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Edited by

J. R. PATADIA

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ON THE DIVISION BY 5 OF PERIODS OF ELLIPTIC FUNCTIONS AND THE EXISTENCE OF AN OUTER-AUTOMORPHISM OF S_6*

R.Sridharan

Introduction

The systematic study of division of periods of elliptic functions undertaken by Abel [1] in the nineteenth century has its implicit origin in the well-known and centuries' old problem of the ancient Greeks whether a regular heptagon can be inscribed in a circle with ruler and compass alone, which was solved in a spectacular way by Gauss. As is well-known, Gauss, the prince among mathematicians, in his great work *Disquisitiones Arithmeticae*, published in 1801, ([4], an English translation, we quote sometimes from) solved a much more general question in Section 7 of this book by showing that a necessary and sufficient condition that a regular n sided polygon can be inscribed in a circle with ruler and compass alone is that n must be of the form $2^r p_1 \ldots p_t$, where $r \geq 0$ is any integer and p_i, primes of the form $2^{2^m}+1$ for $m \geq 0$.

In fact, in March 30, 1797, as a student, Gauss had already shown that the next prime p after 5 for which a regular p-sided polygon can be inscribed in the above mentioned manner in a circle is $p = 17$ and he did explicitly achieve such a construction! While introducing in section 7, entitled *Equations Defining Sections of Circle* of his book, which deals with the proof of his general theorem, he made the following rather cryptic statement "*The principles of the theory which we are going to explain actually extend much farther than we will indicate* (*for they can be applied not only to circular functions, but just as well to other transcendental functions, e.g. to those which depend on the integral* $\int \frac{dx}{\sqrt{1-x^4}}$)". Towards the end of this section, he adds a warning against any one attempting to achieve "geometric" constructions for sections other than

* The text of the Presidential Address (Technical) delivered at the 76^{th} Annual Conference of the Indian Mathematical Society held at the Sardar Vallabhbhai National Institute of Technology (SVNIT), Surat - 395 007, Gujarat, during the period December 27 - 30, 2010.

the ones suggested by his theory (e.g, sections into 7,11,13,19 etc parts) and so spend his time uselessly.

The "principle" Gauss used to prove his beautiful theorem for the circle can be roughly described in retrospect as "*Galois Theory of Cyclotomic Extensions*".

Abel, then a young mathematician, who read the book of Gauss was first bemused by Gauss' remark that a similar theorem holds for transcendental functions associated to the integrals like $\int \frac{dx}{\sqrt{1-x^4}}$. Of course, it did not take very much long for this genius to understand what Gauss had in mind, namely that a theorem similar to that for a circle can be proved for the Lemniscate curve given by the equation $(x^2+y^2)^2 = x^2 - y^2$, whose arc length turns out to be (as is easily seen by using the polar equation $r^2 = \cos 2\theta$ for the curve) the integral $\int \frac{dr}{\sqrt{1-r^4}}$. Incidentally, the Lemniscate curve can once again be traced back to the Greeks (especially to Eudoxus), who considered such a curve on the 2-sphere and the curve had been called the *Hippopede* ("horseshoe"). This curve was rediscovered by the Bernoulli brothers in the seventeenth century and who, as was usual with them, had unending quarrels between them regarding priorities!

The Lemniscate curve became an object of intense study for the mathematicians of the seventeenth and eighteenth centuries. The Italian mathematician Fagnano spent his life time studying various properties of this curve (At his wish, this curve was engraved on his tomb stone). Fagnano proved in particular that the arc length of this curve can be bisected with ruler and compass (which work came to the notice of Euler in Germany, who proved that Fagnano's result can be generalised to an *addition theorem* for the so-called elliptic integrals which paved the way to a systematic study of "elliptic integrals" by Legendre, but both of them missed the point that "inverting" such integrals led to new transcendental functions called elliptic functions, a remarkable discovery due independently to Abel and Jacobi, and due independently, also to Gauss, much earlier than them!). Fagnano also proved that the arc of the lemniscate can be divided into 5 equal parts in a similar manner. Gauss, who never published much of his work had in fact in a note ([5], entry 62, March 21, 1797) in his diary (which was published only 30 years after his death) also mentions the possibility of division into 5 equal parts of the Lemniscate curve. But by the time Gauss's diary was published, Abel and Jacobi had already defined elliptic functions and worked extensively on various aspects of these functions. In particular, Abel had proved that under the same conditions on the integer n

that Gauss had for the division of the circle into n equal parts, the Lemniscate can be also divided into n equal parts!

Abel, in a series of papers in the Crelle journal systematically developed the theory of elliptic functions ; taking up in one of his papers the question of division of periods, he deduced in particular that for the elliptic function associated with inverting the integral $\int \frac{dx}{\sqrt{1-x^4}}$, division by an integer n is possible if n is of the form $2^r p_1 \ldots p_t$ with $p_i = 2^{2^{m_i}} + 1$ prime for all i, the same condition that Gauss had given for the division of the circle. (It is incidentally worth remarking in this context that the Italian mathematician G.Libri who was a contemporary of Abel, continuing to work on the problem of division of the Lemniscate into equal parts, started by Fagnano and trying to extend it on the lines of Fagnano, submitted a paper to the Paris Academy, which, according to Libri, was placed next to that of Abel, and was ignored! His paper was published in the Crelle later [8]). Abel in his researches also proved that, in general, the problem of division by n of periods of elliptic functions leads to algebraic equations over $\mathbb{Q}$ which in general are not even solvable by radicals for $n \geq 5$.

G.Halphen, in the third of his three volumes on *Traité des fonctions elliptiques et de leurs applications*, this last volume being published posthumously in Paris in 1891 [6], discusses the question of division of periods of elliptic functions by primes and in particular, the problem in full detail for $p = 5, 7, 11$ (which are in some sense special, though we shall not explain here why!) In particular, in chapter I of his third volume, he discusses the question of division by 5 of periods of the Weierstrass elliptic function $\wp$ and shows (as was in fact done by Brioschi earlier) that if

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

the problem of division of its periods depends on the solutions of the sixth degree equation

$$x^6 - 5g_2x^4 - 40g_3x^3 - 5g_2^2x^2 - 8g_2g_3x - 5g_3^2 = 0,$$

and a quadratic equation. If one denotes by α, one of the roots of the above, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$, the rest, Halphen obtains by a non trivial and lengthy computation, the polynomial equation

$$x^5 + \frac{10}{3}x^3 + 5x - \frac{8}{3}g_3\sqrt{J-1} = 0,$$

(where $J = \frac{g_2^3}{\Delta}$, Δ denoting the discriminant $4g_2^3 - 27g_3^2$ of the cubic $4x^3 - g_2x - g_3$), whose roots $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ are given by

$$\beta_i = \alpha\alpha_i + \alpha_{i+1}\alpha_{i+4} + \alpha_{i+2}\alpha_{i+3}$$

(the indices to be read mod 5), which is the so-called *Brioschi-resolvent* of this sixth degree polynomial. (We note incidentally that P.Gordan in a beautiful paper reduced the "general" equation of the 5^{th} degree to the above Brioschi normal form by solution of linear and quadratic equations - unlike Brioschi who had used a cubic equation (cf [10] for a recasting and appreciation of Gordan's work)

The first section of this article is devoted to the derivation of the sixth degree equation mentioned above (though by a method not exactly that of G.Halphen, but based on ([9], p. 234)) and the second section gives a brief statement of the result leading to the Brioschi resolvent of degree 5.

The final section of this article which is perhaps the only, hopefully new contribution of this article is the construction of an outer-automorphism of S_6 by a rather wild guess, motivated by the definition of β_i defined by Halphen. In fact, denoting the symmetric group on 6 symbols denoted by $\infty, 0, 1, 2, 3, 4$, we define a map ϕ on the set of transpositions (∞, i), $0 \leq i \leq 4$ with values in S_6 by setting $\phi((\infty, i)) = (\infty, i)(i+1, i+4)(i+2, i+3)$, ($i$ being read modulo 5) and show that ϕ can be extended to an automorphism of S_6 by using a description, more generally of any S_n (due to G.E. Moore), by generators and relations (cf [2]).

In a lighter vein, I would like to confess that when I look back at the introduction, the Marathi aphorism, included in the introduction of the *Vākyapadīya* of Bhartrhari edited with explanatory notes by the renowned scholars K.V. Abhyankar and V.P.Limaye and published in the "Poona University Sanskrit Series" comes to my mind. These erudite scholars, while comparing the size of the actual text with the total size of their appendices say that they hope that it does not illustrate the Marathi proverb which when translated into Sanskrit reads अङ्गुष्ठमात्रः पुरुषः कूर्चा नाभ्यवलम्बिनः (in English, meaning "The man is just thumb high, but has a beard reaching up to his navel"). I am afraid that this proverb applies more fittingly to the present article when one compares the mathematical contents of this article with the length of the introduction!

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1. A sixth degree polynomial equation associated to the division of periods of elliptic functions

Let $\wp$ denote the Weierstrass elliptic function which is a doubly periodic meromorphic function with the $\mathbb{R}$-linearly independent periods ω_1, ω_2 defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where the summation is over all elements ω of the form $m_1\omega_1 + m_2\omega_2$; m_1, m_2 not both zero. The problem of division by 5 of the periods ω is just the study of the values of $\wp(z)$ for $z = \frac{m_1\omega_1 + m_2\omega_2}{5}$. We recall that the pair $(\wp(z), \wp'(z))$, $z \in \mathbb{C}$ parametrizes the *elliptic curve* in the plane given by the equation

$$y^2 = 4x^3 - g_2 x - g_3,$$

where g_2, g_3 are explicit complex numbers defined in terms of the periods ω_1, ω_2. We do not need their explicit definition and so do not bother to write them down. In view of the periodicity of $\wp$, if we exclude the case $m_1 = m_2 = 0$ which corresponds to the "point at ∞" of the above curve, the values $0 \leq m_i \leq 4$ give rise to 24 values of $\wp\left(\frac{m_1\omega_1+m_2\omega_2}{5}\right)$, of which once again, due to the fact that $\wp$ is an even function of z (as is obvious), there are only 12 distinct values. If we denote by ω any typical period, it is clear that in order to determine these 12 values, it is enough to determine the (distinct) values of $\wp\left(\frac{\omega}{5}\right)$ and $\wp\left(\frac{2\omega}{5}\right)$. Let us set $x = \wp\left(\frac{\omega}{5}\right) + \wp\left(2\frac{\omega}{5}\right)$ and $x' = \wp\left(\frac{\omega}{5}\right)\wp\left(\frac{2\omega}{5}\right)$. The values of $\wp\left(\frac{\omega}{5}\right)$ and $\wp\left(\frac{2\omega}{5}\right)$ are then determined by the quadratic equation

$$y^2 - xy + x' = 0.$$

We will show that x is determined by the 6^{th} degree equation

$$x^6 - 5g_2x^4 - 40g_3x^3 - 5g_2^2x^2 - 8g_2g_3x - 5g_3^2 = 0,$$

and x' from the equation (for such an x) by

$$x^3 - 6xx' + \frac{1}{2}g_2x + g_3 = 0.$$

The fact that x satisfies the required sixth degree equation is proved as follows :

We recall the duplication formula for the function $\wp$ which can be written as

$$\wp(2z) = \frac{(\wp^2(z) + \frac{1}{4}g_2)^2 + 2g_3\wp(z)}{4\wp^3(z) - g_2\wp(z) - g_3},$$

which comes from the relation $\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$ and some simplification. This tells us that

$$\begin{aligned} x &= \wp\left(\frac{\omega}{5}\right) + \wp\left(\frac{2\omega}{5}\right) \\ &= \wp\left(\frac{\omega}{5}\right) + \frac{(\wp^2(\frac{\omega}{5}) + \frac{1}{4}g_2)^2 + 2g_3\wp(\frac{\omega}{5})}{4\wp^3(\frac{\omega}{5}) - g_2\wp(\frac{\omega}{5}) - g_3} \\ &= y + \frac{(y^2 + \frac{1}{4}g_2)^2 + 2g_3 y}{4y^3 - g_2 y - g_3} \end{aligned}$$

with $y = \wp\left(\frac{\omega}{5}\right)$

However, this equation between x and y is satisfied if y is also replaced by $y = \wp\left(\frac{2\omega}{5}\right)$. In other words, $\wp\left(\frac{2\omega}{5}\right)$ is also a root of the same equation since $x = \wp\left(\frac{2\omega}{5}\right) + \wp\left(\frac{\omega}{5}\right) = \wp\left(\frac{4\omega}{5}\right) + \wp\left(\frac{2\omega}{5}\right)$ by the periodicity and evenness of $\wp$. Hence the polynomial

$$\left(y^2 + \frac{1}{4}g_2\right)^2 + 2g_3 y + (y - x)\left(4y^3 - g_2 y - g_3\right)$$

is divisible by the polynomial $y^2 - yx + x'$. If we divide the above polynomial by $y^2 - yx + x'$, the resulting remainder, which is a linear polynomial in y must therefore have both its coefficient of y and the constant term zero. This gives by a computation, the following equations

$$x^3 - 6xx' + \frac{1}{2}g_2 x + g_3 = 0 \ldots (*)$$

$$5x'^2 - x'(x^2 - \frac{1}{2}g_2) + g_3 x + \frac{g_2^2}{16} = 0 \ldots (**)$$

Eliminating x' in (**), using (*), we get the equation

$$x^6 - 5g_2x^4 - 40g_3x^3 - 5{g_2}^2x^2 - 8g_2g_3x - 5g_3^2 = 0$$

Given any root of the above equation (whose roots are distinct for general g_2 and g_3), we can get x' from (*) and finally, $y = \wp\left(\frac{\omega}{5}\right)$ or $\wp\left(\frac{2\omega}{5}\right)$ are then the two distinct roots of the quadratic equation

$$y^2 - xy + x' = 0$$

giving the desired 12 distinct values.

We reiterate that G.Halphen gives a slightly different method of arriving at the 6$^{\text{th}}$ degree equation above. We use the method given in the book of Molk and Tannery ([9] p. 233), correcting incidentally a few misprints (which indeed were rather troublesome) in their proof.

We note incidentally that if $g_3 = 0$, the equations above reduce to the quartic equation $x^4 - 5g_2x^2 - {g_2}^2 = 0$ and a quadratic equation. For instance, for the special value $g_2 = \frac{1}{4}$, this shows that the division of the Lemniscate curve into

5 equal parts can be achieved by ruler and compass alone. (This is essentially the proof Abel gives in his paper. In general, the sixth degree equation is not solvable by radicals.)

2. The Brioschi quintic associated to the sixth degree equation of the previous section

Motivated by the work of Kronecker, Brioschi in fact wrote down a resolvent of degree 5 for the sixth degree equation of the earlier section. G.Halphen, in his third volume on elliptic functions constructs (partly influenced by Brioschi's work) the Brioschi resolvent. The resolvent is in fact the following polynomial of degree 5 given by

$$f(x) = x^5 - \frac{10}{3}x^3 + 5x - \frac{8}{3}\sqrt{J-1},$$

where $J = \frac{g_2^3}{\Delta}$ (Δ denoting the discriminant of the cubic polynomial $4x^3 - g_2x - g_3$, is the absolute invariant of the elliptic curve $y^2 = 4x^3 - g_2x - g_3$). We do not include Halphen's proof here, since it is quite lengthy, and since in any case we are not going to use it in the rest of the article. We would however like to remark that P.Gordan in a remarkable paper proved that the general equation of the fifth degree can always be brought to the remarkable normal form (all whose coefficients except the constant term, being rational constants) only by quadratic and linear substitutions (cf [10] for a "modern" version of his beautiful proof).

We shall need as a vague motivation for the next section, the transformation of the roots of the sixth degree equation used by Halphen to get the Brioschi normal form, which we describe now. We denote by $\alpha, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ the (distinct) roots of the sextic. By an intricate analysis, Halphen proves that if we define $\beta_i, 0 \leq i \leq 4$, by setting

$$\beta_i = \alpha\alpha_i + \alpha_{i+1}\alpha_{i+4} + \alpha_{i+2}\alpha_{i+3},$$

(where the indices i are read modulo 5), the β_i are indeed the roots of the Brioschi quintic. As we remarked, his proof (modelled essentially after Brioschi's) is quite intricate and lengthy. Perhaps, it is possible to simplify Halphen's proof, but we are not in a position to do so now.

3. An outer automorphism of S_6

Let us denote by $\infty, 0, 1, 2, 3, 4$ six distinct symbols and consider the group of bijections of these elements which is an S_6. Our aim in this section is to construct an automorphism of S_6 which is outer, i.e. is not given by conjugation by any element of S_6. Since any conjugation must take any transposition to another, it would be enough to exhibit an automorphism which takes a transposition to a product of (here, in fact three) distinct transpositions. (It is a remarkable property of the number 6 that it is the only natural number for which the number of transpositions (which is 15) is the *same* as the number of permutations which are product of three disjoint transpositions!) The group S_6 is thus indeed the only symmetric group which admits outer automorphisms. This amazing result goes back to O. Hölder ([7], p.343). In fact, the group of all automorphisms of S_6 is 1440, there being only one non trivial conjugate class of outer automorphisms of S_6.

To construct such an automorphism, we first define a map ϕ on the set of transpositions of the form $(\infty, i), 0 \leq i \leq 4$, with values in S_6, defined by

$$\phi((\infty, i)) = (\infty, i)(i+1, i+4)(i+2, i+3),$$

where the indices are to be read modulo 5.

We shall show that ϕ can be extended to an automorphism of S_6 and this is the required automorphism (Note that the definition of ϕ comes from a rather naive imitation of the β_i in terms of the α_i). Once the extension of ϕ has been guaranteed, it is rather immediate that this extension which we shall continue to denote by ϕ must be injective since its kernel has to be either the identity subgroup of S_6 which is what we want, or S_6 or A_6; both the latter possibilities are impossible since the image of ϕ cannot obviously reduce to identity, nor can it reduce to a subgroup of two elements, which is once again clear. Since ϕ is injective, it should be surjective!

The crucial point is therefore to check that ϕ admits (indeed a unique) extension to an endomorphism of S_6. In order to prove this, we first denote by γ the transposition $(\infty, 0)$ and $\gamma_i, 0 \leq i \leq 3$, the transpositions $(i, i+1)$ (the indices being as usual always read modulo 5). Obviously $\gamma, \gamma_i, 0 \leq i \leq 3$ generate S_6. The relations among these five elements are given by the following

- $\gamma^2 = \gamma_i^2 = 1$ for $0 \leq i \leq 3$
- $\gamma_i \gamma_j = \gamma_j \gamma_i$ for $0 \leq i < j - 1 \leq 3$
- $\gamma \gamma_i = \gamma_i \gamma$ for $1 \leq i \leq 3$
- $(\gamma \gamma_0)^3 = 1$
- $(\gamma_i \gamma_{i+1})^3 = 1$ for $0 \leq i \leq 3$

This is a trivial checking which we shall not give. We define $\phi(\gamma_i), 0 \leq i \leq 3$ by forcing multiplicativity of ϕ. Indeed, we have for $i \neq j$ ($i < j$ say)

$$(i, j) = (\infty, i)(\infty, j)(\infty, i),$$

so that, ϕ should be defined by

$$\begin{aligned}\phi((i,j)) &= \phi((\infty,i))\phi((\infty,j))\phi((\infty,i))\\ &= (\infty,i)(i+1,i+4)(i+2,i+3)(\infty,j)(j+1,j+4)(j+2,j+3)\\ &\quad(\infty,i)(i+1,i+4)(i+2,i+3),\end{aligned}$$

so that with $j = i+1$, we have, in particular

$$\phi((i,i+1)) = (\infty,i+3)(i,i+4)(i+1,i+2).$$

Explicitly, one then has

$$\begin{aligned}\phi((0,1)) &= (\infty,3)(0,4)(1,2)\\ \phi((1,2)) &= (\infty,4)(0,1)(2,3)\\ \phi((2,3)) &= (\infty,0)(1,2)(3,4)\\ \phi((3,4)) &= (\infty,1)(0,4)(2,3)\end{aligned}$$

Setting $\gamma' = \phi((\infty,0))$ and $\gamma'_i = \phi((i,i+1))$ for $0 \le i \le 3$, we show that γ' and γ'_i satisfy the relations which should be the same as among the γ and γ_i and this would finish the proof that ϕ indeed extends to an endomorphism of S_6, provided we grant the result of G.E Moore referred to in the introduction.

We do not check all the relations since it would be a rather tedious task both to write and to read. We shall check some sample relations. The conditions $\gamma'^2 = \gamma_i'^2 = 1$ are clear. For instance $\gamma_0'^2 = \phi((0,1))^2 = (\infty,3)(0,4)(1,2)(\infty,3)(0,4)(1,2) = 1$ holds trivially since the R.H.S $= ((\infty,3)(0,4)(1,2))^2 = id$.

We check for example $(\gamma'\gamma'_0)^3 = 1$. In fact $\gamma'\gamma'_0 = (\infty,0)(1,4)(2,3)(\infty,3)(0,4)(1,2) = (1,3,0)(\infty,2,4)$ is the product of two 3 cycles and hence we are through. Similarly,

$$\begin{aligned}\gamma'\gamma'_1 &= (\infty,0)(1,4)(2,3)(\infty,4)(0,1)(2,3)\\ &= (0,4)(1,\infty)\\ \gamma'_1\gamma' &= (\infty,4)(0,1)(2,3)(\infty,0)(1,4),(2,3)\\ &= (1,\infty)(0,4).\end{aligned}$$

which are the same!

For $0 \le i < j-1 \le 3$, it is easily checked that $\gamma'_i\gamma'_j = \gamma'_j\gamma'_i$, which are both products of disjoint transpositions.

For any i, with $0 < i \le 3$, $\gamma'_i\gamma'_{i+1}$ is the product of the (disjoint) 3-cycles $(\infty,i,i+2)$ and $(i+1,i+4,i+3)$ and hence $(\gamma'_i\gamma'_{i+1})^3 = 1$.

Indeed, one can check all the other required relations between the γ', γ'_i, where $0 \le i \le 3$ and this proves that ϕ does indeed admit an extension as an endomorphism of S_6 to S_6. This finishes the proof. We end however with the following remarks.

4. **Remarks**

1. We note that the transpositions $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ generate a subgroup of S_6 which fixes the symbol ∞ and hence is in particular a non-transitive subgroup of S_6. However $\gamma_0', \gamma_1', \gamma_2', \gamma_3'$ by their very definition are easily seen to generate a subgroup of S_6 isomorphic to S_5 which is transitive on the symbols $\infty, 0, 1, 2, 3, 4$. This illustrates the general principle that if an automorphism of S_6 takes a non-transitive subgroup onto a transitive subgroup, then it *cannot* be inner!
2. The existence of outer automorphisms of S_6 is quite an ubiquitous phenomenon and occurs in various contexts. I end this article by referring very briefly and sketchily to one such. In a paper [11] of G. van der Geer, where he discusses a certain compactification of the quotient space $S_2^* = S_2/\Gamma_2(2)$, where S_2 is the Siegel upper-half space of degree 2 and $\Gamma_2(2) = \ker(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/2))$; he shows that the 1-dimensional boundary components correspond bijectively to totally isotropic 1-dimensional subspaces of $\mathbb{F}_2^4$ equipped with the standard skew symmetric form and the 0-dimensional boundary components correspond bijectively to 2-dimensional isotropic subspaces. The group $\mathrm{Sp}(4,\mathbb{F}_2)$ operates on six maximal sets of five disjoint 1-dimensional boundary components and this action establishes an isomorphism of $\mathrm{Sp}(4,\mathbb{F}_2)$ with S_6. Curiously enough, the configuration of the boundary components given on p.324 of his paper has already been considered in a different connection by Tutte (in 1958), namely, in connection with the cycle structure of S_6 and has been called the *Tutte graph* by H.S.M. Coxeter in [3] (Lect 7, p.153). This graph is related to the existence of outer automorphisms of S_6.

References

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A QUESTION OF MODELS*

SHIVA SHANKAR

1. **Introduction**

I am grateful for this opportunity to speak here.

I am going to speak about a question that arises in our attempt to construct a model for some phenomenon in the natural world or of an engineering system. This question resembles in form the following basic question in algebraic geometry:

Let $\mathcal{A} = \mathbb{C}[X_1, \ldots, X_n]$, $\mathfrak{i} \subset \mathcal{A}$ an ideal, and let $\mathcal{V}(\mathfrak{i})$ be the variety of $\mathfrak{i}$ in $\mathbb{C}^n$ (i.e. the set of common zeros of all the polynomials in $\mathfrak{i}$). The question is: if $\mathfrak{i}$ and $\mathfrak{j}$ are two ideals, when is $\mathcal{V}(\mathfrak{i}) = \mathcal{V}(\mathfrak{j})$? The answer is the Hilbert Nullstellensatz which states that $\mathcal{V}(\mathfrak{i}) = \mathcal{V}(\mathfrak{j})$ if and only if $\sqrt{\mathfrak{i}} = \sqrt{\mathfrak{j}}$.

Instead, suppose that $\mathcal{A} = \mathbb{C}[D_1, \ldots, D_n]$, the ring of constant coefficient partial differential operators, where $D_j = \frac{1}{\imath}\frac{\partial}{\partial x_j}$, $\imath = \sqrt{-1}$. Given $p(D)$ in $\mathcal{A}$, a distribution f in $\mathcal{D}'$ is a *zero* of $p(D)$ if $p(D)f = 0$. Let $\mathcal{B}_{\mathcal{D}'}(p) = \{f \in \mathcal{D}' | p(D)f = 0\}$ be the set of all its zeros, in other words the kernel of the $\mathcal{A}$-module map $p(D) : \mathcal{D}' \to \mathcal{D}'$. More generally, instead of $\mathcal{D}'$, one can consider zeros in some $\mathcal{A}$-submodule $\mathcal{F}$ of $\mathcal{D}'$. Even more generally, given $p(D) = (p_1(D), \ldots, p_k(D))$ in $\mathcal{A}^k$, let $\mathcal{B}_{\mathcal{F}}(p)$ be the kernel of the $\mathcal{A}$-module map $p(D) : \mathcal{F}^k \to \mathcal{F}$ given by $p(D)f = \sum_j p_j(D)f_j$, where $f = (f_1, \ldots, f_k)$ is in $\mathcal{F}^k$. Given a submodule $\mathcal{P}$ of $\mathcal{A}^k$, $\mathcal{B}_{\mathcal{F}}(\mathcal{P}) = \cap_{p\in\mathcal{P}}\mathcal{B}_{\mathcal{F}}(p)$ is the set of common zeros in $\mathcal{F}$ of all the $p(D)$ in $\mathcal{P}$.

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The question I wish to address is: given two submodules $\mathcal{P}$ and $\mathcal{Q}$ of $\mathcal{A}^k$, when is $\mathcal{B}_{\mathcal{F}}(\mathcal{P}) = \mathcal{B}_{\mathcal{F}}(\mathcal{Q})$? This is the Nullstellensatz problem for systems of PDE [7].

2. **Models**

To explain the meaning of this question, I must first say a few words about models and the class of models that a theory focuses its attention on.

A model is a picture of reality, and the closer it is to reality the better the picture it will be, and the more effective will be the theory that describes this model. A model seeks to represent a certain phenomenon or system whose attributes are certain qualities that are changing with space, time etc. The closest we can get to this reality, this phenomenon, is to take all possible variations of the attributes of the phenomenon, itself, as the model. This collection, considered all together in our minds, is, in engineering parlance, the *behaviour* of the phenomenon or the system [10]. Paraphrasing Wittgenstein, one might even declare that the behavior is all that is the case!

A priori, any evolution of these attributes could perhaps have occured, but the *laws* of the system prescribe those evolutions that actually do. I shall consider behaviours described by local laws, i.e. laws expressed by differential equations (here by local I mean that the variation of the attributes of the phenomenon at a point depends on the values of the attributes in arbitrarily small neighbourhoods of the point, and not on points far away).

This is a familiar situation in physics, mathematics and engineering. For instance:

1. Planetary motion. A priori, the earth could perhaps have traversed any trajectory around the sun, but Kepler's laws restrict it to travel an elliptic orbit, sweeping equal areas in equal times, and such that the square of the period of revolution is proportional to the cube of the major semi-axis. Kepler's laws are of course local, namely Newton's equations of motion.
2. Magnetic fields. A priori, any function B could perhaps have been a magnetic field, but in fact must satisfy the law that there do not exist magnetic monopoles in the universe. This law is also local:

$$\mathsf{div}\ B = 0$$

3. Complex analysis. Complex analysis studies those functions which admit a convergent power series about each point. This is the law that defines the subject.

This law is again local, viz. the Cauchy-Riemann equations.

As it is impossible to list every possible evolution of the attributes of a phenomenon - usually there are infinitely many - we instead list the laws the phenomenon satisfies - usually these can be effectively listed. This list of laws is a model of the phenomenon, a more concise and efficient model than a listing of every instance of the phenomenon. An important case when all laws can be generated by a finite list is when the phenomenon is linear and shift invariant. Such behaviours arise as kernels of maps given by systems of constant coefficient differential operators, and are therefore also called *differential kernels*.

Thus suppose that the attributes of a system are described by some k-tuple of smooth functions, $f = (f_1, \ldots, f_k)$, $f_j : \mathbb{R}^n \to \mathbb{C}$. Then a law the system obeys is a partial differential operator $p(D) = (p_1(D), \ldots, p_k(D))$ such that $p(D)f = \sum p_j(D)f_j = 0$. In other words a law is an operator $p(D) : (\mathcal{C}^\infty)^k \to \mathcal{C}^\infty$, and to say that f obeys this law is to say that $p(D)f = 0$, or that f is a zero of $p(D)$. The phenomenon itself is

$$\mathcal{B} = \bigcap_{\text{all laws } p} \{f | p(D)f = 0\}$$

the common zeros of all the laws governing the phenomenon.

Shift-invariance implies that $p(D)$ belongs to the ring $\mathcal{A} = \mathbb{C}[D_1, \ldots, D_n]$ of constant coefficient partial differential operators on $\mathbb{R}^n$. Linearity implies that the totality of laws the system obeys is the submodule $\mathcal{P}$ of $\mathcal{A}^k$ generated by all the laws $p(D)$. Thus the sum of two laws is a law, and differentiating a law further is again a law. As $\mathcal{A}$ is Noetherian, all these laws are generated by a finite number of laws. If they be l in number, then writing these l laws as rows of a matrix gives an operator

$$P(D) : (\mathcal{C}^\infty)^k \longrightarrow (\mathcal{C}^\infty)^l$$

whose kernel is precisely the behaviour of the system.

There is nothing very special about the $\mathcal{A}$-module $\mathcal{C}^\infty$, and I shall more generally consider phenomena whose attributes are elements in the $\mathcal{A}$-modules $\mathcal{D}'$, $\mathcal{C}^\infty$, the space $\mathcal{S}'$ of temperate distributions, the Schwartz space $\mathcal{S}$, the spaces $\mathcal{E}'$ and $\mathcal{D}$ of compactly supported distributions and smooth functions, and certain spaces of periodic functions.

The meaning of the Nullstellensatz problem is now clear: given two sets of laws, expressed by the submodules $\mathcal{P}$ and $\mathcal{Q}$ of $\mathcal{A}^k$, when are they models for the same phenomenon? In other words, when is $\mathcal{B}_\mathcal{F}(\mathcal{P}) = \mathcal{B}_\mathcal{F}(\mathcal{Q})$?

3. **Algebraic Structure**

The answer to the above Nullstellensatz question for PDE depends on the $\mathcal{A}$-module structure of the space $\mathcal{F}$. For the spaces $\mathcal{D}'$, $\mathcal{C}^\infty$ and $\mathcal{S}'$, this description is the *Fundamental Principle* of Malgrange-Palamodov which I now explain.

This result states that every image $(\mathcal{D}')^k \xrightarrow{P(D)} (\mathcal{D}')^l$ is also a kernel, in fact the kernel of $(\mathcal{D}')^l \xrightarrow{Q(D)} (\mathcal{D}')^m$, where the m rows of $Q(D)$ generate all the relations between the rows of $P(D)$. Thus the sequence

$$(\mathcal{D}')^k \xrightarrow{P(D)} (\mathcal{D}')^l \xrightarrow{Q(D)} (\mathcal{D}')^m$$

is exact.

The Fundamental Principle answers the solvability question for systems of PDE: given g in $(\mathcal{D}')^l$, is there an f in $(\mathcal{D}')^k$ such that $P(D)f = g$? The principle asserts - yes, if and only if $Q(D)g = 0$ [1,5].

Example: Consider the curl operator

$$\mathsf{curl} = \begin{pmatrix} 0 & -\imath D_z & \imath D_y \\ \imath D_z & 0 & -\imath D_x \\ -\imath D_y & \imath D_x & 0 \end{pmatrix}$$

where $D_x = \frac{1}{\imath}\frac{\partial}{\partial x}$ etc. All the relations between its rows is generated by $(\imath D_x, \imath D_y, \imath D_z)$. Thus $\mathsf{curl}\, f = g$ is solvable for a given g in $(\mathcal{D}')^3$ if and only if $\mathsf{div}\, g = 0$. This is just the exactness of the sequence

$$(\mathcal{D}')^3 \xrightarrow{\mathsf{curl}} (\mathcal{D}')^3 \xrightarrow{\mathsf{div}} \mathcal{D}' \tag{3.1}$$

Similarly the relations between the rows of the gradient operator $(\imath D_x, \imath D_y, \imath D_z)^t$ are generated by the rows of curl. Then again by the Fundamental Principle $\mathsf{grad}\, f = g$ is solvable for a given g in $(\mathcal{D}')^3$ if and only if $\mathsf{curl}\, g = 0$. It similarly follows that $\mathsf{div}\, f = g$ can be solved for every g, so that

$$\mathcal{D}' \xrightarrow{\mathsf{grad}} (\mathcal{D}')^3 \xrightarrow{\mathsf{curl}} (\mathcal{D}')^3 \xrightarrow{\mathsf{div}} \mathcal{D}' \to 0$$

is exact. □

I wish to emphasise the role played by the space $\mathcal{F}$ where the solutions are located (lest it be thought that this is a problem in algebra). For instance it turns out that not only is (1) exact, but that also its restriction to $\mathcal{D}$

$$\mathcal{D}^3 \xrightarrow{\mathsf{curl}} \mathcal{D}^3 \xrightarrow{\mathsf{div}} \mathcal{D}$$

is exact. Now, as the columns $c_1(D), c_2(D), c_3(D)$ of curl are $\mathcal{A}$-dependent, i.e. as $D_x c_1(D) + D_y c_2(D) + D_z c_3(D) = 0$, the $\mathcal{D}'$ image of curl is also the $\mathcal{D}'$ image of the operator $\mathsf{crl} = (c_1(D), c_2(D))$ given by the first two columns of curl. For suppose g is in $\mathcal{D}'$. Let f in $\mathcal{D}'$ be such that $\imath D_z f = g$ (by the Fundamental Principle there is such an f - indeed if $p(D)$ is nonzero, then there is always an f such that $p(D)f = g$ - which is to say that $\mathcal{D}'$ is a divisible $\mathcal{A}$-module). Hence $c_3(D)g = c_3(D)\imath D_z f = -(\imath D_x c_1(D) + \imath D_y c_2(D))f = c_1(D)(-\imath D_x f) + c_2(D)(-\imath D_y f)$, which is in the image of the operator crl.

In other words, the sequence

$$(\mathcal{D}')^2 \xrightarrow{\mathsf{crl}} (\mathcal{D}')^3 \xrightarrow{\mathsf{div}} \mathcal{D}'$$

is also exact. On the other hand its restriction to $\mathcal{D}$

$$\mathcal{D}^2 \xrightarrow{\mathsf{crl}} \mathcal{D}^3 \xrightarrow{\mathsf{div}} \mathcal{D}$$

is *not* exact [9].

In the above I used the Fundamental Principle in the simplest situation, that for a single PDE. It asserted the surjectivity of $\mathcal{D}' \xrightarrow{p(D)} \mathcal{D}'$, viz. that $\mathcal{D}'$ is a divisible $\mathcal{A}$-module. So is $\mathcal{C}^\infty$ - this is the basic existence theorem of Ehrenpreis-Malgrange. The space $\mathcal{S}'$ of temperate distributions is also a divisible $\mathcal{A}$-module - this is a famous result of Hörmander-Łojasiewicz, and in particular it implies the existence of fundamental solutions that are temperate.

But the Fundamental Principle says much more: the sequence $\mathcal{F}^k \xrightarrow{P(D)} \mathcal{F}^l \xrightarrow{Q(D)} \mathcal{F}^m$ is identical to the sequence $\mathsf{Hom}_{\mathcal{A}}(\mathcal{A}^k, \mathcal{F}) \xrightarrow{P(D)} \mathsf{Hom}_{\mathcal{A}}(\mathcal{A}^l, \mathcal{F}) \xrightarrow{Q(D)} \mathsf{Hom}_{\mathcal{A}}\,(\mathcal{A}^m, \mathcal{F})$, and to say this is exact when $\mathcal{A}^m \xrightarrow{Q^t(D)} \mathcal{A}^l \xrightarrow{P^t(D)} \mathcal{A}^k$ is exact is to say that $\mathcal{F}$ is an injective $\mathcal{A}$-module. Indeed, $\mathcal{D}'$ and $\mathcal{C}^\infty$ are injective cogenerators, whereas $\mathcal{S}'$ is injective although not a cogenerator. On the other hand $\mathcal{S}$, $\mathcal{E}'$ and $\mathcal{D}$ are not injective, not even divisible $\mathcal{A}$-modules. Instead, $\mathcal{S}$ is a flat $\mathcal{A}$-module while $\mathcal{E}'$ and $\mathcal{D}$ are faithfully flat [2,4,5,9].

I also consider the following spaces of periodic functions: let T be the torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$. The space $\mathcal{C}^\infty(\mathsf{T}) := \mathcal{C}^\infty(\mathbb{R}^n/2\pi\mathbb{Z}^n)$ of smooth functions on it is a Fréchet space, and also a topological $\mathcal{A}$-module. For positive integers N_1 dividing N_2, the space $\mathbb{R}^n/2\pi N_2\mathbb{Z}^n$ is a covering space of $\mathbb{R}^n/2\pi N_1\mathbb{Z}^n$, and the natural $\mathcal{A}$-module morphism $\mathcal{C}^\infty(\mathbb{R}^n/2\pi N_1\mathbb{Z}^n) \to \mathcal{C}^\infty(\mathbb{R}^n/2\pi N_2\mathbb{Z}^n)$ identifies the first space with a closed subspace of the second. Define $\mathcal{C}^\infty(\mathsf{PT}) := \varinjlim \mathcal{C}^\infty(\mathbb{R}^n/2\pi N\mathbb{Z}^n)$; it is a strict direct limit of Fréchet spaces and is a locally convex bornological and barrelled topological vector space. The elements of $\mathcal{A}$ act continuously on it, so that $\mathcal{C}^\infty(\mathsf{PT})$ is also a topological $\mathcal{A}$-module. It turns out that elements of $\mathcal{C}^\infty(\mathsf{PT})$ can be considered as functions on the inverse limit $\mathsf{PT} := \lim\limits_{\leftarrow} \mathbb{R}^n/2\pi N\mathbb{Z}^n$, which is a compact topological group called the *protorus*.

Finally, let $\mathcal{C}^\infty(\mathsf{T})[x_1,\ldots,x_n]$ and $\mathcal{C}^\infty(\mathsf{PT})[x_1,\ldots,x_n]$ be subalgebras of $\mathcal{C}^\infty(\mathbb{R}^n)$ obtained by adjoining the coordinate functions $x_1,\ldots,x_n$. These are not injective $\mathcal{A}$-modules themselves (unless $n=1$) but because they contain (dense) injective submodules we can solve the Nullstellensatz problem for these algebras [3].

4. **The category of differential kernels**

The above results on PDE, homological algebraic in nature, originate with an observation of Malgrange, that the differential kernel $\mathcal{B}_\mathcal{F}(\mathcal{P})$ is isomorphic as an $\mathcal{A}$-module to $\mathsf{Hom}_\mathcal{A}(\mathcal{A}^k/\mathcal{P},\mathcal{F})$ (see below). This implies that there is a category of differential kernels somewhat like the category of affine varieties, and that some of the questions about the latter can be carried over to questions about the former. A most basic fact about the category of affine varietes is that there is a duality between this category and the category of finitely generated nilpotent-free $\mathbb{C}$-algebras, namely the Hilbert Nullstellensatz. The question I have asked is the corresponding statement for differential kernels.

So let $\mathcal{R} = \mathsf{End}_\mathcal{A}(\mathcal{F})$ be the $\mathcal{A}$-algebra of all $\mathcal{A}$-linear endomorphisms of $\mathcal{F}$. Then $\mathcal{F}$ is an $\mathcal{R}$-module via $rf = r(f)$ for f in $\mathcal{F}$, in fact $\mathcal{F}$ is an $\mathcal{A}$-$\mathcal{R}$ bimodule. If $\mathcal{M}$ is an $\mathcal{A}$-module, the $\mathcal{A}$-module $\mathsf{Hom}_\mathcal{A}(\mathcal{M},\mathcal{F})$ is also an $\mathcal{A}$-$\mathcal{R}$ bimodule by setting $r\phi = r\circ\phi$ (composition).

There is now (as in the category of affine varieties) a contravariant functor

$$\begin{array}{rcl}\mathsf{Hom}_\mathcal{A}(\ \ ,\mathcal{F}) : \mathcal{A}-\mathsf{Mod} & \longrightarrow & \mathcal{R}-\mathsf{Mod}\\ \mathcal{M} & \mapsto & \mathsf{Hom}_\mathcal{A}(\mathcal{M},\mathcal{F})\\ \mathcal{M}\stackrel{\phi}{\to}\mathcal{N} & \mapsto & \mathsf{Hom}_\mathcal{A}(\mathcal{N},\mathcal{F})\stackrel{\circ\phi}{\to}\mathsf{Hom}_\mathcal{A}(\mathcal{M},\mathcal{F})\end{array}$$

The canonical $\mathcal{A}$-isomorphism $\mathsf{Hom}_\mathcal{A}(\mathcal{A}^k,\mathcal{F}) \simeq \mathcal{F}^k$ is also an $\mathcal{R}$-isomorphism, hence

$$\begin{array}{rcl}\mathsf{Hom}_\mathcal{A}(\mathcal{A}^k/\mathcal{P},\mathcal{F}) & \simeq & \mathcal{B}_\mathcal{F}(\mathcal{P})\\ \phi & \mapsto & (\phi(\overline{e}_1),\ldots,\phi(\overline{e}_k))\end{array}$$

is also an $\mathcal{A}$-$\mathcal{R}$ isomorphism (the $\overline{e}_i$ s are the images of the standard basis of $\mathcal{A}^k$ in $\mathcal{A}^k/\mathcal{P}$).

Thus, there is a contravariant functor between differential kernels in $\mathcal{F}$ and the full subcategory of finitely generated $\mathcal{A}$-modules.

By a fundamental result of Oberst [4] this functor establishes a *categorical duality* when $\mathcal{F}$ is $\mathcal{D}'$ or $\mathcal{C}^\infty$ (more precisely he shows that $\mathcal{D}'$ and $\mathcal{C}^\infty$ are *large* injective cogenerators). This raises the

Question: Which is the sub-family of finitely generated $\mathcal{A}$-modules that is in duality (via the above functor) with differential kernels in other spaces? This is, once again, the question:

What is the equivalent of the Hilbert Nullstellensatz for systems of PDE?

5. The Nullstellensatz for systems of PDE

Let $\mathcal{P}$ be a submodule of $\mathcal{A}^k$, and $\mathcal{F}$ an $\mathcal{A}$-module. Let $\bar{\mathcal{P}}_{\mathcal{F}}$ be the submodule of $\mathcal{A}^k$ consisting of all those $p(D)$ whose kernel (in $\mathcal{F}$) contains $\mathcal{B}_{\mathcal{F}}(\mathcal{P})$. Clearly this submodule contains $\mathcal{P}$, and in fact is the largest submodule of $\mathcal{A}^k$ whose kernel in $\mathcal{F}$ equals $\mathcal{B}_{\mathcal{F}}(\mathcal{P})$. Call this submodule the *Willems closure* of $\mathcal{P}$ in $\mathcal{F}$. Call $\mathcal{P}$ *closed* with respect to $\mathcal{F}$ if it equals its closure. This is completely analogous to the familiar Galois correspondence between ideals and varieties. The notion of closure here is the analogue of the radical of an ideal, and its calculation is the analogue of the Hilbert Nullstellensatz.

The calculation of this closure for the spaces of Section 3 is:

Theorem 5.1. *[3,7,8]: (i) Every submodule $\mathcal{P}$ is closed with respect to $\mathcal{D}'$ or $\mathcal{C}^\infty$ (this is the Fundamental Principle together with the cogenerator property).*
(ii) Let $\mathcal{P} = \bigcap_{j=1}^{t} \mathcal{Q}_j$ be an irredundant primary decomposition of $\mathcal{P}$ in $\mathcal{A}^k$, where $\mathcal{Q}_j$ is $\mathfrak{p}_j$-primary. Suppose that the affine varieties $\mathcal{V}(\mathfrak{p}_1), \ldots, \mathcal{V}(\mathfrak{p}_s)$ contain real *points (i.e. intersect $\mathbb{R}^n$) and that $\mathcal{V}(\mathfrak{p}_{s+1}), \ldots, \mathcal{V}(\mathfrak{p}_t)$ do not. Then the Willems closure of $\mathcal{P}$ with respect to $\mathcal{S}'$ is $\bigcap_{j=1}^{s} \mathcal{Q}_j$, so that $\mathcal{P}$ is closed with respect to $\mathcal{S}'$ if and only if the variety of every associated prime of $\mathcal{A}^k/\mathcal{P}$ contains real points.*
(iii) In the notation of (ii), suppose that the affine varieties $\mathcal{V}(\mathfrak{p}_1), \ldots, \mathcal{V}(\mathfrak{p}_s)$ contain integral *points (respectively* rational *points) and that $\mathcal{V}(\mathfrak{p}_{s+1}), \ldots, \mathcal{V}(\mathfrak{p}_t)$ do not. Then the Willems closure of $\mathcal{P}$ with respect to $\mathcal{C}^\infty(\mathsf{T})[x_1, \ldots, x_n]$ (respectively $\mathcal{C}^\infty(\mathsf{PT})[x_1, \ldots, x_n]$) is $\bigcap_{j=1}^{s} \mathcal{Q}_j$, so that $\mathcal{P}$ is closed with respect to $\mathcal{C}^\infty(\mathsf{T})[x_1, \ldots, x_n]$ (respectively $\mathcal{C}^\infty(\mathsf{PT})[x_1, \ldots, x_n]$) if and only if the variety of every associated prime of $\mathcal{A}^k/\mathcal{P}$ contains integral (respectively rational) points.*
(iv) Let $\pi : \mathcal{A}^k \to \mathcal{A}^k/\mathcal{P}$ be the canonical projection. Then the closure of $\mathcal{P}$ with respect to $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$ is $\pi^{-1}(T(\mathcal{A}^k/\mathcal{P}))$, where $T(\mathcal{A}^k/\mathcal{P})$ is the submodule of torsion elements of $\mathcal{A}^k/\mathcal{P}$. Thus $\mathcal{P}$ is closed with respect to any of these spaces if and only if $\mathcal{A}^k/\mathcal{P}$ is torsion free (or equivalently if and only if $\mathcal{P}$ is 0-primary).

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GEOMETRY AND DYNAMICS OF KLEINIAN GROUPS *

MAHAN MJ

Abstract: This is a slightly expanded version of a talk given at the 76th Annual conferences of the Indian Mathematical Society during 27-30, December 2010 at Surat.

Contents

1. Fuchsian Groups

A Kleinian group G is a discrete subgroup of $PSL_2(\mathbb{C}) = Mob(\widehat{\mathbb{C}}) = Isom(\mathbb{H}^3)$. This gives us three closely intertwined perspectives on the field:

(1) Studying discrete subgroups G of the group of Mobius transformations $Mob(\widehat{\mathbb{C}})$ emphasizes the *Complex Analytical/Dynamic* aspect.
(2) Studying discrete subgroups G of $PSL_2(\mathbb{C})$ emphasizes the *Lie group/ matrix group* theoretic aspect.

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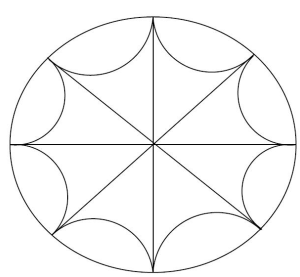
(3) Studying discrete subgroups G of $Isom(\mathbb{H}^3)$ emphasizes the *Hyperbolic Geometry* aspect.

We shall largely emphasize the third perspective. Since G is discrete, we can pass to the quotient $M^3 = \mathbb{H}^3/G$. Thus we are studying hyperbolic structures on 3-manifolds.

In order to obtain some examples, we first move one dimension down and look at discrete subgroups G of the group of Mobius transformations $Mob(\Delta) = Mob(\mathbb{H})$ of the unit disk (which is conformally equivalent to the upper half plane). These are called Fuchsian Groups, and were discovered by Poincare. The natural metric of constant negative curvature on the upper half plane is given by $ds^2 = \frac{dx^2+dy^2}{y^2}$. This is called the *hyperbolic metric.* The resulting space is denoted as $\mathbb{H}^2$.

The associated conformal structure is exactly the complex structure on $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. It turns out that orientation preserving isometries of $\mathbb{H}^2$ are exactly the conformal automorphisms of $\mathbb{H}^2$. The boundary circle S^1 compactifies Δ. This has a geometric interpretation. It codes the 'ideal' boundary of $\mathbb{H}^2$, consisting of asymptote classes of geodesics. The topology on S^1 is induced by a metric which is defined as the angle subtended at $0 \in \Delta$. The geodesics turn out to be semicircles meeting the boundary S^1 at right angles.

We now proceed to construct an example of a discrete subgroup of $Isom(\mathbb{H}^2)$. The genus two orientable surface can be described as a quotient space of an octagon with edges labelled $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2{-}1, b_2^{-1}$, where the boundary has the identification induced by this labelling. In order to construct a metric of constant negative curvature on it, we have to ensure that each point has a small neighborhood isometric to a small ball in $\mathbb{H}^2$. To ensure this it is enough to do the above identification on a regular hyperbolic octagon (all sides and all angles equal) such that the sum of the interior angles is 2π. To ensure this, we have to make each interior angle equal $\frac{2\pi}{8}$. The infinitesimal regular octagon at the tangent space to the origin has interior angles equal to $\frac{3\pi}{4}$. Also the ideal regular octagon in $\mathbb{H}^2$ has all interior angles zero. See figure below.

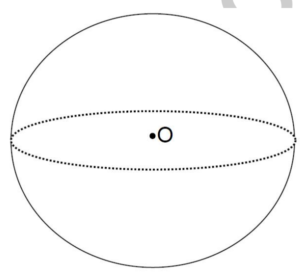


Hence by the Intermediate value Theorem, as we increase the size of the octagon from an infinitesimal one to an ideal one, we shall hit interior angles all equal to $\frac{\pi}{4}$ at some stage. The group G that results from side-pairing transformations corresponds to a Fuchsian group, or equivalently, a discrete faithful representation of the fundamental group of a genus 2 surface into $Isom(\mathbb{H}^2)$. We let ρ denote the associated representation.

2. **Kleinian Groups**

We now move back to $\mathbb{H}^3$. The hyperbolic metric is given by $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$ on upper half space. Note that the metric blows up as one approaches $z = 0$. Equivalently we could consider the ball model, where the boundary $S^2 = \widehat{\mathbb{C}}$ consists of ideal end-points of geodesic rays as before. The metric on $\widehat{\mathbb{C}}$ is given by the angle subtended at $0 \in \mathbb{H}^3$.

Since $Isom(\mathbb{H}^2) \subset Isom(\mathbb{H}^3)$, we can look upon the discrete group G we constructed above also as a discrete subgroup of $Isom(\mathbb{H}^3)$.



In the above picture two things need to be observed.
1) the orbit $G.o$ accumulates on the equatorial circle. This is called the *limit set* Λ_G.
2) The complement of Λ_G consists of two round open discs. On each of these disks, G acts freely (i.e. without fixed points) properly discontinuously, by conformal automorphisms. Hence the quotient is two copies of the 'same' Riemann surface (i.e. a one dimensional complex analytic manifold). The complement $\widehat{\mathbb{C}} \setminus \Lambda_G = \Omega_G$ is called the *domain of discontinuity* of G.

We proceed with slightly more formal definitions.

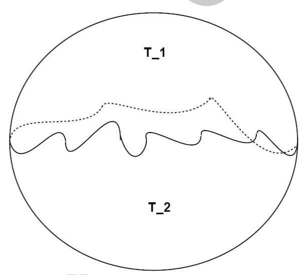
Suppose that G is abstractly isomorphic to the fundamental group of a finite area hyperbolic surface S^h, and $\rho : \pi_1(S^h) \to PSL_2(\mathbb{C})$ is a representation with image G. Suppose further that ρ is *strictly type-preserving*, i.e. $g \in \pi_1(S^h)$ represents an element in a peripheral (cusp) subgroup if and only $\rho(g)$ is parabolic. In this situation we shall refer to G as a *surface Kleinian group.* A recurring theme in the context of finitely generated, infinite covolume Kleinian groups is that the general theory can be reduced to the study of surface Kleinian groups. Equivalently, we study the representation space $Rep(\pi_1(S^h), PSL_2(\mathbb{C}))$.

Regarding G as a subgroup of $Mob(\widehat{\mathbb{C}})$, the *dynamics* of the action of G on $\widehat{\mathbb{C}}$ emerges. The *limit set* Λ_G of G is defined to be the set of accumulation points of the orbit $G.o$ in $\widehat{\mathbb{C}}$ for some (any) $o \in \mathbb{H}^3$. The limit set is the locus of chaotic dynamics of the action of G on $\mathbb{C}$. The complement $\widehat{\mathbb{C}} \setminus \Lambda_G = \Omega_G$ is called the *domain of discontinuity* of G.

On the other hand regarding G as a subgroup of $Isom(\mathbb{H}^3)$, we obtain a quotient hyperbolic 3-manifold $M = \mathbb{H}^3/G$ with fundamental group G.

A major problem in the theory of Kleinian groups is to **understand the relationship between the dynamic and the hyperbolic geometric descriptions** of G.

2.1. **Degenerate Groups.** The **Ahlfors-Bers simultaneous Uniformization Theorem** states that given any two conformal structures τ_1, τ_2 on a surface, there is a discrete subgroup G of $Mob(\widehat{\mathbb{C}})$ whose limit set is *topologically* a circle, and whose domain of discontinuity quotients to the two Riemann surfaces τ_1, τ_2. See figure below.



The limit set is a quasiconformal map of the round circle. These *(quasi Fuchsian)* groups can be thought of as *deformations* of Fuchsian groups (Lie group theoretically) or quasiconformal deformations (analytically). Ahlfors and Bers proved that these are precisely all quasiconvex surface Kleinian groups.

The convex hull CH_G of Λ_G is the smallest closed convex subset of $\mathbb{H}^3$ invariant under G. It can be constructed by joining all pairs of points on the limit set by bi-infinite geodesics and iterating this construction. The quotient of CH_G by G, which is homeomorphic to $S^h \times [0,1]$, is called the *Convex core* $CC(M)$ of $M = \mathbb{H}^3/G$.

The 'thickness' of $CC(M)$ for a quasi Fuchsian surface Kleinian group, measured by the distance between $S^h \times \{0\}$ and $S^h \times \{1\}$ is a geometric measure of the complexity of the quasi Fuchsian group G.

The most intractable examples of surface Kleinian groups are obtained as limits of quasi Fuchsian groups. In fact, it has been recently established by

Minsky et al. [15] [3] that the set of all surface Kleinian groups (or equivalently all discrete faithful representations of a surface group in $PSL_2(\mathbb{C})$) are given by quasiFuchsian groups and their limits. This is known as the Bers density conjecture.

To construct limits of quasi Fuchsian groups, one allows the thickness of the convex core $CC(M)$ to tend to infinity. There are two possibilities:
a) Let only τ_1 degenerate. i.e. $I \to [0, \infty)$ **(simply degenerate case)**
b) Let both τ_1, τ_2 degenerate, i.e. $I \to (-\infty, \infty)$ **(doubly degenerate case)**

Thurston's Double Limit Theorem [27] says that these limits exist. A fundamental question in **relating the geometric and dynamic aspects** of Kleinian groups is the following.

Question 2.1. *(Thurston) How does the limit set behave for the limiting manifold?*

In the doubly degenerate case the limit set is all of $\widehat{\mathbb{C}}$.

In the next section we outline our approach and solution to this problem.

3. Extensions of Maps to Ideal Boundaries

Starting with [19], [18] and [16], we investigated the following question:

Question 3.1. *Let G be a hyperbolic group in the sense of Gromov acting freely and properly discontinuously by isometries on a hyperbolic metric space X. Does the inclusion of the Cayley graph $i : \Gamma_G \to X$ extend continuously to the (Gromov) compactifications?*

A positive answer to Question 3.1 gives us a precise handle on Question 2.1. In this generality the question first appears in [17] (see also the Geometric Group Theory Problem List [4]). As of date no counterexample is known.

However, special cases of Question 3.1 have been raised earlier in the context of Kleinian groups.
•1 In Section 6 of [7] (now published as [8]), Cannon and Thurston propose the following.

Conjecture 3.2. *Suppose a surface group $\pi_1(S)$ acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries. Then the inclusion $\tilde{i} : \widetilde{S} \to \mathbb{H}^3$ extends continuously to the boundary.*

The authors of [7] point out that for a simply degenerate group, this is equivalent to asking if the limit set is locally connected.
•2 In [12], McMullen makes the following more general conjecture:

Conjecture 3.3. *For any hyperbolic 3-manifold N with finitely generated fundamental group, there exists a continuous, $\pi_1(N)$-equivariant map*

$$F : \partial\pi_1(N) \to \Lambda \subset S^2_\infty$$

where the boundary $\partial\pi_1(N)$ is constructed by scaling the metric on the Cayley graph of $\pi_1(N)$ by the conformal factor of $d(e,x)^{-2}$, then taking the metric completion. (cf. Floyd [10]*)*

In [22] and [26] we provide a complete positive answer to both Conjectures 3.2 and 3.3.

As a consequence we also establish in [22] the following Theorem which proves a long-standing conjecture in the theory of Kleinian groups [1] [7].

Theorem 3.4. *Connected limit sets of finitely generated Kleinian groups are locally connected.*

In the next subsection, after describing the history of these problems, we shall give more details about the structure of limit sets and their relation to the geometry of surface Kleinian groups.

3.1. **History and Solution of the Problem.** In [1], Abikoff (1976) claimed to prove that limit sets of simply degenerate surface Kleinian groups were never locally connected. Thurston and Kerckhoff found a flaw in his proof in about 1980.

The first major result that started this entire program was Cannon and Thurston's result [7] for hyperbolic 3-manifolds fibering over the circle with fiber a closed surface group.

Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber F. Let $\widetilde{F}$ and $\widetilde{M}$ denote the universal covers of F and M respectively. Then $\widetilde{F}$ and $\widetilde{M}$ are quasi-isometric to $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. Now let $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ denote the standard compactifications. In [7] Cannon and Thurston show that the usual inclusion of $\widetilde{F}$ into $\widetilde{M}$ extends to a continuous map from $\mathbb{D}^2$ to $\mathbb{D}^3$. This was extended to Kleinian surface groups of bounded geometry without parabolics by Minsky [13].

An alternate approach (purely in terms of coarse geometry ignoring all local information) was given by the author in [19] generalizing the results of both Cannon-Thurston and Minsky. We proved the Cannon-Thurston result for hyperbolic 3-manifolds of bounded geometry without parabolics and with freely indecomposable fundamental group. A different approach based on Minsky's work was given by Klarreich [11].

Bowditch [6] [5] proved the Cannon-Thurston result for punctured surface Kleinian groups of bounded geometry. In [25] we gave an alternate proof of Bowditch's results and simultaneously generalized the results of Cannon-Thurston, Minsky, Bowditch, and those of [19] to all 3 manifolds of bounded geometry whose cores are incompressible away from cusps. The proof has the advantage that it reduces to a proof for manifolds without parabolics when the 3 manifold in question has freely indecomposable fundamental group and no accidental parabolics.

In the expository paper [24] we give our proof of the results of Cannon and Thurston [7], Minsky [13], and Bowditch [6] using the ideas of [19] and [25].

In [14] Minsky established a bi-Lipschitz model for all punctured torus Kleinian groups. McMullen [12] proved the Cannon-Thurston result for punctured torus groups, using Minsky's model for these groups [14].

In [21] we identified a large-scale coarse geometric structure involved in the Minsky model for punctured torus groups (and called it **i-bounded geometry**). *i-bounded geometry* can roughly be regarded as that geometry of ends where the boundary tori of Margulis tubes have uniformly bounded diameter. We gave a proof for models of *i-bounded geometry*. In combination with the methods of [25] this was enough to bring under the same umbrella all known results on Cannon-Thurston maps for 3 manifolds whose cores are incompressible away from cusps.

In [20] we further generalized possible geometries allowing us to push our techniques through to establish the Cannon-Thurston property.

In the mean time, in the proof of the celebrated Ending Lamination Conjecture, Minsky [15] and Brock-Canary-Minsky [3] established a bi-Lipschitz model for all surface Kleinian groups.

In [22], we used the Minsky model of [15] to prove that all hyperbolic 3-manifolds homotopy equivalent to a surface satisfy the conditions imposed in the geometries dealt with in [20]. This establishes the Cannon-Thurston property for all surface Kleinian groups and proves Conjecture 3.2. It follows that surface Kleinian groups have locally connected limit sets. Combining this result with a reduction Theorem of Anderson and Maskit [2], we prove that connected limit sets of finitely generated Kleinian groups are locally connected (Theorem 3.4). Finally in [26] we extend the techniques of [22] to cover handlebody groups and prove Conjecture 3.3.

We then gave explicit descriptions of the boundary identifications of [22] in terms of *ending laminations* in [23] and [9]. This finally yields a rather complete and satisfactory solution to Question 2.1.

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C^*-ALGEBRAS, UNIFORM BANACH ALGEBRAS AND A FUNCTIONAL ANALYTIC META THEOREM*

SUBHASH J. BHATT

Abstract: C^*-algebras and uniform Banach algebras exhibit certain structural analogy inspired by the C^*-property and the square property of their respective norms. This is further reflected in analogies between hermitian Banach *-algebras and Banach algebras commutative modulo the radical, between *-regular Banach *-algebras and regular commutative Banach algebras, between Banach *-algebras with unique C^*-norm and commutative Banach algebras with unique uniform norm and between Frechet *-algebras with a C^*-enveloping algebra and commutative Frechet Q-algebras.These are discussed.

1. Introduction

Prof. Hans Raj Gupta, ex-honorary Professor of Mathematics, Panjab University, Chandigarh contributed significantly towards development of teaching and research in India. His outstanding research in Number Theory has been reported in more than 150 papers published in scientific journals and four monographs. With a deep sense of appreciation and regards to Prof. Gupta, the present lecture (and the subsequent paper based on the lecture) is being

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2000 *Mathematics Subject Classification.* Primary 46Hxx, 46Lxx; Secondary 43Axx.

Key words and phrases: C^*-algebras, uniform Banach algebras, Harmonic Analysis, Complex Analysis.

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offered in this 76^{th} IMS Conference, held at SVNIT, Surat. We shall be dealing with the following two objects in Functional Analysis.

(I) Let H be a Hilbert space. Let $B(H)$ be the vector space of all bounded linear operators on H. It is an algebra with composition as multiplication, carries operator adjoint as a natural involution, and carries the operator norm as the natural norm. A *C^*-algebra* is a norm closed involutive subalgebra of $B(H)$.

(II) Let X be a compact Hausdorff space. Let $C(X)$ be the vector space of all continuous complex valued functions on X necessarily bounded. It is an algebra with pointwise multiplication and carries the sup norm $\|\cdot\|_\infty$. A *uniform Banach algebra (uB- algebra)* is a norm closed (not necessarily involutive) subalgebra of $C(X)$.

Thus a C^*-algebra is necessarily involutive, not necessarily commutative; a uB-algebra is necessarily commutative, not necessarily involutive. These two classes of algebras have developed along two different directions; the C^*-algebras proceeded charting the non-commutative real analytical universe, the uB-algebras proceeded along the commutative complex analytical world. Both have their different philosophies and methodologies. In spite of this, it appears that there is some structural analogy between these two classes of algebras. The purpose of the present paper is to uncover this by formulating a meta theorem and searching for supports for this.

A meta theorem is a theorem on theorems; and the following is what Stefan Banach, who was immortalized by his espaces de type (B), has to say, in a rather suggestive way, about meta theorems.

"A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies."

(quoted by the reviewer Prof. Maria Jesus de la Puente (Madrid) in review No. 1185.14053 Zentalblatt MATH. of the paper: Viro Oleg, "From the sixteenth Hilbert problem to tropical geometry", Jpn. J. Math. (3)3, No.2, (2008) 185-214)

We begin with the following.

Definition 1.1. A *Banach algebra* $(A, \|\cdot\|)$ is a linear associative algebra A over complex scalars equipped with a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach space and the norm satisfies the norm inequality $\|xy\| \leq \|x\|\|y\|$ for all x, y in A; i.e., the norm is sub multiplicative.

In the absence of a multiplicative identity in A, there is a standard procedure for adjoining identity to A getting a unital algebra A_e in which A is imbedded as an ideal of codimension 1 [65],[63]. It is well known that Gelfand and his collaborators Naimark, Raikov, Silov, Ditkin contributed significantly in the initial period and put the theory on solid foundation begining with 1941. The influence of Gelfand has been so much that even after almost 30 years, Bonsall and Duncan in their famous text [33] greatly resisted themselves calling these algebras Gelfand Algebras.

However it is less known that the first paper on Banach algebras was written by Nagumo [73]. Nagumo formally defined a metric ring and initiated the investigation of its analytical aspect, viz. its general linear group consisting of invertible elements. Simultaniously Yoshida published two papers [85] on the same aspect. The work of Nagumo and Yoshida was acknowledged by Mazur in his famous note [69] announcing what is now known as the Gelfand-Mazur Therorem. Then came up Gelfand's announcements of his results on Banach algebras [54]. The work of Gelfand appears to have been inspired by his theses supervisor Kolmogorov [55]. Full length paper appeared as [56] in 1941. And the rest is a history. The book [57] (which is in fact a compilation of original papers) exhibits the spirit of the masters who created the theory. The books [67, 78, 73, 33, 44, 74, 75] have been standard comprehensive expositions of general theory of Banach Algebras at respective times; where as [87, 65, 63] are texts, the last recent one also containing a large number of carefully chosen exercises.

The highly economical proofs by Gelfand of Mazur's Theorem mentioned above and of Wiener's Theorem on absolutely convergent Fourier series are often cited as the first success of Banach Algebras.

As stated by Bonsall and Duncan [33], the axioms of a Banach algebra are happily chosen; they are simple enough to include so many examples from Function Theory, Linear Operator Theory, Harmonic Analysis and Mathematical Physics; at the same time, they are tight enough to support a rich and exciting theory. A characteristic feature of Banach algebras is a rich interplay between Algebra and Analysis. As Rickart [78] put it, Banach Algebra is a subject with its head in algebra and feet in analysis.

There are two classes of Banach algebras that have been extensively studied ultimately resulting into well developed disciplines themselves.

(a) **Uniform Banach Algebras (uB-algebras)** : A *uB-algebra* is a Banach algebra $(A, \|\cdot\|)$ such that the norm satisfies the square property $\|x^2\| =$

$\|x\|^2$ for all x in A. By the Gelfand Theory for commutative Banach algebras, a uB-algebra is concretely realized as a uniformly closed subalgebra of $C_0(X)$, the sup norm Banach algebra of all continuous complex valued functions vanishing at infinity on a locally compact Hausdorff space X with point wise operations. Traditionally a uB-algebra is assumed to contain a multiplicative identity, in which case it is realized as a uniformly closed subalgebra containing the constant function 1 of the algebra of all continuous functions on a compact Hausdorff space. We cite [53, 66, 82] as references for uniform Banach algebras.

(b) C^***-algebras** : A *Banach* **-algebra* is a Banach algebra $(A, \|\cdot\|)$ with an involution $x \in A \to x^* \in A$ which is a conjugate linear continuous anti automorphism of period 2. A Banach *-algebra is a C^**-algebra* if the norm satisfies the C^*-property $\|x^*x\| = \|x\|^2$ for all x in A. One of the corner stone of the theory is the Gelfand-Naimark Theorem which characterizes a C^*-algebra A as an operator norm closed *-subalgebra of the C^*-algebra $B(H)$ of all bounded linear operators on a Hilbert space H with the operator norm and the operator adjoint as the involution. There are several excellent books on Operator Algebras; we mention [49, 50, 83, 77].

The uB-algebras and the C^*-algebras have developed independently with different objectives. The uB-algebras, which are commutative, have aimed at recapturing and advancing the classical theory of functions in abstract functional analytic set up; and they have considerably succeeded in recapturing holomorphy [59]. Thus uB-algebras constitute Function Theory without Theory of Functions. On the other hand, the development of C^*-algebras, they being non-commutative in general, has been inspired by Linear Operator Theory in Hilbert Space, Mathematical Formulation of Quantum Theory as well as Infinite Dimensional Group Representation Theory. The C^*-algebras have resulted in providing new perspectives to much of classical mathematics like measure theory, topology, geometry; and have become a vehicle for uncovering much of the noncommutative universe including volatile fields like Noncommutative Integration (Von Neumann algebras), Noncommutative Probability, Noncommutative Topology and Noncommutative Geometry.

Notwithstanding the differences in the philosophies as well as the methodologies in the study of these classes of Banach algebras, the apparent similarity in the defining conditions of norms on them, viz. $\|x^2\| = \|x\|^2$ in case of the uniform norm and $\|x^*x\| = \|x\|^2$ in case of the C^*-norm suggest the following.

META THEOREM : There exists a structural analogy between (certain, and not all, aspects of) C^*-algebras and uB-algebras; and this extends to analogies between the following pairs of algebras.

(1) hermitian Banach *-algebras and Banach algebras commutative modulo the radical;

(2) C^*-unique Banach *-algebras and Banach algebras with unique uniform norm;

(3) *-regular Banach *-algebras and regular commutative Banach algebras;

(4) topological *-algebras with a C^*-enveloping algebra and commutative topological Q-algebras.

In fact this list might continue. Several theorems supporting this meta theorem can be listed, formulated, and proved. The purpose of the present paper is to illuminate this and to discuss some questions it leads to. This idea was initially formulated in [7],[8]; and developed in subsequent papers. This is certainly not intended to be a survey on this theme. It should be stressed that the analogy discussed here is different from the well established point of view that C^*-algebras are the non commutative $C(X)$ algebras.

2. **Structure of semi norms**

(2.1) From the very begining, there were efforts to examine whether the defining axioms of C^*-algebra can be further weakened. Initially symmetry was assumed in the definition of a C^*-algebra, the redundancy of which was achieved leading to Shirali-Ford Theorem asserting that a Banach *-algebra is symmetric iff it is hermitian. A Banach *-algebra $(A, \|\cdot\|)$ is a C^*-algebra if $\|x^*x\| = \|x\|\|x^*\|$ holds for all $x \in A$ [33]. Araki and Elliot [2] showed that $\|xy\| \leq \|x\|\|y\|$, $\|x^*\| = \|x\|$ for all x, y follow from other axioms for C^*-algebras. The efforts towards weakening the conditions on a C^*-norm resulted into the following due to Sebestyen [70],[81]. A *linear semi norm* on an algebra A is a non-negative real valued function on A such that $p(x+y) \leq p(x)p(y), p(\lambda x) = |\lambda|p(x)$ for all x, y in A, all λ in the complex scalars $\mathbb{C}$. If A is involutive, then p is $*-$ *invariant* if $p(x^*) = p(x)$ for all $x \in A$. Further p is *sub multiplicative* if $p(xy) \leq p(x)p(y)$ for all x, y in A.

Theorem 2.1. [81] *Let p be a linear semi norm on an involutive algebra A such that p satisfies the C^*-property $p(x^*x) = p(x)^2$ for all $x \in A$. Then p is sub multiplicative and *-invariant.*

Can a similar theorem be formulated for a uniform semi norm? Let p be a linear semi norm on an algebra A. Then p has *square property* if for all x in A, $p(x^2) = p(x)^2$. Then p is sub multiplicative if A is commutative [11], or if

A is a Banach algebra, not necessarily commutative [6]. The following noble result along this line was proved independently in [46] as well as in [62], the one in the latter case is in little more general case; the methods in both cases are different.

Theorem 2.2. *Let p be a linear semi norm on an algebra A having the square property. Let $N(p) = \{x \in A : p(x) = 0\}$. Then p is sub multiplicative and the quotient algebra $A/N(p)$ is commutative.*

Does Theorem 2.1 hold if the C^*-property is replaced by the condition that $p(x^*x) = p(x^*)p(x)$ for all $x \in A$? Can Theorem 2.2 be improved in case of *-algebras; e.g., assuming the square property only for all normal elements? Also notice that any two C^*-semi norms on a *-algebra are equal, if they are equivalent; analogously any two uniform semi norms on an algebra are equal, if they are equivalent.

(2.2) The following determines all C^*-semi norms and all uniform semi norms.

Theorem 2.3. *(A)*[4] *Let A be a Banach *-algebra. Let $\pi(C^*(A))$ be the primitive ideal space of the enveloping C^*-algebra $C^*(A)$ of A. Let p be a semi norm on A. Then p is a C^*-semi norm iff p is of the form $p_\gamma(.)$ for a closed subset γ of $\pi(C^*(A))$, where $p_\gamma(x) = \sup\{\|\widehat{x}(P) = x + P\| : P \in \gamma\}$.*

(B)[9] *Let A be a Banach algebra. Let $\triangle(A)$ be the Gelfand space of A. Let p be a uniform semi norm on A. Then p is of the form $p_K(.)$ for a closed subset K of $\triangle(A)$, where $p_K(x) = \sup\{|\phi(x)| : \phi \in K\}$.*

3. Spectral radius r and Ptak spectral function s

(3.1) [33] Undoubtedly the most important concept in Banach algebras is spectrum. The spectrum $sp_A(x)$ of an element $x \in A$ consists of all complex numbers $\lambda \in \mathbb{C}$ such that $(\lambda 1 - x)$ is not invertible in A or in A_e, the algebra obtained by adjoining identity to A, according as A has identity or not. It is a fundamental result that $sp_A(x)$ is a nonempty compact set in the complex plane,with the result, the spectral radius $r_A(x) := \sup\{|\lambda|, \lambda \in sp_A(x)\} = r(x)$ (say) is well defined and finite. In the case of a Banach *-algebra A, the Ptak function $s(x) := r(x^*x)^{1/2}, x \in A$, is known to play an important role.The complete C^*-norm on a C^*-algebra A is algebraically determined as $\|x\| = s(x)$ and is thus unique; analogously the complete uniform norm on a uB-algebra A is algebraically determined as $\|x\| = r(x)$, and is unique. Thus the analytical structure in each of a C^*-algebra and a uB-algebra is completely algebraically determined; where as in case of a general Banach algebra A, a connection between algebra and topology is revealed by the spectral radius formulie $r(x) = \lim \|x^n\|^{1/n} = \inf p(x)$, p varying over all unital norms equivalent to the complete norm.

(3.2) This raises an interesting problem. Can we algebraically characterize those algebras that are Banach algebras with some norm? A semi simple algebra is a Banach algebra with some norm if and only if it is a homomorphic image of a Banach algebra [14]; and by a celebrated theorem due to Johnson, the topology of a semi simple Banach algebra is uniquely determined [33]. Of course, an algebra is normable iff it admits a radially bounded absolutely convex multiplicative sub semigroup [33], with the result, unlike a vector space, not every algebra is normable [44]. A *Q-normed* algebra is a normed algebra in which the quasi-regular elements form an open set; these algebras admit almost all algebraic niceties of Banach algebras. An algebra is *Q-normable* iff it admits a radially bounded absolutely convex multiplicative sub semigroup consisting of quasi-regular elements [74]. A *spectral semi norm* on an algebra A is a semi norm p such that $r(x) \leq p(x)$ for all $x \in A$. A *spectral algebra* is an algebra A to gather with a spectral semi norm. Palmer [74, 75] has exhibited that very many aspects of general theory of Banach algebras carry over to this large class of algebras. Spectral algebras, in particular C^*-spectral algebras, ensure spectral invariance which is closely linked with closure under appropriate functional calculi [25],[26]. These properties manifest the differential structure in a C^*-algebra [27].

(3.3) The deeper influence of the functions r and s on the structure of a Banach algebra is revealed by the following.

Theorem 3.1. [33, 7]*(A) (Ptak) Let A be a Banach *-algebra.*

(i) The function s has C^*-property $s(x^*x) = s(x)^2$ for all $x \in A$.

(ii) s is a linear semi norm iff s is (weakly) sub additive iff A is hermitian. In this case, s is a spectral semi norm and it turns out that $s = m$ the Gelfand-Naimark semi norm, which is the greatest C^*-semi norm.

(B)[33, 3] *Let A be a Banach algebra.*

(i) The spectral radius r has the square property $r(x^2) = r(x)^2$ for all $x \in A$.

(ii) (Aupetit-Zemanek) r is a linear semi norm iff r is (weakly) sub additive iff $A/radA$ is commutative. In this case, r is a spectral semi norm and it turns out that r is the greatest uniform semi norm.

(3.4) The following shows that these uniquely determine r and s giving their intrinsic characterizations.

Theorem 3.2. [15] *Let p be a spectral linear semi norm on a Banach algebra A.*

(a) If p satisfies the square inequality, then p is equivalent to r, $kerp = radA$ and $A/radA$ is commutative. If p has square property, then $p = r$.

(b) Let A be a *-algebra. If p satisfies the C^*-inequality, then p is equivalent to s, $kerp = sradA$ and A is hermitian. If p has C^*- property, then $p = s$.

Several interesting questions arise from this comparison. The Aupatit-Zemanek spectral characterizations of commutativity [3] further states that for a Banach algebra $(A, \|\cdot\|)$, $A/radA$ is commutative iff r is uniformly continuous on A iff r is Lipschitzian on A iff r is (weakly) sub multiplicative on A. Further if A has identity, then above hold iff there is a neighbourhood V of identity and $c > 0$ such that $|r(x) - r(y)| \leq c\|x - y\|$ for all x, y in V iff r is (weakly) sub additive on V iff r is (weakly) sub multiplicative on V. Analogously does it hold that a Banach *-algebra A is hermitian iff s is uniformly continuous on A iff s is uniformly continuous on some neighbourhood of identity iff s is Lipschitzian on A? Notice that if A is hermitian, then s is sub multiplicative; and further, if the involution is isometric, then $|s(x)-s(y)| \leq \|x-y\|$ for all x, y in A; but the converse does not hold [7]. The submultiplicativity of s fails to imply the hermiticity. The disc algebra $A(D)$ with the involution $f^*(z) = f(z^-)^-, z \in D$, provides an example. We refer to [3] for more commutativity criteria whose hermiticity analogues can be sought.

(3.5) The functions r and s are just two examples of intrinsically defined spectral functions on a Banach algebra A. Some other spectral functions are the following [7]. Let $En(A)$ consists of all sub multiplicative norms on A equivalent to the given complete norm; $Eun(A) = \{p \in En(A) : p(1) = 1\}$; and if A is involutive with continuous involution, $Eun^*(A) = \{p \in Eun(A) : p(x^*) = p(x)$ for all x $\}$.

$s'(x) = \limsup(s(x^n)^{1/n})$,

$k(x) = \inf\{p(x) : p \in Eun^*(A)\}$,

$k'(x) = \limsup(k(x^n)^{1/n})$

$r_P(x) = \inf\{p(x)$: p is a sub multiplicative norm on A $\}$, called the *permanent radius* of x,

$s_P(x) = r_P(x^*x)^{1/2}$, the *permanent Ptak function*

$m(x) =$ the Gelfand - Naimark pseudo norm (the greatest C^* -semi norm),

$m'(x) = \lim m(x^n)^{1/n}$

$\nu(x) =$ the *numerical radius* of x;

$\nu'(x) = \limsup(\nu(x^n)^{1/n})$

$s_\nu(x) = \nu(x^*x)^{1/2}$;

$s'_\nu(x) = \limsup(s_\nu(x^n)^{1/n})$

Several other such intrinsic spectral functions can be constructed. Much of the general theory of Banach algebras constitute inter-relations between the algebraic-topological structure of A and analytical properties of such spectral functions. A less known spectral radius formula is $r(x) = k'(x)$ for all $x \in A$ [7]. Given a Banach algebra $(A, \|\cdot\|)$, $r \leq \nu \leq \|\cdot\|$, $(1/e)\|x\| \leq \nu(x) \leq \|x\|$. What are the Banach algebras in which ν is sub multiplicative?

(3.6) For a Banach algebra A, let $N_r = \{x \in A : r(x) = 0\}$ the quasi-nilpotent elements of A. Then $radA \subset N_r$; and the following discusses when the equality occurs. Analogously, in a Banach *-algebra A, $m'(x) \leq m(x) \leq s(x)$ for all x; hence $N_s := \{x \in A : s(x) = 0\} \subset sradA = \{x \in A : m(x) = 0\} \subset N'_m := \{x \in A : m'(x) = 0\}$. Let $K'_m := \{x = h + ik : h = h^*, k = k^*, m'(h) = 0 = m'(k)\}$; $K_r := \{x = h + ik : h = h^*, k = k^*, r(h) = 0 = r(k)\}$; $N_{s'} := \{x \in A : s'(x) = 0\}$.

Theorem 3.3. *(A)* [3] *Let A be a Banach algebra. Then N_r is closed under addition iff N_r is closed under multiplication iff $N_r = radA$.*

(B) [7] *Let A be a Banach *-algebra with continuous involution. Then the following hold.*

(1) $K'_m = sradA$.

(2) N'_m is closed under addition iff N'_m is closed under multiplication iff $N'_m = sradA$.

(3) If N_s is closed under addition, then N_s is a closed *-ideal of A; $N_s = K_r$; and $radA \subset N_s \subset sradA$.

(4) If N'_s is closed under addition, then $N'_s = K_r$; and if $radA = N_r$, then N'_s is a *-subalgebra of A and $radA = N_r \subset N'_s \subset N_s \subset sradA$.

(5) If A is commutative, then $s = s'$; and if further N'_s is closed under addition, then $N'_s = sradA$.

(C) There exists a Banach *-algebra A for which N_s is a closed *-ideal, but $N_s \neq sradA$.

(D) There exists a Banach *-algebra A for which $N_s = sradA$, but A is not hermitian.

If N'_s is closed under addition, is it a closed *-ideal of A? If N_s is closed under multiplication, is it closed under addition? Does $\lim s(x^n)^{1/n}$ exist for all x in A? The following brings out the role of s' and m' in combining the two ends of our meta theorem viz. hermiticity and commutativity modulo radical.

Theorem 3.4. [7] *Let A be a Banach *-algebra.*

(I) The following are equivalent.

(1) A is hermitian and $A/sradA$ is commutative.

(2) s is a uniform semi norm on A.

(3) s' is a C^*-semi norm on A.

(4) s' is a uniform semi norm on A.

(5) $m = s'$.

(6) $r = m$.

(7) $r = s$.

(8) $A/radA$ is commutative and hermitian.

(II) The following are equivalent.

(1) A is hermitian.

(2) $s' = r$.

(3) $m' = s'$.

(4) $m' = r$.

(5) $s(x) = r(x)$ for all normal elements x.

(6) $m'(x) = s'(x)$ for all hermitian elements x.

(7) $m'(x) = r(x)$ for all normal elements x.

(8) $m'(x) = r(x)$ for all hermitian elements x.

This raises several issues that are believed to be unsettled.

Characterize Banach *-algebras satisfying any of the following four statements: (i) s' is sub additive.(ii) s' is uniformly continuous. (iii) s' is Lipschitzian. (iv) $s' = s$. Let A be a Banach *-algebra. Then for all $x \in A$, $r(x) = k'(x)$; and if x is normal, then $r(x) = k(x)$[7]. Further $m = k$ iff $s = k$ iff k is a C^*-semi norm iff k has C^*-property. Also, A is hermitian iff $s = k$ for all normal elements. Let A be hermitian. Is $s = k$? Let $A/sradA$ be commutative; is $r = k$? Note that $r = k$ iff k has square property; and this forces $A/sradA$ to be commutative [7].

4. **Linear topological properties**

(4.1) The presence of the ring structure together with sub multiplicativity of norm on the underlying Banach space structure in a Banach algebra A is likely to simplify the linear topological structure of A; and the envisaged analogy between the C^*-algebras and the uB-algebras should also get reflected in their Banach space properties. The following illustrates this.

Theorem 4.1. *(A) Let $(A, \|\cdot\|)$ be a C^*-algebra.*

(i) If A is reflexive, then A is finite dimensional.

(ii)[49]*(Kadison) Let $\phi : A \to B$ be a unital linear isometry between C^*-algebras. Then $\phi(x^2) = \phi(x)^2$ for all x; ϕ is hermitian and ϕ is a Jordan isomorphism.*

(iii)([28] *If A (not necessarily assumed complete in $\|\cdot\|$ neither assumed unital) is a pre-Hilbert space, then A is isomorphic $\mathbb{C}$.*

(B) Let $(A, \|\cdot\|)$ be a uB-algebra.

(i) (P.G.Dixon) If A is reflexive, then A is finite dimensional.

(ii)[73] *Let $\phi : A \to B$ be a unital linear isometry between uB-algebras. Then $\phi(x^2) = \phi(x)^2$ for all x; ϕ is a Jordan isomorhism, and hence is an isomorphism.*

(iii)[86] *If A (not necessarily assumed complete in $\|\cdot\|$ or unital) is a pre-Hilbert space, then A is isomorphic to $\mathbb{C}$.*

A normed division algebra $(A, \|\cdot\|)$ over reals is isomorphic to the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$ or the quarternions $\mathbb{H}$. This is the noble Gelfand-Mazur Theorem comparable in nobility with say Lioville Theorem in Complex Analysis. In fact stronger versions of (iii) of above hold.

Theorem 4.2. *(A)* [86] *Let A be a real algebra. Let $\|\cdot\|$ be a Pythagorean norm on A. Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ if either of the following hold. (a) A has identity 1, $\|1\| = 1$, $\|x^2\| \leq \|x\|^2$ for all x in A; or (b) $\|x^2\| = \|x\|^2$ for all x in A.*

(B) [28]*Let A be a real *-algebra with a Pythagorean norm $\|\cdot\|$.Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ if either of the following hold. (a) A has identity 1; $\|1\| = 1; \|x^*x\| \leq \|x\|^2$ for all x; and for any x in A, $x^*x = 0$ implies $x = 0$: or (b) $\|x^*x\| = \|x\|^2$ for all x in A.*

(4.2) We consider the extreme points of the unit ball. Let $(A, \|\cdot\|)$ be a Banach algebra. Let $S = \{x \in A : \|x\| \leq 1\}$ be the unit ball in A. Let $extS$ be the set of all extreme points of S. Then S has extreme points if either A has identity or A is the dual of a Banach space (Krein-Milmann); however the converse does not hold. In case of C^*-algebras, one has the following.

Theorem 4.3. *Let A be a C^*-algebra.*

(i)[80] *S has an extreme point iff A has identity. Then $x \in extS$ iff $(1 - x^*x)S(1 - xx^*) = 0$.*

(ii)[45] *(Russo-Dye-Palmer) $S = \overline{co}(extS) = \overline{co}(U(A))$, $U(A)$ denoting the unitary group of A.*

There ought to be a uB-algebra analogue of this.

(4.3) (the second dual) It is a well known construction [33] due to Arens that the second dual space A^{**} of a Banach algebra A becomes a Banach algebra with two distinct multiplications each of which extends, through the canonical imbedding, the multiplication of A. The algebra A is *Arens regular* if both these Arens products coincide.

Theorem 4.4. *A C^*-algebra is Arens regular; and A^{**} is isomorphic to $W^*(A)$ the W^*-algebra generated by A.*

Is a uB-algebra Arens regular having second dual isomorphic to a dual uB-algebra? The uB-algebras that are duals of Banach algebras are analogues of W^*-algebras in the present frame work; they do not seem to have been studied systematically. Notice that a closed subalgebra of an Arens regular Banach algebra is Arens regular, hence a uB-algebra is Arens regular.

(4.4) (Dunford-Pettis Property) A Banach space E has *DP-property* if every weakly compact operator on E is compact. The commutative C^*-algebra $C(X)$ has DP-property; the C^*-algebra $K(H)$ of compact operators on an infinite dimensional Hilbert space fails to have DP. The C^*-subalgebras and the quotient C^*-algebras of a C^*-algebra with DP have DP.

Theorem 4.5. [38] *The following are equivalent in a C^*-algebra A.*

(i) A has DP.

(ii) The dual A^* has DP.

(iii) For some closed ideal I of A, both I and A/I have DP.

(iv) Every irreducible representation of A is finite dimensional.

(v) The second dual A^{**} is a type I W^*-algebra.

We do not know a uB-algebra analogue of this.

5. **Unitization**

(5.1) Let A be any algebra. Let $A_e = A \bigoplus \mathbb{C}1$ be the algebra obtained by adjoining identity to A - an abstract one-point compactification. If $\|\cdot\|$ is any (semi)norm on A, its natural extension to A_e is the l^1- (semi)norm $\|(x,\lambda)\|_1 = \|x\| + |\lambda|$. However if $\|\cdot\|$ is a C^*-(semi)norm or a uniform (semi) norm, then $\|\cdot\|_1$ need not be a C^*-(semi)norm or a uniform (semi)norm respectively. Then one considers the operator seminorm on A_e defined as $\|(x,\lambda)\|_{op} = \sup\{(\|(x+\lambda)y\| : \|y\| \leq 1)\}$. The following is a folklore.

Theorem 5.1. *(A) If $\|\cdot\|$ is a C^*-norm on A, then $\|\cdot\|_{op}$ is a C^*-norm on A_e and $\|\cdot\|_{op}|_A = \|\cdot\|$. If $(A, \|\cdot\|)$ is a C^*-algebra, then $(A_e, \|\cdot\|_{op})$ is a C^*-algebra.*

(B) If $\|\cdot\|$ is a uniform norm on A, then $\|\cdot\|_{op}$ is a uniform norm on A_e and $\|\cdot\|_{op}|_A = \|\cdot\|$. If $(A, \|\cdot\|)$ is a uB-algebra, then $(A_e, \|\cdot\|_{op})$ is a uB-algebra.

Of course, these are very elementary matters; however certain issues do not seem to be clear.

(a) Given a Banach algebra $(A, \|\cdot\|)$, which are all sub multiplicative norms $|.|$ on A_e that extend the given norm $\|\cdot\|$? All such norms necessarily satisfy

$\|\cdot\|_{op} \leq |.| \leq \|\cdot\|_1$. However $\|\cdot\|_{op}$ need not be a norm; even if it is a norm, it need not be complete. Also, $\|\cdot\|_{op}|_A$ need not coincides with $\|\cdot\|$. When $\|\cdot\|_{op}|_A = \|\cdot\|$, the $\|\cdot\|$ on A is called *regular*; if $\|\cdot\|_{op}|_A$ is equivalent to $\|\cdot\|$, then $\|\cdot\|$ is *weakly regular*. The following gives some idea on this.

Theorem 5.2. *Let $(A, \|\cdot\|)$ be a Banach algebra.*

(1) [16] *The semi norm $\|\cdot\|_{op}$ is a norm on A_e iff the left annihilator lanA $= 0$, where lanA $= \{x \in A : xA = 0\}$. Thus if $\|\cdot\|_{op}$ is a norm on A_e for one norm $\|\cdot\|$, then $|.|_{op}$ is a norm on A_e for all norms $|.|$.*

(2) [16] *The norm $\|\cdot\|_{op}$ is complete on A_e iff $\|\cdot\|$ is weakly regular iff $\|\cdot\|_{op}$ is equivalent to $\|\cdot\|_1$ on A_e.*

(3) [58] *If $\|\cdot\|$ is regular, then $\|(x,\lambda)\|_{op} \leq \|(x,\lambda)\|_1 \leq 6e\|(x,\lambda)\|_{op}$ for all (x,λ) in A_e.*

(b) It is interesting to look for properties of A which are not shared by A_e. Here are two instances. If a Banach algebra $(A, \|\cdot\|)$ is a Hilbert space, A_e is never a Hilbert space. If A admits an orthogonal basis [61], A_e never admits an orthogonal basis.

(5.2) Taking multipliers of a non unital Banach algebra A constitute another process of adjoining identity - an abstract Stone - Cech compactification. If A is a C^*-algebra, then its multiplier algebra $M(A)$ is the maximal C^*-algebra with 1 in which A sits as an essential ideal; and in any faithful representation of A on a Hilbert space, $M(A)$ is realized as the idealizer of A. On the other hand, multiplier algebra of a uB-algebra does not seem to have been studied in details, probably because traditionally a uB-algebra is always assumed to be unital.

6. Characterizations

(6.1) A Banach *-algebra $(A, \|\cdot\|)$ is C^*-*equivalent* if there exists a C^*-norm on A equivalent to $\|\cdot\|$. Analogously a Banach algebra $(A, \|\cdot\|)$ is *uB-equivalent* if there exists a uniform norm on A equivalent to $\|\cdot\|$. The following illustrates our point of view. For numerical range theory in Banach algebras, we refer to [34],[35].

Theorem 6.1. *(A) A unital Banach *-algebra $(A, \|\cdot\|)$ is C^*- equivalent iff A is hermitian and for some $c > 0$, $\|h\| \leq cr(h)$ for all $h = h^*$ in A iff for some $c > 0$, $\|x\| \leq cs(x)$ for all $x \in A$ iff there exists $c > 0$ such that $\|x\|^2 \leq c\|x^*x\|$ for all $x \in A$ iff the set $\{e^{ih} : h = h^*\}$ is bounded iff $s(x) = \nu(x)$ for all x.*

Further $(A, \|\cdot\|)$ is a C^*-algebra iff A is hermitian and the numerical range $V(A, \|\cdot\|, h) = \overline{co}sp(h)$ for all $h = h^*$ in A.

(B) A Banach algebra $(A, \|\cdot\|)$ is uB-equivalent iff for some $c > 0$, $\|x\| \leq cr(x)$ for all x iff for some $c > 0$, $\|x\|^2 \leq c\|x^2\|$ for all x in A iff $V(A, \|\cdot\|, x) = \overline{co}sp(x)$ for all $x \in A$ iff $\|e^x\| = r(e^x)$ for all x iff ν has the square property iff $V(A, \|\cdot\|, a^2) \subset \{z^2 : z \in V(A, \|\cdot\|, a)\}$ for all a.

A couple of remarks are in order.

(i) By a result of Bjork [32], given a commutative semi simple Banach algebra $(A, \|\cdot\|)$, if there exists a number $q < 1$ and a positive integer $n \geq 2$ such that $\|x^n\| \leq q^n \|x\| r(x)^{n-1}$ for all $x \in A$, then A is uB-equivalent. Is it true that a Banach *-algebra $(A, \|\cdot\|)$ is C^*-equivalent if for some $q < 1$, some $n \geq 2$, it holds that $\|x^n\| \leq q^n \|x\| s(x)^{n-1}$ for all $x \in A$?

(ii) By a result of Cuntz [42], a Banach *-algebra A is C^*-equivalent if for each $h = h^*$ in A, the closed subalgebra C_h generated by h is C^*-equivalent. Let A be a Banach algebra such that for each $x \in A$, the closed subalgebra C_x generated by x is uB-equivalent. Is A uB-equivalent? We refer to [47] for a collection of results on characterizations of C^*-algebras, the uB-analogues of some of which could be asked for.

(iii) The numerical radius ν in a Banach algebra $(A, \|\cdot\|)$ satisfies $\nu(x^n) \leq n!(e/n)^n \nu(x)^n$ for all x and for all n. If A is a C^*-algebra, then $\nu(x^n) \leq \nu(x)^n$; and if A is a uB-algebra, then $\nu(x^n) = \nu(x)^n$ for all n and for all x. Let $(A, \|\cdot\|)$ be a Banach algebra satisfying the numerical radius power inequality $\nu(x^n) \leq \nu(x)^n$ for all x and for all n. Is A a uB-algebra? Further if A is a Banach*-algebra, is A a C^*-algebra?

(6.2) The following transparently describes the analogy between the structures of a C^*-algebra and of a uB-algebra. A *locally convex algebra* is an algebra A with a Hausdorff topology τ such that A is a locally convex space in which the ring multiplication is separately continuous. Let $B(\tau)$ be the collection of all closed, bounded absolutely convex idempotent sets in A. If A is involutive, let $B^*(\tau)$ consists of B in $B(\tau)$ such that B is *-closed.

Theorem 6.2. [8] *(A)Let A be a unital *-algebra. Let τ be a Hausdorff topology on A. The following (I) and (II) are equivalent.*

(I) A satisfies the following.

(1) A is a complete locally convex algebra with continuous involution

(2) A is a Q -algebra.

(3) A is an algebra with continuous inversion.

(4) A is hermitian.

(5) The collection $B^*(\tau)$ admits greatest member (say B) under inclusion.

(II) There exists a norm $\|\cdot\|$ on A such that $\|\cdot\|$ determines the topology of A; $(A, \|\cdot\|)$ is a C^*-algebra; and $B = \{x \in A : \|x\| \leq 1\}$.

(B) Let A be a unital algebra. Let τ be a Hausdorff topology on A. The following (I) and (II) are equivalent.

(I) A satiesfies the following.

(1) A is a complete locally convex algebra.

(2) A is a Q- algebra.

(3) A is an algebra with continuous inversion.

(4) The collection $B(\tau)$ admits a greatest member (say B) under inclusion.

(II) There exists a norm $\|\cdot\|$ on A such that $\|\cdot\|$ determines the topology τ on A; $(A, \|\cdot\|)$ is a uB-algebra; and $B = \{x \in A : \|x\| \leq 1\}$.

The above theorem is a companion to Kolmogorov Theorem regarded as the first theorem about locally convex spaces [79] stating that a topological vector space is normable iff it is locally convex and locally bounded. It follows that a Banach *-algebra A is C^*-equivalent iff the collection $B^*(\tau)$ has greatest member under inclusion; analogously a commutative Banach algebra is uB-equivalent iff the collection $B(\tau)$ has greatest member.

In spite of this, the structural analogy between C^*-algebras and uB-algebras fails also in certain aspects. We aim to discuss some of these with a view to uncover refined versions of the analogy.

7. Non-uniqueness of incomplete norms

(7.1) The analogy between C^*-algebras and uB-algebras fail dramatically regarding the existence of incomplete norms. Let $(A, \|\cdot\|)$ be a C^*-algebra. Let $|.|$ be any norm (not necessarily complete) on A having C^*-property $|x^*x| = |x|^2$ for all x. Then $\|\cdot\| = |.| = s$. On the other hand, let $(B, \|\cdot\|)$ be a uB-algebra. If $|.|$ is any complete uniform norm on B, then $\|\cdot\| = |.| = r$. There can be non-uniform complete norms on B other than $\|\cdot\|$; e.g. on $C(X)$,$\|f\| = \sup\{|x(s)+x(t)|/2 + |x(s)-x(t)|/2 : s, t \in X\}$ defines a complete Banach algebra norm, distinct from the sup norm. On the other hand, B can admit in-complete uniform norms other than $\|\cdot\|$. On the disc algebra $A(D)$ consisting of functions continuous on the closed unit disc D and analytic in its interior U, which is a uB-algebra with norm $\|f\|_\infty = \sup\{|f(z)| : z \in D\}$, the norms $|.|_r, 0 < r < 1, |f|_r = \sup\{|f(z)| : 0 < |z| < r\}$ are all distint incomplete uniform norms on $A(D)$. A C^*-algebra $(A, \|\cdot\|)$ has *minimum property* (MP)in the sense that $\|\cdot\| \leq |.|$ for any algebra norm $|.|$; where as a uB-algebra need not have MP as is exhibited by the disc algebra $A(D)$.

(7.2) Thus some topological or algebraic conditions are required on $(B, \|\cdot\|)$ so that $|.| = \|\cdot\|$ holds. If $|.|$ is a spectral uniform norm on B, then $\|\cdot\| = |.|$. The following illustrates this in the case of $C(X)$ the existence or non-existence of norms on which has been an intricate problem; part 3(i) is a solution by Dales of the famous problem of Kaplansky; see [44].

Theorem 7.1. *(1) Let* $|.|$ *be any algebra norm on* $C(X)$ *for an infinite compact space* X*. Then* $\|\cdot\|_\infty \leq |.|$*.*

(2) Let $|.|$ *be any linear norm on* $C(X)$*.*

(a) If $|.|$ *satisfies the* C^**-inequality or* $|.|$ *is a* C^**-norm, then respectively* $|.|$ *is equivalent to the sup norm* $\|\cdot\|_\infty$ *or* $|.| = \|\cdot\|_\infty$*.*

(b) If $|.|$ *satisfies the square inequality or* $|.|$ *is a uniform norm, then respectively* $|.|$ *is equivalent to* $\|\cdot\|_\infty$ *or* $|.| = \|\cdot\|_\infty$*.*

(c) If the norm $|.|$ *is unital* $|1| = 1$ *and has the absolute value property* $||x|| = |x|$ *for all* x *(respectively absolute value inequality), then* $|.| = \|\cdot\|_\infty$ *(respectively* $|.|$ *is equivalent to* $\|\cdot\|_\infty$*).*

(3) On $C(X)$*, there exists norms* $|.|$ *such that either of the following hold.*

(i) $|.|$ *is an algebra norm not equivalent to* $\|\cdot\|_\infty$*.*

(ii) $|.|$ *is a Banach algebra norm,* $|1| = 1$*,* $|.| \neq \|\cdot\|_\infty$*.*

(7.3) This suggests to consider the following class of Banach algebras.

Definition 7.2. A Banach algebra $(A, \|\cdot\|)$ has *unique uniform norm property (UUNP)* if it admits exactly one uniform norm.

Notice that $(A, \|\cdot\|)$ admits a greatest uniform semi norm $\|x\|_\infty = \sup\{|\phi(x)| : \phi \in \triangle(A)\} \leq r(x)$; and $\|\cdot\|_\infty$ is a norm iff r is a norm iff A is semi simple and commutative iff A admits a uniform norm. In general, $\|\cdot\|_\infty = r$ iff A is commutative modulo the radical. One may wonder if there is an analogy between C^*-algebras and uB-algebras with UUNP. This suggests looking to more general Banach algebras with UUNP; a non-commutative involutive analogue of which is the following.

Definition 7.3. A Banach *-algebra (not necessarily commutative) $(A, \|\cdot\|)$ has *unique* C^**-norm property (UC*NP)* if it admits exactly one C^*-norm.

A Banach *-algebra $(A, \|\cdot\|)$ admits greatest C^*-semi norm (Gelfand-Naimark pseudo norm) $m(x) = \sup \|\pi(x)\|$, π varying over all non-degenerate *- representations on Hilbert spaces; $m(x) \leq s(x)$; and m is a norm iff A is *-semisimple (need not be hermitian) iff A admits a C^*-norm. Further A is hermitian iff $m = s$.

The UUNP is of recent origin [17],[18]; though UC*NP has been considered in the literature [4] in connection with the following

Definition 7.4. Let $(A, \|\cdot\|)$ be a Banach *-algebra having $Prim(C^*(A))$ as the primitive ideal space (with the hull - kernel topology) of the enveloping

C^*-algebra $C^*(A)$. The algebra A is **-regular* if given a closed subset F of $Prim(C^*(A))$ and P not in F, there exists $x \in A$ such that $\{\widehat{x}(Q) = x + Q : Q \in F\} = \{0\}$, $\widehat{x}(P) = x + P \neq 0$.

The property *-regularity corresponds to the familiar (Silov) regularity [63] in commutative Banach algebras.

Definition 7.5. Let $(A, \|\cdot\|)$ be a commutative Banach algebra having the Gelfand space $\triangle(A)$ with the Gelfand topology. Then A is called *regular* if given a (Gelfand) closed subset F of $\triangle(A)$ and an ϕ not in F, there exists $x \in A$ such that $\{\psi(x) : \psi \in F\} = \{0\}$ and $\phi(x) \neq 0$.

Notice that a *-semi simple *-regular, not necessarily commutative, Banach *-algebra has UC*NP [4]. A commutative semi simple Banach algebra has UUNP if it is regular [17]; and it is regular iff the Gelfand topology coincides with the hull kernel topology on $\triangle(A)$. A commutative *-semi simple Banach *-algebra is regular iff it is hermitian and *-regular [4]; where as it has UUNP iff it is hermitian and has UC^*NP [18].

(7.4) All these suggest to expect a structural analogy between Banach *-algebras with UC^*NP and Banach algebras with UUNP; as well as between *-regular Banach *-algebras and commutative regular Banach algebras. However given a commutative Banach algebra A and a (not necessarily commutative) Banach *-algebra B, one must keep in mind the following intrinsic differences between them that can affect the desired similarities. The *enveloping uB-algebra $U(A)$ of A* is the Hausdorff completion of A in the spectral radius; analogously the *enveloping C^*-algebra $C^*(B)$ of B* is the Hausdorff completion of A in the greatest C^*-semi norm m.

(i) The algebra $C^*(B)$ has UC*NP; where as $U(A)$ need not have UUNP.

(ii) The algebra $C^*(B)$ is *-regular;where as $U(A)$ need not be regular.

(iii) The algebra A is inverse closed in $U(A)$; where as B need not be inverse closed in $C^*(B)$, unless it is hermitian.

(iv) The algebra A is unital iff $U(A)$ is unital; this is not necessarily true for B and $C^*(B)$. The paper [4] contains an example of a commutative non-unital B such that $C^*(B)$ is unital.

(v) The space $Prim(C^*(B))$ is identified with the space of irreducible *-representations, up to equivalence, of B on Hilbert spaces; where as the space $\triangle(A)$ is identified with the space of all irreducible, up to similarity, representations of A on vector spaces. An irreducible representation π of A is a *-representation iff the corresponding ϕ in $\triangle(A)$ is hermitian.

(vi) Every closed ideal of $C^*(A)$ has *-semi simple quotient;where as not every closed ideal of $U(A)$ has semi simple quotient.

(7.5) With this, we continue exploring the meta theorem further.

Theorem 7.6. *(A)*[4] *Let A be a *-semisimple Banach *-algebra. The following are equivalent.*

(1) A has UC*NP.

(2) Given a proper closed subset F of PrimA, there exists $x \in A$ such that $x \neq 0$; $\{\widehat{x}(P) : P \in F\} = \{0\}$.

(3) Every set of *-uniqueness for A in $Prim(C^*(A))$ is dense in $Prim(C^*(A))$.

(4) For every proper closed subset F of the space $FacC^*(A)$ of factor representations of $C^*(A)$, there exist $x \in A$ such that $x \neq 0$, $\{\widehat{x}(P) : P \in F\} = \{0\}$

(5) Every A-separating set in $Prim(C^*(A))$ is $C^*(A)$ - separating.

(6) Every faithful *-representation of A can be extended to a faithful *-representation of $C^*(A)$.

(7) A subset D of $Prim(C^*(A))$ is dense in $Prim(C^*(A))$ iff $\psi(D)$ is dense in the space $Prim_*(A)$ of primitive ideals of A. Here $\psi(P) = P \cap A$ for all $P \in Prim(A)$.

(8) For any non-zero closed *-ideal I of $C^*(A)$, $I \cap A \neq \{0\}$.

(B) [18] *Let A be a semi simple commutative Banach algebra. The following are equivalent.*

(1) A has UUNP.

(2) $U(A)$ has UUNP; and every A -separating subset of $\triangle(U(A))$ is also $U(A)$ separating.

(3) $U(A)$ has UUNP; and any closed subset F in $\triangle(U(A))$ which is a set of A - uniqueness, is also a set of $U(A)$-uniqueness.

(4) $U(A)$ has UUNP; and for any non-zero semi simple closed ideal I of $U(A)$, $I \cap A \neq \{0\}$.

In view of the complexity of the norm behavior of a uB-algebra as compared to that in a C^*-algebra, the property UUNP turns out to be more stringent than UC*NP. In fact, UUNP arises in the investigation of spectral properties of a commutative Banach algebras A associated with incomplete algebra norms [17]. Properties closely related with UUNP in this context are the *spectral extension property (SEP)* and *unique semi simple norm property (USNP)*[17]. The algebra A has SEP if for any algebra norm $\|.\|$ on A, $r(x) \leq \|x\|$ for all $x \in A$; i.e. $\|.\|$ is a Q-norm; where as A has USNP if A admits exactly one, up to equivalence, semi simple norm. A norm is *semi simple* if it results into a semi simple algebra upon completion. The following exhibits their relevance with UUNP and further explains why UUNP is more stringent.

Theorem 7.7. [17] *Let A be a semi simple commutative Banach algebra. The following are equivalent.*

(1) The algebra A has UUNP.

(2) Every semi simple norm on A is spectral.

(3) The spectral radius r is minimum semi simple norm on A.

(4) The Silov boundary $\partial(A)$ is the smallest closed set of uniqueness for A.

(5) Given a closed set $F \subset \triangle(A)$, $\partial(A)$ not containing in F, there exists $x \in A$ such that $r(x) > 0$ and $\phi(x) = 0$ for all $\phi \in F$.

(6) Every multiplicative linear functional in $\partial(A)$ admits a multiplicative linear extension to any semi simple super algebra of A.

(7) For all $x \in A$, $r(x) = \inf\{r_{A_p}(x) : p$ is a semi simple norm on A $\}$, A_p being the completion of A in p.

Thus UUNP is equivalent to semi simple SEP. The spectral properties USNP and SEP make sense in non commutative Banach algebras also. What is the non commutative analogue of above theorem? A commutative Banach algebra A has USNP iff A has UUNP and A is uB- equivalent. Further a commutative A has SEP iff A has UUNP and $r_s = r_p$. It is an open problem whether UUNP implies SEP [18]. The following brings out a curious feature associated with UUNP and UC*NP. This was discovered first in case of Beurling algebra $L^1(G,\omega)$ in [21] as well as in weighted measure algebra [22].

Theorem 7.8. [43] *(A) Let A be a semi simple commutative Banach algebra. Then A admits either only one uniform norm or an infinite number of uniform norms.*

(B) Let A be a *-semi simple commutative Banach *-algebra. Then A admits either only one C^*-norm or an infinite number of C^*-norms.

Does above (B) hold for non commutative Banach *-algebras? Note that for a non abelian locally compact group G, in general the non commutative Banach *-algebra $L^1(G)$ admits at least two C^*- norms viz. one coming from the group C^* - algebra $C^*(G)$ and the other from the reduced group C^*-algebra $C_r^*(G)$.

(7.6) The following exhibits the analogy between *-regularity in non commutative Banach *-algebras and regularity in commutative Banach algebras. It also put UUNP in proper perspective in general theory of commutative Banach algebras.

Theorem 7.9. *(A)* [4] *Let A be a *-semi simple Banach *- algebra. The following are equivalent.*

(1) A is *-regular.

(2) For any closed set F in $Fac(C^*(A))$ (= the space of factor representations of $C^*(A)$) and any $I \in Fac(C^*(A))$, I not in F, there exists $x \in k(F)$ such that x is not in I.

(3) The map $\psi : Prim(C^*(A)) \to Prim_*(A)$,$\psi(I) = I \cap A$ is a homeomorphism.

(4) For any m-closed ideal I of $C^*(A)$, $I \cap A$ is dense in A.

*(5) For any m- closed ideal I of A,A/I has UC*NP (respectively is *-regular).*

*(6) For any m - closed ideal I of $C^*(A)$, $A/I \cap A$ has UC*NP (respectively is *-regular).*

(B)[18] *Let A be a semi simple commutative Banach algebra. The following are equivalent.*

(1) A is regular.

(2) $U(A)$ is regular and for any semi simple ideal I of A, A/I has UUNP.

(3) $U(A)$ is regular and for any semi simple ideal I of $U(A)$,

$I = k(h(A \cap I))$.

Further, each of above is implied by the following.

(4) $U(A)$ is regular and for any semi simple ideal I of $U(A)$, $A \cap I$ is dense in I.

If $U(A)$ is an N-algebra in the sense of Silov (i.e. every closed ideal gives semi simple quotient),then (1) implies (4).

Though regularity is an analogue of *-regularity, it is more restrictive. A C^*-algebra is *-regular; where as a uB-algebra need not be regular. In a commutative Banach algebra (in particular a uB-algebra) A, the structure space $\triangle(A)$ is very large containing all irreducible vector space representations as compared to $Prim(C^*(A))$ in case of a Banach *-algebra. This also results into the Silov boundary $\partial(A)$ many a times properly contained in $\triangle(A)$, a phenomena that can support analytic structure, hence making available several incomplete uniform norms. Analogous situation does not arise in a C^*-algebra. This leads to refined notions of regularity.

Definition 7.10. Let A be a semi simple commutative Banach algebra.

(1) A is *weakly regular* if given a proper closed set F in $\triangle(A)$, there exists x in A such that $x \neq 0$, $\phi(x) = 0$ for all $\phi \in F$.

(2) A is *U-regular* if given a closed set F in $\triangle(A)$ and $\phi \in \partial(A)$, ϕ not in F; there exist $x \in A$ such that $\phi(x) \neq 0$ and $\psi(x) = 0$ for all $\psi \in F$.

The following brings out their role.

Theorem 7.11. [18] *Let A be a semi simple commutative Banach algebra.*

(1) A is weakly regular iff $\partial(A) = \triangle(A)$ and A has UUNP.

(2) A has UUNP iff A is weakly U-regular in the sense that given any closed set $F \subset \triangle(A)$ *having* $\partial(A) \cap (\triangle(A) \setminus F) \neq \phi$*, there exists* $x \neq 0$ *in* A *such that* $\psi(x) = 0$.

(3) Let A be a Banach **-algebra. Then A is weakly regular iff A is hermitian and has UC*NP iff A is hermitian and has UUNP.*

Thus U-regularity sharply corresponds to *-regularity; and UUNP corresponds to weak *-regularity (defined analogously to weak regularity) which is UC*NP in the light of Theorem 7.6. This leads to the following questions. Is $U(A)$ U-regular for any A? Is it true that A is U-regular iff for any semi simple closed ideal I of $U(A)$, $I \cap A$ is dense in I iff for any semi simple r - closed (i.e. closed in the complete uniform norm on $U(A)$) ideal I of A, A/I has UUNP (respectively is U-regular) iff for any semi simple closed ideal I of $U(A)$, the quotient $A/A \cap I$ has UUNP (respectively is U-regular? For behaviour of UUNP and UC^*NP, as well as of regularity and *-regularity with respect to sub algebra, ideals and quotients, we refer to [18],[4].

8. **Relevance to Harmonic Analysis and Complex Analysis**

(8.1)[18] Let G be a locally compact abelian group. The commutative Banach $*$-algebra $L^1(G)$ is regular, hence has UUNP. The measure algebra $M(G)$ has UUNP iff $M(G)$ is hermitian iff $M(G)$ is regular iff G is discrete. Let $M_{00}(G)$ consist of μ in $M(G)$ whose Fourier - Stieltjes transforms vanish off the dual group; where as $M_0(G)$ consists of μ in $M(G)$ the restriction of whose Fourier - Stieltjes transforms to the dual group define continuous functions vanishing at infinity on the dual group. These are closed convolution sub algebras of $M(G)$; and $L^1(G) \subset M_{00}(G) \subset M_0(G) \subset M(G)$. For non discrete G, $M_{00}(G)$ has UUNP; and $M_0(G)$ fails to have UUNP. The convolution Banach algebras $DecM(G)$ and $DM(G)$ consisting of measures $\mu \in M(G)$ defining decomposable convolution operators respectively on $M(G)$ and $L^1(G)$ both have UUNP.

(8.2) Let G be non abelian. Then G is called *C^*-unique* if the non commutative Banach *-algebra $L^1(G)$ has UC*NP. A C^*-unique G is amenable; and a connected G is amenable iff $L^1(G)$ has exactly one C^*-norm that is invariant under isometric *-automorphisms of $L^1(G)$ [37].We refer to [75] and references there in for C^*-unique and *-regular locally compact groups.

(8.3) The UUNP acquires a greater significance in the context of Harmonic Analysis on locally compact abelian groups and semigroups with weights. A *weight function* on a locally compact abelian group G is a continuous function $\omega : G \to \mathbb{R}$ such that for all s,t in G, (i) $\omega(s) > 0$ and (ii) $\omega(s+t) \leq \omega(s)\omega(t)$.

Then $\omega(s)$ presumably represents the frequency or size of s in G. The *Beurling algebra* is the convolution Banach algebra of measurable functions $f$on G such that $\|f\|_\omega := \int_G |f(s)|\omega(s)d\lambda < \infty$, λ denoting the Haar measure on G. It is semi simple [19]; though in the non abelian case, this is not known. A *generalized character* is a continuous homomorphism from G to the multiplicative group of non zero complex numbers. Let $H(G)$ be the space of all generalized characters with compact open topology. By [20], if G is compactly generated, then $H(G)$ is a locally compact abelian group with the operation $\alpha(g) + \beta(g) = \alpha(g)\beta(g)$ for all $g \in G$; $H(G)$ coincides with the dual group $\widehat{G}$ iff G is compact; and with G to be second countable or discrete, $H(G) \cong \triangle(C_c(G))$ the Gelfand space of the convolutin algebra of continuous functions with compact supports and with the inductive topology. The Gelfand space of $L^1(G,\omega)$ is the space $H(G,\omega)$ of ω - bounded generalized characters on G. Generally, in the case of Euclidean groups, the part of $H(G,\omega)$ not in the dual group $\widehat{G}$ is a source of analyticity; hence non uniqueness of uniform norm. By [20],[21], $L^1(G,\omega)$ is regular iff $L^1(G,\omega)$ has UUNP iff $L^1(G,\omega)$ has minimum uniform norm. For a symmetric weight ω, $L^1(G,\omega)$ is a Banach *-algebra; and by [43], it has UUNP iff it has UC^*NP. The weighted measure algebra $M(G,\omega)$ consists of Radon measures μ on G such that $\omega\mu \in M(G)$. Then $FS(\mu) : H(G,\omega) \to \mathbb{C}$, $FS(\mu)(\alpha) = \int \alpha(s)d\mu(s)$ defines the *generalized Fourier - Stieltjes transform* of $\mu \in M(G,\omega)$. By [22], $M(G,\omega)$ has UUNP iff G is discrete and $L^1(G,\omega)$ has UUNP iff $M(G,\omega)$ is regular. The convolution subalgebra $M_{00}(G,\omega)$ is regular iff it has UUNP iff $L^1(G,\omega)$ has UUNP; where as it is believed that $M_0(G,\omega)$ has UUNP iff G is discrete and $L^1(G,\omega)$ has UUNP. For a non abelian locally compact groups G, the UC*NP as well as *-regularity for the non commutative Beurling algebra $L^1(G,\omega)$ have not been investigated in details. So is the case with commutative Beurling algebras on abelian semi groups. Here is a glimpse of what could be happening there. Each of the Banach algebras $L^1(\mathbb{R}^+,\omega)$ and $l^1(\mathbb{Z}^+,\omega)$ does not have UUNP for any weight ω [18].

(8.4) Harmonic Analysis on groups G (and on semigroups S) with weights have several interesting aspects; e.g. (i) generalization of Analysis on G to Analysis on (G,ω), considering the latter as an intrinsic object; (ii) discovering the influence of ω on the analysis on G; (iii) developing new tools, e.g. for some ω, $L^p(G,\omega)$ could be a Banach algebra; as well as (iv) discovering further intimate relation with Complex Analysis. We illustrate (iv) by exhibiting how the complex analytic structure of the planner disc and the annulus can be abstractly functional analytically recaptured - an example in Function Theory with out Theory of Function. For each $k \in \mathbb{N}$, let $\omega_k : \mathbb{Z} \to \mathbb{R}^+$ be a weight sequence on the integers $\mathbb{Z}$. Let $\omega = \{\omega_k\}$ and let $l^1(\mathbb{Z}, \omega) = \cap l^1(\mathbb{Z}, \omega_k)$ a Frechet algebra with topology defined by the sequence of norms $\| \cdot \|_{\omega_k}$, $\|f\|_{\omega_k}$ $= \sum_n |f(n)|\omega_k(n)$. This discrete Beurling - Frechet algebra can be abstractly characterized. A *Laurent series generated Frechet algebra* is a Frechet m-convex algebra A with a defining sequence of semi norms $\{p_k\}$ such that (i) A is topologically generated by $\{x, x^{-1}\}$ for some x in A; and (ii) any y in A can be expressed as $y = \sum_{n\in\mathbb{Z}} \lambda_n x^n$ with $\sum |\lambda_n| p_k(x^n) < \infty$ for each $k \in \mathbb{N}$. It has *unique expression property* if the expression in (ii) is unique.

Theorem 8.1. [24] *A Laurent series generated Frechet algebra with unique expression property is homeomorphically isomorphic to a discrete Beurling - Frechet algebra $l^1(\mathbb{Z}, \omega)$ on the integers $\mathbb{Z}$ defined by a sequence of weights.*

This leads to the following characterizations of several Banach and Frechet algebras associated with an annulus $T(r_2, r_1) = \{z \in \mathbb{C} : r_2 < |z| < r_1\}$. A *uniform Frechet algebra (uF-algebra)* is a Frechet algebra whose topology is defined by a sequence of uniform semi norms.

Theorem 8.2. [24] *Let A be a Laurent series generated Frechet algebra with unique expression property and having Laurent series generator x. Let $p_k(.)$ be a sequence of sub multiplicative semi norms defining the topology of A.*

(1) If $sp_A(x)$ is open subset of $\mathbb{C}$, then A is isomorphic to the uF-algebra $H(T(r_2, r_1))$, $0 \le r_2 < r_1 \le \infty$, of functions holomorphic in the open annulus $T(r_2, r_1)$ and with compact open topology. Further

(1a) $0 < r_2$ iff 0 belongs to the interior of the complement of $sp_A(x)$; and

(1b) $r_1 < \infty$ iff $sp_A(x)$ is bounded.

(2) If the $int(sp_A(x))$ is empty, then A is isomorphic to the Wiener Frechet algebra $W(T_r, \omega)$ consisting of continuous functions f on $T_r = \{z \in \mathbb{C} : |z| = r\}$ such that $\|f\|_{\omega_k} := \sum_{n\in\mathbb{Z}} |F(f_r)(n)|\omega_k(n) < \infty$ for a sequence $\{\omega_k\}$ of weight functions on $\mathbb{Z}$. Here $F(f_r)(n)$ denote the Fourier coefficients of $f_r(z) = f(rz)$. Further assume the following.

(2a) There exists $M_1 > 0$, $M_2 > 0$ and a sequence $\{m_k\}$of natural numbers such that $M_1(1 + |n|)^{m_k} \le p_k(x^n) \le M_2(1 + |n|)^{m_k}$ for all n;

(2b) For some n_0,$\{p_k(x^{n_0})\}$ is bounded.

Then A is isomorphic to the Frechet algebra $C^\infty(\mathbb{T})$ of the C^∞-functions on the unit circle $\mathbb{T}$.

(3) If for all k, $p_k(x^n) = p_k(x)^n (n \ge 0)$ and $p_k(x^n) = p_k(x^{-1})^{-n} (n \le 0)$, then A is isomorphic to any of the Frechet algebras $H(T(r_2, r_1))$ or $H(T[r_2, r_1))$

or $H(T(r_2, r_1])$ or the Banach algebra $H(T[r_2, r_1])$. Further if $sp_A(x)$ is compact, then A is isomorphic to the Banach algebra $H(T[r_2, r_1])$ consisting of functions continuous on the closed annulus $T[r_2, r_1]$ and holomorphic in its interior $T(r_2, r_1)$.

(4) If A is a uF-algebra, then A is isomorphic to $H(T(r_2, r_1))$.

Above theorem recaptures the classical complex analysis of Laurent series representations. We consider an analogue of Taylor series representations. A *power series generated Frechet algebra* is a Frechet m-convex algebra with a defining sequence of semi norms $\{p_k\}$ such that A is topologically generated by some x and each y in A can be expressed as $y = \sum_{n\in\mathbb{N}} \lambda_n x^n$ with $\sum |\lambda_n| p_k(x^n) < \infty$.

Theorem 8.3. [29] *Let A be a power series generated Frechet algebra of power series having unique expression property.*

(1) Then A is isomorphic to either the algebra of all power series in one indeterminate or to the semi group Beurling - Frechet algebra $l^1(\mathbb{Z}^+, \omega) = \cap l^1(\mathbb{Z}^+, \omega_k)$ for a sequence $\{\omega_k\}$ of weights on $\mathbb{Z}^+$.

(2) Either $sp_A(x)$ is totally disconnected or A is isomorphic to $l^1(\mathbb{Z}^+, \omega)$.

(3) Assume that $sp_A(x)$ is not totally disconnected and that A is semi simple.Then the following hold.

(3a) Let A be a Q-algebra.Suppose that the following hold.

(i) There exists $M_1 > 0$, $M_2 > 0$ and a sequence $\{m_k\}$of natural numbers such that $M_1(1+|n|)^{m_k} \leq p_k(x^n) \leq M_2(1+|n|)^{m_k}$ for all n in $\mathbb{Z}^+$;

(ii) For some n_0,$\{p_k(x^{n_0})\}$ is bounded.

Then A is isomorphic to the Frechet algebra $A^\infty(\mathbb{T})$ consisting of C^∞-functions on the unit circle having all negative Fourier coefficients zero.

(3b)Assume that A is not a Q-algebra. Then A is isomorphic to the holomorphic function algebra $H(U)$ of the open disc U or to the entire function algebra E both with the compact open topology depending on whether $sp_A(x)$ is bounded or unbounded respectively.

It should be of interest to look to multivariate analogues of these.

(8.5) [18] In view of the possible failure of UUNP in Banach algebras having Gelfand spaces with analytic structure, it is instructive to have examples of varients of holomorphic function algebras having UUNP. For $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$,let $|z|_\infty = \max(|z_1|, ..., |z_n|)$, $|z|_2 = (|z_1|^2 + ... + |z_n|^2)^{1/2}$. Let $\triangle^n = \{z \in \mathbb{C}^n : |z|_\infty < 1\}$ be the open poly disc and $B^n = \{z \in \mathbb{C}^n : |z|_2 < 1\}$ be the open unit ball. Then the poly disc algebra $A(\overline{\triangle^n})$ consisting of functions continuous on $\overline{\triangle^n}$ that are holomorphic in $\triangle^n$, the ball algebra $A(\overline{B^n})$ consisting of functions continuous on $\overline{B^n}$ that are holomorphic in B^n, as well as the algebras $H^\infty(\triangle^n)$ and $H^\infty(B^n)$ consisting of bounded holomorphic functions all with respective sup norms are all Banach algebras that fail to have UUNP. The following are their variants constructed by thickening the Silov boundary. Let $0 < r < 1$. Let $\triangle^n_r = \{z \in \mathbb{C}^n : |z|_\infty < r\}$, $B^n_r = \{z \in \mathbb{C}^n : |z|_2 < r\}$. Each of the sup norm Banach algebras $A_r(\triangle^n) = \{f \in C(\overline{\triangle^n} : f$ is holomorphic in $\triangle^n_r\}$,$A_r(B^n) = \{f \in C(\overline{B^n} : f$ is holomorphic in $B^n_r\}$,$H^\infty_r(\triangle^n) = \{f \in C_b(\triangle^n)$: f is holomorphic in $\triangle^n_r\}$, and $H^\infty_r(B^n) = \{f \in C_b(B^n) : f$ is holomorphic in $B^n_r\}$ have UUNP; and non of them is even weakly regular.

9. **Some other instances**

(9.1) We consider some other instances wherein the analogy between C^*-algebras and uB-algebras fail. Whenever this happens, it is most likely to happen with a uB-algebra with an analytic structure in its Gelfand space; and in such cases, additional assumptions on a uB-algebra like regularity or UUNP pushing it away from analyticity may be required to obtain results analogous to those for C^*-algebras. The following illustrates this.

Theorem 9.1. [23] *(A) Let A be a non unital C^*-algebra. Then each element of A is a topological devisor of zero.*

(B) Let A be a regular non unital uB-algebra. Then each element of A is a topological devisor of zero.

Theorem 9.2. *(A) Let A be a C^*-algebra. If A is an integral domain, then A is isomorphic to $\mathbb{C}$.*

(B) [18] *Let A be a uB-algebra with UUNP. If A is an integral domain, then A is isomorphic to $\mathbb{C}$.*

Notice that the disc algebra $A(D)$ is a uB-algebra that is an integral domain, and it does not have UUNP.

(9.2) These classes of algebras behave distinctly also with respect to bounded approximate identity (bai).

Theorem 9.3. *(A) A C^*-algebra admits a bai.*

(B) A uB-algebra need not admit a bai.

Indeed, let $A = A(D)$ be the disc algebra.Let $B = \{f \in A(D) : f(0) = 0\}$. The functional ϕ, $\phi(f) = f'(0)$ is continuous on B. Let (e_α) be a bai for B. Now $g(z) = z$ is in B. Hence $g = \lim ge_\alpha$ gives $0 = (ge_\alpha)'(0) = \phi(ge_\alpha) \to \phi(g) = g'(0) = 1$. Let A be a non-unital uB-algebra having UUNP. Does it admit a bai?

(9.3) Curiously enough, C^*-algebras and uB-algebras behave distinctly with respect to quotient.

Theorem 9.4. *(A) Let A be a C^*-algebra. Let I be a closed ideal of A. Then I is a *-ideal; the quotient algebra A/I is *-semi simple; and with the quotient norm, it is a C^*-algebra.*

(B) Let A be a uB - algebra. Let I be a closed ideal of A. The quotient algebra A/I may fail to admit a uniform norm.

Again in the disc algebra $A(D)$, let $I = \{f \in A(D) : f(0) = f'(0) = 0\}$ a closed ideal. Then the quotient $A(D)/I$ is the algebra algebraically generated by $\{1, x\}$ satisfying $x^2 = 0$. Thus the quotient fails to admit a uniform norm; in fact, it is not semi simple. Suppose A is a uB-algebra, if necessary, having UUNP. Let I be a semi simple closed ideal in A. Is A/I with quotient norm a uB-algebra?

This phenomenon is related with the variety structure of these two classes of algebras. A *variety of Banach algebras* [48] is a class of Banach algebras that is closed under taking closed subalgebras, quotients by closed ideals, l^∞ - sum and images under isometric isomorphisms. The class of C^*-algebras form a variety; where as the class of uB -algebras do not form a variety. Interestingly enough, the class Q - of quotients of uB - algebras do form a variety contained in the variety of operator algebras (i.e. Banach algebras isomorphic to closed subalgebras of $B(H)$); and not every commutative operator algebra is quotient of a uB - algebra. This suggests to examine whether there exists a structural analogy between C^*-algebras and quotients of uniform Banach algebras. Not much seem to be known about this.

The properties $*$-regularity and UC*NP, as well as regularity and UUNP, do not behave well with respect to quotients. Let A be a *-semi simple Banach *-algebra. Let I be a *-ideal of A. If A is *-regular (respectively C*- unique), then I is *-regular (respectively C^*-unique); and if I is γ-closed, then A/I is *-regular. However A/I need not be C^*-unique, even if A is C^*-unique. Also, there exists A and I such that both I and A/I are C^*-unique, but A is not C^*-unique [4]. Now let A be a commutative Banach algebra and I be a closed ideal. If A is regular, then both I and A/I are regular; however if A has UUNP, A/I need not have UUNP. Indeed let $\triangle_r = \{z \in \mathbb{C} : |z| \leq r\}$. Let $A = \{f \in C(\triangle_2) : f \in H(int\triangle_1)\}$ a uB - algebra with sup norm; and $I = \{f \in A : f = 0 \text{ on } \triangle_1\}$. Then A has UUNP, but A/I is the disc algebra which admits infinitely many uniform norms.

(9.4) Spectral invariance is another aspect where both these classes of Banach algebras behaves distinctly.

Theorem 9.5. *(A) Let B be a closed *-subalgebra of a C^*-algebra A. Then for any $x \in B$, $sp_B(x) = sp_A(x)$.*

(B) A closed sub algebra B of a uB-algebra A need not be spectrally invariant in A.

The above is exhibited by $A = C(\mathbb{T})$, $B = A(\mathbb{T}) = \{f \in C(\mathbb{T}) :$ f has analytic extension in the open unit disc U $\}$.

Is spectral invariance achieved in a closed sub algebra with UUNP of a uB-algebra with UUNP?

(9.5) With respect to tensor product, C^*-algebras and uB-algebras behave differently, and that too in a singular way.

Theorem 9.6. *(A) Let A and B be C^*-algebras. Let $A \bigotimes B$ be their algebraic tensor product.*

(1) Any C^*-norm on $A \bigotimes B$ is a cross-norm.

*(2) The *-algebra $A \bigotimes B$ need not have UC*NP.*

(B) [18] *Let A and B be uB-algebras. Let $A \bigotimes B$ be their algebraic tensor product.*

(1) Not every uniform norm on $A \bigotimes B$ is a cross-norm.

(2) The algebra $A \bigotimes B$ has UUNP if both A and B have UUNP.

For C^*-algebras A and B, the failure of C^*- uniqueness for $A \bigotimes B$ gave rise to the theory of nuclear C^*-algebrtas. A C^*-algebra A is *nuclear* if for any C^*-algebra B, $A \bigotimes B$ has UC*NP. Nuclear C^*-algebras have turned out to be a most important class of C^*-algebras. The following shows that in the definition of nuclearity, replacing UC*NP by the stronger property *-regularity does not produce a new class of C^*-algebras.

Theorem 9.7. [60] *(1) Let A be a C^*-algebra. Then A is nuclear iff $A \bigotimes B$ is *-regular for all C^*-algebras B.*

*(2) Let A and B be C^*-algebras. Then $A \bigotimes B$ is *-regular iff $A \bigotimes B$ has UC*NP and the completion $A \bigotimes_\alpha B$ in the minimum C^*-norm α has Tomiyama property (F).*

Property (F) : The set of all product states $\phi \bigotimes \psi$ where ϕ and ψ are pure states of A and B respectively separates the closed ideals of $A \bigotimes_\alpha B$.

Theorem 9.8. *(A)* [60] *Let A and B be *-semi simple Banach *-algebras.*

*(1) $A \bigotimes B$ has UC*NP iff A, B and $C^*(A) \bigotimes C^*(B)$ have UC*NP.*

(2) $A \bigotimes B$ is *-regular iff A, B and $C^*(A) \bigotimes C^*(B)$ are *-regular.

(B) Let A and B be semi simple commutative Banach algebras. Let α be a sub multiplicative norm on $A \bigotimes B$ such that the completion $A \bigotimes_\alpha B$ is semi simple.

(1)[18] *$A \bigotimes B$ has UUNP iff each of A and B has UUNP. In this case, $A \bigotimes_\alpha B$ has UUNP.*

(2) [84] *Let α dominates the injective cross norm λ. The $A \bigotimes_\alpha B$ is regular iff A and B are regular.*

It should be noted that for semi simple commutative Banach algebras, the completion $A \bigotimes_\alpha B$ may fail to be semi simple, though $A \bigotimes B$ is semi simple.

(9.6)(4.2.25,p473,[74]) Here is a surprising tensor product characterization of uniform Banach algebras. Let A be a Banach algebra. Let $m : A \bigotimes A \to A$ be the multiplication map $m(x \otimes y) = xy$. Let λ denote the injective cross norm on $A \bigotimes A$. Then A is called *injective* (respectively *geometrically injective*) if m induces a continuous (respectively contractive) linear map $A \bigotimes_\lambda A \to A$. Geometric injectivity of A is equivalent to the statement that $A \bigotimes_\lambda B$ is a Banach algebra for any Banach algebra B.

Theorem 9.9. *(1) An injective Banach algebra is isomorphic to an operator algebra (i.e. a closed, not necessarily involutive, sub algebra of $B(H)$ for some Hilbert space H).*

(2) A geometrically injective Banach algebra is isomorphic to a uniform Banach algebra.

Is there an analogous characterization of C^*-algebras?

10. **Automatic Continuity**

(10.1) Automatic continuity [44] is a fascinating aspect of Banach and Frechet algebras. The most basic result is that a multiplicative linear functional on a Banach algebra is continuous. A corner stone of the theory is B.E.Johson's theorem stating that a semi simple Banach algebra admits a unique complete norm topology. The following slightly recasted version of known results illustrates the meta theorem in the context of automatic continuity.

Theorem 10.1. *(A) (1)*[78] *Let A be a Banach *-algebra. If A admits a C^*-norm, then A admits a unique complete norm topology.*

(2) [83] *Let A be a Banach *-algebra. Let B be a Banach *-algebra that admits a C^*-norm. Let $\phi : A \to B$ be a *-homomorphism.Then ϕ is continuous.*

(3) [83] *Let A be a C^*-algebra. Let B be a Banach *-algebra. Let $\phi : A \to B$ be a one one *-homomorphism (not necessarily onto). Then $\|\phi(x)\| \geq \|x\|$ for all x.*

(B) (1) Let A be a Banach algebra that admits a uniform norm. Then A admits a unique complete norm topology.

(2) Let A be a Banach algebra. Let B be a Banach algebra that admits a uniform norm. Let $\phi : A \to B$ be a homomorphism. Then ϕ is continuous.

(3) [31] *Let A be a weakly regular uB-algebra. Let B be a Banach algebra. Let $\phi : A \to B$ be a one one homomorphism. Then $\|\phi(x)\| \geq \|x\|$ for all x.*

In above (B)(3), the assumption of weak regularity can not be omitted. Let $A = A(D)$ the disc algebra. For $0 < r < 1$, let $B = C(T_r)$ the algebra of all continuous functions on $T_r = \{z \in \mathbb{C} : |z| = r\}$. Let $\phi : A \to B$ be the restriction map $\phi(f) = f|_{T_r}$. The algebra A is not weakly regular. Can we replace weak regularity by the UUNP? The reference [44] is an encyclopedic treatise on automatic continuity a closer examination of which may illuminate the theme of the present paper further.

11. Frechet *-algebras with a C^*-enveloping algebra and commutative Frechet Q - algebras

(11.1) Let A be a Frechet *-algebra. Let $E(A)$ be the enveloping σ-C^*-algebra of A constructed as follows [10]. Let $R(A)$ be the collection of all (continuous) bounded operator *-representations $\pi : A \to B(H_\pi)$. Let $R'(A)$ consists of all topologically irreducible π in $R(A)$. Let $K(A)$ be the collection of all continuous *-semi norms on A. For $p \in K(A)$, let $R_p(A)$ consist of all $\pi \in R(A)$ such that for some $k = k_\pi > 0$, it holds that $\|\pi(x)\| \leq kp(x)$ for all x; and $R'_p(A) = R_p(A) \cap R'(A)$. Let $r_p(x) = \sup\{\|\pi(x)\| : \pi \in R_p(A)\}$. Then r_p is a C^*-semi norm; and $E(A)$ is the Hausdorff completion of A in the family of C^*-semi norms $\{r_p : p \in K(A)\}$. Then A is *an algebra with a C^*-enveloping algebra* if $E(A)$ is a C^*-algebra.

On the other hand, given a commutative Frechet algebra B, let $\triangle(B)$ be the Gelfand space consisting of all continuous complex homomorphisms on B. For each $p \in K_p(B)$, let $\triangle_p(B)$ be the collection of all ϕ in $\triangle(B)$ such that$|\phi(x)| \leq p(x)$ for all $x \in B$. Let $|x|_p = \sup\{|\phi(x)| : \phi \in \triangle_p(A)\}$. Then the enveloping uniform Frechet (uF) - algebra $U(B)$ of B is the Hausdorff completion of B in the family of uniform semi norms $|.|_p, p \in K_c(B)$.

The following describes a structural analogy between these two classes of algebras, which is a reflection of previous considerations.

Theorem 11.1. *(A)* [10],[12] *Let A be a Frechet algebra.Let $\{p_\alpha\}$ be a sequence of semi norms defining the topology of A.The following are equivalent.*

(1) A is an algebra with a C^*-enveloping

(2)A admits a greatest (automatically continuous) C^*-semi norm.

(3) The set $P_c(A)$ consisting of all continuous positive linear functionals f satisfying, for some α, $|f(x)| \leq p_\alpha(x)$ for all $x \in A$, is equicontinuous.

(4) The set $B_c(A)$ of all extreme points of $P_c(A)$ is equicontinuous.

(5) (in the commutative case)The hermitian Gelfand space $\triangle^h(A)$ is equicontinuous.

(6) Every *-representation of A on a Hilbert space maps A necessarily into bounded operators.

(B) [9] *Let A be a commutative Frechet algebra.The following are equivalent.*

(1) A is a Q-algebra.

(2) A admits a greatest continuous uniform semi norm.

(3) The Gelfand space $\triangle(A)$ is equicontinuous (equivalently compact).

(4) The Frechet algebra $U(A)$ is a Banach algebra.

An involutive topological Q - algebra is an algebra with a C^*-enveloping algebra, though the converse does not hold. It is believed that a hermitian Frechet algebra with a C^*-enveloping algebra is a Q-algebra. Notice that a hermitian Frechet algebra in which each self adjoint element has bounded spectrum is necessarily a Q-algebra [10]. Let A be a Frechet *-algebra. For $x \in A$, let the *hermitian spectrum* $sp^h_A(x)$ of x consists of all complex numbers λ such that for every irreducible *-representation (π, H_π) of A, the operator $-\lambda^{-1}\pi(x)$ is not quasi regular in $B(H_\pi)$. The *hermitian spectral radius* of x is $r^h(x) = \sup\{|\lambda| : \lambda \in sp^h_A(x)\}$; and the *hermitian Ptak function* is $s^h_A(x) = r^h_A(x^*x)^{1/2}$. A Banach *-algebra is hermitian iff for all $x \in A$, $sp^h_A(x) = sp_A(x)$.

Theorem 11.2. *(A)* [30] *Let A be a Frechet *-algebra. The following are equivalent.*

(1)A is an algebra with a C^*-enveloping algebra.

(2) There exists a continuous semi norm p on A and $k > 0$ such that $r^h(x) \leq kp(x)$ for all x.

(3) The number 0 is an interior point of $\{x \in A : \pi(x)$ is quasi regular in $B(H_\pi)$ for every irreducible *-representation $(\pi, H_\pi)\}$.

(B) [9] *Let A be a commutative Frechet algebra.The following are equivalent.*

(1) A is a Q - algebra.

(2) There exists a continuous semi norm p and a $k > 0$ such that $r(x) \leq kp(x)$ for all x.

(3) The number 0 is an interior point of the set of all quasi regular elements of A.

A uF -algebra is a uB-algebra iff it is a Q-algebra; similarly a $\sigma - C^*$-algebra is a C^*-algebra iff it is a Q-algebra.

(11.2)A uF - algebra is commutative; and is realized as an inverse limit of a sequence of uB-algebras. A $\sigma - C^*$- *algebra* is a Frechet *-algebra whose topology is defined by a sequence of C^*-semi norms. It is realized as an inverse limit of a sequence of C^*-algebras. Thus at the level of Frechet algebras, $\sigma - C^*$-algebras corresponds to uF-algebras in the present consideration. The uF-algebras have provided the correct frame work for capturing complex analytic structure. This is exhibited by the following, which is a companion to Theorems 8.2 and 8.3. A uF -algebra is a *strongly uniform algebra* if for each kernel ideal I, the quotient A/I is a uF - algebra with the quotient topology. The part (I) of the following is essentially due to Kramn.

Theorem 11.3. [59] *(I) Let A be a uniform Frechet algebra.*

(1) Assume that A is strongly uniform; the Gelfand space $\triangle(A)$ is locally compact;and that there exists an integer $n \in N$ such that for each $\phi \in \triangle(A)$, the Chevally dimension of ϕ in $\triangle(A)$ $\dim_\phi \triangle(A) = n < \infty$. Then $\triangle(A)$ can be given the structure of an n-dimensional Stein space so that A is homeomorphically isomorphic, via the Gelfand map, to the algebra $H(\triangle(A))$ of holomorphic functions on $\triangle(A)$ with compact open topology.

(2) Let A be strongly uniform and a Schwartz space. Let $\triangle(A)$ be locally compact and connected. Then A is homeomorphically isomorphic to $H(X)$ for a Stein manifold X iff there is $n \in N$ such that for each $\phi \in \triangle(A)$, ker ϕ is locally topologically n-generated.

(3) Let $\triangle(A)$ be locally compact and connected. Then A is homeomorphically isomorphic to $H(X)$ for a Riemann surface X iff ker ϕ is a principle ideal for all $\phi \in \triangle(A)$.

(II) Let A be a semi simple commutative Frechet algebra.

(1) If A is rationally singly generated, and if the Silov boundary of A is empty, then A is homeomorphically isomorphic to $H(G)$ for an open subset G in the complex plane.

(2) If A is singly generated; the Silov boundary of A is empty; and if for each $X \in A$, its image under Gelfand transform is unbounded; then Ais homeomorphically isomorphic to the algebra $H(\mathbb{C})$ of all entire functions on the complex plane $\mathbb{C}$.

A very important problem in Frechet algebras is the Michael Problem : Is every multiplicative linear functional on a Frechet algebra (equivalently on a uF-algebra) is continuous? A heroic effort [51] to solve this problem has led discovering its close connection with the deep Poincare-Fatau-Bieberbach phenomemon in Several Complex Variables. Notice that every * representation of a Frechet *-algebra into bounded Hilbert space operators is continuous.

(11.3) The determination of real smooth structure has turned out to be a difficult problem. Let M be a compact differential manifold. Let $C^\infty(M)$ be the algebra of smooth functions on M with the topology of uniform convergence on M of functions and all their derivatives.The algebra $C^\infty(M)$ determines M; and vice versa. The algebra $C^\infty(M)$ is a Frechet algebra (not a uniform Frechet algebra) sitting as a dense *-subalgebra of the C^*-algebra $C(M)$. It is not known when is a commutative Frechet algebra isomorphic to $C^\infty(M)$; though Allan Connes has recently obtained the following characterization of $C^\infty(M)$ in terms of geometric data involving a spectral triple. A *spectral triple* (A, H, D) consists of a (unital) *-algebra A of bounded linear operators on a Hilbert space H togather with a (generally unbounded) self adjoint operator D on H such that (1) for all $a \in A$, the commutator $[D, a]$ is bounded; and (2) for each λ not in the spectrum of D, $(\lambda 1 - D)$ has compact inverse. In Allain Connes philosophy [41, 52], a spectral triple represents a non commutative Riemannian geometric data.

Theorem 11.4. [39] *Let (A, H, D) be a spectral triple with A commutative satisfying the five axioms of Non-commutative Geometry* [40]*(in a slightly stronger form). Then there exists a compact oriented smooth manifold M such that $A = C^\infty(M)$.*

(11.4) This brings us to the formalism of Non commutative Differential Topology and Noncommutative Geometry [41, 52]. In the light of the fundamental Gelfand - Naimark Theorems on C^*-algebras, a non commutative unital C^*-algebra A represents a topological data - a virtual non commutative compact space, virtual in the sense the underlying space can not be represented in the language of set theory; non commutative in the sense that it is represented as a non commutative algebra. In analogy with the pair $(C^\infty(M), C(M))$, a differential structure on such a space is expected to be specified by a dense *-sub algebra of A which is a Frechet algebra admitting smoothness properties like spectral invariance, K - theory isomorphism, closure under appropriate functional calculi,derivation structure etc. On one hand, several concrete smooth subalgebras of specific C^*-algebras have been greatly analyzed; on the other hand, there have been efforts to develop a general theory of smooth sub algebras of C^* -algebras [36],[64],[27]. This may take us too far away into a terrain that is uncharted, at least from the point of view of the present paper.

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Notes added in proof

(1) In connection with theorem 5.2, Palmer[74] has noted, in the review of [58], that the universal constant $6e$ *can be improved to* $1 + 2e$*. this can be further improved 3 by a recent result due to Arhippainen and Muller, "Norms on unitization on Ba- nach algebras revisited," Acta Math. Hunger. 114 (3)(2007) 201-204.*

(2) By [16] A commutative Banach algebra A has UUNP if the unitization A has UUNP. On the other hand Barnes[4] has claimed that given a non unital Banch * *algebra B having* UC^*NP, B_e *has* C^*NP *iff* $C^*(B)$ *does not have identity, and he has applied this to construct an example of B having* UC^*NP *such that* B_e *does not have* UC^*NP*. On the other hand Dabhi and Dedania [43] proved that a non unital commutative B has* UC^*NP *iff* B_e *has* UC^*NP*. This is proved for a not necessarily commutative* B*, recently by Dedania and Kanani, "A non unital *-algebra has UC*NP iff its unitization has* UC^*NP*", preprint, showing a gap in the arguments of Barnes. All these give more insight in to the modest looking process of unitization.*

(3) The following illustrates (8.4)(iii), Yu. N. Kuznetsova, G. Molitor Brown, "Harmonic analysis of weighted Lp algebras", arXiv:1103.3610v1 [math.FA] 18 March 2011.

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A HASSE PRINCIPLE FOR QUADRATIC FORMS*

V. SURESH

Let k be any field with char $(k) \neq 2$. A *quadratic form* q over k is a homogeneous polynomial of degree 2 with coefficients in k. For a given quadratic form

$$q = \sum_{1 \leq i \leq j \leq n} a_{ij} X_i X_j, \quad a_{ij} \in k,$$

we attach a symmetric matric

$$S(q) = \begin{pmatrix} a_{11} & a_{12}/2 & . & a_{1n}/2 \\ . & a_{22} & . & a_{2n}/2 \\ . & . & . & . \\ . & . & . & a_{nn} \end{pmatrix}.$$

We say that q is *non-degenerate* if the determinant of $S(q)$ is non-zero. We say that two quadratic forms q_1 and q_2 are *isometric* if there exists a non-singular matric A with coefficients in k such that

$$S(q_2) = AS(q_1)A^t.$$

Since char$(k) \neq 2$, given a quadratic form q over k, we can find a non-singular matric A such that the matrix $AS(q)A^t$ is a diagonal matrix. Hence every quadratic form over k is isometric to a quadratic form $q(X_1, \cdots, X_n) = a_1X_1^2 + \cdots + a_nX_n^2$, with $a_i \in k$. Such a q is called a *diagonal* form and it is denoted by $< a_1, \cdots, a_n >$. From the definition it follows that a diagonal form $< a_1, \cdots, a_n >$ is non-degenerate if and only if all a_i's are non-zero. If q is a quadratic form in n variables, then n is also called the *dimension* of q.

Given two quadratic forms $q_1(X_1, \cdots, X_n)$ and $q_2(Y_1, \cdots Y_m)$, we define their *orthogonal* sum as the quadratic form $q(Z_1, \cdots Z_{n+m}) = q_1(Z_1, \cdots, Z_n) + q_2(Z_{n+1}, \cdots, Z_{n+m})$. The orthogonal sum of q_1 and q_2 is denoted by $q_1 \perp q_2$.

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We say that a quadratic form q is *isotropic* over k if there exists a non-zero vector $x = (x_1, \cdots x_n) \in k^n$ such that $q(x) = 0$. If q is not isotropic, then it is called *anisotropic*. For example the quadratic form $q_1(X, Y) = X^2 + Y^2$ is anisotropic over the field of real numbers $\mathbf{R}$ and the quadratic form $q_2(X, Y) = X^2 - Y^2$ is isotropic over any field. The quadratic form $X^2 - Y^2$ is denoted by $\mathbf{H}$ and is called the hyperbolic plane.

A classical theorem of Witt asserts that every quadratic form q is isometric to $q_{an} \perp \mathbf{H} \perp \cdots \perp \mathbf{H}$, where q_{an} is anisotropic and it is uniquely determined by q.

Thus to study the isometry classes of quadratic forms over a given field k, it is enough to study the isometry classes of anisotropic quadratic forms.

If k is a p-adic field or more generally a complete discrete valuated field, then it is easy to see when a given form is anisotropic over k. The classical Hasse-Minkowski theorem gives a method to determine when a given form over a number field is anisotropic.

In this article we explain this classical Hasse-Minkowski theorem and also the recent developments. We refer the reader to ([2], [4], [7]) for the theory of quadratic forms.

Hasse-Minkowski theorem

Let $\mathbf{C}$ be the field of complex numbers. Let q be a quadratic form over $\mathbf{C}$ in at least 2 variables. Then it is easy to see that q is isotropic over $\mathbf{C}$.

Let $\mathbf{R}$ be the field of real numbers. Then it is easy to see that every non-singular quadratic form q over $\mathbf{R}$ is isometric to $q = < \pm 1, \cdots, \pm 1 >$. The quadratic form q is isotropic if and only if there is at least one positive 1 and one negative 1.

Let $\mathbf{F}$ be a finite field of characteristic not equal to 2. Let q be a quadratic form over $\mathbf{F}$. Then it is not difficult to see that if the dimension of q is at least 3, then q is isotropic.

Let $\mathbf{Q}$ be the field of rational numbers and p a prime number. Let m be a non-zero integer. Then we can write $m = p^r m_1$ for some $r \geq 0$ and m_1 an integer which is not divisible by p. Let $v_p(m) = r$. We define a metric $|\cdot|_p$ on $\mathbf{Q}$ as follows :

$$\left|\frac{m}{n}\right|_p = \left(\frac{1}{p}\right)^{v_p(m) - v_p(n)}$$

The completion of $\mathbf{Q}$ with respect to the metric $|\cdot|_p$ is denoted by $\mathbf{Q}_p$ and $\mathbf{Q}_p$ is also a field. Let $\mathbf{Z}_p = \{a \in \mathbf{Q}_p \ : \mid a \mid_p \leq 1\}$. Then $\mathbf{Z}_p$ is an integral domain with field of fractions $\mathbf{Q}_p$. Any element of $\mathbf{Z}_p$ can be written as

$$a_0 + a_1p + a_2p^2 + a_3p^3 + \cdots$$

with $a_i \in \{0, 1, \cdots, p-1\}$. There is a unique maximal ideal $m_p = p\mathbf{Z}_p$ of $\mathbf{Z}_p$ and $\mathbf{Z}_p/m_p$ is the finite field $\mathbf{F}_p$ with p elements.

Using the fact that every quadratic form over a finite field in at least 3 variables is isotropic and Hensell's Lemma it follows that every quadratic form over $\mathbf{Q}_p$ in at least 5 variables is isotropic.

Let q be a quadratic form over $\mathbf{Q}$. We know that $\mathbf{Q} \subset \mathbf{R}$ and $\mathbf{Q} \subset \mathbf{Q}_p$ for all p. Suppose that q is isotropic over $\mathbf{Q}$, then it is isotropic over $\mathbf{R}$ and $\mathbf{Q}_p$ for all p. To decide whether a quadratic form is isotropic over $\mathbf{Q}$ or not is very difficult. The Hasse-Minowski's theorem gives a method to determine whether a quadratic form q over $\mathbf{Q}$ is isotropic or not. We have the following

Theorem 1. *(Hasse-Minkowski) A quadratic form over $\mathbf{Q}$ is isotropic if and only if it is isotropic over $\mathbf{Q}_p$ for all p and over $\mathbf{R}$.*

A *number field* is a finite extension of either $\mathbf{Q}$. There is a similar result for number fields also.

Let q be a quadratic form over $\mathbf{Q}$. Then q is isometric to $< m_1, \cdots, m_n >$, for some $m_i \in \mathbf{Z}$. Suppose the dimension of q is at least 5. Then we know that q is isotropic over $\mathbf{Q}_p$ for all p. If some $m_i = 0$, then q is isotropic. Assume that $m_i \neq 0$ for all i. To check q is isotropic or not it is enough to check over $\mathbf{Q}_p$ for all p and over $\mathbf{R}$. Since $n \geq 5$, q is isotropic over $\mathbf{Q}_p$ for all p. Thus it is enough to check over $\mathbf{R}$. We know that q is isotropic over $\mathbf{R}$ if and only if $m_i > 0$ and $m_j < 0$ for some i and j. Hence q is isotropic over $\mathbf{Q}$ if and only if $m_i > 0$ and $m_j < 0$ for some i and j.

Complete discrete valuated fields

Let K be a complete valuated field (for example $\mathbf{Q}_p$), R its ring of integers (e.g. $\mathbf{Z}_p$) and k its residue field (e.g. $\mathbf{F}_p$). Assume that the characteristic of k in not equal to 2. Let $\pi \in R$ be a parameter. Then every element in K^* can be written as $\pi^r u$ for some integer r and unit u in R. For any unit $u \in R$, let $\overline{u}$ denote its image in k.

Let q be a quadratic form over K. If q is not non-degenerate, then q is isotropic. Assume that q is non-degenerate. Then q is isometric to

$$< u_1, \cdots, u_r > \perp < v_1\pi, \cdots, v_s\pi >$$

for some units $u_i, v_j \in R$. Since R is a complete discrete valuation ring and the char$(k) \neq 2$, q is isotropic K if and only if either $< \overline{u}_1, \cdots, \overline{u}_r >$ or $< \overline{v}_1, \cdots, \overline{v_s} >$ is isotropic over k.

Thus if we know when a form is isotropic over k, we can say when a form is isotropic over K. If every quadratic form in at least $n+1$ variables over k is isotropic, then, every quadratic form in at least $2n+1$ variables over F has a non-trivial zero.

Hasse principle for rational function fields

It is interesting to look for a Hasse principle (local-global principle) for isotropy of quadratic forms over function fields. This has deep consequences. Let F be a field and Ω be a set of discrete valuations of F. For $v \in \Omega$, let F_v denote the completion of F at v.

We say that quadratic forms over F satisfy *Hasse Principle* with respect to Ω if every quadratic form q over F which is isotropic over F_v for all $v \in \Omega$, is isotropic over F.

Let k be a field and $F = k(t)$ be the rational function field in one variable over F. Let Ω_k be the set of all discrete valuations of F which are trivial on k. Elements of Ω_k correspond to monic irreducible polynomials in $k[t]$ and $\frac{1}{t}$. For $v \in \Omega_k$, we have $F_v = \ell((\pi))$, where $\ell = \kappa(v)$ is the residue field at v. i.e. if v corresponds to a monic irreducible polynomial $p(t) \in k[t]$, then $\kappa(v) = k[t]/(p(t))$ and if v corresponds to $\frac{1}{t}$ then $\kappa(v) = k$.

Let q be a quadratic form over $k(t)$.

1) Suppose $\dim(q) = 2$. After scaling, we may assume that $q = <1, -f(t)>$, for some $f(t) \in k[t]$. For any field F (of characteristic not 2), a quadratic form $<1, -\lambda>$ is isotropic if and only if λ is a square in F. Thus $q = <1, -f(t)>$ is isotropic if and only if $f(t) \in k(t)^{*2}$. It is not difficult to see that $f(t) \in k(t)^{*2}$ if and only if $f(t) \in k(t)_v^{*2}$ for all $v \in \Omega_k$. In particular q is isotropic over $k(t)$ if and only if q is isotropic over $k(t)_v$ for all $v \in \Omega_k$.

2) Suppose $\dim(q) = 3$. After scaling, we may assume that $q = <1, -f(t), -g(t)>$, $f(t), g(t) \in k(t)^*$.

Let F be any field of characteristic not equal to 2 and $\lambda, \mu \in F^*$. We have a quaternion algebra $H = H(\lambda, \mu)$: $i^2 = \lambda$, $j^2 = \mu$, $ij = -ji$. We know that $q = <1, -\lambda, -\mu>$ is isotropic if and only if the quaternion algebra $H(\lambda, \mu)$ is isomorphic to the matrix algebra $M_2(F)$.

Suppose that $q = <1, -f(t), -g(t)>$ is isotropic over $k(t)_v$ for all $v \in$. Let $H = H(f(t), g(t))$. Then $H \otimes k(t)_v \simeq M_2(k(t)_v)$ for all $v \in \Omega_k$. There are methods to show that in this case $H \simeq M_2(k(t))$. In particular q is isotropic.

Thus Hasse principle holds for 2 and 3 dimensional forms over $k(t)$ with respect to Ω.

There are examples due to Rowen-Sivatski-Tignol of rank 6 quadratic forms over $\mathbf{Q}_p(t)$ which are anisotropic over $\mathbf{Q}_p(t)$, but isotropic over $\mathbf{Q}_p(t)_v$ for all $v \in \Omega_{\mathbf{Q}_p}$ (cf. [1]).

In fact let $q = < -a, -p, ap, t, a(p-t), -at(p-t) >$, where $a \in \mathbf{Z}_p^*$ is a non-square and p is an odd prime(e.g. $p = 3$ and $a = -1$). Then q is anisotropic over $\mathbf{Q}_p(t)$, but isotropic over $\mathbf{Q}_p(t)_v$ for all $v \in \Omega_{\mathbf{Q}_p}$. Thus Hasse Principle fails for 6-dimensional forms over $\mathbf{Q}_p(t)$ with respect to $\Omega_{\mathbf{Q}_p}$.

The set $\Omega_{\mathbf{Q}_p}$ misses all discrete valuations of $\mathbf{Q}_p(t)$ extending the p-adic valuation on $\mathbf{Q}_p$. Let Ω be the set of all discrete valuations of $\mathbf{Q}_p(t)$.

Theorem 2. *(Colliot - Thélène - Parimala - Suresh) Hasse principle holds for quadratic forms over $F = k(t)$ with respect to Ω.*

As we have seen above for dimension 3 quadratic forms, given isotropic vectors over each completion we do not produce an isotropic vector over $\mathbf{Q}_p(t)$. However we use various techniques from Brauer groups, Galois groups, arithmetic geometry and patching techniques which were developed in last 60 years.

u-invariant

In 1953 Kaplansky defined an invariant attached to a field k, called the *u-invariant*, denoted by $u(k)$, as

$$u(k) = \sup \{ \dim(q) \mid \ q \text{ anisotropic quadratic form over k} \}.$$

As we mentioned above $u(\mathbf{C}) = 1$, $u(\mathbf{R}) = \infty$, $u(\mathbf{F}) = 2$ for a finite field $\mathbf{F}$ and $u(\mathbf{Q}_p) = 4$. In fact for any finite extension k of $\mathbf{Q}_p$, $u(k) = 4$.

Let k be a field and $k((t))$ the field of Laurent power series. Since $k((t))$ is a complete discrete valuated field, if $u(k) = N$, then $u(k((t))) = 2N$ (Hensel's Lemma).

A theorem of Tsen-Lang asserts that $u(\mathbf{C}((X_1))((X_2))\cdots((X_n))) = 2^n$ and $u(\mathbf{F}((X_1))((X_2))\cdots((X_n))) = 2^{n+1}$, for a finite field $\mathbf{F}$.

Question. Does $u(k) < \infty$ imply $u(k(t)) < \infty$?

This problem is open and seems to be very difficult for general fields. Let us look at the fields which are of interest from the point of view of Number Theory.

Kaplansky conjectured that If k is a p-adic field, then $u(k(t)) = 8$. Since $u(k) = 4$, it is easy to see that $u(k(t)) \geq 8$

Till late 90's it was not even known whether $u(\mathbf{Q}_p(t))$ is finite. Let K be a finite extension of $\mathbf{Q}_p(t)$. We have the following :

Merkurjev (1997) : $p \neq 2$, $u(K) \leq 26$.

Hoffmann-Van Geel(1997): $p \neq 2$, $u(K) \leq 22$.

Parimala-Suresh (1998): $p \neq 2$, $u(K) \leq 10$.

Theorem 3. *(Parimala - Suresh (2007)) Let $K/\mathbf{Q}_p(t)$ be a finite extension. If $p \neq 2$, then $\mathbf{u}(\mathbf{K}) = \mathbf{8}$.*

The proof of the above theorem uses many results from Brauer groups, Galois cohomology groups and Witt groups. The above theorem for $\mathbf{Q}_p(t)$ can also be deduced from the Hasse principle which we have mentioned before.

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REAL ELEMENTS AND SCHUR INDICES OF A GROUP*

AMIT KULSHRESTHA AND ANUPAM SINGH

Abstract: In this article we try to explore the relation between real conjugacy classes and real characters of finite groups at more refined level. This refinement is in terms of properties of groups such as strong reality and total orthogonality. In this connection we raise several questions and record several examples which have motivated those questions.

1. **Introduction**

Let G be a finite group. It is a basic statement in the character theory of finite groups that the number of irreducible complex characters is same as the number of conjugacy classes in G. Further this statement can be refined to the number of real irreducible characters being equal to the number of real conjugacy classes. However it is very well known that the real characters come from two kinds of representations: *orthogonal* and *symplectic* [21, Section 13.2 Prop. 39]. Schur himself explored this topic and now this is very closely related to Schur index computation. In last 10 years the computation of Schur indices have been almost completed for finite groups of Lie type (for example see [6]).

In this article we raise the question to further divide real conjugacy classes in two parts as to match the sizes of partitions with the number of orthogonal characters and symplectic ones. In section 2 we give basic definitions and ask

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Dedication: This article is dedicated to Mrs. Isha Gondhali.

several questions concerning relations between conjugacy classes and characters. In section 3 we define Schur indices of a representation. In section 4 we define canonical involution on a group algebra and mention known results on its restriction to the simple components. In the following section we provide examples which provide strength to the questions raised earlier. This is the main objective of this article. In the section 6 we mention the Lie algebra defined for a group algebra which makes use of real conjugacy classes. Results mentioned in this article are either calculations using computer algebra system GAP or have been collected from various sources related to real conjugacy classes. Some of the questions raised in this article are already known to experts (for example see [7]).

2. **Conjugacy Classes vs. Representations for a group**

Let G be a group. An element $g \in G$ is called *real* if there exists $t \in G$ such that $tgt^{-1} = g^{-1}$. An element $g \in G$ is called *involution* if $g^2 = 1$. An element $g \in G$ is called *strongly real* if it is a product of two involutions in G. Further notice that if an element is (strongly) real then all its conjugates are (strongly) real. Hence reality (i.e. being real) and strong reality are properties of conjugacy classes. The conjugacy classes of involutions and more generally strongly real elements are obvious examples of real classes. However converse need not be true.

Example 2.1. Take $G = Q_8$, the finite quaternion group. Then we see that $iji^{-1} = j^{-1}$ hence j is real but it is not strongly real.

A *representation* of a group G is a homomorphism $\rho : G \to GL(V)$ where V is a vector space over a field k. The representation ρ is called *irreducible* if V and $\{0\}$ are the only subspaces W of V satisfying $\rho(G).W \subseteq W$. Let V^* denote the dual vector space of V. The *dual representation* of ρ is the representation $\rho^* : G \to GL(V)$ given by $\rho^*(g) = {}^t\rho(g^{-1})$, where ${}^t\rho(g^{-1})$ is the transpose of $\rho(g^{-1})$. To a representation ρ its associated *character* $\chi : G \to k$ is defined by $\chi(g) = trace(\rho(g))$. In this section we only consider complex representations (i.e. $k = \mathbb{C}$). It is a classical theorem that the number of conjugacy classes in G is equal to the number of irreducible complex characters [21, Section 2.5 Theorem 7]. A theorem due to Brauer [13, Chapter 23] asserts that the number of real conjugacy classes is same as the number of irreducible real characters (i.e. the complex characters which take real values only).

Let $\rho : G \to GL(V)$ be an irreducible complex representation of G and χ be the associated character. If χ takes a complex value then the representation ρ is not isomorphic to its dual ρ^* and the vector space V can not afford a

G-invariant non-zero bilinear form. If χ is real then $\rho \cong \rho^*$ and in this case V admits a non-zero G-invariant bilinear form. This form can be either symmetric or skew-symmetric depending on whether (V, ρ) is defined over $\mathbb{R}$ or not (see [21, Section 13.2]). Equivalently the image of G sits inside O_n in the first case and inside Sp_{2n} in the second case. To each character χ one associates *Schur indicator* $\upsilon(\chi)$ which is defined as follows:

$$\upsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

In fact $\upsilon(\chi) = 0, \pm 1$ and $\upsilon(\chi) = 0$ if and only if χ is not real, $\upsilon(\chi) = 1$ if χ is orthogonal and symplectic otherwise [21, Prop. 39]. Hence the real characters come in two classes: orthogonal type and symplectic type. This gives a natural division of real characters. We now have following questions:

Question 2.2. Let G be a finite group.

(1) Can we naturally divide real conjugacy classes of G in two parts so that the number of one part is same as the number of orthogonal representations (we are specially interested in groups of Lie type)?
(2) Is it true that if a group G has no symplectic character (i.e. all self-dual representations are orthogonal) then all real elements are strongly real and vice versa?

A careful look at the examples in section 5 will suggest that these questions are indeed interesting to ask.

We now restrict our attention to those groups in which all elements are real. Such groups are called *real* or *ambivalent*. Tiep and Zalesski classify all real finite quasi-simple groups in [26]. For real groups all characters are real valued. Two interesting subclasses of the real groups are the following:

- Subclass in which all elements are strongly real. Groups belonging to this class are called *strongly real* groups.
- Subclass in which all characters are orthogonal. Groups belonging to this class are called *totally orthogonal* or *ortho-ambivalent*.

An element $g \in G$ is called *rational* if g is conjugate to g^i whenever g and g^i generate the same subgroup of G. A group is called *rational* if every element of g is rational.

Theorem 2.3. *(*[21] *13.1-13.2). A group G is rational if and only if all its characters are $\mathbb{Q}$-valued.*

In fact, the number of isomorphism classes of irreducible representations of G over $\mathbb{Q}$ is same as the number of conjugacy classes of cyclic subgroups of G.

Example 2.4. Let $G = S_n$, the symmetric group. Then every element of G is a product of two involutions and hence strongly real. It is also totally orthogonal. All its character are integer valued. In fact, it is a rational group.

Example 2.5. The group Q_8 has four 1-dimensional representation which are orthogonal and one symplectic representation. This group is neither strongly real nor totally orthogonal.

Let us denote the class of finite groups which are real by $\mathcal{R}$, the class of real groups which have their Sylow 2-subgroup Abelian by $\mathcal{S}$, the class of strongly real groups by $\mathcal{SR}$ and the class of totally orthogonal groups by $\mathcal{TO}$. Then using the results of [27] and [1] we have the following:

$$\mathcal{S} \subset \mathcal{SR} \subset \mathcal{R} \text{ and } \mathcal{S} \subset \mathcal{TO} \subset \mathcal{R}.$$

In particular they prove that if G is real then it is generated by its 2-elements and if G is totally orthogonal then it is generated by involutions. In view of above relations we ask the following question:

Question 2.6. Find the class $\mathcal{SR} \cap \mathcal{TO}$.

We know that the class $\mathcal{SR} \cap \mathcal{TO}$ contains $\mathcal{S}$. The containment though, is far from being equality. The dihedral group D_4 of order 8 is strongly real as well as totally orthogonal but its Sylow 2-subgroup, which is D_4 itself, is not Abelian.

It is therefore natural to ask which strongly real groups are totally orthogonal and vice versa. We have made many calculations using GAP and it seems the class $\mathcal{SR} \cap \mathcal{TO}$ is very close to the class $\mathcal{TO}$, though $\mathcal{SR}$ and $\mathcal{TO}$ are not identical. There is a group of order 32 which is strongly real but not totally orthogonal. This group G has the following properties:

(1) It is not simple. It has normal subgroups of all plausible orders -2, 4, 8 and 16.
(2) Exponent of G is 4.
(3) Derived subgroup of G of order 2.
(4) The group G is a semidirect product of $C_2 \times Q_8$ and C_2. Here C_2 denotes the cyclic group of order 2 and Q_8 denotes the quaternion group of order 8.
(5) This group has one 4-dimensional character and sixteen 1-dimensional characters. The 4-dimensional character assumes non-zero value only on one non-identity conjugacy class.
(6) It is an extra-special group of order 32.

It is worth noting that this is the only group of order 63 or smaller which is strongly real and not totally orthogonal. All totally orthogonal groups till this order are strongly real.

3. Schur Indices of groups

Let G be a finite group. Let k be a field such that $char(k)$ does not divide $|G|$. The *group algebra* of G over k is

$$kG = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in k \right\}$$

with operations defined as follows:

$$\sum_{g\in G} \alpha_g g + \sum_{g \in G} \beta_g g = \sum_{g\in G} (\alpha_g + \beta_g) g$$

$$\alpha \left(\sum_{g \in G} \alpha_g g \right) = \sum_{g \in G} \alpha \alpha_g g$$

$$\left(\sum_{g\in G} \alpha_g g \right) \cdot \left(\sum_{h \in G} \beta_h h \right) = \sum_{t \in G} \left(\sum_{g \in G} \alpha_g \beta_{g^{-1}t} \right) t$$

It is a classical result of Maschke that kG is a semisimple algebra. Hence by using Artin- Wedderburn theorem one can write it as a product of simple matix algebras over division algebras, i.e.,

$$kG \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where D_i's are division algebras over k with center, say L_i, a finite field extension of k. We know that D_i over L_i is of square dimension, say m_i^2. Further each simple component $M_{n_i}(D_i)$ corresponds to an irreducible representation of G over k. The number m_i is called the *Schur index* of the corresponding representation. One can write $1 = e_1 + \cdots + e_r$ using above decomposition where e_i's are idempotents. In fact,

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g.$$

where χ_i is the character of the representation corresponding to $M_{n_i}(D_i)$. For this reason often the Schur index is denoted as $m_k(\chi_i)$. The following are important questions in the subject:

Question 3.1.
(1) Find out Schur indices $m_k(\chi_i)$ and L_i for the representations of a group G.
(2) Determine the division algebras D_i appearing in the decomposition.

The problem of determining Schur indices for groups of Lie type has been studied extensively in the literature notably by Ohmori [17, 18], Gow [8, 9], Turull [24] and Geck [6]. Geck also gives a table of the all known results on page 21 in [6]. However answer to the second question is much more difficult, e.g., Turull does it for $SL_n(q)$ in [24]. The group algebra is well studied over field $k = \mathbb{C}, \mathbb{R}, \mathbb{Q}$ or $\mathbb{F}_q$. For example, $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, as there is only one finite dimensional division algebra over $\mathbb{C}$ which is $\mathbb{C}$ itself (so is over $\mathbb{F}_q$). Moreover the simple components in this decomposition correspond to a finite dimensional representation of G over $\mathbb{C}$. However in the case of $\mathbb{R}$ we know that the finite dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{H}$ and $\mathbb{C}$ hence the corresponding Schur index is 1; 2 or 1 respectively. In this case,

$$\begin{aligned}\mathbb{R}G \cong & M_{n_1}(\mathbb{R}) \times \cdots \times M_{n_l}(\mathbb{R}) \times M_{n_{l+1}}(\mathbb{H}) \times \cdots M_{n_{l+s}}(\mathbb{H}) \times \\ & M_{n_{l+s+1}}(\mathbb{C}) \times \cdots \times M_{n_{l+s+p}}(\mathbb{C}).\end{aligned}$$

In particular we see that the irreducible representations of G are of three kinds which are called orthogonal, symplectic and unitary as they can afford a symmetric bilinear form, an alternating form or a hermetian form respectively. These correspond to the Schur indicator $v(\chi)$ (defined in section 2) being 1, -1 or 0 respectively. Hence the question of calculating Schur indices over $\mathbb{R}$ is related to determining the types of representations: orthogonal, symplectic or unitary.

4. **Canonical Involution on the Group Algebra**

One can define an involution σ on kG as follows:

$$\sigma(\sum \alpha_g g) \mapsto \sum \alpha_g g^{-1}.$$

This involution is called the *canonical involution.* We can define a symmetric bilinear form $T : kG \times kG \to k$ by $T(x, y) = tr(l_{x\sigma(y)})$ where l_x is the left multiplication operator on kG and tr denotes its trace. We note that $tr(l_e) = n$ and $tr(l_g) = 0$ for $g \neq e$. Hence the elements of group G form an orthogonal basis and the form $T \simeq n < 1, 1, \cdots, 1 >$. The following result is proved in [20] (chapter 8 section 13): If the form $n < 1, 1, \cdots, 1 >$ is anisotropic over k (for example $k = \mathbb{R}$ or $\mathbb{Q}$) the involution σ restricts to each simple component of kG. In fact, in the case (ref. [2] theorem 2) G is real the involution σ restricts to each simple component of kG.

In general if $kG \cong A_1 \times A_2 \times \cdots \times A_r$ and $k = \mathbb{C}$ then either σ restricts to a component A_i or it is a switch involution on $A_i \times A_j$ where $A_i \cong A_j$. However when $k = \mathbb{R}$ the involution σ restricts to each component A_i, say σ_i, and is of either first type or second type. It is of the first type when A_i is $\cong M_n(\mathbb{R})$ or

$M_m(\mathbb{H})$ and of the second type when the component is isomorphic to $M_l(\mathbb{C})$. Moreover when we tensor this with $\mathbb{C}$ the first type gives the component over $\mathbb{C}$ the one to which σ restricts and the second type over $\mathbb{R}$ gives the one which correspond to the switch involution over $\mathbb{C}$.

Let us assume now that the canonical involution σ restricts to all components of kG, i.e., $(kG, \sigma) \cong \prod_i (A_i, \sigma_i)$ where A_is are simple algebras over k with involution. For example, this happens when $k = \mathbb{R}$ or when G is real. Algebras with involution (A, σ) have been studied in the literature (see [14]) extensively for its connection with algebraic groups. They are of two kinds: *involution of the first kind* is one which restricts to the center of A as identity and the *involution of second kind* restrict to the center of A as order 2 element. Further the involution of the first kind are of two types called *orthogonal type* and *symplectic type* the second kind is also called *unitary type*.

A group is called *ortho−ambivalent* with respect to a field k if the canonical involution σ restricts to all of its simple components as orthogonal involution (of first kind). The following results are proved in the thesis of Zahinda (ref. [29] Chapter 2): An ortho-ambivalent group is necessarily ambivalent and in fact it is totally orthogonal (see Proposition 2.4.2 in [29]). The notion of ortho-ambivalence over k is equivalent to ortho-ambivalence over $\mathbb{C}$. Further the question that which 2 groups are ortho-ambivalent is analysed.

5. **Some Examples**

Here we write down some examples and some GAP calculations we did.

5.1. **Symmetric and Alternating Groups.** Conjugacy classes in S_n are in one-one correspondence with partitions of n. Every conjugacy class in S_n is strongly real and hence real. All characters of S_n are real and moreover orthogonal.

Let $g \in A_n$. Then if $\mathcal{Z}_{S_n}(g) \subset A_n$ then $g^{S_n} = g^{A_n} \cup (xgx^{-1})^{A_n}$ where x is an odd permutation. In case $\mathcal{Z}_{S_n}(g) \not\subset A_n$ then $g^{S_n} = g^{A_n}$. In [19], Parkinson classified real elements in A_n. Let $n = n_1 + n_2 + \cdots + n_r$ be a partition and C be a conjugacy class corresponding to that in S_n contained in A_n. Then, C is non-real in A_n if and only if

(1) each n_i distinct,
(2) each n_i odd and
(3) $\frac{1}{2}(n - r)$ is odd.

And hence the number of conjugacy classes in A_n is equal to the number of real even partitions + twice the number of non-real even partitions. In fact (see [3, 19], A_n is ambivalent if and only if $n = 1, 2, 5, 6, 10, 14$.

For example in the case of $n = 7$, the partitions are given by $1^7, 1^5 2, 1^4 3$, $1^3 2^2, 1^2 23, 13^2, 2^2 3, 34, 25, 124, 16, 7, 1^2 5, 1^3 4, 12^3$. Out of which 1^7, $1^4 3$, $1^3 2^2$, 13^2, $2^2 3$, 124, 7, $1^2 5$ correspond to elements in A_7. By above criteria the only non-real class corresponds to the partition 7. Hence there are 7 real conjugacy classes in A_7 out of total $7 + 2.1 = 9$ conjugacy classes. Using GAP we verified following statements about A_7.

(1) All but one conjugacy classes are real.
(2) All real conjugacy classes are strongly real.
(3) All real characters are orthogonal.

We summarise some GAP calculations below for A_n:

n	total classes	real classes	st real	orthogonal characters	symplectic characters	unitary characters
5	5	5	5	5	0	0
6	7	7	7	7	0	0
7	9	7	7	7	0	2
8	14	10	10	10	0	4
9	18	16	16	16	0	2
10	24	24	24	24	0	0
14	72	72	72	72	0	0

In [23] section 3, Suleiman proved that in alternating groups an element is real if and only if it is strongly real. Moreover in A_n every real character is orthogonal, i.e., the Schur index of A_n is 1. This follows from a work of Schur which is quoted in a paper of Turull (see [25], Theorem 1.1). This result of Schur says that the Schur index of every ireducible representation of A_n (for each n) is 1. Thus A_n has no symplectic characters. Hence for A_n,

$$|\text{strongly real classes}| = |\text{real classes}| = |\text{real characters}| = |\text{orthogonal characters}|.$$

Here the first equality is from Suleiman, second is obvious and third one is the Schur index computation by Schur himself.

Now one can ask similar questions for the covers of these groups namely $\tilde{S}_n$ and $\tilde{A}_n$. We summarise some GAP calculations below for $\tilde{A}_n$:

n	total	real classes classes	st real	orthogonal characters	symplectic characters	unitary characters
4	7	3	2	2	1	4
5	9	9	2	5	4	0

We summarise some GAP calculations below for $\tilde{S}_n$:

n	total	real classes classes	st real	orthogonal characters	symplectic characters	unitary characters
4	8	6	6	6	0	2
5	12	8	6	7	1	4

Here we see an example of group, say the Schur cover of A_5 for which there are only 2 strongly real classes and all 9 real classes, while it has 5 orthogonal characters and remaining 4 symplectic ones.

5.2. $GL_n(q)$**.** The group $GL_n(q)$ has the property that all real elements are strongly real (ref. [28]). It doesn't have irreducible symplectic representations, i.e., all self-dual irreducible representations are orthogonal ([5] Theorem 4). In [11] and [12] the precise number of the real elements are calculated. In [16], Macdonald gives an easy way to enumerate conjugacy classes.

Theorem 5.1 (Macdonald)**.** *Conjugacy classes in $GL_n(q)$ are in one-one correspondance with a sequence of polynomials $u = (u_1, u_2, \cdots)$ satisfying:*

(1) *a partition of n, $\upsilon = 1^{n_1}2^{n_2}\cdots$, i.e., $|\upsilon| = \sum_i in_i = n$,*
(2) *$u_i(t) = a_{n_i}t^{n_i} + \cdots + a_1t + 1 \in \mathbb{F}_q[t]$ for all i with $a_{n_i} \neq 0$.*

Hence the number of conjugacy classes in $GL_n(q)$ is

$$\sum_{\{\upsilon:|\upsilon|=n\}} c_\upsilon = \sum_{\{\upsilon:|\upsilon|=n\}} \prod_{n_i>0} (q^{n_i} - q^{n_i-1}).$$

Theorem 5.2 ([11])**.** *Real conjugacy classes in $GL_n(q)$ are in one-one correspondance with a sequence of polynomials $u = (u_1, u_2, \cdots)$ satisfying:*

(1) *a partition of n, $\upsilon = 1^{n_1}2^{n_2}\cdots$, i.e., $|\upsilon| = \sum_i in_i = n$,*
(2) *$u_i(t) = a_{n_i}t^{n_i} + \cdots + a_1t + 1 \in \mathbb{F}_q[t]$ for all i with $a_{n_i} \neq 0$.*
(3) *$u_i(t)$ self-reciprocal.*

Hence the number of real conjugacy classes in $GL_n(q)$ is

$$\sum_{\{\upsilon:|\upsilon|=n\}} \prod_{n_i>0} n_{q,n_i}$$

where n_{q,n_i} is the number of polynomials $u_i(t)$ of above kind of degree n_i over field $\mathbb{F}_q$.

Hence in $GL_n(q)$ the number of strongly real elements is same as the number of orthogonal characters.

5.3. $SL_2(q)$**.** In this case if q is even all $q+1$ classes are real as well as strongly real. If q is odd, there are exactly 2 strongly real classes. In the case $q \equiv 1$ $(mod\ 4)$ all $q+4$ classes are real and if $q \equiv 3$ $(mod\ 4)$ only q out of $q+4$ are real (in fact, exactly unipotent ones are not real). Hence we can say that the groups $SL_2(q)$ are ambivalent if and only if q is a sum of (at most) two squares.

In the case q is even all characters are orthogonal. However if q is odd there is always a symplectic character. One can refer to the calculations of Schur indices in [22] Theorem 3.1. Hence one can conclude that the group $SL_2(q)$ is ortho-ambivalent if and only if q is even.

5.4. $SL_n(q)$**.** The real and strongly real classes are calculated in [11]. Turull calculated Schur indices of characters of $SL_n(q)$ in [24] over $\mathbb{Q}$ and also determined the division algebras appearing in the decomposition of group algebra. The following is known from the work of Ohmori, Gow, Zelevinsky, Turull, Geck etc. For the group $SL_n(q)$ if n is odd or $n \equiv 0 \ (mod\ 4)$ or $|n|_2 > |p-1|_2$ then all real characters are orthogonal. In the case $2 \leq |n|_2 \leq |p-1|_2$ there are symplectic representations.

5.5. **Orthogonal Group.** In [28], Wonenburger proved that every element of orthogonal group is a product of two involutions. Hence these groups are strongly real. In [10], Gow proved (Theorem 1) that all characters of $O_n(q)$ are orthogonal.

6. The Lie Algebra $\mathcal{L}(G)$

Since kG is an associative algebra we can define $[x, y] = xy - yx$ which makes it a Lie algebra. The subspace $\mathcal{L}(G)$ generated by $\hat{g} = \{g - g^{-1} | g \in G\}$ is Lie subalgebra. This Lie algebra associated to a finite group is studied in [4] and called Plesken Lie algebra. They prove,

Theorem 6.1. *The Lie algebra $\mathcal{L}(G)$ admits the decomposition:*

$$\mathcal{L}(G) \cong \bigoplus_{\chi \in \mathbb{R}} o(\chi(1)) \oplus \bigoplus_{\chi \in \mathbb{H}} sp(\chi(1)) \oplus {\bigoplus_{\chi \in \mathbb{C}}}' gl(\chi(1))$$

where the sums are over different kind of irreducible characters (with obvious meaning) and the last sum is $'$ed meaning we have to take only one copy of $gl(\chi(1))$ for χ and χ^{-1}.

They also prove that the Lie algebra $\mathcal{L}(G)$ is semisimple if and only if G has no complex characters and every character of degree 2 is of symplectic type, i.e., G is ambivalent and every non-linear character is symplectic. They also classify when $\mathcal{L}(G)$ is simple.

7. Group Algebra and Real Characters

We have $\mathbb{R}G$ a group algebra with involution σ which restricts to each simple component of it. On one hand we see that $\mathcal{Z}(\mathbb{R}G) = \oplus_g \mathbb{R}c_g$ where sum on the right hand side is over conjugacy classes and $c_g = \sum_{t\in G} g^t$ and on other hand we have $\mathcal{Z}(\mathbb{R}G) \cong \oplus\mathcal{Z}(M_n(D)) \equiv \oplus_{\chi\in\mathbb{R}}\mathbb{R}\oplus_{\chi\in\mathbb{C}}\mathbb{C}\oplus_{\chi\in\mathbb{H}}\mathbb{R}$. We know that center of a semisimple algebra is an étale algebra and σ restricts to it. In fact, in the first situation we have $\mathcal{Z}(\mathbb{R}G) \equiv \oplus_{g\sim g^{-1}}\mathbb{R}c_g \oplus (\mathbb{R}c_g \oplus \mathbb{R}c_{g^{-1}})$ where on the first component σ restricts as trivial involution and on the second component it becomes as a switch involution (hence of the second kind). In the second isomorphism we know that σ restricts to the trivial map on the $\mathbb{R}$ and $\mathbb{H}$ components and is of the second kind on $\mathbb{C}$ components. Hence counting the components where σ restricts as first kind gives us the number of real conjugacy classes is same as the number of real plus symplectic representations.

However applying the same trick to $\mathbb{Q}G$ doesn't give the corresponding result regarding rational representations. As it will also count the odd degree field extensions of $\mathbb{Q}$. It will be interesting to find such a proof.

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THE ELEMENT SPLITTING OPERATION FOR GRAPHS, BINARY MATROIDS AND ITS APPLICATIONS *

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Abstract: The element splitting operation is a useful technique in graph theory and has important applications. We take the review of these results. This operation can be extended from graphs to binary matroids. The formulation of this operation in binary matroids is considered here. We highlight few significant properties of this operation in matroids and give some of its applications.

1. **The Element Splitting Operation On Graphs (Slater [10])**

Let G be a graph and u be a vertex of G with deg $u \geq 2n-2$. Let H be the graph obtained from G by replacing u by two adjacent vertices u_1 and u_2, and if x is adjacent to u in G, written x adj u, then make x adj u_1 or x adj u_2 (but not both) such that deg $u_1 \geq n$ and deg $u_2 \geq n$. The transition from G to H is called an "n-element splitting operation".

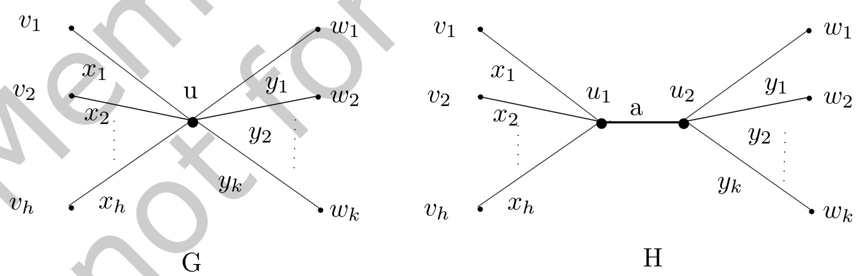

Figure 1. Graph G and its splitting

Key words and phrases: Graph, splitting operation, binary matroid, graphic matroid.

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Let $T = \{x_1, x_2, ..., x_h\}$, the set of edges incident at u_1. We denote H by G_T. Note that if G is connected then the new graph is connected.
The connectivity of a graph G is the minimum number of vertices whose removal results in a disconnected or a trivial graph. A gragh is n-connected if its connectivity is at least n.

Theorem 1.1. **[10]** *If G is n-connected and H arises from G by n-element splitting, then H is n-connected.*

The n-connected graphs ($n \geq 1$) can be characterized in terms of the element splitting operation. Addition of an edge in a graph means adding an edge between two non-adjacent vertices. Subdivision of an edge means replacing the edge by a path of length two.

Now consider the classes of 1-connected, 2-connected, and 3-connected graphs.

Theorem 1.2. **[10]** *The class of 1-connected graphs is the class of graphs obtained from K_2 by finite sequences of edge addition and 1-element splitting.*

Hedetniemi [5] proved the following constructive characterization of the two connected graphs.

Theorem 1.3. *A graph is 2-connected if and only if it can be obtained from K_3 by a finite sequence of subdivisions, vertex additions and edge additions.*

Slater [10] obtained the following result.

Theorem 1.4. *The class of 2-connected graphs is the class of graphs obtained from K_3 by finite sequences of edge addition and 2-element splitting.*

Tutte [15] characterized 3-connected graphs in terms of edge additions and 3-element splitting. For $n \geq 4$ the wheel W_n is defined to be the graph $K_1 + C_{n-1}$.

Theorem 1.5. **[15]** *A graph is* 3*-connected if and only if G is a wheel or can be obtained from a wheel by a sequence of operations of edge addition and* 3*-element splitting.*

The following figure shows that the graph G can be obtained from W_5 using the above two operations.

Slater [10], proved that the class of 3-connected graphs is the class of graphs obtained from K_4 by finite sequence of edge addition, 3-element splitting and 3-line splitting. Further, he gave classification of 4-connected graphs in terms of 4-element splitting and some other operations (see[10]).

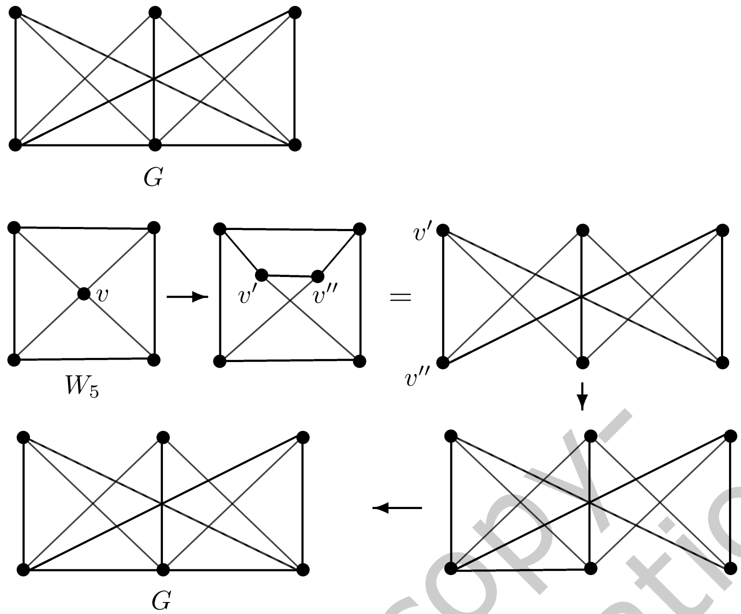

FIGURE 2. Illustration for Tutte's theorem

2. Basic Concepts in Matroid Theory

We generalize the notion of the element splitting operation on graphs to binary matroids. The study of matroids is a branch of discrete mathematics with basic links to graphs, lattices, codes and projective geometry. They are of fundamental importance in combinatorial optimization and their applications extend into electrical engineering and statics. Matroids can be defined in many different but equivalent ways (see [7]).

A *matroid* M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying
(1) $\phi \in \mathcal{I}$
(2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$
(3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

If M is a matroid $(E, \mathcal{I})$, then M is called a matroid on E. The members of $\mathcal{I}$ are called the independent sets of M, and E is the ground se of M. A maximal independent set in M is called a basis of M. A subset of E that is not in $\mathcal{I}$ is called dependent. A minimal dependent set in M will be called a circuit of M.

We can associate matroids to matrices and graphs in the following ways. Let A be an $m \times n$ matrix over a field F, and E be the set of column labels of A. Let $\mathcal{I}$ be the set of subsets X of E for which the set of columns labelled by X is linearly independent in the vector space $V(m, F)$. Then $(E, \mathcal{I})$ is a matroid. This matroid is called the *vector matroid* of A and it is denoted by $M[A]$. Let E be the set of edges of a graph G and let $\mathcal{I} = \{X \subseteq E(G) | X \text{ is cycle free }\}$. Then the pair $(E(G), \mathcal{I})$ is a matroid. This matroid is called the cycle matroid of G and is denoted by $M(G)$.

Two matroids M_1 and M_2 are isomorphic, written $M_1 \cong M_2$, if there is a bijection ψ from $E(M_1)$ to $E(M_2)$ such that, for all $X \subseteq E(M_1)$, $\psi(X)$ is independent in M_2 if and only if X is independent in M_1.

If a matroid M is isomorphic to the vector matroid of a matrix A over a field F, then M is said to be representable over F and A is called a representation for M over F. A matroid M is said to be binary if it is representable over the field $GF(2)$. A matroid M is called *graphic* if M is isomorphic to the cycle matroid of a graph.

Let $M = (E, \mathcal{I})$ be a matroid and $X \subseteq E$. Then the matroid $M \backslash X = (E - X, \mathcal{I}\,')$ where $I\,' = \{Y \mid Y \subseteq E - X,\ Y \in \mathcal{I}\}$ is called the deletion of X from M. If $X \subseteq E$ and B_0 is a maximal independent subset of X. Then the matroid $M/X = (E - X, \mathcal{I}\,')$ where $\mathcal{I}\,' = \{Y \mid Y \subseteq E - X,\ Y \cup B_0 \in \mathcal{I}\}$ is called the contraction of X from M.

A minor of M is a matroid obtained from it by a sequence of deletions and contractions of subsets of $E(M)$. A matroid M is said to be connected if for every pair of distinct elements of $E(M)$, there is a circuit containing both.

The operations of subdivision of an edge and addition of an edge have been extended to matroids as follows: A matroid M is a single element extension of a matroid N if M has an element e such that $M \setminus e = N$. M is a series extension of N if M has a 2-elemnt cocircuit $\{x, y\}$ such that M/x is N. Shikare and Waphare [13] extended Hedetniemi's theorem on graphs to matroids. Indeed, they proved the following theorem.

Theorem 2.1. **[13]** *Let M be a nonempty matroid that is not a coloop. Then M is connected if and only if M is a loop or can be obtained from a loop by a sequence of operations each consisting of a single-element extension or a series extension.*

3. **Element Splitting operation for binary matroids [1]**

The element splitting operation for binary matroids is defined in terms of the matrices representing the matroids.

Let M be a binary matroid on aset E and let A be a matrix over $GF(2)$ that represents M. Suppose that T is a subset of $E(M)$. Let A_T be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to the elements of T where it takes the value 1, and then adjoining an extra column (corresponding to a) with this column being zero everywhere except in the last row where it takes the value 1. Suppose M_T be the vector matroid of the matrix A_T. The transition from M to M_T is called the element splitting operation.

Proposition 3.1. *(i) Let $r(M)$ denote the rank of M. Then $r(M_T) = r(M) + 1$.*
(ii) Suppose T and U are two subsets of E. Then $(M_T)_U = (M_U)_T$.
(iii) If T and U are disjoint subsets of E then $(M \setminus U)_T = (M_T) \setminus U$.

4. Bases of the Matroid M_T

The following theorem characterizes the bases of the matroid M_T in terms of the bases of the matroid M.

Theorem 4.1. [1] *A subset B' of E is a base of M_T if and only if one of the following two conditions hold:*
(1) $B' = B \cup \{a\}$ where B is a base of M.
(2) $B' = B \cup \alpha$ where B is a base of M, $\alpha \in E - B$ and the fundamental circuit of M contained in $B \cup \{\alpha\}$ contains an odd number of elements of T.

The element splitting operation may not preserve the connectedness of the binary matroid M. In the following theorem, we provide a sufficient condition for the splitting operation to preserve the connectedness of the matroid.

Theorem 4.2. *Let M be a connected binary matroid on a set E and $T \subseteq E$ such that there is a circuit of M containing an odd number of elements of T. Then the matroid M_T is connected.*

5. The Element splitting operation on Graphic matroids

The element splitting operation on a graphic matroid may not yield a graphic matroid. For $|T| = 2$, Dalvi, Borse and Shikare [2] proved the following theorem.

Theorem 5.1. *The element splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if it has no minor isomorphic to $M(K_4)$ where K_4 is the complete graph on 4 vertices.*

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MODIFIED HERMITE POLYNOMIALS OF SEVERAL VARIABLES

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Abstract: In this paper modified Hermite polynomials (MHP) and their generalizations in two and three variables are given. Many interesting results of MHP of three variables have been obtained. Finally we extend MHP of three variables to MHP of four variables.

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Key words and phrases: Hermite polynomials, modified Hermite polynomials.

1. Introduction

In 1998, Khan and Abukhammash [2] defined Hermite polynomials of two variables $H_n(x, y)$ suggested by S.F. Ragab's Laguerre polynomials of two variables, as follows:

$$H_n(x, y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{n!(-y)^r H_{n-2r}(x)}{r!(n-2r)!} \tag{1.1}$$

The definition (1.1) is equivalent to the following explicit representation of $H_n(x, y)$:

$$H_n(x, y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{n}{2}-r\right]} \frac{(-n)_{2r+2s}(2x)^{n-2r-2s}(-y)^r(-1)^s}{r!s!} \tag{1.2}$$

The generating function for $H_n(x, y)$ is as follows:

$$\exp(2xt - (y + 1)t^2) = \sum_{n=0}^{\infty} \frac{H_n(x, y)t^n}{n!} \tag{1.3}$$

We consider the modified form of Hermite polynomials (MHP) $H_n^\alpha(x)$ defined by the generating function

$$\exp(\alpha x t - t^2) = \sum_{n=0}^{\infty} \frac{H_n^\alpha(x) t^n}{n!}, \tag{1.4}$$

from which $H_n^\alpha(x)$ can be obtained in terms of the following series

$$H_n^\alpha(x) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r (\alpha x)^{n-2r}}{r!(n-2r)!} \tag{1.5}$$

Now we consider MHP of two variables defined as follows:

$$H_n^{\alpha,\beta}(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-\beta y)^r H_{n-2r}^\alpha(x)}{r!(n-2r)!} \tag{1.6}$$

from which the following generating function for $H_n^{\alpha,\beta}(x,y)$ can be obtained:

$$\exp(\alpha x t - (1+\beta y)t^2) = \sum_{n=0}^{\infty} \frac{H_n^{\alpha,\beta}(x,y) t^n}{n!}. \tag{1.7}$$

It may be noted that for $\alpha = 2$, $\beta = 1$ MHP of two variables $H_n^{\alpha,\beta}(x,y)$ reduces to Hermite polynomials of two variables $\mathbf{H}_n(x,y)$ due to Khan and Abukhammash [2]. Thus, we obtain

$$H_n^{2,1}(x,y) = H_n(x,y) \tag{1.8}$$

Next we consider MHP of three variables defined as follows:

$$H_n^{\alpha,\beta,\gamma}(x,y,z) = n! \sum_{r=0}^{\left[\frac{n}{3}\right]} \frac{(\gamma z)^r H_{n-3r}^{\alpha,\beta}(x,y)}{r!(n-3r)!}, \tag{1.9}$$

The generating function for $H_n^{\alpha,\beta,\gamma}(x,y,z)$ is

$$\sum_{n=0}^{\infty} \frac{H_n^{\alpha,\beta,\gamma}(x,y,z) t^n}{n!} = \exp\left(\alpha x t - (\beta y + 1)t^2 + \gamma z t^3\right) \tag{1.10}$$

Some other generating function for $H_n^{\alpha,\beta,\gamma}(x,y,z)$ are as follows

1. $$\sum_{n=0}^{\infty} (C)_n \frac{H_n^{\alpha,\beta,\gamma}(x,y,z) t^n}{n!} = (1-\alpha x t)^{-C} F_{0,0,0}^{3,0,0} \left[\begin{array}{ll} (C,2,3,2) :: - : - & ; \\ - \quad :: - : - & ; \end{array} -\frac{-\beta y t^2}{(1-\alpha x t)^2}, -\frac{t^2}{(1-\alpha x t)^2}, \frac{\gamma z t^3}{(1-\alpha x t)^3} \right] \tag{1.11}$$

2. $$\sum_{n=0}^{\infty} \frac{H_{n+k}^{\alpha,\beta,\gamma}(x,y,z) t^n}{n!} = \exp\left[\alpha x t - (1+\beta y)t^2 + \gamma z t^3\right] \times H_k^{\alpha,\beta,\gamma}\left(x - \frac{2(1+\beta y)t}{\alpha} + \frac{3\gamma z t^2}{\alpha}, y - \frac{3\gamma z t}{\beta}, z\right) \tag{1.12}$$

3. $\sum_{s,n=0}^{\infty} \frac{H_{n+r+s}^{\alpha,\beta,\gamma}(x,y,z)t^n u^s}{s!n!} =$

$$\exp\left[\alpha xt - 2(1+\beta y)ut + 3\gamma zu^2t + 6\gamma zuvt - (1+\beta y)t^2 + \gamma zt^3\right]$$
$$\times \exp\left[\alpha xu - 2(1+\beta y)uv + 3\gamma zt^2u - (1+\beta y)u^2 + \gamma zu^3\right]$$
$$\times \sum_{r=0}^{\infty} H_r^{\alpha,\beta,\gamma}\left(x - \frac{2(1+\beta y)t}{\alpha} + \frac{3\gamma zt^2}{\alpha} + \frac{3\gamma zu^2}{\alpha},\right.$$
$$\left. y - \frac{3\gamma zu}{\beta} - \frac{3\gamma zt}{\beta}, z\right) \tag{1.13}$$

and the partial diferential equation for $H_n^{\alpha,\beta,\gamma}(x,y,z)$ is

$$\left(x\frac{\partial}{\partial x} - \frac{2(1+\beta y)}{\alpha^2}\frac{\partial^2}{\partial x^2} - \frac{3\gamma z}{\alpha\beta}\frac{\partial^2}{\partial x \partial y} - n\right) H_n^{\alpha,\beta,\gamma}(x,y,z) = 0 \tag{1.14}$$

In this paper we consider the modified Hermite polynomials (MHP) of several variables. Section 2 deals with some special properties of three variables modified Hermite polynomials (3VMHP). Section 3, gives the relationship of 3VMHP with modified Legendre plynomials. Finally in Section 4, some concluding remarks are given.

2. Special Properties of 3VMHP $H_n^{\alpha,\beta,\gamma}(x,y,z)$

I. $\sum_{n=0}^{\infty} H_n^{\alpha,\beta,\gamma}\left((x_1+x_2+x_3),y,z\right)$

$$= \sum_{r=0}^{n}\sum_{s=0}^{(n-r)} \frac{n! H_{n-r-s}^{\alpha,\beta,\gamma}(x_1,y,z)\, H_r^{\alpha,\beta,\gamma}(x_2,y,z)\, H_s^{\alpha,\beta,\gamma}(x_3,-y,-z)}{(n-r-s)!r!s!} \tag{2.1}$$

II. $H_n^{\alpha,\beta,\gamma}\left(x,(y_1+y_2+y_3),z\right) H_k^{\alpha}(2x)$

$$= \sum_{r=0}^{n}\sum_{s=0}^{(n-r)} \frac{n! H_{n-r-s}^{\alpha,\beta,\gamma}(x,y_1,z)\, H_r^{\alpha,\beta,\gamma}(x,y_2,z)\, H_s^{\alpha,\beta,\gamma}(x,y_3,-z)}{(n-r-s)!r!s!} \tag{2.2}$$

III. $H_n^{\alpha,\beta,\gamma}\left(x,y,(z_1+z_2+z_3)\right)$

$$= \sum_{r=0}^{n}\sum_{s=0}^{(n-r)} \frac{n! H_{n-r-s}^{\alpha,\beta,\gamma}(x,y,z_1)\, H_r^{\alpha,\beta,\gamma}(x,y,z_2)\, H_s^{\alpha,\beta,\gamma}(-x,-y,z_3)}{(n-r-s)!r!s!} \tag{2.3}$$

IV.$H_n^{\alpha,\beta,\gamma}(x,y,z)$

$$= \sum_{k=0}^{[\frac{n}{2}]} \sum_{r=0}^{[\frac{n-3r}{2}]} \frac{n! H_{n-3r-2k}^{\alpha,\beta,\gamma}(1,y,z) x^{n-3r-2k} (1+\beta y)^k (x^2-1)^k \left[\gamma z(1-x^3)\right]^r}{(n-3r-2k)!\ k!\ r!} \tag{2.4}$$

$$\mathbf{V}.H_n^{\alpha,\beta,\gamma}(\lambda x,y,z) = \sum_{k=0}^{n} \frac{n! H_{n-k}^{\alpha,\beta,\gamma}(x,y,z)(\alpha x)^k(\lambda-1)^k}{(n-k)!\ k!} \tag{2.5}$$

$$\mathbf{VI}.H_n^{\alpha,\beta,\gamma}(x,\lambda y,z) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n! H_{n-2k}^{\alpha,\beta,\gamma}(x,y,z)(1-\lambda)^k(\beta y)^k}{(n-2k)!\ k!} \tag{2.6}$$

$$\mathbf{VII}.H_n^{\alpha,\beta,\gamma}(x,y,\lambda z) = \sum_{k=0}^{[\frac{n}{3}]} \frac{n! H_{n-3k}^{\alpha,\beta,\gamma}(x,y,z)(\lambda-1)^k(\gamma z)^k}{(n-3k)!\ k!} \tag{2.7}$$

VIII.$H_n^{\alpha,\beta,\gamma}(\lambda x,\mu y,\theta z)$

$$= \sum_{r=0}^{n} \sum_{r=0}^{[\frac{n-3r}{2}]} \frac{n! H_{n-3r-2s}^{\alpha,\beta,\gamma}(x,y,z)\, H_r^{\alpha,\beta,\gamma}((\lambda-1)x,(\mu-1)y,(\theta-1)z)}{(n-3r-2s)!\ k!} \tag{2.8}$$

3. Relationship with Legendre Polynomials

Curzon [1] in 1913, obtained many relations connecting the Hermite polynomials $H_n(x)$ and the Legendre polynomials $P_n(x)$ with n usually not restricted to be integral. One of the simplest of his relation in which n is to be an integer is [3]

$$P_n(x) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp(-t^2) t^n H_n(xt) dt \tag{3.1}$$

In 1998 Khan and Abukhammash extended this relation to two variables as follows [2]

$$P_n(x,y) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp(-t^2) t^n H_n(xt, \frac{y}{t^2}) dt \tag{3.2}$$

which in terms of modified Hermite polynomials can be defined as follows

$$P_n^{\alpha,\beta}(x,y) = \frac{\alpha}{n!\sqrt{\pi}}\int_0^\infty \exp(-t^2)t^n H_n^{\alpha,\beta}(xt, \frac{\beta y}{t^2})dt \tag{3.3}$$

where $P_n^{\alpha,\beta}(x,y)$ is the Modified Legendre polynomial of two variables.

These polynomials $P_n^{\alpha,\beta}(x,y)$ can be defined as

$$P_n^{\alpha,\beta}(x,y) = \sum_{r=0}^{[\frac{n}{2}]} \frac{(-\beta y)^r}{r!} P_{n-2r}^{\alpha}(x) \tag{3.4}$$

Since by (1.9)

$$H_n^{\alpha,\beta,\gamma}(x,y,z) = \sum_{r=0}^{[\frac{n}{3}]} \frac{n!\ (rz)^r\ H_{n-3r}^{\alpha,\beta}(x,y)}{r!\ (n-3r)!}$$

We have

$$\frac{\alpha}{n!\sqrt{\pi}}\int_0^\infty \exp(-t^2)t^n H_n^{\alpha,\beta,\gamma}\left(xt, \frac{\beta y}{t^2}, \frac{\gamma z}{t^3}\right)dt$$

$$= \frac{\alpha}{n!\sqrt{\pi}}\int_0^\infty \exp(-t^2)t^n \sum_{r=0}^{[\frac{n}{3}]} n!\left(\frac{\gamma z}{t^3}\right)^r H_{n-3r}^{\alpha,\beta}\left(xt, \frac{\beta y}{t^2}\right)dt$$

$$= \sum_{r=0}^{[\frac{n}{3}]} \frac{(\gamma z)^r}{r!} \frac{\alpha}{(n-3r)!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^{n-3r} H_{n-3r}^{\alpha,\beta}\left(xt, \frac{\beta y}{t^2}\right)dt$$

$$= \sum_{r=0}^{[\frac{n}{3}]} \frac{(\gamma z)^r}{r!} P_{n-3r}^{\alpha,\beta}(x,y) \text{ using (3.3)}$$

Thus we obtain

$$P_n^{\alpha,\beta,\gamma}(x,y,z) = \frac{\alpha}{n!\ \sqrt{\pi}}\int_0^\infty \exp(-t^2)t^n H_n^{\alpha,\beta,\gamma}\left(xt, \frac{\beta y}{t^2}, \frac{\gamma z}{t^3}\right)dt \tag{3.5}$$

where $P_n^{\alpha,\beta,\gamma}(x,y,z)$ is the Modified Legendre's polynomial of three variables.

These polynomials $P_n^{\alpha,\beta,\gamma}(x,y,z)$ can be defined as

$$P_n^{\alpha,\beta,\gamma}(x,y,z) = \sum_{r=0}^{[\frac{n}{3}]} \frac{(\gamma z)^r}{r!} P_{n-3r}^{\alpha,\beta}(x,y) \tag{3.6}$$

For $\alpha = 2$, $\beta = \gamma = 1$, Eq. (3.5) gives

$$P_n(x,y,z) = \frac{2}{n!\ \sqrt{\pi}}\int_0^\infty e^{-t^2} t^n H_n\left(xt, \frac{y}{t^2}, \frac{z}{t^3}\right)dt \tag{3.7}$$

where $P_n(x,y,z)$ are Legendre's polynomials of three variables.

These polynomials $P_n(x,y,z)$ can be defined as

$$P_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{3}\right]} \frac{z^r}{r!} \, P_{n-3r}(x,y) \tag{3.8}$$

Now we obtain the expansion of three variable MHP as follows:

Since

$$\exp(\alpha x t) = \exp((1+\beta y)t^2)\exp(-\gamma z t^3) \sum_{n=0}^{\infty} \frac{H_n^{\alpha,\beta,\gamma}(x,y,z)t^n}{n!}$$

It follows that

$$\exp\left(\alpha x t - (1+\beta y)t^2 + \gamma z t^3\right) = \sum_{n=0}^{\infty} \frac{H_n^{\alpha,\beta,\gamma}(x,y,z)t^n}{n!}$$

or,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\alpha x t)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(1+\beta y)^n t^{2n})}{n!} \sum_{n=0}^{\infty} \frac{(-\gamma z)^n t^{3n}}{n!} \sum_{n=0}^{\infty} \frac{H_n^{\alpha,\beta,\gamma}(x,y,z)t^n}{n!} \\ &= \sum_{n,r,s=0}^{\infty} \frac{H_n^{\alpha,\beta,\gamma}(x,y,z)(1+\beta y)^s(-\gamma z)^r t^{n+2s+3r}}{n!r!s!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{3}\right]} \sum_{s=0}^{\left[\frac{n-3r}{2}\right]} \frac{H_{n-2s-3r}^{\alpha,\beta,\gamma}(x,y,z)(1+\beta y)^s(-\gamma z)^r t^n}{(n-2s-3r)!r!s!}\end{aligned}$$

Equating the coefficient of t^n, we get

$$x^n = \sum_{r=0}^{\left[\frac{n}{3}\right]} \sum_{s=0}^{\left[\frac{n-3r}{2}\right]} \frac{n! H_{n-2s-3r}^{\alpha,\beta,\gamma}(x,y,z)(1+\beta y)^s(-\gamma z)^r}{\alpha^n (n-2s-3r)!r!s!} \tag{3.9}$$

4. Concluding Remarks

In this paper we have obtained many interesting results of modified Hermite polynomials of three variables. Here we extend modified Hermite polynomials of three variables to four variables.

We define modified Hermite polynomials of four variables by series as follows:

$$H_n^{\alpha,\beta,\gamma,\delta}(x,y,z,w) = \sum_{r=0}^{\left[\frac{n}{4}\right]} \frac{n! \, (\delta w)^r \, H_{n-4r}^{\alpha,\beta,\gamma}(x,y,z)}{r! \, (n-4r)!} \tag{4.1}$$

Now consider the sum

$$\sum_{n=0}^{\infty} \frac{H_n^{\alpha,\beta,\gamma,\delta}(x,y,z,w)t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{4}\right]} \frac{(\delta w)^r \, H_{n-4r}^{\alpha,\beta,\gamma}(x,y,z)t^n}{n! \, (n-4r)!}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\delta w)^r \; H_n^{\alpha,\beta,\gamma}(x,y,z)t^{n+4r}}{n!r!}$$

$$= \sum_{n=0}^{\infty} \frac{H_n^{\alpha,\beta,\gamma}(x,y,z)t^n}{n!} \; \sum_{r=0}^{\infty} \frac{(\delta w)^r \; t^{4r}}{r!}$$

$$= \exp\left(\alpha xt - (1+\beta y)\, t^2 + \gamma z t^3\right) \; \exp(\delta w t^4)$$

Therefore we obtain the generating relation for four variables modified Hermite polynomials

$$\sum_{n=0}^{\infty} H_n^{\alpha,\beta,\gamma,\delta}(x,y,z,w)\frac{t^n}{n!} = \exp\left[\alpha xt - (1+\beta y)t^2 + \gamma z t^3 + \delta w t^4\right] \qquad (4.2)$$

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STABILITY AND BOUNDEDNESS OF IMPULSIVE DELAY DIFFERENTIAL SYSTEMS

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Abstract: In this paper, the criteria of stability and boundedness are established for product impulsive delay differential system. The sufficient conditions have been obtained using piecewise continuous Lyapunov function with Razumikhin technique.

1. **Introduction**

The impulsive delay differential systems are a natural generalization of functional differential systems without impulses and of impulsive ordinary differential systems without delay. They are adequate mathematical models of various real processes and phenomena, characterized by the fact that their state changes by jumps and by the dependence of the process on its history at each moment of time. Since time delay exists in many fields of our society such as control technology, communication networks and biological population management, significant progress has been made in the theory of impulsive delay differential systems in the recent years (see [6, 9, 11] and references therein). Stability and boundedness of impulsive differential systems have been studied by many authors during last few years [1-4, 6-11].
There are some delay differential systems which cannot be solved though they represent physical phenomenon; and there are many occasions when we need to study their qualitative behavior such as stability and boundedness. In such cases, Lyapunov-Razumikhin Technique helps us in studing the qualitative behavior of impulsive systems without actually solving them. So, the technique has been successfully used by many authors to study the stability of impulsive

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functional differential systems [4,6,10].
The aim of this paper is to establish the criterion of uniform eventual stability, uniform asymptotic stability and boundedness for product impulsive delay differential system. The sufficient conditions have been obtained using Lyapunov-Razumikhin Technique. The paper is organized as follows. Section 2 introduces some preliminary definitions and notations. In section 3, sufficient conditions for uniform eventual stability and uniform asymptotic stability of impulsive delay differential system have been established. In section 4, we establish some sufficient conditions for boundedness of solutions of impulsive delay differential systems.

2. **Preliminaries**

Let $\mathbb{R}$ be set of reals and $\mathbb{R}_+ = [0,\infty)$. Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be n and m dimentional Euclidean spaces. Set $PC(\mathbb{R}_+,\mathbb{R}^n) = \{x : \mathbb{R}_+ \to \mathbb{R}^n : x$ is continuous function except at $t = t_k \in \mathbb{R}_+$ and $x(t_k^+), x(t_k^-)$ exists in $\mathbb{R}^n$ with $x(t_k^+) = x(t_k)\}$ and $PC(\mathbb{R}_+,\mathbb{R}^m) = \{y : \mathbb{R}_+ \to \mathbb{R}^m : y$ is continuous function except at $t = t_k \in \mathbb{R}_+$ and $y(t_k^+), y(t_k^-)$ exists in $\mathbb{R}^m$ with $y(t_k^+) = y(t_k)\}$. For any $x \in PC(\mathbb{R}_+,\mathbb{R}^n)$, define the sup-norm $\|x\| = \sup_{s\in\mathbb{R}_+} | x(s) |$, where $| \, . \, |$ is a norm in $\mathbb{R}^n$ and similarly for $y \in PC(\mathbb{R}_+,\mathbb{R}^m)$. Then $PC(\mathbb{R}_+,\mathbb{R}^n)$ and $PC(\mathbb{R}_+,\mathbb{R}^m)$ are Banach spaces. Let $\tau > 0$ be a given constant and $PC([-\tau,0],\mathbb{R}^n)$ and $PC([-\tau,0],\mathbb{R}^m)$ be linear spaces equipped with the norm $\|.\|$ defined as $\|\phi\| = \sup_{-\tau\le s\le 0} \|\phi(s)\|$.
Consider the following impulsive delay differential system

$$\begin{aligned}
x^{'}(t) &= f(t,x_t) + g(t,y_t), \ t \neq t_k, \\
y^{'}(t) &= h(t,x_t,y_t), \ \ t \neq t_k, \\
\Delta x|_{t=t_k} &= A_k(x_{t_k^-}) + B_k(y_{t_k^-}), \\
\Delta y|_{t=t_k} &= C_k(x_{t_k^-}, y_{t_k^-}), \ k = 1,2,3,...,n \\
(x(t_0), y(t_0)) &= (\phi_1,\phi_2),
\end{aligned} \tag{2.1}$$

where $f : \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n, \ g : \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^m) \to \mathbb{R}^n, \ h : \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n) \times PC([-\tau,0],\mathbb{R}^m) \to \mathbb{R}^m, \ A_k : PC([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n, \ B_k : PC([-\tau,0],\mathbb{R}^m) \to \mathbb{R}^n, \ C_k : PC([-\tau,0],\mathbb{R}^n) \times PC([-\tau,0],\mathbb{R}^m) \to \mathbb{R}^m, \ \phi_1 \in PC([-\tau,0],\mathbb{R}^n), \ \phi_2 \in PC([-\tau,0],\mathbb{R}^m), \Delta x(t) = x(t) - x(t^-), \ \Delta y(t) = y(t) - y(t^-), \ x_t, x_{t^-} \in PC([-\tau,0],\mathbb{R}^n)$ and $y_t, y_{t^-} \in PC([-\tau,0],\mathbb{R}^m)$ are defined as $x_t(s) = x(t+s), \ x_{t^-}(s) = x(t^- + s), \ y_t(s) = y(t+s), \ y_{t^-}(s) = y(t^- + s)$ for $-\tau \le s \le 0$ respectively.
A function $(x(t), y(t)) \in PC(\mathbb{R}_+,\mathbb{R}^n) \times PC(\mathbb{R}_+,\mathbb{R}^m)$ is called the solution of system (2.1) if it satisfies the system (2.1). In general, the solution $(x(t), y(t))$

of (2.1) is piecewise right-continuous function with the point of discontinuity of first kind i.e. if solution hits the hypersurface at $t = t_k,\ k \in \mathbb{N}$ then the following relations are satisfied:

$$x(t_k^+) = x(t_k) \text{ and } y(t_k^+) = y(t_k), k \in \mathbb{N}.$$

Throughout the article, we assume that the following conditions hold.
(i) $f, g, h, A_k, B_k, C_k,\ k \in \mathbb{N}$ satisfy all the sufficient conditions for the global existence and uniqueness of solutions for $t \geq t_0$.
(ii) $0 \leq t_0 < t_1 < t_2 < ... < t_k < ...$ with $t_k \to \infty$ as $k \to \infty$.

Definition 2.1. A function $V : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ is said to belong to class $\mathcal{V}_0$ if
(i) V is continuous in each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^m$;
(ii) $V(t, x, y)$ is locally Lipschitizian in all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and $V(t, 0, 0) = 0$ for $t \in \mathbb{R}_+$;
(iii) For each $(t, x, y) \in [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^m$, we have $\lim_{(t,x,y)\to(t_k^-,x,y)} V(t, x, y) = V(t_k^-, x, y)$.

Definition 2.2. Given a function $V : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, the upper right hand derivative of V with respect to system (2.1) is defined as

$$\begin{aligned} D^+V(t, \zeta(0), \eta(0)) \quad = \quad & \lim_{s\to 0^+} \frac{1}{s}\{V(t + s, \zeta(0) + s(f(t, \zeta) + g(t, \eta)), \eta(0) + \\ & sh(t, \zeta, \eta)) - V(t, \zeta(0), \eta(0))\}, \end{aligned}$$

for $(t, \zeta, \eta) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n) \times PC([-\tau, 0], \mathbb{R}^m)$.

Definition 2.3. The zero solution of system (2.1) is said to be uniformly eventually stable if for $\epsilon > 0,\ \exists\ \delta = \delta(\epsilon) > 0$ and $\lambda = \lambda(\epsilon) > 0$ such that $\|x(t) + y(t)\| < \epsilon$ for $\|\phi_1 + \phi_2\| < \delta$ and $t \geq t_0 \geq \lambda(\epsilon)$.

Definition 2.4. The zero solution of system (2.1) is said to be uniformly asymptotically stable if it is uniformly stable and for any $\epsilon > 0,\ \exists\ \delta > 0$ and $T = T(\epsilon) > 0$ such that $\|\phi_1 + \phi_2\| < \delta$ implies $\|x(t) + y(t)\| < \epsilon$ for $t \geq t_0 + T$.

Definition 2.5. The system (2.1) is said to be uniformly bounded if $\forall\ \alpha > 0,\ \exists\ \beta = \beta(\alpha) > 0$ such that $\|\phi_1 + \phi_2\| < \alpha$ implies $\|x(t) + y(t)\| < \beta$ for all $t \geq t_0,\ t_0 \in \mathbb{R}_+$.

Definition 2.6. The system (2.1) is said to be quasi-uniformly ultimately bounded if $\forall \alpha > 0,\ \exists\ B > 0$ and $T = T(\alpha) > 0$ such that $\|\phi_1 + \phi_2\| < \alpha$ implies $\|x(t) + y(t)\| < B,\ \forall\ t \geq t_0 + T, t_0 \in \mathbb{R}_+$.

Definition 2.7. The system (2.1) is said to be uniformly ultimately bounded if definitions 2.5 and 2.6 hold together.

We define the following notations:

$$K = \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : w \text{ is strictly increasing and } w(0) = 0\}.$$

$$K_1 = \{\phi \in C(\mathbb{R}_+, \mathbb{R}_+) : \phi \text{ is increasing and } \phi(s) < s \text{ for } s > 0\}.$$

$$S_1(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho, \rho > 0\}.$$

$$S_2(\rho) = \{y \in \mathbb{R}^m : \|y\| < \rho, \rho > 0\}.$$

3. **Stability**

In this section, we shall present some sufficient conditions for the uniform eventual stability and uniform asymptotic stability for the impulsive differential problem (2.1).

Theorem 3.1. *Assume that there exist a function* $V \in \mathcal{V}_0$, $a, b \in K$ *and* $\psi \in K_1$ *such that*
(i) $b(\|x + y\|) \leq V(t, x, y) \leq a(\|x + y\|)$ *for* $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$;
(ii) $D^+V(t, \zeta(0), \eta(0)) \leq g(t)w(V(t, \zeta(0), \eta(0)))$ *for all* $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, *whenever* $V(t + s, \zeta(s), \eta(s)) \leq \psi^{-1}(V(t, \zeta(0), \eta(0)))$ *for* $s \in [-\tau, 0]$;
(iii) $V(t_k, \zeta(0) + A_k(\zeta) + B_k(\eta), \eta(0) + C_k(\zeta, \eta)) \leq \psi(V(t_k^-, \zeta(0), \eta(0))), k \in \mathbb{N}$;
(iv) There exist a constant $A > 0$ *such that* $\int_{t_{k-1}}^{t_k} g(s)ds < A$ *and* $\int_{\mu}^{\psi^{-1}(\mu)} \frac{ds}{w(s)} \geq A$ *for any* $\mu > 0$, $k \in \mathbb{N}$ *and* $\tau \leq t_k - t_{k-1} \leq \alpha$, α *is a constant.*
Then the zero solution of system (2.1) is uniformly eventually stable.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \delta(\epsilon) > 0$, $\lambda(\epsilon) > 0$ such that $\delta < a^{-1}(\psi(b(\epsilon)))$.
We shall prove that $\|\phi_1 + \phi_2\| < \delta$ implies $\|x(t) + y(t)\| < \epsilon$, $\forall\, t \geq t_0 \geq \lambda(\epsilon)$.
Let $\nu(t) = V(t, x(t), y(t))$. We shall prove that

$$\nu(t) \leq \psi^{-1}(a(\delta)),\ t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{3.1}$$

Firstly, we shall show that

$$\nu(t) \leq \psi^{-1}(a(\delta)),\ t \in [t_0, t_1). \tag{3.2}$$

Clearly, $\nu(t_0) = V(t_0, \phi_1, \phi_2) \leq a(\|\phi_1 + \phi_2\|) \leq a(\delta) < \psi^{-1}(a(\delta))$.
If (3.2) is not true, then there exist some $\bar{t} \in (t_0, t_1)$ such that

$$\nu(\bar{t}) > \psi^{-1}(a(\delta)) > a(\delta) \geq \nu(t_0 + s),\ s \in [-\tau, 0].$$

From the continuity of $\nu(t)$ in $[t_0, t_1)$, there exist some $t^* \in (t_0, \bar{t})$ such that

$$\begin{aligned} \nu(t^*) &= \psi^{-1}(a(\delta)), \\ \nu(t) &> \psi^{-1}(a(\delta)),\ t^* < t \leq \bar{t}, \\ \nu(t) &\leq \psi^{-1}(a(\delta)),\ t_0 + s < t \leq t^*, s \in [-\tau, 0] \end{aligned} \tag{3.3}$$

and there also exist $\hat{t} \in [t_0, t^*)$ such that

$$\begin{aligned} \nu(\hat{t}) &= a(\delta), \\ \nu(t) &\geq a(\delta),\ \hat{t} < t \leq t^*. \end{aligned} \tag{3.4}$$

Then, we obtain for any $t \in [\hat{t}, t^*]$,

$$\nu(t+s) \leq \psi^{-1}(a(\delta)) \leq \psi^{-1}(\nu(t)),\ s \in [-\tau, 0],$$

by condition (ii), we get

$$D^+\nu(t) \leq g(t)w(\nu(t)) \text{ for } t \in [\hat{t}, t^*].$$

Integrating in $[\hat{t}, t^*]$, we get

$$\int_{\nu(\hat{t})}^{\nu(t^*)} \frac{du}{w(u)} \leq \int_{\hat{t}}^{t^*} g(t)dt \leq \int_{t_0}^{t_1} g(t)dt < A. \tag{3.5}$$

On the other hand, from (3.3), (3.4) and condition (iv), we have

$$\int_{\nu(\hat{t})}^{\nu(t^*)} \frac{du}{w(u)} = \int_{a(\delta)}^{\psi^{-1}(a(\delta))} \frac{du}{w(u)} \geq A. \tag{3.6}$$

This is a contradiction to the inequality (3.5). So inequality (3.2) holds i.e. (3.1) is true for $k = 1$.
Now assume that (3.1) holds for $k = 1, 2, ..., m$, i.e.

$$\nu(t) \leq \psi^{-1}(a(\delta)),\ t \in [t_{k-1}, t_k), k = 1, 2, ..., m. \tag{3.7}$$

We shall prove that (3.1) holds for $k = m + 1$, i.e.

$$\nu(t) \leq \psi^{-1}(a(\delta)),\ t \in [t_m, t_{m+1}). \tag{3.8}$$

On contrary, suppose that (3.8) is not true. From(iii) and (3.7), we have

$$\begin{aligned} \nu(t_m) &\leq \psi(\nu(t_m^-)), \\ &< \psi(\psi^{-1}(a(\delta))), \\ &= a(\delta). \end{aligned} \tag{3.9}$$

We define

$$\bar{t} = \inf\{t \in [t_m, t_{m+1}) : \nu(t) > \psi^{-1}(a(\delta))\}.$$

From (3.9), $\bar{t} \neq t_m$. By continuity of $\nu(t)$ in the interval $[t_m, t_{m+1})$, we have

$$\begin{aligned} \nu(\bar{t}) &= \psi^{-1}(a(\delta)), \qquad (3.10) \\ \nu(t) &\leq \psi^{-1}(a(\delta)),\ t_m < t \leq \bar{t}. \end{aligned}$$

From (3.9), we have

$$\nu(t_m) < a(\delta) < \psi^{-1}(a(\delta)) = \nu(\bar{t}),$$

which implies that there exist some $t^* \in (t_m, \bar{t})$ such that

$$\nu(t^*) = a(\delta)$$

and

$$\nu(t^*) \leq \nu(t) \leq \nu(\bar{t}),\ t \in [t^*, \bar{t}].$$

Then $t+s \in [t_{m-1}, \bar{t}]$ for $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$, since $\tau \leq t_k - t_{k-1} \leq \alpha$. From (3.7), we get for $t \in [t^*, \bar{t}]$,

$$\nu(t+s) \leq \psi^{-1}(a(\delta)) = \psi^{-1}(\nu(t^*)) \leq \psi^{-1}(\nu(t)), s \in [-\tau, 0].$$

Then by condition (ii), we get

$$D^+\nu(t) \leq g(t)w(\nu(t)),$$

which leads to a contradiction as discussed above. Therefore (3.1) holds for $k = m+1$. Hence, by method of induction, the result (3.1) is true and we have,

$$\nu(t) \leq \psi^{-1}(a(\delta)) < b(\epsilon),\ t \geq t_0 \geq \lambda(\epsilon).$$

By (i), we have

$$\|x+y\| \leq b^{-1}(\nu(t)) < b^{-1}(b(\epsilon)) = \epsilon,\ t \geq t_0 \geq \lambda(\epsilon).$$

Thus, the zero solution of (2.1) is uniformly eventually stable. □

Theorem 3.2. *Let all the conditions of Theorem 3.1 be satisfied except (iv), which is replaced by*
(v) $r = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty,\ M_1 = \sup_{t\geq 0}\int_t^{t+r} g(s)ds < \infty$ *and* $M_2 = \inf_{q>0}\int_{\psi(q)}^{q} \frac{ds}{w(s)} > M_1$.
Then, the zero solution of (2.1) is uniformly asymptotically stable.

Proof. Obviously, the conditions of Theorem 3.2 imply the conditions of Theorem 3.1 hold, so the impulsive delay differential system (2.1) is uniformly eventually stable and hence uniformly stable.
This implies that given $\eta > 0, \exists\ \delta > 0$ such that $\psi^{-1}(a(\delta)) < b(\eta)$ and $\|\phi_1 + \phi_2\| < \delta$ implies $\|x(t) + y(t)\| < \eta$ for $t \geq t_0$.
Moreover,

$$\nu(t) \leq \psi^{-1}(a(\delta)) < b(\eta),\ t \geq t_0 + s, s \in [-\tau, 0], \tag{3.11}$$

where $\nu(t) = V(t, x(t), y(t))$.
Let $t \geq t_0,\ t \in [t_{m-1}, t_m)$ for some $m \in \mathbb{N}$. Let $\epsilon > 0$ and assume without loss of generality that $\epsilon < \eta$.
Define $M = M(\epsilon) = \sup\{\frac{1}{w(s)} : \psi(b(\epsilon)) \leq s \leq a(\eta)\}$ and note that $0 < M < \infty$.
For $b(\epsilon) \leq q \leq a(\eta)$, we have
$\psi(b(\epsilon)) \leq \psi(q) < q \leq a(\eta)$, so $M_2 \leq \int_{\psi(q)}^{q} \frac{ds}{w(s)} \leq M(q - \psi(q))$, from which we obtain

$\psi(q) \le q - \frac{M_2}{M} < q - d$, where $d = d(\epsilon) > 0$ is chosen such that $d < \frac{M_2 - M_1}{M}$.
Let $N = N(\epsilon)$ be the smallest positive integer for which $a(\eta) < b(\epsilon) + Nd$ and define $T = T(\epsilon) = r + (N-1)(r+\tau)$, we shall prove that $\|\phi_1 + \phi_2\| < \delta$ implies $\|x(t) + y(t)\| < \epsilon$ for $t \ge t_0 + T$.
Given $0 < A \le a(\eta)$ and $j \in \mathbb{N}$, we shall show that
(a) if $\nu(t) \le A$ for $t \in [t_j - \tau, t_j)$, then $\nu(t) \le A$ for $t \ge t_j$;
(b) if in addition $A \ge b(\epsilon)$, then $\nu(t) \le A - d$ for $t \ge t_j$.
Firstly, we prove (a).
If (a) does not hold, then there exist some $t \ge t_j$ such that $\nu(t) > A$.
Let

$$t^* = \inf\{t \ge t_j : \nu(t) > A\}.$$

Thus $t^* \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}$, $k \ge j$. By (iii), $\nu(t_k) \le \psi(\nu(t_k^-)) \le \psi(A) < A$, then $t^* \in (t_k, t_{k+1})$.
Moreover, $\nu(t^*) = A$ and $\nu(t) \le A$ for $t \in [t_j - \tau, t^*]$.
Let

$$\bar{t} = \sup\{t \in [t_k, t^*] : \nu(t) \le \psi(A)\}.$$

As $\nu(t^*) = A > \psi(A)$, then $\bar{t} \in [t_k, t^*]$, $\nu(\bar{t}) = \psi(A)$ and $\nu(t) \ge \psi(A)$ for $t \in [\bar{t}, t^*]$.
Thus for $t \in [\bar{t}, t^*]$, $s \in [-\tau, 0]$, we have $\psi(\nu(t+s)) \le \psi(A) \le \nu(t)$,
that is, $\nu(t+s) \le \psi^{-1}(\nu(t))$. Then by (ii), the following inequality holds:

$$D^+\nu(t) \le g(t)w(\nu(t)).$$

Integrating in $[\bar{t}, t^*]$, we get

$$\int_{\nu(\bar{t})}^{\nu(t^*)} \frac{du}{w(u)} \le \int_{\bar{t}}^{t^*} g(u)du \le \int_{t_k}^{t_{k+1}} g(u)du \le M_1.$$

On the other hand

$$\int_{\nu(\bar{t})}^{\nu(t^*)} \frac{du}{w(u)} = \int_{\psi(A)}^{A} \frac{du}{w(u)} \ge M_2 > M_1,$$

which is a contradiction to above inequality and so (a) holds. Next, we prove (b).
Assume for the sake of contradiction that there exist some $t \ge t_j$, such that $\nu(t) > A - d$.
Then define $r_1 = \inf\{t \ge t_j : \nu(t) > A - d\}$ and $k \ge j$ be chosen so that $r_1 \in [t_k, t_{k+1})$.
Since $b(\epsilon) \le A \le a(\eta)$, then $\psi(A) < A - d$.
So from (a) and condition (iii), we have $\nu(t_k) \le \psi(\nu(t_k^-)) \le \psi(A) < A - d$.

Thus $r_1 \in (t_k, t_{k+1})$. Moreover $\nu(r_1) = A - d$ and $\nu(t) \leq A - d$ for $t \in [t_k, r_1)$. Let

$$\bar{r} = \sup\{t \in [t_k, r_1) : \nu(t) \leq \psi(A)\}.$$

As $\nu(r_1) = A - d > \psi(A) \geq \nu(t_k)$, then $\bar{r} \in [t_k, r_1]$, $\nu(\bar{r}) = \psi(A)$ and $\nu(t) \geq \psi(A)$ for $t \in [\bar{r}, r_1]$.

Thus for $t \in [\bar{r}, r_1], s \in [-\tau, 0]$, we have $\psi(\nu(t+s)) \leq \psi(A) \leq \nu(t)$, that is, $\nu(t+s) \leq \psi^{-1}(\nu(t))$. Then by (ii), the following inequality holds:

$$D^+\nu(t) \leq g(t)w(\nu(t))$$

Integrating in $[\bar{r}, r_1]$, we get

$$\int_{\nu(\bar{r})}^{\nu(r_1)} \frac{du}{w(u)} \leq M_1.$$

But on the other hand

$$\int_{\nu(\bar{r})}^{\nu(r_1)} \frac{du}{w(u)} = \int_{\psi(A)}^{A-d} \frac{du}{w(u)} = \int_{\psi(A)}^{A} \frac{du}{w(u)} - \int_{A-d}^{A} \frac{du}{w(u)}.$$

Since $b(\epsilon) \leq A \leq a(\eta)$, we have $\psi(b(\epsilon)) \leq \psi(A) < A - d \leq a(\eta)$.

Thus $\frac{1}{w(u)} \leq M$ for $A - d \leq u \leq A$. So, we get

$$\int_{\nu(\bar{r})}^{\nu(r_1)} \frac{du}{w(u)} \geq M_2 - \int_{A-d}^{A} M du = M_2 - Md > M_2 + M_1 - M_2 = M_1.$$

This is a contradiction, so (b) holds. Now we define the indices $k^{(i)}$ for $i = 1, 2, ..., N$ as follows. Let $k^{(1)} = m$ and for $i = 2, ..., N$, let $k^{(i)}$ be chosen such that $t_{k^{(i)}-1} < t_{k^{(i-1)}} + \tau \leq t_{k^{(i)}}$. Then from condition (v), we have $t_{k^{(1)}} = t_m \leq t_{m-1} + r \leq t_0 + r$ and for $i = 2, ..., N$,

$t_{k^{(i)}} \leq t_{k^{(i)}-1} + r < t_{k^{(i-1)}} + \tau + r$. Combining these inequalities, we get $t_{k^{(N)}} \leq t_0 + r + (r + \tau)(N - 1) = t_0 + T$.

Next we claim that for each $i = 1, 2, ..., N$, $\nu(t) \leq a(\eta) - id$ for $t \geq t_{k^{(i)}}$.

Since by (3.11), $\nu(t) \leq b(\eta) \leq a(\eta)$ for $t \in [t_0 - \tau, t_{k^{(1)}})$, then by setting $A = a(\eta)$ in our argument proved in (b), we get $\nu(t) \leq a(\eta) - d$ for $t \geq t_{k^{(1)}}$, which establish the base case.

We now proceed by induction and assume $\nu(t) \leq a(\eta) - jd$ for $t \geq t_{k^{(j)}}$ for some $1 \leq j \leq N-1$. Let $A = a(\eta) - jd$, then $b(\epsilon) \leq A \leq a(\eta)$. Since $t_{k^{(j)}} \leq t_{k^{(j+1)}} - \tau$, then $\nu(t) \leq A$ for $t \in [t_{k^{(j+1)}} - \tau, t_{k^{(j+1)}})$ and so $\nu(t) \leq A - d = a(\eta) - (j+1)d$ for $t \geq t_{k^{(j+1)}}$. So our claim is proved by induction.

When $j = N - 1$, we get $\nu(t) \leq a(\eta) - Nd < b(\epsilon)$, $t \geq t_{k^{(N)}}$.

Since $t_0 + T \geq t_{k^{(N)}}$, by condition (i), we get $\|x(t) + y(t)\| \leq \epsilon$, $t \geq t_0 + T$.

Thus, the impulsive delay differential system (2.1) is uniformly asymptotically stable. $\square$

4. **Boundedness**

In this section, we shall present some sufficient conditions for boundedness of impulsive differential problem (2.1).

Theorem 4.1. *Assume that there exist a function $V \in \mathcal{V}_0,\ a, b, c \in K$ such that*
(i) $b(\|x+y\|) \leq V(t,x,y) \leq a(\|x+y\|)$ for $(t,x,y) \in \mathbb{R}_+ \times S_1^c(\rho) \times S_2^c(\rho)$;
(ii) $D^+V(t,\zeta(0),\eta(0)) \leq -c(\|x(t)+y(t)\|)$ for all $t \in [t_{k-1}, t_k),\ k \in \mathbb{N}$, whenever $V(t+s,\zeta(s),\eta(s)) \leq P(V(t,\zeta(0),\eta(0)))$ for $s \in [-\tau, 0]$ where $P(u)$ is continuous function on $\mathbb{R}_+$, non-decreasing in u and $P(u) > u$ for $u > 0$;
(iii) $V(t_k^+, \zeta(0) + A_k(\zeta) + B_k(\eta), \eta(0) + C_k(\zeta,\eta)) \leq V(t_k^-, \zeta(0), \eta(0)), k \in \mathbb{N}$.
Then the system (2.1) is uniformly bounded.

Proof. Let $\alpha \geq \rho$ be given. Choose $\beta = \beta(\alpha) > 0$ so that $\beta > \max\{\alpha, b^{-1}(a(\alpha))\}$. Let $t_0 \in \mathbb{R}_+$. Consider the solution $(x(t,t_0,\phi_1,\phi_2), y(t,t_0,\phi_1,\phi_2))$ of (2.1) with $\|\phi_1+\phi_2\| < \alpha$ and let $\nu(t) = V(t,x(t),y(t))$. Clearly, $\|x(t_0)+y(t_0)\| = \|\phi_1+\phi_2\| < \alpha < \beta$.
Claim: $\|x(t)+y(t)\| < \beta,\ t \geq t_0$.
If this is not true, then there exist some solution $(x(t), y(t))$ of (2.1) with $\|\phi_1+\phi_2\| < \alpha$ and $t^* > t_0$ such that

$$\|x(t^*)+y(t^*)\| \geq \beta.$$

Then there exist $s_1, s_2,\ t_0 \leq s_1 < s_2 \leq t^*$ such that

$$\|x(s_1)+y(s_1)\| \geq \alpha,\ \|x(s_1^-)+y(s_1^-)\| \leq \alpha,\ \|x(s_2^-)+y(s_2^-)\| \geq \beta$$

and

$$\|x(t)+y(t)\| < \beta,\ \|x(t)+y(t)\| \geq \alpha,\ s_1 < t < s_2^-. \tag{4.1}$$

Firstly, we show

$$\nu(s_1^+) < b(\beta).$$

If $s_1 \neq t_k$, then $\|x(s_1^-)+y(s_1^-)\| = \alpha$ and by (i),

$$\nu(s_1^-) \leq a(\|x(s_1^-)+y(s_1^-)\|) = a(\alpha) < b(\beta).$$

If $s_1 = t_k$ for some k, then $\|x(s_1^-)+y(s_1^-)\| \leq \alpha$ and

$$\nu(s_1^-) \leq a(\|x(s_1^-)+y(s_1^-)\|) \leq a(\alpha) < b(\beta).$$

Thus by (iii), $\nu(s_1^+) < b(\beta)$. Next, we wish to show that

$$\nu(t^+) < b(\beta),\ t \in [s_1, s_2]. \tag{4.2}$$

Suppose that this is not true and let

$$\mu = \inf\{s_1 \leq t \leq s_2^- : \nu(t^+) \geq b(\beta)\}.$$

We discuss two cases:
(A) $\mu \neq t_k, k = 1, 2, ...$
Since $\nu(t)$ is continuous at μ,

$$\nu(\mu) = b(\beta).$$

Thus for $h > 0$ with $\mid h \mid$ small enough that the inequality

$$\nu(\mu + h) > b(\beta)$$

holds, which implies that

$$D^+\nu(\mu) = \lim_{h\to 0^+} \sup \frac{1}{h}[\nu(\mu + h) - \nu(\mu)] \geq 0. \tag{4.3}$$

It is clear from the choice of μ that $P(\nu(\mu)) > \nu(\mu) \geq \nu(\mu + s),\ s \in [-\tau, 0]$.
Thus from (ii) and (4.1), we get

$$D^+\nu(\mu) \leq -c(\|x(\mu) + y(\mu)\|) \leq -c(\alpha) < 0,$$

which contradicts (4.3).
(B) $\mu = t_k$ for some $k = 1, 2, ...$
We must have $\nu(t_k^+) = b(\beta)$. In fact, if $\nu(t_k^+) > b(\beta)$, then by (iii), $\nu(t_k^-) > b(\beta)$.
Since $\nu(t)$ is right continuous at t_k, it follows that there exist $\mu_1 > t_k$ such that $\nu(\mu_1) \geq b(\beta)$, which contradicts the choice of μ.
Now for $h > 0$ with $\mid h \mid$ small enough so that $t_k + h \in (t_k, t_{k+1})$,

$$\nu(t_k + h) > b(\beta),$$

i.e.

$$\nu(\mu + h) > b(\beta).$$

Therefore,

$$\begin{aligned} D^+\nu(\mu) &= \lim_{h\to 0^+} \sup \frac{1}{h}[\nu(\mu + h) - \nu(\mu)] \\ &\geq \lim_{h\to 0^+} \sup \frac{1}{h}[\nu(t_k + h) - \nu(t_k)] \geq 0. \end{aligned} \tag{4.4}$$

Since $P(\nu(\mu)) > \nu(\mu) > \nu(\mu + s),\ s \in [-\tau, 0]$,
we obtain, using (ii),

$$D^+\nu(\mu) \leq -c(\|x(\mu) + y(\mu)\|) \leq -c(\alpha) < 0,$$

which contradicts (4.3). Therefore (4.2) holds. On the other hand, using (i), we obtain

$$\nu(s_2^-) \geq b(\|x(s_2^-) + y(s_2^-)\|) \geq b(\beta),$$

which contradicts (4.2). Thus $\|x(t) + y(t)\|) < \beta,\ t \geq t_0$.
Hence the system (2.1) is uniformly bounded. $\square$

Theorem 4.2. *Suppose that all the conditions of Theorem 4.1 hold except (ii) which is replaced by*
(iv) $D^+V(t,\zeta(0),\eta(0)) \le M - c(\|x(t)+y(t)\|$ *for all* $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, *whenever* $V(t+s,\zeta(s),\eta(s)) \le P(V(t,\zeta(0),\eta(0)))$ *for* $s \in [-\tau, 0]$ *where* $P(u)$ *is continuous function on* $\mathbb{R}_+$*, non-decreasing in* u *and* $P(u) > u$ *for* $u > 0$.
Then the system (2.1) is uniformly ultimately bounded.

Proof. Firstly, we prove uniform boundedness of the system (2.1).
Let $\rho > 0$ be sufficiently large that

$$M - c(\rho) < 0.$$

Let $\alpha > \max\{\rho, c^{-1}(M)\}$ be given. Choose $\beta > \max\{\alpha, b^{-1}(a(\alpha))\}$ for any $t_0 \in \mathbb{R}_+$ and $\|\phi_1+\phi_2\| < \alpha$. Clearly, $\|\phi_1+\phi_2\| < \alpha \le \beta$. Let $\nu(t) = V(t,x(t),y(t))$. Now, suppose that there exist a solution $x(t) = x(t,t_0,\phi_1,\phi_2)$, $y(t) = y(t,t_0,\phi_1,\phi_2)$ of (2.1) with $\|\phi_1+\phi_2\| < \alpha$ and $t^* > t_0$ such that

$$\|x(t^*)+y(t^*)\| \ge \beta.$$

Then there exist $s_1, s_2,\ t_0 \le s_1 < s_2 \le t^*$ such that

$$\|x(s_1)+y(s_1)\| \ge \alpha,\ \|x(s_1^-)+y(s_1^-)\| \le \alpha,\ \|x(s_2^-)+y(s_2^-)\| \ge \beta$$

and

$$\|x(t)+y(t)\| < \beta,\ \|x(t)+y(t)\| \ge \alpha,\ s_1 \le t < s_2^-. \tag{4.5}$$

We can easily prove as in the Theorem 4.1 that

$$\nu(s_1^+) < b(\beta)$$

Next, we show that

$$\nu(t^+) < b(\beta),\ t \in [s_1, s_2]. \tag{4.6}$$

Suppose that this is not true and let

$$\sigma = \inf\{s_1 < t \le s_2^- : \nu(t^+) \ge b(\beta)\}.$$

We discuss two cases:
(A) $\sigma \ne t_k, k = 1, 2, ...$, we can see that

$$\nu(\sigma) = b(\beta) \text{ and } D^+\nu(\sigma) \ge 0. \tag{4.7}$$

Also, there exist an $s_1^* \in (s_1, s_2]$ such that

$$\nu(s) \le P(\nu(\sigma)),\ s_1^* \le s \le \sigma.$$

Since $\|x(\sigma)+y(\sigma)\| \ge \alpha$, we get by (iv),

$$\begin{aligned} D^+\nu(\sigma) &\le M - c(\|x(\sigma)+y(\sigma)\|) \le M - c(\alpha) \\ &< M - c(\max\{\rho, c^{-1}(M)\}) = M - \max\{c(\rho), M\} \le 0, \end{aligned}$$

which is a contradiction to (4.7).
(B) $\sigma = t_k$ for some $k = 1, 2, ...$
We get a contradiction by the similar argument as in the proof of Theorem 4.1.
Thus (4.6) is true. On the other hand, using (i), $\nu(s_2^-) \geq b(\|x(s_2^-) + y(s_2^-)\|) \geq b(\beta)$.
Thus $\|x(t) + y(t)\| < \beta,\ t \geq t_0$ for any solution $(x(t), y(t))$ of (2.1) with $\|\phi_1 + \phi_2\| < \alpha$ and system (2.1) is uniformly bounded. The uniform boundedness of system (2.1) means that there exist a positive number B such that for each $t_0 \in \mathbb{R}_+$,

$$\|\phi_1 + \phi_2\| < \alpha \text{ implies } \|x(t) + y(t)\| < B,\ t \geq t_0. \tag{4.8}$$

Now we consider the solution $(x(t), y(t))$ of (2.1) with $\|\phi_1 + \phi_2\| < \alpha$ where α is arbitrary number and $\alpha > \rho$. Then there exist a positive number $\beta = \beta(\alpha) > \max\{B, b^{-1}(a(\alpha))\}$ such that $\|x(t) + y(t)\| < \beta,\ t \geq t_0$.
Let $P : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, nondecreasing function on $\mathbb{R}_+$ and $P(u) > u$ for $u > 0$. We set

$$\lambda = \inf\{P(u) - u : b(B) \leq u \leq b(\beta)\}.$$

Then

$$P(u) > u + \lambda \text{ as } b(B) \leq u \leq b(\beta) \tag{4.9}$$

and we choose an integer N such that

$$b(B) + (N - 1)\lambda > b(\beta). \tag{4.10}$$

Choose $T = N\lambda/c(\rho)$ and define

$$t_m = t_0 + m\frac{\lambda}{c(\rho) - M},\ m = 0, 1, 2, ..., N. \tag{4.11}$$

Then $t_N = t_0 + T$. We shall show that

$$\|x(t) + y(t)\| < B,\ t \geq t_0 + T. \tag{4.12}$$

Suppose that this is not true, then there exist a $t^* > t_0 + T$ such that $\|x(t^*) + y(t^*)\| \geq B$. In view of (4.8),

$$\|x(t) + y(t)\| \geq \rho,\ t \in [t_0, t^*]. \tag{4.13}$$

Claim:

$$\nu(t) < b(\beta),\ t \in [t_0, t^*]. \tag{4.14}$$

Suppose this is not true and let

$$\xi = \inf\{t_0 \leq t \leq t^* : \nu(t) \geq b(\beta)\}.$$

If $\xi \neq t_k,\ k = 1, 2, ...$ then by definition of $\xi,\ \nu(\xi) = b(\beta)$. Thus for $h > 0$ with $\mid h \mid$ small enough, the inequality $\nu(\xi + h) \geq b(\beta)$ holds and so

$$D^+\nu(\xi) = \lim_{h\to 0^+} \sup \frac{1}{h}[\nu(\xi + h) - \nu(\xi)] \geq 0. \tag{4.15}$$

It is clear form the choice of ξ that $P(\nu(\xi)) > \nu(\xi) \geq \nu(s),\ t_0 \leq s \leq \xi$ and from (4.13), we have $\mid x(\xi) + y(\xi) \mid > \rho$. Then from (iv), we have

$$D^+\nu(\xi) \leq M - c(\|x(\xi) + y(\xi)\|) \leq M - c(\rho) < 0,$$

which contradicts (4.15).

If $\xi = t_k$ for some $k = 1, 2, ...,$ then by the same argument as in the proof of theorem 4.1, we get a contradiction. Thus (4.14) holds. We next show that

$$\nu(t) < b(B) + (N - m - 1)\lambda,\ t \in [t_m, t^*],\ m = 0, 1, 2, ..., N - 1. \tag{4.16}$$

We prove it by induction. From (4.10) and (4.14), we have $\nu(t) < b(B) + (N - 1)\lambda,\ t \in [t_0, t^*]$,

which shows that the result holds for $m = 0$.

Suppose result holds for some integer $m, 0 \leq m < N - 1$, that is,

$$\nu(t) < b(B) + (N - m - 1)\lambda,\ t \in [t_m, t^*]. \tag{4.17}$$

Firstly, we shall show that there exist $\hat{t} \in [t_m, t_{m+1}]$ such that

$$\nu(\hat{t}) < b(B) + (N - m - 2)\lambda. \tag{4.18}$$

If this is not true, then

$$\nu(t) \geq b(B) + (N - m - 2)\lambda,\ t \in [t_m, t_{m+1}]. \tag{4.19}$$

By (4.14) and (4.19),

$$b(B) \leq \nu(t) \leq b(\beta)\ t \in [t_m, t_{m+1}]. \tag{4.20}$$

We consider two cases:

(A) $t \neq t_k, k = 1, 2, ...,$ for all $t \in [t_m, t_{m+1}]$. Using (4.9), (4.17) and (4.19), we obtain

$$\begin{aligned} P(\nu(t)) &\geq \nu(t) + \lambda \geq b(B) + (N - m - 1)\lambda \\ &> \nu(s),\ t_m \leq s \leq t, t \in [t_m, t_{m+1}]. \end{aligned}$$

By (4.13), we see that $\|x(t) + y(t)\| \geq \rho$ for $t \in [t_m, t_{m+1}] \subset [t_0, t^*]$. Thus, it follows from (iv) that

$$\begin{aligned} \nu(t) &\leq \nu(t_m) - \int_{t_m}^{t_{m+1}} [c(\|x(s) + y(s)\|) - M]ds \\ &< b(B) + (N - m - 1)\lambda - (c(\rho) - M)[t_{m+1} - t_m] \\ &\leq \nu(t_{m+1}), \end{aligned}$$

which is a contradiction.
(B) There exist $t_{k_i} \in [t_m, t_{m+1}]$ for $k_i \in \{1, 2, ..., k, ...\}, i = 1, 2, ...l; l \geq 1$.
By (4.9), (4.17) and (4.19),

$$\begin{aligned} P(\nu(t)) &\geq \nu(t) + \lambda \geq b(B) + (N - m - 1)\lambda \\ &> \nu(s),\ t_m \leq s \leq t, t \in [t_m, t_{m+1}]. \end{aligned}$$

Since $\|x(t) + y(t)\| \geq \rho$ for $t \in [t_m, t_{m+1}]$ and by (iv), we have

$$\begin{aligned} \int_{t_m}^{t_{m+1}} D^+\nu(s)ds &\leq -\int_{t_m}^{t_{m+1}} [c(\|x(s) + y(s)\| - M]ds \\ &\leq -(c(\rho) - M)[t_{m+1} - t_m] = -\lambda. \end{aligned} \tag{4.21}$$

From (iii), we have

$$\nu(t_{k_i}^-) - \nu(t_{k_i}^+) \geq 0,\ i = 1, 2, ..., l \tag{4.22}$$

Therefore,

$$\begin{aligned} \int_{t_m}^{t_{m+1}} D^+\nu(s)ds &= \int_{t_{k_j}}^{t_{m+1}} D^+\nu(s)ds + \int_{t_{k_{j-1}}}^{t_{k_j}} D^+\nu(s)ds + ... \\ &+ \int_{t_{k_1}}^{t_{k_2}} D^+\nu(s)ds + \int_{t_m}^{t_{k_1}} D^+\nu(s)ds \\ &\geq \nu(t_{m+1}) - \nu(t_m). \end{aligned} \tag{4.23}$$

In view of (4.17), (4.19), (4.22) and (4.23), we have

$$\nu(t_{m+1}) \leq \nu(t_m) - \lambda < b(B) + (N - m - 1)\lambda - \lambda \leq \nu(t_{m+1}),$$

which is a contradiction and therefore there exist a $\hat{t} \in [t_m, t_{m+1}]$ such that (4.18) holds.
Now we shall show that (4.18) implies

$$\nu(t) < b(B) + (N - m - 2)\lambda,\ \hat{t} \leq t \leq t^*,\ \hat{t} \in [t_m, t_{m+1}].$$

Suppose this is not true and let

$$\eta = \inf\{\hat{t} \leq t \leq t^* : \nu(t) \geq b(B) + (N - m - 2)\lambda\}.$$

We can obtain a contradiction by the same argument as in the proof of inequality (4.14). Hence we have

$$\nu(t) < b(B) + (N - m - 2)\lambda,\ t \in [t_{m+1}, t^*].$$

By induction we see that (4.16) is true for any $m = 0, 1, 2, ..., N - 1$. Therefore, we obtain

$$\nu(t) < b(B),\ t \in [t_{N-1}, t^*]. \tag{4.24}$$

On the other hand, using (i),

$$\nu(t^*) \geq b(\|x(t^*) + y(t^*)\|) \geq b(B),$$

which contradicts (4.24). Therefore, the system (2.1) is a uniformly ultimately bounded. The proof is completed. □

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A NOTE ON AN EXTENSION OF ABSOLUTE RIESZ SUMMABILITY FACTORS OF AN INFINITE SERIES AND ITS APPLICATIONS

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Abstract: In the present note, we proved a theorem on a newly defined absolute summability method of an infinite series. Some well known results are the special case of our theorem.

1. **Introduction**

It is well known that H. Bor did the pioneering work in the study of $\left|\overline{N}, p_n\right|_k$ and $\left|\overline{N}, p_n; \delta\right|_k$ summability methods of an infinite series. For the parameter $\delta \geq 0$, Bor [2], extended $\left|\overline{N}, p_n\right|_k$ to $\left|\overline{N}, p_n; \delta\right|_k$ summability. Later on, Suilaman [9] extended $\left|\overline{N}, p_n\right|_k$ to $\left|\overline{N}, p_n, \phi_n\right|_k$ summability, for the sequence (ϕ_n) of positive real constants. He has also shown that, if $\phi_n = \frac{P_n}{p_n}$, the $\left|\overline{N}, p_n, \phi_n\right|_k$ summability becomes $\left|\overline{N}, p_n\right|_k$ but $\left|\overline{N}, p_n; \delta\right|_k$ summability cannot be reduces from $\left|\overline{N}, p_n, \phi_n\right|_k$. We introduced a parameter $\tau \geq 0$ and defined a new summabilty method $\left|\overline{N}_p, \phi_n; \tau\right|_k$ of infinite series.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with (s_n) as the sequence of its partial sums. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty, \text{as n} \to \infty,\ (\mathrm{P_{-i} = p_{-i} = 0,\ i \geq 1}).$$

The sequence - to - sequence transformation $t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$ defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n), generated by the

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sequence of coefficients (p_n). We say that the series $\sum_{n=0}^{\infty} a_n$ is summable $\left|\overline{N}_p, \phi_n; \tau\right|_k$, $k \geq 1$ and $\tau \geq 0$, if

$$\sum_{n=1}^{\infty} \phi_n^{\tau k+k-1} \left|t_n - t_{n-1}\right|^k < \infty,$$

where (ϕ_n) be any sequence of positive real constants.

Remarks:

1. For $\tau = 0$, the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $\left|\overline{N}, p_n; \phi_n\right|_k$ summability due to Sulaiman [9].
2. For $\tau = 0$ and $\phi_n = \frac{P_n}{p_n}$, the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $\left|\overline{N}, p_n\right|_k$ summability due to Bor [2].
3. For $\phi_n = \frac{P_n}{p_n}$, the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $\left|\overline{N}, p_n; \tau\right|_k$ summability due to Bor [2].
4. For $\tau = 0$, $\phi_n = n$, and for all n, the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $|R, p_n|_k$ summability due to Bor [1].
5. For $\phi_n = n$ and for all n, the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $|R, p_n; \tau|_k$, summability due to Sulaiman [8].
6. For $\phi_n = \frac{P_n}{p_n}$ and $p_n = 1$ for all n, the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $|C, 1; \tau|_k$ summability which on substituting $\tau = 0$ becomes $|C, 1|_k$ due to Flett [7].

2. **Main Result**

The aim of this paper is to prove a new result by considering $\left|\overline{N}_p, \phi_n; \tau\right|_k$ summability. In fact, we prove the following result.

Theorem A : Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \text{ as n} \to \infty. \tag{2.1}$$

If (X_n) be an almost increasing sequence and the sequence (λ_n) and (β_n) are such that

$$|\Delta\lambda_n| \leq \beta_n \tag{2.2}$$

$$\beta_n \to 0 \text{ as n} \to \infty \tag{2.3}$$

$$\sum_{n=1}^{\infty} nX_n \left|\beta_n\right| < \infty \tag{2.4}$$

$$\left|\lambda_n\right| X_n = O(1), \ n \to \infty. \tag{2.5}$$

$$\sum_{n=1}^{m} \phi_n^{\tau k+k-1} \left(\frac{p_n}{P_n}\right)^k \left|s_n\right|^k = O(X_m) \text{ as m} \to \infty \tag{2.6}$$

and

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\tau k-1}\frac{1}{P_{n-1}}=O\left[\left(\frac{P_v}{p_v}\right)^{\tau k}\frac{1}{P_v}\right]. \tag{2.7}$$

where (ϕ_n) be a sequence of positive real constants such that $\left(\frac{\phi_n p_n}{P_n}\right)$ is non-increasing sequence, then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\overline{N}_p,\phi_n;\tau\right|_k$, $k\geq 1$ and $0\leq\tau<\frac{1}{k}$.

3. **Lemma** [4]

Under the conditions on (X_n), (λ_n) and (β_n) of the Theorem A, the following conditions holds:

$$n\beta_n X_n = O(1) \text{ as n} \to \infty \tag{3.1}$$

and

$$\sum_{n=1}^{\infty}\beta_n X_n < \infty \tag{3.2}$$

4. **Proof of the Theorem A**

Let (t_n) be the sequence of $(\overline{N},p_n)$ means of the series $\sum_{n=0}^{\infty} a_n\lambda_n$, and

$$t_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v s_v=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{i=0}^{v}a_i\lambda_i=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v,$$

then for $n\geq 1$, we get

$$t_n-t_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v\lambda_v. \tag{4.1}$$

Using Able's transformation to the right hand side of 4.1, we get

$$\begin{aligned} t_n-t_{n-1}&=\frac{p_n s_n\lambda_n}{P_n}-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v s_v\lambda_v+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v s_v\Delta\lambda_v\\ &=t_{n,1}+t_{n,2}+t_{n,3}\ ,\ \text{say.}\end{aligned}$$

Since $|t_{n,1}+t_{n,2}+t_{n,3}|^k\leq 3^k\left(|t_{n,1}|^k+|t_{n,2}|^k+|t_{n,3}|^k\right)$.

In order to prove the Theorem A, it is sufficient to show that

$$\sum_{n=1}^{\infty}\phi_n^{\tau k+k-1}|t_{n,z}|^k<\infty,\ \text{for z} = 1,2,3.$$

We have

$$\sum_{n=1}^{m} \phi_n^{\tau k+k-1} \left|t_{n,1}\right|^k$$
$$= \sum_{n=1}^{m} \phi_n^{\tau k+k-1} \left|\frac{p_n s_n \lambda_n}{P_n}\right|^k$$
$$= O(1) \sum_{n=1}^{m} \phi_n^{\tau k+k-1} \left(\frac{p_n}{P_n}\right)^k \left|s_n\right|^k (\left|\lambda_n\right|)^{k-1} \left|\lambda_n\right|$$
$$= O(1) \sum_{n=1}^{m} \phi_n^{\tau k+k-1} \left(\frac{p_n}{P_n}\right)^k \left|s_n\right|^k \left|\lambda_n\right|, \text{ by } (2.5)$$
$$= O(1) \sum_{n=1}^{m-1} \Delta \left|\lambda_n\right| \sum_{v=1}^{n} \phi_v^{\tau k+k-1} \left(\frac{p_v}{P_v}\right)^k \left|s_v\right|^k + O(1) \left|\lambda_m\right| \sum_{n=1}^{m} \phi_n^{\tau k+k-1} \left(\frac{p_n}{P_n}\right)^k \left|s_n\right|^k$$
$$= O(1) \sum_{n=1}^{m-1} \left|\Delta\lambda_n\right| X_n + O(1) \left|\lambda_m\right| X_m, \text{ by } (2.6)$$
$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \left|\lambda_m\right| X_m, \text{ by } (2.2)$$
$$= O(1) \text{ as m} \to \infty, \text{ by } (3.2) \text{ and } (2.5)$$

Again

$$\sum_{n=2}^{m+1} \phi_n^{\tau k+k-1} \left|t_{n,2}\right|^k$$
$$= \sum_{n=2}^{m+1} \phi_n^{\tau k+k-1} \left|\frac{-p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v\right|^k$$
$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\tau k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{v=1}^{n-1} p_v \left|s_v\right| \lambda_v\right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n}\right)^{\tau k+k-1} \left(\frac{P_n}{p_n}\right)^{\tau k-1} \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} p_v \left|s_v\right|^k \left|\lambda_v\right|\right\} \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} p_v \left|s_v\right|^k \left|\lambda_v\right| \sum_{n=v+1}^{m+1} \left(\frac{\phi_n p_n}{P_n}\right)^{\tau k+k-1} \left(\frac{P_n}{p_n}\right)^{\tau k-1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v}\right)^{\tau k+k-1} p_v \left|s_v\right|^k \left|\lambda_v\right| \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\tau k-1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \phi_v^{\tau k+k-1} \left(\frac{p_v}{P_v}\right)^{\tau k+k-1} \left(\frac{P_v}{p_v}\right)^{\tau k-1} \left|s_v\right|^k \left|\lambda_v\right|, \text{ by } (2.7)$$

$= O(1)\sum_{v=1}^{m} \phi_v^{\tau k+k-1}\left(\frac{p_v}{P_v}\right)^k |s_v|^k |\lambda_v|$
$= O(1)$ as m $\to \infty$, as in case of $|\mathrm{t_{n,1}}|$.

Finally we have,

$$\sum_{n=2}^{m+1} \phi_n^{\tau k+k-1} |t_{n,3}|^k$$
$$= \sum_{n=2}^{m+1} \phi_n^{\tau k+k-1} \left|\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v\right|^k$$
$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\tau k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{v=1}^{n-1} p_v v\,|s_v|\,|\Delta\lambda_v|\right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n}\right)^{\tau k+k-1} \left(\frac{P_n}{p_n}\right)^{\tau k-1} \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} p_v |s_v|^k (v\beta_v)^k\right\} \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n}\right)^{\tau k+k-1} \left(\frac{P_n}{p_n}\right)^{\tau k-1} \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} p_v |s_v|^k (v\beta_v)^k\right\}$$
$$= O(1) \sum_{v=1}^{m} p_v |s_v|^k |(v\beta_v)| \sum_{n=v+1}^{m+1} \left(\frac{\phi_n p_n}{P_n}\right)^{\tau k+k-1} \left(\frac{P_n}{p_n}\right)^{\tau k-1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \phi_v^{\tau k+k-1} \left(\frac{p_v}{P_v}\right)^k |s_v|^k (v\beta_v),\ \text{by } (2.7)$$
$$= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| \sum_{i=1}^{v} \phi_i^{\tau k+k-1} \left(\frac{p_i}{P_i}\right)^k |s_i|^k + O(1)\ m\beta_m \sum_{i=1}^{m} \phi_i^{\tau k+k-1} \left(\frac{p_i}{P_i}\right)^k |s_i|^k$$
$$= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)|\, X_v + O(1) \sum_{v=1}^{m-1} X_v\, |\Delta(\lambda_{v+1})| + O(1)\ m\beta_m\ X_m,$$
$$= O(1) \sum_{v=1}^{m-1} v\,|\beta_v|\, X_v + O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_v + O(1)\ m\beta_m\ X_m,$$
$= O(1)$ as m $\to \infty$, by (2.4), (3.2) and (3.1).

Thus we get
$$\sum_{n=1}^{\infty} \phi_n^{\tau k+k-1} |t_{n,z}|^k < \infty,\ \text{for z} = 1, 2, 3.$$
Which completes the proof of the Theorem A.

5. **Applications**

If we consider the special cases, then the following results are the consequences of our Theorem A. We also use the concept that, every increasing sequence is almost increasing but converse need not be true [6]

1. If we take $\tau = 0$ in our Theorem A, then the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $\left|\overline{N}, p_n, \phi_n\right|_k$ and the condition (2.7) reduces to

$$\sum_{n=v+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O\left(\frac{1}{P_v}\right), \tag{5.1}$$

which always holds. In this case, Theorem A reduces to Result 1 as follow.

Result 1. If the sequences (p_n), (λ_n), (β_n) , (X_n) and (ϕ_n) are as in Theorem A, satisfying conditions (2.1) to (2.6), then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\overline{N}, p_n, \phi_n\right|_k$, $k \geq 1$.

2. If we put $\tau = 0$ and $\phi_n = \frac{P_n}{p_n}$ in Theorem A, then the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $\left|\overline{N}, p_n\right|_k$ and the condition (2.6) reduces to

$$\sum_{n=1}^{m} \left(\frac{p_n}{P_n}\right) |s_n|^k = O(X_m) \text{ as m} \to \infty \tag{5.2}$$

while the condition (2.7) reduces to (5.1) and the condition that $\left(\frac{\phi_n p_n}{P_n}\right)$ is non-increasing becomes redundant. In this case, Theorem A reduces to Result 2 as follow.

Result 2. (Bor [3]) If the sequences (p_n), (λ_n), (β_n) and (X_n) are as in Theorem A, satisfying conditions (2.1) to (2.5) plus (5.2), then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

3. If we put $\phi_n = \frac{P_n}{p_n}$ in Theorem A, then the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $\left|\overline{N}, p_n; \tau\right|_k$ and the condition (2.6) reduces to

$$\sum_{n=1}^{m} \left(\frac{p_n}{P_n}\right)^{\tau k-1} |s_n|^k = O(X_m) \text{ as m} \to \infty \tag{5.3}$$

In this case, Theorem A reduces to Result 3 as follow.

Result 3. If the sequences (p_n), (λ_n), (β_n) and (X_n) are as in Theorem A, satisfying conditions (2.1) to (2.5), (2.7) plus (5.3), then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\overline{N}, p_n; \tau\right|_k$, $k \geq 1$, $\tau \geq 0$ and $0 \leq \tau < \frac{1}{k}$.

4. If we put $\phi_n = n$ in Theorem A, then the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $|R, p_n; \tau|_k$ and the condition (2.6) reduces to

$$\sum_{n=1}^{m} n^{\tau k+k-1} \left(\frac{p_n}{P_n}\right)^k |s_n|^k = O(X_m) \text{ as m} \to \infty \tag{5.4}$$

In this case, Theorem A reduces to Result 4 as follow.

Result 4. If the sequences (p_n), (λ_n), (β_n) and (X_n) are as in Theorem A, satisfying conditions (2.1) to (2.5), (2.7) plus (5.4), then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|R, p_n; \tau|_k$, $k \geq 1$ and $0 \leq \tau < \frac{1}{k}$.

5. If we put $\phi_n = \frac{P_n}{p_n}, p_n = 1$ for all values of n in Theorem A, then the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $|C, 1; \tau|_k$ and the condition (2.6) reduces to

$$\sum_{n=1}^{m} n^{\tau k-1} |s_n|^k = O(X_m) \text{ as m} \to \infty \tag{5.5}$$

In this case, Theorem A reduces to Result 5 as follow.

Result 5. If the sequences (λ_n), (β_n) and (X_n) are as in Theorem A, satisfying conditions (2.2) to (2.5), (2.7) plus (5.5), then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|C, 1; \tau|_k$, $k \geq 1$ and $0 \leq \tau < \frac{1}{k}$.

6. If we put $\tau = 0$, $\phi_n = \frac{P_n}{p_n}, p_n = 1$ for all values of n in Theorem A, then the summability $\left|\overline{N}_p, \phi_n; \tau\right|_k$ reduces to $|C, 1|_k$ and the condition (2.6) reduces to

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n} = O(X_m) \text{ as m} \to \infty \tag{5.6}$$

In this case, Theorem A reduces to Result 6 as follow.

Result 6.(Mishra & Srivastava [4]) If the sequences (λ_n), (β_n) and (X_n) are as in Theorem A, satisfying conditions (2.1) to (2.5) plus (5.6), then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|C, 1|_k$, $k \geq 1$.

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ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract: The paper is devoted to the development of specific properties of curvature tensors, square length of the Ricci tensor if the manifold satisfies the codazzi type Ricci tensor then it is R-harmonic manifold as well as Ricci - symmetric.

1. **Introduction**

Let (M^n, g), $n \geq 3$ be a C^∞ connected semi-Riemannian manifold and ∇ be its Levi-Civita connection, the Riemannian curvature R, the Concircular curvature tensor C, the conharmonic curvature tensor L, the Projective curvature tensor P and the Conformal curvature tensor $\tilde{C}$ of (M^n, g) are defined by ([1],[2])

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.1}$$

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)} [g(Y,Z)X - g(X,Z)Y] \tag{1.2}$$

$$L(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)} [S(Y,Z)X - S(X,Z)Y - g(X,Z)QY + g(Y,Z)QX] \tag{1.3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)} [g(Y,Z)QX - g(X,Z)QY] \tag{1.4}$$

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y] \tag{1.5}$$

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respectively, where Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, $r = tr\,Q$ is the scalar curvature and $X,\ Y,\ Z \in \chi(M)$, $\chi(M)$ being the Lie-algebra of the vector fields of (M^n, g).

In (1989) Matsumota [3] introduced the notion of an LP-Sasakian manifold. In [4] the authors defined the same notion independently and obtained many results.

In section 2, we defined LP-Sasakian manifold and review some formulas and in section 3, the main results of this paper have been obtained.

2. **Preliminaries**

Let M^n be n-dimensional differentiable manifold with (ϕ, ξ, η)-structure, where ϕ is a $(1,1)$ tensor field, ξ is a vector field, and η is a 1-form on M^n, such that

$$\eta(\xi) = -1 \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi \tag{2.2}$$

$$\phi\xi = 0,\ rank(\phi) = n-1,\ \eta(\phi X) = 0 \tag{2.3}$$

Then M^n admits a Lorntzian metric g, such that

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.4}$$

and M^n is said to admit a Lorentzian almost para-contact structure (ϕ, ξ, η, g).

In this case we have

$$g(X, \xi) = \eta(X) \tag{2.5}$$

$$\Phi(X,Y) = g(X, \phi Y) = g(\phi X, Y) = \Phi(Y, X) \tag{2.6}$$

where ∇ is the covariant differentiation with respect to the Lorentzian metric g makes a time like unit vector field, that is $g(\xi, \xi) = -1$.

The manifold M^n equipped with Lorentzian almost para-contact metric structure (ϕ, ξ, η, g) is said to be a Lorentzian almost para-contact metric manifold [3]. A Lorentzian almost para-contact metric manifold M^n, equipped with the structure (ϕ, ξ, η, g) is called Lorentzian para-contact metric manifold [3], if

$$\Phi(X,Y) = \frac{1}{2}\left[(\nabla_X \eta)Y + (\nabla_Y \eta)X\right] \tag{2.7}$$

A Lorentzian almost para-contact metric manifold M^n, equipped with the structure (ϕ, ξ, η, g) is called LP-Sasakian manifold [3], if

$$\nabla_X \phi)(Y) = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X \tag{2.8}$$

In an LP-Sasakian manifold the 1-form η is closed, also in [3] it is proved that if a n-dimensional (M^n, g) admits a time like unit vector field ξ such that 1-form η associated to ξ is closed then

$$\nabla_X \nabla_Y \eta)Z = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z) \tag{2.9}$$

Further, on such LP-Sasakian manifold (M^n, g), the following relations hold

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.10}$$

$$R(X,\xi)\xi = -X - \eta(X)\xi \tag{2.11}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.13}$$

An LP-Sasakian manifold (M^n, g) is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.14}$$

for any vector fields X, Y, where a, b are functions on M^n ([1],[2]).
In particular if $b = 0$ then η-Eisntein reduces to Einstein manifold.
Further, since ξ is a killing vector, S and r remain constant under it.
Then $\underset{\xi}{\hbar} S = 0$ and $\underset{\xi}{\hbar} r = 0$, where $\hbar$ denotes Lie-derivative.

3. Main Results

Definition 3.1. A differentiable manifold (M^n, g), $n \geq 3$ is said to be ξ-concircularly flat, if $C(X,Y)\xi = 0$.

Theorem 3.2. *Let (M^n, g), $n \geq 3$ be an LP-Sasakian manifold M^n is ξ-concicularly flat if and only if the scalar curvature $r = n(n-1)$.*

Proof. For ξ-Concircularly flat LP-Sasakian manifold M^n, we have from (1.1)

$$R(X,Y)\xi = \frac{r}{n(n-1)}\left[g(Y,\xi)X - g(X,\xi)Y\right] \tag{3.1}$$

using (2.5), (2.10) in (3.1), we get

$$\left(1 - \frac{r}{n(n-1)}\right)\eta(Y)X + \left(-1 + \frac{r}{n(n-1)}\right)\eta(X)Y = 0$$

putting$Y = \xi$, yields $r = n(n-1)$.
Conversely, if $r = n(n-1)$, putting $Z = \xi$ in (1.2) and using (2.10), we get $C(X,Y)\xi = 0$.
That is the manifold is ξ-concircularly flat.
This complete the proof of theorem (3.2). □

Definition 3.3. A differentiable manifold (M^n, g), $n \geq 3$ is said to be ξ-conhormonically flat, if $L(X,Y)\xi = 0$.

Theorem 3.4. *Let (M^n, g), $n \geq 3$ be a ξ-conhormonically flat LP-Sasakian manifold. Then the manifold (M^n, g) is an η-Einstein manifold.*

Proof. For ξ-conhormonically flat LP-Sasakian manifold, from (1.3), we have

$$R(X,Y)\xi = \frac{1}{(n-2)}\left[S(Y,\xi)X - S(X,\xi)Y - g(X,\xi)QY + g(Y,\xi)QX\right], \quad (3.2)$$

using (2.10),(2.12) and (2.5) in (3.2), we have

$$QX = \left(\frac{1}{n-2}\right)X + \left(\frac{n}{n-2}\right)\eta(X)\xi$$

That is manifold M^n is an η-Einstein manifold.
where $a = \frac{1}{n-2}$, $b = \frac{n}{n-2}$
This complete proof of the theorem (3.4). □

Corollary 3.5. *In a ξ-conhormonically flat LP-Sasakian manifold M^n, $n \geq 3$ the scalar curvature vanishes.*

Theorem 3.6. *If ξ-conhormonically flat LP-Sasakian manifold M^n, $n \geq 3$ satisfying the cyclic Ricci tensor condition, then the manifold is a space form or manifold of constant curvature.*

Proof. Let ξ-conhormonically flat LP-Sasakian manifold satisfying the cyclic Ricci tensor condition, then we obtain

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \quad (3.3)$$

putting $Y = Z = e_i$ in equation(3.3) and taking summation over i, $1 \leq i \leq 3$, we get

$$(\nabla_X S)(e_i,e_i) + 2(\nabla_{e_i} S)(e_i,X) = 0 \quad (3.4)$$

Now $(\nabla_X S)(e_i,e_i) = \nabla_X S(e_i,e_i) - 2S(\nabla_X e_i, e_i)$ where $r = \sum_i S(e_i,e_i)$ and $\{e_i\}$ is an orthonormal basis, we have $\nabla_X e_i = 0$, then

$$(\nabla_X S)(e_i,e_i) = \nabla_X r \quad (3.5)$$

Since $S(X,Y) = g(QX,Y)$ then

$$(\nabla_Z S)(X,Y) = g((\nabla_Z Q)X,Y)$$

putting $Y = Z = e_i$ and taking summation over i, $1 \leq i \leq 3$, we get

$$(\nabla_{e_i} S)(X,e_i) = \frac{1}{2}dr(X) \quad (3.6)$$

using (3.5), 3.6) in (3.4), we get r is constant. This complete the proof of theorem (3.6). □

From (2.14), we get

$$r = an - b \text{ and } s(\xi, \xi) = b - a \tag{3.7}$$

where r is the scalar curvature and Q be the symmetric enomorphism of tangent space at a point corresponding to the Ricci tensor S.
Let l^2 be the square length of the Ricci tensor, then from [5]

$$l^2 = S(Qe_i, e_i) \tag{3.8}$$

where $\{e_i\}$, $i = 1, 2, ..., n$ be orthonormal basis of the tangent space at the point.
From (2.14), we get

$$S(Qe_ie_i) = ar + b\delta(\xi, \xi) \tag{3.9}$$

using (3.7), (3.8) in (3.9), we get

$$l = \sqrt{a^2n + b^2 - 2ab}$$

Theorem 3.7. *In a η-Einstein LP-Sasakian manifold the length of the Ricci tensor S is given by $\sqrt{a^2n + b^2 - 2ab}$.*

Again form (2.14), we get

$$S(X, \xi) = (a - b)\eta(X)$$

This shows that $(a - b)$ is an eigen value of the Ricci tensor S and ξ is an eigen vector corresponding to the value $(a - b)$. Suppose V be any vector field orthogonal to ξ, such that

$$\eta(V) = 0 \tag{3.10}$$

Also from (2.14), we get

$$S(X, V) = ag(X, V) + \eta(X)\eta(V) \tag{3.11}$$

From (3.10) and (3.11), we get $S(X, V) = ag(X, V)$
This shows that different an eigen value of the Ricci tensor S corresponding to the eigen vector V.

Theorem 3.8. *In a η-Einstein LP-Sasakian manifold the Ricci tensor S has two distinct eigen values $(a - b)$ and a of which former is simple and latter is of multiplicity $n - 1$.*

4. An LP-Sasakian η-Einstein Manifold with Codazzi type of Ricci tensor.

Again from (2.14), we get

$$(\nabla_X S)(Y,Z) = b\,[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)] \tag{3.12}$$

By cyclic rotation of (3.12)and using (2.6, (2.7)), we get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = b\,\{[(\nabla_X \eta)Y + (\nabla_Y \eta)X]\ \eta(Z)$$

$$[(\nabla_X \eta)Z + (\nabla_Z \eta)X]\,\eta(Y) + [(\nabla_Y \eta)Z + (\nabla_Z \eta)Y]\ \eta(X)\}$$

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{3.13}$$

This show that an LP-Sasakian η-Einstein manifold has Cyclic Ricci tensor. Suppose that manifold has Ricci tensor of Codazzi type [8], that is

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$$

then from (3.2), we get $(\nabla_X S)(Y,Z) = 0$ i.e. $\nabla S = 0$.

Theorem 4.1. *If η-Einstein LP-Sasakian manifold has Ricci tensor of Codazzi type, then manifold is Ricci symmetric.*

Corollary 4.2. *If η-Einstein LP-Sasakian manifold has Ricci tensor of Codazzi type, then manifold is R-hormonic, i.e.* $(divR)(X,Y,Z) = 0$

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EXISTENCE THEORY FOR FIRST ORDER RANDOM DIFFERENTIAL INCLUSION

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Abstract: In this paper, existence theorems for the first order ordinary random differential inclusions are proved for convex case of random differential inclusion.

1. STATEMENT OF THE PROBLEM

Let (Ω, A, μ) be a complete $\sigma-$finite measure space and let R be the real line. Let $P(R)$ denote the class of all non-empty subsets of R with property p. Given a closed and bounded interval $J = [0, T]$ and given two measurable functions $q_0, q_1 : \Omega \to R$, consider the first order random differential inclusion (RDI),

$$\left.\begin{aligned} x'(t,\omega) \in &\quad F(t, x(t,\omega),\omega) \;\text{ a.e. }\; t \in J \\ x(0,\omega) = &\quad q_0(\omega) \\ x(a,\omega) = &\quad q_1(\omega) \;\text{ if } a \in R \end{aligned}\right\} \tag{1.1}$$

for all $\omega \in \Omega$, where $F : J \times R \times \Omega \to \mathcal{P}_p(R)$.

By a random solution of the RDI (1.1) on $J \times \Omega$ we mean a measurable function $x : \Omega \to AC^1(J, R)$ satisfying for each $\omega \in \Omega$, $x'(t,\omega) = \nu(t,\omega)$ for some measurable $\nu : \Omega \to L^1(J, R)$ such that $\nu(t,\omega) \in F(t, x(t,\omega),\omega)$ a.e. $t \in J$, where $AC^1(J, R)$ is the space of continuous real-valued functions whose first derivative is absolutely continuous on J.

In the this paper, we will prove existence result for convex case of first order random differential inclusions.

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2. AUXILIARY RESULTS

Let $F : J \times R \times R \times \Omega \to \mathcal{P}_p(R)$ be a multi-valued mapping. Then for only measurable function $x : \Omega \to AC^1(J, R)$, let

$$S_F(\omega)(x) = \{\nu \in \mathcal{M}(\Omega, M(J, R)) \mid \nu(t, \omega) \in F(t, x(t, \omega), \omega) \ \text{ a.e. } \ t \in J\} \quad (2.1)$$

and

$$S^1_F(\omega)(x) = \{\nu \in L^1(\Omega, M(J, R)) \mid \nu(t, \omega) \in F(t, x(t, \omega), \omega) \ \text{ a.e. } \ t \in J\} \quad (2.2)$$

This is our set of selection functions for F on $J \times R \times \Omega$. The integral of the random multi-valued function F is defined as

$$\int_0^t F(s, x(s, \omega), \omega) ds = \left\{ \int_0^t \nu(s, \omega) ds : \nu \in S^1_F(\omega)(x) \right\}$$

Furthermore, if the integral $\int_0^t F(s, x(s, \omega), \omega) ds$ exists for every measurable function $x : \Omega \to C(J, R)$, then we say the multi-valued mapping F is Lebesgue integrable on J. We need the following definitions in the sequel.

Definition 2.1. A multi-valued mapping $F : J \times R \times \Omega \to \mathcal{P}_{cp}(R)$ is called strong random Caratheodory if for each $\omega \in \Omega$

(i) $(t, \omega) \mapsto F(t, x, \omega)$ is jointly measurable for each $x, y \in R$, and
(ii) $x \to F(t, x, \omega)$ is Hausdorff continuous almost everywhere for $t \in J$.

Again, a strong random Caratheodory multi-valued function F is called strong L^1-Caratheodory if

(iii) For each real number $r > 0$ there exists a measurable function $h_r : \Omega \to L^1(J, R)$ such that for each $\omega \in \Omega$

$$\|F(t, x, \omega)\|_{\mathcal{P}} = sup\{|\mathsf{u}| \ : \ u \in F(t, x, \omega)\} \le h_r(t, \omega) \ \text{ a.e. } \ t \in J$$

for all $x \in R$ with $|x| \le r$.

Then we have the following lemmas which are well-known in the literature.

Lemma 2.2 (Lasota and Opial)**.** *Let E be a Banach space. If* $\dim(E) < \infty$ *and $F : J \times E \times \Omega \to \mathcal{P}_{cp}(E)$ is strong L^1-Caratheodory, then $S^1_F(\omega)(x) \neq \phi$ for each $x \in E$.*

Lemma 2.3 (Caratheodory theorem)**.** *Let E be a Banach space. If $F : J \times E \to \mathcal{P}_{cp}(E)$ is strong Caratheodory, then the multi-valued mapping $(t, x) \mapsto F(t, x(t))$ is jointly measurable for any measurable $E-$valued function x on J.*

3. EXISTENCE RESULT

We employ the following random fixed-point theorem for completely continuous multi-valued mappings in Banach spaces.

Theorem 3.1 (Dhage). *Let $(\Omega, \mathcal{A})$ be a measurable space, X a separable Banach space and let $Q : \Omega \times X \to \mathcal{P}_{cp,cv}(X)$ be continuous and condensing multi-valued random operator. Furthermore, if the set $\xi = \{u \in M(\Omega, X) \mid \lambda(\omega)u \in Q(\omega)u\}$ is bounded for all measurable functions $\lambda : \Omega \to R$ with $\lambda(\omega) > 1$ on Ω, then $Q(\omega)$ has a random fixed point, i.e., there is measurable function $\xi : \Omega \to X$ such that $\xi(\omega) \in Q(\omega)\xi(\omega)$ for all $\omega \in \Omega$.*

We consider the following set of hypotheses in the sequel.

(A_1) $F(t, x, \omega)$ is compact-convex subset of R for all $(t, x, \omega) \in J \times R \times \Omega$.

(A_2) F is strong random Caratheodory.

(A_3) There exists a measurable function $\gamma : \Omega \to L^1(J, R)$ with $\gamma(t, \omega) > 0$ a.e. $t \in J$ and a continuous nondecreasing function $\psi : R^+ \to (0, \infty)$ such that for each $\omega \in \Omega$

$$\|F(t, x, \omega)\|_{\mathcal{P}} \leq \gamma(t, \omega)\psi(|x|) \quad \text{a.e.} \quad t \in J$$

for all $x \in R$.

Theorem 3.2. *Assume that the hypotheses $(A_1) - (A_2)$ hold. Furthermore, if*

$$\int_C^\infty \frac{dr}{\psi(r)} > T\|\gamma(\omega)\|_{L^1} \tag{3.1}$$

for all $\omega \in \Omega$, where $C = |q_0(\omega)| + T|q_1(\omega)|$, then the RDI (1.1) has a random solution in $C(J, R)$ defined on $J \times \Omega$.

Proof. Let $X = C(J, R)$. Define a multi-valued operator $Q : \Omega \times X \to \mathcal{P}(X)$ by

$$\begin{aligned} Q(\omega)x &= \{u \in \mathcal{M}(\Omega, X) \mid u(t, \omega) = q_0(\omega) + q_2(\omega)t \\ &\quad + \int_0^t (t - s)\nu(s, \omega)ds, \nu \in S_F^1(\omega)(x)\} \\ &= (\mathcal{L} \circ S_F^1(\omega))(x) \end{aligned} \tag{3.2}$$

where $\mathcal{K} : \mathcal{M}(\Omega, L^1(J, R)) \to \mathcal{M}(\Omega, C^1(J, R))$ is a continuous operator defined by

$$\mathcal{K}\nu(t, \omega) = q_0(\omega) + q_2(\omega)t + \int_0^t (t - s)\nu(s, \omega)ds. \tag{3.3}$$

Clearly, the operator $Q(\omega)$ is well defined in view of hypothesis (A_2). We shall show that $Q(\omega)$ satisfies all the conditions of Theorem 3.1.

Step I : First, we show that Q is closed valued multi-valued random operator on $\Omega \times X$. Observe that the operator $Q(\omega)$ is equivalent to the composition $\mathcal{K} \circ S^1_F(\omega)$ of two operators on $L^1(J,R)$, where $\mathcal{L} : \mathcal{M}(\Omega, L^1(J,R) \to X$ is the continuous operator defined by (3.3).

Next, we show that $Q(\omega)$ is a multi-valued random operator on X. First, we show that the multivalued map $(\omega, x) \mapsto S^1_F(\omega)(x)$ is measurable. Let $f \in \mathcal{M}(\Omega, L^1(J,R))$ be arbitrary. Then we have

$$\begin{aligned}
d(f, S^1_F(\omega)(x)) &= \inf\left\{\|f(\omega) - h(\omega)\|_{L^1} : h \in S^1_F(\omega)(x)\right\}\\
&= \inf\left\{\int_0^T |f(t,\omega) - h(t,\omega)|dt : h \in S^1_F(\omega)(x)\right\}\\
&= \int_0^T \inf\left\{|f(t,\omega) - z| : z \in F(t, x(t,\omega), \omega)\right\} dt\\
&= \int_0^T d\left(f(t,\omega), F(t, x(t,\omega),\omega)\right) dt.
\end{aligned}$$

But by hypothesis (A_2), the mapping $F(t, x(t,\omega), \omega)$ is measurable. Now the function $z \mapsto d\left(z, F(t,x,\omega)\right)$ is continuous and hence the mapping

$$(t, x, \omega, f) \mapsto d\left(f(t,\omega), F(t, x(\eta(t),\omega),\omega)\right)$$

is measurable from $J \times X \times \Omega \times L^1(J,R)$ into R^+. Now the integral is the limit of the finite sum of measurable functions, and so, $d(f, S^1_F(\omega)(x))$ is measurable. As a result, the multi-valued mapping $(\cdot,\cdot) \to S^1_{F(\cdot)}(\cdot)$ is jointly measurable.

Define the multi-valued map ϕ on $J \times X \times \Omega$ by

$$\phi(t,x,\omega) = (\mathcal{K} \circ S^1_F(\omega))(x)(t) = \int_0^t (t-s)F(s, x(s,\omega),\omega)ds.$$

We shall show that $\phi(t,x,\omega)$ is continuous in t in the Hausdorff metric on R. Let $\{t_n\}$ be a sequence in J converging to $t \in J$. Then we have

$$\begin{aligned}
&d_H(\phi(t_n, x, \omega), \phi(t,x,\omega))\\
&= d_H\left(\int_0^{t_n}(t_n - s)F(s, x(s,\omega),\omega)ds, \int_0^t (t-s)F(s,x(s,\omega),\omega)ds\right)
\end{aligned}$$

$$
\begin{aligned}
&= d_H\left(\int_0^{t_n}(t_n-s)F(s,x(s,\omega),\omega)ds, \int_0^{t}(t_n-s)F(s,x(s,\omega),\omega)ds\right)\\
&\;+\; d_H\left(\int_0^{t}(t_n-s)F(s,x(s,\omega),\omega)ds, \int_0^{t}(t_n-s)F(s,x(s,\omega),\omega)ds\right)\\
&= d_H\left(\int_J \mathcal{X}_{[0,t_n]}(s)(t-s)F(s,x(s,\omega),\omega)ds, \int_J \mathcal{X}_{[0,t]}(s)(t-s)F(s,x(s,\omega),\omega)ds\right)\\
&\;+\; \int_0^{t} d_H\left((t_n-s)F(s,x(s,\omega),\omega)ds, (t-s)F(s,x(s,\omega),\omega)\right)ds\\
&= \int_J \left|\mathcal{X}_{[0,t_n]}(s)-\mathcal{X}_{[0,t]}(s)\right| |(t-s)|\;\|F(s,x(s,\omega),\omega)\|_{\mathcal{P}}ds\\
&\;+\; \int_0^{t} |(t_n-s)-(t-s)|\;\;\|F(s,x(s,\omega),\omega)\|_{\mathcal{P}}\,ds\\
&= \int_J \left|\mathcal{X}_{[0,t_n]}(s)-\mathcal{X}_{[0,t]}(s)\right| T\|F(s,x(s,\omega),\omega)\|_{\mathcal{P}}ds\\
&\;+\; \int_0^{T} |(t_n-s)-(t-s)|\;\|F(s,x(s,\omega),\omega)\|_{\mathcal{P}}ds\\
&= \int_J \left|\mathcal{X}_{[0,t_n]}(s)-\mathcal{X}_{[0,t]}(s)\right| \gamma(s,\omega)\psi(|x(s,\omega)|)ds\\
&\;+\; \int_0^{T} |t_n-t|\;\|F(s,x(s,\omega),\omega)\|_{\mathcal{P}}ds\\
&= \int_J \left|\mathcal{X}_{[0,t_n]}(s)-\mathcal{X}_{[0,t]}(s)\right| \gamma(s,\omega)\psi(\|x(\omega)\|)ds\\
&\;+\; \int_0^{T} |t_n-t|\;\|F(s,x(s,\omega),\omega)\|_{\mathcal{P}}ds\\
&\to 0 \;\text{ as }\; n\to\infty.
\end{aligned}
$$

Thus the multi-valued map $t \mapsto \phi(t,x,\omega)$ is continuous and hence, by Lemma 3.2, the map $(t,x,\omega) \mapsto \int_0^t (t-s)F(s,x(s,\omega),\omega)ds$ is measurable. Again, since the sum of two measurable multi-valued functions is measurable, the map

$(t,x,\omega) \mapsto q_0(\omega) + q_2(\omega)t + \int\limits_0^t (t-s)F(s,x(s,\omega),\omega)ds$ is measurable. Consequently, $Q(\omega)$ is a random multi-valued operator on $[a,b]$.

Step II : Next, we show that $Q(\omega)$ is totally bounded and continuous on bounded subsets of X for each $\omega \in \Omega$. Let S be a bounded subset of X. Then there is real number $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. First, we show that $Q(\omega)$ is a continuous multi-valued random operator on X. Let $\{x_n\}$ be a sequence in S converging to a point x. Then by Hausdorff continuity of the multi-valued mapping $F(t,x,\omega)$ in x and by the dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} Q(\omega)x_n(t) &= q_0(\omega) + q_2(\omega)t + \lim_{n\to\infty} \int\limits_0^t (t-s)F(s,x_n(s,\omega),\omega)ds \\
&= q_0(\omega) + q_2(\omega)t + \int\limits_0^t \lim_{n\to\infty} (t-s)F(s,x_n(s,\omega),\omega)ds \\
&= q_0(\omega) + q_2(\omega)t + \int\limits_0^t (t-s)F(s,x(s,\omega),\omega)ds \\
&= Q(\omega)x(t)
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a Hausdorff continuous multi-valued random operator on X.

Next we show that $Q(\omega)$ is totally bounded operator on X for each $\omega \in \Omega$. Let $\{y_n(\omega)\}$ be a sequence in $\bigcup Q(\omega)(S)$ for some $\omega \in \Omega$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in X.

Case I : First, we show that $\{y_n(\omega)\}$ is uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, we have $\nu_n(\omega) \in S^1_F(\omega)(x_n)$ for some $x_n \in S$ such that

$$y_n(t,\omega) = q_0(\omega) + q_2(\omega)t + \int\limits_0^t (t-s)\nu_n(s,\omega)ds, \qquad t \in J$$

Therefore,

$$|y_n(t,\omega)| \leq |q_0(\omega)| + T|q_1(\omega)| + \int\limits_0^t |(t-s)|\ |\nu_n(s,\omega)|ds$$

$$\leq |q_0(\omega)| + T|q_1(\omega)| + \int_0^t |(t-s)|\ \|F(s, x_n(s,\omega),\omega)\|_{\mathcal{P}} ds$$

$$\leq |q_0(\omega)| + T|q_1(\omega)| + \int_0^t T\gamma(s,\omega)\psi(\|x_n(\omega)\|)$$

$$\leq |q_0(\omega)| + T|q_1(\omega)| + T\|\gamma(\omega)\|_{L^1}\psi(r)$$

for all $t \in J$. Taking the supremum over t in the above inequality yields,

$$\|y_n(\omega)\| \leq |q_0(\omega)| + T|q_1(\omega)| + T\|\gamma(\omega)\|_{L^1}\psi(r)$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)(X)$.

Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(X)$. Let $t, \tau \in J$. Then, for each $\omega \in \Omega$, we have

$$\begin{aligned}
|y_n(t,\omega) - y_n(\tau,\omega)| &= \left| \int_0^t (t-s)\nu_n(s,\omega)ds - \int_0^\tau k(\tau,s,\omega)\nu_n(s,\omega)ds \right| \\
&\leq \left| \int_0^t (t-s)\nu_n(s,\omega)ds - \int_0^t (\tau-s)\nu_n(s,\omega)ds \right| \\
&\quad + \left| \int_0^t (\tau-s)\nu_n(s,\omega)ds - \int_0^\tau (\tau-s)\nu_n(s,\omega)ds \right| \\
&\leq \left| \int_0^t |(t-s)-(\tau-s)|\ |\nu_n(s,\omega)|ds \right| \\
&\quad + \left| \int_\tau^t |(\tau-s)|\ |\nu_n(s,\omega)|ds \right| \\
&\leq \int_0^\tau |(t-s)-(\tau-s)|\ \|F(s,x_n(s,\omega),\omega)\|_{\mathcal{P}} ds \\
&\quad + \left| \int_\tau^t |(\tau-s)|\ \|F(s,x_n(s,\omega),\omega)\|_{\mathcal{P}} ds \right| \\
&\leq \int_0^\tau |(t-\tau)|\gamma(s,\omega)\psi(\|x(\omega)\|)ds + \left| \int_\tau^t T\gamma(s,\omega)\psi(\|x(\omega)\|)ds \right|
\end{aligned}$$

$$\leq \int_0^T |(t-\tau|\gamma(s,\omega)\psi(r)ds + |p(\tau-\omega) - p(\tau,\omega)|$$

where $p(t,\omega) = \int_0^t |(t-\tau|\gamma(s,\omega)\psi(r)ds$, From the above inequality, it follows that

$$|y_n(t,\omega) - y_n(\tau,\omega)| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(X)$. Now $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous for each $\omega \in \Omega$, so it has a cluster point in view of Arzela-Ascoli theorem. As a result, $Q(\omega)$ is a compact multi-valued random operator on X. Thus $Q(\omega)$ is a continuous and totally bounded and hence completely continuous multi-valued random operator on X.

Step III : Next, we show that $Q(\omega)$ has convex values on X for each $\omega \in \Omega$. Again, let $u_1, u_2 \in Q(\omega)x$. Then there are $\nu_1, \nu_2 \in S_F^1(\omega)(x)$ such that $u_1(t) = q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)\nu_1(s,\omega)ds, \;\; t \in J$ and $u_2(t) = q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)\nu_2(s,\omega)ds, \;\; t \in J$.

Now for any $\lambda \in [0,1]$

$$\begin{aligned}\lambda u_1(t,\omega) + (1-\lambda)u_2(t,\omega) &= \lambda \left(q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)\nu_1(s,\omega)ds \right) \\ &\quad + (1-\lambda)\left(q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)\nu_2(s,\omega)ds \right) \\ &= q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)\left[\lambda\nu_1(s,\omega) + (1-\lambda)\nu_2(s,\omega)\right] ds.\end{aligned}$$

Since $S_F^1(\omega)$ has convex values on X, we have that $\nu(t,\omega) = \lambda u_1(t,\omega) + (1-\lambda)u_2(t,\omega) \in S_F^1(\omega)(x)(t)$ for all $t \in J$. Hence, $\lambda u_1 + (1-\lambda)u_2 \in Q(\omega)x$ and consequently $Q(\omega)x$ is convex for each $x \in X$. As a result, $Q(\omega)$ defines a multi-valued random operator $Q : \Omega \times X \to \mathcal{P}_{cp,c\nu}(X)$.

Step IV : Finally, we show that the set ξ is bounded. Let $u \in \mathcal{M}(\Omega, C(J,R))$ such that $\lambda u(t,\omega) \in Q(\omega)u(t)$ on $J \times \Omega$ for all $\lambda > 1$. Then there is a $\nu \in S_F^1(\omega)(u)$ such that

$$u(t,\omega) = \lambda^{-1}q_0(\omega) + q_2(\omega)t + \lambda^{-1}\int_0^t (t-s)\nu(s,\omega)ds$$

for all $t \in J$ and $\omega \in \Omega$. Therefore,

$$\begin{aligned}
|u(t,\omega)| &\le |q_0(\omega) + q_2(\omega)t| + \int\limits_0^t |(t-s)|\ |\nu(s,\omega)|ds \\
&\le |q_0(\omega)| + T|q_1(\omega)| + \int\limits_0^t T\|F(s,u(s,\omega))\|_{\mathcal{P}} ds \\
&\le |q_0(\omega)| + T|q_1(\omega)| + \int\limits_0^t T\gamma(s,\omega)\psi(|u(s,\omega)|)ds
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$.

Let $m(t,\omega) = \sup_{s\in[0,t]} |u(s,\omega)|$. Then, we have $|u(t,\omega)| \le m(t,\omega)$ for all $(t,w) \in J \times \Omega$. Furthermore, there is a point $t^* \in [0,t]$ such that $m(t,\omega) = |u(t^*,\omega)|$. Hence, we have

$$\begin{aligned}
m(t,\omega) = |u(t^*,\omega)| &\le |q_0(\omega)| + T|q_1(\omega)| + \int\limits_0^t T\gamma(s,\omega)\psi(|u(s,\omega)|)ds \\
&\le C + \int\limits_0^t T\gamma(s,\omega)\psi(m(s,\omega))ds
\end{aligned}$$

where $C = |q_0(\omega)| + T|q_1(\omega)|$. Put

$$w(t,\omega) = C + \int\limits_0^t T\gamma(s,\omega)\psi(m(s,\omega))ds$$

Differentiating w.r.t. t,

$$\left.\begin{array}{ll} w'(t,\omega) = & T\gamma(t,\omega)\psi(m(t,\omega)) \\ w(0,\omega) = & C \end{array}\right\} \tag{3.4}$$

for all $t \in J$ and $\omega \in \Omega$.

From the above expression, we obtain

$$\left.\begin{array}{ll} \frac{w'(t,\omega)}{\psi(w(t,\omega))} & \le T\gamma(t,\omega) \\ w(0,\omega) & = C \end{array}\right\} \tag{3.5}$$

Integrating the above inequality from 0 to t,

$$\int\limits_0^t \frac{w'(s,\omega)}{\psi(w(s,\omega))}ds \le \int\limits_0^t T\gamma(t,\omega)ds.$$

By change of the variables,

$$\int_{C}^{w(t,\omega)} \frac{dr}{\psi(r)} \leq T\|\gamma(\omega)\| < \int_{C}^{\infty} \frac{dr}{\psi(r)}$$

Now an application of the mean value theorem yields that there is a constant $M > 0$ such that

$$|u(t,\omega)| \leq m(t,\omega) \leq w(t,\omega) \leq M$$

for all $t \in J$ and $\omega \in \Omega$. Hence by Theorem 3.1, the RDI (1.1), has a random solution on $J \times \Omega$.

This completes the proof. $\square$

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CERTAIN CLASSES OF UNIVALENT FUNCTIONS HAVING NEGATIVE COEFFICIENTS WITH RESPECT TO CONJUGATE AND SYMMETRIC CONJUGATE POINTS

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Abstract: The classes $TS_c^*(\alpha,\beta)$ and $TC_c(\alpha,\beta)$ of starlike and convex univalent functions of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$ with respect to conjugate points having negative coefficients are considered. Sharp results concerning coefficients and distortion theorems are proved. Also it has been shown that these classes are closed under arithmetic and convex linear combinations. Finally, the radii of starlikeness for the class $TS_c^*(\alpha,\beta)$ and convexity for the class $TC_c(\alpha,\beta)$ are derived. The paper is ended with a remark about the classes $TS_{sc}^*(\alpha,\beta)$ and $TC_{sc}(\alpha,\beta)$ of starlike and convex functions of order α and type β with respect to symmetric conjugate points.

1. **Introduction**

Let T denote the class of those analytic and univalent functions defined in the unit disk $|z| < 1$ whose non- zero coefficients from second on, are negative; that is, an analytic and univalent function f is in T if and only if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

Let $\alpha(0 \leq \alpha < 1)$ and $\beta(0 < \beta \leq 1)$ be real numbers. Consider the following subclasses of T.

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(i) $TS_s^*(\alpha,\beta)$, the class of starlike functions of order α and type β with respect to symmetric points:

$$\left|\left[\frac{zf'(z)}{f(z)-f(-z)}-1\right]\Big/\left[\frac{zf'(z)}{f(z)-f(-z)}+(1-2\alpha)\right]\right|<\beta,$$

(ii) $TS_c^*(\alpha,\beta)$, the class of starlike functions of order α and type β with respect to conjugate points:

$$\left|\left[\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}-1\right]\Big/\left[\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}+(1-2\alpha)\right]\right|<\beta,$$

(iii) $TS_{sc}^*(\alpha,\beta)$, the class of starlike functions of order α and type β with respect to symmetric conjugate points:

$$\left|\left[\frac{zf'(z)}{f(z)-\overline{f(\overline{-z})}}-1\right]\Big/\left[\frac{zf'(z)}{f(z)-\overline{f(\overline{-z})}}+(1-2\alpha)\right]\right|<\beta.$$

Further, $f\in T$ is in, the class of convex functions of order α and type β with symmetric points, if and only if $zf'\in TS_s^*(\alpha,\beta)$. Similarly, the classes $TC_c(\alpha,\beta)$ and $TC_{sc}(\alpha,\beta)$ of convex functions of order α and type β with respect to, respectively, the conjugate points and symmetric conjugate points are defined.

The subclasses TS_s^*, starlike functions with symmetric points; TS_c^*, starlike functions with conjugate points and TS_{sc}^*, starlike functions with respect to symmetric conjugate points were introduced and studied in [8], [3], [1], [2], see also [10], [11], [7], [8], whereas the classes $TS^*(\alpha,\beta)$ and in $TC(\alpha,\beta)$, respectively, of starlike and convex functions of order α and type β, were introduced and studied in Gupta and Jain [4], see also [5].

In an earlier paper [6], we introduced and studied the classes $TS_s^*(\alpha,\beta)$ and $TC_s(\alpha,\beta)$. In the present paper, the classes $TS_c^*(\alpha,\beta), TS_{sc}^*(\alpha,\beta), TC_c(\alpha,\beta)$ and $TC_{sc}(\alpha,\beta)$ have been studied. The results relating to these classes in respect of coefficient inequalities, distortion theorems and the other related findings have been presented. Finally, it has been observed that the results arrived at for the class $TS_{sc}^*(\alpha,\beta)$ are exactly the same as for the class $TS_s^*(\alpha,\beta)$ discussed in [6].

2. **Class** $TS_c^*(\alpha,\beta)$ *and* $TC_c(\alpha,\beta)$

We first discuss the results concerning coefficient inequalities for the classes $TS_c^*(\alpha,\beta)$and $TC_c(\alpha,\beta)$.

Theorem 2.1. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$ is in class $TS_c^*(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty}\{n(1+\beta) + 2(\beta(1-2\alpha)-1)\}|a_n| \le \beta|3-4\alpha| - 1.$$

This result is sharp.

Proof. Let $|z| = 1$. Then

$$\begin{aligned} & \left|zf'(z) - f(z) - \overline{f(\overline{z})}\right| - \beta\left|zf'(z) + (1-2\alpha)(f(z) + \overline{f(\overline{z})}\right| \\ = & \left|z + \sum_{n=2}^{\infty}(n-2)|a_n|z^n\right| - \beta\left|(3-4\alpha)z - \sum_{n=2}^{\infty}\{n + 2(1-2\alpha)\}|a_n|z^n\right| \\ \le & \ 1 - \beta|3-4\alpha| + \sum_{n=2}^{\infty}\{n(1+\beta) + 2(\beta(1-2\alpha)-1)\}|a_n| \le 0. \end{aligned}$$

Hence, by the maximum modulus theorem, $f \in TS_c^*(\alpha, \beta)$.

For the converse, assume that

$$\left|\left[\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}} - 1\right] \Big/ \left[\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}} + (1-2\alpha)\right]\right| < \beta.$$

Since $|Re(z)| \le |z|$, for all z, we have

$$Re\left\{\frac{z + \sum_{n=2}^{\infty}(n-2)|a_n|z^n}{(3-4\alpha)z - \sum_{n=2}^{\infty}(n + 2(1-2\alpha))|a_n|z^n}\right\} < \beta.$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}$ is real and then let $z \to 1$, through real values, we obtain

$$\sum_{n=2}^{\infty}\{n(\beta+1) + 2(\beta(1-2\alpha)-1)\}|a_n| - \beta|3-4\alpha| + 1 \le 0$$

which gives the required condition.

Finally, the function

$$f(z) = z - \sum_{n=2}^{\infty}\frac{\beta|3-4\alpha| - 1}{n(1+\beta) + 2(\beta(1-2\alpha)-1)}z^n$$

is an external function for the theorem. □

Theorem 2.2. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$ is in class $TC_c(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n\{n(1+\beta) + 2(\beta(1-2\alpha)-1)\}|a_n| \le \beta\ |3-4\alpha| - 1.$$

This result is sharp.

Proof. It immediately follows by using Theorem 2.1 and the fact that $f \in TC_c(\alpha,\beta)$ if and only if $zf' \in TS_c^*(\alpha,\beta)$. □

The followings are the distortion results.

Theorem 2.3. *If $f \in TS_c^*(\alpha,\beta)$, then*

$$r - \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}r^2 \le \left|f(z)\right| \le r + \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}r^2, \qquad |z| = r \tag{2.3.1}$$

and

$$1 - \frac{\beta|3-4\alpha|-1}{2\beta(1-\alpha)}r \le \left|f'(z)\right| \le 1 + \frac{\beta|3-4\alpha|-1}{2\beta(1-\alpha)}r, \qquad |z| = r. \tag{2.3.2}$$

Proof. By Theorem 2.1, we have

$$\sum_{n=2}^{\infty} |a_n| \le \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}.$$

Therefore

$$|f(z)| \le r + \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}r^2$$

and

$$|f(z)| \ge r - \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}r^2,$$

which combined together verifies (2.3.1).

Further, note that

$$1 - r\sum_{n=2}^{\infty} n|a_n| \le |f'(z)| \le 1 + r\sum_{n=2}^{\infty} n|a_n|.$$

But, in view of Theorem 2.1, we have

$$\sum_{n=2}^{\infty} n|a_n| \le \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)},$$

which when used in the above yields (2.3.2). □

Remark 2.4. The bounds in (2.3.1) and (2.3.2) are sharp, since the equalities are attained for the function:

$$f(z) = z - \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}z^2, (z = \pm r).$$

Theorem 2.5. *If $f \in TC_c(\alpha,\beta)$, then*

$$r - \frac{\beta|3-4\alpha|-1}{8\beta(1-\alpha)}r^2 \le \left|f(z)\right| \le r + \frac{\beta|3-4\alpha|-1}{8\beta(1-\alpha)}r^2, \qquad |z| = r \tag{2.4.1}$$

and

$$1 - \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}r \le \left|f'(z)\right| \le 1 + \frac{\beta|3-4\alpha|-1}{4\beta(1-\alpha)}r, \qquad |z| = r \tag{2.4.2}$$

with equalities for $f(z) = z - \frac{\beta|3-4\alpha|-1}{8\beta(1-\alpha)} z^2, \quad (z = \pm r)$.

Proof. It follows by using the same technique as in the proof of Theorem 2.3. □

We shall now prove that the class $TS_c^*(\alpha,\beta)$ is closed under arithmetic mean and convex linear combinations, and state similar results without proof for the class $TC_c(\alpha,\beta)$.

Theorem 2.6. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ are in $TS_c^*(\alpha,\beta)$, then $h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty} |a_n + b_n| z^n$ is also in $TS_c^*(\alpha,\beta)$.*

Proof. The proof follows directly by appealing to Theorem 2.1. In fact, f and g being in $TS_c^*(\alpha,\beta)$, we have

$$\sum_{n=2}^{\infty} \{n(1+\beta) + 2(\beta(1-2\alpha)-1)\}|a_n| \le \beta|3-4\alpha| - 1 \tag{2.5.1}$$

and

$$\sum_{n=2}^{\infty} \{n(1+\beta) + 2(\beta(1-2\alpha)-1)\}|b_n| \le \beta|3-4\alpha| - 1. \tag{2.5.2}$$

It is sufficient, for h to be a member of $TS_c^*(\alpha,\beta)$, to show

$$\frac{1}{2}\sum_{n=2}^{\infty} \{n(1+\beta) + 2(\beta(1-2\alpha)-1)\}|a_n + b_n| \le \beta|3-4\alpha| - 1,$$

which follows immediately by the use of (2.5.1) and (2.5.2). □

Theorem 2.7. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ are in $TC_c(\alpha,\beta)$, then $h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty} |a_n + b_n| z^n$ is also in $TC_c(\alpha,\beta)$.*

Theorem 2.8. *Let*

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{\beta|3-4\alpha|-1}{n(1+\beta) + 2(\beta(1-2\alpha)-1)} z^n, (n = 2, 3, ...).$$

Then $f \in TS_c^*(\alpha,\beta)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{(\beta|3-4\alpha|-1)\lambda_n}{n(1+\beta) + 2(\beta(1-2\alpha)-1)} z^n \\ &= z - \sum_{n=2}^{\infty} t_n z^n. \end{aligned}$$

Then

$$\sum_{n=2}^{\infty}\Big\{\frac{n(1+\beta)+2(\beta(1-2\alpha)-1)}{\beta|3-4\alpha|-1}\Big\}t_n=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\le 1.$$

Thus, by Theorem 2.1, $f\in TS_c^*(\alpha,\beta)$.

Conversely, suppose $f\in TS_c^*(\alpha,\beta)$. Again, by Theorem 2.1, we have

$$|a_n|\le\frac{\beta|3-4\alpha|-1}{n(1+\beta)+2(\beta(1-2\alpha)-1)}\qquad n=2,3,.....$$

Setting

$$\lambda_n=\frac{n(1+\beta)+2[\beta(1-2\alpha)-1]}{\beta|3-4\alpha|-1}|a_n|,\qquad n=2,3,.....$$

and $\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n$, we have

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$$

which completes the proof. $\square$

Theorem 2.9. *Let*

$$f_1(z)=z\,and\,f_n(z)=z-\frac{\beta|3-4\alpha|-1}{n\{n(1+\beta)+2(\beta(1-2\alpha)-1)\}}z^n,(n=2,3,....).$$

Then $f\in TC_c(\alpha,\beta)$ *if and only if it can be expressed in the form* $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$, *where* $\lambda_n\ \ge 0$ *and* $\sum_{n=1}^{\infty}\lambda_n=1$.

Finally, in this section, we determine the radii of starlikeness and convexity for the functions in the classes $TS_c^*(\alpha,\beta)$ and $TC_c(\alpha,\beta)$.

Theorem 2.10. *Let* $f\in TS_c^*(\alpha,\beta)$. *Then* f *is starlike in the disc* $|z|<r=r(\alpha,\beta)$, *where*

$$r(\alpha,\beta)=\inf_n\left\{\frac{n(1+\beta)+2(\beta(1-2\alpha)-1)}{n(\beta|3-4\alpha|-1)}\right\}^{\frac{1}{n-1}}.$$

Proof. Noting

$$\left|\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}-1\right|\le\frac{|z|+\sum_{n=2}^{\infty}(n-2)|a_n||z|^n}{2(|z|-\sum_{n=2}^{\infty}|a_n||z|^n)},$$

we find that $\left|\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}\right|\le 1\ for|z|<1\ if$

$$\sum_{n=2}^{\infty}n|a_n||z|^{n-1}<1.$$

Hence f is starlike if

$$|z| \leq \left\{ \frac{n(1+\beta)+2(\beta(1-2\alpha)-1)}{n(\beta|3-4\alpha|-1} \right\}^{\frac{1}{n-1}}, n = 2, 3, ...$$

which completes the proof . □

Theorem 2.11. *Let $f \in TC_c(\alpha, \beta)$. Then f is convex like in the disc $|z| < r = r(\alpha, \beta)$, where*

$$r(\alpha, \beta) = \inf_n \left\{ \frac{n(1+\beta)+2(\beta(1-2\alpha)-1)}{n^2(\beta|3-4\alpha|-1)} \right\}^{\frac{1}{n-1}}.$$

3. Class $TS^*_{sc}(\alpha, \beta)$

In this section, we consider the class $TS^*_{sc}(\alpha, \beta)$ .Given below the coefficient inequality for this class.

Theorem 3.1. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in class $TS^*_{sc}(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\}|a_n| \leq \beta|3-4\alpha| - 1.$$

This result is sharp.

Proof. Let $|z| = 1$. Then

$$\begin{aligned}
& |zf'(z) - f(z) + \overline{f(\overline{-z})}| - \beta|zf'(z) + (1-2\alpha)(f(z) - \overline{f(\overline{-z})})| \\
= \ & |z + \sum_{n=2}^{\infty}(n-1+(-1)^n)|a_n|z^n| - \beta|(3-4\alpha)z \\
- \ & \sum_{n=2}^{\infty}\{n + (1-2\alpha)(1-(-1)^n)\}|a_n|z^n| \\
\leq \ & 1 - \beta(3-4\alpha) + \sum_{n=2}^{\infty}\{n(1+\beta) + (1-(-1)^n)(\beta(1-2\alpha)-1)\}|a_n| \leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, $f \in TS^*_{sc}(\alpha, \beta)$.

For the converse, assume that

$$\left| \left[\frac{zf'(z)}{f(z) - \overline{f(\overline{-z})}} - 1 \right] \Big/ \left[\frac{zf'(z)}{f(z) - \overline{f(\overline{-z})}} + (1-2\alpha) \right] \right| < \beta.$$

Since $|Re(z)| \leq |z|$, for all z, we have

$$Re \left\{ \frac{z + \sum_{n=2}^{\infty}(n-1+(-1)^n)|a_n|z^n}{(3-4\alpha)z - \sum_{n=2}^{\infty}[n + (1-2\alpha)(1-(-1)^n)]|a_n|z^n} \right\} < \beta.$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z) - \overline{f(\overline{-z})}}$ is real and letting $z \to 1$, through real values, we obtain the required condition.

Finally, the function

$$f(z) = z - \sum_{n=2}^{\infty} \frac{\beta|3-4\alpha|-1}{n(1+\beta)+(1-(-1)^n)(\beta(1-2\alpha)-1)} z^n$$

is an extremal function for the theorem. □

Remark 3.2. The coefficient inequality for the class $TS^*_{sc}(\alpha,\beta)$ obtained in Theorem 3.1 is exactly the same as for the class $TS^*_s(\alpha,\beta)$ obtained in [6]. Consequently, the result concerning distortion, closure property and radius of starlikeness for the classwhich, infact, are determined by an application of the coefficient inequality, are the same as those for the class $TS^*_s(\alpha,\beta)$ determined in [6]; and similarly the results concerning for the class $TC_s(\alpha,\beta)$. For the sake of brevity, we omit the details.

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EXTENDING THE LAPLACE TRANSFORM II CONVERGENCE

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Abstract: A linear space $\mathcal{P}$ was previously introduced in order to extend the classical Laplace transform. A convergence structure on $\mathcal{P}$ is now introduced and investigated.

1. Introduction

The Laplace transform has been found to be useful in many areas of applied mathematics, engineering, and physics. In [1], the Laplace transform was extended to a space $\mathcal{P}$ which is isomorphic to $\mathcal{D}_0'$, the space of transformable distributions of Zemanian [3] whose elements are supported on the interval $[0, \infty)$.

The construction of the space $\mathcal{P}$ only requires elementary calculus and classical Laplace transform theory found in Churchill [2].

In this note, we introduce a convergence structure on $\mathcal{P}$. An Inversion Theorem and a Uniqueness Theorem are established. We also show that applying the Laplace transform, differentiation, and multiplying by a polynomial are continuous operations. In the final section, we show that $\mathcal{P}$ is isomorphic to $\mathcal{D}_0'$ by defining a linear bijective bicontinuous map from $\mathcal{P}$ onto $\mathcal{D}_0'$.

2. Preliminaries

In this section, we give a brief review of the material from [1] that will be needed in the sequel.

Let $\mathcal{A}$ denote the set of all piecewise continuous functions of exponential growth on $(-\infty, \infty)$ which vanish on $(-\infty, 0)$.

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The convolution of two functions $f, g \in \mathcal{A}$ is given by

$$(f * g)(t) = \int_0^t f(t-x)g(x)dx.$$

Let H denote the Heaviside function. That is, $H(t) = 1$ for $t \geq 0$ and zero otherwise.

For each $n \in \mathbb{N}$, we denote by H^n the function $H * \ldots * H$ where H is repeated n times.

Now, the space $\mathcal{P}$ is defined as follows.

$$\mathcal{P} = \left\{ \frac{f}{H^k} : f \in \mathcal{A},\, k \in \mathbb{N} \right\}$$

Two elements of $\mathcal{P}$ are equal, denoted $\frac{f}{H^n} = \frac{g}{H^m}$, if and only if $H^m * f = H^n * g$.

The generalized derivative and multiplication by "t" is defined as follows. Let $F = \frac{f}{H^k} \in \mathcal{P}$. Then,

a) $DF := \frac{f}{H^{k+1}}$
b) $T \bullet F := \frac{-kf}{H^{k-1}} + \frac{Tf}{H^k}$, $k \geq 2$ and $T(t) = tH(t)$, $t \in \mathbb{R}$.

The Laplace transform of $F = \frac{f}{H^k} \in \mathcal{P}$ is given by

$$\mathcal{L}F(s) = s^k \mathcal{L}f(s), \text{ where } \mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)dt.$$

The extended Laplace transform satisfies many of the same properties as the classical Laplace transform.

Let $F, G \in \mathcal{P}, \alpha, \beta \in \mathbb{C}$, and $a \in \mathbb{R}$. Then,

a) $\mathcal{L}(\alpha F + \beta G)(s) = \alpha \mathcal{L}F(s) + \beta \mathcal{L}G(s)$
b) $\mathcal{L}(D^n F)(s) = s^n \mathcal{L}F(s) \qquad (n = 0, 1, 2, \ldots)$
c) $\mathcal{L}(T \bullet F)(s) = -\frac{d}{ds}\mathcal{L}F(s)$
d) $\mathcal{L}(\tau_a F)(s) = e^{-as}\mathcal{L}F(s)$, where $F = \frac{f}{H^k}$, $\tau_a F = \frac{\tau_a f}{H^k}$, and $\tau_a f(t) = f(t-a)$.

3. Convergence

Definition 3.1. A sequence $\{F_n\} \in \mathcal{P}$ is said to converge to $F \in \mathcal{P}$, denoted $\mathcal{P}\text{-}\lim_{n\to\infty} F_n = F$, provided there exist $\gamma \in \mathbb{R}$, $k \in \mathbb{N}$, and $f, f_n \in \mathcal{A}$ $(n \in \mathbb{N})$ such that

$$\text{(i)} \quad F = \frac{f}{H^k},\ F_n = \frac{f_n}{H^k}, \quad n \in \mathbb{N}$$

and

$$\text{(ii)} \quad \sup_{-\infty < t < \infty} e^{\gamma t} |f_n(t) - f(t)| \to 0 \text{ as } n \to \infty.$$

Theorem 3.2. (Unique Limits) *Let $F, G, F_n \in \mathcal{P}$ $(n \in \mathbb{N})$. If $\mathcal{P}\text{-}\lim\limits_{n\to\infty} F_n = F$ and $\mathcal{P}\text{-}\lim\limits_{n\to\infty} F_n = G$, then $F = G$.*

Proof. Since $\mathcal{P}\text{-}\lim\limits_{n\to\infty} F_n = F$, there exist $f_n, f \in \mathcal{A}$, $k \in \mathbb{N}$, and $\gamma \in \mathbb{R}$ such that

$$F_n = \frac{f_n}{H^k}, \tag{3.1}$$

$F = \frac{f}{H^k}$, and $\sup\limits_{-\infty<t<\infty} e^{\gamma t}|f_n(t) - f(t)| \to 0$ as $n \to \infty$. Thus,

$$f_n \to f \text{ uniformly on compact sets as } n \to \infty. \tag{3.2}$$

Now, $\mathcal{P}\text{-}\lim\limits_{n\to\infty} F_n = G$. Thus, there exist $g_n, g \in \mathcal{A}$, $m \in \mathbb{N}$, and $\sigma \in \mathbb{R}$ such that

$$F_n = \frac{g_n}{H^m}, \tag{3.3}$$

$G = \frac{g}{H^m}$, and $\sup\limits_{-\infty<t<\infty} e^{\sigma t}|g_n(t) - g(t)| \to 0$ as $n \to \infty$. Thus,

$$g_n \to g \text{ uniformly on compact sets as } n \to \infty. \tag{3.4}$$

Without loss of generality we may assume $m = k + p$. Now from (3.1) and (3.3), $g_n = f_n * H^p$, $n \in \mathbb{N}$. Thus,

$$\begin{aligned} g - f * H^p &= g - f_n * H^p + f_n * H^p - f * H^p \\ &= g - g_n + (f_n - f) * H^p. \end{aligned}$$

By (3.2), (3.4), and the above, we obtain

$$g = f * H^p.$$

So,

$$G = \frac{g}{H^m} = \frac{f * H^p}{H^k * H^p} = \frac{f}{H^k} = F$$

□

Theorem 3.3. (Inversion) *Let $F \in \mathcal{P}$. Then, there exist $\sigma \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$,*

$$F = \mathcal{P}\text{-}\lim_{n\to\infty} \frac{\frac{1}{2\pi i} H(t) \int_{\gamma-in}^{\gamma+in} e^{st} \frac{\mathcal{L}F(s)}{s^k} ds}{H^k}, \text{ for any } \gamma > \sigma.$$

Proof. Notice that there exists $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}, k \geq k_0$, $F = \frac{f_k}{H^k}$, where $f_k \in C^{(2)}(\mathbb{R}) \cap \mathcal{A}$ and $f_k(0) = f_k'(0) = 0$. For if $F = \frac{g}{H^m}$, $g \in \mathcal{A}$, then for each $k \geq m+3$, $F = \frac{g * H^{k-m}}{H^m * H^{k-m}}$ and $g * H^{k-m} \in C^{(2)}(\mathbb{R}) \cap \mathcal{A}$, $(g * H^{k-m})(0) = (g * H^{k-m})'(0) = 0$.

Now, let $k_0 \in \mathbb{N}$ as above. For a fixed $k \geq k_0$, let $F = \frac{f_k}{H^k}$, where f_k satisfies the above conditions and $f_k(t) = O(e^{\sigma t})$ as $t \to \infty$.

For $n = 1, 2, \ldots$, define

$$g_n(t) = \frac{1}{2\pi i} H(t) \int_{\gamma - in}^{\gamma + in} e^{st} \frac{\mathcal{L}F(s)}{s^k} ds, \text{ where } \gamma > \sigma. \tag{3.5}$$

Since f_k satisfies the conditions stated at the beginning of the proof, there exists a constant $A > 0$ such that

$$\left| \frac{\mathcal{L}F(s)}{s^k} \right| = |\mathcal{L}f_k(s)| \leq \frac{A}{|s|^2}, \quad \text{Re } s \geq \gamma. \tag{3.6}$$

Thus, by (3.5) and (3.6), there exists a constant $M > 0$ (independent of n) such that

$$|g_n(t)| \leq Me^{\gamma t}, \text{ for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N}. \tag{3.7}$$

Moreover, g_n is piecewise continuous on $(-\infty, \infty)$ and vanishes on $(-\infty, 0)$, thus, $g_n \in \mathcal{A}$, $n \in \mathbb{N}$. By the Inversion Theorem for the classical Laplace transform [2],

$$f_k(t) = \lim_{n \to \infty} g_n(t) = \frac{1}{2\pi i} H(t) \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{\mathcal{L}F(s)}{s^k} ds, \text{ for } t \in \mathbb{R}. \tag{3.8}$$

Let $B > 0$. By using (3.6) and (3.8), for $|t| \leq B$ there exists a constant $C > 0$ such that

$$|g_n(t) - f_k(t)| \leq C \int_{|x| > n} \frac{dx}{x^2}.$$

Thus,

$$\sup_{-\infty < t < \infty} e^{-(\gamma+1)t} |g_n(t) - f_k(t)| \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$F = \mathcal{P}\text{-}\lim_{n \to \infty} F_n, \text{ where } F_n = \frac{g_n}{H^k} \quad (n \in \mathbb{N}).$$

This establishes the theorem. □

The following Uniqueness Theorem follows directly from the previous Inversion Theorem and Theorem 3.2.

Uniqueness Theorem. Let $F, G \in \mathcal{P}$ such that $\mathcal{L}F(s)$ exists for Re $s > \sigma_1$ and $\mathcal{L}G(s)$ exists for Re $s > \sigma_2$. If $\mathcal{L}F(s) = \mathcal{L}G(s)$ on $\{s : \text{Re } s = \gamma\}$, where $\gamma > \max\{\sigma_1, \sigma_2\}$, then $F = G$.

The next theorem shows that applying the Laplace transform, differentiation, and multiplying by a polynomial are continuous operations on $\mathcal{P}$.

Theorem 3.4. *Let $F, F_n \in \mathcal{P}$ $(n \in \mathbb{N})$ such that $\mathcal{P}\text{-}\lim_{n \to \infty} F_n = F$. Then,*

a) $\mathcal{L}F_n \to \mathcal{L}F$ uniformly on compact sets in some half-plane Re $s \geq \alpha$ as $n \to \infty$.

b) $\mathcal{P}\text{-}\lim_{n \to \infty} DF_n = DF$.

c) $\mathcal{P}\text{-}\lim_{n\to\infty} Q \bullet F_n = Q \bullet F$*, where* Q *is a polynomial.*

Proof. Let $F, F_n \in \mathcal{P}$ $(n \in \mathbb{N})$ such that $\mathcal{P}\text{-}\lim_{n\to\infty} F_n = F$. Parts a) and b) follow directly from the definitions. Now, by using the definition, it follows that $\mathcal{P}\text{-}\lim_{n\to\infty} T \bullet F_n = T \bullet F$. Thus, using this and linearity, part c) follows. □

Examples 3.5. Let $f(t) = \sum_{k=0}^{\infty} H(t-k)$ and $f_n(t) = \sum_{k=0}^{n} H(t-k)$ for $n = 0, 1, 2, \ldots$. Then for $n = 0, 1, 2, \ldots$.

$$e^{-t}|f(t) - f_n(t)| = e^{-t} \sum_{k=n+1}^{\infty} H(t-k) \le te^{-t} \text{ for } t \ge n \text{ and zero otherwise.}$$

Let $F = \frac{\sum_{k=0}^{\infty} \tau_k H}{H}$ and $F_n = \frac{\sum_{k=0}^{n} \tau_k H}{H}$, for $n = 0, 1, 2, \ldots$.

Then, $F, F_n \in \mathcal{P}$ $(n \in \mathbb{N})$ and $\mathcal{P}\text{-}\lim_{n\to\infty} F_n = F$. So, by part a) of the previous theorem,

$$\begin{aligned} \mathcal{L}F(s) &= \lim_{n\to\infty} \mathcal{L}F_n(s) \\ &= \lim_{n\to\infty} s \sum_{k=0}^{n} \frac{e^{-ks}}{s} \\ &= \frac{1}{1-e^{-s}} = (1/2)\, e^{s/2} \operatorname{csch}(s/2), \quad \text{for Re } s > 0. \end{aligned}$$

4. **Transformable Distributions**

In this section, we will show that the space $\mathcal{P}$ is isomorphic to the space of transformable distributions supported on the half-line. It is assumed that the reader is familiar with the theory of distributions [3].

The k^{th} order distributional derivative operator is denoted by D^k. The following well known facts will be used throughout this section.

For any $k, n \in \mathbb{N}$ and $f \in C(\mathbb{R})$,

a) $D^k H^k = \delta$.
b) $D^k(H^n * f) = D^k H^n * f = H^n * D^k f$.

Denote the space of transformable distributions [3] supported on the half-line $[0, \infty)$ by $\mathcal{D}'_0$. That is, $\nu \in \mathcal{D}'_0$ if and only if there exist a $k \in \mathbb{N}$ and $f \in \mathcal{A} \cap C(\mathbb{R})$ such that $\nu = D^k f$. This can be seen by observing that $\nu \in \mathcal{D}'_0$ if and only if there exist $\sigma \in \mathbb{R}$, polynomial Q, and a function g such that g is analytic in the half-plane Re $s > \sigma$, $|g(s)| \le Q(|s|)$ for Re $s > \sigma$, and $\nu = D^k f$, where deg $Q = k - 2$ and

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{g(s)}{s^k} \, ds, \quad \gamma > \sigma.$$

A sequence $\{\nu_n\}$ in $\mathcal{D}'_0$ converges to $\nu \in \mathcal{D}'_0$, denoted $\nu_n \to \nu$ in $\mathcal{D}'_0$, provided there exist a $k \in \mathbb{N}, \xi \in \mathbb{R}$, and $f, f_n \in \mathcal{A} \cap C(\mathbb{R})\,(n \in \mathbb{N})$ such that $D^k f_n = \nu_n$, $D^k f = \nu$ and $e^{\xi t} f_n \to e^{\xi t} f$ uniformly on $\mathbb{R}$ as $n \to \infty$.

By showing that the range of the Laplace transform on $\mathcal{P}$ is identical to the range of the Laplace transform on $\mathcal{D}'_0$, it was shown in [1] that $\mathcal{P}$ is algebraically isomorphic to $\mathcal{D}'_0$. In this section, we will give an explicit map Λ from $\mathcal{P}$ to $\mathcal{D}'_0$ and show that Λ is a linear bijective bicontinuous mapping of $\mathcal{P}$ onto $\mathcal{D}'_0$.

Define a mapping $\Lambda : \mathcal{P} \to \mathcal{D}'_0$ by $\Lambda(\frac{f}{H^k}) = D^k f$.
Recall that we can assume that $f \in C(\mathbb{R})$.

Suppose that $\frac{f}{H^k} = \frac{g}{H^n}$.
Then,
$$H^n * f = H^k * g. \tag{4.1}$$
By applying D^{n+k} to (4.1), we obtain
$$D^k f = D^n g.$$
Therefore,
$$\Lambda\left(\frac{f}{H^k}\right) = \Lambda\left(\frac{g}{H^n}\right).$$
Thus, Λ is well-defined.

Theorem 4.1. *Λ is a linear bijective bicontinuous map from $\mathcal{P}$ onto $\mathcal{D}'_0$.*

Proof. Let $F = \frac{f}{H^k}$, $G = \frac{g}{H^n}$, $\lambda \in \mathbb{C}$. Then,
$$\begin{aligned}\Lambda(\lambda F + G) &= \Lambda\left(\tfrac{\lambda f * H^n + g * H^k}{H^{k+n}}\right) = D^{k+n}(\lambda f * H^n + g * H^k)\\ &= \lambda D^k f * D^n H^n + D^n g * D^k H^k = \lambda D^k f + D^n g = \lambda\Lambda(F) + \Lambda(G).\end{aligned}$$
That is, Λ is linear.

Now, suppose $\Lambda(F) = \Lambda(G)$. That is, $D^k f = D^n g$, with $n = k + p$.
Then,
$$\begin{aligned}g = \delta * g = D^n H^n * g &= H^n * D^n g = H^n * D^k f = D^k H^k * (H^p * f)\\ &= \delta * (H^p * f) = H^p * f.\end{aligned}$$
Thus,
$$\frac{f}{H^k} = \frac{H^p * f}{H^p * H^k} = \frac{g}{H^n}.$$
That is, Λ is injective.

Since Λ is clearly surjective, only continuity remains to be established.
Let $F_n, F \in \mathcal{P}$ such that $F_n \to F$ in $\mathcal{P}$. That is, $F_n = \frac{f_n}{H^k}$, $F = \frac{f}{H^k}$ $(f_n, f \in \mathcal{A} \cap C(\mathbb{R}))$ such that $e^{\gamma t} f_n \to e^{\gamma t} f$ uniformly on $\mathbb{R}$ as $n \to \infty$. This gives,
$$D^k f_n \to D^k f \text{ in } \mathcal{D}'_0 \text{ as } n \to \infty.$$
That is, $\Lambda(F_n) = D^k f_n \to D^k f = \Lambda(F)$ in $\mathcal{D}'_0$ as $n \to \infty$. Thus, Λ is continuous.

Now, suppose $\nu_n \to \nu$ in $\mathcal{D}'_0$ as $n \to \infty$. That is, $\nu_n = D^k f_n, \nu = D^k f$ $(f_n, f \in \mathcal{A} \cap C(\mathbb{R}))$ and $e^{\xi t} f_n \to e^{\xi t} f$ uniformly on $\mathbb{R}$ as $n \to \infty$. That is,

$$\frac{f_n}{H^k} \to \frac{f}{H^k} \text{ in } \mathcal{P} \text{ as } n \to \infty.$$

Thus,

$$\Lambda^{-1}(\nu_n) = \Lambda^{-1}(D^k f_n) = \frac{f_n}{H^k} \to \frac{f}{H^k} = \Lambda^{-1}(D^k f) = \Lambda^{-1}(\nu) \text{ in } \mathcal{P} \text{ as } n \to \infty.$$

Hence, Λ^{-1} is continuous and the proof is complete. $\square$

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APPLICATION OF FRACTIONAL CALCULUS TO OBTAIN GENERATING FUNCTIONS

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Abstract: In this paper, we consider the concept of fractional derivative techniques to obtain the linear generating and the bilinear generating functions of some of the identities and gives the generalizations.

1. Introduction

In this paper we apply the concept of fractional derivatives to obtain generating functions. In the process we build up a fractional derivative operator which play the role of augmenting parameters in the hypergeometric functions involved. We then employ this operator in identities involving infinite series, and this exercise culminates in linear as well as bilinear generating functions for a variety of special functions.

The simplest approach to a definition of a fractional derivative commences with the formula

$$\frac{d^\alpha}{dx^\alpha}\{e^{ax}\} = D_x^\alpha\{e^{ax}\} = a^\alpha e^{ax}, \tag{1.1}$$

where α is an arbitrary (real or complex) number.

In 1731, Euler extended the derivative formula [9] to the general form

$$D_x^\mu\{x^\lambda\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, x^{\lambda-\mu}, \tag{1.2}$$

where μ is an arbitrary complex number.

The literature contains many examples of the use of fractional derivatives in the theory of hypergeometric functions, in solving ordinary and partial differential equations and integral equations, see for example Banerji and Moshen [1,2], Banerji Moshen and Binsaad [3], Erdelyi [5], Oldham and Spanier [6],

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Ross [7] and Srivastava and Buschman [8]. Although other methods of solution are usually available, the fractional derivative approach to these problems often suggests methods that are not so obvious in a classical formulation.

The following theorems [9] on term-by-term fractional differentiation embodies, in an explicit form, the definition of a fractional derivative of an analytic function:

Theorem 1.1. *If a function $f(z)$, analytic in the disc $|z| < p$, has the power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^n,\ |z| < p, \tag{1.3}$$

$$\begin{aligned} D_z^{\mu}\ \left\{z^{\lambda-1}\ f(z)\right\} &= \sum_{n=0}^{\infty} a_n\ D_z^{\mu}\ \left\{z^{\lambda+n-1}\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}\ z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{a_n\ (\lambda)_n}{(\lambda-\mu)_n}\ z^n, \end{aligned} \tag{1.4}$$

provided that Re $(\lambda) > 0$, Re $(\mu) < 0$ and $|z| < p$.

Theorem 1.2. *Under the hypotheses surrounding equation (1.3),*

$$\begin{aligned} D_z^{\mu}\ \left\{z^{\lambda-1}\ f(z)\right\} &= \sum_{n=0}^{\infty} a_n\ D_z^{\mu}\ \left\{z^{\lambda+n-1}\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}\ z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{a_n\ (\lambda)_n}{(\lambda-\mu)_n}\ z^n, \end{aligned} \tag{1.5}$$

provided that Re $(\lambda) > 0$ and $|z| < p$.

The following fractional derivative formulas are useful in deriving generating functions :

$$\begin{aligned} &D_z^{\lambda-\mu}\ \left\{z^{\lambda-1}(1-az)^{-\alpha}\ (1-bz)^{-\beta}\ (1-cz)^{-\gamma}\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ z^{\mu-1}\ F_D^{(3)}\ [\lambda,\ \alpha,\ \beta,\ \gamma;\ \mu;\ az,\ bz,\ cz] \end{aligned} \tag{1.6}$$

Re $(\lambda) > 0$, $|az| < 1$, $|bz| < 1$, $|cz| < 1$;

$$\begin{aligned} &D_y^{\lambda-\mu}\ \left\{y^{\lambda-1}(1-y)^{-\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\beta; & \\ & \dfrac{x}{1-y} \\ \gamma; & \end{matrix}\right]\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ y^{\mu-1}\ F_2\left[\alpha,\ \beta,\ \lambda;\ \gamma,\ \mu;\ x,\ y\right], \end{aligned} \tag{1.7}$$

Re $(\lambda) > 0$, $|x| + |y| < 1$.

2. LINEAR GENERATING FUNCTIONS

Gegenbauer polynomials of two variables and one parameter are defined by [4]

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x,y;\alpha)t^n \;=\; (\alpha-2xt+yt^2)^{-\lambda} \tag{2.1}$$

Take $\alpha=1$ in Eq. (2.1) and consider the identity

$$(1-2xt+yt^2)^{-\lambda} \;=\; (1+yt^2)^{-\lambda}\left(1-\frac{2xt}{1+yt^2}\right)^{-\lambda}. \tag{2.2}$$

To obtain a generating function from identity (2.2), we first rewrite it as

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(-1)^n(1-2xt)^{-\lambda-n}(yt^2)^n \;=\; (1+yt^2)^{-\lambda}\left(1-\frac{2xt}{1+yt^2}\right)^{-\lambda},$$

$$|yt^2|<1, \left|\frac{2xt}{1+yt^2}\right|<1. \tag{2.3}$$

Now multiply both sides of Eq. (2.3) by $x^{\alpha-1}$ and apply the fractional derivative operator $D_x^{\alpha-\beta}$, we get

$$D_x^{\alpha-\beta}\left\{\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(-1)^n\,x^{\alpha-1}\,(1-2xt)^{-\lambda-n}(yt^2)^n\right\}$$

$$=\;(1+yt^2)^{-\lambda}D_x^{\alpha-\beta}\left\{x^{\alpha-1}\left(1-\frac{2xt}{1+yt^2}\right)^{-\lambda}\right\}. \tag{2.4}$$

Interchanging the order of differentiation and summation, which is valid when $\mathrm{Re}(\alpha)>0$ and $|yt^2|<|(1-2xt)|$, Eq. (2.4) yields

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,(\lambda)_n}{n!}D_x^{\alpha-\beta}\Big\{x^{\alpha-1}\,(1-2xt)^{-\lambda-n}(yt^2)^n\Big\}$$

$$=\;(1+yt^2)^{-\lambda}D_x^{\alpha-\beta}\Big\{x^{\alpha-1}\left(1-\frac{2xt}{1+yt^2}\right)^{-\lambda}\Big\}$$

$$\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,x^{\beta-1}\sum_{n=0}^{\infty}\frac{(-1)^n\,(\lambda)_n}{n!}{}_2F_1\left[\lambda+n,\alpha;\,\beta;\,2xt\right](yt^2)^n$$

$$=\;\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,x^{\beta-1}(1+yt^2)^{-\lambda}{}_2F_1\left[\lambda,\alpha;\,\beta;\,\frac{2xt}{1+yt^2}\right],\; |2xt|<\min\{1,|1+yt^2|\} \tag{2.5}$$

or

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,(\lambda)_n}{n!}\,{}_2F_1\left[\lambda+n,\alpha;\,\beta;\,2xt\right](yt^2)^n=(1+yt^2)^{-\lambda}\,{}_2F_1\left[\lambda+n,\alpha;\,\beta;\,\frac{2xt}{1+yt^2}\right]. \tag{2.6}$$

Again consider identity (2.3) and multiply both sides by $x^{\alpha-1}(1-x)^{-\rho}$ and apply the operator $D^{\alpha-\beta}$ and then reverse the order of differentiation and summation, which is valid when $\mathrm{Re}(\alpha)>0$, we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,(\lambda)_n}{n!}\,D_x^{\alpha-\beta}\Big\{x^{\alpha-1}\,(1-x)^{-\rho}\,(1-2xt)^{-\lambda-n}\Big\}(yt^2)^n$$

$$=\ (1+yt^2)^{-\lambda}D_x^{\alpha-\beta}\Big\{x^{\alpha-1}(1-x)^{-\rho}\Big(1-\frac{2xt}{1+yt^2}\Big)^{-\lambda}\Big\}. \tag{2.7}$$

Further assuming that $\left|\dfrac{2xt}{1+yt^2}\right|<1$, and using Theorems 1.1 and 1.2 in Eq. (2.7), we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,(\lambda)_n}{n!}\,F_1\Big[\alpha,\rho,\lambda{+}n;\ \beta;\ x,2xt\Big](yt^2)^n$$

$$=\ (1+yt^2)^{-\lambda}\,F_1\Big[\alpha,\rho,\lambda;\ \beta;\ x,\frac{2xt}{1+yt^2}\Big] \tag{2.8}$$

3. BILINEAR GENERATING FUNCTIONS

In this section, we obtain the bilinear generating functions.

Consider the identity

$$\Big[(1-\sqrt{xt})(1+\sqrt{xt})+(1-\sqrt{yt})(1+\sqrt{yt})\Big]^n\ =\ 2^n\Big[1-\tfrac{t}{2}(x+y)\Big]^n \tag{3.1}$$

or

$$\sum_{r=0}^{n}\binom{n}{r}\Big[(1+\sqrt{xt})(1-\sqrt{xt})\Big]^r\Big[(1+\sqrt{yt})(1-\sqrt{yt})\Big]^{n-r}\ =\ 2^n\Big[1-\frac{t}{2}(x+y)\Big]^n$$

or

$$\sum_{r=0}^{n}\binom{n}{r}(1-xt)^r(1-yt)^{n-r}\ =\ 2^n\Big[1-\frac{t}{2}(x+y)\Big]^n \tag{3.2}$$

Multiplying both sides of Eq(3.2) by $x^{\alpha-1}\,y^{\gamma-1}$ and apply the fractional derivative operators $D_x^{\alpha-\beta}\,D_y^{\gamma-\delta}$, we obtain

$$D_x^{\alpha-\beta}D_y^{\gamma-\delta}\left\{x^{\alpha-1}y^{\gamma-1}\left[\sum_{r=0}^{n}\binom{n}{r}(1-xt)^r(1-yt)^{n-r}\right]\right\}$$

$$=\ D_x^{\alpha-\beta}D_y^{\gamma-\delta}\left\{x^{\alpha-1}y^{\gamma-1}\left[2^n\Big[1-\frac{t}{2}(x+y)\Big]^n\right]\right\}$$

or

$$D_x^{\alpha-\beta}D_y^{\gamma-\delta}\left\{\sum_{r=0}^{\infty}\binom{n}{r}x^{\alpha-1}(1-xt)^r y^{\gamma-1}(1-yt)^{n-r}\right\}$$

$$= 2^n D_x^{\alpha-\beta} x^{\alpha-1} D_y^{\gamma-\delta} \left\{ y^{\gamma-1} \left[1 - \frac{t}{2}(x+y) \right]^n \right\}. \tag{3.3}$$

Now inter changing the order of differentiation and summation, which is valid when $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\gamma) > 0$ Eq. (3.3) yields

$$\sum_{r=0}^{n} \binom{n}{r} D_x^{\alpha-\beta} D_y^{\gamma-\delta} \left\{ x^{\alpha-1}(1-xt)^r y^{\gamma-1}(1-yt)^{n-r} \right\}$$

$$= 2^n D_x^{\alpha-\beta} x^{\alpha-1} D_y^{\gamma-\delta} \left\{ y^{\gamma-1} \left[1 - \frac{t}{2}(x+y) \right]^n \right\}$$

or

$$\sum_{r=0}^{n} \binom{n}{r} D_x^{\alpha-\beta} \left\{ x^{\alpha-1}(1-xt)^r \right\} D_y^{\gamma-\delta} \left\{ y^{\gamma-1}(1-yt)^{n-r} \right\}$$

$$= 2^n D_x^{\alpha-\beta} x^{\alpha-1} D_y^{\gamma-\delta} \left\{ y^{\gamma-1} \left[1 - \frac{t}{2}(x+y) \right]^n \right\} \tag{3.4}$$

Now using Theorem 1.1 and 1.2 in Eq. (3.4), we obtain

$$\sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(\alpha)}{\Gamma(\beta)} x^{\beta-1} {}_2F_1[-r, \alpha;\ \beta;\ xt] \frac{\Gamma(\gamma)}{\Gamma(\delta)} y^{\delta-1} {}_2F_1[-n+r, \gamma;\ \delta;\ yt]$$

$$= 2^n x^{\beta-1} y^{\delta-1} \frac{\Gamma(\alpha)\ \Gamma(\gamma)}{\Gamma(\beta)\ \Gamma(\delta)} F_2 \left[-n, \alpha, \gamma;\ \beta, \delta;\ \frac{xt}{2}, \frac{yt}{2} \right].$$

or

$$\sum_{r=0}^{n} \binom{n}{r} {}_2F_1[-r, \alpha; \beta; xt]\, {}_2F_1[-n+r, \gamma; \delta; yt] = 2^n F_2 \left[-n, \alpha, \gamma;\ \beta, \delta;\ \frac{xt}{2}, \frac{yt}{2} \right]. \tag{3.5}$$

4. CONCLUDING REMARKS

In Sections 2 and 3, we have obtained some linear and bilinear generating functions. In this section we obtain the generalization of the generating function (2.6) and (3.5) in the following form.

Consider the identity (2.3) and multiply both sides by $x^{\alpha-1}$ and apply the fractional derivative operators $D_x^{a_1-b_1} \dots D_x^{a_p-b_q}$, we get

$$D_x^{a_1-b_1} \dots D_x^{a_p-b_q} \left\{ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (-1)^n x^{\alpha-1} (1-2xt)^{-\lambda-n} (yt^2)^n \right\}$$

$$= (1+yt^2)^{-\lambda} D_x^{a_1-b_1} \dots D_x^{a_p-b_q} \left\{ x^{\alpha-1} \left(1 - \frac{2xt}{1+yt^2} \right)^{-\lambda} \right\}$$

Now by using the already quoted procedure we obtain the following generalization of generating function (2.6)

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,(\lambda)_n}{n!}\ {}_{p+1}F_q\left[\begin{matrix}\lambda+n,\,a_1,\cdots,a_p; & \\ & 2xt \\ b_1,\cdots,b_q; & \end{matrix}\right]$$

$$= (1+yt^2)^{-\lambda}\ {}_{p+1}F_q\left[\begin{matrix}\lambda,\,a_1,\cdots,a_p; & \\ & \dfrac{2xt}{1+yt^2} \\ b_1,\cdots,b_q; & \end{matrix}\right] \tag{4.1}$$

Next consider the identity (3.2) and multiply both sides y $x^{\alpha-1}\,y^{\gamma-1}$ and apply the fractional derivative operators $D_x^{a_1-b_1}\dots D_x^{a_p-b_q}\ D_y^{\alpha_1-\beta_1}\dots D_y^{\alpha_l-\beta_m}$ we get

$$D_x^{a_1-b_1}\dots D_x^{a_p-b_q}\ D_y^{\alpha_1-\beta_1}\dots D_y^{\alpha_l-\beta_m}\left\{\sum_{r=0}^{\infty}\binom{n}{r}x^{\alpha-1}y^{\gamma-1}(1-xt)^r(1-yt)^{n-r}\right\}$$

$$= D_x^{a_1-b_1}\dots D_x^{a_p-b_q}\ D_y^{\alpha_1-\beta_1}\dots D_y^{\alpha_l-\beta_m}\left\{2^n\,x^{\alpha-1}\,y^{\gamma-1}\left[1-\frac{t}{2}(x+y)\right]^n\right\}.$$

Again by using the already quoted procedure we obtain the following generalization of generating relation (3.5)

$$\sum_{r=0}^{\infty}\binom{n}{r}\ {}_{p+1}F_q\left[\begin{matrix}-r,\,(a_p); & \\ & xt \\ (b_q); & \end{matrix}\right]\ {}_{l+1}F_m\left[\begin{matrix}-n+r,\,(\alpha_l); & \\ & yt \\ (\beta_m); & \end{matrix}\right]$$

$$= 2^n\ {}_{p+l+1}F_{q+\beta+1}\left[-n,(a_p),(\alpha_l);\,(b_q),(\beta_m);\ \frac{xt}{2},\frac{yt}{2}\right] \qquad (4.2).$$

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A SIMPLE PROOF OF W. N. BAILEY'S $_2\psi_2$ BILATERAL BASIC HYPERGEOMETRIC SERIES TRANSFORMATION FORMULA

K. R. VASUKI AND C. CHAMARAJU

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Abstract: In this paper, we give an alternative proof of W. N. Bailey's $_2\psi_2$ bilateral basic hypergeometric series transformation formula.

1. Introduction

As usual for any complex number a, we define

$$(a)_n := \prod_{k=0}^{n-1}(1-aq^k)$$

where q is any complex number and

$$(a)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The basic hypergeometric series $_{r+1}\varphi_r$ is defined by

$$_{r+1}\varphi_r\left[\begin{matrix} a_1, a_2, ..., a_{r+1} \\ b_1, b_2, ..., b_r \end{matrix}; z\right] := \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_{r+1})_n}{(q)_n(b_1)_n(b_2)_n\cdots(b_r)_n}z^n$$

where, $|z|<1$ and $a_1, a_2, \cdots, a_{r+1}, b_1, b_2, \cdots, b_r$ are arbitrary except that of course $(b_j)_n \neq 0$, $1 \leqslant j \leqslant r$, $n \geq 0$. The bilateral basic hypergeometric series $_r\psi_r$ is defined by

$$_r\psi_r\left[\begin{matrix} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_r \end{matrix}; z\right] := \sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n(b_2)_n\cdots(b_r)_n}z^n,$$

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where $\left|\dfrac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r}\right| < |z| < 1$ and $(a)_{-m} = \frac{(-1)^m q^{m(m+1)/2}}{a^m (q/a)_m}$. One of the most important transformation formula for basic hypergeometric series is the Sear's ${}_3\varphi_2$ basic hypergeometric transformation formula [22]:

$$ {}_3\varphi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; \frac{de}{abc}\right] = \frac{(e/b)_\infty (de/ac)_\infty}{(e)_\infty (de/abc)_\infty}\ {}_3\varphi_2\left[\begin{matrix} b, d/a, d/c \\ d, de/ac \end{matrix}; \frac{e}{b}\right]. \tag{1.1} $$

Similarly, one of most interesting summation theorem for bilateral basic hypergeometric series is the following Ramanujan's ${}_1\psi_1$ summation theorem [21].

$$ {}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}; z\right] = \frac{(az)_\infty (q/az)_\infty (b/a)_\infty (q)_\infty}{(z)_\infty (b/az)_\infty (q/a)_\infty (b)_\infty}, \qquad |b/a| < |z| < 1. \tag{1.2} $$

G. H. Hardy [14, pp. 222,223] described it as "a remarkable formula with many parameters". The first published proof of (1.2) appears to by W. Hahn [13] and M. Jackson [17]. Other proofs have been given by G. E. Andrews [2], [3], Andrews and R. Askey [4], Askey [5], M. E. H. Ismail [16], N. J. Fine [12], K. Mimachi [20], K. Venkatachaliengar [7], [25], S. Corteel and J. Lovejoy [10], A. J. Yee [26] S. H. Chan [9], Z. G. Liu [19], K. W. J. Kadell [18].

Using Ramanujan ${}_1\psi_1$ summation formula and the q-binomial theorem W. N. Bailey [6] proved an interesting transformation of the series ${}_2\psi_2$:

$$ {}_2\psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; z\right] = \frac{(az)_\infty (d/a)_\infty (c/b)_\infty (dq/abz)_\infty}{(z)_\infty (d)_\infty (q/b)_\infty (cd/abz)_\infty}\ {}_2\psi_2\left[\begin{matrix} a, abz/d \\ az, c \end{matrix}; \frac{d}{a}\right]. \tag{1.3} $$

Recently K. R. Vasuki [24] has given an alternative proof of this transformation formula and demonstrated its diverse uses leading to sums of squares theorem, Lambert series identities related to Dedekind eta functions and q-gamma and q-beta identities and also C. Adiga and Vasuki [1] have employed (1.3) to deduce theorem on representation of a integer as sum of triangular numbers. The identity (1.3) has also been employed by Denis [11], N. A. Bhagirathi [8], Vasuki [23] to establish continued fractions of ratios ${}_2\psi_2$ with its contiguous functions. Motivated by these in Section 2 of this paper, we give an alternative simple proof of (1.3) by employing (1.1). Our proof is similar to the proof of (1.2) by Z. G. Liu[19].

We close this section by recalling the following simple facts of q-shifted factorials, which are required in the next section.

$$ (aq^m)_{-m} = \frac{1}{(a)_m}, \tag{1.4} $$

$$(q^{m+1})_{-m} = \frac{1}{(q)_m}, \tag{1.5}$$

$$(a)_\infty = (a)_{-m}(aq^{-m})_\infty, \tag{1.6}$$

$$\left(\frac{bq^{-m}}{a}\right)_\infty = (-1)^m b^m a^{-m} q^{-m(m+1)/2}\left(\frac{aq}{b}\right)_m \left(\frac{b}{a}\right)_\infty \tag{1.7}$$

and

$$(a)_{k-m} = (a)_{-m}(aq^{-m})_k. \tag{1.8}$$

2. **Proof of (1.3)**

For any positive integer m, we consider the series

$$\sum_{n=-m}^{\infty} \frac{(aq^m)_n(b)_n(c)_n}{(q^{m+1})_n(d)_n(e)_n}\left(\frac{de}{abc}\right)^n.$$

Letting $n = k - m$ in the above series and then employing (1.8), we have

$$\sum_{n=-m}^{\infty} \frac{(aq^m)_n(b)_n(c)_n}{(q^{m+1})_n(d)_n(e)_n}\left(\frac{de}{abc}\right)^n$$

$$= \frac{(aq^m)_{-m}(b)_{-m}(c)_{-m}}{(q^{m+1})_{-m}(d)_{-m}(e)_{-m}}\left(\frac{de}{abc}\right)^{-m} \sum_{k=0}^{\infty} \frac{(a)_k(bq^{-m})_k(cq^{-m})_k}{(q)_k(dq^{-m})_k(eq^{-m})_k}\left(\frac{de}{abc}\right)^k. \tag{2.1}$$

Changing b by bq^{-m}, c by cq^{-m}, d by dq^{-m} and e by eq^{-m} in (1.1), we obtain

$${}_3\varphi_2\left[\begin{matrix} a, bq^{-m}, cq^{-m} \\ dq^{-m}, eq^{-m}\end{matrix}; \frac{de}{abc}\right] = \frac{(e/b)_\infty(deq^{-m}/ac)_\infty}{(eq^{-m})_\infty(de/abc)_\infty}\ {}_3\varphi_2\left[\begin{matrix} bq^{-m}, dq^{-m}/a, d/c \\ dq^{-m}, deq^{-m}/ac\end{matrix}; \frac{e}{b}\right]. \tag{2.2}$$

Using this in (2.1), and then employing (1.4)-(1.7), we obtain

$$\sum_{n=-m}^{\infty} \frac{(aq^m)_n(b)_n(c)_n}{(q^{m+1})_n(d)_n(e)_n}\left(\frac{de}{abc}\right)^n$$

$$= \frac{(e/b)_\infty(de/ac)_\infty(aq/d)_m(d/c)_m}{(a)_m(q/c)_m(de/abc)_\infty(e)_\infty} \sum_{n=-m}^{\infty} \frac{(b)_n(d/a)_n(dq^n/c)_n}{(q^{m+1})_n(d)_n(de/ac)_n}\left(\frac{e}{b}\right)^n.$$

Now, letting $m \to \infty$ in the above, we obtain

$${}_2\psi_2\left[\begin{matrix} b, c \\ d, e\end{matrix}; \frac{de}{abc}\right] = \frac{(e/b)_\infty(de/ac)_\infty(aq/d)_\infty(d/c)_\infty}{(a)_\infty(q/c)_\infty(de/abc)_\infty(e)_\infty}\ {}_2\psi_2\left[\begin{matrix} b, d/a \\ d, de/ac\end{matrix}; \frac{e}{b}\right].$$

Replacing a by de/bcz, b by a, c by b, d by c and e by d in the above, we obtain (1.3). This completes the proof.

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DISTRIBUTIONAL GENERALIZED MODIFIED STRUVE TRANSFORM

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Abstract: In this paper, a transform involving generalized modified Struve function in the kernel is extended to a class of (generalized functions) distributions and the properties of a test function space and its dual are studied. Transformable generalized functions are defined and an analyticity theorem, inversion theorem and uniqueness theorem are also proved for the distributional generalized modified Struve transform. It is shown that several classes of differential equations can be solved with the help of this distributional generalized modified Struve transform.

1. **Introduction**

The generalized modified struve transform in classical sense is studied by Pathak [4]

$$F(s) = \{F_{q,a,c}f(t)\}(s) = \int_0^\infty e^{-qst}\Omega(a,c;st)f(t)dt \tag{1.1}$$

under two cases $|q| > 1$ and $|q-1| > 1$ which are based on the following two integrals by Babister [1,p.120],

$$\int_0^\infty e^{-pz}z^{b-1}\Omega(a,c;kz)dz = \Gamma(b)p^{-b}B(b,a;c;kp^{-1}), \tag{1.2}$$

$(Re\ b > 0, Re\ p > 0, Re\ p > Re\ k, |p| > |k|)$ where $B(a,b;\ c;z)$ is the nonhomogeneous hypegeometric function given by Babister [1,p.167] and

$$\int_0^\infty e^{-pz}z^{b-1}\Omega(a,c;kz)dz = -\Gamma(b)(p-k)^{-b}B(b,c-a;c;\frac{k}{k-p}), \tag{1.3}$$

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($Re\ b > 0, Re\ p > 0, Re\ p > Re\ k, |p-k| > |k|$), where $\Omega\,(a, c; z)$ is the generalized modified struve function explored by Babister [1, p.96] and is defined as under:

$$\Omega(a,c;z) = i(2\pi)^{-1}e^{i\pi a}\Gamma(1-a)\Gamma(c)\Gamma(1+a-c)\omega, \tag{1.4}$$

where

$$\omega = i(2^{-c}/\pi)e^{-i\pi(a+c)}e^{z/2}[(1-e^{2\pi ia})\int_0^{(1+)} e^{zu/2}(1+u)^{a-1}(1-u)^{c-a-1}du$$

$$+\{1-e^{2i\pi(c-a)}\}\int_0^{(-1+)} e^{zu/2}(1+u)^{a-1}(1-u)^{c-a-1}du].$$

It satisfies a nonhomogeneous differential equation and, for $c = 2a$, we have

$$\Omega(a,2a;z) = \Gamma(a+1/2)(z/4)^{1/2-a}e^{z/2}L_{a-1/2}(z/2), \tag{1.5}$$

where $L_\nu(z)$ is the modified Struve function [see 2,p.38].

By specializing parameters various other particular cases of $\Omega\,(a, c; z)$ have been obtained by Babister [1, pp.104-107; 113-114].

In this paper a simple generalization of generalized modified Struve transform (1.1) defined in classical sense by

$$F(s) = F_{q,\lambda,a,c}\,\{f(t)\}\,(s) = \int_0^\infty (st)^\lambda e^{-qst}\Omega(a,c;st)f(t)dt \tag{1.6}$$

where s, q, λ are complex numbers ($Re\ \lambda > 0$) and a, c are constants, is extended to a certain class of (generalized functions) distributions and the properties of a test function space and its dual with analyticity theorem are studied. Also the inversion and uniqueness theorems are proved for the generalized modified Struve transform of generalized functions. It is shown that several classes of differential equations can be solved with the help of this generalized modified Struve transform of generalized functions. We first prove an inversion formula for (1.6) in the classical sense and define a differential operator and examine the behavior of the result of its n^{th} operation on the kernel of the transform (1.6), as this will be needed in our study.

2. Classical Inversion Theorem and Differential Operator

Following is the classical inversion theorem.

Theorem 2.1. *Let*

$$F_{q,\lambda,a,c}(f) = F(p) = \int_0^\infty K(px)f(x)dx, \tag{2.1}$$

where $K(px) = (px)^\lambda \exp(-qpx)\Omega(a,c;px)$. *Then*

$$\frac{1}{2}\left[f(x+)+f(x-)\right] = \lim_{T\to\infty}\frac{1}{(2\pi i)}\int_{\sigma-iT}^{\sigma+iT} x^{-s}M(s)\psi(s)ds, \tag{2.2}$$

where

$$\psi(s) = [q^{(\lambda+1-s)}/\Gamma(\lambda+1-s)].\left[B\left(\lambda+1-s,a;c;q^{-1}\right)\right]^{-1}, (|q|>1) \quad (2.3)$$

$$= [-(q-1)^{(\lambda+1-s)}/\Gamma(\lambda+1-s)].\left[B\left(\lambda+1-s,c-a;c;(1-q)^{-1}\right)\right]^{-1}, (|q-1|>1) \quad (2.4)$$

and

$$M(s) = \int_0^\infty p^{-s}F(p)dp, \quad (2.5)$$

provided that

i) the integrals $\int_0^\infty t^{\sigma-1}f(t)dt$ *and* $\int_0^\infty t^{-s}F(t)dt$ *are absolutely convergent* $(s=\sigma+iT,\ 0<T<\infty)$,

ii) $f(t)$ *is of bounded variation in the neighborhood of point* $t=x$,

iii) $f(t) = O(t^\lambda)$, *for small t,* $Re(\lambda+1-s)>0$, $Re(pq)>0, Re(pq)>Re$ p *and* $|q|>1$ *or* $|q-1|>1$,

iv) $K(p_0t)f(t)$ *is bounded for* $t\geq 0$ *and* $Re\ p>p_0>0$.

Proof. Multiplying both sides of (2.1) by p^{-s} and integrating with respect to p from 0 to ∞ we get

$$M(s) = \int_0^\infty p^{-s}F(p)dp = \int_0^\infty p^{-s}dp\int_0^\infty (px)^\lambda \exp(-qpx)\Omega(a,c;px)f(x)dx$$

$$= \int_0^\infty x^{s-1}f(x)dx\int_0^\infty (px)^{\lambda-s}\exp(-qpx)\Omega(a,c;px)d(px). \quad (2.6)$$

On changing the order of integration, which will be shown to be justified by evaluating the second integral on the right by means of the formulae 4.158 and 4.159 from Babister [1, §4.17, p.120] as given by (1.3) and (1.4), we obtain the following respective values of M(s) as follows

$$M(s) = [q^{-(\lambda+1-s)}.\Gamma(\lambda+1-s)].\left[B\left(\lambda+1-s,a;c;q^{-1}\right)\right]\int_0^\infty x^{s-1}f(x)dx, \quad (2.7)$$

$(Re(\lambda+1-s)>0, Req>1)$,

$$= [-(q-1)^{-(\lambda+1-s)}\Gamma(\lambda+1-s)]B\left(\lambda+1-s,c-a;c;(1-q)^{-1}\right)\int_0^\infty x^{s-1}f(x)dx, \quad (2.8)$$

$(Re(\lambda+1-s)>0, Req>1, |q-1|>1)$.

Thus

$$\int_0^\infty x^{s-1}f(x)dx = \psi(s)M(s), \quad (2.9)$$

where $\psi(s)$ and $M(s)$ have the values given by (2.3), (2.4) and (2.7), (2.8) respectively. Applying Mellin's inversion formula to (2.9) we get (2.2) provided that the conditions given by (1.2) and (1.3) are satisfied. □

Now since the modulus of the integrand in (2.6) is $p^{-\sigma}\exp(-qpx)\,|\Omega(a,c:px)|$ $|f(x)|$ and from Babister [1, p. 108, (4.105)], $\Omega(a,c:x)=O(1)$ for small values of x, it follows that the right hand side of (2.6) is absolutely convergent double integral under the conditions (iii) and (iv) of the Theorem 2.1. Consequently we may change the order of integration and integrate first with respective to x.

2.1. **Differential Operator.** Let

$$u=x^{\lambda}e^{-qx}\Omega(a,c;x)=x^{\lambda}e^{-(q-1)x}e^{-x}\Omega(a,c;x).$$

Then,

$$x^{-\lambda}ue^{(q-1)x}=e^{-x}\Omega(a,c;x)$$

and

$$\frac{d}{dx}\left[x^{-\lambda}ue^{(q-1)x}\right]=\frac{d}{dx}e^{-x}\Omega(a,c;x)=-\frac{(c-a)}{(c)}e^{-x}\Omega(a,c+1;x);$$

as also

$$\begin{aligned}\frac{d^m}{dz^m}\left[e^{-z}\Omega(a,c;z)\right] &= (-1)^m\frac{(c-a)_{(m)}}{(c)_{(m)}}e^{-z}\Omega(a,c+m;z)\\ &= (-1)^m\frac{\Gamma(c-a+m)\Gamma(c)}{\Gamma(c-a)\Gamma(c+m)}e^{-z}\Omega(a,c+m;z)\end{aligned}$$

for $m=1,2,3,\ldots$, by Babister [1, p.103]. Therefore

$$\frac{d}{dx}.\frac{d}{dx}\left[x^{-\lambda}ue^{(q-1)x}\right]=\frac{(c-a)_{(2)}}{(c)_{(2)}}e^{-x}\Omega(a,c+2;x)$$

and

$$x^{\lambda}e^{-(q-1)x}\frac{d}{dx}\left[\frac{d}{dx}\left[x^{-\lambda}ue^{(q-1)x}\right]\right]=\frac{(c-a)_{(2)}}{(c)_{(2)}}x^{\lambda}e^{-qx}\Omega(a,c+2;x).$$

Let us define an operator $A_{\lambda,q,x}$ by

$$A_{\lambda,q,x}\phi(x)=x^{\lambda}e^{-(q-1)x}D_x\{D_x[x^{-\lambda}e^{(q-1)x}\phi(x)]\}$$

$$=[D_x^2+2[(q-1)x-\lambda]x^{-1}D_x+[(q-1)^2x^2-2\lambda(q-1)x+\lambda(\lambda+1)]x^{-2}]\phi$$

$$=x^{-2}\left\{x^2D_x^2+2[(q-1)x-\lambda]xD_x+[(q-1)^2x^2-2\lambda(q-1)x+\lambda(\lambda+1)]\right\}\phi, \tag{2.10}$$

where $D_x=D=d/dx$. Therefore, we have

$$A_{\lambda,q,t}[(st)^{\lambda}e^{-qst}\Omega(a,c;st)]=\frac{(c-a)_{(2)}}{(c)_{(2)}}s^2(st)^{\lambda}e^{-qst}\Omega(a,c+2;st)$$

and for $n=0,1,2,3,\ldots$

$$A^n_{\lambda,q,t}[(st)^{\lambda}e^{-qst}\Omega(a,c;st)]=\frac{(c-a)_{(2n)}}{(c)_{(2n)}}s^{2n}(st)^{\lambda}e^{-qst}\Omega(a,c+2n;st). \tag{2.11}$$

For large t,

$$A^n_{\lambda,q,t}[(st)^\lambda e^{-qst}\Omega(a,c;st)] \cong \frac{\Gamma(c)}{\Gamma(a)}\left[\frac{(c-a)_{(2n)}}{(c)_{(2n)}}\right].s^{2n}e^{-(q-1)st}(st)^{\lambda+a-c-2n}$$

and for small t,

$$A^n_{\lambda,q,t}[(st)^\lambda e^{-qst}\Omega(a,c;st)] \cong \left[\frac{(c-a)_{(2n)}}{(c)_{(2n)}}\right].s^{2n}e^{-qst}(st)^\lambda$$

since for large z, $\Omega(a,c,z) \cong \frac{\Gamma(c)}{\Gamma(a)}e^z(z)^{a-c}[1+O(|z|^{-1})]$ by Erdelyi [2], and for small z, $\Omega(a,c;z) \cong O(1)$ by Babister [1, p.108, 4.105].
Hence $A^n_{\lambda,q,t}[(st)^\lambda e^{-qst}\Omega(a,c;st)] < \infty$ when $Re\ s>0, q>1$, $Re\ \lambda \geq 0$ and $Re\ c > Re\ a > 0$.

3. Test Function Space $H_{\alpha,\beta}(I)$ and Its Dual $H'_{\alpha,\beta}(I)$

Let us define functional $\rho^{\lambda,q}_{\alpha,\beta,n}$; $n=0,1,2,3\ldots$ on certain smooth functions $\phi(t),(0<t<\infty)$, by

$$\rho^{\lambda,q}_{\alpha,\beta,n}(\phi) = \sup_{0<t<\infty}\left|e^{-\alpha t}t^{\beta+2n}A^n_{\lambda,q,t}\phi(t)\right|. \tag{3.1}$$

Let $H_{\alpha,\beta}(I)$ be the space of all those complex-valued smooth functions $\phi(t)$ defined on $I=(0,\infty)$ for which $\rho^{\lambda,q}_{\alpha,\beta,n}(\phi)$ is finite for all $n=0,1,2,3\ldots$, where α,β are suitably fixed real numbers and λ, a complex number with $Re\ \lambda \geq 0$, $|q|>1$ and $|q-1|>1$. For any complex number γ, we have $\rho^{\lambda,q}_{\alpha,\beta,n}(\gamma\phi) = |\gamma|.\rho^{\lambda,q}_{\alpha,\beta,n}(\phi)$ and $\rho^{\lambda,q}_{\alpha,\beta,n}(\phi+\psi) \leq \rho^{\lambda,q}_{\alpha,\beta,n}(\phi)+\rho^{\lambda,q}_{\alpha,\beta,n}(\psi)$, $\phi,\psi \in H_{\alpha,\beta}(I)$.
Hence $\rho^{\lambda,q}_{\alpha,\beta,n}$ is a semi norm on $H_{\alpha,\beta}(I)$. Again $\rho^{\lambda,q}_{\alpha,\beta,0}(\varphi)=0$ & implies $\varphi(t)$ is zero element in $H_{\alpha,\beta}(I)$. Hence $\rho^{\lambda,q}_{\alpha,\beta,0}$ is a norm. Thus the collection $\left\{\rho^{\lambda,q}_{\alpha,\beta,n}\right\}_{n=0}^{\infty}$ is a countable multinorm on $H_{\alpha,\beta}(I)$ equipped with topology generated by $\left\{\rho^{\lambda,q}_{\alpha,\beta,n}\right\}_{n=0}^{\infty}$.Thus $H_{\alpha,\beta}(I)$ is a countably multinormed space.

Lemma 3.1. *For every fixed s such that $0<Re\ s<\frac{\alpha}{1-q}$, $|q|>1$ or $|q-1|>1$ and $\beta+Re\ \lambda>0$, $K(st)\in H_{\alpha,\beta}(I)$ where $K(st)=(st)^\lambda e^{-qst}\Omega(a,c;st)$.*

Proof. From (2.11) and (3.1), we have

$$\rho^{\lambda,q}_{\alpha,\beta,n}K(st) = \sup_{0<t<\infty}\left|e^{-\alpha t}t^{\beta+2n}A^n_{\lambda,q,t}K(st)\right|$$

$$= \sup_{0<t<\infty}\left|e^{-\alpha t}t^{\beta+2n}\frac{(c-a)_{(2n)}}{(c)_{(2n)}}s^n(st)^\lambda e^{-qst}\Omega(a,c+2n;st)\right|.$$

Now, for large t and fixed s we have $|st|\to\infty$ and

$$\left|e^{-\alpha t}t^{\beta+2n}\frac{(c-a)_{(2n)}}{(c)_{(2n)}}s^n(st)^\lambda e^{-qst}\Omega(a,c+2n;st)\right|$$

$$= \left|e^{[-\alpha+(1-q)s]t}\frac{(c-a)_{(2n)}}{(c)_{(2n)}}t^\beta s^{-n}(st)^{\lambda+a-c}\right|$$

which tends to zero as $t \to \infty$ since s is fixed with $0 < Re\ s < \frac{\alpha}{1-q}$, $|q| > 1$ or $|q-1| > 1$ and $Re\ c > Re\ a > 0$. For small t,

$$\left| e^{-\alpha t} t^{\beta+2n} \frac{(c-a)_{(2n)}}{(c)_{(2n)}} s^n (st)^\lambda e^{-qst} \Omega(a, c+2n; st) \right|$$
$$= \quad \left| e^{[-\alpha-qy]t} t^\beta \frac{(c-a)_{(2n)}}{(c)_{(2n)}} s^{-n} (st)^{\lambda+2n} \right|$$
$$= \text{a finite number, as } t \to 0, \text{since } \beta + Re\lambda > 0, |q| > 1.$$

Hence $\rho^{\lambda,q}_{\alpha,\beta,n}[K(st)] < \infty$ for $n = 0, 1, 2, 3 \ldots,$ which shows that $K(st) \in H_{\alpha,\beta}(I)$. Thus **(I)** Members of $H_{\alpha,\beta}(I)$ are complex-valued smooth functions defined on I and $H_{\alpha,\beta}(I)$ is a Frechet space. **(II)** If $\{\phi_\nu\}_{\nu=1}^\infty$ converges in $H_{\alpha,\beta}(I)$ to zero, then for every non-negative integer m $\{D^m\phi_\nu\}_{\nu=1}^\infty$ converges to zero function uniformly on every compact subset of I, and so $H_{\alpha,\beta}(I)$ is a testing function space satisfying all necessary conditions for it to be such a space. The collection of all continuous linear functional on $H_{\alpha,\beta}(I)$ is called dual space of $H_{\alpha,\beta}(I)$ and is denoted by $H^{'}_{\alpha,\beta}(I)$. Members of $H^{'}_{\alpha,\beta}(I)$ are (distributions) generalized functions. Since $H_{\alpha,\beta}(I)$ is complete, $H^{'}_{\alpha,\beta}(I)$ is also complete. □

4. **Properties of** $H_{\alpha,\beta}(I)$

As in Zemanian [5, pp. 32-36], $D(I)$ is a space which contains those complex-valued smooth functions $\phi(t)$ defined on $0 < t < \infty$ which have compact supports and $E(I)$ is the space of all complex-valued smooth functions on I. We now compare $H^{'}_{\alpha,\beta}(I)$ and $H_{\alpha,\beta}(I)$ with $D(I)$ and $E(I)$ and their duals and list some of the properties:

Property 4.1 From the definition of spaces $D(I)$ and $E(I)$ we see that $D(I) \subset H_{\alpha,\beta}(I) \subset E(I)$. Since $D(I)$ is dense in $E(I)$, it follows that $H_{\alpha,\beta}(I)$ is dense in $E(I)$. Consequently, the restriction of $f \in H^{'}_{\alpha,\beta}(I)$ to $D(I)$ is in $D^{'}(I)$.

Property 4.2 If $0 < \alpha_1 < \alpha_2$ then $H_{\alpha_1,\beta}(I) \subset H_{\alpha_2,\beta}(I)$ and the topology of $H_{\alpha_1,\beta}(I)$ is stronger than the topology induced on it by the topology of $H_{\alpha_2,\beta}(I)$. Hence the restriction of any $f \in H^{'}_{\alpha_2,\beta}(I)$ to $H_{\alpha_1,\beta}(I)$ is in $H^{'}_{\alpha_1,\beta}(I)$. Also convergence in $H^{'}_{\alpha_2,\beta}(I)$ implies convergence in $H^{'}_{\alpha_1,\beta}(I)$.

Property 4.3 $H_{\alpha,\beta}(I)$ is a dense subspace of $E(I)$, whatever be the choice of α and β. Indeed $D(I) \subseteq H_{\alpha,\beta}(I) \subseteq E(I)$ and since $D(I)$ is dense in $E(I)$ so in $H_{\alpha,\beta}(I)$.

Moreover, convergence of any sequence in $H_{\alpha,\beta}(I)$ implies its convergence in $E(I)$. $E'(I)$ is a subspace of $H^{'}_{\alpha,\beta}(I)$ for any permissible values of α and β.

Property 4.4 The differential operator $t^r A^r_{\lambda,q,t}$ $(r = 1, 2, 3 \ldots)$ are continuous linear mapping of $H_{\alpha,\beta}(I)$ into itself.

The adjoint operator $B^r_{\lambda,q,t}$ of $t^r A^r_{\lambda,q,t}$ is a generalized differential operator on $H^{'}_{\alpha,\beta}(I)$ into $H^{'}_{\alpha,\beta}(I)$ and is defined by $\left\langle B^r_{\lambda,q,t} f, \phi \right\rangle = \left\langle f, t^r A^r_{\lambda,q,t}(\phi) \right\rangle$.

5. The Distributional Generalized Modified Struve Transform

We shall call f a generalized modified Struve transformable generalized function if f is a member of $H^{'}_{\alpha,\beta}(I)$ for some suitably fixed real number β and for some positive real number α. According to §4, Property 4.2, f is then a member of $H^{'}_{\alpha',\beta}(I)$ if for every $0 < \alpha' < \alpha$. This implies that there exists a positive real number σ_f (possibly $\sigma_f = +\infty$) such that $f \in H^{'}_{\alpha,\beta}(I)$ for every $0 < \frac{\alpha}{1-q} < \sigma_f$ and $f \notin H^{'}_{\alpha,\beta}(I)$ for every $0 < \sigma_f < \frac{\alpha}{1-q}$.

Definition 5.1. Let $f \in H^{'}_{\alpha,\beta}(I)$ for some fixed real numbers α and β, with $Re\ s > 0$ and $\beta + Re\ \lambda > 0$. The distributional generalized modified Struve transformation of generalized function f denoted by $F^{'}(s) = F^{'}_{q,\lambda,a,c}\left\{f(t)\right\}(s)$, is defined by $F^{'}(s) = F^{'}_{q,\lambda,a,c}\left\{f(t)\right\}(s) = \langle f(t), K(st)\rangle$ where $K(st) = (st)^{\lambda} e^{-qst}$ $\Omega(a, c; st)$ and $s \in \Omega_f$ where $\Omega_f = \{s : 0 < Res < \sigma_f, -\pi < \arg s < \pi\}$.

Lemma 5.2. *Let α and α' be real numbers with $\alpha < \alpha'$, then for $q > 1$, $Re\ z \geq \alpha', z \neq 0, -\pi < \arg z < \pi$ and $0 < t < \infty$, we have*

$$\left|e^{-\alpha t}(zt)^{\lambda+\beta+2n} e^{-qtz}\Omega(a, c + 2n; tz)\right| \leq C.(1 + z^{Re\mu})$$

where C is constant with respective to t and z and $\mu = \lambda + \beta + a - c$.

Proof. Since $z \neq 0$, and $-\pi < \arg z < \pi$, from the series expansion and asymptotic properties of generalized modified Struve function, we see that for $|z| \leq 1$, there exist a constant $M_{\lambda,a,c}$ independent of z such that

$$\left|z^{\lambda+\beta+2n} e^{-qz}\Omega(a, c + 2n; z)\right| < M_{\lambda,a,c}$$

and another constant $N_{\lambda,a,c}$ independent of z such that for $|z| > 1$

$$\begin{aligned}\left|z^{\lambda+\beta+2n} e^{-qz}\Omega(a, c + 2n; z)\right| &< N_{\lambda,a,c}.z^{Re(\lambda+\beta+a-c)}.e^{Re(1-q)z} \\ &= N_{\lambda,a,c}.z^{Re\mu}.e^{Re(1-q)z}\end{aligned}$$

where $\mu = \lambda + \beta + a - c$. Consequently, for $Re\ z > \alpha'$ and $0 < t < \infty$, there exist constant $B^{a,c}_{\lambda}$ independent of z and x such that

$$\left|e^{-\alpha t}(zt)^{\lambda+\beta+2n} e^{-qtz}\Omega(a, c + 2n; tz)\right| \leq B_{\lambda,a,c}.(1+z^{Re\mu})(1+t^{Re\mu}).e^{-\alpha-Re(q-1)z}$$

Also for

$$q > 1, Rez > \alpha' > \alpha; (1 + t^{Re\mu}).e^{-\alpha+Re(1-q)z}$$

is uniformly bounded on $0 < t < \infty$ by another constant $C_{\lambda,a,c}$ and so

$$\left|e^{-\alpha t}(zt)^{\lambda+\beta+2n} e^{-qtz}\Omega(a, c + 2n; tz)\right| \leq C_{\lambda,a,c}.(1 + z^{Re\mu}).$$

This completes the proof of lemma. □

Theorem 5.3. *(Analyticity Theorem) Let*

$$F'(s) = F^{'}_{q,\lambda,a,c}\left\{f(t)\right\}(s) = \langle f(t), K(st)\rangle \ \text{for } s \in \Omega_f.$$

Then $F'_{q,\lambda,a,c}(s)$ is analytic function on Ω_f and

$$\frac{\partial}{\partial s} F'_{q,\lambda,a,c}(s) = \left\langle f(t), \frac{\partial}{\partial s} K(s,t)\right\rangle. \tag{5.1}$$

Proof. The proof is standard and follows from Zemanian [5]. □

6. Inversion Theorem for the Distributional Generalized Modified Struve Transform

In this section we establish an inversion formula for the distributional generalized modified Struve transformation, which determines the restriction to $D(I)$ of any Struve-transformable generalized function from its generalized modified Struve transform. From this we will obtain a uniqueness theorem, which states that two generalized Struve-transformable generalized functions having the same transformation must have the same restriction to $D(I)$. First we shall prove some lemmas and theorems which will be used for proving the main inversion theorem.

Lemma 6.1. *The function u^{s-1} as a function of u is a member of $H_{\alpha,\beta}(I)$ if Re $s \geq 1-\beta$ and $\alpha > 0$.*

Proof. It is clear that u^{s-1} is differentiable function of u. Consider

$$\begin{aligned}
&\sup_{0<u<\infty} \left|e^{-\alpha u} u^{\beta+2n} A^n_{\lambda,q,u} u^{s-1}\right| \\
&= \sup_{0<u<\infty} \left| e^{-\alpha u} u^{\beta+2n} \sum_{r=0}^{2n} {}^{2n}C_r[(q-1)u - P^i]^r u^{-r} D^{2n-r} u^{s-1}\right| \\
&= \sup_{0<u<\infty} \left| e^{-\alpha u} u^{\beta+2n} \sum_{r=0}^{2n} {}^{2n}C_r[(q-1)u - P^i]^r u^{-r} \frac{(s-1)!}{(2n-r-s+1)!} u^{s-1-2n+r}\right| \\
&= \sup_{0<u<\infty} \left| e^{-\alpha u} u^{\beta+s-1} \sum_{r=0}^{2n} {}^{2n}C_r[(q-1)u - P^i]^r \frac{(s-1)!}{(2n-r-s+1)!}\right| < \infty
\end{aligned}$$

Therefore $\rho^{\lambda,q}_{\alpha,\beta,n}(u^{s-1}) < \infty$, under the conditions stated in Lemma for $n = 0, 1, 2, 3\ldots$ □

Lemma 6.2. *Let β be suitably fixed real number and $f \in H'_{\alpha,\beta}(I)$, then*

$$\int_0^\infty x^{-s}\langle f(u), K(xu)\rangle\, dx = \left\langle f(u), \int_0^\infty x^{-s} K(xu) dx\right\rangle. \tag{6.1}$$

Proof. By using the technique of Riemann sums and following Zemanian [5], one can easily prove the result. □

Lemma 6.3. *Let*

$$F'_{q,\lambda,a,c}(f) = F'(s) \text{ for } 0 < Re\ s < \sigma_f,$$

let $\phi \in D(I)$ *and set* $\psi(s) = \int_0^\infty y^{-s}\phi(y)dy$ *where* $s = \sigma + iT$ *and* σ *is fixed such that* $\max(0,\ 1-\beta) < \sigma < \sigma_f$. *Then for any fixed real number* r *with* $0 < r < \infty$

$$\int_{-T}^{T} \left\langle f(u), u^{s-1}\right\rangle \psi(s)dT = \left\langle f(u), \int_{-T}^{T} u^{s-1}\psi(s)dT\right\rangle \tag{6.2}$$

Proof. The proof can be carried on as in Malgonde and Saxena [3] and Zemanian [5, §3.5 and pp.65-66]. □

Lemma 6.4. *Let* α *and* β *be real numbers such that* $\alpha > 0$, $\beta + Re\ \lambda \geq 0$ *and fix* $\sigma \underline{\underline{\Delta}} Re\ s$ *such that* $(1-\beta) \leq \sigma < \alpha/(1-q)$. *Also let* $\phi \in D(I)$. *Then*

$$\frac{1}{\pi}\int_0^\infty \phi(y)(u/y)^\sigma . \frac{\sin(T.\log(u/y))}{u.\log(u/y)}dy \text{ converges to } \phi(u)$$

in $H_{\alpha,\beta}(I)$ *as* $T \to \infty$.

Proof. Setting $log(u/y) = t$, we will prove that

$$\theta_T(u) = e^{-\alpha u}u^{\beta+2n}(\pi)^{-1}A^n_{\lambda,q,u}\int_{-\infty}^{\infty}[\phi(ue^{-t})e^{(\sigma-1)t} - \phi(u)]\frac{\sin(tT)}{t}dt$$

converges uniformly to zero in $0 < u < \infty$ as, $T \to \infty$ for $n = 0, 1, 2, 3 \ldots$. Since ϕ is smooth and is of bounded support, we have by differentiating under the integral sign $\theta_T(u) = I_1(u) + I_2(u) + I_3(u)$, where

$$I_1(u) = (\pi)^{-1}e^{-\alpha u}.u^\beta \int_{-\infty}^{-\delta}\left\{e^{(\sigma-1)t}u^{2n}A^n_{\lambda,q,u}\phi(ue^{-t}) - u^{2n}A^n_{\lambda,q,x}\phi(u)\right\}\frac{\sin(tT)}{t}dt,$$

and, $I_2(u)$ and $I_3(u)$ are the same integrals with intervals of integration $(-\delta, \delta)$ and (δ, ∞) respectively. As in Zemanian [5] we can prove now that $I_1(u)$, $I_2(u)$ and $I_3(u)$ tend to zero uniformly as $r \to \infty$, which proves the lemma. □

Theorem 6.5. *(Inversion Theorem) Let*

$$F'_{q,\lambda,a,c}(f) = F'(s) \text{ for } 0 < Res < \sigma_f.$$

Then in the sense of convergence in $D'(I)$

$$f(y) = \lim_{T\to\infty} (2\pi i)^{-1}\int_{\sigma-iT}^{\sigma+iT} y^{-s}M(s)\phi(s)ds, \tag{6.3}$$

where σ is any fixed real number such that $0 < \sigma < \sigma_f$, $M(s) = \psi(s)$ is as given by (2.7) and (2.8) and

$$\phi(s) = \int_0^\infty x^{-s} F'(x) dx. \tag{6.4}$$

Proof. Let $\phi \in D(I)$, and choose real numbers α and β such that $\alpha > 0$, $(\beta + Re\ \lambda) \geq 0$ and $0 < (1-\beta) \leq \sigma < \alpha/(1-q) < \sigma_f$. Our object is now to show that

$$\lim_{T\to\infty} \left\langle (2\pi i)^{-1} \int_{\sigma - iT}^{\sigma + iT} y^{-s} M(s)\phi(s) ds, \phi(y) \right\rangle = \langle f, \phi \rangle \tag{6.5}$$

Now the integral on s is a continuous function of y and therefore the left hand side without the limit notation can be rewritten as:

$$(2\pi)^{-1} \int_0^\infty \phi(y) \int_{-T}^{T} y^{-s} M(s)\phi(s) dT dy.$$

Since $\phi(y)$ is of bounded support and the integrand is a continuous function of (y, T), the order of integration may be changed. This yields

$$(2\pi)^{-1} \int_{-T}^{T} M(s) \left\{ \int_0^\infty x^{-s} \langle f(u), K(xu) \rangle \right\} \int_0^\infty y^{-s} \phi(y) dy dT$$

which, by Lemma 6.2, is equal to

$$(2\pi)^{-1} \int_{-T}^{T} \langle f(u), u^{s-1} \rangle \int_0^\infty y^{-s} \phi(y) dy dT$$

provided $Re(\lambda + 1 - s) > 0$, $Re\ q > 0$, $|q| > 1$ or $|q-1| > 1$ and $(\beta + Re\ \lambda) > (c-a) > 0$. By Lemma 6.3,

$$\begin{aligned} &(2\pi)^{-1} \int_{-T}^{T} \langle f(u), u^{s-1} \rangle \int_0^\infty y^{-s} \phi(y) dy dT \\ = &\left\langle f(u), (2\pi)^{-1} \int_{-T}^{T} u^{s-1} \int_0^\infty \phi(y) y^{-s} dy dT \right\rangle. \end{aligned}$$

The order of integration for the repeated integral herein may be changed because again $\phi(y)$ is of bounded support and the integrand is continuous function of (y, T). Upon doing this, we obtain

$$\begin{aligned} &\left\langle f(u), (2\pi)^{-1} \int_0^\infty \phi(y) \int_{-T}^{T} u^{s-1} y^{-s} dT dy \right\rangle \\ = &\left\langle f(u), (\pi)^{-1} \int_0^\infty \phi(y)(u/y)^\sigma . \frac{\sin(T\log(u/y))}{u.\log(u/y)} dy \right\rangle. \end{aligned}$$

The last expression tends to $\langle f(u), \phi(u) \rangle$ as $T \to \infty$ because $f \in H'_{\alpha,\beta}(I)$ and according to Lemma 6.4, the testing function in the last expression converges to $\phi(u)$ in $H_{\alpha,\beta}(I)$. This completes the proof. □

Theorem 6.6. *(Uniqueness Theorem). Let* $F'(s) = F'_{q,\lambda,a,c}(f)$ *for* $0 < Re\ s < \sigma_f$ *and* $G'(s) = F'_{q,\lambda,a,c}(g)$ *for* $0 < Re\ s < \sigma_g$ *and let* $F'(s) = G'(s)$ *for* $0 < Re\ s < \min(\sigma_f, \sigma_g)$*. Then in the sense of equality in* $D'(I)$*,* $f = g$*.*

Proof. The proof is trivial and follows from Malgonde and Saxena [3]. □

7. Distributional Solution to a Class of Differential Equations

The distributional generalized modified Struve transform can be used to solve certain boundary value problems. From Property 4.4, we see that the operator $t^r A^r_{\lambda,q,t} (r = 0, 1, 2, 3...)$ is a continuous linear mapping of $H_{\alpha,\beta}(I)$ into itself. Its adjoint operator $B^r_{\lambda,q,t}$ is a continuous linear mapping of $H'_{\alpha,\beta}(I)$ into itself and is defined by

$$\left\langle B^r_{\lambda,q,t} f, \phi \right\rangle = \left\langle f, t^r A^r_{\lambda,q,t} \phi \right\rangle. \tag{7.1}$$

So we observe that

$$\begin{aligned}
F'_{q,\lambda,a,c}\{(B^r_{\lambda,q,t} \frac{(c)_{(2r)}}{(c-a)_{(2r)}} f)\}(s) &= \left\langle B^r_{\lambda,q,t} \frac{(c)_{(2r)}}{(c-a)_{(2r)}} f(t), K(st) \right\rangle, \text{ by (5.1)} \\
&= \left\langle \frac{(c)_{(2r)}}{(c-a)_{(2r)}} f(t), t^r A^r_{\lambda,q,t} K(st) \right\rangle \\
&= \left\langle f(t), \frac{(c)_{(2r)}}{(c-a)_{(2r)}} t^r A^r_{\lambda,q,t} (st)^{\lambda} e^{-qst} \Omega(a, c; st) \right\rangle \\
&= \left\langle f(t), (st)^{\lambda+r} e^{-qst} \Omega(a, c+2r; st) \right\rangle \\
&= F'_{q,\lambda+r,a,c+2r} \{f(t)\} (s)(f)
\end{aligned}$$

Thus

$$F'_{q,\lambda,a,c} [\, B^r_{\lambda,q,t} \frac{(c)_{(2r)}}{(c-a)_{(2r)}} f \,] = F'_{q,\lambda+r,a,c+2r}(f) \tag{7.2}$$

We can exploit the relation (7.2) to solve a differential equation, with certain boundary conditions, of course of the type

$$B^r_{\lambda,q,t} [\, \frac{(c)_{(2r)}}{(c-a)_{(2r)}} f \,] = g, \tag{7.3}$$

where g is a known generalized function belonging to $H'_{\alpha,\beta}(I)$ and is to be determined. On applying distributional generalized modified Struve transform to (7.3) and using (7.2), we get $F'_{q,\lambda+r,a,c+2r}(f) = F'_{q,\lambda,a,c}(g) = G(s)$, say.
Hence $f = (F'_{q,\lambda+r,c+2r})^{-1}[G(s)]$ which gives a solution of (7.3); where

$$(F'_{q,\lambda+r,c+2r})^{-1}[G(s)] = \frac{1}{2\pi i} [\lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} \psi(s) M(s) y^{-s} ds \,],$$

in which $\psi(s)$ is as given by (2.3) and (2.4) and $M(s) = \int_0^\infty p^{-s} F'_{q,\lambda,a,c}(p) dp$; $s = \sigma + iT$; σ be a real with $\sigma_f < \sigma < \infty$ and $F'_{q,\lambda,a,c}(p) = \langle f(t), K(pt) \rangle$.

Here we have not given the form of $B^r_{\lambda,q,t}$. However, these can be calculated by the method of integration by parts. For $r = 1$, $B^r_{\lambda,q,t}$ is given by
$B_{\lambda,q,t}(f) = [\ (\lambda+1)[\lambda+(3/2)]t^{-1} + [2\lambda+(7/2)]D + tD^2]f$
or

$$tB_{\lambda,q,t}(f) = [(\lambda+1)[\lambda+(3/2)]+[2\lambda+(7/2)]tD+t^2D^2]f = (t^{-\lambda}Dt^{1/2}Dt^{\lambda+3/2})f$$

which is the adjoint operator of $tA^r_{\lambda,q,t}$ and for $r = 1$ is given by

$$tA_{\lambda,q,t}(f) = \{\lambda[\lambda+(1/2)] - [2\lambda-(1/2)]tD + t^2D^2\}f = (t^{\lambda+3/2}Dt^{1/2}Dt^{-\lambda})f.$$

Particular Cases. In view of the general nature of the kernel involved in (1.6), for $\lambda = 0$, we have been able to extend theory of the transform (1.1) and other well-known classical transforms to a certain spaces of (generalized functions) distributions.

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TRIANGULAR SQUARES: SOME NEW IDENTITIES

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Abstract: This paper studies a problem in the theory of figurate numbers: identifying and investigating those numbers which are polygonal in two ways - triangular and square. The author has discovered many identities relating to these and other associated numbers.

The ancient Greeks, particularly the Pythagoreans, linked arithmetic with geometry thus initiating the theory of figurate numbers. They introduced the idea of polygonal numbers - numbers represented by evenly spaced dots arranged symmetrically in the form of triangles, squares, pentagons, etc. The n^{th} figurate or polygonal number of order g is the non-negative integer given by $f_g(n) = \frac{n}{2}[\{(g-2)(n-1)\}+2], n = 0, 1, 2, ...$ Plutarch (1^{st} cent. A.D.) remarked that every triangular number taken 8 times and then increased by 1 gives a square i.e., $8 * \frac{n(n+1)}{2} + 1 = (2n+1)^2$. Diophantus ($3^{\mathrm{rd}}$ century A.D.) was interested in the problem of finding the number of ways in which any given number can be a polygonal number. [3]

I shall identify and examine those numbers which are polygonal in two ways - triangular $\left\{\frac{n(n+1)}{2}\right\}$ and square (m^2). Here n represents the number of dots in the side of the equilateral triangle and, m the number of dots in the side of the square. For example, $36 = \frac{8.9}{2} = 6^2$ is such a number represented in two ways below.

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1. Method of Discovery

Euler, in Example I at the end of his paper titled *'An easy rule for Diophantine problems which are to be resolved quickly by integral numbers'*[1], gives in brief a method to find these infinitely many numbers. Lafer [4] chooses an alternative approach. I shall follow Euler and describe his method in some detail.

Let the nth triangular number $T_n = \frac{n(n+1)}{2} = m^2 = S_m$ be the mth square. Of the two consecutive numbers n and $n+1$, one is even and the other is odd. They are necessarily co-prime and so if n is even, then $gcd(\frac{n}{2}, n+1) = 1$, or if n is odd then $gcd(n, \frac{n+1}{2}) = 1$. Whichever of the two numbers is even, it contains only odd powers of 2.

If n is even, then setting $n+1 = x^2$ i.e. $n = x^2 - 1$, $\frac{n}{2} = y^2$ i.e. $n = 2y^2$ and equating n in terms of x and y give:

$$x^2 - 2y^2 = 1 \tag{1.1}$$

Alternately, if n is odd then putting $n = x^2$, $(n+1)/2 = y^2$ i.e. $n = 2y^2 - 1$ and equating n in terms of x and y lead to:

$$x^2 - 2y^2 = -1 \tag{1.2}$$

In both cases, $T_n = (xy)^2 = m^2 = S_m = N_k$. Solutions of Pell equations (1.1) & (1.2) would give all N_k.

Now, $x^2 - 2y^2 = (x - y\sqrt{2})(x + y\sqrt{2})$. The underlying idea behind the method of solution here is that every solution gives a factorization of 1 involving conjugate surds with $\sqrt{2}$. If the least non-zero positive solution (x_1, y_1), called the fundamental solution, is found then that will generate all solutions. (3, 2) is the fundamental solution of (1.1) and (1, 1) that of (1.2). We see that:

$1 = 1^k = (-1)^{2k} = ((3 - 2\sqrt{2})(3 + 2\sqrt{2}))^k = ((1 - \sqrt{2})(1 + \sqrt{2}))^{2k}$;

$-1 = (-1)^{(2k-1)} = ((1 - \sqrt{2})(1 + \sqrt{2}))^{(2k-1)}$

Hence, the general solution of (1.1) and that of (1.2) respectively is:

$$x_k + y_k\sqrt{2} = (3+2\sqrt{2})^k = (1+\sqrt{2})^{2k} \tag{1.3}$$

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})^{(2k-1)} \tag{1.4}$$

Combining the even and odd power solutions in (1.3) and (1.4), we get the single formula for all solutions of (1.1) & (1.2):

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})^k \tag{1.5}$$

We also have the following explicit formulae:

$$x_k = \frac{(1+\sqrt{2})^k + (1-\sqrt{2})^k}{2}; \tag{1.6}$$

$$y_k = \frac{(1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}}. \tag{1.7}$$

The recurrence relations beginning with $x_0 = 1$, $y_0 = 0$ define them:

$$x_{k+1} = 2x_k + x_{k-1} \tag{1.8}$$

$$y_{k+1} = 2y_k + y_{k-1} \quad k \geq 1 \tag{1.9}$$

Recurrence relations can be very easily translated into generating functions and we have:

$$f(x) = \frac{1+x}{(1-2x-x^2)} = 1 + 3x + 7x^2 + \cdots \tag{1.10}$$

$$f(y) = \frac{1}{(1-2y-y^2)} = 1 + 2y + 5y^2 + \cdots \tag{1.11}$$

These **Lucas-type formulae** involving binomial coefficients are adapted from Weisstein [6]:

$$x_k = \sum_{r=0}^{\left[\frac{k}{2}\right]} 2^r \binom{k}{2r} \tag{1.12}$$

$$y_k = \sum_{r=1}^{\left[\frac{k+1}{2}\right]} 2^{r-1} \binom{k}{2r-1} \tag{1.13}$$

Having computed the values of x and y, we can get the required values of n and m. For the even values of subscript k of n, we use $n = x^2 - 1 = 2y^2$, and for the odd, we take $n = x^2 = \sqrt{2}y^2 - 1$. For m, we simply need to multiply the corresponding values of x and y.

I found these relations between $x_k, n_k; y_k, m_k$:

$$\frac{(n_{k+2r-1} - n_k)}{x_{2r-1}} = \frac{(m_{k+2r-1} + m_k)}{y_{2r-1}} = x_{2k+2r-1};$$
$$\frac{(n_{k+2r} - n_k)}{2y_{2r}} = \frac{(m_{k+2r} + m_k)}{x_{2r}} = y_{2k+2r}$$
$$\frac{(m_{k+2r-1} - m_k)}{x_{2r-1}} = y_{2k+2r-1}; \frac{(m_{k+2r} - m_k)}{y_{2r}} = x_{2k+2r}$$

The following explicit formulae for n_k and m_k are due to Euler:

$$n_k = \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k - 2}{4}; \tag{1.14}$$
$$m_k = \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{4\sqrt{2}}. \tag{1.15}$$

The following recurrence relations, with initial values 0 and 1, are also given by Euler:

$$n_{k+1} = 6n_k - n_{k-1} + 2 \tag{1.16}$$
$$m_{k+1} = 6m_k - m_{k-1}, \ k \geq 1 \tag{1.17}$$

These generating functions define them:

$$f(u) = \frac{(1+u)}{(1-u)(1-6u+u^2)} = 1 + 8u + 49u^2 + \cdots \tag{1.18}$$
$$f(v) = \frac{1}{(1-6v+v^2)} = 1 + 6v + 35v^2 + \cdots \tag{1.19}$$

I derived these Lucas-type formulae involving binomial coefficients from (1.12) & (1.13):

$$n_k = \sum_{r=1}^{k} 2^{r-1} \binom{2k}{2r} \tag{1.20}$$
$$m_k = \sum_{r=1}^{k} 2^{r-2} \binom{2k}{2r-1} \tag{1.21}$$

I also found the following summation formulae:

$$2\sum_{r=1}^{k} n_r = \left\{ \sum_{r=1}^{k} 2^{r-1} \binom{2k+1}{2r} \right\} - k \tag{1.22}$$
$$2\sum_{r=1}^{k} m_r = \left\{ \sum_{r=1}^{k} 2^{r-1} \binom{2k+1}{2r+1} \right\} + k \tag{1.23}$$

Now we come to the triangular squares. The following formula is straight forward:

$$N_k = m_k^2 = \left(\frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{4\sqrt{2}}\right)^2 = \frac{(17+12\sqrt{2})^k + (17-12\sqrt{2})^k - 2}{32} \tag{1.24}$$

This recurrence relation with initial values 0 and 1 is already known:

$$N_{k+1} = 34N_k - N_{k-1} + 2 \tag{1.25}$$

Plouffe [5] gives this generating function for the square triangular numbers:

$$f(z) = \frac{(1+z)}{(1-z)(1-34z+z^2)} = 1 + 36z + 1225z^2 + \cdots \tag{1.26}$$

A product formula for the k$^{\text{th}}$ triangular square is recorded by Weisstein [6]:

$$N_k = 2^{2k-5} \prod_{n=1}^{2k} \left(3 + cos\frac{n\pi}{k}\right) \tag{1.27}$$

Since $cos(\pi - \theta) = cos(\pi + \theta)$, and $cos\pi = -1, cos2\pi = 1$; we have

$$\prod_{n=k+1}^{2k} \left(3 + cos\frac{n\pi}{k}\right) = 2\prod_{n=1}^{k} \left(3 + cos\frac{n\pi}{k}\right)$$

Hence, $N_k = \left(2^{k-2}\prod_{n=1}^{k}\left(3 + cos\frac{n\pi}{k}\right)\right)^2$ i.e.,

$$m_k = 2^{k-2} \prod_{n=1}^{k} \left(3 + cos\frac{n\pi}{k}\right) \tag{1.28}$$

The above-noted formulae give these values of $x_k, y_k, T_{n_k}, S_{m_k}$ and the associated square triangular numbers N_k:

k	x_k	y_k	T_{n_k}	$= S_{m_k}$	$= N_k$
0	1	0	T_0	S_0	0
1	1	1	T_1	S_1	1
2	3	2	T_8	S_6	36
3	7	5	T_{49}	S_{35}	1225
4	17	12	T_{288}	S_{204}	41616
5	41	29	T_{1681}	S_{1189}	1413721
6	99	70	T_{9800}	S_{6930}	48024900
7	239	169	T_{57121}	S_{40391}	1631432881
8	577	408	T_{332928}	S_{235416}	55420693056
9	1393	985	$T_{1940449}$	$S_{1372105}$	1882672131025
10	3363	2378	$T_{11309768}$	$S_{7997214}$	63955431761796

The values assumed by y_k are known as Pell Numbers which are related to square triangular numbers:

$$(y_k(y_k + y_{k-1}))^2 = \frac{\{(y_k + y_{k-1})^2 - (-1)^k\} * (y_k + y_{k-1})^2}{2},\ k \geq 1$$

There exist infinitely many primitive Pythagorean triples $< a, b, c >$ of positive integers satisfying $a^2+b^2 = c^2$. These are given by $a = 2st,\ b = s^2-t^2,\ c = s^2 + t^2;\ s, t \in \mathcal{Z}^+,\ s > t,\ (s, t) = 1,\ s \not\equiv t\ (\ mod\ 2)$. Hatch [2] studied '**special triples**' with $|a - b| = 1$ to uncover connection between them and triangular squares. Then, $s^2 - t^2 - 2st = \pm 1 \Rightarrow (s + t)^2 - 2s^2 = \mp 1$ giving (1a) & (1b).

One can verify that if $< g, g + 1, h >$ is a special primitive Pythagorean triple, then so is $< 3g + 2h + 1, 3g + 2h + 2, 4g + 3h + 2 >$. Beginning with the primitive triple $< 0, 1, 1 >$, we successively derive $< 3, 4, 5 >, < 20, 21, 29 >, < 119, 120, 169 >, \cdots$ This construction exhausts all such triples. They can also be obtained from any of the following four relations:

$$(x_k x_{k-1})^2 + \{x_k x_{k-1} + (-1)^k\}^2 = \left(\frac{x_{2k-1} + x_{2k-2}}{2}\right)^2;\ k \geq 1 \tag{1.29}$$

$$(2y_k y_{k-1})^2 + \{2y_k y_{k-1} - (-1)^k\}^2 = (y_{2k-1})^2; k \geq 1 \tag{1.30}$$

$$\left(\frac{n_k - n_{k-1} - 1}{2}\right)^2 + \left(\frac{n_k - n_{k-1} + 1}{2}\right)^2 = \left(\frac{n_k + n_{k-1} + 1}{2}\right)^2;\ k \geq 1 \tag{1.31}$$

$$\left(\frac{7m_{k-1} - m_{k-2} - 1}{2}\right)^2 + \left(\frac{7m_{k-1} - m_{k-2} + 1}{2}\right)^2 = (m_k - m_{k-1})^2;\ k \geq 1,\ m_{-1} = -1 \tag{1.32}$$

It can also be shown that there exist infinitely many primitive Pythagorean triples having the property $a = T_k, b = T_{k+1}$ and $c = T_{(k+1)^2}$. We saw earlier: T_{x^2} is a perfect square for infinitely many values of $n = x^2$. In fact, if $< g, g+1, h >$ forms a Pythagorean triple, then so does $< T_{2g}, T_{2g+1}, (2g+1)h >$ proving the infinity of triangular squares as on putting $n = h - g - 1,\ m = g - \frac{h-1}{2}$, we get $\frac{n(n+1)}{2} = m^2$.

2. New Identities: Part A

I studied the triangular squares and the associated numbers and discovered this relation between neighbouring triangular numbers that are perfect squares:

$$n_{k+1} = 3n_k + 2 * \sqrt{2n_k(n_k + 1)} + 1 \tag{2.1}$$

$$n_{k-1} = 3n_k - 2 * \sqrt{2n_k(n_k + 1)} + 1 \tag{2.2}$$

I deduced from (2.2) the following relation:

$$2\{(n_k - n_{k-1})^2 + 1\} = (n_k + n_{k-1} + 1)^2 \tag{2.3}$$

I found the following relation between two consecutive squares that are triangular numbers:

$$m_{k+1} = 3m_k + \sqrt{(8m_k^2 + 1)} \tag{2.4}$$

$$m_{k-1} = 3m_k - \sqrt{(8m_k^2 + 1)} \tag{2.5}$$

We can deduce from (2.5) the following relation:

$$2(m_k - m_{k-1})^2 = (m_k + m_{k-1})^2 + 1 \tag{2.6}$$

I discerned these general recurrence relations regarding n_k and m_k :

$$n_{k+d} = \{(m_{d+1} - m_{d-1})n_k\} - (n_{k-d} - 2n_d);\ d \geq 1. \tag{2.7}$$

$$m_{k+d} = \{(m_{d+1} - m_{d-1})m_k\} - m_{k-d};\ d \geq 1. \tag{2.8}$$

Note: $(m_{d+1} - m_{d-1}) = 2x_{2d};\ 2n_d = (x_{2d} - 1)$.

Thus (1.16) and (1.17) become special cases of (2.7) and (2.8) respectively.

I found identities involving $2k+1$ number of consecutive values of n and m:

$$\sum_{j=0}^{2k} n_{i+j}(-1)^j = x_{k+1}.y_k(2n_{i+k} + 1) + (-1)^k n_{i+k} - r; \tag{2.9}$$

$r = 1$ if $2k+1 \equiv 3\ (mod\ 4)$, $r = 0$ if $2k+1 \equiv 1\ (mod\ 4)$; $i \geq 1,\ k \geq 1$.

$$\sum_{j=0}^{2k} n_{i+j} = (m_k + m_{k+1})n_{i+k} + 2\sum_{j=1}^{k} n_j,\ i \geq 1,\ k \geq 1. \tag{2.10}$$

$$\sum_{j=0}^{2k} m_{i+j}(-1)^j = 2x_{k+1}.y_k(m_{i+k}) + (-1)^k m_{i+k},\ i \geq 1,\ k \geq 1. \tag{2.11}$$

$$\sum_{j=0}^{2k} m_{i+j} = (m_k + m_{k+1})m_{i+k},\ i \geq 1,\ k \geq 1. \tag{2.12}$$

Putting i =1 in (2.10) and (2.12), we get these sum formulae:

$$\sum_{j=0}^{2k} n_{1+j} = (\ m_k + m_{k+1})\ n_{k+1} + 2\sum_{j=1}^{k} n_j,\ k \geq 1. \tag{2.13}$$

$$\sum_{j=0}^{2k} m_{1+j} = (\ m_k + m_{k+1})\ m_{k+1},\ k \geq 1. \tag{2.14}$$

I also discovered general formulae for $2k$ number of consecutive values of n and m:

$$\sum_{j=0}^{2k-1} n_{i+j}(-1)^{j+1} = m_k(n_{i+k} - n_{i+k-1}),\ i \geq 1,\ k \geq 1. \tag{2.15}$$

$$\sum_{j=0}^{2k-1} n_{i+j} = m_k(n_{i+k-1} + n_{i+k} + 1) - k,\ \ i \geq 1, k \geq 1. \tag{2.16}$$

$$\sum_{j=0}^{2k-1} m_{i+j}(-1)^{j+1} = m_k(m_{i+k} - m_{i+k-1}),\ i \geq 1,\ k \geq 1. \tag{2.17}$$

$$\sum_{j=0}^{2k-1} m_{i+j} = m_k(m_{i+k-1} + m_{i+k}),\ i \geq 1,\ k \geq 1. \tag{2.18}$$

Putting $i = 1$ in (2.16) and (2.18), we get these sum formulae:

$$\sum_{j=0}^{2k-1} n_{1+j} = m_k(n_k + n_{k+1} + 1) - k,\ k \geq 1 \tag{2.19}$$

$$\sum_{j=0}^{2k-1} m_{1+j} = m_k\ (m_k + m_{k+1}),\ k \geq 1. \tag{2.20}$$

We could get (2.20) by combining the next two results:

$$\sum_{j=1}^{k} m_{2j-1} = (m_k)^2,\ k \geq 1. \tag{2.21}$$

Proof. : We can prove it by induction. The identity is obviously true for n=1. Suppose the identity is true for n=k. Then, $\sum_{j=1}^{k}\ m_{2j-1}\ = (\ m_k)^2$
Hence,

$$\sum_{j=1}^{k+1} m_{2j-1} = \sum_{j=1}^{k} m_{2j-1} + m_{2k+1} = (\ m_k)^2 + m_{2k+1}$$

$$= \left(\frac{(3+2\sqrt{2}\ )^{\mathrm{k}} - (3-2\sqrt{2}\ )^{\mathrm{k}}}{4\ \sqrt{2}}\right)^2 + \frac{(3+2\sqrt{2}\ )^{\mathrm{2k+1}} - (3-2\sqrt{2}\ )^{\mathrm{2k+1}}}{4\ \sqrt{2}}$$

$$= \frac{(3+2\sqrt{2}\ )^{\mathrm{2k}} - 2 + (3-2\sqrt{2}\ )^{\mathrm{2k}}}{32} + \frac{(3+2\sqrt{2}\ )^{\mathrm{2k+1}} - (3-2\sqrt{2}\ )^{\mathrm{2k+1}}}{4\ \sqrt{2}}$$

$$= \frac{(3+2\sqrt{2}\ )^{\mathrm{2k}} + 4\ \sqrt{2}(3 + 2\sqrt{2}\ )^{\mathrm{2k+1}}}{32}$$

$$+ \frac{(3-2\sqrt{2}\ )^{\mathrm{2k}} - 4\ \sqrt{2}(3-2\sqrt{2}\ )^{\mathrm{2k+1}}}{32} - \frac{2}{32}$$

$$= \frac{(3+2\sqrt{2}\ )^{2k}[1+4\ \sqrt{2}(3+2\sqrt{2})]}{32}$$
$$+ \frac{(3-2\sqrt{2}\ )^{2k}[1-4\ \sqrt{2}(3-2\sqrt{2}]}{32} - \frac{2}{32}$$
$$= \frac{(17+12\sqrt{2}\ )^{k}[17+12\ \sqrt{2}]}{32} + \frac{(17-12\sqrt{2}\ )^{k}[17-12\ \sqrt{2}]}{32} - \frac{2}{32}$$
$$= \frac{(17+12\sqrt{2}\ )^{k+1} + (17-12\sqrt{2}\ )^{k+1} - 2}{32} = N_{k+1} \ = \ (\ m_{k+1})^2$$

Thus if it is true for $n = k$ then it is true for $n = k + 1$ also. Hence true for all n. □

$$\sum_{j=1}^{k} m_{2j} \ = \ m_k m_{k+1},\ k \geq 1 \tag{2.22}$$

Proof. We prove it by induction again. The result is true for $n = 1$. Suppose it is true for $n = k$. Then, $\sum_{j=1}^{k} \ m_{2j} \ = \ m_k m_{k+1}$
Hence,

$$\sum_{j=1}^{k+1} m_{2j} \ = \ \sum_{j=1}^{k} \ m_{2j} + \ m_{2k+2} \ = m_k m_{k+1} + \ m_{2k+2}$$
$$= \frac{(3+2\sqrt{2}\ )^{k} - (3-2\sqrt{2}\ )^{k}}{4\sqrt{2}} * \frac{(3+2\sqrt{2}\ )^{k+1} - (3-2\sqrt{2}\ )^{k+1}}{4\ \sqrt{2}}$$
$$+ \frac{(3+2\sqrt{2}\ )^{2k+2} - (3-2\sqrt{2}\ )^{2k+2}}{4\ \sqrt{2}}$$
$$= \frac{(3+2\sqrt{2}\ )^{2k+1} - \left(3-2\sqrt{2}\right) - (3+2\sqrt{2}) + (3-2\sqrt{2}\ )^{2k+1}}{32}$$
$$+ \frac{(3+2\sqrt{2}\ )^{2k+2} - (3-2\sqrt{2}\ )^{2k+2}}{4\ \sqrt{2}}$$
$$= \frac{(3+2\sqrt{2}\ )^{2k+1} \ -6 \ + (3-2\sqrt{2}\ )^{2k+1}}{32} + \frac{(3+2\sqrt{2}\ )^{2k+2} - (3-2\sqrt{2}\ )^{2k+2}}{4\ \sqrt{2}}$$
$$= \frac{(3+2\sqrt{2}\ )^{2k+1} \ + 4\ \sqrt{2}(3+2\sqrt{2}\ )^{2k+2} \ -6}{32}$$
$$+ \frac{(3-2\sqrt{2}\ )^{2k+1} - 4\ \sqrt{2}\ (3-2\sqrt{2}\ )^{2k+2}}{32}$$
$$= \frac{(3+2\sqrt{2}\ )^{2k+1}\ [1+\ 4\ \sqrt{2}\left(3+\ 2\sqrt{2}\right)] - 6}{32}$$
$$+ \frac{(3-2\sqrt{2}\ )^{2k+1}\ [1-4\ \sqrt{2}\ \left(3-\ 2\sqrt{2}\right)]}{32}$$

$$= \frac{(3+2\sqrt{2}\ )^{2k+1}\ [17+\ 12\ \sqrt{2}\ ]-6\ \ +(3-2\sqrt{2}\ )^{2k+1}\ [17-12\ \sqrt{2}]}{32}$$
$$= \frac{(3+2\sqrt{2}\ )^{2k+3}-6\ \ +(3-2\sqrt{2}\ )^{2k+3}}{32}$$
$$= \frac{(3+2\sqrt{2}\ )^{k+1}-\ (3-2\sqrt{2}\ )^{k+1}}{4\ \sqrt{2}} * \frac{(3+2\sqrt{2}\ )^{k+2}-\ (3-2\sqrt{2}\ )^{k+2}}{4\ \sqrt{2}}$$
$$= m_{k+1}m_{k+2}$$

Thus if it is true for $n = k$ then it is true for $n = k + 1$ also. Hence true for all n. $\square$

I found these interesting sum formulae:

$$6\sum_{j=0}^{[\frac{k-1}{2}]} m_{2k-4j-1} = m_k m_{k+1},\ \mathrm{k} \geq 1;\ [\ ]\ \text{is the greatest integer function.} \tag{2.23}$$

$$6\sum_{r=1}^{k} \mathrm{m_{2r}} = 36 \sum_{\mathrm{r=1}}^{[\frac{\mathrm{k+1}}{2}]} \mathrm{m_{2k-4r+3}} = {m_k}^2 + {m_{k+1}}^2 - 1 \tag{2.24}$$

I found identities expressing n and m in terms of all preceding values:

$$n_i = 5n_{i-1} + 4\sum_{j=1}^{i-2} n_j + (2i-1),\ \ i \geq 1 \tag{2.25}$$

$$m_i = 5\ m_{i-1} + 4\sum_{j=1}^{i-2} m_j\ \ +1,\ \ i \geq 1 \tag{2.26}$$

These identities are more interesting:

$$m_{2r-1}\ \ =\ \ (m_r\ )^2 - (m_{r-1}\ )^2 \tag{2.27}$$

Proof. We can establish it by simply manipulating the defining formula.

$$R.H.S. = \left(\frac{(3+2\sqrt{2}\ )^{\mathrm{r}}-\ (3-2\sqrt{2}\ )^{\mathrm{r}}}{4\ \sqrt{2}}\right)^2 - \left(\frac{(3+2\sqrt{2}\ )^{\mathrm{r}-1}-\ (3-2\sqrt{2}\ )^{\mathrm{r}-1}}{4\ \sqrt{2}}\right)^2$$
$$= \frac{(3+2\sqrt{2}\ )^{2\mathrm{r}}-2+\ \ (3-2\sqrt{2}\ )^{2\mathrm{r}}}{32} - \frac{(3+2\sqrt{2}\ )^{2\mathrm{r}-2}-2+\ \ (3-2\sqrt{2}\ )^{2\mathrm{r}-2}}{32}$$

$$= \frac{(3+2\sqrt{2}\,)^{2r}-(3+2\sqrt{2}\,)^{2r-2}+(3-2\sqrt{2}\,)^{2r}-(3-2\sqrt{2}\,)^{2r-2}}{32}$$

$$= \frac{(3+2\sqrt{2}\,)^{2r-2}[(3+2\sqrt{2}\,)^{2}-1]+(3-2\sqrt{2}\,)^{2r-2}[(3-2\sqrt{2}\,)^{2}-1]}{32}$$

$$= \frac{(3+2\sqrt{2}\,)^{2r-2}(16+12\sqrt{2}\,)+(3-2\sqrt{2}\,)^{2r-2}(16-12\sqrt{2}\,)}{32}$$

$$= \frac{(3+2\sqrt{2}\,)^{2r-2}(4+3\sqrt{2}\,)+(3-2\sqrt{2}\,)^{2r-2}(4-3\sqrt{2}\,)}{8}$$

$$= \frac{(3+2\sqrt{2}\,)^{2r-2}(2\sqrt{2}+3\,)+(3-2\sqrt{2}\,)^{2r-2}(2\sqrt{2}-3)}{4\sqrt{2}}$$

$$= \frac{(3+2\sqrt{2}\,)^{2r-1}-(3-2\sqrt{2}\,)^{2r-1}}{4\sqrt{2}} = m_{2r-1}$$

$= L.H.S.$

□

$$m_{2r} = m_r(m_{r+1} - m_{r-1}) \tag{2.28}$$

Proof. We establish it with the help of the method employed above.

$$R.H.S. = \frac{(3+2\sqrt{2}\,)^{r}-(3-2\sqrt{2}\,)^{r}}{4\sqrt{2}}\left(\frac{(3+2\sqrt{2}\,)^{r+1}-(3-2\sqrt{2}\,)^{r+1}}{4\sqrt{2}} - \frac{(3+2\sqrt{2})^{r-1}-(3-2\sqrt{2})^{r-1}}{4\sqrt{2}}\right)$$

$$= \frac{(3+2\sqrt{2})^{r}-(3-2\sqrt{2})^{r}}{4\sqrt{2}}\left(\frac{(3+2\sqrt{2})^{r-1}[(3+2\sqrt{2})^{2}-1]}{4\sqrt{2}} - \frac{(3-2\sqrt{2})^{r-1}[(3-2\sqrt{2})^{2}-1]}{4\sqrt{2}}\right)$$

$$= \frac{(3+2\sqrt{2})^{r}-(3-2\sqrt{2})^{r}}{4\sqrt{2}}\left(\frac{(3+2\sqrt{2})^{r-1}(4+3\sqrt{2})}{\sqrt{2}} - \frac{(3-2\sqrt{2})^{r-1}(4-3\sqrt{2})}{\sqrt{2}}\right)$$

$$= \frac{(3+2\sqrt{2})^{r}-(3-2\sqrt{2})^{r}}{4\sqrt{2}}[(3+2\sqrt{2})^{r-1}(2\sqrt{2}+3)-(3-2\sqrt{2})^{r-1}\left(2\sqrt{2}-3\right)]$$

$$= \frac{(3+2\sqrt{2})^{2r}-(3-2\sqrt{2})^{2r}}{4\sqrt{2}} = m_{2r}$$

$= L.H.S.$

□

$$n_{2r-1} = (n_r - \ n_{r-1})^2 \tag{2.29}$$

Proof.

$$R.H.S. = \left(\frac{(3+2\sqrt{2})^{\mathrm{r}} + (3-2\sqrt{2})^{\mathrm{r}} - 2}{4} - \frac{(3+2\sqrt{2})^{\mathrm{r}-1} + (3-2\sqrt{2})^{\mathrm{r}-1} - 2}{4} \right)^2$$

$$= \left(\frac{(3+2\sqrt{2})^{\mathrm{r}} - (3+2\sqrt{2})^{\mathrm{r}-1}}{4} + \frac{(3-2\sqrt{2})^{\mathrm{r}} - (3-2\sqrt{2})^{\mathrm{r}-1}}{4} \right)^2$$

$$= \left(\frac{(3+2\sqrt{2})^{\mathrm{r}-1}(\ 3+2\sqrt{2}-1)}{4} + \frac{(3-2\sqrt{2})^{\mathrm{r}-1}(3-2\sqrt{2}-1)}{4} \right)^2$$

$$= \left(\frac{(3+2\sqrt{2})^{\mathrm{r}-1}(\ 1+\sqrt{2})}{2} + \frac{(3-2\sqrt{2})^{\mathrm{r}-1}(1-\sqrt{2})}{2} \right)^2$$

$$= \frac{(3+2\sqrt{2})^{2\mathrm{r}-2}(1+\sqrt{2})^2}{4} + \frac{(3-2\sqrt{2})^{2\mathrm{r}-2}(1-\sqrt{2})^2}{4}$$

$$+ \ 2 * \ \frac{(3+2\sqrt{2})^{\mathrm{r}-1}(\ 1+\sqrt{2})}{2} * \frac{(3-2\sqrt{2})^{\mathrm{r}-1}(1-\sqrt{2})}{2}$$

$$= \frac{(3+2\sqrt{2})^{2\mathrm{r}-1} + (3-2\sqrt{2})^{2\mathrm{r}-1} - 2}{4} = n_{2\mathrm{r}-1}$$

$$= L.H.S.$$

□

$$2(n_{2r-1} + 1) = (n_r + n_{r-1} \ + 1)^2 \tag{2.30}$$

$$n_{2r} = (2n_r + 1)^2 - 1, \ r \geq 1 \tag{2.31}$$

Proof.

$$R.H.S. = (2n_r + 1)^2 - 1$$

$$= \left(2\frac{(3+2\sqrt{2}\ )^{\mathrm{r}} + (3-2\sqrt{2}\ )^{\mathrm{r}} - 2}{4} + 1 \right)^2 - 1$$

$$= \ \left(\frac{(3+2\sqrt{2}\ )^{\mathrm{r}} + (3-2\sqrt{2}\ )^{\mathrm{r}} - 2 + 2}{2} \right)^2 - 1$$

$$= \frac{(3+2\sqrt{2}\ )^{2\mathrm{r}} + (3-2\sqrt{2}\ )^{2\mathrm{r}} - 2}{4} = n_{2r} = L.H.S.$$

□

$$n_{3r} = n_r\,(4n_r{+}3)^2\,,\ r \geq 1. \tag{2.32}$$

$$m_{3r} = m_r(2^5{m_r}^2 + 3),\ r \geq 1. \tag{2.33}$$

Proof.

$$\begin{aligned}
R.H.S. &= \frac{(3{+}2\sqrt{2})^{\mathrm{r}}{-}(3{-}2\sqrt{2})^{\mathrm{r}}}{4\sqrt{2}}\left\{32\left(\frac{(3{+}2\sqrt{2})^{\mathrm{r}}{-}(3{-}2\sqrt{2})^{\mathrm{r}}}{4\sqrt{2}}\right)^2{+}3\right\}\\
&= \frac{1}{4\sqrt{2}}\{(3{+}2\sqrt{2})^{\mathrm{r}}{-}(3{-}2\sqrt{2})^{\mathrm{r}}\}\{(3{+}2\sqrt{2})^{2\mathrm{r}}{+}(3{-}2\sqrt{2})^{2\mathrm{r}}+1\}\\
&= \frac{1}{4\sqrt{2}}\{(3{+}2\sqrt{2})^{3\mathrm{r}}{-}(3{-}2\sqrt{2})^{3\mathrm{r}}\} = m_{3r}\\
&= L.H.S.
\end{aligned}$$

□

I discovered this **general identity for** $m_{(2n+1)r}$:

$$\begin{aligned}
m_{(2n+1)r} &{=} 2^{5n}\ {m_r}^{\ 2n+1}{+}2^{5(n-1)}(2n{+}1)\ {m_r}^{\ 2n-1}\\
&+[(2n+1)\{\sum_{k=1}^{n-1} 2^{5(n-k-1)}\ \frac{P_k}{(k{+}1)!}\ {m_r}^{\ 2(n-k)-1}\}],\\
&\text{where the product } P_k{=}\prod_{j=1}^{k} 2\,(n{-}k)\,{+}j{-}1
\end{aligned} \tag{2.34}$$

I detected some interesting relations between n and m:

$$n_i + n_{i-2j+1} \ = 2(m_j - m_{j-1})(m_{i-j+1} - m_{i-j}) - 1, j \geq 1,\ i \geq 2j \tag{2.35}$$

$$m_i + m_{i-2j+1} \ = (m_j - m_{j-1})(n_{i-j+1} - n_{i-j}),\ j \geq 0,\ i \geq 2j \tag{2.36}$$

$$n_{i+j} + n_{i-j} = (2n_j\ {+}1)(3m_i\ - m_{i-1}\ ) - 1 \tag{2.37}$$

$$n_{i+j} - n_{i-j} = \ 8\ m_j m_i;\ i > j > 0 \tag{2.38}$$

$$m_{i+2j-1} - m_{i-2j+1} = 2m_{2j-1}(2n_i + 1),\ j \geq 0, i \geq 2j \tag{2.39}$$

$$n_i = m_i + 2\sum_{j=1}^{i-1} m_j,\ i \geq 1 \tag{2.40}$$

3. New Identities: Part B

We discussed the numbers n and m till now. I now take up the triangular squares.
Neighbouring triangular squares can be expressed in terms of each other:

$$N_{k+1} = 6 * \sqrt{N_k(8N_k+1)} + 17N_k \ + 1 \tag{3.1}$$

$$N_{k-1} = 17N_k - \ 6 * \sqrt{N_k(8N_k+1)} + 1 \tag{3.2}$$

As N_k is both a square and triangular number, the quantity under the square root above is an integer. (3.1) and (3.2), which on addition result in (1.2), will be proved indirectly.

I deduced this relation from (3.2):

$$\{9\{8(N_k - N_{k-1})\}^2 + 1\} = \{8(N_k \ + N_{k-1}) + 1\}^2 \tag{3.3}$$

Proof. We establish it by using definition and the identity to come later with proof.

$$R.H.S. = \left(8\frac{(17+12\sqrt{2})^{\mathrm{k}} + (17-12\sqrt{2})^{\mathrm{k}} - 2}{32}\right.$$

$$\left. + 8\frac{(17+12\sqrt{2})^{\mathrm{k}-1} + (17-12\sqrt{2})^{\mathrm{k}-1} - 2}{32} + 1\right)^2$$

$$= \left(\frac{(17+12\sqrt{2})^{\mathrm{k}-1}(\ 17+12\sqrt{2}+1)}{4} + \frac{(17-12\sqrt{2})^{\mathrm{k}-1}(17-12\sqrt{2}+1)}{4}\right)^2$$

$$= 9\left(\frac{(17+12\sqrt{2})^{\mathrm{k}-1}(\ 3+2\sqrt{2})}{2} + \frac{(17-12\sqrt{2})^{\mathrm{k}-1}(3-2\sqrt{2})}{2}\right)^2$$

$$= 9\left(\frac{(3+2\sqrt{2})^{2\mathrm{k}-1}}{2} + \frac{(3-2\sqrt{2})^{2\mathrm{k}-1}}{2}\right)^2$$

$$= 9\left\{\frac{(3+2\sqrt{2})^{2(2\mathrm{k}-1)} + 2 + (3-2\sqrt{2})^{2(2\mathrm{k}-1)}}{4}\right\}$$

$$= 9\left\{8\frac{(3+2\sqrt{2})^{2(2\mathrm{k}-1)} - 2 + (3-2\sqrt{2})^{2(2\mathrm{k}-1)}}{32} + 1\right\}$$

$$= 9\left\{8\frac{(17+12\sqrt{2})^{2\mathrm{k}-1} + (17-12\sqrt{2})^{2\mathrm{k}-1} - 2}{32} + 1\right\}$$

$$= 9\left\{8N_{2k-1} + 1\right\} = 9\left\{8(N_k - N_{k-1})^2 + 1\right\} = L.H.S.$$

□

This relation can be verified easily using (2.23) & (2.24):

$$N_k + N_{k-1} = 6 * \sqrt{N_k N_{k-1}} + 1 \qquad (3.4)$$

I discovered this **general recurrence relation**:

$$N_{k+d} = \{(m_{2d+1} - m_{2d-1})N_k\} - N_{k-d} + 2N_d, \ d \geq 1. \qquad (3.5)$$

Note: $(m_{2d+1} - m_{2d-1}) = 2x_{4d}$; $2N_d = 2(m_d)^2$

Thus (1.25) becomes a special case of (3.5).

The following relation holds between four consecutive triangular squares:

$$N_{k+1} - N_{k-2} = 35(N_k - N_{k-1}) \qquad (3.6)$$

We now have identities involving many more consecutive triangular squares:

$$\sum_{j=0}^{4k-2} N_{i+j}(-1)^j = M_k(16N_{i+k}+1) + N_{i+k}; \ i,k \geq 1. \qquad (3.7)$$

$M_k = 2\sum_{i=1}^{k} m_{4i-3}$; alternatively, $M_k = 2\left(N_{2k-1} - \sum_{i=1}^{k-1} m_{4i-1}\right)$.

$$\sum_{j=0}^{4k} N_{i+j}(-1)^j = M'_k(16N_{i+k+1}+1) + N_{i+k}; \ i,k \geq 1. \qquad (3.8)$$

$M'_k = 2\sum_{i=1}^{k} m_{4i-1}$; alternatively, $M'_k = 2(N_{2k} - \sum_{i=1}^{k} m_{4i-3})$.

$$\sum_{j=0}^{2k-1} N_{i+j}(-1)^{j+1} = M_{2k}(N_{i+k} - N_{i+k-1}), \ i \geq 1, \ k \geq 2. \qquad (3.9)$$

$M_{2k} = N_k - 2\sum_{j=0}^{[\frac{k-2}{2}]} m_{2k-4j-3}$; [ ] denotes the greatest integer function; $k \geq 2$.
Alternatively, $M_{2k} = \left|\sum_{j=1}^{k} m_{2j-1}(-1)^j\right|$; $k \geq 2$.
We have these recursive formulae with $M_0 = 0$ and $M_2 = 1$ for deriving M_{2k}:
$M_{2k+2} = M_{2k-2} + 32N_k + 2$; $k \geq 1$.
$M_{4k-2} - 1 = M_{2k-2} * (m_{2k+1} - m_{2k-1})$; $k \geq 2$; $M_{4k} = M_{2k} * (m_{2k+1} - m_{2k-1})$; $k \geq 2$

Putting $i = 1$ in (3.9) gives the difference of the first $2k$ triangular squares:

$$\sum_{j=0}^{2k-1} N_{1+j}(-1)^{j+1} = M_{2k}(N_{1+k} - N_k), \ i \geq 1, \ k \geq 2. \qquad (3.10)$$

We have these general sum formulae:

$$\sum_{j=0}^{2k} N_{i+j} = m_{2k+1} N_{i+k} + 2\sum_{j=1}^{k} N_j, \ i \geq 1, \ k \geq 1. \qquad (3.11)$$

$$\sum_{j=0}^{2k-1} N_{i+j} \;=\; M_{2k}(N_{i+k-1} + N_{i+k}) + 4N_{k-1},\ i \geq 1,\ k \geq 2. \tag{3.12}$$

Putting $i = 1$ in the preceding identities gives the sum of first $2k + 1/2k$ triangular squares:

$$\sum_{j=0}^{2k} N_{1+j} = m_{2k+1}N_{1+k} + 2\sum_{j=1}^{k} N_j,\ k \geq 1. \tag{3.13}$$

$$\sum_{j=0}^{2k-1} N_{1+j} = \{N_k - 2\sum_{j=0}^{\left[\frac{k-2}{2}\right]} m_{2k-4j-3}\}(\ N_k + N_{k+1}) + 4N_{k-1},\ k \geq 2 \tag{3.14}$$

$$\sum_{j=0}^{2k-1} N_{1+j} = \left|\sum_{j=1}^{k} m_{2j-1}(-1)^j\right| (N_k + N_{k+1}) + 4N_{k-1},\ k \geq 2. \tag{3.15}$$

I found this relation expressing N in terms of all preceding values:

$$N_k = 33N_{k-1} + 32\sum_{j=1}^{k-2} N_j + (2k-1),\ k \geq 1 \tag{3.16}$$

I like these three identities:

$$N_{2r-1} = (N_r - N_{r-1})^2 \tag{3.17}$$

Proof.

$$\begin{aligned}
R.H.S. &= \left(\frac{(17+12\sqrt{2})^r + (17-12\sqrt{2})^r - 2}{32} - \frac{(17+12\sqrt{2})^{r-1} + (17-12\sqrt{2})^{r-1} - 2}{32}\right)^2 \\
&= \left(\frac{(17+12\sqrt{2})^{r-1}(\ 17+12\sqrt{2}-1)}{32} + \frac{(17-12\sqrt{2})^{r-1}(17-12\sqrt{2}-1)}{32}\right)^2 \\
&= \left(\frac{(17+12\sqrt{2})^{r-1}(\ 4+3\sqrt{2})}{8} + \frac{(17-12\sqrt{2})^{r-1}(4-3\sqrt{2})}{8}\right)^2 \\
&= \frac{(17+12\sqrt{2})^{2r-1} + (17-12\sqrt{2})^{2r-1} - 2}{32} = n_{2r-1} = L.H.S.
\end{aligned}$$

□

$$N_{2r} = 4N_r(8N_r + 1),\ r \leq 1 \tag{3.18}$$

Proof.

$$
\begin{aligned}
RHS &= 32{N_r}^2 + 4N_r \\
&= 32\left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{32}\right)^2 + 4\left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{32}\right) \\
&= \left(\frac{(17+12\sqrt{2})^{2r}+(17-12\sqrt{2})^{2r}+4+2(17+12\sqrt{2})^r(17-12\sqrt{2})^r}{32}\right. \\
&\quad \left. -\frac{4(17-12\sqrt{2})^r-4(17+12\sqrt{2})^r}{32}\right) + \left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{8}\right) \\
&= \left(\frac{(17+12\sqrt{2})^{2r}+(17-12\sqrt{2})^{2r}-2}{32}\right) - \left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{8}\right) \\
&\quad + \left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{8}\right) \\
&= N_{2r} = L.H.S.
\end{aligned}
$$

$\square$

$$N_{3r} = N_r\,(32N_r+3)^2,\ r \geq 1. \tag{3.19}$$

These **propositions** discovered empirically are not difficult to prove:

(1) $N_{\frac{p+1}{2}j} \equiv 0 \ \left(mod\ \ p^2\right) \leftrightarrow p \equiv \pm 3,\ \pm 5\ \ (mod\ 16),\ p$ - odd prime;
(2) $N_{\frac{p-1}{2}j} \equiv 0 \ \left(mod\ p^2\right) \leftrightarrow p \equiv -1, \pm 7\ \ (mod\ 16),\ p$ - odd prime;
(3) $N_{\frac{p-1}{4}j} \equiv 0 \ \left(mod\ p^2\right) \leftrightarrow p \equiv 1\ \ (mod\ 16),\ p$ - odd prime.

Proof of (i): If $p \equiv \pm 3,\ \pm 5\ \ (mod\ 16)$, then $p\ \equiv \pm 3\ (mod\ 8)$, so $\left(\frac{2}{p}\right) = -1$, where the notation used is the Legendre's symbol, meaning that 2 is a quadratic non-residue of p. So, $a^{\frac{n-1}{2}} \equiv -1\ (mod\ p)$.
Set $\alpha = (3+2\sqrt{2})$ and $\beta = (3-2\sqrt{2})$. Thus with the above notation, $\alpha^p = (3+2\sqrt{2})^p \equiv 3^p + 2^{3p/2} \equiv 3 + 2^{3/2}.\left(2^{\frac{p-1}{2}}\right)^3 \equiv 3 - 2^{3/2} \equiv \beta\ (mod\ p)$. In the same way, $\beta^p \equiv \alpha\ (mod\ p)$.
Hence, with the formula (1.24) for N_k, we have $N_{\frac{p+1}{2}j} = \frac{\alpha^{2\left(\frac{p+1}{2}j\right)}+\beta^{2\left(\frac{p+1}{2}j\right)}-2}{32} \equiv \frac{\alpha^{pj}.\alpha^j+\beta^{pj}.\beta^j-2}{32}\ (mod\ p) \equiv \frac{\beta^j.\alpha^j+\ \alpha^j.\beta^j-2}{32} \equiv \frac{2(\alpha.\beta)^j-2}{32} \equiv 0\ (mod\ p)$. Thus $p|N_{\frac{p+1}{2}j}$ and N_k being a square, it is divisible by p^2.

The other propositions can be similarly proved.
Note: $\frac{p+1}{2}, \frac{p-1}{2}, \frac{p-1}{4}$ *may not necessarily the smallest subscript.* p^2 *may divide* N_k *with smaller subscript* k *than the noted subscripts but in that case* k *must*

divide the larger subscript. For example, in case of $p = 197, 199$ *we have* $\frac{p+1}{2} = \frac{p-1}{2}$*=99, but* 197^2 *and* 199^2 *both divide* N_9*. Here 9 divides 99.*
For $j = 0, 1, 2, \cdots$, triangular squares satisfy these congruences:
$N_{6j} \equiv 0 \ (mod\ 10)$; $N_{6j+3} \equiv 5 \ (mod\ 10)$;
$N_{6j+1} \equiv N_{6j+5} \equiv 1 \ (mod\ 10)$; $N_{6j+2} \equiv N_{6j+4} \equiv 6 \ (mod\ 10)$.
Since $N_{2j+1} \equiv 1 \ (mod\ 2^2) \equiv 1 \ (mod\ 3^2), \therefore N_{2j+1} \equiv 1 \ (mod\ 6^2)$.
Since $N_{2j} \equiv 0 \ (mod\ 2^2) \equiv 0 \ (mod\ 3^2),\ \therefore N_{2j} \equiv 0 \ (mod\ 6^2)$.
Since $N_{3j} \equiv 0 \ (mod\ 5^2) \equiv 0 \ (mod\ 7^2),\ \therefore N_{3j} \equiv 0 \left(mod\ 35^2\right)$.
Hence, $N_{6j} \equiv 0 \ \left(mod\ 210^2\right)$. Further, $N_{6j} \equiv 0 \ (mod\ 11^2)$. $\therefore N_{6j} \equiv 0 \left(mod\ 2310^2\right)$.
The above congruences can be checked by invoking the known fact that $\{N_k\}$ is periodic modulo every modulus and computing its period say modulo 3; 7; 10; 210, etc.

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ON PROXIMALITY PAIRS

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Abstract: For two nonempty subsets A and B of a metric space (X, d), we call (A, B) a distance pair or proximinal pair or proximality pair if there exist $a_0 \in A, b_0 \in B$ such that $d(a_0, b_0) = d(A, B) \equiv inf\{d(x, y) : x \in A, y \in B\}$. The paper gives some sufficient conditions under which the pair (A, B) is a distance pair. The underlying spaces are real linear metric spaces.

Let A and B be two nonempty subsets of a metric space (X, d). If $A \cap B \neq \phi$ then $d(A, B) \equiv inf\{d(x, y) : x \in A, y \in B\} = 0$ but the converse need not be true. More generally, if $d(A, B) = r$ then there does not necessarily exist a pair of points $a \in A, b \in B$ such that $d(a, b) = r$. If such a pair $(a, b) \in A \times B$ exists then the pair (A, B) is called a proximinal pair or distance pair or proximality pair and the points $a \in A, b \in B$ are called proximinal points. If there exists at most one pair of proximinal points for A, B then the pair (A, B) is called a semi-Chebyshev pair. If the pair (A, B) is proximinal as well as semi-Chebyshev then it is called a Chebyshev pair. For the case, when one of the two sets is reduced to a single point, the problem of finding proximinal points reduces to the problem of best approximation. Accordingly, a subset A of X is called a distance set or proximinal set if $(A, \{x\})$ is a distance pair for every element x of X i.e. there exists $a \in A$ such that $d(x, a) = d(x, A) \equiv inf\{d(x, y) : y \in A\}$. A is called Chebyshev if such an a is also unique for each $x \in X$. It may be noted that proximinal points are the points mutually nearest to each other

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from the respective sets but converse is in general not true (see [15]). Some conditions under which the converse is also true have been discussed by Bradley and Willner [3], Cheney and Goldstein [4] and Pai [15], [16]. The study of proximinal points was introduced by M. Nicolescu [13](see Singer [22], p.385) in 1938 and subsequently this study was taken up by E.W.Cheney and A.A. Goldstein, J.J.Dionisio, V.Klee, G. Köthe, Bor-Luh-Lin, T.D.Narang, D.V. Pai, Ivan Singer, J.W. Tukey and few others (see the survey article [11] by the author).

Practical motivation for the consideration of the proximinal points is the following convex minimization problem (see [14]).

Suppose K_1, K_2 are disjoint convex sets in a space E and $\phi : K_1 \times K_2 \to \mathbb{R}^+$ is a functional which is convex in each individual variable $k_i, i = 1, 2$ (the other variable being fixed). Then under what conditions, for fixed $k_i^*, i = 1, 2$, the minimization of ϕ in each individual variable ensures the minimization of ϕ in both the variables k_1 and k_2. Pai[14] gave an answer to this question for the case $\phi(k_1, k_2) = \|k_1 - k_2\|$.

The study of distance sets have been used in characterization of reflexivity of normed linear spaces (A Banach space E is reflexive if and only if for any bounded closed convex subset A and closed convex subset B of E , (A,B) is a distance pair - see [7]), finite dimensionality of normed linear spaces (A normed linear E is finte dimensional if and only if for any two linearly bounded closed convex subsets A and B of E, (A,B) is a distance pair-see [7]), smoothness of normed linear spaces (A normed linear space E is smooth if and only if for each pair A,B of convex subsets of E, proximinal points are the points mutually nearest to each other-see [15]) and for solution of linear inequalities (see [4]).

This study of distance sets is also related to fixed point theory which is an indispensable tool for solving the equation $Tx = x$ for a mapping T. If the fixed point equation Tx = x does not possess a solution then we search for an element x such that x is in proximity to Tx in some sense. Specifically , given non-empty subsets A and B of a metric space (X, d) , a non-self mapping $T : A \to B$ does not necessarily have a fixed point. So , it is desirable to determine an approximate solution x such that the error $d(x, Tx)$ is minimum. Since $d(x, Tx) \geq d(A, B)$, an absolute optimal approximate solution is an element x for which the error $d(x, Tx)$ assumes the least possible value $d(A, B)$. As a result, best proximity pair theorems provide sufficient conditions for the existence of an optimal approximate solution x, known as a best proximity point of the mapping T satisfying the condition $d(x, Tx) = d(A, B)$. Interestingly, a best proximity point becomes a fixed points if the mapping T is a self mapping. Many best proximity point theorems are available in the literature (see e.g.[17]-[20],and the references cited therein)

Some sufficient conditions for a pair to be proximinal, semi-Chebyshev and Chebyshev are available in the literature (see[3], [4], [6]-[14], [22] and [25]). For characterization of proximinal points, one may refer to [3], [11], [15], [16]. Initiating the study of proximinal points,M. Nicolescu [13] proved the following:

Theorem 1. *If A and B are two compact subsets of a metric space (X, d) then (A, B) is a distance pair.*

Proof. By the definition of $d(A, B)$ there exist sequences $< a_n >$ and $< b_n >$ in A and B respectively such that $d(a_n, b_n) \to d(A, B)$. Since A and B are compact, there are subsequences $< a_{n_i} >$, $< b_{n_i} >$ such that $< a_{n_i} >\to a \in A$ and $< b_{n_i} >\to b \in B$. Then $d(a, b) = d(\lim a_{n_i}, \lim b_{n_i}) = \lim d(a_{n_i}, b_{n_i}) = d(A, B)$.Hence (A, B) is a distance pair. $\square$

Ivan Singer gave a generalization of Theorem 1 (see [22]-Theorem 2.3, p.385) by taking the sets A and B to be boundedly compact (a set is said to boundedly compact if each bounded sequence in it has a convergent subsequence) closed sets but it was shown in [8] that Singer's generalization is not correct.

The following correct version of Singer's theorem, which is a generalization of Theorem 1 was given by the author in [8]:

Theorem 2. *If A and B are nonempty subsets of a metric space (X, d), A is compact and B is boundedly compact closed set then (A, B) is a distance pair.*

It may be remarked (see [11]) that if A is compact and B is closed then (A, B) need not be a distance pair.

Tukey [23] constructed two closed convex subsets A and B of a Hilbert space such that (A, B) is not a distance pair. Klee [5] showed that in a Hilbert space there exist two linearly bounded (A subset A of a vector space is linearly bounded if A intersects each line in a bounded set) closed convex sets A and B such that (A, B) is not a distance pair. In fact, in every infinite-dimensional normed linear space E, there exist two linearly bounded closed convex subsets A and B such that (A, B) is not a distance pair. Consequently, a normed linear space is finite dimensional if and only if for any two linearly bounded closed convex subsets A and B, (A, B) is a distance pair. It is easy to see (see [7]) that in each normed linear space E there exist two locally compact closed convex subsets A and B such that (A, B) is not a distance pair. However, the following result concerning distance pairs is well known (see Köthe [6], p.345):

Theorem 3. *Let E ba a normed linear space. If A is weakly compact convex subset of E and B is a weakly locally compact closed convex subset of E, then* (A, B) *is a distance pair.*

Proceeding on similar lines, we prove the following theorem on distance pairs in linear metric spaces by using Schauder's fixed point theorem:

Theorem 4. *Let* (E, d) *be a strictly convex linear metric space and A a closed convex locally compact subset of E, B a compact convex subset of E then* (A, B) *is a distance pair.*

Before proving the result, we recall a few definitions and known results.

A metric space (X, d) is called a **linear metric space** if (i) X is a linear space, (ii) addition and scalar multiplication in X are continuous, and (iii) d is translation invariant i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$.

A linear metric space (X, d) is said to be **locally convex** if it has a base of convex neighbourhoods.

A linear metric space (X, d) is said to be **strictly convex** [1] if $d(x, 0) \leq r, d(y, 0) \leq r$ imply $d(\frac{x+y}{2}, 0) < r$ unless $x = y$. $x, y \in X$ and r is any positive real number.

A linear metric space (X, d) is said to be **uniformly convex** [1] if there corresponds to each pair of positive numbers (ε, r) a positive number δ such that if x, y lie in X with $d(x, y) \geq \varepsilon, d(x, 0) < r + \delta, d(y, 0) < r + \delta$ then $d(\frac{x+y}{2}, 0) < r$.

Clearly, every uniformly convex linear metric space is strictly convex but converse is not true (see [1]).

Let M be a subset of a metric space (X, d) and $x \in X$. An element $m_0 \in M$ satisfying $d(x, m_0) = d(x, M)$ is called a **best approximation** to x in M. The set $P_M(x) = \{m_0 \in M : d(x, m_0) = d(x, M)\}$ is called the set of best approximinants to x in M. The set M is said to be **proximinal** if $P_M(x) \neq \phi$ for each $x \in X$ and is called **Chebyshev** if $P_M(x)$ is exactly singleton for each $x \in X$. The set-valued map P_M is called the **nearest point map**.

A set M in a metric space (X, d) is said to be

(i) **approximatively compact** if for every $x \in X$ and every sequence $< y_n >$ in M satisfying $\lim d(x, y_n) = d(x, M)$, there exists a subsequence $< y_{n_i} >$ converging to an element of M.

(ii) **spherically compact** if for each $x \in X \setminus M$ there is a number $r > d(x, M)$ such that the set $\{y \in M : d(x, y) \leq r\}$ is compact.

(iii) **boundedly compact** if every bounded sequence in M has a convergent subsequence.

Lemma 5. ([21], [24]) *In a strictly convex linear metric space each ball is convex*

Proof. Let (X,d) be a strictly convex linear metric space. Firstly, we show that each closed ball B with centre at the origin i.e. $B = \{u \in X : d(u,0) \leq r\}$, $r \geq 0$ in X is convex. Suppose B is not convex. Then there exist $x, y \in B$, $x \neq y$ such that $(x,y) \cap B^C \neq \phi$, where B^C is complement of B in X. Let $A = \{t \in (0,1) : tx + (1-t)y \in B^C\}$. Then A is non-empty open subset of the real line R. Let B be a component of A. Then there exist , $\alpha, \beta \in R$ such that $\alpha < \beta$ and $B = (\alpha, \beta)$ .Write $z_1 = \alpha x + (1-\alpha)y$ and $z_2 = \beta x + (1-\beta)y$.Then it is to easy to see that z_1 , z_2 are distinct points of ∂B (the boundary of B) and $(z_1, z_2) \cap B = \phi$, a contradiction to the strict convexity of X. Hence B is convex. Thus closed balls with centre at the origin are convex and so the open balls with centre at the origin are also convex. Since every ball is a translate of a ball with centre at the origin, the result follows. □

Corollary 6. *A strictly convex linear metric space is locally convex.*

Lemma 7. [24] *In a strictly convex linear metric space every locally compact closed convex set is a Chebyshev set.*

Using Lemma 2, we prove the following:

Lemma 8. *If M is a closed convex locally compact subset of a strictly convex linear metric space (E,d) then the nearest point mapping P_M is continuous.*

Proof. Suppose $x_o \in E$ is arbitrary and $< x_n >$ is a sequence in E such that $< x_n > \to x_0$. To show $< P_M(x_n) > \to P_M(x_0)$. Let $\varepsilon > 0$ be given. Then there exists a positive integer m_0 such that $d(x_n, x_0) < \varepsilon$ for all $n, m \geq m_0$. For $n \geq m_0$,

$$\begin{aligned} d_n \equiv \inf_{y \in M} d(x_n, y) &\leq d(x_n, x_0) + \inf_{y \in M} d(x_0, y) \\ &\leq \varepsilon + d_0. \end{aligned}$$

Consider

$$\begin{aligned} d(x_0, P_M(x_n)) \leq d(x_0, x_n) &+ d(x_n, P_M(x_n)) \\ &\leq \varepsilon + d_n \\ &\leq d_0 + 2\varepsilon \end{aligned}$$

for all $n \geq m_0$. Therefore $P_M(x_n) \in B[x_0, d_0 + 2\varepsilon] \cap M \equiv C(\varepsilon)$ for all $n \geq m_0$. The diameters of these sets $C(\varepsilon)$ must tend to zero as $\varepsilon \to 0$ for otherwise there would be a nearest point to x_0 in M different from $P_M(x_0)$ contradicting Lemma 2. So,$P_M(x_n) \in C(\varepsilon)$ converges to $P_M(x_0) = \cap_{\varepsilon > 0} C(\varepsilon)$ as $\varepsilon \to 0$. □

Proof of Theorem 4: Let $P_A : B \to A$ and $P_B : A \to B$ be nearest point maps. Then by Lemma 3, P_A and P_B are continuous and so the composite map $T = P_B P_A : B \to B$ is continuous. Since B is a compact, convex subset of the locally convex space E, T has a fixed point b in B by Schauder's fixed point theorem i.e. $T(b) = b$. Then

$$b = P_B P_A(b) = P_B(a), Where\ a = P_A(b).$$

Hence $d(a, b) = d(P_A(b), b) = d(A, B)$.

Since a boundedly compact closed convex set M in a strictly convex linear metric space (E, d) is a Chebyshev set (see [2]) and the nearest point map P_M is continuous (see [2]), we have

Theorem 9. *Let (E, d) be a strictly convex linear metric space and A a boundedly compact closed convex subset of E, B a compact convex subset of E then (A, B) is a distance pair.*

Since a spherically compact convex set M in a strictly convex linear metric space (E, d) is a Chebyshev set (see [2]) and the nearest point map is continuous (see [2]), we have

Theorem 10. *Let (E, d) be a strictly convex linear metric space and A a spherically compact convex subset of E, B a compact convex subset of E then (A, B) is a distance pair.*

Since an approximatively compact convex set M in a strictly convex linear metric space (E, d) is a Chebyshev set (see [2]) and the nearest point map is continuous (see [2]), we have

Theorem 11. *Let (E, d) be a strictly convex linear metric space and A an approximatively compact convex subset of E, B a compact convex subset of E then (A, B) is a distance pair.*

Since a closed convex subset of a complete uniformly convex linear metric space is approximatively compact (see [1]) and a uniformly convex linear metric space is strictly convex, we have

Theorem 12. *Let (E, d) be a complete uniformly convex linear metric space and A a closed convex subset of E, B a compact convex subset of E then (A, B) is a distance pair.*

Note: For Banach spaces, this result was proved by Pai [14].

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GRAPH THEORETIC MODELS FOR SOCIAL NETWORKS - A BRIEF SURVEY

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Abstract: Graph theory serves as an efficient model in social network analysis and provides a vocabulary, which consists of several primitive concepts. Several properties of a social network can be quantified and measured in terms of graph theoretic parameters. In addition, graph theory gives a representation of a social network as a model, which often helps researchers to obtain certain patterns sitting in the network that might otherwise go unnoticed. Another major reason for substantial growth in Research in Social network Analysis in recent years is the availability of efficient heuristic algorithms for analyzing large graphs. In this paper we provide a brief survey of graph theoretic models for representing a social network, graph theoretic parameters required for analyzing the network along with specific applications and a case study. We also present the formal mathematical definitions of the various graph theoretic parameters and a few major results, which are relevant for the study of social networks.

1. **Introduction**

Social Network Analysis is a multidisciplinary research area involving Social, Mathematical, Statistical and Computer Sciences. Graph theory, which is one of the pillars of discrete mathematics, constitutes the central point of reference of social network analysis. It is universally applicable in modeling social relationships. Data on social relationships are transformed into graphs and evaluated by using various graph theoretic parameters. Thus graph theory forms a basis for functional interdisciplinary approach and linguistic coherence in the area of Social Network Analysis.

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A social network consists of a set of elements such as individuals, families, households etc together with a social relationship that exists between the elements. The following are some examples of typical social networks arising from relationships between the elements of the network.

- Friendship network
- Network of citations between academic papers
- Collaboration network among scientists

Recent years have witnessed a substantial growth in social network research, varying from analysis of small groups to networks of very large size. The basic approach is to create models of social networks, find properties that characterize the structure of the networks, ways to measure them and predict behavior of networks on the basis of the measured structural properties. Modern computer technologies are used to empower social science to do massive surveys on all kinds of social networks. Researchers look for parameters which will eventually lead to a better understanding of how relations work, information flows and organizations collaborate. Many types of relationships can be revealed by Social Network Analysis, such as information/knowledge brokers in a network, information bottlenecks in a network etc. Social network tools are used even to understand how an epidemic is spreading over a network of people. They are also used to improve performance, innovation, information flows, relation buildings etc., in an organization.

In this paper we provide a brief survey of graph theoretic models for representing a social network, graph theoretic measures required for analyzing the network along with specific applications and case study. The growth of social network analysis during the past two decades is so rapid, that it is almost impossible to cover all the important concepts in such a small survey. We have chosen a few concepts which we feel are fundamental in this field.

There is extensive literature on the use of graph models in social networks and two of the pioneering basic books in this area of study are by Harary et al. [19] and Wasserman and Faust [40]. Another recent book is by Scott [38]. The book by Easley and Kleinberg [13] gives an extensive coverage of topics in social network analysis. The book by Cross and Parker [9] is about using the latest social network theories to improve the networks of organizations. The recent book by Lewis [29] gives an excellent account of the development of network science along with applications of networks to various disciplines ranging from computer science, business, public health, internet virus countermeasures, social network behavior, biology and physics.

2. Graphs and Directed Graphs

In this section we present some of the basic definitions in graph theory which are essential for formulating basic network properties. For detailed study of basics in graph theory we refer to the books Chartrand and Lesniak [7] and West [41]. A relationship between the elements of a social network is said to be symmetric if for any two elements x, y in the network, whenever x is related to y, then y is related to x. In general, a social relationship between the elements of a social network may or may not be symmetric. For example, collaboration among scientists is a symmetric relation, whereas a relation such as A goes to B for help or advice is not symmetric. A social network can be modeled as a graph or a directed graph, depending on whether the relationship under consideration is symmetric or not.

Definition 2.1. A graph G consists of a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set whose elements are called points or vertices or nodes or actors and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called lines or edges or links of the graph G. If $\{u, v\} \in E(G)$, then u and v are called adjacent vertices. We also say that the edge $\{u, v\}$ is incident with the vertices u and v. Two edges are called adjacent edges if they are incident with a common vertex.

Definition 2.2. A directed graph D is a pair (V, A) where V is a finite nonempty set and the elements of A are ordered pairs of distinct vertices of V. The elements of A are called arcs. If $(u, v) \in A$, then u is called the initial vertex and v is called the terminal vertex of the arc (u, v).

There are occasions, when the standard definition of a graph may not serve as an appropriate model. For example to model a social network in which we consider more than one relationship between the nodes, we have to allow more than one edge joining the same pair of nodes. The resulting structure is called a *multigraph* and edges joining the same pair of nodes are called *multiple edges*.

If we associate with each edge of a graph a real number $w(e)$ then G is called a *weighted graph*. The number $w(e)$ is called the *weight of the edge e*. Weighted graphs occur frequently in applications of graph theory. For example in a friendship network, the weight of an edge may indicate the intensity of friendship. In a communication network the weight of an edge may represent the maintenance cost of the communication link.

It is customary to represent a graph or a directed graph by a diagram. In the case of a graph each vertex is represented by a small dot and each edge is represented by a line segment joining the two vertices with which the edge is incident. In the case of a directed graph an arc (u, v) is represented by a directed line segment where the direction is indicated by means of an arrow from u to v.

If $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, d\}\}$, then $G = (V, E)$ is a graph with four vertices and three edges. Also $D = (V, A)$ where $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (1, 3), (3, 1)\}$ is a directed graph with four vertices and four arcs. The diagrams of the graph G and the directed graph D are given in Figures 1(a) and 1(b) respectively. The figure given in 1(c) represents a multigraph. The vertices 1 and 3 are joined by three edges and the vertices 1 and 2 are joined by two edges. Figure 1(d) represents a weighted graph in which each edge is assigned a weight.

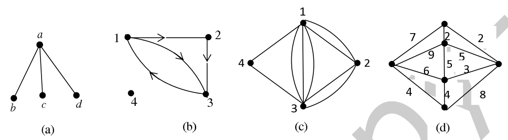

Figure 1: A graph, a directed graph, a multigraph and a weighted graph

A graph in which any two distinct vertices are adjacent is called a *complete graph*. The complete graph on n vertices is denoted by K_n. The graph on n vertices whose edge set is empty is called the *null graph* or *totally disconnected graph* and is denoted by $\overline{K_n}$. A graph G is called a *bipartite graph* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2 and (V_1, V_2) is called a *bipartition* of G. If further G contains every edge joining the vertices of V_1 to the vertices of V_2, then G is called a *complete bipartite graph*. If V_1 contains m vertices and V_2 contains n vertices then the complete bipartite graph G is denoted by $K_{m,n}$.

Bipartite graphs occur in a natural way in the study of social networks. For example an affiliation network is represented by a bipartite graph. An affiliation network consists of two types of nodes, a set of actors and a set of events and a node representing an actor is joined to a node representing an event if the actor has participated in the event. Thus in an affiliation network the set of actors and the set of events form a bipartition. Another example of an affiliation network consists of members belonging to different professional societies and a node representing an individual is joined to a node representing a professional society if the individual is a member of the society. Research on affiliation network has two major objectives, namely understanding of the relational structures among actors through their joint involvement in events and understanding of the relational structures of events attracting common participants. For a wide range of applications on affiliation networks we refer to Chapter 9 of Wasserman and Faust [40].

Examples of a complete graph, a complete bipartite graph and a bipartite graph representing an affiliation network are given in Figure 2.

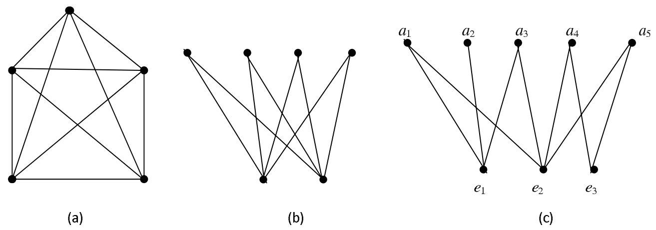

Figure 2: Complete graph K_5, Complete bipartite graph $K_{4,2}$ and a bipartite graph of an affiliation network

Eubank et al. [14] used bipartite graphs to represent a very large realistic network. In this study the authors considered a social contact network represented as a bipartite graph with two types of nodes, namely, a set of people and a set of locations and edge joins a person and a location if the person visited the location on a particular day. Such networks are useful in the study of controlling large scale epidemics in urban areas.

Graph theoretic models in social networks arise from data collected from the members of a social group. We now give an example of a real life social network arising from a set of data. Data has been collected from a set of 30 students of Kalasalingam University in a Post Graduate class who have already spent one year together. Each student was asked to provide the list of students of his class with whom he/she shares his/her personal problems. Clearly the relation is not symmetric and hence the collected data gives a directed graph on 30 nodes, which is given in Figure 3.

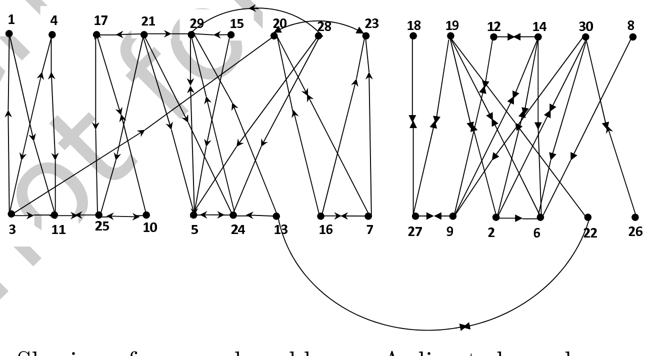

Figure 3: Sharing of personal problems - A directed graph representation

The analysis of this directed graph in the context of reciprocity in social networks is given in Section 6.

There are several fundamental theorems in graph theory which are useful and relevant for social network analysis. We mention a few such results.

A collection $S_1, S_2, \ldots, S_k, k \geq 1$, of finite nonempty sets is said to have a system of distinct representatives (SDR) if there exists a set $\{s_1, s_2, \ldots, s_k\}$ of distinct elements such that $s_i \in S_i$.

Hall [17] proved the following theorem.

Theorem 2.3. [17] *A collection $S_1, S_2, \ldots, S_k, k \geq 1$, of finite sets has a system of distinct representatives if and only if the union of any j of these sets contains at least j elements, for each j such that $1 \leq j \leq k$.*

The above theorem is directly related to the following well known *Marriage Problem*: Given a set of boys and a set of girls where each girl knows some of the boys, under what conditions can all girls get married, each to a boy she knows? It follows from the above theorem that if there are k girls, then the Marriage Problem has a solution if and only if every subset of j girls $(1 \leq j \leq k)$ collectively know at least j boys.

Another well known theorem, popularly known as the *Friendship Theorem* [12] states that in any party in which every pair of people have exactly one common friend, there is one person who is the friend of all the members in the party. The resulting graph of the friendship relation consists of some number of triangles sharing a common vertex.

3. Degrees and Degree Equitable Sets

The *degree* of a vertex v_i in a graph G is the number of edges incident with v_i. The degree of v_i is denoted by $d_G(vi)$ or $deg\ v_i$ or simply $d(v_i)$. A vertex v of degree 0 is called an *isolated vertex*. A vertex v of degree 1 is called an *end vertex*. For any graph G, we define $\delta(G) = \min\{deg\ v : v \in V(G)\}$ and $\Delta(G) = \max\{deg\ v : v \in V(G)\}$. If all the vertices of G have the same degree r, then $\delta(G) = \Delta(G) = r$ and in this case G is called a *regular graph* of degree r. A regular graph of degree 3 is called a *cubic graph*. In the case of directed graphs we have *indegree* and *outdegree* for any vertex. The indegree $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The outdegree $d^+(v)$ of v is the number of arcs having v as its initial vertex.

In the case of graphs the degree of any vertex v indicates the status of v in the corresponding network. In the case of directed graphs the outdegree of a vertex indicates the expansiveness or its capacity for sociability and the indegree represents its popularity or potential for influence or leadership. Hence in the context of a social network it is significant to collect the set of all vertices whose degrees are almost equal. Motivated by this observation Anitha et al. [3] introduced the concept of *degree equitable sets*.

Let $G = (V, E)$ be a graph. A subset S of V is called a *degree equitable set* if the degrees of any two vertices in S differ by at most one. For example consider the graph given in Figure 4. Let $S_1 = \{a, d, g, f\}$ and $S_2 = \{b, c, e\}$. Clearly every vertex in S_1 has degree 1. Also every vertex in S_2 has degree 3 or 4 and hence the degrees of any two vertices in S_2 differ by at most 1. Thus both S_1 and S_2 are degree equitable sets. Further any superset which contains S_1 is not a degree equitable set and in such a case we say that S_1 is a maximal degree equitable set. Similarly S_2 is also maximal degree equitable sets. We observe that S_1 has 4 elements and S_2 has 3 elements.

The maximum cardinality of a maximal degree equitable set in G is called the *degree equitable number* of G and is denoted by $D_e(G)$. The minimum cardinality of a maximal degree equitable set in G is called the *lower degree equitable number* of G and is denoted by $d_e(G)$. For the graph G given in Figure 4, $d_e(G) = 3$ and $D_e(G) = 4$. Several properties of the parameters $D_e(G)$ and $d_e(G)$ are given in Anitha et al. [3]. Anitha [4] in her doctoral thesis also considered the problem of partitioning the vertex set into the minimum number of subsets such that each subset is a degree equitable set in G. Such a partition gives a natural grouping of the actors of a social network such that each subset is a set of actors having almost equal status in the network.

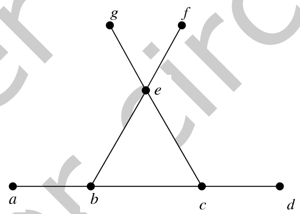

Figure 4: A graph G with $d_e(G) = 3$ and $D_e(G) = 4$

Degree equitableness may also be thought of as a measure of cohesiveness. For example, suppose S is a set of actors in a network containing two actors u and v with $deg\ u = 5$ and $deg\ v = 100$. The actor v, because of his high status in the network, can impose his views on the members of S while actor u may not even be able to express his views. Thus in an organization if the leader wants to form a committee for recommending appropriate decisions on sensitive issues, it is desirable that the committee forms a degree equitable set in the network.

In a similar way in a directed graph one can define outdegree equitable set and indegree equitable set and define four parameters, namely, the maximum cardinality of an outdegree equitable set, the minimum cardinality of a maximal outdegree equitable set, the maximum cardinality of an indegree equitable set and the minimum cardinality of a maximal indegree equitable set. Graph theoretical investigations of the properties of the above four parameters for directed graphs, the problem of partitioning the vertex set of a directed graph into a minimum number of subsets such that each subset is either outdegree equitable or indegree equitable and applications of these concepts to networks which are represented as directed graphs are potential areas for further research.

4. **Giant Components**

The concept of connectedness is basic in the study of social networks. A path in a network G is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ such that any two consecutive vertices $u_i u_{i+1}$ is an edge of G. We say that the path connects the vertices u_1 and u_k. A graph G is said to be *connected* if every pair of its nodes are connected. A graph which is not connected has several connected components. A connected graph and a disconnected graph with five components are given in Figures 5(a) and 5(b) respectively.

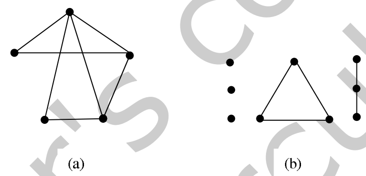

Figure 5: A connected graph and a disconnected graph with 5 components

Identification of connected components in a large network such as collaboration network among scientists leads to interesting internal structure within components. Again in a large network of datasets, often there exists a connected component that contains a significant fraction of all the vertices. Such a component is called a *giant component*. When a network contains a giant component in most cases it contains only one.

The notion of giant component can be used for analyzing even small networks. An interesting example of the role of a giant component in a small network is considered by Bearman and Moody [5]. This paper deals with the study of the romantic relationships in an American high school over an 18-month period. The focus of study in this paper is the spread of sexually transmitted diseases. Surprisingly the graph in this investigation had a single large component. A student who may have had a single partner during the period of investigation may be a part of this large component and thus he is a part of many paths of potential transmission. The tools of social network analysis lead to the discovery of the presence of such a large component, which is otherwise invisible.

We have efficient algorithms in graph theory for partitioning a graph into connected components. For example, Hopcroft and Tarjan [23] have used the standard depth-first algorithm (DFS) to find the connected components. Once all the components are identified, the giant component, if it exists, can be obtained.

Another concept which is closely related to connected graphs is the concept of connectivity. A cut vertex of a graph is a vertex whose removal increases the number of components. A cut edge of a graph is an edge whose removal increases the number of components. For the graph given in Figure 6, the vertices 1, 2 and 3 are cut vertices and the edges $\{1, 2\}$ and $\{3, 4\}$ are cut edges. The vertex 5 is not a cut vertex.

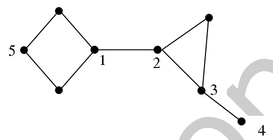

Figure 6: Cut vertices and Cut edges

The vertex connectivity and the edge connectivity give a measure of the extent to which a graph is connected. The *vertex connectivity* of a graph G is the minimum number of vertices whose removal results in a disconnected graph or trivial graph. The *edge connectivity* λ of G is the minimum number of edges whose removal results in a disconnected graph.

A cut vertex v in a social network corresponds to an actor whose role in the network is crucial in the sense that his presence in the network keeps the entire network as a single unit and in his absence the network breaks into several connected subgraphs. Moreover in the network any communication between two actors in two different connected subgraphs of $G - v$ always uses the actor v as an intermediate vertex. Hence the presence of a cut vertex, especially in communication networks is rather undesirable, since the failure of this specific node leads to a collapse in communication between the remaining nodes of the network. In such a case it is desirable to look into the largest component in $G - v$, where communication is still possible. Such considerations lead to other measures of vulnerability of networks such as toughness, integrity and tenacity.

5. **Distance Related Concepts**

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. The *diameter* of a connected graph G is the maximum distance between two vertices of G and is dented by $diam(G)$. The *eccentricity* $e(u)$ of a vertex u in a graph G with vertex set V is $\max\{d(u,v) : v \in V\}$. Thus the eccentricity of a vertex u is the distance from u to a vertex which is farthest from u in the graph G. The *radius* $r(G)$ of G is $\min\{e(u) : u \in V\}$ and any vertex v such that $e(v) = r(G)$ is called a *centre* of G. Thus from a central vertex we can reach every other vertex of the graph G by a path of length at most $r(G)$. A graph G along with the eccentricities of its vertices is given in Figure 7. The two vertices with eccentricity 3 are the central vertices of the graph.

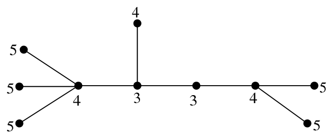

Figure 7: Graph with two central vertices

There are several other measures of centrality such as median, centroid etc., and these centrality measures have important applications in facility location problems. Facility location problem deals with the task of choosing a site in a geographical network subject to some criterion. For example an emergency facility such as a hospital or fire station must be located at a vertex that will minimize the response time between the facility and the location of a possible emergency, which can be any other vertex of the network. Hence such a facility must be located at a vertex of minimum eccentricity namely, a central vertex. However for a service facility such as a post office, a shopping mall or a bank, the location must be such that the average distance that a person serviced by the facility has to travel is minimized. This is equivalent to minimizing the total distance from all the other vertices of the network to the service facility. A vertex v having the property that the sum of the distances from v to all the other vertices in the network is minimum is called a *median vertex*, and such a vertex is the ideal place for locating a service facility.

The concept of distances in large networks has resulted in the discovery of the *Small-World Phenomenon.* This is also known as *six degrees of separation.* The first experimental study of this notion was performed by Stanley Milgram and his colleagues and the findings were reported in Milgram [32] and Travers and Milgram [39]. The aim of their experiment was to test the speculative idea that people are connected in the global friendship network by a path of relatively small length. For this purpose they started with a collection of 296 randomly chosen volunteers to try forwarding a letter to a target person. The volunteers were asked to forward the letter to someone they knew on a first-name basis with the same instructions to the successors, in order to eventually reach the target as quickly as possible. Each letter thus passed through the hands of a sequence of friends in succession, terminating at the target vertex. Thus we have a path in the global friendship network. Among the 64 paths that succeeded in reaching the target, the median length was six. In the above experiment, the conclusion was arrived at without full knowledge of the global friendship network. This has been confirmed in settings where we have the full data on the network structure, see for example, Grossman and Ion [16].

In Mathematics Paul Erdös who has published around 1500 papers in his career was taken as a central figure in the collaboration network where the vertices correspond to mathematicians and edges connecting pairs who have jointly authored a paper. A mathematician's Erdös number is the distance from him/her to Erdös in this graph. For example, any mathematician having a joint paper with Erdös has Erdös number 1. Interestingly most mathematicians have Erdös number at most 4 or 5. We observe that in graph theoretic terminology, in the collaboration network of mathematicians, the distance from the node represented by Erdös to any other mathematician in the network is at most 5. Even when the collaboration graph was extended by including co-authorship with other sciences, the Erdös numbers of other scientists are only slightly larger. Thus the Small-World Phenomenon which was discovered by Milgram without the full knowledge of the global friendship network has been confirmed even in a situation where we have full data on the complete network structure.

The network of mathematicians leading to the concept of Erdös number given above is an example of an academic social network. The problem of selecting most influential nodes in an academic social network based on certain criterions relevant to academic environment such as number of citations, working location of authors, cross reference and coauthorship, is another interesting problem. Several social networks arise in a natural way in this context. A node in the network is either an author or an article. An edge between two authors denotes that they have coauthored a research article. An edge between article and author indicates that the researcher is an author of the article. An edge between two articles indicates that one of the articles has been cited in the other article. The problem of identifying most influential researchers in such a network is an interesting problem, since they are most likely the trend setters for new innovations. Structural properties of such networks have been investigated by Newman [34] and Jensen and Neville [24]. A study of the spread of influence in such networks is reported in Kempe et al. [26] and Domingos and Richardson [11]. Liben-Nowell and and Kleinberg [30] attempt to predict future interactions between actors using the network topology. Junapudi et al. [25] have discussed methods for identifying most influential nodes using different relevant criterions.

6. **Reciprocity in Directed Graphs**

In this section we discuss the concept of reciprocity in directed graphs. Rao [35] investigated the concept of reciprocity in social networks and marital networks, which were constructed using the data collected from the households of a village in West Bengal in India. Rao and Bandobadyay [37] have given several measures of reciprocity. Rao [36] has given several methods for standardising the number of reciprocal pairs in a network, so that the resulting reciprocity measure can be used for comparing the reciprocities of two different social networks. Reciprocity refers to the presence of the arc (i, j) when (j, i) is an arc. If (i, j) is an arc and (j, i) is not, we may call (i, j) a one way or unreciprocated arc and such arcs usually present hierarchical or patron-client relationships, whereas reciprocated arcs indicate some sort of balance. For example, the social network of a village determined by *help* relation, where the vertices are the households and an arc joins i and j if i goes to j for help at times of crisis gives a directed graph. If the frequency of help is also taken into account we get a weighted directed graph where the weight of the arc (i, j) represents the number of times i goes to j for help during a fixed period of investigation.

We now illustrate the concept of reciprocity with the directed graph given in Figure 2, which represents the social network constructed using a set of data collected from 30 students of a post-graduate class of Kalasalingam University in India. When the data were collected from students, they had already spent one year together and hence all of them knew each other very well. Each student was asked to provide the list of students in the class with whom he/she shares his/her personal problems and the same were cross checked for validation. Since sharing of personal problems is not a symmetric relation, the above data gives a directed graph $D = (V, A)$ where the vertex set V is the set of 30 students and (u, v) is an arc in A if u shares his personal problem with v. The resulting directed graph is given in Figure 2. The number of arcs in the directed graph is 71 and the number of reciprocal pairs of arcs is 22. Since the maximum number of arcs in a directed graph on 30 vertices is $30 \times 29 = 870$, the density of the directed graph is $71/870 = 0.0815$. Among the 30 vertices in the directed graph, 13 vertices represent male students and 17 vertices represent female students.

The set of male students and the set of female students are respectively $M = \{1, 3, 4, 5, 10, 11, 15, 17, 21, 24, 25, 28, 29\}$ and $F = V - M$. Out of the total of 22 reciprocal pairs of arcs, 7 pairs were between males and 15 pairs were between females. Interestingly there is no reciprocal pair of arcs between a male and a female. In fact out of the total of 71 arcs in the directed graph, only 3 arcs have one end as male and the other end as female, showing that sharing of personal problems between a male and a female was quite rare. Taking into account the fact that Kalasalingam University is located in a remote rural area in the country and the cultural background is such that boys and girls do not move freely, it is not surprising to note the presence of very few arcs joining two nodes, one representing a male and the other representing a female. Also the presence of a higher number of reciprocal pairs in the subnetwork of female students shows that mutual personal sharing is more frequent among female students. This is again not surprising and perhaps female students are more comfortable in sharing their personal problems with their friends in the class rather than with their parents and relatives at home. The vertex 29 has in-degree 6 and the vertices 24 and 6 have each in-degree 5, indicating that more students approach them for personal sharing. Thus the students representing the nodes 29, 24 and 6 are showing concern to others and are popular within the group. Interestingly these three nodes have outdegree just 1 or 2. The vertices 15, 21 and 28 have indegree 0 and all of them have positive outdegrees, indicating that they go to others for sharing their problems and none comes to them. Further there is no vertex in the network having both indegree and outdegree 0, and there is no isolated vertex in the network.

7. Structural Balance in Signed Graphs

In a social network links which indicate relationships such as collaboration, sharing of information and friendship are positive ties. However in most network settings, there are also negative links that work. Some relations are friendly while some others are hostile. Interactions between people sometimes lead to controversy or even outright conflict. In this section we deal with an important study of structural balance in a network having both positive and negative links. The concept of structural balance indicates a nice connection between local and global network properties. This concept captures the way in which local effects, which involve only a few members, can have global consequences in the whole network.

The principles underlying structural balance have its roots in the theory of social psychology, initiated by Heider [22]. The concept was generalized and extended to signed graphs by Harary [19], Cartwright and Harary [6] and Davis [10]. If we look at any two vertices in a signed graph the edge between them maybe positive or negative depending on whether the corresponding members are friends or enemies. However, when we look at a set of three vertices we have four possible configurations as given in Figure 8.

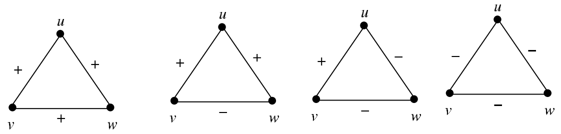

Figure 8

Figure 8(a) represents a natural situation in which all the three people are mutual friends. The situation given in Figure 8(c) is also natural in the sense that it represents two friends having a common enemy. The other two possibilities introduce some psychological stress or strain in the system. A triangle with two positive links and one negative link as given in Figure 8(b) corresponds to a person u who has friends v and w but v and w are enemies. In this situation u experiences a psychological strain. Similarly in a situation given in Figure 8(d) all three members are mutual enemies. Based on the above reason Harary called Figures 8(a), 8(c) as balanced and 8(b), 8(d) as unbalanced. In the unbalanced situation there is always a course motivating towards change in one of the ties, leading to balance. Harary [18] generalized the notation of structural balance for signed graph by proposing the following definition.

A signed graph G is said to be *balanced* if every cycle in G has an even number of negative ties. Harary obtained the following nice characterization for balance in signed graphs.

Theorem 7.1. *A signed graph G is balanced if and only if the vertex set V can be partitioned into two subsets V_1 and V_2 (one of them may be empty) such that all ties within V_1 and all ties within V_2 are positive and all ties between V_1 and V_2 are negative.*

Thus the above notion of balance predicts only dichotomy as its basic social structure. For example, one interpretation of the above theorem is that a political system in a country can provide stable governance if there is a two party system in the country. Davis [10] observed that human groups often break up into more than two mutually hostile subgroups. He proposed another concept of balance for signed graphs which we call weakly balanced signed graph. A signed graph is weakly balanced if there is no cycle with exactly one negative edge. Thus weak balance imposes less restriction and hence we have a broader range of weakly balanced network structure as seen in the following theorem.

Theorem 7.2. *A signed graph is weakly balanced if and only if the vertex set can be partitioned into two or more subsets such that every positive edge joins two vertices of the same subset and every negative edge joins two vertices of two different subsets.*

8. **Dominating Sets and Structural Equivalence**

One of the fastest growing areas within graph theory is the study of domination and related subset problems such as covering, independence and irredundance. An excellent treatment of fundamentals of domination is given in the book by Haynes et al. [20] and a survey of several advanced topics is given in the book edited by Haynes et al. [21]. The concept of domination has important applications in the context of social network. For example, in a network where the actors are people in an organization and if we wish to form a committee consisting of a few members, then it is desirable that every member not in the committee knows at least one member in the committee. This ensures that members not in the committee can easily pass on information to the members of the committee. Such a subset of members who are in the committee is a dominating set in the network. If a member in the committee does not know any other member in the committee, then he is isolated within the committee. Hence it is desirable that every member in the committee knows at least one other member in the committee. This gives the concept of total domination, which was introduced by Cockayne et al. [8]. We now present the formal definitions of these concepts.

A set $S \subseteq V$ is said to be a *dominating set* of G if every vertex in $V - S$ is adjacent to some vertex in S. A dominating set S of G is called a *total dominating set* if no vertex in S is an isolated vertex within S. The minimum number of vertices in a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$ and the minimum number of vertices in a total dominating set of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$.

For example for the graph G given in Figure 9, the set $S = \{v_1, v_4, v_7, v_{10}\}$ is a dominating set since every vertex not in S is adjacent to a vertex in S. In fact there is no dominating set for G having less than four vertices and $\gamma(G) = 4$. However, S is not a total dominating set since every vertex in S is not adjacent to another vertex in S. The set $S_1 = \{v_1, v_2, v_5, v_6, v_9, v_{10}\}$ is a total dominating set of G and $\gamma_t(G) = 6$.

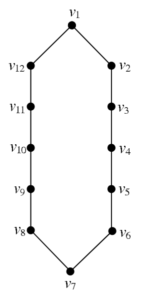

Figure 9: Graph with $\gamma(G) = 3$ and $\gamma_t(G) = 6$

We now proceed to give an application of dominating set in the context of a social network. Social scientists are often interested in structural equivalence of actors. A set S of two or more actors having identical relations with others in the network is called a *structural equivalence set*. In other words S is structurally equivalent if $N(u) = N(v)$ for any two vertices u and v in S. Here $N(u)$ stands for the set of all vertices which are adjacent to u and is called the neighbor set of u. Structurally equivalent actors have a competitive relation. For example two farmers who market their produce to the same set of retailers are structurally equivalent and are in stiff competition to sell their products. Structurally equivalent actors are completely substitutable for one another. Social Network theorists have developed algorithms for finding a maximal structurally equivalent set and a detailed study of structural equivalence is presented in Chapter 9 of Wasserman and Faust [40]. Kelleher [27] in her doctoral thesis presented research on dominating sets in social network graphs. Kelleher and Cozzens [28] have shown that structurally equivalent sets can be found using the properties of dominating sets.

We now present two other models of dominating sets which have a good potential for application. Consider a graph $G = (V, E)$ representing a social network. Let S be a subset of V and let $v \in S$. Then the number of neighbors of v in $V - S$ gives a measure of the outside support that v has in the network. Similarly if $v \in V - S$, then the number of neighbors of v in S gives a measure of the influence that v has over the members in S. Consider for example the situation where V represents the set of elected representatives of a political party and the leader of the party wants to identify a subset S of V for governance. If an actor x in S has substantially more number of supporters outside S when compared to other actors in S, then the actor x may exert pressure on the leader. Also if an actor y who is not in S has large number of his neighboring vertices in S when compared to other vertices in $V - S$, then he may be able to thrust his point of view on the numbers of S. Hence the leader of the party may like to have either one or both of the following properties for the subset S.

(i) Any two members in S have almost equal number of supporters in $V - S$.
(ii) Any two members in $V - S$ have influence on almost equal number of members in S.

Motivated by this situation Anitha et al. [2] introduced the following concept of equitable domination. Let D be a dominating set of a graph G. Then D is called an *equitable dominating set* of type 1 if for any two vertices v, w in D, the number of neighbors of v and w outside D differ by at most one. Also D is called an equitable dominating set of type 2 if for any two vertices x and y in $V - D$, the number of neighbors of x and y in D differ by at most one. The minimum cardinality of an equitable dominating set of G of type 1 (type 2) is called the *1-equitable* (it 2-equitable) *domination number* of G and is denoted by $\gamma_{eq1}(G)(\gamma_{eq2}(G))$. If D is an equitable dominating set of type 1 and type 2, then D is called an equitable dominating set and the *equitable domination number* of G is defined to be the minimum cardinality of an equitable dominating set and is denoted by $\gamma_{eq}(G)$. For basic properties of the above domination parameters we refer to Anitha et al. [2].

9. Cliques, Clubs and Clans

In a global social network there may be dense subgroups which might be perhaps a group of friends or a group of people living in a specific locality. Dense subgraphs often represent cohesive grouping of nodes that are the natural focal points for studying the network structure, dynamics and evolution. An efficient algorithm for extracting dense subgroups in large networks has been proposed by Gibson et al. [15]. There are several terminologies that describe such cohesive subgroups in a network. Researchers in social network introduced several concepts to describe closely knit groups in a network. In standard graph theory a familiar cluster concept is given by cliques of a graph. A *clique* is a maximal complete subgraph of G. Another cluster type definition of subgraphs was introduced by Luce [31]. An *n-clique* L of a graph G is a maximal subgraph of G such that for any two points u, v in L, the distance between u and v in G is at most n. Clearly an n-clique is a global concept based on the total structure of the network. Within the subgraph L the distance between points can be larger than n. Hence the concept of n-clique does not imply the tightness or even connectedness of the set. Alba [1] introduced a more satisfactory subclass of n-cliques, which was later termed as clans by Mokken [33].

An *n-clan* M of a graph G is an n-clique of G such that for any pair of points u, v of M, the distance between u and v, computed within the subgraph M, is at most n. Consequently an n-clan is connected and has diameter at most n. Instead of the restriction of n-cliques to n-clans Mokken [33] introduced the concept of n-club. An *n-club* N of a graph G is a maximal subgraph of G of diameter n. It follows immediately form the definition that every n-club is contained in some n-clique and every n-clan is an n-club. These concepts can be extended for directed graphs also. Thus the maximum order of a clique, club or clan indicates the size of closely knit subgroups in the network.

10. **Future Research Directions**

In the previous sections we have introduced several specific subsets such as dominating sets, degree equitable sets, cliques, clans, clubs and also several graph theoretic measures which are quite relevant in social network analysis. The problem of identifying such subsets and the computation of most of these parameters are notoriously hard combinatorial problems. Since this is a multidisciplinary research area involving social science, graph theory, statistics and computer science, there are several possible directions for further research. One obvious direction is to analyze small and large social networks arising from real data sets, using graph theoretic tools so as to bring out hidden patterns sitting in the network and leading to specific conclusions towards improving the performance of the network. Social network researchers can take advantage of a large collection of freely available computer software for computing the various graph theoretic measures and subsets of specific type. Since a social network is simply a graph that represents something real, the abstract graph theoretic tools can be efficiently used to explain the behavior of a real system and on the other hand social network problems may also lead to the discovery of new graph theoretic tools whose theory can be independently developed, and thus the two areas of research complement each other. To give an example, the concept of structural equivalence defined by the social scientists can be used to define new graph theoretic measures and the abstract theory developed on these concepts can be applied again to real networks. We hope to see more of this type of complementary research involving social network and graph theory.

11. **Conclusion**

In this chapter we have given a brief survey, which by no means is exhaustive, on graph theoretic models for social networks. Several specific subsets and graph theoretic parameters which are relevant in the study of social networks along with a few research directions and a case study have been presented. Readers can construct social networks based on small data collected from organizations or massive data from other sources, apply the graph theoretic tools and arrive at conclusions, leading to a better understanding of the system and ways for improving the performance of the system.

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76$^{\text{TH}}$ IMS CONFERENCE : A BRIEF REPORT

A. K. SHUKLA

The 76$^{\text{th}}$ Annual Conference of the Indian Mathematical Society was held at Sardar Vallabhbhai National Institute of Technology (SVNIT), Surat - 395 007, Gujarat, during the period December 27 - 30, 2010 under the Presidentship of Prof. R. Sridharan, CMI, Chennai. The Conference was attended by more than 500 delegates from all over the country.

Padma-Bhusan Prof. M.S. Narasimhan, IISc, Bangalore inaugurated the conference. The Inaugural function was held in the forenoon of December 27, 2010 in the specially prepared Samiyana in the campus of Sardar Vallabhbhai National Institute of Technology, Surat. Prof. S. G. Dani, TIFR, Mumbai was the guest of honour. The inaugural function was presided by Prof. R. Sridharan, President of the Indian Mathematical Society. Prof. P. L. Patel, Director (I/C), SVNIT, delivered the welcome address. Prof. V. M. Shah, General Secretary of the Indian Mathematical Society, spoke about the Society and expressed his sincere and profuse thanks, on behalf of the Society, to the host for organizing the Conference. Prof. Satya Deo, The Academic Secretary of the Society, reported about the academic programmes of the Conference.

Prof. V. M. Shah reported about A. Narasinga Rao Memorial Prize. The Prize is awarded to K. R. Vasuki for his research paper adjudged to be the best research paper published in the Journal of the Indian Mathematical Society / The Mathematics Student in the year 2008.

Prof. V. M. Shah also reported about P. L. Bhatnagar Memorial Prize for 2010. The Prize was awarded to the top scorer Gaurav D. Patil (Pune), the top scorer of the Indian Team at the International Mathematical Olympiad, 2010.

It was the unique emotional event of the inaugural session this time when **Mr. K. Subramanian Iyer**, former *Asst. Librarian of the IMS Library* at the Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai, *was felicitated for his long and dedicated services from February 1951 to October 2009*, till his attainment of the age of Eighty eight (88) years. A Citation and an amount of Rs. 2000/- was presented to him in absentia as he could not remain present on health ground.

The Inaugural function ended with a vote of thanks proposed by Dr. A. K. Shukla, Head, Department of Applied Mathematics & Humanities, SVNIT, Surat and the Local Organizing Secretary of the Conference.

The Academic Sessions of the conference began with the Presidential General Address by Prof. R. Sridharan , the President of the Society on "An Encomium of Hermann Weyl" in the Main Hall, which was presided over by Prof. V. M. Shah, the senior most past president of the Society present in the Conference. The rest of the academic programmes were held in the Main Hall, LT-1, LT-2, ADM Drawing Hall, MED Lecture Hall, AMD Seminar Hall.

One Plenary Lecture, Four Award Lectures and Nine invited Lectures were delivered by eminent mathematicians during the conference. Besides these, there were Seven symposia on various topics of mathematics and computer science. A Paper Presentation Competition for various IMS and other prizes was held (without parallel session) in which 33 papers were presented. In all about 167 papers were presented for general reading during the conference on various disciplines of mathematics and computer science.

The Annual Meeting of the Council of the Indian Mathematical Society was held on December 26, 2010 at 6.00 p.m. at the Guest House, Sardar Vallabhbhai National Institute of Technology (SVNIT), Surat 395 007, Gujarat (which was adjourned) and the adjourned Meeting was held on December 29, 2010 at 4.30 p.m. at the Guest House, Sardar Vallabhbhai National Institute of Technology (SVNIT), Surat 395 007, Gujarat. The Annual General Body Meeting of the Indian Mathematical Society was held on December 30, 2010 at 12:15 p.m. in the Samiyana in the Campus of the Sardar Vallabhbhai National Institute of Technology (SVNIT), Surat. Since the President Prof. R. Sridharan had to leave before the meeting, the General Body meeting was presided by the senior most past president Prof. V. M. Shah present in the meeting.

After the General Body Meeting, the Valedictory function was held. During the valedictory function held on December 30, 2010, the IMS and other prizes were awarded to the winners of the Paper Presentation Competition. It was also announced that the next annual conference of the Society will be held at the SRTM University, Nanded, (Maharashtra).

A Cultural Evening was also arranged on December 28, 2010. The conference was supported by the NBHM, SVNIT, DST, CSIR, DRDO, Laxmi Diamond Pvt. Ltd, KRIBHCO, SVNIT Alumni Association, Shajanand Group of companies, Kunj Enterprise, and Suraj Corporation.

The details of the academic programmes of the Conference follows.

Details of the Plenary Lecture:

1. Prof. M. S. Narasimhan (FRS, I I Sc, Banagalore) delivered a Plenary Lecture on "ALGEBRAIC GEOMETRY AND ANALYSIS". Chairperson : Prof. N. K. Thakare.

Details of the Memorial Award Lectures:

1. The 24$^{\text{th}}$ P. L. Bhatnagar Memorial Award Lecture was delivered by Shiva Shankar (CMI, Chennai) on *"Fourier Series, Arithmetic and Control Theory "*. Chairperson : R. Sridharan.
2. The 21$^{\text{st}}$ V. Ramaswamy Aiyer Memorial Award Lecture was delivered by A. Ojha (DPMIIIT, Jabalpur) on *"Certain applications of mathematics in path planning"*. Chairperson : N. K. Thakare.
3. The 21$^{\text{st}}$ Srinivasa Ramanujan Memorial Award Lecture was delivered by Mahan Mj (School of Mathematical Sciences, Ramkrishna Mission Vivekanand University, Belur Math, Howrah) on *"Geometry and dynamics of Kleinian groups"*. Chairperson : Satya Deo.
4. The 21$^{\text{st}}$ Hansraj Gupta Memorial Award Lecture was delivered by Subhash J. Bhatt (Dept. of Mathematics, S. P. University, Vallabh Vidyanagar, Gujarat). on *"C*-algebras, uniform Banach algebras and a functional analytic meta theorem"*. Chairperson : S. B. Nimse.

Details of various prizes awarded by the Society :

1. **A. Narasinga Rao Memorial Prize.**
   This Prize was awarded to K. R. Vasuki for his research paper entitled *"On certain Ramanujan - Weber type Modular functions"* published in

the J. Indian Math, Soc. 75 (2008), 173 - 192 and adjudged to be the best research paper published in the Journal of the Indian Mathematical Society / The Mathematics Student in the year 2008.

2. P. L. Bhatnagar Memorial Prize.
For the year 2010, this prize was awarded to Gaurav D. Patil (Pune) for being the top scorer for the Indian Team (25 points) at the 51$^{\text{st}}$ International Mathematics Olympiad held at Astana (Kazakstan) during July 02 - 14, 2010.

3. Various prizes for the Paper Presentation Competition:
A total of 33 papers were presented for Paper Presentation Competition for Six IMS Prizes, AMU Prize and VMS Prize. There were 23 entries for six IMS prizes, one entry for the AMU prize and 9 entries for the VMS prize.
Professors N. K. Thakare (Chairperson), Geetha S. Rao, Huzoor Khan, S. Bhoosnurmath and J. R. Patadia were the judges.
Following is the result for the award of various prizes:
IMS Prize - Group-1: 07 Presentations. Prize awarded to :
Smruti Mane, Pune University, Pune.
IMS Prize - Group-2 : 02 Presentations. Prize not awarded.
IMS Prize - Group-3 : No Entry. Prize not awarded.
IMS Prize - Group-4 : 04 Presentations. Prize awarded to :
Pratibha Manohar, Rajasthan University, Jaipur.
IMS Prize - Group-5 : 05 Presentations. Prize awarded to :
Dilip Kumar, Center for Mathematical Sciences, Pala, Kerala.
IMS Prize - Group-6 : 02 Presentations. Prize awarded to : Keya A. Shah, Sarvajanik College of Engineering, Surat.
AMU Prize : No Presentations. Prize not awarded
V M Shah Prize : 06 Presentations : Prize awarded to :
Renukadevi Dyavanal, Karnataka University, Dharwad.

Details of Invited Lectures delivered :
One hour Invited talks :

- V. Suresh (University of Hyderabad) on *"A Hasse Principle for quadratic forms".*
- Madhavan Mukund (CMI, Chennai) on *"Interplay between Automata Theory and Mathematical Logic".*
- V. D. Sharma (IIT Bombay, Powai, Mumbai) on *"Hyperbolic systems of PDEs and the associated wave phenomena".*

Half an hour Invited talks :

- C. S. Rajan (TIFR, Mumbai) on *"Spectrum and Arithmetic"*.
- Anupam Singh (IISER, Pune) on *"Real and Strongly Real Classes in finite linear groups"*.
- M M Shikare (Pune University) on *"The element splitting operation for graphs, matroids and its applications"*.
- M. S. Mahadeva Naika (Central Collage, Bangalore University, Bangalore) on *"Recent work on Modular equations and its applications"*.
- R. D. Giri (University of Bombay, Mumbai) on *"Coding Theory for every one - semigroup approach"*.
- B. M. Pandeya (IT, BHU) on *"CMC Modules over Noetherian rings"*.

Details of the Symposia organized :

1. On *"Summability Operators and Applications & Method of Approximation"*
   Convener: Huzoor H. Khan (AMU, Aligarh).
   Speakers :
   - Huzoor H. Khan (AMU, Aligarh) : *"Approximation by Summability Operators"*.
   - Shyam Lal (BHU, Varanasi) : *"Approximation of functions of generalized Lipschitz class by product summability operators "*.
   - J. K. Maitra (R. D. University, Jabalpur) : *"On Spline Modules"*.
   - Mridula Dube (R. D. University, Jabalpur) : *"Trigonometric Splines"*.
   - S. K. Upadhyaya (IT, BHU, Varanasi) : *"The Bessel wavelet transformation involving Hankel convolution"*.
2. On *"Modeling in Environmental Fluid-mechanics"*
   Convener : Girija Jayaraman (I. I. T. Delhi, Delhi).
   Speakers :
   - Swaroop Nandan Bora (IIT Guwahati) : *"Linear Water Wave Propagation over a porous sea bed "*.
   - B V Ratishkumar (IIT Kanpur) : *"Convection in Porous Media and Applications - A Numerical perspective"*.
   - Om P. Sharma (IIT Delhi) : *"Modeling aerosol distributions in atmospheric flows "*.
   - Girija Jayraman (IIT Delhi): *"Modeling in Environm- ental Fluid Mechanics - State of the art"*.
3. On *"Recent advances in Analysis and Operator Theory"*
   Convener : Manjul Gupta (IIT Kanpur, Kanpur).

Speakers :

- M. A. Sofi (University of Kashmir, Srinagar) : *"Factoring certain operator ideal equations determini- ing finite dimentionality"*.
- TSSRK Rao (ISI Bangalore) : *"Cental subspaces of Banach spaces"*.
- Manjul Gupta (IIT Kanpur, Kanpur) : *" "*.

4. On *"Mathematics and Information Technology"*
   Convener : G. N. Prasanna (IIIT, Bangalore).
   Speakers :
   - Madhavan Mukund (CMI, Chennai) : *"Formal varification"*.
   - Raja (TIFR, Mumbai) : *"Computarized Proof Assistants"*.
   - Sakti Balan (Infosys) : *"Peptide Computer - Towards a formal Model for Peptide computing "*.
   - G. N. Prasanna (IIIT, Bangalore): *"Mathematics and Information Technology: Vistas and opportunities "*.
5. On *"Cohomology of Transformation groups"*.
   Convener : Satya Deo (HRI, Allahabad).
   Speakers :
   - P. Sankaran (IMSc, Chennai): *"Vector field problem for certain homogeneous spaces"*.
   - H. K. Mukherjee (NEHU, Shillong) : *"Homology and Dynamical systems "*.
   - S. K. Roushon (TIFR, Mumbai): *"The Ferrel-Jones Isomorphism Conjecture"*.
   - Satya Deo (HRI, Allahabad) : *"Spaces of finite cohomological dimension and finifistic spaces "*.
6. On *"Special Functions and their Applications"*
   Convener : Arun Verma (IIT Roorkee, Roorkee).
   Speakers :
   - A. K. Agarwal (Punjab University, Chandigarh) : *"Matrix Theoretic interpretations of some basic series identities"*.
   - D. D. Somasekhar (Mysore University, Mysore) : *"On some summation formulas and their applications"*.
   - Vivek Sahai (Lucknow University, Lucknow) : *"p, q-representations of Lie algebra gl(2) and p, q-Mellin integral transformation"*.
   - S. Bhargava (Mysore University, Mysore): *"On the inversion of Ramanujan's Cubic Analogue of Jacobian Elliptic Integral"*.
   - A. Shukla (SVNIT, Surat) : *"Recent development and study in general class of polynomials"*.

- Arun Verma (IIT Roorkee, Roorkee) : *"Summation and Transformations of balanced Terminating Hypergeometric series".*

7. On *"Around Continued fractions".*
   Convener : K C Prasad (University of Ranchi, Ranchi).
   Speakers :
   - K C Prasad (University of Ranchi, Ranchi) : *"Some results in Diophantine Approximations".*
   - S. Bhargava (Mysore University, Mysore) : *"Some continued fractions of Ramanujan".*
   - S. P. Singh (TDPG College, Jaunpur) : *"On Hypergeometric function and Ramanujan's continued fractions".*
   - S. K. Agrawal (Vinoba Bhave University, Hazaribag, Jharkhand) : *"Chakrawala method of Bhaskara and continued fraction algorithm".*
   - K. R. Vasuki (University of Mysore, Mysore) : *"Two Modular Equations for Squares of the Cubic-functions with Applications".*

A. K. Shukla,
Local Organizing Secretary,
76^{th} Annual Conference of the India Mathematical Society-2010
Sardar Vallabhbhai National Institute of Technology,
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REPORT ON IMS SPONSORED LECTURES

The details of the IMS Sponsored Lectures/Popular Talks held is as under:

1. Speaker : Prof. Gautami Bhowmik.
   (Universite de Lille, France).

   Title of the Lecture : Some aspects of Additive Combinatorics.

   Day & Date : March 04, 2011 at 3 p.m.

   Venue : Seminar Hall - 1, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai.

   Organizer : Prof. K. Parthasarathy.

Abstract: In additive combinatorics, subsets of additive groups are described and usually quantitative measures of these structures are given. This study involves the use of tools from diverse areas of mathematics.

In this Lecture, some examples of recent years will be mentioned to show the use of classical methods such as exponential sums and Fourier analysis as well as some newer ones like geometry of numbers.

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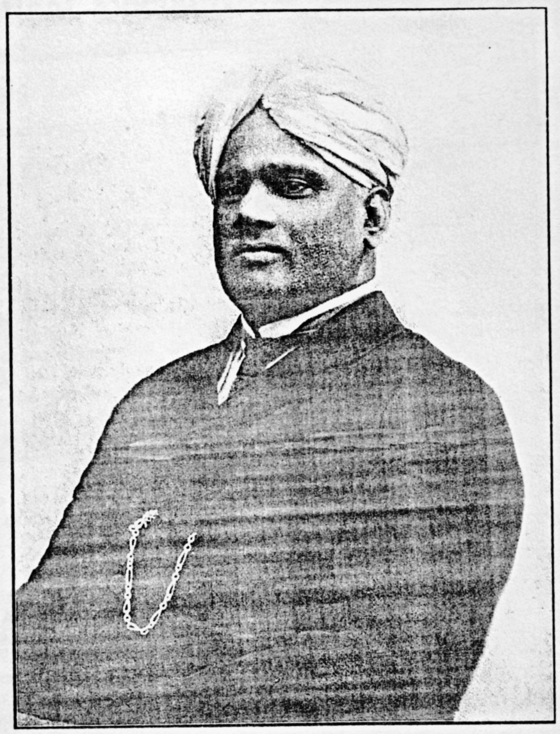
ADDRESS TO MR. V. RAMASWAMI AIYAR, M.A., FOUNDER OF THE INDIAN MATHEMATICAL SOCIETY

FELLOW IMS MEMBERS OF 1926



MR. V. RAMASWAMI AIYAR, M.A.

Address to Mr. V. Ramaswami Aiyar, M. A., Founder of the Indian Mathematical Society.

SIR :

The Indian Mathematical Society under whose auspices we are assembled in Conference to-day completed its eighteenth year of existence last year, and according to Indian reckoning attained its majority. On this important occasion, we, the Members of the Society, feel it our pleasant duty and privilege to present you, its Founder, with this Address in token of our great appreciation of your labour of love.

In this connection, it may not be out of place to review briefly the history of the Society. On the 25th of December 1906, you conceived the happy idea of addressing a few friends interested in Mathematics to form an "Analytic Club" for the purpose of securing facilities for advanced study in that subject by way of mathematical books and periodicals. About twenty gentlemen responded to the invitation, and the formation of the Club was announced in the Madras papers on the 4th of April 1907. The original idea was only to subscribe for periodicals, purchase the latest books connected with mathematical research, and arrange for their circulation among the members. But even in those early days, you, Sir, cherished larger and higher ideals for the future of the Society—such as greater Equipment, the conduct of a Mathematical Journal, and the formation of a Central Library. Notwithstanding your arduous duties in other directions, you spared no pains to promote the growth and usefulness of the Institution. By issuing circular after circular, you had a suitable Constitution framed for the Society, and placed its affairs on a secure basis before you laid down the office of Secretary. It may not be out of place to mention that with the true scholar's freedom from provincial bias, you, a

Madrasi, arranged to locate the Head-quarters of the Society at Poona, in the Bombay Presidency—a step which gave our Society the unique position of being an All-India Institution.

It is impossible to overstate the difficulties that you had to encounter in nurturing your infant creation and enabling it to grow to lusty youth. The members were scattered all over India. They were not acquainted with one another sufficiently enough to arrive at a common understanding and to tone down the inevitable differences of opinion. Undaunted by these difficulties and in spite of the pressure of your official duties, you carried on the work of the Society successfully, arguing, remonstrating and pleading; and though the work had often to be done in the late hours of the night, you were never dilatory in your efforts to secure the speedy realization of the aims of the Society. In a few years, you succeeded in building up a Library and starting a Journal. In 1907, the membership of the Society was only 20. It had no Journal. At present, it has 219 members and 49 subscribers ; its Journal is running through its 16th volume ; its Library has a large stock of books ; it subscribes for a decent number of mathematical journals, and is in intimate touch with similar Societies in the world.

We are proud of this achievement ; and if, to-day, the Society occupies a position well-recognized in India and in the mathematical world outside, we feel, Sir, that it is in no small measure due to your initial labours to place it in working order and to promote its advancement in all directions. The Society has already enrolled you as an Honorary Member and at this public Conference, in this eventful year of its existence, we beg to tender our loving tribute of appreciation and gratitude to you, its Founder, for all that you have done to bring it to its present position of vigorous life and activity. When you retired from the Secretary's post, there was a general feeling that you should be elected President of the Society, but with your usual modesty xviii

you declined the office not only then but on every subsequent occasion. We are glad that you have at last yielded to our wishes, and consented to be the President for this term.

May the Society flourish under your care and may a long and happy life be vouchsafed to you!

We subscribe ourselves,

Sir,

Your Fellow-Members of the Indian Mathematical Society.

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GENESIS OF FOUNDING OF THE SOCIETY : BY V. RAMASWAMI AIYAR IN HIS PRESIDENTAL ADDRESS OF 1926

The Presidential Address

DR. SEAL, LADIES AND GENTLEMEN :

The Indian Mathematical Society and its Committee have honoured me this year, in a three-fold manner, for my humble services, done over 18 years ago, in founding the Society: first, they have made me an Honorary Member of the Society ; secondly, they have raised me to the position of the President of the Society; thirdly, they have presented me with a beautiful address, which makes this a red-letter day in my life.

Under the constitution the Committee can elect as Honorory Members persons distinguished in Mathematics, or in eminent positions and interested in the objects of the Society. I do not see that the Committee have made a strict application of the Rule in electing me an Honorary Member. They have simply expressed an exceeding good-will towards me as founder of the Society. By their act, they have put me in one list with such eminent *Acharyas* of the Science as Baker and Hardy, Whittaker and Whitehead, and such a distinguished son of India as Dr. Paranjpye. To such a list I feel my name has been added very unworthily.

Since our last Conference we have lost from this list, by death, the brilliant name of Sir Asutosh Mukherji, M.A., D.L., C.S.I., who was fitly styled as *Saraswathi*. He was eminent in many a field of knowledge besides Mathematics, and his great services in the cause of scientific advancement in India can never be forgotten.

The Presidentship of our Society has been held before me by such distinguished scholars as Prof. Wilkinson and Mr. Balak Ram. They represented the Bombay and the Panjab areas of our membership, respectively. By an unwritten convention, the Presidentship, it appears, should go now to one from the Madras side. When I was offered the place, I expressed my dissent from this convention as Mr. Balak Ram was holding it very worthily ; and I also said that, if the convention must be followed, the honour should go to a veteran mathematician like Mr. Naraniengar. But Mr. Seshu Aiyar, the energetic Secretary, would not be deflected from his purpose. He said that Mr. Naraniengar was still wanted as Editor of our Journal, that I must take the Presidentship before it can go to younger men, and that the Committee's wish to see me honoured in this way, must be respected. I found myself not strong enough to resist further, and I accepted the nomination.

The Committee have further honoured me with their very kind address read to-day. In every previous Conference I have been mentioned to be the founder of the Society and therefore a special address to me now, on the fortunate circumstance of my being the founder, is really superfluous. But, as an expression of the good-will of the members towards me, I value it greatly and shall cherish it deeply. It is moreover a happy circumstance, Sir, that it is given to me to receive this address, standing on Mysore soil. For it was in Bangalore and Mysore that, years ago, I had a short spell of service as a Mathematics teacher—a period which I remember with great pleasure. I wish to thank the Committee and members of the Society very sincerely for these several honours which they have bestowed upon me.

When I accepted the Presidentship I did not realize that the task of a Presidential Address at this Conference was to be mine so soon. I am very ill-fitted for the task. My life has been cast in a different sphere of work and away from any Mathematical Library. I own but few books and even these I have left behind in one of my previous stations. Both Messrs. Wilkinson and Balak Ram have, in previous Conferences, given us most scholarly addresses, the level of which it is impossssible for me to reach. Casting about for ideas I have thought of making up my address by placing before members some of my own mathematical reminiscences, which, if you are indulgent enough, may show the nature of the interests that the founder possessed ; and, along with them, a few observations of a general character regarding the progress we can make for attaining the general objects of the Society. It is an amateur's address, and I hope for your utmost indulgence.

3

Before I proceed further, it is my duty to pay a tribute of respect to Messrs. Balak Ram, Seshu Aiyar and Ross for their services in general and as members of the last Committee.

When our Society was formed in 1907 its membership was confined to the Madras and Bombay areas. But soon afterwards Mr. Balak Ram was introduced as a member by Dr. Paranjpye. Mr. Balak Ram was a fine product of the young University of Punjab, and, ever since he joined, he has been a pillar of strength to our Society. He soon got in for us many members from Punjab. The Calcutta Mathematical Society was under formation at the time and there were reasons to fear that the Punjab area would go over to the Calcutta Society as Punjab formed a part of the extensive area of the old Calcutta University. But Mr. Balak Ram made that area safe for our Society. He has performed for us the mathematical annexation of the Punjab. Not only so, but during his Presidentship our Society reached a world-wide standing coming in contact with all the sister institutions in the world. Our Journal made a marked progress in his time, in keeping with the advance in our Society's standing. To crown all, he has made us a princely benefaction of Rs. 1,000 to our Library. To this benefaction I shall have occasion to refer again. It is one that our Society can never forget. I am so glad that Mr. Balak Ram, though he has laid down the office of President, continues to be a member of the Committee.

I have next to express my great regret that Mr. Seshu Aiyer has gone out from the Committee. He has given his best services to the Committee now for over 15 years. He had a rare touch with the men in all districts of Madras who were engaged in mathematical teaching in High Schools and Colleges. Year after year, also, there were batches of young men who studied Mathematics under him in the Presidency College, Madras. He has brought in many of them as members of our Society. To Mr. Seshu Aiyer is due the credit of organizing our first Conference in Madras, which proved a great success. When he proposed the Presidentship to me, he kept away from me the fact that he was going to be off from the

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Committee. Had I known it, it is nearly certain that I would have preferred to be outside the Committee, too But, a good tactician, he has fixed me in first, and then got himself out, mentioning our rule of rotation, and various circumstances as excuse. But though outside the Committee, he has promised me, that his endeavours on behalf of our Society will continue unrelaxed. The Government have honoured him with the title of Rao Bahadur which he deserves by his work for our Society alone.

I regret also that Prof. Ross ceases to be in the Committee under the rule of rotation. His breadth of mathematical scholarship, especially in the field of Analysis, is second to none. His genial character and his fine spirit of fellowship with Indians are known to all who have come in contact with him ; and I feel that, by his retirement, the Committee have lost a most valuable member.

I must now turn to the promised mathematical reminiscences of the founder. In 1893 I took the M.A. degree. In the following year, about August, I was a candidate for the Mysore Civil Service Examination and failed. But I then received an enquiry from Mr. Bhabha, Inspector-General of Education in Mysore, asking if I would act as Professor of Mathematics in the Central College, Bangalore, in the place of Mr. T. R. Venkataswami Naidu, who was taking a short leave. I was glad to accept the offer. This made me a colleague of Mr. M. T. Naraniengar and we became intimate friends. Mr. Naraniengar had passed the M. A. in the year previous to mine. We were both deeply interested in Mathematics. Geometry was our *forte* (in a modest way), and curves, our particular fancy. Mr. Naraniengar and myself used to take long walks together in the mornings, in the bracing climate of Bangalore, discussing, I think, nothing but Mathematics. Ball's *Mathematical Recreations* had recently come out and contained a lot to interest us. I wonder how often we stopped in the road to draw and discuss a figure, careless of mankind passing by. After Mr. Venkataswami Naidu rejoined duty, I continued to remain in Bangalore, meeting him frequently. He was a fine conversationalist with shrewd remarks on nearly every topic. We had a specia affinity as both of us were old boys of the Coimbatore College. I regret he is no longer among us.

When, years later, I felt the need of a Mathematical Society in India, and sent my proposal to form one, my efforts were warmly supported by both Messrs. Venkataswami Naidu and Naraniengar. Not only were they among our foundation members, but they both came into our first Managing Committee, contributing to its strength. Our Society, when born, showed signs of rapid progress, and, not only did I know that the Journal of our Society would be started soon, but I knew also precisely who was going to be its distinguished Editor. Mr. Naraniengar has carried on the work of our Journal very ably during 18 years and, for this labour of love, we cannot be too thankful.

Proceeding with my reminiscences, in 1895 Mathematics classes were opened for B.A. in the Maharaja's College, Mysore, and I was appointed to a permanent post in the staff of the College, as Lecturer in the subject. Mr. J. Weir was the Principal of the College and Professor of Mathematics. I became his assistant. Mr. Weir got me introduced as a member of the Edinburgh Mathematical Society and I felt very proud to become a member of that body. The proceedings of the Society, which I received, gave me my first glimmer of hope that a Mathematical Society like the Edinburgh could, perhaps, be formed in India. I spent a happy year in Mysore, teaching Mathematics I remember very well the room in the college that was given to us, Assistant-masters and Lecturers. It was an institution by itself. Men teaching different subjects, we met daily in the interval for lunch. In that company which included the Sanskrit and Kanarese Pandits, I used to feel very happy, and realised that every branch of learning was a means for a great culture : and Mathematics was surely, but only, one such branch.

One very interesting reminiscence of that year is my becoming private tutor in Mathematics to Miss Bhabha, daughter of the

Inspector-General of Education, who was studying for the Matric. Miss Bhabha, I learn, afterwards became Lady Dorab Tata. I have not met her as Lady Tata and the image in my mind is that of Miss Bhabha only. Miss Bhabha did everything gracefully. Her note-books were a model of neatness. Once, when I was taking her over the theorem of Pythagoras and discussing some *extra* proof of it, she brought to my notice an interesting little figure, which had a neat property. She proved it one way and I showed another way of doing it. Between these lay the proof of a particular case of Euclid, I. 47. Varying the figure a little, both our modes of proof applied for proving a like property of the new figure; and that meant, we found out a new proof of Euclid I. 47. I communicated this proof to Mr. Venkatasami Naidu at Bangalore, who sent me an observation thereon. This led me to see that the principle of the proof that Miss Bhabha and myself had found had a wider application, and that it yielded a proof, *within the First Book of Euclid*, of the theorem that a regular figure of *n* sides described on the hypotenuse of a right-angled triangle was equal to the sum of the regular figures of *n* sides described on the other two sides.

My service in Mysore did not continue long. I appeared for the Madras Civil Service Examination of 1895 and came out successful—thanks to my friend Mr. Belwadi Dasappa, the Lecturer in English, who coached me in that subject in which I was very deficient. I left Mysore with great regret, for, not only my work as Mathematics Lecturer, but the life with the students, the scenery and traditions of Mysore, and the work of its statesmen, all filled me with interest; and Mysore was to me emblematic of all the glory of our motherland. I joined the Madras Civil Service as a Deputy Collector at the end of 1895.

Taking a step forward now of about 12 years we come to 1907, the year in which our Society was founded. But, to give an account of the circumstances which led to the formation of our Society, we must go back to the previous year 1906.

The year 1906, I think, is an important date in the intellectual history of the world. It was the year in which Einstein made his great discovery of the principle of Relativity. It was also the year, probably, in which our own great mathematician, Ramanujan, unknown as yet, was making some of his discoveries. However that be, there was a feeling in India that it was the time of a large awakening. There was considerable political agitation then, owing to the partition of Bengal. But men also saw that, before India could become great, we needed advancement in many different directions. Our great countryman, Sir J. N. Tata, had these problems in mind and had laid the foundation of a considerable industrial and intellectual advancement. But his great scheme of a Central Research Institute for India made no provision for mathematical advancement. One day I put myself the question " Can I not be of some help in advancing the interests of Mathematics in India." The spirit of the times made me think seriously about the question. I wanted to form a Mathematical Society which might be something like the Edinburgh Society. I obtained the Calendars of the Madras and Bombay Universities and made a list of all the men who had taken the M.A. degree, or a first class in B.A., or were doing work as Professors and Assistant Professors in the various colleges. The list was very encouraging. There were men of distinction like Dastur, Sanjana, Apte and Paranjpye in the Bombay Presidency. There were men like Hanumantha Rao, my own teacher Swaminatha Aiyar, Ramachandra Rao, Naraniengar, Ramesam, Venkataswami Naidu, and so on, in my Presidency. On perusal of the long list, it occurred to me that, if by some magic, I could only put all these names into a Society, then, with Euclid, I might say, Q. E. F. (what is required is now done). But I did not see my way for this magic. I was a Deputy Collector and not a Professor of a College whom his fellow mathematicians would know. I sometimes thought that, if I were the great Asutosh Mukherji of Bengal, I could make a trumpet call, bringing all the mathematicians together at once. But I was unknown ; and any call by me, I felt, would fall flat on the public ear. So I was in a state of hesitation as to the action I should take.

In that year I saw a copy of the quarterly journal of Mathematics, sent me by a friend. The matter which it contained was mostly beyond me but there was a little bit which delighted me And I began to wonder how many such delightful bits, we in India may be missing by not seeing the leading journals. This made me more eager than ever to try to form a Mathematical Society. At length came the Christmas Day of December 1906. I was put up in a village *Chavidi*, in Gooty Division, Anantapur District. Thinking calmly, I felt that God will prosper any earnest endeavour made in a self-less cause. I was able to see my plan clearly and I wrote a little letter asking friends to join and form a small Mathematical Society. The letter was simply conceived and had no ambitious programme. It has been published in Vol. XI of the Journal of our Society. I sent copies of the letter to one friend after another and I received an abundant response during three months. Then on the 4th of April 1907, I was able to announce in the Madras dailies that the Society was formed, with a strength of 20 members, under the name " The Indian Mathematical Club," and I became one of its Joint Secretaries along with Mr. Naraniengar and I continued so till 1910. The strength of the Society had grown then to about 120. This period of service in the cause of Mathematics is one which gives me great pleasure to look back upon, and your generous recognition of it to-day fills me with thankfulness that it was given to me to render the service.

After the formation of our Society in 1907, I wrote to the leading mathematicians of Bengal (with copies of my original letter and letter of announcement) asking for their support. I got a prompt response from Professor Mahendra Nath De, but not from others. Prof. De informed me that my letters to others had been received, but the mathematicians of Bengal were considering the formation of a separate Society in Calcutta. I was feeling disappointed that Bengal did not join us. Before long, however, I received a kind letter from Sir Gurudas Bannerji, the eminent Judge, as well as mathematician, acknowledging my letter, commending our action, and stating that they in Bengal preferred to have a Society formed in Calcutta itself, to gain our common objects the better, and that plans for this were ready. He said that

India was such a vast country that there was ample room for both the Societies to function and he wished all prosperity to our endeavours. With such blessings from Sir Gurudas we were very glad to see our sister Society come into existence in Calcutta, in 1908, for promoting our common objects. The thought has sometimes occurred to me, Sir, that if I have been the positive founder of our Society, I am also a sort of *negative* founder of the sister Society at Calcutta.

The year 1910 holds a special place in my mathematical reminiscences being that in which I met Ramanujan, that brightest flower of Indian Mathematics which has been plucked away so soon from the bosom of Mother India. Ramanujan, as yet unknown to the world, came to me at Tirukoilur, with a letter of introduction from Mr. K. S. Patrachariar, member of our Society. The letter said that Ramanujan had great abilities in Mathematics which, Mr. Patrachariar suggested, I would be in a better position to appreciate. Ramanujan stayed with me 3 days. I then had a view of those remarkable note-books in which he had recorded the numerous results of his discovery. There was little or no explanation given but the results were of a very striking character. I put him some test questions in order to find out if his methods were really based on sound principles. I remember one such question. Taking the harmonic series $1/2 + 1/4 + 1/6 + 1/8 + \ldots$ formed with the even natural numbers and the harmonic series $1 + 1/3 + 1/5 + \ldots$ formed with the odd natural numbers, my question was to show that the sum of $4n$ terms of the former series exceeded the sum of n terms of the latter series by an amount which constantly diminished as n increased and vanished as n became infinite. I asked Ramanujan to let me know if this was true. He soon said it was. I asked him for his method. He gave it and I observed it did not require the use of the Euler Constant. When I told him that my method required the use of the Euler Constant, he observed that both the series were plain enough in their meaning and did not involve the idea of a logarithm or the Euler's Constant and therefore, intrinsically, the result should be capable of proof without the Euler Constant, a remark which impressed me as showing a broad out-look which we discussed further. I put Ramanujan other questions involving delicate points of convergence which I had come to know in correspondence with Prof. Hardy. Ramanujan was able to grasp them quickly. I then felt convinced that Ramanujan's abilities were of a high order and that the results in his note-books must be genuine work. In the end I asked Ramanujan to let me know whether there was anything I could do for him. He said he was badly in need of employment and suggested that I might put him as a teacher in one of the Elementary Schools belonging to the Taluk Board of which I was President. I told Ramanujan that it would be sacrilege on his talents to put him to teach in an Elementary Board School but that I had been thinking out if I could not be of help to him in a real manner.

The idea that had struck me was that I should try to get Ramanujan a University Research Scholarship. I devoted anxious thought as to how this was to be done. Ramanujan was a failed F. A. and how was he to get a scholarship for research, from the University ? In the end, I wrote a letter to Mr. Seshu Aiyar, then at Madras, a letter written both for him and Mr. Narayana Aiyar, the Treasurer of our Society. I said in that letter that Ramanujan had extraordinary abilities in Mathematics though he was a failed F.A., and that a University Research Scholarship should be secured for him; that for this purpose, his abilities should be brought to the personal notice of every member of the Mathematical Board of Studies (I had then the honor to be on the Board), and, with our united support, we should make an impression on the Syndicate. I wrote also that, till a scholarship was secured, some job must be found for Ramauujan to live by in Madras ; and that till he got a job, friends should maintain him somehow. I asked in particular that Ramanujan's merits should be promptly brought to the notice of Dewan Bahadur R. Ramachandra Rao. Such was the substance of my letter. Mr. Ramachandra Rao, our past President, within whose bosom there beats the heart of a true son of India, generously helped Ramanujan to remain in Madras at a time of pinch. My friend Mr. Narayana Aiyar removed the pinch by finding him a clerkship in the Port Trust Office of which he was Manager. The getting of a research scholarship for Ramanujan was, however, very difficult. But Sir Francis Spring, President of the Port Trust, and Dr. G. T. Walker, F.R.S., came to discover that there was a mathematical genius rotting in the Port Trust Office and, thanks to *their* intervention, the Research Scholarship, that I had anxiously hoped for was eventually secured for Ramanujan. After becoming a Research Scholar, Ramanujan became the architect of his own further advance by those first letters that he wrote to Prof. Hardy, giving some hundreds of his results These led Prof. Hardy to realize that Ramanujan was a mathematician of a high order; and he warmly took the steps required to get Ramanujan to go over to England for study and made his genius world known. It is the duty of our Society to keep Ramanujan's memory ever fresh, and, coupled with it, the name of Prof. Hardy.

Our Society, having attained 18 years of age, is probably now in a self-conscious stage of its progress and it behoves us to see what we can do to further advance the objects of the Society. I would first advocate that each of our younger members should try to get into intimate touch with a few others and keep an active correspondence on matters mathematical which interest them. Our members are scattered over a large area and it is necessary that we should cultivate close touch, each with as many others as possible, through the post. I may give a bit of my own experience. Two years ago, Mr. M. Bhimasena Rao wrote to me his first letter, asking me to look into a paper which he was then contributing to our Conference of 1924. Since then our correspondence grew up and I have collected Mr. Bhimasena Rao's letters, received so far into 4 little volumes which are here before you. These volumes contain some of the most highly interesting work in Mathematics that I know of and show Mr. Bhimasena Rao to be a gifted geometer. Mr. Bhimasena Rao has with him two volumes into which he has collected the letters I wrote to him. The principal subject of our study has been a generalization of the pedal circle theory. By means of it we have been able to extend the theory of pedal and contact circles *in plano* to a spherical triangle. I think I may try to give a general idea of the subject.

Nearly a hundred years ago Feuerbach discovered the theorem that the nine points circle of a triangle touched the inscribed and escribed circles. The beauty of this theorem has attracted many to try and find out proofs, and several proofs have been given. A notable advance was made by Dr. Hart some 40 years later when he found that the inscribed and escribed circles of a spherical triangle were all touched by another circle. This is known as Dr. Hart's circle. Some 25 years later, about the nineties of the last century, M'Cay discovered an extension of Feuerbach's theorem. It is to the effect that if the join of a pair of isogonal conjugates pass through the circum-centre their pedal circle touches the nine points circle. In 1896 I discovered an extension of M'Cay's Theorem, forming so to say, an extension of the second order of Feuerbach's Theorem and published it in the proceedings of the Edinburgh Mathematical Society. In 1901 I made a further order. I was intending to write a paper about it, but it never came about. In 1906 I published a case of it, forming an extension of Feuerbach's Theorem of the third order, as a question in the *Educational Times*. In 1907 I published, also as a question in the *Educational Times*, another case deduced from my theorem of 1901, which forms an extension of the second order of Feuerbach's Theorem. About 1910, Mr. Bhimasena Rao made his discovery of a new circle connected with a pair of isogonal conjugates which he called the contact circle. He showed that the contact circle of a pair of isogonal conjugates touched the nine-points circle if their join passed through the Symmedian point. This is an interesting extension of Feuerbach's Theorem, analogous to M'Cay's. Mr. Bhimasena Rao also discovered the theorem that the pedal circle of a point P touched the nine-points circle if the sum of the angles PAB, PBC, PCA was equal to one right angle. Mr.

Bhimasena Rao afterwards found that this theorem was anticipated by Mr. C. E. Youngman. In 1917 I gave an extension of this theorem, namely, the pedal circle of any point P cut the nine point circle of ABC at an angle equal to the complement of the sum of the angles PAB, PBC, PCA. In this way several extensions of Feuerbach's Theorem, *in plano,* have been discovered. But it does not appear that any similar extensions of Dr. Hart's theorem on the sphere have been found out, though it is now some 60 years since Dr. Hart made his discovery. In our generalized pedal circle theory, we have found that it is possible to define 'the pedal circle,' as well as 'the contact circle', with respect to any pair of isogonal conjugates in a spherical triangle, in such a way that we have several analogous extensions of the property of Dr. Hart's circle. Mr. Bhimasena Rao has worked up the extension to the sphere, of my third order extension *in plano* in a very able manner; and we are now standing confronted by the question of extending to the sphere my fourth order extension *in plano.*

Mr. Bhimasena Rao who is a son of the Mysore soil is holding a small place in the Educational service of his State. When I think of his great abilities I am put in mind of Ramanujan as a clerk in the Port Trust Office. May I make an appeal to His Highness the Maharaja of Mysore, who has sent us a gracious message to-day, that Mr. Bhimasena Rao may be placed in a better position to make his best contributions to Mathematics? Is it necessary, Sir, that I should appeal to His Highness, is it not sufficient if I appeal to you, the learned Vice-Chancellor of the Mysore University, to give this matter your kind consideration?

The next point that I wish to place before our members, is now that our Society has grown strong during 18 years, we should try to see that not only certain branches of Mathematics, but the whole field, should be cultivated by our members, based on the principle of study in close association, that I have spoken of. The whole field is recognizable as falling into so many divisions, and sub-divisions, which are not water-tight compartments but have vital connection with one another. I should like to place before members the idea that there should be a few of us studying each of these divisions or fields. Taking any one of these fields the progress that can be made in its study will be generally as follows. First one must be eager to enter the field. At this stage he is a mere *entrant*, an *embryo*. Then, by study, he develops into a *learner*. After this stage, with some enlightenment, he becomes an *interpreter* of the subject. In this stage he sees that many of the things learnt are not essential, and are mere cobwebs in the mind. He sweeps them out and takes hold of the essential. Finally, one rises to the stage of a *master*. Here he brings a fresh light of his own into the subject, becomes a discoverer, and extends the scope of the field. We can rarely rise to be masters, but we can all be learners, and possibly rise to be interpreters, by close study. All these stages of progress are symbolized by the letters of the word MILE. M is the stage of the master; I the stage of the interpreter; L the stage of the learner; and E the stage of the entrant, or embryo in the subject. Applying this philosophy, in order to cultivate any particular branch of Mathematics in our Society, I would like to catch some of you, younger members of the Society, as entrants, or embryos, and put you into a compartment, stock around you all the books on the subject, find a senior member, if possible, to serve you as interpreter, and find you also a master, if possible. I want to see created in our Society many such groups of interested members, of all grades of advance, compact little groups, call them Gurukulas if you like, to study the different fields of Mathematics, so that none is neglected in our Society. Success can only be slowly realized, but we can make an effort from now to secure our progress in all branches, so to say, by MILE and MILE, that is, by means of such compact groups consisting of masters, interpreters, learners and entrants. It may be said we have not got masters, or interpreters, for many subjects. But the Committee have power to appoint Honorary Members and we can cast about the wide world for finding such men to assist us.

May we discuss Prof. Eddington as an illustration of the above theory of progress? The subject of Relativity is itself new and so, we can be sure, that, not many years ago, he was but an entrant, in the field. This eager learner soon made his own the wealth of thought created by Einstein and Minkowski, De Sitter and Weyl. He saw all this wealth in clear light, and became an interpreter; at which stage he has given the world that fascinating exposition of the subject, contained in his "Space,Time and Gravitation." From this stage he has had further illumination, become a master, and reached a deeper synthesis of the subject, an advance upon Weyl's theory. It is a most difficult branch of Mathematics, but, it is necessary that some of our members should try to get into its mysteries. There *are* members gifted with an insight into the subject, Mr. S. V. Ramamurthi, for example; and it gives me great hope that even this abstruse subject can be successfully cultivated in India by a band of our members

While we should organise for the study of all branches of Mathematics in some such manner, the Journal of our Society must develop also. At present, in our Journal, different classes of matter, the Higher and the Elementary, are put in together : and they both experience a discomfort in thus travelling together. I think we should early separate our Journal into an Advanced part and an Elementary part. Neither of these should be sacrificed for the other. The former should be reserved for original work and for notes relating to Advanced Mathematics. The latter should be devoted to Notes, Reviews, Questions and Solutions and should be a journal for entrants, learners and interpreters, in the better known fields of Mathematics forming the subjects of study in Colleges. In this way our Journal will become more effective in its purpose and more appealing to all classes of our members.

Then comes the big question of the expansion of our Library. Our funds are sufficient for every purpose, but the Library. I think we may proceed to improve our Library as follows. We have the princely gift of Rs. 1,000 from Mr. Balak Ram. We should ear-mark the amount for a shelf in Mr. Balak Ram's name. The shelf may be devoted to books in a few subjects, say, Quaternions, Differential Geometry, Relativity, and Statistics which, we guess, have been specially attractive to the donor. Every book in the shelf should have a label showing the particular shelf it belongs to and contain a photogravure of its donor. We should form a shelf in the name of Ramanujan and put into it all the books relating to the fields traversed by him, Theory of Numbers, Theory of Modular Equations, and so on. The Government of Mysore are making a liberal annual grant to the Indian Institute of Science. We should appeal to them to make us also a small gift annually. In that case we can start one or more shelves in the name of His Highness the Maharaja, every book in them showing by its label that it is a gift from the Mysore State and bearing the figure of its gracious Ruler.

The position of our Society, as regards the Library, reminds me of a recent experience of mine. I went into a village one morning and saw a middle-aged woman drawing water from a well. She had some 15 pots, placed around her feet, on the platform, and was engaged in filling them. I asked the woman what she wanted so much water for. She said she was keeping a *tsalivendra* close by. *Tsalivendra* in Telugu means a place where water is supplied to people, as a charitable act, to quench thirst. I asked her if there was need for so much water for the *tsalivendra*. She said, "Not 15 pots, Sir, but even 150 will not suffice on the *shandy* days." Then I asked her who was the good man who was doing the charity. She understood my mistake and said meekly that she was doing it all herself. Then suddenly, as if she were wrong and I were right, she said, "Sir, I said wrongly, the good man who is doing this charity is the Lord Venkateswara of yonder temple, and I am doing it in His service." I praised the good woman and said she must be feeling very happy with the work. She admitted it was a happiness to her. When I was about to go, the good woman said she wanted to make a request and asked me to kindly look into the well. I looked in and saw that the water lay very deep below and told the woman that it must be hard to draw water from that depth. The woman said she did not mind the hard work and, to explain her point, she let down her bucket into the well and showed me that it can only half fill. She expressed her distress that the hot weather was coming and even this little water would go away. She asked me that the well should be deepened. Sir, our Society is like that distressed woman, who was willing to work in God's service And as she appealed to me, I wish to appeal to States and Governments, and all the wealthy in the land, to help us to get a good Library. I appeal to them, to keep the Well of Knowledge full for us, so that we may draw and distribute, and serve to strengthen our national life; and that we may hold the banner of Mathematics aloft, as the Motherland marches to glory with the rest of the world.

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ORIGINAL LETTER FROM V. RAMASWAMI AIYER SOLICITING SUPPORT FROM FELLOW MATHEMATICS LOVING PERSONS FOR ESTABLISHING "THE ANALYTIC CLUB" AND THE LIST OF 20 FOUNDING MEMBERS

V. RAMASWAMI AIYAR

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PROGRESS REPORT.

1. *The Founder of Our Society.*—The honor of being the Founder of our Society belongs to Mr. V. Ramaswami Aiyar (whose photo accompanys the present number of the Journal), now Special Deputy Collector of Ramnad. His first letter proposing the formation of the Society, dated 25th December 1906, is contained in his letter, dated 3rd April 1907, announcing the formation of the Society under the name '*The Analytic Club.*' The latter was published in the leading Madras dailies of the 4th April 1907; and it is but fitting that we place it on record in the pages of our Journal. It will be observed from the dates, that our Society has just completed the twelfth year of its existence. The discovery of Mr. S. Ramanujan, B. A., F. R. S., one of the greatest living Mathematicians of the world, marks this period of its life. The founder of the Society devoutly hopes that the objects with which the Society was formed will be fullfilled to a greater degree in the future periods of its existence, and expects that all members will unite their best endeavours in the realization of this hope. The letter announcing the formation of the Society is printed below.

2. *Mr. S. Ramanujan, F. R. S.* This distinguished Mathematician has returned to Madras in somewhat indifferent health, after a prolonged stay at Cambridge. By his unique mathematical talents and by the amount of useful and original work, he has raised India in the estimation of the outside world. We extend our most cordial welcome to him and most fervently pray that he may be soon restored to his full vigour to prosecute his glorious work in the scientific world. The committee hope to show their mark of appreciation further by a suitable announcement in the next issue of the Journal.

3. *Mr. T. J. Mirchandani B. A., B. Sc.*—Demonstrator, D. J. S. College, Karachi, has been elected a member of our Society.

POONA,
1st April 1919.

D. D. KAPADIA,
Hon. Joint Secretary.

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Mr. V. Ramaswamiaiyer's Letter, dated 3rd April 1907.

Sir,—I recently sent a proposal to some gentlemen interested in mathematics suggesting the formation of a small Mathematical Society. The proposal ran as follows :—

Sir,—I believe several friends interested in Mathematics have felt the present lack of facilities for seeing mathematical periodicals and books. This is a very great disadvantage we are suffering,

I propose therefore that a few friends may at once join and form a small Mathematical Society and subscribe for all the important Mathematical periodicals and, as far as possible, for all important books in Higher Mathematics.

We may call the Society "The Analytic Club" for the present, and have it in view to give it a broader basis with a suitable name by and by.

Our work immediately will be to obtain all the important periodicals and new books and circulate them to members. I shall be glad to undertake the duties of Secretary for the first year and do my best to promote the object in view.

If half a dozen members can be counted upon to join immediately and each subscribe Rs. 25 per annum, we shall be able to make a good start. The Annual Subscription may perhaps be somewhat less, say, if a dozen members can be had ; but even a dozen members paying Rs. 25 per annum would not suffice to enable the Club to obtain the more important books appearing every year. I propose therefore that the subscription be Rs. 25 per annum.

It appears to me necessary also that members should be prepared for a further sacrifice, and I propose that each member should send the journals and books he receives on to the next, and the last to the Secretary, at his own cost. This in effect would be to add about Rs. 5 more to one's subscription. I hope friends interested in mathematics will not consider this a too heavy sacrifice—at any rate initially, in giving the Club a start.

Will you kindly write to me if you are in favour of the proposal ? In case you are, I request you will send me your subscription of Rs. 25 for 1907, as early as possible, so that we may make a start at once.

This is only a tentative scheme and we may try it for a year and then introduce necessary changes.

I propose to consider the Club formed as soon as three friends have agreed to the proposal, making with me four members. Thereafter all business requiring determination by the Club can be done by circulation. Requesting the favour of an early reply. I remain, Sir, yours truly,

(Signed) V. Ramaswami Aiyar.

2. In response to this proposal (which I was able to send only to a very limited number of persons) the undermentioned have written to me consenting to become members of the proposed Society ;

Messrs :

R. N. Apte, M.A., LL. B., F.R A.S., Prof. of Math., Rajaram College, Kolhapur.

M. V. Arunachala Sastri, M. A., Assistant Professor, Nizam's College, Hyderabad.

K. Chinnatambi Pillai, B.A., Assistant Professor, Christian College, Madras.

B. Hanumanta Rao, B.A., Prof. of Mathematics, College of Engineering, Madras.

D. K. Hardikar, B.A., Professor of Mathematics, Nizam's College, Hyderabad.

G. Kasturiranga Aiyangar, M.A., Lecturer, Maharajah's College, Mysore.

B. Krishnamachari, M.A., Asst. Superintendent, Accountant-General's Office, Madras.

A. V. Kuttikrishna Menon, M. A., Teachers' College, Saidapet.

V. Madhava Rao, M.A., Prof. of Mathematics, Maharaja's College, Vizianagaram.

M. T. Narayana Aiyangar, M A, Prof. of Mathematics, Central College, Bangalore.

R. P. Paranjpye, B. SC., M.A., Principal and Professor of Mathematics, Fergusson College, Poona.

R. Ramachandra Rao, B. A., Collector of Kurnool.

K. J. Sanjana, M A., Principal and Professor of Mathematics, Samaldas College, Bhavnagar, Kathiawar.

P. V. Seshu Aiyar, B.A., Lecturer, Government College, Kumbhakonam.

S. P. Singaravelu Mudaliar, B A., Asst. Professor of Mathematics, Christian College, Madras.

R. Swaminatha Aiyar, B.A., Treasury Deputy Collector, Coimbatore.

T. R. Venkataswami Nayudu, *B A., Professor of Mathematics, Maharaja's College, Mysore.

K. Krishnan Nayar, B.A., B.C.E., District Board Engineer, Mangalore.

S. A. Subramania Aiyar, B.A, B.C.E., Executive Engineer, Madras P. W. D., Madanapalli.

With me it makes 20 members.

3. I beg to declare on behalf of all those that have joined, that the Society is now formed, under the proposed name "The Analytic Club" for the time being; and I shall be its Secretary provisionally.

4. My thanks are due to the gentlemen who have joined. for the support they have given me in starting the Club. The membership has already exceeded my modest anticipations, and many more, I think, will be joining. This renders some changes and a better organization at once necessary. I shall soon be submitting to members proposals for a simple Constitution for the Society according to which the affairs of the Society will be managed by a committee consisting of a President, a few

office-bearers and some additional members. From the support that I have received in this respect also, I feel we shall have a Committee giving the greatest possible confidence.

5. I think, the Society should now aim at being a combination of all Professors, Assistant Professors and Lecturers, and men of like interest in Mathematics or in the promotion of Mathematical study, in whatever province of India, with a view to the creation of a Central Mathematical Library from which members can obtain all mathematical, including astronomical, books and journals through the post. Every journal should be sent in circulation to members who have made a general application for the same ; and every book and journal should be sent to members on special application, as often as applied for. All this, of course, subject to rules that shall be made in this behalf.

6. If this conception of the immediate aims of the Society be approved, our Library should preferably be in a place which is postally a good centre for all India. In this respect, Poona is, next to Bombay, the most central place for all India. It is practically Bombay as regards the rest of India and it is a centre as regards Bombay Presidency itself. Further, I am glad to be able to mention that Prof. Paranjpye will be willing to take charge of the Library, provision being made for the discharge of purely mechanical work through Assistant Librarians. My draft constitution will be found to provide for this. I trust Madras members will endorse my suggestions and show their full appreciation of the reasons which make Poona, which contains that noble institution, the Fergusson College, so felicitous a centre for our Society. Further, by voting Poona as our centre, we Madrassees will convince the rest of India that we do not look at the matter from a purely provincial point of view. And it is my deep hope that a combination formed on these lines may be of fruitful consequences in the future.

7. Further, we shall have to get more than one copy of most books, and journals, and probably several copies of some journals, for the sake of convenience. This being the case, some of the extra copies we get can be placed in Branch Libraries in Madras, Allahabad, &c., as membership extends, so that members in these Provinces may get them quicker on their application.

8. The subscription of Rs. 25 has been felt heavy by some of those whom I addressed. Among the provisions in the draft constitution that I shall submit, will be one enabling the Committee to reduce the subscription in the case of any particular member to Rs. 15, and to continue this concession as long as they deem fit. I trust that this provision will be passed and that it will sufficiently meet all cases where such encouragement may be necessary. The scheme ought to be taken advantage of, more specially by the younger gentlemen interested in mathematics, many of whom will be the future occupants of the mathematical chairs of our colleges,—and I hope none of them will feel deterred from applying for membership at once.

9. Intending members will communicate with the undersigned.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts of the Society for the year 1918.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Balance from 1917	2,997	4	4	Ordinary working expenses	231	5	9
Subscriptions from Members	2,457	4	0	Books and Journals	443	3	9
Subscriptions for Journal	212	0	0	Printing Journal	640	2	0
Interest on Investments	111	0	3	Library	286	4	0
Miscellaneous	33	10	0	Closing Balance	4,210	3	1
Total ...	5,811	2	7	Total ...	5,811	2	7

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I have examined the Treasurer's books and vouchers, as also the monthly Statements submitted by the Secretary, Assistant Secretary and Assistant Librarian and declare the above accounts to be correct.

N. Ranganathan,
Auditor.

5th April 1919.

Madras,
22nd March 1919.

S. Narayana Aiyar,
Honorary Treasurer.

FORM IV

(See Rule 8)

1.	Place of Publication.	PUNE.
2.	Periodicity of publication.	QUARTERLY.
3.	Printer's Name. Nationality. Address.	DINESH BARVE. INDIAN. PARASURAM PROCESS, 38/8, ERANDWANE, PUNE - 411 004, INDIA.
4.	Publisher's Name. Nationality. Address.	V. M. SHAH. INDIAN. GENERAL SECRETARY, THE INDIAN MATHEMATICAL SOCIETY, (DEPARTMENT OF MATHEMATICS, THE M. S. UNIVERSITY OF BARODA), "ANANT", 7 , JAYANAGAR SOCIETY, R. V. DESAI ROAD, VADODARA– 390 001, GUJARAT, INDIA.
5.	Editor's Name. Nationality. Address.	J. R. PATADIA. INDIAN. (DEPARTMENT OF MATHEMATICS, THE M. S. UNIVERSITY OF BARODA), 5 , ARJUN PARK, NEAR PATEL COLONY, BEHIND DINESH MILL, SHIVANAND MARG, VADODARA – 390 007, GUJARAT, INDIA.
6.	Names and addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital.	THE INDIAN MATHEMATICAL SOCIETY.

I, V. M. Shah, hereby declare that the particulars given above are true to the best of my knowledge and belief.

V. M. SHAH

Dated: 23^{rd} March, 2012. Signature of the Publisher

Published by V. M. Shah for the Indian Mathematical Society, type set by Rupesh Shewale at Department of Mathematics, University of Pune, Pune 411 007 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune 411 041 (India). Printed in India.

Back volumes of both the periodicals, except for a few numbers out of stock, are available.

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