep research, profound insight, vivid imagination, striking surprises and supreme mathematical beauty.

Milnor is widely known for his pioneering discoveries of exotic spheres, Milnor fibration, Milnor-Thurston kneading theory, Hauptvermuting, Milnor conjecture in knot theory, Algebraic K-theory, Hairy ball theorem, combinational group theory, Milnor number, holomorphic dynamics. He published more than 140 research/expository articles.

Honours and prizes:

- Field Medal in 1962 at the age of 31.
- US National Medal of Science in 1967.
- Wolf Prize in 1989.
- L.P. Steele Prize from A.M.S. in 1982 for his contribution to mathematical research.
- Steele Prize from A.M.S. in 2004 for Mathematical Exposition.
- Steele Prize from A.M.S. in 2011 for life time achievement.
- Abel Prize in 2011 for life time achievement.

John Milnor is the only Mathematician to get all the three A.M.S. Steele Prizes.

Announcing this year's Abel Prize, the committee noted:

"Milnor is a wonderfully gifted expositor of sophisticated mathematics. He has often tackled difficult, cutting edge subjects where no account in book form existed. Adding novel insights, he produced a stream of timely yet lasting works with masterly lucidity. Like an inspired music composer, who is also charismatic performer, John Milnor is both a discoverer and an expositor."

Milnor's work is so vast and so deep that it is practically impossible for me to do justice in giving brief description of his work. However, I will try to give a brief account of some of his more significant contributions. My effort will be to be as less technical as possible.

I will focus on the following:

1. Non-diffeomorphic structures on spheres.
2. Hauptvermutung.
3. Hairy ball theorem and parallelizability of S^n.
4. Total curvature of a knot.
5. Polynomial growth rate of a group.
6. Algebraic K-theory.
7. Dynamical Systems and Milnor-Thurston kneading theory.

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6. Algebraic K-theory.
7. Dynamical Systems and Milnor-Thurston kneading theory.

2. Non-diffeomorphic structures on sphere

Although Milnor is famous for many results, the one for which he is best known is the construction of seven dimensional "**exotic sphere**" *i.e.* a 7-dim differentiable manifold which is homeomorphic to S^7 but not diffeomorphic to S^7 with standard differential structure.

In order to appreciate the exotic spheres, it is desirable to understand manifold and differential structure on a manifold.

Manifold: An $n-$*dimensional* manifold M^n is a topological space such that $\forall x \in M^n, \exists$ an open neighbourhood U of x and an homeomorphism ϕ from U to an open subset of $\mathbb{R}^n.(U, \phi)$ is called a **chart** or **local coordinate system** around the point x.

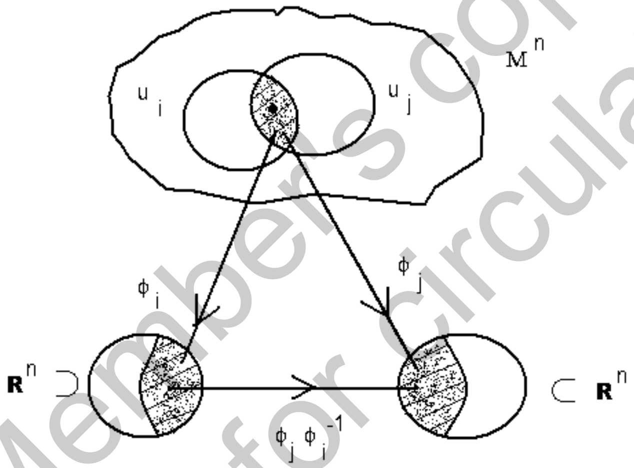

FIGURE 1. coordinate change map

Differential Structure: Consider a collection Φ of charts (U_i, ϕ_i) around each $x \in M^n$. Then Φ (called an atlas) is said to give rise to a differential structure on M^n, if whenever two charts (U_i, ϕ_i) and (U_j, ϕ_j) overlap, the map $\phi_j\phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is differentiable.

$\phi_j\phi_i^{-1}$ is called **coordinate change map**.

A maximal atlas on M^n is called a differential structure on M^n. By giving a differential structure, one can do calculus on M^n. Loosely speaking, two manifolds (shapes) are said to be **homeomorphic**, if one can be obtained from other by a continuous deformation (allowed to stretch/shrink/bend but not allowed to tear/cut/patch). For example,

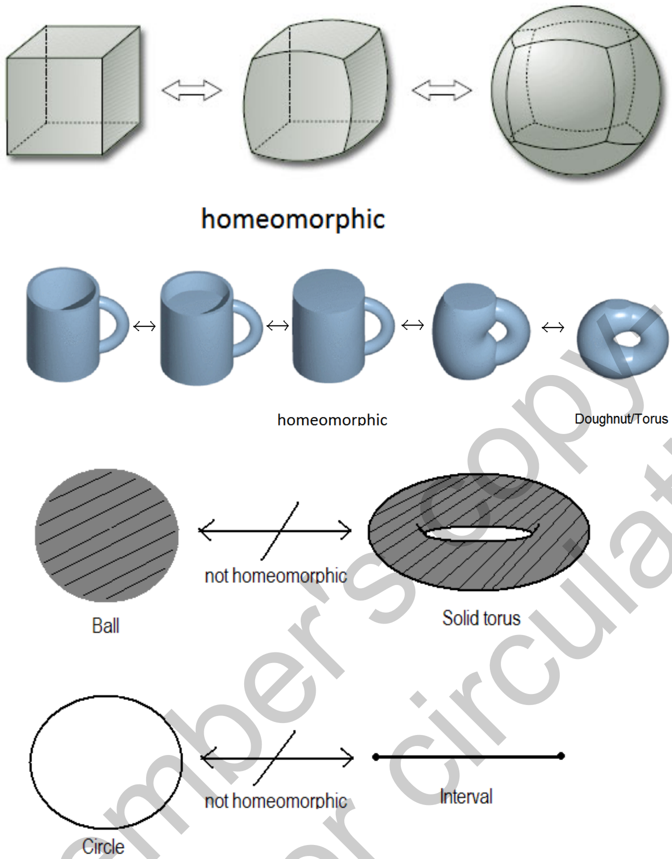

FIGURE 2. homeomorphic and non-homeomorphic spaces

Loosely speaking, two manifolds/shapes are said to be diffeomorphic if one can be deformed to other smoothly (differentially). A smooth (differentiable) deformation is stronger than continuous deformation in the sense that if one follows a steady path in one manifold (shape), then the corresponding path in the second deformed manifold (shape) should not have any sudden changes of speed or direction, folds, corners, sharp bends. Rigorously, a continuous map $f : M_1^m \to M_2^n$ is said to be smooth/differentiable, if for each $x \in M_1^m$, for each chart (U, ϕ) around x, and for each chart (V, ψ) around $f(x)$, the map $\psi f \phi^{-1} : \phi(U) \to \psi(V)$ is smooth/differentiable.

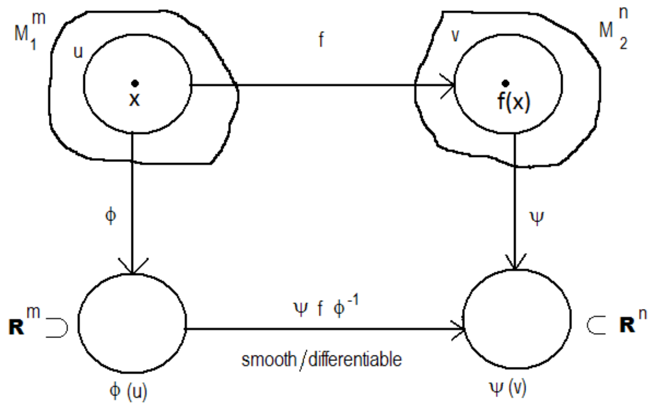

FIGURE 3. Differentiability of f

A differential manifold M_1^m is said to diffeomorphic to M_2^m, if $\exists$ a differentiable map $f : M_1^m \to M_2^n$ which has inverse f^{-1} which is also differentiable. Supose on a topological manifold M^n, two differential structures D_1 and D_2 are given. Let us denote by (M^n, D_1), the differential manifold M^n with differential structure D_1 and by (M^n, D_2), the differential manifold M^n with differential structure D_2. The two differential structures D_1 and D_2 on M^n are said to be equivalent, if there exists a diffeomorphism

$$f : (M^n, D_1) \to (M^n, D_2),$$

otherwise D_1 and D_2 are said to be non-diffeomorphic structures on M^n.

In 1956, Milnor [9] constructed an extraordinary object, a 7-dimensional oriented differential manifold M^7, which is homeomorphic to S^7 but not diffeomorphic to S^7. He called this manifold "exotic sphere".

It means that the "exotic sphere" M^7 can be continuously deformed to standard 7-sphere S^7, but the continuous deformation is not smooth/differentiable or cannot be made smooth /differentiable. One's intuition may suggest that given a continuous deformation from M^7 to S^7, one may look at points where the continuous deformation fails to be smooth and then modify the deformation at those points to make it smooth by "ironing out the kink" or by "smoothening the corners". But in higher dimension (> 6), the points where a continuous deformation fails to be smooth may form a complicated manifold of dimension ≥ 3 and due to this when one irons out the crease/kink at one place, one may simply push the crease/kink somewhere else.

In fact only in good situations, we may be successful in ironing out crease/kinks at all points without creating any more crease/kinks. Only in such situations, a continuous deformation may be converted to differential deformation. Milnor himself could not believe his finding at the beginning. His own immediate reaction to the discovery of exotic sphere was as follows:

"*When I first came upon such an example in the mid 50's, I was very puzzled and did not know what to make out of it. At first, I thought I had found a counter example to the generalized Poncaré conjecture (A closed n-dimensional manifold, which is homotopocally equivalent to S^n, is homeomorphic to S^n) in dimension 7. But a careful study showed that the new manifold constructed was homeomorphic to S^7. Thus there exists a differential structure on S^7 not diffeomorphic to the standard differential structure on S^7.*"

2.1. **Construction of exotic sphere** M^7**:** One of the first examples of an exotic sphere found by Milnor in 1956 was the following:

$$M^7 = \frac{B^4 \times S^3 \sqcup B^4 \times S^3}{\sim},$$

where $\sim$ identifies $(a, b) \in \partial(B^4 \times S^3) = S^3 \times S^3$ in one copy with $(a, a^2ba^{-1)})$ in the other copy. Here, we identify each S^3 with the group of unit quaternious. The resulting manifold has a natural differential structure.

Milnor showed that $\exists$ a differential function $f : M^7 \to \mathbb{R}$ having only two critical points which are non-degenerate. This shows that M^7 is homeomorphic to S^7 by the following Result proved by Milnor

Result 1.1: If $\exists$ a differential function $F : M^n \to \mathbb{R}$ having only two critical points which are non-degenerate, then M^n is homeomorphic to S^n.

He further showed that M^7 is not a boundary of any smooth manifold of dim. 8 with vanishing 4^{th} Betti number and has no orientation reversing diffeomorphism. To establish these results, he associated an invariant $\lambda(M^7)$ residue class modulo 7 to M^7 and showed that $\lambda(M^7) \neq 0$, which implies the above mentioned two facts. Either of these two facts about M^7 implies that M^7 cannot be diffeomorphic to S^7.

In his next paper "Differentiable structures on spheres" [10], Milnor determined minimum number of distinct differential structures on $S^{4k-1}, k = 2, 4, 5, 6, 7, 8$. His argument does not work for $k = 1, 3$. He used the following construction.

Consider $h : S^n \to [-1, -1]$ given by

$$h(y_0, ..., y_n) = y_n.$$

Clearly, h has only two critical points $(0, ...0, \pm 1)$ which are non-degenerate. Milnor constructed a diffeomorphism

$f : S^m \times S^n \to S^m \times S^n$ with $f(x, y) = (x', y')$ such that $h(y) = h(y'), \forall (x, y)$. Using the diffeomorphism f, he constructed an $(m + n + 1)$-dimensional manifold

$M(f)$ as

$$\frac{\mathbb{R}^{m+1}\times S^n \sqcup S^m \times \mathbb{R}^{n+1}}{\sim},$$

where $(tx, y) \sim (x', t'y')$. Here $t' = \frac{1}{t}, 0 < t < \infty$. Milnor proved the following:

Theorem 1.1: $M(f)$ is differentiable manifold homeomorphic to S^{m+n+1}, but not diffeomorphic to S^{m+n+1} for $m+n+1 = 4k-1,\ k = 2,4,5,6,7,8,\ m+1$ and $n+1$ being multiple of 4.

Sketch of the proof of Theorem 1.1: Consider $g : M(f) \to [-1, 1]$ given as $g(tx, y) = \frac{h(y)}{\sqrt{1+t^2}}$, in the first coordinate system $g(x', t'y') = \frac{t'h(y')}{\sqrt{1+t'^2}}$, in the second coordinate system

Milnor showed that g is a differential map having just two critical points which are non-degenerate. Therefore by the Result 1.1, he concluded that $M(f)$ is homeomorphic to S^{m+n+1}. In order to show that $M(f)$ is not diffeomorphic to S^{m+n+1}, Milnor constructed invariant $\lambda(M^n)$ as follows:

If M is a $4k$-dimensional oriented differentiable manifold without boundary having Pontrjagin classes $p_1, ..., p_k$ ($p_i \in H^{4i}(M; \mathbb{Q})$ *are specific elements which help in determining oriented bordism*), then the index $I(M)$ is defined as the index (*the number of positive terms minus the number of negative terms, when the form is diagonalized over* $\mathbb{R}$) of the quadratic form over $H^{2k}(M; \mathbb{Q})$ given by the formula

$$\alpha \to (\alpha \cup \alpha)[M],$$

where $[M] : H^{4k}(M, \mathbb{Q}) \to \mathbb{Q}$ is the homomorphism determined by the orientation of M. By Hirzebruch index theorem

$$I(M) = L_k(p_1, ..., p_k)[M],$$

where L_k is certain polynomial in $p_1, ...p_k$. For example

$$L_1 = p_1/3, L_2 = (7p_2 - p_1^2)/45$$

Hirzebruch expressed the coefficients s_k of p_k in L_k in terms of Bernoulli number B_k as follows

$$s_k = 2^{2k}(2^{2k-1} - 1)B_{2k}/(2k)!$$

For example $s_1 = 1/3, s_2 = 7/45, s_3 = 62/945, s_4 = 127/4725.$

Now, let M^{4k-1} be a differentiable oriented manifold having same rational homology as S^{4k-1} such that M^{4k-1} is boundary of a differentiable oriented $4k$-manifold W^{4k}.The inclusion map $j : W^{4k} \to (W^{4k}, M^{4k-1})$ induces homomorphism

$$j* : H^i(W^{4k}, M^{4k-1}; \mathbb{Q}) \to H^i(W^{4k}; \mathbb{Q})$$

which will be an isomorphism for $0 < i < 4k-1$, since $H^i(M^{4k-1}; \mathbb{Q}) = H^i(S^{4k-1}; \mathbb{Q}) = 0, i \neq 0, 4k-1$. Hence the Pontrjagin class $p_1, ...p_k$ of W^{4k} can be lifted to $H^*(W^{4k}, M^{4k-1}; \mathbb{Q})$ through j^*. Milnor defined the **invariant** $\lambda(M^{4k-1})$ as

$$\lambda(M^{4k-1}) = ([I(W^{4k}) - L_k({j^*}^{-1} p_1, ..., j^* p_k^{-1}, 0)[W]])(\mathrm{mod}\, 1),$$

where $\left[W^{4k}\right]$ stands for the homomorphism from $H^{4k}(W^{4k}, M^{4k-1}; \mathbb{Q})$ to $\mathbb{Q}$ associated to the orientation of W^{4k} and $I(W^{4k})$ denotes the index of the quadratic form $\alpha \to (\alpha \cup \alpha)[W^{4k}], \alpha \in H^{2k}(W, M; \mathbb{Q})$.

Milnor showed that

1. If the orientation of M^{4k-1} is reversed, then λ changes its sign
2. $\lambda(M_1^{4k-1} \# M_2^{4k-1}) \equiv (\lambda(M_1^{4k-1}) + \lambda(M_2^{4k-1}))$ (mod 1). Here $\#$ denotes the connected sum.
3. λ is invariant of h-cobordism class of M^{4k-1}.

(Two closed oriented n-manifolds M_1 and M_2 are h**-cobordant** if the disjoint union $M_1 \sqcup (-M_2)$ is the boundary of some oriented manifold W^{n+1} with both M_1 and $(-M_2)$ being deformation retract of W^{n+1}).

Milnor proved that for $m + n + 1 = 4k - 1, k = 2, 4, 5, 6, 7, 8,\ \lambda(M(f)) \neq 0$, when $m \neq n$ with $m = 4r - 1, n = 4(k - r) - 1$ or $m = n = 4r - 1$. This shows that $M(f)$ possesses no orientation reversing diffeomorphism onto itself. Therefore, $M(f)$ cannot be diffeomorphic to $S^{4k-1}, k = 2, 4, 5, 6, 7, 8$. (the reader may consult [18] for cup product, Pontragin classes, Poincare'duality, index, polynomial L_k, orientation class etc.) #

Next, Milnor proved the following interesting result which gives minimum number of distinct differential structures on $S^{4k-1}, k = 2, 4, 5, 6, 7, 8$.

Result 1.2: Given k, suppose r is an integer satisfying $\frac{k}{3} < r \leq \frac{k}{2}$. Then $\exists$ a differentiable manifold M^{4k-1} homeomorphic to S^{4k-1} for which $\lambda(M^{4k-1})$ is congruent modulo 1 to $s_r s_{k-r}/s_k$ times some integer with all prime factors less than $2(k - r)$. Further, minimum number of distinct differential structures on $M^{4k-1}, k = 2, 4, 5, 6, 7, 8$, is equal to the denominator of $s_r s_{k-r}/s_k$ after cancelling all prime factors less than $2(k - r)$ from the denominator.

Theorem 1.2: There exists at least

(i) 7 distinct differential structures on S^7
(ii) 127 distinct differential structures on S^{15}
(iii) 73 distinct differential structures on S^{19}
(iv) 23.89.691 on S^{23}
(v) 8191 on S^{27}
(vi) 31.151.3617 on S^{31}

Proof (i) $k = 2$. Choose $r = 1$. Clearly $s_1 = \frac{2^2(2-1)B_2}{2!} = \frac{1}{3}$ and $s_2 = \frac{2^4(2^3-1)B_4}{4!} = \frac{7}{45}$. Hence, $s_1 s_1/s_2 = \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{7}{45}} = \frac{5}{7}$.
Therefore minimum number of distinct differential structures on S^7 is 7.

(ii) $k = 4$. Choose $r = 2$.

$$\frac{s_2 s_2}{s_4} = \frac{\frac{7}{45} \times \frac{7}{45}}{\frac{127}{4725}} = \frac{49 \times 7}{127 \times 3}$$

Now, cancelling all the prime factors less than $2(k - r) = 4$, we get 127 as denominator. Therefore there will be at least 127 distinct differential structures on S^{15}.

(iii) $k = 5$. Choose $r = 1$, which satisfies $\frac{k}{3} < r \leq \frac{k}{2}$.

$$s_5 = \frac{2^{10}(2^9 - 1)B_{10}}{10!} = \frac{73 \times 4}{66 \times 81 \times 5}$$

Therefore,

$$\frac{s_1 s_4}{s_5} = \frac{127 \times 33}{35 \times 2 \times 73}$$

cancelling all the prime factors $< 2(k - r) = 8$ from the denominator, we get 73 in the denominator. Hence $\exists$ at least 73 distinct differential structures on S^{19}. Note that by choosing $r = 2$, which also satisfies $\frac{k}{3} < r \leq \frac{k}{2}$

$$\frac{s_2 s_3}{s_5} = \frac{31 \times 11}{73 \times 5}$$

Here, also cancelling all prime factors $< 2(5 - 2) = 6$, the denominator is still 73.

(iv) $k = 6$. Choose $r = 3$, which satisfies $\frac{k}{3} < r \leq \frac{k}{2}$.

$$s_6 = \frac{2^{12}(2^{11} - 1)B_{12}}{12!} = \frac{2 \times 23 \times 89 \times 691}{1365 \times 11 \times 25 \times 7 \times 243}$$

Therefore,

$$\frac{s_3 \times s_3}{s_6} = \frac{\frac{62}{45\times21} \times \frac{62}{45\times21}}{\frac{2\times23\times89\times691}{1365\times11\times25\times7\times243}}$$

$$= \frac{62 \times 62 \times 1365 \times 11 \times 25 \times 7 \times 243}{45 \times 45 \times 21 \times 21 \times 2 \times 23 \times 89 \times 691}$$

$$= \frac{62 \times 62 \times 195 \times 11 \times 25 \times 243}{45 \times 45 \times 3 \times 3 \times 2 \times 23 \times 89 \times 691}$$

cancelling all the prime factors less than $2(k - r) = 6$, we get denominator $23 \times 89 \times 691$.

(vi) $k = 7$. Choose $r = 3$, as $\frac{k}{3} < 3 \leq \frac{k}{2}$.

$$s_7 = \frac{2^{14}(2^{13} - 1)B_{14}}{14!} = \frac{8191}{3 \times 13 \times 3 \times 11 \times 5 \times 9 \times 7 \times 45}$$

Therefore,

$$\frac{s_4 \times s_3}{s_7} = \frac{127 \times 62 \times 13 \times 11}{7 \times 5 \times 8191}$$

cancelling all prime factors less than $2(k - r) = 8$ from the denominator, we get 8191 as denominator.

(vii) $k = 8$. One can choose $r = 3$ or 4. Let us take $r = 3$

$$s_8 = \frac{2^{16} \times 32767 \times B_{16}}{16!} = \frac{2^{16} \times 151 \times 7 \times 31 \times 3617}{510 \times 16!}$$

Therefore,

$$\frac{s_3 \times s_5}{s_8} = \frac{\frac{62}{45\times21} \times \frac{73\times4}{66\times81\times5}}{\frac{2^{16}\times151\times7\times31\times3617}{510\times16!}}$$

$$= \frac{62 \times 73 \times 4 \times 510 \times 16!}{2^{16} \times 151 \times 7 \times 31 \times 3617 \times 45 \times 21 \times 66 \times 81 \times 5}$$

After cancelling prime factors of numerator and denominator and then cancelling all the prime factors $< 2(k-r) = 10$ from the denominator, one gets $151 \times 31 \times 3617$ as denominator.
Choosing $r = 4$

$$\frac{s_4 \times s_4}{s_8} = \frac{127 \times 127 \times 510 \times (16)!}{4725 \times 2^{16} \times 151 \times 7 \times 31 \times 3617 \times 4725}$$

After cancelling prime factors of numerator and denominator and then cancelling all the prime factors $< 2(k-r) = 8$ from the denominator, one gets $31 \times 151 \times 3617$ as denominator. #

In his next paper titled "Groups of homotopy spheres - I", Milnor [11] calculated n^{th} homotopy sphere cobordism group, $n \neq 3$.

Homotopy sphere: A compact oriented and differentiable manifold M^n is defined to be a homotopy sphere, if M^n is closed and has the homotopy type of the sphere S^n.

h-cobordism: Two closed oriented and differentiable manifolds M_1^n and M_2^n are defined to be h-cobordant, if $M_1^n \sqcup (-M_2^n) = \partial W$; and M_1^n and $(-M_2^n)$ are deformation retracts of W.
The h-cobordism classes of homotopy n-sphere form an abelian group under the connected sum operation. This group is denoted by Θ_n and is called the n-th homotopy sphere cobordism group. Clearly, $\Theta_1 = \Theta_2 = 0$. The main result of his paper "Groups of homotopy spheres - I" is as follows.

Result 1.3: For $n \neq 3, \Theta_n$ is finite. The order of Θ_n, for $n = 1, 2, 3, ..., 18$, are given as follows

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Θ_n	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24

Remark 1.1: By Smale [25], every homotopy n-sphere $n \neq 3, 4$ is homeomorphic to S^n. Smale has also proved that two homotopy n-spheres, $n \neq 3, 4$ are h-cobordant if and only if they are diffeomorphic. Therefore, for $n \neq 3, 4$, the group Θ_n can be described as the set of all diffeomorphism classes of differentiable structures on topological n-spheres S^n. Thus the table given in the theorem gives number of distinct differentiable structures on $S^n, n \neq 3, 4$.

Remark 1.2: According to the generalized Poincaré conjecture, every homotopy n-sphere is isomorphic to S^n in the chosen category i.e. homeomorphism/ diffeomorphism/ piece-wise linear isomorphism. In the topological category, it is true for all n. In piece-wise linear category, it is true for all $n \neq 4$ and for $n = 4$ it is equivalent to differential category. In the differential category, it is true for $n = 1, 2, 3, 5$ and 6. For $n = 4$, it is unsettled. In dim 3, it is equivalent in all the three categories. Poincaré original conjecture (i.e. for $n = 3$) has been proved by Perelman [20] in 2002. Poincaré conjecture for $n = 4$ in the differential category is equivalent to saying that there is no exotic sphere in dim 4.

3. **Hauptvermutung**

One of the techniques to study manifolds or a curved geometrical shapes is to triangulate them. A triangulation breaks the given manifold into small pieces (triangles or tetrahedron or their higher dimensional version,) which are simple to understand. Thus some informations can be obtained about manifold from the informations about smaller and simpler part/pieces. Certainly these small pieces (triangles) need to be glued together by some admissible rules to give desired triangulation.

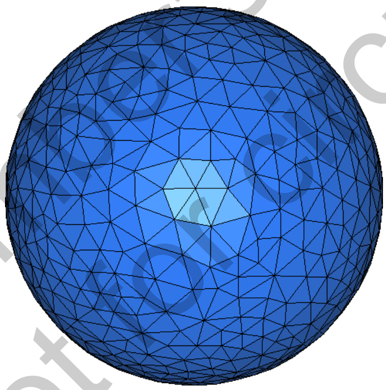

FIGURE 4. A triangulation of S^2

Given a triangulation, one can further divide triangles into smaller triangles. The resulting triangulation is called a **refinement** of earlier triangulation. A refinement of a triangulation has same essential properties as the original triangulation. There are two natural questions regarding triangulation of manifolds.

Q 1. Can every manifold be triangulated?

Q 2. Suppose a manifold is triangulated in two different ways. Do these two triangulations T_1 and T_2 necessarily have a common refinement?

Answer: In dimension 2, both questions have positive answer. Hauptvermutung is a conjecture for polyhedron/manifold. According to Hauptvermutung for polyhedron, any two simplicial complexes, which are homeomorphic, have subdivisions such that with these subdivisions the two simplicial complexes are piecewise linearly homeomorphic. According to Hauptvermutung for manifolds, if a manifold has two distinct triangulations T_1 and T_2, then T_1 and T_2 have common refinement.

Till 1960, Hauptvermutung was known to be true for manifolds upto 3 and for polyhedron of dimension upto 2. In 1961, Milnor gave counterexample to Hauptvermutung [12,21] for simplicial complexes of dimension atleast 6, which was open problem since 1908.

In order to appreciate Milnor's work on Hauptvermutung, it is essential to know simplicial complex and lens manifold.

Simplicial Complex: Loosely speaking, a simplicial complex is a topological space constructed by gluing together points, line segments, triangles, tetrahedrons and their higher dimensional counterparts (called simplex) in an "admissible" manner.

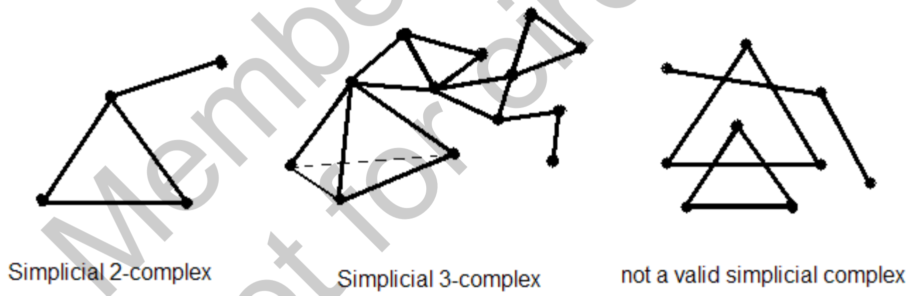

FIGURE 5. valid and invalid simplicial complexes

The highest dimension of simplex (triangle/tetrahedron) present in a simplicial complex is called the dimension of the simplicial complex. By admissible gluing, we mean

(i) any face of a simplex in a simplicial complex K must also be in K

(ii) If σ_1 and σ_2 are two simplexes in K, which are not disjoint, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Lens manifold L(p,q): Let p and q be relatively prime positive integer, $p > q$. Let $\mathbb{Z}_p$ be the cyclic group of order p generated by $\omega = e^{2\pi i/p}$. Consider the action of $\mathbb{Z}_p$ on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ given by

$$T(z_1, z_2) = (\omega z_1, \omega^q z_2).$$

The action is differentiable without any fixed point, so that the quotient $S^3/\mathbb{Z}_p$ is a differentiable manifold of dimension 3 which is denoted by $L(p, q)$ and is called **Lens manifold**.

A simplicial complex structure on Lens manifold $L(p, q)$ can be given as follows. Let P be the convex hull of the set

$$\{(\omega^j, 0) : 0 \leq j < p\} \cup \{(0, \omega^k) : 0 \leq k < p\}$$

in $\mathbb{C}^2$. The boundary ∂P of the solid polyhedron P is a simplicial complex which is homeomorphic to S^3. The simplicial complex structure on $L(p, q)$ can now be obtained by subdividing barycentrically twice and then dividing out the action of $\mathbb{Z}_p$. This simplicial structure on $L(p, q)$ is compatible with the differential structure on $L(p, q)$.

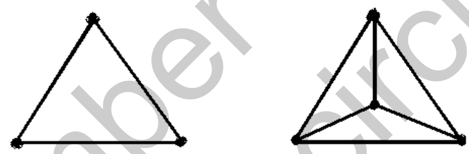

FIGURE 6. Barycentric subdivision of a triangle

If $L(p, q)$ and $L(p', q')$ are homotopically equivalent, then $p = p'$. Two lens manifolds are combinatorially equivalent $\Leftrightarrow$ they are homeomorphic. Also $L(p, q)$ and $L(p, q')$ are homotopically equivalent $\Leftrightarrow \exists$ an integer x such that $x^2 \equiv \pm qq' (\text{mod } p)$. Further, $L(p, q) \times \mathbb{R}^n$ is diffeomorphic to $L(p, q') \times \mathbb{R}^n$ for $n > 3$, if $\exists$ an integer x such that $x^2 = \pm qq' (\text{mod } p)$.

3.1. **Milnor's construction of simplicial complexes as counter example to Hauptvermutung.** Given a natural number n, consider the finite simplicial complex X_q of dimension $n + 3$ obtained from $L(7, q) \times \triangle^n$ by adjoining a cone over $L(7, q) \times \partial\triangle^n$. Milnor [12] proved the following

Theorem 2.1: For $n \geq 3$, the simplicial complex X_1 is homeomorphic to X_2. But no finite cell subdivisions of simplicial complexes X_1 and X_2 can make X_1 and X_2 isomorphic (i.e. piece-wise linearly homeomorphic).

Sketch of proof: Suppose $n > 3$. If we remove the vertex of the cone in X_q, the remaining part is clearly homeomorphic to $L(7,q) \times \mathbb{R}^n$. Therefore, X_q is homeomorphic to one point compactification of $L(7,q) \times \mathbb{R}^n$. Also $L(7,1) \times \mathbb{R}^n$ is diffeomorphic to $L(7,2) \times \mathbb{R}^n$, as $3^2 \equiv 1 \times 2 (\mathrm{mod}\ 7)$. Therefore X_1 is homeomorphic to X_2.

In order to show that there are no finite cell subdivisions of X_1 and X_2 which are isomorphic, Milnor constructed an invariant $\triangle_h(X_q) \in \mathbb{C}$ for a given homomorphism $h : \mathbb{Z}_p \to \mathbb{C}$ and proved that if some finite cell subdivisions of X_1 and X_2 make X_1 isomorphic to X_2 then $\triangle_h(X_1) = \triangle_h(X_2)$. He actually calculated $\triangle_h(X_1)$ and $\triangle_h(X_2)$ and found that they are unequal. #

Subsequently D. Sullivan [26] disproved the Hauptvermutung version for manifolds as well. In 1982, Michael Freedman [4] discovered a 4-dimensional manifold that can not be triangulated at all.

4. Hairy Ball theorem and parallelizable Spheres

At each point x of S^2, we can draw tangent plane T_x consisting of all tangent vectors at x. Consider two mutually perpendicular nonzero tangent vectors e_{1x} and e_{2x} in T_x with origin at x with assigned positive directions to e_{1x} and e_{2x}.

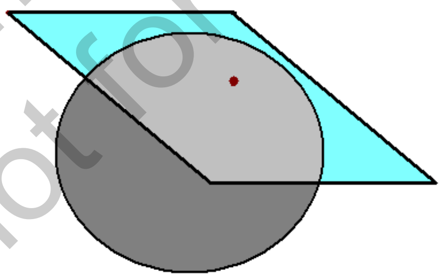

FIGURE 7. Tangent plane on S^2

Question 3.1: Can we choose such mutually perpendicular tangent vectors e_{1x} and e_{2x} with assigned positive directions at each $x \in S^2$ in such a way that they vary continuously when the points x on S^2 move continuously?

If this is true, S^2 will be said to be parallelizable. Rigorously, we say that a differentiable manifold M^n is **parallelizable**, if $\forall x \in M^n$, there exists n mutually perpendicular nonzero tangent vectors $e_{1x}, ..., e_{nx} \in$ in the tangent space T_x such that $e_{1x}, ..., e_{nx}$ move continuously as x move continuously on M^n.

Question 3.2: For what values of n, S^n is parallelizable?
In 1912, Brouwer proved that for S^2, it is not possible. In fact for S^2, it is not possible even to choose just one nonzero tangent vector at each $x \in S^2$ continuously. This result is known as **Hairy Ball theorem**. Loosely one can say that a hairy ball cannot be continuously combed without having a hair bunch at some point.

FIGURE 8. Hairy ball and hairy torus

Later, it was proved using homology and homotopy that Hairy ball theorem is true for all even dimensional sphere S^{2n} and is false for odd dimensional sphere S^{2n+1}.

Milnor [14], in 1978, gave a very elementary proof of Hairy ball theorem (without using homology and homotopy) which can be understood by a graduate student.

However, for S^1, it is clearly possible to have non-zero tangent vector $e^{i(\theta+\frac{\pi}{2})}$ at $e^{i\theta} \in S^1$ continuously. For S^3, we can think an element $(x, y, z, w) \in S^3$ in terms of unit quaternions $x + yi + zj + wk,\ i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,\ x^2 + y^2 + z^2 + w^2 = 1$. With this identification, given any unit quaternion $q = x + yi + zj + wk$ in S^3, we can choose continuously three mutually perpendicular quaternions $qi,\ qj$ and qk each of them being perpendicular to q. Hence S^3 is parallelizable.

Similarly, any element of S^7 can be considered as unit octonions/cayley numbers.
$x = x_0e_0 + x_1e_1 + ... + x_7e_7,\ x_i \in \mathbb{R}$

$e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k,$
$\varepsilon_{ijk} = 1$, if $ijk = 123,\ 145,\ 176,\ 246,\ 257,\ 347,\ 365$
$e_i e_0 = e_0 e_i = e_i$
$e_0 e_0 = e_0$
$\sum_{i=0}^{7} x_i^2 = 1$

In the same way, given any $x \in S^7$, one can choose continuously seven non zero tangent vectors in the tangent space T_x, showing that S^7 is also parallelizable.

Bott and Milnor [2], in 1958, proved the following magical result about parallelizability of S^{n-1} (Kervaire also proved this independently).

Theorem 3.1: The sphere S^{n-1} is parallelizable only for $n-1 = 1, 3$ or 7.

Sketch of proof: First he proved that $\mathbb{R}^n$ possesses a bilinear product operation without zero divisor only for $n = 1, 2, 4$ and 8. Given such a product in $\mathbb{R}^n$, he defined $\theta : S^{n-1} \to GL_n$ as $\theta(x) = f_x$, where $f_x : \mathbb{R}^n \to \mathbb{R}^n$ is given by $f_x(y) = xy$. Also, $GL_n \approx \Omega BGL_n$ (upto homotopy equivalence), where ΩBGL_n is the loop space of the classifying space of GL_n. Further, $[S^{n-1}, GL_n] \underset{iso}{\approx} [S^{n-1}, \Omega BGL_n] \underset{iso}{\approx} [S^n, BGL_n]$. Thus the map $\theta : S^{n-1} \to GL_n$ gives rise to a map from $S^n \to BGL_n$ which gives rise to a GL_n- bundle on S^n. It can be verified that the n^{th} Stiefel-Whitney class $\omega \neq 0$ for the GL_n- bundle over S^n thus obtained.

Wu proved earlier that a GL_m- bundle over S^n with $\omega_n \neq 0$ can exist only if n is a power of 2. Milnor, modified Wu's result by showing that such a GL_m- bundle over S^n with $\omega_n \neq 0$ can exist only if $n = 1, 2, 4, 8$. For this, he first proved a Lemma stating that a GL_m- bundle over S^{4k} has $\omega_{4k} = 0$ whenever $k \geq 3$.

Suppose S^{n-1} is parallelizable. Therefore, $\forall x \in S^{n-1}, \exists n-1$ linearly independent tangent vectors $v_{1x},\ v_{2x}, \ldots.., \ v_{(n-1)x}$ at x. This gives a map θ from S^{n-1} to the Stiefel manifold of n-frames in $\mathbb{R}^n$ as

$$\theta(x) = (x,\ v_{1x}, \ldots..\ v_{(n-1)x}).$$

The Stiefel manifold of n-frames in $\mathbb{R}^n$ can be identified with GL_n. This gives a map from S^{n-1} to GL_n which in turn gives rise to a GL_n- bundle over S^n with $\omega_n \neq 0$, showing that $n-1$ has to be 1, 3, 7. #

5. Total Curvature of a knot

The **curvature** of a curve at a point P on the curve is $1/r$, where r is the radius of the circle that best approximates the curve near P. It indicates the curvedness of the curve near P. The larger is the curvature at P, the more the curve is curved near P. The smaller is the curvature at P, the more flat the curve is near P.

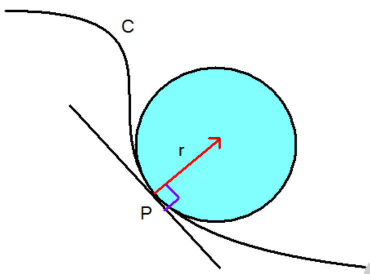

FIGURE 9. Curvature of a curve

The total curvature of a curve is the sum of the curvature at different points of the curve (more precisely through integration). For example a circle of radius a has curvature $\frac{1}{a}$ at every point, so that the total curvature of the circle is $2\pi a \times \frac{1}{a}$ $(= \int_0^{2\pi} \frac{1}{a}.ad\theta) = 2\pi$.

Knot is a closed curve in the space ($\mathbb{R}^3$), simplest knot being circle, which is often called unknot or improper knot.

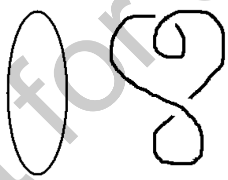

FIGURE 10. unknot

A knot which cannot be transformed to circle in an admissible way is called a **non-trivial knot**. The simplest non-trivial knot is a **trefoil-knot**.
Milnor's first paper [13] of his life was "On the total curvature of knots" published in 1950 at the age of 18 years only. German mathematician Werner Fenchel showed in 1929 that the total curvature of a closed curve always exceeds 2π. Milnor [13] and Fary [3] independently proved the following:

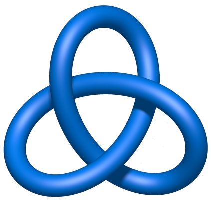

FIGURE 11. trefoil knot

Fary-Milnor Theorem 4.1: If a knot is a non-trivial knot, its total curvature always exceeds 4π.

Loosely speaking, it means that the rope of a knot (if a knot is thought to be made of rope) has to be turned at least two times around to produce a non-trivial knot. Disregarding the parts of the knot where it crosses itself, the plane projection of the trefoil knot will have total curvature 4π. In the crossing, where one branch has to lifted, there has to be some additional curvature in the direction of the paper. Due to additional curvature because of this, the total curvature of trefoil knot is little more than 4π.

Note 4.1: Converse of Milnor's theorem is not true. There are trivial knots also whose total curvature exceeds 4π.

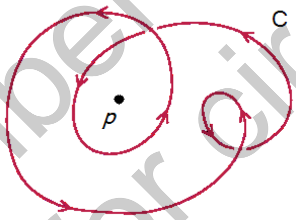

FIGURE 12. unknot

Note 4.2: The proof of Milnor's theorem on the curvature of knot does not involve very deep mathematics, but it is very elegant. The mathematical community was surprised with the extraordinary talent of an 18 years old boy giving such a wonderful result.

6. Polynomial growth of groups

Milnor's contribution to the problem of growth rate of groups is also significant. In order to understand Milnor's work on growth rate of group, we need to know nilpotent, virtually nilpotent, solvable groups and growth rate of a group.

Nilpotent group: A group G is said to be nilpotent, if G has a central series of finite length. By a **central series** of a group G, we mean a sequence $\{A_i\}$ of a normal subgroups of G such that
$\{1\} = A_0 \leq A_1 \leq ..... \leq A_i \leq ..... \leq A_n = G$ with $\frac{A_{i+1}}{A_i} \leq Z(\frac{G}{A_i})$, where
$Z(H) = \{z \in H : zh = hz,\ \forall h \in H\}$
If G is nilpotent, then the smallest n, such that G has a central series of length n, is called the **nilpotency class** of G.
Examples 5.1:

(1) Every abelian group is nilpotent.
(2) Quaternion group Q_8 is nilpotent of nilpotency class 2, as $\{1\} \subset \{1, -1\} \subset Q_8$ is central series of smallest length 2.
(3) A finite p-group is nilpotent (A p-group is a group G such that each element of G has order some power of p).

Vitually nilpotent: A group G is called virtually nilpotent if $\exists$ a nilpotent subgroup of G having finite index. All finite group, virtually abelian groups, nilpotent groups are virtually nilpotent. Semi direct product $N \rtimes H$ of a nilpotent subgroup of G and a finite subgroup is virtually nilpotent. (G is called **semi-direct product** of N and H, if N is normal subgroup, H is a subgroup of G, $G = NH = HN$ and $N \cap H = \{e\}$.)

Solvable group: A group G is called a solvable group, if its derived series

$$G \rhd G^{(1)} \rhd G^{(2)} \rhd ....$$

eventually reaches trivial subgroup $\{1\}$ after finitely many steps. Here $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$, the commutator subgroup of $G^{(i-1)}$. The least n such that $G^{(n)} = \{1\}$ is called the **derived length** of the solvable group G.

Examples 5.2:

(i) All abelian groups and all nilpotent groups are solvable.
(ii) S_3 is solvable but not nilpotent.
(iii) All groups of order < 60 are solvable because the smallest non-abelian simple group is A_5 of order 60.

Growth rate: Given a group G, let T be a symmetric set ($x \in T \Rightarrow x^{-1} \in T$) of generators. Every element g of G can be written as a product $x_1x_2.....x_k$ of elements x_i of generator set T. Loosely speaking, growth rate counts the number of elements of G that can be written as product of members of T with length $\leq n$. Let

$B_n(G,T) = \{g \in G : g = x_1.....x_k,\ x_i \in T,\ k \leq n\}$

$|B_n(G,T)|$ denotes the number of elements in $B_n(G,T)$. The equivalence class of $|B_n(G,T)|$ is called the **growth rate of** G, where equivalence relation is defined as follows. Given $a(n)$ and $b(n)$two non-decreasing positive functions

$$a(n) \sim b(n), \text{ if } \exists \text{ a constant } c \text{ such that}$$

$$a(\frac{n}{c}) \leq b(n) \leq a(cn)$$

For example, $p^n \sim q^n$, if $p, q > 1$.

Polynomial growth rate: A group G is said to have polynomial growth rate if $|B_n(G,T)| \leq c(n^k + 1)$, for some constant c and $k < \infty$. The smallest value of k is called the **order of polynomial growth**.

Exponential growth rate: A group G is said to have exponential growth rate if

$$|B_n(G,T)| \geq a^n \text{ for } a > 1.$$

A free group (non-abelian) with 2 generators a and b has growth rate 4.3^{n-1} (there are 4 choices for first letter of the word and 3 choices for each subsequent letter of the word).

It is well known that any finitely generated abelian group has polynomial growth rate. Milnor [16] and Wolf [27] extended this result for any finitely generated virtually nilpotent group. They, further, characterised finitely generated solvable group in terms of growth rate as follows

Result (Milnor-Wolf 1968) 5.1: A finitely generated solvable group has exponential growth rate $\Leftrightarrow$ it is not virtually nilpotent.
In 1981, Gromov [6] characterised finitely generated group in terms of polynomial growth rate as follows.

Result (Gromov 1981) 5.2: A finitely generated group G has polynomial growth rate $\Leftrightarrow$ G is virtually nilpotent.

7. Algebraic K-theory

A very important class of invariants were invented in the 1950's in the form of K-groups attached to a given ring which helped significantly in solving many problems in homotopy theory, surgery theory and problems of Algebraic geometry. It was pioneered by Grothendieck in the context of algebraic geometry and later by Atiyah and Hirzebruch in the context of topology.

$\mathbf{K_0(A)}$ (Grothendieck group) [1]: Given a ring A, K_0 is the functor that takes the ring A to the Grothendieck group of the set of isomorphism classes of its finitely generated projective A-modules, regarded as a monoid under direct sum. It is denoted by $K_0(A)$. If A is a field F then $K_0(F) \approx \mathbb{Z}$, since projective modules over a field F are vector spaces.

$\mathbf{K_1(A)}$ (Whitehead group) [1]: $K_1(A)$ is the abelianization of the infinite general linear group $GL(A)$, which is defined as the direct limit of $GL_n \hookrightarrow GL_{n+1}$.

$$K_1(A) = \frac{GL(A)}{[GL(A), GL(A)]}$$

Whitehead in 1950, first defined $K_1(A) = \frac{GL(A)}{E(A)}$, where $E(A)$ is the group generated by elementary matrices. Whitehead showed that $E(A) = [GL(A), GL(A)]$. When A is a field F, then $K_1(F)$ becomes isomorphic to F^*, the group of units of F.

$\mathbf{K_2(A)}$ (Milnor): It was far from clear how to define higher K-group. Finally Milnor [15], in 1970, gave proper definition of $K_2(A)$ which gave way for further higher K-group $K_n(A)$, which were subsequently given by Quillen (1974) [22].

For a given field F, Milnor defined $K_2(F)$ as follows

$$K_2(F) = F^* \otimes_z F^* / < a \otimes (1-a) : a \neq 0, 1 >,$$

where F^* is multiplicative group and $< a \otimes (1-a) : a \neq 0, 1 >$ is the ideal generated by elements $a \otimes (1-a)$ with $a \neq 0, 1$.

$K_2(F)$ can also be defined as the center of the Steinberg group $St(F)$. Unstable Steinberg group $St_r(F)$ of order r is defined as the group with the generators $x_{ij}(\lambda),\ 1 \leq i,\ j \leq r,\ i \neq j,\ \lambda \in F$ subject to the following Steinberg relations

$$x_{ij}(\lambda) x_{ij}(\mu) = x_{ij}(\lambda + \mu)$$

$$[x_{ij}(\lambda), x_{jk}(\mu)] = x_{ik}(\lambda\mu),\ i \neq k$$

$$[x_{ij}(\lambda), x_{kl}(\mu)] = 1,\ i \neq l,\ j \neq k$$

The Steinberg group $St(F)$ is the direct limit of the system $St_r(F) \to St_{r+1}(F)$. Mapping $\phi : St(F) \to GL(F)$ defined as

$$\phi(x_{ij}(\lambda)) = e_{ij}(\lambda)$$

is a group homomorphism, where $e_{ij}(\lambda)$ is the elementary matrix with λ at the $(i, j)^{th}$ place and 0 elsewhere. Since the elementary matrices generate the commutator subgroup of $GL(F)$, ϕ is onto the commutator subgroup with $Ker\phi = K_2(F)$ and cokernel $\phi = K_1(F)$. One has

$$1 \to K_2(F) \to St(F) \to GL(F) \to K_1(F) \to 1$$

an exact sequence connecting $K_1(F)$ and $K_2(F)$.

Later, Quillen [22] gave suitable definition of higher K-groups as follows. For $n > 0$

$$K_n(R) \overset{def}{=} \pi_n(BGL(R)^+),$$

where $BGL(R)^+$ is given by Quillen's plus construction.
For $n = 2$, it coincided with Milnor's K_2 group.

8. **Dynamical Systems and Milnor-Thurston kneading theory**

Apart from Milnor's contribution in topology and algebra, he made significant contribution to the field of dynamical systems also during the last 2 decades.

Given a quadratic polynomial

$$f(x) = x^2 + c, \ c \in \mathbb{C},$$

one can create a dynamical system by taking an initial point x_0, then computing the value of $x_1 = f(x_0)$, then the next value $x_2 = f(f(x_0))$, then $x_3 = f(f(f(x_0)))$ and so on. The sequence

$$x_0, x_1, x_2, \ldots.., x_n, \ldots.$$

is called the **orbit** of x_0 under the iteration of $x^2 + c$ and x_0 is called the **seed of the orbit**. One can ask
Question 7.1: Where does the sequence $x_0, x_1, x_2, \ldots.., \ldots$ end up? Does it land on a fixed point, or bounce back and forth between two points, or escape to ∞?

Example 7.1: To determine the orbit of 0.

(1) Taking $c = 1, \ x_0 = 0$, the orbit of 0 under the iteration of $x^2 + 1$ is

$$0, \ 1, \ 2, \ 5, \ 26, \ 26^2 + 1, \text{ bigger,....}$$

Thus the orbit $\to \infty$.

(2) For $c = -1$, the orbit of the seed 0 is

0, -1, 0, -1, 0, -1,.......

which bounces back and forth between 0 and -1. The orbit is cycle of period 2.

(3) For $c = 0$, the orbit of 0 is

0, 0, 0, 0,.........

Thus, for a given arbitrary $c \in \mathbb{C}$, the orbit of $x_0 = 0$ under the iteration of $x^2 + c$ escapes to ∞ or it does not escape to ∞.

Mandelbrot Set: It consists of all $c \in \mathbb{C}$ such that the orbit of $x_0 = 0$ under $x^2 + c$ does not escape to ∞.

Remark 7.1: The orbit of 0 gives information about the orbits about other points under $x^2 + c$. Therefore the orbit of 0 is important to study.

FIGURE 13. Mandelbrot Set

Mandelbrot [23] set consists of a main body, which looks like a heart (cardioid) lying on its side. It is called the **main cardioid**. Attached to the main cardioid are **myriads** (decorations) called **primary bulbs**. Every primary bulb has infinitely many smaller myriads and spokes like antennas attached.

If c lies in the interior of a primary bulb, then the orbit of 0 gets closer and closer (i.e. attracted) to a cycle of period n, which remains same for all c in the interior of a primary bulb. Such n is called the **period** of the primary bulb. The number of spokes in the antenna attached to each primary bulb is equal to the period of the bulb.

Filled Julia Set: For a given $c \in \mathbb{C}$, the collection of all seeds x_0 whose orbit does not escape to ∞ is called filled Julia set corresponding to c and is denoted by J_c.

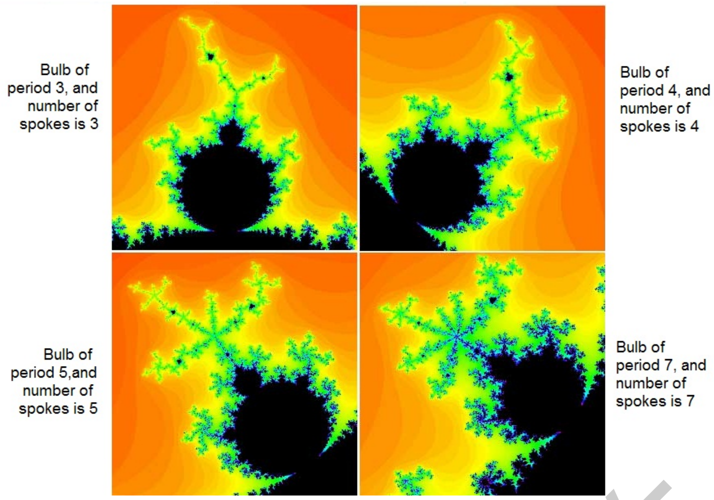

FIGURE 14. Various bulbs and their periods

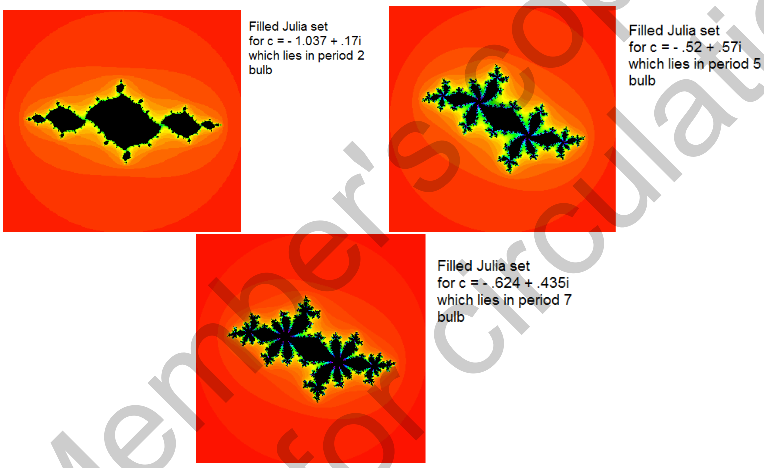

FIGURE 15. Julia Set

The following result shows the connection between Mandelbrot set and Julia set.
Result 7.1: Mandelbrot set M consists of all $c \in \mathbb{C}$ such that the filled Julia set J_c is connected. If $c \notin M$, then J_c consists of infinitely many path components each of them being single point.
Such J_c are called **Cantor sets** and look like clouds of discrete points.
The following result gives interesting property of Julia set.
Result 7.2: If c lies in a primary bulb with period n then the filled Julia set J_c is connected having infinitely many junction points and each junction point has n regions attached to it.

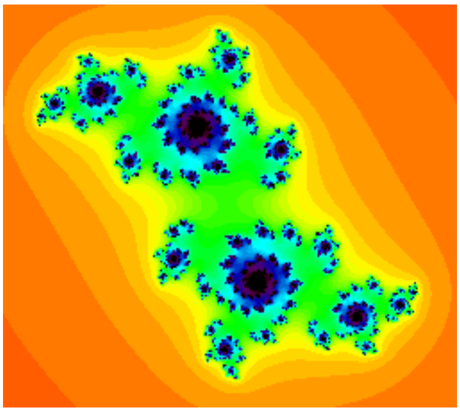

FIGURE 16. Julia set is a Cantor Set

Milnor [17] gave two conjectures regarding Mandelbrot set.

Milnors Hairiness conjecture (1989) 7.1: For a quadratic function $z \to z^2 + c$, if $c \in [-2, \frac{1}{4}]$ is a Feigenbaum parameter value, then rescaling of Mandelbrot set near c converges in Hausdorff metric on compact sets to the whole complex plane.

Milnors conjecture on self similarity of Mandelbrot set 7.2: The little Mandelbrot sets around Feigenbaum point of stationary type have asymptotically the same shape.
These conjectures were proved by Mikhail Lyubich in 1999 [8].

Milnor-Thurston Kneading Theory:
Kneading theory is a tool developed by Milnor and Thurston [19] for studying the topological dynamics of piecewise monotone self maps on the interval [0,1]. Kneading theory is most easily understood in the special case of unimodal maps. Milnor made a detailed study of iteration maps on Riemann surfaces and on higher dimensional complex structures.
A **unimodal map** $f : [0, 1] \to [0, 1]$ is a continuous map such that (i) $\exists$ some $c \in (0, 1)$ such that f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$ and (ii) $f(0) = f(1) = 0$. A prototypical example of unimodal map is given by $f(x) = \mu x(1 - x)$.
Given a unimodal map f, in order to study the iterates of f, consider $f^0 =$ identity, $f^1 = f, f^2 = f \circ f, \ldots.., f^n = f \circ f^{n-1}$. The **orbit** $O_f(x)$ of f at x is defined as the set $\{x, f(x), f^2(x), f^3(x), \ldots..\}$. The orbit $O_f(x)$ is said to be **periodic** of period n, if n is the least positive integer with $f^n(x) = x$. A periodic

orbit of $O_f(x)$ of f at x of period n is said to be **stable periodic** orbit if for some $y \in O_f(x),\ |Df^n(y)| \leq 1$. In this case, $\forall y \in O_f(x),\ |Df^n(y)| \leq 1$. If $O_f(x)$ is stable periodic of period n, then $\exists$ a neighbourhood U of $y \in O_f(x)$ such that $\lim_{m\to\infty} f^{mn}(z) = y$ for some $z \in U$ except possibly, if $|Df^n(y)| = 1$. If f has three continuous derivatives, then **Schwarzian derivative** $S(f)$ at x is defined as

$$S(f) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

If $S(f) < 0,\ \forall x \in [0,1] - c$, then f has at the most one stable periodic orbit in $(0,1)$.

A point y is said to be **attracted to a periodic point** x, if $|f^n(y) - f^n(x)| \to 0$, as $n \to \infty$. Similarly a point y is said to be **attracted to a fixed point** x, if $f^n(y) \to x$ as $n \to \infty$. A fixed point x of f is defined to be **attracting fixed point** if every point y in a small neighbourhood of x is attracted to x. A fixed point y of f is defined to be **repelling fixed point** if $\exists\ \epsilon > 0$ such that $\forall y$ with $|y - x| \leq \epsilon,\ y \neq x,\ \exists\ n > 0$ such that $|f^n(y) - x| > \epsilon$. A periodic point x is said to be **attracting periodic point** if for every point y in a small neighbourhood of $x,\ |f^n(y) - f^n(x)| \to 0$.

Singer [24] proved that if $f : [0,1] \to [0,1]$ is a continuous map with $S(f) < 0$ and with n critical points then f has at the most $n + 2$ attracting periodic points. In particular a unimodal map f with $S(f) < 0$ can have at the most three attracting periodic points.

Li and Yorke [7] proved that a continuous map $f : [0,1] \to [0,1]$ with a periodic point of period 3 has periodic point of all periods. For such map $f, \exists$ an uncountable set of points such that if x and y are two distinct points in this set then $f^n(x)$ and $f^n(y)$ are arbitrarily close to each other for some n and farther than some positive distance for other sufficiently large integer m. Thus such maps show chaotic behaviour.

Consider a unimodal map $f : [0,1] \to [0,1]$. Let

$$\Gamma = \{x \in [0,1] :\ f^i(x) = c \text{ for some } i \geq 0\}$$

clearly $\Gamma = \bigsqcup_{i\geq 0} \Gamma_i$, where

$$\Gamma_i = \{x \in [0,1] : f^i(x) = c \text{ but } f^j(x) \neq c,\ j < i\}.$$

Let γ_i denote the cardinality of Γ_i. Clearly $\gamma_i \leq 2^i, \forall i$, since the degree of the equation $f^i(x) - c = 0$ is 2^i. The **cutting invariant** of f is the formal power series

$$\gamma(t) = \sum_{i=0}^{\infty} \gamma_i t^i \in \mathbb{Z}[[t]]$$

Similarly, the **lap invariant** $l(t)$ of f is defined as

$$l(t) = \sum_{i=0}^{\infty} l_{i+1} t^i \in \mathbb{Z}[[t]],$$

where l_i is the number of monotone pieces/laps of f^i, $i \geq 1$. Since $l_i = 1 + \sum_{j=0}^{i-1} \gamma_j$, we get

$$l(t) = \frac{1 + \gamma(t)}{1 - t}.$$

Further, $s = \lim_{i \to \infty} \sup l_i^{1/i} \in [1, 2]$ is the reciprocal of the radius of convergence of the power series $l(t)$ and $\gamma(t)$. The quantity s is called the **growth number** of f. The **topological entropy** $h(f)$ of f is a non-negative number (or ∞) which measures the dynamical complexity of f. When $h(f) > 0$, we say that f is **chaotic**. Consider first n points on the orbit $O_f(x) = \{x, f(x), f^2(x), ..., \}$ of each x. Suppose x can be distinguished from y if $d(f^i(x), f^i(y)) \geq \epsilon$, for some $i < n$. A subset S of [0,1] is called (n, ϵ)**-seperated**, if $\forall x, y \in S$, $\exists\, i$ with $0 \leq i \leq n$ such that $d(f^i(x), f^i(y)) \geq \epsilon$. By $s(n, \epsilon)$, we mean the maximum cardinality of an (n, ϵ) seperated set. Roughly speaking, if $s(n, \epsilon)$ grows like e^{nh}, when ϵ is small, then **topological entropy** is h. Rigorously,

$$h(f) \stackrel{def}{=} \lim_{\epsilon \to 0} \left(\lim_{n \to \infty} \sup \frac{1}{n} . \log s(n, \epsilon) \right)$$

It can be shown that $h(f) = \log s$, where $s = \lim_{n \to \infty} \sup \gamma_i^{1/i}$. For example, consider the doubling map $f : [0, 1] \to [0, 1]$. Since f doubles the distance between nearby points, we can distinguish points which are at a distance $\epsilon / 2^n$ apart and hence $s(n, \epsilon) \approx 2^n / \epsilon$. Therefore,

$$\frac{1}{n} \log(s(n, \epsilon)) \approx \frac{n \log 2 - \log \epsilon}{n} \to \log 2$$

Hence $h(f) = \log 2$.
It is easy to see that $h(f^n) = nh(f)$, $n \geq 0$.
Let $x \in [0, 1] \smallsetminus \Gamma$. The **kneading coordinate of** x is defined as the sequence $\theta(x) \in \{+1, -1\}^{\mathbb{N}}$ given by

$$\theta_0(x) = \begin{cases} +1, \text{ if } x < c \\ -1, \text{ if } x > c \end{cases} \quad \text{and } \theta_i(x) = \theta_{i-1}(x) \theta_0(f^i(x)), i \geq 1$$

In short $\theta_i(x)$ is +1 or -1 according as f^{i+1} is locally increasing or locally decreasing at x. Consider the power series

$$\theta(x, t) = \sum_{i=0}^{\infty} \theta_i(x) t^i \in \mathbb{Z}[[t]].$$

Note that $\theta(0, t) = \dfrac{1}{1 - t}$ and $\theta(1, t) = \dfrac{-1}{1 - t}$. For any $x \in [0, 1]$, define

$$\theta(x^+) = \lim_{y\to x^+} \theta(y)$$
$$\theta(x^-) = \lim_{y\to x^-} \theta(y),$$

where limits are taken through elements y in $[0,1]\backslash\Gamma$. Define

$$\theta(x^\pm, t) = \sum_{i=0}^{\infty} \theta_i(x^\pm)t^i$$

The **kneading determinant** $D(t)$ **of** f is defined as

$$D(t) = \theta(c^-, t) \in \mathbb{Z}[[t]]$$

$D(t)$ is also called **kneading invariant of** f. It expresses the discontinuity of the kneading coordinate across $x = c$.

Example 7.2: To determine $D(t), r(t)$ and $l(t)$ for the unimodal map $f : [0,1] \to [0,1]$ given by $f(x) = 3.84x(1-x)$ for $c = \frac{1}{2}$.

The orbit of $c = \frac{1}{2}$ satisfies $f^{3i+1}(c) > c,\ f^{3i+2}(c) < c$ and $f^{3i+3}(c) < c, \forall i \geq 0$. Hence, (abbreviating +1 and -1 by + and -),

$$\theta_0(f^i(c^-)) = (+,-,++,-,++,-,++,-,++,.....)$$
$$\theta(c^-) = (+,-,--,+,+,+,-,-,-,+,+,+,....)$$

Therefore,

$$\begin{aligned} D(t) &= 1 - t - t^2 - t^3 + t^4 + t^5 + t^6 - t^7 - t^8 - t^9 + ... \\ &= (1 - t - t^2)/(1 + t^3) \end{aligned}$$

The Kneading determinant $D(t)$ and the cutting invariant $\gamma(t)$ are related as follows

$$D(t)\gamma(t) = \frac{1}{1-t}$$

Therefore, in the above example of unimodal map $f : [0,1] \to [0,1]$ given by $f(x) = 3.84x(1-x)$,

$$\gamma(t) = \frac{1+t^3}{(1-t-t^2)(1-t)} = \frac{1+t^3}{1-2t+t^3}$$

Hence,

$$\begin{aligned} l(t) &= \frac{1}{1-t}(1+\gamma(t)) = \frac{1}{1-t}\left[1 + \frac{1+t^3}{1-2t+t^3}\right] \\ &= \frac{2}{1-t}\left(\frac{1-t+t^3}{1-2t+t^3}\right) \\ &= \frac{2(1-t+t^3)}{(1-3t+2t^2+t^3-t^4)} \\ &= 2 + 4t + 8t^2 + 16t^3 + 30t^4 + 54t^5 + 94t^6 + ... \end{aligned}$$

This shows that f^7 has 94 laps.

The topological entropy of a unimodal map f is given by the following result.

Theorem 7.1: The topological entropy $h(f)$ of a unimodal map f is positive if and only if $D(t)$ has a zero in $|t| < 1$. In this case $h(f) = \log \frac{1}{r}$, where r is the smallest zero of $D(t)$ in [0,1), and $D(t)$ has no zeros in $|t| < r$.

Sketch of the proof: We know that $h(f) = \log \dfrac{1}{r}$, where r is the radius of convergence of the power series $\gamma(t)$. Clearly, $D(t)$ has radius of convergence 1. Hence, $D(t)$ and $\gamma(t)$ are analytic in $|t| < r$. Therefore $D(t)$ can have no zeros in $|t| < r$. If $\frac{1}{r} > 1$, then $\gamma(t)$ has a pole at $t = r$, since all the coefficients of $\gamma(t)$ are positive. Letting $t \to r$ from below in

$$D(t)\gamma(t) = \frac{1}{1-t}$$

gives $D(r) = 0$. #

Example 7.3 To find out the topological entropy of the unimodal map $f : [0, 1] \to [0, 1]$ given by $f(x) = 3.84x(1 - x)$ with $c = \dfrac{1}{2}$.

As per the Example 7.2,

$$D(t) = \frac{1 - t - t^2}{1 + t^3}$$

Hence $D(t)$ has a unique zero in [0,1), which is $\dfrac{\sqrt{5}-1}{2}$. Therefore the topological entropy of f is given by

$$\begin{aligned} h(f) &= \log \tfrac{1}{r} = \log \tfrac{2}{(\sqrt{5}-1)} \\ &= \log \tfrac{\sqrt{5}+1}{2}. \end{aligned}$$

Suppose $f : [0, 1] \to [0, 1]$ is a unimodal map having only finitely many periodic orbits of each period. The **Artin-Mazur zeta function** $\zeta(t)$ of f is defined as

$$\zeta(t) = \exp \sum_{k \geq 1} n(f^k) \frac{t^k}{k},$$

where $n(f^k)$ is the number of fixed points of f^k. One of the most important results of Milnor and Thurston [19], in kneading theory, is a relationship between zeta function and the kneading determinant $D(t)$ given as below.

Result (Milnor, Thurston, 1988) 7.3: Suppose f is a differentiable unimodal map with finitely many periodic orbits of each period. Suppose all but finitely many of its periodic orbits are unstable. Then

$$(\zeta(t))^{-1} = D(t) \prod_P K_P(t),$$

where the product is over the periodic orbits P of f which are not unstable together with the fixed point 0 and the polynomial $K_P(t)$ are as follows

(i) For the fixed point $0, K_P(t) = (1 - t)^2$, if the fixed point is stable (on the right) and $K_P(t) = (1 - t)$ otherwise

(ii) For each other non-unstable periodic orbit P of period k,
   (a) $K_P(t) = 1 - t^k$, if P is stable on one side only
   (b) $K_P(t) = 1 - t^{2k}$, if P is stable and the derivative of f^k is negative at the points of P
   (c) $K_P(t) = (1 - t^k)^2$, if P is stable and the derivative of f^k is non-negative at the points of P

In particular $\zeta(t)$ has radius of convergence $\gamma = \frac{1}{s}$, where s is the growth number of f.

Example 7.4: To determine the number of fixed points $n(f^k)$ of f^k for the unimodal map $f : [0, 1] \to [0, 1]$ given by $f(x) = 3.84x(1 - x)$ with $c = \frac{1}{2}$.
As per the Example 7.2, we have

$$D(t) = \frac{1 - t - t^2}{1 + t^3}.$$

clearly, f has stable period 3 orbit at the points where f^3 has negative derivative (and there are no other stable periodic orbits since f has negative Schwarzian derivative). The fixed point at $x = 0$ is unstable, because $|Df(0)| > 1$. Hence $K_P(t) = 1 - t$ for the fixed point 0. Therefore,

$$\begin{aligned}(\zeta(t))^{-1} &= \frac{1 - t - t^2}{1 + t^3}(1 - t)(1 - t^6) \\ &= 1 - 2t + 2t^4 - t^6\end{aligned}$$

Hence,

$$\begin{aligned}\frac{t\zeta'(t)}{\zeta(t)} &= t.\frac{d}{dt}\left(\frac{1}{1 - 2t + 2t^4 - t^6}\right)(1 - 2t + 2t^4 - t^6) \\ &= (2t - 8t^4 + 6t^6)(1 - (2t - 2t^4 + t^6))^{-1} \\ &= 2t + 4t^2 + 8t^3 + 8t^4 + 12t^5 + \ldots\ldots .\end{aligned}$$

Since $\zeta(t) = exp \sum_{k\geq 1} n(f^k)\frac{t^k}{k}$,

$$\log \zeta(t) = \sum n(f^k)\frac{t^k}{k},$$

so that

$$\begin{aligned}\frac{t\zeta'(t)}{\zeta(t)} &= t\sum_{k\geq 1} n(f^k)t^{k-1} \\ &= \sum_{k\geq 1} n(f^k)t^k.\end{aligned}$$

This implies that

$$\sum_{k\geq 1} n(f^k)t^k = 2t + 4t^2 + 8t^3 + 8t^4 + 12t^5 + \ldots .$$

From this equation, one can determine $n(f^k)$ for each k.

References

[1] 1 Bass, H., *Algebraic K-theory,* Benjamin, 1968.

[2] 2 Bott, R. and Milnor, J. *On parallelizable of spheres,* Bull. Amer. Math. Soc. 64, 1958.

[3] 3 Fary, I., *Sur la courbure totale d'une courbe gauche faisant un noeud,* Bulletin de la Société Mathématique de France 77, 1949, 128–138.

[4] 4 Fredman, Michael, *The Topology of four dimensional manifolds,* J. Differential Geometry Vol. 17, Number 3 (1982), 357–453.

[5] 5 Gowers, W.T. *The work of John Milnor,* A popular presentation of the work of John Milnor immediately following the formal announcement of Abel Prize 2011 to John Milnor.

[6] 6 Gromov, M., *Groups of polynomial growth and expanding maps,* publications Mathematiques, I. H. E. S., 53, 1981.

[7] 7 Li, T.Y. and Yorke, J.A., *Period three implies chaos,* Amer. Math. Monthly Vol. 82, 1975, 985–992.

[8] 8 Michail Lyubich, *Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness conjectute,* Annals of Mathematics, Vol. 149, 1999, 319–420.

[9] 9 Milnor, J., *On manifolds homeomorphic to the 7-sphere,* Ann. of Maths. Vol. 64, No. 2, September 1956.

[10] 10 Milnor, J., *Differentiable Structures on spheres,* American Journal of Mathematics, 1959.

[11] 11Milnor, J., *Groups of homotopy spheres: I,* Annals of Math. Vol 77, No. 3, May, 1963.

[12] 12 Milnor, J., *Two complexes which are homeomorphic but combinatorially distinct,* Ann. of Math. Vol. 74, No. 3, Nov., 1961.

[13] 13 Milnor, J., *On the total curvature of knots,* Ann. of Math. Vol. 52, No. 2, 1950.

[14] 14 Milnor, J., *Analytical proof of the "Hairy Ball Theorem" and Brouwer fixed point theorem,* The American Mathematical Monthly, Vol. 85, No. 7, 1978.

[15] 15 Milnor, J., *Algebraic K-theory and quadratic forms,* Inventiones Mathematicae 9, 1970.

[16] 16 Milnor, J., *Growth of infinitely generated solvable groups,* J. Diff. Geom., (2), 1968, 447–449.

[17] 17 Milnor, J., *Self similarity and hairiness on the Mandelbrot set, Computers in Geometry and Topology,* Lect. Notes in Pure and Applied Math. 114 (1989), 211–257.

[18] 18 Milnor, J. and Stasheff, J. D., *Characteristic classes,* Annals of Mathematics Studies, No. 76, Princeton University press, 1974.

[19] 19 Milnor, J. and Thurston, W., *On iterated maps of interval. Dynamical system,* Lecture Notes in Math., 1342, 465-563, Springer, Berlin, 1988.

[20] 20 Perelman, G. , *The entropy formula for the Ricci flow and its geometric applications,* arXIV: math.DG/0211159 v1, November 11, 2002.

[21] 21 Plitt, B., *Milnor's counterexample to the Hauptvermutung* (Seminar talk).

[22] 22Quillen, D., *Higher Algebraic K-theory 1,* Lecture Notes in Math 341, Berlin, New York SV, 85–147.

[23] 23 Robert L. Devaney , *Unveiling the Mandelbrot set,* plus.maths.org.

[24] 24 Singer, D., *Stable orbits and Bifurcations of Maps of the Interval,* SIAM J. Appl. Math., Vol. 35, No. 2 (1978), 260–267.

[25] 25 Smale, Stephen , *Generalized Poincaré's conjecture in dimensions greater than four,* Ann. of Math, (2), 74, 1961, 391–406.

[26] 26 Sullivan, D., *Triangulating Homotopy Equivalence,* Thesis, Princeton University 1966.

[27] 27 Wolf J. A., *Growth of finitely generated solvable groups and curvature of Riemanniann manifolds,* J. of Diff. Geom., 2(1968), 421–446.

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78^{TH} IMS CONFERENCE : A BRIEF REPORT

The 78^{th} Annual Conference of the Indian Mathematical Society was held at Banaras Hindu University (BHU), Varanasi - 221 005 (UP), during January 22 - 25, 2013 under the Presidentship of Prof. Huzoor H. Khan, Department of Mathematics, Aligarh Muslim University, Aligarh - 202002 (UP), India. The conference was attended by more than 260 delegates from all over country including foreign delegates.

The Chief Guest Prof. Gopal Prasad (Bott Chair Professor, University of Michigan, USA) inaugurated the Conference and welcomed all the delegates at Varanasi. The Inaugural function was held in the forenoon of January 22, 2013 at 11.00 a.m. in Swatantrata Bhawan and was presided over by Prof. Huzoor H. Khan, President of the Indian Mathematical Society. Prof. A. K. Srivastava, Head, Department of Mathematics, Faculty of Science, Banaras Hindu University, and Local Organizing Secretary, 78th IMS Conference, delivered the welcome address. On behalf of the General Secretary Prof. V. M. Shah (who was absent), Prof. N. K. Thakare, Ex. Vice-Chancellor, North Maharashtra University, Jalgaon and the Editor, Journal of the Indian Mathematical Society, spoke about the Society and on behalf of the Society expressed his sincere and profuse thanks to the host for organizing the Conference. Prof. Satya Deo, Academic Secretary of the Society, reported about the academic programmes of the conference. Prof. N. K. Thakare also reported about (i) A. Narasinga Rao Memorial Prize and (ii) P. L. Bhatnagar Memorial Prize - the details of these Prizes are given later on.

Dr. S. B. Nimse, Vice Chancellor, SRTM University, Nanded, was felicitated for his getting elected to the Office of the **General President** of the Indian Science Congress Association for 2015.

Prof. Huzoor H. Khan delivered his Presidential address (General) on " *Importance of Mathematics at School Level*". The function ended with a vote of thanks by the host.

The Academic Sessions of the conference began with the Presidential Technical Address by Prof. Huzoor H. Khan, the President of the Society, on "*Growth and Polynomial approximation of entire functions - A brief survey*" which was presided over by Prof. N. K. Thakare, the senior most past president of the Society present in the Conference.

Two Plenary Lectures, **Four** Memorial Award Lectures and **Sixteen** Invited Lectures were delivered by eminent mathematicians during the conference. Besides this, there were **Five** symposia on various topics of mathematics.

A Paper Presentation Competition for various IMS and other prizes was held (without parallel sessions) in which 20 papers were presented. The result of this Competition is declared in the Valedictory Function and appears later on in this report. In all about 100 papers were accepted for presentation during the conference on various disciplines of mathematics and computer science.

A Cultural Evening was also arranged on January 23, 2013. The conference was supported by the NBHM, DST, CSIR, DRDO and UGC (through its unassigned grant).

The Annual General Body Meeting of the Indian Mathematical Society was held at 12.15 p.m. on January 25, 2013 in Seminar Complex, Faculty of Science, of the Banaras Hindu University, Varanasi, which was followed by the Valedictory Function in which the IMS and other prizes were awarded to the winners of the Paper Presentation Competition. The minutes of the General Body Meeting are circulated through the IMS News Letter No. 30 of August 2013. It was also announced that Prof. Geetha S. Rao, (Retired. Professor, Ramanujan Institute of Advanced study in Mathematics, Chennai) will be the next President of the Society with effect from April 1, 2013 and that the next Annual Conference of the Society will be held at the Rajagiri College of Engineering and Technology, Cochin, Kerala, during December 28-31, 2013 with Prof. Vinod Kumar, Head, Department of Mathematics of the College as the Local Organizing Secretary.

The details of the academic programme of the Conference are as follows.

Details of the Plenary Lecture:

1. Gopal Prasad (Bott Chair Professor, University of Michigan, USA) delivered a Plenary Lecture on "*APPLICATIONS OF NUMBER THEORY TO GEOMETRY*". Chairperson : Satya Deo.

2. V. Srinivas (T I F R, Mumbai) delivered a Plenary Lecture on "*BLOCH-BELINSON CONJECTURES*". Chairperson : J. K. Verma.

Details of the Memorial Award Lectures:

1. The 26^{th} P. L. Bhatnagar Memorial Award Lecture was delivered by Basudeb Datta (I. I. Sc., Bangalore) on " *Triangulations of Projective spaces*". Chairperson : S. D. Adhikari.

2. The 23^{rd} V. Ramaswami Aiyar Memorial Award Lecture was delivered by Ajay Kumar (University of Delhi, Delhi) on " *From Fourier series to Harmonic analysis on locally compact groups*". Chairperson : Geetha S. Rao.

3. The 23^{rd} Srinivasa Ramanujan Memorial Award Lecture was delivered by J. K. Verma (IIT Bombay, Mumbai) on "*Linear Diophantine equations and Stanely's solution of the Anand-Dumir-Gupta Conjecture*". Chairperson : Ravi Kulkarni.

4. The 23^{rd} Hansraj Gupta Memorial Award Lecture was delivered by B. N. Waphare (Pune University, Pune) on "*Dismantlability and Reducibility in Discrete Structures*". Chairperson : J. R. Patadia.

Details of various prizes awarded by the Society :

1. A. Narasinga Rao Memorial Prize.

This prize for 2010 was not recommended by the Committee appointed for the purpose, and hence is not awarded to any one.

2. P. L. Bhatnagar Memorial Prize.

For the year 2012, this prize is presented **jointly** to **Mr. Debdyuti Banerjee** (Hoogly, 033-26324711, *chandana.snbv@gmail.com*) and **Mr. Prafulla Susil Dhariwal** (Pune, 0-9096542402, *prafullasd@hotmail.com*) - the top scorers from the Indian Team with 28 points each at the 53rd International Mathematical Olympiad (IMO) that was held during July 04-16, 2012 at Mar del Plata, Argentina.

3. Various prizes for the Paper Presentation Competition:

A total of 20 papers were received for Paper Presentation Competition for Six IMS Prizes, AMU Prize and VMS Prize. There were 14 entries for six IMS prizes, 05 entries for the AMU prize and 01 entries for the VMS prize.

Professors Satya Deo (Chairperson), S. S. Khare, J. R. Patadia, Nita Shah and Sunita Gakkhar were the judges. Following is the result for the award of various prizes :

IMS Prize - Group-1:	03 Presentations. Prize is awarded to: Leetika Kathuria, Chandigarh, *kathurialeetika@gmail.com*
IMS Prize - Group-2:	01 Presentation. Prize not awarded.
IMS Prize - Group-3 :	No presentation.
IMS Prize - Group-4:	02 Presentations. Prize not awarded.
IMS Prize - Group-5:	01 Presentation. Prize not awarded.
IMS Prize - Group-6:	02 Presentations. Prize is awarded to: B. B. Upadhyay, Varanasi, *bhooshanbhu@gmail.com*
AMU Prize :	03 Presentations. Prize is awarded to: M. N. Shobha, Karnataka, *shobha.me@gmail.com*

V M Shah Prize : 01 Presentation. Prize is awarded to :
V. N. Misra, Surat, *vishnunarayanmishra@gmail.com*

Details of Invited Lectures delivered :

- S. D. Adhikari (HRI, Allahabad) on *"Some early Ramsay type theorems in combinatorial number theory: some generalizations and open questions".*
- S. B. Nimse (S.R.T.M. University, Nanded) on *"Riemann Zeta function and Riemann hypothesis".*
- Zafar Hasan (AMU, Aligarh) on *"Analogies between electromagnetism and gravitation in relativity theory".*
- P. Sundaram (Pollachi, Tamilnadu) on *"Recent trends in intuitionistic fuzzy topology".*
- Sarita Thakar (Shivaji University, Kolhapur) on *"The general solution of Lane Emden type equation".*
- Nita Shah (Gujarat University, Ahmedabad) on *"Transportation problem using Tonelli Lagrangians".*
- Vijay Gupta (Netaji Subhash Inst. of Technology, New Delhi) on *"Recent developments on linear operators ".*
- Arindama Singh (IIT Madras, Chennai) on *" Elongated Diagonalization'.*
- M. Srivastava (R. D.University, Jabalpur) on *"Some results on intuitionistic fuzzy topological spaces".*
- Pankaj Jain (South Asian University, Delhi) on *"Properties and inequalities in certain generalized Lorentz spaces".*
- C. S. Dalawat (HRI, Allahabad) on *" Some refinements of Serre's mass formula in prime degree ".*
- Asma Ali (AMU, Aligarh) on *" Commuting maps and their generalizations".*
- Santwana Mukopadhyay (BHU, Varanasi) on *"Thermo-elasticity with phase lags Future trends and challanges ".*
- Debjit Hazarika (Tezpur University) on *"On some recent applications of topology".*
- Harish Chandra (BHU, Varanasi) on *" Local Spectral theory and composition operators".*
- Jorge Jimenez Urroz (Madrid, Spain) on *" ".*

Details of the Symposia organized :

1. On **"Algebraic and Differential Topology"** Convener: S. S. Khare (NEHU, Shillong).

Speakers:

- A. R. Shastri (IIT Bombay, Powai, Mumbai) : *"Brouwer's invariance of Domain : A short proof"*.
- S. S. Khare (NEHU, Shillong) : *"Finite group action, Bordism and fixed point set: Recent trends"*.
- H. K. Mukherji (NEHU, Shillong) : *"On homology and dynamical systems"*.
- K. Gangopadhyay (IISER, Mohali) : *"Classification of complex hyperbolic isometries"*.
- K. Ramesh (IIT Bombay, Powai, Mumbai) : *"Detecting exotic structures on locally symmetric space"*.

2. On **"Lie Theory".** Convener: Punita Batra (HRI, Allahabad).

Speakers:

- Ravi Kulkarni (IIT Bombay, Powai, Mumbai): *"z-classes in groups and geometry"*.
- P. K. Ratnakumar (HRI, Allahabad): *"An uncertainty principle on Heisenberg group"*.
- S. Senthamarai Kannan (CMI, Chennai): *"An analogue of Bott's theorem for Schubert varieties"*.

3. On **"Wavelets and their applications".** Convener: S. K. Upadhyay (BHU, Varanasi).

Speakers:

- R. S. Pathak (BHU, Varanasi) : *"Properties of wavelet transforms"*.
- B. N. Mandal (BHU, Varanasi) : *"Evaluation of singular integrals by using Daubechies real function"*.
- M K Ahmad (BHU, Varanasi): *"Construction of wavelet frames and its applications"*.
- K. N. Rai (BHU, Varanasi): *"Wavelets and its applications"*.
- Romesh Kumar (BHU, Varanasi): *"Weighted compostion operators"*.
- S. K. Upadhyay (BHU, Varanasi): *"Watson wavelet transform"*.

4. On **"Approximation theory and Optimization".** Convener: Q. H. Ansari (AMU, Aligarh).

Speakers:

- P. Shunmugaraj (IIT Kanpur, Kanpur): *"Convergence of Slices and proximinality"*.

- V. Indumathi (Pondicheri University) : *"Semi continuity properties of metric projection".*
- P. Veeramani (IIT Madras, Chennai): *"Best proximity points".*
- S. K. Misra (BHU, Varanasi): *"On approximately star shaped functions and approximate variational inequalities".*
- Q. H. Ansari (AMU, Aligarh): *"Triple heiraarchical variational inequalities".*

5. On **"Biomathematics".** Convener : A. K. Misra (BHU, Varanasi).

Speakers:

- Sikha Singh (IIT Kanpur, Kanpur) : *"Mathematical modeling malaria: effects of ecological factors".*
- Sunita Gakkhar (IIT Roorkee, Roorkee) : *"Some models in eco epidemiology".*
- O. P. Misra (Jiwaji University, Gwalior): *"Modelling transmission dynamics of Influenza in two economic groups".*
- Joydeep Dhar (IITM, Gwalior): *"Mathematical modeling and prediction of the growth of mosquito vector : A real case study".*
- A. K. Misra (BHU, Varanasi): *"Modelling the dynamics of algal bloom and its control: effect of time delay".*

Panel Discussion : On **"Current trends in Mathematical Research in India".**

The Panel discussion was held in the Conference Room of the IMSC building of BHU. The following mathematicians were present and participated:
Professors N. K. Thakare, Satya Deo (Convener and Moderator), Huzoor Khan, S. B. NImse, M. K. Ahmad, Ajay Kumar, Ravi Kulkarni, J. K. Verma, Harish Chandra, C. S. Dalawat, A. R. Shastri, Sarita Thakar, Pramod Kumar, P. K. Ratnakumar, B. N. Waphare, S. N. Bagchi, Bankateshwar Tiwari, Krishnendu Gangopadhyay.

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I , N. K. Thakare, hereby declare that the particulars given above are true to the best of my knowledge and belief.

N. K. THAKARE
Signature of the Publisher

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CONTENTS

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