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Edited by

J. R. PATADIA

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UNSINKABLE MATHEMATICS THAT IS SINKING*

P. K. BANERJI

Have we anytime enquired or ever questioned ourselves as how fair do we teach, have we even questioned how fair is our research. Both the teaching and research are two parallel streams, yet sometimes we draw a separable line between them. Both to be administrated properly, require expression, orientation, and innovation. We are here to serve the scientific community and care for it, without which our progress at the best be the halting and our achievements, of routine nature.

The convention, that the President of the Indian Mathematical Society should address the annual gathering of mathematicians, enjoins me to stand before you today, and also conventionally, the President elect to speak on a non-specialized field, the general address. I propose to address you on a subject (or theme) with which we all are connected and concerns us all. At the same time, is in conformity with the main objective of the Society, namely,*to advance and promote cause of mathematics in India.* I am convinced that without addressing ourselves to what I may call the *characteristic value* of academics and academic development, our progress can at best be halting and our achievements, of a routine nature. The medias both print and electronic have hardly designed or have ever spoken for either teaching or research. For some time the news papers advertised and television aired the slogan on environment saying **only eleven hundred twenty four tigers are left in India, it is a matter of concern.** Have you ever heard one saying or asking *how many good teachers and researches are left in India and we are seriously concerned.* Teachers' Training Institutes and B. Ed. Colleges are mushrooming in the country, central government and state governments are approving them and recognizing them without any plan and objective. In schools, teachers do not teach, they narrate the same as can be found in certain books

* The text of the Presidential Address (General) delivered at the 77th Annual Conference of the Indian Mathematical Society held at Swami Ramanand Teerth Marathwada University (SRTM University), Vishnupuri, Nanded - 431 606, Maharashtra, during the period December 27 - 30, 2011.

(most of the time verbatim) which too are not among classical names, teachers are only finishing the courses without teaching mathematics. Similar are the cases in colleges and universities (this is commonly said). Governments are opening new universities, upgrading some and giving recognitions to many. Vice-President of the Planning Commission, Motek Singh Ahluvaliya opined the same during his inaugural speech at the West Zone Vice-Chancellor's conference in Udaipur, Rajasthan, on 29 July 2011. He said, quote, instead of opening new universities and putting efforts to make it to international level, efforts should be made to convert existing potential universities as centre of excellence. Economic or financial burden should not be on students and guardians. A similar concept was expressed by Prof. Peeyush Chandra (IIT, Kanpur), in his Presidential Address (General) at the 75$^{\text{th}}$ *Annual Conference of the IMS during December 2009.*

If you are not teaching, good students will not born and so you cannot have good qualitative mind which results into a vacuum in advance studies. Teaching and research both need orientation, clarity of ideas, and innovation. Both require proper expressions, both vocal and written, which require command on language and need the art of explanation. Research is a profession and full time occupation, which is required to be injected into every student we meet as a teacher and to your own research scholars. According to a recent (February 2011, published April 2011) report of the Global Innovation Index (GII), Geneva, despite acquiring the title of rising star of economic and financial management, India is far behind many countries of the world in research and new findings. The report of GII is that from the industrial field to every domain in the society, India has gone down in index from 56 to 62, Switerland is at number one. Whereas Singapore, Hong Kong, and Finland are placed second, third, fourth and fifth, respectively. Mr. N. R. Narainmurthy, Advisor, Infosis, in an interview (at Ahemdabad) on 19 July 2011, said that IITs do not have the glamour that used to be there during 60s and 70s. There used to be new researches those years, but now they remain as coaching institutes. He was talking at IIT, Gandhinagar. I personally feel and I am sure all of you will agree (at least partially, if not completely) that teaching is an art, a very natural phenomenon with which some are born while others devote to acquire the art, which is exactly similar to a painter who draws his imaginations on the canvas. Despite this we can create good teachers from among our own scholars, for which we will have to be ourself an artist and know avenues to build teachers.

Do not separate these two entities, they have one common value, that is, value for orientation. The hairline difference is (may be thought of) that in teaching we (usually) explain the already existing concepts through our expertization, whereas in research we attempt to establish a relation between some already existing concepts and our innovated ideas; that is how so many theories are invented. Searching for some new and lucid techniques for explaining already existing mathematical concepts, while teaching, will raise awareness among students and in this way they will experience an avenue leading them later to research. In connection with higher education and research, on 13 July 2011, a survey report that is conducted in Jiwaji University, Gwalior, was published. The survey report is alarming and such is the situation in the country. There is a great decline in the traditional research, which according to educationists, is the temptation to earn more in less time and thus students, after graduation opt for professional courses (which earn money for them). Research in Physics, Mathematics, Chemistry, and in biological (or life sciences) sciences, these days, do not attract students; instead they prefer commerce, arts, and social sciences. The report, if read within lines, reflects only one aspects regarding decline in research in science, and that is, they do not foresee job opportunities and thereby feel insecured. Both the central and the state Governments are not encouraging appointments and the higher education.

Score of scholars come to conference to present papers for which they are allowed maximum ten minutes. They (maximum of them) come like *orphans* (it sounds harsh, yet true) without a proper guidance from the supervisors, who (otherwise) have not guided them or trained them to present their paper, through overhead projector or power point or on blackboard, they have not been supervised or directed adequately regarding correct language, proper pronunciation, proper audition and speech, each of which are the prerequisites for communication. If we do not communicate, we cannot teach and do research. They are not trained how to present a paper of 15 pages in 10 minutes or (on the other hand) a 2 pages paper in 10 minutes. Train them today to become one of you in years to come.

I do not believe in Power Point presentation, I do not believe in Overhead Projector presentation if anyone of them is used to project and be read verbatim. However, they are best used to draw an outline of the lecture or shows a rigid calculation which you have, otherwise, short time to write and explain.

A catastrophic storm, called impact factor, is fast approaching to wreck mathematics. The damage has already begun which is cutting the coast line of mathematics. The impact factor has been widely adopted as a proxy for the quality of journal. It is used by libraries to guide purchase and renewal decisions, by researchers deciding where to publish, by promotion committees labouring under the assumption that publication in a higher impact factor journal represents better work, and by editors and publishers as a means to evaluate and promote their journals. It has been widely criticized on variety of grounds, see Arnold, D. N. and Fowler, K. K. and references therein. Arnold and Fowler writes, impact factor as a flawed statistics. For one thing, the distribution of citations among papers is highly skewed, thus the mean for the journal tends to be misleading. For another, the impact factor only refers to citations within the first two years after publication (a particular serious deficiency for mathematics, in which around 90 percent of citations occur after two years). In this connection, I refer to my own lecture text, *Why should we do Research and What Should we do in Research : The Research Methodology,* which was delivered at the National Meet of Research Scholars, a DST Orientation Programme, held at IIT Roorkee in December 2009. I would also like to advise you all to refer to a survey article, *Impact Factor and Mathematics : A Debate,* by our research group, which was submitted to the Open Mathematics Journal, Bentham Science Publishers, USA.

Some time in December 2009, during my one day visit, I along with Prof. Rajiv Srivastava and Prof. Sunder Lal of The Institute of Basic Sciences, B. R. Ambedkar University, Agra, discussed the current decline of mathematics and thought we do something to motivate others to contribute their ideas to save mathematics from falling. In what resulted is the *text of the mail* that I sent to some mathematicians in the country and I did receive affirmative responses, while many never responded.

Dear Colleague

Around the world both mathematics teaching and research are flourishing with the advancing mathematical ideas, but such is not happening in India and even if a part of such is happening that is/are restricted to small section. Mass is not entailed, enthusiasm is lost, how can we endure working for mathematics in a situation as it is in the present time. During the last forty fifty years we have seen how classical problems have been solved and new theories are advented and moreover, mathematics and physics have rediscovered each other. Young researchers and teachers are either not aware of such advancement or they do not have interest to do so or else those who have the orientation are not secured. They feel hesitant and do not receive support for career in mathematics. Even as a research scholar, no one feel secured and we supervisors, after helping them to earn the degree of Ph. D. remain unanswered to get them obtain a job or furthering their research.

In India mathematics is considered a last choice for students, not because it has lost its glamour or usefulness, but because of the attitude of negligence of the government which reflected the indian society to believe it. Particularly, the state government run colleges and universities are in poor state, which is very pathetic. The state-of-the-art of mathematics raises concern among us and many have also, at the same time, did think to do something, but could not be done. Even now, though late we are, we can contribute together much to save mathematics instead of leaving it on governmental ventilator, do not let it die. This mail is the result of many verbal discussions of the writer of this mail with several prominent indian mathematicians, who have a common feeling to enhance mathematics teaching and

research . I, therefore, appeal to you all, as an individual, to join this venture. No one alone can complete this thinking. We will succeed, may not be over night but very soon, so let us begin, let us bring back the respect for mathematics.
We appeal

(1) to abolish the ***Guest Faculty tradition,*** *which has become a common word in state colleges and universities, where one is hired for rupees 150-200 per class and that too for only those classes, which are actually held.*
(2) to introduce ***Teaching Assistantship Programme,*** *where a promising candidate is appointed for at least two years, and renewal for another two years subject to satisfactory performance. After two years the concerned college or the university should negotiate with state government to sanction posts. The concerned government should support and assist.*
(3) to introduce ***Hiring of Mathematics Teachers.*** *By this we mean hiring of good teachers from X university by the Y university on tenure track appointment. Modelities are to be decided.*
(4) that the state run colleges and universities should be given free hand to appoint qualified teachers without the interference of the concerned government. Modelities are to be decided.
(5) to introduce ***Mathematics Awareness Programme*** *among college and university teachers both in teaching and research.*

These are only few. You are invited to write your views. Write to me personally. When many such are received, they will be compiled and consolidated and will be mailed to you again. We may decide later to meet.

I look forward to your joining this venture. Awaiting your response.
P. K. BANERJI

Now, from this platform I want to share this concept with you all and appeal to contribute your cooperation, in a manner that is appropriate, for this academic cause, to stop mathematics from decline, to stop it from evaporating, and to prevent it from sinking.

Most of this onerous responsibility will develop on the shoulders of the younger generation where we look for leaders and the rank. Young mathematicians will have to rise to the occasion to perform, to prove their credentials, to make others feel dependent on you, develop a faith in mathematics, in order to save it from decline and enjoy teaching and research. All of us should distinguish ourselves by undertaking an unusal or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. I am confident of the success of various academic sessions and the conference. My profound regards to the big family of the Indian Mathematical Society and I urge to those, who have not yet had the membership of the Society, to join our family. I am beholden to my colleagues of the Indian Mathematical Society for giving me this unique privilege of presiding over its 77th Session. This kind of talk is more flexible than presenting a paper, where I have the freedom to indulge in informatics, whimsicalities, and a little impression, and more so, a lot to tell of things tried and things that failed. Finally, thanks for your forbearance and tolerance to listen to me. P.

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THREE MOST GENERALIZED INTEGRALS OF THE 20^{TH} CENTURY*

P. K. BANERJI

Abstract: The Fundamental Theorem of Calculus remained the potential factor for the development of generalized integration processes in the twentieth century. The integrals, discussed in this text and presented to a house of heterogeneous mathematicians, are named after their principal investigators Denojy, Perron, Henstock, and McShane, each generalizes some aspect of the Lebesgue integral. These new integrals focus on a different property of the Lebesgue integral and thus their definitions are radically different. However, it turns out that all three integrals are equivalent. This text intends, very briefly, to present these integration process. The matter for this text has been drawn from a number of different sources owing to the fact that recent and past developments related to these integrals exists in research articles. This text is presented to all young scholars who for many reasons have not experienced any revolution regarding these integrals. What required for detailed reading of these integrals is a thorough understanding of basic real analysis such as that found in Rudin's *Principles of Mathematical Analysis*, DePree and Swartz's *Introduction to Real Analysis,* Apostol's *Mathematical Analysis*, and Goldberg's *Real Analysis.* Wherever and whenever required, notations and terminologies used are explained. Proofs of theorems are avoided.

1. **The Denjoy integral**

A. Denjoy, in 1912, developed an integration process that satisfies the following theorem.

Theorem 1.1. *Let $F : [a, b] \to R$ be continuous. If F is differentiable nearly everywhere on$[a, b]$, then F' is integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$ for each $x \in [a, b]$.*

He called the process of computing the values of his integral *totalization* and established that every derivative met the criteria for this process and that the original function was recovered. This process of totalization is found to be

* The text of the Presidential Address (Technical) delivered at the 77th Annual Conference of the Indian Mathematical Society held at Swami Ramanand Teerth Marathwada University (SRTM University), Vishnupuri, Nanded - 431 606, Maharashtra, during the period December 27 - 30, 2011.

a complicated process, which admits the use of transfinite numbers (see Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, pp.998–99, Oxford University Press, New York, (1972). Hobson [5]. N. Lusin, within few months, could establish a connection between this new integral and the notion of *generalized absolute continuity.* Most of researchers follow this approach.

The Lebesgue integral offers the following advantage over the Riemann integral. (1) A function does not need to be continuous at any point in order to be Lebesgue integrable. (2) If $\{f_n\}$ is a sequence of uniformly bounded, Lebesgue integrable functions that converges pointwise to f on $[a, b]$, then f is Lebesgue integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$. (3) If F has a bounded derivative on $[a, b]$, then F' is Lebesgue integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$ for each $x \in [a, b]$. An extension of (3) is, (4). Let F be a continuous function defined on $[a, b]$. If f is differentiable nearly everywhere on $[a, b]$ and if F' is Lebesgue integrable on $[a, b]$, then $\int_a^x F' = F(x) - F(a)$ for each $x \in [a, b]$.

Moreover, an integral with the property (Theorem 1.1) is said to recover a function from its derivative, and further, any integral that satisfies Theorem 1.1 should include the Lebesgue integral. This means, any function that is Lebesgue integrable should be integrable in the new sense and the integral should be equal. We know that, a function $f : [a, b] \to R$ is Lebesgue integrable on $[a, b]$ if and only if there exists an *absolutely continuous function* $F : [a, b] \to R$ such that $F' = f$ almost everywhere on $[a, b]$. The Denjoy integral is a simple generalization of his characterization of the Lebesgue integral.

Definition 1.1. A function $f : [a, b] \to R$ is Denjoy integrable on $[a, b]$ if there exists a generalized absolutely continuous function in the restricted sense (on E, set of real numbers) $F : [a, b] \to R$ such that $F' = f$ almost everywhere on $[a, b]$. The function f is Denjoy integrable on a measurable set $E \subseteq [a, b]$ if $f\mathcal{X}_E$ ($\mathcal{X}_E$ is the characteristic function on set E) is the Denjoy integrable on $[a, b]$.

We know the result [2, p.103] which provides a condition for a function to be generalized absolutely continuous function in the restricted sense on an integral, that is, Let $F : [a, b] \to R$ *be continuous on* $[a, b]$. *If* F *is differentiable nearly everywhere on* $[a, b]$, *the* F *is generalized absolutely continuous function in the restricted sense on* $[a, b]$. By virtue of this the Denjoy integral of a function is uniquely determined upto an additive constant. If the condition $F(a) = 0$ is added, then the function F is unique, which is denoted by $\int_a^x f$. All the usual properties of an integral are possessed by the Denjoy integral, see [2, p. 109]. The *difference* between the Lebesgue and the Denjoy integral is that the indefinite latter integral is generalized absolutely continuous in the restricted sense rather than absolutely continuous. This is an after the fact comparison, owing to the fact of the involvement of indefinite integrals rather than integrands. Following remarks will offer the clarification to pronounce the *difference* between a *Lebesgue* and a *Denjoy integrable function.*

Remark 1.1. A measurable function f is Lebesgue integrable on $[a, b]$ if and only if $|f|$ is Lebesgue integrable on $[a, b]$.Let f is Denjoy integrable on $[a, b]$, but not Lebesgue integrable on $[a, b]$. Then $|f|$ in not Denjoy integrable on $[a, b]$. Because, if $|f|$ is Denjoy integrable on $[a, b]$, then by virtue of the property : *Let* $f : [a, b] \to R$ *be Denjoy integrable on* $[a, b]$ *. If* f *is non-negative on* $[a, b]$*, then* f *is Lebesgue integrable* on $[a, b]$, it is Lebesgue integrable on $[a, b]$ which implies that f is also Lebesgue integrable on $[a, b]$. For this reason, the Denjoy integral is called a *non-absolute integral.* This means that, it does not follow that $|f|$ is Denjoy integrable on $[a, b]$ given that f **is** Denjoy integrable on $[a, b]$.

Remark 1.2. We know, if a function is Lebesgue integrable on $[a, b]$, then it is Lebesgue integrable on every measurable subset of $[a, b]$, but not subset of $[a, b]$. Let f is Denjoy integrable on $[a, b]$, but not Lebesgue integrable on $[a, b]$ (we quoted same above). Then by virtue of the property : *Let* $f : [a, b] \to R$ *be Denjoy integrable on* $[a, b]$ *. If* f *is Denjoy integrable on every measurable subset of* $[a, b]$*, then* f *is Lebesgue integrable* on $[a, b]$, there exists a measurable subset of $[a, b]$ on which f **is not** Denjoy integrable.

Above mentioned differences make it possible (that is why they are important) for the Denjoy integral to possess properties which the Lebesgue integral does not have. The theorem that follows expresses the *close relationship* between the Denjoy and Lebesgue integrals.

Theorem 1.2. *If* $f : [a, b] \to R$ *is Denjoy integrable on* $[a, b]$ *then for each* $\epsilon > 0$ *there exists a measurable set* $E \subseteq [a, b]$ *such that* $\mu([a, b] - E) < \epsilon$ *,* f *is Lebesgue integrable on* E *and* $\int_E f = \int_a^b f$.

Following two theorems (properties rather) are for the the Denjoy integral which are *not valid* for the Lebesgue integral.

Theorem 1.3. *Let* $f : [a, b] \to R$ *is Denjoy integrable on each interval* $[c, d] \subseteq (a, b)$*. If* $\int_c^d f$ *converges to a finite limit as* $c \to a^+$ *and* $d \to b^-$ *then* f *is Denjoy integrable on* $[a, b]$ *and* $\int_a^b f = \lim\limits_{c\to a^+, d\to b^-} \int_c^d f$.

Theorem 1.4. *Let* E *be bounded, closed set with bounds* a *and* b *and let* $\{(a_k, b_k)\}$ *be the sequence of intervals contiguous to* E *in* $[a, b]$*. Let* $f : [a, b] \to R$ *be Denjoy integrable on* E *and on each* $[a_k, b_k]$*. If the series* $\sum\limits_{k=1}^{\infty} \omega \left(\int_a^x f \ , [a_k, b_k] \right)$ *converges,* *then* f *is Denjoy integrable on* $[a,b]$ *and* $\int_a^b f = \int_a^b f\ \mathcal{X}_E + \binom{\infty}{k=1}\omega \int_{a_k}^{b_k} f$.

Prior concluding this section we would desire to read an interesting mathematical concepts of the Denjoy integral. Suppose that for a given function f , we know it is the derivative of a continuous function F on $[a,b]$. In computing $F(b)-F(a)$, with the knowledge of first year calculus, we may work out various techniques. But when none of the techniques work, we appeal to the definition of the Riemann integral and, for large values of n , compute

$$\sum_{i=1}^{n} f\left(a+\frac{(b-a)i}{n}\right)\frac{b-a}{n}.$$

the limiting value being $F(b)-F(a)$. We, on the other hand, note that this approach favours if f is Riemann integrable on $[a,b]$, but there are derivatives which are not Riemann integrable. However, if f is Lebesgue integrable, then simple functions can be used to approximate the value of $F(b)-F(a)$. The definition of the Denjoy integral, however, does not convey as to how do we find $F(b)-F(a)$, and that is why the definition is called a *descriptive definition* of the Denjoy integral. The original definition of Denjoy was a *constructive definition* through a process of Lebesgue integrations and limit operations. In the early days of the theory of Lebesgue integral, a Lebesgue integrable function was called *summable*. The best source for curious reader is Natanson [12] and/or Saks [15], Royden [14], and Pfeffer [13] among others.

2. The Perron integral

In 1914, another extension of the Lebesgue integral was developed by O. Perron and has shown that this integral also had the property that every derivative was integrable. This work was independent of Denjoy, and thus has a different flavour. We will try to discuss the property of the Lebesgue integral which was chosen by Perron to generalize. We will begin by introducing notion of *major* and *minor* functions, in terms of which Perron integral is defined. These functions are defined by virtue of *upper* and *lower derivatives*.

Definition 2.1. Let $F:[a,b]\to R$. The upper right and lower right and the upper left and lower left derivatives of F at $x\in[a,b]$ are denoted by $D^+F(x)$ and $D_+F(x)$, and $D^-F(x)$ and $D_-F(x)$, respectively. The function F is differentiable at $x\in(a,b)$ if all four derivatives are finite and equal (we know of them through calculus).The upper and lower derivatives of F at $x\in[a,b]$ are denoted by $\overline{D}F(x)$ and $\underline{D}F(x)$, respectively such that

$$\overline{D}F(x)=\max\{D^+F(x),D^-F(x)\}$$

and

$$\underline{D}F(x) = \min\{(D_+F(x)), D_-F(x)\}.$$

Definition 2.2. Let $f : [a, b] \to R_e$ (extended real numbers). (a) A function $U : [a, b] \to R$ is a *major function* of f on $[a, b]$ if $\underline{D}U(x) > -\infty$ and $\underline{D}(x) \geq f(x)$ for all $x \in [a, b]$. (b) A function $V : [a, b] \to R$ is a *minor function* of f on $[a, b]$ if $\overline{D}V(x) < +\infty$ and $\overline{D}(x) \leq f(x)$ for all $x \in [a, b]$.

By virtue of the property, that if $F : [a, b] \to R$ and $E \subseteq [a, b]$ and if $\overline{D}F < +\infty$ or $\underline{D}F > -\infty$ nearly everywhere on E then F is of generalized bounded variation in the restricted sense on E, the functions U and V are of generalized bounded variation in the restricted sense on $[a, b]$. This implies that U and V are measurable and differentiable almost everywhere on $[a, b]$.

Theorem 2.1. *A measurable function $f : [a, b] \to R_e$ is Lebesgue integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there exists absolutely continuous major and minor functions U and V of f on $[a, b]$ such that $U_a^b - V_a^b < \epsilon$.*

Here U_a^b is used to mean $U(b) - U(a)$, which and similar for V are used when major and minor functions are involved. In proving (we are not detailing the proof) this theorem, corresponding to $\epsilon/2$, we choose functions u and v [1] such that $U(x) = \int_a^x u$ and $V(x) = \int_a^x v$ are absolutely continuous on $[a, b]$, and also $\underline{D}U(x) \geq u(x)$ and $\overline{D}V(x) \leq v(x)$ are valid for all $x \in [a, b]$,and $\underline{D}U(x) > -\infty, \underline{D}U(x) \geq f(x)$ and $\overline{D}V(x) < +\infty$, $\overline{D}V(x) \leq f(x)$ for all $x \in [a, b]$. Hence, U *is a major function and V is a minor function of f*. By avoiding (or cancelling) the condition of the absolute continuity of major and minor functions, we achieve the generalization of the Lebesgue integral. We also recall that F is differentiable at c if and only if $\underline{D}F(c)$ and $\overline{D}F(c)$ are finite and equal.

We take a pause for the definition of the Perron integral by mentioning little more for U and V. For the functions U and V , discussed above, the function $U - V$ is non-decreasing on $[a, b]$, and it follows that $V_a^b < U_a^b$ and that $0 \leq U_c^d - V_c^d \leq U_a^b - V_a^b$ whenever $[c, d]$ is a subinterval of $[a, b]$. In particular,

$$-\infty < \sup\{V_a^b\} \leq \inf\{U_a^b\} < \infty,$$

where supremum (l. u. b.) is taken over all minor functions of f and the infimum (g. l. b.) is taken over all major functions of f .

Definition 2.3. A function $f : [a, b] \to R_e$ is Perron integrable on $[a, b]$ if f has at least one major and one minor function on $[a, b]$ and the numbers

$$\inf\{U_a^b : U \text{ is a major function of } f \text{ on } [a, b]\}$$
$$\sup\{V_a^b : V \text{ is a minor function of } f \text{ on } [a, b]\}$$

are equal. This common value is the Perron integral of f on $[a, b]$ and is denoted by $\int_a^b f$.

Moreover, the function f is Perron integrable on a measurable set $E \subseteq [a, b]$, if $f\mathcal{X}_E$ is Perron integrable on $[a, b]$. In particular, a Lebesgue integrable function is Perron integrable and the integrals are equal. Following theorem is a consummate result of the Definition 2.3.

Theorem 2.2. *A function $f : [a, b] \to R_e$ is Perron integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there exists a major function U and a minor function V of f on $[a, b]$ such that $U_a^b - V_a^b < \epsilon$.*

Whereas the following theorem expresses the relationship between Perron integrability and subintervals.

Theorem 2.3. *Let $f : [a, b] \to R_e$ and $c \in (a, b)$.*

(a) If f is Perron integrable on $[a, b]$, then f is Perron integrable on every subinterval of $[a, b]$.

(b) If f is Perron integrable on each of the intervals $[a, c]$ and $[c, b]$, then f is Perron integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Before proceeding to the conclusion, it is worth mentioning that it is sufficient to consider finite-valued functions only to study the Perron integral, which will avoid difficulties when it comes to adding functions as $f + g$ may not be defined if both f and g assume infinite values. Let major and minor functions are continuous, for which following definition is perfect.

Definition 2.4. A function $f : [a, b] \to R_e$ is P_c integrable on $[a, b]$ if f has at least one continuous major and one continuous minor function $[a, b]$ and the numbers

$$\inf\{U_a^b : U \text{ is a continuous major function of } f \text{ on } [a, b]\}$$

and

$$\sup\{V_a^b : V \text{ is a continuous minor function of } f \text{ on } [a, b]\}$$

are equal. Symbol P_c stands for Perron continuous.

3. The Henstock integral

The Denjoy and Perron integrals of the preceding two sections are the generalizations of the Lebesgue integral which recover a continuous function from its derivative. In this section we describe the development of a generalization of the Riemann integral. A short Historical Remark, that follows, may be interesting [1]. This new integral, referred to as the Henstock integral, does not have a standard name. It is referred to a Henstock -Kurzweil integral, the generalized Riemann integral and the gauge integral.

Historical Remark: The gauge integral was originally introduced by the Czech mathematician Kurzweil [6], but he did not give a detailed study of the properties of the integral that he used in the context of differential equations. The integral was rediscovered independently by Henstock [3], where he develops the basic properties of the integral and established the convergence theorems for the same, see [4] for details. The books, McLeod [8] and Mawhin [7] contain complete account of this integral. We have the knowledge (for that matter we may refer to [1, p.152]) that the general version of the Fundamental Theorem of Calculus is false for both the Riemann integral and the Lebesgue integral, which require the assumption that the derivative F' be integrable in order to obtain the basic formula $\int_a^b F' = F(b)-F(a)$. This motivated mathematicians in the early 20^{th} century to develop an integration theory for which the Fundamental Theorem of Calculus is valid in its full generality. This problem was solved by Denjoy and Perron that is discussed in Sections 1 and 2 . The gauge integral is equivalent to Perron integral; however, the Denjoy and Perron integrals are technically much more difficult to describe than the gauge integral. The Perron, and, therefore the gauge, integral is more general than the Lebesgue integral and, in fact, a function f is Lebesgue integrable if and only if it is absolutely gauge integrable.

In order to define the Henstock integral, it is necessary to specify the terms that are allowed in the Riemann sum. We discuss tagged partition, tagged interval and related terminologies.

Definition 3.1. Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A *tagged interval* $(x, [c, d])$ consists of an interval $[c, d] \subseteq [a, b]$ and a point $x \in [c, d]$. A tagged interval is *subordinate* to δ if $[c, d] \subseteq (x - \delta(x), x + \delta(x))$.

We will use letter $\mathcal{P}$ to denote *finite collections of non-overlapping tagged intervals*, where let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be such a collection in $[a, b]$. If δ is a constant, or more generally if δ is bounded away from 0, then the collection of tagged partitions subordinate to δ is no different than the collection of tagged partition used in the definition of the Riemann integral. It is also important to note the distinction between the phrases $\mathcal{P}$ is subordinate to δ and $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to δ. It is worth mentioning that tagged partition always contain a finite number of intervals and that the tag of each interval is a point in the interval.

Proposition 3.1. *If δ is a positive function defined on the interval $[a, b]$ then there exists a tagged partition of $[a, b]$ subordinate to δ.*

This proposition ensures the existence of tagged partition of $[a,b]$ that are subordinate to δ for each positive function δ on $[a,b]$. We now write the definition of the Henstock integral.

Definition 3.2. A function $f:[a,b]\to R$ is Henstock integrable on $[a,b]$ if there exists a real number L with the property : for each $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P})-L|<\epsilon$ whenever $\mathcal{P}$ is tagged partition of $[a,b]$ that is subordinate to δ. The function f is Henstock integrable on a measurable set $[a,b]$ if $f\mathcal{X}_E$ is Henstock integrable on $[a,b]$.

From what we have mentioned above, it is clear that every Riemann integrable function is Henstock integrable and that the integrals are equal. Not many theorems we intend to write and some those we have mentioned here are not provided with proofs. Curious reader may look for those details in the books we have listed in the Reference at the end of this text. We observe, through a theorem on this integral, that the characteristic function[1] of the rational numbers is the standard example of a bounded function that is not Riemann integrable on $[0,1]$. The same theorem further, clarifies that a function which is Henstock integrable does not have to be continuous at any point.

Following two theorems are regarding the basic properties of the Henstock integral and the the need for a Cauchy criterion for a function to be Henstock integrable, those we study in the case of Riemann integral.

Theorem 3.1. *A function $f:[a,b]\to R$ is Henstock integrable on $[a,b]$ if and only if for each $\epsilon>0$ there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P}_1)-f(\mathcal{P}_2)|<\epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[a,b]$ that are subordinate to δ.*

Theorem 3.2. *Let $f:[a,b]\to R$ and $c\in(a,b)$.*

(a) If f is Henstock integrable on $[a,b]$, then f is Henstock integrable on every subinterval of $[a,b]$.

(b) If f is Henstock integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is Henstock integrable on $[a,b]$ and $\int_a^b f=\int_a^c f+\int_c^b f$.

We mention now a lemma which is used frequently in describing theory of the Henstock integral. The lemma was first noted by Henstock in his initial paper (1961) on this integral, but refers to Saks [15] for the idea behind it. Thus, this is called *Saks-Henstock lemma.*

Lemma 3.1. *Let $f:[a,b]\to R$ be Henstock integrable on $[a,b]$. Let $F(x)=\int_a^x f$ for each $x\in[a,b]$, and let $\epsilon>0$. Suppose that δ is a positive function on $[a,b]$ such*

[1] $A\subseteq E$. Then the function $\mathcal{X}_E$ represnts the characteristic function of A

that $|f(\mathcal{P}) - F(\mathcal{P})| < \epsilon$ *whenever* $\mathcal{P}$ *is tagged partition of* $[a, b]$ *that is subordinate to* δ *. If* $\mathcal{P}_0 = \{x_i, [c_i, d_i] : 1 \leq i \leq n\}$ *is subordinate to* δ*, then*

$$|f(\mathcal{P}_0) - F(\mathcal{P}_0)| < \epsilon$$

and

$$\sum_{i=1}^{n} |f(x_i)(d_i - c_i) - (F(d_i) - F(c_i))| \leq 2\epsilon.$$

We note that for the Riemann and Lebesgue integrals, the integrability of f is absolutely integrable. This is not the case for the Henstock integral (a function f is absolutely integrable if both f and $|f|$ are integrable). We consider the occupation of improper integrals in the context of the Henstock (or gauge) integral. In calculus of Riemann integral functions such as $f(t) = 1/\sqrt{t}$, with singualritites, must be given a special treatment. It requires an extension of the Riemann integral, sometimes called the Cauchy - Riemann integral, to properly handle such functions. Functions with several singularities on an interval become very difficult to treat. Theorem that follows will show that, see [1, p.173] for detailed proof, it is not necessary to give such functions special treatment in the theory of the Henstock integral.

Recall that if $f : [a, b] \to R$ has a singularity at a in the sense that $\lim\limits_{x\to a} |f(x)| = \infty$ and if f is Riemann integrable on $[c, b]$ for each $c > a$, then f is said to be Cauchy - Riemann integrable over $[a, b]$ if and only if $\lim\limits_{c\to a^+} \int_c^b f$ exists and, in this case, the Cauchy-Riemann integral of f over $[a, b]$ is defined to be this limit.

Theorem 3.3. *Let* $f : [a, b] \to R$ *be integrable over* $[c, b]$*, for all* $a < c < b$*. Then* f *is integrable over* $[a, b]$ *if and only if*

$$\lim_{c\to a^+} \int_c^b f$$

exists. In this case

$$\int_a^b f = \lim_{c\to a^+} \int_c^b f.$$

One may also, in this connection, refer to [12].

4. **Note on the Mcshane integral**

The title of this section very clearly spells that I will be very brief than the preceding three sections. Not that I am pretending this frankness, it is true through core of my heart, and do you know the truth ? Because mathematics, that too of this abstract nature, can be tolerated for very short time in one slot. Let us begin.

The definitions of the Denjoy and Perron integrals are matural generalizations of a property of the Lebesgue integral. The Henstock integral is clearly an extension of the Riemann integral, but it appears to be very different from the Lebesgue integral. Henstock integral is an extension of a characterization of the Lebesgue integral.

A measurable function f to be a Lebesgue integrable, it is necessary and sufficient for $|f|$ to be Lebesgue integrable. While discussing Henstock integral in Section 3, we did notice Henstock integrable functions f for which $|f|$ is not Henstock integrable. However, a non-negative Henstock integrable function is Lebesgue integrable, see following theorem.

Theorem 4.1. *Let $f : [a, b] \to R$ is Henstock integrable on $[a, b]$.*

(a) If f is bounded on $[a, b]$, then f is Lebesgue integrable on $[a, b]$.
(b) If f is non-negative on $[a, b]$, then f is Lebesgue integrable on $[a, b]$.
(c) If f is Henstock integrable on every measurable subset of $[a, b]$, then f is Lebesgue integrable on $[a, b]$.

It became necessary then to modify the definition of the Henstock integral in order to obtain the Lebesgue integral as a limit of Riemann sums so that the non-absolutely integrable functions are eliminated. McShane observed that a simple alteration in the definition of the Henstock (or the gauge integral, as its synonym) integral produces exactly the classical Lebesgue integral in the definition of tagged interval (see, Definition 3.1 and Lemma 3.1), McShane dropped the requirement that tag x_i must belong to the interval $[c_i, d_i], 1 \leq i \leq n$. Exposure about this may be seen in McShane [10, 11] and also in [9]. Infact, *McShane proved that the Lebesgue integral is indeed equivalent to a modified version of the Henstock integral.* We may notice that a function is McShane integrable if and only if its absolute value is McShane integrable, and due to this, we say that the McShane integral is equivalent to the Lebesgue integral.

It is clear that every McShane integrable function is Henstock integrable and that the integrals are equal. In particular, every McShane integrable function is measurable. Moreover, I would appeal to all young mathematics scholars and curious readers to verify that the proof of each property of the Henstock integral that is not valid for the Lebesgue integral and cannot be extended to the McShane integral. We intend to conclude this section and the text in general, by mentioning some theorems (as usual, we do not give proofs).

Theorem 4.2. *If $f : [a, b] \to R$ is a simple function, then f is McShane integrable on $[a, b]$, and that $(M) \int_a^b f = (L) \int_a^b f$.*

Theorem 4.3. *A function $f : [a, b] \to R$ is Lebesgue integrable on $[a, b]$ if and only if it is McShane integrable on $[a, b]$. The value of the integral is the same in both cases.*

Theorem 4.4. *Let $F : [a, b] \to R$, let D be the set of all $x \in [a, b]$ for which $F'(x)$ exists and let $c \in D$.*

(1) If F is strongly differentiable at c, then $F'|_D$ is continuous at c.
(2) Suppose that F is absolutely continuous on $[a, b]$. If $F'|_D$ is continuous at c then F is strongly differentiable at c.

This theorem shows that an indefinite McShane integral is not strongly differentiable almost everywhere.

Indeed I feel relaxed talking to all young scholars present here today. I was recalling an age of mine parallel to you. There has been no attempt in this text to write things new, I have drawn this work from a number of different sources and I am deeply indebted to all those authors, of research papers and books, particularly the books by Gordon and DePree and Swartz. A short list of reference is given, which by no means is exhaustive.

Acknowledgement: There are many individuals from whom I learned mathematics which includes my graduate and doctoral students, from whom, in addition, I learned to teach mathematics. This text is the result of such association and mathematical sentiments.

References

[1] DePree, J. D. and Swartz, C. W., *Introduction to Real Analysis,* John Wiley & Sons, New York, 1988.

[2] Gordon, R. A., *The Integrals of Lebesgue, Denjoy, Perron and Henstock, Vol. 4,* Amer. Math. Soc., Providence, RI, 1994.

[3] Henstock, R., *Definitions of Riemann type of variational integral,* Proc. London Math. Soc., 11 (1961), 402–418.

[4] Henstock, R., *A Riemann type integral of Lebesgue power,* Canadian J. Math., 20 (1968), 79–87.

[5] Hobson, E. W., *The Theory of Functions of a Real Variable and the Theory of Fourier Series,* Vol. 1, 3rd edn., Cambridge University Press, Cambridge, New York, 1927.

[6] Kurzweil, J., *Generalized ordinary differential equations and continuous dependence on a parameter,* Czech. Math.J., 82 (1957), 418–499.

[7] Mawhin, J., *Introduction à L'Analyse,* Université de Louvain, Louvain, 1984.

[8] McLeod, R. M., *The Generalized Riemann Integral,* Math. Assoc. America, Providence, RI, 1980.

[9] McShane, E. J.and Botts, J., *Real Analysis,* Van Nostrand, Princeton, NJ, 1959.

[10] McShane, E. J., *A unified theory of integration,* Amer. Math. Monthly, 80 (1973), 349–359.

[11] McShane, E. J., *Unified Integration,* Academic Press, New York 1983.

[12] Natanson, I. P., *Theory of Functions of Real Variable, Vol. 1 and 2,* Prederick Ungar Publishing Co., New York, 1964.
[13] Pfeffer, W. F., *A note on generalized Riemann integral,* Proc. Amer. Math. Soc., 103 (1988), 1161–1166.
[14] Royden, H. L., *Real Analysis,* 3rd edn., Macmillian, New York, 1988.
[15] Saks, S., *Theory of the integral,* 2nd edn., Revised, Hafner, New York, 1937.

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ABSTRACTS OF THE PAPERS SUBMITTED FOR PRESENTATION AT THE 77^{TH} ANNUAL CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY HELD AT SWAMI RAMANAND TEERTH MARATHWADA UNIVERSITY, VISHNUPURI, NANDED - 431 606, DURING DECEMBER 27 - 30, 2011

A: Combinatorics, Graph Theory and Discrete Mathematics:

A-1: Fuzzy rule based inference system for detection and diagnosis of blood cancer,
Sanjeev Kumar and Amendra Singh, *sanjeevibs@yahoo.co.in*

In this work we design a fuzzy rule based inference system to determine and identify blood cancer. The proposed system accepts the symptoms as input and provides the confirmed disease and stage as the output. It also calculates the membership function for both input as well as the output variables. Domain expert knowledge is gathered to generate rules and stored in the rule base and the rules are fired when there exists appropriate symptoms. In last section a case study is taken to which is showing how the inference system work to detection and diagnosis of blood cancer.

A-2: On upper bound for d-covered triangulation of surfaces,
Anand Kumar Tiwari, *anand@iitp.ac.in*

A triangulation of surfaces in which each edge is incident with a vertex of degree d is called d-covered . We give a technique to obtain d-covered triangulations from known triangulations. As a consequence we show the existence of d-covered triangulations on surfaces of Euler characteristics χ for which the value $n = \dfrac{7+\sqrt{49-24\chi}}{2}$ is a fraction. This fills a gap in the earlier relevant work.

A-3: Dismantable Lattices in which reducible elements are comparable,
A. N. Bhavale and B. N. Waphare, *ashokbhavale@gmail.com*

In order to enumerate the class of Dismantlable lattices with n elements and $n+k-1$ coverings in which all reducible elements are comparable, we obtain the partitions of this class. This work is in respect of Birkhoff's open problem of enumerating all finite lattices which are uniquely determined (upto isomorphism) by their diagrams, considered purely as graphs.

A-4: me-Degree sequences in semigraphs, N. S. Bhave, B. Y. Bam and C. M. Deshpande, *nshsb@yahoo.co.in, bpa.maths@coep.ac.in, hod.maths@coep.ac.in*

We define me-degree of a vertex in semigraphs and give necessary conditions for a sequence to be me-Semigraphical. Applying concepts of design theory to semigraphs, some results are given about me-degree sequences of regular, uniform and complete semigraphs.

A-5: An ideal - based zero-divisor graph of posets, B. Elavarasan, *belavarasan@gmail.com*

The structure of a poset P with a smallest element 0 is looked at from two view points. Firstly, with respect to the Zariski topology, it is shown that $Spec(P)$, the set of all prime ideals of P, is a compact space and $Max(P)$, the set of all maximal ideals of P, is a compact T_1 subspace. Various other topological properties are derived. Secondly, we introduce the ideal-based zerodivisor graph structure of poset P, denoted by $\Gamma_I(P)$. It is shown that if I is an ideal of P then every two vertices in $\Gamma_I(P)$ are connected by a path of length at most 3; and if $\Gamma_I(P)$ contains a cycle then the core K of $\Gamma_I(P)$ is a union of triangles and rectangles.

A-6: Total irregularity strength of some special graphs, D. G. Akka and N. S. Hungund, *neelu_152@yahoo.co.in*

An edge (A vertex) irregular total k-labeling of a graph G is such a labeling of vertices and edges with integers $1, 2, \ldots, k$ that weighs of any two different edges (vertices) are distinct, where the weight of an edge (a vertex) is the sum of the label of the edge (vertex) itself and the label of incident vertices (edges). The minimum k, for which the graph G and is denoted by $tes(G)(tvs(G))$. In this paper, we determine the exact value of the total edge (vertex) irregularity strength of the line graph of path P_n and the upper bound of the total edge (vertex) irregularity strength of the total graph of the star graph $K_{1,n}, n \geq 2$.

A-7: Domination number and the splitting operation for certain classes of graphs, Naiyer Pirouz and M. M. Shinde, *naiyer.pirouz@gmail.com, mms@math.unipune.ac.in*

In a Graph $G = (\nu, E)$, a set $S \subseteq V(G)$ is said to be a dominating set if every vertex in G either belongs to S or is adjacent to some vertex in S. The domination number $\gamma(G)$ is the minimum size of a dominating set in G. In this paper, we explore the realtion between domination number of the splitting graph $G_{x,y}$ in terms of the domination number of the original graph G.

A-8: Hereditary properties for the set of fixed points in pseudo ordered sets, Shiju George, Department of Mathematics, Sacred Heart College Thevara, Cochin.

For an order preserving map f on a pseudo ordered set A (an algebraic structure with a binary relation which is reflexive and antisymmetric), the set of all fixed points of f inherits the properties of A, namely, completeness, chain-completeness and weakly chain-completeness, as in the case of posets.

A-9: A decomposition of K_n and $K_{m,n}$, Ancykutty Joseph and Jinitha Varughese, *ancykuttyjoseph@gmail.com, jinith@gmail.com*

A graph G is H-decomposable if the edge set of G can be partitioned into isomorphic copies of H. Wilson R. M. proved that for every non-empty graph H, there exists infinitely many positive integers p such that K_p is H-decomposable. In this paper we characterize those complete graphs K_n and $K_{m,n}$ which are $K_{1,2} \cup K_{1,3}$- decomposable.

A-10: On independent domination in hypercubes, S. A. Mane and B. N. Waphare, *manesmruti@yahoo.com, bnwaph@math.unipune.ac.in*

Harary and Livingston observed earlier that $\gamma(Q_n) = \gamma_i(Q_n)$ holds when $n \in \{1, 2, 3, 4, 5, 6\}$, but for $n = 5$ we have $\gamma(Q_5) = 7$ and $\gamma_i(Q_5) = 8$. They ask whether $\gamma(Q_n) = \gamma_i(Q_n)$ for any $n \neq 5$. In their discussion of this question they interpret the earlier results of Van Wee to imply that $\gamma(Q_n) = \gamma_i(Q_n)$ for $n = 2^k$, even though what Van Wee's earlier result imply is that $\gamma(Q_n) = \gamma_t(Q_n)$ for $n = 2^k$. In this paper, we prove that $\gamma(Q_n) = \gamma_i(Q_n) = 2^{n-k}$ when $n = 2^k$ and $k \geq 3$. In fact, we give an upper bound of independent domination number $(\gamma_i(Q_n))$ and total domination number $(\gamma_t(Q_n))$ of Q_n. We prove that $\gamma_i(Q_n) \geq 2^{n-k}$ for $2^k - 1 \leq n < 2^{k+1} - 1$ and $k \geq 1$ also $\gamma_t(Q_n) \leq 2^{n-k}$ for $2^k \leq n < 2^{k+1} - 1$ and $k \geq 1$.

B: Algebra, Number Theory and Lattice Theory:

B-1: On bicomplex matrices, Anjali, *swtanjaligos.maths773@gmail.com*

In this paper a detailed study of Bicomplex matrices has been done. A number of results are proved on bicomplex square matrices, conjugates of bicomplex matrices, transpose of bicomplex matrices, bicomplex Hermitian matrices, bicomplex Skew - Hermitian matrices.

B-2: Some identities of fibonacci-like sequences - I and II,
V. K. Gupta and Yashwant K. Panwar, *dr_vkg61@yahoo.com, yashwantpanwar@gmail.com*

In this study, we introduce second order linear sequences. Namely, we define the Fibonacci-Like sequence-I, (FLS - I) and Fibonacci-Like sequence-II, (FLS - II). Afterwards, we investigate some properties of the Fibonacci- Like sequence-I and II, with their Binets formula and give several interesting identities involving them.

B-3: On some new modular equations and their applications to continued fractions, S. Chandan Kumar, M. S. Mahadeva Naika and M. Manjunatha, *msmnaika@rediffmail.com*

In this paper, we obtain some new modular equations of degree 2. We obtain several general formulas for the explicit evaluations of the Ramanujan's theta function. As an application, we establish some new modular relations for Ramanujan-Gollnitz Gordon continued fraction, Ramanujan, A Selberg continued fraction and a continued fraction of Eisenstein. We also establish their explicit evaluations.

B-4: Certain results of zeros of bicomplex polynomials, Gunjan Shukla, *gunn735@gmail.com*

In this Paper, we deal with Bicomplex polynomials and their zeros. For the sake of brevity and clarity, we have dealt with quadratic and cubic Bicomplex polynomials only. We have obtained conditions for the existence of zeros in different types of conjugate pairs. We have also established the relation between the existence of conjugate pairs of zeros with the coefficients of the bicomplex polynomial.

B-5: Zero divisor graphs of lattices with respect to primal ideals,
Vinayak Joshi, B. N. Waphare and H. Y. Pourali, *hosseinypourali@gmail.com*

In this paper, we introduce the concepts of primal and weakly primal ideals in lattices and characterize them in terms of the vertex set of the zero divisor graph. Further, the diameter of the zero divisor graph of a lattice with respect to a non-primal ideal is characterized.

B-6: A note on groups and its cyclic subgroups, J. N. Salunke and A. R. Gotmare, Department of Mathematics, North Maharashtra University, Jalgaon.

Any group is the union of its cyclic subgroups. If a group is non-cyclic then it is the union of its proper subgroups. Any cyclic group cannot be written as the union of its proper subgroups. Union of all proper subgroups of a nontrivial cyclic groups G is a subgroup of G if and only if G is finite and $O(G) = p^n$ for some prime p and $n \in N$. Multiplicative groups $C^*, R^*, R+$ cannot be expressed as countable union of their cyclic subgroups, C^* is cyclic and C^* contains infinitely many elements of finite order and uncountable elements of infinite order.

B-7: On some parameter involving Ramanujan's cubic continued fraction, M. S. Mahadeva Naika, S. Chandankumar and K. Sushan Bairy, msnaika@rediffmail.com

In this paper, we find several new modular relations connecting $\xi(q)$ with $\xi(q^3)$, $\xi(q^5)$, $\xi(q^7)$, $\xi(q^9)$, $\xi(q^{11})$ and $\xi(q^{13})$, where $\xi(q)$ is the parameter associated with Ramanujan's cubic continued fraction introduced by M. S. Mahadeva Naika.

B-8: On bihyperbolic numbers and the fundamental theorem of bihyperbolic algebra, Kalpana Sharma, *kalpanaibsmath@gmail.com*

Extensions of basic complex numbers to higher dimensions have a renewed interest in mathematics, physics and engineering because of their fruitful applications. A couple of interesting commutative algebras of tetranumbers are defined by Bicomplex and Bihyperbolic numbers. Bicomplex numbers are natural extension of complex numbers, whereas Bihyperbolic numbers are natural extension of hyperbolic numbers to four dimensions. In this paper, we present a variety of algebraic properties of Bihyperbolic numbers and also establish the fundamental theorem of Bihyperbolic algebra.

B-9: A note on ZI-Lattices, M. T. Gophane and Vilas S. Kharat, *machchhu@yahoo.co.in, vsk@math.unipune.ac.in*

G. Richter and M. Stern introduced the concept of ZI-lattices. In this paper we have studied some properties of ZI-lattices and F(L) the collection of finite elements in a ZI-lattice L.

B-10: Some results on strong regularity of a near-rings, Manoj Kumar Manoranjan, *manojmanoranjan.kumar@gmail.com*

A right near-ring N is called (i) left (right) strongly regular if for every a there is an x in N such that $a = xa^2(a = a^2x)$ and (ii) left (right) regular if for every a there is an x in N such that $a = xa^2(a = a^2x)$ and $a = axa$. Right regularity and right strong regularity are defined in a symmetric way and the definition of regularity shall be same as for rings. Thus a left regular near-ring is both regular and left strongly regular. For a zero-symmetric near-ring with identity,the notion of left regularity, right regularity and left strong regularity are equivalent. A near-ring is said to be reduced if it does not contain any non-zero nilpotent element and N has I.F.P.(Insertion of Factor Property) if $ab = \theta$ implies $axb = \theta$ for all $x \in N$. We see that a strongly regular near-ring is reduced. In this paper we shall prove some properties if strongly reduced near-rings.We find some characterizations of strong regularity in near-rings which are closely related with strongly reduced near-rings.

B-11: Stability analysis of mutualistic interactions among three species with limited resources for first and second species and unlimited resources for thrid species, A. B. Munde and M. B. Dhakne, *mashok.math@gmail.com*

In this paper, a mathematical model consisting of mutualistic interactions among three species is proposed and analyzed. The local stability analysis of the system is carried out in each of the following three cases (i) The death rate of any one (say third) species is greater than its birth rate. (ii) The death rate of any two (say second and third) species are greater than their birth rate. (iii) The death rate of all the species are greater than their birth rate.

B-12: Some properties of the complement of the zero-divisor graph of a commutative ring, S. Visweswaran, *s_visweswaran2006@yahoo.co.in*

In this article, some properties of the complement of the zero-divisor graph of a commutative ring with identity which admits at least two nonzero zero-divisors are discussed. We prove that if this graph is connected, then its radius equals 2. We show that its girth is equal to 3 when it is connected. Moreover, we study about the cliques of this graph.

B-13: Minimal prime elements in multiplicative lattices, S. B. Ballal and V. S. Kharat, *sbb@math.unipune.ac.in, vsk@math.unipune.ac.in*

We study $\pi(L)$ the set of all minimal prime elements of a reduced lattice L, when it carries hull-kernel topology. Several characterizations of minimal prime elements and properties of hulls, kernels are obtained. For any element a in a reduced lattice L, it is shown that, $h(a^*)$ and $\pi(L/a^*)$ are homeomorphic.

B-14: Generalized Jordan (θ, Φ) derivations in rings, Asma Ali and Shahoor Khan, *asma_ali2@rediffmail.com, Shahoor.khan@rediffmail.com*

Let R, S be any rings and U a Lie ideal of S such that $u^2 \in U$. Let(θ, Φ) be homomorphisms of R into S and M be a 2-torsion free S-bimodule such that

$mRx = (\theta)$ with $m \in M, x \in R$ implies that either $m = \theta$ or $x = \theta$. An additive mapping $F : R \to M$ is called a generalized (θ, Φ) -derivation (resp. generalized Jordon (θ, Φ)- derivation) on U if there exists a (θ, Φ)-derivation $d : S \to M$ such that $F(uv) = F(u)\theta(v) + \Phi(u)d(v)(resp.F(u^2) = F(u)\theta(u) + \Phi(u)d(u))$, holds for all $u, v \in U$. In the present paper, it is shown that if θ is injective on R, then every generalized Jordon (θ, Φ)-derivation F on U is a generalized (θ, Φ)-derivation on U. In fact our result is a wide generalization of the results proved in [Arch. Math. (Brno) 36(2002), 1-6,Theorem], [Tamkang J. Math. 32(2001),247–252,Theorem 3.1 and Theorem 3.2], [Math. J. Okayama Univ. 42(2000), 7–9,Theorem], [Internat. J. Math. Game Theory & Algebra 12(2002), 295–300,Theorem 2.1],[Comm. Algebra 32(2004), 2977–2985,Theorem 2.1], [Proc. Amer. Math. Soc.90(1984), 9–14,Theorem], [Proc. Amer. Math. Soc.104(1988), 1003–1006,Theorem 1], [J. Algebra 127(1989), 218–228,Theorem] and [Glasnik Mat.6(1991), 13–17,Theorem 1] [Proc. Amer. Math. Soc.8(1957), 1104–1110, Theorem 3.3] and [Comm. Algebra 13(1985), 1229–1244,Theorem 2.6].

B-15: A note on p-orthoposets, Vinayak Joshi and Shubhangi Kavishwar, *spk.maths@coep.ac.in*

In this paper, we introduce the concept of generalized Stonian p-orthoposets and study its properties. Further, an equivalent conditions for p-orthoposets to be generalized Stonian are given.

B-16: Admissibility of groups over function fields of p-adic curves, B. Surendranath Reddy, *surendra.phd@gmail.com*

Let K be a field and G a finite group. We say that G is admissible over K if there exists a division ring D central over K and a maximal subfield L of D which is Galois over K with Galois group G. One may ask when is G admissible over K. This question was originally posed by Schacher, who gave partial results in the case $K = Q$. In this paper, we give a necessary conditions for admissibility of a finite group G over function fields of curves over complete discrete value fields. Using this we give an example of a finite group which is not admissible over $Qp(t)$. We also prove a certain Hasse principle for division algebras over such fields.

B-17: Method of infinite ascent applied on $mA^3 + nB^3 = 3C^2$, Susil Kumar Jena, *susil kumar@yahoo.co.uk*

In this paper, we present a technique of generating infnitely many co-prime parametric solutions for (A, B, C) in the Diophantine equation $mA^3 + nB^3 = 3C^2$ for any pair of co-prime integers (m, n) where $(m^2 - n^2) = 3k$, and 3 is not a factor of k. We show how each of these parametric solutions breeds infnitely many co-prime integral solutions for (A, B, C) linked with this Diophantine equation.

B-18: Method of infinite ascent applied on $A^3 \pm nB^2 = C^3$, Susil Kumar Jena, *susil kumar@yahoo.co.uk*

In this paper we will produce different formulae for which the Diophantine equation $A^3 \pm nB^2 = C^3$ will generate infinite number of co-prime integral solutions for (A, B, C) for any positive integer n.

B-19: On Jacobi - Fourier transformation, T. G. Thange and M. S. Chaudhary, *tgthange@gmail.com*

In this paper we are going to construct different testing function spaces Y_a, X_a and their dual spaces for Jacobi-Fourier transformation. Also we define the generalized Jacobi-Fourier transformation on the dual space $X_a^{'}$.

B-20: Inversion and convolution formulae based on partition, N. Balasubramanian, *nbalu nbalu@yahoo.com, n.balasubramanian@gmail.com*

Convolution summations of the so-called additive and multiplicative types are well- known: $g(N) = \sum f(m, n)$ where m and n vary such that $m + n = N$ or $m.n = N$. A new type of vectorial convolution, wherein m, n are vectors of natural number components with N as their fixed dot-product, is presented as arising from functions of the form $F(G(x))$ where F and G are power-series of their respective arguments. So-based convolution and hence resulting inversion formulae are derived, showing a promising direction of prolific results.

C: Real and Complex Analysis (Including Special Functions, Summability and Transforms):

C-1: Certain results on maximum and minimum norm principle for bicomplex holomorphic function, Kumar Sen, *kumarsn84@gmail.com*

Maximum and minimum modulus principles of complex analytic functions are very important concepts in complex analysis and have found immense applications. In this paper, we have attempted to develop similar concept for the bicomplex valued functions of a bicomplex variable in terms of the maximum and minimum moduli of its idempotent component functions. Using classical results like Cauchy integral formula and Cauchy inequality, a holomorphic function defined in a discus is shown to possess maximum norm and minimum norm on the boundary of the discus.

C-2: On certain classes of multivalant functions, R. M. Dhaigude and M. G. Shrigan, *rmdhaigude@gmail.com, mgshrigan@gmail.com*

In this paper we introduce the special class $A(p)$ of multivalent function and obtained coefficient bounds and convolution results belonging to this class.

C-3: Iterated integral transforms and related identities, B. B. Waphare and S. B. Gunjal, *balasahebwapharewaphare@gmail.com, sbgb23gunjal@gmail.com*

In this paper a number of identities involving iterated integral transforms are established, making use of the fact that a function which is a linear combination of the Macdonalds function $K_{\alpha-\beta}(z)$, where z is a complex variable, is a Fourier kernel.

C-4: Some results on generalized hypogeometric functions, S. B. Rao, I. A. Salehbhai and A. K. Shukla, *ajayshukla2@rediffmail.com*

The principal aim of this paper is to study the various properties of generalized Hypergeometric Function. A recurrence relation and integral representation have also been discussed.

C-5: On σ-type zero polynomials, S. J. Rapeli, R. K. Jana and A. K. Shukla, *ajayshukla2@rediffmail.com*

The main object of present paper is to investigate some properties of σ-type polynomials in one and two variables.

C-6: Theorems with application to the modified stieltjes transformations, S. B. Gaikwad, *sbgaikwad_2008@rediffmail.com*

The Asymptotic behaviour of solutions of Mathematical models, Classical or generalized, using Abelian and Tauberian Type Theorems is of great importance and have practical use. In the last two decades many definitions of the asymptotic behaviour of distributions have been presented, elaborated and applied to integral transformations of distributions. In the first part we shall define the space $L'(r)$ and modified Stieltjes Transformation introduced by J. Lavoine and O. P. Misra and Marichev respectively. In the second part of paper we extend Tauberian type Theorems for the distributional Stieltjes Transformation to the distributional Modified Stieltjes transformations and its Tauberian type theorems based on the quasiasymptotic behaviour of distributions at zero and at infinity.

C-7: Perron stieltjes and Henstok stieltjes integrals in L^r,Sanket A. Tikare and M. S. Chaudhary, *sanket.tikare@gmail.com*

The present paper concerns with the introduction of an L^r-derivative with respect to a function and Perron-Stieltjes Integral for L^r-derivative, called PS^r-integral using major and minor functions. Also we define the Henstock-Stieltjes integral for L^r-derivative, called HS^r-integral and study some of its properties. It is shown that Henstock-Stieltjes integral is a particular case of HS^r-integral and HS^r-integral includes PS^r-integral. Finally, we characterize the indefinite HS^r-integral using absolute continuity with respect to a function.

C-8: Some further investigations on the growth properties of special type of differential polynomial, Sanjib Kumar Datta and Sudipta Kumar Pal, *sanjib kr datta@yahoo.co.in, sudipta.pal07@yahoo.com*

Let f be a meromorphic function and $S(f)$ denotes the set of all meromorphic functions $a \equiv a(z)$ in finite complex plane which satisfy $T(r, a) = o(T(r, f))$ as $r \to \infty$. We shall call $a \in S(f)$ as a small function with respect to f. Two meromorphic functions f and g are said to share a iff $E(f = a) = E(g = a)$. In this paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions and special type of a differential polynomial generated by one of the factors improving some earlier theorems on the basis of sharing of values of entire and meromorphic functions.

C-9: A new topology on the bicomplex space, Uma Sharma, *umamathsibs@gmail.com*

In literature, systematic study of Bicomplex Topology has been initiated and few topologies have been defined on C_2. In this paper, we have developed a new topology on C_2 in connection with meromorphic function of bicomplex variable. We have compared this new topology with those existing in literature.

C-10: Two - dimentional generalized offset fractional fourier transform, V. D. Sharma and P. S. Thakare, *vdsharma@hotmail.co.in*

The Offset Fourier Transform and Offset Fractional Fourier Transform are the space-shifted and frequency modulated version of the original transform.They are more general and flexible than the original ones. In this paper two-dimensional Offset Fractional Fourier Transform(2-D offset FRFT) is extended in the distributional generalized sense. Relation between 2-D offset Fourier transform and 2-D offset Fractional Fourier Transform is given. Analyticity theorem for two-diamensional offset Fractional Fourier Transform is also proved.

C-11: Inversion formula for two dimensional generalized fourier-Mellin transform and its Ω-applications, V. D. Sharma and P. D. Dolas, *vdsharma@hotmail.co.in*

The concept of signals and system arise in an extremely wide variety of fields. When the signals are impulsive type (Dirac delta signals $\delta(t)$ then the theory of generalized functions become a powerful tool. Due to shift form invariant property of Fourier-Mellin transform, it is frequently used in number of problems as pattern recognition, image restoration, computer data protection etc. In this paper two dimensional Fourier-Mellin transform extended in the distributional generalized sense. Inversion formula for the two-dimensional Fourier-Mellin transform is proved. Uniqueness theorem and application of Fourier-Mellin transform for solving Partial Differential Equation is also studied.

C-12: Certain results on the growth of a bicomplex entire function, Aijaz Ahmad Najar, *aijaznajar25@gmail.com*

In this paper we have initiated the study of growth properties of a bicomplex entire function. The concept of order is defined and the order of a bicomplex entire function has been estimated in terms of orders of its idempotent components. An estimate for the order of sum and product of two bicomplex entire functions has also been obtained.

C-13: Transformation formulae for poly-basic hypergeometric series, S. P. Singh, *snsp39@yahoo.com*

In this paper, we have established some very interesting transformation formulae for poly-basic hypergeometric series.

C-14: Ramanujan's theta functions and their evaluations for a particular value of q, V. Yadav, *snsp39@yahoo.com*

In this paper, using a modular equation of degree five, we have evaluated Ramanujans theta functions for $q = e^{-\pi}$.

C-15: On certain integral transforms and generalized voigt functions, M.Kamarujjama, Waseem A. Khan and M.A.Pathan,
waseem08 khan@rediffmail.com

In this paper, we first established an integral transform involving Struve function and then discussed a set of new representations of Voigt functions $K(x, y)$ and $L(x, y)$ and their unifications (or generalizations) in terms of familiar special functions of mathematical physics. In many physical problems (for example astrophysical spectroscopy, emission, absorption and transfer of radiation, plasma physics and theory of neutron reaction), a numerical or analytical evaluation of the Voigt functions is required. A set of new expansions from these representations are obtained.

C-16: Integral transforms and generating functions associated

with generalized Horn's functions, M. Kamarujjama, Waseem A. Khan and N. U. Khan, *mdkamarujjama@rediffmail.com,* *waseem08 khan@rediffmail.com, nabi khan1@rediffmail.com*

In this paper, authors present an integral transform involving Whittaker function and modified Bessel function. This paper also deals with a technique of integral transform to obtain generating relation for generalized Horns function ${}^{(k)}H_4^{(n)}$, which are partly bilateral and partly unilateral. Many known and unknown 4 generating functions involving generalized Horns functions of two and three variables are obtained as special cases.

C-17: A class of univalent analytic functions with varying argument of coefficients involving convolution, Prachi Srivastava, *prachi2384@gmail.com*

The object of present paper is to introduce a new class of univalent analytic functions with alternating arguments of its coefficients, which satisfy a subordinate condition involving convolutions, coefficients inequalities, results on growth and distortion theorems, extreme points, integral means inequality and partial sums of functions belonging to this class are obtained. Some consequences are also discussed.

C-18: Fractional integration of the H-functions and a general class of polynomials, Praveen Agarwal, *Goyal_praveen2000@yahoo.co.in*

The aim of the present paper to study and develop the generalized fractional integral operators recently introduce by Saigo. First, author establish two results that give the image of the products of two H-functions and a general class of polynomials in Saigo operators. These results, besides being of very general character have been put in a compact form avoiding the occurrence of infinite series and thus making them useful in applications. Our findings provide interesting unifications and extensions of a number of (new and known) images. For the sake of illustration, we give here exact references to the results (in essence) of five earlier research papers that follow as particular cases of our findings.

C-19: Two complex variants of generalized Hankel-Clifford transformation of generalized functions, S. P. Malgonde and V. R. Lakshmi Gorty, *spmalgonde@rediffmail.com, lakshmi_gorty@rediffmail.com*

In this paper, we consider two variants of the generalized Hankel-Clifford transformation and extend these transformations to certain spaces of generalized functions and prove several results on inversion, uniqueness, boundedness and analyticity. The operational calculus of these transformations is developed and then applied to solve certain class of partial differential equations of the Bessel-Clifford type differential operators.

C-20: An algorithm for computing zero-order generalized Hankel-Clifford transforms, S. P. Malgonde and V. R. Lakshmi Gorty, *spmalgonde@rediffmail.com, lakshmi gorty@rediffmail.com*

Postnikov, in 2003, proposed an algorithm to compute the Hankel transforms of order zero and one by using Haar wavelets. But the proposed method faced problems in the evaluation of zero order Hankel and Hankel-type transform. To overcome this problem an efficient algorithm for evaluating Hankel transform of order zero by using rationalized Haar wavelets was obtained by O. P. Singh. Exact analytical representation of the zero order generalized Hankel-Clifford transform, as series of the Bessel functions multiplied by the wavelet coefficients of the input function, is obtained in this paper. Numerical computations as well graphs are demonstrated using MATLAB software to illustrate the proposed algorithm.

D: Functional Analysis:

D-1: Fixed point theorems in a 2-metric space, D. P. Shukla, *shukladpmp@gmail.com*[0.2 cm]

In this paper, we have established fixed point theorems in 2-metric space self mapping and type of contraction mappings. The results presented in this paper substantially improve and extend the results due to Saha and Day and Rhoades.

D-2: Nn algebraic derivations induced by non surjective composition operators on l_2, Kamlesh Kumar Rai, *kkraib@rediffmail.com*

Derivation is a linear map D from a Banach Algebra A into self satisfying. $D(ab) = D(a)b + aD(b)$ for every $a, b \in A$. It is characterized in 7 theorems. In some cases these operators are similarly defined. Let x be an element of A. Define D_x on A into itself by $D_x(a) = axxa$ for every $a \in A$. It can be easily seen that $D_x(a)$ is a bounded derivation of A induced by x. A derivation D on A into itself is said to be an algebraic derivation if there is a non-zero complex polynomial $p(t)$ such that $p(D) = 0$. Algebraic operators are similarly defined.

D-3: Fixed point theorems in two metric places, N. M. Kavthekar, Department of Mathematics, Y. C. I. S., Satara.

Many authors such as Edelstein, Chatterjea, Kannan, Reich, Rogers and Hardy have generalized the Banach fixed point principle in different directions. D. S. Jaggi and Bal Kishan Das, through rational expression, have generalized Banach's contraction principle. In this paper, we have obtained fixed point theorems in complete two metric space for self map T. Also, we have obtained fixed point theorem for pair of mapping T_1 and T_2 and for sequence of self mappings T_i and T_j.

D-4: Fixed point theorems of self mapping in a complete 2-metric spaces, Nagender Bharadwaj, Department of Mathematics, Govt. P. G. Science College, Rewa, M. P.

In this paper, we have obtained some fixed point theorems in complete metric space using self mapping and mixed type of contraction mappings. The work of this paper extended the results due to Saha and Day and Rhoades.

D-5: A set of projections, Rajiv Kumar Mishra, *dr.rkm65@gmail.com*

In this paper we define a special type of operator called tetrajection on a linear space in analogue to a projection operator. We obtain a set of projections resulting from a given tetrajection and show that this set is closed with respect to multiplication of two projections. We also show that this set is a partially ordered set but not totally ordered.

D-6: Cubic trigonometric B-spline curves with a shape parameter, M. Dube and Reenu Sharma, *reenusharma@rediffmail.com*

In this paper a new kind of splines, called cubic trigonometric B-spline (cubic Trigonometric B-spline) curves with a shape parameter, are constructed over the space spanned by $\{\sin t, \cos t, \sin 2t, \cos 2t, \sin 3t, \cos 3t\}$. As each piece of the curve is generated by four consecutive control points, they posses many properties of the cubic B-spline curves. These trigonometric curves with a uniform knot vector are C^1 continuous. These curves are closer to the control polygon than the cubic B-spline curves. With the increase of the shape parameter, the trigonometric spline curves approximate to the control polygon. The generation of tensor product surfaces by these new splines is straightforward.

D-7: Dunkl transform of integrable Boehmians, A. Singh and P. K. Banerji, *abhijnvu@gmail.com*, *banerjipk@yahoo.com*

In this paper the Dunkl transform for integrable Boehmians is established and its properties are proved and, further, an inversion theorem of the same is established.

E: Differential Equations, Integral Equations and Functional Equations:

E-1: Hybrid fixed point theorem for first order functional integro differential equation, B. D. Karande, *bdkarande@rediffmail.com*

In this paper, we apply a hybrid fixed point theorem for a first order functional integro-differential equation for existence result under mixed generalized Lipschitz and Caratheodory type of conditions.

E-2: Existence theory for periodic boundary value problem of random differential equations, D. S. Palimkar, *dspalimkar@rediffmail.com*

In this paper, an existence of random solution are proved for a periodic boundary value problem of second order ordinary random differential equations. Our investigations have been placed in the space of real-valued functions defined and continuous on closed and bounded intervals of real line together with the applications of the random version of a nonlinear alternative of Leray-Schauder type and an algebraic random fixed point theorem of Dhage. An example is demonstrating the realizations of the abstract theory.

E-3: Comparison theorem for limit point case and limit circle case of singular Sturm-Liouville differential operators, Kishor J. Shinde and S. M. Padhye, *shinde_kj1805@yahoo.co.in*

We extend a result of J. Weidmann on the criteria for determining the limit point and limit circle case for Sturm-Liouville differential operators. This forms a comparison theorem for limit point and limit circle case for Sturm- Liouville differential operators

E-4: On global existence of solutions of nonlocal second order nonlinear functional integrodifferential equation, Rupali Jain, *rupalisjain@gmail.com*

In this paper, we investigate the global existence of solutions to second order initial value problem for an abstract nonlinear functional integrodifferential equation in Banach spaces with nonlocal condition. The results are obtained by using topological transversality theorem known as Leray-Schauder alternative and Pachpatte's inequality.

E-5: Invariants of generalized Emden Fowler equations, Sarita Thakar, Department of Mathematics, Shivaji University, Kohlapur.

The Emden-Fowler equation of index n is studied utilizing the techniques of Lie group analysis. For general n information about the integrability of this equation is obtained. As a special case analysis is carried out for canonical generalized Emden - Fowler equation.

E-6: On a new integral transform and solution of some integral equations, Shaikh Sadikali and M. S. Chaudhary, *sad.math@gmail.com, chaudhary_m_s@rediffmail.com*

In this paper, the convolution theorem and uniqueness theorem for the ELzaki transform is proved. Applicability of ELzaki transform is demonstrated for solving integral equations of convolution type and Abels integral equations.

E-7: Controllability of second order abstract mixed volterra-fredholm functional integrodifferential equations, K. D. Kucche and M. B. Dhakne, *kdkucche@gmail.com*

In this paper we establish the controllability result for second order mixed Volterra-Fredholm functional integrodifferential operator and the Schauder fixed point theorem. An example is provided to illustrate the results obtained.

E-8: A note on certain retarded integral inequalities, Jayashree V. Patil, *jv.patil29@gmail.com*

In this note, we generalize two retarded integral inequalities. One of these inequalities says: If $p > 1$,

$$\omega^p(t) \leq h^p(t) + p\int_0^{\alpha(t)} \left\{ f(s)\omega(s)\left[\omega^{p-1}(s) + \int_0^s g(\gamma)\omega^{p-1}(\gamma)\mathrm{d}\gamma\right] \right. \\ \left. + q(s)\omega(s)\mathrm{d}(s) \right\}$$

then

$$\omega(t) \leq \left\{ (h(t) + (p-1)) \int +0^{\alpha(t)} q(s\mathrm{d}s) \right. \\ \left. \left[1 + (p-1)\int_0^{\alpha(t)} f(s)\exp\left(\int_0^\infty (f(\sigma) + g(\sigma))\mathrm{d}\sigma\right)\mathrm{d}s \right] \right\}^{1/(p-1)}$$

for $t \in [\theta, \infty)$ under suitable conditions on functions ω, α, h, f, g and p on $[0, \infty)$.

E-9: On existence of abstract nonlinear Volterra-Fredholm functional integrodifferential equation with nonlocal condition,
M. B. Dhakne and Poonam S. Bora, *mbdhakne@yahoo.com, poonamsbora@gmail.com*

The aim of this paper is to study the existence of solutions of nonlinear Volterra-Fredholm functional integrodifferential equation with nonlocal condition in Banach space by using the Leray-Schauder Alternative.

E-10: Existence theorem for abstract measure integro-differential equations, S. S. Bellale and B. C. Dhage, *sidhesh.bellale@gmail.com, bcdhage@yahoo.co.in*

In this paper, an existence theorem for a abstract measure integrodifferential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for extremal solutions for Caratheodory as well as discontinuous cases of the nonlinearities involved in the equations.

G: Topology:

G-1: On some generalizations of open and closed topological spaces, Lal Babu Singh and K. Swetank, *nbasu10@gmail.com*

In this article we generalize closed sets in topological spaces. We also introduce some notations in this connection which are very useful in such study. We introduce some of our own notations in context of topological spaces. After introducing new notations we apply them to definitions and results in topology. Pre-open, Pre-closed, closed, B-open sets and some properties are given in terms of new notations. We have also generalized open and closed sets in our way and some results have been also obtained. Examples have been also given illustrating some of our results.

G-2: On pgpr-pre regular and pgpr-pre normal spaces,
P. Gnanachandra and P.Thangavelu, *pgchandra07@rediffmail.com*

Levine initiated the study of generalized closed sets in Topology. Since then topologists have utilized the concept to the various notions of subsets, weak separation axioms, weak regularity, weak normality and covering axioms in general topology. The concepts of g-regular and g-normal spaces were introduced and studied by Munshi. The authors recently introduced pgpr-regular and pgpr-normal spaces. The purpose of this paper is to introduce and study the concepts of pgpr-pre-regular and pgpr-pre-normal spaces.

G-3: On generalizations of fuzzy closed sets and functions,
P. Thangavelu and K. Bageerathi, *thangavelu@karunya.edu*

Fuzzy topology is a generalization of general topology. In 1968, Chang initiated the theory of fuzzy topological spaces. Since then several mathematicians contributed many papers to this area. The connection between the fuzzy open sets and the fuzzy closed sets is given by the standard complement λ' of λ defined by $\lambda'(x) = 1 - \lambda(x)$. Several fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complements in the fuzzy literature. The standard complement is obtained by using the function $C : [0,1] \to [0,1]$ defined by $C(x) = 1 - x$, for all $x \in [0,1]$. In this paper , the concept of fuzzy C-closed subsets of a fuzzy topological space is introduced where $C : [0,1] \to [0,1]$ is a function. In fact the notions of fuzzy closed sets and fuzzy closure with respect to the standard complement are extended to the analog concepts with respect to an arbitrary complement function $C : [0,1] \rightarrow [0,1]$. Using this extended fuzzy closure operator, some sets that are nearer to fuzzy open sets and fuzzy closed sets are studied. Fuzzy continuous functions and fuzzy irresolute functions are also extended in this sense.

G-4: On intuitionalistic L-fuzzy topological spaces, S. G. Dapke, *dapkesada@yahoo.com*

In this paper, an existence theorem for a abstract measure integro-differential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for extremal solutions for Caratheodory as well as discontinuous cases of the nonlinearities involved in the equations, under certain monotonicity. As a generalization of ordinary integro-differential equations, there is a series of papers dealing with the abstract measure integro-differential equations in which ordinary derivative is replaced by the derivative of set functions, namely, the Radon-Nokodym derivative of a measure with respect to another measure. See Dhage, Dhage and Bellale and the references therein paper. The above mentioned papers also include some already known abstract measure differential equations those considered in Sharma and Shendge and Joshi as special cases. The origin of the quadratic integral equations appears in the works of Chandrasekhar's H-equation in radioactive heat transfer, but the study of nonlinear integral equations via operator theoretic techniques seems to have been started by Dhage in the year 1988. Similarly, the study of nonlinear quadratic differential equations is relatively new and initiated by Dhage and ORegan in the year 1988. In the beginning, the development of the subject was slow, but recently this topic has gained momentum and growing very rapidly.

G-5: Generalized minimal closed maps in topological spaces, Suwarnlatha N. Banasode, *suwarn_nam@yahoo.co.in*

In this paper a new class of generalized minimal continuous maps that include a class of generalized minimal irresolute maps are introduced and studied in bitopological spaces. A mapping $f : (X, r_1, r_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(r_i, r_j) - \sigma_k$ generalized minimal continuous (briefly $(r_i, r_j) - \sigma_k - g - m_i$ continuous) map if the inverse image of every σ_k-minimal closed set in (Y, σ_1, σ_2) is $(\tau_i, \tau_j) - g - m_i$ closed set in (X, r_1, r_2).

G-6: On some smooth mappings between smooth fuzzy bitopological spaces, V. E. Nikumbh, *vinayaknikumbh@yahoo.co.in*

Smooth fuzzy topologies are an extension of both, crisp topologies and fuzzy topologies, where not only the sets but also the axiomatic are fuzzified. In this paper, we study some types of smooth functions on smooth fuzzy topological spaces, in the sense of Sostak and Ramadan. The relations among various types of continuity of smooth fuzzy functions on a fuzzy set and at fuzzy point belonging to the fuzzy set are discussed. Further we extend these notions to pairwise smooth mappings between smooth fuzzy bitopological spaces. Also various types of open, closed, continuous mappings are introduced and their interrelations are studied.

G-7: Connection properties in FN-spaces, Suprabha D. Kulkarni and S. B. Nimse, *sup_red_shri@yahoo.co.in*

In this paper, we have studied the connection properties in FN-spaces. We have defined the fuzzy uniform local uniform connectedness and fuzzy uniform local connectedness of fuzzy nearness spaces. We have shown that a FN-space is fuzzy uniformly locally uniformly connected iff its completion is. Also every topological fuzzy locally connected FN-space is fuzzy uniformly locally uniformly connected. We also establish the result for a fuzzy topological space X that X has a fuzzy locally connected regular T_1- extension iff X is s the underlying fuzzy topological space of a FN-space Y which is concrete, regular and fuzzy uniformly locally uniformly connected. Lastly we achieve the equivalence of the concepts of fuzzy uniform local connectedness and fuzzy local connectedness.

G-8: Fixed points for weak contraction in G-metric spaces, C. T. Aage and J. N. Salunke, Department of Mathematics, North Maharashtra University, Jalgaon - 425 001.

In this paper we have proved a fixed point theorem for weak contraction in G-metric spaces. Our result is supported by an example.

G-9: On fuzzy strongly α-continuous maps, M. Shrivastava, J. K. Maitra and M. Shukla, *jkmrdvv@rediffmail.com*

In this paper we have introduced and studied the concept of fuzzy strongly α-continuous maps on fuzzy topological spaces. We have established equivalent conditions for the map to be fuzzy strongly α-continuous map. Further we have observed some properties of fuzzy strongly α-continuous map.

H: Measure Theory, Probability Theory, Stochastic Processes and Information Theory:

H-1: A note on Karl Pearson's coefficient of correlation and its

application to mathematics, M. S. Bhalachandra,
msbhalachandra@yahoo.co.in

It is well known that "$Y = -Z \implies \rho(X,Y) = -\rho(X,Z)$" where ρ is Karl Pearson's coefficient of correlation between X and Y. The first section of the paper discusses the truth of the converse of the above result with the help of trigonometric functions. In the second section of the paper, regression models are developed for estimating the values of Sine and Cosine functions.

I: Numerical Analysis, Approximation Theory and Computer Science:

I-1: Exploring set theoretic operations on level two fuzzy sets,

Gandhi Supriya K., *gandhi.supriya@rediffmail.com,*
gandhi.supriya17@gmail.com

The history of fuzzy sets goes back to the year 1965 when Professor Lotfi Zadeh gave the birth to this term. Ten years later, the same professor introduced a new term called level-2 fuzzy sets. So, it was necessary to distinguish between level-2 fuzzy sets and what was invented before and named as fuzzy sets. For this reason, the pre-existing fuzzy sets were given the name type-1/level1 fuzzy sets. Even these level-1 fuzzy sets were used in many applications since 1965, it was noticed that they have limited capabilities to directly handle data uncertainties,*i.e.*, to model and minimize the effect of uncertainty. On the other hand, level-2 fuzzy sets are much more flexible in modern applications. This is possible only if proper set theoretic construction is done. Hence the author has introduced mathematical interpretation, new geometrical interpretation of level 2 fuzzy sets and their set theoretic structures. This paper provides an introduction of level 2 fuzzy sets (L_2FS). It does this by answering the following questions. What are (L_2FS) are they different from (L_2FS)/T_2FS? How are operations on (L_2FS) defined?

I-2: Beliefs induced by probability density functions,

D. N. Kandekar,*kandekardn@gmail.com*

In early development of belief function theory, frame of discernment is considered as power set of a set. As set contains too many elements *i.e.* sufficiently large then it becomes cumbersome to represent frame of discernment as lattice in order to observe embeddings then it is approximated by an interval. Therefore discrete variables are approximated by continuous variable which mostly represents real life situations. Infinite number of subsets of the interval exists, the power set is generalized to Borel sigma-algebra. Only interested subsets in the frame of discernment are considered as special case, nested intervals. In all these cases, we tried to summarize and find the beliefs induced by underlying probability density functions.

I-3: A two party secret handshake scheme with dynamic matching, Manmohan Singh Chauhan, Preeti Kulshrestha and Sunder Lal,
manmohansingh27@gmail.com, ibspreeti@gmail.com,
sunder lal2@rediffmail.com

Balfanz et al. introduced in 2003, Secret Handshake (SH) protocol which allows two members of a specific group to authenticate themselves secretly to the other whether they belong to a same group or not, in the sense that each party reveals his affiliation to the other only if the other party is also a member of the same group. They also introduced the SH with roles, so that a group member A can specify the role another group member B must have in order to successfully complete the protocol. Ateniese et al in 2007 presented a SH with dynamic matching in which each party can specify both the group and the role the other must have in order to complete a successful handshake. This paper presents a new SH scheme with dynamic matching that is computationally more efficient than existing scheme of Ateniese et al. The proposed scheme is inspired on an identity based key pair derivation algorithm proposed by Sakai et al and uses ZSS signature. The paper also gives security proofs for the new scheme in the random oracle.

I-4: Approximation by iterates of beta operators, P. N. Agrawal, Karunesh Kumar Singh and Vikas Kumar Mishra, *pna_iitr@yahoo.co.in,*
kksiitr.singh@gmail.com, vikas.mishra45@gmail.com

In this paper, we study the degree of approximation by an iterative combination $T_{n,k}$ of the beta operators introduced by Upreti (J. Approx. Theory, (1985) , 85–89) wherein she studied some approximation properties of these operators. Subsequently, Ding-Xuan Zhou (J. Approx. theory 66, (1991), 279–287) discussed the direct and inverse theorems in L^p- approximation for these operators. Since then no other work has appeared for these operators. It turns out that the order of approximation by these operators is, at best, $O(n^{-1})$, howsoever smooth the function may be. In our paper an attempt has been made to improve this rate of convergence by Beta operators using the iterative combination technique.

I-5: Approximation of derivative in a singularity perturbed second-order ordinary differential equation with discontinuous source term subject to mixed type boundary conditions, R. Mythili Priyadharshini and N. Ramanujam, *mythiliroy777@yahoo.co.in*

In this paper, a singularly perturbed second-order ordinary differential equation with discontinuous source term subject to mixed type boundary conditions is considered. A robust-layer- resolving numerical method is suggested. An -uniform error estimates for the numerical solution and also to the numerical derivative are derived. Numerical results are presented, which are in agreement with the theoretical results.

I-6: Selection of machine using fuzzy multiple criteria decision making problem, A. A Muley, N. S. Darkunde and M. R. Fegade, *aniket.muley@gmail.com, darkundenitin@gmail.com, mrfegade@gmail.com*

In this paper, we have proposed the method to solve the machine problem.The main concern of this study is to find the sequence of the machine so that the cost will be minimum or to reduce the expenditure and maximizes the profit of the whole project. To illustrate the feasibility of the proposed method is checked with a numerical example.

J: Operations Research:

J-1: Solving multi objective transportation problem using fuzzy logic, M. R. Fegade and V. A. Jadhav, *mrfegade@gmail.com*

In this paper, we have proposed the method to solve the multi objective transportation problem. In this study we have proposed the algorithm for finding minimum cost and maximize the profit. Here we have proposed the different membership function for checking feasibility of the proposed method with a numerical example.

J-2: Solution of a fuzzy transportation problem, Poonam Shugani, S. S. L. Jain, P. G. College, Vidisha (M.P.), S. H. Abbas, Safia Science College, Bhopal (M.P.) and Vijay Gupta, UIT, RGPV Bhopal (M.P.)

There are various methods, in the literature, for determining the fuzzy optimal solution of a fuzzy transportation problem (transportation problems in which all the parameters are represented by fuzzy numbers). In earlier work we have solved the Fuzzy transportation problem with the help of dual simplex and two phase method. Here we solve a simple balanced fuzzy transportation problem of trapezoidal number with several methods of linear programming problems with the help of Tora package. We also compare the results of linear programming methods with result of Vogel Approximation Methods. Finally, we give illustrative example and their numerical solutions.

J-3: Two stage reinforcement to the security force in a counter insurgency operation, Lambodara Sahu, *lsahucme@gmail.com*

In this paper, an attempt is made to figure out the importance of second stage reinforcement with a good strength at an early stage of combat operation by considering a few case studies.

K: Solid Mechanics, Fluid Mechanics, Geophysics and Relativity:

K-1: Anisotropic bianchi type - I model with varying term, Divya Singh, Department of Mathematics, Govt. Model Science College, Rewa.

Einstein field equations with variable gravitational and cosmological constants are considered in the presence of perfect fluid for Binachi type-I universe by assuming the cosmological term proportional to the hubble parameter. The variation law for vacuum density has recently been proposed by Schiutzhold on the basis of quantum field estimation in the curved expanding background. The cosmological terms tends asymptotically to a genuine cosmological constant and model tends to deSitter universe. We obtain that the present universe is accelerating with a large fraction of cosmological density in the form of cosmological term.

K-2: Anisotropic bianchi type - I cosmological model with varing G and A, Pratibha Shukla, Department of Mathematics, Govt. Model Science College, Rewa.

We investigate Binachi type-I cosmological modles containing perfect fluid satisfying perfect gas equation of state. Gravitational constant G and cosmological constant A are taken to be time dependent. For the solution of the field equations we assume the cosmological term proportional to f_2 where F is volume of the universe). the physical significance of the cosmological models have also been discussed.

K-3: Explanation for superluminal motion observed in large hadron collider, Ravindra D. Rasal, Department of Mathematics, Maharaja Ranjit Singh International Institute.

CERN scientists, contradicting Einstein's Special Theory of Relativity, claimed recently to have detected faster than light Neutrinos in Large Hardon Collider. This phenomenon is explained, in this paper, on the basis of hypothesis that "existence " is associated with a factor which aliens space with the arrow of time. It is expected to figure out in the reactions conducted under extra-ordinary conditions such as created in Lagr Hardon Collider. Transformations representing this phenomenon are obtained by introducing a new factor which has finite magnitude and it drags space with the Arrow of time. It is conceived like a "spin" that can be isolated from other aspects of matter, under appropriate conditions. This type of transformation is qualitatively different from Tachyons and would be helpful in explaining other types of faster than light observations.

K-4: Frw bulk viscuous cosmology in higher dimensional space time, R. S. Rane, P. R. Dhabne and S. D. Katore, *rsrane53@rediffmail.com, katoresd@rediffmail.com.*

The exact solutions of the field equations are obtained by using the gamma law equation of state $p = (\gamma - 1)p$ in which the parameter γ depends on scale factor R. The fundamental form of $\gamma(R)$ is used to analyze a wide range of phases in cosmic history: inflationary phase and radiation-dominated phase. The corresponding physical interpretations of cosmological solutions are also discussed in the framework of higher dimensional space time.

K-5: Einstein rosen bulk viscous cosmological solutions with zero mass scalar field in lyra geometry, S. D. Katore, M. M. Sancheti and J. L. Pawade, *katoresd@rediffmail.com, shaikh 2324ay@yahoo.com.*

In this paper, we have investigated cylindrically symmetric Einstein-Rosen cosmological model with bulk viscosity and zero-mass scalar field in Lyra geometry. The cosmological models are obtained with the help of the special law of variation for Hubbles parameter proposed earlier by Bermann. Some physical properties of the models are discussed.

K-6: Accelerating and decelerating kasner universe with perfect fluid and dark energy, S. D. Katore, S. H. Shaikh, B. B. Chopade and S. A. Bhaskar, *katoresd@rediffmail.com*

We consider a self-consistent system of Kasner Universe cosmology and binary mixture of perfect fluid and dark energy. The perfect fluid is taken to be one obeying the usual equation of state $p = \gamma\rho$ with $\gamma \in [0, 1]$. The dark energy is considered to be either the quintessence or Chaplygin gas. Exact solutions to the corresponding Einstein equations are obtained as a quadrature. Models with power-law and exponential expansion have discussed in detail.

K-7: Bianchi type - VI cosmological models with modified chaplygin gas, S. D. Katore, K. S. Wankhade and S. P. Hatkar, *katoresd@rediffmail.com*

We consider a modified chaplygin gas model in Bianchi type VI universe with equation of state $P = \gamma\rho - \frac{\beta}{\rho^{\alpha}}$. For complete solution of Einstein's field equation we assume that expansion (θ) in the model is proportional to shear (σ) and equation of state of this modified model is valid from the radiation era to $\triangle CDM$ model. State finder and various physical, geometrical properties have also been discussed.

K-8: Generalized dispersion of chiral fluid in a channel bounded by porous layers in the presence of convective current,

N. Rudraiah and S. V. Raghunatha Reddy, UGC-Centre for Advanced Studies in Fluid Mechanics, Department of Mathematics, Bangalore University, Bangalore - 560 001.

The generalized dispersion model is used in this paper to study the dispersion of chiral fluid in channel bounded by porous layers in the presence of a transverse magnetic field. Analytical solutions for velocity are obtained in the presence of transverse magnetic field and it is evaluated for different values of electromagnetic parameter Wem. Concentration distribution is determined analytically using the series solution in the presence of advection of concentration by the chiral fluid. It is shown that the chiral molecules are dispersed relative to a plane moving with the mean speed of velocity of the fluid with a relative diffusion coefficient $K_2(\tau) - \frac{1}{Pe^2}$. This $K_2(\tau) - \frac{1}{Pe^2}$ is numerically computed and the results reveal that it increases with an increase in the Reynolds number Re but decreases with an increase in porous parameter. This decrease in generalized dispersion coefficient $K_2(\tau) - \frac{1}{Pe^2}$, with the increase in σ and Re is shown to be favorable for the formation of artificial organs like synovial joints free from the side effect of haemolysis.

K-9: The $Z = (t/Z)$ - type plane gravitational waves of weakened field equations in plane symmetry, S. R. Bhoyar, V. R. Chirde and A. G. Deshmukh, *sbhoyar68@yahoo.com, vrchirde333@rediffmail.com dragd2003@yahoo.co.in*

In this paper, we propose to obtain the $Z = (t/z)$ - type plane gravitational waves in a set of five vacuum weakened equations in the space-time introduced by Bhoyar and Deshmukh (Int. Jour. Theo. Phy. 2011) having plane symmetry in the sense of Taub [Ann. Math. 53, 472 (1951)]. These vacuum field equations has been suggested as alternatives to the Einstein vacuum field equations of general relativity. Furthermore the physical significance of modified gravitational waves for the space-time is obtained.

K-10: Effects of variation of viscosity and viscous dissipation on oberbeck magnetoconvection in a chiral fluid, N. Rudraiah and N. Sujatha, UGC-Centre for Advanced Studies in Fluid Mechanics, Department of Mathematics, Bangalore University, Bangalore 560 001.

The flow and heat transfer characteristics of Oberbeck convection of a chiral fluid in the presence of the transverse magnetic field, viscous dissipation and variable viscosity are investigated. The coupled non-linear ordinary differential equations governing the flow and heat transfer characteristics of the problem are solved both analytically and numerically. The analytical solutions are obtained using a regular perturbation and numerical solutions obtained using finite difference method. The solution is valid for small values of Buoyancy parameter N and variable viscosity parameter R_1. The analytical results are compared with the numerical results and found good agreement. The role of temperature dependent viscosity and viscous dissipation on velocity, temperature, skin friction and the rate of heat transfer are determined. The results are depicted graphically, from these graphs it is noticed that the velocity is parabolic in nature and increases with an increase in magnetochiral number, M. Physically this is attributed to the fact that magnetochiral number introduces small scale turbulences.

K-11: Bianchi type - V isotropic cosmological model with strange quark matter attached to cosmic string, V. R. Chirde and P. N. Rahate, Gopikabai Sitaram Gawande College Umarkhed, Dist.: Yaotmal-445206, Sarashwati Higher Secondary School, Kinwat.

In this paper we have investigated Bianchi type-V cosmological model with strange quark matter attached to the string cloud in general relativity. The model is obtained with the help of special law of variation for Hubble parameter proposed by Bermann (Nuovo Cimento 74 B: 182, 1983). Also, some physical and kinematics properties of the model are studied. The results are similar to results obtained by Yilmaz (Gen. Rel. Grav. 38:13971406, 2006).

K-12: Bianchi type - III string cosmological models, K. S. Adhav, M. V. Dawande and V. B. Raut, *ati ksadhav@yahoo.co.in, vbraut62@yahoo.com*

Bianchi Type-III space time is considered in presence of cosmic strings in Einsteins general theory of relativity . Exact cosmological models are presented with the help of the relation QUOTE between metric coefficients C and B. Some physical properties of the model in each case are discussed.

K-13: Finite difference analysis of free convection on stroke's problem for a vertical plate in a rotating dissipative fluid with constant heat flux, V. B. Bhalerao and R. M. Lahurikar,
vbhalerao2010@gmail.com, rmlahurikar@gmail.com

Stokes problem for a vertical plate has been considered for an incompressible viscous fluid on taking into account viscous dissipative fluid with constant heat flux. The couple non linear equations are solved by finite difference method. Axial velocity profile, transverse velocity profile and temperature profile are shown graphically where as numerical values of the skin friction and Nusselt number are listed in a table. The effect of different parameters are discussed. It is observed that greater viscous dissipative heat (Ec) causes rise in the transverse skin friction but there is a fall in axial skin friction and heat transfer. An increase in Gr

leads to increase in transverse skin friction and heat transfer but there is a fall in axial skin friction and heat transfer. Due to increase in Pr there is a decrease in transverse skin friction, whereas rise in axial skin friction and heat transfer. If t increases then axial velocity and temperature rise where as transverse velocity falls.

K-14: Geometry of quark and strange quark matter in higher dimensional general relativity, G. S. Khadekar and Rupali Wanjari, *gkhadekar@yahoo.com,rwanjari@gmail.com*

In this paper we study quark matter and strange quark matter in higher dimensional spherical symmetric space-times. We analyze strange quark matter for the different equations of state and obtain the space-time geometry of quark and strange quark matter. We also discuss the features of the obtained solutions in the context of higher-dimensional general relativity.

K-15: Viscous fluid cosmological models with strange quark matter, G. S. Khadekar and N. V. Gharad, *gkhadekar@yahoo.com*

In this study Bianchi type -V cosmological model solutions are obtained for quark matter. For this purpose Einstein field equations are solved in framework of Bianchi type-V space time filled with viscous fluid. Some physical and kinematical quantities for the modes are obtained. Also the nature of the model and features of obtained solutions are discussed.

L: Electromagnetic Theory, Magneto-Hydrodynamics, Astronomy and Astrophysics:

L-1: Numerical solution of radiative transfer equation in gray medium, J. Banerjee, A. K. Shukla and Amit D. Patel, *ajayshukla_svrec@yahoo.co.in*

An attempt is made to obtain numerical solution of radiative transfer equation in presence of absorbing, emitting, non-scattering medium for gray medium.

L-2: Viscous dissipation effects on heat transfer in flow past a continuously moving porous plate, Ashwani Kumar Sinha, Sweta Sinha and Rashmi Sinha, M. M. Mahila College, ARA and Alok kumar, P. G. Dept. of Mathematics, Maharaja College, ARA, Bihar.

An analysis of the effects of viscous dissipative heat on the temperature and rate of heat transfer in the flow past a continuously moving flat porous plate has been carried out. It has been observed that greater viscous dissipative heat causes a rise in the temperature and a fall in the rate of heat transfer for both suction and injection.

L-3: Finite difference analysis of dean's stability problem,
Ashwani Kumar Sinha, M. M. Mahila College, ARA, Bihar.

The stability of the flow of a viscous fluid in a curved channel, due to a transverse pressure gradient, is studied by finite-difference method. Axisymmetric disturbances in a narrow-gap case are considered. Our results agree well with earlier results.

L-4: Generalized spectrum of linear codes associated to schubert varieties, Mahesh S. Waware, S. B. Nimse and Arunkumar R. Patil, Department of Mathematics, Rajarshi Shahu Mahavidyalaya, Latur - 413 512.

Denote by $G(\ell, m)$, the Grassmannian of ℓ-dimensional subspaces of an m-dimensional vector space $\mathbb{F}_q^m$ over the finite field $\mathbb{F}_q$ and $\Omega\alpha(\ell, m)$, the Schubert subvarieties of $G(\ell, m)$. A linear $[n, k]_q$ code is a k-dimensional subspace of the n-dimensional vector space $\mathbb{F}_q^n$. In this paper, we consider the problem of determining generalized spectrum of linear codes associated to Schubert subvarieties of Grassmannians. We make a small begining here by determining the second generalized spectrum (i.e. second weight distribution) of Schubert codes associated to Schubert subvarieties of $G(\ell, m)$ over $\mathbb{F}_2$ in case of $\ell = 2$ and $m = 5$.

L-5: MHD forced convection with laminar pulsating flow in a channel, B. G. Prasad and Raj Shekhar Prasad, *dr.balgangadhar@yahoo.com, shekharraj.2010@gmail.com*

In this paper a mathematical model will be analyzed in order to investigate the effect of magneto hydrodynamics on forced convection with laminar pulsating flow in a parallel plate channel. The analytical expression is given for the velocity temperature and the transient Nusselt number to first order in by using perturbation technique. It is assumed that the relative amplitude of the pulsation is considered as small in compared to unity. The case of uniform flux boundary condition is considered in turn. The peck let number assumed to be large so that the axial heat conduction can be neglected. The effect of various parameters entering into the problem can be discussed with the help of graphs.

L-6: Effects of chemical reaction and radiation on an unsteady mhd flow past an accelerated infinite vertical plate with variable temperature and mass transfer, N. Ahmed, J. K. Goswami and D. P. Barua, *saheel nazib@yahoo.com*

This paper deals with the study of the effects of chemical reaction and radiation on an unsteady MHD flow of an incompressible viscous electrically conducting fluid past an accelerated infinite vertical plate with variable temperature and mass transfer. A uniform magnetic field is assumed to be applied normal to the plate directed into the fluid region. The expressions for the velocity field, temperature field and concentration field and skinfriction in the direction of the flow, coefficient of heat transfer and mass flux at the plate have been obtained in non-dimensional form. The effects of different physical parameters namely chemical reaction parameter K, radiation (absorption) N, Schmidt number S_c, Hartmann number M, Grashof number for mass transfer G_m and time t on the flow and transport characteristics are discussed through graphs and results obtained are physically interpreted. It is seen that the viscous drag is reduced due to the application of transverse magnetic field for larger t and an increase in Grashof number for mass transfer or absorption of thermal radiation results in a growth in the magnitude of the drag force whereas it falls under the effects of chemical reaction and Schmidt number.

L-7: An oscillatory three dimensional flow past an infinite vertical porous plate with soret and dofour effects temperature and mass transfer, N. Ahmed, H. Kalita and D. P. Barua, *saheel nazib@yahoo.com, kalita.himanshu@gmail.com, math byte@yahoo.com*

The problem of an oscillatory three dimensional flow past an infinite vertical porous plate with Soret and Dufour effects is presented. Analytical solutions to the coupled non-linear equations governing the flow and heat and mass transfer are solved by regular perturbation technique. The expression for the velocity field, temperature field, species concentration, current density, the Skin-friction, Nusselt number and Sherwood number at the plate are obtained in non-dimensional forms. The velocity distribution, temperature, chemical species concentration, Coefficient of Skin-friction, Nusselt number and Sherwood number at the plate are demonstrated graphically and the effects of different parameters *viz.* the Suction Reynolds number, the Grashof numbers, the Soret number and the Dufour number on these fields are discussed.

L-8: Strange quark matter coupled to the string cloud in kaluza-klein theory of gravitation, G. S. Khadekar and Nirmala Ramtekkar, *gkhadekar@yahoo.com, nirmala.a.r@gmail.com*

The solution are obtained for quark matter coupled to the string cloud in Kaluza-Klein theory of gravitation. The features of the obtained solutions are also discussed.

L-9: Higher dimensional cosmological model with quark and strange

quark matter, G. S. Khadekar and Rajani Shelote,
gkhadekar@yahoo.com

We study higher dimensional homogeneous cosmological model in the presence of quark and strange quark matter. The dynamical behavior of the model for the strange quark matter equation of state of the form $p = \frac{1}{3}(\rho - 4B_c)$ are studied.

M: Bio-Mathematics:

M-1: A mathematical model for the tumors cell density and immune response,

Sanjeev Kumar and Ravendra Singh,
sanjeevibs@yahoo.co.in

With the recent development of the mathematical models of cancerous growth, we consider a procedure for cancer therapy which consists of interaction between immune response (immune cells) and tumor cells without any specific drug. The cytotoxic T lymphocyte (CTL) and tumor necrosis factor (TNF) cause of the immune response. This process is modeled as a system of tumor cell density (TCD) and tumor necrosis factor (TNF). The purpose of this study is to establish a rigorous mathematical analysis of the model and to explore the density/concentration of tumor cell and immune response (TNF). The result suggests that although TCD capable to growth of tumor but the immune response is block to direct tumor growth.

M-2: Space time dependence of BIS index numbers, their uses in estimating the future of our earth, galaxy and universe,

M. M. Bajaj, Ashwani Kumar Sinha, Rashmi Sinha and V. R. Singh, International Kamdhenu Ahinsa University (I.K.A.U), A 122, Vikaspuri, New Delhi - 110 018.

In this paper, we study
(1) BIS index of a country,
(2) BIS index of a nation, and
(3) BIS index of a state/region/zone/area.

M-3: Life insurance underwriting when insurer is diabetic: a fuzzy approach,

Sanjeev Kumar and Hemlata Jain, *sanjeevibs@yahoo.co.in*

In life insurance medical underwriting, the mortality of the applicants within the period of insurance is assessed on the basis of present risk factors or diseases. Unlike hypertension or overweight, where risk assessment of the medical values is possible directly, the risk assessment of diabetes is a complex problem. If a person is diabetic then to find out the mortality of insurer for life insurance medical underwriting is a complex problem due to multitude of medical risk factors. For life insurance medical underwriting, to handle this complex problem, the insurance companies are in need to have a reliable expert system that can help them to evaluate the mortality of the applicants with in the period of insurance. Here a model is presented, which is based on fuzzy expert system that will help insurance companies to find out the mortality of insurer in the existence of diabetes for life insurance underwriting.

M-4: A fuzzy approach to calculate the health insurance premium of a person having cardiovascular disease, Sanjeev Kumar and Kushal Pal Singh, *sanjeevibs@yahoo.co.in*

Traditional actuarial methodologies have been built upon probabilistic models. They are often driven by stringent regulation of the insurance business. Deregulation and global competition of the last two decades have opened the door for new methodologies, one among them being fuzzy method. The defining property of fuzzy sets is that their elements possibly have partial membership (between 0 and 1), and therefore this always help to design such models where the membership (belongings) is lies between 0 and 1. Hence in this work a fuzzy model is considered for the calculation of premium of health insurance of a person having cardiovascular disease. This work provides a quick overview of developing fuzzy sets methodologies in actuarial science, and the basic mathematics of the fuzzy sets. Further it is also shown how to build and fine-tune fuzzy logic models for changing rules to reflect supplementary data.

M-5: Fuzzy mathematical model for medical diagnosis, Mohd Mahatab and Sanjeev Kumar, *sanjeevibs@yahoo.co.in,*
sanjeevibs@yahoo.co.in

Most mathematical models of physician decision processes offered of date, especially those relative to diagnosis and patient treatment, suffer from the inability to incorporate all useful data on the patient. Pertinent information so neglect or poorly modeled relate to variables that are intrinsically fuzzy but which describe the patient health status. Therefore here we present a mathematical model based on fuzzy set theory for physician aided evaluation of a complete representation of information emanating from the initial interview including patient present symptoms signs observed upon medical diagnosis of the diseases.

N: History and Teaching of Mathematics:

N-1: Effect of conceptual teaching of modern mathematics on +2 students (tribal and non-tribal) of Jharkhand state, G. Rabbani, Ranchi University, P.O. Medical College, Ranchi 834 009.

Ranchi, Bokaro, and Jamshedpur towns of Jharkhand state provide a good number of students to IIT every year, the prestigious technical Institutions of India. Also it is a tribal dominated area. Keeping these facts in mind in this research paper, I have done this experiment "Effect of conceptual teaching of Modern mathematics on +2 students (tribal and non-tribal) of Jharkhand State". The research was done by a standardized test containing all parts of modern mathematics. It was administered on a random sample of 400 students Tribal and non-tribals of Jharkhand state and the result obtained was analyzed mathematically. It was found that there is no significant difference between the two groups if the conceptual teaching is imparted to them.

N-2: History of complex analysis, R. B. Dash, H. K. Sharma, *Rajani bdash@rediffmail, harikishoresharma9@gmail.com*

In most of the books of Complex Analysis there is no mention of historical information about the origin of the subject. However from the life history of different mathematicians and their contribution towards complex analysis in this paper we tried our best to present systematically development of this subject through different years.

N-3: Critical thinking in nepal: an initiative for better teaching and learning environment, B. P. Sapkotta, *buddhisapkota@yahoo.com*

Critical thinking is an emerging trend for classroom teaching, decision making and personnel development. People in different location are using the didactic learning which believes on attention, recall, recitation, single path to understanding, limited learning styles. But critical learning is an approach based on exploration, creativity, critical analysis, indirect paths to understanding, multiple learning styles. Critical thinking is the art of analyzing and evaluating thinking with a view to improve it. It gives an idea to move beyond what we already know, expand the domain of knowledge, and know when a discovery or conclusion constitutes new knowledge. Critical thinking approach and some initiations made in Nepal are discussed in this paper.

Abstracts of the papers for IMS prizes GROUP -1:

IMS-1: On zero divisor graph of Boolean poset, Anagha Khiste, *avanikhiste@gmail.com*

In this paper, we characterize the graphs which can be realizable as zero divisor graphs of Boolean posets. Further, it is provided that if B is a Boolean poset and S is a bounded pseudocomplemented poset such that $S\mathbb{Z}(S) = 1$ then

$\Gamma(B) \cong \Gamma(S)$ if and only if $B \cong S$.

IMS-2: New combinatorial interpretations of two basic series identities, Megha Goyal, *meghagoyal2021@gmail.com*

Analogous to MacMahon's combinatorial interpretations of Rogers-Ramanujan identities, we in this paper, using n-colour partitions of Agarwal and Andrews, provide combinatorial interpretations of two identities one of which was given by L.J. Slater and the other by G. E. Andrews.

IMS-3: On primary ideals in lattices, Nilesh Mundlik, *nileshmundlik@rediffmail.com*

In this paper, we define a prime radical in lattices. Further, we obtain some properties of radical of an ideal in lattices. Also, we define a primary ideal in lattices. We prove that in a 1-distributive complemented lattice radical of a primary ideal is prime. Further, an analogue of first and second uniqueness theorems regarding primary decomposition of an ideal are obtained.

IMS-4: Relationship between abelian and constabelian codes, Pooja Grover, *poojagrover1285@gmail.com*

In this paper, the relationship between abelian codes and corresponding constabelian codes of smaller lengths has been studied. We have proved that for a divisor m of a positive integer n, an abelian code of length n can be written as (direct) sum of either m abelian or constabelian codes.

IMS-5: Congruence relation in lattice of convex sublattices of a lattice, H. S. Ramananda, *ramanandahs@gmail.com*

Let L be a lattice. In this paper, corresponding to a given congruence relation θ of L, a congruence relation Ψ_θ on $CS(L)$ is defined and it is proved that (i) $CS(L/\theta)$ is isomorphic to $\frac{CS(L)}{\Psi_\theta}$; (ii) L/θ and $\frac{CS(L)}{\Psi_\theta}$ are in the same equational class; (iii) if θ is representable in L, then so is Ψ_θ in $CS(L)$.

IMS-6: Zero divisor graph of a reduced lattice, Sachin Sarode, *sarodemaths@gmail.com*

In this paper, we introduced the zero-divisor graph $\Gamma(L)$ of a multiplicative lattice L. It is shown that $\Gamma(L)$ is connected with its diameter ≤ 3. The complete bipartite zero-divisor graphs $\Gamma(L)$ are characterized. An algebraic and a topological characterization is given for the graph $\Gamma(L)$ to be triangulated or hypertriangulated. We show that the clique number of $\Gamma(L)$ and the cellularity of $\mathrm{Spec}(L)$ coincide. In a semi-simple *mpm*-lattice, it turns out that the dominating number of $\Gamma(L)$ is between the density and the weight of $\mathrm{Spec}(L)$. Further, we prove a form of Beck's conjecture for the reduced lattices.

GROUP -2:

IMS-7: On (κ, μ) contact metric manifolds, Avijit Sarkar, *avjaj@yahoo.co.in*

The tensor $h = 1/2L_\epsilon$ denoting the Lie derivative, plays a crucial role to determine the nature of a (κ, μ) contact metric manifold. The object of the present paper is to study (κ, μ) contact metric manifolds for which the tensor h is parallel, recurrent and cyclically parallel. Three-dimensional (κ, μ) contact metric manifolds with recurrent Ricci tensor have been studied. Illustrative examples, related to the results obtained in each section, are also given.

IMS-8: Bicomplex nets and their zones of clustering, Sukhdev Singh, *ssukhdev1209@yahoo.com*

In this paper, we have studied the clustering of the bicomplex nets with respect to the idempotent order topology. We have divided three sections. In section 1, certain basics of the bicomplex numbers and clustering of the nets have been mentioned, In section 2, we have defined different types of the bicomplex nets and also have proved some results related to the conditions required for the clustering of the bicomplex nets. In section 3, we have given a relation between the clustering of the bicomplex net and the ID-confinement of its subnets.

IMS-9: Uniform versions of index for uniform spaces with free involutions, Jaspreet Kaur, *jasp.maths@gmail.com*

In this paper, uniform versions of index for uniform spaces equipped with free involutions are introduced. They are mainly based on B-index defined and studied by C.-T. Yang in 1955, index studied by Conner and Floyd in 1960 and further development well collected by Matousek in his book on using the Borsuk-Ulam theorem in 2003. Examples of uniform spaces with finite B-index but infinite uniform version of index are given. It is also seen that for a uniform space X with a free involution T , a dense T -invariant subspace is capable of determining the uniform version of index of (X, T). In the end, the concept of coloring is carried over to uniform set up and, to a certain extent, connection between uniform versions of colorings and uniform versions of index is also established.

GROUP -3:

IMS-10: A generalization of mittag-leffler density and process, *Pratik V. Shah, pratikshah8284@yahoo.co.in*

An attempt is made to generalize and investigate some properties of Mittag-Leffler density and also discuss the Structural representation of the Mittag- Leffler variable.

GROUP -4:

IMS-11: ϵ-uniformly convergent finite difference scheme for singularly perturbed delay differential equations with twin boundary layers, Pratima Rai, *pratimarai5@gmail.com*

This paper analyzes singularly perturbed delay differential equation with interior turning point. Presence of turning points results into boundary or interior layers in the solution of such type of problems. We have studied the case where the presence of the turning point results into twin boundary layers in the solution of the considered problem. Development of numerical schemes to approximate the solution of such type of problems encounters great difficulties due presence of the perturbation parameter, delay term and the turning point. We derive certain a priori estimates on the solution and its derivatives. Finite difference scheme based on Shishkin meshes is constructed and the resulting scheme is shown to be almost first order accurate in the discrete maximum norm. Numerical results complement the theoretical results and also demonstrate the effect of the delay on the boundary layers.

IMS-12: Periodic solutions of delayed differential equations, Shilpee Srivastava, IIT, Kanpur.

We study the existence of multiple positive T-periodic solutions for the first order delay difference equation of the form $\Delta x(n) = \alpha(n)g(x(n))x(n) - \lambda f(n, x(n-\tau(n)))$. Leggett-Williams multiple fixed point theorem has been employed to prove the results, which are established considering different cases on functions g and f.

IMS-13: Bilinear optimal control for a heat equation, Sonawane Ramdas Baburao, *sonawaneramdas@gmail.com*

We consider the problem of optimal control for a heat equation with Dirichlet boundary conditions. A bilinear control is used to bring the state solutions close to a desired profile under a quadratic cost of control. We establish the existence of an optimal control that minimizes the cost functional. Also, we give the characterization of the optimal control by formally differentiating the cost functional with respect to the control and using adjoint problem.

GROUP -6:

IMS-14: Effective fuzzy clustering algorithms in analyzing medical database, S. R. Kannan, *srkannaniitm@mail.com*

In recent years the use of clustering techniques in medical diagnosis is increasing steadily, because of the effectiveness of clustering techniques in recognizing the systems in the medical database to help medical experts in diagnosing diseases. Such a disease is lung cancer, which is a very common type of cancer among public. As the incidence of this disease has increased significantly in the recent years, fuzzy clustering applications to this problem have increased. The aim of this paper is to analysis the Lung Cancer database for finding three different types of lung cancer in order to assist physician in diagnosing lung cancer. By using kernel function, some effective fuzzy c-means clustering techniques were obtained to diagnose the Lung cancer dataset into three different types of lung cancers. In order to speed up the proposed fuzzy c-means techniques this paper proposes a new algorithm to initialize the cluster centers of fuzzy c-means. In order to evaluate the performance of proposed techniques, the real Wine dataset and IRIS dataset are used for initial experimentation. This paper obtained clustering accuracy of 99%, which is the highest one reached so far. The clustering accuracy was obtained via Silhouette method and Error Matrix.

Abstracts of the papers for AMU prize:

AMU-1: Semi-baer and semi-p.p.rings, Anil S. Khairnar,
anil maths2004@yahoo.com

We have introduced concept of semi-Baer, semi-quasi Baer, semi-p.q. Baer and semi-p.p.rings as a generalization of Baer, quasi Baer, p.q. Baer and p.p. rings respectively. We have given an example of a ring which is semi- Baer but not Baer, and an example of a commutative reduced ring which is not semi-Baer. We have given a condition under which a semi-Baer ring becomes a Baer ring. Also we have proved some results about extensions of semi-Baer and semi-quasi Baer rings to polynomial, power series, Laurent polynomial and Laurent power series rings motivated by the similar results about extensions of Baer and quasi Baer rings. Several examples and counter examples are included that occur in the process of this paper.

AMU-2: On a branch algebra of entire bicomplex dirichlet series, Jogendra Kumar, *jogendra j@rediffmail.com*

A class T of entire functions represented by Bicomplex Dirichlet series has been provided with a Modified Banach algebra structure. T is a Modified Banach algebra which is neither a division algebra nor a B algebra. Invertible and quasi invertible elements in T have been investigated.

AMU-3: Some decomposition theorems for rings and near rings,
Rekha Rani, *linerekha2@yahoo.co.in*

Using commutativity of rings satisfying $(xy)^{n(x,y)} = xy$ proved by Searcoid and MacHale [Amer. Math. Monthly, 93 (1986)], Ligh and Luh [Amer. Math. Monthly, 96 (1989)] gave a direct sum decomposition for rings with the mentioned condition. Further, Bell and Ligh [Math. J. Okayama Univ. 31 (1989)] sharpened the result and obtained a decomposition theorem for rings with the property $xy = (xy)^2 f(x,y)$ where $f(X,Y) \in \mathbb{Z} < X,Y >$, the ring of polynomials in two noncommuting indeterminates over the ring $\mathbb{Z}$ of integers. In the present paper, we continue the study and investigate structure of certain rings and near rings satisfying the following condition: $xy = p(x,y)$ with $p(x,y)$, an admissible polynomial in $\mathbb{Z} < X,Y >$. This condition is naturally more general than the above mentioned conditions. Moreover, we deduce the commutativity of such rings.
Theorem 1 . Let R be a ring such that for each $x, y \in R$ there exists an admissible polynomial $p(X,Y) \in \mathbb{Z} < X,Y >$ for which $xy = p(x,y)$. Then R is a direct sum of a J-ring and a zero ring.
Theorem 2. Let R be a $d-g$ near ring such that for each $x, y \in R$ there exists an admissible polynomial $p(X,Y) \in \mathbb{Z} < X,Y >$ for which $xy = p(x,y)$. Then R is periodic and commutative. Moreover, $R = P \oplus N$, where P, the set of potent elements of R is a subring and N, the set of nilpotent elements of R is a subnear ring with trivial multiplication.

Abstracts of the papers for V. M. Shah prize:

VMS-1: Coefficient inequalities for generalized sakaguchi type functions, Trilok Mathur, *tmathur@bits-pilani.ac.in, trilokmathur@yahoo.co.in*

In this paper we have studied two subclasses $S(\alpha, s, t)$ and $T(\alpha, s, t)$ concerning with generalized Sakaguchi type functions in the open unit disc U, further by using the coefficient inequalities for the classes $S(\alpha, s, t)$ and $T(\alpha, s, t)$ two subclasses $S_\theta(\alpha, s, t)$ and $T_\theta(\alpha, s, t)$ are defined. Some properties of functions belonging to the class $S_\theta(\alpha, s, t)$ and $T_\theta(\alpha, s, t)$ are also discussed.

VMS-2: On approximation of conjugate of signals (functions) belonging to the generalized weighted $W(L_r, \xi(t)), (r \geq 1)$–class by product summability means of conjugate series Fourier series,
Vishnu Narayan Mishra, Saradar Vallabhbhai National Institute of Technology, Ichchhanath, Surat 395 007.

In this paper, a known theorem Nigam and Sharma [2010] dealing with the degree of approximation of conjugate of a function (signal) belonging to a certain class by product summability means of conjugate series of Fourier series has been generalized for the Weighted class $W(L_r\xi(t))$, where $\xi(t)$ is non-negative and increasing function of t, by Product Transform of Conjugate Series of Fourier series.

VMS-3: On the uniqueness problems in the class of meromorphic functions, Veena L. Pujari, Department of Mathematics, Karnatak University, Dharwad - 580 003.

In this paper we study the value distribution and uniqueness of meromorphic functions in certain class of meromorphic functions A. We obtain significant results which improve as well as generalize the result of Yang and Hua [Uniqueness and Value-sharing of meromorphic functions. Ann. Acad. Sci. Fenn Math. 22 (1997), 395–406] in class A.

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SOME DIOPHANTINE PROBLEMS CONCERNING PERFECT POWERS OF INTEGERS*

AJAI CHOUDHRY

"God made the integers, all else is the work of man."

Leopold Kronecker

Professor P. K. Banerji, President of the 77^{th} Annual Conference of the Indian Mathematical Society, Professor J. R. Patadia, Chairman for this session, Professor N. K. Thakare, Dr. S. B. Nimse, Vice-Chancellor, SRTM University, Nanded, Distinguished Office-bearers of the IMS, Members of the IMS, and Distinguished Participants at this 77^{th} Annual Conference of the IMS.

1. Introduction

I am deeply honoured to have received this invitation to deliver the 22$^{\text{nd}}$ Hansraj Gupta Memorial Award Lecture - 2011 at the prestigious 77$^{\text{th}}$ Annual Conference of the Indian Mathematical Society. I sincerely thank the IMS for having given me this opportunity. While I never had the privilege of meeting Prof. Hansraj Gupta, I have something in common with this outstanding number theorist. Both he and I have contributed research papers in The Mathematics Student that have been cited in Richard Guy's book entitled "Unsolved Problems in Number Theory" published by Springer, New York in 2004. The research work of both of us has also been cited in Hardy and Wright's well-known book, "An Introduction to the Theory of Numbers." One of Prof. Gupta's monographs was published in 1940 by the University of Lucknow where I studied some three decades later. With these associations, it is indeed very satisfying for me to give this lecture instituted in the memory of Prof. Gupta.

I wish to thank Prof. Nimse for the excellent arrangements made for my stay. I also wish to thank Prof. Patadia for the kind words he has said about me. As you have just heard in the introduction, I am not a professional mathematician. If therefore you find my presentation somewhat unconventional or unorthodox, please bear with me.

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* The text of the 22nd Hansraj Gupta Memorial Award lecture delivered at the 77th Annual Conference of the Indian Mathematical Society held at Swami Ramanand Teerth Marathwada University (SRTM University), Vishnupuri, Nanded - 431 606, Maharashtra, during the period December 27 - 30, 2011.

I understand this august gathering represents mathematicians specialising in a variety of subjects, not necessarily number theory, so I will begin this presentation at a very general level, discuss some elementary problems concerning equal sums of powers and at the same time try to describe some of the developments in this subject in the recent past.

An equation in one or more unknowns is called a diophantine equation if we have to search for solutions in integers or in rational numbers. The most natural diophantine equations to consider are polynomial equations that may be written generally as

$$f(x_1, x_2, \ldots, x_n) = 0, \tag{1.1}$$

where the coefficients of the polynomial $f(x_j)$ are integers.

Some examples of such equations are given below:

$$\begin{aligned} x^2 + y^2 &= z^2. \\ x^2 - 61y^2 &= 1. \\ x^3 + y^3 &= u^3 + v^3. \\ x^n + y^n &= z^n, \end{aligned} \tag{1.2}$$

where n is an integer ≥ 3. As you are aware, the complete solution of the first two of the above mentioned equations are known, the third equation has been the subject of a familiar anecdote about Ramanujan, and the last one concerns the well-known Fermat's Last Theorem.

We could also consider simultaneous diophantine equations that may be written as follows:

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0, \\ f_2(x_1, x_2, \ldots, x_n) &= 0, \\ &. \\ &. \\ &. \\ f_s(x_1, x_2, \ldots, x_n) &= 0. \end{aligned} \tag{1.3}$$

where again all the coefficients in the polynomials

$$f_j(x_1, x_2, \ldots, x_n), \ j = 1, 2, \ldots, s$$

are integers.

A single diophantine equation or a system of simultaneous equations could be considered as an algebraic curve or as an algebraic surface depending on the number of independent variables, and the problem then is to determine the lattice points, or the rational points, on the curve or the surface.

While polynomial equations are the most natural ones to consider, other types of diophantine equations have also been investigated. In this lecture, however, we restrict ourselves exclusively to polynomial diophantine equations and systems of simultaneous diophantine equations related to perfect powers of integers.

Given a diophantine equation (or a system of such equations), the following principal problems arise:

(i) to decide whether the equation has any solution in integers (or, in rational numbers);
(ii) to decide whether there are infinitely many solutions, or the number of solutions is finite;
(iii) if the number of solutions is finite, to determine them all;
(iv) if the number of solutions is infinite, to find formulae giving them all.

While we have seen some examples of diophantine equations involving equal sums of powers, a very general equation of this type may be written as follows:

$$x_1^k + x_2^k + \cdots + x_m^k = y_1^k + y_2^k + \cdots + y_n^k, \tag{1.4}$$

while simultaneous equations may be written as,

$$x_1^{k_i} + x_2^{k_i} + \cdots + x_m^{k_i} = y_1^{k_i} + y_2^{k_i} + \cdots + y_n^{k_i}, \quad i = 1, 2, \ldots, s, \tag{1.5}$$

where the exponents k, k_i as well as m, n and s are all integers.

We note that the above equations are homogeneous in the variables x_j, y_j, and therefore any rational solution of these equations yields, on appropriate scaling, a solution in integers.

We now consider some specific symmetric diophantine systems of the type (1.5), that is, systems with $m = n$, and we will try to solve them.

2. Some examples of diophantine systems and their reduction to simpler equations

2.1. The diophantine system $\sum_{i=1}^{3} a_i^r = \sum_{i=1}^{3} b_i^r, \quad r = 1, 2, 4$. We now consider the following simple diophantine system solutions of which have been known for over a hundred years:

$$\begin{aligned} a + b + c &= d + e + f, \\ a^2 + b^2 + c^2 &= d^2 + e^2 + f^2, \\ a^4 + b^4 + c^4 &= d^4 + e^4 + f^4. \end{aligned} \tag{2.1}$$

On substituting $c = -a - b,\ f = -d - e$, the first equation of the diophantine system (2.1) is identically satisfied, the second equation reduces to $2(a^2+ab+b^2) = 2(d^2+de+e^2)$, while the third one reduces to $2(a^2+ab+b^2)^2 = 2(d^2+de+e^2)^2$.

Thus it suffices to solve the single equation

$$a^2 + ab + b^2 = d^2 + de + e^2. \tag{2.2}$$

There are several ways of solving this diophantine equation. I present an elegant solution using a well-known identity from the theory of composition of forms. It appears that this method of solving equation (2.2) has not been published earlier. We have the identity,

$$\begin{aligned}(x^2 + xy + y^2)&(u^2 + uv + v^2)\\ &= (xu - yv)^2 + (xu - yv)\{xv + y(u + v)\}\\ &\qquad\qquad + \{xv + y(u + v)\}^2\\ &= a^2 + ab + b^2,\end{aligned} \tag{2.3}$$

where

$$a = xu - yv, \quad b = xv + y(u + v). \tag{2.4}$$

In view of the symmetry, we can interchange u and v when we may rewrite our identity as follows:

$$\begin{aligned}(x^2 + xy + y^2)&(u^2 + uv + v^2)\\ &= (xv - yu)^2 + (xv - yu)\{xu + y(u + v)\}\\ &\qquad\qquad + \{xu + y(u + v)\}^2\\ &= d^2 + de + e^2.\end{aligned} \tag{2.5}$$

where

$$d = xv - yu, \quad e = xu + y(u + v). \tag{2.6}$$

From these identities it immediately follows that a solution of the diophantine equation (2.2), is given by (2.4) and (2.6), and it follows that a solution of the diophantine system (2.1), is given by

$$\begin{aligned} a &= xu - yv, & d &= xv - yu,\\ b &= xv + y(u + v), & e &= xu + y(u + v),\\ c &= -x(u + v) - yu, & f &= -x(u + v) - yv.\end{aligned} \tag{2.7}$$

If there exist numbers a, b, c, d, e, f such that

$$\begin{aligned} a + b + c &= d + e + f,\\ a^2 + b^2 + c^2 &= d^2 + e^2 + f^2.\end{aligned} \tag{2.8}$$

then, it is easy to see that adding a constant, say z, to each of the numbers a, b, c, d, e, f would give another solution of this type. We thus get

$$\begin{aligned} a &= xu - yv + z, & d &= xv - yu + z, \\ b &= xv + y(u+v) + z, & e &= xu + y(u+v) + z, \\ c &= -x(u+v) - yu + z, & f &= -x(u+v) - yv + z \end{aligned} \tag{2.9}$$

as a solution of the simultaneous equations (2.8). In this solution x, y, z, u and v are arbitrary parameters.

2.2. **The diophantine system $\sum_{i=1}^{3} a_i^r = \sum_{i=1}^{3} b_i^r,\ \ r = 1, 3, 4$.** Most diophantine systems will not so readily reduce to a simple single equation as in the previous example. Nevertheless it is often possible to make certain substitutions and reduce the system to a single linear or quadratic equation in a specific variable. For instance, to solve the system of equations,

$$\begin{aligned} a + b + c &= d + e + f, \\ a^3 + b^3 + c^3 &= d^3 + e^3 + f^3, \\ a^4 + b^4 + c^4 &= d^4 + e^4 + f^4, \end{aligned} \tag{2.10}$$

we make the substitutions,

$$\begin{aligned} a &= p\theta + \alpha, & d &= q\theta + \alpha, \\ b &= q\theta + \beta, & e &= r\theta + \beta, \\ c &= r\theta + \gamma, & f &= p\theta + \gamma, \end{aligned} \tag{2.11}$$

where p, q, r, α, β, γ and θ are arbitrary parameters. By transposing all terms to the left-hand side, each equation of the diophantine system (2.10) may be written in the form $\Sigma_j h_j \theta^j = 0$, and the idea is to choose a substitution such that all the coefficients h_j vanish except for two consecutive coefficients in one of the equations. With the substitutions (2.11), the first equation of the diophantine system (2.10) is identically satisfied, and the second equation will also be identically satisfied if the following conditions hold good:

$$(p-q)\alpha^2 + (q-r)\beta^2 + (r-p)\gamma^2 = 0, \tag{2.12}$$

$$(p^2-q^2)\alpha + (q^2-r^2)\beta + (r^2-p^2)\gamma = 0. \tag{2.13}$$

In the equation arising from the third equation of the system (2.10), the coefficients of θ^4 and the constant term are easily seen to be 0, and equating the coefficient of θ to zero, we get the condition,

$$(p-q)\alpha^3 + (q-r)\beta^3 + (r-p)\gamma^3 = 0. \tag{2.14}$$

If we choose α, β, γ such that

$$\alpha\beta + \beta\gamma + \gamma\alpha = 0, \tag{2.15}$$

then, equations (2.12) and (2.14) become identical while equation (2.13) reduces to a linear equation in p, q, r. We can thus solve equations (2.12), (2.13) and (2.14) to get the following solution for p, q, r :

$$\begin{aligned} p &= \alpha(\beta^2-\gamma^2)^2 - \gamma(\alpha^2-\beta^2)^2 - \beta(\gamma^2-\alpha^2)(\alpha^2-2\beta^2+\gamma^2), \\ q &= \beta(\gamma^2-\alpha^2)^2 - \alpha(\beta^2-\gamma^2)^2 - \gamma(\alpha^2-\beta^2)(\beta^2-2\gamma^2+\alpha^2), \\ r &= \gamma(\alpha^2-\beta^2)^2 - \beta(\gamma^2-\alpha^2)^2 - \alpha(\beta^2-\gamma^2)(\gamma^2-2\alpha^2+\beta^2). \end{aligned} \tag{2.16}$$

When α, β, γ, p, q, r, are chosen as above, the first two equations of the diophantine system (2.10) are identically satisfied, while the third equation reduces to a linear equation in θ which is readily solved and we get,

$$\theta = -\frac{3\{(p^2-q^2)\alpha^2 + (q^2-r^2)\beta^2 + (r^2-p^2)\gamma^2\}}{2\{(p^3-q^3)\alpha + (q^3-r^3)\beta + (r^3-p^3)\gamma\}}. \tag{2.17}$$

Thus when α, β, γ satisfy equation (2.15), p, q, r are defined by (2.16) and θ is defined by (2.17), a rational parametric solution of the diophantine system (2.10) is given by (2.11). A solution in integers is now readily obtained on appropriate scaling.

As a numerical example, taking $\alpha = 6,\ \beta = 3,\ \gamma = -2$, we get the solution,

$$\begin{aligned} 5658 + 5583 + (-3254) &= 6738 + (-1329) + 2578, \\ 5658^3 + 5583^3 + (-3254)^3 &= 6738^3 + (-1329)^3 + 2578^3, \\ 5658^4 + 5583^4 + (-3254)^4 &= 6738^4 + (-1329)^4 + 2578^4. \end{aligned} \tag{2.18}$$

2.3. **The diophantine system $\sum_{i=1}^{3} a_i^r = \sum_{i=1}^{3} b_i^r, \quad r = 1, 2, 6$.** When we consider diophantine systems involving higher powers, it is naturally more difficult to find a suitable substitution that would reduce the system to a solvable equation. In fact, a system of equations may not have any nontrivial solution. For instance, it has been proved using the theory of equations and properties of elementary symmetric functions that the system of equations,

$$\begin{aligned} a+b+c &= d+e+f, \\ a^2+b^2+c^2 &= d^2+e^2+f^2, \\ a^k+b^k+c^k &= d^k+e^k+f^k. \end{aligned} \tag{2.19}$$

has no nontrivial solutions when $k = 3$ or $k = 5$. But no such proof exists when $k \geq 6$. We therefore consider the diophantine system,

$$\begin{aligned} a+b+c &= d+e+f, \\ a^2+b^2+c^2 &= d^2+e^2+f^2, \\ a^6+b^6+c^6 &= d^6+e^6+f^6. \end{aligned} \tag{2.20}$$

To give some historical information, a single numerical solution, namely,

$$3^k + 19^k + 22^k = 10^k + 15^k + 23^k, \quad k = 2,\ 6, \tag{2.21}$$

with equal sums of squares as well as of sixth powers was discovered in 1934, but this remained a solitary example till the 1970s when parametric solutions of this type were discovered. It took another 25 years or so before a solution could be found of the diophantine system (2.20).

For the first two equations of the diophantine system (2.20) we have already obtained a solution which is given by (2.9). With the values of a, b, c, d, e, f as given by (2.9), we have

$$\begin{aligned} &a^6+b^6+c^6-d^6-e^6-f^6 \\ &\quad = 3xy(u-v)(u+2v)(2u+v)(x+y) \\ &\qquad \times(pu^2+qu+r), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} p &= (2x^3+3x^2y-3xy^2-2y^3)v \\ &\qquad -10(xyz-x^2z-y^2z), \\ q &= pv, \\ r &= -10z\{v^2(x^2+xy+y^2)+2z^2\}. \end{aligned} \tag{2.23}$$

For a nontrivial solution of the equations (2.20), we must have $pu^2+qu+r=0$. This equation will have a rational solution for u if the discriminant of this equation is a perfect square, that is, the diophantine equation,

$$\begin{aligned} w^2 &= (x-y)^2(x+2y)^2(2x+y)^2v^4+20(x-y)(x+2y)(2x+y) \\ &\quad \times(x^2+xy+y^2)v^3z-300(x^2+xy+y^2)^2v^2z^2 \\ &\quad +80(x-y)(x+2y)(2x+y)vz^3-800(x^2+xy+y^2)z^4 \end{aligned} \tag{2.24}$$

is solvable. It is found by trial that when we take

$$x=4,\ y=5,\ v=7,\ z=-2, \tag{2.25}$$

we get

$$pu^2+qu+r=-54(u-30)(u+37), \tag{2.26}$$

so taking $u=30$ or $u=-37$, we will get a solution. Both these values of u lead to the same solution of the diophantine system (2.20), namely,

$$\begin{aligned} 83+211+(-300) &= (-124)+303+(-185), \\ 83^2+211^2+(-300)^2 &= (-124)^2+303^2+(-185)^2, \\ 83^6+211^6+(-300)^6 &= (-124)^6+303^6+(-185)^6. \end{aligned} \tag{2.27}$$

A few other solutions of the diophantine system (2.20) obtaining by solving equation (2.24) by computer trials, are as follows:

$$43^r+371^r+(-372)^r=140^r+307^r+(-405)^r,$$

$$271^r+387^r+(-562)^r=178^r+461^r+(-543)^r,$$

$$167^r+699^r+(-764)^r=271^r+627^r+(-796)^r,$$

where in each case the equality holds for $r = 1, 2, 6$. Computer trials have the limitation that while we may be able to find several solutions, we are unable to determine whether or not there are any other solutions, or whether there are infinitely many solutions of our diophantine system. We will accordingly consider other approaches to deal with quartic equations of the type (2.24).

Taking $x = 4$, $y = 5$, $z = -2$ in (2.24), we get the quartic equation,

$$\begin{aligned} w^2 &= 33124v^4 + 444080v^3 - 4465200v^2 \\ &\quad +116480v - 780800, \end{aligned} \tag{2.28}$$

and from computer trials we know that $v = 7$, $w = 3618$ is a solution of (2.28).

Fermat gave a method of finding rational solutions of the quartic equation

$$Y^2 = a_0X^4 + a_1X^3 + a_2X^2 + a_3X + a_4 \tag{2.29}$$

when a_4 is a perfect square or one solution is already known. When a_4 is a perfect square, say β^2, we can write the above equation as

$$Y^2 = \left\{\beta + \frac{a_3X}{2\beta} + \frac{(4\beta^2a_2 - a_3^2)X^2}{8\beta^3}\right\}^2 + a_5X^3 + a_6X^4, \tag{2.30}$$

where the values of a_5 and a_6 are readily determined by comparing coefficients, and we thus get $X = -a_5/a_6$ as a rational solution of equation (2.29). When a_4 is not a perfect square but one rational solution of equation (2.29) is given by $X = \alpha$, $Y = \beta$ we can simply write $X = X_1 + \alpha$, when equation (2.29) reduces to a quartic in X_1 with the constant term being β^2, and so we can obtain a new solution as before. We can thus use Fermat's method to obtain additional solutions of equation (2.28) but we are not sure whether we will get infinitely many solutions in this manner of our diophantine system (2.20).

Fermat's method gives numerically large solutions. For instance, applying the method of Fermat, we find that

$$v = \frac{86499221330410}{8392338832699}, \tag{2.31}$$

gives a solution of (2.28). Naturally this leads to solutions in large integers of our diophantine system.

We will revert to the quartic equation (2.28) a little later.

3. The TARRY-ESCOTT problem

We now consider yet another diophantine problem – known as the Tarry-Escott problem – concerning perfect powers of integers. The Tarry-Escott problem of degree k consists of finding two distinct sets of integers $\{a_1, a_2, ..., a_s\}$ and $\{b_1, b_2, ..., b_s\}$ such that

$$\sum_{i=1}^{s} a_i^r = \sum_{i=1}^{s} b_i^r, \quad r = 1, 2, ..., k. \tag{3.1}$$

If the above relations hold, it is easily seen that we also have

$$\sum_{i=1}^{s}(ma_i + d)^r = \sum_{i=1}^{s}(mb_i + d)^r, \quad r = 1, 2, ..., k, \tag{3.2}$$

where m and d are arbitrary constants. This follows readily from the binomial theorem.

It is also easily established that for a non-trivial solution of (3.1) to exist, we must have $s \geq (k+1)$. Solutions of the system of equations (3.1) are called ideal if $s = (k+1)$ and are of particular interest.

In order to reduce the number of equations of the system (3.1), the following simplifying conditions are often imposed:

$$a_i = -b_i, \;\; i = 1, 2, \ldots, s, \quad \text{for } s \text{ odd}, \tag{3.3}$$

or,

$$a_{s+1-i} = -a_i, \;\; b_{s+1-i} = -b_i, \;\; i = 1, 2, \ldots, s/2, \quad \text{for } s \text{ even}. \tag{3.4}$$

Solutions of (3.1) subject to the conditions (3.3) or (3.4) are called symmetric solutions. Solutions of (3.1) that are not symmetric are called non-symmetric.

When (3.1) holds, we write for brevity,

$$a_1, a_2, ..., a_s \stackrel{k}{=} b_1, b_2, ..., b_s. \tag{3.5}$$

Parametric ideal solutions of the Tarry-Escott problem for $k \leq 7$ were found in 1930s and numerical ideal solutions were found for $k = 8$ and $k = 9$ in 1942. Some examples of numerical ideal solutions are given below.

$$\begin{aligned} &1,\ 5,\ 10,\ 24,\ 28,\ 42,\ 47,\ 51 \\ &\stackrel{7}{=} 2,\ 3,\ 12,\ 21,\ 31,\ 40,\ 49,\ 50. \end{aligned} \tag{3.6}$$

$$\begin{aligned} &0,\ 24,\ 30,\ 83,\ 86,\ 133,\ 157,\ 181,\ 197 \\ &\stackrel{8}{=} 1,\ 17,\ 41,\ 65,\ 112,\ 115,\ 168,\ 174,\ 198. \end{aligned} \tag{3.7}$$

$$\begin{aligned} &0,\ 3083,\ 3301,\ 11893,\ 23314,\ 24186, \\ &\quad 35607,\ 44199,\ 44417,\ 47500 \\ &\stackrel{9}{=} 12,\ 2865,\ 3519,\ 11869,\ 23738,\ 23762, \\ &\quad 35631,\ 43981,\ 44635,\ 47488. \end{aligned} \tag{3.8}$$

There was no progress in finding ideal solutions for higher values of k for more than 50 years. In 1999 a single numerical symmetric ideal solution of (3.1) was found for $k = 11$.

This arises from the following identities discovered by computer trials in 1999:

$$22^k + 61^k + 86^k + 127^k + 140^k + 151^k = 35^k + 47^k + 94^k + 121^k + 146^k + 148^k \tag{3.9}$$

where $k = 2, 4, 6, 8, 10$. It immediately follows that

$$\begin{aligned} &\pm 22,\ \pm 61,\ \pm 86,\ \pm 127,\ \pm 140,\ \pm 151 \\ &\stackrel{11}{=} \pm 35,\ \pm 47,\ \pm 94,\ \pm 121,\ \pm 146,\ \pm 148. \end{aligned} \tag{3.10}$$

To obtain additional ideal solutions of the Tarry-Escott problem of degree 11, we will try to solve the following simultaneous equations

$$\sum_{i=1}^{s} A_i^r = \sum_{i=1}^{s} B_i^r, \quad r = 2,\ 4,\ 6,\ 8,\ 10. \tag{3.11}$$

We make the following substitutions:

$$\begin{aligned} A_1 &= 2\,x\,y + x\,z + 2\,y\,z - 7\,z^2, \\ A_2 &= 2\,x\,y - x\,z - 2\,y\,z - 7\,z^2, \\ A_3 &= 2\,x\,y - 2\,x\,z + y\,z + 7\,z^2, \\ A_4 &= 2\,x\,y + 2\,x\,z - y\,z + 7\,z^2, \\ A_5 &= 3\,x\,z + 5\,y\,z, \\ A_6 &= 5\,x\,z - 3\,y\,z, \\ B_1 &= 2\,x\,y + 2\,x\,z + y\,z - 7\,z^2, \\ B_2 &= 2\,x\,y - 2\,x\,z - y\,z - 7\,z^2, \\ B_3 &= 2\,x\,y - x\,z + 2\,y\,z + 7\,z^2, \\ B_4 &= 2\,x\,y + x\,z - 2\,y\,z + 7\,z^2, \\ B_5 &= 5\,x\,z + 3\,y\,z, \\ B_6 &= 3\,x\,z - 5\,y\,z, \end{aligned} \tag{3.12}$$

and denote

$$S_k = \sum_{i=1}^{6} \left(A_i^k - B_i^k\right). \tag{3.13}$$

It can be easily verified using *Mathematica* that

$$\begin{aligned} S_2 &= S_4 = 0, \\ S_6 &= 4320xy\,(x-y)\,(x+y)\,z^4 f(x,\,y,\,z), \\ S_8 &= 5376xy\,(x-y)\,(x+y)\,z^4 f(x,\,y,\,z), \\ &\quad \times (8x^2y^2 + 37x^2z^2 + 37y^2z^2 + 98z^4), \\ S_{10} &= 10080xy\,(x-y)\,(x+y)\,z^4 f(x,\,y,\,z) \\ &\quad \times (32x^4y^4 + 248x^4y^2z^2 + 665x^4z^4 + 248x^2y^4z^2 + 2384x^2y^2z^4 \\ &\qquad + 3038x^2z^6 + 665y^4z^4 + 3038y^2z^6 + 4802z^8). \end{aligned} \tag{3.14}$$

where

$$f(x,\,y,\,z) = 8x^2y^2 - 17x^2z^2 - 17y^2z^2 + 98z^4. \tag{3.15}$$

Hence any solution in integers of the equation

$$8x^2y^2 - 17x^2z^2 - 17y^2z^2 + 98z^4 = 0 \tag{3.16}$$

leads by the relations (3.12) to a solution in integers of the system (3.11).

On substituting

$$x = Xz, \quad y = Yz/(8X^2 - 17), \tag{3.17}$$

equation (3.16) reduces to

$$Y^2 = 136X^4 - 1073X^2 + 1666. \tag{3.18}$$

It is easily found by trials that $(X, Y) = (1, 27)$ is a solution of (3.18). This gives us a solution $(x, y, z) = (1, 3, 1)$ of equation (3.16) which, however, leads to a trivial solution of (3.11). To get nontrivial solutions of (3.11), we have to find other solutions of the quadratic equation (3.18).

4. Reduction of the quadratic equation $Y^2 = A_0X^4 + A_1X^3 + A_2X^2 + A_3X + A_4$ to a cubic equation

We have seen in a couple of instances that our diophantine problem was reduced to finding rational solutions of a quadratic equation of the following type:

$$Y^2 = a_0X^4 + a_1X^3 + a_2X^2 + a_3X + a_4, \tag{4.1}$$

where a_i, $i = 0, 1, \ldots, 4$ are constants. This happens fairly frequently, and we will now try to reduce this equation further. We can, of course, use computer trials to find numerical solutions of the above equation. But finding numerical solutions by computer trials is by itself not adequate. I may also mention here that at present there is no algorithm that enables us to find a rational solution in every case of a quadratic equation of the above type.

We assume that we know one rational solution, say $(X, Y) = (\alpha, \beta)$ of the above equation. We will use this solution and some very simple substitutions to reduce the above quartic equation as outlined below.

Writing $X = X_1 + \alpha$, the above quartic equation (4.1) gets reduced to the following type:

$$Y^2 = b_0X_1^4 + b_1X_1^3 + b_2X_1^2 + b_3X_1 + \beta^2, \tag{4.2}$$

for some values of b_i, $i = 0, \ldots, 3$. Next writing, $X_1 = 1/X_2$, and $Y = Y_1/X_2^2$, and multiplying by X_2^4, equation (4.2) reduces to the type,

$$Y_1^2 = \beta^2X_2^4 + c_1X_2^3 + c_2X_2^2 + c_3X_2 + c_4, \tag{4.3}$$

for some c_i, $i = 1, \ldots, 4$. Next, on dividing this equation by β^2, and replacing Y_1/β by v, we get,

$$v^2 = X_2^4 + d_1X_2^3 + d_2X_2^2 + d_3X_2 + d_4, \tag{4.4}$$

for some d_i, $i = 1, \ldots, 4$. Finally, replacing $X_2 + d_1/4$ by u, the equation reduces to the type

$$v^2 = u^4 + pu^2 + qu + r, \tag{4.5}$$

where p, q, r are determined by the values of a_0, a_1, a_2, a_3, a_4 as well as α, β.

We now make the following substitutions,

$$\begin{aligned} u &= (24V - 3q)/(24U + 4p), \\ v &= (3456U^3 + 864pU^2 - 1728V^2 + 432qV \\ &\quad -8p^3 - 27q^2)/\{48(6U + p)^2\}, \end{aligned} \tag{4.6}$$

when the quadratic equation (4.5) reduces to the following cubic equation:

$$V^2 = U^3 - \frac{(p^2 + 12r)}{48}U + \frac{2p^3 - 72pr + 27q^2}{1728}. \tag{4.7}$$

We also note that equations (4.6) can be solved for U, V to get

$$\begin{aligned} U &= (1/2)u^2 + (1/2)v + (1/12)p, \\ V &= (1/2)u^3 + (1/2)uv + (1/4)pu + (1/8)q, \end{aligned} \tag{4.8}$$

It follows from equations (4.6) and (4.8) that there is a one-to-one correspondence between the points on the quartic curve (4.5) and the cubic curve (4.7), and hence also between points on the quadratic curve (4.1) and the cubic curve (4.7) (except for a finite number of points).

We also note that if we actually work out the values of p, q, r starting from the quartic curve (4.1), and substitute these values in equation (4.7), we get the cubic equation,

$$V^2 = U^3 - IU/(4\beta^4) - J/(4\beta^6) \tag{4.9}$$

where

$$\begin{aligned} I &= a_0a_4 - a_1a_3/4 + a_2^2/12, \\ J &= a_0a_2a_4/6 - a_0a_3^2/16 - a_1^2a_4/16 \\ &\quad + a_1a_2a_3/48 - a_2^3/216, \end{aligned} \tag{4.10}$$

are the invariants of the binary quartic form

$$a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4. \tag{4.11}$$

Finally, on writing $V = y/\beta^3$, $U = x/\beta^2$, equation (4.9) reduces to

$$y^2 = x^3 - Ix/4 - J/4. \tag{4.12}$$

Thus there exists a birational transformation that reduces the quadratic equation (4.1) to the equation (4.12). This cubic equation is defined only by the coefficients of the quadratic equation and is independent of the choice of the initial rational point used to reduce the equation.

5. **Group of rational points on an elliptic curve**

We now consider rational points on the algebraic curve defined by a cubic equation of the type

$$y^2 = x^3 + ax^2 + bx + c, \tag{5.1}$$

where the coefficients a, b, c are rational numbers. This curve will have a singularity if the discriminant of the cubic polynomial on the right-hand side of (5.1) is zero. We are interested in solving the diophantine equation (5.1) which is equivalent to determining rational points on the curve defined by (5.1). If the curve has a singularity, it is straightforward to determine rational points on it by drawing an arbitrary line through the singular point.

We shall therefore assume that the discriminant is not zero. We note that till now there is no known algorithm that will enable us to find a rational point on every equation of type (5.1). At present, we just assume that there are some known rational points on this cubic curve. If there exists a rational point on the cubic curve (5.1) (and the discriminant is nonzero), equation (5.1) gives us an example of what is known as an elliptic curve.

We denote this elliptic curve by E and the set of rational points on it by $E(\mathbb{Q})$. We shall now see that this set $E(\mathbb{Q})$ can be given the structure of an abelian group. The group structure arises in the following manner (see Figure 1).

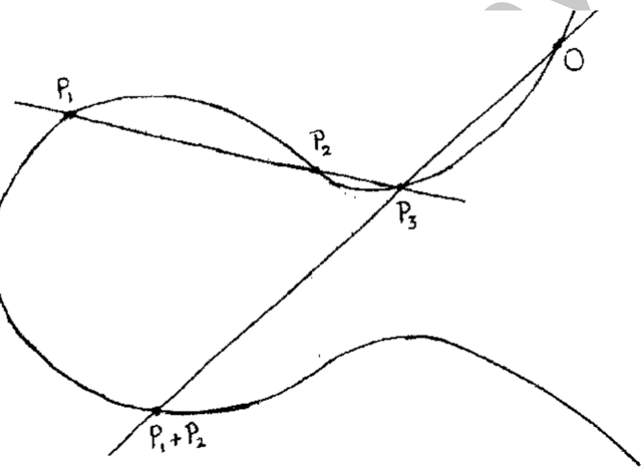

Figure 1: The Group Operation on an Elliptic Curve

: Let O be a fixed rational point on the curve E. We connect two rational points $P_1,\ P_2 \in E(\mathbb{Q})$ by a straight line. Since we are considering the intersections of a straight line with a cubic curve, this line will intersect E in a third point P_3. Moreover, since $P_1,\ P_2$ are rational points, the straight line joining them is represented by a linear equation with rational coefficients, and since equation (5.1) is also rational, it is easily seen that the third point of intersection P_3 must also be rational. Now we connect the rational points P_3 and O by a straight line and denote the third point of intersection of the line joining O and P_3 with E by P_1+P_2. The composition of the rational points P_1 and P_2 resulting in the rational point P_1+P_2 defines an abelian group with O as the identity for the group. It is clear that the operation is commutative since it immediately follows from the definition given above that $P_1+P_2 = P_2+P_1$. We now illustrate the fact that O

is the identity (see Figure 2). To add an arbitrary rational point P on the elliptic curve E to the point O, we draw the line PO to intersect the curve E in the third point Q. Next we join OQ and extend this line segment to intersect the curve E in a third point which is naturally the point P. Thus, by definition $P + O = P$, and similarly $O + P = P$, so it follows that O acts as the identity element.

Next we illustrate the inverse of a point P on the elliptic curve E (see Figure 3). To find the inverse of an arbitrary rational point P on the elliptic curve E, we first draw a tangent to the curve E at the identity O to intersect the curve E at the rational point T. We now draw the line PT and extend it to intersect the curve E in a third point, say Q. We will now show that the point Q is in fact the inverse of P. We find the sum $P + Q$ by drawing the line PQ which intersects the curve E in the point T. Next we draw the line OT and since this is a tangent with double contact at the point O, the third point of intersection of OT with the curve E is to be taken as O. Thus, by definition, $P + Q = O$ and hence Q is indeed the inverse of t

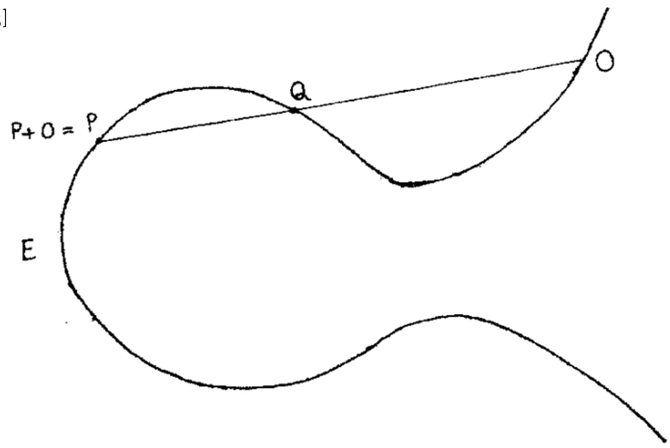

Figure 2: O as the identity element

The verification of the associative law is somewhat cumbersome and it is therefore being omitted. I may mention that algebraic formulae for the addition of two rational points on the elliptic curve E can be worked out. While the formulae are a bit cumbersome, we can use these formulae to verify all the properties of a group, and thus establish algebraically that the rational points on the elliptic curve E indeed form a group.

A rational point $P \in E(\mathbb{Q})$ is said to be of finite order m if

$$mP = P + P + \cdots + P = O \tag{5.2}$$

but $m'P \neq O$ for all integers $1 \leq m' < m$. If such an m exists, then P has finite order; otherwise it has infinite order.

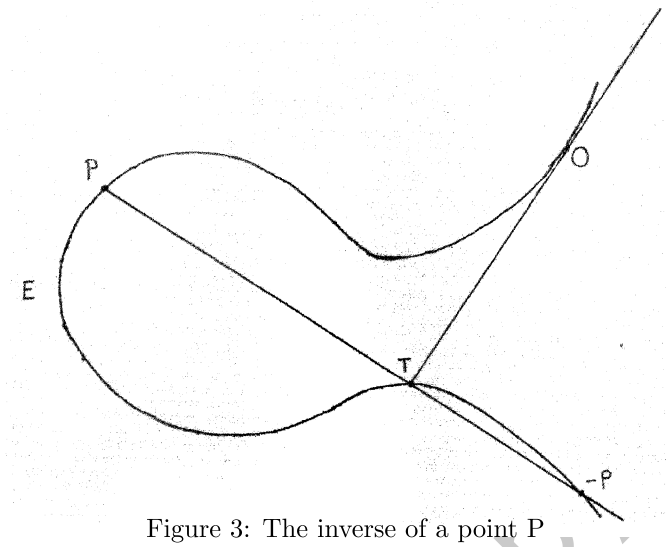

Figure 3: The inverse of a point P

There is a celebrated theorem of Mordell (1922) according to which $E(\mathbb{Q})$ is a finitely generated group. According to Mordell's theorem, there is a finite set of rational points on the curve E such that every other rational point is obtained by combining these points with one another.

According to the fundamental theorem on abelian groups, the group $E(\mathbb{Q})$ is isomorphic to a direct sum of infinite cyclic groups and finite cyclic groups of prime power order. If we denote the additive group of integers by Z, the above isomorphism may be written as

$$E(\mathbb{Q}) \cong \underbrace{Z \oplus Z \oplus \cdots \oplus Z}_{r \text{ times}} \oplus T, \tag{5.3}$$

where T is a finite group. The integer r is called the rank of $E(\mathbb{Q})$. It is conjectured that there is no upper bound for r. The finite group T is called the torsion group and according to a theorem of Mazur, its order does not exceed 16.

We will make use of the above facts as well as another theorem on elliptic curves, namely, the Nagell-Lutz theorem, which is as follows:

Theorem 5.1. *Let*

$$y^2 = f(x) = x^3 + ax^2 + bx + c \tag{5.4}$$

be an elliptic curve with integer coefficients, and let D be the discriminant of the cubic polynomial $f(x)$,

$$D = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2. \tag{5.5}$$

Let $P = (x, y)$ be a rational point of finite order on the above elliptic curve. Then x and y are integers; and either $y = 0$, in which case P is of order two, or else, y divides D.

It follows immediately from the Nagell-Lutz theorem that if a point P on our elliptic curve has co-ordinates that are not integers, then P cannot be a point of finite order. It follows that there are infinitely many points on the elliptic curve under consideration.

We have already seen that if a rational point on the quadratic curve

$$y^2 = ax^4 + bx^3 + cx^2 + dx + e \tag{5.6}$$

is known, then this quadratic equation can be transformed by means of a birational transformation to a cubic equation of the type (5.1). We will now apply the theorems on elliptic curves to the problems we have considered earlier.

6. The diophantine system $\sum_{i=1}^{3} a_i^r = \sum_{i=1}^{3} b_i^r$, $r = 1, 2, 6$.

We have earlier (see Section **2.3**) reduced the diophantine system $\sum_{i=1}^{3} a_i^r = \sum_{i=1}^{3} b_i^r$, $r = 1, 2, 6$ to the quadratic diophantine equation

$$w^2 = 33124v^4 + 444080v^3 - 4465200v^2 + 116480v - 780800, \tag{6.1}$$

which may be re-written as

$$w^2 = 4(91v - 610)(91v^3 + 1830v^2 + 320). \tag{6.2}$$

We now make the following birational transformation,

$$\begin{aligned} v &= \frac{10(61\xi + 2529372)}{91\xi}, \\ w &= \frac{303524640Y}{91\xi^2}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \xi &= \frac{25293720}{91v - 610}, \\ \eta &= \frac{191810710w}{(91v - 610)^2}, \end{aligned} \tag{6.4}$$

when the quadratic equation (6.2) reduces to the cubic equation

$$\eta^2 = \xi^3 + 93025\xi^2 + 2571528200\xi + 17771451984400. \tag{6.5}$$

The known rational point (7, 3618) on the quadratic curve (6.1) corresponds to a rational point P on the elliptic curve (6.5) given by

$$(\xi, \eta) = (8431240/9, 25702635140/27). \tag{6.6}$$

As the point P does not have integer co-ordinates, it follows from the Nagell-Lutz theorem that this is not a point of finite order. Thus there are infinitely many rational points on the elliptic curve (6.5) and these can be obtained by the group law. These infinitely many rational points on the curve (6.5) yield infinitely many rational solutions of equation (6.2), and working backwards, we get infinitely many solutions of the diophantine system (2.20).

7. The TARRY-ESCOTT problem of degree 11

We have earlier (see Section **3**) reduced the Tarry-Escott problem of degree 11 to the quartic equation

$$Y^2 = 136X^4 - 1073X^2 + 1666. \tag{7.1}$$

We already know that there are four rational points on this curve given by $(\pm 1, \pm 27)$. We now apply the following birational transformation to equation (7.1),

$$\begin{aligned} X &= (251\xi + 3\eta - 206690)/(89\xi + 3\eta - 36410), \\ Y &= (243\xi^3 - 766260\xi^2 + 312978600\xi \\ &\quad -9256500\eta + 59116365000) \\ &\quad \times (89\xi + 3\eta - 36410)^{-2}, \end{aligned} \tag{7.2}$$

$$\begin{aligned} \xi &= (-86X^2 - 1430X + 54Y + 2974)/(X-1)^2, \\ \eta &= (14688X^3 - 57942X^2 - 1602XY - 57942X + 4518Y + 179928) \\ &\quad \times (X-1)^{-3}, \end{aligned} \tag{7.3}$$

when equation (7.1) reduces to the cubic equation

$$\eta^2 = \xi^3 + \xi^2 - 1290080\xi + 556370100. \tag{7.4}$$

The rational point $(-1, 27)$ on the quadratic curve (7.1) corresponds to a rational point P on the elliptic curve (7.4) given by

$$(\xi, \eta) = (9460/9, 514250/27). \tag{7.5}$$

As the point P does not have integer co-ordinates, it follows from the Nagell-Lutz theorem, as before, that this is not a point of finite order. Thus there are infinitely many rational points on the elliptic curve (7.4) and these can be obtained by the group law. These infinitely many rational points on the curve (7.4) yield infinitely many rational solutions of equation (7.1). Finally using the relations (3.17), and assigning appropriate values to z in each case, we get infinitely many integer solutions of equation (3.16). These integer solutions of equation (3.16) lead to infinitely many solutions of the diophantine system (3.11).

The point P on the curve (7.4) corresponds to the solution $(x, y, z) = (1, 3, 1)$ of equation (3.16) which, as already noted earlier, leads to a trivial solution of (3.11). To get nontrivial solutions of (3.11), we have to compute other points on the curve (7.4). For instance, the point $2P$, given by

$$(\xi, \eta) = (163579/225, -6090292/3375), \tag{7.6}$$

on the curve (7.4) corresponds to the point

$$(X, Y) = (-457/353, -1968867/124609) \tag{7.7}$$

on the curve (7.1) which leads to the solution

$$(x,\, y,\, z) = (-101911,\, 346293,\, 78719) \tag{7.8}$$

of (3.16) and finally to the following nontrivial solution of our diophantine system:

$$\begin{aligned} &1019171^k + 774217^k + 712866^k + 447858^k + 428496^k + 102257^k \\ &= 1018599^k + 795069^k + 652598^k + 570049^k + 264662^k + 224448^k, \end{aligned} \tag{7.9}$$

where $k = 2,\, 4,\, 6,\, 8,\, 10$.

Two numerically smaller solutions of the diophantine system (3.11) obtained by using computer trials to solve the quadratic equation (3.16) are as follows:

$$\begin{aligned} &107^k + 622^k + 700^k + 1075^k + 1138^k + 1511^k \\ &\quad = 293^k + 413^k + 886^k + 953^k + 1180^k + 1510^k, \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} &257^k + 891^k + 1109^k + 1618^k + 1896^k + 2058^k \\ &\quad = 472^k + 639^k + 1294^k + 1514^k + 1947^k + 2037^k, \end{aligned} \tag{7.11}$$

both identities being valid for $k = 2,\, 4,\, 6,\, 8,\, 10$.

8. **Addition laws on other models of elliptic curves**

The algebraic formulae for the addition law on the cubic elliptic curve can be worked out explicitly. I have, however, not given these formulae as they are cumbersome to write, and that is why we get numerically large solutions using the group law. In recent years there has been an effort to find out other models of elliptic curves for which simple addition laws can be found. Any curve that is birationally equivalent to an elliptic curve given by a cubic equation can be considered as an elliptic curve. In 2007 Edwards found the curve

$$x^2 + y^2 = c^2(1 + dx^2y^2), \tag{8.1}$$

for which he gave the following simple addition formula: If there are two points $P_1 = (x_1,\, y_1)$ and $P_2 = (x_2,\, y_2)$ on the above curve, then the addition of P_1 and P_2 is defined by the simple formula,

$$P_1 + P_2 = \left(\frac{x_1y_2 + x_2y_1}{c(1 + x_1x_2y_1y_2)},\, \frac{y_1y_2 - x_1x_2}{c(1 - x_1x_2y_1y_2)} \right). \tag{8.2}$$

The identity element is $(0,\, c)$ while the inverse of any point $(x,\, y)$ on the Edwards curve is given by $(-x,\, y)$. The Edwards curve is birationally equivalent to an elliptic curve given by a cubic equation, and is therefore to be regarded as an elliptic curve.

There are other curves also for which simple addition formulae have been found. We will not go into details here. I will end this lecture by stating some open problems.

9. SOME OPEN PROBLEMS

1. Are there any nontrivial solutions of the following diophantine equations:

$$x^n + y^n = u^n + v^n, \quad n \geq 5. \tag{9.1}$$

$$x^n + y^n + z^n = u^n + v^n + w^n, \quad n \geq 7. \tag{9.2}$$

2. Are there any ideal solutions of the Tarry-Escott problem of degree 10? More generally, do there exist ideal solutions of the Tarry-Escott problem of an arbitrary degree n?

3. Is there a 3×3 magic square with all 9 entries being perfect squares? The best known results concerning this problem are as follows:

127^2	46^2	58^2
2^2	113^2	94^2
74^2	82^2	97^2

In the above square, while the entries are all perfect squares, and 3 rows, 3 columns and one diagonal have the same magic sum 21609, the other diagonal has a different sum 38307. Below is given an example of a 3×3 magic square with 7 square entries.

373^2	289^2	565^2
360721	425^2	23^2
205^2	527^2	222121

Finally, I would like to thank the Indian Mathematical Society and the Organizing Committee once again. I also wish to thank all of you for attending this lecture.

Notes

I wish to acknowledge the contributions of a number of authors whose books and research papers have been consulted while preparing for the lecture. The introductory material on diophantine equations is primarily based on [3, 16, 17]. The diophantine system (2.1) of Section **2.1** is discussed in [13, pp. 708-710], the diophantine system (2.10) of Section **2.2** is solved in [7, pp. 305-6], the system (2.19) for $k = 3$ is included in a result of Bastien (quoted in [13, p. 712]), the system (2.19) for $k = 5$ is shown to have no nontrivial solutions in [9] while the discussion on the system of equations (2.20) is based on [8] and [13, p. 639]. The material on the Tarry-Escott problem is based on [4, 11, 12], Section **4** is based on [16, p. 77], Section **5** on [18, Chapters $1, 2, 3$] and Section **8** on [1, 2]. Some of the references to open problems are in [4, 14] and the problem of 3×3 magic squares is dealt with in [5, 6, 10]. Several remarks in the lecture are based on various other sources too numerous to mention.

References

[1] D. J. Bernstein and T. Lange, *Faster addition and doubling on elliptic curves,* ASIACRYPT 2007, LNCS 4833, (2007) 29–30.

[2] D. J. Bernstein and T. Lange, *Explicit-Formulas Database,* http://www.hyperelliptic.org/EFD/index.html

[3] F. Beukers and Y. I. Manin, *Diophantine equations,* Nieuw Arch. Wisk. 7 (1989), 3–13.

[4] P. Borwein and C. Ingalls, *The Prouhet-Tarry-Escott Problem revisited,* Enseign. Math. 40 (1994), 3–27.

[5] A. Bremner, *On squares of squares, Acta Arithmetica,* 88 (1999), 289–297.

[6] A. Bremner, *On squares of squares II, Acta Arithmetica,* 99 (2001), 289–308.

[7] A. Choudhry, *Symmetric diophantine systems,* Acta Arithmetica, 59 (1991), 291–307.

[8] A. Choudhry, *On equal sums of sixth powers,* Rocky Mountain Journal of Mathematics, 30 (2000), 843–848.

[9] A. Choudhry, *The system of simultaneous equations* $\sum_1^3 x_i^k = \sum_1^3 y_i^k$, $k = 1, 2$ *and 5 has no non-trivial solutions in integers,* The Mathematics Student, 70 (2001), 85–88.

[10] A. Choudhry, *On squares of squares,* The Mathematics Student, 76 (2007), 247–255.

[11] A. Choudhry and J. Wroblewski, *Ideal solutions of the Tarry-Escott problem of degree eleven with applications to sums of thirteenth powers,* Hardy-Ramanujan Journal, 31 (2008), 1–13.

[12] H. L. Dorwart and O. E. Brown, *The Tarry-Escott problem,* Amer. Math. Monthly, 44 (1937), 613–626.

[13] L. E. Dickson, *History of theory of numbers, Vol.2,* Chelsea Publishing Company, New York, 1952, reprint.

[14] R. K. Guy, *Unsolved Problems in Number Theory,* Third Edition, Springer-Verlag (2004).

[15] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers,* Sixth edition, Oxford University Press, 2008.

[16] L. J. Mordell, *Diophantine Equations,* Academic Press, London, 1969.

[17] A. Schinzel, *Diophantine Equations,* Wiadom. Mat. (2) 12, (1971), 227–232.

[18] J. H. Silverman and J. Tate, *Rational points on elliptic curves,* Springer–Verlag, New York, 1992.

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SPAN OF SPECIFIC MANIFOLDS

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Abstract: A nowhere vanishing vector field s on a manifold M^n is a section $s : M^n \to T(M^n)$ of the tangent bundle $T(M^n)$ on M^n with $s(x) \neq 0, \forall x$. By the span of a manifold M^n, we mean maximum number of linearly independent nowhere vanishing vector fields on M^n. The main objective of this paper is to give an account of vector fields on manifolds, and lower bounds and upper bounds of the span of specific manifolds like Grassmann manifolds, Milnor manifolds, Dold Manifolds, Wall manifolds, Stiefel manifolds and projective Stiefel manifolds.

1. **Introduction**

All the manifolds M^n dealt in shall be smooth, closed (compact without boundary) and connected of dimension n. A **vector field** s on M^n is a (continuous) section $s : M^n \to T(M^n)$ of the tangent bundle $\tau : T(M^n) \to M^n$. A vector field s on M^n is called **nowhere vanishing** if $s(x) \neq 0, \forall x \in M^n$. A set $\{s_1, ..., s_r\}$ of nowhere vanishing vector fields on M^n is called everywhere **linearly independent**, if the set $\{s_1(x), \ldots, s_r(x)\}$ is linearly independent, $\forall x \in M^n$. The maximum number of everywhere linearly independent vector fields on M^n is called the **span of** M^n and is denoted by **span** (M^n). Also, the number span $(M^n \oplus \epsilon^r) - r$ is same, $\forall r \geq 1$ and is called the **stablespan of** M^n, which is denoted by stable span M^n. Clearly,

$$\text{span}(M^n) \leq \ \text{stablespan}(M^n) \leq n.$$

The existence of nowhere vanishing vector field on M^n, conditions for the existence of k-everywhere linearly independent vector fields on M^n expressed in terms of some algebraic invariants, and computation of span of specific manifolds like n-dimensional sphere $\mathbf{S^n}$, n-dimensional real projective space $\mathbb{R}\mathbf{P^n}$, lens spaces, flag manifolds, Grassmann manifolds, Stiefel manifolds, projective Stiefel manifolds,

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Milnor manifolds, Dold manifolds, Wall manifolds etc. have been of long standing interest in Differential Topology. The first general result regarding existence of a nowhere vanishing vector field on M^n is due to Hopf [4] in 1972. He proved that M^n admits a nowhere vanishing vector field if and only if the Euler characteristic $\chi(M^n) = \Sigma(-1)^i b_i,\ b_i = \dim H_i(M^n, \mathbb{Q})$ being the i^{th} Betti number of M^n, summation being from 0 to ∞ and $\mathbb{Q}$ being the field of rationals.

2. Vector fields on $\mathbf{S^n}$ and $\mathbb{R}\mathbf{P^n}$

A deep study of the span of $\mathbf{S^n}$ has been made by Hurwitz [5], Radon [21], Steenrod and Whitehead [30] and Adams [1]. Since $\chi(\mathbf{S^n}) \neq 0$, for even n, one follows that span $(S^n) = 0$. In 1957, Steenrod and Whitehead [30] proved that for odd n with $n+1 = 2^\nu$. odd, span $\mathbf{S^n} < 2^\nu$. In 1923, Hurwitz [5] proved the following.

Theorem 2.1 (Hurwitz)**.** *For odd n with $n+1 = 2^{c+4d}$. odd, let $\rho(n) = 2^c + 8d$. Then*

$$span(\boldsymbol{S}^n) \geq \rho(n) - 1.$$

Proof: For an odd n, $\exists$ exactly $\rho(n)-1$ orthogonal transformations A_i in the orthogonal group $O(n)$ such that

$$A_i^2 = -Id. \text{ and } A_iA_j = -A_jA_i, i \neq j.$$

Let α and β be two odd dimensional vector bundles on X such that the Whitney sum $\alpha \oplus \beta \approx \epsilon^{n+1}$, ϵ^{n+1} being $(n+1)$-dimensional trivial bundle on X. Let $Hom(\alpha, \beta)$ be the manifold of all vector bundle homomorphisms from α to β. Define

$$v_i : X \to Hom(\alpha, \beta)$$

as $v_i(x) = p \circ A_i \circ j$, where $j : \alpha_x \to \mathbb{R}^\mathbf{n}$ is the inclusion map, $p : \mathbb{R}^\mathbf{n} \to \beta_x$ is the projection map, α_x and β_x being the fibers of the bundles α and β over $x \in X$. It can be seen that $\{v_i\}$ is a linearly independent set of the sections of the bundle $Hom(\alpha, \beta) \to X$. Hence span $(Hom(\alpha, \beta)) \geq \rho(n) - 1$. (Here, by span of a vector bundle $\xi : E \to X$, we mean maximum number of everywhere linearly independent sections of the bundle ξ). Replacing X by $\mathbf{S^n}$, α by the tangent bundle $\tau(\mathbf{S}^n)$ of $\mathbf{S}^n$ and β by the one dimensional trivial bundle ϵ^1 over $\mathbf{S^n}$, one gets

$$\text{span}\,(Hom(\tau(\mathbf{S}^n), \epsilon^1)) \geq \rho(n) - 1.$$

This, together with the fact that

$$Hom(\tau(\mathbf{S^n}), \epsilon^1) \approx \tau(\mathbf{S^n}),$$

implies that

$$\text{span}\,(\tau(\mathbf{S^n})) = \text{span}\,(\mathbf{S^n}) \geq \rho(n) - 1. \quad \#$$

In 1962, Adams [1] proved the otherway inequality.

Theorem 2.2 (Adams)**.** *. For n odd with $n+1 = 2^{c+4d}$ odd,*

$$span\,(\boldsymbol{S^n}) \leq \rho(n) - 1.$$

The proof of Adams theorem is extremely complicated and involves deep concepts and tools like S-duality, J-map, cohomology operations in K-theory, spectral sequence in K-theory etc. With Adams theorem, the computation of span ($\mathbf{S^n}$) was complete.

Singhof [27], in 1982, proved the following general result, which will be used at different places in the paper.

Result 2.1 (Singhof). If $p : E \to B$ is a smooth fiber bundle, then

$$\text{span}\,(E) \geq \text{ span}\,(B) \text{ and stablespan}\,(E) \geq \text{stablespan}\,(B).$$

Further, if the fiber F is connected, then

$$\text{stablespan}\,(B) \leq \text{ stablespan}\,(E) \leq \text{stablespan}(F) + \dim B.$$

The following theorem gives span ($\mathbb{R}\mathbf{P^n}$).

Theorem 2.3. *Span*$(\mathbb{R}\boldsymbol{P^n})$ = *span*$(\boldsymbol{S^n})$.

Proof: For odd n, in the proof of the Theorem 1.1, choose $X = \mathbb{R}\mathbf{P^n}$, $\alpha = \gamma_n^1$, the canonical line bundle over $\mathbb{R}\mathbf{P}^n$ and $\beta = \gamma^\perp$, the orthogonal complement of γ_n^1 in ϵ^{n+1}. Then

$$Hom(\gamma_n^1, \gamma^\perp) \approx \tau(\mathbb{R}\mathbf{P^n}).$$

Therefore, span$(\mathbb{R}\mathbf{P^n}) \geq \rho(n) - 1$. Also, the projection map $p : \mathbf{S^n} \to \mathbb{R}\mathbf{P}^n$ is a fiber bundle giving span$(\mathbf{S^n}) \geq$ span$(\mathbb{R}\mathbf{P^n})$ by Result 2.1. Therefore, by Theorem 2.2, span$(\mathbb{R}\mathbf{P}^n) \leq \rho(n) - 1$, showing that

$$\text{span}(\mathbb{R}\mathbf{P^n}) = \rho(n) - 1 = \text{span}(\mathbf{S^n}),$$

for odd n. However, for even n, the fiber bundle $p : \mathbf{S^n} \to \mathbb{R}\mathbf{P^n}$ gives

$$\text{span}(\mathbb{R}\mathbf{P^n}) \leq \text{span}(\mathbf{S^n}) = 0.$$

Therefore, for even n,

$$\text{span}(\mathbb{R}\mathbf{P^n}) = 0 = \text{span}(\mathbf{S^n}). \quad \#$$

3. Span of π-manifolds

Before we go for the study of span of π-manifolds, it is necessary to mention some useful results for general manifolds which will be used at different places in the paper.

Result 3.1. For a manifold M^n, if $W_{n-k}(M^n) \neq 0$, then span$(M^n) \leq k$, $W_{n-k}(M^n)$ being the $(n-k)^{th}$ Stiefel Whitney class of M^n. (cf Steenrod [29]).

Result 3.2. Let M be a manifold of dim $n > 6$, $n \equiv 2(\mod 4)$. Then span$(M) \leq 3$ if and only if $\chi(M)$ and $W_{n-2}(M)$ vanish (cf. Koschorke [15]).

Result 3.3. Let M^n be a connected nonorientable manifold of dimension n with $n \equiv 3(\mod 4)$. If $n = 3$, then span$(M^n) \geq 2$ if and only if $W_1^2(M^n) = 0$. If $n \geq 7$, then span$(M^n) \geq 2$ if and only if $W_1^2(M^n) = 0$ or if $W_1^2(M^n) \neq 0$ and $W_{n-1}(M^n) = 0$ (cf. Koschorke [15] and Randall [22]).

Result 3.4. Let M^n be a connected nonorientable manifold of dimension n with $n \equiv 3(\mod 4)$. If $n \geq 7$, then span$(M^n) \geq 3$ if and only if $\beta^* W_{n-3}(M^n) = 0$, where β^* is the Bockstein homomorphism. (cf. [23]).

A manifold M^n is defined to be **parallelizable**, if the tangent bundle $\tau(M^n)$ is trivial. The Stiefel manifold $V_{n,k}$, consisting of all orthonormal $k-$frames in $\mathbb{R}^n$, $k \leq n$, is parallelizable for all $k > 1$. $\mathbf{S^n}$ is not parallelizable.

By a $\pi-$**manifold**, we mean a manifold M^n immersed in $\mathbb{R}^{n+1}$ with trivial normal bundle. π-manifold is also called stably parallelizable. $\mathbf{S^n}$ is a π-manifold for every n. The Stiefel manifold $V_{n,r}$ is a π-manifold, $\forall n$ and r. Kosinski and Bredon [16, 17] proved the following.

Theorem 3.1. *Let M^n be a π-manifold. Then*

(a) *Either M^n is parallelizable or span(M^n) = span$(\boldsymbol{S^n})$.*

(b) *For even n, M^n is parallelizable if and only if $\chi(M^n) = 0$.*

(c) *For $n = 2q + 1$ with $n \neq 1, 3, 7$, M^n is parallelizable $\Leftrightarrow$ $\hat{\chi_2}(M^n) = 0$, where $\hat{\chi_2}(M^n) = \sum_{i=0}^{q} \dim H_i(M^n; \mathbb{Z}_2) \mod 2$ and is called the Kervairc semi-characteristic of M^n.*

Proof: We give a brief sketch of the proof.

(a) Since M^n is a π-manifold, $\tau(M^n) \oplus \epsilon^1$ is trivial. Let $F = \{(e_1, \ldots.., e_{n+1})\}$ be a framing of $\tau(M^n) \oplus \epsilon^1$ compatible with the given orientation of $\tau(M^n) \oplus \epsilon^1$. Let $x \in M^n$ and $v = x_1 e_1 + \cdots + x_{n+1} e_{n+1}$ be in the fiber at x. Define $F^*(v) = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. Let $s(x)$ be the image of the canonical section of the trivial line bundle ϵ^1 on M^n. Treating $s(x)$ as an element of the top space of the bundle $\tau(M^n) \oplus \epsilon^1$, $F^*(s(x)) \in \mathbf{S^n}$. This defines a map

$$\nu_F : M^n \to \mathbf{S^n}$$

as $\nu_F(x) = F^*(s(x))$. It is easy to see that $\nu_F^*(\tau(\mathbf{S^n})) = \tau(M^n)$, which implies that

$$\text{span}(M^n) \geq \text{span}(\mathbf{S^n}) = \rho(n) - 1.$$

Next, one can show that span$(M^n) \geq k$ if and only if for every framing F of the bundle $\tau(M^n) \oplus \epsilon^1$, $\exists$ a map $f_k : M^n \to V_{n+1,k+1}$ with $\nu_F = \pi f_k$, where

$\pi : V_{n+1,k+1} \to V_{n+1,1} = \mathbf{S^n}$ is the canonical projection. Further, if F and G are two framings of $\tau(M^n) \oplus \epsilon^1$, then

$$\deg(\nu_G) = \deg(\nu_F) + \deg(\pi' \circ g),$$

where $g : M^n \to SO(n+1)$ is a map such that $\forall x \in M^n,\ \nu_G(x) = g(x)\nu_F(x)$ and

$$\pi' : V_{n+1,n+1} = SO(n+1) \to \mathbf{S}^n = V_{n+1,1}$$

is the canonical projection. Also, $\deg(\pi' \circ g)$ is always even and any map $\nu : M^n \to \mathbf{S}^n$ of degree 1 can be lifted to $f : M^n \to V_{n+1,k+1}$ if and only if $\mathrm{span}(\mathbf{S^n}) \geq k$.

Now, suppose either n is even and $\exists$ a framing F with $\deg(\nu_F) = 0$ or n is odd and $\exists$ a framing G with $\deg(\nu_G)$ being even. Therefore, in both cases $\exists$ a framing $\tilde{G}$ such that $\deg(\nu_{\tilde{G}}) = 0$ which implies that

$$\nu_{\tilde{G}} : M^n \to \mathbf{S^n}$$

is homologically trivial. Therefore, $\nu_{\tilde{G}}$ is null homotopic and hence can be lifted to a map

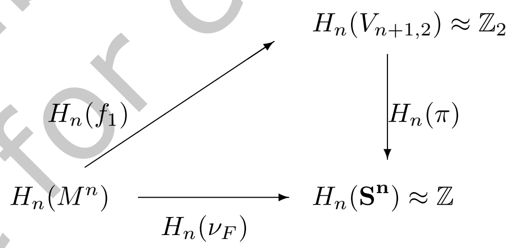
$$f_n : M^n \to V_{n+1,n+1} = SO(n+1).$$

Hence, $\mathrm{span}(M^n) \geq n$, giving $\mathrm{span}(M^n) = n$.

Next, let n be even with some framing F such that $\deg(\nu_F) \neq 0$. We claim that $\mathrm{span}(M^n) = 0$. On the contrary, let $\mathrm{span}(M^n) \geq 1$. Hence $\exists$ a lifting

$$f_1 : M^n \to V_{n+1,2}$$

which induces the following commutative diagram



This shows that $\deg(\nu_F) = \deg(\pi f_1) = 0$, contradicting the hypothesis that $\deg(\nu_F) \neq 0$. Thus $\mathrm{span}(M^n) = 0$.

Now, suppose n is odd $\neq 1, 3, 7$ and F is some framing with $\deg(\nu_F) = 1(\mod 2)$. Then there exists a framing G such that $\deg(\nu_G) = 1$ and hence ν_G can be lifted to a map $f : M^n \to V_{n+1,k+1}$ if and only if $\mathrm{span}(\mathbf{S^n}) \geq k$, which in turn shows that $\mathrm{span}(M^n) \geq k \Leftrightarrow \mathrm{span}(\mathbf{S^n}) \geq k$. This, together with the fact that $\mathrm{span}(M^n) \geq \mathrm{span}(\mathbf{S^n})$ implies that $\mathrm{span}(M^n) = \mathrm{span}(\mathbf{S^n})$. This proves part (a).

(b) and (c): If n is even, then the canonical projection

$$\pi : V_{n+1,n+1} = SO(n+1) \to \mathbf{S^n} = V_{n+1,1}$$

factors through $V_{n+1,2}$ with $H_n(V_{n+1,2}) \approx \mathbb{Z}_2$, which shows that $\deg(\pi \circ g) = 0$. Hence, for any two framings F and G, $\deg(\nu_F) = \deg(\nu_G)$. Also for even n, M^n is parallelizable implies that span $M^n = n$, which in turn implies that $\deg(\nu_F) = 0$, for all framings F. Conversely, if $\deg(\nu_F) = 0$, then span $M^n = n$ and by part (a), one infers that M^n is parallelizable. Thus, we conclude that for even n, M^n is parallelizable $\Leftrightarrow \deg(\nu_F) = 0, \forall$ framing F.

Further, if n is odd $\neq 1, 3, 7$, then M^n is parallelizable $\Rightarrow$ span $M^n = n \Rightarrow$ for every framing F, $\deg(\nu_F)$ must be even. Conversely, $\deg(\nu_F)$ is even for some framing $F \Rightarrow$ span $M^n = n \Rightarrow M^n$ is parallelizable by part (a). Now, let us define

$$d(M^n) = \begin{cases} \deg(\nu_F), & \text{if } n \text{ is even} \\ (\deg(\nu_F)) \mod 2, & \text{if } n \text{ is odd} \end{cases}$$

It is not difficult to verify that for a π-manifold M^n, $n \neq 1, 3, 7$

$$d(M^n) = \hat{\chi}(M^n),$$

where

$$\hat{\chi}(M^n) = \begin{cases} \frac{1}{2}\chi(M^n) & \text{if } n \text{ is even} \\ \sum\limits_{i=0}^{r} \dim H_i(M^n; \mathbb{Z}) \mod 2, & \text{if } n = 2r+1 \end{cases}$$

This gives parts (b) and (c). #

A manifold M^n is defined as an almost **parallelizable manifold** if after removing a point, it becomes parallelizable. $\mathbf{S^n}$ is almost parallelizable, $\forall n$. Note that $\mathbf{S^n}$ is also π-manifold *i.e.* stably parallelizable. It is not difficult to show that a π-manifold is always almost parallelizable. Further, an almost parallelizable manifold is not stably parallelizable if and only if $n = 4k$ and the Pontrjagin class $P_k(M^n) \in H^{4k}(M^n; \mathbb{Z})$ is nonzero. Thomas [32] gave span of an almost parallelizable manifold $M^n, n = 4k$, in 1969.

Theorem 3.2 (Thomas). *Let M^{4k} be an almost parallelizable manifold, $k > 4$ with $P_k(M^{4k}) \neq 0$ and $\chi(M^{4k}) = 0$. Then span$(M^{4k}) = 2k$ or $2k-1$.*

Proof: Let D be a small closed $4k$-disc in M^{4k} around a point x_0. Collapsing $M - D$ to a point, we get $\mathbf{S^{4k}}$. Let $p : M^{4k} \to \mathbf{S^{4k}}$ be the collapse map. Since M^{4k} is almost parallelizable, the tangent bundle $\tau(M^{4k})$ restricted to $M - D$ is trivial. Hence $\exists$ a $4k$-plane bundle ξ on $\mathbf{S^k}$ such that $\tau(M^{4k}) \approx p^*(\xi)$. Also, one knows that for an arbitrary manifold M^n with $4k \leq n$ and $P_k(M^n) \neq 0$,

$\text{span}(M^n) \leq n - 2k$. Therefore, in the present situation, $\text{span}(M^{4k}) \leq 2k$. By naturality, $P_k(M^{4k}) = p^*(P_k(\xi))$, showing that $P_k(\xi) \neq 0$. Barrot and Mahowald [3] proved in 1964 that if $P_k(\xi) \neq 0$, then ξ has precisely $2k-1$ linearly independent sections. But $\tau(M^n) \approx p^*(\xi) \Rightarrow \text{span}(M^{4k}) \geq 2k - 1$. Hence $\text{span}(M^{4k})$ is $2k$ or $2k - 1$. #

4. Span of GRASSMANN manifolds

Grassmann manifold $G(m, n)$ is the manifold of dimension mn consisting of all subspaces of dimension n in $\mathbb{R}^{m+n}$. Clearly $G(1, n) \approx \mathbb{R}P^n$.

Let $r_i : E(r_i) \to G(n_1, n_2)$ be the canonical n_i-bundle, $i = 1, 2$, with

$$E(r_i) = \{((S_1, S_2), v) : \; v \in S_i\}.$$

Lam [18], in 1975, proved that $r_1 \oplus r_2 = \epsilon^{n_1+n_2}$ and the tangent bundle $\tau(G(n_1, n_2)) \approx r_1 \otimes r_2$. The following result helps in estimating stablespan of $G(n_1, n_2)$.

Theorem 4.1.

$$\textit{stablespan}\,(G(n_1, n_2)) \leq \; \textit{stablespan}\,(G(n_1 - 1, n_2 - 1)) + (2n - 3),$$

where $n = n_1 + n_2$.

Proof: The inclusion map $a : \mathbb{R}^{n-1} \to \mathbb{R}^n$ given by

$$a(x_1, x_2, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{n-1}, 0)$$

induces inclusion homomorphism

$$a_1 : G(n_1 - 1, n_2) \to G(n_1, n_2)$$
$$a_2 : G(n_1, n_2 - 1) \to G(n_1, n_2)$$

such that

$$a_i^*(r_j) = \begin{cases} r_j, \; j \neq i \\ r_i' \oplus \epsilon, \; j = i, \end{cases}$$

where r_1' is the canonical $(n_1 - 1)$-plane bundle on $G(n_1 - 1, n_2)$ and r_2' is the canonical $(n_2 - 1)$-plane bundle on $G(n_1, n_2 - 1)$. Since

$$\tau(G(n_1, n_2)) \approx r_1 \otimes r_2,$$

$$\begin{aligned} & a_i^*(\tau(G(n_1, n_2))) \oplus r_i' \\ = & a_i^*(r_1 \otimes r_2) \oplus r_i' \\ = & \begin{cases} ((r_1' \oplus \epsilon) \otimes r_2) \oplus r_1', \; i = 1 \\ (r_1 \otimes (r_2' \oplus \epsilon)) \oplus r_2', \; i = 2 \end{cases} \end{aligned}$$

$$= \begin{cases} (r_1' \otimes r_2) \oplus (r_2 \oplus r_1'), & i = 1 \\ (r_1 \otimes r_2') \oplus (r_1 \oplus r_2'), & i = 2 \end{cases}$$

$$= \begin{cases} \tau(G(n_1 - 1, n_2)) \oplus (n-1)\epsilon, & i = 1 \\ \tau(G(n_1, n_2 - 1)) \oplus (n-1)\epsilon, & i = 2 \end{cases}$$

Therefore,

$$\begin{aligned} \text{stablespan } (G(n_1, n_2)) &\leq (n-1) + \text{ stablespan } (G(n_1 - 1, n_2)) \\ &\leq (n-1) + (n-2) + \text{stabspan } (G(n_1 - 1, n_2 - 1)) \\ &= (2n-3) + \text{stablespan } (G(n_1 - 1, n_2 - 1)). \end{aligned}$$

Corollary 4.1. *stablespan*$(G(n_1, n_2)) \leq (n_1 - 1)(2n_2 + 1) +$ *stablespan* $(\mathbb{R}\boldsymbol{P}^{n_2 - n_1 + 1})$, $n_2 \geq n_1$.

Proof: stablespan $(G(n_1, n_2)) \leq (2n-3) +$ stablespan $(G(n_1 - 1, n_2 - 1))$
$\leq (2n-3) + (2n-7) + \cdots + (2n - (3 + 4(n_1 - 2))) +$ stablespan $(G(1, n_2 - n_1 + 1))$
$= (2n-3)(n_1 - 1) - (4 + 8 + \cdots + 4(n_1 - 2)) +$ stablespan $(\mathbb{R}\mathbf{P}^{n_2 - n_1 + 1})$
$= (n_1 - 1)(2n_2 + 1) +$ stablespan $(\mathbb{R}\mathbf{P}^{n_2 - n_1 + 1})$

Shanahan [26] in 1979 proved that

$$\chi(G(n_1, n_2)) = \begin{cases} \begin{pmatrix} [\frac{n_1 + n_2}{2}] \\ [\frac{n_1}{2}] \end{pmatrix}, & \text{if at most one } n_i \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

The following result gives estimate of the span of $G(n_1, n_2)$.

Theorem 4.2. *If $n_1 n_2$ is odd, then*
span $(G(n_1, n_2)) \geq \rho(n_1 + n_2) - 1$

Proof: Let r_1 and r_2 be the canonical bundles of dimensions n_1 and n_2 on $G(n_1, n_2)$.
Since

$$r_1 \oplus r_2 = \epsilon^{n_1 + n_2},$$

$$\text{span } (Hom(r_1, r_2)) \geq \rho(n_1 + n_2) - 1.$$

Therefore,

$$\begin{aligned} \text{span } (G(n_1, n_2)) &= \text{span } (\tau(G(n_1, n_2))) \\ &= \text{span } (r_1 \otimes r_2) \\ &\geq \text{span } (Hom(r_1, r_2)) \\ &\geq \rho(n_1 + n_2) - 1. \qquad \# \end{aligned}$$

Theorem 4.3. *For $n_1 \geq 2$*

(i) $G(n_1, n_2)$ has positive span $\Leftrightarrow$ both n_1 and n_2 are odd.
(ii) No $G(n_1, n_2)$ is stably parallelizable.

Proof: Suppose both n_1 and n_2 are odd. Then the Euler characteristic $\chi(G(n_1, n_2)) = 0$, showing that span $(G(n_1, n_2)) \geq 1$. Conversely, suppose span $(G(n_1, n_2)) \geq 1$. Therefore, $\chi(G(n_1, n_2)) = 0$, showing that both n_1 and n_2 are odd. Further, $G(n_1, n_2)$ is not parallelizable since i^{th} Stiefel-Whitney class of $G(n_1, n_2)$ does not vanish for some $i > 1$. #

Korbaš [12, 13], Zvengrowski [35, 37], Sankaran [24] and Maitello and Maitello [19] made extensive study of span of Grassmann manifolds. Still, a lot is to be done to estimate span $G(n_1, n_2)$ for general values of n_1 and n_2.

5. Span of projective STIEFEL manifold

The projective Stiefel manifold $X_{n,r}$ is obtained from the Stiefel manifold $V_{n,r}$ by identifying an orthonormal r-frame $v = (v_1, \ldots, v_r)$ with the r-frame $-v = (-v_1, -v_2, \ldots, -v_r)$. In particular, $X_{n,1} = \mathbb{R}\mathbf{P}^{n-1}$.

Let $\xi_{n,r}$ be the line bundle associated with the double covering $V_{n,r} \to X_{n,r}$. Let $\eta_{n,r}$ be the canonical $(n-r)$-plane bundle over $X_{n,r}$ with the total space $E(\eta_{n,r})$ given by

$$\{(x, \{v, -v\}) \in \mathbb{R}^n \times X_{n,r} : \; x \text{ is orthogonal to the space spanned by } v\}.$$

According to Smith [28],

$$r\xi_{n,r} \oplus \eta_{n,r} \approx \epsilon^n.$$

The line bundle $\xi_{n,r}$ over $X_{n,r}$ admits following universal property [28].

Result 5.1. Let X be a CW-complex. Then given a line bundle ξ over X, $n\xi$ admits r everywhere linearly independent sections if and only if $\exists\ g : X \to X_{n,r}$ such that $g^*(\xi_{n,r}) \approx \xi$.

Antoniano, Gitler, Ucci and Zvengrowski [2], using complex k-theory of $X_{n,r}$, proved the following result regarding parallelizability.

Result 5.2. $X_{n,r}$ is parallelizable, if the pair (n, r) is of the form

$$(n, n),\ (n, n-1),\ (2m, 2m-2),\ (4, r),\ (8, r), \text{ or } (16, 8),$$

where m is an arbitrary positive integer.

Lam [18] gave stable tangent bundle of $X_{n,r}$ as follows:

$$\tau(X_{n,r}) \oplus \binom{r+1}{2} \epsilon^1 = nr\xi_{n,r} \tag{5.1}$$

Korbaš, Zvengrowski and Sankaran [14, 25, 35, 36] made an extensive study of the span of $X_{n,r}$. They gave upper bound of the span through the following result.

Theorem 5.1. *Span $(X_{n,r}) \leq$ stablespan $(X_{n,r}) < d-k$, if there does not exist a map*

$$f : X_{n,r} \to X_{nr,nr-k}$$

with $f^*(\xi_{nr,nr-k}) = \xi_{n,r}$. Here $d = \dim(X_{n,r})$.

Proof: Suppose, stablespan $(X_{n,r}) \geq d-k$. Then from the stable tangent bundle of $X_{n,r}$, one infers that $nr\xi_{n,r}$ admits $nr-k$ everywhere linearly independent sections. Therefore, by Result 5.1, $\exists$ a map

$$f : X_{n,r} \to X_{nr,nr-k}$$

such that $f^*(\xi_{nr,nr-k}) = \xi_{n,r}$. #

Simple lower bound for span $(X_{n,r})$ can be found using the following smooth fibrations

$$X_{n,r} \to X_{n,r-1} \to \ \ldots \ \to X_{n,1} = \mathbb{R}\mathbf{P}^{n-1},$$

which gives

$$X_{n,r} \to X_{n,r-1} \to \ \ldots \ \to X_{n,1} = \mathbb{R}\mathbf{P}^{n-1},$$

which gives

$$\rho(n) - 1 = \text{ span}\,(X_{n,1}) \leq \text{ span}\,(X_{n,2}) \leq \cdots \leq \text{ span}\,(X_{n,r})$$

Another lower bound noted by Lam [18] is given as follows

$$\text{span}\,(X_{n,r}) \geq \binom{r}{2}.$$

However, the best lower bound mostly arise in the cases where span $(X_{n,r}) =$ stablespan $(X_{n,r})$.

In such cases, by (5.1), one gets

$$\begin{aligned} \text{stablespan } (X_{n,r}) \quad &= \text{span } (nr\xi_{n,r}) - \binom{r+1}{2} \\ &\geq \text{span } (nr\xi_{n,1}) - \binom{r+1}{2}. \end{aligned}$$

Sankaran and Zvengrowski [25] obtained upper bounds for the span of projective Stiefel manifolds $X_{n,k}$ for $2 \leq k \leq [\frac{n}{2}]$, using stable tangent bundle and complex K-ring of these manifolds and the universal properties enjoyed by these manifolds ($X_{n,k}$ is classifying space for line bundle whose n-fold Whitney sum with itself admits k-everywhere linearly independent sections). Recently Zvengrowski and Korbaš [36] computed span $(X_{n,r})$ for $r = 2, 3, 4$ and for suitable (infinitely many) values of n.

6. Estimation of span of MILNOR manifolds

A Milnor manifold $H(m,n)$ of dimension $m+n-1$ is a submanifold of $\mathbb{R}\mathbf{P}^m \times \mathbb{R}\mathbf{P}^n$ consisting of elements $([x_0,.....,x_m],\ [y_0,.....,y_n])$ in $\mathbb{R}\mathbf{P}^m \times \mathbb{R}\mathbf{P}^n$ such that $\sum_{i=0}^{l} x_i y_i = 0$, where $l = \min(m,n)$. For $m \leq n,\ H(m,n)$ is a fiber bundle over $\mathbb{R}\mathbf{P}^m$ with fiber $\mathbb{R}\mathbf{P}^{n-1}$. Let $a \in H^1(\mathbb{R}\mathbf{P}^m;\mathbb{Z}_2)$ and $b \in H^1(\mathbb{R}\mathbf{P}^n;\mathbb{Z}_2)$ be generators with $a^{m+1} = b^{n+1} = 0$ and $a^i, b^j \neq 0,\ 0 \leq i \leq m,\ 0 \leq j \leq n$. Then the total Stiefel-Whitney class $W(H(m,n))$ of $H(m,n)$ is given by the restriction of

$$(1+a)^{m+1}(1+b)^{n+1}/(1+a+b)$$

to $H(m,n)$ and the Stiefel-Whitney number of $H(m,n)$ for a partition $i_1 + i_2 + \cdots + i_k$ of $m+n-1$ is given by

$$< (a+b) \cup W_{i_1} \cup \cdots \cup W_{i_k},\ [\mathbb{R}\mathbf{P}^m \times \mathbb{R}\mathbf{P}^n] > \in \mathbb{Z}_2,$$

where W_{i_j} is the sum of i_j-dimensional terms in the expansion of

$$(1+a)^{m+1}(1+b)^{n+1}/(1+a+b),$$

$[\mathbb{R}\mathbf{P}^m \times \mathbb{R}\mathbf{P}^n]$ is the fundamental class of $\mathbb{R}\mathbf{P}^m \times \mathbb{R}\mathbf{P}^n$ and $<,>$ is the Kronecker product. Bounding and independence problems for Milnor manifolds $H(m,n)$ have been settled by Khare, Sanghita and Das [10, 11] in 2000-2001. Estimation of bounds of span of $H(m,n)$ has been given by Khare [8] in 2008. Estimation of bounds of span of $H(m,n)$ has been divided in two cases.

Case I: Dim $H(m,n)$ is odd.

Theorem 6.1.

$$\rho(2^\alpha - 1) - 1 \leq \operatorname{span}(H(2^\alpha - 1, 2^\beta - 1)) \leq 2^\alpha - 1,\ \beta \geq \alpha.$$

Proof: The total Stiefel-Whitney class W of $H(2^\alpha - 1, 2^\beta - 1)$ is given by

$$\begin{aligned} W &= \frac{(1+a)^{2^\alpha}(1+b)^{2^\beta}}{(1+a+b)} \mid H \\ &= (1 + \sum_i (a+b)^i) \mid H. \end{aligned}$$

Therefore,

$$(a+b)W_{2^\beta-2} = (a+b)^{2^\beta-1} \neq 0,$$

which implies that

$$\begin{aligned} \text{span } (H(2^\alpha-1, 2^\beta-1)) &\leq (2^\alpha + 2^\beta - 3) - (2^\beta - 2) \\ &= 2^\alpha - 1 \end{aligned}$$

Further, $H(2^\alpha - 1, 2^\beta - 1)$ fibers over $\mathbb{R}\mathbf{P}^{2^\alpha-1}$.

Therefore,

$$\begin{aligned} \text{span } (H(2^\alpha-1, 2^\beta-1)) &\geq \text{span } (\mathbb{R}\mathbf{P}^{2^\alpha-1}) \\ &= \rho(2^\alpha - 1) - 1. \qquad \# \end{aligned}$$

Corollary 6.1. *For $l = 1, 3, 5$ and $l < 2^\beta$*

$$span\ H(l, 2^\beta - 1)) = l$$

Proof: Here $l = 2^\alpha - 1,\ \alpha = 1, 2, 3$. Therefore,

$$\begin{aligned} \rho(l) &= 2^c + 8d, \text{ for } c = 1, 2, 3 \text{ and } d = 0 \\ &= 2^c \\ &= l + 1. \end{aligned}$$

Theorem 6.2. $1 \leq span\ (H(2^\alpha, 2^\beta)) \leq 2^\alpha - 1,\ \alpha \leq \beta$

Proof: In this case

$$\begin{aligned} W(H(2^\alpha, 2^\beta)) &= \frac{(1+a)^{2^\alpha+1}(1+b)^{2^\beta+1}}{(1+a+b)} \mid H \\ &= (1 + ab + \sum_1^\infty ab(a+b)^i)(1+a)^{2^\alpha}(a+b)^{2^\beta} \mid H \end{aligned}$$

Therefore,

$$\begin{aligned} (a+b)W_{2^\beta} &= ab(a+b)^{2^\beta-1} + (a+b)a^{2^\alpha}.ab(a+b)^{2^\beta-2^\alpha-2} + (a+b)b^{2^\beta} \\ &= ab(a+b)^{2^\beta-1} + ab^{2^\beta} \neq 0. \end{aligned}$$

Hence,

$$\text{span}(H(2^\alpha, 2^\beta)) \leq (2^\alpha + 2^\beta - 1) - 2^\beta = 2^\alpha - 1$$

Further, it can be calculated that

$$\chi(H(2^\alpha, 2^\beta)) = 0,$$

which shows that

$$\text{span}\,(H(2^\alpha, 2^\beta)) \geq 1. \qquad \#$$

Theorem 6.1 can be generalized to the Milnor manifolds $H(m, n)$, with

$$\begin{aligned} m &= 2^{\alpha_1} + \cdots + 2^{\alpha_m} - 1, \\ n &= 2^{\beta_1} + \cdots + 2^{\beta_n} - 1, \end{aligned}$$

for $\alpha_m < \cdots < \alpha_1 \leq \beta_n < \cdots < \beta_1$ as follows.

Theorem 6.3. *For $\alpha_m < ..... < \alpha_1 \leq \beta_n < \cdots < \beta_1$*

$$\rho(2^{\alpha_m} - 1) - 1 \leq span\ (H(m, n)) \leq 2^{\alpha_m} - 1$$

Proof: $W(H(m, n))$ is given by

$$(1 + \sum_i (a+b)^i)(1+a)^{m+1}(1+b)^{n+1} \mid H.$$

Therefore,

$$(a+b)W_{m+n-2^{\alpha_m}} = (a+b)^{m+n-2^{\alpha_m}+1} + \sum_{i,j} a^i b^j (a+b)^{m+n-2^{\alpha_m}+1-i-j},$$

where i is the sum of some members of $\{2^{\alpha_1}, \ldots, 2^{\alpha_m}\}$ and j is the sum of some members of $\{2^{\beta_1}, \ldots, 2^{\beta_n}\}$ with $i \neq m+1,\ j \neq n+1$ and $i+j < m+n-2^{\alpha_m}+1$. All the terms with $i \neq m-2^{\alpha_m}+1$ are zero, while the terms with $i = m-2^{\alpha_m}+1$ are nonzero. Number of such nonzero terms is odd giving one nonzero term $a^{m-2^{\alpha_m}+1} b^{n+1-2^{\beta_1}} (a+b)^{2^{\beta_1}} - 1$. Hence,

$$\begin{aligned} \text{span } (H(m,n)) &\leq (m+n-1) - (m+n-2^{\alpha_m}+1) \\ &= 2^{\alpha_m} - 1. \end{aligned}$$

Also, $H(m,n)$ fibers over $\mathbb{R}\mathbf{P}^m$, which implies that

$$\begin{aligned} \text{span } (H(m,n)) &\geq \text{span } (\mathbb{R}\mathbf{P}^m) \\ &= \rho(m) - 1 \\ &= \rho(2^{\alpha_m} - 1) - 1 \qquad \# \end{aligned}$$

Similarly, Theorem 6.2 can be generalized for $H(m,n)$ with

$$m = 2^{\alpha_1} + \ldots. + 2^{\alpha_m}$$
$$\text{and} \quad n = 2^{\beta_1} + \ldots. + 2^{\beta_n},$$

for $\alpha_m < \ldots. < \alpha_1 \leq \beta_n < \ldots. < \beta_1$ as follows.

Theorem 6.4. *For* $\alpha_m < \ldots. < \alpha_1 \leq \beta_n < \ldots. < \beta_1$

$$1 \leq \mathit{span}\ (H(m,n)) \leq 2^{\alpha_m} - 1.$$

Proof:

$$W(H(m,n)) = (1 + \sum_{0}^{\infty} ab(a+b)^i)(1+a)^m (a+b)^n \mid H.$$

Therefore,

$$(a+b)W_{m+n-2^{\alpha_m}} = ab(a+b)^{m+n-2^{\alpha_m}-1} + \sum_{i,j} a^i b^j (a+b)^{m+n-2^{\alpha_m}+1-i-j},$$

where i is the sum of some members of the set $\{2^{\alpha_1}, \ldots.., 2^{\alpha_m}\}$ and j is the sum of some members of the set $\{2^{\beta_1}, \ldots.., 2^{\beta_n}\}$ with $i \neq m,\ j \neq n$ and $i+j < m+n-2^{\alpha_m}+1$. All but two terms namely $a^{m-2^{\alpha_m}} b^n (a+b)$ and $a^{m-2^{\alpha_m}} b^{n-2^{\beta_n}} (a+b)^{2^{\beta_n}+1}$ of the above sum can be paired in such a way that each pair contributes zero. The remaining two terms mentioned above clearly contribute nonzero. Hence

$$(a+b)W_{m+n-2^{\alpha_m}} \neq 0.$$

Therefore,

$$\text{span}(H(m,n)) \leq 2^{\alpha_m} - 1.$$

Also, $H(m,n)$ fibers over $\mathbb{R}\mathbf{P}^m$ with fiber $\mathbb{R}\mathbf{P}^{n-1}$, showing that

$$\chi(H(m,n)) = \chi(\mathbb{R}\mathbf{P}^m).\chi(\mathbb{R}\mathbf{P}^{n-1}) = 0.$$

Hence,

$$\text{span}(H(m,n)) \geq 1. \qquad \#$$

Case II: Dim $H(m,n)$ is even.

Theorem 6.5. *For* $\alpha \le \beta$

$$\rho(2^\alpha - 1) - 1 \le \text{span}\,(H(2^\alpha - 1, 2^\beta)) \le 2^\alpha - 1.$$

Proof: $W(H(2^\alpha - 1, 2^\beta))$ is given by

$$(1 + \sum_i (a+b)^i)(1 + b + b^{2^\beta}) \mid H.$$

Therefore,

$$(a+b)W_{2^\beta - 1} = (a+b)^{2^\beta} + b(a+b)^{2^\beta - 1} \neq 0,$$

so that

$$\text{span}\,(H(2^\alpha - 1, 2^\beta)) \le 2^\alpha - 1.$$

Further, $H(2^\alpha - 1, 2^\beta)$ fibers over $\mathbb{R}\mathbf{P}^{2^\alpha - 1}$, which implies that

$$\text{span}\,(H(2^\alpha - 1, 2^\beta)) \ge \text{span}\,(\mathbb{R}\mathbf{P}^{2^\alpha - 1}) = \rho(2^\alpha - 1) - 1. \quad \#$$

Corollary 6.2. *Span* $(H(l, 2^\beta)) = l$*, for* $l = 1, 3, 7 < 2^\beta$*.*

Proof: $l + 1 = 2^\alpha,\ \alpha = 1, 2, 3$ and $\rho(2^\alpha - 1) - 1 = l.$ #

Theorem 6.6. *For* $\alpha_1 > \cdots > \alpha_m > \beta_1 > \cdots > \beta_n$*,*

$$\text{span}\,(H(2^{\alpha_1} + \cdots + 2^{\alpha_m} - 1,\ 2^{\beta_1} + \cdots + 2^{\beta_n})) = 0$$

Proof: The total Stiefel-Whitney class is given by

$$(1 + \sum (a+b)^i)(1+b)^{2^{\beta_1} + \cdots\cdots + 2^{\beta_n} + 1} \mid H.$$

Therefore, the top Stiefel-Whitney class

$$(a+b)W_{m+n-2} = (a+b)^{m+n-1} + \sum b^j (a+b)^{m+n-1-j} + \sum b.b^j (a+b)^{m+n-2-j},$$

where $m = 2^{\alpha_1} + \cdots + 2^{\alpha_m}$, $n = 2^{\beta_1} + \cdots + 2^{\beta_n}$, j is the sum of any number of members of the set $\{2^{\beta_1}, \ldots, 2^{\beta_n}\}$. It is easy to see that this sum is nonzero, showing that

$$\text{span}\,(H(m-1, n)) = 0. \quad \#$$

Theorem 6.7. *For* $\alpha_m < \cdots < \alpha_1 < \beta_n < \cdots < \beta_1$*,*

$$\rho(2^{\alpha_m} - 1) - 1 \le \text{span}\,(H(m-1, n)) \le 2^{\alpha_m} - 1,$$

where

$$[-0.1in]m = 2^{\alpha_1} + \cdots + 2^{\alpha_m} \text{ and } n = 2^{\beta_1} + \cdots + 2^{\beta_n}.$$

Proof: $W(H(m-1,n))$ is given by

$$(1+\sum(a+b)^i)(1+a)^m(1+b)^{n+1}|H.$$

Therefore,

$$(a+b)W_{m+n-2^{\alpha_m}-1} = (a+b)^{m+n-2^{\alpha_m}} + \sum a^i b^j (a+b)^{m+n-2^{\alpha_m}-i-j} + \sum b.a^i.b^j(a+b)^{m+n-2^{\alpha_m}-i-j-1},$$

where i is the sum of some terms of the set $\{2^{\alpha_1},\ldots,2^{\alpha_m}\}$ and j is the sum of some terms of the set $\{2^{\beta_1},\ldots,2^{\beta_n}\}$, $i \neq m$. All the terms in the above sum which are of the type $a^i b^{even}(a+b)^k$ are equal to $a^m b^n$ and the number of such term is even, since $m > 1$. Thus all these terms together contribute zero. Also terms of the type $a^i b^{odd}(a+b)^l$ are nonzero and are odd in number. Hence

$$(a+b)W_{m+n-2^{\alpha_m}-1} \neq 0.$$

Therefore, span $(H(m-1,n)) \leq 2^{\alpha_m}-1$. Further, $H(m-1,n)$ fibers over $\mathbb{R}\mathbf{P}^{m-1}$, which implies that

$$\text{span}\,(H(m-1,n)) \geq \text{span}\,(\mathbb{R}\mathbf{P}^{m-1}) \geq \rho(2^{\alpha_m}-1)-1.$$

7. Estimation of span of DOLD manifolds

A Dold manifold $\mathbf{P}(m,n)$ of dimension $m+2n$ is the quotient manifold $\mathbf{S}^m \times \mathbb{C}\mathbf{P}^n/\sim$, where $((x_0,\ldots,x_m),\ [z_0,\ldots,z_n])$ is related to $((-x_0,\ldots,-x_m),\ [\bar{z}_0\ldots,\bar{z}_n])$. It is easy to see that $\mathbf{P}(m,0) \approx \mathbb{R}\mathbf{P}^m$ and $\mathbf{P}(0,n) \approx \mathbb{C}\mathbf{P}^n$. Also, $\mathbf{P}(m,n)$ fibers over $\mathbb{R}\mathbf{P}^m$ with fiber $\mathbb{C}\mathbf{P}^n$. Further, total Stiefel-Whitney class of $\mathbf{P}(m,n)$ is given by

$$W(\mathbf{P}(m,n)) = (1+c)^m(1+c+d)^{n+1},$$

where $0 \neq c \in H^1(\mathbf{P}(m,n);\mathbb{Z}_2)$ and d is suitable nonzero element of $H^2(\mathbf{P}(m,n);\mathbb{Z}_2)$ with $c^{m+1} = d^{n+1} = 0$ and $c^i, d^j \neq 0,\ i < m+1,\ j < n+1$. Bounding and indepence problems of Dold manifolds have been settled by Khare [6]. Recently Peter Novotný [20] gave estimation of span of Dold manifolds. Since

$$\mathbb{C}\mathbf{P}^n \to \mathbf{P}(m,n) \to \mathbb{R}\mathbf{P}^m$$

is a fiber bundle,

$$\text{span}\,(\mathbf{P}(m,n)) \geq \text{span}\,(\mathbb{R}\mathbf{P}^m) = \text{span}\,\mathbf{S}^m$$

Further, the canonical map $\mathbf{S}^m \times \mathbb{C}\mathbf{P}^n \to \mathbf{P}(m,n)$ is given by

$$(x,[z]) \to [x,[z]]$$

induces a fiber bundle

$$\mathbb{Z}_2 \to \mathbf{S}^m \times \mathbb{C}\mathbf{P}^n \to \mathbf{P}(m,n).$$

Therefore,

$$\chi(\mathbf{P}(m,n)) = \tfrac{1}{2}\chi(\mathbf{S}^m)\chi(\mathbb{C}\mathbf{P}^n) = \begin{cases} \neq 0, & \text{if } m \text{ is even} \\ = 0, & \text{if } m \text{ is odd} \end{cases}$$

Hence, $\operatorname{span}(\mathbf{P}(m,n)) = 0$, if m is even and $\operatorname{span}(\mathbf{P}(m,n)) \geq 1$, if m is odd. This implies that it is enough to study $\operatorname{span}(\mathbf{P}(m,n))$ for odd m. Novotný proved the following important result.

Theorem 7.1. *Given a pair (m,n) of non-negative integers, let*

$$m = 2^{\nu(n+1)}.k + l, \quad 0 \leq l < 2^{\nu(n+1)}.$$

Then

$$\rho(m) - 1 \leq \text{(stable)span}\, \boldsymbol{P}(m,n) \leq 2^{\nu(n+1)}[2^{\nu(k+1)} + 1] - 2,$$

where $\nu(k)$ is the highest power of 2 dividing k.

Proof: Using Lucas' Theorem on binomial coefficients, it is proved that the coefficient of $c^{m-2^{\nu(n+1)}(2^{\nu(k+1)}-1)}d^{n-(2^{\nu(n+1)}-1)}$ is nonzero. Therefore, Stiefel-Whitney class

$$W_{m-2^{\nu(n+1)}(2^{\nu(k+1)}-1)+2(n-2^{\nu(n+1)}+1)}$$

is nonzero, showing that

$$\begin{aligned}\operatorname{span}(\mathbf{P}(m,n)) &\leq (m+2n) - m + 2^{\nu(n+1)}(2^{\nu(k+1)} - 1) - 2n + 2(2^{\nu(n+1)} - 1) \\ &= 2^{\nu(n+1)}(2^{\nu(k+1)} + 1) - 2.\end{aligned}$$

Also, $\rho(m) - 1 \leq \operatorname{span}\,(\mathbf{P}(m,n))$ follows from the fiber bundle

$$\mathbb{C}\mathbf{P}^n \to \mathbf{P}(m,n) \to \mathbb{R}\mathbf{P}^m$$

and the fact that $\operatorname{span}(\mathbb{R}\mathbf{P}^m) = \rho(m) - 1$. #

Corollary 7.1. *If n is even, then*

$$\rho(m) - 1 \leq \operatorname{span}\,(\boldsymbol{P}(m,n)) \leq 2^{\nu(m+1)} - 1.$$

Further, if n is even and $\nu(m+1) \in \{1,2,3\}$, then

$$\operatorname{span}\,(\boldsymbol{P}(m,n)) = 2^{\nu(m+1)} - 1.$$

Proof: If n is even, then $\nu(n+1) = 0$. Therefore $k = m$, showing that

$$\rho(m) - 1 \leq \text{(stable)span}\,(\mathbf{P}(m,n)) \leq 2^{\nu(m+1)} - 1.$$

Also, if $\nu(m+1) \in \{1,2,3\}$, then

$$\rho(m) - 1 = 2^{\nu(m+1)} - 1. \quad \#$$

Corollary 7.2. *If $n \equiv 1(\mod 4)$ and m is odd, then span* $(\boldsymbol{P}(m,n)) \leq 2^{\nu(m+1)}$. *Further, if $\nu(m+1) \in \{1,2,3\}$, then*

$$2^{\nu(m+1)} - 1 \leq \text{span}\,(\boldsymbol{P}(m,n)) \leq 2^{\nu(m+1)}.$$

Proof: If $n \equiv 1(\mod 4)$, then $\nu(n+1) = 1$, so that $k = \dfrac{m-1}{2}$. This implies that

$$\nu(k+1) = \nu(m+1) - 1.$$

Hence,

$$\begin{aligned}\text{span}\,(\mathbf{P}(m,n)) &\leq 2(2^{\nu(m+1)-1} + 1) - 2 \\ &= 2^{\nu(m+1)}\end{aligned}$$

Further, if $\nu(m+1) \in \{1,2,3\}$ then

$$\rho(m) - 1 = 2^{\nu(m+1)} - 1.$$

Therefore,

$$2^{\nu(m+1)} - 1 \leq \text{span}\,(\mathbf{P}(m,n)) \leq 2^{\nu(m+1)}. \quad \#$$

Theorem 7.2. *If $m,n \equiv 1(\mod 4)$, then*

$$\textit{span}\,(\boldsymbol{P}(m,n)) = 2.$$

Further, if $m \equiv 1(\mod 4)$ and $n \equiv 3(\mod 4)$, then

$$\textit{span}\,(\boldsymbol{P}(m,n)) \geq 3.$$

Proof: If $n \equiv 1(\mod 4)$, then

$$\text{span}\,(\mathbf{P}(m,n)) \leq 2^{\nu(m+1)}.$$

Further, if $m \equiv 1(\mod 4)$, then $\nu(m+1) = 1$, which implies that span $(\mathbf{P}(m,n)) \leq 2$. Since $m+n$ is even,

$$W_1(\mathbf{P}(m,n)) = (m+n+1)c \neq 0,$$

so that $\mathbf{P}(m,n)$ is non-orientable. Also, $\dim \mathbf{P}(m,n) = D = m+2n \equiv 3(\mod 4)$. If $D = 3$ *i.e.* $m = n = 1$, then $W_1^2(\mathbf{P}(1,1)) = 0$. If $D \geq 7$, then $W_{D-1}(\mathbf{P}(m,n))$ can be verified to be zero. Therefore by Result 3.3,

$$\text{span}\,(\mathbf{P}(m,n)) \geq 2.$$

Hence span $(\mathbf{P}(m,n)) = 2$. Next, if $m \equiv 1(\mod 4)$ and $n \equiv 3(\mod 4)$, then $m+2n \equiv 3(\mod 4)$ and $m+n =$ even. Therefore, $W_1(\mathbf{P}(m,n)) = (m+n+1)c \neq 0$, showing that $\mathbf{P}(m,n)$ is non-orientable. Also,

$$D = m + 2n \geq 1 + 2 \times 3 = 7.$$

It can also be verified that $W_{D-3}(\mathbf{P}(m,n)) = 0$. Therefore, by Result 3.4

$$\text{span}\,(\mathbf{P}(m,n)) \geq 3. \quad \#$$

8. Estimation of span of wall manifold

Wall manifold $\mathbf{Q}(m,n)$ [33] is $(m+2n+1)$-dimensional manifold defined as the quotient

$$\mathbf{P}(m,n) \times I/\sim,$$

where $\mathbf{P}(m,n)$ is Dold manifold and $\sim$ is the equivalence relation given by

$$(p,0) \sim (A(p),1).$$

Here, $A : \mathbf{P}(m,n) \to \mathbf{P}(m,n)$ is an autohomeomorphism given by

$$A[x,[z]] = [T(x),[z]],$$

$T : \mathbf{S^m} \to \mathbf{S^m}$ being the reflection in the plane $x_m = 0$. The projection map

$$[[x,[z]],t] \xrightarrow{\beta} t$$

induces fiber bundle

$$\mathbf{P}(m,n) \to \mathbf{Q}(m,n) \xrightarrow{\beta} \mathbf{S}^1 \tag{A}$$

with the fiber $\mathbf{P}(m,n)$. Also, the projection map

$$[[x,[z]],t] \xrightarrow{\gamma} [[x],t]$$

induces another fiber map with fiber $\mathbb{C}\mathbf{P}^n$

$$\mathbb{C}\mathbf{P^n} \to \mathbf{Q}(m,n) \xrightarrow{\gamma} \mathbf{Q}(m,0). \tag{B}$$

Further, the classifying map

$$\mathbf{Q}(m,0) \to \mathbb{R}\mathbf{P}^{m+1}$$

for the bundle $\gamma : \mathbf{Q}(m,n) \to \mathbf{Q}(m,0)$ is covered by the bundle map

$$\theta : \mathbf{Q}(m,n) \to \mathbf{P}(m+1,n)$$

given as

$$\theta([[x,[z]],t]) = [(x_0, \ldots.., x_{m-1}, x_m \cos \pi t, x_m \sin \pi t), [z]].$$

$$\begin{array}{ccc} \mathbf{Q}(m,n) & \xrightarrow{\theta} & \mathbf{P}(m+1,n) \\ \downarrow \gamma & & \downarrow \\ \mathbf{Q}(m,0) & \longrightarrow & \mathbb{R}\mathbf{P}^{m+1} \end{array}$$

Let x be the generator of $H^1(\mathbf{S}^1;\mathbb{Z}_2) \approx \mathbb{Z}_2$ and $\beta^*(x) = \tilde{x} \in H^1(\mathbf{Q}(m,n);\mathbb{Z}_2)$. Let d be a suitable non-zero element of $H^2(\mathbf{P}(m+1,n);\mathbb{Z}_2) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\theta^*(d) = \tilde{d} \in H^2(\mathbf{Q}(m,n);\mathbb{Z}_2)$. Similarly, let $c \in H^1(\mathbb{R}\mathbf{P}^{m+1};\mathbb{Z}_2) \approx H^1(\mathbf{P}(m+1,0);\mathbb{Z}_2) \approx \mathbb{Z}_2$ be the nonzero element and $\gamma^*\theta^*(c) = \tilde{c} \in H^1(\mathbf{Q}(m,n);\mathbb{Z}_2)$. Wall [33], in 1960, showed that $H^*(\mathbf{Q}(m,n);\mathbb{Z}_2)$ is generated by $\tilde{x}, \tilde{c}$ and $\tilde{d}$ with only relations

$$\tilde{x}^2 = 0 = \tilde{d}^{n+1} \text{ and } \tilde{c}^{m+1} = \tilde{c}^m\tilde{x} \tag{C}$$

Wall also gave the total Stiefel-Whitney class of $\mathbf{Q}(m,n)$ for $m > 0$ as

$$W = (1+\tilde{c}+\tilde{x})(1+\tilde{c})^{m-1}(1+\tilde{c}+\tilde{d})^{n+1} \tag{D}$$

Further, Wall showed that $\mathbf{Q}(2^r-1, 2^r.s)$ in each odd dimension $d \neq 2^i - 1$ along with $\mathbb{R}\mathbf{P}^{2i}$ generate the polynomial bordism algebra $\mathbf{N}_*$ over $\mathbb{Z}_2$.

The bounding and independence problems of the Wall manifolds $\mathbf{Q}(m,n)$ have been settled in [7] by us in 2004. Some partial results giving estimation of bounds of span of Wall manifolds are given by us in 2010 in [9].

Theorem 8.1. *(i) Span* $(\boldsymbol{Q}(m,n)) \geq 1,\ \forall m,n.$

(ii) Span $(\boldsymbol{Q}(m,n)) \geq 3,$ *if* $4|m$ *and* $n = 2^s.odd - 1,\ s \geq 2.$

(iii) Span $(\boldsymbol{Q}(m,n)) \leq 2^{\nu(n+1)}[2^{\nu(k+1)}+1]-1,$ *where* $m = 2^{\nu(n+1)}.k + l,\ 0 \leq l \leq 2^{\nu(n+1)}$ *and* $2^{\nu(n+1)}$ *is the highest power of 2 dividing* $n+1$.

(iv) Span $(\boldsymbol{Q}(m,0)) = 1,\ \forall m$

Proof:

(i) The fiber bundle

$$\mathbf{P}(m,n) \to \mathbf{Q}(m,n) \to \mathbf{S}^1$$

gives

$$\chi(\mathbf{Q}(m,n)) = \chi(\mathbf{S}^1)\chi(\mathbf{P}(m,n)) = 0,$$

since $\chi(\mathbf{S}^1) = 0$. Therefore,

$$\text{span}\,(\mathbf{Q}(m,n)) \geq 1.$$

(ii) For $n = 2^s(2q+1) - 1,\ s > 1$ and $m = 4k$, the total Stiefel-Whitney class $W(\mathbf{Q}(m,n))$ is given by

$$(1+\tilde{c}^4)^k(1+\tilde{c}^2+\tilde{d}^2)^{2q+1} + \tilde{x}(1+\tilde{c})^{4k-1}(1+\tilde{c}+\tilde{d})^{2^s(2q+1)} \tag{E}$$

From this, it is clear that $W_1(\mathbf{Q}(m,n)) = \tilde{x} \neq 0$, showing that $\mathbf{Q}(m,n)$ is nonorientable. Since

$$D = \dim(\mathbf{Q}(m,n)) = 4k - 1 + 2^{s+1}.(2q+1),$$

the claim will follow from the result 3.4, if we show that $W_{D-3}(\mathbf{Q}(m,n)) = 0$.

Clearly, each nonzero monomial coming from $(1+\tilde{c}^4)^k(1+\tilde{c}^{2^s}+\tilde{d}^{2^s})^{2q+1}$ would be of the form $\tilde{c}^u\tilde{d}^v$, where $u < 4k+2$ and $v < 2^s.(2q+1)$. But, in addition, u and v would be divisible by 4. Therefore, we would have

$$u \leq 4k \,\text{and}\, v \leq 2^s.(2q+1)-4.$$

This implies that $\tilde{c}^u\tilde{d}^v$ would be in the degree $u+2v \leq 4k+2^{s+1}(2q+1)-8 < D-3$. Thus, $\exists$ no nonzero contribution of $W_{D-3}(\mathbf{Q}(m,n))$ coming from $(1+\tilde{c}^4)^k(1+\tilde{c}^{2^s}+\tilde{d}^{2^s})^{2q+1}$. Similarly, each nonzero monomial coming from

$$\tilde{x}(1+\tilde{c})^{4k-1}(1+\tilde{c}^{2^s}+\tilde{d}^{2^s})^{2q+1}$$

would be of the form $\tilde{x}\tilde{c}^u\tilde{d}^v$, where $u < 4k+2,\ v < 2^s.(2q+1),\ 4|v$ and thus $v \leq 2^s.(2q+1)-4$. But then the class $\tilde{x}\tilde{c}^u\tilde{d}^v$ would be in the degree

$$\begin{aligned} 1+u+2v &< 1+4k+2+2^{s+1}.(2q+1)-8 \\ &= 4k-5+2^{s+1}.(2q+1) < D-3. \end{aligned}$$

Thus, there is no nonzero contribution in $W_{D-3}(\mathbf{Q}(m,n))$ coming from

$$\tilde{x}(1+\tilde{c})^{4k-1}(1+\tilde{c}^{2^s}+\tilde{d}^{2^s})^{2q+1}.$$

So, $W_{D-3}(\mathbf{Q}(m,n))=0$, which proves (ii).

(iii) The fiber bundle

$$\mathbf{P}(m,n) \to \mathbf{Q}(m,n) \to \mathbf{S}^1$$

gives

$$\begin{aligned} \text{span}\ (\mathbf{Q}(m,n)) &\leq \text{stablespan}\ (\mathbf{Q}(m,n)) \\ &\leq \text{stablespan}\ (\mathbf{P}(m,n)) + \dim \mathbf{S}^1 \\ &\leq 2^{\nu(n+1)}[2^{\nu(k+1)}+1]-1, \end{aligned}$$

by Result 2.1 and Theorem 6.1.

(iv) The total Stiefel-Whitney class of $\mathbf{Q}(m,0)$

$$= (1+\tilde{c})^{m+1}+\tilde{x}(1+\tilde{c}^m),$$

as $\tilde{d}=0$ by (C). Therefore,

$$\omega_m(\mathbf{Q}(m,0)) \neq 0,$$

implying that

$$\text{span}\,(\mathbf{Q}(m,0)) \leq m+1-m=1.$$

Hence by (i), we obtain that span $(\mathbf{Q}(m,0))=1$.

Corollary 8.1. *(i) If n is even, then*

$$1 \leq \textit{span}\ (\boldsymbol{Q}(m,n)) \leq 2^{\nu(m+1)}.$$

(ii) If n is even and $m+1=2.odd$, then

$$1 \leq \textit{span}\ (\boldsymbol{Q}(m,n)) \leq 2\ .$$

(iii) If n is odd given by $4.odd-1,\ m=8.odd$, then

$$3 \leq \textit{span}\ (\boldsymbol{Q}(m,n)) \leq 7.$$

(iv) If m and n both are even, then span $(\boldsymbol{Q}(m,n)) = 1$.

Proof:

(i) Follows from (i) and (iii) of Theorem 8.1, for n is even implies that $\nu(n+1) = 0$ and $k = m$.

(ii) Follows from (i) of the Corollary, as $m+1 = 2.odd$ implies that $\nu(m+1) = 1$.

(iii) Follows from (ii) and (iii) of Theorem 8.1, since $\nu(n+1) = 2$ and $8|m$ implies that $m = 2^{\nu(n+1)}.(even)$ so that $\nu(k+1) = \nu(odd) = 0$. Therefore,

$$2^{\nu(n+1)}[2^{\nu(k+1)} + 1] - 1 = 7$$

(iv) n is even implies that $\nu(n+1) = 0$ and $m = k$. Therefore, m is even implies that $\nu(k+1) = 0$. Hence,

$$\text{span}\,(\mathbf{Q}(m,n)) \leq (1+1) - 1 = 1,$$

so that span $(\mathbf{Q}(m,n)) = 1$, by Theorem 8.1. #

Theorem 8.2. $1 \leq$ *span* $(\boldsymbol{Q}(m,n)) \leq m + 1 + 2.\nu(n+1)$

Proof: Since (cf [31]) stablespan $(\mathbb{C}\mathbf{P}^n) = 2\nu(n+1)$ and we have the fiber bundle (B), by applying Result 2.1, we obtain

$$\begin{aligned} \text{span}\ (\mathbf{Q}(m,n)) &\leq \text{stablespan}\ (\mathbf{Q}(m,n)) \\ &\leq \text{stablespan}\ (\mathbb{C}\mathbf{P}^n) + \dim(\mathbf{Q}(m,0)) \\ &\leq 2\nu(n+1) + m + 1. \quad \# \end{aligned}$$

Corollary 8.2. *If n is even, then*

$$1 \leq \textit{span}\,(\boldsymbol{Q}(m,n)) \leq m+1$$

Proof: Proof is straightforward. #

Theorem 8.3. *If $4|m$, $n \equiv 1(\mod 4)$, then*

$$2 \leq \textit{span}\,(\boldsymbol{Q}(m,n)) \leq 3$$

Proof: Let us write $m = 4r$ and $n = 4s+1$. Then the dimension of $\mathbf{Q}(m,n) = 4r + 8s + 3 \equiv 3(\mod 4)$. Further, the total Stiefel-Whitney class $W(\mathbf{Q}(m,n))$ is given by

$$\begin{aligned} &(1+\tilde{c}+\tilde{x})(1+\tilde{c})^{m-1}(1+\tilde{c}+\tilde{d})^{n+1} \\ &= (1+\tilde{c}+\tilde{x})(1+\tilde{c})^{4r-1}(1+\tilde{c}+\tilde{d})^{4s+2} \\ &= [(1+\tilde{c})^{4r} + \tilde{x}(1+\tilde{c})^{4r-1}](1+\tilde{c}+\tilde{d})^{4s+2}. \end{aligned}$$

Therefore, $W_1 = \tilde{x} \neq 0$, showing that $\mathbf{Q}(m,n)$ is nonorientable. Further, $W_1^2 = \tilde{x}^2 = 0$. Hence by the Result 3.3,

$$\text{span}\,(\mathbf{Q}(m,n)) \geq 2.$$

Also, $n+1 = 4s+2$ implies that $\nu(n+1) = 1$. Therefore,

$$m = 4r = 2^{\nu(n+1)}.2r,$$

so that $\nu(k+1) = \nu(2r+1) = 0$. This shows that

$$\text{span}\,(\mathbf{Q}(m,n)) \leq 2.(1+1) - 1 = 3,$$

by Theorem 8.1 (iii). #

Note 8.1 Theorem 8.1 (iii) and Theorem 8.2 give different upper bounds for span of $\mathbf{Q}(m,n)$. One notes that

$$\begin{aligned} 2\nu(n+1) + m + 1 &= 2\nu(n+1) + 2^{\nu(n+1)}.k + l + 1 \\ &= 2\nu(n+1) + 2^{\nu(n+1)}(r.2^{\nu(k+1)} - 1) + l + 1 \\ &> 2^{\nu(n+1)}(2^{\nu(k+1)} + 1) - 1, \end{aligned}$$

for odd $r \geq 3$, where $k+1 = r.2^{\nu(k+1)}$. Thus, for $r \geq 3$, the estimation of upper bound given by Theorem 8.1 (iii) is better than the estimation given by Theorem 8.2. However, only for $r = 1$, the estimation of upper bound given by Theorem 8.2 may be better than the estimation given by Theorem 8.1 (iii), whenever

$$\begin{aligned} &2^{\nu(n+1)}[2^{\nu(k+1)} + 1] - 1 - 2\nu(n+1) - 2^{\nu(n+1)}(2^{\nu(k+1)} - 1) - l - 1 \\ &\quad = 2^{\nu(n+1)+1} - 2\nu(n+1) - l - 2 > 0, \end{aligned}$$

which is true, when $\nu(n+1) = 2$ with $0 \leq l < 1$ or when $\nu(n+1) \geq 3$ with $0 \leq l < 2^{\nu(n+1)}$.

References

[1] Adams, J. F., *Vector fields on Spheres,* Ann. of Math., 75 (1962), 603–632.

[2] Antoniano, E., Gitler, S., Ucci, J., Zvengrowski, P., *On the k-theory and parallelizability of projective Stiefel manifolds,* Bol. Soc. Mat. Mexicana, 31 (1986), 29–46.

[3] Barratt, M. and Mahowald, M., *The metastable homotopy of $O(n)$,* Bull. Amer. Math. Soc., 70 (1964), 758–760.

[4] Hopf, H., *Vektorfelder in n-dimensionalalen Mannigfaltigkeiten,* Math. Ann., 96 (1927), 225–250.

[5] Hurwitz, A., *Über die Komposition der quadratischen Formen,* Math. Ann., 88 (1923), 1–25.

[6] Khare, S. S., *On Dold manifolds,* Topology and its Applications, 33(1989), 297–307.

[7] Khare, S. S., *On bounding and independence of Wall manifolds,* J. Indian Math. Soc., Vol 71 (2004), 195–201.

[8] Khare, S. S., *Span of Milnor manifolds,* J. Indian Math. Soc., Vol. 75 (2008), 47–57.

[9] Khare, S. S., *Span of Wall manifolds,* J. Indian Math. Soc., Vol 77 no 1-4 (2010), 01–12.

[10] Khare, S. S. and Das, A. K. *Which Milnor manifolds bound?,* Indian J. of Pure and Applied Math., 31 (11), (2000), 1503–1513.

[11] Khare, S. S. and Dutta, S., *Independence of bordism classes of Milnor manifolds,* J. Indian Math. Soc., Vol 16 (2001), 1–16.

[12] Korbaš, J., *Vector fields on real flag manifolds,* Ann. Global Anal. Geom., 3 (1985), 173–184.

[13] Korbaš, J., *On the Stiefel-Whitney classes and span of real Grassmannians,* Czechoslovak Math. J., 36 (111), (1986), 541–552.
[14] Korbaš, J., Sankaran, P., Zvengrowski, P., *Span of projective Stiefel manifolds,* Preprint.
[15] Koschorke, U., *Vector fields and other vector bundle morphisms - a singularity approach,* Lecture notes in Math. 847, Springer-Verlag, Berlin, 1981.
[16] Kosinski, A. A., *Differentiable manifolds,* Academic Press, 1993.
[17] Kosinski, A. and Bredon, G., Vector fields on π-manifolds, Ann. of Math. (2), 84 (1966), 85-90."
[18] Lam, K. Y., *A formula for the tangent bundle of flag manifolds and related manifolds,* Trans. Amer. Math. Soc., 213 (1975), 305–314.
[19] Miatello, I. D. and Miatello, R. J., *On stable parallelizability of $\tilde{G}_{k,n}$ and related manifolds,* Math. Ann., 259 (1982), 343–350.
[20] Novotný, P., *Span of Dold manifolds,* Bull. Belg. Math. Soc., Simon Stevin, 15 (2008), 687–698.
[21] Radon, J., *Lineare Scharen orthogonaler Matrizen,* Abh. Math. Sem. Univ. Hamburg, 1 (1922), 1–14.
[22] Randall, D., *CAT 2-fields on nonorientable CAT manifolds,* New York J. Math., 9 (2003), 271–293.
[23] Randall, D., *On indices of tangent fields with finite singularities,* Differential Topology (Siegen 1987), Lecture Notes in Math. 1350, SV, Berlin, (1988), 213–240.
[24] Sankaran, P., *Vector fields on flag manifolds,* Ph.D. thesis, University of Calgary, 1985.
[25] Sankaran, P. and Zvengrowski, P., *Upper bounds for the span of projective Stiefel manifolds,* Recent developments in Algebraic Topology, 173–181, Contemp. Math., 407, Amer. Math. Soc. Providence, RI, 2006.
[26] Shanahan, P., *On the signature of Grassmannians,* Pacific J. Math., 84 (1979), 483–490.
[27] Singhof, W., *Parallelizability of homogeneous spaces I,* Math. Ann., 260 (1982), 101–116.
[28] Smith, L., *Some remarks on projective Stiefel manifolds, immersions of projective spaces and spheres,* Proc. Amer. Math. Soc., 80 (1980), 663–669.
[29] Steenrod, N., *The topology of fibre bundles,* Princeton Univ. Press, Princeton, N.J., 1951.
[30] Steenrod, N. and Whitehead, J. H. C., *Vector fields on the n-sphere,* Proc. Nat. Acad. Sci., U.S.A., 37 (1957) 58–63.
[31] Steer, B., *une interpretation geometrique des nombres de Radon-Hurwitz,* Ann. Inst. Fourier (Grenoble), 17 (1967), 209–218.
[32] Thomas, E., *Vector fields on manifolds,* Bull. Amer. Math. Soc., 75 (1969), 643–683.
[33] Wall, C. T. C., *Determination of the cobordism ring,* Ann. of Math., Vol 72, No. 2 (1960).
[34] Yoshida, T., *A note on vector fields up to bordism,* Hiroshima Math. J., 8 (1978), 63–69.
[35] Zvengrowski, P. and Korbaš, J., *The vector field problem: A survey with emphasis on specific manifolds,* Expo. Math., 12 (1994), 3–30.
[36] Zvengrowski, P., Korbaš and Adams, J. F., *The vector field problem for projective Stiefel manifolds,* Preprint.
[37] Zvengrowski, P., *Recent work on parallelizability of flag manifolds,* Contemporary Math., (Amer. Math. Soc.), (Part II), 58 (1987), 129–137.

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MODULAR FORMS OF HALF-INTEGRAL WEIGHT

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Abstract: In this article we present some of our works on the theory of modular forms of half-integral weight.

1. Introduction

The study of Modular forms has a long history. Early work on this theory goes back to the 18th century. For example, theta functions appear in the work of Bernoulli which are fundamental in the theory. These functions also appear in the work of Euler in connection with identities for the partition function. The Dedekind eta function, which is the building block for modular functions plays an important role in the theory of partition functions. The classical period for the theory of modular forms is the 19th century with the work of Carl Jacobi, who developed the theory of theta functions and used this to prove the "Jacobi's four square theorem".

In the later part of the 19th century, Felix Klein used function-theoretic methods to investigate certain modular functions, called the "j-invariant". This j-invariant has many applications in Mathematics and in particular, Number Theory, Algebraic Geometry and Algebra.

In the early 20th century Srinivasa Ramanujan studied an explicit modular form which is referred to as the *Ramanujan Delta* function. Its Fourier coefficients are denoted as $\tau(n)$ and called as the *Ramanujan Tau* function. He made conjectures about the arithmetic nature and growth of $\tau(n)$. These conjectures were proved later and led to rapid development of the theory of modular forms. Ramanujan also generalised Jacobi's theta function and discovered a new family of similar objects, called Mock Theta functions. Recently there have been a lot of activities exploring the theory behind these mock theta functions.

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brief account of the work of the author on the theory of modular forms of half-integral weight. In the second section, we give a brief account of modular forms of integral weight. The third section contains some preliminaries on modular forms of half-integral weight, followed by the works of W. Kohnen and the author.

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2. Modular forms of integral weight

2.1. **What are modular forms?** To explain in brief, modular forms are analytic functions defined on the complex upper half-plane, having certain automorphic properties with respect to a group and also are analytic on the boundary. Let us explain it a little bit.

Let $\mathcal{H}$ denote the complex upper half-plane and $k \geq 4$ be an even integer. Let $SL_2(\mathbb{Z})$ be the group of all 2×2 matrices having integer entries with determinant 1. This group acts on $\mathcal{H} \cup \mathbb{Q} \cup \infty$ by

$$\gamma \circ z = \frac{az+b}{cz+d},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and

$$\gamma \circ \infty = a/c,$$

$$\gamma \circ (-d/c) = \infty.$$

A modular form is a holomorphic function f defined on $\mathcal{H}$ satisfying the two conditions:

(i) $f(\gamma z) = (cz+d)^k f(z)$, for all $\gamma \in SL_2(\mathbb{Z})$.

(ii) f has a Fourier expansion as follows:

$$f(z) = \sum_{n \geq 0} a_f(n) e^{2\pi i n z}.$$

The integer k is called the weight of the form. It is a fact that the weight is always positive and further it is an even integer ≥ 4. As one of the examples of a modular form, we give the following one, which is called the Eisenstein series:

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (cz+d)^{-k},$$

where the sum varies over all coprime pair (c, d). The function defined above is a modular form of weight k ($k \geq 4$ is an even integer) as described above and has the following Fourier expansion:

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)e^{2\pi inz},$$

where B_k is the kth Bernoulli number defined by $\frac{x}{e^x-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}x^k$ and $\sigma_r(n) = \sum_{d|n} d^r$ is the divisor function.

The space of modular forms for a fixed weight k w.r.t $SL_2(\mathbb{Z})$ is a $\mathbb{C}$-vector space, denoted by M_k. For a modular form f, if $a_f(0) = 0$, then we say that f is a cusp form. If we multiply two modular forms, the weights get added and if we divide two modular forms, the weights get subtracted. In the latter case, the resulting function is a modular function (where we may not get the holomorphic properties and we may have only the meromorphic conditions). For a detailed account of the theory of modular forms of integral weight, we refer to [24],[7, Chapter 3].

Now, E_4^3 and E_6^2 are both modular forms of same weight 12, whose Fourier series are given respectively by

$$E_4^3(z) = 1 + 720q + 179280q^2 + \cdots,$$

$$E_6^2(z) = 1 - 1008q + 220752q^2 + \cdots,$$

where $q = e^{2\pi iz}$. Let

$$\Delta(z) = \frac{1}{1728}(E_4^3(z) - E_6^2(z)) = \frac{1}{1728}(1728q - 41472q^2 + \cdots).$$

i.e.,

$$\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \cdots = \sum_{n\geq 1}\tau(n)q^n.$$

The numbers $\tau(n)$ are integers and Ramanujan made the following conjectures in his celebrated paper [22].

(R1) $\tau(n)$ is multiplicative. i.e., $\tau(mn) = \tau(m)\tau(n)$, if $(m,n) = 1$.
(R2) $\tau(p^{n+1}) = \tau(p^n)\tau(p) - p^{11}\tau(p^{n-1})$, for $n \geq 1$.
(R3) $|\tau(p)| \leq 2\, p^{11/2}$.

Conjectures (R1) and (R2) were proved in 1917 by L.J Mordell [20]. A generalization of these results to forms of higher weight was obtained by E. Hecke, by introducing certain endomorphisms in the vector space. (These operators are now termed as the Hecke operators.) Later, Hecke's student H. Petersson generalised these ideas and also introduced an inner product in the space thereby making it into a Hilbert space. Similar conjectural estimates (like (R3) for the Ramanujan Delta function) for the Fourier coefficients of cusp forms which are eigenfunctions under these Hecke operators were posed by Petersson, and this conjecture is referred to as the Ramanujan-Petersson conjecture. In 1974, P. Deligne [2] proved this generalised Ramanujan-Petersson conjecture, by proving the Weil conjectures. In particular, this proved the original conjecture (R3).

So far we have mentioned only modular forms with respect to the full group $\Gamma = SL_2(\mathbb{Z})$. One can extend this concept to congruence subgroups of Γ. Here we confine only to a particular subgroup, namely $\Gamma_0(N)$. It is a subgroup of Γ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that c is a multiple of N. So, a modular form of weight k on $\Gamma_0(N)$ is a holomorphic function defined on $\mathcal{H}$, satisfying the transformation (ii) (of the definition of a modular form mentioned above) with respect to the group $\Gamma_0(N)$ along with certain conditions at the boundary (called cusp conditions). The vector space of modular forms of weight k for $\Gamma_0(N)$ is denoted by $M_k(N)$ and the space of cusp forms is denoted as $S_k(N)$. One also defines modular forms for $\Gamma_0(N)$ with character, but here we do not go into these details. For a detailed theory we refer to [7].

2.2. **Theory of newforms.** A natural question is 'are there modular forms on $\Gamma_0(N)$ which satisfy (R1) and (R2) (as stated by Ramanujan for the Δ function)?'. The answer to this question in the higher level was obtained by A. O. L. Atkin and J. Lehner in 1969 [1]. The theory obtained by Atkin and Lenher is now referred to as the *Atkin-Lehner theory of newforms.* Atkin and Lehner obtained the theory by removing the modular forms which come from lower levels (N is called the level of the modular form). One can obtain modular forms by duplicating the old forms, namely by considering $f(dz)$, where f is a form on $\Gamma_0(r)$, $r|N$, $r < N$ and $d|N/r$. That is, the function $f(dz)$ is a modular form on $\Gamma_0(N)$. The idea is to remove all such forms, starting from the forms on Γ and consider the orthogonal complement of this space with respect to the Petersson inner product. This orthogonal complement space is denoted as $S_k^{new}(N)$. Here, we consider only the space of cusp forms. The reason is that the full space $M_k(N)$ can be written as a direct sum of the space generated by the Eisenstein series and the space of cusp forms. Further, one has the structure of the space of Eisenstein series. So, it is enough to consider the space of cusp forms. Atkin and Lehner showed that the space $S_k^{new}(N)$ has a basis of eigenforms (these are eigenforms with respect to all the Hecke operators). If f is one such basis element, then its first Fourier coefficient $a_f(1)$ is non-zero, and so by normalising it, we may assume that $a_f(1) = 1$. In this case, one has:

$$a_f(mn) = a_f(m)a_f(n), \quad (m,n) = 1,$$
$$a_f(p^{n+1}) = a_f(p^n)a_f(p) - p^{k-1}a_f(p^{n-1}), \quad n \geq 1.$$

Deligne's theorem gives the following estimate for the Hecke eigenform f:

$$|a_f(n)| \leq \sigma_0(n)n^{(k-1)/2}. \tag{2.1}$$

Associated to a modular form f, one can attach an L-function defined by

$$L(f,s) = \sum_{n\geq 1} \frac{a_f(n)}{n^s},$$

which is a generalization of the Riemann zeta function. Due to its importance in the subject, the study of L-functions is also a very active research area. The multiplicative relations for the Fourier coefficients give the required Euler product expansion of the L-function:

$$L(f,s) = \prod_p (1 - a_f(p)p^{-s} + p^{k-1-2s})^{-1}.$$

The above is the Euler product of the L-function associated to a modular form of weight k on $SL_2(\mathbb{Z})$. For a newform of weight k on $\Gamma_0(N)$, the Euler factor has two terms one corresponding to $p|N$ and the other corresponding to $p \nmid N$. We don't go into these details here. These are called modular L-functions and are part of the general study of L-functions coming from various objects. One may ask a question in the reverse direction. i.e., if an L-function satisfy certain properties (like the one given above), can one say that the L-function comes from a modular form? This question was considered by A. Weil and he gave a solution to this problem. He showed that if an L-function has an Euler product as above and satisfy certain functional equations for a class of twists of L-functions, then under certain growth conditions of the corresponding coefficients of the Dirichlet series representing the L-function, one can conclude that the L-function comes from a modular form of a specific weight and also some specific level. This is called Weil's converse theorem. Earlier this was obtained by Hecke for the full modular group.

3. Modular forms of half-integral weight

Though examples of modular forms of weight 1/2 exist from the 19th century (classical theta function and Dedekind eta function), a systematic theory along the lines of modular forms of integral weight was lacking till the year 1973, when G. Shimura developed the theory in his seminal work [25]. Let us now define the two modular forms of weight 1/2 mentioned above.

$$\begin{aligned} \Theta(z) &= \sum_{n\in\mathbb{Z}} q^{n^2}, \\ \eta(z) &= q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \ \ q = e^{2\pi i z}. \end{aligned} \tag{3.1}$$

Note that the Ramanujan Delta function is the 24th power of the Dedekind eta function:

$$\Delta(z) \ = \ q \prod_{n=1}^{\infty} (1-q^n)^{24} = (\eta(z))^{24}.$$

Recall that if f is a modular form then $f(\gamma z) = (cz+d)^k f(z)$. The ratio $f(\gamma z)/f(z)$, which is equal to $(cz+d)^k$, is called the automorphic factor. Here k should be an integer. If we choose k to be a half-integer, then we have to fix a square root of a complex number. We take the branch of the square root having argument in $(\pi/2, \pi/2]$. So, $\sqrt{z}$ is a holomorphic function on the complex plane with negative real axis removed. Once we fix the square root of z, we define $z^{k/2}$ for k integer as $(\sqrt{z})^k$. When k is half-integral, then the factor $(cz+d)^k$ no longer defines an automorphic factor uniquely. In order to get the right automorphic factor, one uses the transformation property of the theta function. For $\gamma \in \Gamma_0(4)$, the theta function $\Theta(z)$ satisfies the transformation property:

$$\Theta(\gamma z) = \kappa_\gamma (cz+d)^{1/2}\Theta(z), \tag{3.2}$$

where $\kappa_\gamma = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}$ (which is in terms of the generalized quadratic residue symbol, whose value is a fourth root of unity). For a proof of this transformation property, see Chapter 4 of [7]. Shimura used these ideas to define a modular form of half-integral weight. A holomorphic form f defined on $\mathcal{H}$ is called a modular form of weight $k+1/2$, $k \in \mathbb{N}$, on $\Gamma_0(4N)$, if it satisfies the transformation rule:

$$f(\gamma z) = j(\gamma, z)^{2k+1} f(z), \text{ for all } \gamma \in \Gamma_0(4N), \tag{3.3}$$

where $j(\gamma, z) = \kappa_\gamma (cz+d)^{1/2}$. Further, it satisfies the holomorphic conditions at the boundary. The vector space of modular forms of weight $k+1/2$ for the group $\Gamma_0(4N)$ is denoted by $M_{k+1/2}(4N)$ and the subspace of cusp forms is denoted by $S_{k+1/2}(4N)$. As in the case of integral weight modular forms, one has the theory of Hecke operators. However, the Hecke operators act non-trivially only when the index is a square, namely $T(p^2)$. If f is a Hecke eigenform for $T(p^2)$ with eigenvalue λ_p, then we have the following relation:

$$a_f(np^2) + \chi_1(p)\left(\frac{(-1)^k n}{p}\right)p^{k-1}a_f(n) + \chi_1(p^2)p^{2k-1}a_f(n/p^2) = \lambda_p a_f(n),\ n \geq 1,$$

where χ_1 is the principal (trivial) character modulo $4N$. Shimura noticed that one has to consider a different type of Dirichlet series in order to get an Euler product. This consideration led to a correspondence between half-integral weight and integral weight modular forms. Let us explain it very briefly. For details the reader is referred to the article of Shimura [25] or the book of Koblitz [7].

Let t be a squarefree integer such that $(-1)^k t > 0$. For a modular form of weight $k+1/2$ on $\Gamma_0(4N)$, consider the Dirichlet series

$$\sum_{n\geq 1} a_f(|t|n^2)n^{-s}.$$

If f is an eigenfunction under all the Hecke operators $T(p^2)$ with eigenvalues λ_p, then using the relation between the Fourier coefficients as mentioned before, we have

$$\sum_{n\geq 1} a_f(|t|n^2)n^{-s} = a_f(|t|)\prod_p \frac{1-\chi_1(p)\left(\frac{(-1)^k t}{p}\right)p^{k-1-s}}{(1-\lambda_p p^{-s}+\chi_1(p)^2p^{2k-1-2s})}.$$

Now bringing the numerator of the right-hand side to the left-hand side and simplify it we get,

$$\begin{aligned}&\sum_{n\geq 1}\left(\sum_{\substack{d|n\\(d,2N)=1}}\left(\frac{(-1)^k t}{d}\right)d^{k-1}a_f(|t|n^2/d^2)\right)n^{-s}\\&=a_f(|t|)\prod_p\left(1-\lambda_p p^{-s}+\chi_1(p)^2p^{2k-1-2s}\right)^{-1}\\&=a_f(|t|)\sum_{n\geq 1}a_F(n)n^{-s},\end{aligned} \tag{3.4}$$

where $a_F(p) = \lambda_p$. Using the converse theorem of Weil, Shimura proved that the right-hand side Euler product in equation (3.4) is indeed the Dirichlet series associated to a modular form F of weight $2k$ on $\Gamma_0(2N)$. The level of the integral weight form was conjectured by Shimura and was later proved in 1975 by S. Niwa [21]. With these details in background, we describe the Shimura map:

$$f|\mathcal{S}_{t,N}(z) = \sum_{n\geq 1}\left(\sum_{\substack{d|n\\(d,2N)=1}}\left(\frac{(-1)^k t}{d}\right)d^{k-1}a_f(|t|n^2/d^2)\right)q^n. \tag{3.5}$$

Therefore, $f|\mathcal{S}_{t,N} = F \in S_{2k}(2N)$. In 1975, T. Shintani [26] gave a correspondence from the reverse direction. That is, given a modular form of weight $2k$, how to construct a modular form of weight $k+1/2$. In 1980/81, J. -L. Waldspurger [27, 28] gave a remarkable connection between the Fourier coefficients of a half-integral weight Hecke eigenform and the special value of the L-function associated to the corresponding Hecke eigenform of integral weight via the Shimura correspondence. More precisely, if $f \in S_{k+1/2}(4N)$ and $F \in S_{2k}(2N)$ correspond via the Shimura correspondence $\mathcal{S}_{t,N}$, then he showed that

$$|a_f(|t|)|^2 \sim L(F,\chi_t,k),\ t\equiv 1 \pmod 4,\ \text{square-free}\ ,(-1)^k t>0,(t,2N)=1, \tag{3.6}$$

where the right-hand side is the special value at $s = k$ of the twisted L-function of F twisted with the quadratic character $\chi_t = \left(\frac{t}{\cdot}\right)$. This is called the Waldspurger theorem.

3.1. **The work of Kohnen.** During the period 1980–85 W. Kohnen developed the theory of newforms and studied the Shimura and Shintani correspondences (with an explicit version of the Waldspurger theorem) for a subspace of the space of modular forms of half-integral weight on $\Gamma_0(4N)$, where N is an odd square-free natural number. For a detailed account of these results, we refer to [8, 12, 9, 10]. In this section, we describe his work in brief and consider only the space of cusp forms with trivial character. Let $N \geq 1$ be an odd integer. The Kohnen plus space inside the space $S_{k+1/2}(4N)$ is defined as follows:

$$S^{+}_{k+1/2}(4N) = \big\{ f \in S_{k+1/2}(4N) \ : \ a_f(n) = 0 \\ \text{whenever } (-1)^k n \equiv 2, 3 \pmod 4 \big\}. \tag{3.7}$$

Kohnen described this subspace in terms of Hecke and the Atkin-Lehner operators, viz., $f \in S_{k+1/2}(4N)$ is in the plus space if and only if $f|U(4)W(4) = 2^k f$. Here $U(4)$ is the Hecke operator and $W(4)$ is the Atkin-Lehner W operator. He further refined the Shimura correspondence and showed that a refined map takes functions from $S^{+}_{k+1/2}(4N)$ into the space $S_{2k}(N)$. He introduced Hecke operators in the plus space, which differs only when $p = 2$. When N is odd and square-free he developed the theory of newforms which is analogous to the Atkin-Lehner theory of newforms. By explicitly computing the trace formula for the Hecke operators in the space, he showed the existence of an isomorphism between the spaces $S^{+}_{k+1/2}(4N)$ and $S_{2k}(N)$ and further had given a decomposition of the spaces in terms of old and newforms. In order to construct the Shintani map, he defined a kernel function for the lifting maps, describing it in two different ways, one expressed in terms of the Poincaré series of half-integral weight and the other in terms of the holomorphic kernel functions for the periods of modular forms. We list below the main theorems proved by Kohnen in the papers mentioned before.

Theorem 3.1. *(Kohnen)*

(1) *Let N be an odd square-free natural number. The space $S^{+}_{k+1/2}(4N)$ is decomposed as oldforms and newforms as follows:*

$$S^{+}_{k+1/2}(4N) = S^{+,new}_{k+1/2}(4N) \oplus S^{+,old}_{k+1/2}(4N), \tag{3.8}$$

where the above is an orthogonal directsum (with respect to the Petersson scalar product) and $S^{+,old}_{k+1/2}(4N)$ has the following decomposition:

$$S^{+,old}_{k+1/2}(4N) = \bigoplus_{rd|N, d<N} S^{+,new}_{k+1/2}(4d)|U(r^2). \tag{3.9}$$

The space of newforms $S^{+,new}_{k+1/2}(4N)$ has a basis of eigenforms with respect to all the Hecke operators in the plus space. Moreover, the vector spaces $S^{+,new}_{k+1/2}(4N)$ and $S^{new}_{2k}(N)$ are isomorphic under a suitable linear combination of (modified) Shimura maps.

(2) *For a fundamental discriminant D (i.e., $D = 1$ or it is a discriminant of a quadratic field) with $(-1)^k D > 0$, let $\mathcal{S}^{+}_{D,N}$ be the Shimura (Kohnen)*

map defined on $S^+_{k+1/2}(4N)$ *as follows.*

$$f|\mathcal{S}^+_{D,N}(z) = \sum_{n\geq 1}\left(\sum_{\substack{d|n\\(d,N)=1}} \left(\frac{D}{d}\right) d^{k-1} a_f(|D|n^2/d^2)\right) q^n. \tag{3.10}$$

Then $\mathcal{S}^+_{D,N}$ *maps the Kohnen plus space into* $S_{2k}(N)$*. There exists an explicit Shintani correspondence, denoted by* $\mathcal{S}^{+,*}_{D,N}$*, mapping the space* $S_{2k}(N)$ *into the plus space* $S^+_{k+1/2}(4N)$*. Further these two maps are adjoint to each other under the Petersson scalar product and preserve the spaces of newforms. Let* f *be a newform in* $S^{+,new}_{k+1/2}(4N)$ *and* F *be the corresponding newform in* $S^{new}_{2k}(N)$ *under the Shimura (Kohnen) correspondence. Then one has the following explicit form of the Waldspurger theorem:*

$$\frac{|a_f(|D|)|^2}{\langle f, f\rangle} = 2^{\nu(N)}\frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L(F,\chi_D,k)}{\langle F, F\rangle}, \tag{3.11}$$

where χ_D *is the quadratic character* $\left(\frac{D}{\cdot}\right)$*,* $(D,N)=1$*, and* $\nu(N)$ *is the number of distinct prime factors of* N*.*

Remark 3.1. For $N = 1$, the explicit form of the Waldspurger theorm (eq. (3.11)) was obtained by Kohnen and Zagier [12].

3.2. **Works of the author.** In this final section, we shall describe some of our results on the theory of modular forms of half-integral weight. We also indicate some of our recent results which are published or in progress. The works mentioned in this article are obtained in collaboration with M. Manickam, T. C. Vasudevan, Sanoli Gun and Jaban Meher.

3.2.1. *Newform theory and Shimura, Shintani correspondences.* In [17], we generalised the work of Kohnen on the theory of newforms to the full space. For $N \geq 1$ odd and square-free, consider the full space $S_{k+1/2}(4N)$. In this case, S. Niwa, by using the trace formulas, showed that there exists a Hecke equivariant isomorphism between the vector spaces $S_{k+1/2}(4N)$ and $S_{2k}(2N)$. Using this existence of an isomorphism, we develop the theory of newforms on the full space $S_{k+1/2}(4N)$ and moreover, obtain Kohnen's results. For simplicity, we shall explain briefly the case $N = 1$.

Define

$$S^{old}_{k+1/2}(4) = S^+_{k+1/2}(4) + S^+_{k+1/2}(4)|U(4)$$

and $S^{new}_{k+1/2}(4)$ is defined to be the orthogonal complement of $S^{old}_{k+1/2}(4)$ in $S_{k+1/2}(4)$ with respect to the Petersson inner product. It is shown that $S^+_{k+1/2}(4) \oplus S^+_{k+1/2}(4)|U(4)$ is a direct sum. Now by Niwa's theorem, we see that the space $S^{old}_{k+1/2}(4)$ is isomorphic to $S^{old}_{2k}(2) = S_{2k} \oplus S_{2k}|U(2)$. Since the full spaces are isomorphic and we have a direct sum of the old forms, it follows that the spaces of new forms $S^{new}_{k+1/2}(4)$ and $S^{new}_{2k}(2)$ are isomorphic. As a consequence of this, we also get the isomorphism of the spaces $S^{+}_{k+1/2}(4)$ and S_{2k}. By using induction on the number of prime factors of N we get the required isomorphism. We summarise the results obtained in [17] in the following theorem.

Theorem 3.2. *(Manickam-Ramakrishnan-Vasudevan) Let N be an odd square-free natural number. The space $S_{k+1/2}(4N)$ is decomposed as oldforms and new-forms as follows:*

$$S_{k+1/2}(4N) = S^{new}_{k+1/2}(4N) \oplus S^{old}_{k+1/2}(4N), \tag{3.12}$$

where the above is an orthogonal directsum (with respect to the Petersson scalar product) and $S^{old}_{k+1/2}(4N)$ has the following decomposition:

$$S^{old}_{k+1/2}(4N) = \bigoplus_{rd|N, d<N} S^{new}_{k+1/2}(4d)|U(r^2) \oplus \bigoplus_{d|N, rd|2N} S^{+,new}_{k+1/2}(4d)|U(r^2). \tag{3.13}$$

The space of new forms $S^{new}_{k+1/2}(4N)$ (resp.$S^{+,new}_{k+1/2}(4N)$) has a basis of eigenforms with respect to all the Hecke operators in the space (resp. in the plus space). Moreover, the vector spaces $S^{new}_{k+1/2}(4N)$ and $S^{new}_{2k}(2N)$ (resp. $S^{+,new}_{k+1/2}(4N)$ and $S^{new}_{2k}(N)$) are isomorphic under a suitable linear combination of Shimura maps, and therefore 'multiplicity 1' result holds good in both the new forms space $S^{new}_{k+1/2}(4N)$ and $S^{+,new}_{k+1/2}(4N)$.

Our next result is concerning the Shintani map corresponding to the full space. The idea behind getting the adjoint Shintani map is to construct a kernel function which is described in two different ways (one by using the Poincaré series of half-integral weight and the other in terms of certain kernel functions for the periods of cusp forms). The original idea of using these functions was due to Zagier, who first used this method to prove the Doi-Naganuma lifting between Hilbert modular forms and elliptic modular forms. These ideas were later used by Kohnen to prove results in the plus space. We also adopt the same ideas to prove our results. This adjoint liftings together with the theory of newforms developed in [17] give the explicit version of the Waldspurger theorem. Below we summarise the results obtained in our work [18].

Theorem 3.3. *(Manickam-Ramakrishnan-Vasudevan) Let f be a newform in $S^{new}_{k+1/2}(4N)$ and let F be the corresponding normalised newform in $S^{new}_{2k}(2N)$ under the Shimura correspondence $\mathcal{S}_{k,t}$, where $(-1)^k t > 0$, $t \equiv 1 \pmod 4$ square-free, $(t, N) = 1$. Then we have*

$$\frac{|a_f(|t|)|^2}{\langle f, f\rangle} = 2^{\nu(N)-1}\frac{(k-1)!}{\pi^k}|t|^{k-1/2}\frac{L(F,\chi_t,k)}{\langle F, F\rangle}. \tag{3.14}$$

3.2.2. *Other related works.* In his thesis [23], the author had obtained the Shintani map (adjoint to the Shimura map) for more general N. This is useful in getting the explicit form of the Waldspurger theorem. In the author's work with Manickam [14], a study of the Shimura correspondence between the space of Jacobi forms and modular forms of integral weight (for higher levels) was undertaken. Moreover, in the same work, a generalisation of the work of Eichler and Zagier on the correspondence between Jacobi forms and modular forms of half-integral weight has been done. This correspondence plays a crucial role in obtaining a connection between Siegel modular forms and modular forms of integral weight, the so called *Saito-Kurokawa correspondence.* The author had worked on this topic also, but we will not go into the details here. For details we refer to [3, 19, 15].

3.2.3. *Lindelöf hypothesis in weight aspect for the average of the L-functions.* In [11], Kohnen and Sengupta used a formula of Waldspurger to obtain an average Lindelöf hypothesis in weight aspect for the central critical values of the Hecke L-functions attached to Hecke eigen cusp forms. In [16], we used our results obtained in [14] to prove similar results, which generalised the work of Kohnen and Sengupta mentioned above. Below we state our result.

Theorem 3.4. *(Manickam and Ramakrishnan) Let $D < 0$ be a fundamental discriminant such that $(D, N) = 1$ and let $\epsilon > 0$. Then*

$$\sum_f L(f, \chi\left(\frac{D}{\cdot}\right), k) \ll_{\epsilon, D, N} k^{1+\epsilon} \quad (k \to \infty), \tag{3.15}$$

where the sum varies over all newforms in $S_{2k}^{new}(N, \overline{\chi}^2)$, and χ is a Dirichlet character modulo N.

3.2.4. *Representation of integers as sums of an odd number of squares.* Let $r_k(n)$ denote the number of representations of an integer n by k integer squares. When k is even number formulas for $r_k(n)$ were given by Ramanujan and there is a long history for finding formulas for $r_k(n)$. For odd values of k general formulas are not known. Note that

$$\Theta^{2k+1}(z) = 1 + \sum_{n \geq 1} r_{2k+1}(n) q^n \tag{3.16}$$

is a modular form in $M_{k+1/2}(4)$. So applying the Shimura map to $\Theta^{2k+1}(z)$, its image will lie in the space $M_{2k}(2)$, the vector space of modular forms of weight $2k$ on $\Gamma_0(2)$, and if one gets a nice basis for $M_{2k}(2)$, a formula for $r_{2k+1}(n^2)$ can be obtained. In [4], the author in collaboration with Sanoli Gun constructed a basis for $M_{2k}(2)$ using the Dedekind eta-quotients and thereby presented formulas for the numbers $r_{2k+1}(n^2)$. However, when $k \equiv 2 \pmod 4$ in the examples presented in the work, some of the basis elements of $M_{2k}(2)$ were missing in the explicit formulas. It was pointed out by D. Zagier that this fact is due to the existence of a subspace of $M_{k+1/2}(4)$, containing $\Theta^{2k+1}(z)$, which descends to M_{2k} (the space of modular forms of weight $2k$ on $SL_2(\mathbb{Z})$) under a class of Shimura maps. In a recent joint work with Sanoli Gun and M. Manickam [5], we obtained a characterisation of the space of newforms $S^{new}_{k+1/2}(4N)$ (where N is odd and square-free) in terms of the Atkin-Lehner W operator $W(4)$ and using this characterisation proved the observation made by Zagier. As an application, we obtained interesting formulas and congruences for the numbers $r_{2k+1}(n^2)$. Let $S^{(\pm,2)}_{k+1/2}(4N)$ be the subspace such that $f|W(4) = \pm f$. Then we consider the intersection of this subspace with the newform space and denote it by $S^{(\pm,2),new}_{k+1/2}(4N)$. We mention below the main results of [5].

Theorem 3.5. *(Sanoli Gun, M. Manickam and B. Ramakrishnan)*

(1) *The space* $S^{(\pm,2),new}_{k+1/2}(4N)$ *is characterised by*

$$S^{(\pm,2),new}_{k+1/2}(4N) = \Big\{ f \in S^{new}_{k+1/2}(4N) \ : \ a_f(n) = 0 \text{ if } \left(\frac{(-1)^k n}{2}\right) = \mp \left(\frac{2}{2k+1}\right) \Big\}, \tag{3.17}$$

where the symbol $\left(\frac{(-1)^k n}{2}\right)$ *is the Kronecker symbol.*

(2) *Let* $t \equiv 1 \pmod 4$ *be a square-free integer such that* $(-1)^k t > 0$, $(t, N) = 1$ *and* $\left(\frac{t}{2}\right) = \mp \left(\frac{2}{2k+1}\right)$. *Then the Shimura map* $\mathcal{S}_{t,1}$ *maps* $M^{(\pm,2)}_{k+1/2}(4N)$ *into* $M_{2k}(N)$. *In particular, if* $N = 1$ *and* t *is such that* $\left(\frac{t}{2}\right) = -\left(\frac{2}{2k+1}\right)$, *then* $\Theta^{2k+1}(z)|\mathcal{S}_{t,1} \in M_{2k}$.

Using the above theorem, we obtained a formula for $r_{2k+1}(|t|n^2)$ in terms of Fourier coefficients of a basis of M_{2k}. These formulas lead to interesting congruences satisfied by the numbers $r_{2k+1}(n^2)$. For example, we get the following congruences for $r_{2k+1}(p^2)$:

$$\begin{gathered} r_{13}(p^2) + p \equiv \tau(p) \pmod 5, \quad r_{21}(p^2) + 2p \equiv 2p\tau(p) \pmod 5, \\ r_{21}(p^2) \equiv 0 \pmod 7, \quad r_{29}(p^2) \equiv 0 \pmod{29}, \quad r_{37}(p^2) \equiv 0 \pmod{37}. \end{gathered} \tag{3.18}$$

Using the fact that $\tau(n) \equiv n\sigma_9(n) \pmod 5$, we get the following.

$$\begin{aligned} r_{13}(p^2) &\equiv p^2 \pmod 5, \\ r_{21}(p^2) &\equiv \begin{cases} 2 \pmod 5 & \text{if } p^2 \equiv 1 \pmod 5 \\ p-2 \pmod 5 & \text{if } p^2 \equiv -1 \pmod 5. \end{cases} \end{aligned} \tag{3.19}$$

3.2.5. *Recent work.* Recently, in a joint work with M. Manickam and Jaban Meher [13], we obtained the theory of newforms of half-integral weight for the full space on $\Gamma_0(16N)$, N is an odd and square-free natural number. Further, we have defined the Kohnen plus space in the general case and using the theory of newforms in the full space, we have extended the theory of newforms in the Kohnen plus space on $\Gamma_0(2^a N)$, $a = 3, 4, 5$.

References

[1] A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.

[2] P. Deligne, *La conjecture de Weil. I,* Inst. Hautes tudes Sci. Publ. Math. **43** (1974), 273–307.

[3] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55**, Birkhaüser, Boston, 1985.

[4] S. Gun and B. Ramakrishnan, *On the representation of integers as sums of an odd number of squares,* Ramanujan J. **15** (2008), 367–376.

[5] S. Gun, M. Manickam and B. Ramakrishnan, *A canonical subspace of modular forms of half-integral weight,* Math. Ann. **347** (2010), 899–916.

[6] C. G. J. Jacobi, *Fundamenta nova theories functionum ellipticarum*, Königesberg, 1829.

[7] N. Koblitz, *Elliptic Curves and Modular Forms*, Graduate Texts in Mathematics **97**,Springer-Verlag, 1984.

[8] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249–266.

[9] W. Kohnen, *Newforms of half-integral weight,* J. Reine Angew. Math. **333** (1982), 32–72.

[10] W. Kohnen, *Fourier coefficients of modular forms of half-integral weight,* Math. Ann. **271** (1985), 237–268.

[11] W. Kohnen and J. Sengupta, *Waldspurger's formula and central critical values of L-functions of newforms in weight aspect,* Number Theoretic methods (Iizuka 2001), 213–217, Dev. Math. **8**, Kluwer Acad.Publ. Dordrecht, 2002

[12] W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip,* Invent. Math. **64** (1981), 175–198.

[13] M. Manickam, Jaban Meher and B. Ramakrishnan, *On the theory of newforms of half-integral weight on* $\Gamma_0(8N)$ *and* $\Gamma_0(16N)$, Preprint 2012.

[14] M. Manickam and B. Ramakrishnan, *On Shimura, Shintani and Eichler-Zagier correspondences,* Trans. Amer. Math. Soc. **352** (2000),

[15] M. Manickam and B. Ramakrishnan, *On Saito-Kurokawa correspondence of degree two for arbitrary level,* J. Ramanujan Math. Soc. **17** (2002), 149–160.

[16] M. Manickam and B. Ramakrishnan, *An estimate for a certain average of the special values of character twists of modular L-functions,* Proc. Amer. Math. Soc. **133** (2005), 2515–2517.

[17] M. Manickam, B. Ramakrishnan and T. C. Vasudevan, *On the theory of newforms of half-integral weight,* J. Number Theory **32** (1990),

[18] M. Manickam, B. Ramakrishnan and T. C. Vasudevan, *On Shintani Correspondence,* Proc. Indian Acad. Sci, (Math. Sci.) **99** (1989),

[19] M. Manickam, B. Ramakrishnan and T. C. Vasudevan, *On Saito-Kurokawa descent for congruence subgroups,* Manuscripta Math.**81** (1993), 161–182.

[20] L. J. Mordell, *On Mr. Ramanujan's Empirical Expansions of Modular Functions,* Proc. Cambridge Phil. Soc. **19** (1917), 117–124.

[21] S. Niwa, *Modular forms of half-integral weight and the integral of certain theta functions*, Nagoya Math. J. **56** (1975), 147–161.

[22] S. Ramanujan, *On certain arithmetical functions,* Trans. Cambridge Phil. Soc.**22**(1916), 159–184.

[23] B. Ramakrishnan, *On Shimura and Shintani correspondences for modular forms and Jacobi forms and some results on Hecke operators*,Ph. D. Thesis, University of Madras, 1989.

[24] J. -P. Serre, *A course in Arithmetic*, Springer-Verlag, 1976.

[25] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–481.

[26] T. Shintani, *On construction of holomorphic cusp forms of half integral weight,* Nagoya Math. J.**58** (1975), 83–126.

[27] J. -L. Waldspurger, *Correspondence de Shimura,* J. Math. Pures Appl. **59** (1980), 1–132.

[28] J. -L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier,* J. Math. Pures Appl. **60** (1981), 375–484.

[29] A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen,* Math. Ann. **168** (1967), 149–156.

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TOOLS AND TECHNIQUES IN APPROXIMATION THEORY

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Abstract: The tools and techniques employed in abstract approximation theory are listed. Several important applications stem out of the theory and will prove to be useful in mathematical sciences, as well as other subjects.

Functions are the basic mathematical tools for describing objects and processes from the real world. These functions are known explicitly only in some cases. Very frequently, it is necessary to construct approximations to them, based on limited information about the underlying processes. The theory of approximation of functions can be divided into at least three parts: Best Approximation, Good Approximation and Possible Approximation (or the problem of completeness).

P.L. Chebyshev initiated the study of best approximation in 1853. He investigated the problem of best approximation of a real-valued continuous function on a closed line segment $[a, b]$ by algebraic polynomials of degree $\leq n$. The general problem concerns the search for a function from the prescribed class, which has the least deviation from a given function, as measured with respect to a prescribed metric or norm. Questions of existence, uniqueness, characterization and qualitative properties of the minimizing function constitute this study. Questions concerning the operator and selections for the operator of best approximation are answered. There is a computational aspect to this theory which seeks to grind out a sequence of functions which converges to a minimizing function. Estimates about the rapidity of convergence are very important.

Settings. The settings include Metric spaces, Normed Linear spaces, Banach spaces, Normed linear spaces with property (Λ),Normed almost linear spaces and locally convex spaces.

Tools. The tools employed are closed sets, linear subspaces,compact sets, boundedly compact sets, approximately compact sets, convex sets, strictly convex sets,

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uniformly convex sets, tangent functionals or Gateaux derivatives, suns, moons, star-shaped sets, hyperplanes, projections, bases in Banach spaces, unique Hahn-Banach extensions, fixed points, modulus of continuity, modulus of smoothness, selections for operators, Markov, Bernstein and Nikolskii inequalities and errors of approximation.

Techniques. The techniques used are: best approximation,best coapproximation, strong best approximation, strong best coapproximation, best simultaneous approximation, best simultaneous coapproximation, restricted range approximation, monotone approximation, one-sided approximation, Padé approximation, weighted approximation, splines, wavelets, approximation by operators.

Some preliminary definitions are provided next.

Definition In a linear space X, a set A is convex if for every pair of distinct points x, y in the set, the line segment joining them is completely in the set. i.e. for every distinct pair $x, y \,\varepsilon A$,

$$tx + (1-t)y \;\varepsilon\; A, \quad 0 \le t \le 1.$$

Definition In a linear space X, a set A is star-shaped if there exists a point $x \varepsilon A$, called the star-center of A, such that

$$tx + (1-t)y \;\varepsilon\; A, \qquad 0 \le t \le 1, \qquad y \;\varepsilon\; A.$$

Remark Every convex set is star-shaped, the converse is not true.

Example $A = \{0\} \times [0,1] \bigcup [1,0] \times \{0\}$ in $\Re^2$ is star-shaped with star-center $(0,0)$, but not convex.

Definition In a linear space X, a set A is a cone with vertex at $x_0 \varepsilon X$, if $x \varepsilon A$ implies that

$$t\,x + (1-t)\,x_0 \;\varepsilon\; A, \qquad t > 0.$$

Definition In a normed linear space X, a set A is strictly convex, if for every pair of distinct points

$$x, y \;\varepsilon\; A, \qquad \|x\| = 1, \|y\| = 1 \quad \text{implies that} \quad \|(x+y)/2\| < 1.$$

Definition In a normed linear space X, a set A is uniformly convex if to each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that

$$\|x\| = 1,\ \|y\| = 1 \quad \|x - y\| < \epsilon \text{ implies that } \|(x+y)/2\| \;\le\; 1 - \delta(\epsilon).$$

Chebyshev systems

Definition A system of n functions $f_1, f_2, \cdots, f_n$ in $C(Q)$, Q a compact space, endowed with the norm $\|f\| = \max q\varepsilon Q|f(q)|$, is a Chehyshev system on Q if and only if for every set of n distinct points $q_1, q_2, \cdots, q_n$ in Q,

$$\begin{vmatrix} f_1(q_1) & \cdots & f_1(q_n) \\ f_2(q_1) & \cdots & f_2(q_n) \\ \cdots & \cdots & \cdots \\ f_n(q_1) & \cdots & f_n(q_n) \end{vmatrix} \neq 0$$

Chebyshev systems have been used to state and prove the famous Chebyshev alternation theorem characterizing best approximation.

Definition *A* set G in a normed linear space X is boundedly compact if it intersects each closed ball in a compact set. G is approximatively compact if it is convex and closed.

Best approximation and best coapproximation in a normed linear space.

Let G be a nonempty subset of a real normed linear space X.

Definition [51, 46] An element $g_0 \varepsilon G$ is called an element of best approximation (respectively, best coapproximation) of $x \varepsilon X$ by the elements of G if $\|x - g_0\| \leq \|x - g\|$, for all $g \ \varepsilon \ G$ (respectively, $\|g_0 - g\| \leq \|x - g\|$, for all $g \varepsilon G$).

Notation. The set of all such elements of best approximation (best coapproximation) by elements of G, is denoted by $P_G(x)$ $(R_G(x))$, respectively.

G is proximinal, semi-Chebyshev or Chebyshev if $P_G(x)$ $(R_G(x))$ has at least, at most or exactly, one element of best approximation (best coapproximation) to each $x \ \varepsilon \ X$.

The set $D(P_G) = \{x \ \varepsilon \ X : P_G(x) \neq \phi\}$ is called the domain of P_G.

The set $D(R_G) = \{x \ \varepsilon \ X : R_G(x) \neq \phi\}$ is called the domain of R_G.

The mapping $x \to P_G(x)$ is a set-valued map and is called the set-valued metric projection of X onto G.

The mapping $x \to R_G(x)$ is a set-valued map and is called the set-valued cometric projection of X onto G. **Definition** Let G be a linear subspace of a real normed linear space X. Let $x, y \ \varepsilon \ X$. Then x is orthogonal to y,written as $x \perp y$, if $\|x + \alpha y\| \geq \|x\|$,for all $\alpha \ \varepsilon \ \Re$.

Remarks.

(1) $x \perp G$ if $x \perp g$, for all $g \ \varepsilon \ G$, $G \perp x$ if $g \perp x$, for all $g \perp G$.
(2) $g_0 \ \varepsilon \ P_G(x)$ iff $(x - g_0) \perp G$.
(3) $g_0 \ \varepsilon \ R_G(x)$ iff $G \perp (x - g_0)$.
(4) $P_G = R_G$ in inner product spaces and Hilbert spaces.

The metric projection bound of a real normed linear space X, denoted by $MPB(X)$, is defined by M.A. Smith [54] who proved that

$$1 \leq MPB(X) \leq 2.$$

Geetha S. Rao and K. R. Chandrasekaran define the cometric projection bound, denoted by $CMPB(X)$, as follows:

Definition Let X be a real normed linear space. Then
$CMPB(X) = sup\{\|R_G\| \ : \ G$ is a closed proper proximinal subspac of $X\}$,
where
$\|R_G\| \ = \ sup \ \{\|y\| \ : \ y \varepsilon R_G(x), \qquad \|x\| \ \leq \ 1\}.$

Remark $CMPB(X) = 1$. Since $y \ \varepsilon \ R_G(x)$, with $\|x\| \leq 1$, it implies that

$$\|y\| \ \leq \ \|x\| \leq \ 1.$$

Therefore,

(1) $$\|R_G\| \ \leq \ 1.$$

But $R_G(g) \ = \ \{ \ g \ \}$, for all $g \ \varepsilon \ G$ implies that

(2) $$\|R_G\| \ \leq \ 1.$$

(1) and (2) show that $\| \ R_G \ \| \ = \ 1$. Therefore,

$$CMPB(X) \ = \ 1.$$

Remark $P_G \ = \ R_G$ for every linear subspace G of X if and only if

$$MPB(X) \ = \ 1.$$

Consequently, there always exists a subspace G with $P_G \ \neq \ R_G$ in a real normed linear space satisfying the condition $MPBX \ \neq \ 1$.

As illustrated by Papini and Singer [46], best coapproximation is clearly related to best approximation.

Papini and Singer [46]. Let G be a linear subspace of a real normed linear space X and $x \ \varepsilon \ X \backslash G$. Then

$$R_G(x) \ = \ \{ \ g_0 \ \varepsilon \ G : \ g_0 \ \varepsilon \ \bigcap \ P_{<g_0, x>} \ g\},$$

where

$$< g_0, \ x \ > = \ \{ \ ax + (1-a) \ g_0 \ : \ a \ \varepsilon \ R \ \}$$

is the linear manifold spanned by g_0 and x.

Definition Let G be a subset of a normed linear space $X, f \ \varepsilon \ X \backslash G$ and $g_f \ \varepsilon \ G$. Then g_f is called a strongly unique best approximation to f from G,if there exists a constant $k_f \ > \ 0$ such that

(1) $$\| \ f - g_f \ \| \ \leq \ \| \ f - \ g \ \| - k_f \ \| \ g - g_f \ \|, \text{ for all } g \ \varepsilon \ G,$$

where k_f depends only on f.

g_f is called a strongly unique best coapproximation to f from G if there exists a constant $k_f > 0$ such that

$$(1') \qquad \|g - g_f\| \leq \| f - g \| - k_f \| f - g_f \|, \text{ for all } g \varepsilon G,$$

where k_f depends on both f and g_f.

Remark It is clear that if some $k_f > 0$ satisfies inequality (1) (respectively (1')), then every smaller value of k_f will also satisfy inequality (1)(respectively (1')). The maximum of all such numbers $k_f > 0$ is called the strong unicity constant of f and is denoted by $K(f)$.

Geetha S. Rao and R. Saravanan [29]. Obviously, every strongly unique best approximation is a unique best approximation. But a strongly unique best coapproximation need not imply the uniqueness. This will happen if the corresponding strong unicity constant is 1.

Example Let $X = \Re \times \Re$, $G = \Re \times \{ 0 \}$, $x = (0, 1)$ or $x = (0, -1)$.

The following are true:

(1) The point (0,0) is the unique best approximation to x from G under the ℓ_1- norm and ℓ_2- norm:

$$\| (a, b) \|_1 = |a| + |b|, \ a, b \ \varepsilon \ \Re$$

$$\| (a, b) \|_2 = \sqrt{(a^2 + b^2)}, \ a, b \ \varepsilon \ \Re.$$

(2) The set $\{(b, 0) : -1 \leq b \leq 1\}$ consists of best approximations to x from G under ℓ_∞- norm:

$$\|(a, b)\|_\infty = \max\{|a|, |b|\}, \ a, b \ \varepsilon \ \Re.$$

(3) The point $(0, 0)$ is the strongly unique best approximation to x from G under the ℓ_l- norm but not under ℓ_2- norm. In this case,

$$k_f \leq 1 \text{ and the strong unicity constant is 1.}$$

(4) The set $\{(b, 0) : -1 \leq b \leq 1\}$ consists of best coapproximations to x from G under the ℓ_1- norm.

(5) The point (0,0) is the unique best coapproximation to x from G under the ℓ_2-norm and the ℓ_∞- norm.

(6) The set $\{(b, 0) : -1 < b < 1\}$ consists of best coapproximations to x from G under the ℓ_1- norm. Here,

$$k_f = \inf_{a \varepsilon \Re} \{(|a| + 1 - |a - b|)/(|b| + 1)\}.$$

This shows that strongly unique best coapproximation is not unique.

(7) The point (0,0) is not a strongly unique best coapproximation to x from G under the ℓ_∞- norm.

(8) The point (0,0) is the unique strongly best coapproximation to x from G under the ℓ_2- norm. Observe that

$$k_f \leq \inf_{a \varepsilon \Re} \{\sqrt{(a^2+1)} - a\}$$

Note. Remark 8 along with Remark 3 shows that a strongly unique best coapproximation is not equal to a strongly unique best approximation in the inner product space $\Re^2$.

Definition A set G in a linear space X is a cone with vertex at $x_0 \varepsilon X$ if $x \varepsilon G$ implies that

$$t\, x \;+\; (1-t)\, x_0 \;\varepsilon\; G, \qquad (t > 0).$$

The concept of a sun was first introduced by N. Effimov and S. B. Steckin [11].

Definition A set G in a linear space X is a sun if whenever g_0 is a best approximation to some $x \,\varepsilon X \backslash G$, then g_0 is a best approximation to every element on the ray from g_0 through x.

Remarks.

(1) Since every convex set has this property, a sun may be regarded as a generalization of a convex set.

(2) Every boundedly compact Chebyshev set is a sun.

Definition A normed linear space is smooth if there is a unique supporting hyperplane to the unit sphere at each point.
i.e. if $x \;\varepsilon\; X$, with $\|x\| = 1$, there exists a unique functional $f = f_x \;\varepsilon X*$ such that

$$\|\, f \,\| \;=\; 1 \text{ and } f(x) = 1.$$

Remark In smooth spaces, every proximinal sun is convex.

Definition A point of a set A in X which is not an interior point of every segment in A is called an extreme point.

Let $B(X) = \{x \,\varepsilon\, X \;:\, \|x\| \leq 1\}$ and let $\xi(X)$ denote the set of extreme points of $B(X)$.

Definition A cone $K(g_0, x)$ is defined by

$$K(g_0, x) \;=\; \{g \,\varepsilon X \;:\; x^*(g - g_0) > 0, \text{ for all } x^* \;\varepsilon\; \xi(x - g_0)\}$$

where $\xi(x - g_0)$ is the set of extreme points of the set

$$\{x^* \;\varepsilon\; X^* : \|x^*\| \leq 1 \text{ and } x^*(x - g_0) \;=\; \|\, x - g_0 \,\|\}$$

Proposition $K(g_0, x) = \bigcup B(g_0 + \lambda(x - g_0), \lambda \|g_0 - x\|),\ \lambda > 0.$

Definition A set $G \subset X$ is called a Kolmogorov set with respect to best approximation if and only if whenever $g_0 \;\varepsilon\; G$ is a best approximation to $x \;\varepsilon\; X$, then

$$\max x^*(g - g_0) \geq 0, \quad \text{for all } g \ \varepsilon \ G.$$

$$x^* \ \varepsilon \ \xi(x - g_0).$$

or

$$\min x^*(g - g_0) \leq 0, \quad \text{for all } g\varepsilon \ G.$$

$$x^* \ \varepsilon \ \xi(x - g_0).$$

Theorem (B. Brosowski and F. Deutsch [5]**).** Let $G \subset X$. The following are equivalent:

(1) G is a Kolmogorov set with respect to best approximation.
(2) $G \bigcap K(g_0, x) = f$, whenever $g_0 \varepsilon G$ is a best approximation to x.
(3) G is a sun.

In a finite dimensional normed linear space, every Chebyshev set is a sun. In an arbitrary normed linear space, this is not necessarily true. (see [**?**]).

Definition A proximinal set $G \subset X$ is a strict sun if for any $x \varepsilon X \setminus G$ and $y \varepsilon P_G(x), y \ \varepsilon \ P_G((1-\lambda)y + \lambda x)$, for all $\lambda > 0$.

Remark Every strict sun is a sun. The converse need not be true [7, 10].

Example Let $\|(x, y)\| = \max\{|x|, |y|\}$ be a norm in $\Re^2$.
Then $\{(x, y) : x \geq -1, y \geq -1\}$ is a sun but not a strict sun.
For semi-sun, meta-sun, a-sun, b-sun, g-sun, d-sun, etc. (see L.P. Vlasov [55]).

Definition Let $G \subset X$. A point $g_0 \ \varepsilon \ G$ is a lunar point if $x \ \varepsilon \ P_G^{-1}(g_0)$ and $G \bigcap K(g_0, x) \neq \phi$ imply that $g_0 \ \varepsilon \ G \bigcap K(g_0, x)$.

Definition $G \bigcap X$ is a moon if each of its points is a lunar point.

Remarks.

(1) Every sun is a moon. The converse is not true in general (see [1]).
(2) Those spaces in which every moon is a sun are called MS-spaces.
(3) Relations between suns, moons and certain continuity properties of the metric projection have been investigated. In fact, in MS-spaces, suns have been characterized by the ORL continuity (mentioned later) of their metric projections.

Theorem (Singer [51]**).** Let G be a convex subset of X, $x \ \varepsilon \ X \backslash G$, $g_0 \ G$. Then

$$g_0 \ \varepsilon \ P_G(x) \text{ iff there exists } f \ \varepsilon \ X^* \text{such that } \|f\| = 1,$$
$$f(g_0 - g) \geq 0, \quad (g \ \varepsilon \ G), \ f(x - g_0) = \|x - g_0\|.$$

Theorem (Singer [[51] Let X be a real normed linear space, $x \ \varepsilon \ X$ and $r > 0$. For any $f \ \varepsilon X^*$ with $\|f\| = 1$, the hyperplane $H \subset X$ defined by

$$H = \{y \ \varepsilon X : f(x - y) = r\} \textit{ supports the ball} B(x, r).$$

Conversely, for any support hyperplane H of the ball $B(x,r)$, there exists a unique $f\, \varepsilon X^*$ with $\|f\| = 1$ such that

$$H = \{y\, \varepsilon X : f(x-y) = r\}.$$

Theorem (Singer [[51] Let G be a convex subset of X, $x\, \varepsilon X \backslash G$, and $g_0\, \varepsilon\, G$. Then $g_0\, \varepsilon\, P_G(x)$ iff for each $g\, \varepsilon\, G$, there exists

$$f^g\, \varepsilon\, X^* \text{ such that } f^g\, \varepsilon \xi(B(X^*)), f_g(g_0 - g) \geq 0$$

and

$$f_g(x - g_0) = \|x - g_0\|.$$

Definition (Geetha S. Rao and K. R. Chandrasekaran [21]**)** A proximinal set G of X is a strict sun with respect to coapproximation if for every $x\, \varepsilon X \backslash G$, and $g_0\, \varepsilon\, R_G(x)$,

$$g_0\, \varepsilon\, R_G(t\, x + (1-t)\, g_0), \text{ for all } t \geq 0.$$

Definition A nonempty subset G' of X is a Kolmogorov set with respect to best coapproximation if $R_{G'}(x) = R_G(x)$ for every $x\, \varepsilon\, D(R_G)$.
Note. $R_{G'}(x)$ is defined in [20]. From this theorem, it is clear that Kolmogorov sets encompass convex sets, suns and linear subspaces.

Characterization Theorem. *Let G be a proximinal subset of X. The following statements are equivalent:*

(1) $P'_G(x) \subset P_G(x)$.
(2) $g_0\, \varepsilon\, P'_G(x) \to g_0\, \varepsilon P'_G(tx + (1-t)g_0),\ t \leq 0$.
(3) If G is convex, $P'_G(x) = P_G(x)$.
(4) If g is such that $t\, g + (1-t)\, g_0\, \varepsilon\, G$, for every $g\, \varepsilon\, G$, $0 \leq t \leq 0$ and $g_0\, \varepsilon\, P_G(x)$, then $P'_G(x) = P_G(x)$.

Note. If G is a sun, $P'_G(x) = P_G(x)$, G. Nürnberger [43].

Continuity properties of the metric projection and cometric projection

Radial continuity is more general and simple than upper and lower semi-continuity. These concepts require the restriction of the concerned set-valued maps to certain prescribed line segments be upper or lower semi-continuous.

Outer Radial Lower semi-continuity (ORL), Inner Radial Lower semi- continuity (IRL), Outer Radial Upper semi-continuity(ORU) and Inner Radial Upper semi-continuity (IRU), have been defined and used effectively see W. Pollul [47].

Theorem Let G be a nonempty subset of X and $x_0 \varepsilon X$.

If $\|x_0 - \beta g_0 - (1-\beta)g_0\| = dist(x_0, G)$, for all $g_0, g_1 \varepsilon P_G(x_0)$, and $\beta = 1$.

Selections

Definition A selection for R_G is a function $\{s : X \to G$ with $s(x) \varepsilon R_G(x)$, for all $x \varepsilon X$. A linear (continuous) selection for R_G is a selection which is also linear (continuous). (see F. Deutsch and P. Kenderov [8]).

Theorem (Geetha S. Rao and K. R. Chandrasekaran [21]). Let G be a Chebyshev subspace of X. The following are true:

(1) R_G is linear.
(2) R_G^1 is a subspace of X.
(3) R_G^{-1} is a subspace of H such that $X = G \bigoplus H$.
(4) R_G is quasi-linear.: i.e. there exists a constant such that

$$\|R_G(x+y)\| \leq (\|R_G\| + \|R_G(y)\|).$$

Theorem Let G be a proximal hyperplane in X. Then R_G admits a linear selection.

Remark Let G be a proximinal subspace of X. Then $R_G^{-1}(0)$ is a subspace of X if and only if G is a Chebyshev subspace of X and R_G is linear.

The modulus of continuity of the set-valued cometric projection
(Geetha S. Rao, K.R. Chandrasekaran [19]).

Definition Let X be a real normed linear space. For two nonempty subsets A, B of X, the deviation of A from B, denoted by $\delta(A,B)$, is defined by

$$\delta(A,B) = \sup_{x \epsilon A} \inf_{y \epsilon B} \|x-y\| = \sup_{x \epsilon A} dist(x,B).$$

Let G be a linear subspace of X.

For $A \subset X$, let

$$R_G(A) = \bigcup_{x \epsilon A} R_G(x).$$

For $x \epsilon X$, $\epsilon > 0$, let

$$B(x,\varepsilon) = \{y \epsilon X : \|y-x\| < \varepsilon\}.$$

$$S(x,\varepsilon) = \{y \epsilon X : \|y-x\| = \varepsilon\}.$$

For $x \varepsilon D(R_G)$, $\epsilon > 0$, define the modulus of continuity of R_G by

$$W_G(x,\varepsilon) = R_G\ \delta(R_G(B(x,\varepsilon)), R_G(x)).$$

Theorem The following hold:

(1) $R_G(B(x,\varepsilon))$ is not empty whenever $R_G(x)$ is not empty.
(2) $W_G(x,0) = 0$.
(3) $W_G(x,\varepsilon_1) \leq W_G(x,\varepsilon_2)$, where $\varepsilon_1 \leq \varepsilon_2$).

(4) $W_G(tx, |t|\ \varepsilon) = |t| W_G(x, \varepsilon), \quad t\ \varepsilon\ R.$

(5) $W_G(x+g,\ \varepsilon) = W_G(x,\ \varepsilon), \quad g\varepsilon\ G.$

Theorem If G is a proximinal subspace of X and $R_G^{-1}(0)$ is a subspace of X, then $W_G(x,\varepsilon) = \varepsilon$, for every $x\ \varepsilon\ X$ and $\varepsilon > 0$.

Theorem If $x\varepsilon D(R_G)$, then $W_G(x,\varepsilon) = \delta(R_G(S(x),\varepsilon)), R_G(x))$, for $\varepsilon > 0$.

Lipschitz condition

Definition Let G be a linear subspace of X and $x\ \epsilon\ D(R_G)$. R_G satisfies a Lipschitz condition at x, if there exists a constant $C > 0$, depending on x and G, such that

$$\delta(R_G(x), R_G(y)) \leq C\|x-y\|, \text{ for all } y\ \varepsilon\ D(R_G).$$

Each real number C satisfying the above condition is called a Lipschitz constant for R_G at x. R_G satisfies a uniform Lipschitz condition on $M \subset D(R_G)$ if there exists a real number C, depending on M and G, which is a Lipschitz constant for R_G, at each $x\ \epsilon\ M$.

Theorem R_G satisfies a Lipschitz condition at $x\ \epsilon\ D(R_G)$ iff $x\epsilon D(R_G)$ and $\sup W_G(tx, 1) < \infty, \quad t > 0.$

Remark If $x\ \epsilon\ D(R_G)$ and $\sup W_G(tx, 1) < \infty$, then $\sup W_G(tx, 1)$ is the smallest Lipschitz constant for R_G at x.

Monotone approximation [37, 38, 39, 40, 48, 49, 50].

For an increasing continuous function f on $[a, b]$, let

$$E_n^*(f) = \inf_{p_n \varepsilon M_n} \|f - p_n\|,$$

where M_n is the set of all polynomials p_n which satisfy

$$\varepsilon_i p_n^{(ki)}(x) \geq 0, \quad \text{for} \quad a \leq x \leq b, (i = 1, 2, \cdots, p).$$

Let $H_n = \{p_n\ :\ p_n$ is an algebraic polynomial of degree $\leq n\}$. Then $E_n^*(f)$ is the degree of the monotone approximation. The degree of approximation to $f\ \varepsilon\ C[a, b]$ by polynomials from H_n is defined by $E_n(f) = \inf |\ f - p_n|$.

(1) $E_n^*(f)/E_n(f)$ is bounded under which conditions ?
(the bound depends on the range of f).

(2) For sufficiently large n, $E_n^*(f) = E_n(f)$.

Restricted range approximation [36, 52, 53].

Restricted range approximation is best approximation having restricted range.

Definition Let X be a compact subset of $[a, b]$ containing at least $n+1$ points.

$$\text{For } f\ \epsilon\ C(X),\ \|f\| = \max |f(x)|.$$

Let M be an n-dimensional linear subspace of $C[a,b]$ such that the zero function is the only function in M which vanishes at n distinct points of $[a,b]$.

Let $\{\phi_1, \phi_2, \cdots, \phi_n\}$ form a basis for M. Fix two extended real-valued functions ℓ, u defined on X with the following conditions:

(1) ℓ may take the value $-\infty$, but never $+\infty$.-

(2) u may take the value $+\infty$, but never $-\infty$.

(3) $x-\infty = \{x : \ell(x) = -\infty\}, x+\infty = \{x : u(x) = \infty\}$ are open subsets of X.

(4) ℓ is continuous on $x - x = -\infty$ and u is continuous on $x - x = \infty$.

(5) $\ell < u$ for all $x \epsilon X$.

Define K by $K = \{p \varepsilon M : \ell(x) \leq p(x) \leq u(x),\ for\ all\ x \varepsilon X\}$.

Assume further that K consists of more than one function. K is the class of approximants and the functions in K have their ranges restricted by ℓ and u. ℓ and u belong to $C(X)$ and satisfy condition 5.

Relative Chebyshev Centers & Relative Chebyshev Radius [41, 42].

The existence of a best approximation from K corresponding to each $f \epsilon C(X)$ follows immediately from the fact that K is a closed subset of a finite dimensional subspace of $C(X)$ and

$$\inf_{g \epsilon K} \{ \|f-g\| : g \epsilon K, \|g\| \leq 2\|f\|\} = \inf_{g \epsilon K} \|f-g\|.$$

Definition Let A be a non-empty bounded subset and G an arbitrary non-empty subset of a normed linear space X. Each element $g_0 \epsilon G$ which satisfies the equation

$$\sup_{y \epsilon A} \|g_0 - y\| = \inf_{y \epsilon A} \sup_{y \epsilon A} \|x-y\|$$

is a relative Chebyshev center of A in G. The set of all such elements $x_0 \epsilon G$ is denoted by $Z_G(A)$. This set of relative Chebyshev centres of A in G is also called the set of best simultaneous approximations to A from G. The study of Chebyshev centers was initiated by A.L. Garkavi [13].

Several questions concerning the characterization and existence of the relative Chebyshev centers, the nature of Chebyshev centers in certain concrete normed spaces and the continuity properties of the Chebyshev center maps have been investigated.

Definition Let A be a non-empty bounded subset of a normed linear space X. If G is an arbitrary non-empty subset of X, then the relative Chebyshev radius of A in G, denoted by $r_G(A)$, is defined by

$$r_G(A) = \inf_{x \varepsilon G} \sup_{y \varepsilon A} \|x-y\|$$

The (possibly empty) set

$$Z_G(A) = \{x_0 \,\varepsilon\, G : \sup \|x_0 - y\| = r_G(A)\}$$

$$= \{x_0 \,\varepsilon\, G : \sup_{y \varepsilon A} \|x_0 - y\| = \inf_{x \varepsilon G} \sup_{y \varepsilon A} \|x - y\|\}$$

is called the set of relative Chebyshev centres of A in G.

Remark $r_G(A)|$ is the radius of the smallest ball in G (if one exists) which contains A,and each element of $Z_G(A)$ is the center in G of a ball of minimal radius covering A. $Z_G(A)$ is called **the set of best simultaneous approximation to** A **from** G.

(1) When $G = X$, the (absolute) Chebyshev radius $r(A)$ and the (absolute) Chebyshev center $Z(A)$ are obtained.

(2) When A is a singleton, say $A = \{y\}$, then

$$r_G(A) = d(G, y) = \inf_{x \varepsilon G} \|x - y\|$$

and $Z_G(A)$ is the set of best approximations to g in G.
i.e. $Z_G(A) = P_G(y)$, where P_G denotes the metric projection onto G.

Properties of Chebyshev centers.

(1) $Z_G(A)$ is closed in $G, r_G(A) = r_G(A), Z_G(A) = Z_G(A)$.

(2) If G is convex, $Z_G(A)$ is convex.

Theorem Let G be a convex subset and K a compact subset of a normed linear space X, $x_0 \varepsilon G$, $r_0 = r_{\{x_0\}}(K) = \sup \|x_0 - y\|$.
Then $x_0 \,\varepsilon\, Z_G(K)$ iff $x_0 \,\varepsilon\, Z_G(K \bigcap S(x_0, r_0))$, where $S(x_0, r_0) = \{x \varepsilon X : \|x_0 - x\| = r_0\}$.The strict convexity is related to the size of the centers. X is strictly convex iff for each compact set K in X, the center $Z(K)$ is at most a singleton.
Theorem Let X be a Banach space in which closed, convex and bounded sets have unique Chebyshev centers on themselves. The following are equivalent:

(1) For every pair of closed bounded convex sets A and B with $A \subset B, r_A(A) \leq r_B(B)$.

(2) For every closed bounded set A in $X, Z(A) \,\varepsilon\, A$.

(3) Every two dimensional cross section S of the unit ball has radius $r_S(S) \leq 1$.

(4) X is two dimensional.

Let X be a normed linear space and let G be a proximinal subspace of X. The subspace G is homogeneously embedded in X if

$$x \,\varepsilon\, P_G^{-1}(0),\ y, z \,\varepsilon G,\ \|y\| = \|z\|$$

implies that

$$\|x-y\| \;=\; \|y-z\|.$$

Remarks.

(1) $x\;\varepsilon\;P_G^{-1}(0)$ means that $0\varepsilon P_G(x)$.

(2) G is homogeneously embedded in X iff it is a Chebyshev subspace of X and the intersection of every sphere S in X with G is a sphere in X centered at the best approximation of the center of S.

(3) If G is homogeneously embedded in X, then for every convex subset $A \subset G$, $P_A \;=\; P_A P_G$.

(4) If G and F are linear subspaces a normed linear space X and $G \subset F$, the following inclusion and transitivity relations hold:

a) When G is homogeneously embedded in X, then it is also homogeneously embedded in F.

b) When G is homogeneously embedded in F and F is homogeneously embedded in X, then G is homogeneously embedded in X.

Theorem Let G be a proximinal subspace of a normed linear space X. The following implications hold:

a) F is homogeneously embedded in X,
implies that

b) For all $x, y\;\varepsilon\;X$, and $u\varepsilon P_G(x), v\varepsilon P_G(y), Z_G(\{x,y\})\bigcap[u,v] \;\neq\; \phi$
implies that

c) For all $x\;\varepsilon\;X$ and $u\;\varepsilon\;P_G(x)$, $Z_G(\{0,x\})\bigcap[0,u] \;\neq\; \phi$.

Remark If the smooth points of S are dense in S_G then all the statements are equivalent.

Normed Almost Linear Spaces

G. Godini [36] propounded the concept of a normed linear space as a natural framework for the theory of best simultaneous approximation in a normed linear space. She also provided the notion of a dual space of a normed almost linear space, where the functionals are no longer linear but almost linear. **Definition** An almost linear space is a set X together with two maps $S \;:\; X \times X \to X$ and $m \;:\; R \times X \to X$ satisfying the following conditions (L_1) –(L_8) : For every $x, y\;\varepsilon\;X$ and $\alpha\varepsilon R$, denote $S(x,y)$ by $x+y$,
$m(\alpha, x)$ by αx and $-1x$ by $-x$. In the sequel, $x-y$ means $x+(-y)$.

Let $x, y, z\;\varepsilon\;X$ and $a, b\;\varepsilon\;R$.

(L_1) $(x+y)+z \;=\; x+(y+z)$.

(L_2) $x+y \;=\; y+x$.

(L_3) There exists an element $0\varepsilon X$ such that $x+0 \;=\; x$ for each $x\varepsilon X$.

(L_4) $1\;x \;=\; x$.

(L_5) $0\ x\ =\ 0$.
(L_6) $\alpha\ (x+y)\ =\ \alpha\ x+\alpha\ y$.
(L_7) $\alpha\ (\beta\ x)\ =\ (\alpha\ \beta)\ x$.
(L_8) $(\alpha\ +\ \beta)\ x\ =\ \alpha\ x\ +\ \beta\ x$, for $\alpha \geq 0,\ \beta \geq 0$.

Def. A non empty set Y of an almost linear space X is an almost linear subspace of X if for each $y_1, y_2\ \varepsilon\ Y$ and $\alpha \varepsilon R$, $S(y_1,\ y_2) \varepsilon Y$ and $m \varepsilon (\alpha, y_1) \varepsilon Y$. **Note.** An almost linear subspace Y of X is a linear subspace of X if $S\ :\ Y \times Y \to Y$ and $m\ :\ \Re \times Y \to Y$ satisfy all the axioms of a linear space. **Definition** Let X be an almost linear space:

$$V_X\ =\ \{x \varepsilon X\ :\ x - x = 0\} \qquad\qquad W_X\ =\ \{x \varepsilon X\ :\ x = -x\}.$$

Remarks

(1) The set V_X is a linear subspace of X and it is the largest one.
(2) The almost linear space X is a linear space iff $V_X\ =\ X$.
(3) The set W_X is an almost linear subspace of X.
(4) $V_X \bigcap W_X\ =\ \{0\}$.

Definition A norm of an almost linear space X is a functional, denoted by $|||.|||\ :\ X \to \Re$ satisfying (N1) − (N4):
(N1) $|||x+y|||\ \leq\ |||x|||+|||y|||$
(N2) $|||x|||\ =\ 0$ iff $x\ =\ 0$.
(N3) $|||a|||\ =\ |a|\ |||x|||$.
(N4) $|||x|||\ \leq\ |||x+w|||$.
Remark Clearly, $|||x|||\ \geq\ 0$, for all $x\ \varepsilon\ X$.
Definition A normed almost linear space is an almost linear space X together with a norm $|||.|||\ :\ X \to \Re$ satisfying (N1) − (N4).
Definition Let X be an almost linear space.

a) A functional $f\ :\ X \to \Re$ is called an almost linear functional if (1) – (3) hold.

(1) $f(x+y)\ =\ f(x)+f(y)$, $x, y \varepsilon X$.
(2) $f(\alpha x)\ =\ \alpha f(x)$, $a \varepsilon \Re, \alpha \geq 0$, $x \varepsilon X$.
(3) $-f(-x)\ \leq\ f(x)$, $x \varepsilon X$.

Note. A functional $f\ : X \to \Re$ is a linear functional if (1) and (2) hold for all $x, y \varepsilon X$ and $\alpha \varepsilon \Re$.
Notation. X^* is the set of all almost linear functionals defined on an almost linear space X. X^* is an almost linear space, with $S(f_1, f_2)$ and $m(\alpha, f)$ being

the functionals on X defined by

$$S(f_1, f_2)(x) = f_1(x) + f_2(x), \quad m(\alpha, f)(x) = f(ax), \quad x \varepsilon X, \quad f_1, f_2, f \varepsilon X^*.$$

Note. $\Theta \, \varepsilon \, X^*$ is the functional which is 0 at each $x \, \varepsilon \, X$. **Definition** For each $f \varepsilon X^*$, the norm of f, denoted by $|||f|||$, is defined by

$$|||f||| = \sup\{|f(x)| : x \varepsilon B_X\}, \quad \text{where} \quad B_X = \{y \varepsilon X : |||y||| \leq 1\}.$$

Let $X^* = \{f \varepsilon X : |||f||| < \infty\}$. Then X^* together with $||| \cdot |||$ defined for $f \varepsilon X$ by

$$|||f||| = \sup\{|f(x)| : x \varepsilon B_X\}$$

is a normed almost linear space, called the dual of the normed almost linear space X.

Geetha S. Rao and T. L. Bhaskaramurthy [14, 15, 16] discussed several aspects of Approximation in normed almost linear spaces.

Normed Linear Spaces with Property (Λ)

Definition (G. Godini) Let X be a normed linear space. Let G be a subset of X.

Let $G^{\perp} = \{f \varepsilon X^* : f(y) = 0, \text{ for all } y \, \varepsilon \, G\}$.

For $A \subset X^*$, let $A^{\perp} = \{x \, \varepsilon \, X^* : f(x) = 0, \text{ for all } f \, \varepsilon \, A\}$.

For $x \, \varepsilon \, X$, let $A_X(x) = \{f \varepsilon B_{X^*} : f(x) = \|x\|\}$,

where $B_{X^*} = \{f \varepsilon X^* : \|f\| \leq 1\}$.

Observe that $A_X(x)$ is a nonempty convex subset of B_{X^*}.

Let $N(x) = N_X(x) = [A_X(x) \bigoplus \{x\}]$.

Definition The tangent functional or Gateaux derivative at $x, y \varepsilon X$, denoted by $\tau(x, y)$, is defined by

$$\tau(x, y) = \lim_{t \to 0^+} \frac{\|x + t \, y\| - \|x\|}{t}.$$

This limit always exists in the present context.

In 1979, G. Godini showed that

$$0 \leq \tau(x, y) + \tau(x, -y) = 2 dist.(y, N(x)), \text{ for all } x, y \, \varepsilon \, X$$

and defined a normed linear space with property (Λ) as follows :

Definition A normed linear space X has property (Λ) if

$$\tau(x, y) + \tau(x, -y) = 2 \; dist(y, N(x)), \quad \text{for all } x, \; y \varepsilon X, \|x\| = \|y\| = 1.$$

Example $L^1(T, \nu), C^1(Q, \nu)$ have property (Λ).

Theorem Let $x \varepsilon X$, let $ex \; A(x)$ represent the extreme points of $A(x)$. For $x, y \varepsilon X$,

(1) $\tau(x, y) = \max\{f(y) : f \varepsilon \; ex \; A(x)\}$.

(2) $-\tau(x, -y) = \min\{f(y) : f \varepsilon \; exA(x)\}$.

(3) If $C_x = \{z \varepsilon X \ : \ \tau(x, -z) \geq 0\}$ and
$D_x = \{z \varepsilon X \ : \ \tau(x, z) - \tau(x, -z) \geq 0\}$ then $C_x \subset D_x$.

(4) X has the property (Λ) iff $C_x = D_x$, for all $x \varepsilon X$.

Geetha S. Rao and K.R. Chandrasekaran [17] characterize the normed linear spaces with the property (Λ) in terms of best coapproximation as follows :

Theorem The following are equivalent :

(1) X has property (Λ).

(2) For each nonempty subset G of X , each $x \ \varepsilon \ X \setminus G$ and

$$g_0 \ \varepsilon \ G, \tau(g - g_0, m) - \tau(g - g_0, -m) \ \leq \ 2dist(m, N(g, -g_0)),$$

for each $g \ \varepsilon \ G$ and $m \ \varepsilon \ M \ = \ \{t(x - g_0) : t \geq 0\}$ is a necessary and sufficient condition that

$$g_0 \ \varepsilon \ R'_G(x) \text{ respectively, } g_0 \ \varepsilon \ R_G(x)).$$

(3) For each linear subspace G of X, each $x \varepsilon \ X \backslash G$ and $g_0 \ \varepsilon \ G$, $\tau(g - g_0, m) - \tau(g - g_0, -m) \ \leq 2 \ dist.(m, N(g, -g_0))$ for each $g \varepsilon G$ and $m \ \varepsilon \ M \ = \{t(x - g_0) : t \varepsilon \Re\}$ is a necessary and sufficient condition that $g_0 \ \varepsilon \ R_G(x)(*)$.

(4) For each one dimensional subspace G of X, each $x \ \varepsilon \ X \backslash G$ and $g_0 \ \varepsilon \ G$, (*) is a necessary and sufficient condition that $g_0 \ \varepsilon \ R_G(x)$.

References

[1] D. Amir, and F. Deutsch, F., *Suns, moons and quasi-polyhedra*, J. Approx. Theory **6** (1972), 176 – 201.

[2] M.W. Bartelt, and H.W.McLaughlin, *Characterization of strong unicity in approximation theory*, J. Approx. Theory, **9** (1973), 255 – 266.

[3] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), 169 – 172.

[4] J. Blatter, P.D. Morris, and D.E. Wulbert, *Continuity of the set-valued metric projection*, Math. Ann. **178** (1968), 12 – 24.

[5] B. Brosowski, and F. Deutsch, *On geometric properties of suns*, J. Approx. Theory **10** (1974), 245 – 267.

[6] B. Brosowski, and F. Deutsch, *Radial continuity of set-valued metric projections*, J. Approx. Theory **11** (1974) 236 – 253.

[7] B. Brosowski, J.M. Lambert, and P.D. Morris, *Chebyshev sets which are not suns*, Math. Anal. **212** (1974 – 75), 89 – 101.

[8] F. Deutsch, and P. Kenderov, *Continuous selections and approximate selections for set-valued mappings and applications to metric projections*, SIAM J. Math. Anal. **14** (1983), 185 – 194.

[9] J.B. Diaz, and H.W. McLaughlin, *Simultaneous approximation and restricted Chebyshev centers in function spaces*, Approx. Th. & Appl., Proc. Workshop, Technion, Israel Inst. Tech., Haifa (1980), Academic Press, New York (1981).

[10] B. Dunham, *Chebyshev sets in $C[0,1]$ which are not suns*, Canad. Math. Bull. **18** (1975), (1), 35 – 37.

[11] N. Effimov and S.B. Steckin, in, *Some properties of Chebyshev sets*, Dokl. Akad. Nauk SSSR **118** (1958), 17 – 19.

[12] C. Franchetti and M. Furi, *Some characteristic properties of real Hilbert spaces*, Rev. Roumaine Math. Pures Appl. **17** (1972), 1045 – 1048.

[13] Garkavi, A.L., *The best possible net and the best possible cross-section of a set in a normed space*, Amer. Math. Soc. Translations, **39** (1964), 111 – 132.

[14] Geetha S. Rao and Bhaskaramurthi, T.L., *Non-expansive mappings and proximinality in normed almost linear spaces*, Proc.Int.Conf. on "Functional Analysis and Operator Theory", Ed. B.S.Yadav and B.Singh, Lecture Notes in Math. 1511, Springer-Verlag (1992), 80 – 87.

[15] Geetha S. Rao and T.L. Bhaskaramurthi, *Compact almost linear operators in normed almost linear spaces*, J. Math. Sci., U. N. Singh Memorial Volume, **28** (1994), 172 – 193.

[16] Geetha S. Rao and T.L. Bhakaramurthi, *Chebyshev centres, proximinality and farthest points in strong normed almost linear spaces*, Approx. Theory and its Appl. (1997), 99 – 111.

[17] Geetha S. Rao and K.R. Chandrasekaran, *Best coapproximation in normed linear spaces with property (Λ)*, Math Today **2** (1984), 33 – 46.

[18] Geetha S.Rao and K.R. Chandrasekaran, *Some radial continuity properties of the farthest point map*, Pure Appl. Math. Sci. **21** (1985), 71 – 74.

[19] Geetha S. Rao and K.R. Chandrasekaran, *The modulus of continuity of the set-valued cometric projection*, in "Methods of Functional Analysis in Approximation Theory", Eds. C.A. Micchelli, et.al., Birkhäuser Verlag, Basel (1986), 157 – 163.

[20] Geetha S. Rao and K.R. Chandrasekaran, *Some properties of the maps R_G and $R_{G'}$*, Pure Appl. Math. Sci. **23** (1986), 121 – 127.

[21] Geetha S. Rao and K.R. Chandrasekaran, *Characterizations of elements of best coapproximation in normed linear spaces*, Pure Appl. Math. Sci. **26** (1987), 139 – 147.

[22] Geetha S. Rao and S. Elumalai, *Semicontinuity properties of operators of strong best approximation and strong best coapproximation*, Proc. Int. Conf. on "Constructive Function Theory", Varna, Bulgaria (1981), 495 – 498.

[23] Geetha S. Rao and S. Elumalai, *Approximation and strong approximation in locally convex spaces*, Pure Appl. Math. Sci. **19** (1984), 13 – 26.

[24] Geetha S. Rao and S.A. Mariadoss, *Applications of fixed point theorems to best approximations*, Serdica Bulgaricae Math. Publ. **9** (1983), 243 – 248.

[25] Geetha S. Rao and S. Muthukumar, *Semicontinuity properties of the best coapproximation operator*, Math. Today **5** (1987), 37 – 48.

[26] Geetha S. Rao and M. Swaminathan, *Best coapproximation and Schauder Bases in Banach spaces*, **54** (1990), 339 – 354.

[27] Geetha S. Rao and R. Saravanan, *Best simultaneous coapproximation*, Indian J. Math. **40** (1998), 353 – 362.

[28] Geetha S. Rao and R. Saravanan, *Characteization of best uniform coapproximation*, Approx. Theory and its Appl. **15** (1999), 23 – 37.

[29] Geetha S. Rao and R. Saravanan, *Some results concerning best uniform coapproximation*, J. Inequalities in Pure and Appl. Math. Electronic Communication, **3 (2)**, 24, (2002), 1 – 13.

[30] G. Godini, *Semi-continuity properties of set-valued metric projections*, Rev. Roumaine Math. Pures. Appl. **18** (1973), 243 – 249.

[31] G. Godini, *Set-valued mappings*, Rev. Roumaine Math. Pures. Appl. **22** (1977), 53 – 67.

[32] G. Godini, *Geometrical properties of a class of Banach spaces including the spaces* c_0 *and* $L_p(1 \leq p < \infty)$, Math. Ann. **243** (1979), 197 – 212.

[33] G. Godini, *A framework for best simultaneous approximation : normed almost linear spaces*, J. Approx. Theory. **43** (1985), 338 – 358.

[34] G. Godini, G., *Operators in normed almost linear spaces*, Suppl. Rend. Circ. Mat. Palermo Serie **14** (1987) 309 – 328.

[35] R.B. Holmes, *A course on optimization and best approximation*, Lecture Notes in Maths, **257**, Springer-Verlag, N.Y. (1972).

[36] E. Kimchi and N. Richter-Dyn, *Restricted range approximation of* k *– convex functions in monotone norms*, SIAM J. Numerical Analysis **15** (1978), 1030 – 1038.

[37] G.G. Lorentz and K.L. Zeller, *Degree of approximation by monotone polynomials I*, J. Approx. Theory **1** (1968), 501 – 504.

[38] G.G. Lorentz and K.L. Zeller, *Degree of approximation by monotone polynomials II*, J. Approx. Theory **2** (1969), 265 – 269.

[39] G.G. Lorentz and K.L. Zeller, *Monotone approximation by algebraic polynomials*, Trans. Amer. Math. Soc. **149** (1970),1 – 18.

[40] R.A. Lorentz, *Uniqueness of best approximation by monotone Polynomials*, J. Approx. Theory **4** (1971), 401 – 408.

[41] J. Mach, *On the existence of best simultaneous approximation*,J. Approx. Theory **25** (1979) 258 – 265.

[42] J. Mach, *Continuity properties of Chebyshev centers*, J. Approx. Theory **29** (1980) 223 – 230.

[43] G. Nürnberger, *Unicity and strong unicity in approximation theory*, J. Approx. Th. **25** (1979), 54 – 70.

[44] G. Nürnberger and I. Singer, *Uniqueness and strong uniqueness of best approximation by spline subspaces and other subspaces*, J. Math. Anal. Appl. **90** (1982), 171 – 184.

[45] P.L. Papini, *Approximation and strong approximation in normed spaces via tangent functionals*, J. Approx. Th. **22** (1978), 111 – 118.

[46] P.L. Papini and I. Singer, *Best coapproximation in normed linear spaces*, Mh. Math. **88** (1979), 27 – 44.

[47] W. Pollul, Ph.D. thesis (1960).

[48] R.J. Roulier, *Negative theorems on monotone approximation*, Proc. Amer. Math. Soc. **55** (1976), 37 – 43.

[49] R.J. Roulier, *Polynomials of best approximation which are monotone*, J. Approx. Theory **9** (1973), 212 – 217.

[50] R.J. Roulier, *Some remarks on the degree of monotone approximation*, J. Approx. Theory **14** (1975), 225 – 22

[51] I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag (1970).

[52] G.D. Taylor, *On approximations by polynomials having restricted Ranges*, J. SIAM Numerical Analysis **5** (1968), 258 – 268.

[53] G.D. Taylor, *Approximation by functions having restricted ranges*, J. Math. Anal. Appl. **27** (1989), 241 – 248.

[54] M.A. Smith, *On the norms of metric projection*, J. Approx. Theory **31** (1981), 224 – 229.

[55] L.P. Vlasov, *Approximative properties of sets in normed linear spaces*, Russian Math. Surveys **28 (6)** (1973), 1 – 67.

[56] D.E. Wulbert, *Continuity of metric projections*, Trans. Amer. Math. Soc. **134** (1968), 335 – 341.

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DIFFERENTIATION, HOMOGENEOUS OPERATORS AND MATRIX VALUED CO-CYCLES FOR THE MÖBIUS GROUP

GADADHAR MISRA

Abstract: In this short note, we construct a class of functions $\mathfrak{d} : \text{Möb} \times \mathbb{D} \to \mathbb{C}^{k\times k}$ for which the co-cycle property is verified directly. It turns out that these are all the co-cycles for the natural action of the Möbius group on the unit disc. However, while this follows from the joint work of the author with A. Korányi, no elementary proof seem to be available.

Let G be a topological group and $\Omega \subseteq \mathbb{C}^m$ be a locally compact G-space, that is, there is a map $\tau : G \times \Omega \to \Omega$ such that the induced map $g \to \tau_g$, where $\tau_g(w) = \tau(g, w)$ is a continuous homomorphism. As is usual, we will let $g \cdot w$ denote the action $\tau(g, w)$.

Suppose that $U : G \to \mathcal{U}(\mathcal{H})$ is a representation of the group G on the group of unitary operators acting on some Hilbert space $\mathcal{H}$ and that $\varrho : \mathbf{C}(\Omega) \to \mathcal{L}(\mathcal{H})$ is a $*$ - homomorphism of the C^* - algebra of continuous functions $\mathbf{C}(\Omega)$ on the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators acting on the Hilbert space $\mathcal{H}$.

Definition 1. The triple (G, U, ϱ) is said to be a system of imprimitivity if

$$\varrho(g \cdot f) = U(g)^* \varrho(f) U(g),\ f \in \mathbf{C}(\Omega),\ g \in G,$$

where $(g \cdot f)(w) = f(g^{-1} \cdot w)$, $w \in \Omega$.

The notion of imprimitivity was introduced by Mackey (cf. [6]). In this short note, we discuss the case of the unit disc $\mathbb{D}$ and the Möbius group acting transitively on it.

Now, fix $G := G_0/\{\pm I\}$ to be the *Möbius group*, where

$$G_0 := SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : \mid a \mid^2 - \mid b \mid^2 = 1 \right\}$$

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and Ω be the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The group G acts on $\mathbb{D}$ resp. $\bar{\mathbb{D}}$, the closed unit disc by the rule:

$$\tau_\varphi : z \to \frac{az + b}{\bar{b}z + \bar{a}},\ z \in \mathbb{D},\ \varphi \in G.$$

Moreover, this action is transitive, that is, for $\alpha \in \mathbb{D}$, there exists a φ in G with $\varphi(\alpha) = 0$.

Let $L^2(\mathbb{D})$ is the Hilbert space of square integrable (with respect to the Lebesgue measure) functions on the unit disc $\mathbb{D}$ and M, R be multiplication and composition operators on $L^2(\mathbb{D})$ respectively. The map $U : G \to \mathcal{U}(L^2(\mathbb{D}))$ defined by $U_{\varphi^{-1}} = M_{\varphi'} R_\varphi$,, where φ' denotes the derivative of the holomorphic map τ_φ induced by $\varphi \in G$, is a unitary representation of the Möbius group G. Similarly, the map $\varrho : \mathbf{C}(\bar{\mathbb{D}}) \to \mathcal{L}(L^2(\mathbb{D}))$ defined by the formula $\varrho(f) = M_f$ is a $*$ - homomorphism. The triple (G, ϱ, U), with these choices forms a system of imprimitivity in the sense of Mackey.

It is not difficult to characterize all the imprimitivities based on the unit disc with respect to the Möbius group. These are induced by the multiplication operator on the $\oplus_{i=1}^m L^2(\mathbb{D})$. This is just the spectral theorem.

Now, we ask what if we start with a homomorphism of a function algebra rather than a $*$ - homomorphism of a C^* - algebra?

For instance, the multiplication operator on the Bergman space $\mathbb{A}^2(\mathbb{D})$ – the Hilbert space of square integrable holomorphic functions on the unit disc – defines a homomorphism of the disc algebra $\mathcal{A}(\mathbb{D})$, the algebra of continuous functions on $\bar{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$, by the formula $\varrho(f) = M_f$, f in $\mathcal{A}(\mathbb{D})$. As before, $U_{\varphi^{-1}} = M_{\varphi'} R_\varphi$ defines a unitary representation of the Möbius group on the Bergman space. The algebra homomorphism ϱ and the unitary representation U satisfy the imprimitivity relation.

Let $\mathcal{H}$ be a space of functions, say, on the unit disc or the unit circle. Suppose that the homomorphism $\varrho : \mathcal{A}(\mathbb{D}) \to \mathcal{L}(\mathcal{H})$ defined by the rule $\varrho(f) = M_f$, $f \in \mathcal{A}(\mathbb{D})$ is bounded.

Let $J : G \times \mathbb{D} \to \mathbb{C}$ be a *multiplier*, that is,

$$J_{\varphi\psi}(z) = J_\varphi(\psi(z))J_\psi(z),\ \varphi, \psi \in G,\ z \in \mathbb{D}.$$

Let $\mathcal{H}$ be a Hilbert space of functions on Ω. A representation U of the group G on the Hilbert space $\mathcal{H}$ which is of the form $U(\varphi^{-1}) = M_{J_\varphi} R_\varphi$, where M_{J_φ} is the multiplication by J_φ and R_φ is the composition by φ is said to be a *multiplier representation.*

If there is a multiplier representation, say U, of the group G on the Hilbert space $\mathcal{H}$ which is also unitary, then it is easily verified that (G, U, ϱ) is an imprimitivity:

$$\big(M_{J_\varphi} R_\varphi\big)^* \varrho(\varphi \cdot f)(M_{J_\varphi} R_\varphi) = \varrho(f).$$

A Borel map $U : G \to \mathcal{U}(\mathcal{H})$ is said to be a *projective* unitary representation of the group G if

$$U_{\varphi\psi} = c(\varphi, \psi) U_\varphi U_\psi,\ \varphi, \psi \in G$$

for some Borel function $c : G \times G \to \mathbb{T}$, $c(\varphi, e) = 1$, satisfying the cocycle identities:

$$c(\varphi_1, \varphi_2) c(\varphi_1\varphi_2, \varphi_3) = c(\varphi_1, \varphi_2\varphi_3) c(\varphi_2, \varphi_3),\ \varphi_1, \varphi_2, \varphi_3 \in G.$$

The projective unitary representations of G lift to unitary representations of the universal cover $\tilde{G}$ which are well-known [1, Theorem 3.2]. All of them are multiplier representations!

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be *homogeneous* if its spectrum is contained in the closed unit disc and for every Möbius transformation φ, the operator $\varphi(T)$ (defined via the usual holomorphic functional calculus) is unitarily equivalent to T. If T is a contraction, then the homomorphism $p \to p(T)$ is contractive via the vonNeumann inequality, that is, $\|p(T)\| \leq \|p\|_\infty$. Therefore, it extends to a contractive homomorphism ϱ_T of the disc algebra $\mathcal{A}(\mathbb{D})$.

To every homogeneous irreducible operator T there corresponds an *associated projective unitary representation* U [1, Theorem 4.1]of the Möbius group G:

$$U(\varphi)^* \, \varrho(f \cdot \varphi) \, U(\varphi) = \varrho(f),\ \varphi \in G,\ f \in \mathcal{A}(\mathbb{D}).$$

Thus the homomorphism ϱ_T corresponding to an irreducible homogeneous operators T defines an imprimitivity via the *associated* projective unitary representation U of the Möbius group G.

Let $\mathcal{H}$ be a space of functions, say, on the unit disc or the unit circle. Suppose the operator T defined by the rule $(Tf)(x) = xf(x)$, $f \in \mathcal{H}$ is bounded.

If there is a multiplier representation, say U, of the group G on the Hilbert space $\mathcal{H}$, then the operator T is homogeneous and U is the associated representation.

Recall that the multiplier identity for $J : G \times \mathbb{D} \to \mathbb{C}$ with $J_\varphi(z) = \varphi'(z)$ is nothing but the familiar chain rule.

Let us also emphasize that the map $U : G \to \mathcal{O}(\mathbb{D})$, where $\mathcal{O}(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$, defined by the rule $\big(U(\varphi^{-1})f\big)(z) = J(\varphi, z) f(\varphi(z))$ is a homomorphism only if J satisfies the multiplier identity.

Clearly, any power of the derivative $J_\varphi^{(\lambda)}(z) := (\varphi')^\lambda(z)$, $\lambda \in \mathbb{R}$ is well-defined since the derivative φ' is non-vanishing on the unit disc, which is simply connected. Thus $J_\varphi^{(\lambda)}(z)$ will continue to obey the multiplier identity. It is known that these are all the possible complex valued holomorphic multipliers for the Möbius group. We will give a somewhat amusing proof of this fact based on operator theory.

Let $\Omega \subseteq \mathbb{C}$ be a bounded open connected set. In the seventies, Cowen and Douglas [3]introduced the class $B_n(\Omega)$ consisting of those operators T with the property that (i) $T - w$ is onto for $w \in \Omega$, (ii) kernel of $T - w$ is of the fixed dimension $n \in \mathbb{N}$ for $w \in \Omega$ and (iii) span of $\ker(T - w)$, $w \in \Omega$ is dense.

For T in $B_n(\Omega)$, let E_T be the holomorphic Hermitian sub-bundle of the trivial bundle $\Omega \times \mathcal{H}$ defined by

$$\{(w, x) \in \Omega \times \mathcal{H} : x \in \ker(T - w)\}$$

with the natural projection map $\pi : E_T \to \Omega$, $\pi(w, x) = w$. Among other things, Cowen and Douglas show that the unitary equivalence class of the operator T and the class of holomorphic Hermitan vector bundle E_T determine each other. For $T \in B_1(\Omega)$, the curvature

$$\mathcal{K}_T(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log \|\gamma_w\|^2, \; w \in \Omega,$$

where γ_w is an holomorphic determination of the eigenvectors of T, is then seen to be a complete unitary invariant for T since it determines the class of the holomorphic Hermitian line bundle E_T as shown below.

Suppose that E is a holomorphic Hermitian line bundle over a bounded domain $\Omega \subseteq \mathbb{C}^m$. Pick a holomorphic frame γ for the line bundle E and let $\Gamma(w) = \langle \gamma_w \rangle \gamma_w$ be the Hermitian metric. The curvature $\mathcal{K}_E$ is a $(1,1)$ form given by the formula $\mathcal{K}_E(w) = \sum_{i,j=1}^m (\partial_i \bar{\partial}_j \log \Gamma)(w) dz_i \wedge d\bar{z}_j$, for $w \in \Omega$. Clearly, in this case, the $(1,1)$ form $\mathcal{K}(w) \equiv 0$ on Ω if and only if the coefficient matrix $(\partial_i \bar{\partial}_j \log \Gamma)(w) \equiv 0$ implying $\log \Gamma$ is harmonic on Ω. Let F be a second line bundle over the same domain Ω with the metric Λ with respect to a holomorphic frame η. Suppose that the two curvatures $\mathcal{K}_E$ and $\mathcal{K}_F$ are equal. It then follows that $u = \log(\Gamma/\Lambda)$ is harmonic on Ω and thus there exists a harmonic conjugate v of u on any simply connected open subset Ω_0 of Ω. For $w \in \Omega_0$, define $\tilde{\eta}_w = e^{(u(w)+iv(w))/2}\eta_w$. Then clearly, $\tilde{\eta}_w$ is a new holomorphic frame for F, which we can use without loss of generality. Consequently, we have the metric $\Lambda(w) = \langle \tilde{\eta}_w \rangle \tilde{\eta}_w$ for F and we see that

$$\begin{aligned}
\Lambda(w) &= \langle \tilde{\eta}_w \rangle \tilde{\eta}_w \\
&= \langle e^{(u(w)+iv(w))/2} \eta_w e^{(u(w)+iv(w))/2} \eta_w \rangle \\
&= e^{u(w)} \langle \eta_w \eta_w \rangle \\
&= \Gamma(w).
\end{aligned}$$

This calculation shows that the map $\eta_w \mapsto \gamma_w$ defines an isometric holomorphic bundle map between E and F.

Let $\mathcal{H}$ be a space consisting of holomorphic functions on the unit disc $\mathbb{D}$. Suppose, for each $w \in \mathbb{D}$, there exists a vector K_w in $\mathcal{H}$ with the reproducing property:

$$\langle f K_w \rangle = f(w),\, f \in \mathcal{H}.$$

Then K is said to be the reproducing kernel for $\mathcal{H}$. Clearly, K is positive definite – for any $n \in \mathbb{N}$, the $n \times n$ matrix $\left(\!\left(K_{w_i}(w_j) \right)\!\right)$ is positive definite for an arbitrary choice of w_i, $1 \leq i \leq n$. Conversely, given a positive definite kernel K, let $\mathcal{H}_0$ be linear space of consisting of span of the vectors K_w, w in $\mathbb{D}$. Define $\langle K_u K_w \rangle$ to be $K_u(w)$ on $\mathcal{H}_0$ and complete it to obtain a Hilbert space $\mathcal{H}$ consisting of holomorphic functions on $\mathbb{D}$. This Hilbert space then possess the reproducing property with respect to the kernel K.

If the multiplication (by the co-ordinate function) M on the Hilbert space $\mathcal{H}$ possessing a reproducing kernel K is also bounded then it is easy to verify that $\mathcal{M}^* K_w = \bar{w} K_w$. Thus $\gamma(\bar{w}) = K_w$, $w \in \mathbb{D}$ is a holomorphic determination of the holomorphic eigenvectors for the operator M^*. All the operators T in the Cowen-Douglas class $B_1(\mathbb{D})$ are determined by some choice of a positive definite kernel K on $\mathbb{D}$ (cf. [3]). We recall, from [7], the following Theorem and its proof.

Theorem 1. *For T in $\mathrm{B}_1(\mathbb{D})$ with $\varphi_\alpha(T) \cong T$, $\alpha \in \mathbb{D}$, we have*

$$\mathcal{K}_T(\alpha) = (1 - |\alpha|^2)^{-2} \mathcal{K}_T(0),$$

where $-\lambda = \mathcal{K}_T(0) < 0$ is arbitrary.

Proof. Let φ be a bi-holomorphic automorphic of the unit disc $\mathbb{D}$. Consider the operator $\varphi(T)$ obtained via the usual holomorphic functional calculus. (Here, for simplicity, assume the spectrum $\sigma(T)$ of the operator T is a subset of the closed unit disc $\overline{\mathbb{D}}$.) Now,

$$\begin{aligned}
\varphi(T)_{|\ker(\varphi(T)-\varphi(w))^2} &= \varphi\big(T_{|\ker(T-w)^2}\big) \\
&= \varphi \begin{pmatrix} w & h_T(w) \\ 0 & w \end{pmatrix} \\
&= \begin{pmatrix} \varphi(w) & \varphi'(w) h_T(w) \\ 0 & \varphi(w) \end{pmatrix} \\
&\cong \begin{pmatrix} \varphi(w) & |\varphi'(w)| h_T(w) \\ 0 & \varphi(w) \end{pmatrix}.
\end{aligned}$$

Theerfore, it follows that

$$\begin{aligned}
\big(- \mathcal{K}_{\varphi(T)}(\varphi(w)) \big)^{-1/2} &= h_{\varphi(T)}(\varphi(w)) \\
&= |\varphi'(w)| h_T(w) \\
&= |\varphi'(w)| \big(- \mathcal{K}_T(w) \big)^{-1/2}.
\end{aligned}$$

This is really a "change of variable formula for the curvature", which can be obtained directly using the chain rule.

Put $w = 0$, choose $\varphi = \varphi_\alpha$ such that $\varphi(0) = \alpha$. In particular, take $\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$. Then

$$\begin{aligned}\mathcal{K}_{\varphi_\alpha(T)}(\alpha) &= \mathcal{K}_{\varphi_\alpha(T)}(\varphi_\alpha(0)) \\ &= |\varphi'(0)|^{-2}\mathcal{K}_T(0) \\ &= (1 - |\alpha|^2)^{-2}\mathcal{K}_T(0),\ \alpha \in \mathbb{D}. \qquad (1)\end{aligned}$$

Now, suppose $\varphi_\alpha(T) \cong T$ for all φ_α, $\alpha \in \mathbb{D}$. Then $K_{\varphi_\alpha(T)}(w) = \mathcal{K}_T(w)$ for all $w \in \mathbb{D}$. Hence

$$\begin{aligned}(1 - |\alpha|^2)^{-2}\mathcal{K}_T(0) &= K_{\varphi_\alpha(T)}(\varphi_\alpha(0)) \\ &= K_T(\alpha),\ \alpha \in \mathbb{D}. \qquad (2)\end{aligned}$$

(Here the first equality is the change of variable formula given in (1) and the second equality is the statement that unitarily equivalent operators have the same curvature.) □

We note that the corresponding metric γ_E for the bundle E_T is determined upto the absolute value of a holomorphic multiplier and is given by the formula:

$$\gamma_E(\alpha, \bar{\alpha}) = (1 - |\alpha|^2)^{-\lambda},\ \lambda > 0.$$

This corresponds to the reproducing kernel K (obtained via polarization from γ_E):

$$K(\alpha, \beta) = B(\alpha, \beta)^{\frac{\lambda}{2}} = (1 - \alpha\bar{\beta})^{-\lambda},\ \lambda > 0;\ \alpha, \beta \in \mathbb{D},$$

Here $B(\alpha, \beta) = (1 - \alpha\bar{\beta})^{-2}$ is the Bergman kernel for the unit disc $\mathbb{D}$.

The adjoint of the multiplication operator M on the Hilbert space $\mathcal{H}^{(\lambda)}$ corresponding to the reproducing kernel $B^{\frac{\lambda}{2}}$ is a weighted Bergman operator for $\lambda > 1$. The measure in this case is

$$d\mu(z) = (1 - |z|^2)^{\lambda - 2} dz d\bar{z}.$$

For $\lambda = 1$, this operator is the usual shift on the Hardy space. However, for $\lambda < 1$, there is no such measure. The corresponding operator is a shift which is not a contraction and is not subnormal.

Assume that $\mathcal{H}$ is a Hilbert space of holomorphic functions on the unit disc $\mathbb{D}$ possessing a reproducing kernel K. Let $J : G \times \mathbb{D} \to \mathbb{C}$ be a holomophic for fixed g in G. Let $U : G \to \mathcal{L}(\mathcal{H})$ be the map induced by J by the formula $U(\varphi^{-1})(f) = J_\varphi\,(f \circ \varphi)$.

For the map U to be a homomorphism, it is necessary and sufficient that

$$J(\varphi\psi, z) = J(\varphi, z) J(\psi, z \cdot \varphi),\ \varphi, \psi \in G.$$

Define a kernel K by polarizing

$$K_u(u) = J(u, \varphi_u) K_0(0) \overline{J(u, \varphi_u)},\ \varphi \in G,\ u \in \mathbb{D},$$

for every choice of a positive constant $K_0(0)$. Here φ_u is the involutive automorphism in G taking u to 0. Further more, since $U_\varphi^* = U_{\varphi^{-1}}$, the homomorphism U_φ, φ in G, is unitary if and only if $\langle U_\varphi K_u K_w \rangle = \langle K_u U_{\varphi^{-1}} K_w \rangle$, $u, w \in \mathbb{D}$. Hence

$$U_\varphi f(z) = J(\varphi^{-1}, z) f(\varphi^{-1}(z)),\ f \in \mathcal{H},\ \varphi \in G,$$

is easily seen to be a unitary representation of G on the Hilbert space $\mathcal{H}$ determined by the kernel K, whenever it is positive definite.

Theorem 2. *Suppose $J : G \times \mathbb{D} \to \mathbb{C}$ is a holomorphic co-cycle. Then $J(\varphi, z) = (\varphi')^\lambda(z)$, $\lambda \in \mathbb{R}$.*

Proof. Let J be a holomorphic cocycle. Define $K(z, z) = |J(\varphi_z, z)|^2 K(0, 0)$. Assume that the kernel K obtained from $K(z, z)$ via polarization is positive definite on $\mathbb{D} \times \mathbb{D}$. Suppose that the multiplication operator M on the corresponding Hilbert space $\mathcal{H}$ is bounded and M^* is in $B_1(\mathbb{D})$. Then $U(\varphi) : f \mapsto J(\varphi, \cdot) f \circ \varphi$ defines a homomorphism of G into the unitary operators on $\mathcal{H}$. What is more, it is easy to see that $U(\varphi)$ intertwines M with $\varphi(M)$ (cf. [1, theorem 5.1]). Thus the operator M is homogeneous and so is its adjoint M^* which is assumed to be in $B_1(\mathbb{D})$. Therefore, revisiting the proof leading up to the characterization of the homogeneous operators in $B_1(\mathbb{D})$, one sees that $J(\varphi, z)$ must be of the form $(\varphi')^\lambda(z)$, $\lambda \in \mathbb{R}$. □

How do we construct multipliers taking values, say, in $n \times n$ matrices?

Assume that we have found a derivation $\mathfrak{d} : G \times \mathbb{D} \to \mathbb{C}^{n \times n}$ satisfying the multiplier identity, that is,

$$\mathfrak{d}(gh, z) = \mathfrak{d}(g, h(z)) \mathfrak{d}(h, z),\ g, h \in G,\ z \in \mathbb{D}.$$

For $\lambda \in \mathbb{R}$, define the map $\Gamma : G \to \mathcal{E}(\mathcal{O}(\mathbb{D}, \mathbb{C}^n))$ by the rule

$$\big(\Gamma(g^{-1}) f\big)(z) = J_g^{(\lambda)}(z) \mathfrak{d}(g, z) f(g(z)),\ f \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n),\ g \in G.$$

Not only Γ is a homomorphism but any homomorphism must be of this form. It would be therefore desirable to find all the possible derivations $\mathfrak{d}$.

Let $D(g, z)$ be the diagonal matrix whose (j, j) entry is $J^{(-m+j)}(g, z) I_{d_j}$, $0 < j \le m$, $d_0 + \cdots + d_m = n$.

Also, the ratio

$$-\frac{1}{2} \frac{g''(z)}{(g'(z))^{\frac{3}{2}}},\ g \in G,\ z \in \mathbb{D}$$

is independent of z, which we denote by b_g.

Let $Y_i : \mathbb{C}^{d_j} \to \mathbb{C}^{d_{j+1}}$ be a set of m linear transformations and Y be the corresponding shift operator on $\mathbb{C}^n$.

Theorem 3. *The multiplier identity holds for the derivation $\mathfrak{d}$ defined by the rule:*

$$(g,z) \mapsto J^{(\lambda)}(g,z) D(g,z)^{\frac{1}{2}} \exp(-b_g Y) D(g,z)^{\frac{1}{2}},\ g \in G,\ z \in \mathbb{D}.$$

Proof: It is easy to verify that $D(g_1 g_2, z) = D(g_1, g_2(z)) D(g_2, z)$ using the chain rule. Now,

$$\mathfrak{d}(g_1 g_2, z) = D(g_1 g_2, z)^{\frac{1}{2}} \exp(-b_{g_1 g_2} Y) D(g_1 g_2, z)^{\frac{1}{2}}.$$

However, we have

$$\begin{aligned} -b_{g_1 g_2} &= \frac{1}{2} \frac{(g_1 g_2)''(z)}{\big((g_1 g_2)'(z)\big)^{3/2}} \\ &= \frac{1}{2} \frac{\big(g_1'(g_2(z)) g_2'(z)\big)'}{\big(g_1'(g_2(z)) g_2'(z)\big)^{3/2}} \\ &= \frac{1}{2} \frac{g_1''(g_2(z))(g_2'(z))^2 + g_1'(g_2(z)) g_2''(z)}{\big(g_1'(g_2(z)) g_2'(z)\big)^{3/2}} \end{aligned}$$

$$= \frac{1}{2}\Big\{ \frac{g_1''(g_2(z))}{(g_1'(g_2(z)))^{3/2}} g_2'(z)^{1/2} + \frac{g_2''(z)}{(g_2'(z))^{3/2}} (g_1'(g_2(z)))^{-1/2} \Big\}$$

$$= -b_{g_1}(g_2'(z))^{1/2} - b_{g_2}(g_1'(g_2(z)))^{-1/2}.$$

$$\begin{aligned} &\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z) \\ &= D(g_1, g_2(z))^{\frac{1}{2}} \exp(-b_{g_1} Y)) D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} \exp(-b_{g_2} Y) D(g_2, z)^{\frac{1}{2}} \\ &= D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} \exp\big(-b_{g_1} D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2}\big) \\ &\qquad \exp\big(-b_{g_2} D(g_1, g_2(z))^{1/2} Y D(g_1, g_2(z))^{-1/2}\big) D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} \\ &= D(g_1 g_2, z)^{\frac{1}{2}} \exp\big(-b_{g_1}(g_2'(z))^{1/2} Y - b_{g_2} g_1'(g_2(z))^{-1/2} Y\big) D(g_1 g_2, z)^{\frac{1}{2}} \\ &= D(g_1 g_2, z)^{\frac{1}{2}} \exp\big((-b_{g_1}(g_2'(z))^{1/2} - b_{g_2} g_1'(g_2(z))^{-1/2}) Y\big) D(g_1 g_2, z)^{\frac{1}{2}} \\ &= D(g_1 g_2, z)^{\frac{1}{2}} \exp\big(-b_{g_1 g_2} Y\big) D(g_1 g_2, z)^{\frac{1}{2}} \\ &= \mathfrak{d}(g_1 g_2, z) \end{aligned}$$

□

Although it is not stated explicitly, it follows, from [5], that $\mathfrak{d}$ of the Theorem describes all possible multipliers for the Möbius group taking values in the $n \times n$ matrices for suitable choice of $Y_1, \ldots, Y_m$ and $\lambda \in \mathbb{R}$.

References

[1] B. Bagchi and G. Misra, *Homogeneous operators and the projective representations of the Möbius group: a survey*, Proc. Indian Acad. Sci. Math. Sci **111** (2001), no. 4, 415 – 437.

[2] B. Bagchi and G. Misra, *The homogeneous shifts*, J. Funct. Anal. **204**(2003), 293 – 319.

[3] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory,* Acta Math. 141 (1978), 187–261.

[4] A. Korányi and G. Misra, *Homogeneous operators on Hilbert spaces of holomorphic functions*, J. Funct. Anal.**254** (2008), 2419 – 2436.

[5] A. Korányi and G. Misra, *A classification of homogeneous operators in the Cowen-Douglas class*, Adv. Math., **226** (2011) 5338 – 5360.

[6] G. W. Mackey, *The theory of unitary group representations*, Chicago University Press, 1976.

[7] G. Misra, *Curvature and the backward shift operator,* Proc. Amer. Math. Soc., **91** (1984), 105 – 107.

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REDUCTION AND SUMMATION FORMULAE FOR BASIC AND POLYBASIC HYPERGEOMETRIC SERIES OF TWO VARIABLES

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Abstract: In this paper, making use of a known identity and summation formulae for truncated q-series, an attempt has been made to establish certain reduction and summation formulae for basic and polybasic hypergeometric series of two variables.

1. Introduction

For real or complex q with $q < 1$, put

$$[\lambda; q]_\infty = \prod_{n=0}^{\infty}(1 - \lambda q^n)$$

and

$$[\lambda; q]_\mu = \frac{[\lambda; q]_\infty}{[\lambda q^n; q]_\infty} \text{ for arbitrary parameters } \lambda \text{ and } \mu \text{ so that}$$

$$[\lambda; q]_n = \begin{cases} 1, & n = 0 \\ (1-\lambda)(1-\lambda q)\dots\dots\dots & (1-\lambda q^{n-1}) \text{ with } n = 1, 2, 3\dots\dots\dots \end{cases}$$

and

$${}_r\Phi_s\left[\begin{matrix} a_1, a_2\dots\dots\dots, a_r; q; z \\ b_1, b_2, \dots\dots\dots, b_s \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2\dots\dots\dots, a_r; q]_n z^n}{[q, b_1, b_2, \dots\dots\dots, b_s; q]_n}, \qquad (1.1)$$

where $[a_1, a_2\dots\dots\dots, a_r; q] = [a_1; q]_n\ [a_2; q]_n \dots\ \dots [a_r; q]_n$ and $\max\{|z|, |q|\} > 1$ for convergence.

A truncated basic hypergeometric series is defined as :

$${}_r\Phi_s\left[\begin{matrix} a_1, a_2\dots\dots\dots, a_r; q; z \\ b_1, b_2, \dots\dots\dots, b_s \end{matrix}\right] = \sum_{n=0}^{N} \frac{[a_1, a_2\dots\dots\dots, a_r; q]_n z^n}{[q, b_1, b_2, \dots\dots\dots, b_s; q]_n}, \qquad (1.2)$$

The poly basic hypergeometric series of one variable is defined as

$$\Phi\left[\begin{matrix} a_1, a_2\dots\dots\dots, a_r; c_{1,r_1}; \dots; c_{m,1}, \dots, c_{m,r_n}, q, q_1, \dots, q_m; z \\ b_1, b_2, \dots, b_s;\ d_{1.1}, \dots, d_{i,s_1}; \dots; \dots, d_{m,1}; \dots, d_{m,s_m} \end{matrix}\right]$$

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$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2 \ldots\ldots, a_r; q]_n z^n}{[q, b_1, b_2, \ldots\ldots, b_s; q]_n} \prod_{j=1}^{m} \frac{[c_j, \ldots, c_{j,1}; q_j]_n}{[d_j, \ldots, d_{j,1}; q_j]_n} \tag{1.3}$$

where $\max\{|z|, |q|, |q_1|, \ldots\ldots\ldots, |q_m|\} < 1$ for convergence.

Basic hypergeometric series of two variables is defined as (refer [3]):

$$\Phi \begin{bmatrix} (a); (c); (b); q; x, y \\ : (c); (d); (d); ijk \end{bmatrix} =$$

$$\sum_{m,n=0}^{\infty} \frac{\prod_{t=1}^{A} [a_{t;q}]_{m+n}, \prod_{t=1}^{B} [b_t; q]_m, \prod_{t=1}^{B'} [b'_{t;q}]_n x^m y^n,}{\prod_{t=1}^{C} [c_t; q]_{m+n}, \prod_{t=1}^{D} [d_t; q]_m, \prod_{t=1}^{D'} [d'_t; q]_n [q; q]_m [q; q]_n}$$
$$\times q^{im(m-1)/2 + jn(n-1)/2 + kmn} \tag{1.4}$$

where $|x| < 1, |y| < 1$ and i, j, k positive integers.

We shall be in need of the following summation of truncated series (refer [1])

$${}_2\Phi_1 \begin{bmatrix} x, y, q; q \\ xyq \end{bmatrix}_n = \frac{[xq, yq, q]_0}{[q, xyq; q]_0}, \tag{1.5}$$

$${}_4\Phi_3 \begin{bmatrix} \alpha,\ q\sqrt{\alpha}, & -q\sqrt{\alpha}, e; q; 1/e \\ \sqrt{\alpha}\ -\sqrt{\alpha} & \alpha q/e \end{bmatrix} = \frac{[\alpha q, eq; q]_n}{[q, \alpha q/e; q]_n e^n} \tag{1.6}$$

$${}_6\Phi_5 \begin{bmatrix} \alpha,\ q\sqrt{\alpha}; & -q\sqrt{\alpha}, \beta\gamma, \delta, q; q \\ \sqrt{\alpha},\ -\sqrt{\alpha} & \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta \end{bmatrix} = \frac{[\alpha q, \beta q, \gamma q, \delta q; q]_n}{[q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q]_n} \tag{1.7}$$

where $\alpha = \beta\gamma\delta$.

$$\sum_{k=0}^{n} \frac{(1 - \alpha p^k q^k)[\alpha; p]_k [\beta, q]_k \beta^{-k}}{(1 - \alpha)[q; q]_k [\alpha p/\beta; p]_k} = \frac{[\alpha p; p]_n [\beta q; q]_n}{[q; q]_n [\alpha p/\beta; p]_n} \beta^{-n}. \tag{1.8}$$

$$\sum_{k=0}^{n} \frac{(1 - \alpha p^k q^k)(1 - \beta p^k q^k)[\alpha, \beta; p]_k [\gamma, \alpha/\beta\gamma, q]_k q^k}{(1 - \alpha)}$$
$$\times (1 - \beta)[q, \alpha q/\beta; q]_k [\alpha p/\gamma, \beta\gamma p; p]_k = \frac{[\alpha p\beta p; p]_n [\gamma q; \alpha q/\beta\gamma, q]_n}{[q; \alpha q/\beta]_n [\alpha p/\gamma; \beta\gamma p]_n} \tag{1.9}$$

We shall also make use of the following identity :

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(n, k) = \sum_{n,k=0}^{\infty} A(n + k, k). \tag{1.10}$$

2. **Main results**

In this section we shall establish our main reduction formulae.

(I) Multiplying both sides of (1.5) by $\Omega_n Z_n$, summing over n from 0 to ∞ and then applying the identity (1.10) we get :

$$\sum_{r,n=0}^{\infty} \frac{[x,y;q]_r q^r}{[q,xyq;q]_r} \Omega_{r+n} Z^{r+n} = \sum_{n=0}^{\infty} frac[zq,yq;q]_n \Omega_n Z^n [q,xyq;q]_n \tag{2.1}$$

(II) Multiplying both sides of (1.6) by $\Omega_n Z_n$, summing over n from 0 to ∞ and then applying the identity (1.10) we find :

$$\sum_{r,n=0}^{\infty} \frac{[\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},e;q]_r}{[q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/e]_r e^r} \Omega_{r+n} Z^{r+n} = \sum_{n=0}^{\infty} \frac{[\alpha q,eq;q]_n}{[q,\alpha qe;q]_r} \Omega_n (z/e)^n \tag{2.2}$$

(III) Similarly, (1.7) yields

$$\sum_{r,n=0}^{\infty} \frac{[\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},\beta,\gamma,\delta;q]_r q^r}{[q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q]_r} \Omega_{r+n} Z^{r+n}$$

$$= \sum_{n=0}^{\infty} \frac{[\alpha q,\beta q,\gamma q,\delta q;q]_n}{[q,\sqrt{\alpha},\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q]_r} \Omega_n Z^n, \tag{2.3}$$

where $\alpha = \beta\gamma\delta$.

(IV) Similarly, (1.8) yields

$$\sum_{n,k=0}^{\infty} \frac{[\alpha,p,q;pq]_k\,[\alpha;p]_k\,[\beta;q]_k}{[\alpha;pq]_k\,[q;q]_k\,[\alpha p/\beta;p]_k} \Omega_{n+k} \frac{Z^{n+k}}{\beta^k}$$

$$= \sum_{n=0}^{\infty} \frac{[\alpha p;p]_n\,[\beta q;q]_n}{[q;q]_n\,[\alpha p/\beta;p]_n} \Omega_n \left(\frac{Z}{\beta}\right)^n. \tag{2.4}$$

(V) Similarly, (1.9) yields

$$\sum_{n,k=0}^{\infty} \frac{[\alpha pq;pq]_k\,[\beta p/q;p/q]_k\,[\alpha,\beta;p]_k\,[\gamma,\alpha/\beta\gamma,q]_k\,q^k}{[\alpha;pq]_k\,[\beta;q/q]_k\,[q,\alpha q/\beta;q]_k\,[\alpha p/\gamma,\beta\gamma p;p]_k} \Omega_{n+k} Z^{n+k}$$

$$= \sum_{n=0}^{\infty} \frac{[\alpha p,\beta p;p]_n\,[\beta q;q]_n}{[q;\alpha q/\beta;q]_n\,[\alpha p/\gamma,\beta\gamma p;p]_n} \Omega_n Z^n. \tag{2.5}$$

3. **Applications**

(i) Taking $\Omega_r = 1$ in (2.1) we get the transformation,

$$_2\Phi_1[x,y;xyq;q;zq] - (1-z)_2\Phi_1[xq,yq;xyq;q;z], |z| < 1. \tag{3.1}$$

(ii) Choosing $\Omega_r = frac[\alpha_1,\alpha_2,\alpha_3,\ldots\ \ldots,\alpha_r;q]\beta_1,\beta_2,\beta_s;q$ in (2.1) we have :

$$\Phi\begin{bmatrix}\alpha_1,\alpha_2,\ldots,\alpha_r:q; & x, & y;q;z,zq\\ \beta_1,\beta_2,\ldots,\beta_s:-; & xyq & \end{bmatrix}$$

$$=_{r+1}\Phi_{s+1}\begin{bmatrix}\alpha_1,\alpha_2,\ldots,\alpha_r, & xq; & yq;q;z\\ \beta_1,\beta_2,\ldots,\beta_s: & xyq & \end{bmatrix}, |x| < 1. \tag{3.2}$$

(iii) Taking $r=1, s=2, \alpha_1 = xyq, \beta_1 = xq$, in (3.2) we get

$$\Phi\begin{bmatrix} xyq & : & q; & x, & y; & q; & z, & zq \\ xq & yq & :--; & xyq \end{bmatrix} = \frac{1}{[z;q]_\infty} \tag{3.3}$$

(iv) if we take $r=1, s=1, \alpha_1 = xyq, \beta_1 = yq$, in (3.2) we have

$$\Phi\begin{bmatrix} xyq & :q; & x, & y; & q; & z, & zq \\ yq & :-; & xyq \end{bmatrix} = \frac{[xzq;q]_\infty}{[z;q]_\infty} \tag{3.4}$$

(v) Taking $r=1, s=1, \alpha_1 = xyq, \beta_1 = xyq3$ and $z=q$ in (3.2) we find

$$\Phi\begin{bmatrix} xyq; q; x, y; q; q, q^2 \\ xyq^3; \ -; xyq \end{bmatrix} = \frac{[xq^2, yq^2; q]_\infty}{[q, xyq^3; q]_\infty} \tag{3.5}$$

(vi) Choosing $r=1, s=1, \alpha_1 = q^{-m}, \beta_1 = yq$ and $z=q$ in (3.2) we find :

$$\Phi\begin{bmatrix} q^{-m} & : & q; & x, y; & q: & q, & q^2 \\ yq^3 & : & -; & xyq \end{bmatrix} = \frac{[y;q]_m\,(xq)_m}{[x,y,q;q]\infty} \tag{3.6}$$

(vii) If we take $\Omega_t = 1$ in (2.2) we have the following transformation:

$$ {}_4\Phi_3\begin{bmatrix} \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha} & e; & q; & z/e \\ & \sqrt{\alpha}, & -\sqrt{\alpha} & \alpha q/e \end{bmatrix}$$

$$= (1-x)_2\Phi_1\begin{bmatrix} \alpha q, & eq; & q; & z/e \\ \alpha q/e \end{bmatrix},$$

(viii) Choosing $\Omega_r = \frac{[\alpha q/e, q]_r}{[eq;q]_r}$ in (1.2) we have

$$\Phi\begin{bmatrix} \alpha q/e & :q; & \alpha, q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q; & z, & z/e \\ \alpha q, eq & :-; & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e \end{bmatrix}$$

$$= \frac{[\alpha z q/e;q]_\infty}{[z/e;q]_\infty} \tag{3.7}$$

(ix) Taking $\Omega_r = \frac{[\alpha q/e;q]_r}{[\alpha z q^2;q]_r}$ in (1.2)we obtain the following summation formulae

$$\Phi\begin{bmatrix} \alpha q/e & :q; & \alpha, q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q; & z, & z/e \\ \alpha q, eq & :-; & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e \end{bmatrix}$$

$$= \frac{[zq/e;q]_\infty}{[\alpha z q^2 z/e;q]_\infty} \tag{3.8}$$

(x) Putting $\Omega_r = \frac{[\alpha q/e;q]_r}{[\alpha p, eq;q]_r}$ in (1.2) we get

$$\Phi\begin{bmatrix} \alpha q/e & :q; & \alpha, q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e: & q; & z & z/e \\ \alpha q, eq; & :-; & \sqrt{\alpha} & -\sqrt{\alpha} & \alpha q/e \end{bmatrix} = \frac{1}{[z/e;q]_\infty} \tag{3.9}$$

(xi) If we take $\Omega_r = 1$ i (1.3) we get the following transformation:

$$
{}_6\Phi_5\left[\begin{matrix}\alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma & \delta, & q; & zq\\ & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta, & & \end{matrix}\right]
$$

$$
= (1-z)\,{}_4\Phi_3\left[\begin{matrix}\alpha q, & \beta q; & \gamma q, & \delta q, & q; & z\\ \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & & \end{matrix}\right] \tag{3.10}
$$

where $\alpha = \beta\gamma\delta$

(xii) Taking $\Omega_r = 1$ in (1.1) we get

$$
\Phi\left[\begin{matrix}\beta, & \alpha pq; & \alpha, & q, & pq, & p; & z/\beta\\ -; & \alpha; & \alpha p/\beta; & & & & \end{matrix}\right]
$$

$$
= (1-z)\Phi\left[\begin{matrix}\beta q; & \alpha p; & q, & p; & z/\beta\\ -; & \alpha p/\beta; & & & \end{matrix}\right] \tag{3.11}
$$

(xiii) Choosing $\Omega_r = \frac{[\alpha q/\beta;p]_r}{[\alpha p;p]_r\,[\beta q;q]_r}$ in (1.4) we have:

$$
\sum_{n,k=0}^{\infty}\frac{[\alpha p/\beta;p]_{n+k}\,[\alpha pq;pq]_k\,[\alpha;p]_k\,[\beta;q]_k\,z^{n+k}}{[\alpha p;p]_{n+k}\,[\beta q;q]_{n+k}\,[\alpha;pq]_k\,[\alpha p/\beta;p]_k\,[q;q]_k\,\beta^k} = \frac{1}{[z/\beta;q]_\infty} \tag{3.12}
$$

(xiv) Taking $\Omega_r = \frac{[\alpha p/\beta;p]_r}{[\alpha p;p]_r}$ in (1.4) we have

$$
\sum_{n,k=0}^{\infty}\frac{[\alpha p/\beta;p]_{n+k}\,[\alpha pq;pq]_k\,[\alpha;p]_k\,[\beta;q]_k\,z^{n+k}}{[\alpha p;p]_{n+k}\,[\alpha;pq]_k\,[\alpha p/\beta;p]_k\,[q;q]_k\,\beta^k} = \frac{[zq;q]_\infty}{[z/\beta;q]_\infty} \tag{3.13}
$$

(xv) As, $\beta \to \infty$, (3.13) yields:

$$
\sum_{n,k=0}^{\infty}\frac{[\alpha pq;pq]_k\,[\alpha;p]_k\,(-1)^k \overset{k(k-1)}{\sum} z^{n+k}}{[\alpha p;p]_{n+k}\,[\alpha;pq]_k\,[q;q]_k} = [zq;q]_\infty \tag{3.14}
$$

Conclusion : Many more interesting results can be deducted from our main results (2.1) (2.5).

References

[1] Agarwal, R. P., *Generalized Hypergeometric Series and its Application to the Theory of Combinatorial analysis and Partition Theory,* (2004), (Unpublished monograph).

[2] Gasper, G. and Rahman, *Basic Hypergeometric Series,* Cambridge University Press, Cambridge, (1991).

[3] Srivastava, H. M. and Karlsson, P. W., *Multiple Gaussion Hypergeometric Series,* Ellis, Horwood Limited, Halsted Press : A Division of John Wiley and Sons, New York, (1986).

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METHOD OF INFINITE ASCENT APPLIED ON $mA^3 + nB^3 = 3C^2$

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Abstract: In this paper, we present a technique of generating infinitely many co-prime parametric solutions for (A, B, C) in the Diophantine equation $mA^3 + nB^3 = 3C^2$ for any pair of co-prime integers (m, n) where $(m^2 - n^2) = 3k$, and 3 is not a factor of k. We show how each of these parametric solutions breeds infinitely many co-prime integral solutions for (A, B, C) linked with this Diophantine equation.

1. **Introduction**

Diophantine equations would continue to fascinate both the amateurs and the mathematicians alike because, most of them are so simple at the first sight, yet requiring more advanced techniques to analyze and realize them. Until a solution is found, a typical Diophantine problem exhibits the complexity of a jigsaw puzzle; but, once solved, it is so simple to verify. As indicated in [1], the Diophantine equation $A^3 + B^3 = C^2$ was first studied by Euler, and the general solution was found by R. Hoppe. In a recent paper [2], Jena gives a non-recursive formula for which the Diophantine equation $mA^3 + nB^3 = C^2$ generates an infinite number of co-prime integral solutions for (A, B, C) for any pair of co-prime integers (m, n). But, the following Diophantine equation

$$mA^3 + nB^3 = 3C^2 \tag{1.1}$$

has not been studied, yet. Hence, in this paper, we present a technique of finding an infinite number of co-prime parametric solutions for (A, B, C) in the Diophantine equation (1.1) for any pair of co-prime integers (m, n), where $(m^2 - n^2) = 3k$, and 3 is not a factor of k.

Throughout the length of this paper, we will denote different assumptions and conclusions by letters A_i and C_i respectively, where i takes integral values ≥ 1. Similarly, the greatest common factor r of any two non-zero integers α and β will be denoted by $(\alpha, \beta) = r$, where we take the magnitudes of α and β. If they are

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co-prime, it will be denoted by $(\alpha\,,\beta) = 1$. For any two non-zero integers c and d where $1 < |c| < |d|$, if c is a factor of d, then, c divides d, and it will be denoted by $c|d$.

2. A Parametric Solution Of $mA^3 + nB^3 = 3C^2$

We give the main result of this paper in Theorem 2.1 via Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *For any pair of integers p and q*

$$p(4q)^3 + (p-3q)(3p-q)^3 = 3\{\pm(3p^2 - 6pq - q^2)\}^2. \tag{2.1}$$

Proof. L. H. S. of (2.1)

$$\begin{aligned}
&= p(4q)^3 + (p-3q)(3p-q)^3 \\
&= p\cdot 4^3q^3 + (p-3q)(3^3\cdot p^3 - 3\cdot 3^2p^2\cdot q + 3\cdot 3p\cdot q^2 - q^3) \\
&= 64pq^3 + (p-3q)(27p^3 - 27p^2q + 9pq^2 - q^3) \\
&= 64pq^3 + p(27p^3 - 27p^2q + 9pq^2 - q^3) - 3q(27p^3 - 27p^2q + 9pq^2 - q^3) \\
&= 64pq^3 + 27p^4 - 27p^3q + 9p^2q^2 - pq^3 - 81p^3q + 81p^2q^2 - 27pq^3 + 3q^4 \\
&= 27p^4 + (-27-81)p^3q + (9+81)p^2q^2 + (64-1-27)pq^3 + 3q^4 \\
&= 27p^4 - 108p^3q + 90p^2q^2 + 36pq^3 + 3q^4 \\
&= 3(9p^4 - 36p^3q + 30p^2q^2 + 12pq^3 + q^4) \\
&= 3(9p^4 - 36p^3q + 36p^2q^2 - 6p^2q^2 + 12pq^3 + q^4) \\
&= 3(9p^4 + 36p^2q^2 + q^4 - 36p^3q + 12pq^3 - 6p^2q^2) \\
&= 3\{(3p^2)^2 + (-6pq)^2 + (-q^2)^2 + 2(3p^2)(-6pq) + 2(-6pq)(-q^2) \\
&\quad + 2(3p^2)(-q^2)\} \\
&= 3\{\pm(3p^2 - 6pq - q^2)\}^2 \\
&= \text{R. H. S. of (2.1).}
\end{aligned}$$

Hence, Lemma 2.1 is proved. □

Lemma 2.2. *For integers x and y, and non-zero integers m, n, u, v, p, q and k, if we take the following assumptions:*

(A_1) $(m\,,n) = 1$, $(3\,,k) = 1$, *and* $(m^2 - n^2) = 3k$,

(A_2) $p = mu^3$ *and* $(p-3q) = nv^3$,

(A_3) $u = (3nvx + m)$, *and*

(A_4) $v = (6my + n)$, *where n is any odd integer,*

then, for any integer pair $(m\,,n)$, q has infinitely many integral values such that $(mu\,,nv) = (q\,,mu) = (q\,,nv) = (3\,,q) = 1$.

Proof. In two steps Lemma 2.2 can be proved.

Step 1. From (A_2), we get

$$\begin{aligned} p-(p-3q) &= (mu^3 - nv^3) \\ \Longrightarrow 3q &= (mu^3 - nv^3) \\ \Longrightarrow q &= (mu^3 - nv^3)/3 \end{aligned} \tag{2.2}$$

From (2.2), (A_3) and (A_4), we get

$$q = \{m(3nvx+m)^3 - n(6my+n)^3\}/3 \tag{2.3}$$

$$\begin{aligned} \Longrightarrow q &= \{m(27n^3v^3x^3 + 27n^2v^2x^2m + 9nvxm^2 + m^3) \\ &\quad - n(216m^3y^3 + 108m^2y^2n + 18myn^2 + n^3)\}/3 \\ \Longrightarrow q &= \{m(27n^3v^3x^3 + 27n^2v^2x^2m + 9nvxm^2) \\ &\quad - n(216m^3y^3 + 108m^2y^2n + 18myn^2) + (m^4 - n^4)\}/3 \\ \Longrightarrow q &= \{m(27n^3v^3x^3 + 27n^2v^2x^2m + 9nvxm^2) \\ &\quad - n(216m^3y^3 + 108m^2y^2n + 18myn^2)\}/3 + (m^4 - n^4)/3 \\ \Longrightarrow q &= \{m \cdot 9nvx(3n^2v^2x^2 + 3nvxm + m^2) \\ &\quad - n \cdot 18my(12m^2y^2 + 6myn + n^2)\}/3 + (m^4 - n^4)/3 \\ \Longrightarrow q &= \{9mnvx(3n^2v^2x^2 + 3nvxm + m^2) \\ &\quad - 18mny(12m^2y^2 + 6myn + n^2)\}/3 + (m^4 - n^4)/3 \\ \Longrightarrow q &= 9mn\{vx(3n^2v^2x^2 + 3nvxm + m^2) \\ &\quad - 2y(12m^2y^2 + 6myn + n^2)\}/3 + (m^4 - n^4)/3 \end{aligned}$$

$$\begin{aligned} \Longrightarrow q &= 3mn\{vx(3n^2v^2x^2 + 3nvxm + m^2) \\ &\quad - 2y(12m^2y^2 + 6myn + n^2)\} + \{(m^2 - n^2)(m^2 + n^2)\}/3 \end{aligned} \tag{2.4}$$

Putting the value of $(m^2 - n^2)$ from (A_1) in (2.4), we get

$$\begin{aligned} q &= 3mn\{vx(3n^2v^2x^2 + 3nvxm + m^2) \\ &\quad - 2y(12m^2y^2 + 6myn + n^2)\} + \{3k(m^2 + n^2)\}/3 \\ &= 3mn\{vx(3n^2v^2x^2 + 3nvxm + m^2) \\ &\quad - 2y(12m^2y^2 + 6myn + n^2)\} + k(m^2 + n^2) \end{aligned} \tag{2.5}$$

Since, from (A_1), m and n are co-prime and $3|(m^2 - n^2)$, 3 would not be a factor of $(m^2 + n^2)$, because that would require 3 to be a factor of both m and n, which is not possible. Similarly, from (A_1), 3 is not a factor of k. Thus, from (2.5), we see that, for any integer pair $(m\,,n)$ satisfying (A_1), q has infinitely many integral values such that $(3\,,q) = 1$.

Step II. Now, we will prove the rest part of Lemma 2.2 that $(mu\,,nv) = (q\,,mu) =$

$(q\,,nv)=1$.
From (A_1) we get
(C_1) $(3\,,m)=(3\,,n)=1$.
Since $(m\,,n)=1$, from (A_3) we get
(C_2) $(u\,,n)=1$, because any common factor of u and n would divide m, which contradicts one assumption of (A_1).
Similarly, from (A_4), we have
(C_3) $(v\,,m)=1$, because $(m\,,n)=1$.
In (A_3), we have $(v\,,m)=1$ as shown in (C_3). Hence
(C_4) $(u\,,v)=1$.
(C_2) and (C_4) gives
(C_5) $(u\,,nv)=1$.
Since $(m\,,n)=1$, and from (C_3), $(m\,,v)=1$, we get
(C_6) $(m\,,nv)=1$.
From (C_5) and (C_6), we get
(C_7) $(mu\,,nv)=1$; Or, $(mu^3\,,nv^3)=1$.
From (A_3) we get
(C_8) $(3\,,u)=1$ as we have $(3\,,m)=1$ from (C_1).
From (A_4) we get
(C_9) $(3\,,v)=1$ as we have $(3\,,n)=1$ from (C_1).
From (2.2) we get, $3q=(mu^3-nv^3)$. Since mu^3 and nv^3 are co-prime as shown in (C_7), and 3 is neither a factor of mu^3 nor nv^3 as we observed in (C_8) and (C_9) respectively, q would be co-prime to both mu^3 and nv^3. Thus
(C_{10}) $(mu\,,nv)=(q\,,mu)=(q\,,nv)=1$.
Combining Step I and Step II, Lemma 2.2 is proved. □

Theorem 2.1. *For all integers x, y, and non-zero integers m, n, u, v, p, q and k, if we take the following assumptions:*
(A_1) $(m\,,n)=1$, $(3\,,k)=1$, *and* $(m^2-n^2)=3k$,
(A_2) $p=mu^3$ *and* $(p-3q)=nv^3$,
(A_3) $u=(3nvx+m)$, *and*
(A_4) $v=(6my+n)$, *where n is any odd integer,*
then, the Diophantine equation $mA^3+nB^3=3C^2$ would have infinitely many non-zero co-prime integral solutions for $(A\,,B\,,C)$ such that
$A=4uq$,
$B=v(3mu^3-q)$, *and*
$C=\pm(3m^2u^6-6mqu^3-q^2)$,
where

$$q=3mn\{vx(3n^2v^2x^2+3nvxm+m^2)-2y(12m^2y^2+6myn+n^2)\}+k(m^2+n^2).$$

Proof. We will need two steps for proving this theorem.

Step 1. Putting $p = mu^3$ and $(p - 3q) = nv^3$ from (A_2) in (2.1) we get

$$mu^3(4q)^3 + nv^3(3mu^3 - q)^3 = 3\{\pm(3m^2u^6 - 6mu^3q - q^2)\}^2$$

or,

$$m(4uq)^3 + n\{v(3mu^3 - q)\}^3 = 3\{\pm(3m^2u^6 - 6mqu^3 - q^2)\}^2,$$

which gives a solution for $(A\,, B\,, C)$ of the Diophantine equation (1.1) so that

$$A = 4uq \tag{2.6}$$

$$B = v(3mu^3 - q) \tag{2.7}$$

and

$$C = \pm(3m^2u^6 - 6mqu^3 - q^2) \tag{2.8}$$

For any integer pair $(m\,, n)$, under the given assumptions in Theorem 2.1, we make the following conclusions.

From assumption (A_4), we see that v is a function of y, which takes all integral values including zero. Assumption (A_3) together with (A_4) ensures that u is a function of x and y. Assumption (A_2) together with (A_3) and (A_4) shows that

$$q = 3mn\{vx(3n^2v^2x^2 + 3nvxm + m^2) - 2y(12m^2y^2 + 6myn + n^2)\} + k(m^2 + n^2)$$

in accordance with (2.5) of Lemma 2.2. Thus, q is a function of x and y. So, A, B and C are functions of x and y. There are infinitely many solutions for $(A\,, B\,, C)$ for each integral pair $(m\,, n)$, because x and y take all integral values. Here, if we impose the condition (A_1) on these integral pairs $(m\,, n)$, then for each such pair $(m\,, n)$, the Diophantine equation (1.1) has infinitely many co-prime integral solutions for $(A\,, B\,, C)$.

Step II. We have to show that mA^3, nB^3 and $3C^2$ would be pair-wise co-prime. Since, these three terms are related by the equation (1.1), it will be sufficient to prove that mA^3 and nB^3 are co-prime; or, mA and nB are co-prime, which would lead to the proof that A, B and C are pair-wise co-prime.

From (2.6) and (2.7) we get

$$mA = 4muq \tag{2.9}$$

$$nB = nv(3mu^3 - q) \tag{2.10}$$

From (A_4), nv is odd. So, $(4\,, nv) = 1$. From (C_{10}), $(mu\,, nv) = (q\,, nv) = 1$. Hence, we get

(C_{11}) $\quad (4muq\,, nv) = 1$.

From (A_4), nv is odd. Hence, from (2.2), q would be even when mu is odd; or, q is odd when mu is even. Thus

(C_{12}) $\quad (3mu^3 - q)$ is always odd. So, $(4\,, (3mu^3 - q)) = 1$.

From Lemma 2.2, we have $(mu\,,q) = (3\,,q) = 1$. So we have

(C_{13}) $\qquad (muq\,,(3mu^3 - q)) = 1.$

Combining (C_{12}) and (C_{13}) gives

(C_{14}) $\qquad (4muq\,,(3mu^3 - q)) = 1.$

From (C_{11}) and (C_{14}) we get

(C_{15}) $\qquad (4muq\,,nv(3mu^3 - q)) = 1.$

Comparing (C_{15}) with the right hand sides of (2.9) and (2.10), we conclude that $(mA\,,nB) = 1$.

Thus, combining Step I and Step II, our proof of Theorem 2.1 is complete. □

3. Some applications of Theorem 2.1

The Diophantine equation (1.1) has infinitely many co-prime parametric solutions for $(A\,,B\,,C)$ for any integer pair $(m\,,n)$ which satisfies assumption (A_1) in Theorem 2.1. We illustrate this point by applying Theorem 2.1 for two pairs of values taken by $(m\,,n)$.

3.1. Diophantine equation $2A^3 + B^3 = 3C^2$.

The pair $(m\,,n) = (2\,,1)$ satisfies assumption (A_1) in Theorem 2.1. Thus, the Diophantine equation

$$2A^3 + B^3 = 3C^2 \tag{3.1}$$

would have infinitely many co-prime integral solutions for $(A\,,B\,,C)$. Now, from Theorem 2.1, it is easy to show that for $(m\,,n) = (2\,,1)$ we have

$$\begin{aligned}
v &= (12y + 1),\\
u &= (36xy + 3x + 2),\\
q &= 6\{vx(3v^2x^2 + 6vx + 4) - 2y(48y^2 + 12y + 1)\} + 5,\\
A &= 4uq,\\
B &= v(6u^3 - q),\\
\text{and } C &= \pm(12u^6 - 12qu^3 - q^2).
\end{aligned} \tag{3.2}$$

Putting $y = 0$ in (3.2), we get

$$\begin{aligned}
v &= 1,\\
u &= (3x + 2),\\
q &= 6x(3x^2 + 6x + 4) + 5\\
&= (18x^3 + 36x^2 + 24x + 5),\\
A &= 4uq = 4(3x + 2)(18x^3 + 36x^2 + 24x + 5),\\
B &= v(6u^3 - q) = \{6(3x + 2)^3 - (18x^3 + 36x^2 + 24x + 5)\} = \dots\\
&= (144x^3 + 288x^2 + 192x + 43),
\end{aligned}$$

and

$$\begin{aligned} C &= \pm(12u^6 - 12qu^3 - q^2) \\ &= \pm\{12(3x+2)^6 - 12(18x^3+36x^2+24x+5)(3x+2)^3 \\ &\qquad - (18x^3+36x^2+24x+5)^2\} = \dots \\ &= \pm(2592x^6 + 10368x^5 + 17280x^4 \\ &\qquad + 15480x^3 + 7920x^2 + 2208x + 263). \end{aligned}$$

Thus, for any integer x, we have

$$\begin{aligned} &2 \cdot \{4(3x+2)(18x^3+36x^2+24x+5)\}^3 \\ &\qquad + 1 \cdot (144x^3 + 288x^2 + 192x + 43)^3 \\ &= 3 \cdot \{\pm(2592x^6 + 10368x^5 + 17280x^4 \\ &\qquad + 15480x^3 + 7920x^2 + 2208x + 263)\}^2. \end{aligned} \tag{3.3}$$

Similarly, putting $y = 1$ in (3.2), we get

$$\begin{aligned} v &= 13, \\ u &= (39x+2), \\ q &= (39546x^3 + 6084x^2 + 312x - 727), \\ A &= 4uq \\ &= 4(39x+2)(39546x^3 + 6084x^2 + 312x - 727), \\ B &= v(6u^3 - q) \\ &= 13\{6(39x+2)^3 - (39546x^3 + 6084x^2 + 312x - 727)\} = \dots \\ &= 13(316368x^3 + 48672x^2 + 2496x + 775), \end{aligned}$$

and

$$\begin{aligned} C &= \pm(12u^6 - 12qu^3 - q^2) \\ &= \pm\{(12(39x+2)^6 - 12(39546x^3 + 6084x^2 + 312x - 727)(39x+2)^3 \\ &\qquad - (39546x^3 + 6084x^2 + 312x - 727)^2\} = \dots \\ &= \pm(12511088928x^6 + 3849565824x^5 + 493534080x^4 \\ &\qquad + 612963000x^3 + 90408240x^2 + 4596384x - 457969). \end{aligned}$$

So, for any integer x, we have

$$\begin{aligned} &2 \cdot \{4(39x+2)(39546x^3 + 6084x^2 + 312x - 727)\}^3 \\ &\qquad + 1 \cdot \{13(316368x^3 + 48672x^2 + 2496x + 775)\}^3 \\ &= 3 \cdot \{\pm(12511088928x^6 + 3849565824x^5 + 493534080x^4 \\ &\qquad + 612963000x^3 + 90408240x^2 + 4596384x - 457969)\}^2. \end{aligned} \tag{3.4}$$

Now, in (3.3) or (3.4), for each integral value of x, we get a co-prime integral solution for $(A\,,B\,,C)$ linked with the Diophantine equation (3.1). By putting $x=0$ in (3.3) and (3.4), we find co-prime numerical values for $(A\,,B\,,C)$ as $(40,43,\pm 263)$ and $(-5816,10075,\pm 457969)$ respectively, which are two pairs of solutions of (3.1). Since, x has infinitely many integral values, the Diophantine equation (3.1) has infinitely many co-prime integral solutions. An infinite number of parallel identities like (3.3) and (3.4) can be generated by taking different values of integer y in (3.2). Each of these identities confers an infinite number of co-prime integral solutions for $(A\,,B\,,C)$ in the Diophantine equation (3.1).

3.2. **Diophantine equation** $2A^3+5B^3=3C^2$**.**

The pair $(m\,,n)=(2\,,5)$ satisfies assumption (A_1) in Theorem 2.1. Thus, the Diophantine equation

$$2A^3+5B^3=3C^2 \tag{3.5}$$

would have infinitely many co-prime integral solutions for $(A\,,B\,,C)$.
Taking $(m\,,n)=(2\,,5)$, from Theorem 2.1 we get

$$\begin{aligned}
v &= (12y+5),\\
u &= (180xy+75x+2),\\
q &= 30\{vx(75v^2x^2+30vx+4)-2y(48y^2+60y+25)\}-203, && (3.6)\\
A &= 4uq, && (3.7)\\
B &= v(6u^3-q),
\end{aligned}$$

and

$$C=\pm(12u^6-12qu^3-q^2).$$

Putting $y=0$ in (3.6), we obtain

$$\begin{aligned}
v &= 5,\\
u &= (75x+2),\\
q &= (281250x^3+22500x^2+600x-203),\\
A &= 4uq = 4(75x+2)(281250x^3+22500x^2+600x-203),\\
B &= v(6u^3-q) = 5\{6(75x+2)^3-(281250x^3+22500x^2+600x-203)\} = \ldots\\
&= 5(2250000x^3+180000x^2+4800x+251),
\end{aligned}$$

and

$$\begin{aligned}
C &= \pm(12u^6-12qu^3-q^2)\\
&= \pm\{12(75x+2)^6-12(281250x^3+22500x^2+600x-203)(75x+2)^3\\
&\qquad -(281250x^3+22500x^2+600x-203)^2\} = \ldots
\end{aligned}$$

$$= \pm(632812500000x^6 + 101250000000x^5 + 6750000000x^4 + 1411875000x^3 + 98550000x^2 + 2551200x - 20953).$$

Hence, for any integer x we have

$$\begin{aligned} &2 \cdot \{4(75x+2)(281250x^3 + 22500x^2 + 600x - 203)\}^3 \\ &\quad + 5 \cdot \{5(2250000x^3 + 180000x^2 + 4800x + 251)\}^3 \\ &= 3 \cdot \{\pm(632812500000x^6 + 101250000000x^5 + 6750000000x^4 \\ &\quad + 1411875000x^3 + 98550000x^2 + 2551200x - 20953)\}^2. \end{aligned} \tag{3.8}$$

Putting $y = -1$ in (3.6), we get

$$\begin{aligned} v &= -7, \\ u &= -(105x - 2), \\ q &= -(771750x^3 - 44100x^2 + 840x - 577), \\ A &= 4uq = 4(-(105x-2))(-(771750x^3 - 44100x^2 + 840x - 577)) \\ &= 4(105x-2)(771750x^3 - 44100x^2 + 840x - 577), \\ B &= v(6u^3 - q) \\ &= -7\{6(-(105x-2))^3 - (-(771750x^3 - 44100x^2 + 840x - 577))\} \\ &= 7\{6(105x-2)^3 - (771750x^3 - 44100x^2 + 840x - 577)\} = \dots \\ &= 7(6174000x^3 - 352800x^2 + 6720x + 529), \end{aligned}$$

and

$$\begin{aligned} C &= \pm(12u^6 - 12qu^3 - q^2) \\ &= \pm\{12(-(105x-2))^6 \\ &\quad - 12(-(771750x^3 - 44100x^2 + 840x - 577))(-(105x-2))^3 \\ &\quad - (-(771750x^3 - 44100x^2 + 840x - 577))^2\} = \dots \\ &= \pm(4764784500000x^6 - 544546800000x^5 + 25930800000x^4 \\ &\quad + 8165115000x^3 - 494802000x^2 + 9532320x - 387553). \end{aligned}$$

Hence, for any integer x we have

$$\begin{aligned} &2 \cdot \{4(105x-2)(771750x^3 - 44100x^2 + 840x - 577)\}^3 \\ &\quad + 5 \cdot \{7(6174000x^3 - 352800x^2 + 6720x + 529)\}^3 \\ &= 3 \cdot \{\pm(4764784500000x^6 - 544546800000x^5 + 25930800000x^4 \\ &\quad + 8165115000x^3 - 494802000x^2 + 9532320x - 387553)\}^2. \end{aligned} \tag{3.9}$$

Now, in (3.7) or (3.8), for each integral value of x, we get a co-prime integral solution for (A, B, C) linked with the Diophantine equation (3.5). By putting

$x = 0$ in (3.7) and (3.8), we find co-prime numerical values for (A, B, C) as $(-1624, 1255, \pm 20953)$ and $(4616, 3703, \pm 387553)$ respectively, which are two pairs of solutions of (3.5). Since, x has infinitely many integral values, the Diophantine equation (3.5) has infinitely many co-prime integral solutions. An infinite number of parallel identities like (3.7) and (3.8) can be generated by taking different values of integer y in (3.6). Each of these identities gives an infinite number of co-prime integral solutions for (A, B, C) linked with the Diophantine equation (3.5).

4. Conclusion

There may still exist other parametric solutions, or singular solutions for the Diophantine equation (1.1). The present paper is not an attempt to address this question. Our study of this Diophantine equation is the beginning of an investigation of a more generalized Diophantine equation

$$mA^3 + nB^3 = kC^2$$

where m , n and k are non-zero integers, and are pair-wise co-prime. We hope that the results of this paper would be an inspiration for the prospective researchers to delve further in that direction.

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References

[1] M. D. Hirschhorn, *Two Diophantine Equations Studied by Ramanujan*, Math. Mag., Vol. 71, No. 5 (Dec., 1998), 386–388.

[2] S. K. Jena, *Method of Infinite Ascent applied on* $mA^3 + nB^3 = C^2$, The Math. Stud., Vol 79, Nos. 1-4 (2010), 187–192.

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THERMAL INSTABILITY IN WALTERS' B' ELASTICO-VISCOUS FLUID IN POROUS MEDIUM IN HYDROMAGNETICS

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Abstract: The thermal instability of a layer of Walters' B' elastico-viscous fluid in porous medium acted on by a uniform magnetic field is considered. For stationary convection, the Walters' B' elastico-viscous fluid behaves like an ordinary (Newtonian)fluid. The magnetic field is found to have stabilizing effect whereas medium permeability has destabilizing effect. The magnetic field introduces oscillatory modes in the system. A sufficient condition for the non-existence of overstability is also obtained.

1. **Introduction**

A detailed account of thermal convection in hydromagnetics has been given by Chandrasekhar [11] . Bhatia and Steiner [4] have studied the thermal instability of a Maxwellian viscoelastic fluid in the presence of magnetic field while the thermal convection in Oldroydian viscoelastic fluid in hydromagnetics has been considered by Sharma [6] . Sharma and Kumar [7] have considered the thermal convection in Rivlin-Ericksen fluid in hydromagnetics.

The medium has been considered to be non-porous in all the above studies. Aggarwal and Goel [12]

have studied the shear flow instability of viscoelastic fluid in porous medium. Lapwood [2] has studied the stability of convective flow in hydrodynamics using Rayleigh's procedure. Wooding

[5] has considered the Rayleigh instability of a thermal boundary layer in flow through porous medium.

The gross effect when the fluid slowly percolates through the pores of the rock is represented by the well known Darcy's law.

The problem of thermal convection in fluids

in a porous medium is of importance in geophysics, soil science, ground water hydrology and astrophysics. The development of geothermal

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power resources has increased general interest, in the properties of convection in porous media. The effect of a magnetic field on the stability of such a flow is of interest in geophysics, particularly in the study of Earth's core where the Earth's mantle, which consists of conducting fluid, behaves like a porous medium which can become convectively unstable as a result of differential diffusion. The other application of the results of flow through a porous medium in the presence of a magnetic field is in the study of the stability of a convective flow in the geothermal region.In the physical world, the investigation of flow of Walters' (Model B') fluid through porous medium has become an important topic due to the recovery of crude oil from the pores of reservoir rocks. Flows in porous region are a creeping flow. When a fluid permeates a porous material, the actual path of the individual particles cannot be followed analytically. When the density of a stratified layer of a single-component fluid decreases upwards, the configuration is stable. This is not necessarily so far a fluid consisting of two or more components which can diffuse relative to each other. The reason lies in the fact that the diffusivity of heat is usually much greater than the diffusivity of a solute. A displaced particle of fluid thus, loses any excess heat more rapidly than any excess solute. The resulting buoyancy force may tend to increase the displacement of the particle from its original position and thus cause instability.

There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's constitutive relations. Two such classes of elastico-viscous fluids are Rivlin-Ericksen fluid [10]and Walters' B' fluid [3] . Walters has proposed the constitutive equations of such elastico-viscous fluids. Srivastava and Singh [9] have studied the unsteady flow of a dusty elastico-viscous Rivlin-Ericksen fluid through channels of different cross-sections in the presence of time-dependent pressure gradient. In another study, Garg et al. [1] have studied the rectilinear oscillations of a sphere along its diameter in conducting dusty Rivlin-Ericksen fluid in the presence of uniform magnetic field. Sharma and Kango [8] have studied the thermal convection in Rivlin-Ericksen elastico-viscous fluid in porous medium in Hydromagnetics. A very little work has been done on the stability of such fluids.

Keeping in mind the relevance and growing importance of non-Newtonian fluids in geophysical fluid dynamics, chemical technology and industry; the present paper attempts to study the thermal instability in Walters' B' elastico-viscous fluid in porous medium in hydromagnetics.

2. **Formulation of the problem and perturbation equations**

Here we consider an infinite horizontal layer of an electrically conducting Walters' (Model B') elastico-viscous fluid confined between the planes $z = 0$ and $z = d$ in porous medium

of porosity ϵ and medium permeability k_1, acted on by a uniform vertical magnetic field $\mathbf{H} = (0, 0, H)$ and gravity force $g = (0, 0, -g)$. This layer is heated from below and the surfaces $z = 0$ and $z = d$ are maintained at constant temperature T_0 and $T_1 (T_1 > T_0)$, so that a uniform temperature gradient $\beta(|dT/dz|$ is maintained. The

schematic of the problem is shown in the figure 1 below:

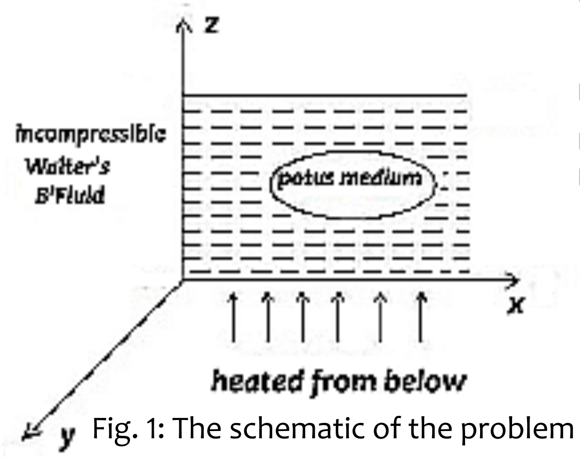

Fig. 1: The schematic of the problem

The hydromagnetic equations for thermal convection in Walters' B' elastico-viscous fluid in porous medium in hydromagnetics are

$$\frac{1}{\epsilon}[\frac{\partial \mathbf{q}}{\partial}t + (\mathbf{q} \cdot \Delta)\mathbf{q}] = \frac{\nabla p}{\rho_0} + \mathbf{g}\left(1 + \frac{\delta \rho}{\rho_0}\right) + \frac{\mu_e}{4\pi\rho_0}(\nabla \times \mathbf{H}) \times \mathbf{H} - \frac{1}{k_1}\left(\nu - \nu' \frac{\delta}{\delta t}\right)\mathbf{q}, \tag{2.1}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{2.2}$$

$$E\frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla)T = k\nabla^2 T, \tag{2.3}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{2.4}$$

$$\epsilon \frac{\rho \mathbf{H}}{\rho t} = (\mathbf{H} \cdot \nabla)\mathbf{q} + \eta \nabla^2 \mathbf{H}, \tag{2.5}$$

Where $\rho, p, T, \mu_e, \nu, \nu', k$ and η stand for density, pressure, temperature, magnetic permeability, the kinematic viscosity, the kinematic viscoelasticity, thermal diffusivity and electrical resistivity respectively and

$$E = \epsilon + (1-\epsilon)\rho_s C_s/\rho_0 C,$$

Here $E = \epsilon + (1-\epsilon)\rho_s C_s/(\rho_0 C)$ is a constant; ρ_s, C_s and ρ_0, C_s stand for density and heat capacity for solid and fluid, respectively.

The equation of state is

$$\rho = \rho_0[1-\alpha(T-T_0)], \tag{2.6}$$

where the suffix zero refers to values at the reference level $z = 0$ and α is called the thermal coefficient of expansion. Let $\mathbf{q} = (u, v, w), \delta\rho, \delta p, \theta$ and $\mathbf{H} = (h_x, h_y, h_z)$ denote the perturbations in velocity $(0, 0, 0)$, density ρ, pressure p, temperature T and magnetic field $\mathbf{H} = (0, 0, H)$ respectively. The change in density $\delta\rho$, caused by the perturbation θ in temperature, is given by

$$\delta\rho = -\alpha\rho_0\theta. \tag{2.7}$$

Then the equations (2.1) - (2.5), on linearization, give

$$\frac{1}{\epsilon}\frac{\partial \mathbf{q}}{\partial t} = -\frac{\nabla\delta p}{\rho_0} - \mathbf{g}\alpha\theta + \frac{\mu_e}{4\pi\rho_0}(\nabla\times\mathbf{h})\times\mathbf{H} - \frac{1}{k_1}\left(\nu - \nu'\frac{\partial}{\partial t}\right)\mathbf{q}, \tag{2.8}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{2.9}$$

$$E\frac{\partial\theta}{\partial t} = \beta w + k\nabla^2\theta, \tag{2.10}$$

$$\nabla\cdot\mathbf{h} = 0, \tag{2.11}$$

$$\epsilon\frac{\partial\mathbf{h}}{\partial t} = (\mathbf{H}\cdot\nabla)\mathbf{q} + \epsilon\eta\nabla^2\mathbf{h}. \tag{2.12}$$

The boundaries are taken to be free as well as perfect conductors of heat and the adjoining medium is electrically non-conducting. The case of two free surfaces is a little artificial except in the case of stellar atmospheres. However, this assumption allows us to obtain the analytical solution without affecting the essential features of the problem. The boundary conditions appropriate for the problem are

$$W = \frac{\partial^2 w}{\partial z^2} = 0,\ \theta = 0 \,\text{at}\ s = 0 \text{ and } z = d, \tag{2.13}$$

And $\mathbf{h}$ is continuous with an external vacuum field.
Equations (2.8)-(2.12) give

$$\frac{1}{\epsilon}\frac{\partial}{\partial t}\nabla^2 w = g\alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta - \left(\frac{1}{k_1}\right)\left(\nu - \nu'\frac{\partial}{\partial t}\right)\nabla^2 w + \left(\frac{\mu_e H}{4\pi\rho_0}\right)\frac{\partial}{\partial z}\nabla^2 h_z, \tag{2.14}$$

$$\left(E\frac{\partial}{\partial t} - k\nabla^2\right)\theta = \beta w, \tag{2.15}$$

$$\epsilon\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)h_z = H\frac{\partial w}{\partial z}, \tag{2.16}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2.17}$$

3. Dispersion relation

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, h_z] = [W(z), \Theta(z), K(z)] \exp(ik, x + ik, y + nt), \tag{3.1}$$

Where k_x, k_y are wave numbers along the x and y directions, respectively, $k = \sqrt{k_x^2, k_y^2}$ is the resultant wave number and n is, in general, a complex constant. Using expression (3.1), eqns. (2.14)-(2.16), in non-dimensional form, become

$$\frac{\sigma}{\epsilon}(D^2 - \alpha^2)W + \left(\frac{g\alpha d^2 \alpha^2}{\nu}\right)\Theta - \left(\frac{\mu_e H d}{4\pi\rho_0 \nu}\right)(D^2 - \alpha^2)DK$$
$$= -\frac{1}{p_1}(1 - F\sigma)(D^2 - \alpha^2)W, \tag{3.2}$$

$$(D^2 - \alpha^2 - Ep_1\sigma)\Theta = \left(\frac{\beta d^2}{k}\right)W \tag{3.3}$$

$$(D^2 - \alpha^2 - p_2\sigma)K = -\left(\frac{Hd}{\epsilon\eta}\right)DW, \tag{3.4}$$

where we have put $\alpha = kd, \sigma = \frac{nd^2}{\nu}, \frac{x}{d} = x^*, \frac{y}{d} = y^*, \frac{z}{d} = z^*$ and $D = \frac{d}{dz^*}, p_1 = \nu/k$ is the Prandtl number, $p_2 = \nu/\eta$ is the magnetic Prandtl number, $p_1 = k_1/d^2$ is the dimensionless medium permeability and $F = \nu'/d^2$ is the dimensionless kinematic viscoelasticity.

Elimination Θ, K between Equations (3.2)-(3.4), we obtain

$$(D^2 - \alpha^2))(D^2 - \alpha^2 - Ep_1\sigma)(D^2 - \alpha^2 - p_2\sigma)\left[\frac{\sigma}{\epsilon} + \frac{1}{p_1}(1 - F\sigma)\right]W$$

$$= R\alpha^2(D^2 - \alpha^2 - p_2\sigma)W = \frac{Q}{\epsilon}(D^2 - \alpha^2)(D^2 - \alpha^2 - Ep_1\sigma)D^2W, \tag{3.5}$$

Where $R = g\alpha\beta d^4/(\nu k)$ is the Rayleigh number and $Q = \mu_e H^2 d^2/(4\pi\rho_0\nu\eta)$ is the Chandrasekhar number. Now the boundary conditions (2.13) reduces to

$$W = D^2W = 0, \Theta = D^2\theta = 0, DK = D^3K = 0 \,\text{at}\, z^* = 0 \,\text{and}\, z^* = 1. \tag{3.6}$$

Dropping the stars for convenience and using the above boundary conditions, it can be shown that all the even order derivatives of W vanish on the boundaries and hence the proper solution of Equation (3.5) characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{3.7}$$

where W_0 is a constant. Substituting (3.7) in equation (3.5), we obtain the dispersion relation

$$R_1 = \frac{(1+x)(1+x+iEp_1\sigma_1)}{x(1+x+ip_2\sigma_1)}\left[(1+x+ip_2\sigma_1)\left\{\frac{I\sigma_1}{\epsilon}+(1-I\pi^2F\sigma_1)\right\}+\frac{Q_1}{\epsilon}\right], \tag{3.8}$$

Where $R_1 = \frac{R}{\pi^4}, Q_1 = \frac{Q}{\pi^2}, x = \frac{\alpha^2}{\pi^2}, p = \pi^2 p_1$ and $i\sigma_1 = \frac{\sigma}{\pi^2}$, it being remembered that σ can be complex.

4. The stationary convection

For the case of stationary convection, $\sigma = 0$ and equation (3.2) reduces to

$$R_1 = \frac{(1+x)}{x}\left[\frac{(1+x)}{p}+\frac{Q_1}{\epsilon}\right] \tag{4.1}$$

We thus find that for stationary convection the viscoelastic parameter F vanishes with σ and the Rivlin-Ericksen elastico-viscous fluid behaves like an ordinary (Newtonian) fluid.

To study the effects of the magnetic field and medium permeability, we examine the

natures of $\frac{dR_1}{dQ_1}$ and $\frac{dR_1}{dp}$. Equation (4.1) yields

$$\frac{dR_1}{dQ_1} = \frac{1}{\epsilon}\frac{1+X}{x} \tag{4.2}$$

$$\frac{dR_1}{dp} = \frac{(1+x)^2}{xp^2} \tag{4.3}$$

which imply that the magnetic field has a stabilizing effect whereas the medium permeability has a destabilizing effect on thermal convection in Rivlin-Ericksen fluid in porous medium in hydromagnetics for the stationary convection.

The critical Rayleigh number for the onset of instability is determined by the condition $\frac{dR_1}{dx} = 0$, which yields

$$x = \sqrt{1+\frac{PQ_1}{\epsilon}} \tag{4.4}$$

The corresponding value of R_1 is

$$R_{1c} = \frac{1}{p}\left[1+\sqrt{1+\frac{PQ_1}{\epsilon}}\right] \tag{4.5}$$

5. Stability of the system and oscillatory modes

Multiplying Eq. (2.12) by W^*, the complex conjugate of w, integrating over the range of z and making use of Eq. (2.13) and (2.14) together with the boundary conditions (2.16), we obtain

$$\left[\frac{\sigma}{\epsilon}+\frac{1}{p_1}(1+f\sigma)\right]I_1-\left(\frac{g\alpha k\alpha^2}{\nu\beta}\right)[I_2+Ep_1\sigma^* I_3]+\left(\frac{\mu_e\eta\epsilon}{4\pi\rho_0\nu}\right)[I_4+p_2\sigma^* I_3], \quad (5.1)$$

Where

$$I_1=\int_0^1(|DW|^2+\alpha^2|W|^2)dz,\ I_2==\int_0^1(|D\Theta|^2+\alpha^2|\Theta|^2)dz,\ I_2=\int_0^1|\Theta|^2dz,$$

$$I_4=\int_0^1(|D^2K|^2+2\alpha^2|DK|^{2+}\alpha^4|K|^2)dz,\ I_5=\int_0^1(|DK|^2+\alpha^2|K|^2)dz \quad (5.2)$$

and σ^* is the complex conjugate of σ . The integrals $I_1,\cdots I_5$ are all positive definite.

Putting $\sigma=\sigma_r+I\sigma_1$ in Eq. (5.1) and equating the real and imaginary parts, we obtain

$$\left[\left(\frac{1}{\epsilon}+\frac{F}{p_1}\right)I_1-\left(\frac{g\alpha\beta\alpha^2}{\nu\beta}\right)Ep_1I_3+\left(\frac{\mu_e\eta\epsilon}{4\pi\rho_0\nu}\right)p_2I_5\right]\sigma_r$$

$$=-\left(\frac{1}{p_1}\right)I_1+\left(\frac{g\alpha k\alpha^2}{\nu\beta}\right)I_2-\left(\frac{\mu_e\eta\epsilon}{4\pi\rho_0\nu}\right)I_4, \quad (5.3)$$

And

$$\left[\left(\frac{1}{\epsilon}+\frac{f}{p_1}\right)I_1+\left(\frac{g\alpha k\alpha^2}{\nu\beta}\right)Ep_1I_3-\left(\frac{\mu_e\eta\epsilon}{4\pi\rho_0\nu}\right)p_2I_5\right]\sigma_1=0 \quad (5.4)$$

It is evident from (5.3) that σ_r is positive or negative. The system is, therefore, stable or unstable. It is clear from (5.4) that σ_t may be zero or non-zero, meaning thereby that the modes may be non-oscillatory or oscillatory. The oscillatory modes are introduced due to the presence of magnetic field which were non-existent in its absence.

6. The case of overstability

If at the outset of instability a stationary pattern of motion prevails, then one says that the 'principle of the exchange of

stabilities' is valid and that instability sets in as stationary cellular convection or secondary flow. On

the other hand, if at the onset of instability oscillatory motions prevails, then it is called the case of 'overstability'.

Here we discuss the possibility of whether instability may occur as an overstability. When the marginal state is oscillatory, we must have $\sigma_r=0$, $\sigma_1\neq 0$. Since for overstability, we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (3.2) will admit of solutions with σ_1 real. Equating the real and imaginary parts of (3.2), we obtain

$$R_1(\alpha-1) = \frac{\alpha^2}{p} - p_2\sigma_1^2\left(\frac{\alpha}{\epsilon}\right) - Fp_2\sigma_1^2\pi^2\left(\frac{\alpha}{p}\right) - Ep_1\sigma_1^2\left(\frac{\alpha}{\epsilon}\right) - EFp_1\sigma_1^2\pi^2\left(\frac{\alpha}{p}\right) - Ep_1p_2\sigma_1^2\left(\frac{\sigma_1^2}{p}\right) + Q_1\alpha, \tag{6.1}$$

and

$$R_1p_2(\alpha-1) = \frac{\alpha_3}{\epsilon} + \frac{1}{p}\left[F\pi^2\alpha^3 + p_2\alpha^2 + Ep_1\alpha^2 - EFp_1p_2\sigma_1^2\pi^2\alpha\ \right] - Ep_1p_2\sigma_1^2\left(\frac{\alpha}{\epsilon}\right) + Q_1Ep_1\alpha, \tag{6.2}$$

Where $1+x=\alpha$ Eliminating R_1 between (6.1) and (6.2), we obtain

$$\left[p_2^2\left(\frac{\alpha}{\epsilon}\right) + Fp_2^2\pi^2\left(\frac{\alpha}{p}\right) + \frac{Ep_1p_2^2}{p}\right]\sigma_1^2 + \left[\frac{\alpha^3}{\epsilon} + \frac{F\pi^2\alpha^3}{p} + \frac{Ep_1\alpha^2}{p} + Q_1\alpha(Ep_1-p_2)\right] = 0 \tag{6.3}$$

Equation (6.3) is of the form

$$A\sigma_1^2 + B = 0, \quad A > 0 \tag{6.4}$$

Where A is the coefficient of σ_1^2 and B is constant term, in Eq. (6.3). Since σ_1 is real for overstability, the value of σ_1^2 is positive.

Equation (6.4) shows that this is clearly impossible if $B > 0$. Therefore $B > 0$ gives sufficient condition for the non-existence of overstability which yields

$$Ep_1 > p_2, \tag{6.5}$$

which implies

$$k < \eta\left[\epsilon + (1-\epsilon)\frac{\rho_s C_s}{\rho_0 C}\right]. \tag{6.6}$$

Inequality (6.6) is, therefore, a sufficient condition for the non-existence of overstability, the violation of which does not necessarily imply occurrence of overstability.

7. Conclusion

The study of viscoelastic fluids in porous medium may find applications in geophysics and chemical technology. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's constitutive relations. Walters' B' fluid is one such class of elastico-viscous fluids.A layer of electrically conducting Walters' B' elastico-viscous fluid heated from below in porous medium has been considered in the presence of a uniform vertical magnetic field. For stationary convection, the Walters' B' elastico-viscous fluid behaves like an ordinary (Newtonian) fluid. The onset of instability is postponed with the increase in magnetic field. The medium permeability has destabilizing effect on the system. It is also found that the magnetic field introduces oscillatory modes in the system which were non-existent in its absence. The results obtained herein hold good for the Maxwellian as well as Oldroydian viscoelastic fluid (Bhatia and Steiner [4], Sharma [6]). A sufficient condition for the non-existence of overstability is also obtained, the violation of which does not necessarily imply the occurrence of overstability.

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References

[1] Garg et al, *Rectilinear oscillations of a sphere along its diameter in conducting dusty Rivlin-Ericksen fluid in the presence of uniform magnetic field* Proc. Nat. Acad. Sci. India A64 (1994), 355.

[2] E. R. Lapwood, *Convection of a fluid in a porous medium,* Proc. Camb. Phil. Soc. 44 (1948), 508–519.

[3] K. Walters, *The motion of an elastico-viscous liquid contained between coaxial Cylinders,* Quart. J. Mech. Apple. Math., 13 (1960), 444–461.

[4] P. K. Bhatia and J. M. Steiner, *Thermal instability in a viscoelastic fluid layer in hydromagnetics,* J. Math. Anal. Appl. 41 (1973), 271–283.

[5] R. A. Wooding, *Rayleigh instability of thermal boundary layer in flow through porous medium,* J. Fluid Mech. Vol. 9 (1960), 183–192.

[6] R. C. Sharma, *Thermal instability of a viscoelastic fluid in hydromagnetics,* Acta Physics Hungrica, 38 (1975), 293.

[7] R. C. Sharma and P. Kumar, *Thermal instability in Rivlin-Ericksen elastico- viscous fluids in hydromagnetics,* Z. Naturforsch. 52a (1997) 369.

[8] R. C. Sharma and S. K. Kango, *Thermal convection in Rivlin-Ericksen elastico- viscous fluid in porous medium in hydromagnetics,* Czech. J. Phys. 49 (1999), 197–203.

[9] R. K. Srivastava and K.K. Singh, *Unsteady flow of a dusty elastico-viscous Rivlin-Ericksen fluid through channels of different cross-sections in the presence of time-dependent pressure gradient,* Bull. Cal. Math. Soc. 80 (1988) 286.

[10] R. S. Rivlin and J. L. Ericksen, *Stress-deformation relations for isotropic materials,* J. Rational Mech. Anal. 4 (1955), 323–329.

[11] S.Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability,* Dover Publication, New York, 1981.

[12] S. C. Aggarwal and A. K. Goel, *Shear flow instability of viscoelastic fluid in porous medium,* Indian J. of Pure and Appl. Math. 29 (1998), 969-981

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THE LAPLACE DERIVATIVE

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Abstract: Laplace derivative is a generalization of Peano derivative. Here we study first order Laplace derivative and show by an example that Laplace derivative is more general than ordinary derivative. Also we prove Rolle's theorem, Darboux theorem and some other theorems for Laplace derivative using only undergraduate level calculus.

Definition: Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue integrable in a neighbourhood of $x \in \mathbf{R}$. If for some $\delta > 0$, $\lim_{s\to\infty} s^2 \int_0^\delta e^{-st}[f(x+t) - f(x)]dt$ exists; then this limit is called the right hand Laplace derivative of f at x and is denoted by $LD^+f(x)$. Similarly the left hand Laplace derivative is

$$LD^-f(x) = \lim_{s\to\infty}(-s^2)\int_0^\delta e^{-st}[f(x-t) - f(x)]dt.$$

If $LD^+f(x) = LD^-f(x)$ then the common value is called the Laplace derivative of f at x and is denoted by $LDf(x)$.

Laplace derivative was first introduced in [3] and the properties of this derivative were studied in [2]. The symmetric version of Laplace derivative was introduced and explored in [1].

Lemma 1. [3, Lemma 5.1] *The above definitions are independent of* δ*.*

Proof. Write $\phi(t) = f(x+t) - f(x)$ and let $0 < \delta_1 < \delta_2$. Integrating by parts we get

$$\left|s^2\int_{\delta_1}^{\delta_2} e^{-st}\phi(t)dt\right| \le \left|s^2 e^{-s\delta_2}\int_{\delta_1}^{\delta_2}\phi(t)dt\right| + \left|s^3\int_{\delta_1}^{\delta_2}\{e^{-st}\int_{\delta_1}^{t}\phi(u)du\}dt\right|.$$

As $s \to \infty$, the first term in the right hand side vanishes. Suppose $M = \sup_{\delta_1\le t\le\delta_2}\int_{\delta_1}^{t}\phi(u)du$. So the second term in the right hand side is less than or

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equal to $s^2M(e^{-s\delta_1} - e^{-s\delta_2})$, which also tends to zero as $s \to \infty$. This shows that, $\lim_{s\to\infty} s^2 \int_{\delta_1}^{\delta_2} e^{-st}\,[f(x+t) - f(x)]\,dt = 0$. Hence

$$\lim_{s\to\infty} s^2 \int_0^{\delta_1} e^{-st}\,[f(x+t) - f(x)]\,dt = \lim_{s\to\infty} s^2 \int_0^{\delta_2} e^{-st}[f(x+t) - f(x)]dt.$$

This completes the proof.

Theorem 2. [3, Theorem 5.5] *If the ordinary derivative $f'(x_0)$ exists then $LDf(x_0)$ exists with equal value.*

Proof. Suppose $f'(x_0)$ exists. So, $\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$. Hence for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{f(x_0+h) - f(x_0)}{h} - f'(x_0)\right| < \epsilon$$

whenever $|h| < \delta$. This gives

(1) $\quad |f(x_0+h) - f(x_0) - hf'(x_0)| < \epsilon|h| \quad$ whenever $|h| < \delta$.

Therefore,

$$\begin{aligned}
&\left|s^2 \int_0^{\delta} e^{-st}[f(x_0+t) - f(x_0) - tf'(x_0)]dt\right| \\
&\le s^2 \int_0^{\delta} \left|e^{-st}[f(x_0+t) - f(x_0) - tf'(x_0)]\right| dt \\
&\le s^2 \int_0^{\delta} \epsilon|t|e^{-st}dt = \epsilon + o(1), \text{ as } s \to \infty.
\end{aligned}$$

This shows,

$$\lim_{s\to\infty} s^2 \int_0^{\delta} e^{-st}\,[f(x_0+t) - f(x_0) - tf'(x_0)]\,dt = 0.$$

Or,

$$\lim_{s\to\infty} s^2 \int_0^{\delta} e^{-st}\,[f(x_0+t) - f(x_0)]\,dt = f'(x_0).$$

It follows that, $LD^+f(x_0) = f'(x_0)$ and similarly it can be proved that $LD^-f(x_0) = f'(x_0)$. Hence $LDf(x_0)$ exists and $LDf(x_0) = f'(x_0)$.

Example 3. Let $f(x) = \begin{cases} \cos\frac{1}{x^2} & : \quad x \neq 0 \\ 0 \quad : & \quad x = 0. \end{cases}$

Clearly $f^{'}(0)$ does not exist. We prove that $LDf(0) = 0$ by showing that $\lim_{s\to\infty} s^2 \int_0^{\delta} e^{-st} \cos\frac{1}{t^2}dt = 0$. Let $g(x) = \begin{cases} x^3 \sin\frac{1}{x^2}, & \text{for } x \neq 0, \\ 0 \quad \text{for } x = 0. \end{cases}$

Then $g^{'}(x) = \begin{cases} 0, \quad \text{for } x = 0, \\ 3x^2 \sin\frac{1}{x^2} - 2\cos\frac{1}{x^2}, & \text{for } x \neq 0. \end{cases}$

Therefore $\cos\dfrac{1}{x^2} = \dfrac{3}{2}x^2\sin\dfrac{1}{x^2} - \dfrac{1}{2}g'(x)$. Let $\delta > 0$ be fixed. Choose $\epsilon, 0 < \epsilon < \delta$. Then there is $\delta_1 > 0$ such that $\delta_1 < \delta$ and $\left|t\sin\dfrac{1}{t^2}\right| < \dfrac{\epsilon}{3}$ for $0 < |t| \le \delta_1$. So,

$$\left|s^2\int_0^{\delta_1} e^{-st}t^2\sin\frac{1}{t^2}dt\right| \le \frac{\epsilon}{3}s^2\int_0^{\delta_1} e^{-st}tdt = \frac{\epsilon}{3} + o(1) \text{ as } s\to\infty. \tag{2}$$

$$\left|s^3\int_0^{\delta_1} e^{-st}t^3\sin\frac{1}{t^2}dt\right| \le \frac{\epsilon}{3}s^3\int_0^{\delta_1} e^{-st}t^2dt = \frac{2\epsilon}{3} + o(1) \text{ as } s\to\infty. \tag{3}$$

Now from (2) and (3)

$$\begin{aligned}
&\left|s^2\int_0^{\delta_1} e^{-st}\cos\frac{1}{t^2}dt\right| \\
&= \left|\frac{3s^2}{2}\int_0^{\delta_1} e^{-st}t^2\sin\frac{1}{t^2}dt - \frac{s^2}{2}\int_0^{\delta_1} e^{-st}g'(t)dt\right|. \\
&= \left|\frac{3s^2}{2}\int_0^{\delta_1} e^{-st}t^2\sin\frac{1}{t^2}dt - \frac{s^2}{2}\left[e^{-s\delta_1}g(\delta_1) + s\int_0^{\delta_1} e^{-st}g(t)dt\right]\right|. \\
&\le \frac{\epsilon}{2} + o(1) + \frac{1}{2}\frac{s^2}{e^{s\delta_1}}|g(\delta_1)| + \frac{\epsilon}{3} + o(1) \text{ as } s\to\infty. \\
&< \epsilon + \frac{1}{2}\frac{s^2}{e^{s\delta_1}}|g(\delta_1)| + o(1) \text{ as } s\to\infty.
\end{aligned}$$

Also, $s^2\displaystyle\int_{\delta_1}^{\delta_2} e^{-st}\cos\frac{1}{t^2}dt = o(1)$ as $s\to\infty$. Hence,

$$\left|s^2\int_0^{\delta} e^{-st}\cos\frac{1}{t^2}dt\right| \le \epsilon \text{ as } s\to\infty.$$

Since ϵ is arbitrary, $\displaystyle\lim_{s\to\infty} s^2\int_0^{\delta} e^{-st}\cos\frac{1}{t^2}dt = 0$.

Theorem 4. *If f is maximum or minimum at x_0 and $LDf(x_0)$ exists then $LDf(x_0) = 0$.*

Proof. Suppose f is maximum at x_0 then there exists $\delta > 0$ such that $f(x_0 + t) \le f(x_0)$ and $f(x_0 - t) \le f(x_0)$ $\forall\ 0 \le t < \delta$. So for any $s > 0$,

$$s^2\int_0^{\delta} e^{-st}[f(x_0+t) - f(x_0)]dt \le 0.$$

Letting $s\to\infty$ we get, $LD^+f(x_0) \le 0$. Again for any $s > 0$,

$$s^2\int_0^{\delta} e^{-st}[f(x_0-t) - f(x_0)]dt \le 0.$$

Hence,

$$(-s^2)\int_0^{\delta} e^{-st}[f(x_0-t) - f(x_0)]dt \ge 0.$$

Letting $s \to \infty$ we get, $LD^- f(x_0) \geq 0$. Since $LDf(x_0)$ exists $LDf(x_0) = 0$.
If f is minimum at x_0, the proof is similar.

Theorem 5. *(Darboux Theorem) Let f be continuous on $[a, b]$ and LDf exist on $[a, b]$. If $LDf(a) > 0 > LDf(b)$ then there is $c \in (a, b)$ such that $LDf(c) = 0$.*

Proof. Since $LDf(a) > 0$, for any $s > 0$,

$$s^2 \int_0^\delta e^{-st} \left[f(a+t) - f(a)\right] dt > 0.$$

This shows that $f(a + t) - f(a) > 0$ for some positive measure set in $[0, \delta]$. So there are points $x > a$ such that $f(x) > f(a)$. Hence a is not a maximum point of f. Now $LDf(b) < 0$ gives, for any $s > 0$,

$$(-s^2) \int_0^\delta e^{-st} \left[f(b-t) - f(b)\right] dt < 0.$$

Or,

$$s^2 \int_0^\delta e^{-st} \left[f(b-t) - f(b)\right] dt > 0.$$

Hence $f(b - t) - f(b) > 0$ for some positive measure set in $[0, \delta]$. So there are points $x < b$ such that $f(x) > f(b)$. So b is also not a maximum point of f. Now f is continuous on $[a, b]$ so maximum point of f is somewhere in (a, b). Let f is maximum at $c \in (a, b)$. Thus by Theorem 4, $LDf(c) = 0$.

Theorem 6. *(Rolle's Theorem) Suppose*

(i) f *is continuous in* $[a, b]$
(ii) LDf *exists in* (a, b)
(iii) $f(a) = f(b)$.

Then there exists $\xi \in (a, b)$ such that $LDf(\xi) = 0$.

Proof. Suppose f is not a constant function, otherwise there is nothing to prove. As f is continuous in $[a, b]$ so f attains a maximum and a minimum in $[a, b]$. Suppose f is maximum at $c \in [a, b]$ and minimum at $d \in [a, b]$. Clearly $c \neq d$ and by (iii) c or d must belong to (a, b). Hence Theorem 4 ensures that $\xi = c$ or $\xi = d$. This completes the proof.

Corollary 7. *(Lagrange's Mean Value Theorem) Suppose f is continuous on $[a, b]$ and $LDf(x)$ exists in (a, b). Then there is $c \in (a, b)$ such that $f(b) - f(a) = (b - a)LDf(c)$.*

Corollary 8. *(Monotonicity Theorem) Suppose f is continuous on $[a, b]$ and $LDf(x)$ exists and is positive in (a, b). Then f is monotone increasing in $[a, b]$.*

Corollary 9. *(Cauchy's Mean Value Theorem) Suppose f and g are continuous on $[a,b]$, $LDf(x)$ and $LDg(x)$ exist in (a,b) and $LDg(x) \neq 0$ in (a,b). Then there is $c \in (a,b)$ such that* $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{LDf(c)}{LDg(c)}$.

Theorem 10. *Let f be continuous on $[a,b]$ and $x \in (a,b)$. If LDf exists and continuous at x then $f'(x)$ exists and $f'(x) = LDf(x)$.*

Proof. Clearly LDf exists in a neighbourhood of x. So for sufficiently large n, by Corollary 7 we get

$$g_n(x) = n\left[f(x+\frac{1}{n}) - f(x)\right] = LDf(x+h_n), \quad 0 < h_n < \frac{1}{n}.$$

Since $LDf(x)$ is continuous at x, the right hand ordinary derivative

$$Rf'(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} LDf(x+h_n) = LDf(x).$$

Similarly we can prove that the left hand ordinary derivative $Lf'(x) = LDf(x)$.

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References

[1] A. Garai and S. Ray, *On The Symmetric Laplace derivative,* Acta Mathematica Hungarica, **133** (1–2) (2011), 166–184

[2] S. N. Mukhopadhayay and S. Ray, *On Laplace Derivative,* Analysis Mathematica, **36** (2010), 131–153.

[3] R. E. Sevtic, *The Laplace Derivative,* Comment. Math. Univ. Carolinae, **42.2** (2001), 331–343.

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THE LAPLACE DERIVATIVE II

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Abstract: Peano derivative is a generalization of ordinary derivative. Peano derivative is also generalized in many ways. Laplace derivative is one of such generalization (refer [1], [2], [5]). In [5], Sevtic introduced Laplace derivative and showed that it is more general than Peano derivative and in [2] it is shown that Laplace derivative shares many properties with Peano derivative. In the previous paper [4] we have discussed first order Laplace derivative. Here we have studied Higher order Laplace derivative. We have shown directly that Laplace derivative is more general than ordinary derivative. Also Taylor's Theorem and some properties of Laplace derivative are presented. As in [4], the proofs of this article are simple and elementary.

1. **Higher order Laplace derivative**

Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue integrable in a neighbourhood of $x \in \mathbf{R}$. If there are real numbers $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ such that

$$\lim_{s\to\infty} s^{n+1} \int_0^\delta e^{-st} \left[f(x+t) - \sum_{i=0}^{n-1} \frac{t^i}{i!}\alpha_i \right] dt$$

exists for some $\delta > 0$, then this limit is called the right hand Laplace derivative of f at x of order n and is denoted by $LD_n^+ f(x)$. Similarly if there are real numbers $\beta_0, \beta_1, \cdots \beta_{n-1}$ such that

$$\lim_{s\to\infty} (-1)^n s^{n+1} \int_0^\delta e^{-st} \left[f(x-t) - \sum_{i=0}^{n-1} \frac{(-t)^i}{i!}\beta_i \right] dt$$

exists for some $\delta > 0$, then this limit is called the left hand Laplace derivative of f at x of order n and is denoted by $LD_n^- f(x)$. It can be proved (see [2]) that if $LD_n^+ f(x)$ (respectively $LD_n^- f(x)$) exists, then $LD_i^+ f(x)$ (respectively $LD_i^- f(x)$) exists for $i = 1, 2, \cdots, n-1$. If $LD_n^+ f(x)$ and $LD_n^- f(x)$ exist and $LD_i^+ f(x) = LD_i^- f(x)$ for $i = 1, 2, \cdots, n$, then f is said to have nth order Laplace derivative $LD_n f(x)$. If f is Lebesgue integrable we simply write f is integrable.

Theorem 1.1. *If $f^{(k)}(x_0)$ exists, then $LD_k f(x_0)$ exists with equal value.*

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Proof. We prove this theorem by induction. For $k=1$, this is the Theorem 2 of [4]. Suppose for $k=r, r\geq 1$, the theorem is true and $f^{(r+1)}(x_0)$ exists. So $f^{(r)}(x)$ exists finitely in $(x_0-\sigma, x_0+\sigma)$ for some $\sigma>0$ and by induction hypothesis $f^{(r)}(x)=LD_r f(x)$ for $x\in(x_0-\sigma,x_0+\sigma)$. Choose δ such that $0<\delta<\frac{\sigma}{2}$. Let

$$I_{r+2}(f,s)=s^{r+2}\int_0^\delta e^{-st}\left[f(x_0+t)-\sum_{i=0}^{r}\frac{t^i}{i!}f^{(i)}(x_0)\right]dt.$$

Integrating by parts

$$I_{r+2}(f,s)=-s^{r+1}e^{-s\delta}\left[f(x_0+\delta)-\sum_{i=0}^{r}\frac{\delta^i}{i!}f^{(i)}(x_0)\right]+I_{r+1}(f',s).$$

So as $s\to\infty$ the right hand side approaches to $LD_r(f')(x_0)=(f')^{(r)}(x_0)=f^{(r+1)}(x_0)$. Hence $\lim_{s\to\infty} I_{r+2}(f,s)$ exists and equal to $f^{(r+1)}(x_0)$. So $LD^+_{r+1}f(x_0)=f^{(r+1)}(x_0)$. The proof for left hand derivative is similar. □

Lemma 1.1. *Let $\delta>0$ be fixed and f be integrable in $[-\delta,\delta]$ then for all $s>0$,*

(i) $$\int_0^\delta\int_0^t e^{-st}f(u)du dt=\frac{1}{s}\left[\int_0^\delta e^{-st}f(t)dt-e^{-s\delta}\int_0^\delta f(t)dt\right].$$

(ii) $$\int_0^\delta\int_0^t e^{-st}f(-u)du dt=\frac{1}{s}\left[\int_0^\delta e^{-st}f(-t)dt-e^{-s\delta}\int_0^\delta f(-t)dt\right].$$

Proof. We prove (i), the proof of (ii) is similar. For $0\leq t\leq\delta$ define $g(t)=-\frac{1}{s}\left[e^{-s\delta}-e^{-st}\right]$ then $g(\delta)=0$ and $g'(t)=-e^{-st}$. Now integrating by parts we get

$$\int_0^\delta g(t)f(t)dt=g(\delta)\int_0^\delta f(t)dt-\int_0^\delta\left[g'(t)\int_0^t f(u)du\right]dt.=\int_0^\delta e^{-st}\int_0^t f(u)du dt$$
$$=\int_0^\delta\int_0^t e^{-st}f(u)du dt.$$

This completes the proof. □

Example: Let $f(x)=\begin{cases}\cos\frac{1}{x^2} & : \quad x\neq 0,\\ 0 & : \quad x=0\end{cases}$

and $F(x)=\dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1}f(t)dt.$

Clearly $F^{(n)}(t)$ exists in $\mathbf{R}$ and $F^{(n)}(t)=f(t)$. So $F^{(n+1)}(0)$ does not exists. But we shall prove that $LD_{n+1}F(0)$ exists. Now

$$\begin{aligned}LD^+_{n+1}F(0) &= \lim_{s\to\infty} s^{n+2}\int_0^\delta e^{-st}F(t)dt.\\ &= \lim_{s\to\infty}\left\{s^{n+2}\left[\frac{e^{-st}}{-s}F(t)\right]_{t=0}^{\delta} + s^{n+1}\int_0^\delta e^{-st}\frac{1}{(n-2)!}\right.\\ &\qquad \left.\int_0^t (t-u)^{n-2}f(u)dudt\right\}.\\ &= \frac{1}{(n-2)!}\lim_{s\to\infty} s^{n+1}\int_0^\delta e^{-st}\int_0^t (t-u)^{n-2}f(u)dudt.\end{aligned}$$

Now integrating by parts $(n-2)$ times and performing the limit $s\to\infty$ at every time we get

$$LD^+_{n+1}F(0) = \lim_{s\to\infty} s^3\int_0^\delta e^{-st}\int_0^t f(u)dudt.$$

Using Lemma 1.1 (i)

$$LD^+_{n+1}F(0) = \lim_{s\to\infty} s^2\int_0^\delta e^{-st}f(t)dt,$$

so by Example 3 of [4]

$$LD^+_{n+1}F(0) = 0.$$

Again

$$\begin{aligned}LD^-_{n+1}F(0) &= \lim_{s\to\infty}(-1)^{n+1}s^{n+2}\int_0^\delta e^{-st}F(-t)dt.\\ &= \lim_{s\to\infty}(-1)^{2n+1}s^{n+2}\int_0^\delta e^{-st}\left\{\frac{1}{(n-1)!}\int_0^t (t-u)^{n-1}f(-u)du\right\}dt.\\ &= (-1)\lim_{s\to\infty}\{s^{n+2}\left[\frac{e^{-st}}{-s}\frac{1}{(n-1)!}\int_0^t (t-u)^{n-1}f(-u)du\right]_{t=0}^{\delta}\\ &\qquad + s^{n+1}\int_0^\delta e^{-st}\frac{1}{(n-2)!}\int_0^t (t-u)^{n-2}f(-u)dudt\}.\\ &= (-1)\frac{1}{(n-2)!}\lim_{s\to\infty} s^{n+1}\int_0^\delta e^{-st}\int_0^t (t-u)^{n-2}f(-u)dudt.\end{aligned}$$

Again integrating by parts $(n-2)$ times more and performing the limit $s\to\infty$ at every time we get

$$LD^-_{n+1}F(0) = \lim_{s\to\infty}(-1)s^3\int_0^\delta e^{-st}\int_0^t f(-u)dudt.$$

Using Lemma 1.1(ii)

$$LD^{-}_{n+1}F(0) = \lim_{s\to\infty}(-1)s^2\int_0^{\delta} e^{-st}f(-t)dt.$$

Now from Example 3 of [4]

$$LD^{-}_{n+1}F(0) = 0.$$

So, $LD_{n+1}F(0)$ exists and equals to zero.

Theorem 1.2. *If $f^{(n-1)}(x)$ exists in a neighbourhood of x and $LD_nf(x)$ exist, then $LD_nf(x) = LD_r(f^{(n-r)})(x)$ for $r = 1, 2, \ldots, (n-1)$.*

Proof. Since $f^{(n-1)}(x)$ exists in a neighbourhood of x, there is a $\delta > 0$ such that $f^{(n-1)}(x)$ exists in $(x-\delta, x+\delta)$. Write

$$I_n(f,s) = s^{n+1}\int_0^{\delta} e^{-st}\left[f(x+t) - \sum_{i=0}^{n-1}\frac{t^i}{i!}f^{(i)}(x)\right]dt.$$

Integrating by parts we get

$$I_n(f,s) = -s^n e^{-s\delta}\left[f(x+\delta) - \sum_{i=0}^{n-1}\frac{\delta^i}{i!}f^{(i)}(x)\right] + I_{n-1}(f',s).$$

So

$$LD_nf(x) = \lim_{s\to\infty} I_n(f,s) = \lim_{s\to\infty} I_{n-1}(f',s) = LD_{n-1}(f').$$

Repeating the process $(n-r-1)$ times $r > 0$ we get

$$LD_nf(x) = \lim_{s\to\infty} I_r(f^{(n-r)},s) = LD_r(f^{(n-r)}(x))$$ □

Lemma 1.2. *If LD_nf is bounded in $[a,b]$, then LD_rf is also bounded in $[a,b]$ for $r = 1, 2, \ldots, (n-1)$.*

Proof. Suppose k and K be such that $k < LD_nf(x) < K \quad \forall x \in [a,b]$. So for any $x \in [a,b]$ there is a positive integer $N_x > 1$ such that for $s \geq N_x$,

$$k < s^{n+1}\int_0^{\delta} e^{-st}\left[f(x+t) - \sum_{i=0}^{n-1}\frac{t^i}{i!}LD_if(x)\right]dt < K.$$

or,

$$k < N_x s^n\int_0^{\delta} e^{-st}\left[f(x+t) - \sum_{i=0}^{n-2}\frac{t^i}{i!}LD_if(x)\right]dt$$
$$- {N_x}^2 s^{n-1}\int_0^{\delta} e^{-st}\frac{t^{n-1}}{(n-1)!}LD_{n-1}f(x)dt < K.$$

Now letting $s \to \infty$ we get $k < N_x LD_{n-1}f(x) < K$, for $\lim_{s\to\infty} s^{n-1}\int_0^\delta e^{-st}t^{n-1}dt$ is the $(n-2)$-th order ordinary derivative of x^{n-1} at zero so is zero. This shows there is $M > 0$ such that $|LD_{n-1}f(x)| < \dfrac{M}{N_x}$, $\forall x \in [a,b]$ and so $|LD_{n-1}f(x)| < M$, $\forall x \in [a,b]$. This completes the proof. □

Lemma 1.3. *If $LD_1 f$ is bounded in $[a,b]$, then it is Lebesgue integrable on $[a,b]$ and $\int_a^x LD_1 f(t)dt = f(x) - f(a)$, $\forall x \in [a,b]$.*

This is Theorem 3.10 of [3].

Theorem 1.3. *Let $n \geq 2$. If $LD_n f$ exists in $[a,b]$ and $LD_{n-1}f$ is bounded in $[a,b]$, then $LD_{n-1}(LD_1 f)$ exists in $[a,b]$ and $LD_n f(x) = LD_{n-1}(LD_1 f(x))$, $\forall x \in [a,b]$.*

Proof. Let $n = 2$. Then $LD_1 f$ is bounded and by Lemma 1.3, for any $x, x+t \in [a,b]$, $\int_x^{x+t} LD_1 f(t)dt = f(x+t) - f(x)$. Hence, putting $z = u - x$ one gets

$$\begin{aligned} f(x+t) - f(x) - tLD_1 f(x) &= \int_x^{x+t} [LD_1 f(u) - LD_1 f(x)]du \\ &= \int_0^t [LD_1 f(z+x) - LD_1 f(x)]dz \end{aligned}$$

Now, by Lemma 1.1(i)

$$\begin{aligned} LD_2 f(x) &= \lim_{s\to\infty} s^3 \int_0^\delta e^{-st}[f(x+t) - f(x) - tLD_1 f(x)]dt \\ &= \lim_{s\to\infty} s^3 \int_0^\delta e^{-st} \int_0^t [LD_1 f(z+x) - LD_1 f(x)]dzdt. \\ LD_2 f(x) &= \lim_{s\to\infty} s^2 \int_0^\delta e^{-st}[LD_1 f(x+t) - LD_1 f(x)]dt \\ &= LD_1^+(LD_1 f(x)). \end{aligned}$$

Similarly it can be proved that $LD_1^-(LD_1 f(x))$ exists and equal to $LD_2 f(x)$. So $LD_1(LD_1 f(x))$ exists and $LD_1(LD_1 f(x)) = LD_2 f(x)$. Suppose the theorem is true for $n = r \geq 2$. Let $LD_{r+1} f$ exists in $[a,b]$ and $LD_r f$ is bounded in $[a,b]$. Hence by Lemma 1.2 $LD_i f$ is also bounded in $[a,b]$ and

(1) $\quad LD_{i-1}(LD_1 f(x)) = LD_i f(x)$ $\forall x \in [a,b]$ and for $1 \leq i \leq r$.

Now,

$$\begin{aligned}&LD_{r+1}f(x)\\&=\lim_{s\to\infty}s^{r+2}\int_0^\delta e^{-st}\left[f(x+t)-f(x)-\sum_{j=1}^{r}\frac{t^j}{j!}LD_jf(x)\right]dt\\&=\lim_{s\to\infty}s^{r+2}\int_0^\delta e^{-st}\int_0^t\left[LD_1f(z+x)-\sum_{j=1}^{r}\frac{z^{j-1}}{(j-1)!}LD_jf(x)\right]dzdt.\end{aligned}$$

By Lemma 1.1(i) and using (1) we can write

$$\begin{aligned}&LD_{r+1}f(x)\\&=\lim_{s\to\infty}s^{r+1}\int_0^\delta e^{-st}\left[LD_1f(x+t)-LD_1f(x)-\sum_{j=1}^{r-1}\frac{t^j}{j!}LD_j(LD_1f(x))\right]dt.\end{aligned}$$

Thus $LD_r^+(LD_1f(x))$ exists and equals to $LD_{r+1}f(x)$. Similarly we can prove $LD_r^-(LD_1f(x))$ exists and equals to $LD_{r+1}f(x)$. Thus $LD_{r+1}f(x)$ $= LD_r(LD_1f(x))$.This completes the proof by induction. □

The next Theorem is proved in [2] using monotonicity theorem of Laplace derivative. Here we give an alternative proof without using that type advanced theorem.

Theorem 1.4. *Let f be continuous at x and $LD_nf(x)$ exists and continuous at x then $f^{(n)}(x)$ exists with equal value.*

Proof. For $n=1$ the theorem follows from the Thoerem 10 of [4]. Suppose this theorem is true for $n=1,2,\dots,r,\quad r\geq 1$, and we prove it for $n=r+1$. So suppose $LD_{r+1}f$ is continuous at x. So there is an interval $[x-\sigma,x+\sigma]$ in which $LD_{r+1}f(x)$ exists and bounded. Hence by Lemma 1.2 $LD_if(x), i=1,2,...r$ are bounded in $[x-\sigma,x+\sigma]$. Thus by Theorem 1.3 $LD_{r+1}f(u)=LD_r(LD_1f(u))\ \ \forall u\in[x-\sigma,x+\sigma]$. Since $LD_r(LD_1f)$ is continuous at x by induction hypothesis the ordinary r th derivative of LD_1f exists at x and $(LD_1)^{(r)}(x)=LD_{r+1}f(x)$. So $LD_1f(x)$ is continuous at x. This implies $f'(x)$ exists and equals to $LD_1f(x)$. Thus $LD_{r+1}f(x)=f^{(r+1)}(x)$. This completes the proof. □

Theorem 1.5. *Let f be continuous on $[a,b]$ and $c\in(a,b)$. If $g'(c)$ and $LD_1f(c)$ exists, then $LD_1(fg)$ exists at c and*
$LD_1(fg)(c)=g(c)LD_1f(c)+f(c)g'(c)$.

Proof. Suppose $|f(x)|<K,\ \ \forall x\in[a,b]$ and define $h(x)=K-f(x)$. Then $h(x)>0\ \ \forall x\in[a,b]$. Let $h(c)<M<\infty$. So there exists $\delta_1>0$ such that $h(c+t)<M$, for $0<t<\delta_1$. Now suppose $g'(c)>0$. So there is $\delta_2>0$ such that

$g(c+t) - g(c) > 0$ for $0 < t < \delta_2$. Let $\delta = Min\{\delta_1, \delta_2\}$. Now for any $s > 0$ we have

$$s^2 \int_0^\delta e^{-st} \left[g(c+t) - g(c)\right] h(c+t) dt < Ms^2 \int_0^\delta e^{-st} \left[g(c+t) - g(c)\right] dt.$$

Letting $s \to \infty$, we get

$$\limsup_{s\to\infty} s^2 \int_0^\delta e^{-st}[g(c+t) - g(c)]h(c+t)dt \le Mg'(c).$$

Since this is true for arbitrary M, $0 < h(c) < M$,

$$\limsup_{s\to\infty} s^2 \int_0^\delta e^{-st}[g(c+t) - g(c)]h(c+t)dt \le h(c)g'(c).$$

Similarly assuming $0 < m < h(c)$ it can be proved that

$$\liminf_{s\to\infty} s^2 \int_0^\delta e^{-st}[g(c+t) - g(c)]h(c+t)dt \ge h(c)g'(c).$$

Hence the limit exists and

(2) $$\lim_{s\to\infty} s^2 \int_0^\delta e^{-st}[g(c+t) - g(c)]h(c+t)dt = h(c)g'(c).$$

Writing,

$$g(c+t)h(c+t) - g(c)h(c) = [g(c+t) - g(c)]h(c+t) + g(c)[h(c+t) - h(c)]$$

we get

(3) $$LD_1(gh)(c) = h(c)g'(c) + g(c)LD_1h(c).$$

Now $gh = Kg - gf$ and $LD_1h(c) = -LD_1f(c)$. So (3) gives

$$LD_1(gf)(c) = g(c)LD_1f(c) + f(c)g'(c).$$

Now if $g'(c) \le 0$, take $\phi(x) = (g'(c)+1)x - g(x)$ and we use ϕ instead of g. So the proof is completed. □

Theorem 1.6. *(* **Taylor's Theorem** *) Suppose*

(i) f *is continuous in* $[a, a+h]$

(ii) $LD_{n-1}f$ *is continuous in* $[a, a+h]$

(iii) LD_nf *exists in* $(a, a+h)$

Then there exists $\theta \in (0,1)$ *such that*

$$f(a+h) = \sum_{i=0}^{n-1} \frac{h^i}{i!} f^{(i)}(a) + \frac{h^n}{n!} LD_nf(a+\theta h)$$

Proof. Since $LD_{n-1}f$ is continuous in $[a, a+h]$ by Theorem 1.4, $f^{(n-1)}$ exists and continuous in $[a, a+h]$. By Theorem 1.2 $LD_n f(x) = LD_1(f^{(n-1)}(x))$, $\forall x \in (a, a+h)$. Let

$$F(x) = \sum_{i=0}^{n-1} \frac{(a+h-x)^i}{i!} f^{(i)}(x) + \frac{(a+h-x)^n}{h^n}\left[f(a+h) - \sum_{i=0}^{n-1} \frac{h^i}{i!} f^{(i)}(a)\right].$$

By Theorem 1.5 $F(x)$ is Laplace differentiable in $(a, a+h)$. Then $F(x)$ satisfy all the conditions of Theorem 6 of [4] in $[a, a+h]$. So there exists $\theta \in (0, 1)$ such that $LD_1 F(a + \theta h) = 0$ and our result is proved. □

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References

[1] A. Garai and S. Ray, *On The Symmetric Laplace derivative,* Acta Math. Hungarica, **133**(1-2)(2011),166–184.

[2] S. N. Mukhopadhyay and S.Ray, *On Laplace Derivative,* Analysis Mathematica, **36** (2010), 131–153.

[3] S. Ray and T. K. Garai, *On Laplace Continuity,* Real Analysis Exchange 37(2), (2011/2012), 299–310.

[4] S. Ray and A. Garai, *The Laplace derivative,* The Math. Student **81**, Nos. 1-4,(2012).

[5] R. E. Sevtic, *The Laplace Derivative,* Comment.Math.Univ.Carolinae, **42.2** (2001), 331–343.

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TRIADS OF INTEGERS WITH EQUAL SUMS AND EQUAL PRODUCTS

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Abstract: This paper gives two methods of generating an arbitrarily large number of distinct triads of integers with equal sums and equal products. One of these methods can be used to generate an arbitrarily large number of such triads in positive integers only.

The problem of finding positive integers with equal sums and equal products is mentioned in [2, p. 271] and [3]. A numerical example of 10 such triads is given in [1] and Schinzel [4] has proved the existence of an arbitrarily large number of such triads. Schinzel's proof is not constructive and it seems that till now an effective way of constructing a numerical example of an arbitrarily large number of such triads has not been published.

In this paper we give two parametric solutions that generate an arbitrarily large number of distinct triads of integers with equal sums and equal products. The first solution is very simple but it necessarily generates triads that include both positive and negative integers. The second solution provides a way of generating triads of positive integers with the desired property.

For the first solution, we write

$$x = (r+1)^2/r, \quad y = -r^2/(r+1), \quad z = -1/\{r(r+1)\}, \tag{1}$$

where r is an arbitrary parameter. It is readily verified that

$$x + y + z = 3, \quad xyz = 1, \tag{2}$$

and so, given any positive integer n, on assigning suitable rational values $r_1, r_2, \ldots, r_n$, to r, we get n triads of rational numbers, $\{x_i, y_i, z_i\}$, $i = 1, 2, \ldots, n$, with the same sum 3 and the same product 1. The numbers in the i^{th} triad will be distinct from the numbers in the j^{th} triad if $(x_i - x_j)(x_i - y_j)(x_i - z_j) \neq 0$. Further for all of the triads to be well-defined, we take $r_i(r_i + 1) \neq 0$ for each i with $1 \leq i \leq n$. It now follows that if the numbers

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r_i, $i = 1, 2, \ldots, n$, are so chosen that

$$(r_i - r_j)(r_i + r_j + 1)(r_i r_j - 1)(r_i r_j + r_i + 1)(r_i r_j + r_j + 1)(r_i r_j + r_i + r_j) \neq 0, \quad (3)$$

for any $i, j, 1 \leq i \leq n, 1 \leq j \leq n, i \neq j$, we get n distinct triads of rational numbers and hence, on appropriate scaling, we can obtain n distinct triads of integers with equal sums and equal products.

A second solution is obtained by writing

$$\begin{aligned} f_1(p, q, r) &= \frac{\phi_1^2(p, q, r)}{\phi_2(p, q, r)\phi_3(p, q, r)}, \\ f_2(p, q, r) &= \frac{pqr\phi_2^2(p, q, r)}{\phi_1(p, q, r)\phi_3(p, q, r)}, \\ f_3(p, q, r) &= \frac{\phi_3^2(p, q, r)}{\phi_1(p, q, r)\phi_2(p, q, r)}, \end{aligned} \quad (4)$$

where p, q, r are arbitrary parameters and

$$\begin{aligned} \phi_1(p, q, r) &= p^2q - 3pqr + pr^2 + q^2r, \\ \phi_2(p, q, r) &= p^2 - pq - pr + q^2 - qr + r^2, \\ \phi_3(p, q, r) &= p^2r + pq^2 - 3pqr + qr^2. \end{aligned} \quad (5)$$

It is readily verified that

$$\begin{aligned} f_1(p, q, r) + f_2(p, q, r) + f_3(p, q, r) &= p + q + r, \\ f_1(p, q, r)f_2(p, q, r)f_3(p, q, r) &= pqr. \end{aligned} \quad (6)$$

If we take arbitrary rational numbers x_1, y_1, z_1 and define

$$x_2 = f_1(x_1, y_1, z_1), \ \ y_2 = f_2(x_1, y_1, z_1), \ \ z_2 = f_3(x_1, y_1, z_1), \quad (7)$$

then, in view of the relations (6), we get

$$x_2 + y_2 + z_2 = x_1 + y_1 + z_1, \ \ x_2y_2z_2 = x_1y_1z_1. \quad (8)$$

We may now define

$$x_3 = f_1(x_2, y_2, z_2), \ \ y_3 = f_2(x_2, y_2, z_2), \ \ z_3 = f_3(x_2, y_2, z_2), \quad (9)$$

when we get, as before,

$$\begin{aligned} x_3 + y_3 + z_3 &= x_2 + y_2 + z_2 = x_1 + y_1 + z_1, \\ x_3y_3z_3 &= x_2y_2z_2 = x_1y_1z_1. \end{aligned} \quad (10)$$

By continuing this process, we obtain n triads of rational numbers, $\{x_i, y_i, z_i\}$, $i = 1, 2, \ldots, n$, with the same sum $x_1 + y_1 + z_1$ and the same product $x_1y_1z_1$ where n is any arbitrary positive integer. All of these n triads are defined in terms of the parameters x_1, y_1 and z_1 which are arbitrary rational numbers.

We will now prove that given any arbitrary positive integer n, we can choose x_1, y_1 and z_1 suitably so that the above process yields n distinct triads of positive rational numbers with equal sums and equal products.

We note that

$$\begin{aligned} f_1(p,\,q,\,r) - p &= \frac{(p-q)(r-p)\psi(p,\,q,\,r)}{\phi_2(p,\,q,\,r)\phi_3(p,\,q,\,r)}, \\ f_2(p,\,q,\,r) - q &= \frac{q(p-r)^2\psi(p,\,q,\,r)}{\phi_1(p,\,q,\,r)\phi_3(p,\,q,\,r)}, \\ f_3(p,\,q,\,r) - r &= \frac{(p-r)(r-q)\psi(p,\,q,\,r)}{\phi_1(p,\,q,\,r)\phi_2(p,\,q,\,r)}, \end{aligned} \tag{11}$$

where

$$\psi(p,\,q,\,r) = p^3r - 3p^2qr - pq^3 + 6pq^2r - 3pqr^2 + pr^3 - q^3r. \tag{12}$$

We further note that we can assign positive real values to $p,\,q,\,r$ such that $\psi(p,\,q,\,r) = 0$. For instance, taking $q = 2,\ r = 5$, we get $\psi(p,\,2,\,5) = 5p^3 - 30p^2 + 87p - 40$, and the cubic equation $\psi(p,\,2,\,5) = 0$ has a real root $p = 0.55673259...$. With these values of $p,\,q,\,r$, we get $\psi(p,\,q,\,r) = 0$, and further,

$$\phi_1(p,\,q,\,r) = 17.83...,\ \phi_2(p,\,q,\,r) = 15.41...,\ \phi_3(p,\,q,\,r) = 37.07... . \tag{13}$$

If we now take $y_1 = 2,\ z_1 = 5$, and assign to x_1 a rational value that is sufficiently close to 0.55673259..., then the value of $\psi(x_1,\,y_1,\,z_1)$ can be made sufficiently close to 0, and in view of the relations (11) and (13), the values of $x_2,\,y_2,\,z_2$ as defined by (7) will be rational and sufficiently close to the values of $x_1,\,y_1,\,z_1$ respectively, and hence also sufficiently close to the aforementioned values of $p,\,q,\,r$ respectively. We can now repeat the process as before and get rational values for $x_3,\,y_3,\,z_3$ which will again be sufficiently close to the values of $p,\,q,\,r$ respectively. By repeating this process n times, we get n triads of positive rational numbers that have equal sums and equal products.

We will now show that by suitably choosing the initial value for x_1, we can ensure that the n triads obtained above are all distinct. We note that on taking $y_1 = 2,\ z_1 = 5$, and treating x_1 as a parameter, the values of $x_i,\,y_i,\,z_i,\ i = 2,\,3,\,\ldots,\,n$, are given by rational functions of x_1, and so all equations of the type $x_i = x_j,\ y_i = y_j,\ z_i = z_j,\ i \neq j$ are expressible as polynomial equations in x_1. These polynomial equations have at most a finite number of rational roots. To ensure that we get n distinct triads of positive rational numbers, it suffices to choose an initial rational value of x_1 that is different from any of these rational roots and is also sufficiently close to 0.55673259... . The process described above then yields n distinct triads of positive rational numbers with equal sums and equal products, and on appropriate scaling, we can obtain n distinct triads of positive integers with the same property.

As a numerical example, taking $x_1 = 5/9 = 0.555...$, $y_1 = 2$, $z_1 = 5$, we get the following two triads,

$$\begin{aligned} x_2 &= 417605/750649 &= 0.556..., \\ y_2 &= 3120002/1563201 &= 1.995..., \\ z_2 &= 1806005/360961 &= 5.003..., \end{aligned}$$

and

$$\begin{aligned} x_3 &= \frac{43380480884930823344114004236773646902789918167622138805}{78300579005052796783939322721579187217247930722568530329} \\ &= 0.554..., \\ y_3 &= \frac{326333761304841208748458843051335839343612966946276396322}{162500108877126068318843934260527847184074343375136207761} \\ &= 2.008..., \\ z_3 &= \frac{187874544393196403204518769614213502925929809663033826405}{37625145174992005651461310667302188599218316608408992961} \\ &= 4.993... . \end{aligned}$$

It is readily verified that

$$\begin{aligned} x_i + y_i + z_i &= 68/9, \quad i = 1, 2, 3, \\ x_i y_i z_i &= 50/9, \quad i = 1, 2, 3. \end{aligned}$$

We thus have three triads of positive rational numbers x_i, y_i, z_i, $i = 1, 2, 3$, with the same sum 68/9 and the same product 50/9. The integers in the resulting triads, kx_i, ky_i, kz_i, $i = 1, 2, 3$, where k is suitably chosen, are too large and are therefore not being given explicitly.

References

[1] L. L. Foster and G. Robins, *Solution to Problem E 2872,* Amer. Math. Monthly 89 (1982), 499–500.

[2] R. K. Guy, *Unsolved problems in number theory,* third edition, Springer-Verlag (2004).

[3] J. G. Mauldon, *Problem E 2872,* Amer. Math. Monthly 88 (1981), 148.

[4] A. Schinzel, *Triples of positive integers with the same sum and the same product,* Serdica Math. J. 22 (1996), 587–588.

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LENGTH INEQUALITIES FOR VECTORS IN NORMED LINEAR SPACES

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Abstract: An important problem in approximation theory in linear spaces is to find, for a given set of vectors or a linear subspace and a direction vector: a vector in the subspace that is closer in length to the directional vector than the given set of vectors. This is related to the classical problems of Chebyshev centres, best approximations and central subspaces. In this survey article we explore these concepts and show their geometric significance.

1. Introduction

Let X be a normed linear space over the filed of real numbers. In this paper we consider certain length or norm inequalities that are related to proximinality and the notion of central subspaces from [2]. These questions are motivated by the work of Garkavi [4] and the study of intersection properties of balls by Lindenstrauss, [11]. Accordingly the paper has been divided into two sections. Section 2 deals with questions related to best approximations and Section 3 deals with central subspaces and intersection properties of balls. We have kept the style at an informal and elementary level, supplying more detailed arguments, in order to make the article more accessible.

We recall that a subspace $Y \subset X$ is said to be proximinal, if for every $x \in X$, there exists a $y \in Y$ such that $d(x, Y) = \|x - y\|$. In Section 2 we start motivating this notion by relating it to the notion of functionals in the dual space X^*, that attain the norm. These ideas then lead to the proof of Garkavi's theorem for subspaces of finite codimension, namely, if $Y \subset Z \subset X$ are closed subspaces such that the quotient space $X|Y$ is finite dimensional and Y is a proximinal subspace of X, then Z is a proximinal subspace of X.

Motivated by a proof of Garkavi's theorem (given in Section 2), for a Banach space X and a closed subspace $Y \subset X$, we introduce the notion of a Garkavi pair. Consider a Banach space property p whose description involves Y and X. We say that (Y, X) is a Garkavi pair for the property p, if Y has the property p in X and for any closed subspace Z such that $Y \subset Z \subset X$, if the quotient space $Z|Y$ has

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the property p in $X|Y$, then Z has property p in X. If p is the property 'Y is proximinal in X', then in this notation Garkavi's theorem can be interpreted as for a proximinal subspace $Y \subset X$, (Y, X) is a Garkavi pair for p.

This is different from the generally studied '3-space'-problem, in which one considers a Banach space property, satisfied by a subspace $Y \subset X$ and the quotient space $X|Y$, is also satisfied by X.

In [11], Lindenstrauss studied the relation between extension of linear operators and intersection properties of balls. Motivated by this, the notion of a central subspace was introduced in [2]. A closed subspace $Y \subset X$ is said to be a central subspace, if for any $y_1, \ldots, y_n \in Y$ and $x \in X$, there is a $y \in Y$ such that $\|y - y_i\| \leq \|x - x_i\|$ for all i. In Section 3 we show that this is related to the notion of Chebyshev centres of finite sets. We also give a description of central subspaces of $C(K)$, the space of continuous functions on a compact Hausdorff space K. For the property 'central subspace', we show that (c_0, ℓ^∞) is a Garkavi pair.

2. Best Approximations

Let X be a normed linear space and $Y \subset X$ be a subspace, i.e., Y is also a vector space under the operations of addition and multiplication by real numbers. Let $x \in X$ and $x \notin Y$. An interesting question in Analysis, which is at the heart of several applications is : is there a vector in Y that is closest to x?

Thus we are looking for a vector $y_0 \in Y$ such that $\|x - y_0\| \leq \|x - y\|$ for all $y \in Y$. This is what we mean by 'finding a vector in Y closer in length in the direction x'. When we can find such a vector we write $d(x, Y) = \|x - y_0\|$. y_0 is said to be a best approximation to x, in Y. When such a best approximation can be found for all $x \notin Y$, we say that Y is a proximinal subspace.

For $x \notin Y$, it can be seen that there exists a sequence $\{y_n\}_{n \geq 1} \subset Y$ such that $lim\|x - y_n\| = d(x, Y)$. Such a sequence is called an approximating sequence for x. Even if there is no best approximation for x in Y, in concrete situations there are algorithms known for finding 'good' approximating sequence for x.

If Y is a finite dimensional subspace, a well known result from Analysis called the Bolzano-Weierstrass theorem ensures that there is a $y_0 \in Y$ such that we may assume $lim\|y_0 - y_n\| = 0$, so that $d(x, Y) = \|x - y_0\|$. Thus any finite dimensional subspace is a proximinal subspace. In what follows, most often our subspaces are of infinite dimension. It is easy to see, as the 'distance' function is continuous, that proximinal subspaces are closed.

Example 2.1. Let X be a normed linear space. Let $f : X \to \mathbb{R}$ be a linear functional. We assume that $f \neq 0$. Now let us suppose that f satisfies the following length inequality, $|f(x)| \leq \|x\|$ for all x, in other words, f contracts the length. If $B(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}$ denotes the closed ball with center at x_0 and radius r ($B(0,1)$ is called the closed unit ball), then this is equivalent to $f(B(0,1)) \subset [-1,1]$.

Suppose further there is a vector x_0 with $f(x_0) = 1 = \|x_0\|$. In this case we say that f attains its norm at x_0 and

$$sup_{\{x:\|x\|=1\}}|f(x)| = f(x_0) = 1.$$

Consider the subspace $Y = ker(f) = \{x \in X : f(x) = 0\}$. We now claim that for any z with $f(z) \neq 0$, there is a $y_0 \in Y$ such that $d(z,Y) = \|z - y_0\|$. Put $y_0 = z - f(z)x_0$. $f(y_0) = f(z) - f(z)f(x_0) = 0$, so that $y_0 \in Y$. Now for any $y \in Y$,

$$\|z - y\| \geq |f(z - y)| = |f(z) - f(y)| = |f(z)| = \|f(z)x_0\| = \|z - y_0\|.$$

Thus $d(z,Y) = \|z - y_0\| = |f(z)|$. Hence Y is proximinal in X.

Now conversely suppose $Y = ker(f)$ is proximinal, for a non-zero linear functional $f : X \to \mathbb{R}$. First we will show that for some $M > 0$, $|f(x)| \leq M\|x\|$ for all $x \in X$. This is clearly equivalent to $f(B(0,1))$ is a bounded set. Since f is linear, this is also equivalent to, for some positive integer n_0, $f(B(0,\frac{1}{n_0}))$ is a bounded set. Suppose this statement is false. That is, for every positive integer n, $f(B(0,\frac{1}{n}))$ is not a bounded set. Again since f is linear and $f(0) = 0$, these sets are symmetric, convex and not bounded, so all of them coincide with $\mathbb{R}$. Let $y_n \in B(0,\frac{1}{n})$ such that $f(y_n) = 1$ for all n. Since f is a non-zero functional, also $f(X) = \mathbb{R}$. So let $z \in X$ be such that $f(z) = 1$. Clearly $y_n - z \in Y$ and it is easy to see that $y_n - z \to z$. Since $f(z) = 1$ we get a contradiction to the fact that Y is a closed set. Therefore for some $M > 0$, $|f(x)| \leq M\|x\|$ for all $x \in X$.

Thus by replacing f by an appropriate scalar multiple, if necessary, we may and do assume that $|f(x)| \leq \|x\|$, for all $x \in X$. We will next show that there is a x_0 such that $f(x_0) = 1 = \|x_0\|$. Consider the quotient vector space $X|Y$. This is again a vector space. Elements of this vector space are denoted by $[x]$ for $x \in X$ and one can define the length of the vector $\|[x]\| = d(x,Y)$. Now the function $\phi : X|Y \to \mathbb{R}$ defined by $\phi([x]) = f(x)$, for $[x] \in X|Y$, is a liner map and if $\phi([x]) = 0$ for $[x] \in X|Y$, then as $f(x) = 0$ we have that $[x]$ is the zero vector of $X|Y$. Thus ϕ is a one-to-one, linear and hence an onto map. So corresponding to $1 \in \mathbb{R}$ there is a vector $[x] \in X|Y$ with $\|[x]\| = 1$ and $\phi([x]) = 1$. But this means, $f(x) = 1$ and $d(x,Y) = 1$. Since Y is proximinal, $1 = d(x,Y) = \|x - y_0\|$ for some $y_0 \in Y$. Now $1 = \|x - y_o\| = f(x - y_0)$. Hence we have our claim with $x_0 = x - y_0$.

For a subspace $Y \subset X$, the dimension of the quotient vector space $X|Y$ is called the codimension of Y. Thus in the above example, we have a proximinal subspace Y of codimension one. In any normed linear space X, there is such a subspace Y. As already noted when X is finite dimensional, this is easy to establish. In the infinite dimensional case, this is a consequence of the well known Hahn-Banach theorem, that states that given any non-zero vector x_0 there is a functional f with $f(x_0) = \|x_0\|$ and $|f(x)| \leq \|x\|$ for all $x \in X$. See [7].

Another interesting question is that, 'how many' proximinal subspaces $Y \subset X$ of codimension one can one produce? That there always 'plenty' , in the sense of norm denseness in the dual space X^*, of the set of all norm attaining functionals, under completeness assumption on X, i.e., it is a Banach space. This is a consequence of the celebrated Bishop-Phelps theorem. See [7].

Now we can take this process one more step forward and ask 'for any normed linear space X, is there always a proximinal subspace $Y \subset X$ such that the quotient space $X|Y$ is of dimension 2? Nobody knows!, this a long standing open problem in this area.

Next we want to consider a more general situation. Let $f_i : X \to \mathbb{R}$ be n-many, linearly independent functionals. Suppose $|f_i(x)| \leq \|x\|$ for all x and for $1 \leq i \leq n$. Let $Y = \{x : f_i(x) = 0 \; 1 \leq i \leq n\}$. Then Y again is a closed linear subspace.

Like before we want to know when is Y a proximinal subspace?

In order to understand this problem, from our experience with the case of one functional, we know that one needs to consider quotient spaces. Let X be a normed linear space and let $Y \subset X$ be a closed subspace. Consider the quotient vector space $X|Y$. Here one can define the norm (or length) of a vector $[x]$, by, $\|[x]\| = d(x, Y) = inf_{\{y \in Y\}} \|x - y\|$. One can verify that this definition satisfies the properties of a norm and thus $X|Y$ is also a normed linear space. Also as $0 \in Y$, $\|[x]\| \leq \|x\|$.

Proposition 2.1. *Let X be a normed linear space and let $Y \subset X$ be a proximinal subspace. Suppose $Z \subset X$ is another closed subspace such that $Y \subset Z \subset X$. Assume further that $Z|Y$ is a proximinal subspace of $X|Y$. Then Z is a proximinal subspace of X.*

Proof. Let $x \in X$ and $x \notin Z$. It is easy to see that $[x] \notin Z|Y$. Since $Z|Y$ is a proximinal subspace of $X|Y$, there exists a $z_0 \in Z$ such that

$$d([x], Z|Y) = \|[x] - [z_0]\| = \|[x - z_0]\| = d(x - z_0, Y).$$

Now since $Y \subset X$ is a proximinal subspace, there exists a $y_0 \in Y$ such that $d(x - z_0, Y) = \|x - z_0 - y_0\|$. Note that $z_0 + y_0 \in Z$. We claim that this vector is a best approximation.

For any $z \in Z$,

$$\|x - z\| \geq \|[x - z]\| = \|[x] - [z]\| \geq \|[x] - [z_0]\| = d(x - z_0, Y) = \|x - (z_0 + y_0)\|.$$

Therefore Z is a proximinal subspace of X. □

We are now ready to prove the famous theorem due to A. L. Garkavi, [4], [15].

Theorem 2.1. *Let X be a normed linear space. Let Y, Z be closed subspaces of X such that Y is a proximinal subspace of X and $Y \subset Z$. Assume further that $X|Y$ is a finite dimensional space. Then Z is also a proximinal subspace of X.*

Proof. Since $X|Y$ is finite dimensional, $Z|Y \subset X|Y$ is also finite dimensional. By our earlier remarks, $Z|Y$ is a proximinal subspace of $X|Y$. Therefore all the conditions in the statement of the above Proposition are satisfied. Hence we have that Z is a proximinal subspace of X. □

We are now ready to answer the question posed earlier. Let $\mathbb{R}^n$ denote the vector space of n-tuples of real numbers. This space is of dimension n.

Corollary 2.1. *Let X be a normed linear space. Let $f_i : X \to \mathbb{R}$ be n-many, linearly independent functionals. Suppose $|f_i(x)| \leq \|x\|$ for all x and for $1 \leq i \leq n$. Let $Y = \{x : f_i(x) = 0, 1 \leq i \leq n\} = \cap_1^n ker(f_i)$. Suppose Y is proximinal. Then $ker(f_i)$ is a proximinal subsubspace for $1 \leq i \leq n$. More over for any k many indices $\{j_1, \ldots, j_k\}$ among $\{1, 2, .., n\}$, $\cap_1^k ker(f_{j_i})$ is a proximinal subspace.*

Proof. Define $\Phi : X|Y \to \mathbb{R}^n$ by $\Phi([x]) = (f_1(x), .., f_n(x))$. It is easy to see that Φ is a well defined and linear map. If $\Phi([x]) = 0$, then $f_i(x) = 0$ for $1 \leq i \leq n$, so that $x \in Y$. Thus $[x]$ is the zero vector of $X|Y$. This shows that Φ is a linear, one-to-one map. Hence $X|Y$ is a finite dimensional space. Now for $1 \leq i \leq n$, $ker(f_i) \subset Y \subset X$ satisfies the hypothesis of the above theorem. Therefore $ker(f_i)$ is a proximinal subspace.

By the same token, we have $\cap_1^k ker(f_{j_i}) \subset \cap_1^n ker(f_i) = Y$. So by the Garkavi's theorem again, $\cap_1^k ker(f_{j_i})$ is a proximinal subspace. □

Remark 2.1. Let f, g be two continuous linear functional such that $ker(f), ker(g)$ are proximinal subspaces. In general it is not known when does $ker(f+g)$ or more generally $ker(\alpha f + \beta g)$ is a proximinal subspace ?, for all scalars α, β. Note that if $Y = ker(f) \cap ker(g)$ is a proximinal subspace, then since $Y \subset ker(\alpha f + \beta g) \subset X$, by Garkavi's theorem we have that $ker(\alpha f + \beta g)$ is proximinal subspace.

There are very many ways in which these ideas can be taken forward. The property we have considered here is 'proximinality'. This is of particular interest since in general (in infinite dimensional spaces) the intersection of two proximinal subspaces need not be a proximinal subspace. See [10] and [12] for several applications involving best approximations and convex optimization.

One generic way of formulating these problems is to consider a property, say p, that is valid for finite dimensional spaces. Now suppose X is a Banach space, $Y \subset Z \subset X$ are closed subspaces such that Y has property p in X and the quotient space $X|Y$ is finite dimensional, does Z have the property p? We will pose one such question in the next section.

We conclude this section with an example of a subspace that is not proximinal in the space c_0 of null sequences of real numbers.

Example 2.2. Let $c_0 = \{\{\alpha_n\}_{n\geq 1} : lim\alpha_n = 0\}$. For $\{\alpha_n\}_{n\geq 1} \in c_0$ define $\|\{\alpha_n\}_{n\geq 1}\| = sup_{n\geq 1}|\alpha_n|$. This is a well defined quantity and is a norm. Define $\phi : c_0 \to \mathbb{R}$ by $\phi(\{\alpha_n\}_{n\geq 1}) = \sum_1^\infty \frac{1}{2^n}\alpha_n$, for $\{\alpha_n\}_{n\geq 1} \in c_0$. This is also well defined, linear functional. One can verify that $|\phi(\{\alpha_n\}_{n\geq 1})| \leq \|\{\alpha_n\}_{n\geq 1}\|$, for all $\{\alpha_n\}_{n\geq 1} \in c_0$. Let $Y = ker(\phi)$. We claim that Y is not a proximinal subspace.

From Example 2.1, it is enough to show that there is no vector $\{\alpha_n\}_{n\geq 1} \in c_0$ such that $\phi(\{\alpha_n\}_{n\geq 1}) = 1 = \|\{\alpha_n\}_{n\geq 1}\|$. Suppose there is such a sequence. We have, $\sum_1^\infty \frac{1}{2^n}\alpha_n = 1$. From this it is easy to see that $|\alpha_n| = 1$ for all n. But as $\{\alpha_n\}_{n\geq 1}$ is a null sequence, this is not possible. Therefore Y is a closed subspace of codimension 1 that is not a proximinal subspace.

Let ℓ^∞ denote the vector space of bounded sequences of real numbers. Once again, for $\{\alpha_n\}_{n\geq 1} \in \ell^\infty$, $\|\{\alpha_n\}_{n\geq 1}\| = sup_{n\geq 1}|\alpha_n|$ is a norm making it a Banach space. c_0 is a closed subspace here. It is easy to see that $\phi' : \ell^\infty \to \mathbb{R}$ defined by $\phi'(\{\alpha_n\}_{n\geq 1}) = \sum_1^\infty \frac{1}{2^n}\alpha_n$, for $\{\alpha_n\}_{n\geq 1} \in \ell^\infty$ is a well defined, linear functional. One can again verify that $|\phi'(\{\alpha_n\}_{n\geq 1})| \leq \|\{\alpha_n\}_{n\geq 1}\|$ for all $\{\alpha_n\}_{n\geq 1} \in \ell^\infty$. Let $Y' = ker(\phi')$. Now for the constant sequence $1 \in \ell^\infty$, $\phi'(1) = 1$. Thus by Example 2.1, Y' is a proximinal subspace of ℓ^∞. It is also an interesting exercise to show that c_0 is a proximinal subspace of ℓ^∞.

Remark 2.2. Arguments similar to the one given above, easily lead to a more general statement that, if a $\alpha \in \ell^1$ has infinitely many non-zero coordinates, then it fails to attain its norm on c_0.

Now one can ask in which Banach spaces, every closed subspace of codimension 1 is proximinal? A consequence of a well known theorem of R. C. James, is that those are precisely reflexive spaces, [7]. It turns out that in reflexive spaces, every closed subspace is proximinal. Thus we have the situation that if every closed subspace of codimension 1 of X is proximinal, then every closed subspace of X is a proximinal subspace.

Let X be a Banach space. A closed subspace $Y \subset X$ is said to be factor reflexive, if the quotient space $X|Y$ is reflexive. Garkavi's theorem for factor reflexive spaces, states that for closed subspaces, $Y \subset Z \subset X$ such that Y is proximinal in X and $X|Y$ is reflexive, then Z is proximinal in X. Since $X|Y$ is reflexive we have that $Z|Y$ is a proximinal subspace of $X|Y$, thus this version of Garkavi's theorem again follows from Proposition 2.1. See [15].

There has been a lot of work related to these ideas, see [8], [9], [5] and [3].

3. Central subspaces, Chebyshev centres and intersection properties of balls

Here is another concept of length inequalities. See [2].

Definition 3.1. Let X be a Banach space. A closed subspace $Y \subset X$ is said to be a central subspace, if for any $x \in X$ and for every finite set of vectors, $y_1, ..., y_n \in Y$, there is a $y \in Y$ such that $\|y - y_i\| \leq \|x - y_i\|$, for $1 \leq i \leq n$.

Thus here again we want a vector in Y, closer in length to the given set of finite vectors, in the direction of x.

It is easy to see that in the Euclidean space every closed subspace is a central subspace. It can be shown that in a finite dimensional normed space X, if every subspace is central then the space is the Euclidean space. Thus certain length inequalities can be used to distinguish Euclidean spaces among finite dimensional spaces. The monograph by D. Amir ([1]) contains several geometric conditions that will allow one to distinguish Euclidean spaces or more generally Hilbert spaces.

Question 1. *Let $Y \subset Z \subset X$ be Banach spaces such that $X|Y$ is the Euclidean space of dimension n. Suppose Y is a central subspace of X. Is Z a central subspace of X?*

In any Banach space, a subspace of dimension one, is always a central subspace. It is an interesting exercise to show that $c_0 \subset \ell^\infty$ is a central subspace.

Given a finite set $x_1, ..., x_n \in X$, $x_0 \in X$ is said to be a Chebyshev centre for this finite set, if

$$inf_{\{x \in X\}} max_{\{1 \leq i \leq n\}} \|x - x_i\| = max_{\{1 \leq i \leq n\}} \|x_0 - x_i\|.$$

Proposition 3.1. *Let $Y \subset X$ be a central subspace. Suppose $y_1, .., y_n \in Y$ has a Chebyshev centre in X. Then it has a Chebyshev centre (relative to Y) in Y.*

Proof. Let x_0 be a Chebyshev centre for $y_1, .., y_n$ in X. Since Y is a central subspace, there is a $y_0 \in Y$ such that $\|y_0 - y_i\| \leq \|x_0 - y_i\|$ for $1 \leq i \leq n$. Now for $y \in Y$,

$$\begin{aligned} max_{\{1 \leq i \leq n\}} \|y - x_i\| &\geq inf_{\{x \in X\}} max_{\{1 \leq i \leq n\}} \|x - x_i\| \\ &= max_{\{1 \leq i \leq n\}} \|x_0 - x_i\| \\ &\geq max_{\{1 \leq i \leq n\}} \|y_0 - y_i\|. \end{aligned}$$

Therefore $inf_{\{y \in Y\}} max_{\{1 \leq i \leq n\}} \|y - y_i\| = max_{\{1 \leq i \leq n\}} \|y_0 - y_i\|$. □

Next proposition (from [2]) again deals with quotient spaces and involves both the concepts studied here.

Proposition 3.2. *Let X be a Banach space with closed subspaces Y, Z such that $Y \subset Z \subset X$. Suppose Y is a proximinal subspace of X and Z is a central subspace. Then $Z|Y$ is a central subspace of $X|Y$.*

Proof. Let $[z_1], [z_2], ..., [z_n] \in Z|Y$ and $[x_0] \in X|Y$. Since Y is a proximinal subspace of X, there exists $y_i \in Y$ such that

$$d(x_0 - z_i, Y) = \|x_0 - z_i - y_i\| = \|[x_0] - [z_i]\|, \text{ for } 1 \leq i \leq n.$$

Since Z is a central subspace, for $\{z_i + y_i\}_{\{1\leq i\leq n\}}$ and $x_0 \in X$, there exists a $z_0 \in Z$ such that $\|z_0 - z_i - y_i\| \leq \|x_0 - z_i - y_i\|$ for $1 \leq i \leq n$. Now

$$\|[z_0] - [z_i]\| \leq \|z_0 - z_i - y_i\| \leq \|x_0 - z_i - y_i\| = \|[x_0] - [z_i]\|.$$

Thus $Z|Y$ is a central subspace of $X|Y$. □

Remark 3.1. Let $Y \subset c_0$ be as in Example 2.2. Consider $Y \subset c_0 \subset \ell^\infty$. Now Y is not a proximinal subspace of ℓ^∞ (as it is not a proximinal subspace of c_0) and c_0 is a central subspace of ℓ^∞. However $c_0|Y \subset \ell^\infty|Y$, being a space of dimension one, is a central subspace. Thus proximinality is not a necessary condition in the above proposition.

Let X be a normed linear space. We recall that for $x_0 \in X$ and $r > 0$, $B(x, r) = \{x \in X : \|x - x_0\| \leq r\}$ is the closed ball with center at x_0 and radius r. If $Y \subset X$ is a subspace then for $y_0 \in Y$, one can consider the closed ball $B(y_0, r)$ in Y or the bigger ball $\{x \in X : \|x - y_0\| \leq r\}$ in X. For notational simplicity, we write $B(y_0, r)$ in both cases. Here is an equivalent formulation of central subspaces in terms of intersection properties of balls.

Proposition 3.3. *A closed subspace $Y \subset X$ is a central subspace if and only if for any finite set $\{B(y_i, r_i)\}_{1\leq i\leq n}$ of closed balls in Y, if their intersection is non-empty in X, then it is non-empty in Y.*

Proof. Suppose $Y \subset X$ is a central subspace. Let $\{B(y_i, r_i)\}_{1\leq i\leq n}$ be closed balls in Y and suppose $\|x_0 - y_i\| \leq r_i$ for $1 \leq i \leq n$ and for some $x_0 \in X$. Since Y is a central subspace, there exists a $y_0 \in Y$ such that, $\|y_0 - y_i\| \leq \|x_0 - y_i\| \leq r_i$ for $1 \leq i \leq n$. Thus the balls intersect in Y.

Conversely suppose that for any finite set $\{B(y_i, r_i)\}_{1\leq i\leq n}$ of closed balls in Y, if their intersection is non-empty in X, then it is non-empty in Y. Let $x_0 \in X$. Consider the closed balls $B(y_i, \|x_0 - y_i\|)_{\{1\leq i\leq n\}}$. Clearly x_0 is in all of them. Thus by hypothesis, there exists a $y_0 \in Y$, such that $\|y_0 - y_i\| \leq \|x_0 - y_i\|$. □

Example 3.1. Consider $\mathbb{R}^3$ with the maximum norm, $\|(x_1, x_2, x_3)\| = max_{\{1\leq i\leq 3\}}|x_i|$. Let $Y = \{(x_1, x_2, -x_1 - x_2)\}$. We claim that Y is not a central subspace. Let $x_0 = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and let $y_1 = (-2, 1, 1), y_2 = (1, 1, -2), y_3 = (1, -2, 1)$. Note that $\|x_0 - y_i\| = \frac{3}{2}$ and is the only point in $\cap_1^3 B(y_i, \frac{3}{2})$. Therefore Y is not a central subspace.

Consider $C([0, 1])$ the vector space of all real-valued continuous functions defined on $[0, 1]$, equipped with the norm, $\|f\| = sup_{\{t\in[0,1]\}}|f(t)|$. Next, we describe all central subspaces Y of $C([0, 1])$. Our necessary and sufficient condition on Y does not involve any information about $C([0, 1])$!! In particular these arguments work in the case of continuous functions on any compact Hausdorff space.

Theorem 3.1. *A closed subspace $Y \subset C([0, 1])$ is a central subspace if and only* *if every finite family of pair-wise intersecting closed balls in Y have non-empty intersection in Y.*

Proof. Let $\{B(f_i, r_i)\}_{\{1 \leq i \leq n\}}$ be a finite collection of pair-wise intersecting closed balls in $C([0,1])$. We will show that $\cap_i^n B(f_i, r_i) \neq \emptyset$. Since the balls pair-wise intersect, for any $1 \leq i, j \leq n$, $\|f_i - f_j\| \leq r_i + r_j$. Thus for $t \in [0,1]$, $|f_i(t) - f_j(t)| \leq \|f_i - f_j\| \leq r_i + r_j$, for all $1 \leq i, j \leq n$. Therefore $-r_i - r_j \leq f_i(t) - f_j(t) \leq r_i + r_j$ for all $t \in [0,1]$ and $1 \leq i, j \leq n$. Now define a function f_0 on $[0,1]$ by $f_0(t) = min_{\{1 \leq i \leq n\}}\{f_i(t) + r_i\}$. Then f_0 is a continuous function and $-r_i \leq f_0(t) - f_i(t) \leq r_i$ for all $t \in [0,1]$ and $1 \leq i \leq n$. Thus $f_0 \in \cap_1^n B(f_i, r_i)$.

Now suppose $Y \subset C([0,1])$ be a central subspace. Let $\{B(y_i, r_i)\}_{\{1 \leq i \leq n\}}$ be a finite family of pair-wise intersecting balls in Y. If we consider them as balls in $C([0,1])$ with centers from Y, they will still be pair-wise intersecting. Thus from what we have proved above, there exists a $f_0 \in C([0,1])$ such that $\|f_0 - y_i\| \leq r_i$ for $1 \leq i \leq n$. Now since Y is a central subspace by Proposition 3.3, there exists a $y_0 \in Y$ such that

$$\|y_0 - y_i\| \leq \|f_0 - y_i\| \leq r_i, \text{ for } 1 \leq i \leq n.$$

Conversely suppose that every finite family of pair-wise intersecting closed balls in Y have non-empty intersection in Y. We will again use Proposition 3.3. Let $\{B(y_i, r_i)\}_{\{1 \leq i \leq n\}}$ be closed balls in Y and there exists a $x_0 \in C([0,1])$ such that $\|x_0 - y_i\| \leq r_i$ for $1 \leq i \leq n$. In particular

$$\|y_i - y_j\| \leq \|y_i - x_0\| + \|y_j - x_0\| \leq r_i + r_j.$$

Therefore the balls $B(y_i, r_i)$ pair-wise intersect in Y. Hence by hypothesis there exists a $y_0 \in \cap_1^n B(y_i, r_i)$. Thus by Proposition 3.3 again we get that Y is a central subspace of X. □

In the next proposition we show that (c_0, ℓ^∞) is a Garkavi pair for the property of being a central subspace. These ideas also work in the more general setting of Lindenstrauss spaces from [11] but is too technical to be included here.

Proposition 3.4. *Let $c_0 \subset Z \subset \ell^\infty$ be a closed subspace such that $Z|c_0$ is a central subspace of $\ell^\infty|c_0$. Then Z is a central subspace of ℓ^∞.*

Proof. As already noted, c_0 is a central subspace of ℓ^∞. Suppose $Z|c_0$ is a central subspace of $\ell^\infty|c_0$. It follows from the arguments given during the proofs of Proposition 3.3 and Theorem 3.1 that in $Z|c_0$ any finite collection of pair-wise intersecting closed balls have non-empty intersection. It now follows from Theorem 6.1 in [11] and the remarks made in the introduction in [14] that in Z, any finite collection of pair-wise intersecting closed balls have non-empty intersection. It follows again from the arguments given during the proof of Theorem 3.1 that Z is a central subspace of ℓ^∞. □

These ideas developed from the seminal work of Lindenstrauss, from the memoir [11] has numerous applications in approximation theory, convex optimization and are at the heart of the theory of geometry of Banach spaces. See [13] for some of these results for spaces over the complex filed.

References

[1] D. Amir, *Characterizations of inner product spaces*, Operator Theory: Advances and Applications, 20. Birkhuser Verlag, Basel, 1986.

[2] P. Bandyopadhyay and T. S. S. R. K. Rao, *Central subspaces of Banach spaces*, J. Approx. Theory 103 (2000) 206-222.

[3] Darapaneni Narayana and T. S. S. R. K. Rao, *Transitivity of proximinality and norm attaining functionals*, Colloq. Math. 104 (2006) 1-19.

[4] A. L. Garkavi , *On the best approximation by the elements of infinite dimensional subspaces of a certain class*, Mat. Sb. 62 (1963) 104-120.

[5] G. Godefroy and V. Indumathi, *Proximinality in subspaces of c_0*, J. Approx. Theory 101 (1999) 175-181.

[6] R. B. Holmes, *A course on optimization and best approximation*, Lecture Notes in Mathematics, No 257, Springer-Verlag .

[7] R. B. Holmes, *Geometric functional analysis and its applications*, Graduate Texts in Mathematics, No. 24. Springer-Verlag, New York-Heidelberg, 1975. x+246 pp

[8] V.Indumathi, *Proximinal subspaces of finite codimension in general normed linear spaces*, Proc. London Math. Soc. 45 (1982) 435-455.

[9] V. Indumathi, *On transitivity of proximinality*, J. Approx. Theory 49 (1987) 130-143.

[10] W. Krabs, *Optimization and approximation*, Translated by Philip M Anselone and Ronald B Guenther, John Wiley and Sons,Chichester-New York-Brisbane-Toronto, 1979, xii + 220 pp.

[11] J. Lindenstrauss, *Extension of compact operators*,Mem. Amer. Math. Soc. No. 48 1964, 112 pp.

[12] Efe A.Ok, *Real Analysis with Economic Applications*, Princeton University Press, 2007.

[13] T. S. S. R. K. Rao, *Chebyshev centres and centrable sets*, Proc. Amer. Math. Soc. 130 (2002) 2593-2598.

[14] T. S. S. R. K. Rao, *Three-space problem for some classes of L^1-predulas*, J. Math. Anal. Appl. 351 (2009) 311-314

[15] I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*,Die Grundlehren der mathematischen Wissenschaften, Band 171,Springer-Verlag, Berlin, 1970.

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GENERALIZED MATROIDS

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Abstract: The aim of this paper is to introduce the notion of generalized matroid and to formulate some simple properties of generalized matroids and to study, based on this concept, suitable generalizations of the concept of strong maps.

1. Introduction

The matroid terminology and notation will follow [9]. For an introduction on matroids, see also [5, 6, 7, 8]. In particular, a *matroid* M is an ordered pair $(E, \mathcal{O})$ such that $\mathcal{O}$ is a collection of subsets, called *open sets* of M, of a finite set E, called the *ground set* of M, such that $\emptyset$ is an open set, unions of open sets are open and if O_1 and O_2 are open sets and $x \in O_1 \cap O_2$, there exists an open set O_3 such that

$$(O_1 \cup O_2) - (O_1 \cap O_2) \subseteq O_3 \subseteq (O_1 \cup O_2) - \{x\}.$$

If $M = (E,\mathcal{O})$ is a matroid on E and the *inner of* A is $A^o =: \{x \in A \mid \exists O \in \mathcal{O}, x \in O \subseteq A\}$ [4] and the *closure* of A is $\bar{A} = \{x \in E :$ for every open set O in M that contains x, $O \cap A \neq \emptyset\}$ [4], a set $A \subseteq E$ is called a *feeble-open set* in M [4] if there exists an open set $O \in \mathcal{O}$ such that $O \subseteq A \subseteq \bar{O}$ and the *feeble-inner* of A is defined as $A_o =: \{x \in A \mid \exists$ a feeble-open set $O \in \mathcal{O}, x \in O \subseteq A\}$, a set $A \subseteq E$ is called a *P-open set* in M [2] if $A \subseteq \overline{A}^o$ and a set $A \subseteq E$ is called a *β-open set* in M [2] if $A \subseteq \overline{\overline{A}^o}$. For details on these notions, see [1, 3].

The purpose of this paper is to formulate some simple properties of generalized matroids and to study, based on this concept, suitable generalizations of the concept of strong maps.

2. Generalized matroids

We begin this section by defining the notion of generalized matroid.

Definition 2.1. A *generalized matroid* (simply *GM*) $\mathcal{M}$ is an ordered pair $(E,\mathcal{O})$ such that $\mathcal{O}$ is a collection of subsets of a set E such that $\emptyset \in \mathcal{O}$ and unions of elements in $\mathcal{O}$ is in $\mathcal{O}$.

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Let $\mathfrak{G}(E)$ denotes the collection of all GM's on E. Clearly, every matroid is a generalized matroid, but the converse needs not be true.

Example 2.1. Consider the set $E = \{a, b\}$ and $\mathcal{O} = \{\emptyset, \{a\}, E\}$. Then $(E, \mathcal{O})$ is a GM, but not a matroid.

Definition 2.2. Let E be a set and m a map from the power set $\mathcal{P}(E)$ to itself. We suppose that m is *monotonic*, *i.e.* if $A \subseteq B \subseteq E$, then $m(A) \subseteq m(B)$. We denote by $M(E)$ the collection of all monotonic maps $m : \mathcal{P}(E) \to \mathcal{P}(E)$. A set $A \subseteq E$ is called *m-open* if $A \subseteq m(A)$.

It obvious that $\emptyset$ is an m-open set and it is easy to show that arbitrary union of m-open sets is an m-open set. Thus m-open sets constitute a GM. Also its clear that we obtain the following important cases: the collection of all open sets A where $m(A) = A^o$, the collection of all feeble-open sets A where $m(A) = \overline{A^o}$, the collection of all P-open sets A where $m(A) = \overline{A}^o$ and the collection of all β-open sets A where $m(A) = \overline{\overline{A}^o}$.

Next, we show that the method of considering m-open sets for some $m \in M(E)$ can produce all GM's on E.

Lemma 2.1. *If $\mathcal{M}$ is a GM on E, then there is $m : \mathcal{P}(E) \to \mathcal{P}(E)$ such that $\mathcal{M}$ is the collection of all m-open sets. Furthermore,*

$$m(\emptyset) = \emptyset, m(A) \subseteq A, m(m(A)) = m(A) \text{ for all } A \subseteq E.$$

Proof Define $m(A)$ to be the union of all $M \in \mathcal{M}$ such that $M \subseteq A$. Then $m(A) \in \mathcal{M}$, $m(\emptyset) = \emptyset$ and $m(A) \subseteq A$. As $M \in \mathcal{M}$, $m(M) = M \supseteq M$ and so the elements of $\mathcal{M}$ are $m-$open sets, while $A \subseteq m(A)$ implies that $m(A) = A$ and $A \in \mathcal{M}$. Finally $m(A) \in \mathcal{M}$ implies that $m(m(A)) = m(A)$.
We write M_m for the collection of all $m - open$ sets.

Another way for obtaining GM's is to let $\psi : E \to \mathcal{P}(E)$ satisfying $x \in V$ for every$V \in \mathcal{P}(E)$. We say that $V \in \mathcal{P}(E)$ is a *generalized neighbourhood (simply GN) of* $x \in E$ and ψ is a *generalized neighbourhood system (simply GNS)* on E. We denote by $\Psi(E)$ the collection of all GNS's on E.

Lemma 2.2. *Let ψ be a GNS on E and suppose that $M \in \mathcal{M}$ if and only if $M \subseteq E$ such that:*

$$if\ x \in M, \text{ then there is } V \in \psi(x) \text{ such that } V \subseteq M.$$

Then $\mathcal{M}$ is a GM.

Proof Clearly $\emptyset \in \mathcal{M}$. If $M = \cup_{i \in \triangle} M_i$ and $G_i \in \mathcal{M}$ for $i \in \triangle$, then for $x \in M$ there exists $i \in \triangle$ such that there is $V \in \psi(x)$ satisfying $V \subseteq M_i$. Obviously $V \subseteq M$, so $M \in \mathcal{M}$.

Let us write $\mathcal{M} = \mathcal{M}_\psi$ in this case. Conversely:

Lemma 2.3. *If $\mathcal{M}$ is a GM on E, there is $\psi \in \Psi(E)$ satisfying $\mathcal{M} = \mathcal{M}_\Psi$. We can suppose that Ψ fulfils:*

$$V \in \mathcal{M} \textit{ for some } V \in \psi(x),\ x \in E. \qquad ((*))$$

Proof Let us define $V \in \psi(x)$ if and only if $x \in V \in \mathcal{M}$. Then clearly $\psi \in \Psi(E)$ fulfils (*). If $M \in \mathcal{M}$ and $x \in M$, then $x \in M \in \psi(x)$. Suppose that $x \in M$ implies the existence of $V \in \psi(x)$ such that $V \subseteq M$. Then M is the union of these sets $V \in \mathcal{M}$ so that $M \in \mathcal{M}$.

Let us write $\psi = \psi_\mathcal{M}$ for this ψ and $\psi \in \Psi_\mathcal{M}(E)$ if and only if ψ fulfils (*). Observe that possibly $\psi_\mathcal{M}(x) = \emptyset$ for $\mathcal{M} \in \mathfrak{G}(E)$, $x \in E$. More generally, we can define $\psi_\mathcal{A} \in \Psi(E)$ if $\mathcal{A} \subseteq \mathcal{P}(E)$ (thus $\mathcal{A}$ need not be a GM) : $V \in \psi_\mathcal{A}(x)$ for $x \in E$ if and only if $x \in V \in \mathcal{A}$.

Another construction is the following one: let $\mathcal{M} \in \mathfrak{G}(E)$ and $m \in M(E)$ be such that $m(A) \subseteq A$ for $A \subseteq E$. We write $V \in \psi(m, \mathcal{M})(x)$ for $x \in E$ if and only if $V = m(M)$ for some $M \in \mathcal{M}$ such that $x \in M$. Then clearly $\psi(m, \mathcal{M}) \in \Psi(E)$.

If $\mathcal{M}$ is a GM on E and $A \subseteq E$, we write $i_\mathcal{M}A$ for the largest $M \in \mathcal{M}$ such that $M \subseteq A$: it is the union of all $M \subseteq A$, $M \in \mathcal{M}$. We also call $\mathcal{M}$-open sets the elements of $\mathcal{M}$ and $\mathcal{M}$-closed sets their complements. $c_\mathcal{M}A$ is by definition the smallest $\mathcal{M}$-closed set containing A, i.e. the intersection of all $\mathcal{M}$-closed sets containing A. If $\mathcal{M} = \mathcal{M}_m$ for some $m \in M(E)$, then we write i_m for $i_{\mathcal{M}_m}$ and c_m for $c_{\mathcal{M}_m}$. We define similarly i_ψ and c_ψ for $\psi \in \Psi(E) : i_\psi = i_{\mathcal{M}\psi}$, $c_\psi = c_{\mathcal{M}\psi}$.

For $\psi \in \Psi(E)$ again, we define two more elements of $M(E)$: let $l_\psi A$ consist of all $x \in E$ such that there is $V \in \psi(x)$ satisfying $V \subseteq A$ and let $m_\psi A$ be the set of all those $x \in E$ for which $V \in \psi(x)$ implies $V \cap A \neq \emptyset$.

Lemma 2.4. *For every $\psi \in \Psi(E)$ and $A \subseteq E$, we have $l_\psi, m_\psi \in M(E)$, $m_\psi A = E - l_\psi(E - A)$, $i_\psi A \subseteq l_\psi A$ and $m_\psi A \subseteq c_\psi A$.*

Proof The first equality is obvious. If $x \in i_\psi A$, then there is $M \in \mathcal{M}_\psi$ such that $x \in M \subseteq A$. Then there is $V \in \psi(x)$ satisfying $V \subseteq M \subseteq A$ and $x \in l_\psi A$. The last equality results by considering the complements.

In general, $l_\psi A \neq i_\psi A$ and $m_\psi A \neq c_\psi A$.

Example 2.2. Let $E = \mathbb{R}$ and $\psi(x) = \{(x-1, x+1)\}$ for every $x \in \mathbb{R}$. Then for $A = (a, b)$, $b - a > 2$, clearly $l_\psi A = \{a+1, b-1\}$ while $\mathcal{M}_\psi = \{\emptyset, E\}$ and $i_\psi A = \emptyset$.

However, we can prove the following result:

Lemma 2.5. *If $\psi \in \Psi_\mathcal{M}(E)$ for every GM $\mathcal{M} = \mathcal{M}_\psi$ on E, then $l_\psi = i_\psi$ and $m_\psi = c_\psi$.*

Proof If $x \in l_\psi A$ for some $A \subseteq E$, then there is $V \in \psi(x)$ such that $x \in V \subseteq A$. By assumption $V \in \mathcal{M}_\psi$ and so $V \subseteq i_\psi A$ and $x \in i_\psi A$. The inclusion $l_\psi A \subseteq i_\psi A$ follows from Lemma 2.4, while the other equality results from considering the complements.

3. Generalized strong maps

We begin this section with the following definition:

Definition 3.1. Let $\mathcal{M}_1 = (E_1, \mathcal{O}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{O}_2)$ be GM's. A map $f : E_1 \to E_2$ is *($\mathcal{M}_1, \mathcal{M}_2$)−strong* if the inverse image of every element in $\mathcal{M}_2$ is in $\mathcal{M}_1$. We write $f : \mathcal{M}_1 \to \mathcal{M}_2$.

In the literature, we find many generalized strong definitions belonging to this type. For example if $\mathcal{M}_1$ and $\mathcal{M}_2$ are matroids, then $(\mathcal{M}_1, \mathcal{M}_2)-$strong is the strong in the classical sense, $(F\mathcal{M}_1, \mathcal{M}_2)-$strong is the feeble strong map in the sense of [2, 4], $(P\mathcal{M}_1, \mathcal{M}_2)-$strong is the pre-strong in the sense of [2], $(F\mathcal{M}_1, F\mathcal{M}_2)-$strong is the hesitant maps in the sense of [2, 4],
$(\beta\mathcal{M}_1, \mathcal{M}_2)-$strong is the $\beta-$*strong* in the sense of [4] and $(P\mathcal{M}_1, P\mathcal{M}_2)-$strong is the prehesitant in the sense of [4].

We obtain another (more general) kind of generalized strong maps if we consider two GM's $\psi_1 \in \Psi(E_1)$ and $\psi_2 \in \Psi(E_2)$.

Definition 3.2. A map $f : E_1 \to E_2$ is $(\psi_1, \psi_2)-$*strong* if for every $x \in E_1$ and $V_2 \in \psi_2(f(x))$, there is $V_1 \in \psi_1(x)$ such that $f(V_1) \subseteq V_2$.

A more general concept is:

Proposition 3.1. *A $(\psi_1, \psi_2)-$strong map is $(\mathcal{M}_{\psi_1}, \mathcal{M}_{\psi_2})-$strong.*

Proof If $M_2 \in \mathcal{M}_{\psi_2}$, $x \in E_1$ and $f(x) \in M_2$, then there exists $V_2 \in \psi_2(f(x))$ such that $V_2 \subseteq M_2$. By hypothesis, there exists $V_1 \in \psi_1(x)$ such that $f(V_1) \subseteq V_2$. Thus $f(V_1) \subseteq M_2$ and $V_1 \subseteq f^{-1}(M_2)$. Therefore $f^{-1}(M_2) \in \mathcal{M}_{\psi_1}$.

The following example shows that the converse of the preceding Lemma need not be true.

Example 3.1. Let $E = \{a, b, c\}$, $\mathcal{O}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{O}_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then $(E, \mathcal{O}_1)$ and $(E, \mathcal{O}_2)$ are matroids on E. If c is the closure operation and o is the interior operation, then consider $\psi_1 = \psi_1(c_o, \mathcal{M}_1)$ and $\psi_2 = \psi_2(c_o, \mathcal{M}_2)$. Then $\mathcal{M}_{\psi_1} = \{\emptyset, E\}$. In fact, if $M_1 \in \mathcal{M}_{\psi_1}$ and $a \in M_1$ then $\{a\} \in \mathcal{O}_1$ and $c_o\{a\} = \{a, c\}$ imply $c \in M_1$. Finally $c \in M_1$ implies $M_1 = E$ as E_1 is the only $\mathcal{O}_1$-open set containing c.

Similarly, $\mathcal{M}_{\psi_2} = \{\emptyset, E\}$. Thus $f = id_E$ is $(\mathcal{M}_{\psi_1}, \mathcal{M}_{\psi_2})-$strong. However f is not $(\psi_1, \psi_2)-$strong as $f(a) \in \{a\} \in \mathcal{O}_2$ and $c_o\{a\} = \{a, b\}$ in $(E, \mathcal{O}_2)$. Thus $\{a, b\} \in \psi_2(a)$. But if $V \in \psi_1(a)$, then $V \supseteq c_o\{a\} = \{a, c\} \subsetneq \{a, b\}$.

In order to obtain a sort of converse of the preceding Proposition, observe that:

Lemma 3.1. *If $\mathcal{M}$ is a GM on E and $\psi = \psi_{\mathcal{M}}$, then $\mathcal{M}_\psi = \mathcal{M}$.*

Proof If $M \in \mathcal{M}$ and $x \in M$, then $M \in \psi_{\mathcal{M}}(x)$ and hence $M \in \mathcal{M}_\psi$. Conversely, if $M \in \mathcal{M}_\psi$ and $x \in M$, then there exists $V \in \psi(x)$ such that $V \subseteq M$. By hypothesis, $V \in \mathcal{M}$. and as M is the union of these sets $V \in \mathcal{M}$, we have $M \in \mathcal{M}$.
Now we can prove

Proposition 3.2. *If f is $(\mathcal{M}_{\psi_1}, \mathcal{M}_{\psi_2})-$strong and $\psi_2 = \psi_{1\mathcal{M}_2}$ for some GM $\mathcal{M}_2$, then f is $(\psi_1, \psi_2)-$strong.*

Proof Let $x \in E_1$ and $V_2 \in \psi_2(f(x))$. Then by hypothesis $f(x) \in V_2$ and $V_2 \in \mathcal{M}_2 = \mathcal{M}_{\psi_2}$ by Proposition 3.1. Thus $x \in f^{-1}(V_2) \in \mathcal{M}_{\psi_1}$. Hence there exists $V_1 \in \psi_1(x)$ such that $V_1 \subseteq f^{-1}(V_2)$ and so $f(V_1) \subseteq V_2$. Therefore, f is $(\psi_1, \psi_2)-$strong.

We end this section with the following characterization of $(\psi_1, \psi_2)-$strong maps.

Theorem 3.1. *If $f : E_1 \to E_2$ is a map, $\psi_1 \in \Psi(E_1)$ and $\psi_2 \in \Psi(E_2)$, then the following are equivalent:*

(1) f is $(\psi_1, \psi_2)-$strong.
(2) $f(m_{\psi_1} A) \subseteq m_{\psi_2} f(A)$ for $A \subseteq E_1$.
(3) $m_{\psi_1} f^{-1}(B) \subseteq f^{-1}(m_{\psi_2} B)$ for $B \subseteq E_2$.

Proof

(1) $\Longrightarrow$ (2). If $x \in m_{\psi_1} A$, then $f(x) \in m_{\psi_2} f(A)$, otherwise there would exist $V_2 \in \psi_2(f(x))$ such that $V_2 \cap f(A) = \emptyset$. By hypothesis, there would be $U \in \psi(x)$ such that $f(U) \subseteq V_2$ and $f(U) \cap f(A) = \emptyset$, hence $U \cap A = \emptyset$, a contradiction.

(2) $\Longrightarrow$ (3). Let $A = f^{-1}(B)$. Then by (2)

$$f(m_{\psi_1} A) \subseteq m_{\psi_2} f(A) = m_{\psi_2} f(f^{-1}(B)) \subseteq m_{\psi_2} B,$$

hence $m_{\psi_1} f^{-1}(B) \subseteq f^{-1}(m_{\psi_2} B)$.

(3) $\Longrightarrow$ (1). If $V_2 \in \psi_2(f(x))$, then let $B = E_2 - V_2, f(x) \notin m_{\psi_2} B$, hence $x \notin f^{-1}(m_{\psi_2} B)$. By (3), $x \notin m_{\psi_1} f^{-1}(B)$ and so there exists $U \in \psi_1(x)$ such that $U \cap f^{-1}(B) = \emptyset$ and thus $f(U) \cap B = \emptyset$ and $f(U) \subseteq V_2$.

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References

[1] Talal Al-Hawary, *Characterizations of Certain Matroids via Flats,* Journal of Automata, Languages and Combinatorics, 7(3), 2002, 295–301.

[2] Talal Al-Hawary, *Characterization of Matroids via OFR-sets,* Turkish J. Math. 24, 2001, 1–11.

[3] Talal. Al-Hawary, *Decompositions of strong maps between matroids,* Italian J. Pure Appl. Math. No. 20, 2007, 9-18.

[4] Talal Al-Hawary, *Feeble matroids,* Italian J. Pure Appl. Math. 14, 2003, 87–94.

[5] J. Kung, *A Source Book in Matroid Theory,* Birkhauser, Boston Inc., Basel, 1986.

[6] J.G. Oxley, *Matroid Theory,* Oxford University Press, Oxford, 1992.

[7] K. Truemper, *Matroid Decomposition,* Academic Press, INC., New York, 1992.

[8] N. White, ed., *Matroid Applications,* Cambridge University Press, Cambridge, 1992.

[9] W. White, ed., *Theory of Matroids,* Cambridge University Press, New York, 1986.

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ON GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract: The object of the present paper is to characterize generalized Sasakian-space-forms satisfying certain curvature conditions on W_2-curvature tensor. In this paper we study W_2 - semisymmetric, W_2 - flat, $\xi - W_2$ - flat, W_2- recurrent generalized Sasakian-space-forms. Also $W_2.S = 0$ and $W_2.R = 0$ on generalized Sasakian-space-forms are studied.

1. Introduction

Studying almost Hermitian manifold, Alfred Gray, a well known geometrician, formulated a principle according to which the so called curvature identities for the Riemann-Christoffel tensor are the keys to understanding differential-geometric properties of such manifolds [14]. Many papers are devoted to the study of geometric consequences of these identities and to their analogous for almost contact metric structures. As a continuation of this line of research, we consider in the present paper some curvature properties of generalized Sasakian-space-forms regarding W_2 - curvature tensor.

A generalized Sasakian-space-form was defined by P. Alegre, D. E. Blair and A. Carriazo in [1] as that almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \tag{1.1}$$

where f_1, f_2, f_3 are some differential functions on M^{2n+1} and

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$
$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$
$$R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,$$

for any vector fields X, Y, Z on M^{2n+1}. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well known Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$. It is known that any

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three dimensional (α, β) - trans-Sasakian manifold with α, β depending on ξ is a generalized Sasakian-space-form [2]. P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon give results in [3] about B. Y. Chen′s inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid analyse in [4] and [5] the CR-submanifolds of generalized Sasakian-space-forms. In [10], U. K. Kim studied conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms. U. C. De and A. Sarkar [7] have studied generalized Sasakian-space-forms regarding projective curvature tensor.

On the other hand, Pokhariyal and Mishra [12] have introduced new tensor fields, called W_2 and E-tensor fields, in a Riemannian manifolds and studied their relativistic significance. Further, Pokhariyal [13] has studied some properties of this curvature tensor in a Sasakian manifold. Recently, Matsumoto, Ianus and Mihai [11] have studied P-Sasakian manifolds admitting W_2 and E-tensor fields. De and Sarkar [8] have studied P-Sasakian manifolds admitting W_2-tensor field. Yildiz and De [15] have studied W_2-curvature tensor in Kenmotsu manifolds.

The W_2-curvature tensor of type $(0, 4)$ is defined by

$$\begin{aligned} &'W_2(X, Y, Z, U) \\ &= {}'R(X, Y, Z, U) + \frac{1}{n-1}[g(X, Z)S(Y, U) - g(Y, Z)S(X, U)], \end{aligned} \tag{1.2}$$

where $'R$ is a Riemannian curvature tensor of type $(0, 4)$ defined by

$$'R(X, Y, Z, U) = g(R(X, Y)Z, U)$$

and S is the Ricci tensor of type $(0, 2)$. Motivated by the above studies, in the present paper, we study flatness and symmetry property of generalized Sasakian-space-forms regarding W_2-curvature tensor. The present paper is organized as follows

In this paper, we study the W_2-curvature tensor of generalized Sasakian-space-forms. In section 2, some preliminary results are recalled. In section 3, we study W_2- semisymmetric generalized Sasakian-space-forms. Section 4 deals with W_2 - flat generalized Sasakian-space-forms. $\xi - W_2$ -flat generalized Sasakian-space-forms are studied in section 5. In section 6, W_2 -recurrent generalized Sasakian-space- forms are studied. Section 7 is devoted to study generalized Sasakian-space-forms satisfying $W_2.S = 0$. The last section contains the generalized Sasakian-space-forms satisfying $W_2.R = 0$.

2. **Preliminaries**

If on an odd dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \phi(\xi) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, g(X, \xi) = \eta(X), \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifold [6] and the structure (ϕ, ξ, η, g) is called an almost contact metric structure to M^{2n+1}. In view of equations (2.1), (2.2) and (2.3), we have

$$g(\phi X, Y) = -g(X, \phi Y), g(\phi X, X) = 0, \tag{2.4}$$

$$(\nabla x\eta)(Y) = g(\nabla x\xi, Y). \tag{2.5}$$

Again we know that [1] in a $(2n+1)$-dimensional generalized Sasakian-space-form

$$\begin{aligned} R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y \\ - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{2.6}$$

for all vector fields X, Y, Z on M^{2n+1}, where R denotes the curvature tensor of M^{2n+1}.

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\,\eta(X)\eta(Y), \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\,\eta(X)\,\xi, \tag{2.8}$$

$$r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3). \tag{2.9}$$

We also have for a generalized Sasakian-space-forms

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi, X)Y = -R(X, \xi)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \tag{2.11}$$

$$\eta(R(X, Y)Z) = (f_1 - f_3)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)], \tag{2.12}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.13}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.14}$$

where Q is the Ricci operator, *i.e.* $g(QX, Y) = S(X, Y)$.

A generalized Sasakian-space-form is said to be η -Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.15}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on M^{2n+1}.

For a $(2n+1)$ - dimensional $(n > 1)$ almost contact metric manifold the W_2 - curvature tensor is given by [12]

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{2n}[g(X, Z)QY - g(Y, Z)QX]. \tag{2.16}$$

The W_2 -curvature tensor for a generalized Sasakian-space-form is given by

$$W_2(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + \frac{1}{2n}[\eta(X)QY - \eta(Y)QX], \tag{2.17}$$

$$\eta(W_2(X,Y)\xi) = 0, \tag{2.18}$$

$$W_2(\xi,Y)Z = -W_2(Y,\xi)Z = -(f_1 - f_3)\eta(Z)Y + \frac{1}{2n}\eta(Z)QY], \tag{2.19}$$

$$\eta(W_2(\xi,Y)Z) = -\eta(W_2(Y,\xi)Z) = 0 \tag{2.20}$$

and

$$\eta(W_2(X,Y)Z) = 0. \tag{2.21}$$

3. W_2 - semisymmetric generalized sasakian-space-forms

Definition 3.1. A $(2n+1)$- dimensional $(n > 1)$ generalized Sasakian-space-form is said to be W_2 - semisymmetric [7] if it satisfies $R.W_2 = 0$, where R is the Riemannian curvature tensor of the space-forms.

Theorem 3.1. *A* $(2n+1)$ *- dimensional* $(n > 1)$ *generalized Sasakian-space-form satisfies* $R(\xi, X), W_2 = 0$ *if and only if* $f_1 = f_3$ *or* $W_2 = 0$.

Proof: Let us consider

$$(R(\xi, X).W_2)(Y, Z)U = 0. \tag{3.1}$$

The above equation can be written as

$$\begin{aligned} &R(\xi,X)W_2(Y,Z)U - W_2(R(\xi,X)Y,Z) - W_2(Y,R(\xi,X)Z)U \\ &- W_2(Y,Z)R(\xi,X)U = 0. \end{aligned} \tag{3.2}$$

In view of equation (2.11) the above equation reduces to

$$\begin{aligned} &(f_1 - f_3)[g(X, W_2(Y,Z)U)\xi - \eta(W_2(Y,Z)U)X \\ &- g(X,Y)W_2(\xi,Z)U + \eta(Y)W_2(X,Z)U - g(X,Z)W_2(Y,\xi)U \\ &+ \eta(Z)W_2(Y,X)U - g(X,U)W_2(Y,Z)\xi + \eta(U)W_2(Y,Z)X] \\ &= 0. \end{aligned} \tag{3.3}$$

Now, taking the inner product of above equation with ξ and using equation (2.2), we get

$$\begin{aligned} &(f_1 - f_3)[g(X, W_2(Y,Z)U) - \eta(W_2(Y,Z)U)\eta(X) \\ &- g(X,Y)\eta(W_2(\xi,Z)U) + \eta(Y)\eta(W_2(X,Z)U) - g(X,Z)\eta(W_2(Y,\xi)U) \\ &+ \eta(Z)\eta(W_2(Y,X)U) - g(X,U)\eta(W_2(Y,Z)\xi) + \eta(U)\eta(W_2(Y,Z)X)] \\ &= 0, \end{aligned} \tag{3.4}$$

which on using equations (2.18), (2.20) and (2.21) gives

$$(f_1 - f_3)\,{}'W_2(Y, Z, U, X) = 0. \tag{3.5}$$

This shows that either $f_1 = f_3$ or $W_2 = 0$.
Converse part is obvious from equations (3.1) and (3.5).

4. W_2 -flat generalized sasakian-space-forms

Theorem 4.1. *A $(2n+1)$ - dimensional $(n > 1)$ generalized Sasakian-space-form satisfying $W_2 = 0$ is an η - Einstein manifold.*

Proof: Suppose $W_2 = 0$ on generalized Sasakian-space-form, then from equation (2.16), we have

$$\begin{aligned} &R(X,Y)Z \\ &= -\frac{1}{2n}\left[(2nf_1 + 3f_2 - f_3)\{g(X,Z)Y - g(Y,Z)X\}\right. \\ &\left.-(3f_2 + (2n-1)f_3)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi\right], \end{aligned} \tag{4.1}$$

Taking the inner product of above equation with U, we get

$$\begin{aligned} &{}'R(X,Y,Z,U) \\ &= -\frac{1}{2n}\left[(2nf_1 + 3f_2 - f_3)\{g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\}\right. \\ &\left.-(3f_2 + (2n-1)f_3)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(U)\right]. \end{aligned} \tag{4.2}$$

Putting $Y = Z = e_i$ in above equation and taking summation over i,
$1 \le i \le 2n+1$, we get

$$S(X,U) = (2nf_1 + 3f_2 - f_3)g(X,U) + (-3f_2 - (2n-1)f_3)\eta(X)\eta(Y), \tag{4.3}$$

which shows that M^{2n+1} is an η - Einstein manifold. This completes the proof.

Also in view of theorem (3.1) and theorem (4.1), we can state as follows

Theorem 4.2. *W_2 - semisymmetric generalized Sasakian-space-form is either an η - Einstein manifold or $f_1 = f_3$.*

5. $\xi - W_2$-flat generalized sasakian-space-forms

Definition 5.1. A $(2n+1)$ - dimensional $(n > 1)$ generalized Sasakian-space-form is said to be $\xi - W_2$ - flat [16] , if $W_2(X,Y)\xi = 0$ for all $X, Y \in TM$.

Theorem 5.1. *If a $(2n+1)$ - dimensional $(n > 1)$ generalized Sasakian-space-form is $\xi - W_2$ - flat, then it is an η- Einstein manifold.*

Proof: Suppose generalized Sasakian-space-form is $\xi - W_2$ - flat, *i.e.* $W_2(X,Y)\xi = 0$.
Then in view of equation (2.17), we have

$$(f_1 - f_3)[\eta(Y)X - \eta(X)Y] + \frac{1}{2n}[\eta(Y)QX - \eta(X)QY] = 0, \tag{5.1}$$

which by putting $Y = \xi$ takes the form

$$QX = 2n(f_1 - f_3)[2\eta(X)\xi - X], \tag{5.2}$$

Now, taking the inner product of above equation with U, we get

$$S(X,U) = -2n(f_1 - f_3)[g(X,U) + (-2)\eta(X)\eta(U)], \tag{5.3}$$

which shows that generalized Sasakian-space-form is an η -Einstein manifold. This completes the proof.

6. W_2 - recurrent generalized sasakian-space-forms

Definition 6.1. A non-flat Riemannian manifold M^{2n+1} is said to be W_2 - recurrent if its W_2 - curvature tensor satisfies the condition

$$\nabla W_2 = A \otimes W_2, \tag{6.1}$$

where A is non-zero 1-form.

Theorem 6.1. *If a* $(2n+1)$ *- dimensional* $(n > 1)$ *generalized Sasakian-space-form is* W_2 *- recurrent, then either* $f_1 = f_3$ *or* $W_2 = 0$.

Proof: We define a function $f^2 = g(W_2, W_2)$ on M^{2n+1}, where the metric g is extended to the inner product between the tensor fields. Then we have

$$f(Yf) = f^2 A(Y),$$

This can be written as

$$Yf = f(A(Y)), (f \neq 0). \tag{6.2}$$

From above equation, we have

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of above equation is identically zero and $f \neq 0$ on M^{2n+1}. Then

$$dA(X,Y) = 0, \tag{6.3}$$

i.e. 1-form A is closed.

Now from

$$(\nabla_Y W_2)(Z,U)V = A(Y)W_2(Z,U)V,$$

we have

$$(\nabla_X \nabla_Y W_2)(Z,U)V = \{XA(Y) + A(X)A(Y)\}W_2(Z,U)V, \tag{6.4}$$

In view of equations (6.3) and (6.4) , we have

$$\begin{aligned}(R(X,Y).W_2)(Z,U)V &= [2dA(X,Y)]W_2(Z,U)V \\ &= 0.\end{aligned} \tag{6.5}$$

Thus in view of theorem (3.1), we have either $f_1 = f_3$ or $W_2 = 0$.

Also in consequence of theorem (4.1), we can state as follows

Theorem 6.2. *A W_2 - recurrent generalized Sasakian-space-form is an η - Einstein manifold.*

7. Generalized sasakian-space-forms satisfying $W_2.S = 0$

Theorem 7.1. *A $(2n+1)$ - dimensional $(n > 1)$ generalized Sasakian-space-form satisfying $W_2.S = 0$ is an η -Einstein manifold.*

Proof: Let us consider generalized Sasakian-space-form satisfying $W_2(\xi, X).S = 0$.
In this case we can write

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0. \tag{7.1}$$

In view of equation (2.19) above equation reduces to

$$\begin{aligned}&-(f_1 - f_3)[\eta(Y)S(X,Z) + \eta(Z)S(X,Y)] \\ &+\frac{1}{2n}[\eta(Y)S(QX,Z) + \eta(Z)S(QX,Y)] = 0.\end{aligned} \tag{7.2}$$

Now, putting $Z = \xi$ in above equation, we get

$$S(QX, Y) = 2nS(X, Y). \tag{7.3}$$

By virtue of equation (2.7) above equation takes the form

$$S(X, Y) = 2ng(X, Y) - 2n\frac{K_1}{K_2}\eta(X)\eta(Y), \tag{7.4}$$

where

$$K_1 = (3f_2 + (2n-1)f_3)(1 - f_1 + f_3)$$

and

$$K_2 = (2nf_1 + 3f_2 - f_3),$$

which shows that M^{2n+1} is an η - Einstein manifold. This completes the proof.

8. Generalized sasakian-space-forms satisfying $W_2.R = 0$

Theorem 8.1. *If a* $(2n+1)$ *- dimensional* $(n > 1)$ *generalized Sasakian-space-form satisfies* $W_2.R = 0$*, then either* $f_1 = f_3$ *or it is an Einstein manifold.*

Proof: Suppose M^{2n+1} satisfying $W_2(\xi, X).R = 0$, then it can be written as

$$\begin{aligned} &W_2(\xi, X)R(Y, Z)U - R(W_2(\xi, X)Y, Z)U \\ &- R(Y, W_2(\xi, X)Z)U - R(X, Y)W_2(\xi, X)U = 0, \end{aligned} \tag{8.1}$$

which on using equation (2.19) takes the form

$$\begin{aligned} &(f_1 - f_3)[-\eta(R(Y, Z)U)X + \eta(Y)R(X, Z)U \\ &+ \eta(Z)R(Y, X)U + \eta(U)R(Y, Z)X] + \frac{1}{2n}[\eta(R(Y, Z)U)QX \\ &- \eta(Y)R(QX, Z)U - \eta(Z)R(Y, QX)U - \eta(U)R(Y, Z)QX] \\ &= 0. \end{aligned} \tag{8.2}$$

Taking the inner product of above equation with ξ, we get

$$\begin{aligned} &(f_1 - f_3)[-\eta(R(Y, Z)U)\eta(X) + \eta(Y)\eta(R(X, Z)U) \\ &+ \eta(Z)\eta(R(Y, X)U) + \eta(U)\eta(R(Y, Z)X)] + \frac{1}{2n}[\eta(R(Y, Z)U)\eta(QX) \\ &- \eta(Y)\eta(R(QX, Z)U) - \eta(Z)\eta(R(Y, QX)U) - \eta(U)\eta(R(Y, Z)QX)] \\ &= 0. \end{aligned} \tag{8.3}$$

Now using equations (2.6), (2.11) and (2.12) in above equation, we get

$$\begin{aligned} &(f_1 - f_3)[(f_1 - f_3)\{\eta(U)(Y)g(X, Z) - \eta(U)\eta(Z)g(X, Y) \\ &- \eta(Z)\eta(X)g(Y, U)\}] + \frac{1}{2n}[(f_1 - f_3)\{\eta(U)\eta(Z)S(X, Y) \\ &- \eta(U)\eta(Y)S(X, Z) + 2n(f_1 - f_3)\eta(Z)\eta(X)g(Y, U)\}] \\ &= 0. \end{aligned} \tag{8.4}$$

Putting $Z = U = e_i$ in above equation and summing over $i, 1 \le i \le 2n+1$, we get

$$(f_1 - f_3)[-(f_1 - f_3)g(X, Y) + \frac{1}{2n}S(X, Y)] = 0, \tag{8.5}$$

which gives either $f_1 = f_3$ or

$$S(X, Y) = 2n(f_1 - f_3)g(X, Y), \tag{8.6}$$

which shows that M^{2n+1} is an Einstein manifold. This completes the proof.

In particular if for a Sasakian-space-form $f_1 - f_3 = 1$, then from theorem (8.1) we have a corollary as

Corollary 8.1. *A* $(2n + 1)$ *- dimensional* $(n > 1)$ *Sasakian-space-form satisfying* $W_2.R = 0$ *is an Einstein manifold.*

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References

[1] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian-space-forms,* Israel J. Math., 141 (2004), 151–183.

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian-space-forms,* Differential Geom. and its application, 26 (2008), 656–666.doi 10.1016/J.difgeo.

[3] P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon, *B. Y. Chen′s innequality for submanifolds of generalized space forms,* Indian J. Pure Appl. Math., 38 (2007), 185–201.

[4] R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid, *CR-Submanifolds of generalized Sasakian- space-forms,* JP J. Geom.Topol., 6 (2006), 151–166.

[5] R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid, *Contact CR-warped product submanifolds in generalized Sasakian-space-forms,* Balkan J. Geom., Appl., 11(2) (2006), 1–10.

[6] D. E. Blair, *Contact manifolds in a Riemannian geometry,* Lecture notes in Math., 509, Springer Verlag, Verlin, (1976).

[7] U. C. De and Avijit Sarkar, *On the projective curvature tensor of generalized Sasakian-space- forms,* Quaestiones Mathematicae, 33(2010), 245–252.

[8] U. C. De and A. Sarkar, *On a type of P-Sasakian manifolds,* Math. Reports, 11(61) 2(2009), 139–144.

[9] U.C. De, R.N.Singh and S.K.Pandey, *On the conharmonic curvature tensor of generalized Sasakian-space-forms,* (Communicated).

[10] U. K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms,* Note di Mathematica, 26 (2006), 55–67.

[11] K. Matsumoto, S. Ianus and I. Mihai, *On P-Sasakian manifolds which admit certain tensor fields,* Publ. Math. Debrecen, 33 (1986), 61–65.

[12] G. P. Pokhariyal and R. S. Mishra, *Curvature tensors and their relativistic significance ,* Yokohama Mathematical Journal, 18 (1970), 105–108.

[13] G. P. Pokhariyal, *Study of a new curvature tensor in a Sasakian manifold,* Tensor N.S., 36 (1982), 222–225.

[14] E. A. Polkina, *Curvature identities for almost contact metric manifolds,* Russian Mathematics (IZ VUZ), 51 (7) (2007), 54–57.

[15] Ahmet Yildiz and U. C. De, *On a type of Kenmotsu manifolds,* Differential Geometry-Dynamical Systems, 12 (2010), 289–298.

[16] G. Zhen, J. L. Cabrerizo, L. M. Fernandez and M. Fernandez, *On ξ-conformally flat contact metric manifolds,* Indian J. Pure Appl. Math., 28 (1997), 725–734.

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EXISTENCE OF A UNIQUE GROUP OF FINITE ORDER

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Abstract: Let n be a positive integer. Then cyclic group $\mathbb{Z}_n$ is the only group of order n if and only if g.c.d$(n, \phi(n)) = 1$, where ϕ denotes the Euler-phi function. In this article we have given another proof of this result using the concept of semidirect product and induction.

1. Introduction

One of the main problem in group theory is to classify groups upto isomorphisms. For example one may ask: How many non-isomorphic groups of order n are there and what are they? The Fundamental theorem of abelian groups says that every finite abelian group is the direct product of cyclic groups of prime power order, which classifies the non-isomorphic abelian groups of order n. But there is no classification of non-isomorphic non-abelian groups of order n. For every natural n, there must be a cyclic group $\mathbb{Z}_n$ of order n. If n is prime, then $\mathbb{Z}_n$ is the only group of order n. Also if $n = pq$, where p and q are distinct primes such that p does not divide $(q-1)$, then by Sylow theorem, $\mathbb{Z}_n$ is the only group of order n. Thus there is a natural question: what are conditions for existence of a unique group of order n. Dieter Jungnickel proved ([2]) that if greatest common divisor (g.c.d.) $(n, \varphi(n)) = 1$, then Z_n is the only group of order n. In this article we have given another proof using the knowledge of the semi direct product and induction.

Definition 1.1. Let H and K be two groups and ϕ a homomorphism from K to $Aut(H)$. Let $G = \{(h, k) | h \in H \ \&\ k \in K\}$. Define multiplication on G as $(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1)(h_2), k_1k_2)$. This multiplication makes G a group of order $| G |=| H || K |$, where $(1, 1)$ is the identity of G and $(h, k)^{-1} = (\phi(k^{-1})(h^{-1}), k^{-1})$ is the inverse of (h, k). The group G is called the semi-direct product of H and K with respect to ϕ.

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Remark 1.1. If ϕ is a trivial homomorphism from K to $Aut(H)$, then $\phi(k_1) = I$ (identity), $\forall k_1 \in K$. Therefore
$(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1)(h_2), k_1k_2) = (h_1h_2, k_1k_2)$. Hence semi direct product of H and K is the direct product of H and K.

Lemma 1.1. *For a given natural number n, if g.c.d. $(n, \varphi(n)) = 1$, then n must be square free, where $\varphi(n)$ is the Euler-phi function.*

Proof. Suppose contrary that $n = p^\alpha m$, where $\alpha > 1$ and p^α does not divide m. Then $\varphi(n) = (p^\alpha - p^{\alpha-1})\varphi(m)$. This shows that $p^{\alpha-1} \mid \varphi(n)$ and $p^{\alpha-1} \mid n$. Hence $p^{\alpha-1}|(n, \varphi(n))$, which is a contradiction. Hence n is square free. □

Proposition 1.1. *If cyclic group Z_n is the only group of order n, then n must be square free.*

Proof. By the Fundamental theorem of abelian groups, we know that if $n = p_1^{n_1} \cdots p_r^{n_r}$, where $p_1, p_2, \cdots, p_r$ are distinct primes, the number of non-isomorphic abelian groups of order n is $p(n_1) \ldots p(n_r)$, where $p(n_i)$ denotes number of partitions of n_i. Since Z_n is the unique group of order n, therefore n must be square free. □

Proposition 1.2. [3] *Let the finite group G have an abelian Sylow p-subgroup P and let N denote $N_G(P)$. Then $P = C_P(N) \times [P, N]$, where $N_G(P), C_P(N)$ and $[P, N]$ denote normalizer of P in G, centralizer of N in P and commutator of P & N respectively.*

Proposition 1.3. [3] *Let p be the smallest prime dividing the order of the finite group G. Assume that G is not nilpotent. Then the Sylow p-subgroups of G are not cyclic.*

Theorem 1.1. [3] *If G is a finite group all of whose Sylow subgroups are cyclic , then G has a presentation $G = \langle a, b \mid a^m = 1 = b^n, b^{-1}ab = a^r\rangle$, where $r^n \equiv 1$ (mod m), m is odd, $0 \le r < m$ and g.c.d$(m, n(r-1)) = 1$.*

Proof of this theorem follows using Proposition 1.2 and Proposition 1.3.

Remark 1.2. Let G be group having a presentation $G = \langle a, b \mid a^m = 1 = b^n, b^{-1}ab = a^r\rangle$. Suppose $H = \{1, a, a^2, \cdots a^{m-1}\}$. Then H must be a normal subgroup of G.

Remark 1.3. Let G be a group of square free order i.e. $|G| = p_1p_2 \cdots p_r$, where $p_1, p_2, \cdots, p_r$ are distinct primes. Since every group of prime order is cyclic, all the Sylow p_i-subgroups of G are cyclic. Thus from Theorem 1.1 and Remark 1.2, it follows that G has a normal subgroup. Therefore the group of square free order has a proper normal subgroup.

Proposition 1.4. *Let G be a group of order $p_1p_2\dots p_s$, where $p_1, p_2, \dots, p_s$ are distinct primes. Then G is solvable.*

Proof. Proof is by induction on s. If $s = 1$, then G is a cyclic group. Thus G must be solvable. Therefore result is true for $s = 1$. Assume that result is true for every $k < s$ i.e. every group of order $p_1p_2\dots p_k$ is solvable. Now we prove the result for $k = s$. Since G is a group of square free order, by the Remark 1.3, it has a normal subgroup. Suppose H is a normal subgroup of G. Then H and G/H are groups of order less than s. By induction assumption H and G/H are solvable group. Since H and G/H are solvable, G is also solvable. Hence by induction hypothesis result is true for every s. □

2. Main Theorem

Theorem 2.1. *Let n be a positive integer. Then the cyclic group Z_n is the only group of order n if and only if g.c.d. $(n, \varphi(n)) = 1$, where φ denotes the Euler-phi function.*

Proof. First assume that cyclic group Z_n is the only group of order n. Then by Proposition 1.3, $n = p_1p_2\dots p_r$ where $p_1, p_2, \dots, p_r$ are distinct primes. So $\varphi(n) = (p_1 - 1)(p_2 - 1)\dots(p_r - 1)$. Suppose g.c.d. $(n, \varphi(n)) \neq 1$. Then there must exist two primes p_i and p_j such that $p_i \mid p_j - 1$, where $i \neq j$ for some $i, j \in \{1, 2, \dots, r\}$. Then $n = p_ip_jm$ (say). Hence there exists a non-Abelian group H of order p_ip_j. Take another group $K = Z_m$. Then semi-direct product of H and K is a non-abelian group of order n, which contradicts our assumption that Z_n is the only group of order n. Therefore g.c.d. $(n, \varphi(n)) = 1$.

Conversely, suppose g.c.d. $(n, \varphi(n)) = 1$. Then by the Lemma 1.1, n must be square free i.e. $n = p_1p_2\dots p_r$ where $p_1, \dots, p_r$ are distinct primes. We show that Z_n is the only group of order n. Proof is by induction on r. If $r = 1$. Then $n = p_1$ i.e. n is a prime. Since every group of prime order is cyclic, Z_n is the only group of order n. Thus result is true for $r = 1$. Suppose by induction hypothesis the result is true for $r = k$ i.e. for $n = p_1p_2\dots p_k$, Z_n is the only group of order n. Now to prove the result for $r = k + 1$ i.e. for $n = p_1p_2\dots p_{k+1}$, Take $H = Z_m$, where $m = p_1p_2\dots p_k$ such that g.c.d. $(m, \varphi(m)) = 1$ and $K = Z_{p_{k+1}}$. Consider the semi direct product of H and K. It exists, because the semi direct product of any two groups exist. Since H is a cyclic group, $o(Aut(H)) = (p_1 - 1)\dots(p_k - 1)$. Then any homomorphism from K to $Aut(H)$ must be a trivial homomorphism, because if not, then g.c.d. $((p_1 - 1)(p_2 - 1)\dots(p_k - 1), p_{k+1}) \neq 1$. So g.c.d. $((p_1 - 1)(p_2 - 1)\dots(p_k - 1), p_{k+1})$ must be equal to p_{k+1}. Therefore $p_{k+1} \mid (p_1 - 1)(p_2 - 1)\dots(p_k - 1)$, so $p_{k+1} \mid (p_1 - 1)(p_2 - 1)\dots(p_{k+1} - 1)$ i.e. p_{k+1} divides $\varphi(n)$, which is not possible because g.c.d. $(n, \varphi(n)) = 1$. Thus homomorphism must be trivial. Therefore by Remark 1.1, semi-direct product of H and K is the direct product of H and K, which is a cyclic group of order $p_1p_2\dots p_{k+1}$, for $p_1, p_2, \dots, p_{k+1}$ are distinct primes. Since H is the only group of order m, $Z_{p_1p_2\dots p_{k+1}}$ is the only group of order $n = p_1p_2\dots p_{k+1}$. Hence by induction hypothesis result is true for every r.

Now let G be any group of order $p_1p_2\dots p_kp_{k+1}$ where $p_1, p_2, \dots, p_k, p_{k+1}$ are all distinct primes. Then we prove the existence of a normal subgroup H of order $p_1p_2\dots p_k$ and a subgroup K of order p_{k+1}. Since all p_i's are primes, by Sylow theorem G has a subgroups of order p_i, $\forall i$. Therefore existence of K is proved.

Now by Proposition 1.4, G is solvable, so commutator G' is either identity subgroup or a proper subgroup of G (because if $G' = G$, then G can not be solvable). If $G' = \{e\}$, where e is the identity of G, then G must be an abelian group. Therefore G is cyclic by the Fundamental theorem of abelian groups. Suppose G' is a proper subgroup of G. Then $o(G') = p_1p_2\dots p_i$, where $1 \leq i < k+1$. So the factor group G/G' is an abelian group of order $p_{i+1}\dots, p_{k+1}$. Therefore by Cauchy's theorem G/G' has a normal subgroup H/G' of order $p_{i+1}\dots, p_k$. Hence H is also a normal subgroup of G and $o(H) = p_1p_2\dots p_k$. This proves the existence of a normal subgroup H. Now consider the semi direct product of H and K, which is isomorphic to G for $o(G) = o(HK)$. Hence the result is proved. □

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References

[1] David S. Dummit and Richard M. Foote, *Abstract Algebra,* John - Wiley and Sons Inc. 1995.

[2] Dieter Jungnickel, *On the uniqueness of the cyclic group of order n*, Amer. Math. Monthly, 99 (6), (1992), 545 – 547.

[3] Derek J. S. Robinson, *A course in the theory of groups,* Springer - Verlag, New York Inc., 1982.

[4] Harvey E. Rose, *A course on finite groups,* Springer - Verlag, London Ltd, 2009.

[5] John S. Rose, *A course on group theory,* Cambr. Univ. Press, (1978).

[6] Ramji Lal, *Algebra Vol I,* Shail Publication, 2002.

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NIETO-LOPEZ THEOREMS IN ORDERED METRIC SPACES

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Abstract: The 2007 relational version of the fixed point result established by Nieto and Rodriguez-Lopez [Acta Math. Sinica (English Series), 23 (2007), 2205-2212] is nothing but a particular case of the 1922 Banach's contraction principle [Fund. Math., 3 (1922), 133-181].

1. **Introduction**

Let X be a nonempty set. Take a metric $d(\ldots)$ over it; as well as a selfmap $T : X \to X$. We say that $x \in X$ is a `Picard point` (modulo $(d;T)$) if

1a) $(T^n x; n \geq 0)$ is d-convergent,

1b) $z := \lim_n T^n x$ is in $Fix(T)$ (`i.e.`, $z = Tz$).

If this happens for each $x \in X$ and

1c) $Fix(T)$ is a singleton ($x, y \in Fix(T) \Rightarrow x = y$),

then T is referred to as a `Picard operator` (modulo d); cf. Rus [15, Ch 2, Sect 2.2].

For example, such a property holds whenever d is complete and T is d-contractive; cf. Section 2. Structural extensions of this fact - when an order $(\leq)$ on X is being added - may be obtained as follows. Call $T, (d, \leq; \alpha)$-`contractive` (where $\alpha > 0$) if

(a01) $d(Tx, Ty) \leq \alpha d(x, y),\ \forall x, y \in X,\ x \leq y$.

If this holds for some $\alpha \in]0, 1[$, we say that T is $(d, \leq)$-`contractive.`

Further, denote

(a02) $(x, y \in X) : x <> y$ iff either $x \leq y$ or $y \leq x$

(x and y are `comparable`).

This relation is reflexive and symmetric; but not in general transitive. Term $(x_n; n \geq 0), <>$-ascending if $x_n <> x_{n+1}$, for all n.

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Theorem 1.1. *Assume that d is complete, T is $(d, \le)$-contractive, and*

(a03) $X(T; <>) := \{x \in X; x <> Tx\}$ *is nonempty,*

(a04) *T is monotone (increasing or decreasing),*

(a05) *for each $x, y \in X$, $\{x, y\}$ has lower and upper bounds.*

In addition, suppose that one of the conditions below holds:

(a06) *T is d-continuous: $x_n \xrightarrow{d} x$ implies $Tx_n \xrightarrow{d} Tx$*

(a07) *if $(x_n; n \ge 0)$ is $<>$-ascending and $x_n \xrightarrow{d} x$ there exists a subsequence $(y_n; n \ge 0)$ of $(x_n; n \ge 0)$ with $y_n <> x, \forall n$.*

Then, T is a Picard operator (modulo d).

Note that, the variant $(a06)$ of Theorem 1.1 is just the 2004 main result in Ran and Reurings [14]; while, the variant $(a07)$ of the same is the 2007 statement obtained by Nieto and Rodriguez-Lopez [10]. According to many authors, these two results are the first extension of the classical 1922 Banach's contraction mapping principle [3] to the realm of (partially) ordered metric spaces. However, the assertion is false: some early statements of this type have been obtained in 1986 by Turinici [18], in the context of quasi-ordered metric spaces.

Now, the Ran-Reurings and Nieto-Rodriguez-Lopez fixed point results found some useful applications to operator equations theory; so, it is a natural question to discuss the "ontological" status of Theorem 1.1. An appropriate answer to its version $(a06)$ was provided in Turinici [19] : the Ran-Reurings theorem is but a particular case of the 1968 fixed point statement in Maia [9]. Concerning the version $(a07)$ of the same, the conclusion to be derived reads (cf. Section 2): the Nieto-Rodriguez-Lopez theorem is but a particular case of the Banach's contraction mapping principle. Further, in Section 3, an extension is given for this result; which, in particular, includes a recent contribution in Agarwal, El-Gebeily and O'Regan [1]. Finally, in Section 4, a Suzuki type variant [17] of Theorem 1.1 is considered.

2. Main result

Let X be a nonempty set. Take a metric $d(\ldots)$ over it; as well as a relation ∇ on X; assumed to be `reflexive` $[x\nabla x, \forall x \in X]$ and `symmetric` $[x\nabla y \Longrightarrow y\nabla x]$. Term the sequence $(x_n; n \ge 0)$ in X, ∇-`ascending` if $x_n \nabla x_{n+1}$, for all n. Given $x, y \in X$, any subset $\{z_1, \ldots, z_k\}$ (for $k \ge 2$) in X with $z_1 = x, z_k = y$, and $[z_i \nabla z_{i+1}, i \in \{1, \ldots, k-1\}]$ will be referred to as a ∇-`chain` between x and y; the class of all these will be denoted as $C(x, y; \nabla)$. Let $\sim$ stand for the attached relation

$(b01)$ $(x, y \in X) : x \sim y$ iff $C(x, y; \nabla)$ is nonempty;

it is an equivalence on X, as it can be directly seen. Finally, take a selfmap T of X; call it, $(d, \nabla; \alpha)$-`contractive` (where $\alpha > 0$) if $(a01)$ is fulfilled, with ∇ in place of $(\leq)$. If this holds for some $\alpha \in]0,1[$, , we say that T is (d,∇)-`contractive`. (A) The following variant of Theorem 1.1 is our starting point.

Theorem 2.1. *Assume that d is complete, T is (d,∇)-contractive, and*
$(b02)$ *T is* ∇-`increasing` $[x\nabla y$ implies $Tx\nabla Ty]$
$(b03)$ $(\sim) = X \times X$ *$[C(x,y;\nabla)$ is nonempty, for each $x,y \in X]$*
$(b04)$ *if $(x_n; n \geq 0)$ is ∇-ascending and $x_n \overset{d}{\longrightarrow} x$ there exists a subsequence $(y_n; n \geq 0)$ of $(x_n; n \geq 0)$ with $y_n \nabla x, \forall n$.*
Then, T is a Picard operator (modulo d).

This result includes Theorem 1.1 if ∇ is identical with $<>$; because $(a04) \Longrightarrow (b02), (a05) \Longrightarrow (b03)$. [In fact, given $x,y \in X$, there exist, by $(a05)$, some $u,v \in X$ with $u \leq x \leq v, u \leq y \leq v$. This yields $x <> u, u <> y$; wherefrom, $x \sim y$]. The remarkable fact to be noted is that Theorem 2.1 (hence the Nieto-Rodriguez-Lopez statement as well) is deductible from the 1922 Banach's contraction mapping principle [3]. Let $e(.,.)$ be another metric over X. Call the selfmap T, $(e;\alpha)$-`contractive` (for some $\alpha > 0$) when
$(b05)$ $e(Tx,Ty) \leq \alpha e(x,y), \forall x,y \in X$;
if this holds for some $\alpha \in]0,1[$, the resulting convention will read as: T is e-`contractive`. The Banach's result is:

Theorem 2.2. *Assume that e is complete and T is e-contractive. Then, T is a Picard operator (modulo e).*

We are now in position to give the announced answer.

Proposition 2.1. *We have Theorem 2.2 $\Longrightarrow$ Theorem 2.1; hence (by the above) the Banach fixed point result implies the Nieto-Rodriguez-Lopez theorem.*

Proof. Let the conditions of Theorem 2.1 hold. We introduce a mapping $e : X \times X \to R_+$ as: for each $x,y \in X$,
$(b06)$ $e(x,y) = \inf[d(z_1,z_2) + \cdots + d(z_{k-1},z_k)]$,
where $\{z_1,\ldots,z_k\}$ (for $k \geq 2$) is a ∇-chain between x and y.

I) Clearly, e is reflexive $[e(x,x) = 0, \forall x \in X]$, symmetric $[e(x,y) = e(y,x)$, $\forall x,y \in X]$ and triangular $[e(x,z) \leq e(x,y) + e(y,z), \forall x,y,z \in X]$. In addition, the triangular property of d gives $d(x,y) \leq d(z_1,z_2) + \cdots + d(z_{k-1},z_k)$, for any ∇-chain $\{z_1,\ldots,z_k\}$ (where $k \geq 2$) between x and y. So, passing to infimum, yields

$$d(x,y) \leq e(x,y), \forall x,y \in X \; [d \text{ is subordinated to } e]. \tag{2.1}$$

Note that e is sufficient in such a case $[e(x,y) = 0 \Longrightarrow x = y]$; hence, it is a (standard) metric on X. Finally, by the very definition of e,

$$d(x,y) \geq e(x,y) \text{ (hence } d(x,y) = e(x,y)), \text{whenever } x\nabla y. \tag{2.2}$$

II) We claim that e is complete on X. Let $(x_n; n \geq 0)$ be an e-Cauchy sequence in X. There exists a strictly ascending sequence of ranks $(k(n); n \geq 0)$, in such a way that $[(\forall n): k(n) < m \Longrightarrow e(x_{k(n)}, x_m) < 2^{-n}]$. Denoting $(y_n = x_{k(n)}; n \geq 0)$, we therefore have $e(y_n, y_{n+1}) < 2^{-n}, \forall n$. Moreover, by the imposed e-Cauchy property, (x_n) is e-convergent iff so is (y_n). To establish this last property, one may proceed as follows. As $e(y_0, y_1) < 2^{-0}$, there exists (for the starting rank $p(0) = 0$) a ∇-chain $\{z_{p(0)}, \ldots, z_{p(1)}\}$ between y_0 and y_1 (hence $p(1) - p(0) \geq 1, z_{p(0)} = y_0, z_{p(1)} = y_1$), such that $d(z_{p(0)}, z_{p(0)+1}) + \cdots + d(z_{p(1)-1}, z_{p(1)}) < 2^{-0}$. Further, as $e(y_1, y_2) < 2^{-1}$, there exists a ∇-chain $\{z_{p(1)}, \ldots, z_{p(2)}\}$ between y_1 and y_2 (hence $p(2) - p(1) \geq 1, z_{p(1)} = y_1, z_{p(2)} = y_2$), such that $d(z_{p(1)}, z_{p(1)+1}) + \cdots + d(z_{p(2)-1}, z_{p(2)}) < 2^{-1}$; and so on. The procedure may continue indefinitely; it gives us a ∇-ascending sequence $(z_n; n \geq 0)$ in X with (cf. (2.2))

$$\sum_n e(z_n, z_{n+1}) = \sum_n d(z_n, z_{n+1}) < \sum_n 2^{-n} < \infty. \tag{2.3}$$

In particular, $(z_n; n \geq 0)$ is d-Cauchy; wherefrom (as d is complete), $z_n \xrightarrow{d} z$ as $n \to \infty$, for some $z \in X$. Combining with $(b04)$, there must be a subsequence $(t_n := z_{q(n)}; n \geq 0)$ of $(z_n; n \geq 0)$ with $t_n \nabla z, \forall n$. This firstly gives (by the above convergence property), $t_n \xrightarrow{d} z$ as $n \to \infty$. Secondly (again combining with (2.2)), $e(t_n, z) = d(t_n, z), \forall n$; so that (by the above relation), $t_n \xrightarrow{e} z$ as $n \to \infty$. On the other hand, (2.3) tells us that $(z_n; n \geq 0)$ is e-Cauchy. Adding the convergence property of $(t_n; n \geq 0)$ gives $z_n \xrightarrow{e} z$ as $n \to \infty$; wherefrom (as $z_{p(n)} = y_n; n \geq 0), y_n \xrightarrow{e} z$ as $n \to \infty$; and our claim follows.

III) Let $\alpha \in]0, 1[$ be the number appearing in the (d, ∇)-contractive property. Given $x, y \in X$, let $\{z_1, \ldots, z_k\}$ (for $k \geq 2$) be a ∇-chain connecting them (existing by $(b03)$). From $(b02), \{Tz_1, \ldots, Tz_k\}$ is a ∇-chain between Tx and Ty. So, combining with the contractive condition,

$$e(Tx, Ty) \leq \sum_{i=1}^{k-1} d(Tz_i, Tz_{i+1}) \leq \alpha \sum_{i=1}^{k-1} d(z_i, z_{i+1}),$$

for all such ∇-chains; wherefrom, passing to infimum, $e(Tx, Ty) \leq \alpha e(x, y)$; `i.e.`, $(b05)$ holds. Summing up, Theorem 2.2 applies to these data; and we are done.

(B) Let X be a nonempty set. Take a metric $d(\ldots)$ over it; and fix some $\gamma > 0$. Define a relation ∇ over X as

$(b07)$ $(x, y \in X): x \nabla y$ if and only if $d(x, y) < \gamma$.

Clearly, ∇ is reflexive and symmetric; but, not in general transitive. Given $x, y \in X$, any subset $\{z_1, \ldots, z_k\}$ (for $k \geq 2$) in X with $z_1 = x, z_k = y$, and $[z_i \nabla z_{i+1}, i \in \{1, \ldots, k-1\}]$ will be referred to as a γ-`chain` between x and y; the class of all these will be denoted as $C(x, y; \gamma)$. Let $(\sim)$ stand for the relation over X attached to ∇ as in ($b01$); as already noted, it is an equivalence over X. When $(\sim) = X \times X$, we say that d is γ-`chainable`. Finally, take a selfmap T of X; call it, γ-`locally` (d, α)-`contractive` (for $\alpha > 0$) when ($a01$) is retainable, with ∇ in place of $(\leq)$; `i.e.`,

($b08$) $d(Tx, Ty) \leq \alpha d(x, y), \forall x, y \in X, d(x, y) < \gamma$.

If this holds for some $\alpha \in]0, 1[$, we say that T is γ-`locally` d-`contractive`.

Theorem 2.3. *Assume that d is complete, T is γ-`locally` d-`contractive`, and d is γ-`chainable`. Then, T is a `Picard operator`.*

Proof We show that Theorem 2.1 is applicable to the data (X, d, ∇). Clearly, ($b02$) and ($b03$) hold (by the choice of ∇). Further, ($b04$) holds too; for, if $(x_n; n \geq 0)$ is ∇-`ascending` and $x_n \xrightarrow{d} x$, there must be some rank $n(\gamma)$ in such a way that $d(x_n, x) < \gamma$ (`i.e.`, $x_n \nabla x$), for all $n \geq n(\gamma)$. The proof is thereby complete.

The obtained result is just the well-known 1961 Edelstein's generalization [5] of the Banach contraction principle [3]. Note that, by the proposed proof, these results are equivalent to each other.

3. Extensions of theorem 1.1

Let X be a nonempty set. Denote by $S(X)$, the class of all sequences (x_n) in X. By a (sequential) `convergence structure` on X we mean, as in Kasahara [7], any part C of $S(X) \times X$ with the properties

($c01$) $x_n = x, \forall n \in N \Longrightarrow ((x_n); x) \in C$

($c02$) $((x_n); x) \in C \Longrightarrow ((y_n); x) \in C$, for each subsequence (y_n) of (x_n).

In this case, $((x_n); x) \in C$ writes $x_n \xrightarrow{C} x$; and reads: x is the C-`limit` of (x_n). The set of all such x is denoted $\lim(x_n)$; when it is nonempty, we say that (x_n) is C-`convergent`; and the class of all these will be denoted $S_c(X)$. Assume that we fixed such an object, with

($c03$) C is separated: $\lim(x_n)$ is a singleton, for each (x_n) in $S_c(X)$;

as usually, we shall write $\lim(x_n) = \{z\}$ as $\lim(x_n) = z$. Let $(\leq)$ be a `quasi-order` (`i.e.`: reflexive and transitive relation) over X; and take a selfmap T of X, with

($c04$) T is progressive ($X(T, \leq) = \{x \in X; x \leq Tx\}$ is nonempty)

($c05$) T is increasing ($x \leq y \Longrightarrow Tx \leq Ty$).

We say that $x \in X(T, \leq)$ is a `Picard point` (modulo $(C, \leq, T)$) if

3a) $(T^n x; n \geq 0)$ is C-`convergent`,

3b) $z = \lim(T^n x)$ is in $\mathrm{Fix}(T)$ and $T^n x \leq z, \forall n$.

If this happens for each $x \in X(T, \leq)$ and

3c) $Fix(T)$ is $(\leq)$-`singleton` $[z, w \in Fix(T), z \leq w \Longrightarrow z = w]$,
then T is called a `Picard operator` (modulo $(C, \leq)$). In this case, each $x^* \in Fix(T)$ fulfils

$$\forall u \in X(T, \leq) : x^* \leq u \Longrightarrow u \leq x^*; \tag{3.1}$$

`i.e.`: x^* is $(\leq)$-maximal in $X(T, \leq)$. In fact, assume that $x^* \leq u \in X(T, \leq)$. By 3a) and 3b), $(T^n u, n \geq 0)$ C-converges to some $u^* \in Fix(T)$ with $T^n u \leq u^*, \forall n$; hence, $x^* \leq u \leq u^*$. Combining with 3c) gives $x^* = u^*$; wherefrom $u \leq x^*$.

Concerning the sufficient conditions for such a property, an early statement of this type was established by Turinici [18]. Here, we propose a different approach, founded on ascending orbital concepts (in short: ao-concepts) and almost metrics. Some conventions are in order. Call the sequence $(z_n; n \geq 0)$ in X, `ascending` if $z_i \leq z_j$ for $i \leq j$; and T-`orbital` when $z_n = T^n x, n \geq 0$, for some $x \in X$; the intersection of these concepts is just the precise one. We say that

3d) $(\leq)$ is $(C; ao)$-`self-closed` when the C-limit of each C-convergent ao-sequence is an upper bound of it,

3e) T is $(C; ao)$-`continuous` if $[(z_n) = ao\text{-sequence}, z_n \xrightarrow{C} z, z_n \leq z, \forall n]$ implies $Tz_n \xrightarrow{C} Tz$.
Further, by an `almost metric` over X we shall mean any map $e : X \times X \to R_+$; supposed to be reflexive, triangular and sufficient. This comes from the fact that such an object has all properties of a metric, excepting symmetry. Call the sequence (x_n), e-`Cauchy` when $[\forall \delta > 0, \exists n(\delta) : n(\delta) \leq n \leq m \Longrightarrow e(x_n, x_m) \leq \delta]$. We then say that

3f) e is $(C; ao)$-`complete`, provided [(for each ao-sequence) e-Cauchy $\Longrightarrow$ C-convergent].

Let $\mathcal{F}(R_+)$ stand for the class of all functions $\varphi : R_+ \to R_+$; and $\mathcal{F}_1(R_+)$, the subclass of all $\varphi \in \mathcal{F}(R_+)$ with $\varphi(0) = 0$ and $\varphi(t) < t$ for all t in $R_+^0 =]0, \infty[$. Given $\varphi \in \mathcal{F}_1(R_+)$ and $s \in R_+^0$, denote

$$\limsup_{t \to s+} \varphi(t) = \inf_{\epsilon > 0} \Phi[s+](\epsilon), \text{ where}$$

$$\Phi[s+](\epsilon) = \sup\{\varphi(t); s < t < s + \epsilon\}, \epsilon > 0.$$

Call $\varphi \in \mathcal{F}_1(R_+)$, a BW-`function` if $\limsup_{t \to s+} \varphi(t) < s$, for all $s \in R_+^0$; see Boyd and Wong [4]. Note that, by this very definition,

$$(\forall \gamma > 0), (\exists \beta > 0), (\forall t) : 0 \leq t < \gamma + \beta \Longrightarrow \varphi(t) < \gamma. \tag{3.2}$$

This yields the property (to be used further)

$$[(t_n, n \geq 0) \subseteq R_+, t_{n+1} \leq \varphi(t_n), \forall n] \Longrightarrow [\lim_n (t_n) = 0]. \tag{3.3}$$

Denote, for $x, y \in X : H(x, y) = \max\{e(x, Tx), e(y, Ty)\}, L(x, y) = \frac{1}{2}[e(x, Ty) + e(Tx, y)], M(x, y) = \max\{e(x, y), H(x, y), L(x, y)\}$. Clearly,

$$M(x, Tx) = \max\{e(x, Tx), e(Tx, T^2x)\}, \forall x \in X. \tag{3.4}$$

Given $\varphi \in \mathcal{F}_1(R_+)$, we say that T is $(e, M, \leq; \varphi)$-`contractive`, when
(c06) $e(Tx, Ty) \leq \varphi(M(x, y)), \forall x, y \in X, x \leq y$;
if this holds for at least one BW-function φ, the resulting convention reads: T is `extended` $(e, M, \leq)$-`contractive`.

Theorem 3.1. *Suppose that T is extended $(e, M, \leq)$-contractive, e is (C, ao)-complete, $(\leq)$ is (C, ao)-self-closed, and T is (C, ao)-continuous. Then, T is a Picard operator (modulo $(C, \leq)$).*

Proof Let $x^*, u^* \in Fix(T)$ be such that $x^* \leq u^*$. By the contractive condition, $e(x^*, u^*) = 0$; wherefrom, $x^* = u^*$; and so, $Fix(T)$ is $(\leq)$-singleton. It remains to show that each $x = x_0 \in X(T, \leq)$ is a `Picard point` (modulo $(C, \leq, T)$). Put $x_n = T^n x, n \geq 0$; and let $\varphi \in \mathcal{F}_1(R_+)$ be the BW-function given by the extended $(e, M, \leq)$-contractivity of T.

I) By the contractive condition and (3.4),

$$e(x_{n+1}, x_{n+2}) \leq \varphi(M(x_n, x_{n+1})) = \varphi[\max\{e(x_n, x_{n+1}), e(x_{n+1}, x_{n+2})\}], \forall n.$$

If (for some n) the maximum in the right hand side is $e(x_{n+1}, x_{n+2})$, then (via $\varphi \in \mathcal{F}_1(R_+)$), $e(x_{n+1}, x_{n+2}) = 0$; so that (as e=sufficient) $x_{n+1} \in Fix(T)$; and we are done. Suppose that this alternative fails: $e(x_{n+1}, x_{n+2}) \leq \varphi(e(x_n, x_{n+1}))$, for all n. This, along with (3.3), yields

$$e(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty.$$

II) We claim that $(x_n,\ n \geq 0)$ is e-Cauchy in X. Denote, for simplicity, $E(k, n) = e(x_k; x_{k+n}), k, n \geq 0$. Let $\gamma > 0$ be arbitrary fixed; and $\beta > 0$ be the number appearing in (3.2); without loss, one may assume that $\beta < \gamma$. By the preceding step, there exists a rank $m = m(\beta)$ such that

$$k \geq m \text{ implies } E(k, 1) < \beta/2 < \beta < \gamma. \tag{3.5}$$

The desired property follows from the relation

$$\forall n \geq 0 : [E(k, n) < \gamma + \beta/2, \text{ for each } k \geq m]. \tag{3.6}$$

The case $n = 0$ is trivial; while the case $n = 1$ is clear, via (3.5). Assume that (3.6) is true, for all $n \in \{1, \ldots, p\}$ (where $p \geq 1$); we want to establish that it holds as well for $n = p + 1$. So, let $k \geq m$ be arbitrary fixed. By (3.5) and the induction argument, $e(x_k, x_{k+p}) = E(k, p) < \gamma + \beta/2$ and $H(x_k, x_{k+p}) = \max\{E(k, 1), E(k + p, 1)\} < \beta/2$. This yields $L(x_k, x_{k+p}) = (1/2)[E(k, p + 1) +$

$E(k+1,p-1)] \le (1/2)[E(k,p)+E(k+p,1)+E(k+1,p-1)] < \gamma+\beta$; wherefrom $M(x_k,x_{k+p}) < \gamma+\beta$; so that, combining with the contractive condition and (3.2),

$$E(k+1,p) = e(x_{k+1},x_{k+p+1}) = e(Tx_k,Tx_{k+p}) \le \varphi(M(x_k,x_{k+p})) < \gamma;$$

which improves the previous evaluation (3.6) of this quantity. By the triangular property and (3.5), one thus gets $E(k,p+1) = e(x_k,x_{k+p+1}) < \gamma+\beta/2$.

III) As e is $(C;ao)$ - complete, (3.6) tells us that $x_n \overset{C}{\longrightarrow} x^*$ for some $x^* \in X$. Moreover, as $(\le)$ is $(ao;C)$-self-closed, $x_n \le x^*$, for all n; hence, $x \le x^*$. Combining with the $C,ao)$-continuity of T, we must have $x_{n+1} = Tx_n \overset{C}{\longrightarrow} Tx^*$; wherefrom (as C is separated), $x^* \in Fix(T)$.

Now letting d be a metric on X, the associated convergence $C = (\overset{d}{\longrightarrow})$ is separated; moreover, the $(C;ao)$-complete property of e is holding whenever d is complete and subordinated to e. Clearly, this last property is trivially assured if $d = e$; when Theorem 3.1 is comparable with the main result in Agarwal, El-Gebeily and ORegan [1]. In fact, a little modification of the working hypotheses allows us getting the whole conclusion of the quoted statement; we do not give details.

4. Conditional versions

(A) Let (X,d) be a metric space. Define a map $F : [0,\infty[\to]0,1]$ as

$(d01)$ $F(t) = 1$, if $0 \le t \le (\sqrt{5}-1)/2$
$F(t) = (1-t)t^{-2}$, if $(\sqrt{5}-1)/2 \le t \le 2^{-1/2}$
$F(t) = (1+t)^{-1}$, if $2^{-1/2} \le t < \infty$.

Note that F is continuous and decreasing over its existence domain. Given the selfmap $T : X \to X$, call it `conditional` (F,α)-`contractive` (where $\alpha \ge 0$) provided

$(d02)$ $[x,y \in X, F(\alpha)d(x,Tx) \le d(x,y)] \Rightarrow d(Tx,Ty) \le \alpha d(x,y)$;
if this holds for some $\alpha \in [0,1[$, then T is called `conditional` F-`contractive.`

The following 2008 result in Suzuki [17] is our starting point.

Theorem 4.1. *Suppose that d is complete and T is conditional F-contractive. Then, T is a Picard operator.*

Actually, the family of conditions $(d02)$ gives a characterization of completeness; see the quoted paper for details. A related statement involving the Kannan type conditions [6] is to be found in Kikkawa and Suzuki [8]. For various extensions of such results we refer to Altun and Erduran [2]; see also Popescu [13]. Note that, in all these statements, the premise of the conditional contractive property $(d02)$

is asymmetric with respect to the couple (x, y); so, it is natural to ask whether a supplementary condition may be added there, with a dual information about the variable y. As we shall see, a positive answer to this is possible, by means of ao-concepts and ordinary metrics.

(**B**) Let X be a nonempty set. Fix a (sequential) convergence structure C on X as in $(c01)+(c02)$; which, in addition, is separated (cf. $(c03)$). Remember that $x_n \xrightarrow{C} x$ reads: x is the C-`limit` of (x_n). The set of all such x is denoted $\lim(x_n)$; when it is nonempty, we say that (x_n) is C-`convergent`. Further, take a quasi-order $(\leq)$ on X; its associated relation $<>$ over X is introduced as in $(a02)$. Let also $T : X \to X$ be a selfmap of X; supposed to be progressive and increasing (cf. $(c04)+(c05)$). We say that $x \in X(T; \leq)$ is a `Picard point` (modulo $(C, \leq, T)$) if **3a)** and **3b)** hold. If this happens for each $x \in X(T, \leq)$ and **3c)** holds,then T is called a `Picard operator` (modulo $(C, \leq)$); note that, in this last case,(3.1) is true.

The structural conditions for obtaining such properties are as follows. We say that

4a) $(\leq)$ is $(C; ao)$-`self-closed`, when the C-limit of each C- convergent ao-sequence is an upper bound of it.

Further, let $d(.,.)$ be a metric on X; it is called

4b) $(C; ao)$-`complete`, provided [(for each ao-sequence) $d-Cauchy \Rightarrow C-convergent$],

4c) $(C; ao)$-`half-continuous`, provided $[(x_n) = ao{-}sequence, x_n \xrightarrow{C} x, x_n \leq x, \forall n]$ implies $[d(x_n, y) \to d(x, y)$, for all $y \in X]$.

Finally, the metrical conditions are to be stated according to the lines below. Denote $G(t) = 1/(1+t), t > 0$; this function is continuous, decreasing and maps $]0, \infty[$ onto $]0, 1[$. Call the selfmap T, `weakly conditional` $(G, <>, \alpha)$-`contractive` (where $\alpha > 0$) provided

$(d03)$ $x <> y, G(\alpha)\max\{d(x, Tx), d(y, Tx)\} \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y)$.

If this holds for at least one $\alpha \in]0, 1[$, the underlying map T is called `weakly conditional` $(G, <>)$-`contractive.`

We are now in position to state the announced answer.

Theorem 4.2. *Assume that T is weakly conditional $(G, <>)$-contractive, d is $(C; ao)$-complete, $(\leq)$ is $(C; ao)$-self-closed, and d is $(C; ao)$-half-continuous. Then T is a Picard operator (modulo $(C, \leq)$).*

Proof Let $\alpha \in]0, 1[$ be the number appearing in the weak conditional contractive property of T. There are three steps to be passed.

I) Let $x^*, u^* \in Fix(T)$ be such that $x^* \leq u^*$. We have $d(x^*, Tx^*) = 0, d(u^*, Tx^*) = d(u^*, x^*)$; hence $G(\alpha)\max\{d(x^*, Tx^*); d(u^*, Tx^*)\} \leq d(x^*, u^*)$. This, by the contractive condition, yields $d(x^*, u^*) \leq \alpha d(x^*, u^*)$; so that (as $d =$ metric), $x^* = u^*$; wherefrom, $Fix(T)$ is $(\leq)$-singleton. It remains to show that each $x = x_0 \in X(T, \leq)$ is a Picard point (modulo $(C, \leq, T)$). Put $x_n = T^n x, n \geq 0$; clearly, $(x_n, n \geq 0)$ is an ao-sequence in X. For each $n \geq 0$,

$$x_n \leq x_{n+1}, G(\alpha) \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_n)\} \leq d(x_n, x_{n+1});$$

so, by the imposed contractive condition,

$$d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}); \text{ for all } n. \tag{4.1}$$

This tells us that $(x_n; n \geq 0)$ is d-Cauchy. As d is $(C; ao)$-complete and $(\leq)$ is $(C; ao)$-self-closed,

$$x_n \xrightarrow{C} z \,(\text{hence } x_n \leq z, \forall n), \text{ for some } z \in X. \tag{4.2}$$

II) Suppose that our sequence is such that
($d04$) for each n, there exists $m > n$ with $x_m = z$.
It follows that a subsequence $(x_{p(n)}; n \geq 0)$ of (x_n) exists with $x_{p(n)} = z$, for all n. This, along with $[x_{p(n)+1} = Tz, \forall n]$ gives (as C is separated) $z = Tz$.

III) Assume in the following that the opposite situation holds:
($d05$) there exists h such that: $x_n \neq z$, for all $n \geq h$.
Fix $k \geq h$; and put $x_k = u$; clearly, $u \leq x_n \leq z$, for all $n \geq k$. From (4.1), $d(x_n, Tx_n) \to 0$ as $n \to \infty$; moreover, as d is (C, ao)-half-continuous, $d(x_n, u) \to d(z, u) > 0$. By simply combining these, there must be some rank $p \geq k$ such that $d(x_n, Tx_n) \leq \alpha d(x_n, u), \forall n \geq p$. On the other hand (for the same ranks), $d(u, Tx_n) \leq d(u, x_n) + d(x_n, Tx_n) \leq (1 + \alpha) d(u, x_n)$; hence, summing up,

$$x_n \geq u, G(\alpha) \max\{d(x_n, Tx_n), d(u, Tx_n)\} \leq d(x_n, u), \forall n \geq p.$$

These, by the contractive condition, give $d(Tx_n, Tu) \leq \alpha d(x_n, u), \forall n \geq p$; so that (passing to limit as $n \to \infty$), $d(z, Tu) \leq \alpha d(z, u)$. By the triangle inequality $d(u, Tu) \leq d(u, z) + d(z, Tu) \leq (1 + \alpha) d(u, z)$, wherefrom (putting these together)

$$u \leq z, G(\alpha) \max\{d(u, Tu), d(z, Tu)\} \leq d(u, z);$$

so that (by the same contractive condition), $d(Tu, Tz) \leq \alpha d(u, z)$. Taking into account the adopted notation, we have $d(Tx_k, Tz) \leq \alpha d(x_k, z), \forall k \geq h$. So, passing to limit as $k \to \infty$, one derives $z = Tz$, and conclusion follows.

In particular, when $(\leq)$ is the trivial quasi-order of X, the obtained result extends, in a partial way, Theorem 4.1. Note that, from the very definition of Picard (modulo $(C, \leq)$) operator, a reduction of Theorem 4.2 to the Banach contraction principle [3] is no longer possible. Moreover, by the same technique, one gets a quasi-ordered version of the main result in Singh, Pathak and Mishra [16]; we do not give details.

References

[1] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces,* Applicable Anal., 87 (2008), 109–116.

[2] I. Altun and A. Erduran, *A Suzuki type fixed-point theorem,* Intl. J. Mathematics Math. Sci., Volume 2011, Article ID 736063.

[3] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales,* Fund. Math., 3 (1922), 133–181.

[4] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions,* Proc. Amer. Math. Soc., 20 (1969), 458–464.

[5] M. Edelstein, *An extension of Banach's contraction principle,* Proc. Amer. Math. Soc., 12 (1961), 7–10.

[6] R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc., 60 (1968), 71–76.

[7] S. Kasahara, *On some generalizations of the Banach contraction theorem,* Pub. Res. Inst. Math. Sci. Kyoto Univ., 12 (1976), 427–437.

[8] M. Kikkawa and T. Suzuki, *Some similarity between contractions and Kannan mappings,* Fixed Point Th. Appl., Volume 2008, Article ID 649749.

[9] M. G. Maia, *Un'osservazione sulle contrazioni metriche,* Rend. Sem. Mat. Univ. Padova, 40 (1968), 139–143.

[10] J. J. Nieto and R. Rodriguez-Lopez, *Existence and uniqueness of xed point in partially ordered sets and applications to ordinary differential equations,* Acta Math. Sinica (English Series), 23 (2007), 2205–2212.

[11] J. J. Nieto, R. L. Pouso and R. Rodriguez-Lopez, *Fixed point theorems in ordered abstract spaces,* Proc. Amer. Math. Soc., 135 (2007), 2505–2517.

[12] A. Petrusel and I. A. Rus, *Fixed point theorems in ordered L-spaces,* Proc. Amer. Math. Soc., 134 (2006), 411–418.

[13] O. Popescu, *Two fixed point theorems for generalized contractions with constants in complete metric space,* Central European J. Math., 7 (2009), 529–538.

[14] A. C. M. Ran and M. C. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proc. Amer. Math. Soc., 132 (2004), 1435–1443.

[15] I. A. Rus, *Generalized Contractions and Applications,* Cluj University Press, Cluj-Napoca, 2001.

[16] S. L. Singh, H. K. Pathak, and S. N. Mishra, *On a Suzuki type general fixed point theorem with applications,* Fixed Point Th. Appl., Volume 2010, Article ID 234717.

[17] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness,* Proc. Amer. Math. Soc., 136 (2008), 1861–1869.

[18] M. Turinici, *Fixed points for monotone iteratively local contractions,* Dem. Math., 19 (1986), 171–180.

[19] M. Turinici, *Ran-Reurings theorems in ordered metric spaces,* J. Indian Math. Soc., 78 (2011), 207–214.

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REPORT ON IMS SPONSORED LECTURES

The details of the IMS Sponsored Lectures/Popular Talks held is as under:

1. Speaker: Prof M. Waldschmidt
(University of Paris 6, France).
Title of the Lecture: Transcendental Numbers.
Day and Date: Friday, December 09, 2011 at 4.00 p.m.
Venue : H. R. I. Auditorium, Allahabad.
Organizer: Prof. Satya Deo.

Abstract: An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1+\sqrt{5})/2$, the roots of unity $e^{2i\pi a/b}$, the roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers. A **transcendental number** is a complex number which is not algebraic.

The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of e. In 1882, the proof by F. Lindemann of the transcendence of π gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of $2^{\sqrt{2}}$ and e^{π}, which was included in Hilbert's seventh problem in 1900, was proved by Gel'fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this lecture we survey the state of the art on known results and open problems.

2. Speaker: Dr. S. S. Benchalli
(Karnatak University, Dharwad).
Title of the Lecture: Fuzzy Logic and Fuzzy set theory.
Day and Date: Wednesday, March 21, 2012.
Venue: R. L. Science Institute, Belgaum.
Organizer: S. N. Banasode.

Abstract: Following I. A. Zadeh [1965], the concept of a fuzzy subset as a generalization of an ordinary subset was explained and introduced. Operations on fuzzy subsets, fuzzy subsets induced by mappings and fuzzy topological spaces were discussed.

3.	Speaker:	Dr. S. B. Nimse (Vice Chancellor, SRTM University, Nanded).
	Title of the Lecture:	Riemann Conjecture and the Millennium Problems.
	Day and Date:	Thursday, 27-9-2012 at 3 p. m.
	Venue:	Department of Mathematics, Dr. B. A. M. University, Aurangabad.
	Organizer:	S. K. Nimbhorkar.

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BOOK REVIEW

MATRIX COMPLETIONS, MOMENTS AND SUMS OF HERMITIAN SQUARES
BY MIHALY BAKONYI AND HUGE J. WOERDEMAN

Princeton University Press; 2011; Cloth binding; xii-520 pages;
ISBN No.978-0-691-12889-4;
Library of Congress Control Number- 2011922035.

Reviewer : N. K. Thakare.

This book is a welcome and unique addition to the research areas of matrix completions, moment problems and sums of Hermitian squares. There are five chapters in the book. In Chapter - I, the authors study cones in the real Hilbert spaces of Hermitian matrices and real valued trigonometric polynomials. The main theme of Chapter II is positive definite and semidefinite completions of partial operator matrices with three cases namely banded case, chordal case and Toeplitz case. After describing the structure of positive semidefinite operator matrices, they study the Hamburger problem based on positive semidefinite completions of Hankel matrices.

In one variable case, the five problems namely (i) Caratheodory interpolation problem, (ii) Moment problem, (iii) Bernstein - Szego moment problem, (iv) Ordered group moment problem and (v) Free group moment problem are closely related. However, these five closely related problems become vastly different when one considers the multivariable case. This is the main thesis of Chapter III. One would like to opine that this area of research on multivariable problems is very active and this enhances the value of this book.

In Chapter IV, the authors deal with contractive completion of partial operator matrices. They solve several operator-valued contractive interpolation problems. The Nehari problems, Nevalinna-Pick interpolation problems via a linearly fractional transform by a unitary matrix, the problem of Nehari and Caratheodory-Fejer for operator-valued functions defined on compact groups are looked into.

In Chapter V, the authors consider various completion problems that are in one way or another closely related to positive semidefinite or contractive completion problems. In particular, they discuss Hermitian completions, ranks of completions, bounds for eigenvalues of Hermitian completions, Euclidean distance matrix completions, normal completions etc.

The detailed notes given at the end of each chapter will be highly helpful to the researcher as well as any fresh beginner. The exercises given at the end of each chapter range in difficulty, and are meant not only to test the understanding of the subject matter included in the chapter but also to enrich the reader in the related results published recently.

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77^{TH} IMS CONFERENCE : A BRIEF REPORT

S. B. NIMSE

The 77^{th} Annual Conference of the Indian Mathematical Society was held at Swami Ramanand Teerth Marathwada University, Vishnupuri, Nanded - 431 606, Maharashtra, during the period December 27 - 30, 2011, under the presidentship of Prof. P. K. Banerji, Jai Narain Vyas University, Jodhpur, Rajasthan. The conference was attended by near about 600 delegates from all over country including foreign delegates.

Hon'ble Shri D. P. Sawant, Minister of State, Higher & Technical Education, Medical Education, Special Assistance and Non-Conventional Energy, Government of Maharashtra, Mumbai, inaugurated the Conference and welcomed all the delegates at Nanded city. The Inaugural function was held in the forenoon of December 27, 2011 at 11.00 a.m. in the Ramanujan Hall of SRTM University. The inaugural function was presided by Prof. P. K. Banerji, president of the Indian Mathematical Society. Dr. S. B. Nimse, Vice-Chancellor, SRTM University and Local Organizing Secretary, 77^{th} IMS Conference delivered the welcome address. On behalf of the General Secretary Prof. V. M. Shah (who was absent), Prof. N. K. Thakare, Ex. Vice-Chancellor, North Maharashtra University, Jalgaon and the Editor, Journal of the Indian Mathematical Society, spoke about the Society and expressed his sincere and profuse thanks, on behalf of the Society, to the host for organizing the Conference. Prof. Satya Deo, Academic Secretary of the Society, reported about the academic programmes of the conference. Prof. N. K. Thakare reported about A. Narasinga Rao Memorial Prize. The Prize, for the year 2009, is awarded to B. B. Waphare for his research paper adjudged to be the best research paper published in the Journal of the Indian Mathematical Society / The Mathematics Student in the year 2009. Prof. N. K. Thakare also reported about P. L. Bhatnagar Memorial Prize for 2011. The Prize is presented to the top scorer Akashnil Dutta (Kolkata), the top scorer of the Indian Team at the International Mathematical Olympiad, 2011. Prof. P. K. Banerji delivered his Presidential address (General) *"Unsinkable Mathematics that is sinking"*. The function ended with a vote of thanks proposed by Dr. J. N. Salunke, Director, School of Mathematical Sciences, SRTM University, Nanded.

The Academic Sessions of the conference began with the Presidential Technical Address by Prof. P. K. Banerji, the President of the Society, on *"Three Most Generalized Integrals of the 20th Century"* which was presided over by Prof. N. K. Thakare, the senior most past president of the Society present in the Conference. The rest of the academic programmes were held in the Ramanujan Hall, Senate Hall, Aryabhatta Hall, Brahmagupta Hall, Bhaskaracharya Hall and the Mahavira Hall.

Two Plenary Lectures, Three Memorial Award Lectures and Nine Invited Lectures were delivered by eminent mathematicians during the conference. Besides this, there were Seven symposia on various topics of mathematics - the two of which were of a unique nature targeted to special groups : One on *"Teaching and research of Mathematics in SAARC Countries"* targeting participation from mathematicians of SAARC countries and deliberations with them; and, the other on *"Careers in Higher Mathematics"* targeting junior college mathematics teachers of the region around Nanded for motivating students for careers in higher mathematics. Details in this regard are given in the following.

A Paper Presentation Competition for various IMS and other prizes was held (without parallel session) in which 20 papers were presented. In all about 160 papers were accepted for presentation during the conference on various disciplines of mathematics and computer science. During the valedictory function held on December 30, 2011, the IMS and other prizes were awarded to the winners of the Paper Presentation Competition. It was also announced that the next annual conference of the Society will be held at the Banaras Hindu University, Varanasi, and Prof. Huzoor H. Khan, Aligarh Muslim University, will be the next President with effect from April 1, 2012.

A Cultural Evening and a sight-seeing was also arranged on December 28, 2011 and December 29, 2011. The conference was supported by the NBHM, INSA, DST, CSIR, UGC and SRTM University.

The details of the academic programmes of the Conference follows.

Details of the Plenary Lecture:

1. Prof. Ravi S. Kulkarni (I. I. T. Bombay, Powai, Mumbai) delivered a Plenary Lecture on *"F-conjugacy, f-characters and f-primitive central characters"*. Chairperson : Prof. Satya Deo.
2. Prof. Raman Parimala (Emory University, U. S. A.) delivered a Plenary Lecture on *"Trace forms"*. Chairperson : Prof. Ravi Kulkarni.

Details of the Memorial Award Lectures:

1. The 25^{th} P. L. Bhatnagar Memorial Award Lecture could not be delivered.

2. The 22^{nd} V. Ramaswami Aiyar Memorial Award Lecture was delivered by Geetha S. Rao (Ramanujan Institute of Advanced Study in Mathematics, Chennai) on *"Tools and Techniques in Approximation Theory"*. Chairperson: P. K. Banerji.
3. The 22^{nd} Srinivasa Ramanujan Memorial Award Lecture was delivered by B. Ramakrishnan (HRI, Allahabad) on *"Modular forms of half- integral weight"*. Chairperson : Satya Deo.
4. The 22^{nd} Hansraj Gupta Memorial Award Lecture was delivered by Ajai Choudhry (Dean, Foreign Service Institute, Delhi) on *"Some Diophantine problems concerning perfect powers of integers"*. Chairperson : J. R. Patadia.

Details of various prizes awarded by the Society :

1. A. Narasinga Rao Memorial Prize.

This prize is awarded to B. B. waphare (Maeer's MIT Arts, Commerce and Science. College, Alandi (D), Pune-412105, Maharashtra, *(balasahebwaphare@gmail.com)* for his research paper entitled *"The pseudo-differential type operator and Hankel type transformation associated with Bessel-type operators"* published in the J. Indian Math, Soc. 76 (2009), 187 - 200 and adjudged to be the best research paper published in the Journal of the Indian Mathematical Society / The Mathematics Student in the year 2009.

2. P. L. Bhatnagar Memorial Prize.

For the year 2011, this prize is presented to Akashnil Dutta (Kolkata) for being the top scorer for the Indian Team (28 points) at the 52^{nd} International Mathematics Olympiad (IMO) that was held during July 13-24, 2011 at Amsterdam, Netherlands.

3. Various prizes for the Paper Presentation Competition:

A total of 20 papers were received for Paper Presentation Competition for Six IMS Prizes, AMU Prize and VMS Prize. There were 14 entries for six IMS prizes, 03 entries for the AMU prize and 03 entries for the VMS prize.

Professors Satya Deo (Chairperson), Huzoor H. Khan, M. A. Sofi and J. R. Patadia were the judges. Following is the result for the award of various prizes:

IMS Prize - **Group-1:** 06 Presentations. Prize is awarded to:
H. S. Ramananda, Joseph Engineering College, Mangalore-575028.

IMS Prize - **Group-2:** 02 Presentations. Prize is awarded to:
Jaspreet Kaur, University of Delhi,Delhi.

IMS Prize - **Group-3 :** 01 Presentation. Prize not awarded.
IMS Prize - **Group-4:** 03 Presentations. Prize is awarded to:
PRATIMA RAI, PUNJAB UNIVERSITY,CHANDIGARH
IMS Prize - **Group-5:** No Presentation. Prize not awarded.
IMS Prize - **Group-6:** 01 Presentation. Prize not awarded.
AMU Prize - 02 Presentations. Prize is awarded to:
ANIL S. KHAIRNAR, A. G. COLLEGE, PUNE.
V M Shah Prize - 02 Presentations. Prize not awarded.

Details of Invited Lectures delivered :

Half an hour Invited talks :

- Chanchal Kumar (IISER, Mohali): *"Alexander Duals of monomial ideals".*
- *Vinayak Joshi (Pune University, Pune):"On zero divisor graphs of posets".*
- *S. S. Khare (NEHU, Shillong): "Span of some specific manifolds".*
- *S. K. Khanduja (Mohali, Chandigarh): "Irreducible polynomials".*
- *V. Lokesh Acharya (Institute of Technology, Bangalore): "An excursion on mathematical means".*
- *K. Srinivas (IMSc, Chennai):"On zeros of Riemann zeta-function: some recent results".*
- *G. S. Khadekar (RTM Nagpur University, Nagpur): "The Universe : What is Man?".*
- *Selby Jose (Ismail Yusuf College, Mumbai):"Generalized Key lemma on Suslin Matrices ".*
- *S. M. Padhey (RLT PG College, Akola):"On Atsuji space, L-spaces, Uniformly approachable spaces and compactness related concepts".*

Details of the Symposia organized :

1. On *"Topology"*
Convener: V. Kannan (Central University, Hyderabad).
Speakers :

- Sharan Gopal (Central University, Hyderabad) : *"Periodic points of homeomorphisms on some compact metric spaces".*
- Zohreh Vaziry(Pune University, Pune): *"Nearness on Boolean frames ".*
- Dieter Leseberg (presented by Zohreh Vaziry) : *"Bounded proximities and related nearness".*
- V. Kannan (Central University, Hyderabad) : *"Extension without further periodic points".*

2. On *"Extremal Graph Theory".*
Convener: B. N. Waphare (University of Pune, Pune).
Speakers :

- B. N. Waphare (University of Pune, Pune): *"Domination and fault tolerance in hypercubes"*.
- S. Arumugam (Kalaslingam University, Krishnankoil): *"Extremal graph theory for metric dimension "*.
- Ambat Vijay Kumar (CUST, Cochin): *"Studies in graph spectra and energy - a survey"*.
- Yashwant Borse (University of Pune, Pune): *"Critically connected graphs and matroids"*.

3. On *"Methods of Approximation"*.
Convener: Smita Naik (Defence Institute of Advanced Technology, Pune).
Speakers :

- Rajendra Deodhar (DIAT, Pune) : *"Neural Networks"*.
- Tanmoy Som (BHU, Varanasi) : *"Concepts of soft sets, soft relation, Fuzzy soft relation and Vague soft sets "*.
- S. V. Ingale (New Arts, Science and Commerce College, Ahmadnagar): *"Uncertainty Quantification in Probabilistic safety analysis using dempster-shafer Theory "*.
- Smita Naik (DIAT, Pune): *"Applications of Mathematical Modelling and Simulation in Defence"*

4. On *"Ramanujan - yesterday and today"*.
Convener: M. A. Pathan (AMU, Aligarh).
Speakers :

- A. K. Agarwal (Punjab University, Chandigarh): *"New Partition Theoretic Interpretations of Rogers-Ramanujan Identities"*.
- S. N. Singh (PU, Jaunpur) : *"On transformation formulae for elliptic hypergeometric series "*..
- M. S. Mahadev Naika (Bangalore University, Bangalore): *"A brief survey on continued fractions of Ramanujan"*.
- S. Ahmad Ali (BBD University, Lucknow): *"On q-series transformations and Ramanujans Partial Theta Function identities"*.

5. On *"Analysis and related fields"*
Convener : Rajiv Srivastava (I. B. S., Agra)
Speakers :

- A. P. Singh (CURAJ, Ajmer) : *"Some applications of approximation theory in complex Dynamics"*.
- Indrajit Lahiri (Kalyani University, Kalyani) : *"Nevanlinna's five and four value theorems and allied results"*.
- Rajiv Srivastava (I. B. S., Agra): *"Bicomplex Analysis: State of Art "*.

- P. K. Banerji (JNV University, Jodhpur): *"Applications of Fractional Calculus"*.
- S. K. Nimbhorkar (Dr. B. A. M. University, Aurangabad): *"Some Concepts from Analysis in Lattices and Posets"*

6. On *"Teaching and research of Mathematics in SAARC Countries"*
Convener: S. B. Nimse (SRTM University, Nanded).
Speakers :

- Dil Bahadur Gurung (Kathmandu University, Nepal): *"Teaching and Research of Mathematics in Nepal"*.
- Buddhi Prasad Sapkota (Tribhuwan University, Nepal) : *"Critical Thinking Strategies in Nepal "*
- Saraswati Acharya (Kathmandu University, Nepal): *"Time Dependent Mathematical Model for Tempreture Distribution on Layered Human Dermal Part"*.
- Mahameed Rasheed (Maldives National University, Maldives) and Ali Arif (Maldives) : *"Current status of higher education in Maldives"*.
- P. K. Banerji (JNV University, Jodhpur) : *"Architects of Mathematical Aristocracy"*
- D. K. Sinha (Kolkata University, Kolkata): *"Whither a roadmap for an evolving SAARC Mathematics?"*
- Zebun Nisa Khan (AMU, Aligarh): *"Teaching of Mathematics"*

7. On it "Careers in Higher Mathematics".
Convener: S. B. Nimse (SRTM University, Nanded).
Speakers :

- Satya Deo (HRI, Allahabad),
- Amin Sofi (University of Kashmir, Kashmir),
- A Vijaykumar (CUST, Cochin).
- S. B. Nimse (SRTM University, Nanded).

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