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*Edited by*

J. R. PATADIA



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J. R. PATADIA

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4. research papers, and
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\ textwidth=11.5cm
\ textheight=19.5cm
\ topmargin= 1.5cm
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# MATHEMATICS EDUCATION IN INDIA: SOME OBSERVATIONS & CONCERNS*

**PEEYUSH CHANDRA**

Honourable Chief guest Professor Thulsi Raman, Honourable Chancellor and the Vice Chancellor of Kalasalingam University, Prof. V. M. Shah, General Secretary of the Indian Mathematical Society (IMS), Prof. Satya Deo Tripathi, Academic Secretary IMS, Prof. S. Arumugam Local Secretary for $75^{th}$ Annual Conference IMS, fellow mathematicians, young researchers, Ladies and gentlemen!

At the outset, I wish to thank the members of IMS for electing me to this august office for the year 2009-10. I feel greatly humbled by this honour and would like to dedicate it to the memory of Prof. P. L. Bhatnagar and Prof. H. C. Khare, who acted as mentors and philosopher guides during my early professional career at Allahabad. I had been involved with the organization of the IMS annual conferences held at Allahabad (1981) and Kanpur (1996) and I assure you that I will continue to serve the society to the best of my ability in the years to come also.

It is heartening to note that the $75^{th}$ Annual Conference of the IMS is being held at Kalasaligam University. I thank the Honourable Chancellor and Vice Chancellor of the university and the Local secretary Prof. Arumugam, who were instrumental in inviting IMS to hold its annual conference this year in this beautiful campus. It may be noted that IMS is holding its annual conference in Tamil Nadu after a long gap - the $49^{\text{th}}$ conference was held in Chennai in the year 1984. It is hoped that in the years to come, IMS will get opportunities to hold its annual conferences at regular intervals in this part of the country. It is a moment of great pride for the mathematics community that the Department of Posts (Philately Division), Government of India, is issuing a commemorative postage stamp on the "Indian Mathematical Society" on the



* The text of the Presidential Address (General) delivered at the Platinum Jubilee 75th Annual Conferenc of the Indian Mathematical Society held at the Kalasalingam University, Anand Nagar, Krishnankoil - 626190, (Via) Srivilliputtur, Dist. Virudunagar, Tamilnadu, India, during December 27-30, 2009.

occasion of the Platinum Jubilee Conference of the Society. We thank the Department of Posts for their gesture.

Today I would like to share some of my concerns and observations about the Mathematics Education - teaching and research - in India. In my opinion, the general scenario for mathematics is not very encouraging. There is an overall decline for the study of the sciences and it is all the more for mathematics, which is considered to be an abstract subject. Though it is a universal phenomenon, the problem is more acute in our country, where societal pressure guides students to go for the study of engineering or medicine and very few join the science stream by choice.

To begin with, I pose a simple question 'What is Mathematics? This will have different answers; and the response would vary from person to person depending upon the level of details and precision achieved. There is no common acceptable definition of a subject which is based on its rigour. Thus, if we confront the students, who have completed their school education, with the question - 'Have you entered the world of Mathematics' the answer will be in affirmative, however most of the mathematicians will think it otherwise. Though the students would have done some mathematics, they are still not prepared to enter the realm of mathematics. Most of the school mathematics prepares students for problem solving. It is all the more true in our country where the students opt for the coaching classes to prepare for the entrance tests of engineering. Here again the style of teaching is focused to 'problem solving' ability only. In general, the students are not imparted logical thinking, hence, once they enter university level mathematics they find that the world of Mathematics is very different from the one encountered by them earlier. It is much more demanding and the level of analytical ability and logical thinking is very different from what they had learnt at the school level. Even the best students at the school level find it difficult to cope with the intricacies of higher mathematics. It is at this stage that they start losing their interest in mathematics. As teachers at the university level we find a need to 'undo' some of the earlier 'learning' and demolishing the structures learnt at that level before rebuilding them. In fact, there is need to realize that raw mathematical skill of problem solving is just not sufficient, but a logical and hard training will be required. Higher mathematics is altogether a much more demanding affair. The teacher needs to inculcate amongst the students a sense of mathematical reasoning, logical thinking and analytical storming. To keep the interest of the students alive, the basic mathematical structures should be created and explored by starting with familiar concepts and, if possible, by relating them to the world around.

We also need to stress the connections between different areas of mathematics. For example, the students may be informed about the close relation between (i) Fourier series and Sturm Liouville boundary value problem, (ii) Linear differential equations and dimension of a vector space, (iii) deformation gradient in continuum mechanics with polar decomposition in linear algebra (iv) functional analysis and industrial problems, etc. This could generate interest of students in different aspects of Mathematics.

We also need to learn how to use technology to better understand mathematics with a vision of transforming the teaching methodologies and then promoting students' learning of mathematical concepts with technology-enabled pedagogy. It is indeed a challenge to utilize technology as a tool in developing a deep understanding of mathematics at all levels. Mathematical experiments can be devised at an early stage, e.g., the students may be asked to write algorithms for (i) square root of a number, (ii) HCF of two numbers, etc.

Having said this, I must say that it is a challenging task for us to bridge the gap between school mathematics and higher mathematics. The eminent mathematicians and active researchers should take a step forward. I request them to visit colleges and universities to narrow down the gap. We have to reach out to the students to explain the beauty of mathematics in simpler terms. The IMS had taken an initiative in this direction sometime back. Under the scheme, the eminent mathematicians are requested to visit university colleges and interact with the faculty and students there. We also need to understand the problems before school teachers, for which frequent meetings / workshops are desirable. Some of us have been involved with basic training programmes of DST and NBHM. I must say that the experience had been very fruitful. In recent years, some workshops were organized in Kanpur for degree college students, who were not necessarily very bright but had come from remote places. I could see lot of motivation in some of them at the end of the programme. I am happy to share with you that some of them did join the M. Sc. programmes at the IITs and universities. More such workshops in different parts of the country will be very helpful in clearing up some of the basic concepts and creating interest for Mathematics amongst the participants. These workshops should also have lectures on history of mathematics and development of some concepts. We need to provide a vibrant environment to cultivate an aptitude for Mathematics in young minds.

Now, while turning to the research scenario, I would like to recall that one of the factors for forming the 'Analytic club' which ultimately culminated in the formation of the IMS - was the 'lack of facilities for advanced study and research in this country'. In the present time, while there is an overall improvement in the research facilities, the research output has not improved significantly. One of the indicators is provided in 'Status of India in Science and Technology as reflected in its publication output in Scopus International database (web of sciences) 1996-2006', Gupta & Dhawan, National Institute of Science, Technology and Development Studies, New Delhi. As per this study, Mathematics is placed in the medium productivity category list and reports 12567 publications from India in Scopus from 1996 to 2006. A point of concern is the comparison of global publication shares from India and China. In 1996, the publication share of China was 2.07% and that of India was 2.53%, but in 2006 it has dramatically reversed to 13.06% and 2.04% respectively. Here we have to ponder why Indian share has gone down where as Chinese share of publication has gone up drastically. This is inspite of the fact that the number of Indian publications has increased in recent years from 2829 for 1996-98 to 4493 for 2004-06. What is more concerning is that in the ranking of top 20 Indian Institutes / universities, only three universities figure. While this study may not be very accurate, it does provide some pointers. It does indicate towards a need to give serious thought to this issue.

The university system provides a mass base for teaching and research; therefore an effort should be made to strengthen the system. The university system has to re-energize itself. They must explore the frontiers of research in mathematics. Mathematics is no more confined to the classical areas. Today, it is finding applications in almost all areas of knowledge. Scientific computing has opened new doors for mathematics. Collaborative research in newer areas is going to be very fruitful. To focus on the front line areas of research, workshops / seminars / symposia should be arranged and the active researchers should be invited to participate so as to make the participants aware of the current research and methodologies. It may not be out of place to bring to the notice of young researchers that in the year 2010, the International Congress of Mathematicians will be held in India at Hyderabad. A number of satellite conferences will also be organized in this connection. Young research fellows should make the best use of this opportunity. Apart from this the faculty and research scholars should arrange discussions / seminars locally. This will not only improve presentation but also make them aware of the research going on in other areas. It may be worth mentioning here that one of such discussion meetings laid the foundation of 'Journal of Fluid Mechanics' one of the top journals today in experimental and theoretical fluid mechanics.

I may also add here that the workshops on research methodologies are the need of the hour. Most of our young researchers lack on this count. They should be told how to initiate research, how to write a research paper, how to make a good presentation, how to look for the relevant literature etc. The last point is very important as with the availability of e-prints of books and research papers, one is tempted to download lot of material without really going through their relevance.

In today's world, we need good mathematicians - be it pure or applied. In particular, our country needs a large number of good mathematics teachers at all the level. We have to work to increase output of quality research publications so as to make an impact on international level. Hence there is an urgent need to ponder on mathematics education in India.

Before I conclude, I would like to draw your attention to the fact that young researchers need encouragement during their presentation in conferences. The mere presence of senior mathematicians during the paper presentation will motivate them to do better. Henceforth I urge the delegates to motivate and guide young researchers by their presence in various Paper reading sessions of the conference.

I wish the conference a grand success. I do hope that the various sessions and the invited lectures will enable us to look beyond our specialization and will enrich our knowledge.

Thank you.

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# MATHEMATICAL MODELING OF HIV DYNAMICS: *IN VIVO*

**PEEYUSH CHANDRA**

## 1. Introduction

In recent years, mathematical modeling has emerged as a powerful tool to get an insight in many real world problems. One such area of investigations is epidemiology (study of infectious diseases). Mathematical models are extensively used in epidemiology to understand the underlying mechanism of a disease and also to suggest control measures. It attempts to answer: how fast a disease spreads in a population? how much of the total population is infected or will be infected? what control measures can be effective? persistence of the disease etc.

There are many infectious diseases (such as, AIDS, Hepatitis, Poliomyelitis, Influenza, Smallpox, Rabies etc.) which spread through virus. The dynamics between virus infection and immune system is very complex and involve different components. In such a scenario, mathematical models provide an essential tool to capture a set of assumptions and to generate new hypothesis. They also allow us to suggest experiments. A particular case is HIV infection which causes AIDS. It is well known that AIDS (Acquired Immune Deficiency Syndrome) is a disease of human immune system. The first case of death due to AIDS was reported in 1981 and the disease has now become a global epidemic. In view of this, several researches are taking place to investigate various aspects of AIDS which is caused by HIV (Human Immunodeficiency Virus)[50].

In last 20 years, a large number of articles pertaining to mathematical models of HIV/AIDS have appeared in the literature. These models can be



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classified into two major categories: Epidemiological models - which assess the spread of infection through a population; and Immunological models - which examine *in vivo* (within host) dynamics of HIV and immune system.

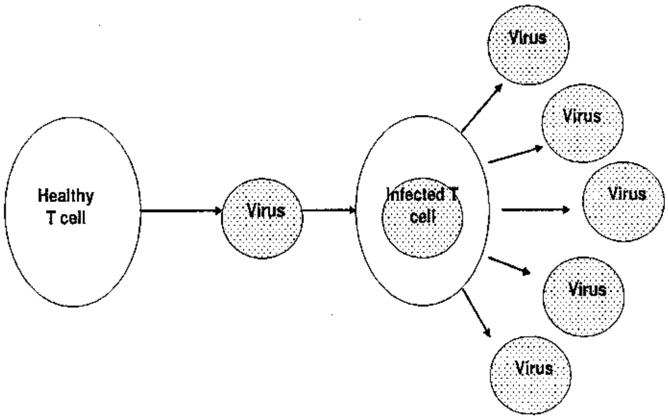
In this article we shall consider some mathematical models related to *in vivo* dynamics of HIV and immune system. The interaction between HIV and immune system is much more complex than many other infections. In order to appreciate these models an understanding of biology of disease virus is desirable. Hence in the following section a brief account of the virus infection is given.

## 2. Biology of Infection

HIV infection generally occurs via transfer of body fluid (such as blood, semen, vaginal fluid etc.) from an infected individual to an uninfected one. It is most commonly spread by sexual intercourse, contaminated needles, use of blood or blood products or by vertical transmission (*i.e.* from infected mother to child: either at time of birth or through breast feeding).

Once inside the body, HIV infects white blood cells that bear CD 4 protein on their surface. These cells include CD $4^+$ T cells, macrophages and dendrite cells but HIV favourably attacks CD $4^+$ T cells [1]. HIV is a retrovirus which contains its genetic material in the form of RNA. The process of infection begins with the attachment of virus with the host cell. The viral envelope is fused and the capsid containing a pair of viral RNA and other important proteins (protease, reverse transcriptase and integrase etc.) enter the cell. This process is called *fusion.* Viral RNA and other proteins are liberated inside the cell and then viral RNA is converted into DNA with the help of reverse transcriptase protein. This is called *reverse transcription.* This viral DNA further moves to the nucleus and integrates itself to host cell genome using protein integrase. At this stage virus may either remain latent for a long period or it may become activated. An activated host cell genome produces viral RNA in the form of polyprotein. The protein protease cleaves these polyproteins into necessary proteins and finally all these proteins and RNA gather near the cell wall and virus buds out of the cell. These viral particles then mature and infect other cells [40]. Sometimes this replication is so explosive that the host cell membranes lyse as newborn virus emerge, killing the host cell.

The body, on detecting the invasion of virus (HIV), reacts to it by stimulating CD $4^+$ T cells which in turn stimulate the Cytotoxic T Lymphocytes (CTLs) to clear the infected CD $4^+$ T cells. The CTLs start proliferating and surround the infected cell and kill it. The problem with HIV is that immune system kills CD $4^+$ T cells which are basically responsible for strong immune




Figer 1 :Schematic diagram of progression of HIV infection.

system. This results in the depletion of CD $4^+$ T cells leading to immunodeficiency [26]. Thus HIV infection not only harms individual target cells, it also disturbs the entire communication network for the body's defence.

There are three phases in HIV infection: primary, latency and AIDS. During primary infection there appear some flu like symptoms with the increase of viremia in blood. At this stage the CTLs response actually controls the virus level which peaks and then decline. Latency period comes after this stage and is of 2 to 10 years duration. During latency period the virus load increases and CD $4^+$ T cells count decreases. Thereafter the stage of AIDS comes when CD $4^+$ T cell count goes below $200\,\mu l^{-1}$ as against the normal count of $\sim 1000$ $\mu l^{-1}$.

Anti retroviral drugs *e.g.*, Fusion inhibitors (FI), Reverse Transcriptase inhibitors (RTI) and Protease inhibitors (PI) have been developed so as to attack the different phases of viral life cycle during infection. FI inhibits the fusion of viral particles in the host cell, which causes low infection rate. RTI inhibits the reverse transcription of viral RNA into DNA, which decreases the number of productively infected cells. PI inhibits the cleaving of polyprotein encompassing reverse transcriptase, protease, integrase, and other matrix proteins into functional units. This causes the production of non-infectious viral particles. In order to get better results in therapy a combination of these drugs is used.



## 3. Basic Models

In the last two decades many mathematical models have been proposed and analysed to understand the dynamics of HIV and immune system as well as to design drug therapy to check the progression of the disease [2, 16, 18, 24, 27, 39, 41, 45]. A review of these models is provided by Nelson and Perelson (1999) [32] and Wodarz and Nowak (2002) [51].

The mathematical models for *in vivo* dynamics of HIV and immune system primarily consider the interaction of CD $4^+$ T cells and virus [23, 25, 28]. They are broadly based on interaction of three populations: uninfected and infected CD $4^+$ T cells and virus population (figure-**??**). The infected cells may further be divided into latently infected cells and actively/productively infected cells. Some researchers have also considered the immune response in their models [5, 25]. Also, there are some models which have emphasized on the mathematical aspect of the modeling and analysis [4, 6].

One of the earliest models for the dynamics of HIV and immune system was proposed by Perelson in 1989 [28] in an edited volume of Lecture notes in Biomathematics. He proposed a general model that can potentially account for many of the immunological consequences of HIV infection. This model involved a large number of ordinary differential equations and model parameters to consider interaction involving various antigen specific T cell sub populations, virus population and antigen. In view of this an analysis was made for a simpler model which considered the interaction among uninfected CD $4^+$ T cells, latently infected CD $4^+$ T cells, actively infected CD $4^+$ T cells and virus population only. The model resulted into a system of coupled nonlinear ordinary differential equation. The model was studied numerically and it reasonably succeeded to explain biological observations reflected during HIV infection [28].

Nowak and Bangham (1996) [23] proposed a model for the dynamics of HIV and immune system. The model primarily considers the interaction of CD$4^+$ T cells and virus, and is based on interaction of three populations: uninfected CD $4^+$ T cells, infected CD $4^+$ T cells and virus population (figure- 1). The model assumes that after infection CD $4^+$ T cells immediately become actively infected and start producing virus. The mathematical formulation is given by following system of coupled nonlinear equations [23]:

$$\frac{dT(t)}{dt} = \Lambda - kV(t)T(t) - \mu T(t), \tag{3.1}$$

$$\frac{dT^*(t)}{dt} = kV(t)T(t) - \delta T^*(t) \tag{3.2}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t). \tag{3.3}$$

where $T(t)$, $T^*(t)$ and $V(t)$ represent densities of uninfected CD $4^+$ T cells, infected CD $4^+$ T cells and virus populations respectively at time $t$. $\Lambda$ is inflow

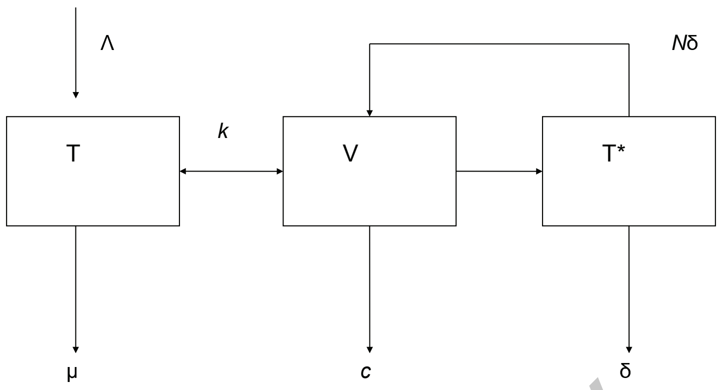



FIGURE 1. Schematic diagram of HIV infection of CD $4^+$ T cells, Here $T$ represent healthy CD $4^+$ T cell, $T^*$ represent infected CD $4^+$ T cell and $V$ represent virus population.



rate of uninfected CD $4^+$ T cells and $\mu$ their natural death rate, hence $1/\mu$ is average life time of uninfected CD $4^+$ T cells. $k$ represents interaction infection rate of T cells with virus, $\delta$ is death rate of infected cells which also includes the possibility of death by bursting of cells hence $\delta \geq \mu$, $c$ is clearance rate of virus and $N$ is total number of viral particles produced by an infected cell during its life time (also known as 'burst size'). All parameters are positive and constant.

The basic assumptions in the model are: $(i)$ the interaction between virus and infected CD $4^+$ T cells is of 'mass action type' (*i.e.* virus particles infects CD $4^+$ T cells at a rate proportional to their densities, $kV(t)T(t)$), $(ii)$ infected cells produce free virus in proportion to their densities, $(iii)$ inflow rate of healthy CD $4^+$ T cells is a constant, $\Lambda$ and they have fixed life span, $(iv)$ average life time for infected cells is $1/\delta$ and that of virus is $1/c$.

In the absence of infection one can easily see that T cells settle down to $\frac{\Lambda}{\mu}$. Suppose that a small amount of virus particles get introduced into the system, then, if the virus can grow and establish an infection or not depends on a condition very similar to that for the spread of an infectious disease in a population of host individuals. The crucial quantity is the *basic reproductive ratio*, $\mathcal{R}_0$, which is defined as the number of newly infected cells that arise from any one infected cell when almost all cells are uninfected.

The stability analysis is performed for the steady states of the model system (3.1)-(3.3), which has following two steady states:

(1) the infection free steady state $E_1 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$ and
(2) the infected steady state $E_2 = \left(\frac{c}{Nk}, \frac{N\Lambda k - \mu c}{N\delta k}, \frac{N\Lambda k - \mu c}{ck}\right)$.

For local stability analysis the system (3.1)-(3.3) is linearized around a steady state, and Jacobian matrix evaluated at steady state is obtained as follows:

$$M = \begin{pmatrix} -\mu - kV_{ss} & 0 & -kT_{ss} \\ kV_{ss} & -\delta & kT_{ss} \\ 0 & N\delta & -c \end{pmatrix},$$

(suffix $ss$ represents corresponding steady state).

The characteristic equation of $M$ evaluated at $E_1$ is given as,

$$(\lambda + \mu)\left(\lambda^2 + (\delta + c)\lambda + c\delta - \frac{N\delta k\Lambda}{\mu}\right) = 0.$$

It can be noted that all the roots of above equation are with negative real part when $\mathcal{R}_0 = \frac{N\Lambda k}{\mu c} < 1$. Hence the infection free steady state $E_1$ is locally asymptotically stable whenever $\mathcal{R}_0 < 1$ and becomes unstable when $\mathcal{R}_0 > 1$ and in that case infected steady state $E_2$ exists.

Further, the characteristic equation of $M$ evaluated at $E_2$ is given as,

$$\lambda^3 + (\mu + k\overline{V} + \delta + c)\lambda^2 + (\mu + k\overline{V})(\delta + c)\lambda + \left(c\delta - \frac{N\delta k\Lambda}{\mu}\right) = 0.$$

Using Routh-Hurwitz criterion we note that all the roots of above equation have negative real parts. Hence $E_2$ is locally asymptotically stable whenever $\mathcal{R}_0 > 1$.

Thus if $\mathcal{R}_0 > 1$ the infection persists and if $\mathcal{R}_0 < 1$ it is cleared. This model gives the basic spike like viral population graph during primary infection. Thereafter the viral population settles to a low state, which is in accordance with experimental observations. An important insight obtained from these models is that HIV does not remain latent during the asymptomatic phase of the infection, but it continuously replicates with high turn-over rate. This enables the virus to evolve at a fast rate [51].

Some variations of the above model (3.1)-(3.3) appear in literature where researchers also consider the proliferation rate of T cells [6, 23, 32]. One such model was proposed by Perelson and Nelson (1999) [32] where proliferation of T cells is considered to be logistic type. It means T cells first increase and then settle to a maximum attainable level of the population and do not grow beyond that. Model equations are as follows [32]:

$$\frac{dT(t)}{dt} = \Lambda - kV(t)T(t) + rT\left(1 - \frac{T}{T_{Max}}\right) - \mu T(t), \tag{3.4}$$

$$\frac{dT^*(t)}{dt} = kV(t)T(t) - \delta T^*(t), \tag{3.5}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t). \tag{3.6}$$

Here $r$ is proliferation rate and $T_{Max}$ is maximum count for T cells where proliferation shuts off. In some other cases, the proliferation rate was also considered to be dependent upon infected cell population [6, 28, 30, 32].

In most of the models the interaction between CD $4^+$ T cells and HIV has been taken to be mass action type. Yet some researchers have modeled the interaction differently: *e.g.* Wang and Song (2007) [49] considered that the interaction may be less than linear in $V$ due to saturation at high virus concentration and may be greater than linear if infectious fraction was very small or if multiple exposures were necessary for infection. Hence they took interaction term as $kT^qV$, where $q > 0$ is constant. The following model was proposed [49]:

$$\frac{dT(t)}{dt} = \Lambda - \mu T(t) + rT(t)\left(1 - \frac{T(t) + T^*(t)}{T_{Max}}\right) - kT^q(t)V(t), \tag{3.7}$$

$$\frac{dT^*(t)}{dt} = kT^q(t)V(t) - \delta T^*(t), \tag{3.8}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t). \tag{3.9}$$

They found that infection free steady state is globally asymptotically stable if it is the only non-negative steady state of the system. They also established the global stability of infected equilibrium under certain restrictions on parameters using the technique given by Li and Muldowney [21].



## 4. Models with Delay

The fact that the biological systems do not react immediately implies that there is an inherent delay in the system. In particular, for HIV infection, once a cell is attacked by virus there are cascade of events which take place inside CD $4^+$ T cells. Therefore, there is a time lag for infected cells to become actively infected. In 1996 Herz et al. [15] introduced a discrete delay parameter to model the intracellular delay in a HIV model and showed that the incorporation of a delay would substantially shorten the estimate for the half-life of free virus. The following model was studied by Culshaw and Ruan (2000) [6] to account for the latently infected cells by using discrete time delay:

$$\frac{dT(t)}{dt} = \Lambda + rT(t)\left(1 - \frac{T(t)+T^*(t)}{T_{Max}}\right) - \mu T(t) - kV(t)T(t), \tag{4.1}$$

$$\frac{dT^*(t)}{dt} = k'V(t-\tau)T(t-\tau) - \mu_I T^*(t), \tag{4.2}$$

$$\frac{dV(t)}{dt} = N\mu_b T^*(t) - kV(t)T(t) - cV(t). \tag{4.3}$$

The model also accounts for the loss of virus due to fusion during infection as observed earlier. $\tau$ is time between infection of a T cell and the emission of viral particles. $\mu_I$ is blanket death rate for infected T cells, $k'$ is the rate at which infected cells become actively infected and $\mu_b$ is lytic death rate of infected T cells. Rest of the parameters are same as defined earlier. Culshaw and Ruan [6] studied stability of the model and found that under certain conditions the model has a capability to show Hopf bifurcation and delay induced oscillations for a critical value of $\tau$.

Song and Cheng [42] proposed a model with time delay in infection. They studied the effect of the time delay on the stability of the endemically infected equilibrium. Criteria were given to ensure that the infected equilibrium is asymptotically stable for all magnitudes of delay. Finally they obtained the condition for the existence of an orbitally asymptotically stable periodic solution. They also accounted for cells that are infected but die before becoming productively infected. Recently, in 2010, Song et al. [43] modified the model of Song and Cheng [42] with saturation response of the infection rate. They studied the stability of equilibrium as well as persistence of populations, and established that Hopf bifurcation occurs under certain condition on parameters.

Jiang et al. (2008) [17] extended the above model by introducing the therapy induced cure in the model and proposed that a fraction $\rho T^*$ of infected cell population will revert to uninfected class due to therapy and proposed the following equivalent model:

$$\frac{dT(t)}{dt} = \Lambda + rT(t)\left(1 - \frac{T(t)+T^*(t)}{T_{Max}}\right) - \mu T(t) - kV(t)T(t) + \rho T^*(t), \tag{4.4}$$

$$\frac{dT^*(t)}{dt} = k'V(t-\tau)T(t-\tau) - \mu_I T^*(t) - \rho T^*(t), \tag{4.5}$$

$$\frac{dV(t)}{dt} = N\mu_b T^*(t) - cV(t). \tag{4.6}$$

The local stability was studied and it was observed that for a critical value of delay parameter $\tau$ the system will undergo Hopf bifurcation. They also found the direction and stability of bifurcating periodic orbit.

Li and Ma (2007) [20] considered the interaction between T cells and virus population to be of Holling-Type II interaction in model (3.1)-(3.3) and studied following delay differential equation model introducing delay in infection:

$$\frac{dT(t)}{dt} = \Lambda - \mu T(t) - \frac{kV(t)T(t)}{1+V(t)}, \tag{4.7}$$

$$\frac{dT^*(t)}{dt} = \frac{kV(t-\tau)T(t-\tau)}{1+V(t-\tau)} - \delta T^*(t) \tag{4.8}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t). \tag{4.9}$$

They established that delay has no effect on local asymptotic stability of infected equilibrium, if it exists and discussed global asymptotic stability for viral free equilibrium of the delay model.

Cai and Li (2009) proposed following model considering the proliferation of infected T cells [4]:

$$\frac{dT(t)}{dt} = \Lambda + rT(t)\left(1 - \frac{T(t)+T^*(t)}{T_{Max}}\right) - \mu T(t) - kV(t)T(t), \tag{4.10}$$

$$\frac{dT^*(t)}{dt} = kV(t-\tau)T(t-\tau) + rT^*(t)\left(1 - \frac{T(t)+T^*(t)}{T_{Max}}\right) - \delta T^*(t), \tag{4.11}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t). \tag{4.12}$$

They established the sufficiency criteria for delay independent stability of infected equilibrium. Then they obtained critical value of delay for which infected steady state undergo Hopf bifurcation.

It is observed that when infection with virus takes place inside T cell the viral RNA may not be completely reverse transcribed. Since the un-integrated virus present in the cell is labile and degrades with time [52, 53], a fraction of infected CD $4^+$ T cells revert back to uninfected class[11, 38]. Srivastava and Chandra (2010) [47] first proposed an ODE model then and modified it to a DDE model to account for this reverting back of the infected cells. The ODE model is as follows:

$$\frac{dT(t)}{dt} = \Lambda - kV(t)T(t) - \mu T(t) + bT^*(t), \tag{4.13}$$

$$\frac{dT^*(t)}{dt} = kV(t)T(t) - (b+\delta)T^*(t), \tag{4.14}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t), \tag{4.15}$$

with initial conditions $T(0) = T_0 > 0$, $T^*(0) = 0$ and $V(0) = V_0 \geq 0$. $b$ is the rate at which the fraction of infected cells revert back to uninfected class and rest of the parameters are as defined earlier.

The above system (4.13)-(4.15) has the following two steady states:

(1) the non-infected steady state $E_1 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$, and
(2) the infected steady state $E_2 = (\overline{T}, \overline{T^*}, \overline{V})$ which exists when $\mathcal{R}_0 = \frac{N\delta k\Lambda}{c\mu(b+\delta)} > 1$, where $\overline{T} = \frac{\Lambda}{\mu\mathcal{R}_0}$, $\overline{T}^* = \frac{\Lambda}{\delta}\left(1 - \frac{1}{\mathcal{R}_0}\right)$ and $\overline{V} = \frac{N\Lambda}{c}\left(1 - \frac{1}{\mathcal{R}_0}\right)$.

Further, the bounded set

$$\Gamma = \left\{(T(t), T^*(t), V(t)) \in \mathbb{R}^3_+ : 0 \leq T(t),\, T^*(t) \leq \Lambda/\mu, V(t) \leq \frac{N\delta\Lambda}{c\mu}\right\},$$

is positively invariant with respect to model system (4.13)-(4.15).

Using Lyapunov-Lasalle Theorem [19], following result is established for the stability of $E_1$:

**Theorem 4.1.** *The non-infected steady state $E_1$ is globally asymptotically stable in $\Gamma$ if $\mathcal{R}_0 \leq 1$ and is unstable if $R_0 > 1$.*

For the infected steady state, stability analysis gave the following results:

**Theorem 4.2.** *The infected steady state $E_2$ is locally asymptotically stable, whenever it exists, i.e., when $\mathcal{R}_0 > 1$.*

For the global stability of infected steady state $E_2$ of model system (4.13)-(4.15) the following lemma is proved for uniform persistence [3]:

**Lemma 4.1.** *$E_1$ is the only omega limit point of the system on the boundary of $\Gamma$ when $\mathcal{R}_0 > 1$. Further, it can not be omega limit point of any orbit which starts in the interior. In this case the system is uniformly persistent.*

Using Li and Muldowney [21] and lemma (4.1), following result is obtained:

**Theorem 4.3.** *If $\mu \leq \delta$, and if $\mathcal{R}_0 > 1$ then the infected steady state $E_2$ is globally asymptotically stable.*

Further a time delay $\tau$ is introduced in the model (4.13)-(4.15) to account for the delay between infection of CD $4^+$ T cell and the emission of virus particles on a cellular level (*i.e.* time from infection of cells with virus to the cells becoming productively infected [6, 15] is considered). Hence the model is modified as follows,

$$\frac{dT(t)}{dt} = \Lambda - kV(t)T(t) - \mu T(t) + bT^*(t), \tag{4.16}$$

$$\frac{dT^*(t)}{dt} = kV(t-\tau)T(t-\tau) - (b+\delta)T^*(t), \tag{4.17}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t), \tag{4.18}$$

with initial conditions $T(\theta) = T_0 > 0$, $V(\theta) = V_0 \geq 0$, and $T^*(\theta) = 0$; where $\theta \in [-\tau, 0]$.

To study the stability of infected steady state $E_2$ we consider the corresponding linearized system

$$\dot{Y}(t) = J_1 Y(t) + J_2 Y(t-\tau), \tag{4.19}$$

where $J_1$ and $J_2$ are constant matrices of corresponding linearized system and $Y(\cdot) = (T(\cdot), T^*(\cdot), V(\cdot))^T$.

The characteristic equation of the linearized system (4.19) is given by

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_1\lambda + B_0)e^{-\lambda\tau} = 0, \tag{4.20}$$

where

$$\begin{aligned} A_2 &= c + \mu + \delta + k\overline{V} + b > 0, \\ A_1 &= (\mu + k\overline{V})(c + b + \delta) + c(b + \delta) > 0, \\ A_0 &= (\mu + k\overline{V})(cb + c\delta) > 0, \\ B_1 &= -k(N\delta\overline{T} + b\overline{V}) < 0, \\ B_0 &= -k(\mu N\delta\overline{T} + bc\overline{V}) < 0. \end{aligned}$$

It may be noted that when $\tau = 0$ all the roots of equation (4.20) have negative real part and for $\tau \neq 0$, it has infinitely many roots. By Rouché Theorem [9] and continuity in $\tau$, equation (4.20) has roots with positive real part if and only if it has purely imaginary root. Now to see if equation (4.20) has purely imaginary roots or not, we put $\lambda = i\omega$ in equation (4.20) to get,

$$m^3 + \alpha m^2 + \beta m + \gamma = 0, \tag{4.21}$$

where $\omega^2 = m$, $\alpha = A_2^2 - 2A_1$, $\beta = A_1^2 - B_1^2 - 2A_2A_0$ and $\gamma = A_0^2 - B_0^2$.

The equation (4.21) will have all the roots with negative real part if and only if Routh Hurwitz criterion is satisfied and in that case equation (4.20) will not have any purely imaginary root.

It is easy to see that all the conditions of Routh Hurwitz criterion are satisfied if $\alpha\beta - \gamma > 0$ (as $\alpha > 0, \gamma > 0$). Hence we have following theorem:

**Theorem 4.4.** *When $\mathcal{R}_0 > 1$, the infected steady state $E_2$ is locally asymptotically stable for the delay system (4.16)-(4.18) for all $\tau > 0$, provided $\alpha\beta - \gamma > 0$.*

It is found that the infection is cleared when the basic reproductive ratio $\mathcal{R}_0 \leq 1$ *i.e.* the uninfected steady state is globally stable, whereas the infection persists when $\mathcal{R}_0 > 1$ and in that case the infected steady state is globally stable provided $\mu \leq \delta$. Biologically the inequality $\mu \leq \delta$ always holds because in the presence of virus the death rate of infected CD $4^+$ T cells will be greater than that of uninfected CD $4^+$ T cells. It is clear from the definition of $\mathcal{R}_0$ that it decreases as the reverting rate, $b$ of infected cells increases, hence $\mathcal{R}_0$ can be low for high parametric value of $b$. Further it is observed that there is no effect of delay in activation of infection in CD $4^+$ T cells on the stability of infected steady state if the condition $\alpha\beta - \gamma > 0$ is satisfied.

## 5. Model with CTL response

As discussed in the introduction, immune system responds to HIV infection via CTLs. In view of this some mathematical models have been studied to understand the effect of immune response [5, 25]. In 2006 Ciupe *et. al.* [5] discussed the dynamics of HIV infection consisting of three distinct phases starting with primary infection, then latency and finally either progression to AIDS or drug treatment. They modeled the dynamics of primary infection and the beginning of latency. The analysis showed that the model with time delay better predicts viral load data as compared to models with no time delay. In 2008 Srivastava and Chandra [46] considered a primary infection model with CTLs response and in accordance with [2] it was assumed that the growth of CTLs response is proportional to the number of infected CD $4^+$ T cells. Thus the following was proposed [46],

$$\frac{dT(t)}{dt} = \Lambda - kV(t)T(t) - \mu T(t), \tag{5.1}$$

$$\frac{dT^*(t)}{dt} = k'V(t)T(t) - \delta T^*(t) - d_x E(t)T^*(t), \tag{5.2}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t) - kV(t)T(t), \tag{5.3}$$

$$\frac{dE(t)}{dt} = pT^*(t) - d_E E(t), \tag{5.4}$$

with $T(0) = T_0 > 0$, $T^*(0) = 0$, $V(0) = V_0 > 0$, $E(0) = 0$.

Here $E(t)$ is the density of CTLs at time $t$. $k'$ is the rate at which infected cells become actively infected thus the ratio $\frac{k'}{k}$ is the proportion of T-cells that become actively infected. $d_x$ is the clearance rate of infected cells by CTLs,

$p$ is taken to be constant proliferation rate and $d_E$ death rate of CTLs. The system (5.1)-(5.4) has following two steady states:

(1) The infection free steady state $E_1 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$, and

(2) The infected steady state $E_2 = \left(\overline{T}, \overline{T}^*, \overline{V}, \overline{E}\right)$, where $\overline{E} = \frac{p\overline{T^*}}{d_E}$, $\overline{V} = \frac{N\delta\overline{T^*}}{c+k\overline{T}}$ and $\overline{T}$ and $\overline{T^*}$ are given by solution of following equations:

$$\Lambda - \frac{N\delta k T^* T}{c+kT} - \mu T = 0, \text{ and } T^* = \frac{d_E}{d_x p}\left[\frac{N\delta k' T}{c+kT} - \delta\right]$$

which exists provided $c\mu + k\Lambda < N\Lambda k'$ *i.e.* $\mathcal{R}_0 = \frac{N\Lambda k'}{c\mu + k\Lambda} > 1$.

Following stability results have been obtained [46]:

**Theorem 5.1.** *The infection free steady state is locally asymptotically stable if* $\mathcal{R}_0 < 1$.

**Theorem 5.2.** *The infected steady state is locally asymptotically stable whenever it exists.*

It is observed that it takes time for immune system to recognize the foreign body and then to react to it. Hence there is a time lag between the invasion of virus and CTLs response to clear it. In view of this a discrete time delay $\tau$ was introduced for the time lag in CTLs response and the modified model is given as follows [46]:

$$\frac{dT(t)}{dt} = \Lambda - kV(t)T(t) - \mu T(t), \tag{5.5}$$

$$\frac{dT^*(t)}{dt} = k'V(t)T(t) - \delta T^*(t) - d_x E(t) T^*(t), \tag{5.6}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t) - kV(t)T(t), \tag{5.7}$$

$$\frac{dE(t)}{dt} = pT^*(t-\tau) - d_E E(t), \tag{5.8}$$

with initial conditions $T(\theta) = T_0 > 0$, $V(\theta) = V_0 > 0$, $E(\theta) = 0$ and $T^*(\theta) = T_0^* > 0$ where $\theta \in [-\tau, 0]$.

The stability of infected steady state $E_2$ is investigated. The linearization of delay system (5.5)-(5.8) is given by

$$\dot{Y}(t) = M_1 Y(t) + M_2 Y(t-\tau), \tag{5.9}$$

where $M_1$ and $M_2$ are constant matrices of corresponding linearized system.

The characteristic equation of the linearized system (5.9) is then given by

$$\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau} = 0, \tag{5.10}$$

where

$$\begin{aligned}
A_3 &= \mu + k\overline{V} + \delta + d_x\overline{E} + c + k\overline{T} + d_E > 0,\\
A_2 &= (\delta + d_x\overline{E} + d_E)(\mu + k\overline{V} + c + k\overline{T}) + (c(\mu + k\overline{V}) + \mu k\overline{T})\\
&\quad + d_E(\delta + d_x\overline{E}) - N\delta k'\overline{T} > 0,\\
A_1 &= d_E(\delta + d_x\overline{E})(\mu + k\overline{V} + c + k\overline{T}) + (\delta + d_x\overline{E} + d_E)(c\mu + ck\overline{V}\\
&\quad + \mu k\overline{T}) - (\mu + d_E)N\delta k'\overline{T} > 0,\\
A_0 &= d_E(\delta + d_x\overline{E})(c(\mu + k\overline{V}) + \mu k\overline{T}) - \mu d_E N\delta k'\overline{T} > 0,\\
B_2 &= pd_x\overline{T^*} > 0,\\
B_1 &= pd_x\overline{T^*}(\mu + k\overline{V} + c + k\overline{T}) > 0,\\
B_0 &= pd_x\overline{T^*}(c(\mu + k\overline{V}) + \mu k\overline{T}) > 0.
\end{aligned}$$

Clearly when $\tau = 0$ all the roots of equation (5.10) have negative real part. Further, for $\tau \neq 0$ equation (5.10) has infinitely many roots. Now to see if equation (5.10) has purely imaginary roots or not, we put $\lambda = i\omega$ in equation (5.10) and separate the real and imaginary parts, which gives

$$\omega^4 - A_2\omega^2 + A_0 = (B_2\omega^2 - B_0)\cos\omega\tau - B_1\omega\sin\omega\tau, \tag{5.11}$$

$$-A_3\omega^3 + A_1\omega = -(B_2\omega^2 - B_0)\sin\omega\tau - B_1\omega\cos\omega\tau. \tag{5.12}$$

Squaring and adding equations (5.11) and (5.12) and then substituting $\omega^2 = m$, we get the following quartic equation:

$$\xi(m) \equiv m^4 + \alpha m^3 + \beta m^2 + \gamma m + \zeta = 0, \tag{5.13}$$

where $\alpha = A_3^2 - 2A_2$, $\beta = A_2^2 - B_2^2 + 2A_0 - 2A_3A_1$, $\gamma = A_1^2 - B_1^2 + 2B_2B_0 - 2A_2A_0$, and $\zeta = A_0^2 - B_0^2$. Using Routh Hurwitz criterion following result is obtained:

**Theorem 5.3.** *For the delay model (5.5)-(5.8) the infected steady state $E_2$ will be locally asymptotically stable for all $\tau \geq 0$ if and only if the following conditions are satisfied*

$$\zeta > 0,\ \ \Lambda = \alpha\beta - \gamma > 0,\ \ \Lambda\gamma - \alpha^2\zeta > 0$$

It is easy to see that $\zeta < 0$ whenever

$$\delta ck\overline{V} < \mu d_x\overline{E}(c + k\overline{T}). \tag{5.14}$$

Hence we have the following lemma,

**Lemma 5.1.** $\xi(m) = 0$ *will have at least one positive root if condition (5.14) is satisfied.*

Under condition (5.14), if $m_0(=\omega_0^2)$ is a positive root of (5.13), then equation (5.10) will have a pair of purely imaginary root $\pm i\omega_0$ corresponding to the delay $\tau$. Now we shall measure the value of $\tau$ for which system (5.5)-(5.8) will remain stable. From equations (5.11) and (5.12) we have,

$$\tau_k = \frac{1}{\omega_0}(2k\pi + \cos^{-1}\Phi), \;\; k = 0, 1, 2, 3, ... \tag{5.15}$$

where $\Phi = \frac{(B_2\omega_0^2 - B_0)(\omega_0^4 - A_2\omega_0^2 + A_0) + B_1\omega_0^2(A_3\omega_0^2 - A_1)}{(B_2\omega_0^2 - B_0)^2 + B_1^2\omega_0^2}$.

Also, differentiating equation (5.10) with respect to $\tau$ and considering real part only we have

$$Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} = \frac{1}{\omega_0^2}\left[3\omega_0^8 + \alpha\omega_0^6 + \beta\omega_0^4 - \zeta\right] > 0, \tag{5.16}$$

since $\alpha, \beta > 0$ and $\zeta < 0$.

Hence the transversality condition is satisfied and using Butler's Lemma [12] we get the following theorem:

**Theorem 5.4.** *If the condition* (5.14) *is satisfied then the infected steady state* $E_2$ *is locally asymptotically stable for* $\tau < \tau_0$ *and it becomes unstable for* $\tau > \tau_0$.

*Further, a Hopf bifurcation occurs for* $\tau = \tau_0$, *that is, a family of periodic solutions bifurcate from the infected steady state* $E_2$ *as* $\tau$ *passes through* $\tau_0$ [13].

In this paper the direction, stability and period of the bifurcating periodic solutions of the system (5.5)-(5.8) from the steady state are also investigated. The method used is based on the normal form theory and center manifold theorem as given by Hassard *et al.* [14].

If the infection persists then the system stabilizes to the infected steady state otherwise infection is cleared. When the delay in the activation of immune response is considered it is found that the infected steady state $E_2$ undergoes the Hopf bifurcation as delay value crosses the critical value. Further, the Hopf bifurcation is found to be subcritical with orbitally unstable periodic solutions *i.e.* the bifurcation occurs when $\tau$ crosses $\tau_0$ to the left. Hence the system shows complex dynamics when delay is taken into consideration.



## 6. Drug Therapy Model

Various drugs such as fusion inhibitor, RT inhibitor and protease inhibitor are used to attack at different phases of virus life cycle. In recent years some models have been proposed for drug therapy also [25, 27].

Perelson et al. [30] used their model to study the effects RT-Inhibitor on viral growth and T cell population dynamics also. In 1999, Ding and Wu [10] studied the effect of RT-Inhibitor and Protease Inhibitor drugs on viral replication. Rong et al. [38] in 2007 incorporated an ‘eclipse phase’ (phase that represent the stage in which infected T cells have not started producing new virus) into a simple HIV model. Analysis of the model indicates that quicker the infected T cells progress from the eclipse stage to the productively infected stage, it is more likely that a viral strain will persist. The model also studied effect of drug therapy and found that long term treatment effectiveness of antiretroviral drugs is often hindered by the frequent emergence of drug resistant virus during therapy. In 2007 Dehghan et al. [8] studied the global stability and other characteristics of a nonlinear differential equation model for HIV infection in the presence of combination therapy using geometric technique [21].

In 2009, Srivastava et al. [45] proposed a mathematical model for HIV infection with RT inhibitor. They considered three populations of CD $4^+$ T cells: $T(t)$ represents density of susceptible CD $4^+$ T cells, $T_1^*(t)$ represents density of infected CD $4^+$ T cells in pre-RT class *i.e.* before reverse transcription, $T^*(t)$ represents density of infected CD $4^+$ T cells in post-RT class in which reverse transcription is completed and the cells are capable of producing virus. The infected cells in pre-RT class $T_1^*(t)$ leave pre-RT class at a rate $\alpha$ to post-RT class. Also due to the presence of RT inhibitor a fraction of cells $\eta\alpha T_1^*(t)$ in pre-RT class reverts back to uninfected class. Remaining $(1-\eta)\alpha T_1^*(t)$ proceeds to post-RT class and become productively infected, where $\eta$ $(0 < \eta < 1)$ is the efficacy of RT inhibitor. Hence we have the following model [45]:

$$\frac{dT(t)}{dt} = \Lambda - kV(t)T(t) - \mu T(t) + (\eta\alpha + b)T_1^*(t), \tag{6.1}$$

$$\frac{dT_1^*(t)}{dt} = kV(t)T(t) - (\mu_1 + \alpha + b)T_1^*(t), \tag{6.2}$$

$$\frac{dT^*(t)}{dt} = (1-\eta)\alpha T_1^*(t) - \delta T^*(t), \tag{6.3}$$

$$\frac{dV(t)}{dt} = N\delta T^*(t) - cV(t), \tag{6.4}$$

with initial values $T(0) = T_0 > 0$, $T_1^*(0) = T^*_{1\,0} > 0$, $T^*(0) = T_0^* > 0$, $V(0) = V_0 > 0$.

The parameter $k$ represents rate of infection of CD $4^+$ T cells (here ‘ infection’ means the attachment and fusion of virus with cell) and $\mu_1$ is death rate of infected cells. $\alpha$ is the transition rate from pre-RT infected CD $4^+$ T cells class to productively infected class (post-RT).

The following result is proved in [45]:

**Theorem 6.1.** *The infection free steady state is locally stable if* $\widehat{T} \leq \overline{T}$*, and it becomes unstable when* $\widehat{T} > \overline{T}$*. Further it is globally asymptotically stable if* $\widehat{T} \leq \overline{T}$*, where* $\overline{T} = \frac{\mu_1+\alpha+b}{N\alpha k(1-\eta)}$ *and* $\widehat{T} = \frac{s}{\mu}$.

The replication rate for HIV is quite high. There are almost $10^{10}$ copies of virus particles are produced everyday in an infected individual [33]. The process of reverse transcription of viral RNA into DNA is highly error prone and the probability that a mutation occur is very high (the number of changes per genome is 0.3 per replication cycle [29, 31]). Since a single mutation of virus or a combination of them can result in drug resistance, there is a reasonable chance of existence of drug resistance strain even before therapy begins [35]. Rong et al. [36, 37] studied the models with both wild type, *i.e.*, drug sensitive viral strain and drug resistant viral strain to understand the dynamics of each strain in the presence of both the RTI and PI. Srivastava [44] modified the model (6.1)-(6.4) to include dynamics of two strain in presence of drug therapy.

## 7. Further Considerations

Some mathematical models with age structure of the cell have also been proposed and age since infection has been used to model the HIV infection. Therefore Nelson et al. [22] proposed an age structured model. This further improves the approach of the modeling and provides more realistic picture of the infection [22].

Tuckwell and Le Corfec (1998) [48] considered the stochastic extension of the model proposed by Phillips [34] by introducing additional noise term. Their model, which is a multi-dimensional diffusion process, assumed that stochastic effects arise in the process of infection of CD $4^+$ T cells and transitions may occur from uninfected to latently or actively infected cells

Nirav Dalal et al. [7] noted that the real life systems are so complex that there is always an error in parameters so instead of considering fixed values of the parameters they perturbed the death rate parameters by white noise and studied the noise induced stability.

As a concluding remark it may be pointed out that in this article an attempt has been made to illustrate the development of mathematical models for the *in vivo* dynamics of HIV and immune system. In general, we are interested in a long time behaviour of these models and hence stability analysis is performed to understand the dynamics of HIV infection and therapy. Mathematical modeling of HIV dynamics *in vivo* is very challenging as system is multifactorial and involves different components. It is also challenging from mathematical point of view to analyse these models. However, it may be pointed out that even a simple model can provide information to measure crucial parameters. Needless to say that more experimental work is needed along with the mathematical models so as to test theories and assumptions in more detail.

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# ABSTRACTS OF THE PAPERS SUBMITTED FOR PRESENTATION AT THE PLATINUM JUBILEE 75$^{\text{TH}}$ ANNUAL CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY HELD AT THE KALASALINGAM UNIVERSITY, ANAND NAGAR, KRISHNANKOIL - 626 190 DURING DECEMBER 27 - 30, 2009

## A: Combinatorics, Graph Theory and Discrete Mathematics

### A-1: Acyclic vertex coloring of graphs of maximum degree $\Delta$.

*S. Varagani, K. Kothapalli, V. Ch. Venkaiah*, Email: *kkishore@mail.iiit.ac.in, venkaiah@hotmail.com, vsatish@research.iiit.ac.in*

An acyclic vertex coloring of a graph is a proper vertex coloring such that there are no bichromatic cycles. The acyclic chromatic number of $G$, denoted $\alpha(G)$, is the minimum number of colors required for acyclic vertex coloring of Graph $G = (V, E)$. In this paper we show that for any Graph $G$ with maximum degree $\Delta$, $\alpha(G) \leq \frac{3\Delta^2+4\Delta+8}{8}$ . This improves the known result of Fertin and Raspaud by a factor of 4/3, while using similar techniques. Our proof investigates the colors in 3-neighborhood, as opposed to the 2-neighborhood considered earlier.

### A-2: Balanced and consistency of total signed graphs.

*Deepa Sinha and Pravin Garg,* Email: *deepasinha2002@gmail.com*

A signed graph ( or sigraph in short) is an ordered pair $S = (S^u, \sigma)$, where $S^u$ is a graph G = (V,E), called the underlying graph of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of $S^u$ into the set $\{+, -\}$, called the signature of S. The $x$- line graph of S denoted by $L_x(S)$ is a graph defined on the line graph $L(S^u)$ of the graph $S^u$ by assigning to each edge $ef$ of $L(S^u)$, the product of signs of the adjacent edges $e$ and $f$ in $S$. For any sigraph $S$, we define $\mu_1$– marking in total sigraph $T(S)$ as $\mu_1 : V(T(S)) \to \{+, -\}$ such that $\mu_1(v_i) = \prod_{ej \in E_{v_i}} \sigma(e_j)$ and $\mu_1(e_i) = \sigma(e_i)$. In this paper,we define total sigraph of a given sigraph and then obtain the following theorems: Theorem 1. The total sigraph $T(S)$ of a sigraph $S$ is balanced if and only if the following conditions hold in S:

(i) if $e$ is a positive edge in $S$ and $u, v$ are the end vertices of $e$, then the number of negative edges incident on both $u$ and $v$ are of same parity,
(ii) if $e$ is a negative edge in $S$ and $u, v$ are the end vertices of $e$, then the number of negative edges incident on both $u$ and $v$ are of opposite parity.

Theorem 2. The total sigraph $T(S)$ of a sigraph $S$ is $\mu_1$–consistent if and only if $S$ is an all-positive sigraph.

**A-3: Fractional metric dimension of graphs.** *S. Arumugam and V. Mathew,* Email: *s.arumugam.klu@gmail.com, varughes_m1@ yahoo.co.in*

A vertex $x$ in a connected graph $G$ is said to resolve a pair $\{u, v\}$ of vertices of $G$ if the distance from $u$ to $x$ is not equal to the distance from $v$ to $x$. A set $S$ of vertices is a resolving set for $G$, if every pair of vertices of $G$ is resolved by some vertex of $S$. The smallest cardinality of a resolving set for $G$, denoted by $dim(G)$, is called the metric dimension of $G$. For the pair $\{u, v\}$, of vertices of $G$ the collection of all resolving vertices is denoted by $R_n\{u, v\}$ and is called the resolving set of $\{u, v\}$. A real valued function $g : V(G) \to [0, 1]$ is a resolving function of $G$, if for every distinct pair $u, v$ of $V(G)$, $g(R_n\{u, v\}) \geq 1$. The fractional metric dimension of $G$ is defined as $dim_f(G) = min\{|g| : g$ is a minimal resolving function of $G\}$, where $|g| = \sum_{v \in V} g(V)$. In this paper, we initiate a study of this parameter.

**A-4: Gamma - domination number gamma lambda(G) of a graph G.** *N. Sridharan, S. Amutha and A. Ramesh Babu,*
*E-mail: amutha.angappan@rediffmail.com*

We introduce a new domination parameter $\gamma\lambda(G)$, where $0 \leq \lambda \leq 1$, and initiate a study $\lambda\frac{1}{2}(G)$. We find sharp bounds for $\lambda\frac{1}{2}(G)$.

**A-5: The middle dominating graph of a graph.** *B.Basavanagoud Sunilkumar, M. Hosamani and I. M. Kadakol,* Email: *bgouder1@yahoo.co.in*

The middle dominating graph $M_d(G)$ of a graph $G$ is defined to be the intersection graph on the vertex set of $M_d(G)$. Where the vertex set of $M_d(G)$ is the union of vertex set of $G$ and the set of all minimal dominating sets of $G$. In this paper, characterizations are given for graphs whose middle dominating graph is connected and $K_p \subseteq Md(G)$. Other properties of middle dominating graphs are also obtained.

**A-6: A criterion for (non-) planarity of the transformation graph $G^{XYZ}$ when $XYZ$ = - + +.** *B. Basavanagoud and P. V. Patil,* Email: *bgouder1@yahoo.co.in*

The transformation graph $G^{-++}$ of $G$ is the graph with vertex set $V(G) \cup E(G)$ in which the vertex $x$ and $y$ are joined by an edge if one of the following conditions holds: $(i)\, x, y \in V(G)$ and $x$ and $y$ are not adjacent in $G$, $(ii)\, x, y \in E(G)$ and $x$ and $y$ are adjacent in $G$, $(iii)$ one of $x$ and $y$ is in $V(G)$ and the other is in $E(G)$, and they are incident in $G$. In this paper we present characterizations of graphs whose transformation graphs $G^{-++}$ are eulerian, outerplanar, maximal outerplanar or minimally nonouterplanar. Further we establish a necessary and sufficient condition for the transformation graphs $G^{-++}$ to have crossing number one or two

**A-7: On graphs with signed domination number zero, positive and negative.** *M. S. Patil and H. B. Walikar,*
*E-mail: patil_2005ms@yahoo.co.in*

A two-valued function $f$ defined on the vertices of a graph $G = (V, E)$, $f : V \rightarrow \{-1, 1\}$ is a signed dominating function $(SDF)$, if the sum of its function values over any closed neighbourhood is at least one. The weight of a signed dominating function is defined to be $w(f) = \sum f(v)$, over all vertices $v \in V$. The signed domination number of a graph $G$, denoted by $\gamma_s(G)$ and $\gamma s(G) = min\{w(f)\}$ where $f$ is Signed dominating function of $G$. In this paper, we characterize the class of graphs $G$ with $\gamma s(G) = n$, where $n$ is any integer and we found both upper and lower bounds on the size of a graph with $\gamma s(G) = n$, where n is any integer.

**A-8: Degree preserving spanning tree set.** *K. Sangavai,* Email: *sangavai_k@yahoo.com*

A spanning tree $T$ of a graph $G$ is said to be a A-Degree Preserving Spanning Tree $(A - DPST)$ if $deg_T(v_i) = deg_G(v_i)$, for all $v_i$ in $A$, a non empty subset of $V$. A subset $A$ of the vertex set $V(G)$ is said to be Degree Preserving Spanning Tree $(DPST)$-set, if A-DPST exists. Any DPST- set $A$ is called a maximal DPST set if, $A \cup \{v\}$ is not a DPST set for every $v \in V - A$. The maximum and minimum of the cardinalities of the DPST-set are called the upper and lower degree preserving numbers and are denoted by $\Psi_P(G)$ and $\psi_P(G)$ respectively. This paper analyzes $\Psi_P(G)$ for some classes of graphs and identifies the conditions for a super set of a DPST-set to be a DPST-set. Characterizations of graphs with some fixed $\Psi_P(G)$ are given. A necessary and sufficient condition for a graph to have $\Psi_P(G) = 1$ is also determined.

**A-9: The L(2, 1)-labeling of some new graphs.** *D. D. Bantva,* Email: *devsi.bantva@gmail.com*

An $L(2,1)$-labeling of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$

and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$. The $L(2, 1)$-labeling number $\lambda(G)$ of (G) is the smallest number $k$ such that (G) has an $L(2, 1)$-labeling with $\{f(v) : v \in V(G)\} = k$ .In this paper we completely determine $\lambda$-number for one point of $n$cycles, middle graphs of path $P_n$ and middle graph cycle $C_n$ .

**A-10: Enumeration of non cordial graphs.** *N. B. Vyas,* Email: *niravbvyas@rediffmail.com*

As all the graphs does not admit cordial labeling, it is very interesting to find out the graphs which are not cordial. Here we carried out a detailed and extensive exploration for the enumeration of a subclass of non cordial graphs.

**A-11: The global convexity graph of a graph.** *K. Karuppasamy,* Email: *k_karuppasamy@yahoo.co.in*

Let $G = (V, E)$ be a graph. A function $g : V \to [0, 1]$ is called a global dominating function $(GDF)$ of $G$, if for every $v \in V$, $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ and $g(\overline{N(v)}) = \sum_{u \notin N(v)} g\bar{(}u) \geq 1$. A $GDFg$ of a graph $G$ is called minimal $(MGDF)$ if for all functions $f : V \to [0, 1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v2V, f$ is not a $GDF$. Analogous to the concept of convexity graph with respect to minimal dominating functions, we introduce the concept of global convexity graph and determine the global convexity graphs for some standard graphs.

**A-12: $b$-chromatic number of kneser graph.** *T. Kavaskar,* Email: *tkavaskar@yahoo.com*

A $b$-coloring of a graph $G$ with $k$ colors is a proper coloring of $G$ using $k$ colors in which each color class contains a color dominating vertex, that is, a vertex which has a neighbor in each of the other color classes. The largest positive integer $k$ for which $G$ has a $b$-coloring using $k$ colors is the $b$-chromatic number $b(G)$ of $G$. The $b$-spectrum of a graph $G$ is the set of positive integers $k, \chi(G) \leq k \leq b(G)$, for which $G$ has a $b$-coloring using $k$ colors. In this paper, we obtain an upper bound for some families of Kneser graphs. In addition we establish the b-spectrum of some families of Kneser graphs.

**A-13: On some new matrices of a graph II.** *S. R. Jayaram and Divya Rashmi S. V.,* Email: *drjayaram2002@gmail.com*
Given a Graph $G = ((V(G), E(G))$, and a subset $T \subseteq E(G)$, $T$ with a given property(Line covering set, Line Dominating set, Line Neighbourhood set), we define a matrix taking a row for each of the minimal set corresponding to the given property and a column for each of the edge of $G$. The elements of the matrix are 1 or 0 respectively as the edge is contained in minimal set or otherwise. That is matrix $(m_{ij})$ has elements $m_{ij}$ and $m_{ij} = 1$ if $i$th row

minimal set contains $j$th vertex and $m_{ij} = 0$ otherwise. This paper initiates a study on these new types of matrices of a graph and we characterize such matrices for some special classes of graphs.

**A-14: The diameter variability of cartesian product of some graphs.** *Chithra M. R, Daniela Ferrero, M. K. Menon, A. Vijayakumar,* Email:*vijay@cusat.ac.in*

The diameter of a graph is an important factor for communication as it determines the maximum communication delay between any pair of processors in a network. It is very important to study whether the diameter is changed or remains unchanged when the edges are added or deleted. In this paper we study the diameter variability of Cartesian product of some graphs.

**A-15: Star-line graphs.** *Seema Varghese and A. Vijayakumar* Email: *seema @cusat.ac.in, vijay@cusat.ac.in*

Let $H$ be connected graph with at least three vertices. The $H$ -line graph, $HL(G)$, of a graph $G$ has all the edges of $G$ as its vertices, two vertices of $HL(G)$ are adjacent if the corresponding edges in $G$ are adjacent and belong to a common copy of $H$. In this paper we show that $H$-line graphs do not admit a forbidden subgraph characterization. We also obtain conditions for planarity of iterated star-line graphs, $K_{1,n}L^k(G)$.

**A-16: Unfolding tree.** *R. B. Gnanajothi, S. M. MeenaRani, E-mail: smmeenarani@gmail.com, gnanajothi_pcs@rediffmail.com*

Neural Network is a tool to solve a problem intelligently. Recursive Neural Network (RNN) is a topology used for processing graphs with labeled edges. The graphs considered are Directed Positional Acyclic Graphs. In object detection problems, the images are represented as graphs. In order to avoid cycles the graphs are transformed into unfolding trees and RNN is applied.

In this paper, concepts of node equivalence and unfolding equivalence are compared. The unfolding trees of certain families of graphs are characterized. A proper choice of source node is suggested to decrease the depth of the minimal unfolding tree.

## B: Algebra, Number Theory and Lattice Theory

**B-1: Prime ideals in 0-distributive posets.** *N. D. Mundlik, E-mail: nileshmundlik@rediffmail.com*

In this paper, we study prime ideals in finite 0-distributive posets. Further, we characterize minimal prime ideals in finite 0-distributive posets.

**B-2: Modern concepts of fuzzy sets.** *A. K. Sinha and Alok Kumar, Dept. of Maths., M. M. Mahila College, Ara; P. G. Dept. of Maths., Maharaja College, Ara.*

The Modern concept of native set theory was introduced by Cantor and Boole in the beginning of 20th century. Let $X$ be a universal set and $A$ be a subset of $X$. Then $A$ can be characterized by using the characteristics function of $A$.

In this definition, the belongingness of an element $x$ in the subset $A$ is either True or False; Black and White; Yes or No etc. This way of representation can only be imaginary; which cannot be realistic. Hence applying this concept to real problems will not give actual solutions. Starting from Russel in 1920 (Russel Paradox), Max Black in 1935, mathematicians analyzed and understood this difficulty. But none could improve the concept. This improvement of the concept is actually the graduation of the belongingness of the element 0 and 1, more or less than between True or False; the Grey area between Black and White; the fuzz between a set and its complement.

Hence in 1965 cyberneticist L. A. Zadeh in his seminar Paper introduced fuzzy sets and applied to his control problems.

**B-3: On generalized derivations in modules.** *Shakir Ali*, Email: *shakir 50@rediffmail.com, shakir.ali.mm@amu.ac.in*

In this paper, we will discuss some commutativity theorems involving generalized $(\alpha, \beta)$-derivations in certain classes of rings.

**B-4: On semidefinite linear complementarity problems.**
*A. Chandrashekaran, T. Parthasarathy and V. Vetrivel,*
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Let $S^n$ denote the set of all real symmetric matrices of order $n$. Let $L : S^n \to S^n$ be a linear map. Given $Q \in S^n$, the semidefinite linear complementarity problem $SDLCP(L, Q)$ is to find a $X \in S^n_+$ (set of all positive semidefinite matrices in $S^n$) such that $L(X) + Q \in S^n_+$ with $\langle X, L(X) + Q\rangle =$ trace $(X(L(X) + Q)) = 0$ or $X(L(X) + Q) = 0$. If we could find such an $X \in S^n_+$ with the above property then $X$ is called a solution to the problem $SDLCP(L, Q)$. In this article we give an algorithm to find an exact solution to the problem $SDLCP(M_A, Q)$, when $A$ is either symmetric positive definite or negative definite matrix. Further we present some results on computing the Lipschitzian constants for the solution maps.

**B-5: On modular pairs in posets.** *V. S. Kharat and R. S. Shewale,* Email: *rss@math.unipune.ernet.in*

In this paper we gave definition of a modular pair by means of the concepts $'L'$ and $'U'$ in posets. The relation between new definition and the existing definitions of a modular pair in poset by Haskins and Gudder, by Thakare, Pawar and Waphare, and by Thakare, Wasadikar and Maeda were studied in different classes of posets. We succeeded in characterizing a modular pair by means of forbidden configuration in general poset. Several characterizations of modular pair were obtained in atomistic and semi-orthogonal posets. We have also introduced $\nabla$-relation in poset.

**B-6: Primeness and semiprimeness in posets.** *V. S. Kharat and K. A. Mokbel,* Email: *vsk@math.unipune.ernet.in, khalidalaghbari@yahoo.com*

The concept of a semiprime ideal in a poset is introduced. Characterizations of semiprime ideals in a poset $P$ as well as characterizations a semiprime ideal to be prime in $P$ are obtained in terms of meetirreducible elements of a lattice of ideals of $P$ and in terms of maximality of ideals. Also, prime ideals in a poset are characterized.

**B-7: $L$-submodule of $L$-module over a ring $R$.** *R. Natarajan and R. Meenakshi, meenakshi_alu@rediffmail.com*

In this paper we introduce $l$ submodule and derive for the quotient $l$ module and give the fundamental theorem, first isomorphism theorem and second isomorphism theorem.

**B-8: On the existence of prime ideals in posets.** *Vinayak Joshi,* Email:*vinayakjoshi111@yahoo.com*

In this paper, we give a simple proof of an analogue Stone's Theorem for general finite posets. Further, we characterize the posets of prime ideals of a pseudocomplemented 1-distributive poset and prove an analogue of Nachbin's Theorem for finite posets.

**B-9: On derivations of semirings.** *M. Chandramouleeswaran,* Email: *moulee59@gmail.com*

The notion of rings with derivations is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. The study of derivations in rings though initiated long back, but got interested only after Posner who in 1957 established two very striking results on derivations in prime rings.

The notion of derivation has also been generalized in various directions, such as Jordan derivation, generalized derivation, generalized Jordan derivation etc. Also there has been considerable interest in investigating commutativity of rings, more often that of prime and semiprime rings admitting these mappings which are centralizing or commuting on some appropriate subsets of the semiring.

We define derivations on semirings and investigate some simple results thus connecting derivations with the structure theory of semirings.

**B-10: On some new schlfli-type mixed modular equations.** *M. S. Mahadeva Naika and K. Sushan Bairy,* Email:*msmnaika@rediffmail.com*

On pages 86 and 88 of his first notebook, Ramanujan recorded eleven Schlfli-type modular equations for composite degrees. Out of eleven, ten have been proved by Berndt using theory of modular forms. In this paper, we establish several new Schlfli-type mixed modular equations.

**B-11: Certain identities for a continued fraction of Eisenstein.** *M. Manjunatha*, Email: *mmanjunathapes@gmail.com*

In this paper, we establish several modular relations between a continued fraction of Eisenstein $H(q)$ and $H(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 39, 47, 55, 71$ and $119$. We also establish two integral representations for $H(q)$

**B-12: Frankl's conjecture and 0-conditions in Eulerian lattices.** *A. Vethamanickam and R. Subbarayan, E-mail: aquarayan@rediffmail.com, rsubbarayan@in.com*

In this paper, we show that Frankl's conjecture holds for Eulerian lattices. We also discuss the concept of equational class of 0-distributive lattices. Finally, we prove that Eulerian lattice with some weaker conditions of 0-distributivity becomes Boolean.

**B-13: Some new schlfli-type mixed modular equations in the quartic theory.** *M. S. Mahadeva Naika and Chandan Kumar S.,* Email: *msmnaika@rediffmail.com*

In this paper, we establish several new Schlflitype modular equations in the theory of signature 4. We establish modular relations between the quartic class invariants $h_n$ with $h_{kn}^2$ and $H_n$ with $H_{kn}^2$. We also establish several new explicit evaluations of the quartic class invariants and quartic singular moduli.

**B-14: Some theorems on Ramanujan's cubic continued fraction**

**and related identities.** *M. S. Mahadeva Naika,* Email: *msmnaika@rediff mail.com*

On page 366 of his lost notebook, Ramanujan has recorded cubic continued fraction and several theorems analogous to Rogers-Ramanujan continued fractions. In this paper, we establish several interesting results of cubic continued fraction which are analogous to Rogers-Ramanujan continued fractions.

**B-15: Riemann hypothesis: a collective study.** *Miss. Usha K. Sangale,* Email: *usha.sagale@rediffmail.com*

This paper is devoted to the open problem of 20th century which got pulled over to 21st century the Riemann Hypothesis . Here is an attempt to focus the importance of the still unsolved problem Riemann Hypothesis which we term as RH. Around 1859, Riemann Conjectured that "all the nontrivial zeros of the zeta function $\varsigma(z)$ are on the critical line, $\mathrm{Re}(z) = 1/2$"then onwards, mathematical world accepted this conjecture as RH. Many mathematicians have worked on it. Some of them claimed to having solved the RH. This paper collects many of the works together. Many mathematicians have derived its applications by assuming RH is true. Few mathematicians have extended the RH, labelling it as the Extended Riemann Hypothesis(ERH). Assuming ERH, many mathematicians have reached to a number of its applications in various fields such as algebra, computers, quantum mechanics, cryptography, non-commutative geometry, signal processing etc. This paper gathers few of them. This is an attempt to view RH in a collective manner.

**B-16: The $P_2^{'}$ -property of linear transformations on Eucledean Jordan algebras.** *I. Jeyaraman and V. Vetrivel* Email: *i* & *jeyaraman@yahoo.co.in, vetri@iitm.ac.in*

Motivated by the concepts of $P_2^{'}$-Property in standard linear complementarity problems and semi definite linear complementarity problems, we introduce and study $P_2^{'}$-Property of linear transformations on Euclidean Jordan algebras. We show that $P_2^{'}$-Property implies strict semimonotonicity property, $Q$- property and GUS on the cone of square $K$. We prove that strictly co-positive on $K$ implies the $P_2^{'}$-Property. We show by an example that $P_2^{'}$-Property need not imply strictly co-positive on $K$, even for $Z$-transformations. However, we establish the equivalence between them for Lyapunov like transformations and normal $Z$-transformations.

**B-17: Some results on complete regular rings with a rank function.** *M. K. Manoranjan, Dept. of Maths., B. N. M. V. College, Madhepura 852113.*

This paper is concerned with the structure of von Neumann regular rings which are complete with respect to the weakest metric derived pseudo rank functions known as the $N^*$-metric. We study the existence and uniqueness of rank functions on a simple von Neumann regular and with the structure of regular ring with respect to the $N^*$-metric induced by a rank function. The completeness of a regular ring with respect to a rank function is shown to be right and Lift-injective ring.

**B-18: Some sufficient conditions of a given series with rational terms converging to an irrational number or a transcdental number.** *Yun Gao, Jining Gao, Shanghai Putuo University, North Carolina State University.*

In this paper, we generalize Edors's result on irrationality of some specific fast convergent series, and give out various application examples even in dynamical system, base on it, we propose a conjecture on the irrationality of E series.

## C: Real and Complex Analysis (Including Special Functions, Summability and Transforms)

**C-1: On certain transformation of bilateral poly-basic hypergeometric series.** *Roselin,* Email: *roselimaths@gmail.com*

A well known series identity of Santosh is used to obtain many transformations of bilateral poly-basic hypergeometric series. The inner sums of both sides of the identity are replaced by terminating quadratic, cubic and quartic series to establish transformations of terminating asymmetric bilateral poly-basic hypergeometric series in terms of the same series.

**C-2: In the degree of approximation of signals (functions) belonging to $W(L_p, \xi(t)), (p \geq 1)-$ class by $(C,1)(E,1)$ means of its Fourier series.** *V. N. Mishra,* Email: *vishnu_narayanmishra@yahoo.co.in*

In this paper, we generalize a theorem of Lal and Singh, on the degree of approximation of function belonging to the weighted $W(L_p, \xi(t)), (p \geq 1)-$ class, by using product summability $(C,1)(E,1)$ means of its Fourier series.

**C-3: Henstock-Kurzweil fractional Fourier transform.** *M. S. Chaudhary and P. N. Mane,* Email: *poonam.mane9394@gmail.com*

Here we consider the Fractional Fourier transform as a Henstock-Kurzweil integral. In recent years, the Fractional Fourier transform has been the focus of many research papers. Here we deducted many of its properties from those of Fourier transform by a simple change of variables . In this paper some sufficient conditions are given for it to be continuous. Here we also give frequency differentiation and time dissertation. Lastly convolutions and inversion theorems are established.

**C-4: An integral involving the product of an incomplete gamma function and generalised h-function of '$r'$ variable.** *T. M. Vasudevan, Nambisan and Prathima J., E-mail: pamrutharaj@yahoo.co.in*

This paper contains an integral involving the product of an incomplete gamma function and $A$-function of '$r'$ variables. The Laplace transform of the $A$-function of '$r'$ variables as a preliminary result.

**C-5: On gauge Laplace transform.** *M. S. Chaudhary and Sanket A. Tikare,* Email: *sanket.tikare@gmail.com*

The Laplace transform is considered as a Gauge(Henstock-Kurzweil) integral. Different existential conditions are given. Elementary properties and analyticity are discussed. Existence of Gauge Laplace transform of some functions have been studied. The inversion theorem is established using generalized differentiation.

**C-6: Convolution of Hankel type transform involving Bessel type functions and its application.** *B. B. Waphare,*
Email: *bbwaphare@mitpune.com*

In this paper, we have constructed a convolution of the Hankel type transform. As one of its applications, a formula of infinite interval of a product of Bessel type functions of first kind is established.

**C-7: Note on Laguerre transform in two variables.** *I. A. Salehbhai, R. K. Jana and A. K. Shukla, SVNIT, Surat–395 007, Gujarat.*

Laguerre polynomials play an important role in the field of science, engineering, numerical mathematics,quantum mechanics, communication theory and numerical inverse Laplace transform. Some explicit evaluation of integrals involving Laguerre polynomials are required in applied areas of mathematical and physical sciences. In 1960, Debnath first introduced Laguerre transform of one variable and its properties; he also discussed its applications in study of heat conduction, diffusion equation and to the oscillations of a very long and heavy chain with variable tension. Debnath et al. reported this work in their book. Recently Shukla et al introduced Laguerre transform in two variables. In present paper, we discussed some properties of Laguerre Transform in two variables.

**C-8: Omitted values and dynamics of meromorphic function.**
*Tarakanta Nayak and Jian-Hua-Zheng, Department of Mathematical Sciences, Tsinghna University, Beijing–100084, Peoples' republic of China.*

For a transcendal meromorphic function $f$ having at least two poles or one pole which is not an omitted value, let $f$ have at least one omitted value. A complete classification of all the multiply connected Fatou components of $f$ is made in this article. As a corollary to this, it follows that the Julia set of $f$ is not totally disconnected in every situation except when all the omitted values are contained in a single Fatou component. A necessary and sufficient condition for the existence of a dense subset of singleton buried components in the Julia set is established for functions with two omitted values. The conjecture that a meromorphic function has at most two completely invariant Fatou components is confirmed except in the case when $f$ has a single omitted value, no critical value and is of infinite order. Non existence of both Baker wandering domains in variant Herman rings are proved. For Functions with exactly one pole, we show that Herman rings of period two also do not exist. Eventual connectivity of each wandering domain is proved to exist. Some relevant examples are discussed.



**C-9: Generalized Laguerre transform of functions of two variables.** *T. G. Thange, E-mail: thange.tukaram@gmail.com*

The work in this paper consist of an extension of Generalized Laguerre transform of functions of two variables with the use of Zemanians technique related to the transformation arising from orthonormal series expansions. Inversion formula is also obtained.

**C-10: On Mellin type integral transform.** *Shaikh Sadikali Latif and M. S. Chaudhary,* Email: *chaudhary_m_s@rediffmail.com*

The purpose of this paper is to develop the class of Mellin type integral transformable functions. Further sufficient condition for the existence of Mellin type integral transform is derived. Relation between Mellin type integral transform and Laplace transform is obtained. Inversion theorem for Mellin type integral transform is proved. It is shown that the differential operator $(x.d/dx)^2$ is removed by this integral transform provided some initial values of the function are known. Some operational properties of this integral transform are also proved.

**C-11: Boehmians and homogeneous distributions.** *V. Karunakaran and C. Prasanna Devi, E-mail: pras_2@yahoo.co.in*

The concept of homogeneous distributions is extended to the context of a suitable Boehmian space. It is shown that every homogeneous distribution can be viewed as a convolution quotient of homogeneous functions.

**C-12: On the direct sum and the direct product of Boehmians.**
*V. Karunakaran and R. Angeline Chella Rajathi, jesangel_sweety@yahoo.co.in*

In this paper we define the notion of an external direct sum of two Boehmian spaces and an internal direct sum of sub Boehmian spaces and address the consistency problem. We shall also illustrate these ideas with examples. We shall use these concepts to define the Fourier transform and the Laplace transform as a bijective bicontinuous map of the direct sum of two Boehmian spaces onto itself. We shall also extend the concept of direct product of distributions to the context of Boehmians.

**C-13: A plancherel type theorem for Boehmians.** *V. Karunakaran and R. Bhuvaneswari, duvibharathi@yahoo.co.in*

In this paper we construct a Boehmian space containing Fourier transforms of elements of the classical Sobelev space $H^s$ and define their Fourier-Plancherel transform. The image is also characterized as Locally $H^s$ distributions. This in particular characterizes the Fourier-Plancherel transforms of Locally $H^s$ distributions (which are known only as Ultra distributions) as convolution quotients of $L^2_{\mu_s}$ functions where $\mu_s$ is the weighted Lebesgue measure used in the definition of Sobelev spaces.

**C-14: Region of variability for functions with positive real part.**
*Vasudevarao Allu,* Email: *alluvasu@iitm.ac.in*

For $\gamma \in \mathbb{C}$ such that $|\gamma| < \pi/2$ and $0 < \beta < 1$, let $P_{\gamma,\beta}$ denote the class of all analytic functions $P$ in the unit disk $\mathbb{D}$ with $P(0) = 1$ and $Re(e^{i\gamma}P(z)) > \beta\cos\gamma$ in $\mathbb{D}$. For any fixed $z_0 \in \mathbb{D}$ and $\lambda \in \bar{\mathbb{D}}$, we shall determine the region of variability $V_P(z_0, \lambda)$ for $R\int_0^{z0} P(\xi)d\xi$ when $P$ ranges over the class $P(\lambda) = \{P \in P_{\gamma,\beta} : P'(0) = 2(1-\beta)\lambda e^{i\gamma}\cos\gamma\}$.

As a consequence, we present the region of variability for some subclasses of univalent functions. We also graphically illustrate the region of variability for several sets of parameters. This talk is based on the joint work with Prof. S. Ponnusamy.

**C-15: Shape preserving $GC^2$ rational cubic trigonometric spline.** *M. Dube and S. Sanyal,* Email: *mriduladube@yahoo.com*

In this paper we give the construction of a $GC^2$ trigonometric rational cubic interpolant, and establish its existence and uniqueness.We also construct a rational $B$-spline, show that it forms a partition of unity and give the $B$-spline curve for the same. By adjusting the weights in the rational $B$-spline curve we can bring the rational segment pretty close to the control polygon.

**C-16: E-valued Borel exceptional values of meromorphic functions.** *Smt. V. L. Pujari, vipujari@gmail.com*

In this paper, we deal with the Borel exceptional values for the E-valued meromorphic functions and introduce the order of multiplicity for Borel exceptional values and obtain a basic inequality. From this we deduce several well-known results and also obtain some new significant results in this direction. The results hold good for finite and infinite order for meromophic functions.

**C-17: On the generalized fractional calculus operators involving the $H$-function of several variables.** *U. K. Saha and L. K. Arora, Department of Mathematics, North Eastern Regional Institute of Science and Technology, Nirjuli (Itanagar)–791 109, Arunachal Pradesh, India.*

In the present paper we introduce two very general key formulas in fractional calculus involving the H-function of several variables of Srivastava and Panda. The operators developed here also contain the Erdelyi-Kober operators of fractional calculus. Similar formulas for the one-dimensional -function are already known and are implied by our results. The fractional calculus operators studied here are quite general in character, and therefore, would widely be applicable to several classes of higher transcendental functions.

## D: Functional Analysis

**D-1: Common fixed point of nonself hybrid contractions using quasi S- commutativity.** *P. K. Shrivastava, J. S. T. Govt. P. G. College, Balaghat 481001.*

The purpose of this paper is to introduce the concept of quasi sequential commutativity of Hybrid pair$(T, F)$ and to prove it is more general than the concept of sequential commutativity introduced by Pathak and Mishra. We thus generalize the results of Pathak and Mishra, who have indicated that the result of Chang admits a counter example and presented a correct version of the result. Our paper in turn generalizes several known results in this direction. We suggest an essential restriction to impose for defining sequential commutativity. We also give examples to justify our claims.

**D-2: Convergence theorems of iterative schemes for non expansive mappings.** *D. P. Shukla,* Email: *shukladpmp@gmail.com*

In this paper introduces a new idea for common fixed point of non expansive mapping using iterative schemes and we prove that fixed point theorems for weakly convergence of sequence in Banach space which satisfies opial's condition.

**D-3: Coincidence & common fixed point theorems for hybrid contraction in symmetric space.** *N. P. S. Bawa, Y. K. Vijaywar, P. K. Shriwastava, Govt. Model Science College, Rewa,(MP), India.*

Using compatibility of type (N) and (E-A) property of hybrid pair of mapping we obtain some new coincidence and common fixed point theorems under strict contractive conditions in symmetric(semi-metric)space.

**D-4: Numerical range of certain composition operators on $l^2$.** *Kamlesh Kumar Rai, E-mail: kkraib@rediffmail.com*

Approach : Let $l^2$ be the Hilbert space of all square summable sequences of complex numbers under the standard inner product on it. Let $C\phi$ be a non-surjective composition operator on $l^2$ induced by a function $\phi$ on $N$ into itself. Numerical range of $C\phi$ is denoted by $W(C\phi)$ and is defined by $W(C\phi) = \{< C\phi(f), f >: |f| = 1\}$.

Findings : In this paper numerical range of positive composition operator, self adjoint composition operator and certain other composition operator are characterized . In some cases these operators are polynomially compact while in others these are not polynomially compact. In this paper each proposition is followed by suitable examples of each case.

## E: Differential Equations, Integral Equations and Functional Equations

**E-1: Numerical solution of fuzzy differential equations by Runge-Kutta Nystrom method.** *K.Kanagarajan and M. Sambath,* Email: *kanagarajank@gmail.com*

In this paper we give a numerical algorithm for solving fuzzy differential equations based on Seikkalas derivative of fuzzy process. We discuss in detail a numerical method based on Runge-Kutta Nystrom method of order 3. The algorithm is illustrated by solving some fuzzy differential equations.

**E-2: On existence results of second order Volterra integrodifferential inclusions.** *S. D. Kendre and M. B. Dhakne,* Email:*ksubhash@math.unipune.ernet.in*

The objective of the paper is to study the existence of mild solutions of second order Volterra integrodifferential inclusions in a real Banach space by using semigroup theory and suitable fixed point theorem when the multivated map has convex values.

**E-3: A new formula for fractional derivative.** *Kuldeep Singh Gehlot, Dept. of Maths., Bangur Govt. P. G. College, Pali, M. D. S. University, Ajmer.*

In this paper I calculated a new formula for Fractional Derivative in terms of Backward Difference and Forward Difference. Also find the Fractional Derivative of, and General Class of Polynomial with the help of New formula, named as Gehlot's Formula for Fractional Derivative.

**E-4: Stability analysis of Mixed Euler method.** *Sarita Thakar and Sachin S. Wani* Email: *saritathakar@yahoo.com, sachin.swani@gmail.com*

A difference method called Mixed Euler method is proposed for one dimensional non linear Burgers Equation with homogeneous Dirichlets boundary conditions. We proved that the method is stable in $L_2$ norm. Numerical solutions of one dimensional non linear Burgers equation obtained by Mixed Euler method are bounded and are better than the solution obtained by Pade's approximation. Numerical solutions are compared with the exact solution for different values of $t$ and constant of diffusivity $k$.

## F: Geometry

**F-1: $\pi^*$ on squaring a circle.** *Shiv Dayal.* Email: *shivonpi@gmail.com*

There is a fundamental problem of mathematics "squaring the circle" In other words, how to draw a circle with equal area of square. The above problem has been created only, as $\pi$ is irrational, rather a transcendental. Since $\pi$ is a numerical constant express the relationship between the circumference of the circle and its diameter on a flat plane surface. So with the help of series of experiment a number which is quite closer to the approximated value of $\pi$ is determined. The experiments were performed Buffon's needle probability theory, volume of sphere, squaring the circle etc.and get a number 3.24 denoted by $\pi^*$.

**F-2: Sasakian manifolds with Pseudo projective curvature tensor.** *H. G. Nagaraja, hgnraj@yahoo.com*

We consider Pseudo projective curvature tensor on a Sasakian manifold $(M^n, g)$ satisfying Einstein Field Equations. We prove that every Sasakian manifold with Pseudo curvature tensor $(M^n, g)$ is locally isometric with unit sphere and if the pseudo projective curvature tensor is conservative then the integral curves of the characteristic vector field in $(M^n, g)$ are geodesic.

**F-3: On Curvature inheriting symmetry in Finsler spaces.** *Chayan Kumar Mishra, Department of Mathematics and Statistics, Dr. Ram Manohar Lohia Awadh University, Faizabad–224 001,(UP).*

K. L. Duggal has studied Curvature inheritance symmetry in Riemannian space with application of fluid space time. (Ricci curvature inheriting symmetry of semi-Riemannian manifolds were introduced by K. L. Duggal and R. Sharma ). In this paper we study Curvature inheritance symmetry and Ricci-Inheriting symmetry in Finsler spaces and investigated some results.

**F-4: On the fractional calculus operator associated with generalized heat polynomial $p_{n,v}(x, t)$.** *Lal Sahab Singh, Department of Mathematics and Statistics, Dr. Ram Manohar LohiaAwadh University, Faizabad–224 001,(UP).*

The theme of the present paper is to establish a fractional calculus formula for the product of - function due to Inayat Hussain [1987] and in general class of Heat Polynomial $P_{n,v}(x, t)$ due to Singh & Singh [1989] based upon generalized fractional integration and differentiation operator's of arbitrary complex order involving Apple function $F_3$ due to Saigo and Meada (1996). The results are obtained in compact form containing the Riemann-Liuville, Evedlyi-Kober and Saigo operators of fractional calculus.

## G: Topology

### G-1: Generalized minimal continuous maps in topological spaces.

*S. S. Benchalli and S. N. Banasode, Karnatak University, Dharwad–580 003.*

In this paper a new class of generalized minimal continuous maps in topological spaces are introduced and their related theorems have been proved. A mapping $f : (X, \tau) \to (Y, \sigma)$is said to be generalized minimal continuous (briefly $-m_i$ continuous )map if the inverse image of every minimal closed set in $Y$ is a $g - m_i$ closed set in $X$. Also, as an analogy of $gc-$ irresolute maps, generalized minimal irresolute are introduced and characterized in topological spaces.

### G-2: On fuzzy irresolute maps.

*J. K. Maitra, M. Shrivastava and M. Shukla, E-mail: jkmrdvv@rediffmail.com*

In this paper we have studied fuzzy irresolute maps and have characterized several conditions for a map $f : X \to Y$, where $X$ and $Y$ are fuzzy topological spaces to be fuzzy irresolute map. Also we have established some significant properties of fuzzy irresolute maps.

### G-3: Topology induced by relations.

*V. Renukadevi, E-mail: renu_siva@yahoo.com*

We generalize a characterization of continuous functions between closure spaces. Also, we will study the weaker separation axioms defined via relational concepts.

### G-4: Weakly compatible maps and fixed point in uniform spaces.

*A. K. Goyal and Neha Jain, Department of Mathematics, M. S. J. Govt. College, Bharatpur–321 001,(Rajasthan), India.*

Some result on common fixed point by taking a control function /c for weakly compatible maps defined on a sequential complete Hausdorff uniform space have been obtained. Our result generalized several known corresponding results in this direction.

### G-5: On fuzzy minimal open and fuzzy maximal open sets in

**fuzzy topological spaces.** *S. S. Benchalli and Basavaraj M. Ittanagi,* Email: *dr.basavarajsit@gmail.com*

In this paper a new class of sets called fuzzy minimal open sets and fuzzy maximal open sets in fuzzy topological spaces are introduced and studied. A nonzero fuzzy open set $A(\neq 1)$ of a fuzzy topological space $X$ is said to be fuzzy minimal open (resp. fuzzy maximal open) set if any fuzzy open set which is contained (resp. contains) in $A$ is either 0 or $A$ itself (resp. either 1 or $A$ itself). Some properties of the new concepts have been studied.

**G-6: On $(SP, P)$ regular spaces.** *Govindappa Navalagi and Girish Sajjanshettar,* Email: *sajjangm@yahoo.com*

The aim of the paper is to introduce and study weaker forms of regularity, in topology viz $(SP, P)$ - regular spaces and some basic properties of these spaces using semipreclosed and preopen sets.

**G-7: Projection maps on product of smooth fuzzy topological spaces.** *R. Roopkumar and C. Kalaivani,* Email: *kalai_posa@yahoo.co.in*

In this paper, we define projection maps on product of fuzzy subsets as fuzzy proper functions. We prove that this projection maps are smooth fuzzy continuous, weakly smooth fuzzy continuous and $\alpha$-weakly smooth fuzzy continuous. We also prove the following results.
(1) If $F : A \to \Pi B_j$, then $F = [P_k o F]$, where $P_k : \Pi B_j \to B_k$ be the $k$ th projection map for every $k \in J$,
(2) Let $F : (A, \sigma) \to (\Pi A_j, \tau)$ be proper function such that $F = [Fj]$, where $F_j : (A, \sigma) \to (A_j, \tau_j)$, for every $j \in J$. If $F$ is smooth fuzzy continuous on $A$, then $F_j$ is smooth fuzzy continuous for every $j \in J$ on $A$. But the converse is not true.
(3) If $F$ is weakly smooth fuzzy continuous on $A$, then $F_j$ is weakly smooth fuzzy continuous for every $j \in J$ on $A$. But the converse is not true.

Finally, we state some sufficient conditions under which the converses of the statements (2), (3) are true.

**G-8: A study of some local properties in fuzzy nearness spaces.** *Suprabha D. Kulkarni, and S. B. Nimse, E-mail: sup_red_shri@yahoo.co.in*

In this paper we have studied some local properties of fuzzy nearness spaces. We have shown the equivalence of fuzzy completeness and fuzzy local completeness of a fuzzy nearness space. We also have shown that a fuzzy regular chained T1 FN-space is compact iff it is fuzzy uniformly locally compact and fuzzy uniformly chained.

**G-9: Some allied open functions in topology.** *G. Navalagi and S. V. Gurushantanavar, E-mail: gurushnatanavar@yahoo.co.in*

In 1986, D. Andrijevic had defined and studied the concepts of semipreopen sets and semipreclosed sets in topology. In 2002, Navalagi has defined and studied classes of various continuous functions, open functions and closed functions in between topological spaces using semipreopen sets and semiproclosed sets. In this paper, we define and study new classes of functions called s-semipreopen functions and contra s-semipreopen functions using semipreopen sets, semiopen sets and semiclosed sets. Also, we characterize their basic properties.

**G-10: On fuzzy rw-closed sets and fuzzy rw-open sets in fuzzy topological spaces.** *R. S. Wali, E-mail: rswali@rediffmail.com*

In this paper, a new class of sets called fuzzy rw-closed sets and fuzzy rw-open sets in fuzzy topological spaces are introduced and studied. A fuzzy set $\alpha$ of a fuzzy topological space $X$ is said to be fuzzy regular w-closed (briefly, fuzzy rw-closed) if $cl(\alpha) \leq \sigma$ whenever $\alpha \leq \sigma$ and $\sigma$ is fuzzy regular semiopen in fuzzy topological space $X$. The complement of fuzzy rw-closed set is called a fuzzy regular w-open (briefly, fuzzy rw-open) set in fuzzy topological spaces X. Some properties of new concept have been studied.

**G-11: Pre-open sets.** *K. Thanalakshmi, and Renukadevi,* Email: *thanalakshmi_kasi@yahoo.co.in, renu_siva2003@yahoo.com*

In this paper we discuss the properties of $\delta$-preopen sets and the relations of $\delta$-preclosure and $\delta$-preinterior operators with the closure and interior operators of some well known generalized topologies.

**G-12: Related fixed point theorem on two complete metric spaces.** *Zaheer K. Ansari, Manish Sharma and Arun Garg, E-mail: msharma0905@yahoo.co.in*

S. Banach in 1922, established his famous fixed point theorem, popularly known as Banach's contraction theorem. Since then in next eight decades there are a large number of research papers appeared which concerned the fixed point theorems of mappings defined from some space X to itself under various contractive conditions.

The concept of related fixed point by taking two different complete metric spaces $(X, d)$ and $(Y, \rho)$ and two mappings $T : (X, d) \to (Y, \rho), S : (Y, \rho) \to (X, d)$ are established by Brian Fisher in 1981 and he proved that under certain contractive conditions the composite mapping $ST$ has a unique fixed point in

$X$ and $TS$ has a unique fixed point in $Y$. A related fixed point theorem on two complete metric spaces is obtained which generalizes the result of Brian Fisher.

**G-13: Some applied applications of $\alpha$-open sets.** *G. B. Navalagi and T. Prabhu Prakash, K. L. E. society's G. H. college, Haveri 581110.*
In 1965, O. Njastad defined and studied the concept of $\alpha$-sets. Later, these are called as $\alpha$-open sets in topology. In 1983, A.S. Mashhour et. al have defined the concepts like $\alpha$-closed sets $\alpha$-closure of a set, $\alpha$continuity, $\alpha$-openness and $\alpha$-closed ness in topology. In this paper, we study the various applications of $\alpha$-open sets, $\alpha$closed sets, $\alpha$ g-open and $\alpha$ g-closed sets by defining various continuous open and closed functions.

## H: Measure Theory, Probability Theory and Stochastic Processes, and Information Theory

(No Papers)

## I: Numerical Analysis, Approximation Theory and Computer Science

**I-1: An M/G/1 queue with impatient customers.** *K. Nagoor Sultan Kani, K. Kalidass and Kasturi Ramanath, E-mail: dassmaths@gmail.com*

This paper studies the steady state behaviour of an $M/G/1$ queue with impatient customers. A customer who finds the server busy may join the queue with probability $\alpha$ or may decide to leave the system with probability $(1-\alpha)$ where $0 \leq \alpha \leq 1$. The steady state system size distribution, queue size distribution, expected waiting time in the system and in the queue are obtained.Some special cases are also discussed.

**I-2: Steady state analysis of an M/G/1 queue with finite capacity and impatient customers.** *N. Radha, K. Kalidass and Kasturi Ramanath, E-mail: dassmaths@gmail.com*

In this paper, we study the $M/G/1$ queue with a finite buffer capacity and impatient customers. We obtain the system size distribution and queue size distribution in the equilibrium state. We obtain expressions for the expected waiting time of a customer in the system and in the queue.

**I-3: An M/G/1 retrial queue with impatient cutomers, a second optional service and different vacation policies.**
*K. Kalidass, Kasturi Ramanath, E-mail: dassmaths@gmail.com*

In this paper, we study a single channel queueing system with Poisson input and two phases of heterogeneous service. A first essential service is provided to all arriving customers. Upon completion of this service a customer either leaves the system with probability $(1-r)$ or opts for a second optional service with probability r. A customer, who finds the server busy, he either leaves the system with probability $(1-\alpha)$ or join an orbit with probability $\alpha$ where $1 \leq \alpha \leq 1$. From the orbit the customer makes repeated attempts(retrials) to obtain service. We assume the inter retrials times are exponential random variables. We also assume that upon completion of a service, the server either remains in the system with probability $\beta_0$ or leaves the system for an ith type of vacation with probability $\beta_i (1 \leq i \leq M)$ where $\sum_{i=0}^{M} \beta_i = 1$. We obtain the expressions for the steady state system size distribution, the steady state orbit size distribution, expected waiting time in the system and in the orbit. We obtain a stochastic decomposition of the system size distribution. We discuss some particular cases.

**I-4: A single server queue with service interruption under Bernoulli schedule and a general vacation time.** *S. Baskar, Rajalakshmi Rajagopal, S. Palaniammal, Department of Mathematics, VMKV Engineering College, Salem, (TN).*

We consider a single server queue with Poisson input, two types of services one without interruption and another with interruption that is after attending interruption the server starts his service with Bernoulli schedule and a general vacation time, here the server provides one of the services to the arriving customers. After completion of service the server may attend interruption service. After that server either goes for a vacation with probability $p$, $(0 \leq p \leq 1)$ or may continue to serve the next unit, if any, with probability $(1-p)$. Otherwise the server remains in the system until a customer arrives. For this model, we obtain the steady state system size distribution.

## J: Operations Research

**J-1: Reinforcement to the security force in a counter insurgency operation.** *Lambodara Sahu, E-mail: lsahucme@gmail.com*

A sharp increase in incidences of insurgency like Naxalites. activity and terrorism in the recent times in both rural and urban areas is becoming a cause of deep concern to most of the defence analysts all over the world. Even security force engaged in a counter insurgency operation faces many major problems like identification of insurgents and their ultimate captive/ destruction because of insurgents. Freedom of movement among the innocent public. Therefore, they encompass the entire spectrum from postulating reforms at various levels to redesigning the existing combat tactics and strategies. In order to evaluate various tactical doctrines certain battle field models like Lanchester.s equations are very essential.

Sometimes in a counter insurgency operation security force is outnumbered to insurgents due to the frequent change in strategies of insurgents. Therefore, reinforcement to the outnumbered security force is significant to minimize casualties and to finish that operation within a stipulated time frame.

In this paper, an attempt is made to figure out the importance of reinforcement at an early stage of combat operation and the strength of reinforcement by considering a few case studies.

### J-2: Nature and assumptions of economic theory.

*Kamini Sinha, A. K. Sinha, M. M. Mahila College, Ara,(Bihar).*

Economic theory, like all theories, may be thought of either as a language or as a set of substantive empirical propositions. As a language, economic theory should be useful, while as a set of substantive propositions, it should contain propositions which are capable of being tested because they attempt to make certain predictions. The definition of a demand curve is theory as a language. However, the statement that the demand curve slopes negatively downward to the right-is theory as substantial empirical propositions which are capable of being tested. It has empirically observable consequences, whereas the definition of demand curve does not. Economic theory as a language coincides with Alfred, Marshall rengine of analysis which helps investigator to construct a language which will furnish or provide maximum possible substantive propositions.

## K: Solid Mechanics, Fluid Mechanics, Geophysics and Relativity

### K-1: Anisotropic bulk viscous cosmological models with particle creation.

*G. P. Singh and A. Y. Kale,* Email: *gpsingh@mth.vnit.ac.in*

This paper is devoted to the study of role of particle creation and bulk viscosity on evolution of homogeneous and isotropic model of the universe represented by Bianchi type I space time metric. Particle creation and bulk viscosity have been considered as separable irreversible process. In order to discuss physical and geometrical behaviour of the model, a new set of exact solutions of Einstein's field equation have been obtained in non-causal, truncated and full causal theories. Dynamical behaviours of models have also been discussed.

**K-2: Tilted bianchi typeI stiff fluid cosmological model.** *S. K. Sahu, E-mail: subrat_sahoo2002@rediffmail.com*

The distribution of heat conduction filled with stiff fluid is investigated in a new scalar tensor theory of gravitation proposed by Saez and Ballester (Phys. Lett. A. 113:467, 1986) when the space-time is described by a tilted Bianchi type - $I$ metric.Exact solutions to the field equations are derived when the metric potentials are functions of cosmic time only. Some physical and geometrical properties of the solutions are also discussed.

**K-3: Propagation of exploding uniform strong cylindrical shock wave in condensed medium (water).** *R. P. Yadav, M. K. Mishra, R. kumar and P. S. Rawat,* Email: *rpyadav93pphysics@yahoo.co.in*

The very well-known theory of the shock dynamics, Chester - Chisnell-Whitham method is applied to study the exploding i.e. diverging wave in condensed medium, the water is assumed to be the simplesed form of condensed medium. The analytical shock velocity and shock strength are obtained for two eases viz, (i) when propogates freely i.e effect of overtaking disturbances is negligible and (ii) when it moves under the influence of overtaking disturbances.

Finally, flow variables of the medium perturbed by strong cylindrical shock are obtained and discussed with the help of figures and tables for both the cases. The results obtained here are compared with those obtained by Sharma et.al.

**K-4: A new class of homogeneous and isotropic cosmological models with bulk viscosity.** *G. P. Singh and A. Y. Kale,* Email: *gpsingh@mth.vnit.ac.in*

The effect of bulk viscosity on the evolution of the homogeneous and isotropic universe represented by FRW space time metric is investigated in the context of open thermodynamic system, which allow particle creation. A new class of physically plausible models has been taken into consideration. The dynamical behaviour of the models has also been discussed.

**K-5: A mathematical model for the flow control at low values of magnetic Prandtl numbers.** *T. V. S. Sekhar, R. Sivakumar, and S. Vimala, E-mail: vimlaks@pec.edu*

Numerical simulations have been performed to investigate the flow of an electrically conducting fluid around a cylinder under the influence of an external magnetic field for low values of the magnetic Prandtl number without using quasi-static approximation. The magnetic field is applied in the direction of the fluid flow. Moreover, the fluid flow and magnetic field are uniform and parallel at large distances from the cylinder. The governing partial differential equations are solved in the verticity stream function form using finite differences. The second order derivatives are approximated by second order accurate central differences and nonlinear convective terms by first order upwind scheme to ensure diagonal dominance. The resulting algebraic system is solved using the multigrid technique with coarse-grid correction to enhance the convergence rate. Defect correction technique is then employed to achieve second order accuracy. The results are obtained from high grid resolution of 512 x 512 for Reynolds numbers $Re = 10, 20$ and 40. It is found that the adverse pressure gradient is effectively reduced and hence separation is fully suppressed. Also it is observed that for low values of magnetic field, the viscous,pressure and total drag coefficients are lower than that of the flow with zero magnetic field. On the other hand, at moderate and higher magnetic fields, all these drag coefficients increase with increase of magnetic field. These results are in agreement with experimental findings.



**K-6: On Toyoki Koga's solution of the dirac equation and the anisotropy of the electron field: spacetime algebra approach.**

*S. K. Pandey, Department of Mathematics, Cochin University of Science and Technology, Cochin–682 022, Kerala, India.*

We review Toyoki Koga's solution of the Dirac equation and study a particular case by employing the techniques of spacetime algebra .The mathematical ambiguity in Koga's approach is removed and it is shown that there exists a solution to the Dirac-Hestenes equation that describes an anisotropic electron field.

**K-7: Effect of irregularity on the propagation of SH Waves in monoclinic-magnetoelastic layer.** *A. Chattopadhyay, S. Gupta, S. A. Sahu and A. K. Singh, E-mail: abhi.5700@gmail.com*

The present paper discusses the study of propagation of magnetoelastic SH waves in an internal monoclinic layer, with rectangular irregularity in lower interface, sandwiched between two isotropic half-spaces. The dispersion relation has been obtained. The dispersion relation is in assertion with the standard SH wave equation, for isotropic layer lying in between two isotropic half spaces, in the absence of magnetic field and irregularity. This study shows the remarkable effect of wave number, size of irregularity and magnetic field on the phase velocity.

**K-8: Solution by group invariant method of instability phenomenon with powerlaw nonlinearity.** *M. N. Mehta, N. B. Desai, M. S. Joshi, E-mail: mitesh_josho2003@yahoo.com*

The present paper analytically discusses the phenomena of flow of two immiscible liquids through homogeneous porous media with capillary mean pressure by employing a Group theoretic analysis method. The problem has great importance in petroleum technology. The underlying basic assumption made in the present analysis is that the individual pressure of the two flowing phases may be replaced by their common mean pressure. The governing equation of instability phenomenon is obtained in the form of partial differential equation with power-law nonlinearity. This is solved by Group invariant method. Group invariant method is generalization of similarity transformation method. Numerical calculation and its graphical representation is obtained by using Mat lab coding.

**K-9: Effect of laser radiation and non-uniform electric field on rayleigh- taylor instability at the interface in a finite thickness electrorheological couple stress poorly conducting fluid.**
*G. Chandrashekara and N. Rudraiah*, Email: *chandrashekarg.4@gmail.com, rudraiahn@hotmail.com*

In these days there is an urgent need of power supply from unconventional sources of energy which is economically affordable and environmental friendly. The United Nations, through International Atomic Energy Agency (IAEA), is proposing to generate adequate electric power from Inertial Fusion Energy (IFE) by fusing the Deuterium - Tritium (DT), the two hydrogen isotopes in Inertial Fusion Target (IFT), which is also one of the environmental friendly and unconventional methods of generating electric power. The nuclear picture of DT is such that there will be a strong repulsive force between them which prevents natural fusion of DT. Therefore, there is a need of external agency like laser radiation to fuse them. Therefore high intensity laser radiation will be impinged at the ablative surface of IFE target to fuse DT. This radiation no doubt fuses DT but causes asymmetry in IFT which is one of the reasons to reduce the efficiency of extraction of IFE. To reduce this asymmetry for efficient extraction of IFE, there should be mechanisms to control surface instability of the type Rayleigh Taylor Instability (RTI). Most of the available works in the literature in reducing the growth rate of RTI are treated DT has Newtonian fluid. The DT in IFT is precisely not Newtonian, because it has the characteristic of complex fluids and hence following Eringen (1966), we treated it has special case micropolar fluid which deforms and produces a spin which in turn setsup an antisymetric stress known as coupe stress forming couple stress fluid. This DT which is suspended in IFT with the couple stress and in the presence of electric field can be regarded as Electrorheological fluid (ERF).

In this paper we study the RTI at the interface between dense electrorheological fluid (i.e. couple stress poorly conducting fluid) accelerated by a lighter ERF in the presence of an electric field and laser radiation. A simple theory based on fully developed approximations is used to derive the growth rate of RTI. The cutoff and maximum wave numbers and corresponding maximum frequencies are obtained. It is shown that the effect of couple stress parameter and the electric field reduces the growth rate considerably compared to non-conducting fluid in the absence of electric field. We conclude that the combined effect of the couple stress and the electric field are favorable to control the asymmetry of the IFT.

**K-10: Mathematical model for stability of electrorheological couple stress poorly conducting viscous parallel fluid flow in a channel.** *B. M. Shankar and N. Rudraih,* Email: *rudraiahn@hotmail.com*

The linear stability of electrorheological poorly conducting couple stress viscous parallel fluid flow in a channel in the presence of a non-uniform electric field generated by embedded electrodes of different potentials at the boundaries is studied using the mathematical model of liapounov technique supplemented with Schwarz's inequality. This study has potential applications in the effective design of devices in electronics, electrical, mechanical and biomedical engineering. Some of these devices involving fluids contain organic, inorganic and microsubstances dissolving in fluids make that fluid poorly conducting having electrical conductivity several orders of magnitude smaller than that in naturally conducting fluid. Experiments conducted in a poorly conducting fluids show that their conductivity increases with temperature and with concentration of freely suspended particles making their conductivity non-uniform. These freely suspended particles spin producing microrotation forming micropolar fluids whose theory was developed by Eringen (1996). A particular case of micropolar fluid theory when micro rotation balances the natural vorticity of a poorly conducting fluid in the presence of non-uniform electric field is called Electrorhelogical fluid (ERF). In the present paper the effects of non- uniform electric field and electrical conductivity and the couple stress on the stability of ERF is studied. An estimate of the eigen value, ci, of the solution of stability equation is obtained. From this a sufficient condition for stability is obtained which shows that strengthening or weakening of the stability criterion is dictated by competition between electric field and couple stress poorly conducting fluid mechanisms as well as the effect of the ratio of couple stress parameter to viscosity of the fluid. It is shown that these effects enable an accurate dictation of stability of poorly conducting couple stress fluid and unambiguous steady of interaction of electric field with couple stress.

**K-11: Longitudinal dispersion of tracer partials in chiral fluid in channel bonded by rigid boundaries.** *S. V. Raghunatha Reddy and N. Rudraiah,* Email: *rudraiahn@hotmail.com*

To our knowledge not much work has been done on chiral fluid like turpentine, sugar solution, benzene and body fluids. In spite of their importance in chemical engineering and biomedical engineering, particularly in the design of artificial organs like synovial joints, coronary artery diseases. Proper functioning of these organs depends on the dispersion of synovial fluid in synovial joints and the dispersion of physiological fluids in arteries. It is known that the results obtained on dispersion mechanism using Newtonian fluids in the artificial organs are not proper. In the artificial organs the fluids behave as chiral fluids and dispersion in chiral fluid in artificial organs has been studied in this paper using Aris dispersion model which is an improvement over Taylor's dispersion model.

The present paper seeks an analytical solution of dispersion of tracers in chiral fluid between two rigid plates. It is shown that the tracers will disperse relative to a plane moving with mean speed of flow with an effective dispersion coefficient which decreases with decreasing in value of the chiral parameter.

**K-12: The dispersion of atmospheric aerosols in the presence of non-uniform electric field.** *Devaraju N. and N. Rudraiah,* Email: *rudraiahn@hotmail.com*

In the present paper, we investigate the uncertainties of increase, stay uniform or decrease of aerosols in the atmosphere using the phenomena of dispersion of aerosols in the atmosphere in the presence of an electric field. Aerosols are the dust, organic and inorganic substances particles fixed in the atmosphere. These organic and inorganic substances will have poor electrical conductivity $\sigma$ which increases with an increase in the concentration of these fixed particles. The difference in concentration produces the variation of $\sigma$ which in turn releases the charges. These charges produce an electric field known as concentration electric field. This electric field not only produces a current, but also produces a force. This current acts as sensing and the force acts as actuation. These are the two important properties, which makes atmospheric fluid as a smart material. The effect of electric field on the dispersion of aerosols is studied regarding the mixture of atmospheric fluid and aerosols as fluid-saturated porous media using Taylor's (1953) model. Analytical solutions for velocities and concentration distributions are obtained using a regular perturbation technique. It is shown that the aerosols are dispersed relative to a plane moving with the mean speed of atmospheric fluid as well as the mean speed of agglomeration of aerosol with a relative diffusion coefficient,$D_r^\beta$ called Taylor dispersion coefficient. Here $\beta = f$ for atmospheric fluid and$\beta = s$ for aerosols. This $D_r^\beta$ is numerically computed for $\beta = f$ and also for $\beta = s$,and the results reveal that $D_r^\beta$ increases with an increase in Electric number (ratio of electric energy to kinetic energy), but decreases with an increase in porous parameter $\sigma$. A physical explanation for decrease in $D_r^\beta$with an increase in $\sigma$ and We is given.

**K-13: Positive solutions for a nonlocal boundary value problem involving a pair of second order dynamic equations with a singular interface.** *P. K. Baruah and D. K. K. Vamsi,*
*E-mail: baruahpk@gmail.com*

In this paper we consider a nonlocal boundary value problem involving a pair of dynamic equations satisfying matching interface conditions at the singular interface. Using Krasnoselskii's fixed point theorem results on existence of positive solutions for this nonlocal boundary value problem are proved.

**K-14: Effect of irregularity on the propagation of torsional surface waves in an initially stressed anisotropic poro-elastic layer.**
*S. Gupta, A. Chattopadhyay, D. K. Majhi,* Email: *shishir_ism@yahoo.com*

The present paper has been framed to study the torsional surface wave propagation in an initially stressed anisotropic poro-elastic layer over a semi-infinite heterogeneous half space with linearly varying rigidity and density due to irregularity at the interface. The irregularity has been taken in the half-space in the form of a parabola. It is observed that torsional surface waves propagate in this assumed medium. In the absence of irregularity the velocity of torsional surface wave has been obtained. Further, it has been seen that for a layer over a homogeneous half space, the velocity of torsional surface waves coincides with that of Love waves.

**K-15: Aerodynamic design by shape optimization.** *Indira Narayanaswamy, Aeronautical Development Agency, ADA, P. B. No. 1718, Vimanpura Post, Bangalore–560 017.*

An aerodynamic design using shape optimization technique is presented which produces the desired airfoil with minimum drag at specified free stream condition, subject to geometric and aerodynamic constraints. The design procedure is based on gradient based optimization using Method of Feasible Directions (MFD) for constraints handling, coupled with a Full Potential Solver, with corrections for skin friction drag. The method is demonstrated using RAE 2822 airfoil geometry as baseline geometry, wherein the objective is drag minimization, subject to constraints on lift, pitching moment, leading edge radius and thickness to chord ratio. Also, an inverse design problem is being studied where an airfoil having specified target pressure distribution is generated using Numerical optimization procedure, subject to design constraints. The gradient based optimizer is used to minimize the pressure discrepancies between the target and baseline airfoils. The design procedure converges very quickly making the method very attractive for practical aerospace applications.

**K-16: A method for finding the invariants of non-conservative dynamical system via canonical transformation.** *Ashwini N. Sakalkar, Department of Applied Science, R. C. Patel Institute of Technology, Shirpur, Dist. Dhule, (Maharashtra).*

This study is concerned with the derivation of conservation laws of non-conservative dynamical systems with finite numbers of degrees of freedom. A family of canonical transformations is generated that preserves the form of Hamilton's equations of motion for some Hamiltonian like function. This function consists of the Hamiltonian function in the transformed co-ordinates and an additional function dependent on the non-conservative force.

Noether's theorem is applied to the transformed system to determine invariant of the system in the original variables. These invariants are not derivable by applying Noether's theorem to the system describes in the old co-ordinates. The result is applied to generate constants of motion for generic example of dissipative gyroscopic non-conservative dynamical system.

## L: Electromagnetic Theory, Magneto-Hydrodynamics Astronomy And Astrophysics

(No Papers)

## M: Bio-mathematics

**M-1: Spherically symmetric cosmic strings and domain walls**

**in bimetric relativity.** *A. K. Ronghe and S. D. Deo, Department of Mathematics, S. S. L. Jain P. G. College, Vidisha, (MP)–464 001.*

Spherically symmetric Kantowski - Sachs space - time is studied in the context of Rosen's (1973) bimetric relativity with the source of matter is cosmic strings and domain walls respectively. It is observed that in this theory Spherically symmetric Kantowski - Sachs space - time does not accommodate cosmic strings as well as domain walls. Hence, further we obtained the vacuum solutions and resulting space - time represents the Robertson - Walker flat metric.

**M-2: Non-obligate and obligate models of mutualisms in multispecies community:a mathematical approach.** *B. Rai and M. Singh, University of Allahabad, Allahabad, (UP), India.*

One of the most important questions from ecological point of view concerns the conditions under which long term survival of all the species is assured, and it is our aim to show here that not with standing the complex dynamics which may occur, it is often possible to give rather simple and complete answer to the question for an important class of models (mutualistic models). We propose and analyseo a mathematical model governed by a system of four autonomous differential equations in view of

1. the number of species that must interact to maintain the mutualism,
2. the degree of obligateness, and
3. the cost of Mutualism, with special reference to persistence.
4. $\dot{u} = uh(u, x, y_1, y_2)$
   $\dot{y} = axg(u, x) - y_1, p_1(u, x)-, y_2 p_2(u, x)$
   $\dot{y}_1 = y_1[-s_1(u, y_1) - q_1(z) + c_1 p_1(u, x)]$
   $\dot{y}_2 = [-s_2(u, y_2) - q_2(y) + c_2 p_2(u, x)]$

**M-3: Synchronization of chaos in a food web in ecological systems.** *Anuraj Singh, Dept. of Maths., IIT Roorkee, Roorkee–247 667.*

Synchronization is an important phenomenon commonly observed in nature. It is also often artificially induced because it is desirable for a variety of applications in physics, biology and applied sciences as well as engineering. The chaos synchronization of a three-species food web model is investigated theoretically and numerically. By using suitable coupling, chaos synchronization of the model system is studied. The sufficient condition of achieving synchronization is derived by using Lyapunov stability theory. Numerical simulations are presented to demonstrate the effectiveness and feasibility of the analytical results. Chaos can be controlled to achieve stable synchronization in food webs.

**M-4: A mathematical study of incubation delay on the dynamics of an eco-epidemiological model.** *S. Chaudhury, Krishna Pada Das, J. Chattopadhyay,* Email: *krishna r@isical.ac.in, joydev@isical.ac.in*

The effect of time delay on eco-epidemiological model is an important issue from the mathematical and biological point of view. In this paper we have modified a system studied by Mandal et al.,2009 (Mandal, A.K., Kundu, K., Chatterjee, P., Chattopadhy, J.,2009. An eco-epedemiological study with parasite attack and alternative prey, Journal of Biological Systems, 17, 269-282) by incorporating a incubation delay. It is observed that under certain parametric condition the system remains stable around the interior equilibrium for all delay and if this condition does not hold there exists a critical value of $\tau$ below which system is stable and above which system enters into $Hopf$ bifurcation. Our numerical results show that delay has important role in eliminating infected prey and saving predator population from extinction. In lower infection rate, it is observed that stable system becomes unstable through eradication of infected prey for increasing $\tau$. But for a higher infection rate, there exists a threshold value of the delay above which the system is unstable and below which the system is stable leading to the persistence of all species. Incubation delay induced oscillations may be controlled if alternative food is increased. It is also observed that alternative food has important role in controlling infected prey in presence of incubation delay.

**M-5: A model of disease outbreak with immunity and delayed incubation.** *G. P. Sahu, Joydip Dhar, ABV-IIIT and M, Gwalier.*

In this paper, a mathematical model is proposed with three classes of population, namely, susceptible, incubated and infected. The incubation period is defined as the time from exposure to onset of disease, and when limited to infectious disease, corresponds to the time from infection with a microorganism to symptom development. Immunity (natural or caused by vaccination) plays an important role in recovery of a disease, due to strong immunity a portion of incubated class rejoins susceptible class without being infected. In many diseases population stay for fixed time in incubated class before joining diseased class or rejoining susceptible class. The stability behaviour of the trivial, disease free and endemic steady states are studied, it is found that the instability of disease free state leads to the existence of the endemic state. The possibility of Hopf-bifurcation of the endemic equilibrium is studied, considering the transfer rate from susceptible to incubated population as bifurcation parameter. Finally, threshold values of bifurcation parameter are determined numerically for a particular set of values for both with and without delay in incubation class.

**M-6: Mathematical study of the role of delay on a plankton ecosystemin the presence of a toxic producing phytoplankton.**
*Swati Khare, O. P. Misra, Joydip Dhar, Malwa Institute of Technology and Management, Gwaliar, (MP).*

A mathematical model is proposed to study the role of delay on a plankton ecosystem in the presence of a toxic producing phytoplankton. The model includes three state variables viz., nutrient concentration, phytoplankton biomass and zooplankton biomass. Again release of toxic substance by phytoplankton species reduces the growth of zooplankton and this plays an important role in plankton dynamics. In this paper, at the first stage we introduced a delay (time-lag) in the digestion of nutrient by phytoplankton. In the second stage, we introduce time-lag in the digestion of phytoplankton by zooplankton. The model at both these stages is studied by conducting the stability analysis of all the feasible equilibria and also the bifurcation analysis of the interior equilibrium of the systems has been carried out. Again axial equilibrium of both the systems are shown to be stable, if the growth rate of phytoplankton biomass due to the availability of nutrient is less than the fraction of natural death rate of phytoplankton biomass to the constant input rate of nutrient. Further, the boundary equilibrium and interior equilibrium are proved to be locally asymptotically stable under certain conditions.Hence, we conclude that the supply rate of nutrient and delay parameter play an important role in shaping the dynamics of the two systems.

**M-7: Spreading of carrier dependent infectious diseases: a mathematical model.** *A. K. Misra, Department of Mathematics, Faculty of Science, BHU, Varanasi–221 005, (UP).*

A non-linear mathematical model for spread of carrier dependent infectious diseases is proposed and analyzed. It is assumed that the disease spreads by direct contact between susceptible and infective as well as due to the presence of carriers. It is further assumed that carriers follow logistic growth in the environment however they also grow due to the human related activities, which makes the environment conducive for their growth. Equilibria of the model have been obtained and their stability have been discussed.

## N: History And Teaching Of Mathematics.

**N-1: A brief study of contributions of NASIR AL-DIN-AL-TUSI to mathematics and astronomy.** *Amit Kumar and Ashwini Kumar Sinha, Department of Mathematics, R. P. S. College, Chakeyaj, Mahnor, BBA Bihar University, Muzaffarpur,(Bihar).*

In this paper we have attempted to take a view of unforgeable contribution of the all time great astronomer, mathematician and philosopher NASIR AL-DIN-AL-TUSI(1201-1274)

**N-2: Global mathematics- amendments in fundamental mathematics.** *S. K. Mahajan, Angel Valley School, Opp. 32 Bangalows, Hospital Sector, Bhilai, (CG).* A discussion on global mathematics is carried out through reconsideration and rethinking with 2-elements triangle theory applied to various mathematical and engineering applications. Some of these are tested and illustrated in the book on Global Mathematics. It is that introduction of global mathematics marks the beginning of new era in carrying out researches in mathematics, basic science, engineering and entrepreneurial development.

## Abstracts of the Papers for the IMS Prizes

**IMS-1: Fluid flow in narrowing systems.** *Amit Patel,*
*E-mail: ptl_amit@yahoo.com*

Narrowing of pipeline network is an important aspect in drinking water distribution systems, Sewage system and in oil-well techniques. The deposition causing narrowing has been replaced by using sinusoidal model with axial velocity. In the present paper, an attempt is made to obtain velocity of fluid in pipeline network during the deposition causing narrowing system by applying Integral transforms, (Finite Hankel and Laplace Transforms) method. MAPLE 11.02 has also been used for plotting the graphs.

**IMS-2: Analytic solutions for contaminant transport in surface-subsurface flows.** *Senthil G., E-mail: senthilg1@gmail.com*

A general analytical model based on a two-dimensional advection-diffusion equation with the exchange between three zones - open channel (fast zone) and saturated porous beds (slow zones) is proposed and each of the three zones is considered to be well mixed. The model is based on the method of moments.

The exchange between the three domains is modelled through boundary conditions that ensure the continuity of concentration and flux at the interface. Flow and transport in the open channel and slow zones are diffusion and advection driven. An exponentially velocity profile of the slow zone links the parabolic velocity profile of the open channel at the top and with an exponentially velocity profile of the zone at the bottom. The analytical expressions for the zeroth, first and second moments of concentration are derived using the Aris' method of moments. After obtaining the moments of concentrations, the exact expressions for the mean particle velocity and effective dispersion coefficient are derived. The results suggest that the mean particle velocity increases with an increase in friction velocity and it is sensitive towards the ratio of the depths of two of the slow zones. It is evident from the results that the effective dispersion coefficient decreases with increase in relative diffusivity. Further, for a fixed relative diffusivity, the effective diffusion coefficient increases with increase in the friction velocity.

**IMS-3: Viscoplastic squeeze films between rectangular slippary plates.** *K. P. Vishwanath, Email: shastry_vishwanath@yahoo.co.in*

Mathematical modeling of squeeze ow problems using non-Newtonian fluids has been a fundamental area of interest to many researchers. In the class of non-Newtonian fluids, there has been an increasing interest in using time-independent fluids with yield stress as lubricants. Some of them which fit into this class of fluids are Bingham, Casson and Herschel-Bulkley models. The present work deals with in- vestigation of a rectangular squeeze film bearing using Bingham fluid as a lubricant including the effects of surface slip. The thickness of the plug core formed between the two plates has been determined numerically for various values of the Bingham number and the non-dimensional slip coefficient of the surface. The flow is confined to the region between the core and the rectangular plates. Numerical solutions have been obtained for the bearing performances such as pressure distribution and load capacity for different values of Bingham number and slip coefficient. The effects of non-dimensional slip coefficient on the load capacity of the bearing are investigated through the non-Newtonian effects of the fluid.

**IMS-4: Connectivity of the generalised mycielskians.** *S. Francis Raj, E-mail: francisraj_s@yahoo.com*

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph $G$

into a new graph $\mu(G)$, which is called the Mycielskian of $G$. A generalisation of this transformation is the generalised Mycielskian $\mu m(G)$. This paper investigates the vertex-connectivity and edge-connectivity of $\mu m(G)$. We show that the vertex-connectivity $\kappa(\mu m(G)) = min\{\delta(\mu m(G)), m\kappa(G)+1\}$ and that the edge-connectivity of the generalised Mycielskian is same as the minimum degree of it.

**IMS-5: Sums and partial sums of generalized factorial.** *D. S. Dilip,* Email: *mr.dilipmaths@rediffmail.com*

In this paper, author obtain some results on generalized polynomial factorial using generalized difference operator of first kind $\triangle_l$ and second kind $\triangle_{l,l}$, for the positive real $l$. Also we derive the formulae for the sums and partial sums of generalized polynomial factorial in number theory using inverse operators. Suitable examples are provided to illustrate the main results.

**IMS-6: Iwasawa $\lambda$ invariants and $\Gamma$ transforms of $\rho$ adic measures.** *Rupam Barman, E-mail: rupamb@tezu.ernet.in*

In this paper we determine a relation between the $\lambda$ invariants of a $\rho$-adic measure on $\mathbb{Z}_p^n$ and its $\Gamma$-transform. Along the way we also determine $\rho$-adic properties of certain Mahler coefficients.Finally, we relate the coefficients of a $\rho$-adic measure $\alpha$ on $\mathbb{Z}_p^2$ to the $\lambda$-invariant of the Iwasawa series of the $\Gamma$ transform of $\alpha$

**IMS-7: Theory of generalized difference operator of the $n$th kind and its applications in number theory.** *V. Chandrasekar, P. G. and Research Dept. of Maths., Sacred Heart College, Tirupattur–635 001.*

In this paper, the authors extend the theory of the generalised difference operator $\Delta_l$ to the Generalized difference operator of the $n$th kind $\Delta_{l_1,l_2,\ldots,l_n}$. We also present the discrete version of Leibnite Theorem, Binomial Theorem, Newton's formula with reference to $\Delta_{l_1,l_2\ldots,l_n}$. Also by defining its inverse to establish a formulae for the sum of general partial sums of the higher power of arithmetic progression in number theory.

**IMS-8: On the nets of bicomplex numbers.** *S. Singh, E-mail: ssukhdev1209@yahoo.com*

This paper is a sequel to our earlier work on order topologies on the bicomplex space. In this paper, we have discussed the forms of various basis elements which generate the order topologies on the bicomplex space. We have also defined nets of the bicomplex numbers, called as bicomplex nets. Finally, we have explored condition under which a bicomplex net can be eventually in different types of basis elements.

**IMS-9: Min-max and max-min graph saturation parameters.**
*S. Sudha,* Email: *sudha_ akce@yahoo.co.in*

Let $P$ be a graph theoretic property concerning subsets of the vertex set $V$, which is either hereditary or super hereditary. We associate with this property $P$ five graph theoretic parameters namely a minimum parameter, a max-min parameter, a min-max parameter, a maximum parameter and a partition parameter. We initiate a study of this parameters for the properties of independence, domination and irredundance.

**IMS-10: On the edge partition of a line graph.** *Pravas K., E-mail: pravask@cusat.ac.in*

Gallai and Anti-Gallai graphs of a graph $G$ are spanning subgraphs of the Line graph of $G$. Here we introduce an algorithm to partition the edge set of the Line graph of a graph $G$ into the Gallai and Anti-Gallai graph of $G$.

**IMS-11: Modeling of consciousness: a model based approach.**
*K. Reji Kumar,* Email: *rkkextra@yahoo.co.in*

In this paper we discuss the basic properties of mathematical models and explain the difference between mathematical models and ordinary models. In an attempt to model human consciousness, it has been realized that the fundamental units of consciousness are models. An axiomatic definition of consciousness is presented and an algebra for models is developed.

**IMS-12: The 2-dissection representation of Ramanujan's continued fraction.** *Roselin,* Email: *roselimaths@gmail.com*

Continued fractions can be represented in terms of power series. Ramanujan has listed some continued fractions and he represented them in terms of power series. Ramanujan has given dissections for some continued fractions. The m-dissections of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the presentation of $P$ as $P = P_0 + P_1 + \ldots + P_{m-1}$ where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$. In this paper, we give the 2-dissection of Ramanujan's continued fraction.

**IMS-13: On dominator colorings in graphs.** *K. Raja Chandrasekar, E-mail: rajamath84@yahoo.com*

A dominator coloring of a graph $G$ is a proper coloring of $G$ in which every vertex dominates every vertex of at least one color class. The minimum number of colors required for a dominator coloring of $G$ is called the dominator chromatic number of $G$ and is denoted by $\chi_d(G)$. In this paper we obtain several results on this coloring parameter.

**IMS-14: Degree equitable chain of a graph.** *A. Anitha,* Email: *anithavalli@tce.edu*

Let $G = (V, E)$ be a graph. A subset $S$ of $V$ is called a degree equitable set if $|degu - degv| \leq$ for all $u, v \in S$. The set $S$ is called an independent degree equitable set if $S$ is both independent and degree equitable. In this paper we introduce two inequality chains, each consisting of six parameters, called degree equitable chain and independent degree equitable chain. We also prove that these two chains can be realized as standard domination chains of suitable graphs.

**IMS-15: Convexity graphs.** *Sithara Jerry, Kalaslingam University, Anand Nagar, Krishnankoil–626 190, (TN), India.*

Let $F$ denote the set of all minimal dominating functions of a graph $G$. The equivalence relation $\rho$ defined on $F$ by $f \rho g$ if $f$ and $g$ have the same positive set and same boundary set, gives a partition of $F$ into a finite number of equivalence classes $X_l, X_2, \cdots X_t$. The convexity graph $C(G)$ of $G$ is defined as follows: $V(C(G)) = \{X_1, X_2, \cdots, X_t\}$ and $X_i$ is adjacent to $X_j$ if there exist $f_i \in X_i$ and $f_i \in X_j$ such that any convex combination of $f_i$ and $f_j$ is a minimal dominating function. In this paper we determine the convexity graphs of some standard graphs. We also prove that the convexity graph of any connected graph contains $K_3$.

## Abstracts of the Papers for the AMU Prize

(No Papers)

## Abstracts of the Papers for the V. M. Shah Prize.

### VMS-1: Certain results on bicomplex Dirichlet series.

*Jogendra Kumar, E-mail: jogendra_j@rediffmail.com*

This paper deals with the Bicomplex analogue of the classical Dirichlet series. A uniqueness theorem for the representation of Bicomplex Dirichlet series has been obtained. Region of Convergence and Region of absolute convergence of the Bicomplex Dirichlet series have been obtained and conditions for their equality have been evolved. The singularities of Bicomplex Dirichlet series have been obtained. Integral representation of a specific Bicomplex Dirichlet series has also been obtained.

### VMS-2: On the value distribution of differential polynomials.

*Mrs. R. S. Dyavanal, Dept. of Maths., Karnatak University, Dharwad 580003.*

In this paper,we deal with the value distribution of differential polynomials generated by a transcendental meromorphic function and thus obtain two inequalities that are significant improvements and generalizations of some earlier results. These inequalities are useful in value sharing of $f$ and its differential polynomials. As a very particular case we deduce the results of Kit Wing Yu, Lahiri And Shymali Dewan .

### VMS-3: Fractional integration and fractional differentiation of generalized m-series.

*Manoj Sharma, E-mail: manoj240674@yahoo.co.in*

In this paper a new Special function called as Generalized M-series is introduced. This series is a particular case of Fox's H-function and it is a Generalized case of M-series defined by the author. The Generalized M-series is interesting because the ${}_pF_q$-hypergeometric function and the Generalized Mittag-Leffler function follow as its particular cases and these functions have recently found essential applications in solving problems in physics, biology, engineering and applied sciences. Let us note that the Generalized Mittag-Leffler function occurs as solution of fractional integral equations in those area. In this paper we have obtained formulas for the fractional integral and fractional differential of the Generalized M-series.

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# TRANSFORM ANALYSIS AND DISTRIBUTION SPACES

**M. S. CHAUDHARY**

At the outset, I thank the Indian Mathematical Society and the organizers of the 75th annual conference of IMS, held at Kalaslingum University, Anand Nager, Krishnankoil (TN) during December $27-30$, 2009, for inviting me to deliver the $20th$ Ramaswamy Aiyer Memorial Award lecture.

## INTRODUCTION

Integral transforms provide a powerful technique for solving initial and boundary value problems arising in applied mathematics, mathematical physics and engineering. They have been successfully used for last two centuries. Their use is still predominant in advanced study and research. There are many important integral transforms including Fourier, Laplace, Mellin, Hankel, Hilbert, Stieltjes etc. Still new integral transforms are being introduced in mathematical literature.

The integral transform of a function defined on a subset $\Omega$ of $\mathbb{R}$ is denoted by $I(f(x))(y) = F(y)$ and is defined by

$$I(f(x))(y) = F(y) = \int_{\Omega} K(x,y)f(x)dx, \qquad (*)$$

where $K(x, y)$, a function of two variables $x$ and $y$ with $x \in \Omega$ and $y$ is in a suitable set, called the kernel of the transform. The integral in $(*)$ may be Lebesgue or generalized Riemann integral. Naturally for the existence of the integral in $(*)$ we have to impose certain conditions on the function $f$. This is a serious restriction on the integral transformation. For example, in the case of Fourier transform, if the integral is in Lebesgue sense then the Fourier transform of important functions such as sine, cosine functions do not exist. Not only that the function $f$ but $F(y)$ obtained in $(*)$ is also in the restricted class. Arbitrary function $F(y)$ on a

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suitable set of $y$ may not be transform of some function f. For example, the Laplace transform of a function $f(x)$ is analytic but every analytic function need not be Laplace transform of some function. So there is a need to generalize the Laplace transform to a larger class which includes Laplace transformable functions.

Much work has been done in extending large number of integral transforms to the classes of generalized functions using different methods. In the present lecture I will discuss some of the work done by our group.

There are mainly three methods for extending integral transforms to the classes of generalized functions.

**(A) Direct or Kernel method.**

The procedure in this method is to construct a testing function space $V(I)$ containing the kernel $K(s,t)$. The dual space $V'(I)$ consists of continuous linear functionals on $V(I)$ which are called generalized functions. The generalized transform $F(s)$ of $f \in V'(I)$ is now defined as direct application of $f$ to the kernel $K(s,t)$,

$$F(s) = \langle f, K(s,t)\rangle,\ f \in V'(I).$$

**(B) Indirect or Adjoint method.**

In this method extension is done using the Parseval equation related to the classical integral transformation.

**(B1) Automorphism.**

Let $T$ be the integral transform. Construct a testing function space $V(I)$ such that $T : V(I) \to V(I)$ is an automorphism. Then the adjoint operator $T^{-1}$ is the generalized integral transform on $V'(I)$. That is when $f \in V'(I)$, $\Phi = T(\phi) \in V(I)$ then

$$\langle T'f, \Phi\rangle = \langle f, \Phi\rangle, \phi \in V(I) \text{ or } \langle T'f, \Phi\rangle = \langle f, T^{-1}\Phi\rangle.$$

**(B2) Isomorphism.**

Let $T$ be any integral transform. Construct the testing function space $S(I)$ such that $T\phi$ exists for all $\phi \in S(I)$. Let $\check{S}(I) = \{T(\phi)|\phi \in S(I)\}$ that is $\check{S}(I)$ consists all $T$ transforms of the members in $S(I)$. By showing $T$ an isomorphism from $S(I)$ onto $\check{S}(I)$ we can claim that $\check{S}(I)$ is also a testing function space. Then the adjoint operator of $T^{-1}$ is the generalized $T$ integral transform denoted by $T'$ on $S'(I)$. That is

$$\langle T'f, \Phi\rangle = \langle f, \Phi\rangle, \phi \in S, \Phi \in \check{S}, f \in S'.$$

Gelfand and Shilov extended some transform pairs by using this method. We extend Fourier-Jacobi transform by this technique.

**(C) Convolution method.**

This method treats an integral transform $T$ of the form $(Tf)(x) = \int_0^\infty K(x-t)f(t)dt$. $0 < x < \infty$. So that it is the convolution of the kernel $K$ and the unknown function $f$.

Convolution is an operation which is meaningful for the distribution whose support is bounded on the left. In particular for the members of $D^{'}(0,\infty)$, we define $T$ on $D^{'}(0,\infty)$ by $\breve{T}h = \breve{K} * h$, where $*$ denotes the distributional convolution.

$$\langle f * g,\, \phi\rangle = \langle f(t) \times g(\tau),\, \phi(t+\tau)\rangle \;=\; \langle f(t),\, \langle g(\tau),\, \phi(t+\tau)\rangle\rangle.$$

## 1. THE COMPLEX LAPLACE-HANKEL TRANSFORMATION OF GENERALIZED FUNCTIONS

We have extended the Laplace-Hankel transformation to a certain class of generalized functions by generalizing the Parsevals equation [3]

The notation and terminology of this work follows that [51] and [49].

For any real number $a$ and $\lambda > \frac{1}{2}$ the spaces $LH_{a,\lambda}$, $L\overset{\wedge}{H}_{a,\lambda}$ and $L\overset{\tilde{\vee}}{H}_{a,\lambda}$ have been constructed such that the mapping $\Phi = LH_\lambda(\phi) \to 2\pi i\phi$ is an isomorphism from $L\overset{\tilde{\vee}}{H}_{a,\lambda}$ onto $LH_{a,\lambda}$, where $LH_\lambda$ denotes the conventional Laplace-Hankel transformation and $\overset{\vee}{\phi}(x,y) = \phi(-x,y)$, $\phi \in L\overset{\vee}{H}_{a,\lambda}$ and $\overset{\vee}{\phi} \in LH_{a,\lambda}$.

For any generalized function $f \in LH'_{a,\lambda}$ the Laplace-Hankel transform $LH'_\lambda(f)$ was defined by

$$\langle LH'_\lambda(f), LH_\lambda(\phi)\rangle = 2\pi i\langle f, \overset{\vee}{\phi}\rangle \tag{1.1}$$

where $LH_\lambda(\phi) \in L\overset{\tilde{\vee}}{H}_{a,\lambda}$ and $\overset{\vee}{\phi} \in LH_{a,\lambda}$.

The generalized Laplace-Hankel transformation $L\overset{\tilde{\vee}}{H}_{a,\lambda}$ is an isomorphism from $LH'_{a,\lambda}$ onto $L\overset{\tilde{\vee}}{H}_{a,\lambda}$, where $LH'_{a,\lambda}$ and $L\overset{\vee}{H}_{a,\lambda}$ denote the dual spaces of $LH_{a,\lambda}$ and $L\overset{\tilde{\vee}}{H}_{a,\lambda}$ respectively [9].

In contrast to this, we define the Laplace-Hankel transform $F(u,v)$ of generalized function $f$ directly as the application of $f(x,y)$ to $e^{-ux}\sqrt{vy}J_\lambda(vy)$, i.e.,

$$F(u,v) = \langle f(x,y), e^{-ux}\sqrt{vy}J_\lambda(vy)\rangle. \tag{1.2}$$

For this purpose, we have constructed the testing function space $B_{\lambda,a,b}$ on $0 < x < \infty$, , $0 < y < \infty$ which contains the kernel $e^{-ux}\sqrt{vy}J_\lambda(vy)$ for all $(u,v)$ in some restricted domain.

### 1.1. Testing Function Space $B_{\lambda,a,b}$.

For any fixed real numbers $a$ and $\lambda$ and fixed positive real number $b$, an infinitely differentiable complex valued function $\phi(x,y)$ defined on $I = \{(x,y)|0 < x < \infty,\ 0 < y < \infty\}$ belongs to $B_{\lambda,a,b}$ if

$$B'_{a,b,k,k'}(\phi) = \sup_{0<x<\infty,\ 0<y<\infty} |e^{ax-by} y^{-\lambda-\frac{1}{2}} D_x^k S_{\lambda,y}^{k'} \phi(x,y)| < \infty$$
$$\text{for all } k, k' = 0, 1, 2, 3, \cdots$$
$$\text{where } D_x^k = \frac{\partial^k}{\partial x^k} \text{ and } S_{\lambda,y}^{k'} = \left( y^{-\lambda-\frac{1}{2}} \frac{\partial}{\partial y} y^{2\lambda+1} \frac{\partial}{\partial y} y^{-\lambda-\frac{1}{2}} \right)^{k'}.$$

Observe that $B_{\lambda,a,b}$ is a vector space over the field of complex numbers, $\phi(x,y) = 0$ on $I$ is the zero element of $B_{\lambda,a,b}$. Since $B^{\lambda}_{a,b,0,0}$ is norm, the countable collection of seminorms $\{B^{\lambda}_{a,b,k,k'}\}^{\infty}_{k,k'=0}$ is a countable multinorm for $B_{\lambda,a,b}$, [51, p. 8]. The topology of $B_{\lambda,a,b}$ is generated by the multinorm $\{B^{\lambda}_{a,b,k,k'}\}$, [51, p. 8]. Hence it becomes a countably multinormed space.

1.1.1. **Convergence in** $B_{\lambda,a,b}$**.**

We say that a sequence $(\phi_\nu)^{\infty}_{\nu=1}$ where $\phi_\nu \in B_{\lambda,a,b}$ converges in $B_{\lambda,a,b}$ to $\phi$, if for each pair of non-negative integers $k$ and $k'$, $B^{\lambda}_{a,b,k,k'}(\phi_\nu - \phi) \to 0$ as $\nu \to \infty$. A sequence $\{\phi_\nu\}^{\infty}_{\nu=1}$ is said to be a Cauchy sequence in $B_{\lambda,a,b}$ if $B^{\lambda}_{a,b,k,k'}(\phi_\nu - \phi_\mu) \to 0$ as $\nu$, $\mu \to \infty$ independently for each pair of $k$ and $k'$.

Every Cauchy sequence in the space $B_{\lambda,a,b}$ is convergent, *i. e.*, $B_{\lambda,a,b}$ is sequentially complete. Moreover, it is a locally convex Hausdorff topological vector space. Further, it is a testing function space. The space $D(I)$ is a subspace of $B_{\lambda,a,b}$ and the convergence concept in $D(I)$ is stronger than that in $B_{\lambda,a,b}$ . Therefore, the restriction of any $f \in B'_{\lambda,a,b}$ to $D(I)$ is in $D'(I)$, where $B'_{\lambda,a,b}$ is the space of all continuous linear functionals on $B_{\lambda,a,b}$.

If $a \le c$ and $0 < d < b$ then $B_{\lambda,c,d}$ is a subspace of $B_{\lambda,a,b}$ and the topology of $B_{\lambda,c,d}$ is stronger than the induced topology on it by $B_{\lambda,a,b}$. Hence the restriction of $f \in B'_{\lambda,c,d}$ to $B_{\lambda,c,d}$ is in $B'_{\lambda,c,d}$.

1.2. **Countable-union space** $B(\lambda, w, z)$**.**

Let $w$ denote either a real number or $-\infty$ and $z$ denote either a positive real number or $+\infty$. Choose two monotone sequences of real numbers $\{a_\nu\}^{\infty}_{\nu=1}$ and $\{b_\nu\}^{\infty}_{\nu=1}$ such that $a_\nu \to w^+$ and $b_\nu > 0$, $b_\nu \to z^-$. Then $\{B_{\lambda,a_\nu,b_\nu}\}^{\infty}_{\nu=1}$ is a sequence of testing function spaces such that $B_{\lambda,a_\nu,b_\nu} \subset B_{\lambda,a_{\nu+1},b_{\nu+1}}$ and the topology of $B_{\lambda,a_\nu,b_\nu}$ is stronger than the induced topology on it by $B_{\lambda,a_{\nu+1},b_{\nu+1}}$ for all $\nu$. The space $B(\lambda, w, z)$ is defined to be $B(\lambda, w, z) = \bigcup_{\nu=1}^{\infty} B_{\lambda,a_\nu,b_\nu}$.

Observe that $B(\lambda, w, z)$ is a vector space.

1.2.1. **Convergence in** $B(\lambda, w, z)$**.**

A sequence $\{\phi_\nu\}^{\infty}_{\nu=1}$ is said to converge in $B(\lambda, w, z)$ to $\phi$ if all the $\phi_\nu$ and $\phi$ belong to $B_{\lambda,a_\nu,b_\nu}$ for some particular $\nu$ and $(\phi_\nu)^{\infty}_{\nu=1}$ converges to $\phi$ in $B_{\lambda,a_\nu,b_\nu}$ [51, p. 15].

Since all the $B_{\lambda,a_y,b_y}$ are sequentially complete, $B(\lambda, w, z)$ is also complete.

If $w < u$ and $0 < \nu < z$ then $B(\lambda, u, \nu)$ is a subspace of $B(\lambda, w, z)$ and the convergence in $B(\lambda, u, v)$ implies the convergence in $B(\lambda, w, z)$. Therefore, the restriction on any $f \in B'(\lambda, w, z)$ to $B(\lambda, u, \nu)$ is in $B'(\lambda, u, \nu)$.

**Result (1.1):** Let $\Omega$ be a subset of $\mathbb{C}^2$ defined as follows:

$$\Omega = \{(u, v)|\mathrm{Re}u > a, |\mathrm{Im}v| < b, v \neq 0 \text{ or a negative integer}\}.$$

For any $(u, v) \in \Omega,\ e^{-ux}\sqrt{vy}J_\lambda(vy)$ is a member of $B_{\lambda,a,b}$ whenever $\lambda > -\frac{1}{2}$.

**Remark 1.1.** *Since* $w$ *and* $z$ *are members of* $B(\lambda, w, z)$*, whenever,* $\mathrm{Re}\, u > w$, $|\mathrm{Imv}| <$ z, $\mathrm{v} \neq 0$ *or a negative integer and* $\lambda > -\frac{1}{2}$.

**Result (1.2)** If $f(x, y)$ is a locally integrable function on $I$ such that $e^{-ax+by}y^{\lambda+\frac{1}{2}}f(x, y)$ is absolutely integrable on $I$, then $f(x, y)$ generates a regular generalized function $f$ in $B'_{\lambda,a,b}$ through the definition $\langle f, \phi\rangle = \int\limits_0^\infty \int\limits_0^\infty f(x, y)\phi(x, y)dxdy$, for all $\phi \in B_{\lambda,a,b}$.

1.3. **Generalized Laplace-Hankel transformation.**

Let $\lambda$ be restricted as $-\frac{1}{2} < \lambda < \infty$. A generalized function $f$ is said to be complex Laplace-Hankel transformable if $f \in B'(\lambda, w, z)$ for some real numbers $w$ and $z$, where $z$ is positive. Let $\sigma_f$ be infimum of all such $w$ and $\rho_f$ be supremum of all such $z$. Define $\Omega_f$ in $\mathbb{C}^2$ as

$\Omega_f = \{(u, v) \in \mathbb{C}^2|\mathrm{Re}u > \sigma_f, |\mathrm{Im}\ v| < \rho_f, v \neq 0 \text{ or a negative integer }\}$

If $f$ is a Laplace-Hankel transformable generalized function, then we see that there exist a pair of real numbers $\sigma_f$ and $\rho_f$, $(\rho_f > 0)$ such that $f \in B'_{\lambda,w,z}$ for all $w > \sigma_f$and $z > \rho_f$.

The complex Laplace-Hankel transform $LH'_\lambda(f)$ of $f$ is defined as the application of $f(x, y)$ to the kernel $e^{-ux}\sqrt{vy}J_\lambda(vy)$, *i.e.*,

$F(u, v) = (LH'_\lambda(f))(u, v) = \langle f(x, y), e^{-ux}\sqrt{vy}J_\lambda(vy)\rangle$ where $(u, v) \in \Omega_f$.

Since $e^{-ux}\sqrt{vy}J_\lambda(vy) \in B_{\lambda,a,b}$ for all $a > \sigma_f$ and $0 < b < \rho_f$, the R.H.S. of the above equation has meaning.

$\Omega_f$ is called the region of definition for $F(u, v)$ and $\sigma_f$ and $\rho_f$ are called the abscissas of definitions.

**Analyticity theorem**

If $L(H'_\lambda(f))(u, v) = F(u, v)$ for all $(u, v) \in \Omega$, then $F(u, v)$ is analytic on $\Omega_f$ and $DF(u, v) = \langle f(x, y), D(e^{-ux}\sqrt{vy}J_\lambda(vy))\rangle$, where, $D = \frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$. For proving this result we have used the Hartog's theorem [6, p. 140].

**Inversion theorem**

Let $F(u, v) = (LH'_\lambda(f))(u, v)$ for all $(u, v) \in \Omega_f$. Then in the sense of convergence in $D'(I)$,

$$f(x, y) = \lim_{r,r'\to\infty} \frac{1}{2\pi i} \int\limits_{\sigma-ir}^{\sigma+ir} \int\limits_0^{r'} e^{ux}\sqrt{vy}J_\lambda(vy)F(u, v)dvdu,$$

where $\sigma$ is any fixed real number such that $\sigma > \sigma_f$.

**Uniqueness theorem**

Suppose $(LH'_\lambda(f))(u,v) = F(u,v)$ for all $(u,v) \in \Omega_f$ and $(LH'_\lambda(g))(u,v) = G(u,v)$ for all $(u,v) \in \Omega_g$. If $\Omega_f \cap \Omega_g$ is not empty and if $F(u,v) = G(u,v)$ on $\Omega_f \cap \Omega_g$ then $f = g$ in the sense of equality in $D'(I)$.

## 2. HANKEL TYPE TRANSFORM OF DISTRIBUTION

The Hankel type transformation of a function $f(x)$ is defined by

$$f(y) = \int_0^\infty \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) f(x) dx, \tag{2.1}$$

where is $J_\lambda(x)$ the Bessel function of first kind of order $\lambda$.

The inversion formula for this transform is given by

$$f(y) = \int_0^\infty \left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{x,y}) F(y) dy. \tag{2.2}$$

We have extended this transformation to a class of distributions and used it to solve a distributional integral equation. [10]

For any real number $a$, let $L_{+,a}$ be the space of all infinitely differentiable functions $\phi(t)$ on $I$ such that

$B_{a,k}(\phi) = \sup_{t\in I} |e^{at} D^k(\phi)t| < \infty$ , $k = 0, 1, 2, 3, \cdots$ where $D = \frac{d}{dt}$.

$L_{+,a}$ is a testing function space whose topology is generated by $\{B_{a,k}\}_{k=0}^\infty$. $e^{-st} \in L_{+,a}$. If Res > a [51, p. 90], $L'_{+,a}$ denotes the dual space of $L_{+,a}$ . If $f \in L_{+,a}$, we define the Laplace transform of $f$ by

$$F(s) = \langle f(t), e^{-st} \rangle \ \mathrm{Re\, s} > \mathrm{a}. \tag{2.3}$$

If $f$ is a locally integrable function on $I$ and if $\int_0^\infty e^{-at}|f(t)| < \infty$ , then $f$ generates a regular generalized function in $L'_{+,a}$ through the definition :

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t) dt, \quad \phi \in L_{+,a}. \tag{2.4}$$

### 2.1. **The testing function space $H_{a,\lambda}$ and its dual.**

For any real numbers $a$ and $\lambda$, let $H_{a,\lambda}$ be the collection of all infinitely differentiable functions $\phi(x)$ defined on $I$ such that for every nonnegative integer $k$ ,

$$T_k(\phi) = T_k^{\lambda,a}(\phi) = \sup_{\substack{x\in I \\ <\infty}} |e^{-ax} \Delta^k_{\lambda,x} \phi(x)|, \tag{2.5}$$

$$where, \Delta^k_{\lambda,x} = \left[D x^{-\lambda+1} D x^\lambda\right]^k ; \ D = \frac{d}{dx}.$$

Note that $\{T_k\}_{k\to 0}^{\infty}$ is a separating collection of seminorms and $H_{a,\lambda}$ is a sequentially complete locally convex topological vector space.

Let $H'_{a,\lambda}$ be denote the dual of $H_{a,\lambda}$. The restriction of $f \in H'_{a,\lambda}$ to $D(I)$ is in $D'(I)$.

For $\lambda > -\frac{1}{2}$ and $y > 0$, $\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \in H_{a,\lambda}$ and the operation $\Delta_{\lambda,x}$ satisfies the property

$$\Delta_{\lambda,x}^k \left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{x,y})\right] = (-1)^k y^k \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{x,y}) \tag{2.6}$$

for any nonnegative integer $k$.

### 2.2. The distributional Hankel type transform.

For $f \in H'_{a,\lambda}$ and $\lambda > -\frac{1}{2}$, we define the distributional Hankel type transform $F = h_{\lambda,f}$ of $f$ by

$$F(y) = (h_{\lambda,f})(y) = \langle f(x), \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{x,y})\rangle. \tag{2.7}$$

Using convergence in $H_{a,\lambda}$ and equation (2.6) we can establish results:

**Theorem 2.1.** *For $f \in H'_{a,\lambda}$ and $\lambda > -\frac{1}{2}$, let $F(y)$ be defined by* (2.7)*. Then*

$$\frac{d}{dy}F(y) = \langle f(x), \frac{\partial}{\partial y}\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{x,y})\rangle. \tag{2.8}$$

**Theorem 2.2.** *Let $F(y)$ be the distributional Hankel transform of $f \in H'_{a,\lambda}$ as defined by (2.7). Then $F(y)$ satisfies the inequality*

$$|F(y)| \le \begin{cases} Ky^{\lambda}; & 0 < y < 1 \\ Ky^{p}; & 1 < y < \infty \end{cases} \tag{2.9}$$

*where $p$ is a sufficiently large real number and $k$ is chosen appropriately.*

For the proof of this result we use the theorem 18.1 of [51].

### 2.3. Inversion theorem for distributional Hankel type transformation.

Let $f \in H'_{a,\lambda}$ and $F(y)$ be the distributional Hankel type transform of $f$ defined by (2.7). Then for $\lambda > -\frac{1}{2}$ in the sense of convergence in $D'(I)$

$$f(x) = \lim_{R\to\infty} \int_0^R \left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})F(y)dy. \tag{2.10}$$

Now we give a result which is a useful application of distributional Hankel type transform to distributional integral equation.

**Lemma 2.1.** *Let* $0 < b < 1$ *and* $\lambda > -\frac{1}{2}$ *and* $f \in H'_{a,\lambda} \cup L'_{+,b}$. *If* $h_{\lambda,f}$ *is a distributional Hankel type transform of* $f$ *defined by (2.7), then* $h_{\lambda,f}$ *generates a regular generalized function in* $L'_{+,b}$ *and the distributional Laplace transform* $L[h_{\lambda,f}]$ *of* $h_{\lambda,f}$ *is given by*

$$L[h_{\lambda,f}](p) = p^{-\lambda,-1} F(\frac{1}{p}), \tag{2.11}$$

*where* $F(p) = \langle f(x), e^{-px} \rangle$ *for* $b < \mathrm{Re\,p} < \frac{1}{\mathrm{b}}$ *is the distributional Laplace transform of* $f$.

We have used these facts to solve the distributional integral equation

$$f + kh_\lambda(f) = g. \tag{2.12}$$

In the sense $H'_{a,\lambda} \cap L'_{+,b}$, where $0 < b < 1$, $a > 0$ and $g$ is a known distribution in $H'_{a,\lambda} \cap L'_{+,b}$ and $k \neq -1$.

Apply the distributional Laplace transformation $L$ to equation (2.12), we get

$$L(f) + kL(h_\lambda(f)) = L(g).$$

Using (2.11) we get

$$L(f)(p) + kp^{-\lambda-1} L(f)(\frac{1}{p}) = L(g)(p), \tag{2.13}$$

if $b <$ Re $p < \frac{1}{b}$.
Replacing $p$ by $\frac{1}{p}$ we have

$$L(f)(\frac{1}{p}) + kp^{\lambda+1} L(f)(p) = L(g)(\frac{1}{p}), \tag{2.14}$$

if $b <$ Re $p < \frac{1}{b}$.
Eliminating $L(f)(\frac{1}{p})$ from (2.13) and (2.14) we get

$$\begin{aligned} L(f)(p) &= \frac{1}{(1-k^2)} \left[ L(g)(p) - kp^{-\lambda-1} L(g)(\frac{1}{p}) \right] \\ &= \frac{1}{(1-k^2)} \left[ L(g)(p) - kL(h_\lambda g)(\frac{1}{p}) \right] \end{aligned}$$

*i.e.*,

$$L(f) = \frac{1}{(1-k^2)} L(g - k\, h_\lambda(g)),$$

which implies that

$$f = \frac{1}{(1-k^2)} L(g - k\, h_\lambda(g)).$$

## 3. DISTRIBUTIONAL ABELIAN THEOREMS FOR THE GENERALIZED STIELTJES TRANSFORM

Let $F(s)$ be a Laplace transform of a function $\phi(y)$.

$$F(s) = \int_0^\infty H^{1,0}_{0,1}\left[su|_{(b_1,B_1)}\right]\phi(u)du, \tag{3.1}$$

where the kernel is written in terms of the H-function [26] which has been studied in detail in [7]. For $b_1 = 0, B_1 = 1$, it reduces to $e^{-su}$.

Again, let $\phi(u)$ be the generalized Laplace transform of the function $f(t)$ given by

$$\phi(u) = \int_0^\infty H^{1,1}_{1,2}\left[(tu)^\lambda \Big|^{d_1,D_1}_{(e_1,E_1);(e_2,E_2)}\right] f(t)dt. \tag{3.2}$$

Observe that by a suitable choice of the parameters $d_1, e_1, e_2$ etc., the kernel of (3.2) reduces to some constant multiplied by confluent hypergeometric function $\mathbf{1F_1(-tu)}$ which is the kernel of the generalized Laplace transform introduced in [21]. After iteration and using result [26, 2.6.8], we get

$$F(s) = \int_0^\infty S_{-1}H^{1,2}_{2,2}\left[(\frac{t}{s})^\lambda \Big|^{(d_1,D_1),(1-b_1B_1,\lambda B_1)}_{(e_1,E_1);(e_2,E_2)}\right] f(t)dt. \tag{3.3}$$

Hence $F(s)$, $s$ not lying on the negative real axis, given by (3.3) is the generalized Stieltjes transform. Note that (3.3) covers almost all the existing generalizations of Stieltjes transform.

We have established the Abelian theorems which relate the behavior of generating function $F(s)$ as $s \to 0$ $(s \to \infty)$ to the behavior of determining function $f(t)$ as $t \to 0\,(s \to 0)$ [13].

### 3.1. **Classical Abelian theorems.**

**Theorem 3.1. Initial value Abelian theorem**

*If*

*(I) the generalized Stieltjes transform $F(s)$ of a complex-valued function $f(t)$ is defined by (3.3),*

*(II) $\lim\limits_{t\to 0} \frac{f(t)}{t^\eta} = \alpha$, where $\eta$ is a real number and $\alpha$ is a complex number,*

*(III) $\frac{f(t)}{t^\eta}$ is bounded on $y \le t < \infty$, for all $y > 0$.*

*(IV)* $-1-\lambda P_1 < \eta < -1-\lambda Q_1$, *where* $P_1$ *and* $Q_1$ *are complex numbers defined as*

$$P_1 = \min\left(Re\frac{b_j}{B_j}\right), \quad j = 1, 2, \cdots, m$$
$$Q_1 = \max\left(Re\frac{a_{j-1}}{A_j}\right), \quad j = 1, 2, \cdots, n.$$

*Then* $\lim_{x\to 0} s^{-\eta} F(s)\lambda G(\lambda,\eta) = \alpha$,

*where* $G(\lambda,\eta) =$

$$\frac{\Gamma\left(1-e_2-E_2\left(\frac{\eta}{\lambda}+\frac{1}{\lambda}\right)\right)}{\Gamma\left(e_1+E_1\left(\frac{\eta}{\lambda}+\frac{1}{\lambda}\right)\right)\ \Gamma\left(1-d_1-D_1\left(\frac{\eta}{\lambda}+\frac{1}{\lambda}\right)\right)\ \Gamma(b_1+B_1-B_1(\eta+1))}$$

**Theorem 3.2. Final value Abelian theorem**

*If*

*(I)* $-1-\lambda P_1 < \eta < -1-\lambda Q_1$ *where* $P_1$ *and* $Q_1$ *are complex numbers defined as*

$$P_1 = \min\left(Re\,\frac{b_j}{B_j}\right), \quad j = 1, 2, \cdots, m$$
$$Q_1 = \max\left(Re\,\frac{a_{j-1}}{A_j}\right), \quad j = 1, 2, \cdots, n,$$

*(II) the complex-valued function* $f(t)$ *satisfies the following condition* $\lim_{t\to 0} \frac{f(t)}{t^\eta} = \alpha$ *for* $\eta$ *is a real number and* $\alpha$ *is a complex number, and*

*(III)* $\frac{f(t)}{t^\eta}$ *is bounded on* $0 \le t \le y$, *for all* $y > 0$, *then* $\lim_{s\to\infty} s^\eta F(s)\lambda G(\lambda,\eta) = \alpha$,

*where* $G(\lambda,\eta) =$

$$\frac{\Gamma\left(1-e_2-E_2\left(\frac{\eta}{\lambda}+\frac{1}{\lambda}\right)\right)}{\Gamma\left(e_1+E_1\left(\frac{\eta}{\lambda}+\frac{1}{\lambda}\right)\right)\ \Gamma\left(1-d_1-D_1\left(\frac{\eta}{\lambda}+\frac{1}{\lambda}\right)\right)\ \Gamma(b_1+B_1-B_1(\eta+1))}.$$

### 3.2. Extension of classical Abelian theorem to the Space $J_{c,d}$.

The space $J_{c,d}$ is fully discussed in [51, 8.6]. $J_{c,d}$ is the space of all smooth functions $\phi(t)$ on $0 < t < \infty$ and the seminorm is defined as

$$\delta_{c,d,k}(\phi) = \delta_k(\phi) = \sup_{0<t<\infty} |\lambda_{c,d}(logt)(tD_t^k)\sqrt{t}\phi(t)| < \infty,\ k = 0, 1, 2, \cdots,$$

where

$$\lambda_{c,d}(logt) = \begin{cases} t^d\ ; & 0 < y < 1 \\ t^c\ ; & 1 \le y < \infty. \end{cases}$$

$J_{c,d}$ is a complete countably multinormed space. If $f \in J'_{c,d}$, the dual space of $J_{c,d}$ , then the Stieltjes transform $F$ of $f$ is defined by $F(s) = \langle f(t), s^{-1} H \left(\frac{t}{s}\right)^{\lambda} \rangle$ for any complex $s$ not lying on the negative real axis.

**Theorem 3.3.** *The kernel* $k(s,t) = s^{-1} H \left(\frac{t}{s}\right)^{\lambda}$ *is a member of* $J_{c,d}$ *if* $c + \frac{1}{2} + \lambda Q_1 < 0$ *and* $d + \frac{1}{2} + \lambda P_1 < 0$*, where*

$$P_1 = \min \left( Re \frac{b_j}{B_j} \right), \quad j = 1, 2, \cdots, m,$$

$$Q_1 = \max \left( Re \frac{a_{j-1}}{A_j} \right), \quad j = 1, 2, \cdots, n.$$

To extend the preceding results to the space $J_{c,d}$ we require the notion of the value of the distribution at a point. This concept is introduced in [24].

**Definition 3.1.** *Let $T$ be a distribution defined in a neighborhood of a point $x_0$ . We say that $T$ has a value $c$ at $x_0$, i.e., $T(x_0) = c$ if the distributional limit $T_{\lambda - 0}(x_0 + \lambda x)$ exists in a neighborhood of $x_0$ and if it is a constant function $c$.*

**Theorem 3.4. Distributional initial value Abelian theorem**

*If*

*(I)* $f \in J'_{c,d}$ ,

*(II)* $\frac{f(t)}{t^n} \to \alpha$ *as* $t \to 0^+$ *in the sense of Lojasiewicz, and*

*(III)* $-1 - \lambda P_1 < \eta < -1 - \lambda Q_1$ *, where $P_1$ and $Q_1$ are complex numbers defined as*

$$P_1 = \min \left( Re \frac{b_j}{B_j} \right), \quad j = 1, 2, \cdots, m,$$

$$Q_1 = \max \left( Re \frac{a_{j-1}}{A_j} \right), \quad j = 1, 2, \cdots, n,$$

*then* $\lim\limits_{s \to 0} S^{-\eta} F(S) \lambda G(\lambda, \eta) = \alpha,$

*where* $G(\lambda, \eta) =$

$$\frac{\Gamma\left(1 - e_2 - E_2\left(\frac{\eta}{\lambda} + \frac{1}{\lambda}\right)\right)}{\Gamma\left(e_1 + E_1\left(\frac{\eta}{\lambda} + \frac{1}{\lambda}\right)\right) \Gamma\left(1 - d_1 - D_1\left(\frac{\eta}{\lambda} + \frac{1}{\lambda}\right)\right) \Gamma\left(b_1 + B_1 - B_1(\eta + 1)\right)}.$$

Now for extension of final value Abelian theorem to the space $J'_{c,d}$ we need some notations:

$D$- The space of smooth functions having compact support.

$I$- The interval $(0, \infty)$.

$E(I)$- The space of smooth functions on $I$.

$E'(I)$-The space of distributions having compact support with respect to $I$.

Note that $J_{c,d}$ is dense in $E$, since $D \subset J_{c,d} \subset E$.

**Theorem 3.5. Distributional final value Abelian theorem**

*If*

(I) *$f \in J'_{c,d}$ and $f$ can be decomposed into $f = f_1 + f_2$ , where $f_1$ is an ordinary function and $f_2 \in E'(I)$ ; $f_1$ satisfies the hypothesis of Theorem* 3.2*,*

(II) *$\frac{f(t)}{t^n} \to \alpha$ as $t \to \infty$ in the sense of Lojasiewicz,*

(III) *$-1 - \lambda P_1 < \eta < -1 - \lambda Q_1$, where $P_1$ and $Q_1$ are complex numbers defined as*

$$P_1 = \min\left(Re\,\frac{b_j}{B_j}\right), \quad j = 1, 2, \ldots, m$$

$$Q_1 = \max\left(Re\,\frac{a_{j-1}}{A_j}\right), \quad j = 1, 2, \ldots, n$$

(IV) *$\lambda Q_1 < q < \lambda P_1$ and $-\eta - 1 - q < 0$, where $q$ is a nonnegative integer such that for all*

$$\phi \in D(I), |\langle f, \phi\rangle| \le c \sup_{0<t<\infty} |D_t^q\,\phi(t)|,$$

$$\text{then} \quad \lim_{s\to\infty}, \; s^{-\eta}F(S) = \lim_{s\to\infty} \frac{f(t)}{t^\eta \lambda G(\lambda, \eta)}.$$

**Remark 3.1.** *The generalized Stieltjes transform given by (3.3) covers almost all existing generalizations. Hence this result of the Abelian theorem unifies all the results of the Abelian theorems given so far, e.g. if we choose the parameters $\lambda = 1$, $d_1 = e_2$, $D_1 = E_2$, $b_1 = e$, $B_1 = 1$, $e_1 = 0$, $E_1 = 1$ and use the result* [26, 1.7.3] *then*

$$F(s) = \int_0^\infty s^{-1} H_{1,1}^{1,1}\left[\frac{t}{s}\Big|_{(0,1)}^{(-e,1)}\right] f(t)dt$$

$$\text{and} \quad \frac{H(s)s^{-e}}{|(e+1)} = F(s) = \int_0^\infty \frac{f(t)}{(s+t)^{e+1}}dt.$$

This generalization was considered by Misra [28] and his result is a particular case of our result of the Abelian theorem with $G(I, \eta) = \frac{1}{|(\eta+1)\,|(\rho-\eta)}$ [28, 3.2].

## 3.3. Application of Abelian theorem.

John W. Dettman [14] has shown that if $P_1$ and $P_2$ as defined given below

$$P_2 : \frac{\partial^2}{\partial t^2}\nu(x,t) = p(x,D)\nu(x,t), t > 0,$$
$$\nu(x,0) = \phi(x),$$
$$\nu_t(x,0) = 0,$$
$$P_3 : \frac{\partial^2}{\partial y^2}\omega(x,y) + p(x,D)\omega(x,y) = 0, y > 0,$$
$$\omega(x,0) = \phi(x),$$

are the sets of initial boundary value problems for partial differential equations then their solutions are related as follows:

$$\omega(x,y) = \frac{2y}{\pi}\int_0^\infty \frac{\nu(x,z)}{y^2+z^2}dz,$$

$$\nu(x,t) = \frac{t}{2i}\lim_{\epsilon\to 0}\left[\frac{\omega(x,\sqrt{-t^2-i\epsilon})}{\sqrt{-t^2-i\epsilon}} - \frac{\omega(x,\sqrt{-t^2+i\epsilon})}{\sqrt{-t^2+i\epsilon}}\right].$$

Here, we show that as a function of $z$, $\nu(x,z) \in J'_{c,d}$. Hence $\omega(x,y)$ is the distributional generalized Stieltjes transform of $\nu(x,z)$ ,*i.e.*,

$$\omega(x,y) = \langle \nu(x,z), \frac{2y}{\pi(y^2+z^2)}\rangle$$

or

$$\frac{\pi}{2}\omega(x,y) = \langle \nu(x,z), \frac{y}{(y^2+z^2)}\rangle.$$

We have related the distributional solution $\nu(x,t)$ of $P_2$ with the solution $\omega(x,y)$ of $P_3$.

**Theorem 3.6.** *If $\nu(x,z)$ is the distributional solution of $P_2$ then $\omega(x,y)$ is the solution of $P_3$ .*

## 4. ON STIELTJES TRANSFORM OF BANACH SPACE-VALUED DISTRIBUTIONS

Silva [41] has discussed the theory for vector-valued distributions. Zemanian [52] has presented the theory of Banach-space valued distributions. Motivated by his work, we have extended the generalized Stieltjes transformation

$$F(s) = s^{m\rho-1}\overline{|(\rho)}\int_0^\infty f(t)(s^m+t^m)^{-\rho}dt \quad (m,\rho>0)$$

to certain Banach-valued distributions and derived an inversion formula for the same.

Various generalizations of the Stieltjes transform defined by the equation

$$F(s) = \int_0^\infty f(t)(s+t)^{-1}dt \tag{4.1}$$

have been studied by many mathematicians from time to time. One such is given by

$$F(s) = s^{m\rho-t}\overline{|(\rho)}\int_0^\infty f(t)(s^m+t^m)^{-\rho}dt \quad (m,\rho>0) \tag{4.2}$$

whenever the integral on the right hand side converges for a complex $s$ with $\mathrm{Re}\phi(s) > 0$.

Motivated by the work of Zemanian we studied the transform defined by (4.2) for Banach space-valued distributions. Also, we have extended the following complex inversion theorem for the Banach space-valued distributional transform [44].

**Theorem 4.1.** *Let $x^{mu-m}f(x) \in L(0,\infty)$ and $f(x)$ be of bounded variation in the neighborhood of the point $x = t$. Let $F(s)$ be defined by*

$$F(s) = s^{m\rho-1}\overline{|(\rho)}\int_0^\infty f(t)(s^m + t^m)^{-\rho}dt \quad (m, \rho > 0)$$

*and $s^{ml-m}F(S) \in L(0,\infty)$, where $l = \sigma + iT$. Then*

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Q(l)M(l)t^{-ml+m-1}dl,$$

*where $Q(l) = \frac{m}{|(1-l)|(s-1+l)}$ and $M(l) = \int_0^\infty s^{ml-m}F(s)ds$ provided $1-\rho < Re(l) < 1$ and $M(l)$ is absolutely convergent.*

4.1. **The spaces $J_{c,d}(A)$ and $[J_{\omega,z}(A); B]$.**

Let $A$ and $B$ be Banach spaces. Given any two real numbers $c$ and $d$, $J_{c,d}(A)$ is defined as the linear space of all $A$- valued smooth functions $\psi(t)$ on $I$ into $A$ such that

$$i_k(\psi(t)) = i_{c,d,k}(\psi(t)) = \sup_{t\in i} ||\lambda_{c,d}(t)(t\frac{d}{dt})^k\sqrt{t}\psi(t)||_A < \infty,\ k = 0, 1, 2\cdots,$$

$$\text{where} \quad \lambda_{c,d}(t) = \begin{cases} t^d: & 0 < y < 1, \\ t^c: & 1 \leq < \infty. \end{cases}$$

The locally convex topology of $J_{c,d}(A)$ is defined by the family of seminorms $\{i_k\}_{k=0}^{\infty}$.

For $A = \mathbb{C}$ we write $J_{c,d}(A) = J_{c,d}$. It is easy to see that $D(A) \subset J_{c,d}(A)$ and consequently $D \subset J_{c,d}$.

Now, let $\{c_j\}_{j=1}^{\infty}$ be a strictly decreasing sequence in $\mathbb{R}$ tending to $\omega$, where either $\omega \in \mathbb{R}$ or $\omega = -\infty$. Similarly, let $\{d_j\}_{j=1}^{\infty}$ be a strictly increasing sequence in $R$ tending to $z$, where, either $z \in R$ or $z = -\infty$ in $i, z \in i$.

The space defined by $J_{\omega,z}(A) = \bigcup_{j=1}^{\infty} J_{c_j,d_j}(A)$ is the inductive limit of the $J_{c_j,d_j}(A)$. It is a normal $\rho - type$ testing function space. Thus any continuous mapping $f$ of $J_{\omega,z}(A)$ into $\beta$ is called an $[A, B]-valued$ generalized function on $I$, where the symbol $[A, B]$ denotes the linear space of all continuous linear mappings of $A$ into $B$.

For a Banach space $B$, any $f \in [D(A); B]$ will be called a Banach space-valued distribution. Upon setting $A = \mathbb{C}$, $f$ becomes a $B - valued$ distribution and when $A = B = \mathbb{C}$, $f$ becomes a scalar distribution. The simple or weaker topology for $[D(A); B]$ is generated by the collection of seminorms $\{\gamma_\psi\}_\psi$, where $\psi$ traverse $D(A)$ and $\gamma_\psi(f) = ||\langle f, \psi\rangle||_B$.

Now, we shall define the space $[J_{\omega,z}(A), B]$.

If $\psi \in J_{\omega,z}, a \in A$, then $\psi a = a\psi$ is the function from $I$ into $A$ that assigns to each $t \in I$ the value $\psi(t)a$. Clearly $\psi a \in J_{\omega,z}(A)$. We denote by $J_{\omega,z} \odot A$ the subspace of $J_{\omega,z}(A)$ consisting of elements of the form $\psi a$.

If $g \in [J_{\omega,z}; \mathbb{C}]$ and $a \in A, ga = ag$ is defined by

$$\langle ga, \psi\rangle = \langle g, \psi\rangle \; a, \;\; \psi \in J_{\omega,z}. \tag{4.3}$$

Clearly, $ga \in [J_{\omega,z}; A]$.

$[J_{\omega,z}; \mathbb{C}] \odot A$ denotes the subspace of $[J_{\omega,z}; A]$ consisting of all linear combinations of elements of the form $ga$. Similarly if $g \in [J_{\omega,z}; [A, B]]$ and $a \in A$ , we define $ga$ by equation (3) again. Now $ga \in [J_{\omega,z}; B]$. $[J_{\omega,z}; [A, B]] \odot A$ denotes the subspace of $[J_{\omega,z}; B]$ consisting of all linear combinations of elements of the form $ga$. Thus every $f \in [J_{\omega,z}(A), B]$ uniquely defines a $g \in [J_{\omega,z}(A, B)]$ by means of the equation $\langle g, \psi\rangle a = \langle f, \psi a\rangle$, $\psi \in J_{\omega,z} a \in A$.

### 4.2. **Generalized Stieltjes transform.**

It can be shown that if

$$c < m\rho - \frac{1}{2} \text{ and } d > -\frac{1}{2}, \;\; k(s,t) = \overline{|(\rho)}S^{m\rho-1}(S^m + t^m)^{-\rho} \in J_{c,d}$$

where $m, \rho > 0$ and $-\pi < args < \pi$.
Now, let $f \in [J_{\omega,z}, B]$, and we define the generalized Stieltjes transform $F(s)$ of $f$ by $F(s) = \langle f(t), k(s,t)\rangle$.
The L.H.S. $F(s)$ defined as above is an analytic function and

$$\frac{d}{ds}F(s) = \langle f(t), \frac{d}{ds}k(s,t)\rangle.$$

Similarly, if $y \in [J_{\omega,z}(A); B]$, the generalized Stieltjes transform of $fy$,

$$Y(s) = \langle fy(t), \, k(s,t)\rangle,$$

where $fy \in [J_{\omega,z}; [A; B]]$. The above definition is meaningful because corresponding to $y$ we have a unique $fy$ as discussed in the previous section. Note that $Y(s)$ is an $[A; B] - valued$ function.

### 4.3. **An inversion theorem.**

We prove an inversion theorem.

**Theorem 4.2.** *If $y \in [J_{\omega,z}(A);B)]$ where $c < m\rho - \frac{1}{2}$ and $d > -\frac{1}{2}$, and let $Y(s)$ be the generalized Stieltjes transform of Banach space-valued distribution $y$ , then in the sense of weak convergence in $[D(A);B]$, for $m - d - \frac{1}{2} < m\sigma < m - c - \frac{1}{2}$.*

$$y(x) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} Q(l)M(l)x^{-ml+m-1}dl,$$

$$\textit{where} \quad l = \sigma + iT,$$

$$Q(l) = \frac{m}{\Gamma(1-l)\,\Gamma(e-1+l)},$$

$$\textit{and} \quad M(l) = \int_0^\infty s^{ml-m}Y(s)ds.$$

## 5. ON FOURIER-JACOBI TRANSFORMATION

We have constructed test function spaces $Z^{\alpha}_{\alpha,\beta}\Psi_{\alpha,\alpha,\beta}$ and the dual spaces. We define Fourier-Jacobi transformation on the dual space using Parseval relation following the technique of Gelfand and Shilov [18] and study some operation transform formulae related to the Laplace differential operator $\Delta_{\alpha,\beta}$ and its adjoint [11].

We recall some results from Mizony [29]. For $f \in L'(\mathbb{R}_+, W_{\alpha,\beta}(t)dt)$, $\mathrm{Re}(\alpha) > -1$, $\lambda \in \mathbb{R}_+$, then the Fourier-Jacobi transformation of $f$ given by

$$F_{\alpha,\beta}[f](\lambda) = F(\lambda) = 2^{2e+0.5}[\overline{\Gamma(\alpha+1)}]^{-1} \int_0^\infty f(t)\phi_{\alpha,\beta}(\lambda,t)W_{\alpha,\beta}(t)dt, \tag{5.1}$$

where

$$w = W_{\alpha,\beta}(t) = \sinh^{2\alpha+1}(t)\cosh^{2\beta+1}(t) \tag{5.2}$$

is a measure on $\mathbb{R}_+$ related to a pair $(\alpha, \beta)$ of complex numbers.

For $\alpha \neq -1, -2, -3, \ldots \quad \alpha, \beta \in \mathbb{C}, i\lambda \notin \mathbb{Z}$ and $t \in \mathbb{R}_+$,

$$\phi_{\alpha,\beta}(\lambda,t) = {}_2F_1((\rho+i\lambda)/2), ((\rho-i\lambda)/2; \alpha+1; -\sinh^2(t)) \tag{5.3}$$

is the Jacobi function of first kind which is one of the solutions of the differential operator equation

$$(\Delta_{\alpha,\beta} + \lambda^2 + \rho^2)f = 0, \tag{5.4}$$

where the Laplacian operator

$$\Delta_{\alpha,\beta} = w^{-1}DwD = D^2 + (w'/w)D \text{ with } D = \frac{d}{dt} \text{ and } \rho = \alpha + \beta + 1.$$

The particular cases of (5.1) are:

(i) For $\alpha = \beta = -0.5$, (5.1) is a Fourier transform .

(ii) When $\alpha = (n-2)/2, n \in \mathbb{N}^*$, $\beta = -0.5$ (5.1) gives a Fourier-Jacobi spherical transformation associated with the group $SO_0(n,1)$.

(iii) For $\alpha = n-1, n \in \mathbb{N}^*$, $\beta = 0$ then (5.1) reduces to the Fourier spherical transformation related to the group $SU(n,1)$.

(iv) If $\alpha = 2n-1$, and $\beta = 1$, we get (5.1) in the form of fourier spherical transformation associated with $SP(n,1)$, $n = 2, 3, 4, \dots$

(v) $\alpha = 7$, and $\beta = 3$, (5.1) becomes a transformation associated with exceptional group $F4(-,20)$.

The classical inversion formula for the Fourier-Jacobi transform is as below: When $\mathrm{Re}(\alpha) > -0.5$, $|\mathrm{Re}(\beta)| < \mathrm{Re}(\alpha+1)$ and $f \in L^1(\mathbb{R}_+, [C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda)]^{-1}d\lambda)$, then

(5.5)

$$F'_{\alpha,\beta}[f](t) = f(t) = \sqrt{2}\left[\Gamma(\alpha+1)\right]^{-1} \times \int_0^\infty f(\lambda)\phi_{\alpha,\beta}(\lambda,t)/\left[C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda)\right] d\lambda$$

where

(5.6)

$$C_{\alpha,\beta}(\lambda) = 2^\rho \Gamma(i\lambda/2)\,\Gamma[(1+i\lambda)/2)] \times \{\Gamma[(\alpha+\beta+1+i\lambda/2)]\,\Gamma[(\alpha-\beta+1+i\lambda/2)]\}^{-1}$$

is Harishchandra function.
The Bessel Parseval formula is as given below:

When $\alpha, \beta \in \mathbb{R}, |\beta| < \alpha+1$ then the Fourier-Jacobi transformation is an isometric isomorphism from $L^2(\mathbb{R}_+, 2^{2\rho}W_{\alpha,\beta}(t)dt)$ onto $L^2(\mathbb{R}_+, |C_{\alpha,\beta}(t)|^{-2}d\lambda)$ and for all $f, g \in L^2(\mathbb{R}_+, 2^{2\rho}W_{\alpha,\beta}(t)dt)$

$$\int_0^\infty f(t)g(t)2^{2\rho}W_{\alpha,\beta}(t)dt = \int_0^\infty F(\lambda)G(\lambda)|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda. \tag{5.7}$$

Observe that for $\mathrm{Re}(\alpha) > -\frac{1}{2}$

$$W_{\alpha,\beta}(t) = o(1) \text{ as } t \to 0+ \text{ and } W_{\alpha,\beta}(t) = O(e^{2\rho t}) \text{ as } t \to \infty$$

so that $W_{\alpha,\beta}(t) = o(e^{(2\rho+1)t})$ as $t \to \infty$.

The differential operator $\Delta^k_{\alpha\beta}$ for any $k = 0, 1, 2, \dots$ can be written in the form

$$\Delta^k_{\alpha\beta}(\phi) = \sum_{j=1}^{2k} b_j(t)\frac{d^j\phi}{dt^j}, \tag{5.8}$$

where $b_{2k} = 1$ and remaining $b_j(t)$ are functions of $w'/w$ and its derivatives. Also $w'/w = O(1)$ as $t \to \infty$. The result of Erdelyi [16, p. 76] says that for the large values of $t$ the function $\phi_{\alpha,\beta}(\lambda, t)$ can be put in the form

$$\lambda_1 \sinh^{-\rho-i\lambda}(t) + \lambda_2 \sinh^{-\rho+i\lambda}(t) + O(\sinh t)^{-\rho-i\lambda-2} + O(\sinh t)^{-\rho+i\lambda-2}, \tag{5.9}$$

where $\lambda_1, \lambda_2$ are constants and $i\lambda$ is not an integer. For small values of $t, W_{\alpha,\beta}(t)$ and $\phi_{\alpha,\beta}(\lambda, t)$take values close to 0 and 1 respectively.

5.1. **The spaces $Z^a_{\alpha,\beta}$ and $Z'^a_{\alpha,\beta}$.**

Let $I = (0, \infty)$ and $-\rho < \alpha < \infty$ , $Z^a_{\alpha,\beta} = Z_a$ be a space consisting of complex-valued infinitely differentiable functions $\theta(t)$ on $I$ such that the functionals

$$\eta^a_k(\theta) = \eta^a_{\alpha,\beta,k}(\theta) = \sup_{t\in I} |e^{at}\Delta^k_{\alpha,\beta} W_{\alpha,\beta}(t)\theta(t)| \tag{5.10}$$

For each $k = 0, 1, 2, \ldots$ assume finite values. The topology of $Z_a$ is determined by $\{\eta^a_k\}^\infty_{k=0}$. $Z_a$ is a countably multinormed space and is complete and hence a Frechet space. $Z_a$ is a testing function space.

The collection $Z'_a$ of continuous linear functionals on $Z_a$ is defined as dual space of $Z_a$. With the usual addition and multiplication by a complex number $Z'_\alpha$ is a linear space.

**Remark 5.1.** *(I) $D(I) \subset Z_a$. Convergence in $D(I)$ implies the convergence in $Z_a$. Restriction of any member of $Z'_a$ to $D(I)$ is a member of $D'(I)$. Hence $Z'_\alpha$ is the space of distributions and is complete due to the completeness of $Z_a$.*
*(II) $Z_a \subset E(I)$ . Moreover $Z_a$ is dense in $E(I)$ because $D(I) \subset Z_a \subset E(I)$ and $D(I)$ is dense in $E(I)$. Topology of $Z_a$ is stronger than the topology induced on it by $E(I)$. Then every generalized function on $I$ with compact support is in $Z'_\alpha$.*
*(III) When $f(t)$ is locally integrable function such that $f(t)[w(t)e^{at}]^{-1}$ is absolutely integrable on $I$, then $f(t)$ generates a regular member of $Z'_a$ through the definition $\langle f, \theta\rangle = \int\limits_0^\infty f(t)\theta(t)dt, \quad \theta \in Z_a$.*

5.2. **The space $\psi_{a,\alpha,\beta}$ of Fourier-Jacobi transforms.**

**Proposition 5.1.** *If $-\rho < \alpha < \infty, \alpha, \beta$ real and if $\theta \in Z_a$ then the Fourier-Jacobi transform*

$$F_{\alpha,\beta}[f](\lambda) = F(\lambda) = 2^{2e+0.5}[\overline{|(\alpha + 1)}]^{-1} \int\limits_0^\infty f(t)\phi_{\alpha,\beta}(\lambda, t)W_{\alpha,\beta}(t)dt$$

*exists.*

**Definition 5.1.** *For $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\lambda \in \mathbb{R}$ we construct a space $\psi_{a,\alpha,\beta} = \psi_a$ as follows*

$$\psi_a = \left\{F_{\alpha,\beta}(\theta)|C_{\alpha,\beta}(\lambda)|^{-2} \diagup \theta^{\in Z_a}\right\}, \tag{5.11}$$

*where $C_{\alpha,\beta}(\lambda)$ is the Harishchandra function defined by (5.6) . $\psi_a$ is a linear space under the usual addition and a multiplication by a number.*

We define a seminorm on $\psi_a$ as below:

If $\phi \in \psi_a$ then

$$\gamma^a_k(\phi) = \gamma^\alpha_{a,\beta,k}(\phi) = \gamma^\alpha_{a,\beta,k}(F_{\alpha,\beta}(\theta)|C_{\alpha,\beta}(\lambda)|^{-2}) = \eta^a_k(\theta) \tag{5.12}$$

for $\theta \in Z_a$ some satisfying $\phi(\lambda) = F_{\alpha,\beta}(\theta)|C_{\alpha,\beta}(\lambda)|^{-2}$.

**Remark 5.2.** *$\gamma_0^a$ is a norm on $\psi_a$; so that $\{\gamma_k^a\}_{k=0}^{\infty}$ is a countable multinorm on $\psi_a$. The countably multinormed space $\psi_a$ is also complete and hence a Frechet space. The topology of $\psi_a$ is determined by the multinorm $\{\gamma_k^a\}_{k=0}^{\infty}$.*

**Definition 5.2.** *We define the space $Z_{*a}$ as follows: $w(t)\theta(t) \in Z_{*a}$ if and only if $\theta \in Z_a$.*

It is clear that $Z_{*a}$ is a linear space under usual addition and multiplication by a complex number. We define a seminorm $\eta_k^{*a}$ on $Z_{*a}$ by

$$\eta_k^{*a}(\xi) = \eta_k^{*a}(w^{-1}\xi) = \eta_k^a(\theta), \tag{5.13}$$

where $\xi = w(t)\theta(t) \in Z_{*a}$ and $\theta \in Z_a$.

$\eta_k^{*a}$ is a norm hence the sequence $\{\eta_k^{*a}\}_{k=1}^{\infty}$ is a countable multinorm on $Z_{*a}$. $Z_{*a}$ is a complete and hence a Frechet space.

Note: $Z_{*a}$ is a testing space.

**Definition 5.3.** *The collection of linear and continuous functionals on $Z_{*a}$ is defined as the dual space of $Z_{*a}$ and is denoted by $Z'_{*a}$.*

The topology of $Z_{*a}$ is generated by the sequence of multinorm $\{\eta_k^{*a}\}_{k=1}^{\infty}$ . Since $Z_{*\alpha}$ is a testing function space, $Z'_{*\alpha}$ is the space of generalized functions and is complete due to the completeness of $Z_{*a}$.

For any $\xi(t) = w(t)\theta(t) \in Z_{*a}$ with some $\theta(t) \in Z_a$ and for any nonnegative integer $r$ , we set

$$\mu_r^*(\xi) = \operatorname*{Max}_{0\le k\le r} \eta_k^*(\xi) = \operatorname*{Max}_{0\le k\le r} \eta_k(w^{-1}\xi) = \operatorname*{Max}_{0\le k\le r} \eta_k(\theta) = \mu_r(\theta).$$

Then for each $f \in Z'_{*a}$ there exist a constant $C$ and a nonnegative integer $r$ such that

$$|\langle f(t), w(t)\theta(t)\rangle| \le \mu_r(\theta).$$

If $f(t)$ is locally and absolutely integrable on $I$ then $f(t)$ generates a regular generalized function in $Z'_{*a}$ by the formula

$$\langle f(t), w(t)\theta(t)\rangle = \int_0^{\infty} f(t)w(t)\theta(t)d(t), \quad \theta \in Z_a.$$

**Proposition 5.2.** *The mapping $S : \psi_a \to Z_{*a}$ defined by*

$$S(F_{\alpha,\beta}(\theta)|C_{\alpha,\beta}(\lambda)|^{-2}) = 2^{2p}\theta(t)w(t)$$

*is an isomorphism.*

5.3. **Generalized Fourier-Jacobi transformation.**

**Definition 5.4.** *For $\alpha, \beta \in \mathbb{R}, \alpha > -1, \lambda \in \mathbb{R}_+, \theta \in Z_a$, we define a Fourier-Jacobi transform $F'_{\alpha,\beta}(f)$ of a generalized function $f \in Z'_{*a}$ by a formula*

$$\langle F'_{\alpha,\beta}(f), F_{\alpha,\beta}(\theta)|C_{\alpha,\beta}(\lambda)|^{-2}\rangle = \langle f, \theta(t)2^{2\rho}w(t)\rangle \tag{5.14}$$

Since $\theta \in Z_a, 2^{2\rho}w(t)\theta(t) \in Z_{*a}$. Therefore, since $f \in Z'_{*a}$ the right-hand side of (4.1) has sense. Again since the mapping

$$F_{\alpha,\beta}(\theta)|C_{\alpha,\beta}(\lambda)|^{-2} \to 2^{2\rho}w(t)\theta(t)$$

is an isomorphism from $\psi_a$ to $Z_{*a}$, then the mapping of $f \to F'_{\alpha,\beta}(f)$ from $Z'_{*a}$ to $\psi'_a$ is an isomorphism. Our definition of Fourier-Jacobi transform is consistent with the Parseval equation.

Let $F'_{\alpha,\beta}[f] = g(\lambda)$ and $F_{\alpha,\beta}[\theta] = \phi(\lambda)$ then $f(t) = F^{'-1}_{\alpha,\beta}[g]$ and $\phi = F^{'-1}_{\alpha,\beta}[\theta]$. In view of this (4.1) becomes

$$\langle F^{'-1}_{\alpha,\beta}[g], F^{-1}_{\alpha,\beta}[\theta]2^{2\rho}w(t)\rangle = \langle g(\lambda), \phi(\lambda),\ |C_{\alpha,\beta}(\lambda)|^{-2}\rangle \tag{5.15}$$

**Theorem 5.1.** *If $f \in Z_a$ and $\alpha < \rho$ then for a nonnegative integer $k$*

$$\langle(\Delta'_{\alpha,\beta})^k f, \phi_{\alpha,\beta}(\lambda,t)/w\rangle = (-1)^k(\lambda^2+\rho^2)^k\langle f, \phi_{\alpha,\beta}(\lambda,t)/w\rangle \tag{5.16}$$

*where $\Delta'_{\alpha,\beta}$ is an adjoint operator defined as*

$$\langle \Delta'_{\alpha,\beta} f, \phi\rangle = \langle f, (w^{-1}\Delta_{\alpha,\beta}w)\phi\rangle. \tag{5.17}$$

## 6. DISTRIBUTIONAL GENERALIZED JACOBI TRANSFORMATION

In the paper [12], an extension of generalized Jacobi transform [40] with the use of Zemanian's technique related to the transformations arising from orthonormal series expansions [51] , Inversion formula, Characterization theorems and application of differential operator in solving certain type of differential equations have been discussed.

Sharma [40] defined generalized Jacobi transform as

$$u_n(x,y,t) = \int_c^d u(x,y,z,t)P_n^{(\alpha,\beta)}(z)dz \tag{6.1}$$

With the inversion formula

$$u(x,y,z,t) = \sum_{n=0}^{\infty} \frac{(z-c)^\beta \cdot (d-z)^\alpha}{\delta_n} u_n(x,y,t))P_n^{(\alpha,\beta)}(z), \tag{6.2}$$

where

$$\delta_n = \frac{(d-c)^{\alpha+\beta+1}\overline{|(\alpha+n+1)}\,\overline{|(\beta+n+1)}}{n!(\alpha+\beta+2n+1)\,\overline{|(\alpha+\beta+n+1)}} \tag{6.3}$$

and the kernel $P_n^{(\alpha,\beta)}(z)$ is a Jacobi polynomial

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (z-1)^{n-m}(z+1)^m. \tag{6.4}$$

Using the procedure explained by Rainville [37, p. 258] to the differential equation.

$$\frac{d}{dz}\left[(z-c)(d-z)\frac{d}{dz}P_n^{(\alpha,\beta)}(z)\right] + \{(d\beta + c\alpha) - (\alpha+\beta)z\}$$

$$\frac{d}{dz}P_n^{(\alpha,\beta)}(z) + n(\alpha+\beta+n+1)P_n^{(\alpha,\beta)}(z) = 0 \tag{6.5}$$

we get

$$\int_c^d (z-c)^\beta (d-z)^\alpha P_n^{(\alpha,\beta)}(z) P_m^{(\alpha,\beta)}(z)dz = 0, \ m \neq n \tag{6.6}$$

and for $m = n$

$$\int_c^d (z-c)^\beta (d-z)^\alpha [P_n^{(\alpha,\beta)}(z)]^2 dz = \delta_n, \tag{6.7}$$

where $\delta_n$ is as given in (6.3). Thus when $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ form an orthogonal set over $I = (c-d), c < d$ on the real line; with respect to weight function

$$w(z) = (z-c)^\beta (d-z)^\alpha. \tag{6.8}$$

Further we select $\alpha, \beta$ real such that $\alpha > -1$, $\beta > -1$.

For $c = -1, d = -1$ we get distributional Jacobi transform as a special case. Further for $c = -1, d = -1$ and specializing $\alpha, \beta$ we get extensions related to Legendre, Gegenbour, Chebyshev transformations [51, p. 258–269] as particular cases.

### 6.1. The spaces $J(I)$ and $J'(I)$.

On comparing the generalized Rodrigue formula [40]

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{n!(d-c)^n (z-c)^\beta (z-d)^\alpha} \frac{d^n}{dz^n}\left[(z-c)^\beta (z-d)^\alpha \{(z-c)(z-d)\}^n\right]$$

with that in Erdelyi [16, p. 164] we get the quadratic polynomial

$$X = (z-c)(z-d) \tag{6.9}$$

whose coefficients are independent of $n$. We define

$$\eta = [w(z)]^{-1/2} D X w(z) D [w(z)]^{-1/2},$$

where $D \equiv \frac{d}{dz}$. That is

(6.10) $\eta = (z-c)^{-\beta/2}(d-z)^{\alpha/2}D(-1)(z-c)^{\beta+1}(d-z)^{\alpha+1}D(z-c)^{-\beta/2}(d-z)^{-\alpha/2}.$

For a non-negative integer $n$, let us define

$$\psi(z) = \sqrt{\frac{w(z)}{\delta_n}}P_n^{(\alpha,\beta)}(z). \tag{6.11}$$

The differential equation (6.5) can be written down in the form

$$D\left[(z-c)^{\beta+1}(d-z)^{\alpha+1}DP_n^{(\alpha,\beta)}(z)\right]+n(\alpha+\beta+n+1)(z-c)^{\beta}(d-z)^{\alpha}P_n^{(\alpha,\beta)}(z) = 0.$$

Comparing this self adjoint form of the differential equation with equation (3) of Erdelyi [16, p. 167], we can have sequence $\{\lambda_n\}_{n=0}^{\infty}$ of eigenvalues where

$$\lambda_n = n(\alpha+\beta+n+1). \tag{6.12}$$

Definitely $|\lambda_n| \to \infty$ as $n \to \infty$ and

$$\eta\psi_n = \lambda_n\psi_n, \quad n = 0, 1, 2, 3, \ldots. \tag{6.13}$$

The sequence $\{\phi_n\}_{n=0}^{\infty}$ of smooth functions in $L_2(I)$ form a complete orthonormal system in it.

Let $J(I)$ be the collection of all complex-valued smooth functions $\phi(z)$ defined on $I$ such that

(i) for any non-negative integer $k$,

$$\gamma_k(\phi) = \gamma_0(\eta^k\phi(z)) = \left[\int_c^d |\eta^k\phi(z)|^2 dz\right]^{1/2} < \infty \tag{6.14}$$

ii) for each pair of non-negative integers $n, k$

$$(\eta^k\phi, \psi_n) = (\phi, \eta^k\psi_n). \tag{6.15}$$

Every number of the sequence $\{\psi_n\}_{n=0}^{\infty}$ of eigenfunctions, is a number of $J(I)$.

The operator $\eta$ is a continuous linear mapping from $J(I)$ into itself. Continuity is established from the fact that

$$(\eta\phi_\nu, \psi_n) = (\phi_\nu, \eta\psi_n) = \lambda_n(\phi_\nu, \psi_n) \to 0 \text{ as } \nu \to \infty, \quad \text{whenever } \{\phi_\nu\}_{\nu=0}^{\infty}$$

converges to the zero function in $J(I)$ .

$J(I)$ is a linear space under addition and multiplication by a complex number. Clearly $\gamma_0$ is a norm and $\{\gamma_k\}_{k=0}^{\infty}$ is a separating collection of seminorms, hence is a countable multinorm on $J(I)$. We equip $J(I)$ with the topology generated by $\{\gamma_k\}_{k=0}^{\infty}$. Thus $J(I)$ is a countably multinormed space. Every Cauchy sequence in $J(I)$ converges in it hence $J(I)$ is complete and therefore it is a Frechet space. Also $J(I)$ is a testing function space.

6.2. **The Dual Space $J'(I)$.**

The collection of all linear continuous functionals on $J(I)$ is the dual space $J'(I)$. Since $J(I)$ is a testing function space, $J'(I)$ is a space of generalized functions. As usual, the number that $f \in J'(I)$ assigns to any $\phi \in J(I)$ is denoted by $\langle f, \phi \rangle$. We define

$$(f, \phi) = \langle f, \overline{\phi} \rangle, \quad \phi \in J(I) \tag{6.16}$$

Obviously for any complex number $a$,

$$(af, \phi) = a(f, \phi) = (f, \overline{a}\phi). \tag{6.17}$$

Definitely $J'(I)$ is a linear space. Since $J(I)$ is complete, $J'(I)$ is also complete by the Theorem [51].

We now define the generalized differential operator $\overline{\eta}'$ on $J'(I)$ Through

$$(f, \eta\phi) = \langle f, \overline{\eta\phi} \rangle = \langle \overline{\eta}' f, \overline{\phi} \rangle = (\overline{\eta}' f, \phi).$$

Since $\eta$ is self adjoint $\overline{\eta}' = \eta$. Thus

$$(\eta f, \phi) = (f, \eta\phi), \; f \in J'(I), \, \phi \in J(I). \tag{6.18}$$

We can easily prove the property that $\eta : J'(I) \to J'(I)$ is a continuous linear mapping by making use of the fact that $\eta$ is a continuous linear mapping of $J(I)$ into itself.

6.3. **Some Properties of $J(I)$ and $J'(I)$.**

(I) $J(I) \subset L_2(I)$ when we identify each function in $J(I)$ with the corresponding equivalence class in $L_2(I)$. Convergence in $J(I)$ implies the convergence in $L_2(I)$.

(II) $D(I) \subset J(I)$. Convergence in $D(I)$ implies the convergence in $J(I)$. The topology of $D(I)$ is stronger than that induced on it by $J(I)$. The restriction of any member $f$ in $J'(I)$ to $D(I)$ is a member of $D'(I)$. Moreover, convergence in $J'(I)$ implies the convergence in $D'(I)$. Hence in Zemanian [51] sense, the members of $J'(I)$ are distributions.

(III) $J(I) \subset \varepsilon(I)$. Furthermore if $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in $J(I)$ to the limit, say $\phi$ then $\{\phi_\nu\}_{\nu=1}^{\infty}$ also converges in $\varepsilon(I)$ to the same limit $\phi$.

(IV) Since $D(I) \subset J(I) \subset \varepsilon(I)$ and $D(I)$ is dense in $\varepsilon(I)$, $J(I)$ is also dense in $\epsilon(I)$. The topology of $J(I)$ is stronger than the topology induced on $J(I)$ by $\varepsilon(I)$. Hence $\varepsilon'(I)$ is subspace of $J'(I)$.

(V) We make $L_2(I)$ a subspace of $J'(I)$, by defining the number that $f \in L_2(I)$ assigns to any $\phi \in J(I)$ as

$$(f, \phi) = \int_c^d f(z)\overline{\phi(z)} dz \tag{6.19}$$

Since $J(I)$ is a subspace of $L_2(I)$, it is clear that $J(I)$ is imbedded in $J'(I)$. Also $f$ is linear and continuous on $J(I)$.

(VI) If $f(z) = \eta^k g(z)$ for some $g \in L_2(I)$ and some $k$ then $f \in J'(I)$.

Indeed, $\eta$ is a linear and continuous mapping of $J'(I)$ into itself and $L_2(I) \subset J'(I)$ implies that $f \in J'(I)$.

(VII) For each $f \in J'(I)$ there exists a positive constant $C$ and a non-negative integer $r$ such that for every $\phi \in J(I), |(f,\phi)| \le C\rho_r(\phi)$, where $\rho_r = \max\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ and $C, r$ depend on $f$ but not on $\phi$.

6.4. **Orthogonal series expansion of a generalized function in $J'(I)$.**

The following fundamental theorem provides an orthonormal series expansion of any $f \in J'(I)$ with respect to $\psi_n$ which in turn yields an inversion formula for the generalized integral transformation.

**Theorem 6.1.** *Every $f \in J'(I)$ has series expansion*

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\psi_n \tag{6.20}$$

*which converges in $J'(I)$.*

The orthogonal series expansion (3.1) gives inversion formula for a distributional generalized Jacobi transform $\Im$ defined by

$$\Im f = F(n) = (f, \psi_n) = \langle f, \psi_n \rangle, \quad f \in J'(I),\ n = 0, 1, 2, \ldots. \tag{6.21}$$

Thus $\Im$ is a mapping of $J'(I)$ into the space of complex-valued functions $F(n)$ defined on $n$ and

$$\Im^{-1} F(n) = f = \sum_{n=0}^{\infty} F(n)\psi_n. \tag{6.22}$$

$\Im$ is a continuous linear mapping.

**Theorem 6.2.** *(Uniqueness) Let $f, g \in J'(I)$ and $\Im f = F(n), \Im g = G(n)$, satisfy $F(n) = G(n)$ for every $n$, then $f = g$ in the sense of equality in $J'(I)$.*

6.5. **Characterization of Distributional Generalized Jacobi Transforms.**

We now characterize the functions $F(n)$ which are generalized Jacobi transforms of distributions in $J'(I)$.

**Theorem 6.3.** *For complex numbers $b_n$, the series*

$$\sum_{n=0}^{\infty} b_n\psi_n \tag{6.23}$$

*converges in $J'(I)$ if and only if there exists a non-negative integer $q$ such that $\sum_{\lambda_n \ne 0} |\lambda_n|^{-2q}|b_n|^2$ converges . Moreover if $f$ denotes the sum (4.1) in $J'(I)$ then $b_n = (f, \psi_n)$.*

**Theorem 6.4.** *$f$ is a member of $J'(I)$ if and only if there exists some non-negative integer $m$ and a $g \in L_2(I)$ such that*

$$f = \eta^m g + \sum_{n=0}^{\infty} b_n \psi_n, \tag{6.24}$$

*where $b_n$ are complex numbers.*

6.6. **Application of Differential operator.**

Using the series expansion (6.22) of $f \in J'(I)$ we can write for a non-negative integer $k$,

$$\eta^k f = \sum_{n=0}^{\infty} (f, \psi_n) \eta^k \psi_n = \sum_{n=0}^{\infty} (f, \psi_n) \lambda_n^k \psi_n. \tag{6.25}$$

Now consider the differential equation

$$P(\eta) f = g, \tag{6.26}$$

where $P$ is a polynomial, $g \in J'(I)$ is known and $f \in J'(I)$ is unknown. Since $f \in J'(I)$,

$$P(\eta) f = P(\eta) \sum_{n=0}^{\infty} (f, \psi_n) \psi_n = \sum_{n=0}^{\infty} (f, \psi_n) P(\eta) \psi_n = \sum_{n=0}^{\infty} (f, \psi_n) P(\lambda_n) \psi_n.$$

Applying $\Im$ to (6.26) we get $P(\lambda_n)(f, \psi_n) = (g, \psi_n)$.
Case (i). $P(\lambda_n) \neq 0$. Then $(f, \psi_n) = [P(\lambda_n)]^{-1}(g, \psi_n)$.

We now apply $\Im^{-1}$ to both the sides we get

$$f = \sum_{n=0}^{\infty} (g, \psi_n) [P(\lambda_n)]^{-1} \psi_n. \tag{6.27}$$

By the characterization theorem and the uniqueness theorem the solution in (6.27) exists and is unique.
Case (ii). $P(\lambda_n) = 0$ for some $\lambda_n$. Let

$$P(\lambda_{n_k}) = 0 \text{ for} k = 1, 2, 3, \ldots, m.$$

Then in this case solution will exist in $J'(I)$ if and only if

$$G(n_k) = 0, \text{ for } k \neq 1, 2, 3, \ldots, m.$$

The solution will be then

$$f = \sum_{P(\lambda_n) \neq 0} (g, \psi_n) [P(\lambda_n)]^{-1} \psi_n, \tag{6.28}$$

which is not unique in $J'(I)$. We may add to (6.28) any complementary solution $f_c = \sum_{s=0}^{\infty} a_s \psi_{n_s}$,

where $a_s$ are arbitrary numbers.

## 7. FRACTIONAL FOURIER TRANSFORM OF DISTRIBUTIONS OF COMPACT SUPPORT

The fractional Fourier transform (fractional FT) $R^{\alpha}$ is an extension of the ordinary Fourier transform and depends on a parameter $\alpha$ that can be interpreted as assort of rotation by an angle $\alpha$ in the position frequency plane [1]. Pathak [30] gave a comprehensive account of the Fourier transform on various spaces of distributions including distributions of the compact support. Motivated by the above we have extended fractional FT to the distributions of compact support [4].

This theory is similar to that of Fourier transform of distributions of compact support and can be viewed as the extension of that theory. As such, the theory presented here to some extent generalizes the Fourier-Laplace transform in view of the fact that the theories of the Fourier analysis can be obtained as special cases of the theory being presented. We particularly note here that for the parameter $\alpha = \pi/2$ the fractional FT reduces to the ordinary Fourier transform.

**Definition 7.1. The fractional FT**. *The one dimensional fractional FT with parameter $\alpha$ of $f(x)$ denoted by $R^{*}f(x)$, performs a linear operation, given by the integral transform*

$$[R^{\alpha}f(x)](\xi) = F_{\alpha}(\xi) = \int_{-\infty}^{\infty} K_{\alpha}(x,\xi)f(x)dx, \tag{7.1}$$

*where the kernel*

$$K_{\alpha}(x,\xi) = (2\pi i \sin\alpha)^{-1/2}\exp(i\alpha/2)\exp(i/2\sin\alpha((x^2+\xi^2)\cos\alpha - 2x\xi)),$$

[1].

$K_{\alpha}(x,\xi)$, is the propagator of the non-stationary Schrdinger equation for harmonic oscillator, which is well known equation in quantum mechanics (where $\alpha = \omega t$ relates to time $t$ and classical frequency $\omega$, and $\xi$ is a position at the moment $t$). Changing gradually the angle $\alpha$, the fractional FT permits to look at the continuous transformation of an input function $f(x)$ to its Fourier image $F_{\pi/2}(\xi)$ for $\alpha = \pi/2$, then to $f(-x)$ for $\alpha = \pi$ and to $F_{\pi/2}$ for $\alpha = 3\pi/2$. The fractional FT at angles equal to $2\pi n$ ($n$ is an integer) corresponds to the identity operator [1].

It is possible to recover the function $f$ by means of the inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\alpha}(\xi)\overline{K}_{\alpha}(x,\xi)d\xi, \tag{7.1a}$$

where

$$\overline{K}_\alpha(x,\xi) = 2C_{2\alpha}(C_{2\alpha})^{-1}\exp\left(-iC_{2\alpha}((x^2+\xi^2)\cos\alpha - 2x\xi)\right)$$
$$= (1/\sin\alpha)(2\pi i\sin\alpha)^{1/2}\exp\left(-i\alpha/2\right)$$
$$\exp\left(-i\alpha/2\ \sin\alpha((x^2+\xi^2)\cos\alpha - 2x\xi)\right).$$

As obtained below.

The one dimensional fractional FT is given by (7.1), where the kernel $K_\alpha(x,\xi)$ can be written as

$$K_\alpha(x,\xi) = C_{1\alpha}\exp\left(-iC_{2\alpha}((x^2+\xi^2)\cos\alpha - 2x\xi)\right)$$
$$= C_{1\alpha}\exp(iC_{2\alpha}x^2\cos\alpha)\exp(iC_{2\alpha}\xi^2\cos\alpha)\exp(-2iC_{2\alpha}x\xi)$$

As obtained below:

The one dimensional fractional FT is given by (7.1), where the kernel $K_\alpha(x,\xi)$ can be written as

$$K_\alpha(x,\xi) = C_{1\alpha}\exp\left(-iC_{2\alpha}((x^2+\xi^2)\cos\alpha - 2x\xi)\right)$$
$$= C_{1\alpha}\exp(iC_{2\alpha}x^2\cos\alpha)\exp(iC_{2\alpha}\xi^2\cos\alpha)\exp(-2iC_{2\alpha}x\xi),$$

where $C_{1\alpha} = (2\pi i\sin\alpha)^{1/2}\exp(i\alpha/2)$ and $C_{2\alpha} = 1/2\sin\alpha$.

Therefore we can write (7.1) as

$$\exp(-iC_{2\alpha}\xi^2\cos\alpha)F_\alpha(\xi) = \int_{-\infty}^{\infty} f(x)C_{1\alpha}\exp(iC_{2\alpha}x^2\cos\alpha)\exp(-2iC_{2\alpha}x\xi)dx$$
$$= \int_{-\infty}^{\infty} g(x)\exp(-2iC_{2\alpha}x\xi)dx$$
$$= F[g(x)](2C_{2\alpha}\xi).$$

The Fourier transform of $g(x)$ with argument $2C_{2\alpha}\xi = \eta$ (say),
where $g(x) = f(x)C_{1\alpha}\exp(iC_{2\alpha}x^2\cos\alpha)$ (say).
Thus, $\exp(-iC_{2\alpha}\xi^2\cos\alpha)F(\alpha)(\eta/2C_{2\alpha}) = F[g(x)](\eta)$.

Invoking Fourier inversion, we can write

$$g(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} G(\eta) \exp(ix\eta) d\eta$$

where $G(\eta) = \exp(-2iC_{2\alpha}\eta^2/4C_{2\alpha}^2 \cos\alpha) F_\alpha(\eta/2C_{2\alpha})$ (say). Therefore,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\alpha(\xi) \left[C_{1\alpha} \exp(iC_{2\alpha}x^2 \cos\alpha)\right]^{-1} \exp(-iC_{2\alpha}\xi^2 \cos\alpha) \\ &\qquad \exp(2iC_{2\alpha}x\xi) 2C_{2\alpha} d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\alpha(\xi) \overline{K}_\alpha(x,\xi) d\xi, \end{aligned}$$

where

$$\begin{aligned} \overline{K}_\alpha(x,\xi) &= 2C_{2\alpha}(C_{1\alpha})^{-1} \exp\left(-iC_{2\alpha}((c^2+\xi^2)\cos\alpha - 2x\xi)\right) \\ &= (1/\sin\alpha)(2\pi i \sin\alpha)^{1/2} \exp(-i\alpha/2) \\ &\qquad \exp\left(-i/2\sin\alpha((x^2+\xi^2)\cos\alpha - 2x\xi)\right). \end{aligned}$$

**Definition 7.2. The test function space $E$.** *An infinitely differentiable complex valued func tion $\phi$ on $\mathbb{R}^n$ belongs to $E(\mathbb{R}^n)$ if for each compact set $K \subset S_\alpha$ where*

$$S_a = \{x :\in \mathbb{R}^n, |x| \le a,\ a > 0\},\ K \in \mathbb{R}^n,$$

$$\gamma_{E,k}(\phi) = \sup_{x \in K} |D^k \phi(x)| < \infty.$$

Note that the space is $E$ complete and therefore $E$ is a Frechet space. Moreover, we say that $f$ is a fractional Fourier transformable if it is a member of $E'$, the dual space of $E$.

**The fractional FT on $E'$.** It is easily seen that for each $\xi \in \mathbb{R}^n$ and $0 \le \alpha \le \pi/2$, the function $K_\alpha(x,\xi)$ belongs to $E(\mathbb{R}^n)$ as a function of $x$.

Hence, the fractional FT of $f \in E'(\mathbb{R}^n)$ can be defined by

$$[R^\alpha f(x)](\xi) = F_\alpha(\xi) = \langle f(x), K_\alpha(x,\xi)\rangle, \tag{7.2}$$

where the right-hand side has a meaning as the application of $f \in E'$ to $K_\alpha(x,\xi) \in E$.

It can be extended to the complex space as an entire function given by

$$[R^\alpha f(x)](\zeta) = F_\alpha(\zeta) = \langle f(x), K_\alpha(x,\zeta)\rangle \tag{7.3}$$

The right-hand side is meaningful because for each $\zeta \in C^n, K_\alpha(x,\zeta) \in E$, as a function of $x$.

The basic properties of the fractional FT are contained in the following theorem.

**Theorem 7.1.** *Let $f \in E'(\mathbb{R}^n)$ and its fractional FT be defined by (7.3). Then $(R^\alpha f)(\zeta)$ is analytic on $C^n$ if the supp $f \subseteq S_a = \{x : x \in \mathbb{R}^n, |x| \le a,\ a > 0\}$ and* *for each $\epsilon > 0$, there exists a constant $C_1$ and a positive integer $k$ such that for $0 \leq \alpha \leq \pi/2$,*

$$|F_\alpha(\zeta)| \leq C_1 C_{1\alpha} C_{2\alpha}^k (1 + 2(a\cos\alpha + |\zeta|))^k \exp\left(C_{2\alpha}((a+\epsilon)^2 + |Im\zeta|^2)\cos\alpha - 2(a+\epsilon)|Im\zeta|\right) \tag{7.4}$$

*Moreover, $F_\alpha(\zeta)$ is differentiable and*

$$D_\zeta^k F(\zeta) = \langle f(x), D_\zeta^k K_\alpha(x,\zeta)\rangle. \tag{7.5}$$

The next theorem provides the converse of the above theorem.

**Theorem 7.2.** *Every entire function $F$ satisfying*

$$|F_\alpha(\zeta)| \leq C_1 C_{1\alpha} C_{2\alpha}^k (1 + 2(a\cos\alpha + |\zeta|))^k \exp\left(C_{2\alpha}((a+\epsilon)^2 + |Im\zeta|^2)\cos\alpha - 2(\alpha+\epsilon)|Im\zeta|\right)$$

*is the fractional FT of the distribution $f$ belonging to $E'$ with support of $f \subset S_\alpha$ and $0 \leq \alpha \leq \pi/2$.*

**Theorem 7.3. (Inversion)** *Let $f \in E'(\mathbb{R})$, $0 \leq \alpha \leq \pi/2$ and supp $f \subseteq S_a = \{x : x \in \mathbb{R}^n, |x| \leq a,\ a > 0\}$ and $F_\alpha(\zeta)$ be the distributional fractional FT of $f$ as defined by*

$$[R^\alpha f(x)](\zeta) = F_\alpha(\zeta) = \langle f(x), K_\alpha(x,\zeta)\rangle.$$

*Then for each $\phi \in D(\mathbb{R})$ we have*

$$\lim_{N\to\infty}\left\langle \frac{1}{2\pi}\int_{-N}^{N} \overline{K}_\alpha(x,\zeta)F_\alpha(\zeta)d\zeta, \phi(x)\right\rangle = \langle f,\phi\rangle.$$

**Theorem 7.4. (Uniqueness)** *If $[R^\alpha f(x)](\zeta) = F_\alpha(\zeta)$ and $[R^\alpha g(x)](\zeta) = G_\alpha(\zeta)$ for $0 \leq \alpha \leq \pi/2$ and supp $f \subseteq S_a = \{x : x \in \mathbb{R}^n, |x| \leq a,\ a > 0\}$ and supp $g \subseteq S_a = \{x : x \in \mathbb{R}^n, |x| \leq a,\ a > 0\}$ If $F_\alpha(\zeta) = G_\alpha(\zeta)$ then $f = g$ in the sense of equality in $D'(I)$.*

We now establish a convolution theorem for distributions.

**Theorem 7.5.** *Let $f \in S(\mathbb{R})$ and $g \in E(\mathbb{R})$. Then $f * g \in S(\mathbb{R})$ and for $0 \leq \alpha \leq \pi/2$,*

$$R^\alpha(f * g) = R^\alpha(f)R^\alpha(g).$$

For proving this result we use the results from [30].

## 8. QUASIASYMPTOTIC EXPANSION OF DISTRIBUTIONS WITH APPLICATIONS TO THE STIELTJES TRANSFORM

By following the approach of Drozinov [15], Vladmirov and Zavialov [46], we investigate the quasiasymptotic expansion of distributions. We give Abelian type results for the ordinary asymptotic behavior of the distributional modified Stieltjes transform of a distribution with appropriate quasiasymptotic expansion.

Pilipovic and Stankovic in [32, 33] followed the definition of the distributional Stieltjes transform given by Lavoine and Misra [23]. It enabled them to use the strong theory of the space of tempered distributions $S'$. In fact, they slightly generalized the definition of Lavoine and Misra. Using the notion of the quasiasymptotic behavior of distributions from $S'_+ = \{f \in S' : supp f \subset [0, \infty)\}$, introduced by Zaviolov [48] , they obtained more general results than in [8, 22, 23] for the asymptotic behaviour of the distributional Stieltjes transform at $\infty$ and $0^+$.

McClure and Wong [27, 47] studied the asymptotic expansion of the generalized Stieltjes transform of some classes of locally integrable functions characterized by their asymptotic expansions at $\infty$ and $0^+$.

Our approach to the asymptotic expansion of the distributional modified Stieltjes transform studied in the paper [19] is quite different from the approach given by McClure and Wong [27, 47].

We give the definition of the quasiasymptotic expansion at $\infty$ of a distribution from $S'_+$ given by Drozinov and Zavialov [15, p. 385]. They also gave the definition of the quasiasymptotic expansion at $0^+$ of an element from $S'_+$. We give the definition of space $M'(r)$, Stieltjes transformation, Modified Stieltjes transformation $T_{r+1}$, Generalized Modified Stieltjes transformation $\overline{T_{r+1}}$. This enables us to obtain the asymptotic expansion at $\infty$ and at $0^+$ of the modified Stieltjes transforms of appropriate distributions from $S'_+$ .

Domains defined by McClure and Wong [47] and by us on which the modified Stieltjes transform is defined do not contain each other.

A positive continuous function $L$ defined on $(0, \infty)$ is called slowly varying at $\infty(0^+)$ if for every $k > 0$

$$\lim_{t\to\infty} \frac{L(kt)}{L(t)} = 1, \qquad \left( \lim_{t\to 0^+} \frac{L(kt)}{L(t)} = 1 \right).$$

We denote by $\sum_\infty(\sum_{0^+})$ the set of all slowly varying functions at $\infty(0^+)$. If $L$ is a slowly varying functions at $\infty(0^+)$, then for every $\epsilon > 0$ there is $A_\epsilon > 0$ so that

$$x^{-\epsilon} < L(x) < x^{\epsilon},\ (x^{-\epsilon} > L(x) > x^{\epsilon}) \text{ if } x > A_\epsilon,\ (0 < x < A_\epsilon).$$

This Property of $L$ and corresponding properties of $S_m$, [47] imply the following assertion. Let $G \in L^1_{loc}$, $\operatorname{supp} G \subset [0, \infty], \alpha > -1$ and $G(x) \sim x^\alpha L(x)$ as

$x \to \infty (x \to 0^+)$. Then

$$G(kx)/(k^\alpha L(k)) \to S_+^\alpha,\ k \to \infty,\ \text{in } S'_+ \text{ for } t > \alpha + 1.$$

Recall, for $\alpha > -1$, $x_+^\alpha = H(x)x^\alpha; H$ is Heaviside's function. (The symbol $\sim$ is related to the ordinary asymptotic behaviour.)

The following scale of distributions from $S'$ has been used in investigations of the quasiasymptotic behaviour of distributions.

$$f_{\alpha+1} = \begin{cases} \frac{Ht^\alpha}{\overline{|(\alpha+1)}}, & \alpha > -1 \\ D^n f_{\alpha+n+1}, & \alpha > -1, \alpha + n > -1 \end{cases}$$

where $D$ is the distributional derivative.

**Definition 8.1. The Quasi-asymptotic Behaviour of Distribution (q. a. b.) at infinity.** *If $T$ is a distribution from $S'_+$ such that the distributional limit $\lim_{k\to\infty} \frac{T(kx)}{\rho(k)} = \gamma(x)$ exists in $S'$, $(\gamma(x) \neq 0)$ then, $T$ is said to have the quasiasymptotic behaviour at infinity related to the regularly varying function $\rho(k) = k^\alpha L(k)$ with limit $\gamma$. We can write this as $T \overset{q}{\sim} \gamma$ in $S'$ as $x \to \infty$.*

Here $\rho$ is regularly varying at infinity and limit $\gamma \in S'$, is of the form $\gamma(x) = Cf_{\alpha+1}(x)$ for a non zero constant $C$.

Let $f \in S'_+$. It is said that $f$ has the q. a. b. at $\infty$ $(0^+)$ with the limit $g \neq 0$ with respect $k^\alpha L(k), L \in \sum_\infty$ $((1/k)^\alpha L(1/k), L \in \sum_0), \alpha \in \mathbb{R}$, if

$$\left\{ \begin{array}{c} \lim_{k\to\infty} \langle \frac{f(kt)}{k^\alpha L(k)}, \phi(t) \rangle = \langle g(t), \phi(t) \rangle, \phi \in S \\ \lim_{k\to\infty} \left\langle \frac{f(k/t)}{(1/k^\alpha)\, L\,(1/k)}, \phi(t) \right\rangle = \langle g(t), \phi(t) \rangle, \phi \in S \end{array} \right\}. \tag{8.1}$$

We need the following structural theorem [45, 32].

**Structural theorem** *Let $f \overset{q}{\sim} g$ at $\infty$ $(0^+)$ with respect to $k^\alpha L(k)((1/k)^\alpha L(1/k))$. Then there exists $F \in L^1_{loc}$, $supp\, F \subset [0, \infty), C \neq 0$ and $m \in N_0, m + \alpha > -1$, such that*

$$\begin{aligned} &f = D^m F, F(k) \sim Ck^{m+\alpha} L(k), k \to \infty \\ &\left(F(1/k) \sim C(1/k)^{m+\alpha} L(1/k), k \to \infty\right). \end{aligned} \tag{8.2}$$

We remark that the q. a. b. at $0^+$ is a local property while the q. a. b. at $\infty$ is a global property of an $f \in S'_+$.

**Definition 8.2.** *Let $\alpha \in \mathbb{R}$ and $L \in \sum_\infty$ $(L \in \sum_0)$. We put*

$$(f_L)_{\alpha+1} = \begin{cases} H(t)L(t) \cdot t^\alpha / \overline{|(\alpha+1)}, & \alpha > -1 \\ D^n (f_L) - \alpha + n + 1, & \alpha < -1, x + n > -1 \end{cases} \tag{8.3}$$

*where $n$ is the smallest natural number such that $\alpha + n > -1$. Obviously, $(f_L)_{\alpha+1} \overset{q}{\sim} f_{\alpha+1}$ at $\infty$ $(0^+)$ with respect to $k^\alpha L(k)((1/k)^\alpha L(1/k))$.*

**Definition 8.3.** *We say that an $f \in S'_+$ has the closed q. a. e. at $\infty(0^+)$ of order $(\alpha, L) \in R \times \sum_\infty$ $((\alpha, L) \in R \times \sum_0)$ and of length $l$, $0 \leq l < \infty$, with respect to*

$$k^{\alpha-1}L_0(k)((1/k)^{\alpha+1} \times L_0(1/k))$$

*if $f$ has the q. a. b. at $\infty(0^+)$ with respect to $k^\alpha L(k)$ and if there exist*

$$\alpha_i \in R,\, L_i \in \sum_\infty (L_i \in \sum_{0^+}), c_i \in C, \quad i = 1, 2, \ldots, N,$$

$$\alpha_1 \geq \alpha_2 \geq \alpha_N \ (\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_N)$$

*and that $f$ is of the form*

$$f(t) = \sum_{i=1}^{N} c_i(f_{L_i})_{\alpha_{i+1}}(t) + h(t) \tag{8.4}$$

*such that*

$$\left.\begin{array}{c} \lim\limits_{x\to\infty} \langle \dfrac{h(kt)}{k^{\alpha-1}L_0(k)}, \phi(t)\rangle = 0, \phi \in S \\ \left(\lim\limits_{k\to\infty} \langle \dfrac{h(k/t)}{(1/k)^{\alpha+1}L_0(1/k)}, \phi(t)\rangle = 0, \phi \in S\right) \end{array}\right\}. \tag{8.5}$$

Obviously, we shall assume that $c_i \neq 0, i = 1, 2, \ldots, N$ and that $\alpha_N \geq \alpha - l\,(\alpha_N \leq \alpha + l)$. Since the sum of the two slowly varying functions is a slowly varying one, we can and we shall always assume that in the representation (8.4)

$$\alpha_1 > \alpha_2 > \ldots > \alpha_N \ (\alpha_1 < \alpha_2 < \ldots < \alpha_N).$$

$$\text{Namely } (f_{L_j})_{\beta+1} + (f_{L_k})_{\beta+1} = (f_{L_j+L_k})_{\beta+1}.$$

$(f_{L_j})_{\beta_1+1}$ and $(f_{L_k})_{\beta_1+1}$ have the same q. a. b. at $\infty(0^+)$ iff $\beta_1 = \beta_2$ and $L_j \sim L_k$.

**Theorem 8.1.** *Let $f \in S'_+$ satisfy conditions of above definition and assume that there are two representation of $f$,*

$$f(t) = \sum_{i=1}^{N} c_i(f_{L_i})_{\alpha_{i+1}} + h(t) \text{ and } f(t) = \sum_{i=1}^{M} \tilde{c}_i(f_{L_i})_{\widetilde{\alpha_{i+1}}} + \tilde{h}(t)$$

*for which all the assumptions given above hold. Then*

$$M = N,\, \alpha_1 = \tilde{\alpha}_1, \ldots, \alpha_N = \tilde{\alpha}_N, L_1 \sim \tilde{L_1}, \ldots, L_N \sim \tilde{L}_N,\, \alpha_1 = \alpha, L_1 \sim L.$$

8.1. **Space** $M'(r)$**.** :

$M'(r)$, $r \in R\backslash(-N)$ denotes the space of all generalized functions $f \in S'_+(R)$ such that there exist a $k \in N_0$ and a locally integrable function $F$, *suppF* $\subset [0, \infty)$. Then the elements of $M'(r)$ are of the form

$$f = t^{-r}D^kF, \tag{8.6}$$

where $F$ is continuous on $[0, \infty)$ and

$$\int_R |F(t)|(t+\beta)^{-r-1-k}dt < \infty, \text{ for } \beta > 0 \tag{8.7}$$

$$i.\ e.,\ M'(R) = \{\psi \in C^{\infty}(0,\infty)/\psi = t^r\phi; \phi \in S([0,\infty))\}$$

with the multinorm

$$||\psi||_{\alpha} = \sup\{t^r\varphi^{(i)}(t)(1+t)^k;\ i \le k,\ t \in [0,\infty),\ k \in N_0\}.$$

The Stieltjes transformation $S_r(f)(s)$, $r \in R\backslash(-N)$ is a complex valued function, defined by

$$S_r(f(t))(s) = \int_0^{\infty} \frac{f(t)}{(s+t)^{r+1}} dt, \tag{8.8}$$

$$\text{where}\, s \in C\backslash(-\infty,0],\ 0 < t < \infty,\ r \in R\backslash(-N).$$

Modified Stieltjes transformation introduced by Marichev as

$$T_{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \left(1+\frac{x}{y}\right)^{-\alpha} \frac{1}{y} f(y)dy, \tag{8.9}$$

$x \in C\backslash(-\infty,0]$, $0 < y < \infty$, $\alpha \in R\backslash(-N)$. It can be written as

$$T_{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \frac{y^{\alpha-1}f(y)}{(x+y)^{\alpha}} dy. \tag{8.10}$$

Putting $r = \alpha - 1, f(t) = y^{\alpha-1}f(y)$ in (8.8) we get,

$$S_{\alpha-1}(y^{\alpha-1}f(y))(x) = \int_0^{\infty} \frac{y^{\alpha-1}f(y)}{(x+y)^{\alpha}} dy,\ x \in C\backslash(-\infty,0]. \tag{8.11}$$

By equation (8.10) and (8.11), we have

$$T_{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} S_{\alpha-1}(y^{\alpha-1}f(y))(x),\ \ , x \in C\backslash(-\infty,0],$$

By interchanging $x$ by $z$ and $\alpha$ by $r+1$

$$\Gamma(r+1) T_{r+1}(f)(z) = S_r(y^r f)(z), z \in C\backslash(-\infty,0] \tag{8.12}$$

8.2. **Modified Stieltjes transformation $T_{r+1}$.**

Let us define the $T_{r+1}$ - transformation $r \in R\backslash(-N)$. Assume that $f \in M'(r)$, that is $f = t^{-r}D^kF$ and $\int_R |F(t)|(t+\beta)^{-r-1-k}dt < \infty$ for $\beta > 0$ holds. Modified Stieltjes transformation defined as follows,

$T_{r+1}(f), r \in R\backslash(-N)$ is a complex valued function defined by

$$\Gamma(r+1) T_{r+1}(f)(s) = (r+1)_k \int_0^{\infty} \frac{F(t)}{(s+t)^{r+1+k}} dt$$

$r \in R\backslash(-N), s \in C\backslash(-\infty, 0], 0 < t < \infty.$

8.3. **Generalized Modified Stieltjes Transformation $\tilde{T}_{r+1}$.**

The $\tilde{T}_{r+1}$ - transformation of a distribution $f \in S'_{+}(R)$ is complex valued function $\tilde{T}_{r+1}(f)$ defined by

$$\overline{|(r+1)}\tilde{T}_{r+1}(f)(s) = \lim_{\omega \to \infty} \langle f(t), \eta(t)(s+t)^{-r-1} \exp(-\omega t) \rangle.$$

$\omega \in R, s \in \Lambda \subset (C\backslash(-\infty, 0], ), \eta \in A(s),$

where $\Lambda$ is the set of complex number for which this limit exists and $A(s)$ is the family of all smooth functions $\eta$, defined on $R$ for which there exists $\epsilon = \epsilon_{\eta,s} > 0$ such that $0 \leq \eta(t) \leq 1, t \in R, \eta(t) = 1$ if $t$ belongs to the $\epsilon$-neighborhood of $\overline{R}_{+}, \eta(t) = 0$ if it belongs to the complement of the $2\epsilon$ - neighborhood of $\overline{R}_{+}$ $\epsilon > 0$ is arbitrary if $Im\, s \neq 0$ and $0 < 2\epsilon < \max \mathrm{Re}\, s$, if $Im\, s = 0$. Moreover, assume $|D^p\eta(t)| \leq C_p, t \in R$. If $\eta(t) \in A(s), s \in (C\backslash\overline{R}_{-})$, then

$$\eta(t)(s+t)^{-r-1} \exp(-\omega t) \in S(R) \text{ for } \omega \in R_{+}, r \in R.$$

**Theorem 8.2.** *Let $f \in S'_{+}$ and $f = D^m F$. If $f$ has the q. a. b. at $\infty$ with respect to $k^{\alpha}L(k)$, then it follows that*

$$\frac{F(kt)}{k^{\alpha+m}L(k)} \to Cf_{\alpha+m+1} \text{ in } S'_p \text{ for } p > \alpha+m+1 \text{ as } k \to \infty. \tag{8.13}$$

In the case of q. a. b. at $0^{+}$

$$\frac{F(t/k)}{(1/k)^{\alpha+m}L(1/k)} \to Cf_{\alpha+m+1} \text{ in } S'_p \text{ for } p > \alpha+m+1 \text{ as } k \to \infty.$$

If $f \in M'(r)$ and $f = t^{-r}D^m F$ then $F \in S'_{r+m+1}$. For given $z \in C\backslash(-\infty, 0]$, we denote by $A(z)$ the space of all $\eta \in C^{\infty}$, such that $0 \leq \eta \leq 1, \eta(t) = 1$ for $t > -\epsilon, \eta(t) = 0$ for $t < -2\epsilon$, where $\epsilon > 0$ is arbitrary if $z \notin (0, \infty)$ and $z \in (0, \infty)$ then we choose $\epsilon$ such that $0 < 2\epsilon < z$. Clearly, for a given $z \in C\backslash(-\infty, 0]$ and every $\eta \in A(z)$,

$$t \to \eta(t)(t+z)^{-r-m-1} \in S_p \text{ for } p < r+m+1. \tag{8.14}$$

Let $f \in M'(r)$. We have $(x > 0, t > 0)$,

$$\overline{|(r+1)}(T_{r+1}f)(tx) = x(r+t)\int_0^{\infty} \overline{|(r+2)}(T_{r+2}f)(xu)du,$$

and if

$$\overline{|(r+2)}(T_{r+2}f)(x) \sim x^{-(r-\alpha)-1}L(x). \tag{8.15}$$

then

$$\overline{|(r+1)}(T_{r+1}f)(x) \sim \frac{(r+1)}{(1-\alpha)}x^{-(r-\alpha)}L(x).$$

**Theorem 8.3.** *Let $f$ have the closed q. a. e at $\infty$ of order $(\alpha, L)$ and of length $l$ related to $k^{\alpha-l}L_0(k)$. Let $\alpha, r \in R\backslash(-N)$. Then* *(i)* $f \in M'(r), (f_L)_{\alpha+1} \in M'(r)$;

*(ii)* *If we put* $\overline{|(r+1)}T_{r+1}(f_L)_{\alpha+1}(x) = S_{\alpha,L}(x)$, *then for* $L \sim \tilde{L}, S_{\alpha,L}(x) \sim S_{\alpha,L}x \sim \frac{\overline{|(r-\alpha)}}{|(r+1)} x^{\alpha-r}L(x), x \to \infty$;

*(iii)*

$$\overline{|(r+1)}T_{r+1}(f)(x) - C\frac{\overline{|(r-\alpha)}}{\overline{|(r+1)}}x^{\alpha-r}L(x) = 0\,(x^{\alpha 1-r}L_0(x)), x \to \infty. \tag{8.16}$$

**Theorem 8.4.** *Let $f$ have the closed q. a. e at $0^+$ of order $(\alpha, L)$ and of length $l$ with respect to $(1/k)^{\alpha+l}L_0(1/k)$. If $(\alpha+l) < r$ and $f \in M'(r)$, then*

$$\overline{|(r+1)}T_{r+1}(f)(x) - C\frac{\overline{|(r-\alpha)}}{\overline{|(r+1)}}x^{\alpha-\gamma}L(x) = 0(x^{\alpha+1-r}L_0(x)), x \to 0. \tag{8.17}$$

**Theorem 8.5.** *Let $f$ satisfy the conditions of Theorem 8.3. Let all the slowly varying functions in Theorem 8.3 be equal to 1. Then*

*(i)* $A_{fr}(z) = \overline{|(r+1)}(T_{r+1}f)(x) - C\frac{\overline{|(r-\alpha)}}{|(r+1)}\frac{z^{\alpha-r}}{z^{\alpha-l-r}}$ *is bounded analytic function in any* $\Lambda_{\alpha,\epsilon}, \alpha > 0, \epsilon > 0$;

*(ii)* $A_{f,r}(z)$ *converges uniformly to zero in* $\Lambda_{\alpha,\epsilon}$ *when* $|z| \to \infty$.

The proof uses the Montel theorem [5, p. 5].

**Theorem 8.6.** *Let $f$ satisfy the conditions of Theorem 8.4. Let all the slowly varying functions in Theorem 8.4 be equal to 1. Let*
$A_{f,r}(z) = \{\overline{|(r+1)}(T_{r+1}f)(z) - \frac{\overline{|(r-\alpha)}}{|(r+1)}z^{\alpha-r}\}/z^{\alpha+l-r}$.
*Then*

*(i)* $A_{f,r}(z)$ *is a bounded function in* $\Lambda_{0,\epsilon} \bigcap B(0,R_0)$, $\epsilon > 0, R_0 > 0$ *where* $B(0,R_0) = \{z; |z| < R_0\}$;

*(ii)* $A_{f,r}(z)$ *converges uniformly to zero in* $\Lambda_{0,\epsilon}$ *when* $|z| \to 0$.

## 9. TAUBERIAN-TYPE THEOREMS WITH APPLICATIONS TO THE STIELTJES TRANSFORMATION

First we defined the space $L'(r)$ and modified Stieltjes transformation introduced by Lavoine and Misra [23] and Marichev [25], respectively. Next we extended Tauberian-type theorems for the distributional Stieltjes transformations to the distributional modified Stieltjes transformations [20].

We give the definition of the quasiasymptotic expansion at $0^+$ and the quasiasymptotic behaviour of distributions at infinity introduced in [15]. We gave the definition of space $L'(r)$, classical Stieltjes transformation, modified Stieltjes transformation, and generalized modified Stieltjes transformation. This enables us to obtain the Tauberian-type theorems of quasiasymptotic behaviour of distribution at infinity. Again we have given sufficient conditions under which the behaviour at infinity of the modified Stieltjes transformation $\overline{|(r+1)}(T_{r+1}f)(x), r \in$

$\mathbb{R}\backslash(-\mathbb{N}), f \in L'(r)$ determines the quasiasymptotic behaviour of $f$ at infinity. For more properties of slowly varying function, we refer the reader to [39].

**Definition 9.1. (Quasiasymptotic behaviour (q.a.b.))**

*If $T$ is a distribution from $S'_+$ such that the distributional limit*

$$\lim_{k\to\infty} \frac{T(kx)}{\rho(k)} = \gamma(x) \tag{9.1}$$

*exists in $S'(\gamma(x) \neq 0)$, then $T$ is called the quasiasymptotic behaviour of distribution (q.a.b.) at infinity related to the regular function $\rho(k) = k^\alpha L(k)$ with the limit $\gamma$; write this as*

$$T \overset{q}{\sim} \gamma \text{ in } S' \text{ as } x \to \infty \tag{9.2}$$

*Here $\rho$ is regularly varying at infinity and the limit $\gamma$ from $S'_+$ is of the form*

$$\gamma(x) = cf_{\alpha+1}(x), \tag{9.3}$$

*where $c$ is a constant.*

Let $f \in S'_+$. It is said that $f$ has the q.a.b. at $\infty$ with the limit $g \neq 0$ with respect to

$$k^\alpha L(k),\ L \in \sum_\infty \left( (\frac{1}{k})^\alpha L(\frac{1}{k}), L \in \sum_0 \right), \quad \alpha \in \mathbb{R},$$

$$\text{If } \lim_{k\to\infty} \langle \frac{f(kt)}{k^\alpha} L(k), \phi(t)\rangle = \langle g(t), \phi(t)\rangle, \phi \in S \tag{9.4}$$

The following definition is taken from [48].

**Definition 9.2.** *A function $\rho : (\alpha, \infty) \to \mathbb{R}$ is called regularly varying function at infinity if it positive, measurable, and there exists a real number $\alpha$ such that for each $x > 0$,*

$$\lim_{k\to\infty} \frac{\rho(kx)}{\rho(k)} = x^\alpha; \tag{9.5}$$

*where the number $\alpha$ is called the index of $\rho$.*

9.1. **Space $L'(r)$.**

We extend the definition of the space $I'(r)$ given in [31] and using the same idea, we have given the definition of space $L'(r)$.

$L'(r)$, $r \in \mathbb{R}\backslash(-\mathbb{N})$, the space of all distributions $f \in S'_+(\mathbb{R})$ such that there exist $k \in \mathbb{N}_0$ and locally integrable function $F$, supp $F \subset [0, \infty)$ so that $f$ is of the form

$$f = t^{-r} D^k F \tag{9.6}$$

and there exist $c = c(F)$ and $\epsilon = \epsilon(F) > 0$ such that

$$|F(x)| \leq c(1 + x)^{r+k-\epsilon},\ x \geq 0. \tag{9.7}$$

The Stieltjes transformation $S_r(f)(s), r \in \mathbb{R}\backslash(-\mathbb{N})$, is a complex-valued function, defined by

$$S_r(f(t))(s) = \int_0^\infty \frac{f(t)}{(s+t)^{r+1}} dt \; s \in \mathbb{C}\backslash(-\infty, 0], \tag{9.8}$$

$$0 < t < \infty, r \in \mathbb{R}\backslash(-\mathbb{N})$$

Modified Stieltjes transformation introduced by Marichev is defined as

$$T_\alpha(f(x)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(1 + \frac{x}{y}\right)^{-\alpha} \frac{1}{y} f(y) dy, \; x \in \mathbb{C}\backslash(-\infty, 0], \tag{9.9}$$

$$0 < y < \infty, r \in \mathbb{R}\backslash(-\mathbb{N}).$$

It can be written as

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha} dy, \quad x \in \mathbb{C}\backslash(-\infty, 0] \tag{9.10}$$

Setting $r = \alpha - 1, f(t) = y^{\alpha-1} f(y)$ in (9.8) , we get

$$S_{\alpha-1}(y^{\alpha-1} f(y))(x) = \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha} dy, \quad x \in \mathbb{C}\backslash(-\infty, 0] \tag{9.11}$$

Using (9.10) and (9.11), we obtain the relationship between (9.8) and (9.10),

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} S_{\alpha-1}(y^{\alpha-1} f(y))(x), \quad x \in \mathbb{C}\backslash(-\infty, 0] \tag{9.12}$$

By changing $x$ by $z$ and $\alpha$ by $r+1$, it follows that

$$\Gamma(r+1) T_{r+1}(f)(z) = S_r(y^r f)(z), \quad z \in \mathbb{C}\backslash(-\infty, 0], \;\; \gamma \in \mathbb{R}\backslash(-\mathbb{N}) \tag{9.13}$$

9.2. **Modified Stieltjes transformation $T_{r+1}$.**

The modified Stieltjes transformation $T_{r+1}(f), r \in \mathbb{R}\backslash(-\mathbb{N})$, is a complex-valued function defined by

$$\Gamma(r+1) T_{r+1}(f)(s) = (r+1)_k \int_0^\infty \frac{F(t)}{(s+t)^{r+1+k}} dt, \; r \in \mathbb{R}\backslash(-\mathbb{N}), \tag{9.14}$$

$$s \in \mathbb{C}\backslash(-\infty, 0], 0 < t < \infty,$$

where $(\alpha)_k = (\alpha)(\alpha+1) \cdots (\alpha+k-1)$.

9.3. **Generalized modified Stieltjes transformation $\tilde{T}_{r+1}$.** .

The $\tilde{T}_{r+1}$ -transformation of a distribution $f \in S'_{+}(\mathbb{R})$ is complex-valued function $\tilde{T}_{r+1}(f)$ defined by

$$\overline{|(r+1)}\tilde{T}_{r+1}(f)(s) = \lim_{\omega\to\infty} \langle f(t), \eta(t)(s+t)^{-r-1}\exp(-\omega t)\rangle \omega \in \mathbb{R}, \quad s \in \Lambda \subset (\mathbb{C}\backslash(-\infty, 0]), \eta \in A(s). \tag{9.15}$$

Here $\Lambda$ is the set of complex numbers for which this limit exists and $A(s)$ is the family of all smooth functions, defined on $\mathbb{R}$ for which there exists $\varepsilon = \varepsilon_{\eta,s} > 0$ such that

$$0 \leq \eta(t) \leq 1, t \in \mathbb{R}, \eta(t) = 1,$$

if belongs to the $\varepsilon$-neighbourhood of $\overline{\mathbb{R}}_{+}\eta(t) = 0$ or if it belongs to the complement of the $2\varepsilon$-neighbourhood of $\overline{\mathbb{R}}_{+}$ $\varepsilon > 0$ is arbitrary if $Im\, s \neq 0$ and $0 < 2\varepsilon < \max \operatorname{Re} s$, if for some $Im\, s = 0$ and $|D^p\eta(t)| \leq c_p$, $t \in \mathbb{R}$. If $\eta(t) \in A(s), s \in \mathbb{C}\backslash\mathbb{R}$, then

$$\eta(t)(s+t)^{-r-1}\exp(\omega t) \in S(\mathbb{R}) \text{ for } \omega \in \mathbb{R}_{+}, r \in \mathbb{R}.$$

9.4. **Tauberian-type theorems for modified Stieltjes transformation.**

The following theorem is from [36].

**Theorem 9.1.** *Suppose that for some $m > 0$ and $x \to \infty$,*

$$\int_0^\infty \frac{d\phi(\lambda)}{(\phi+x)^{m+1}} \sim \int_0^\infty \frac{d\varphi(\lambda)}{(\phi+x)^{m+1}} \tag{9.16}$$

*and the following conditions are satisfied.*

*(1) Functions $\phi$ and $\varphi$ are defined for $x > 0$ and are nondecreasing.*
*(2) $\lim_{x\to 0} \phi(x) = \infty$*
*(3) For any $c > 1$, there are constants $\gamma$ and $N, 0 < \gamma < m, N > 0$ such that for any $x > y > N, \phi(x)/\phi(y) \leq c(x/y)$. Then for $\lambda \to \infty, \phi(\lambda) \sim \varphi(\lambda)$.*

Now we are ready to prove the first Tauberian result.

**Theorem 9.2.** *Suppose that $s > 1, r + m - s > 0, f \in L'(r)$, and $F$ is a nondecreasing function. Moreover, let*

$$\overline{|(r+1)}(T_{r+1}f)(x) \sim \frac{\overline{|(s)}}{\overline{|(r+1)}} \frac{L(s)}{x^s}, x \to \infty \tag{9.17}$$

*where $L$ is a slowly varying function at $\infty$ defined in some interval $[A, \infty)$ , such that $x^{r+k+s}L(x)$ is a nondecreasing function. Then $f$ has the quasiasymptotic behaviour at $\infty$ related to $k^{r-s}L(k)$ with the limit $cx^{r-s}$ , where $c \neq 0$.*

### 9.5. **Tauberian-type results related to the quasiasymptotic behaviour.**

For the quasiasymptotic behaviour of all original $f$ and for the ordinary asymptotic of the corresponding function $\overline{|(r+1)}(T_{r+1}f)$ , we need the following theorem and Lemmas 4.2, 4.3, and 4.4 [36].

**Theorem 9.3.** *Let $a_{n,m}, n, m \in \mathbb{N}$ be a matrix of complex numbers.*

*(i) If $a_{n,m}$ converges uniformly in $m \in \mathbb{N}$ to $a_m$ as $n \to \infty$ and $\lim\limits_{m\to\infty} a_m$ exists, then*

$$\lim_{n\to\infty} \lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty} \lim_{n\to\infty} a_{n,m} = \lim_{\substack{n\to\infty \\ m\to\infty}} a_{n,m} \tag{9.18}$$

*(ii) If $\lim\limits_{n\to\infty} a_{n,m}$ exists for every $m \in \mathbb{N}$, $\lim\limits_{m\to\infty} a_{n,m}$ exists for every $n \in \mathbb{N}$, $\lim\limits_{n,m\to\infty} a_{n,m}$ exists, then $a_{n,m}$ converges uniformly in $n \in \mathbb{N}$ as $m \to \infty$.*

**Theorem 9.4.** *If $f \in S'_+$ and $f$ has the quasiasypmtoic behaviour at $\infty$ related to $k^\nu L(k), r-1 < \nu < r, r \in \mathbb{R}\backslash(-\mathbb{N})$, then the double sequence*

$$\left\langle \frac{\overline{\overline{|(r+k+2)}(T_{r+k+2}F_1)_+(xm)}}{m^{\nu+k+1}L(m)}, L_{n,r,k+1,x}\phi^{k+1}(x) \right\rangle, \tag{9.19}$$

$$m, n \in \mathbb{N}, \phi \in S$$

*converges uniformly in $n \in \mathbb{N}$ as $m \to \infty$, where $k \in \mathbb{N}, k+\nu > 0$, and $F_1(x) = \int\limits_0^x F(t)dt,\ x \in \mathbb{R}$.*

*(Here $L_{p,q,e,x}$ is defined as in* [36, page 147]*).*

**Theorem 9.5.** *Let $f \in L'(r)$ and $\overline{\overline{|(r+1)}(T_{r+1}f)_+}\,(x)$ have the quasiasymptotic at $\infty$ related to $k^\alpha L(k), -1 < \alpha < 0$. If for any $\phi \in S$, the double sequence (9.19) converges uniformly in $n \in \mathbb{N}$ as $m \to \infty$ , then $f$ has the quasiasymptotic at $\infty$ related to $k^{\alpha+r}L(k)$.*



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# RANDOM GEOMETRICAL CONFIGURATIONS AND RANDOM VOLUMES*

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Abstract. This paper will show the connections among many topics in Mathematics, Statistics, Astrophysics and Information Theory. Mathematical topics covered are hypergeometric functions, generalized functions such as G and H-functions, functions of matrix argument and the associated Jacobians of matrix transformations, Mellin-Barnes integrals, fractional calculus and Mittag-Leffler functions, volumes of geometrical objects, representations of volumes as determinants etc. The statistical topics covered are statistical distributions, multivariate analysis, likelihood ratio test criteria, Dirichlet densities, structural representations of random variables, pathway models, random volumes, random parallelotopes, and matrix-variate distributions. The astrophysics topics covered are fractional reaction-diffusion equations and their solutions, Tsallis statistics and superstatistics, and non-extensive statistical mechanics. Information theory topics covered are Shannon's entropy, generalized entropies, optimization of entropies and derivation of pathway and other models. Connections among all these topics are established in this paper.



## 1. INTRODUCTION

We start with the discussion of elementary geometrical objects such as vectors as directed line segments, geometrical interpretations of linear independence of vectors, convex hull created by a pre-selected set of $p$ vectors in $q$-dimensional Euclidean space, with $q \geq p$, and the $p$-parallelotope created by these $p$ vectors in $q$-space. Then we look into the problem of assigning probability measures to these $p$ vectors, thereby making the volume of the $p$-parallelotope a random volume. Then we look at the representation of these volumes as determinants and then assign general matrix-variate probability measures to the matrices of the determinants. Then treat these densities as real-valued scalar functions of matrix argument and study their properties.

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General Mellin and inverse Mellin transform technique will be used to derive the distributions. This will also produce certain structural decompositions of random variables, establishing connections to multivariate test statistics, generalized Dirichlet models. All these will be shown to be particular cases of the general matrix-variate pathway model of Mathai (2005). Pathway model and connection to hypergeometric functions will be established. It will be shown that Tsallis statistics and superstatistics are available from the scalar version of the pathway model. As byproducts, the distributions of quadratic forms and bilinear forms will be put on a general framework, thus extending the current theories based on Gaussian distributions. It will also be shown that superstatistics are available from a Bayesian procedure in statistical analysis, and all these are available by optimizing certain generalized entropy measures. General representations of these densities will produce Mellin-Barnes representations and will bring in Meijer's G and H-functions. Particular cases will bring in Mittag-Leffler functions, which are directly connected to fractional differential equations and reaction-diffusion problems in physics. It will be shown that the path leading from a Mittag-Leffler density is to a Lévy density rather than to a Gaussian density. Lévy and Linnik densities are connected to non-Gaussian stochastic processes and time series.

Thus, this paper will establish connections among the following topics: (1): Elementary special functions, ${}_1F_0, {}_0F_1, {}_0F_0$ through a pathway parameter; (2): Volume of random parallelotope; (3): Likelihood ratio test statistics in multivariate analysis; (4): Pathway models; (5): Generalized Dirichlet models; (6): Distributions of quadratic and bilinear forms; (7): Tsallis statistics, superstatistics in Statistical Mechanics; (8): Mittag-Leffler, Laplace, Lévy, Linnik densities and processes; (9): Generalized entropies in Information Theory.

To start with consider two points in 2-space

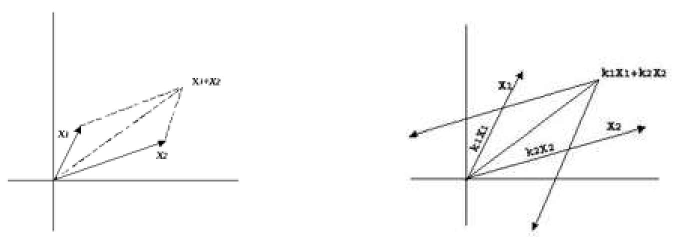


Figure 1.1(a) 1.1(b): Parallelogram in 2−space

$X_1 = (x_{11}, x_{12}), X_2 = (x_{21}, x_{22})$ - two vectors in 2-space. The vectors are shown in Figure 1.1(a), and Figure 1.1(b) gives the geometrical construction to show that if the first two vectors are separated by an angle $\theta \neq 0$, meaning linearly independent, then any third vector in 2-space has to be linearly dependent on the first two. Geometry of linear independence, $\theta \neq 0$

Area $A$ of the parallelogram is available from elementary trigonometry as

$$\begin{aligned} A &= \|X_1\| \, \|X_2\| \sin\theta = \|X_1\| \, \|X_2\| \sqrt{1-\cos^2\theta} \\ &= \sqrt{(\|X_1\|)^2(\|X_2\|)^2 - (X_1.X_2)^2} \end{aligned}$$

where $X_1.X_2$ denotes the dot product between the two. This area $A$ can also be written as a determinant.

$$\begin{aligned} X &= \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Rightarrow \\ |XX'| &= \begin{vmatrix} X_1X_1' & X_1X_2' \\ X_2X_1' & X_2X_2' \end{vmatrix}, \; X_1.X_2 = X_2.X_1 \\ &= \begin{vmatrix} (\|X_1\|)^2 & X_1.X_2 \\ X_2.X_1 & (\|X_2\|)^2 \end{vmatrix} \\ &= (\|X_1\|)^2(\|X_2\|)^2 - (X_1.X_2)^2 \\ &= A^2 \end{aligned}$$

Thus the square of the area is the determinant of the matrix $XX'$.

This result also holds if $X_1$ and $X_2$ are two vectors in $q$-space. Then $X_1 = (x_{11}, ..., x_{1q})$ and $X_2 = (x_{21}, ..., x_{2q})$.

Consider $p$, $p \leq q$, linearly independent vectors $X_1, ..., X_p$ in $q$-space, taken in a given order. Let $X$ be a $p \times q$, $q \geq p$, matrix with the rows $X_1, ..., X_p$. That is, $X = (x_{ij}) = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix}$ is a $p \times q, q \geq p$ matrix of full rank $p$.

Let $O$ be the origin of a rectangular coordinate system so that $\overrightarrow{OX_1}, ..., \overrightarrow{OX_p}$ are ordered set of $p$ linearly independent vectors. These vectors almost surely can create a convex hull, $p$-simplex as well as a $p$-parallelotope. Then from elementary geometry $v^2 = |XX'|$ is the square of the volume or hyper-volume of the $p$-parallelotope.

How to assign probability measures to $X_1, ..., X_p$?

The classical approach to assign probabilities to these points is to assume isotropy and statistical independence. Isotropy implies that the joint density of $X_i$'s is purely a function of $||X_i|| = (x_{i1}^2 + ... + x_{iq}^2)^{\frac{1}{2}}$, the Euclidean length of $X_i$, which is invariant under rotation of the coordinate system or under orthonormal transformations on $X_i,\ i = 1, 2, ..., p$.

Statistical independence implies that the product probability property (PPP) holds. This means that the joint density of $X_i,\ i = 1, ..., p$ is the product of the marginal or individual densities or the joint distribution function is the product of the individual distribution functions. Classical approaches may be seen from Miles (1971),Ruben (1979), Santaló (1976) and others.

Mathai (1999) has shown that the distributional aspects of the volume of the $p$-parallelotope $|XX'|^{\frac{1}{2}} = v$ can be studied when $X$ has a general rectangular matrix-variate distribution belonging to the following categories:

$$f_1(X) = c_1|XX'|^{\gamma}|I - aXX'|^{\beta}, a > 0, I - aXX' > O \tag{1.1}$$

$$f_2(X) = c_2|XX'|^{\gamma}|I + bXX'|^{\beta}, b > 0, \tag{1.2}$$

$$f_3(X) = c_3|XX'|^{\gamma}\mathrm{e}^{-a\mathrm{tr}(XX')}, a > 0 \tag{1.3}$$

where $a, b$ are scalars and $c_1, c_2, c_3$ are the normalizing constants in the sense

$$\int_X f_1(X)\mathrm{d}X = \int_X f_2(X)\mathrm{d}X = \int_X f_3(X)\mathrm{d}X = 1$$

where $\mathrm{d}X$ denotes the wedge product of the differentials in $X$, that is, when $x_{ij}$'s are distinct,

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge ... \wedge \mathrm{d}x_{pq}.$$

Details of the developments in this area may be seen from Mathai (1999a).

## 2. A PATHWAY MODEL

All the three models in (1.1),(1.2) and (1.3) can be combined into one functional form through a pathway parameter introduced in Mathai (2005). Consider the model

$$f_4(X) = c_4|XX'|^{\gamma}|I - a(1-\alpha)XX'|^{\frac{\eta}{1-\alpha}}, a > 0, \eta > 0, \tag{2.1}$$

for $I - a(1-\alpha)XX' > O$. In (2.1) when $\alpha < 1$ the form is that in (1.1); $f(X)$ is in the real matrix-variate type-1 beta family.

When $\alpha > 1, 1 - \alpha = -(\alpha - 1), \alpha > 1$ and hence the form switches to that in (1.2), which is a real matrix-variate type-2 beta model.

When $\alpha \to 1$,

$$\lim_{\alpha \to 1} |I - a(1-\alpha)XX'|^{\frac{\eta}{1-\alpha}} = \mathrm{e}^{-a\eta\ \mathrm{tr}(XX')} \tag{2.2}$$

the model in (2.1) switches into that of (1.3).

The right side in (2.2) is a hypergeometric function of matrix argument belonging to the category

$${}_0F_0(\ ;\ ; -a\eta XX') = \mathrm{e}^{-a\eta\ \mathrm{tr}(XX')}. \tag{2.3}$$

The second factor in (2.2) for $\alpha < 1$ belongs to the family ${}_1F_0$. That is,

$$|I - a(1-\alpha)XX'|^{\frac{\eta}{1-\alpha}} = {}_1F_0(\frac{\eta}{\alpha - 1};\ ; a(1-\alpha)XX') \tag{2.4}$$

for $\alpha < 1, a > 0, ||a(1-\alpha)XX'|| < 1$, where $\|(\cdot)\|$ denotes a norm of $(\cdot)$.

For $\alpha < 1, a > 0, O < a(1-\alpha)XX' < I$ one can expand the right side of (2.4) into a power series in terms of zonal polynomials.

$$|I - a(1-\alpha)XX'|^{\frac{\eta}{1-\alpha}} = \sum_{k=0}^{\infty}\sum_{K}(\frac{\eta}{\alpha-1})_K \frac{C_K(a(1-\alpha)XX')}{k!}, \tag{2.5}$$

for $||a(1-\alpha)XX'|| < 1, K = (k_1, ..., k_p), k_1 + ... + k_p = k$ where, for example,

$$(b)_K = \prod_{j=1}^{k}(b - \frac{j-1}{2})_{k_j},$$

where for example the notation $(a)_k$ stands for the Pochhammer symbol,

$$(a)_m = a(a+1)...(a+m-1),\ a \neq 0,\ (a)_0 = 1,\ m = 0, 1, 2, ...,$$

and $C_K(\cdot)$ is the zonal polynomial of order $k$. For the definition and properties of zonal polynomials see Mathai, Provost and Hayakawa (1995). When $p = 1$, (2.5) reduces to the ordinary binomial expansion on scalar variables. That is, for $p = 1$,

$$[1 - a(1-\alpha)x^2]^{\frac{\eta}{1-\alpha}} = \sum_{k=0}^{\infty}(\frac{\eta}{\alpha-1})_k \frac{[a(1-\alpha)x^2]^k}{k!} \tag{2.6}$$

for $|a(1-\alpha)x^2| < 1$. It can be seen that almost all real matrix-variate distributions in current use are special cases of (2.1).

One can introduce location and scale parameter matrices into the model in (2.1) and write, for $p \times q, q \geq p$ matrices, $X$ and $M$ of full rank $p$,

$$\begin{aligned} f_5(X) &= c_5|A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}}|^{\gamma} \\ &\quad \times|I - a(1-\alpha)A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}} \end{aligned} \tag{2.7}$$

where $c_5$ is the normalizing constant, $A = A' > O, B = B' > O, A$ is $p \times p, B$ is $q \times q$, $A$ and $B$ are constant positive definite matrices, and $M$ is a $p \times q$ constant matrix. $A^{\frac{1}{2}}$ denotes the positive definite square root of $A$, $a > 0$ and $\alpha$ is the pathway parameter. For keeping non-negativity of the determinant in (2.7) we need the condition

$$I - a(1-\alpha)A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}} > 0$$

where, for example, the notation $A < Y < B \Rightarrow A = A' > O, B = B' > O, Y = Y' > O, Y - A > O, B - Y > O$. In (2.7) the constant matrix $M$ can act as a relocation matrix or as the mean value matrix so that $E(X) = M$ where $E$ denotes the expected value.

2.1. **The normalizing constants.** This requires certain transformations and the knowledge of the corresponding Jacobians. Almost all matrix transformations in the current use and the corresponding Jacobians are available from Mathai (1997). In our presentation here, we will use a few multi-linear and some nonlinear transformations and the corresponding Jacobians. In order to save space, the derivations of these Jacobians are not given here. Details are available from Mathai (1997). Put

$$Y = A^{\frac{1}{2}}(X-M)B^{\frac{1}{2}} \Rightarrow \mathrm{d}Y = |A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\mathrm{d}X,$$

see Mathai (1997) for the Jacobian. Since $f_5(X)$ is assumed to be a density,

$$1 = \int_X f_5(X)\mathrm{d}X = c_5|A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\int_Y |YY'|^{\gamma}|I - a(1-\alpha)YY'|^{\frac{\eta}{1-\alpha}}\mathrm{d}Y.$$

Now, make a transformation $Y = TQ$ where $T$ is a $p\times p$ lower triangular matrix with real positive diagonal elements and $Q$ is an element of the Stiefel manifold or a $p \times q$ semi-orthonormal matrix. For the associated Jacobian see Mathai (1997). Then $YY' = TT'$ and make the transformation $Z = YY'$. After integrating out over the Stiefel manifold we have

$$\begin{aligned} 1 &= \int_X f_5(X)\mathrm{d}X \\ &= c_5|A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}\int_Z |Z|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}|I-a(1-\alpha)Z|^{\frac{\eta}{1-\alpha}}\mathrm{d}Z \end{aligned} \tag{2.8}$$

where, for example,

$$\Gamma_p(\beta) = \pi^{\frac{p(p-1)}{4}}\Gamma(\beta)\Gamma(\beta-\frac{1}{2})...\Gamma(\beta-\frac{p-1}{2}), \Re(\beta) > \frac{p-1}{2} \tag{2.9}$$

is the real matrix-variate gamma and $\Re(\cdot)$ denotes the real part of $(\cdot)$.

When $p = 1$, (2.9) yields the ordinary gamma function $\Gamma(\beta)$ with $\Re(\beta) > 0$. Further evaluation of the integral in (2.8) depends on the value of $\alpha$. If $\alpha < 1$ then (2.8) belongs to the real matrix-variate type-1 beta. Then integrating out with the help of a type-1 beta integral we have from (2.8), for $q \geq p, \alpha < 1$,

$$1 = c_5|A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}[a(1-\alpha)]^{-p(\gamma+\frac{q}{2})}\frac{\Gamma(\gamma+\frac{q}{2})\Gamma_p(\frac{\eta}{1-\alpha}+\frac{p+1}{2})}{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2})}$$

for $\Re(\gamma+\frac{q}{2}) > \frac{p-1}{2}$. Hence,

$$c_5 = |A|^{\frac{q}{2}}|B|^{\frac{p}{2}}[a(1-\alpha)]^{p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\frac{q}{2})\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2})}{\pi^{\frac{pq}{2}}\Gamma_p(\gamma+\frac{q}{2})\Gamma_p(\frac{\eta}{1-\alpha}+\frac{p+1}{2})} \tag{2.10}$$

for $\alpha < 1, \eta > 0, a > 0, \Re(\gamma+\frac{q}{2}) > \frac{p-1}{2}, q \geq p, p, q = 1, 2, ..., A = A' > 0, B = B' > 0$. For $\alpha > 1$, write $1-\alpha = -(\alpha-1), \alpha > 1$ then the integral in (2.8) goes into a type-2 beta form. Then integrating out with the help of a real matrix-variate type-2 beta integral we have for $\alpha > 1$,

$$c_5 = \frac{|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}[a(\alpha-1)]^{p(\gamma+\frac{q}{2})}\Gamma_p(\frac{q}{2})\Gamma_p(\frac{\eta}{\alpha-1})}{\pi^{\frac{pq}{2}}\Gamma_p(\gamma+\frac{q}{2})\Gamma_p(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2})} \tag{2.11}$$

for $\alpha > 1, \Re(\gamma+\frac{q}{2}) > \frac{p-1}{2}, \Re(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}) > \frac{p-1}{2}, a > 0, \eta > 0$, $A = A' > O, B = B' > O$. When $\alpha \to 1$ we have

$$\lim_{\alpha\to 1}|I-a(1-\alpha)Z|^{\frac{\eta}{1-\alpha}} = \mathrm{e}^{-a\eta\ \mathrm{tr}(Z)}$$

and hence, evaluating with the help of a real matrix-variate gamma integral

$$\int_Z |Z|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}\mathrm{e}^{-a\eta\ \mathrm{tr}(Z)}\mathrm{d}Z = (a\eta)^{-p(\gamma+\frac{q}{2})}\Gamma_p(\gamma+\frac{q}{2})$$

for $\Re(\gamma+\frac{q}{2}) > \frac{p-1}{2}$. Hence for $\alpha \to 1$,

$$c_5 = \frac{|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}(a\eta)^{p(\gamma+\frac{q}{2})}\Gamma_p(\frac{q}{2})}{\pi^{\frac{pq}{2}}\Gamma_p(\gamma+\frac{q}{2})} \tag{2.12}$$

for $\Re(\gamma+\frac{q}{2}) > \frac{p-1}{2}, a > 0, \eta > 0$.

If the elements of $X$ in (2.7) are in the complex domain then also all the steps from (2.7) to (2.12) can be carried out with the help of the corresponding Jacobians. Jacobians in the complex domain may also be seen from Mathai (1997). A model corresponding to (2.7) in the complex domain, along with its properties and connections to various other fields, is studied in Mathai and Provost (2005). We will not look into this aspect here.

## 3. DENSITY OF THE VOLUME CONTENT

The squared volume of the $p$-parallelotope in the $q$-space,

$$v^2 = |A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}}|. \tag{3.1}$$

What is the density of $u = v^2$ in (3.1)? This can be evaluated by looking at the $h$-th moment of $v^2$ for an arbitrary $h$. That is, the expected value of $(v^2)^h$, is given by

$$E(v^2)^h = \int_X (v^2)^h f_5(X)\mathrm{d}X.$$

This is available from the normalizing constants in (2.10), (2.11) and (2.12) by observing that the only change is that the parameter $\gamma$ is changed to $\gamma + h$. Hence from (2.10) for $\alpha < 1$,

$$E(v^2)^h = [a(1-\alpha)]^{-ph}\frac{\Gamma_p(\gamma+\frac{q}{2}+h)}{\Gamma_p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2})}{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2}+h)} \tag{3.2}$$

for $\alpha < 1, \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}, \Re(\gamma + \frac{q}{2} + h) > \frac{p-1}{2}, \eta > 0, a > 0$. We can open up the gammas in (3.2) by using the definition in (2.9). Let $u_1 = [a(1-\alpha)]^p v^2$. Then

$$E(u_1^h) = \prod_{j=1}^{p} \frac{\Gamma(\gamma + \frac{q}{2} - \frac{j-1}{2} + h)}{\Gamma(\gamma + \frac{q}{2} - \frac{j-1}{2})} \frac{\Gamma(\gamma + \frac{q}{2} + \frac{\eta}{1-\alpha} + \frac{p+1}{2} - \frac{j-1}{2})}{\Gamma(\gamma + \frac{q}{2} + \frac{\eta}{1-\alpha} + \frac{p+1}{2} - \frac{j-1}{2} + h)} = E(v_1^h)E(v_2^h)...E(v_p^h) \tag{3.3}$$

where $v_j$ is a real scalar type-1 beta random variable with the parameters $(\gamma + \frac{q}{2} - \frac{j-1}{2}, \frac{\eta}{1-\alpha} + \frac{p+1}{2}), j = 1, ..., p$, and further, $v_1, ..., v_p$ are statistically independently distributed. Hence structurally

$$u_1 = v_1 v_2 ... v_p \tag{3.4}$$

and the density of $u_1$ is that of $v_1 ... v_p$. This is available by treating (3.3) as coming from the Mellin transform of the density $g_1(u_1)$ of $u_1$. Even though the density $g_1(u_1)$ is unknown,its Mellin transform is available from (3.3) for $h = s - 1$. Hence from the unique inverse Mellin transform

$$g_1(u_1) = u_1^{-1} \frac{1}{2\pi i} \int_L [E(u_1^h)] u_1^{-h} \mathrm{d}h$$

where $i = \sqrt{-1}$, $L$ is a suitable contour and $E(u_1^h)$ is given in (3.3). But the structure in (3.3) is that of a Mellin transform of a G-function of the type $G_{p,p}^{p,0}(\cdot)$. Hence

$$g_1(u_1) = u_1^{-1} \left[ \prod_{j=1}^{p} \frac{\Gamma(\gamma + \frac{q}{2} + \frac{\eta}{1-\alpha} + \frac{p+1}{2} - \frac{j-1}{2})}{\Gamma(\gamma + \frac{q}{2} - \frac{j-1}{2})} \right] \times G_{p,p}^{p,0} \left[ \Big|_{\gamma + \frac{q}{2} - \frac{j-1}{2}, j=1,...,p}^{\frac{\eta}{1-\alpha} + \frac{p+1}{2}, j=1,...,p} \right], 0 < u_1 < 1 \tag{3.5}$$

where a general G-function is defined as the following Mellin-Barnes type integral:

$$G_{p,q}^{m,n}(z|_{b_1,...,b_q}^{a_1,...,a_p}) = \frac{1}{2\pi i} \int_L \frac{\{\prod_{j=1}^{m} \Gamma(b_j + s)\}\{\prod_{j=1}^{n} \Gamma(1 - a_j - s)\}}{\{\prod_{j=m+1}^{q} \Gamma(1 - b_j - s)\}\{\prod_{j=n+1}^{p} \Gamma(a_j + s)\}} z^{-s} \mathrm{d}s. \tag{3.6}$$

Existence conditions, various types of contours $L$, various properties and applications may be seen from Mathai (1993). Special cases can be written in explicit forms in terms of elementary special functions. Such reduction formulae may also be seen from Mathai (1993), Mathai and Haubold (2008), Mathai, Saxena and Haubold (2010), or from other books on special functions.

For $\alpha > 1$, from (2.11), $u_2 = [a(\alpha - 1)]^p v^2$ is a product of $p$ statistically independently distributed real scalar type-2 beta random variables and hence the density of $u_2$ can be written in terms of a G-function of the type $G_{p,p}^{p,p}(\cdot)$.

Similarly for $\alpha \to 1$, $u_3 = (a\eta)^p v^2$ is structurally a product of $p$ statistically independently distributed real gamma random variables and the density of $u_3$ can be written in terms of a G-function of the type $G_{0,p}^{p,0}(\cdot)$.

## 4. CONNECTION TO LIKELIHOOD RATIO CRITERIA

Consider the case $\alpha < 1$. From (3.3) and (3.4) one can see a structural representation in the form of a product of $p$ statistically independently distributed type-1 beta random variables. Such a structure can also arise from a determinant of the type

$$\lambda = \frac{|G_1|}{|G_1 + G_2|} \tag{4.1}$$

where $G_1$ and $G_2$ are independently distributed real matrix-variate gamma variables with the same scale parameter matrix.Such matrix representations are recently examined in Mathai (2007).A particular case of a real matrix-variate gamma matrix is a Wishart matrix. When $G_1$ and $G_2$ are independently distributed Wishart matrices $\lambda$ in (4.1) corresponds to the likelihood ratio test criterion or a one-to-one function of it in multivariate statistical analysis. A large number of test criteria based on the principle of maximum likelihood have the structure in (4.1), with a representation as in (3.4) in the null case or when the statistical hypothesis is true. Distribution of the test statistic, when the null hypothesis is true, is known as the null distribution. Thus, a large number of null distributions, associated with the tests of hypotheses on the parameters of one or more multivariate normal populations, have the structure in (4.1) and thus a large number of cases are covered by our discussion above. Hence a direct link can be established between the volume of a random $p$-parallelotope and the $\lambda$-criterion in (4.1).

Various types of generalizations of the Dirichlet distribution are recently studied by the Palai group and a large number of papers are published on these generalizations, see for example, Jacob et al (2004, 2005)and Kurian et al (2004). In these papers there are several theorems characterizing or uniquely determining these generalized Dirichlet models through a product of independently distributed real scalar type-1 beta random variables. In all these cases one can establish a direct link from (3.4) to (4.1) providing explicit representations of the densities in terms of generalized Dirichlet integrals. As byproducts, these can also produce several results connecting G-functions and Dirichlet integrals. One such characterization theorem is recently explored by Seemon Thomas. This area is still wide open and we will not go into the details here.

## 5. QUADRATIC FORMS

For $p = 1$ and $q > 1$, (3.1) gives

$$Q = (X - M)B(X - M)' \tag{5.1}$$

the quadratic form.

Current theory of quadratic form in random variables is based on the assumption of a multivariate Gaussian population. But from (2.7) we have a generalized quadratic form in $X$, where $X$ has the density in (2.7), and the density of $Q$ is available from the step in (2.8) to (3.6). The theory of quadratic form in random variables can be extended to the density in (2.7), rather than confining the study to Gaussian population. For the study of quadratic and bilinear forms see Mathai and Provost (1992),and Mathai, Provost and Hayakawa (1995). When $p = 1$ the quadratic form in (5.1) can be given many interpretations. Let $B = V^{-1}$ where $V$ is the covariance matrix in $X$, that is, $V = \text{cov}(X) = E[(X-\mu)'(X-\mu)]$ where $X-\mu$ is $1\times q, q > 1$. Since $Y = V^{-\frac{1}{2}}(X-\mu)'$ is the standardized $X$, $YY' = y_1^2 + ... + y_q^2 = (X - \mu)V^{-1}(X - \mu)'$ is the square of the generalized distance between $X$ and $\mu$. Also $(X-\mu)V^{-1}(X-\mu)' = c > 0$ is the ellipsoid of concentration in $X$ or a scalar measure of the extent of dispersion in $X - \mu$. The components in $X$ can be given physical interpretations. Consider a growing and moving rain droplet in a tropical cloud. Then $x_1$ could be the surface area, $x_2$ could be the energy content, $x_3$ the velocity in a certain direction and so on. The components in this case have joint variations.

When $p = 1$ one can derive the pathway density in (2.7) through an optimization of a certain measure of entropy.

## 6. OPTIMIZATION OF ENTROPY

In statistical mechanics and astrophysics problems, physicists usually would like to come up with the underlying density by optimizing a measure of entropy. Entropy or a measure of uncertainty or information in a probability scheme started with Shannon's measure, which for a discrete distribution $P = (p_1, ..., p_k), p_i > 0, i = 1, ..., k, p_1 + ... + p_k = 1$ and for a continuous case are $S(P)$ and $S(f)$ respectively, where

$$S(P) = -c\sum_{i=1}^{k} p_i \ln p_i,\ S(f) = -c\int_X f(X)\ln f(X)\mathrm{d}X \tag{6.1}$$

where $c$ is a constant and $f(X)$ is a density or $f(X)$ is a real-valued function of $X$, where $X$ may be a scalar or vector or matrix. If $S(f)$ is maximized subject to the condition that the expected value of $X$, namely, $E(X)$ is fixed we end up with the exponential density when $X$ is a scalar variable. If the first two moments are fixed then we end up with the real normal density in the scalar case.

There are various generalizations to Shannon entropies $S(P)$ and $S(f)$. A discussion of the classical generalizations is given in Mathai and Rathie (1975). Recent generalizations are Tsallis entropy, $T_\alpha(f)$ (Tsallis (1988)) and $M_\alpha(f)$ of Mathai (Mathai and Haubold (2007))

$$T_\alpha(f) = \frac{\int_X [f(X)]^\alpha \mathrm{d}X - 1}{1-\alpha},\ M_\alpha(f) = \frac{\int_X [f(X)]^{2-\alpha}\mathrm{d}X - 1}{\alpha - 1},\ \alpha \neq 1. \tag{6.2}$$

If $M_\alpha(f)$ is optimized subject to the conditions that $f(X)$ is a density and that the mean value is fixed then a particular case of Mathai's pathway model is obtained. Tsallis statistics of Tsallis (1988) and superstatistics of Beck and Cohen (2003) are special cases there, see Mathai and Haubold (2007). $T_\alpha(f)$ and $M_\alpha(f)$ are generalized entropies in the sense that when $\alpha \to 1$, (6.2) goes to Shannon entropy in (6.1).

If $X$ is a $q \times 1$ vector of real random variables and if Mathai's entropy is optimized subject to the conditions $\int_X f(X)\mathrm{d}X = 1$ and the expected value of the ellipsoid of concentration is fixed, that is, $E(X-\mu)'V^{-1}(X-\mu)$ is fixed then if we apply calculus of variation principle then the Euler equation becomes

$$\frac{\partial}{\partial f}[f^{2-\alpha} - \lambda_1 f + \lambda_2 (X-\mu)'V^{-1}(X-\mu)f] = 0 \Rightarrow$$

$$f(X) = c[1 - a(1-\alpha)(X-\mu)'V^{-1}(X-\mu)]^{\frac{1}{1-\alpha}}, a > 0 \tag{6.3}$$

where $\frac{\lambda_2}{\lambda_1}$ is taken as $a(1-\alpha), a > 0, (\frac{\lambda_1}{2-\alpha})^{\frac{1}{1-\alpha}}$ is taken as $c$. Note that

$$f(X) \to c^* \mathrm{e}^{-a(X-\mu)'V^{-1}(X-\mu)} \text{ as } \alpha \to 1 \tag{6.4}$$

which is the multivariate Gaussian density for $a = \frac{1}{2}$. If two moment-like conditions are imposed and if Mathai's entropy is optimized then one can obtain the pathway model

(6.5)

$$f_1(X) = c_1[(X-\mu)'V^{-1}(X-\mu)]^\gamma [1 - a(1-\alpha)\{(X-\mu)'V^{-1}(X-\mu)\}^\delta]^{\frac{1}{1-\alpha}}$$

$$\to c_2[(X-\mu)'V^{-1}(X-\mu)]^\gamma \mathrm{e}^{-a\ [(X-\mu)'V^{-1}(X-\mu)]^\delta},\ \alpha \to 1. \tag{6.6}$$

When $q = 1$, $\mu = 0$ and $V = I$, then (6.5) can yield the real scalar case of the pathway model

$$f_3(x) = c_3|x|^\gamma [1 - a(1-\alpha)|x|^\delta]^{\frac{1}{1-\alpha}}, a > 0, 1 - a(1-\alpha)|x|^\delta > 0 \tag{6.7}$$

from where a large number of scalar densities in current use are available as special cases. For $\gamma = 0, a = 1, \delta = 1, x > 0$ in (6.7)we have Tsallis statistics of Tsallis (1988). For $\gamma = 0, a = 1, \delta = 1, x > 0, \alpha > 1$ in (6.7) we have superstatistics of Beck and Cohen (2003). For $\gamma = 0, a = 1, \delta = 1, x > 0$ in (6.7) we have the power law

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f_3}{c_3}\right) = -\left(\frac{f_3}{c_3}\right)^{\alpha}. \tag{6.8}$$

In Mathai and Haubold (2007) it is pointed out that (6.7) provides a distributional pathway, entropic pathway and differential pathway.

6.1. **Bayesian procedures.** In terms of Bayesian framework, superstatistics of statistical mechanics can be described as follows: Consider the conditional density $f(x|\theta)$ of a real scalar random variable $x$ at a given value of the parameter $\theta$. Let $g(\theta)$ be the prior density of $\theta$. Then the unconditional density of $x$ is given by

$$h(x) = \int_{\theta} f(x|\theta)g(\theta)\mathrm{d}\theta. \tag{6.9}$$

If compatible $f(x|\theta)$ and $g(\theta)$ are taken then $h(x)$ can be in a nice form. Incidently, such compatible densities were studied by Mathai and Saxena in the 1960's. If $f(x|\theta)$ and $g(\theta)$ are gamma or generalized gamma type then $h(x)$ will have the form

$$h(x) = c(1 + bx^{\delta})^{-\eta} \tag{6.10}$$

where $b > 0, \eta > 0$. By taking $b = a(\alpha - 1), a > 0, \alpha > 1, \delta = 1$ and $\eta = \frac{1}{\alpha-1}$ one has the superstatistics of Beck and Cohen (2003) for $x > 0$. If $\delta \neq 1$ then the stretched exponential of Beck (2006) is available. Observe that this type of Bayesian considerations involving exponential functions can produce only a form in (6.10) with $b > 0$ and $\eta > 0$ which is a very special case of the pathway model in (6.7) for $\alpha > 1$ only. A much more general format in this Bayesian formulation is indicated in Mathai and Haubold (2007).

6.2. **Laplacian density and stochastic processes.** The real scalar case of the pathway model in (6.7), when $\alpha \to 1$, takes the form

$$f_6(x) = c_6|x|^{\gamma}\mathrm{e}^{-a|x|^{\delta}}, -\infty < x < \infty, a > 0. \tag{6.11}$$

The density in (6.11) for $\gamma = 0, \delta = 1$ is the simple Laplace density. For $\gamma = 0$ we have the symmetric Laplace density. A general Laplace density is associated with the concept of Laplacianness of quadratic and bilinear forms. For the concept of Laplacianness of bilinear forms, corresponding to the chisquaredness of quadratic forms, and for other details see Mathai (1993a) and Mathai, Provost and Hayakawa (1995). Laplace density is also connected to input-output type models. Such models can describe many of the phenomena in nature. When two particles react with each other and energy is produced, part of it may be consumed or converted or lost and what is usually measured is the residual effect. The water storage in a dam is the residual effect of the water flowing into the dam minus the amount taken out of the dam. Grain storage in a sylo is the input minus the grain taken out. It is shown in Mathai (1993b) that when we have gamma type input and gamma type output the residual part $z = x - y, x =$ input variable, $y =$ output variable then the special cases of the density of $z$ is a Laplace density. In this case one can also obtain the asymmetric Laplace and generalized Laplace densities, which are currently used very frequently in stochastic processes, as special cases of the input-output model. Some aspects of the matrix version of the input-output model is also described in Mathai (1993b).

## 7. MITTAG-LEFFLER DENSITY AND PROCESSES

Recently there is renewed interest in Mittag-Leffler function as a model in many applied areas due to many reasons, one being that it gives a thicker tail compared to the exponential model. Fractional differential equations often lead to Mittag-Leffler functions as solutions, especially when dealing with fractional equations in reaction-diffusion problems. A large number of such situations are illustrated in Mathai and Haubold (2008), and Mathai, Saxena and Haubold (2010). This author has shown the connection of Mittag-Leffler function to pathway model, Tsallis statistics, superstatistics etc and a pathway is established for going to binomial form, exponential form etc. A review paper on Mittag-Leffler functions and their applications by Haubold, Mathai and Saxena is on the internet now. The Mittag-Leffler density, associated with a Mittag-Leffler function is the following:

$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^\beta}\sum_{k=0}^{\infty}\frac{(\beta)_k}{k!}\frac{(-x^\alpha)^k}{\delta^k\Gamma(\alpha k+\alpha\beta)}, 0 \le x < \infty, \delta > 0, \beta > 0. \tag{7.1}$$

It has the Laplace transform

$$L_f(t) = [1+\delta t^\alpha]^{-\beta}. \tag{7.2}$$

If $\delta$ is replaced by $\delta(q-1)$ and $\beta$ by $\beta/(q-1), q > 1$ and if we consider $q$ approaching to 1 then we have

$$\lim_{q\to 1} L_f(t) = \lim_{q\to 1}[1+\delta(q-1)t^\alpha]^{-\frac{\beta}{q-1}} = \mathrm{e}^{-\delta\beta\ t^\alpha}. \tag{7.3}$$

But this is the Laplace transform of a constant multiple of a positive Lévy variable with parameter $\alpha, 0 < \alpha \le 1$, with the multiplicative constant being $(\delta\beta)^{\frac{1}{\alpha}}$, and thus the limiting form of a Mittag-Leffler distribution is a Lévy distribution. Writing $f(x)$ as a Mellin-Barnes integral we have the following:

(7.4)
$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^{\beta}\Gamma(\beta)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\beta-s)}{\Gamma(\alpha\beta-\alpha s)}\left(\frac{x^{\alpha}}{\delta}\right)^{-s}\mathrm{d}s, 0<c<\beta$$

(7.5)
$$= \frac{1}{\Gamma(\beta)}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\Gamma(\beta-\frac{1}{\alpha}+\frac{s}{\alpha})\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\delta^{\frac{1}{\alpha}}\alpha\Gamma(1-s)}\left(\frac{x}{\delta^{\frac{1}{\alpha}}}\right)^{-s}\mathrm{d}s, 1-\alpha\beta<c_1<1$$

(7.6)
$$= \frac{1}{\Gamma(\beta)}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\Gamma(\beta-\frac{1}{\alpha}+\frac{s}{\alpha})\Gamma(1+\frac{1}{\alpha}-\frac{s}{\alpha})}{\delta^{\frac{1}{\alpha}}\Gamma(2-s)}\left(\frac{x}{\delta^{\frac{1}{\alpha}}}\right)^{-s}\mathrm{d}s.$$

Hence if $f(x)$ is a density then we can take the kernel in the Mellin-Barnes integral as the $(s-1)$-th moment $E(x^{s-1})$. Thus

$$E(x^{s-1}) = \frac{1}{\Gamma(\beta)\delta^{\frac{1}{\alpha}}}\frac{\Gamma(\beta-\frac{1}{\alpha}+\frac{s}{\alpha})\Gamma(1+\frac{1}{\alpha}-\frac{s}{\alpha})}{\Gamma(2-s)\delta^{-\frac{s}{\alpha}}}.$$

Then $s=1$ should give 1. The right hand side gives 1 and hence $f(x)$ in (7.1) is a density function. This is called the generalized Mitta-Leffler density. Note that since the support is $0 \le x < \infty$ one cannot integrate the series to show that the function in (7.1) is a density function. The above representation through the Mellin-Barnes integral, noticing that it is a non-negative function, interpretation as the $(s-1)$-th moment and then showing that at $s=1$ the quantity gives 1 is a novel way of establishing that it is a density.

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# NAVIER-STOKES-BRINKMAN MODEL FOR NUMERICAL SIMULATION OF FREE SURFACE FLOWS

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Abstract. In this paper, a two-dimensional free surface flow model of incompressible viscous fluid is presented. The unknowns are the velocity and pressure fields in the porous-liquid region, together with a function defining the volume fraction of liquid at the free surface. A volume of fluid capturing method is used for free surface flow computation and simulation. The model employs an explicit projection method on a Cartesian staggered gird to solve the complete two-dimensional Navier-Stokes-Brinkman system. The governing Navier-Stokes- Brinkman equations are solved in time by finite volume method. Boundary conditions are imposed efficiently. In order to show the ability of VOF method we used to simulate different test problems such as $2-D$ dam-break and wave propagation. A bi-conjugated gradient stabilizer (Bi-CGSTAB) method with a pre-conditioning procedure is used to solve the resulting matrix system. The validity of present model is verified based on the compressions with the existing theoretical and experimental results. Close agreements between numerical and experimental results show the robustness of the new numerical model.

**keywords :** volume of fluid, incompressible flow, finite volume method, submerged porous breakwater, free surface flows

## 1. INTRODUCTION

Free surface hydrodynamic flows has significant industrial and environmental importance. Nowadays it is appear in many engineering and mathematics problems. However the problems are difficult to simulate due to the existence of the arbitrarily moving surface boundary conditions and deformable computational domains. These difficulties make analytical approaches incompatible to deal with free surface problems. Numerical method can be used in order to overcome this



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problem. In the last years, many models and different numerical methods have been developed for the simulation of free surface flows in different areas. The marker and cell Harlow and Welch (1965) and volume of fluid Hirt and Nichols (1981) methods are two of the most flexible and powerful models for simulating such flows, in which the N-S equations are solved on a fixed Eulerian grid. In the former, marker particles are used to define free surface, while in the latter governing equations are solved for the volume fraction of the fluid. They have been successfully applied to a wide variety of flow problems involving free surfaces. In unsteady free surface flows, the location of the free surface changes continually as the solution progresses and its evolution has to be determined. In the case of steady free surface flows, the free surface is not known a *aprior* and the objective is to locate it correctly. The quantitative description of such problems depends on successfully locating the free surface.

Principles of shallow water theory Cunge (1980) and Chaudhry (1993) can not be used to simplify the analysis of many free surface flow problems when the streamline curvature is not small and a non-hydrostatic pressure distribution exists. Examples of such flows in hydraulic engineering are (i) the flow over a spillway, (ii) dam-break flow, (ii) a hydraulic jump, (iv)the flow past a floor slot, etc. The two-dimensional flow equations in a vertical plane can be numerical solved for a detailed study of the mechanics of free surface flows without having to make any simplifying assumptions regarding the flow structure. The marker-and-cell method by Harlow and Welch (1965) was the first method in this regard. In this method, massless marker particles are introduced in all the cells initially containing the fluid. Different techniques have also been developed for free surface tracking. Notable among them is the SOLA-VOF method by Hirt and Nicholse (1981) in which the Volume of fluid (VOF) technique is used for the surface tracking with solution algorithm (SOLA) from MAC method. This method uses a simple mass balance equation for time stepping a variable $F$, defined as the fractional volume of the cell occupied by the fluid. Strelkoff (1986) reported an exploratory numerical study on dam-break flow for dry bed conditions using the VOF technique. Qingcao and Drewes (1994) used the same technique in their turbulence flow model for free and forced hydraulic jumps. Lemos and Martins (1996) used the VOF technique in their numerical study of the impact of solitary waves on obstacles. Bradford and Katopodes (1998) developed a non-hydrostatic turbulence model for overland flow using the modified MAC method to the SMAC method for solving the governing equations and a height function method for free surface tracking. Recently, the present authors Lemi Guta and S.Sundar (2010) have adapted a VOF technique to study the interaction of viscous waves with a submerged porous structure. One of the unique advantages of these VOF techniques over the other methods, such as the height function method, is their ability to track multiple free surfaces in a convenient way. The robustness of the VOF techniques allows them to solve a spectrum of problems not accessible to most other surface-tracking methods.

Our aim is to assess the applicability of the VOF technique to complicated problems, and comparing the present numerical results with the experimental data and with previous numerical results available in literature.

## 2. GOVERNING EQUATIONS

As our focus is on basic mechanics and surface tracking, we consider two-dimensional flow where all possible forces acting on the fluid are taken into account. The governing equations for these two-dimensional incompressible Navier-Stokes Brinkman system in the dimensionless form are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$\tilde{s}\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}(\tilde{\mu}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \frac{1}{ReDa}u + \frac{1}{F_r^2}\tilde{f}_x \tag{2.2}$$

$$\tilde{s}\frac{\partial v}{\partial t} + \frac{\partial v^2}{\partial y} + \frac{\partial uv}{\partial x} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re}(\tilde{\mu}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) - \frac{1}{ReDa}v + \frac{1}{F_r^2}\tilde{f}_y \tag{2.3}$$

where $\tilde{\mu} = 1$, $\tilde{s} = 1$, $\tilde{f} = f_{NS}$, $K^{-1} = 0$ for the pure liquid region(Region 1) and $\tilde{\mu} = \frac{\mu_{eff}}{\mu}$, $\tilde{s} = s, \tilde{f} = f_B$, $K^{-1} = \epsilon_p k^{-1}$ for porous region (Region 2). The dimensionless parameters: Reyonald's, Darcy's and Froude numbers are given by $Re = \frac{Ud}{\nu}$, $Da = \frac{\tilde{k}}{d^2}$ and $F_r = \frac{U}{\sqrt{gd}}$ respectively with $\tilde{k} = \frac{k}{\epsilon_p}$. The external body force $\tilde{f} = ge_g$, g is gravitational acceleration and $e_g$ is unit vector pointed in the direction of the gravitational force, $\epsilon_p$ is porosity and $k$ is intrinsic permeability.

Several boundary conditions are needed for modeling surface waves given in Fig.1. At the moving free surface $\eta(x,t)$, the prescribed kinematic boundary condition is

$$v_{y=\eta} = \frac{\partial \eta}{\partial t} + u \cdot \nabla \eta \tag{2.4}$$

In terms of the dynamic boundary condition on free surface, stresses on a free surface must be balanced in normal and tangential conditions. The normal stress is equal to the atmospheric pressure and the tangential stress is zero. We ignore surface tension in the dynamic boundary conditions. Hence, the dynamic boundary condition in the normal direction was expressed by Huang(1998) as

$$p = \frac{\eta}{F_r^2} + \frac{2(1 + (\frac{\partial \eta}{\partial x})^2)}{Re(1 - (\frac{\partial \eta}{\partial x})^2)}\frac{\partial v}{\partial y} \tag{2.5}$$

Further, the dynamics boundary condition in the tangential direction can be presented as

$$2(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})\frac{\partial \eta}{\partial x} - (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})((\frac{\partial \eta}{\partial x})^2 - 1) = 0 \tag{2.6}$$

A no-slip boundary condition is used for viscous flows. For solid walls, the impermeability condition is specified, i.e. velocity normal to the wall is zero. At the inflow, the velocity components are specified by either analytical solutions or laboratory conditions. At the outflow boundary, the Sommerfeld radiation condition is implemented to minimize wave reflection into the computational domain.

The initial flow field is assumed to be still, so velocity components are both zero throughout the flow field. The hydrostatic pressure is used to initialized the pressure field.

## 3. NUMERICAL METHOD

A finite volume method is chosen to solve the governing equations. Under a Cartesian staggered grid framework, the computational domain is discretized by $Lx$ and $Ly$ cells in the $x$-,and $y$-directions, and the grid indexes $i$, and $j$ are also introduced, respectively.

### 3.1. **Interior Cells.**

Since the interior cells has a fixed grid size $(\delta x, \delta y)$, a finite volume scheme is used to discretize the momentum equations , the governing equations are integrated over the control volumes. Then, the integral form of $u$ and $v$ components of momentum equations are written as

$$\int_V (\frac{\partial u}{\partial t} + \frac{1}{ReDa}u - \frac{1}{F_r^2}\tilde{f}_x)dV - \int_{\partial V} (\frac{\tilde{\mu}}{Re}\nabla u - uU - pe_1)\cdot nds = 0 \tag{3.1}$$

$$\int_V (\frac{\partial v}{\partial t} + \frac{1}{ReDa}v - \frac{1}{F_r^2}\tilde{f}_y)dV - \int_{\partial V} (\frac{\tilde{\mu}}{Re}\nabla v - vU - pe_2)\cdot nds = 0 \tag{3.2}$$

Without loss of generality consider the x-component of momentum equation. Integrating $\frac{\partial u}{\partial t} + \frac{\tilde{k}}{ReDa}u - \frac{1}{F_r^2}\tilde{f}_x$ over the control volume $ABCD$

$$\int_{ABCD} (\frac{\partial u}{\partial t} + \frac{1}{ReDa}u - \frac{1}{F_r^2}\tilde{f}_x)dV \approx (\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\delta t} + \frac{1}{ReDa}u_{i,j} - \frac{1}{F_r^2}\tilde{f}_x)\cdot \Delta V \tag{3.3}$$

where $U = (u, v)$ and $\Delta V$ is the volume of control cell $ABCD$ and $\delta t$ is time subinterval. To obtain a discretized form for surface integral, we have to integrate over the four faces of control volume $ABCD$ of figure 3. Let $w_1 = \frac{\tilde{\mu}}{Re}\nabla u - uU - pe_1$, then we obtain

$$\int_{\partial ABCD} w_1 \cdot nds = \int_{AB} w_1 \cdot nds + \int_{BC} w_1 \cdot nds + \int_{CD} w_1 \cdot nds + \int_{DA} w_1 \cdot nds \tag{3.4}$$

We approximate the given integral over each phase of control volume $ABCD$. Accordingly, $u$-component of discretized momentum equations can be written as

$$a_{i,j}u_{i_j} = \sum_{s=1}^{2}\sum_{\gamma=\pm 1} a_{((i,j)+\gamma e_s)}u_{((i,j)+\gamma e_s)} + S_Q \tag{3.5}$$

where

$$
\begin{aligned}
a_{i-1,j} &= \frac{2\tilde{\mu}\delta t}{Reh_i(h_i+h_{i+1})} - \frac{2\delta t\tilde{u}_{i-\frac{1}{2},j}}{h_i+h_{i+1}}\\
a_{i,j-1} &= \frac{2\delta t}{l_j(h_i+h_{i+1})C_{i,j-\frac{1}{2}}} + \frac{2\delta t\tilde{v}_{i+\frac{1}{2},j-1}}{l_j}\\
a_{i+1,j} &= \frac{2\tilde{\mu}\delta t}{Reh_{i+1}(h_i+h_{i+1})}\\
a_{i,j+1} &= \frac{2\delta t}{l_j(h_i+h_{i+1})C_{i,j+\frac{1}{2}}}\\
a_{i,j} &= a_{i-1,j} + a_{i+1,j} + a_{i,j-1} + a_{i,j+1} - S_p\\
S_p &= 1 + \frac{\delta t}{ReDa}
\end{aligned}
$$

$$S_Q = \frac{\delta t}{F_r^2}\tilde{f}_x + (1+\frac{\delta t}{l_j}\tilde{v}_{i+\frac{1}{2},j})\tilde{u}_{i,j} + \tilde{t}\frac{\tilde{v}_{i+\frac{1}{2},j-1}}{l_j}\tilde{u}_{i,j-1} - \frac{2\delta t P_{i+1,j}}{h_i+h_{i+1}} + \frac{2\delta t P_{i,j}}{h_i+hi+1}$$

The Navier-Stokes-Brinkman equations clearly determine the respective velocity components so that their roles are plainly defined. This leaves the continuity equation, which does not contain the pressure, to determine the pressure. To do this, the most common method is based on combining the two equations. Thus we used Chorin(1968) projection method to derive pressure Poisson equation given by

$$\Delta p^{(n+1)} = \frac{\nabla \cdot v^*}{\delta t} \tag{3.6}$$

where $v^*$ is an intermediate velocity ( the tentative velocity field ).

3.2. **Free-Surface Cells.**

The most important property of the flow that is studied in this paper is the presence of a free surface. The flow is mainly determined by this free surface, therefore a proper discretization is required to capture its dynamics. To obtain numerical solutions of the governing equations, the whole computational domain is divided into Cartesian cells. Since the free surface calculation is a challenging point in the numerical study, an appropriate approach to treat free surface variation has to satisfy two requirements. First of all, it has to capture free surface profiles. The other point is to impose free surface boundary conditions at the captured free surface.

3.3. **Volume Fluid Method.**

We adopt the Volume -of- Fluid (VOF) method Hirt and Nichols (1981) to evaluate the free surface variation. The main principle of VOF is to define another variable called the VOF function, $F(x, y, t)$. It is also defined at the center of computational cell. Its value varies from 0 to 1. Due to a free surface, computational cell may contain fluids or not. A cell full of fluid is called a full cell. The VOF functions, in a full cell are defined as 1. On the other hand, if a cell does not contain any fluid, it will be called an empty cell with $F$ defined as 0 for an empty cell. In addition, a cell containing a free surface is called a surface cell with $F$ varying between 0 and 1 for a surface cell;

$$F(i, j) = \frac{volume1}{volume2} \tag{3.7}$$

where *volume*1 is volume of the fluid in a given cell and *volume*2 is volume of the corresponding cell. $F = 1$ means the cell is full of water, $F = 0$ is air cell, while $0 < F < 1$ means the cell contains the free surface. Free surface boundary conditions have imposed at surface cells. The free surface is time dependent, so the VOF functions, $F$, has to be calculated at each time step and satisfies the convection equation,

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0 \tag{3.8}$$

For incompressible fluid flow equation (3.8) combined with continuity equation (2.1) to yield the equation

$$\frac{\partial F}{\partial t} + \nabla \cdot (F\vec{u}) = 0 \tag{3.9}$$

where $\vec{u} = (u, v)$. In addition to equation (3.9), Hirt and Nichols (1981) derived the donor - acceptor method to solve the formula. Let a horizontal velocity component, $u$ be located at the center of a control face. If $u$ is positive, then the cell in the left side of the control face is called a donor. On the other hand, the cell in the right hand side is called an acceptor. The variation of $F$ is evaluated using the above given formula (3.9), which may be integrated in the control volume of the single $(i, j)$ cell

$$\int_V \frac{\partial F}{\partial t} dV + \int_V \nabla \cdot (\vec{u}F) dV = 0$$

$$\Rightarrow \int_V \frac{\partial F}{\partial t} dV + \int_{\partial V} (\vec{u} \cdot \vec{n}) F ds \quad = \quad 0 \tag{3.10}$$

For the sides *west* and *east* the above integral yields $\frac{\Delta F}{\delta t}\Delta V + Ful_j = 0$, while for the sides *south* and *north* we get $\frac{\Delta F}{\delta t}\Delta V + Fvh_i = 0$. The discretization of the convective part of the volume fraction equation needs special attention. Its discretization should neither produce diffusion nor unbounded values. So the scheme must satisfy that the computed fluxes of Volume fractions do not underflow or overflow the cells. In a strong convective flow from *west* to *east*, the central difference scheme treatment is unstable because the west face should receive much stronger influencing from west node ($W$) than from central node ($P$). Thus, the upwind differencing scheme or donor cell differencing scheme, which is unconditional stable and always produce a bounded solution takes into account the flow direction when determining the value at a cell face. Let $sgn(u)$ be the sign of velocity $u$, whether the flow is in the positive direction (*west* to *east*) or negative direction (*east* to *west*). Since $V = h_i l_j$, using first order upwind discretization we have

$$\Delta F_{i,j}|_{i+\frac{1}{2},j} = -(1 + sgn(u))\frac{u \cdot \delta t}{2h_i}F_{i,j} + (1 - sgn(u))\frac{u \cdot \delta t}{2h_i}F_{i+1,j} \tag{3.11}$$

where $\Delta F_{i+\frac{1}{2},j}$ denotes the increment in $F_{i,j}$ due to the fluxes over the *east* phase and in which $u = u_{i+\frac{1}{2},j}$. Similar expression can be derived for the change in $F_{i,j}$ due to the contribution of the other sides, yielding

$$\Delta F_{i,j}|_{i-\frac{1}{2},j} = (1 + sgn(u))\frac{u \cdot \delta t}{2h_i}F_{i-1,j} - (1 - sgn(u))\frac{u \cdot \delta t}{2h_i}F_{i,j} \tag{3.12}$$

This denotes the changes in $F$ due to the flux over the *west* phase and in which $u = u_{i-\frac{1}{2},j}$. For the cell $(i,j)$ evaluate at $(i, j + \frac{1}{2})$ we also have an expression

$$\Delta F_{i,j}|_{i,j+\frac{1}{2}} = -(1 + sgn(v))\frac{v.\delta t}{2l_j}F_{i,j} + (1 - sgn(v))\frac{v \cdot \delta t}{2l_j}F_{i,j+1} \tag{3.13}$$

where $\Delta F_{i,j+\frac{1}{2}}$ denotes the increment in $F_{i,j}$ due to the fluxes over the *north* phase and in which $v = v_{i,j+\frac{1}{2}}$. For the change in $F_{i,j}$ due to the contribution of the *south* phase is also written as

$$\Delta F_{i,j}|_{i,j-\frac{1}{2}} = (1 + sgn(v))\frac{v \cdot \delta t}{2l_j}F_{i,j-1} - (1 - sgn(v))\frac{v \cdot \delta t}{2l_j}F_{i,j} \tag{3.14}$$

Assembling the changes in $F$ from all four sides and updating $F$ yields the new distribution of the liquid in the computational domain. In order to ensure that the total amount of fluid does not change, *i.e.* $\sum_{i,j} F_{i,j}$ remains constant, the value of $F$ in the surface cells will be modified. After the new distribution of $F$ has been calculated the labelling of *empty*, $surface$ and $full$ cells is performed again. The boundary conditions on the free surface will be set again, after which the following time step can be taken.

3.4. **Matrix Solver.**

We solve non-symmetric systems of linear algebraic equations corresponding to each of momentum equations, and a system of linear algebraic equations corresponding to pressure correction equation. ILU preconditioned Bi-CGSTAB is used in the first case, while ILU preconditioned CG is used in the second case. Bi-CGSTAB is a fast and smoothly converging variant of Bi-CG for the solution of non-symmetric linear systems. The entire algorithm is implemented in C++. SpharseLib + + 1.7 and IML ++ are used to solve systems of linear algebraic equations. Visualization is done in Matlab.

## 4. RESULTS AND DISCUSSIONS

In this section, we validated the proposed model by considering different test cases and benchmark problem for the simulation of free surface flow. This include; no flow problem and broken dam problem. In case of broken dam problem, the position of the leading fluid front versus time and height of the water column versus time are compared with experimental result of W.J Moyce and J.C.Martin(1952).

### 4.1. Comparison Between Theoretical Solitary Wave Solution and Numerical Results.

The incident solitary waves were generated using a numerical piston-type wavemaker develop by Huang (1998). To generate a solitary wave, the plate displacement, $x$, as a function of time must be determined. According to Yasuda (1994) the water surface elevation is given by

$$\eta(x,t) = Hsech^2[\sqrt{\frac{3H}{4d^3}}(x-ct)] \tag{4.1}$$

where $H$ is the wave height, $d$ the water depth, and $c = \sqrt{g(d+H)}$ the phase speed. Two numerical experiments are conducted: The first test (test 1) has a wave height-to-water depth ratio of $Hi/d = 0.35$, while the second test (test 2) has $Hi/d = 0.175$. In both cases the water depth is kept at 16.0 cm. The breakwater height to water depth ratio is also kept as a constant of $h/d = 0.625$. The numerical results for surface profiles are compared with the theoretical solitary wave solution given by Yasuada(1994). The comparisons of the numerical results with the theory are shown in Fig. 2. In both tests, the agreement between the theoretical solution and the numerical results for surface profile is nearly perfect.

### 4.2. No-flow Problem.

In order to validate our algorithm for free surface flow, first we have used no-flow problem as a benchmark problem.

The dimensionless model of no-flow problem in the fluid domain $\Omega$, is given by,

$$-\frac{1}{R_e}\Delta u + (u\cdot\nabla)u + \nabla p = f$$

$$\nabla\cdot u = 0$$

where $f = (-\delta(1-x)^2, 0)$ with boundary conditions, $u = 0$ on solid walls. $u \cdot n = 0$ and $n \cdot T(u,p) \cdot \tau = 0$ , on free surface. where $T$ is the stress tensor, $n$ is the normal and $\tau$ is tangent to the free surface. The exact solution of this problem is given by $u \equiv 0, p = \frac{\delta}{3}(1-x)^3 - \frac{\delta}{12}$. We calculate the numerical results for $Re = 100$ and $\delta = 0.001$ and compared with the exact solution. The boundary is taken on unit square length. The errors of the pressure and velocity in 2-Norm and the results of GAO(2009) are given in tables (1) and (2). GAO(2009) has obtained results by using finite volume method with hybrid unstructured triangular collocated grids. GAO's results show that as the number of unknowns increases, the errors of

TABLE 1. Comparison of errors of velocity values

| SI.No. | No.of unknowns | Our results | GAO results |
|---|---|---|---|
| 1 | 60 | $1.7396e^{-}12$ | $2.183247e^{-}06$ |
| 2 | 170 | $1.2104e^{-}10$ | $1.570504e^{-}07$ |
| 3 | 560 | $4.3414e^{-}10$ | $1.204146e^{-}08$ |
| 4 | 2230 | $1.2610e^{-}09$ | $9.758404e^{-}10$ |
| 5 | 8400 | $5.8824e^{-}10$ | $8.219707e^{-}11$ |
| 6 | 33400 | $2.0346e^{-}10$ | $7.085978e^{-}12$ |
| 7 | 139800 | $4.8110e^{-}11$ | $7.075600e^{-}13$ |

velocity and pressure decrease. Our results show that for the velocity the accuracy is high and it is independent of number of unknowns. For the pressure the accuracy

TABLE 2. Comparison of errors of pressure values

| SI.No. | No.of unknowns | Our results | GAO results |
|---|---|---|---|
| 1 | 10 | $2.5311e^{-}05$ | $1.544124e^{-}04$ |
| 2 | 25 | $1.3331e^{-}05$ | $3.827905e^{-}05$ |
| 3 | 75 | $4.6311e^{-}06$ | $6.527376e^{-}06$ |
| 4 | 275 | $1.3485e^{-}06$ | $1.378508e^{-}06$ |
| 5 | 1050 | $4.0010e^{-}07$ | $5.943974e^{-}07$ |
| 6 | 4225 | $1.0176e^{-}07$ | $1.485844e^{-}07$ |
| 7 | 7500 | $3.2455e^{-}08$ | $3.714520e^{-}08$ |

is higher than GAO's results.

4.3. **The Broken Dam Problem.**

This is a very popular and typical test case to validate many numerical schemes for the simulations of free surface flows. The problem demonstrate the ability of the method to compute transient fluid flow with free surface. It consists of simple initial configurations and simple initial and boundary conditions. The experimental results are available in W.J. Moyce(1952). Consider a rectangular column of water with a width of $L = 5.1m$ and a height of $h = 10.2m$ . The lines $x = 0$ , $y = 0$ and $x = 25$ consists of the solid wall. The first one is made in a sample tank, and the second in a tank with breakwater. In the simulation, the upper and the right boundary of the water columns are consider as the free surface boundary. Initially, $100 \times 100$ grid points are considered. The gravity with $g = 9.81\ m/s^2$ acts downwards, the initial velocity is set to zero. The initial hydrodynamic pressure $p_0$ is also considered to be zero. The surface tension forces are neglected. When the right wall(dam) is removed, the water column collapses under the influence of gravity. The dynamic viscosity of the fluid is $\mu = 0.5$ kg/m.s . No slip boundary condition is used on the solid walls. Unfortunately measurements of the exact interface shape are unavailable, but others data as the speed of the wave front and the reduction of the column height are presented. The particles, plotted successively in time, are shown in Fig. 4. The numerical results can be compared with the experimental data given in W.J. Moyce(1952). In Fig. 5, the position of the leading fluid front versus time and height of the water column versus time are compared. The figure shows a good agreement between the numerical and experimental result.



4.4. **Viscosity Effect on the Moving Solitary Wave.**

After having verified the accuracy of the numerical scheme, the interaction of a solitary wave and a submerged breakwater was studied. We consider numerical tank of length 20m and depth 0.2m. The wave with height 0.075m was propagated toward the right direction. The viscous solitary wave travels at Reynolds number 1000. The whole simulation time is $25s$. Fig.6 shows that the evolution of moving solitary wave. We learned from the figure, a solitary wave decays due to the energy dissipation caused by the viscosity. Accordingly due to the fluid viscosity, the wave height of moving solitary wave decreases as time goes. Fig.7 shows the time history of the wave height. The wave height decays due to fluid viscosity as the wave travels. It reveals that our numerical results agree very well with the analytic solution. The established numerical model simulates the viscous damping in the moving solitary wave successfully. On the other hand, the viscous effect does not only reduce the wave height but also slow down the wave velocity. Russell(1838) experimental data for a solitary wave cited by Keulegan(1948) is also shown in Fig.8 to provide more information for comparison. The numerical values shown in Fig.8 where calculated with , $H_i = 0.028$ , $\nu = 1.14 \times 10^{-6} m^2/s, g = 9.81 m/s^2$ and d=0.2m, to be consistent with Russells experimental conditions. We noted that our numerical results lie between the Mei(1983) results and Russell(1838) experimental data.

## 5. CONCLUSION

In this study, a numerical method based on the finite volume method and Volume of fluid capturing technique is used to simulate the two-dimensional free surface flow for incompressible viscous fluid interact with permeable submerged breakwater. The unknowns are the velocity and pressure fields in the porous-liquid region, together with a function defining the volume fraction of liquid at the free surface. The model employs an explicit projection method on a Cartesian staggered gird to solve the complete two-dimensional Navier-Stokes Brinkman system. The governing NavierStokes- Brinkman equations are solved in time by finite volume method. The velocity field is integrated in time without enforcing incompressibility in the first step. In the second step, a Poisson equation of pressure is used to satisfy incompressibility condition. The free surface boundary condition and the interfacial boundary conditions between the water and porous media are imposed efficiently. We validate the proposed numerical model with the theoretical and experimental solutions. The numerical experiments are performed with and without porous breakwater. Close agreements between numerical and experimental results show the robustness of the new numerical model.



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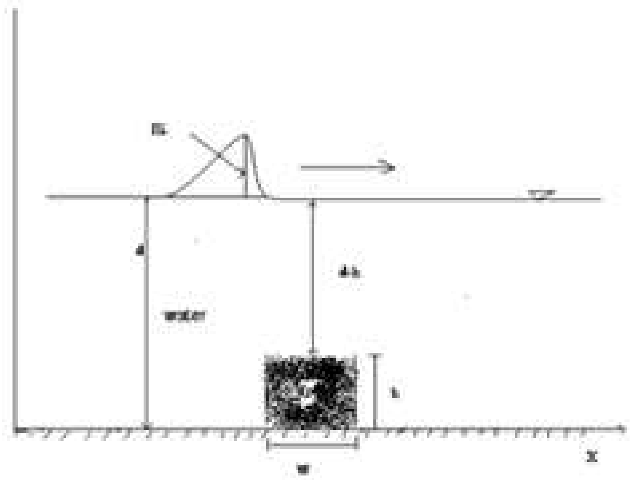


FIGURE 1. Schematic diagram of wave propagating over a submerged porous breakwater

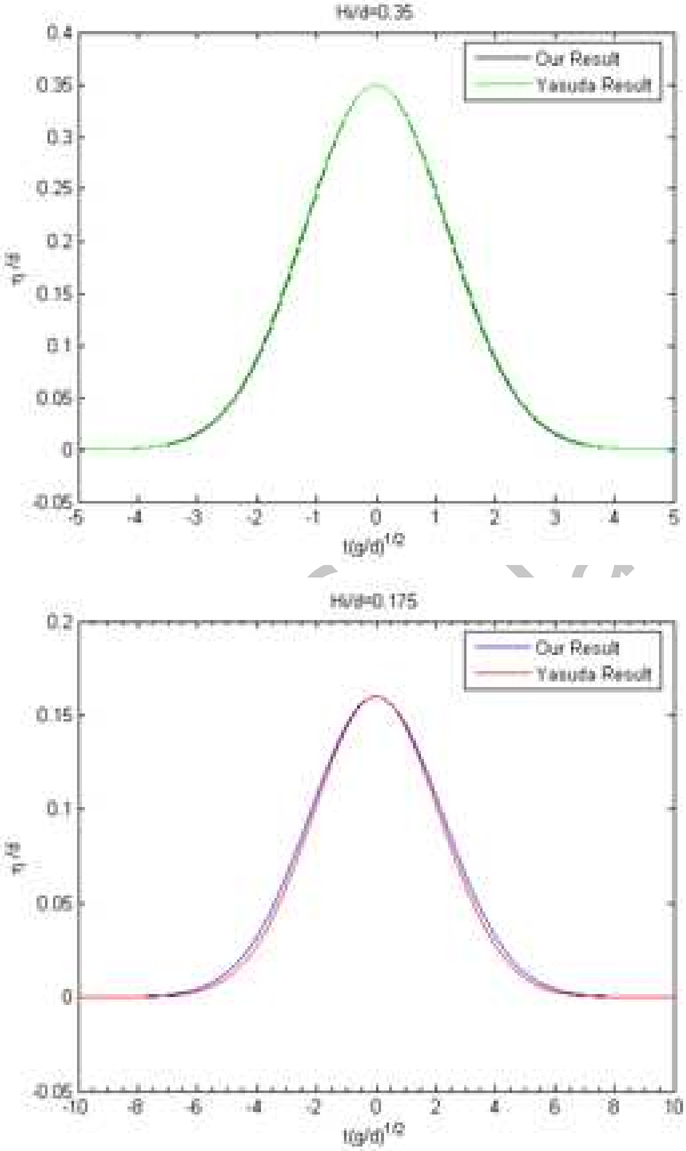


FIGURE 2. Comparison of solitary waves profiles with Yasuda (1994) results for different value of $H_i/d$

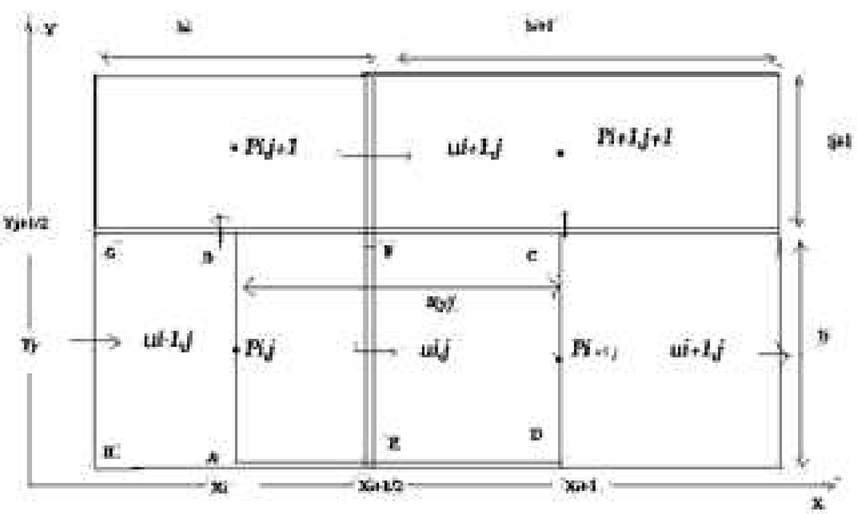


FIGURE 3. : Schematic diagram of a Cartesian staggered grid system on the xy plane

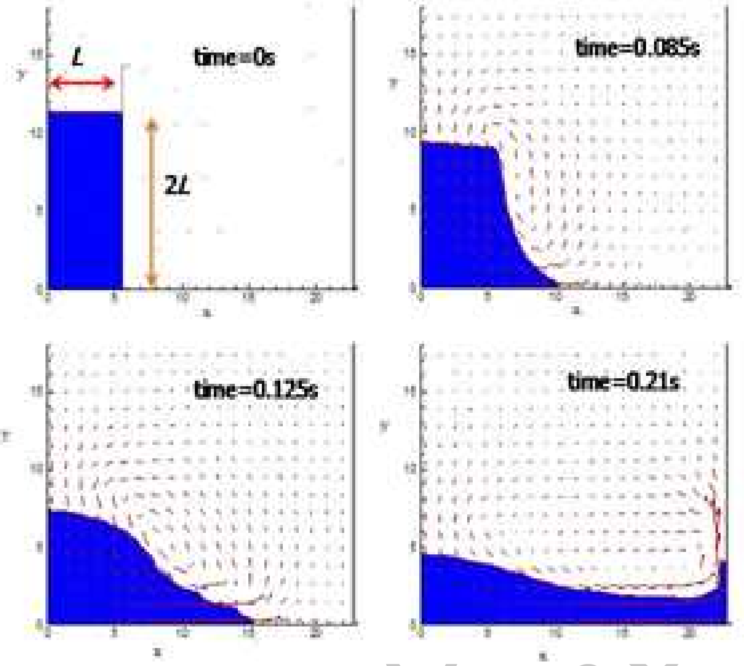


FIGURE 4. Velocity vectors and fluid configuration for the broken dam at times t=0s, t=0.085s, t=0.125s and t=0.21s

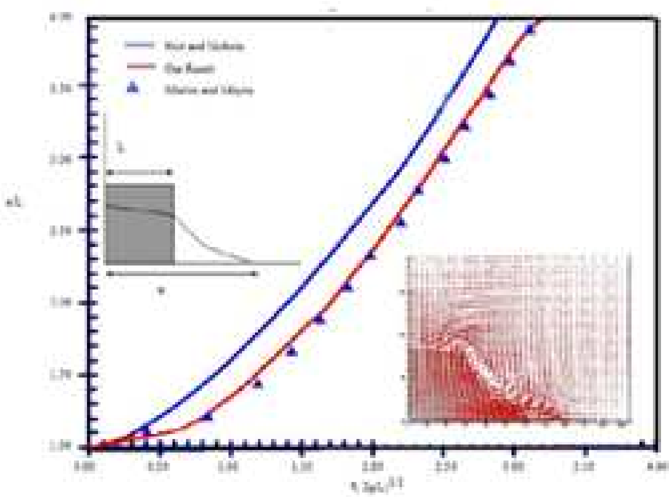

FIGURE 5. Time history of leading edge of the wave for comparison of present numerical result with experimental data

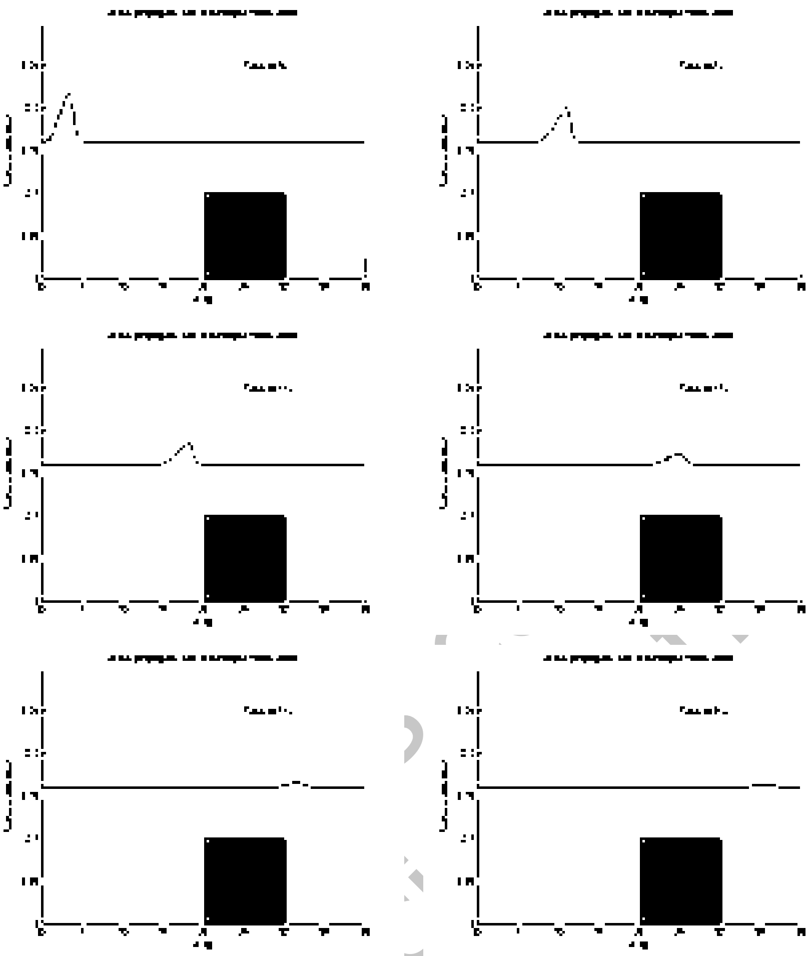

FIGURE 6. The evolution of a moving solitary wave over porous breakwater for $H_i/d = 0.35$

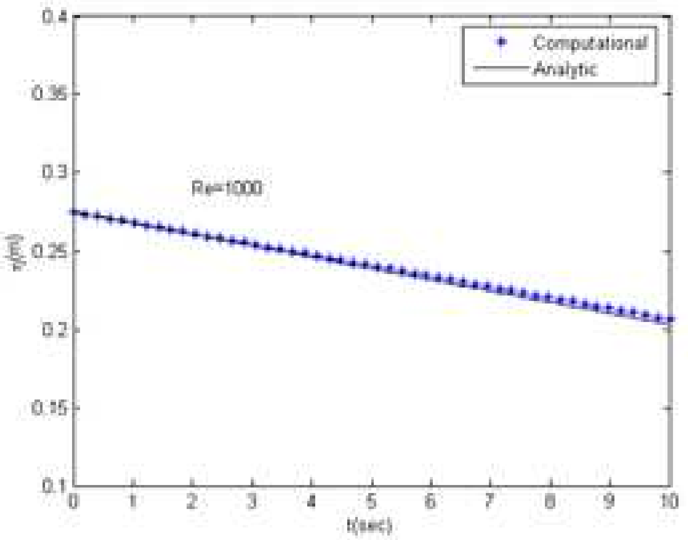


FIGURE 7. The wave height decays due to fluid viscosity as the wave travels

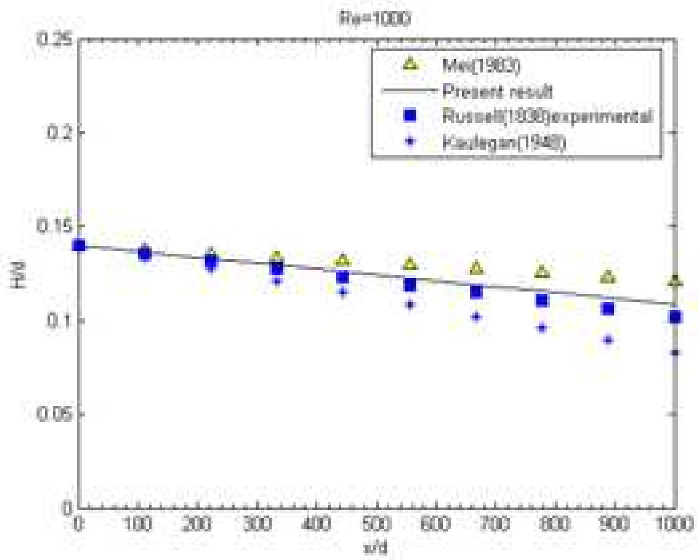


FIGURE 8. Attenuation of a solitary wave of Hi=0.04 propagating in a channel with a flat bottom

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# INVARIANTS OF INCIDENCE MATRICES

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## 1. Introduction

Study of algebraic invariants of integer matrices like ranks over appropriate fields, and more generally Smith normal forms over principal ideal domains, row span over (finite) fields, etc., have been of interest in several problems in design theory, coding theory, algebraic graph theory, finite incidence geometries and representation theory of finite groups of Lie type. The matrices considered usually codify incidence relations between (finite) sets.

An *incidence geometry* $\mathcal{X}$ is a triple $(P, L, I)$, where $P$ and $L$ are disjoint (possibly empty) sets and $(P \times L) \cup (L \times P)$ is a symmetric, nonreflexive relation on $P \cup L$. Elements of $P$ are called the *points* of $\mathcal{X}$, those of $L$ are called the *lines* of $\mathcal{X}$, *and* $I$ is called the *incidence relation.* We say that $x \in P$ and $l \in L$ are *incident if* $(x, l) \in I$. In this case, we write $xIl$. *We assume here through out that* $P$ *and* $L$ *are finite sets and each element of* $P \cup L$ *is incident with at least two of its elements.*

An *incidence matrix* $A = A(\mathcal{X})$ (respectively, a *generalized incidence matrix* ) of $\mathcal{X}$ *over a ring* $R$ is the matrix whose rows are indexed by $L$, columns by $P$ and for $l \in L$ and $p \in P$, the $(l, p)^{th} -$ entry of it is 1 (respectively, a nonzero element of $R$) or 0 according as $p$ is incident with $l$ or not. Note that $A$ depends on the ordering of rows and columns.

Viewing $A$ as a matrix over a principal ideal domain $R$, we can find invertible matrices $U$ and $V$ with entries from $R$ such that $S(A) = UAV = [D|\underline{O}]^{tr}$, where $D = diog\left(s_1, \cdots, s_{|P|}\right)$ and $\underline{O}$ is the zero matrix of size $(|L| - |P|) \times |P|$ (assuming $|P| \leq |L|$). See; [20]. The elements $s_i$ are unique up to units in $R$ and are called the *invariant factors* of $A$. The matrix $S(A)$ is called the *Smith normal form* of $A$ (SNF, for short). Let $d_i$ be the greatest common divisor of all minors of $A$ of order $i$. Then, $d_i = s_1 \cdots s_{i-1}$. If $\langle A \rangle$ denotes the $\mathbb{Z}$- row span of $A$, then

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$\mathbb{Z}^{|P|}/\langle A\rangle \simeq \mathbb{Z}^{|P|}/s_{|P|}\mathbb{Z}^{|P|}$. Clearly, rank of $A$ over rationals is the number of nonzero $s_i$'s and, for any prime $p$, $p$- rank of $A$ is the number of $s_i$ incongruent to zero $\bmod p$ . Thus SNF is a more complete invariant, but usually more difficult to calculate. Unlike the set of eigen values, the SNF (and so the ranks over prime fields) of $A(\mathcal{X})$ is independent of the ordering of the elements of $P$ and $L$, and depends only on the division ring.

Ranks of incidence systems some times distinguish between nonisomorphic incidence systems (see; [66], [2], [83], §.9 for some examples). Moorehouse and Blokhius ([11]) obtained restrictions on the embeddability of certain geometric objects in projective and polar spaces using the $p-$ ranks of appropriate $(0,1)$ -matrices and showed the nonexistence of ovoids in certain polar spaces ([61], [63], [11]). See; [76] for a study of sets of eigen values of incidence matrices of projective planes.

More information about the incidence system can sometimes be obtained by considering the $R-$ *ary code* of $\mathcal{X}$ for an appropriate choice of the ring $R$; that is the $R-$ linear span $\mathcal{C}_R = \mathcal{C}_R(\mathcal{X})$ of $A(\mathcal{X})$. This is determined by $\mathcal{X}$ and is unique up to the permutation of the coordinate positions.Let $R^P$ denote the $R-$ module of all functions from $P$ to $R$. For $l \in L$, let $P_l = \{x \in P : (x,l) \in I\}$. We sometimes view $\mathcal{C}_R$ as the $R-$ submodule of $R^P$ generated by the characteristic functions of $P_l$, $l \in L$, on $P$. We write the constant function one on $P$ as $\mathbf{1}_P$. If $R$ is a finite field $\mathbb{F}_q$, we write $\mathcal{C}_R$ as $\mathcal{C}_q$ and call it the $q-$ *ary code of* $\mathcal{X}$. We consider the dual $A^\perp$ of subsets $A$ of $R^P$ with respect to the symmetric bilinear form on $R^P$ which has the characteristic functions of elements of $P$ as an orthonormal basis of $R^P$. We recall that the weight of an element of $\mathcal{C}_R$ is the number of its nonzero coordinates. From the point of view of coding theory, one would like to know for both $\mathcal{C}_q$ and its dual $\mathcal{C}_q^\perp$: the dimension, minimum of the weights of nonzero elements and if possible, elements of a given nonzero (particularly the minimum) weight. Further, one would like to understand the $RG-$ module structures $\mathcal{C}_R$ and $\mathcal{C}_R^\perp$ acquire when a group $G$ acts on both $P$ and $L$ preserving incidence.

In this expository article, we describe in Section 2 some incidence matrices and the associated $q-$ ary codes arising in the context of finite permutation groups, give some examples in Section 3 and describe some known results, applications and questions in Section 4.For the concepts of coding theory used here, see; [2].

## 2. Incidence systems and codes from permutation groups

Let $G$ be a finite (simple) group; $M_1$ and $M_2$ be maximal subgroups of $G$ ($M_1 = M_2$, possibly) and $X_i = \{gM_i : g \in G\}$. Let $k$ be a field. The following basic questions are studied in literature for special choices of $G, M_1, M_2$ and $k$. See; [2], [68], [18] for examples.

(A) Determine the structure of $k^{X_i}$ as a $kG-$ module.

(B) Determine $Hom_{kG}\left(k^{X_2}, k^{X_1}\right)$.

(C) Given $f \in Hom_{kG}\left(k^{X_2}, k^{X_1}\right)$, determine the $kG-$ module structure and the words of minimum weight of: (a) the image of $f$; (b) the kernel of $f$ and (c) their orthogonal compliments.

Here are some reasons to study these questions. (i) Among the indecomposable $kG-$ modules, the modules involved in the above may be more tractable because of the associated geometry. The number of indecomposable $kG-$ modules is 'usually' infinite if $p$ divides the order of $G$ (in fact, if, and only if, $G$ contains a copy of $\mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is the characteristic of $k$, see; [44], Theorem 5.4, p.68). (ii) The modules arising above might yield good algebraic codes and yield insights into the geometry they arise from.

The questions (A), (B), (C) above are increasingly intricate as we move *from* the case when the characteristic $p$ (including zero) of $k$ does not divide the order of $G$; *to* the case when $p$ divides the order $G$, but is not the 'characteristic' of $G$ (when defined); *to* the case of the 'characteristic' of $G$. A group $G$ *is of characteristic* $p$, $p$ a prime, if $C_H(O_p(H)) \subseteq O_p(H)$ for all $p-$ local subgroups $H$ of $G$. The characteristic of a group of Lie type is that of the field over which it is defined. Note that some groups may have more than one characteristic associated with it, for example ${}^2A_3(2) \backsimeq U_4(2, 2^2) \backsimeq C_2(3)$ and some groups may not have a characteristic associated with it at all (for example, $A_n$). We concentrate here on the case when $p$ is the characteristic of the group. This case has been very useful in the study of some geometric questions. Study of the $2-$ ary code associated with a symplectic $3-$space of even order has enabled the determination of the intersection pattern of classical ovoids in a projective symplectic 3- space of even order [8]. Study of the $2-$ ary codes associated with a generalized quadrangle of odd order has enabled Brouwer, Bagchi and Wilbrink to show that if a point of a generalized quadrangle of odd order is anti-regular, then so is every point [4].

From the classification of finite simple groups, we now know that apart from the alternating groups and the sporadic groups, all finite simple groups are of Lie type [32]. For these groups, a uniform description of the structure and the representation theory is available in terms of the root system and the field defining the group (see [15], [74] and [13]). In the case when $G$ is a finite simple group of Chevalley type defined over a field $\mathbb{F}_q$ of characteristic $p$, $M_1$ and $M_2$ are maximal proper parabolic subgroups of $G$ and $k$ is a field of characteristic $p$, Curtis theory (see; [23], 6.6, 6.8) reduces the solution of (B) to the same question for the corresponding parabolic subgroups of the (much simpler!) Weyl group associated with $G$. In the spirit of Chevalley, one may hope in this case to get the answers in terms of some combinatorial objects related to the root system and the field $\mathbb{F}_q$ defining $G$. Examples in § 3.4 describe some of these incidence systems in a combinatorial language. In Examples § 3.5, one of the maximal subgroups $M_i$, $i = 1, 2$, is parabolic and the other is not.

Some of the $kG-$ modules above are multiplicity free (see for example, Lemma 2.1 of [9]); that is, its composition factors are pair wise distinct. For any such module, the submodule lattice is finite and each submodule is uniquely determined by its maximal semi-simple quotient. It would be interesting to determine the $kG-$ modules $k^X$ and $f\left(k^X\right)$ amongst the above which are multiplicity free.

## 3. Examples

**(1)** The best understood and the most elegant examples studied come from the six Witt designs $W_{24-i}$ with parameters $(5-i)-(24-i, 8-i, 1)$ and $W_{12-i}$ with parameters $(5-i)-(12-i, 6-i, 1)$, $i = 0, 1, 2$. These designs are uniquely determined by their parameters and can be obtained as successive extensions of the projective plane of order 4 and the affine plane of order 3, respectively. The $2-$ ary code for $W_{24}$ and the $3-$ ary code for $W_{12}$ are self dual. These are called the *extended Golay codes*. For $i = 0, 1, 2$, the automorphism group $M_{24-i}$ of $W_{24-i}$ (respectively $M_{12-i}$ of $W_{12-i}$) is simple, called a Mathieu group. It is best studied in terms of the $2-$ ary (respectively $3-$ary) code associated with $W_{24-i}$ (respectively $W_{12-i}$). The design $W_{24}$ is very central to the construction of a very special $24-$ dimensional lattice called the Leach lattice and so the Conway groups and the Monster simple group [22]. The $2-$ ary code for $W_{23}$ and the $3-$ ary code for $W_{11}$ are perfect. These codes are called *the Golay codes*.

**(2)** The $p-$ ary codes associated with a projective plane of order $n$ for the primes $p$ dividing $n$ are extensively studied. Nonexistence of a projective plane of order 10 is proved by a study of these codes together with a lot of computer calculations. See; [56] for the history of this problem. There are several excellent accounts of the theory for the examples in (1) and (2). See; for example, [10], [2] and references there for (1) and [2], [66], [12] and [68] for (2). We say no more about them .

**(3)** Let $X$ be a set having $n$ elements and $X_r$ be the set of its $r-$ subsets. For $0 \leq m \leq r < s \leq n-1$ such that $r \neq 0$ and $r + s \leq m + n$, let $I_{r,s,m} = \{\{A, B\} \in X_r \cup X_s : |A| \neq |B| \text{ and } |A \cap B| = m\}$. Then, $\mathcal{C}^n_{r,s,m} = (X_r, X_s, I_{r,s,m})$ is an incidence system. Clearly, the symmetric group on $X$ acts on $X_r$ and $X_s$ leaving $I_{r,s,m}$ invariant.

Next example is the $q-$ analogue of this.

**(4)** Let $q = p^t$, $p$ a prime, and $V = \mathbb{F}_q^{n+1}$. For $0 \leq r \leq n-1$, let $\mathcal{P}_r = \mathcal{P}^n_r(q) = \mathcal{P}_r(V)$ be the set of all $(r+1)-$ dimensional subspaces of $V$ (that is, the set of all $r-$ flats in the projective space $\mathcal{P}(V)$ associated with $V$) and $\mathcal{A}_r = \mathcal{A}^n_r(q) =$

$\mathcal{A}_r(V)$ be the set of the cosets of the $r$- dimensional subspaces of $\mathbb{F}_q^n$. We take the dimension of the empty set to be $-1$. Thus, $\mathcal{P}_{-1}(V) = \mathcal{A}_{-1}(V) = \{\phi\}$. Observe that

$$|\mathcal{P}_r^n(q)| = \begin{bmatrix} n+1 \\ r+1 \end{bmatrix}_q = \prod_{1=0}^{r} \left(q^{n+1-i} - 1\right)\left(q^{r+1-i} - 1\right)^{-1}$$

$$\text{and } |\mathcal{A}_r^n(q)| = q^{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_q.$$

For integers $m, s, r$ such that $-1 \leq m \leq r < s \leq n, r \neq -1$ and $r + s - m \leq n$, let $I_{r,s,m} = \{\{A, B\} \in \mathcal{P}_r \cup \mathcal{P}_s : |A| \neq |B| \text{ and } \dim A \cap B = m + 1\}$. For $A \in \mathcal{P}_l^n(q), l \in \{r, s\}$, let $[A]_{r,s,m} = \{B \in \mathcal{P}_r \cup \mathcal{P}_s : AI_{r,s,m}B\}$. Then, the sets $[A]_{r,s,m}$, $(-1 \leq m \leq r)$ partition $\mathcal{P}_r^n(q)$ and

$$|[A]_{r,s,m}| = q^{(r-m)(s-m)} \begin{bmatrix} l+1 \\ m+1 \end{bmatrix}_q \begin{bmatrix} n-l \\ l'-m \end{bmatrix}_q,$$

where $\{l, l'\} = \{r, s\}$. We write the incidence system $(\mathcal{P}_r, \mathcal{P}_s, I_{r,s,m})$ as $\mathcal{P}_{r,s,m}^n(q)$. With a similar definition of incidence, we also consider the incidence system $\mathcal{A}_{r,s,m}^n(q) = (\mathcal{A}_r, \mathcal{A}_s, I_{r,s,m})$. If $r = m = 0$, we drop the subscripts $r$ and $m$ in $\mathcal{P}_{r,s,m}^n(q)$ and $\mathcal{A}_{r,s,m}^n(q)$.

The incidence systems $\mathcal{P}_s^n(q)$ and $\mathcal{A}_s^n(q)$ above are $2-$ designs, called *geometric designs*. Further, if $s = n-1$, the designs are said to be *classical*. Recall that, for $t \geq 1$, an incidence system $(P, L, I)$ is said to be a $t-$ *design with parameters* $(v, k, \lambda)$ if $|P| = v, |\{x \in P : xIl\}| = k$ for each $l \in L$ and for each $t-$ subset $A$ of $P, |\{l \in L : lIa \text{ for each } a \in A\}| = \lambda$. The number of $2-$designs with the same parameters as the geometric designs grows exponentially with linear growth of $n$. See; [47], [49], [57], [58] for $2 \leq s \leq n-2$ and [48] for the case $s = n-1$.

**Remarks:** (a) The linear group $G = GL_{n+1}(q)$ (respectively the affine group $Aff_{n+1}(q)$) acts on the sets $\mathcal{P}_r$ and $\mathcal{P}_s$ (respectively $\mathcal{A}_r$ and $\mathcal{A}_s$), leaving each of the above incidence relations above invariant.Note that several $G-$ invariant incidence relations can be defined on $\mathcal{P}_r \cup \mathcal{P}_s$, $0 \leq r < s \leq n$. Taking $\Lambda \subseteq \{-1, 0, \cdots, r\}$ and defining $A \in \mathcal{P}_r$ and $B \in \mathcal{P}_s$ to be *incident* if $\dim(A \cap B) \in \Lambda$, we get one such example.

(b) Let $G$ be a finite classical group which is either symplectic, unitary or one of the three orthogonal groups, and $\mathcal{S}$ be the polar space associated with $G$. So, $\mathcal{S}$ is the partially ordered set (with inclusion as the ordering) of all isotropic subspaces with respect to the sesquilinear or quadratic form defining $G$. See; [19] for a discussion on polar spaces. Just like $\mathcal{P}_{r,s,m}^n(q)$ above, we can define an incidence system which admits an action of $G$ by replacing $\mathcal{P}_l$, $l \in \{r, s\}$, in $\mathcal{P}_{r,s,m}^n(q)$ by the set of all $l-$ dimensional isotropic subspaces and incidence as in $\mathcal{P}_{r,s,m}^n(q)$. Here $r$ and $s$ can be at most equal to the rank of the polar space.

More generally, for any finite group $G$ of Lie type and any two distinct nodes of its Dynkin diagram, we can define an incidence system admitting an action of $G$ as follows : choose for the point-set and the line-set the residues of flags of corank one corresponding to the chosen two nodes in the spherical building associated with $G$ and declare a 'point' and 'a line' to be 'incident' if their union is a flag. See; [69] for an account of spherical buildings.

Incidence systems in Example (4) are said to be of *Lie type*.

**(5)** For the next three examples, we recall some facts about quadratic forms. A *quadratic form* on a vector space $V$ over a field $k$ is a $k$- valued function $f$ on $V$ such that: (i) $f(\lambda x) = \lambda^2 f(x)$ for all $\lambda \in k$ and $x \in V$; and (ii) the map $b$ taking $(x, y) \in V \times V$ to $f(x+y) - f(x) - f(y)$ is a symmetric bilinear form. We then say that $f$ *polarizes* to $b$. If $k$ is of characteristic two, then $b$ is symplectic; that is, $b(x,x) = 0$ for all $x \in V$. An element $x \in V$ is said to be *singular* if $f(x) = 0$; and *non-singular*, otherwise. The set $Q_f = \{kx \in \mathcal{P}_0(V) : 0 \neq x \in V \text{ and } f(x) = 0\}$ is called the *quadric* defined by $f$. For $A \subseteq V$, let $V^\perp = \{x \in V : b(a,x) = 0 \text{ for each } a \in A\}$. We say that $f$ is *nondegenerate* if $V^\perp$ (with respect to $b$) has no nonzero singular elements. Let $k = \mathbb{F}_q$ and the dimension of $V$ be finite, say $n+1$. Then, nondegeneracy of $f$ on $V$ is equivalent to $V^\perp = \{0\}$ when either $n$ or $q$ is odd; and to $V^\perp = \mathbb{F}_q x$ and $f(x) \neq 0$ when both $n$ and $q$ are even. Given a nondegenerate quadratic form $f$ on $V$, we can find a basis $e_0, \cdots, e_n$ of $V$ such that, for $\lambda_i \in \mathbb{F}_q$, $f\left(\sum\limits_{i=0}^{n} \lambda_i e_i\right) = g(\lambda_0, \cdots, \lambda_n)$, where $g(X_0, \cdots, X_n) \in \mathbb{F}_q[X_0, \cdots, X_n]$ is one of the following :

(i) $n = 2m$, $X_0^2 + X_1 X_{m+1} + \cdots + X_m X_{2m}$;

(ii) $n = 2m-1$, $X_0 X_m + \cdots + X_{m-1} X_{2m-1}$;

(iii) $n = 2m-1$, $g_1(X_0, X_m) + X_1 X_{m+1} + \cdots + X_{m-1} X_{2m-1}$,

where $g_1(X_0, X_m) \in \mathbb{F}_q[X_0, X_m]$ is irreducible over $\mathbb{F}_q$ ([40], Theorem 5.2.4, p.106). We say that $f$ and $Q_f$ are *parabolic, hyperbolic* or *elliptic* according as (i), (ii) or (iii) holds. Further, $|Q_f| = \frac{q^n - 1}{q-1} + \varepsilon q^{\frac{n-1}{2}}$, where $\varepsilon = 1, -1$ or 0 according as $f$ is hyperbolic, elliptic (and so $n$ odd) or parabolic ([40], Theorem 5.2.6, p.109). The orthogonal group

$$G_f = \{g \in GL(V) : f(gx) = f(x) \text{ for each } x \in V\}$$

defined by $f$ has a rank 3 permutation representation on $Q_f$. A basic theorem due to Witt says that if $U$ and $W$ are subspaces of $V$ containing only singular points and are maximal with respect to this property, then $g(U) = W$ for some $g \in G_f$ (see; [1],§ 20). Their common dimension is called the *Witt index* of $f$. It is $m, m$ and $m-1$ in the cases (i), (ii) and (iii) respectively ([40], Corollory 1 to Theorem 5.2.4, p.107).

We are now ready to define the incidence systems.

(i) Let $V = \mathbb{F}_q^{2m+1}$, $f$ be a nondegenerate quadratic form on $V$, $b$ be the associated symmetric bilinear form on $V$ and $Q$ be the quadric in $\mathcal{P}_0(V)$ defined by $f$. For $\mathbb{F}_q x \in Q$, $0 \neq x \in V$, let $\Delta(\mathbb{F}_q x) = Q \cap \mathcal{P}_0(x^\perp)$, where $x^\perp \subset V$ is defined with respect to $b$. This is clearly the union of all lines in $Q$ containing $\mathbb{F}_q x$. We say that the hyperplane $\mathcal{P}(x^\perp)$ of $\mathcal{P}(V)$ is *tangent* to $Q$ at $\mathbb{F}_q x$. Let $\Delta = \{\Delta(\mathbb{F}_q x), \mathbb{F}_q x \in Q\}$.Then, the incidence system $I_{\mathcal{P}} = (Q, \Delta, I)$ with inclusion as the incidence relation admits an action of $G_f$.

(ii) Let $b$ be a nondegenerate symplectic bilinear form on $V = \mathbb{F}_{2^t}^{2n}$. Let $\mathcal{Q}_h$ (respectively $\mathcal{Q}_e$) be the set of all hyperbolic (respectively elliptic) quadrics in $\mathcal{P}_0(V)$ defined by the nondegenerate hyperbolic (respectively elliptic) quadratic forms on $V$ which polarize to $b$. Then $|\mathcal{Q}_h| = \binom{1+q^n}{2}$and $|\mathcal{Q}_e| = \binom{q^n}{2}$ (see; [43], Theorems 22.6.6 and 22.9.2). The symplectic group $G_b$ (that is, the group of all $g \in GL(V)$ such that $b(gx, gy) = b(x, y)$ for all $x, y \in V$ ) acts on the incidence systems $I_h = (\mathcal{P}_0(V), \mathcal{Q}_h, I)$ and $I_\varepsilon = (\mathcal{P}_0(V), \mathcal{Q}_e, I)$ where the incidence relation is inclusion in both the cases.

## 4. Known results

In this section, we present some known facts about the examples in 3, 4 and 5 of §3.

4.1. **The incidence system** $\mathcal{C}^n_{r,s,m}(q)$ **(**$0 \leq m \leq r < s \leq n, r \neq 0$ **and** $r+s-m \leq n$**).** .

Let $M_{r,s}$ denote the incidence matrix of $\mathcal{C}^n_{r,s,r}$.

**Theorem 4.1.** *(Wilson [82] ) Let $r \leq \min\{s, n-s\}$. Then, there exist unimodular integer matrices $P$ and $Q$ such that $PM_{r,s}Q$ is a diagonal matrix with diagonal entries $\binom{s-i}{r-i}$ with multiplicity $\binom{n}{i} - \binom{n}{i-1}$, $(0 \leq i \leq r)$.*

An immediate consequence is

**Theorem 4.2.** *(Wilson [81]) If $m = r \leq \min\{s, n-s\}$ and $p$ a prime, then the $p$- rank of the incidence matrix $M_{r,s}$ of $\mathcal{C}^n_{r,s,r}$ is*

$$\sum_i \binom{n}{i} - \binom{n}{i-1},$$

*where the summation is over those integers $i$ for which $\binom{s-i}{r-i}$ is nonzero $(\operatorname{mod} p)$.*

Frumkin and Yakir [30] gave a different proof of these results using the representation theory of $S_n$. Also see; [60]. For applications of the higher incidence matrices to the theory of $t$- designs and extremal set theory, see [3],[**?**] and [34]. The study of signed $t$- designs using this was initiated in [35].

4.1.1. *The incidence system $\mathcal{P}^n_{r,s,m}(q)$ $(-1 \leq m \leq r < s \leq n, r \neq -1$ and $r+s-m \leq n)$.*

**A**. The $kG-$ module homomorphism $\delta$ from $k^{\mathcal{P}_r}$ to $k\mathbf{1}_{\mathcal{P}_r}$ taking $f \in k^{\mathcal{P}_r}$ to $|\mathcal{P}_r|^{-1}\left(\Sigma_{p\in\mathcal{P}_r} f(p)\right)\mathbf{1}_{\mathcal{P}_r}$ is a splitting of the inclusion map from $k\mathbf{1}_{\mathcal{P}_r}$ to $k^{\mathcal{P}_r}$. Thus, we have the $kG-$ module decomposition $k^{\mathcal{P}_r} = k\mathbf{1}_{\mathcal{P}_r} \oplus Y_{\mathcal{P}_r}$, where $Y_{\mathcal{P}_r}$ is the kernel of $\delta$. Being a permutation module, $k^{\mathcal{P}_r}$ and so $Y_{\mathcal{P}_r}$, are symmetric modules. For $A \in \mathcal{P}_r \cup \mathcal{P}_s$, $\left|[A]_{r,s,m}\right| \equiv 1$ or $0 \pmod p$ according as $r = m$ or $r \neq m$. So, the $q$ - ary code $C_{r,s,m}(n,q)$ (written, in the absence of ambiguity, as just $\mathcal{C}$ or as $\mathcal{C}_s$ when further $r = m = 0$) associated with $P^n_{r,s,m}(q)$ contains $1_{\mathcal{P}_r}$ if $r = m$ and is contained in $Y_{\mathcal{P}_r}$ if $r \neq m$. If $r = m$ and $\mathcal{D} = C \cap Y_{\mathcal{P}_r}$, then $\mathcal{D} \subseteq \sum_{m=0}^{r-1} C_{r,s,m}(n,q)$. Further, when $r = 0$, $D = \mathcal{C}_{0,s,-1}(n,q)$ and both $\mathcal{D}$ and $Y_{\mathcal{P}_0}$ are indecomposable $kG$-modules. We do not know if this is true in general.

**B**. We know the SNF of the incidence matrix of $\mathcal{P}^n_{r,s,m}(q)$, and so the dimension of the $q$- ary code $\mathcal{C}$ (see [83], §.10 and Theorem 4.3). The $kG$- module structure of $C$ is known only in the case $r = 0$ (Proposition 4.4).The $kG$- module structure of $\mathcal{C}^{\perp}$ even for $r = 0$ is not known.

In the case $r = m$, each word of $\mathcal{C}$ of minimum weight is a nonzero scalar multiple of the characteristic function of a subset of $\mathcal{P}_r$ of the form $[B] = \{A \in \mathcal{P}_r : AI_{r,s,m}B\}$ for some $B \in \mathcal{P}_s$. This was independently proved by Smith [72] and Delsarte, et al.[25] in the case $r = 0$ (see; [2], Corollary 5.7.5, p.186 for a proof). For a geometric proof in the case $q$ even, see; [6]. The general case $r = m$ is Theorem 1 of [5]. In particular, $\mathcal{C}$ is generated by its words of minimum weight. It is not known if this is true for $\mathcal{C}^{\perp}$.

Regarding the minimum weight of the dual code, we have:

**Proposition 4.1.** *(Bagchi and Inamdar* [5]*, Theorem 3)*

*If $d$ is the minimum weight of $\mathcal{C}^{\perp}_{r,s;r}(n,q)$, $r > 0$, then*

$$2\left(\frac{q^{n-r}-1}{q^{s-r}-1}\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) \leq d \leq 2q^{n-s}. \tag{4.1}$$

*If the lower bound is attained, then $s = r + 1$.*

Words of weight $2q^{n-r}$ in $\mathcal{C}^{\perp}_{r,s,r}(n,q)$ can be obtained as follows. For two flats $M$ and $N$, let $[M,N]_r$ denote the set of all $r-$ flats $K$ such that $M \subseteq K \subseteq N$. Given an $(r-1)$- flat $A$ and two $(n-s+r)$- flats $B$ and $C$ such that $B \cap C$ is a $(n-s+r-1)$- flat and $A \subseteq B \cap C$, then $\chi_{[A,B]_r} - \chi_{[A,C]_r}$ is an element of $\mathcal{C}^{\perp}_{r,s,r}(n,q)$ of weight $2q^{n-r}$. These words are called *standard words*

**Proposition 4.2.** *(see[* [25] *for (i); Lemma 5 and Proposition 2 of* [5] *for the rest.) Assume that one of the following holds: (i) $r = 0$; (ii) $q = 2$; (iii) $q$ is a prime and $r = s - 1$. Then, the minimum weight of $\mathcal{C}^{\perp}_{r,s,r}(n,q)$ is $2q^{n-s}$ and the words of minimum weight are nonzero scalar multiples of the standard words.*

Bagchi and Inamdar conjectured that the proposition holds for all $r < s$ when $q$ is a prime and pointed out that the conjecture is not true if $q$ is even and $q > 4$ ([5], Proposition 4).

**Proposition 4.3.** *([14], Theorem 1) Let $q$ be even and $\mathcal{C}$ be the $q-$ary code associated with the incidence system $\mathcal{P}^n_{r,s,m}(q)$ or $\mathcal{A}^n_{r,s,m}(q, 1 \leq s < n, n \geq 2$. Then, the minimum weight of $\mathcal{C}^{\perp}$ is $q^{n-s-1}(q+2)$.*

Is $\mathcal{C}^{\perp}$ generated by its words of least weight? Given the state of our current knowledge about these codes, weight enumerator polynomials of $\mathcal{C}$ and $\mathcal{C}^{\perp}$, even in the case $r = 0$, seems to be far away.

**C**. *The incidence system* $\mathcal{P}^n_{0,s,0}$.

Hamada ([37], Theorem 1) obtained the dimension of the $q-$ ary code $\mathcal{C}$ associated with the incidence system $\mathcal{P}^n_{0,s,0}$ by extending an approach of Smith [72] to the case $\mathcal{P}^n_{0,n-1,0}$. Since then, this has been obtained from various points of view: a polynomial approach [33], a representation theoretic approach [9]; and by an application of a general method due to Moorehouse [63] applied to the Veronese embedding [45]. Bardoe and Sin gave a very elegant representation theoretic interpretation of this formula using their determination of the $\mathbb{F}_q G-$ module structure of $\mathbb{F}_q^{\mathcal{P}_0}$(see; Theorems [9] and 4.3). Significantly, they also obtain a basis for each cyclic submodule $\mathbb{F}_q^{\mathcal{P}_0} f$ in terms of the monomials appearing in the unique polynomial over $\mathbb{F}_q$ which defines $f$ and is of degree at most $q-1$ in each of the $(n+1)$ variables [9], Theorem B. We outline this approach below. For some previous work on these lines, see; [26], [27], [51], [54] and [53].

**Structure of** $\mathbb{F}_q^{\mathcal{P}_0}$. Let $A$ denote the set all $t-$ tuples $\underline{a} = (a_0, a_1, \cdots, a_{t-1}) \in \mathbb{Z}^t$ such that $1 \leq a_j \leq n+1$ and $0 \leq \lambda_j = pa_{j+1} - a_j \leq (n+1)(p-1)$ with $a_t = a_0$. Let $\underline{0}_t = (0, \cdots, 0) \in \mathbb{Z}^t$. Consider the partial order on $A \cup \{\underline{0}_t\}$, defined by $\underline{a} \preceq \underline{b}$ if $a_i \leq b_i$ for each $i$. With each $\underline{a} \in A$, we associate a simple $\mathbb{F}_q G-$ module $L(\underline{a})$ as follows.

Consider the truncated polynomial ring

$$\overline{S} = \frac{\mathbb{F}_q[X_0, \cdots, X_n]}{\langle X_i^p : i = 0, \cdots, n\rangle} = \overset{(n+1)(p-1)}{\underset{\lambda=0}{\oplus}} \overline{S}^{\lambda},$$

where $\overline{S}^{\lambda}$, $0 \leq \lambda \leq (n+1)(p-1)$, is the homogenous component of $\overline{S}$ of degree $\lambda$. This is a simple $\mathbb{F}_q G-$ module which remains irreducible on restriction to $GL(n+1, p)$. Further, these modules are self-dual.

For an $\mathbb{F}_q G-$ module $M$ and an integer $j$, $0 \leq j \leq t-1$, we denote by $M^{(p^j)}$ the $j^{th}$ - Frobenius twist of $M$. This means that $M^{(p^j)}$ is the $\mathbb{F}_q G-$ module whose $\mathbb{F}_q-$ vector space structure is the same as that of $M$ and $g \in G$ acts on $M$ as its $j^{th}$- Frobenius power acts on $M$; that is, the action of the matrix obtained by raising each entry of the matrix representing $g$ with respect to some basis of $M$ to its $p^j$-th power. Thus, we can interpret $\overline{S}^{(p^j)}$ as the ring $\overline{S}$ with the variables $X_i$ replaced by $X_i^{p^j}$.For $\underline{a} \in A$, we define $L(\underline{a}) = \otimes_{i=0}^{t-1}(\overline{S}^{\lambda_i})^{(p^i)}$and $L(\underline{0}_t)$ as the trivial representation. These are simple $\mathbb{F}_q G$- modules. For $0 \le \lambda \le (n+1)(p-1)$,

$$(4.2) \qquad \overline{S}^{\lambda} = \sum_{i=0}^{\lfloor \lambda/p \rfloor} (-1)^i \wedge^i \left(V^{*(p)}\right) \otimes S^{\lambda - ip}(V^*).$$

in the Grothendiek ring of $kG$- modules ([28], Proposition 3.12 (2)). So the dimension $d(\underline{a})$ of $L(\underline{a})$ is

$$\prod_{j=0}^{t-1} \sum_{i=0}^{\lfloor (pa_{j+1}-a_j)/p \rfloor} (-1)^i \binom{n+1}{i} \binom{n + pa_{j+1} - a_j - ip}{n}.$$

Here, for a real number $x$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$. Recall that an *ideal* in $A$ is a subset $S$ of $A$ such that if $\underline{a} \in A, \underline{b} \in S$ and $\underline{a} \preceq \underline{b}$, then $\underline{a} \in S$.

**Proposition 4.4.** *(*[9]*, see; Theorem A)*

*(a) The map* $\underline{a} \mapsto L(\underline{a})$ *is a bijection from* $A \cup \{\underline{0}_t\}$ *(respectively, A) to the set of all* $\mathbb{F}_q G-$ *composition factors of* $\mathbb{F}_q^{\mathcal{P}_0}$ *(respectively* $Y_{\mathcal{P}_0}$*). In particular,* $\mathbb{F}_q^{\mathcal{P}_0}$ *is a multiplicity-free* $\mathbb{F}_q G-$*module.*

*(b) For any submodule M of* $Y_{\mathcal{P}_0}$*, let* $A_M = \{\underline{a} \in A : L(\underline{a})$ *is a composition factor of* $M\}$*. Then, the map* $M \mapsto A_M$ *is an isomorphism from the lattice of* $\mathbb{F}_q G-$ *submodules of* $Y_{\mathcal{P}_0}$ *to the lattice of all the ideals of A.*

**Hamada formula.** For $1 \le s \le n-1$, the $\mathbb{F}_q G$- composition factors of $\mathcal{C}_q(s)$ are indexed by $A_s = \{\underline{a} \in \mathbb{A} : s+1 \le a_j \le n+1\}$ (see; [9], § 8.1). So,

**Theorem 4.3.** *(*[36]*, Theorem 1) The dimension* $\phi_s(n, p^t)$ *of* $\mathcal{C}_q(s)$ *is* $\sum_{\underline{a} \in A_s} d(\underline{a})$.

**Remarks on Hamada's formula**. (1) $\phi_s(n, p^t)$ is equal to the number of integers $u$ such that :

(i) $0 \le u \le q^{n+1} - 1$;

(ii) $u$ is a multiple of $(q-1)$;

(iii) $w_q(up^j) \le (n-s)(q-1)$ for $j = 0, \cdots, t-1$.

Here, $up^j$ is reduced modulo $q^{n+1}-1$ and, for an integer $z$, $w_q(z)$ is the $q-$weight of $z$; that is, if $z = \Sigma_{i=0}^{\infty} z_i q^i$ is the $q-$adic expansion of $z$, then $w_q(z) = \Sigma_{i=0}^{\infty} z_i$ (see; [2], Theorem 5.8.1).
(2) $\phi_s(n, p^t)$ is also equal to the number of integers $e$, $1 \le e \le \left(q^{n+1}-1\right)/(q-1)$, such that some multinomial coefficient

$$\frac{(e(q-1))!}{(e_0(q-1))!(e_1(q-1))! \cdots (e_s(q-1))!},$$

with $e_i \in \mathbb{Z}$ and $e_0 + \cdots + e_s = e, e_i \neq 0$, is not congruent to zero modulo $p$ ([12], Theorem 4.8)

(3) Hirschfeld and Shaw [42] have simplified the Hamada formula in the case $q = p$; $s = k$ and $n = 2k$

$$\begin{aligned}\phi_k(2k, p) &= \frac{p^{2k+1} - 1}{p - 1} - \sum_{i=0}^{k-1} (-1)^i \binom{(k-i)(p-1) - 1}{i} \binom{k + (k-i)p}{2k - i} \\ &= \frac{1}{2}\left(\frac{p^{2k+1} - 1}{p - 1} + 1\right).\end{aligned}$$

The second equality is proved in [46], see also [59]. Is there a natural basis for this code?

**The special case.** $\mathcal{P}^n_{0,n-1,0}$

Let $V = \mathbb{F}_q^{n+1}$ and $A$ be the incidence matrix of $\mathcal{P}^n_{0,n-1,0}$. That the $p-$ rank of $A$ is $1 + \binom{n+p-1}{n}^t$ was known prior to Hamada's work. The case $q$ a prime was settled in [62] and the general case in [31]. See also [73]. The elements of $\mathcal{P}_{n-1}(V)$ are of the form $\mathcal{P}(x^\perp)$, $0 \neq x \in V$, where ' $\perp$' can be taken with respect to any nondegenerate billinear form on $V$ satisfying the property that $b(x, y) = 0$ for some $x, y \in V$ implies $b(y, x) = 0$. So, $A$ is symmetric in this case. The complement of $k\mathbf{1}_P$ (as an $\mathbb{F}_q G$- module) in the row span $\mathcal{C}$ of $A$ over $\mathbb{F}_q$ is a simple and projective module for $PSL(n+1, q)$, the Steinberg module. This is isomorphic to $\otimes_{i=0}^{t-1} (\overline{S}^{p-1})^{(p^i)}$ (see; [11]) and corresponds to $\underline{s} = \mathbf{1}_t$.

Let $Q$ be a nondegenerate quadric in $\mathcal{P}(V)$ defined by a quadratic form $f$ on $V$ polarizing to $b$. Let $A_Q$ be the submatrix of $A$ determined by the columns indexed by $Q$.

**Proposition 4.5.** *(Blokhius and Moorehouse*[11]*) If $n \geq 2$, then the $p-$ rank of $A_Q$ is $\left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^t + 1$.*

It is interesting that when $n$ is odd, this formula is independent of the Witt index of the quadric. Also, it is clear that the $2-$ rank of $A_Q$ is $(n+1)^t + 1$ when $q$ is even. If $q$ and $n$ are both even, then $A_Q$ contains a submatrix $A_{\mathcal{T},Q}$ of $p-$ rank $n^t + 1$.In fact, we can take $A_{\mathcal{T},Q}$ to be the one determined by the rows of $A_Q$ indexed by the set $\mathcal{T}$ of all hyperplanes $\mathcal{P}(y^\perp)$, $\langle y \rangle \in Q$, tangent to $Q$; that is, the incidence matrix of the incidence system $I_{\mathcal{P}}$. That the $p-$ rank of $A_{\mathcal{T},Q}$ is $n^t + 1$ follows from the isomorphism of incidence structures proved in the next paragraph and the Hamada formula for $\mathcal{P}^{n-1}_{0,n-2,0}$.

In the case when both $q$ and $n$ are even, the geometry of $\mathcal{P}(V)$ is more intricate. In this case, $V^\perp = \mathbb{F}_q x$ for some $x \in V$ such that $f(x) \neq 0$ and $b$ descends, via the projection map $V \to \overline{V} = V/V^\perp$, to a nondegenerate symplectic bilinear form $\bar{b}$ on $\overline{V}$. Since $V^\perp \subseteq \pi$ for each $\pi \in \mathcal{T}$ and $\left|\mathcal{P}_{n-2}(\overline{V})\right| = |Q|$, the projection above defines a bijection from $\mathcal{T}$ to $\mathcal{P}_{n-2}(\overline{V})$. Further, for each

$y \in V \setminus \{0\}$, $f(x+ty)=0$ for a unique $t \in \mathbb{F}_q$. So, each line in $\mathcal{P}(V)$ through $\langle x \rangle$ contains a unique point of $Q$. Thus, $\mathcal{P}_0\left(\overline{V}\right)$ is in bijective correspondence with $Q$. Further, the above two bijections are incidence preserving. So the incidence system $\left(\mathcal{P}_0\left(\overline{V}\right), \mathcal{P}_{n-2}\left(\overline{V}\right)\right)$ is isomorphic to $I_{\mathcal{P}}$.

Now, for $\pi \in \mathcal{P}_{n-1}(V) \setminus \mathcal{T}$, the restriction $\pi_f$ of $f$ to $\pi$ is a nondegenerate quadratic form on $\pi$. Under the projection $V \to \overline{V}$, $\pi_f$ descends to a nondegenerate quadratic form $\overline{\pi}_f$ on $\overline{V}$ which polarizes to $\overline{b}$ and is of the same Witt index as $\pi_f$. Let $\mathcal{H}$ (respectively $\mathcal{E}$) be the set of all $\pi \in \mathcal{P}_{n-1}(V)$ such that $\pi_f$, and so $\overline{\pi}_f$, are hyperbolic (respectively elliptic). Let $\widehat{\mathcal{H}} = \{Q \cap \pi : \pi \in \mathcal{H}\}$ and $\overline{\mathcal{H}} = \left\{Q_{\overline{\pi}_f} : \pi \in \mathcal{H}\right\}$. Define $\widehat{\mathcal{E}}$ and $\overline{\mathcal{E}}$ similarly. Then, the quotient map $V \to \overline{V}$ induces isomorphisms of incidence systems $\left(Q, \widehat{\mathcal{H}}\right) \simeq \left(\mathcal{P}_0\left(\overline{V}\right), \overline{\mathcal{H}}\right)$ and $\left(Q, \widehat{\mathcal{E}}\right) \simeq \left(\mathcal{P}_0\left(\overline{V}\right), \overline{\mathcal{E}}\right)$. Now $\mathcal{T}$, $\mathcal{H}$ and $\mathcal{E}$ partition $\mathcal{P}_{n-1}(V)$ and the map $\pi \to \overline{\pi}_f$ is an injection from $\mathcal{P}_{n-1}(V) \setminus \mathcal{J}$ to the set of all nondegenerate quadratic forms on $\overline{V}$ (up to nonzero scalar multiple) which polarize to $\overline{b}$. Since $|Q_h| + |Q_e| = |\mathcal{P}_{n-1}(V) \setminus \mathcal{T}|$, it follows that $\overline{\mathcal{H}} = Q_h$ and $\overline{\mathcal{E}} = Q_e$, and so $I_h = \left(\mathcal{P}_0\left(\overline{V}\right), \overline{\mathcal{H}}\right)$ and $I_e = \left(\mathcal{P}_0\left(\overline{V}\right), \overline{\mathcal{E}}\right)$. Let $A_{\mathcal{H}}$ (respectively $A_{\mathcal{H},Q}$) be the submatrix of $A$ (respectively $A_Q$) determined by the rows indexed by $\mathcal{H}$. We similarly define $A_{\mathcal{E}}$ and $A_{\mathcal{E},Q}$. Then, we have the following

**Proposition 4.6.** *(Sastry and Sin*[68]*, Theorem 3.1) Let $q$ and $n$ be even. Then, the $q-$ ary codes associated with the incidence systems $I_h$ and $I_e$ coincide, say $M$; the dimension of $M$ is $(n+1)^t$; and the $q-$ ary row span of $A$ is the direct sum of $M$ and $\mathbb{F}_q\mathbf{1}$. In particular the $p-$ ranks of $A_{\mathcal{H}}, A_{\mathcal{H},Q}, A_{\mathcal{E}}$ and $A_{\mathcal{E},Q}$ are all equal and one less than the $p-$ rank of $A$.*

4.2. **The incidence system $\mathcal{A}^n_{r,s,m}(q)$.** We only consider the case $r = m = 0$. Let $\mathcal{A}_r$ denote the subspace of $k^{\mathbb{F}_q^n}$ generated by the characteristic functions of the $r-$ dimensional affine subspaces of $\mathbb{F}_q^n$ and let $\eta_s(n, p^t)$ be the $p-$ rank of the incidence matrix of $\mathcal{A}^n_{0,s,0}$. Then,

**Theorem 4.4.** [37] $\eta_s(n,p^t) = \phi_s(n,p^t) - \phi_s(n-1,p^t)$.

In particular, $\eta_{n-1}(n,p^t) = \binom{n+p-1}{n}^t$.

**Proof:** We identify the affine space $\mathbb{F}_q^n$ with a subset $Y$ of the point set $X$ of $\mathcal{PG}^n(q)$ so that $X \backslash Y$ is the point set of the $n-1$ dimensional projective space $\mathcal{P}_{n-1}(q)$.An affine subspace of $Y$ of dimension $s$ is the intersection with $Y$ of an $s$ -flat in $X$.Let $C_n$ (respectively $C_{n-1}$, $D_n$) be the subspaces of $k^X$ (respectively $k^{X \backslash Y}$, $k^Y$) generated by $s$- flats in $X$ (respectively $s$- flats in $X \backslash Y$, $s$ - dimensional affine subspaces in $Y$). Then, the restriction map

$$\eta : k^X \to k^Y$$

maps $C_n$ on to $D_n$. We now show that $C_n \cap \ker \eta = C_{n-1}$. Let $x = \sum \lambda_s \chi_s \in C_n \cap \ker \eta$, where the summation is over the $s$ -flats $S$ in $X$.Fix any point $a \in X \backslash Y$.[For any $s$ - flat $S$ in $X$ not containing $a$, its projection $\pi(S) = \langle S, a\rangle \backslash Y$ in $X\backslash Y$ is an $s$ - flat in $\mathcal{P}G(n-1, q)$. But, then $x = \sum \lambda_S \chi_{\pi(S)}$, where the sum is over the $S$ not containing $a$.

4.3. **Tonchev's question and Hamada conjecture.** In [65], Rudolph proved

**Theorem 4.5.** *If the supports of the words of weight $k$ in the dual code of a $q$-ary linear code $\mathcal{C}$ of length $v$ form the block set of a $2-(v,k,\lambda)$ design $\mathcal{D}$, then $\mathcal{C}$ can correct $e$ errors, where*

$$e = \left\lfloor \frac{(v-1)\lambda + (k-1)(\lambda-1)}{2\lambda(k-1)} \right\rfloor.$$

This can be done by the majority logic decoding algorithm (see; [78], § 8). This algorithm evaluates, for each of the $v$ coordinates $i$, $\lambda(v-1)/(k-1)$ linear functionals in $v$ variables and chooses the value which occurs most often as the $i-$th coordinate of the error term in the received message.

Note that $e$ depends only on $v, k$ and $\lambda$. So, for using the above algorithm, the best (that is, the largest) $q-$ ary codes which can correct $e$ errors, where $e$ is as given in Theorem 4.5, are those whose parity check matrix is a generalized incidence matrix $M$ of $\mathcal{D}$ of least possible $p-$ rank. We define the *$p$-dimension of a $t$-design* to be the minimum of the $p$- ranks of its generalized incidence matrices over $\mathbb{F}_{p^l}$ for some $l$. The above considerations lead to

**Question** (Tonchev) Given integers $t, v, k, \lambda$ and a prime $p$, find all $t$ - $(v,k,\lambda)$ designs whose $p$- dimension is the smallest.

**Examples** (Tonchev)

(1) For the Witt design $W_{12}$, 3-rank of its incidence matrix is 11 and the 3-dimension of $W_{12}$ is at most 6.

(2) The $p$- dimension of a $2-\left(\frac{q^{n+1}-1}{q-1}, q^n, q^n - q^{n-1}\right)$, $n \geq 2$ and $q = p^t$, design $\mathcal{D}$ is at least $(n+1)$. Equality holds if, and only if, $\mathcal{D}$ is isomorphic to $\mathcal{P}^n_{0,n-1,-1}$. In particular, the Desarguesian projective plane of order $q$ is characterized as the unique projective plane of order $q$ whose complimentary design has $p$- rank 3.

(3) The $p$- dimension of a $2-\left(q^n, q^n - q^{n-1}, q^n - q^{n-1} - q\right)$ design, $n \geq 2$ and $q = p^t$, is $(n+1)$ and equality holds if, and only if, $\mathcal{D}$ is isomorphic to the complimentary design of $\mathcal{A}^n_{0,n-1,0}(q)$.

In this direction, we have the classical

**Conjecture 4.1.** *(Hamada Conjecture) The $p$- rank of a 2-design $\mathcal{D}$ whose parameters is the same as that of a geometric design $\mathcal{P}^n_{0,s;m}(q)$ or $\mathcal{A}^n_{0,s;m}(q)$, $1 \leq s \leq n-1, m \in \{0,1\}$ and $q = p^t$, is at least as much as that of the corresponding geometric design. Equality holds if, and only if, $\mathcal{D}$ is isomorphic to the geometric design.*

This conjecture indicates that the duals of the geometric codes have good error correcting ability if majority logic decoding algorithm is used. Further, it gives a computationally simpler method to decide if a 2-design is a geometric design. The complexity of computing the $p$- rank of a matrix is a cubic polynomial in the number of rows (or columns), while complexity of finding isomorphism between block designs is as difficult as the graph isomorphism theorem (see; [21], Remark VII.6.6).

This conjecture has been verified in the following cases:

| Parameters | $p$ | $p$- rank | Equality iff | Ref |
|---|---|---|---|---|
| $2-\left(2^{n+1}-1, 2^n, 2^{n-1}\right)$ | 2 | $n+1$ | $\mathcal{P}^n_{0,n-1,-1}(2)$ | [39] |
| $2-\left(2^n, 2^{n-1}, 2^{n-1}-1\right)$ | 2 | $n+1$ | $A^n_{n-1}(2)$ | [39] |
| $2-\left(2^{n+1}-1, 3, 1\right)$ | 2 | $2^{n+1}-n-2$ | $\mathcal{P}^n_1(2)$ | [29] |
| $2-(3^n, 3, 1)$ | 3 | $3^n-1-n$ | $A^n_1(3)$ | [29] |
| $3-(2^n, 4, 1)$ | 2 | $2^n-1-n$ | $A^n_2(2)$ | [75] |

Also, the 3-ranks of the nondesarguesian projective planes of order 9 is 41 and that of the desarguesian plane of order 9 is 37 (see [66], p.115).

However, the 'only if' part of the conjecture is not true. Using the classification of binary self-dual $[32, 16, 8]$ codes, Tonchev ([77]) observed that there are exactly five pairwise nonisomorphic quasi-symmetric $2-(31, 7, 7)$ designs - including $\mathcal{P}_{0,2;0}(4, 2)$ - with block intersections 1 and 3. They all have 2- rank 16. Further, their extensions are the only $3-(32, 8, 7)$ designs - one of which is $A_{0,3;0}(5, 2)$-with even block intersections. They all have 2- rank 16. These designs are supported by extremal doubly even self dual $[32, 16, 8]$-codes (one of them is the 2nd-order Reed - Muller code) and their derived designs above are supported by shortened codes.

There are two affine $2-(64, 16, 5)$ designs that are not geometric but have the same 2-rank 16 as $A_{0,2;0}(3, 4)$. Recently, Jungnickel and Tonchev [47] constructed an infinite family of counter examples to the Hamada conjecture.

Thus, there exist examples of 2-designs which are not geometric but has the same parameters and the same $p$- rank as the geometric design. However, there are no known symmetric designs for which the Hamada conjecture does not hold and the conjecture is verified only for $q = 2$. We remark that Hamada conjecture implies the uniqueness of a projective plane of prime order and that of a cyclic projective plane of prime power order. This follows because in these cases, the $p$ -rank is determined uniquely (see; [66], Proposition 2.3).

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# A NEW FAMILY GRACEFUL ROOTED TREES

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Abstract. In this paper, we show that the Graceful Tree Conjecture is true for a new family of rooted trees. More precisely, we prove the following theorem that the rooted trees obtained by identifying one of the end-vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter $d$, $d \geq 3$, with the property that all the internal vertices of each of such caterpillars are exclusively of *even* degrees, are graceful.

**Keywords:** Trees; Graceful Labeling.

**AMS subject classification(2000):** 05C78.



## 1. INTRODUCTION

A *graceful labeling* of a graph $G$ with $n$ edges is an injection $f : V(G) \to \{0, 1, ..., n\}$ with the property that the resulting edge labels are also distinct where an edge incident with vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a *graceful graph*. The Graceful Tree Conjecture [6], [7]: *All trees are graceful,* has been the focus of many papers for more than four decades. A few families of trees are shown to be graceful, some of which include trees with at most 29 vertices [3], all trees of diameter less than or equal to 5 [5]. It is proved that caterpillars are graceful [7]. Special classes of lobsters are shown to be graceful [4], [9]. Bermond and Sotteau [1], have shown that symmetrical trees are graceful. Sethuraman and Jeba Jesintha [8] have proved that the 'Rooted trees obtained by identifying one of the end-vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter $d$, $d \geq 3$, with the property that all the internal vertices of each of such caterpillars are exclusively of *odd* degrees, are graceful'. For an exhaustive survey on graceful trees refer the dynamic survey by Gallian [2]. In this paper, we show that the Graceful Tree Conjecture is true for another family of rooted trees. More precisely, we prove the following theorem that the rooted trees obtained by identifying one of the end-vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter $d$, $d \geq 3$, with the property that all the internal vertices of each of such caterpillars are exclusively of *even* degrees, are graceful.

## 2. Main result

In this section we prove our main result by using the technique of transferring pendant edges with graceful labels on their ends as introduced in [5]. The following Lemma guarantees the gracefulness of the resultant tree after such transfers.

Let $T$ be a tree with $n$ edges and let $f$ be a graceful labeling of the tree $T$. Then the set of labels assigned to all the edges of the tree $T$ is $\{1, 2, \cdots, n\}$.

Let $T$ be a tree and let $uv \in E(T)$. We denote by $T_{u,v}$ the subtree of $T$ induced by the set, $V(T_{u,v}) = \{w \in V(T)$: *$w = u$ or $v$ is on a $u - w$ path*$\}$.

**Lemma 2.1.** [5]. *Let $T$ be a tree with a graceful labeling $f$ and let $u$ be a vertex adjacent with vertices $u_1, u_2$. Let $T'$ be the subtree of $T$ induced by the set*

$$V(T') = \{V(T) - [V(T_{u,u_1}) \cup V(T_{u,u_2})]\} \cup \{u\} \text{ and let } v \epsilon V(T'), v \neq u$$

.

*(a) If $u_1 \neq u_2$, $f(u_1) + f(u_2) = f(u) + f(v)$ and the tree $T''$ is obtained by gluing the trees $T_{u,u_1}, T_{u,u_2}$ and $T'$ in such a way that the vertex $v$ of the tree $T'$ is identified with vertex $u$ of the trees $T_{u,u_1}, T_{u,u_2}$ (see Figure 1) then $f$ is a graceful labeling of the tree $T''$ too.*

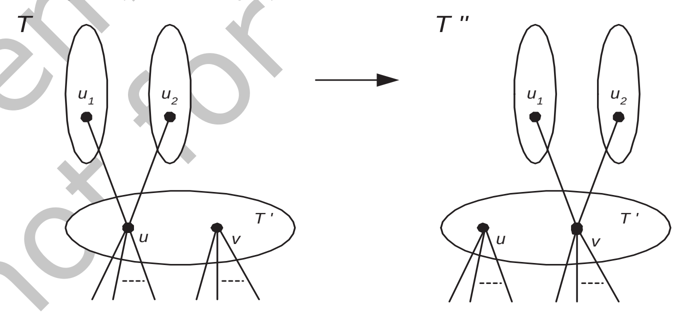


**Figure 1. Transferring subtree at $u$ to $v$.**

*(b) If $u_1 = u_2$ , $2f(u_1) = f(u) + f(v)$ and the tree $T''$ is obtained by gluing the trees $T'$ and $T_{u,u_1}$ in such a way that the vertex $v$ of the tree $T'$ is identified with vertex $u$ of the tree $T_{u,u_1}$, then $f$ is a graceful labeling of the tree $T''$ too.*

We also use the following two Lemmas in the proof of the main result.

**Lemma 2.2.** [5]. *Let $T$ be a tree with $n$ edges. If $f$ is a graceful labeling of $T$ and a mapping $g$ is defined by the equality $g(u) = n - f(u)$ for each vertex $u$, then $g$ is a graceful labeling of $T$ too.*

**Lemma 2.3.** [5]. *Let a tree $T$ with $n$ edges have a graceful labeling $f$ and let $u \in V(T)$ be such that $f(u) = 0$ or $f(u) = n$. Let $H$ be a caterpillar, $V(T) \bigcap V(H) = \emptyset$ and let $v \in V(H)$ be a vertex which has maximal eccentricity or is adjacent to a vertex of maximal eccentricity. If $T'$ is the tree obtained by gluing the trees $T$ and $H$ in such a way that the vertices $u$ and $v$ are identified then $T'$ is a graceful tree too.*

**Theorem 2.1.** *Rooted trees obtained be identifying one of the end-vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter $d, d \geq 3$ with the property that all the internal vertices of each of such caterpillars are exclusively of even degrees, are graceful.*



**Proof.** Let $T$ be a rooted tree obtained by identifying one of the end-vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter $d, d \geq 3$ with the property that all the internal vertices of each of such caterpillars are exclusively of *even* degrees.

**Case 1.** *Root vertex is of odd degree*

For convenience of labeling we describe the rooted tree $T$ of this type as follows:

Let $T$ be a rooted tree with $n$ edges and having $N$ levels of vertices (see Figure 2).

Denote the root vertex at level 0 by $w_0$. We denote the internal vertices at each level $l$, where $1 \leq l \leq (N-1)$ by $w_{(l-1)r+1}, w_{(l-1)r+2}, \cdots, w_{lr}$, ordered from left to right. At level $N$, any two pendant vertices incident with the vertex $w_{(N-2)r+i}$, where $1 \leq i \leq r$, are denoted by $v_i$, where $1 \leq i \leq r$ and $v_{2r-(m-1)}$, where $1 \leq m \leq r$ respectively. At level $l$, where $1 \leq l \leq (N-2)$, one of the end-vertices incident with $w_{(l-1)r+i}$, for $1 \leq i \leq r$, is denoted by $v_{[N-(l-1)]r-(s-1)}$, for $1 \leq s \leq r$.

At level $(N-1)$, excluding the pendant edges $w_{(N-2)r+i}v_i$, for $1 \leq i \leq r$, and $w_{(N-2)r+i}v_{2r-(m-1)}$, for $1 \leq i \leq r$, $1 \leq m \leq r$, all the other remaining pendant edges incident at vertex $w_{(N-2)r+i}$, where $1 \leq i \leq r$, is considered as a star of odd size $a_{(N-2)r+i}$ (say) at $w_{(N-2)r+i}$, for $1 \leq i \leq r$.

Similarly, at level $l$, where $1 \leq l \leq (N-2)$, excluding the pendant edges $w_{(l-1)r+i}v_{[N-(l-1)]r-(s-1)}$, for $1 \leq i \leq r$, $1 \leq s \leq r$, all the other remaining pendant edges incident at $w_{(l-1)r+i}$, for $1 \leq i \leq r$, is considered as a star of odd size $a_{(l-1)r+i}$ (say) at $w_{(l-1)r+i}$, for $1 \leq i \leq r$.

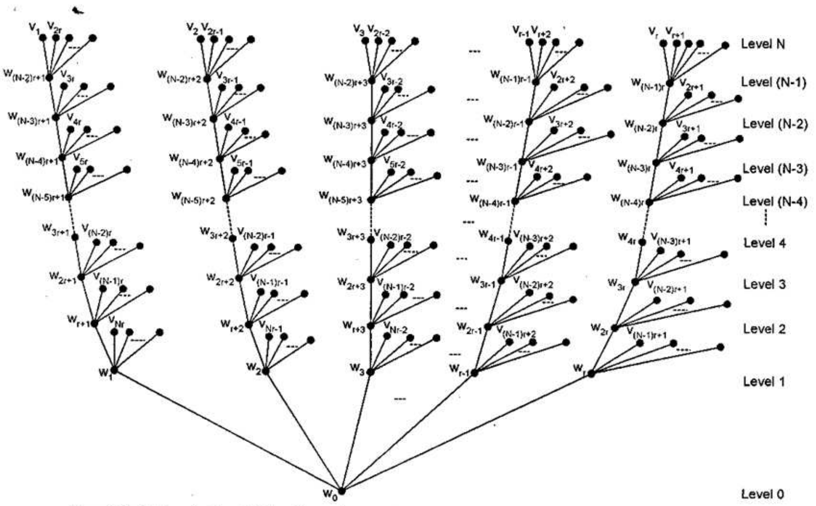


**Figure 2.** Labeled rooted tree $T$ obtained by merging one of the end-vertices adjacent to either of the penultimate vertices of any number of caterpillars, with the property that all the internal vertices of each of such caterpillars, are exclusively of even degrees.

At level $l$, $1 \leq l \leq (N-2)$, there exists a star of odd size $a_j$ at $w_j$ for $1 \leq j \leq (N-2)r$ excluding 1 pendant edge at $w_j$. Similarly, at level $l = (N-1)$, there exists a star of odd size $a_j$ at $w_j$ for $(N-2)r+1 \leq j \leq (N-1)r$ excluding 2 pendant edges at $w_j$. Let $t$ denote the total number of pendant edges of the odd sized stars $a_j$, where $1 \leq j \leq (N-1)r$, incident with the internal vertices $w_j$ of $T$, where $1 \leq j \leq (N-1)r$. Note that $n = (2N-1)r + t$.

In order to give a graceful labeling to the rooted tree $T$ we first obtain the modified tree $T'$ from $T$ by removing the stars of odd size $a_j$, at each vertex $w_j$, where $2 \leq j \leq (N-1)r$, and attaching them at the vertex $w_1$ (see Figure 3).

We denote the end-vertices of the pendant edges which are now attached at $w_1$ as $u_1, u_2, \cdots, u_t$ (see Figure 3). We label the vertices of the modified tree $T'$ as follows:

**Case 1.1.** *$N$ is odd.*

Define

$$\begin{aligned} f(w_{2i}) &= i, && \text{for } 0 \leq i \leq [(N-1)r]/2 \\ f(w_{2i+1}) &= n - i, && \text{for } 0 \leq i \leq [(N-1)r-2]/2 \\ f(v_{2i}) &= f(w_{(N-1)r}) + i, && \text{for } 1 \leq i \leq (Nr-1)/2 \end{aligned}$$

$$\begin{aligned} f(v_{2i+1}) &= f(w_{(N-1)r-1}) - (i+1), \text{ for } 0 \leq i \leq (Nr-1)/2 \\ f(u_i) &= [Nr - \lfloor r/2 \rfloor] + (i-1), \quad \text{for } 1 \leq i \leq t \end{aligned} \tag{2.1}$$

From the above definition it is clear that the vertex labels of $T'$ are distinct

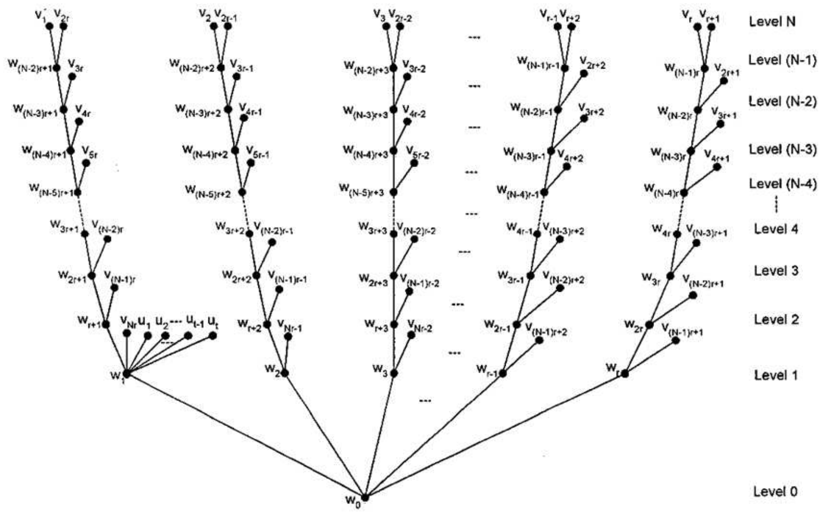


**Figure 3.** Modified Tree $T'$

and the edge labels of $T'$ are also distinct with the edge labels ranging from 1 to $n$. We now distribute the $t$ pendant edges incident at $w_1$ (except the edge $w_1 v_{Nr}$) to the vertices $w_j$, where $2 \leq j \leq (N-1)r$, as in the given rooted tree $T$ with labels. In order to achieve this, for $1 \leq j \leq (N-1)r$, at $w_j$ we leave $a_j$ edges and the remaining edges are transferred to the vertex $w_{j+1}$. This distribution of the pendant edges is done in the following two cases.

**Case 1.1.1.** *$j$ is odd.*

If $j$ is odd, then $j+1$ is even. The transfer of edges from $w_j$ to $w_{j+1}$ is possible since

$$f(w_j) + f(w_{j+1}) = \{n - (j-1)/2\} + \{(j+1)/2\} = n + 1. \tag{2.2}$$

$$\begin{aligned} & f(u_{\sum_{k=1}^{j} \lfloor a_k/2 \rfloor + (\lfloor (j+1)/2 \rfloor + 1)}) + f(u_{t - (\sum_{k=1}^{j} \lfloor a_k/2 \rfloor + \lfloor j/2 \rfloor)}) \\ &= [Nr - \lfloor r/2 \rfloor + (\sum_{k=1}^{j} \lfloor a_k/2 \rfloor + (\lfloor (j+1)/2 \rfloor + 1) - 1)] \\ &+ [Nr - \lfloor r/2 \rfloor + t - (\sum_{k=1}^{j} \lfloor a_k/2 \rfloor + \lfloor j/2 \rfloor) - 1] \\ &= [(2N-1)r + t] + 1 = n + 1. \end{aligned} \tag{2.3}$$

Thus,

$$\begin{aligned} f(w_j) + f(w_{j+1}) &= f(u_{\sum_{k=1}^{j} \lfloor a_k/2 \rfloor + (\lfloor (j+1)/2 \rfloor + 1)}) \\ &\quad + f(u_{t - (\sum_{k=1}^{j} \lfloor a_k/2 \rfloor + \lfloor j/2 \rfloor)}). \end{aligned}$$

Therefore, by Lemma 2.1, we can transfer the edges

$w_j u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor (j+1)/2\rfloor+1)}$, $w_j u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor (j+1)/2\rfloor+2)}$, $\cdots$,

$w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor)}$ from the vertex $w_j$ to the next vertex $w_{j+1}$. After the transfer, $(\lfloor a_j/2\rfloor + 1)$ edges namely, $w_j u_{\sum_{k=1}^{j-1}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor+1)}$, $w_j u_{\sum_{k=1}^{j-1}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor+2)}$, $\cdots$, $w_j u_{\sum_{k=1}^{j-1}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor+\lfloor a_j/2\rfloor+1)}$ are retained at $w_j$ to the *left* of vertex $u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor (j+1)/2\rfloor+1)}$ and $\lfloor a_j/2\rfloor$ edges namely,

$w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor-1))}$, $w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor-2))}$, $\cdots$,

$w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor-\lfloor a_j/2\rfloor))}$ are retained at $w_j$ to the *right* of vertex $u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+\lfloor j/2\rfloor)}$. Therefore, after the transfer, the total number of edges remaining at $w_j$ is $(\lfloor a_j/2\rfloor + 1)+\lfloor a_j/2\rfloor = a_j$.

**Case 1.1.2.** *$j$ is even.*

If $j$ is even, then $j+1$ is odd. The transfer of edges from $w_j$ to $w_{j+1}$ is possible since

$$f(w_j) + f(w_{j+1}) = [j/2] + [n-(j/2)] = n. \tag{2.4}$$

$$\begin{aligned} & f(u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2)+1}) + f(u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+j/2)}) \\ &= [Nr - \lfloor r/2\rfloor + (\sum_{k=1}^{j}\lfloor a_k/2\rfloor + (j/2)+1-1)] \\ &+ [Nr - \lfloor r/2\rfloor + (t - (\sum_{k=1}^{j}\lfloor a_k/2\rfloor + j/2) - 1)] \\ &= 2(Nr - \lfloor r/2\rfloor) + t - 1er \\ &= 2(Nr - [(r-1)/2]) + t - 1 = (2N-1)r + t = n \end{aligned} \tag{2.5}$$

Thus, $f(w_j) + f(w_{j+1}) = f(u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2+1)}) + f(u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+j/2)})$.

Therefore, by Lemma 2.1, we can transfer the edges $w_j u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2+1)}$, $w_j u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2+2)}$, $\cdots$, $w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(\lfloor j/2\rfloor))}$ from the vertex $w_j$ to the next vertex $w_{j+1}$. After the transfer, $(\lfloor a_j/2\rfloor)$ edges namely,

$w_j u_{\sum_{k=1}^{j-1}\lfloor a_k/2\rfloor+(j/2+1)}$, $w_j u_{\sum_{k=1}^{j-1}\lfloor a_k/2\rfloor+(j/2+2)}$,

$\cdots$, $w_j u_{\sum_{k=1}^{j-1}\lfloor a_k/2\rfloor+(j/2+\lfloor a_j/2\rfloor)}$ are retained at $w_j$ to the *left* of vertex $u_{\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2+1)}$ and $(\lfloor a_j/2\rfloor+1)$ edges namely, $w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2-1))}$,

$w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2-2))}$, $\cdots$, $w_j u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+(j/2-(\lfloor a_j/2\rfloor+1)))}$ are retained at $w_j$ to the *right* of vertex $u_{t-(\sum_{k=1}^{j}\lfloor a_k/2\rfloor+j/2)}$. Therefore, after the transfer, the total number of edges remaining at $w_j$ is $\lfloor a_j/2\rfloor+(\lfloor a_j/2\rfloor+1)=a_j$.

**Case 1.2.** *N is even.*

We label the vertices of the modified tree $T'$ as follows:

Define

$$
\begin{aligned}
f(w_{2i}) &= i, && \text{for } 0 \le i \le [(N-1)r-1]/2\\
f(w_{2i+1}) &= n-i, && \text{for } 0 \le i \le [(N-1)r-1]/2\\
f(v_{2i}) &= f(w_{(N-1)r})-i, && \text{for } 1 \le i \le (Nr)/2\\
f(v_{2i+1}) &= f(w_{(N-1)r-1})+(i+1), && \text{for } 0 \le i \le [(Nr)/2]-1\\
f(u_i) &= [Nr-\lfloor r/2\rfloor]+(i-1), && \text{for } 1 \le i \le t
\end{aligned}
\tag{2.6}
$$

From the above definition it is clear that the vertex labels of $T'$ are distinct and the edge labels of $T'$ are also distinct with the edge labels ranging from 1 to $n$. We now distribute the $t$ pendant edges incident at $w_1$ (except the edge $w_1v_{Nr}$) to the vertices $w_j$, where $2 \le j \le (N-1)r$, as in the given rooted tree $T$ with labels. In order to achieve this, for $1 \le j \le (N-1)r$, at $w_j$ we leave $a_j$ edges and the remaining edges are transferred to the vertex $w_{j+1}$. This distribution of the pendant edges is done in the following two cases.

**Case 1.2.1.** *j is odd.*

This case is equivalent to the Case 1.1.1.

**Case 1.2.2.** *j is even.*

This case is equivalent to the Case 1.1.2.

**Case 2.** *Root vertex is of even degree.*

To obtain a graceful tree of this type we repeatedly use Lemmas 2.2 and 2.3.

Therefore, all rooted trees obtained by identifying one of the end-vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter $d, d \ge 3$ with the property that all the internal vertices of each of such caterpillars are exclusively of even degrees are graceful.



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# A NOTE ON RADIUS OF STARLIKENESS OF BOUNDED ANALYTIC FUNCTIONS

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Abstract. Ideas of order of close-to-convexity and convexity with respect to symmetric point have been introduced in this paper. Radius of starlikeness of bounded analytic function has also been obtained.

**keywords :** Starlikeness, convexity, close-to-convexity.

## 1. INTRODUCTION

A function $w = f(z)$ is said to be starlike with respect to origin in the unit disc $D = \{z : |z| < 1\}$ if $w = f(z)$ maps $|z| = r < 1$ into star-shaped domain $\Omega$ of the $w-$plane. Sakaguchi [3] introduced another class of regular functions which are said to be starlike with respect to symmetric points in $D$. Let $S$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are univalent in the unit disc $D$. We define the following class of regular functions in $D$.

**Definition 1.1.** *For $0 \leq \alpha < \frac{1}{2}$, let $S_s^*(\alpha)$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ which are regular and*

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > \alpha \qquad \text{for} \qquad z \in D, \tag{1.1}$$

**Definition 1.2.** *For $0 \leq \alpha < \frac{1}{2}$, let $C_s(\alpha)$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ which are regular and*

$$Re\left\{1 + \frac{zf''(z) - f'(-z)}{f'(z) + f'(-z)}\right\} > \alpha \qquad \text{for} \qquad z \in D. \tag{1.2}$$

*Then the function $f(z) \in C_s(\alpha)$ is said to be convex of order $\alpha$ with respect to symmetric point.*

**Definition 1.3.** *For $0 \le \alpha < \frac{1}{2}$, let $K_s[\alpha, \tau]$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ which are regular and*

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha \qquad \text{for} \qquad z \in D, \tag{1.3}$$

*for some $g(z) \in C_s(\tau)$. Then the function $f(z) \in K_s[\alpha, \tau]$ is said to be close-to-connvex with respect to symmetric point of order $\alpha$ and type $\tau$.*

## 2. THE RESULT

In this section we obtain the region of starlikeness for bounded analytic function as follows :

**Theorem 2.1.** *Let $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + ....$ be an analytic function in $D$ and $0 < |f(z)| \le \alpha$ in $0 < |z| < 1$ where $0 < a_1 \le \alpha < 1$. Then $f(z)$ is starlike function in $|z| < r_0$ where $r_0 = (\alpha - \sqrt{\alpha^2 - a_1^2})/a_1$.*

**Proof:** If $\alpha = a_1$, the function in the theorem becomes $f(z) = a_1 z$. In this case the proof of the theorem is trivial.

For the other cases, we consider

$$F(z) = \Big(\frac{f(z)}{\alpha z} - \frac{a_1}{\alpha}\Big)\Big/\Big(1 - \frac{a_1 f(z)}{\alpha^2 z}\Big), \text{ for } z \neq 0 \tag{2.1}$$

and

$$F(0) = 0.$$

Clearly $$|F(z)| \le 1 \text{ in } D.$$

It is clear that $F(z)$ satisfies Schwarz's condition in $D$. Therefore, by Schwarz's Lemma [3, p.166]

$$\left|\frac{\dfrac{f(z)}{\alpha z} - \dfrac{a_1}{\alpha}}{1 - \dfrac{a_1 f(z)}{\alpha^2 z}}\right| \le |z|, \tag{2.2}$$

for all $z$ in $D$. It is known that

$$|a - b|^2 = |a|^2 - 2\, Re\,(a\bar{b}) + |b|^2, \tag{2.3}$$

where $a$ and $b$ are complex numbers. Squaring both sides of (2.2) and putting $|z| = r < 1$, we have from (2.3)

$$\Big|\frac{f(z)}{z}\Big|^2 - 2a_1\, Re\,\frac{f(z)}{z} + a_1^2 \le \alpha^2 r^2 \Big[1 - \frac{2a_1}{\alpha^2}\, Re\,\frac{f(z)}{z} + \frac{a_1^2}{\alpha^4}\Big|\frac{f(z)}{z}\Big|^2\Big]$$

or

$$\Big(1 - \frac{a_1^2 r^2}{\alpha^2}\Big)\Big|\frac{f(z)}{z}\Big|^2 - (1 - r^2)\, 2a_1\, Re\,\frac{f(z)}{z} \le \alpha^2 r^2 - a_1^2$$

or

$$\Big|\frac{f(z)}{z}\Big|^2 - \frac{(1-r^2)\alpha^2}{\alpha^2 - a_1^2 r^2}\, 2a_1\, Re\,\frac{f(z)}{z} + \frac{(1-r^2)^2\alpha^4 a_1^2}{(\alpha^2 - a_1^2 r^2)^2}$$
$$\leq \frac{\alpha^2(\alpha^2 r^2 - a_1^2)}{\alpha^2 - a_1^2 r^2} + \frac{(1-r^2)^2\,\alpha^4 a_1^2}{(\alpha^2 - a_1^2 r^2)^2}$$

i.e.

$$\Big|\frac{f(z)}{z} - \frac{(1-r^2)\,\alpha^2 a_1}{\alpha^2 - a_1^2 r^2}\Big|^2 \leq \frac{\alpha^2 r^2(\alpha^2 - a_1^2)^2}{(\alpha^2 - a_1^2 r^2)^2}$$

or

$$\Big|\frac{f(z)}{z} - \frac{(1-r^2)\,\alpha^2 a_1}{\alpha^2 - a_1^2 r^2}\Big| \leq \frac{\alpha r(\alpha^2 - a_1^2)}{\alpha^2 - a_1^2 r^2} \tag{2.4}$$

Put

$$G(z) = \frac{f(z)}{z}. \tag{2.5}$$

Since $f(z)|\alpha$ satisfies the hypothesis of Schwarz's lemma [3, p.166], $|G(z)| \leq \alpha$. From a well known result [3, p.68], we have

$$|G'(z)| \leq \alpha\,\frac{1 - \Big|\dfrac{G(z)}{\alpha}\Big|^2}{1-r^2}. \tag{2.6}$$

Differentiating (2.5) and making use of (2.6), we conclude that

$$\Big|f'(z) - \frac{f(z)}{z}\Big| \leq \frac{r\Big[\alpha - \dfrac{1}{\alpha}\Big|\dfrac{f(z)}{z}\Big|^2\Big]}{1-r^2}$$

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| \leq \frac{r\Big[\alpha - \dfrac{1}{\alpha}\Big|\dfrac{f(z)}{z}\Big|^2\Big]}{1-r^2}\,\Big|\frac{z}{f(z)}\Big|. \tag{2.7}$$

Again using the known result [3, p.68], we have from (2.7) that

$$Re\,\frac{zf'(z)}{f(z)} \geq 1 - \frac{r^2\Big[\alpha - \dfrac{1}{\alpha}\Big|\dfrac{f(z)}{z}\Big|^2\Big]}{|f(z)|\,(1-r^2)}$$

$$= \frac{r}{|f(z)|}\Big[\Big|\frac{f(z)}{z}\Big| - \frac{r\Big\{\alpha - \dfrac{1}{\alpha}\Big|\dfrac{f(z)}{z}\Big|^2\Big\}}{1-r^2}\Big]$$

$$\geq \frac{r}{|f(z)|}\Big[Re\,\frac{f(z)}{z} - \frac{r\Big\{\alpha - \dfrac{1}{\alpha}\Big|\dfrac{f(z)}{z}\Big|^2\Big\}}{1-r^2}\Big]$$

$$= \frac{r^2}{\alpha\,|f(z)|(1-r^2)}\Big[\Big|\frac{f(z)}{z}\Big|^2 + \frac{\alpha\,(1-r^2)}{r}\,Re\,\frac{f(z)}{z} - \alpha^2\Big]$$

$$= \frac{r^2}{\alpha\,|f(z)|(1-r^2)}\Big[\Big|\frac{f(z)}{z}\Big|^2 + \frac{\alpha\,(1-r^2)}{r}\,Re\,\frac{f(z)}{z} + \frac{\alpha^2\,(1-r^2)^2}{4r^2} - \frac{\alpha^2\,(1-r^2)^2}{4r^2} - \alpha^2\Big]$$

$$= \frac{r^2}{\alpha\,|f(z)|(1-r^2)}\Big[\Big|\frac{f(z)}{z}+\frac{\alpha\,(1-r^2)}{2r}\Big|^2-\frac{\alpha^2\,(1+r^2)^2}{4r^2}\Big].$$

We have to find the region of $z$ for which $Re\,\dfrac{f'(z)}{f(z)}>0$, i.e. if

$$\Big[\Big|\frac{f(z)}{z}+\frac{\alpha\,(1-r^2)}{2r}\Big|-\frac{\alpha\,(1+r^2)}{2r}\Big]>0. \tag{2.8}$$

Now from (2.4), we have

$$\Big|\Big\{\frac{f(z)}{z}+\frac{\alpha\,(1-r^2)}{2r}\Big\}-\Big\{\frac{(1-r^2)\alpha^2 a_1}{a^2-a_1^2r^2}+\frac{\alpha\,(1-r^2)}{2r}\Big\}\Big|\le\frac{\alpha r(\alpha^2-a_1^2)}{\alpha^2-a_1^2r^2}$$

or

$$\Big|\frac{f(z)}{z}+\frac{\alpha\,(1-r^2)}{2r}\Big|\ge\frac{(1-r^2)\alpha^2 a_1}{a^2-a_1^2r^2}+\frac{\alpha\,(1-r^2)}{2r}-\frac{\alpha r(\alpha^2-a_1^2)}{\alpha^2-a_1^2r^2}$$

$$=\frac{\alpha}{2r(\alpha^2-a_1^2r^2)}\,[\alpha^2+2\alpha a_1 r-(3\alpha^2-a_1^2)r^2-2\alpha a_1 r^3+a_1^2r^4]$$

$$=\frac{\alpha}{2r(\alpha-a_1r)}\,[\alpha+a_1r-3\alpha r^2+a_1r^3]. \tag{2.9}$$

From (2.9) we conclude that (2.8) is true if

$$\frac{\alpha}{2r(\alpha-a_1r)}\,(\alpha+a_1r-3\alpha r^2+a_1r^3)-\frac{\alpha(1+r^2)}{2r}>0$$

*i.e.*, if

$$\frac{\alpha+a_1r-3\alpha r^2+a_1r^3}{\alpha-a_1r}-(1+r^2)>0$$

*i.e.*, if

$$a_1r^2-2\alpha r-a_1>0. \tag{2.10}$$

Let $\phi(r)=a_1r^2-2\alpha r-a_1$ then $\phi'(r)=2(a_1r-\alpha)$ i.e. $\phi'(r)<0$, which in turn implies that $\phi(r)$ is decreasing function of $r$. But $\phi(r)=0$ if $r=\dfrac{\alpha\pm\sqrt{\alpha^2-a_1^2}}{a_1}$.

But $r=\dfrac{\alpha+\sqrt{\alpha^2-a_1^2}}{a_1}>1$, which is not possible. Hence $\phi(r)=a_1r^2-2\alpha r+a_1^2>0$ if $r<\dfrac{\alpha-\sqrt{\alpha^2-a_1^2}}{a_1}$ i.e. $Re\left\{z\,\dfrac{f'(z)}{f(z}\right\}>0$ for $|z|=r$ if $r<\dfrac{\alpha-\sqrt{\alpha^2-a_1^2}}{a_1}$.

Hence we conclude that $f(z)$ is starlike in $|z|<r_0$ where

$$r_0=\frac{\alpha-\sqrt{\alpha^2-a_1^2}}{a_1}.$$

This completes the proof of the theorem.

**Note.** The above type of result can be extended for functions defined in section 1.

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# ON PLICK GRAPHS WITH POINT-COARSENESS NUMBER ONE

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Abstract. The point-coarseness number $\xi'(G)$ of a graph $G$ is the maximum number of point-disjoint non-planar subgraphs of $G$. The Plick graph $P(G)$ of a graph $G$ is obtained from the line graph of $G$ by adding a new point corresponding to each block of the original graph and joining this point to the points of the line graph which correspond to the lines of the original graph. In this paper, we obtain a necessary and sufficient condition for plick graph $P(G)$ to have point-coarseness number one. Also, we have established the same for iterated plick graphs.

**Keywords and phrases :**plickgraph, line-disjoint, point-disjoint, non-planar and point-coarseness.
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## 1. INTRODUCTION

By a graph we mean a finite, undirected graph without loops or multiple lines. For graph theoretic terminology, we refer to Harary [5]. For a graph G, let $V(G)$, $E(G)$, $L(G)$ and $T(G)$ denote the point set, line set, line graph and total graph of $G$ respectively. The middle graph $M(G)$ of a graph $G$ is the graph whose point set is $V(G) \cup E(G)$, and two points of $M(G)$ are adjacent if they are adjacent lines of $G$ or one is a point and the other is a line of $G$ incident with it (see[1]).

For a real number $x$, $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x$, and $\lceil x \rceil$ is the least integer not less than $x$. A planar graph is a graph which can be embeded in a plane. Kuratowski [7] has characterised planar graphs as those graphs which contain no subgraph homeomorphic to $K_5$ or $K_{3,3}$.

In [3], Chartrand et al., introduced the point-partition number denoted by $\pi_n(G)$ for each positive integer $n$. $\pi_n(G)$ is the maximum number of subsets into which the point set of G can be partitioned so that each set induces a graph which contains a subgraph homeomorphic either to $K_{n+1}$ or $K_{\lfloor \frac{n+2}{2} \rfloor, \lceil \frac{n+2}{2} \rceil}$. In particular,



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$\pi_4(G)$ is the maximum number of point-disjoint nonplanar subgraphs of $G$ and is called the point-coarseness of $G$ and is now denoted by $\xi'(G)$.

Danuta[4], Patil and Rengarasu [8], [9] gave a necessary and sufficient condition for iterated line graph $L^n(G)$; $n \geq 1$, iterated middle graph $M^n(G)$; $n \geq 1$ and iterated total graph $T^n(G)$; $n \geq 1$ to have point-coarseness number one.

The *Plick graph* $P(G)$ of a graph $G$ is obtained from the line graph of $G$ by adding a new point corresponding to each block of the original graph and joining this point to the points of the line graph which correspond to the lines of the block of the original graph (see[6]). We write, $P^1(G) = P(G)$; $P^2(G) = P(P(G))$ and in general the $n^{th}$ iterated plick graph as $P^n(G) = P(P^{n-1}(G))$, for $n \geq 2$.

The following will be useful in the proof of our results.

**Remark 1.1.** [6] *For any graph, $L(G)$ is a subgraph of $P(G)$.*

**Theorem 1.1.** [6] *The plick graph $P(G)$ of a graph $G$ is planar if and only if*
(*i*) $\Delta(G) \leq 4$*, and*
(*ii*) *every block of $G$ is either a cycle or a $K_2$.*

**Theorem 1.2.** [2] *A graph $G$ has a planar plick graph if and only if it has no subgraphs homeomorphic to $K_{1,5}$ or $K_4 - x$, where $x$ is a line of $K_4$.*

**Theorem 1.3.** [6] *A graph is a cycle if and only if the plick graph $P(G)$ is a wheel.*

Let us denote by $Q_i$ the set of graphs homeomorphic to $K_4 - x$, $P_4 + K_1$, $K_2 + \bar{K}_3$ and $K_{1,5}$ for $i = 1, 2, 3, 4$, respectively.

## 2. RESULTS

**Theorem 2.1.** *Let $G$ be a planar graph. Then $\xi'(P(G)) = 1$ if and only if $G$ has one of the following properties:*

(*i*) $\Delta(G) = 4$, (*a*) *$G$ is a block and $G$ has no two line-disjoint subgraphs $H_1$, $H_2$( homeomorphic to $G_1$ or $G_2$ ) belonging to $Q_i$, $Q_j$; $i, j = 1, 2, 3$, respectively or* (*b*) *$G$ contains a block incident with at least one cutpoint of degree 4 such that three lines belong to one block and $G$ has no two line-disjoint subgraphs $H_1$, $H_2$ belonging to $Q_i$, $Q_j$; $i, j = 1, 2, 3$, respectively.*

(*ii*) $\Delta(G) = 5$ *and $G$ has no two line-disjointsubgraphs $H_1$, $H_2$ belonging to $Q_i$, $Q_j$; $i, j = 1, 2, 3, 4$, respectively.*

(*iii*) $\Delta(G) = n$; $6 \leq n \leq 9$ *(if $c$ is a cutpoint of degree $n$ and each block contains at most three lines which are incident with $c$, when $n = 8, 9$ or $6 \leq n \leq 8$ (if otherwise), $G$ has exactly one point of degree $n$ and no point of degree $k$; $5 \leq k \leq n$. Further $G$ has no two line-disjoint subgraphs $H_1$; $H_2$ belonging to $Q_i$, $Q_j$; $i, j = 1, 2, 3, 4$, respectively.*

**Proof:** Suppose $G$ is planar, $\Delta(G) = 4$ and satisfies $(i)$.

We consider the following cases.

**case 1.** $\Delta(G) = 4$ and $G$ is a block. Then, by Theorem 1.2, $\xi'(P(G)) \geq 1$. Next, $G$ has no two line-disjoint subgraphs $H_1$, $H_2$ belonging to $Q_i$, $Q_j$; $i, j = 1, 2, 3$, respectively then, $P(G)$ has no two point-disjoint subgraphs $P(H_1)$, $P(H_2)$. In Fig.1. $H_1$, $H_2 \in Q_i, Q_j$; $i, j = 1, 2, 3$, respectively are shown and in Fig.2. their plick graphs are shown. $P(H_1)$ is nonplanar and contains a subgraph homeomorphic to $K_5$. $P(G_1)$, $P(G_2)$; $G_1$, $G_2 \in H_2$ contains nonplanar subgraphs homeomorphic to $K_{3,3}$ or $K_5$ denoted by thick lines in $P(G_1)$ and $P(G_2)$ respectively. Further $P(H_3)$ is also nonplanar and contains a subgraph homeomorphic to $K_{3,3}$ or $K_5$ which is denoted by thick lines or slash lines respectively.

**case 2.** $\Delta(G) = 4$ and $G$ contains a block incident with at least one cutpoint of degree 4 such that three lines belong to one block. Then, by Theorem 1.2, $\xi'(P(G)) \geq 1$. Also, $G$ has no two line-disjoint subgraphs $H_1$, $H_2$ belonging to $Q_i$, $Q_j = 1, 2, 3$, respectively. Then $P(G)$ has no two point-disjoint subgraphs $P(H_1)$, $P(H_1)$; $H_1$, $H_2 \in Q_i, Q_j$; $i, j = 1, 2, 3$, respectively. In case 1,$P(H_1), P(H_2), P(H_3)$ are discussed.

Further, it is easy to verify that $P(H_1) - v$, $P(G_1) - v$, $P(G_2) - v$ where $G_1, G_2 \in H_3$ and $P(H_3) - v$ are planar graphs for each point $v$ of $P(H_1)$, $P(G_1)$, $P(G_2)$, $P(H_3)$ respectively.

Hence $P(G)$ has no two point-disjoint nonplanar subgraphs. Thus $\xi'(P(G)) = 1$.

Suppose (ii) holds. $\Delta(G) = 5$, then, by Theorem 1.1, $\xi'(P(G)) \geq 1$. $G$ has no two line-disjoint subgraphs $H_1, H_2$ beloning to $Q_i, Q_j$; $i, j = 1, 2, 3, 4$, respectively, then $P(G)$ has no two point-disjoint subgraphs $P(H_1)$, $P(H_2)$. $P(H)$ when $H \in Q_i, Q_j$; $i, j = 1, 2, 3$, respectively is considered in the proof of $(i)$. $P(K_{1,5})$ has a subgraph isomorphic to $K_5$. Thus, $P(G)$ has no two point-disjoint nonplanar subgraphs, hence $\xi'(P(G)) = 1$.

Suppose $(iii)$ holds. $G$ is planar and $\Delta(G) = n$, $6 \leq n \leq 9$ (if $c$ is a cutpoint of degree $n$ and each block contains at most three lines which are incident with $c$, when $n = 8, 9$) or $6 \leq n \leq 8$ (if otherwise). Then, by Theorem 1.1, $\xi'(P(G)) \geq 1$. $G$ has exactly one point of degree $n$ and $G$ has no point of degree $k$; $5 \leq k \leq n$, then $P(G)$ has one subgraph isomorphic to $K_n$ and no subgraph isomorphic to $K_k$. $G$ has no two line-disjoint subgraphs $H_1, H_2 \in Q_i, Q_j; i, j = 1, 2, 3, 4$, respectively, then $P(G)$ has no two point-disjoint subgraphs $P(H_1), P(H_2)$. As above, $P(G)$ has no two point-disjoint nonplanar subgraphs, thus $\xi'(P(G)) = 1$.

Conversely, let $\xi'(P(G)) = 1$ for a planar graph $G$.

**(1)** From Theorem 1.1, $\Delta(G) \geq 4$. Notice that $\xi'(K_{10}) = 2$ and $\xi'(K_9) = 1$. Then $n = \Delta(G) \leq 9$ (if $c$ is a cutpoint of degree $n$) and $\Delta(G) \leq 8$ (if otherwise).

**(2)** In any case, if $G$ has at least two line-disjoint subgraphs $H_1$, $H_2$; $Q_i, Q_j$;

$i, j = 1, 2, 3, 4$, respectively, then $P(G)$ contains at least two point-disjoint nonplanar subgraphs, thus $\xi'(P(G)) \geq 2$.

**(3)** Suppose $\Delta(G) = 4$ and $G$ is planar.

We consider the following cases.

**3.1.** Suppose $G$ is a block and contains at least two line-disjoint subgraphs $H_1$, $H_2$; $Q_i$, $Q_j$; $i, j = 1, 2, 3$, respectively. Then, by (2), $\xi'(P(G)) \geq 2$, a contradiction. Hence we obtain property $i(a)$.

**3.2.** Suppose $G$ has at least two blocks each incident with at least one cutpoint of degree 4 such that three lines belong to one block. Then, $\xi'(P(G)) \geq 2$, a contradiction. Hence $G$ must contain only one block incident with at least one cutpoint of degree 4 such that three lines belong to one block.

Now, assume $G$ to be a graph as mentioned in 3.2 with at least two line-disjoint subgraphs $H_1$, $H_2$; $Q_i$, $Q_j$; $i, j = 1, 2, 3$, respectively. Then, by (2), $\xi'(P(G)) \geq 2$, a contradiction.

Also, it is not hard to check that for the graph $G$ mentioned above having only one block with cutpoints of degree 4 such that three lines belong to one block $\xi'(P(G)) = 1$. Thus we obtain property $i(b)$.

**(4)** $\Delta(G) = 5$. Assume $G$ has at least two line-disjoint subgraphs $H_1$, $H_2$; $Q_i$, $Q_j$; $i, j = 1, 2, 3, 4$, respectively. By (2), $P(G)$ has at least two point-disjoint nonplanar subgraphs, a contradiction. Thus we obtain property $(ii)$.

**(5)** $6 \leq \Delta(G) \leq 9$ (if $c$ is a cutpoint of degree $n$) or $6 \leq \Delta(G) \leq 8$ (if otherwise). Suppose $G$ has at least two points of degree $n$; $6 \leq n \leq 9$ or $6 \leq n \leq 8$, then $G$ has at least two subgraphs isomorphic to $K_{1,n}$. Consequently, $P(G)$ has at least two point-disjoint subgraph isomorphic to $K_n$, a contradiction. Also, if $c$ corresponds to a cutpoint of degree $n$, then $P(G)$ contains a subgraph isomorphic to $K_n$ for $n = 6$ or 7. If $n = 8$ or 9, then each block may contain four lines of the lines incident with $c$, thus $P(G)$ contains two nonplanar point-disjoint subgraphs isomorphic to $K_5$, a contradiction. Clearly, $6 \leq \Delta(G) \leq 8$ if the point of maximum degree is not a cutpoint. Further, $G$ must contain exactly one point of degree $n$ and suppose $G$ has at least one point of degree $k$; $5 \leq k \leq n$. Then $G$ has two subgraphs isomorphic to $K_{1,n}$ and $K_{1,k}$. Consequently $P(G)$ has two point-disjoint subgraphs isomorphic to $K_n$ or $K_k$. Hence, $\xi'(P(G)) \geq 2$.

From (1), (2) and (5) we obtain $(iii)$.

We need the following graphs in the proof of our further results.

Let $G_{11}$ be a graph in which all the blocks are $K_2$'s and $\Delta(G_{11}) = 3$. Further $G_{11}$ contains exactly two or three points of degree 3, which are one after the other.

Let $G_{22}$ be a graph in which all the blocks are $K_2$'s and $\Delta(G_{22}) = 3$. Further $G_{22}$ contains exactly four points of degree 3 such that a point of degree 3 is adjacent to all the points of degree 3.

**Theorem 2.2.** *For any connected graph $G$, $\xi'(P^2(G)) = 1$ if and only if one of the following conditions hold in $G$.*
*1. $G$ is a tree with $3 \leq \Delta(G) \leq 4$ and $G$ contains one of the subgraphs $G_{11}$ or $G_{22}$ or a point of degree 4.*
*2. $G$ is unicyclic with cycle $C_n$; $3 \leq n \leq 6$ and $G$ has no subgraph $G_{11}$ or $G_{22}$ or points of degree 4.*

**Proof:** Suppose $\xi'(P^2(G)) = 1$ for a connected graph $G$. Then $P(G)$ is planar and it satisfies the hypothesis of Theorem 2.1. Clearly, $\Delta(G) \leq 4$ and every block of $G$ is either a cycle or a $K_2$.

We consider the following cases.
**Case 1.** Suppose $G$ is a tree.

We consider the following subcases depending on $\Delta(G)$:
**Subcase 1.1.** $\Delta(G) = 1$. Then $G = K_2$. Consequently $P(G) = K_2$ and $P^2(G) = K_2$, which is a contradiction to our assumption.
**Subcase 1.2.** $\Delta(G) = 2$. Then $G$ is a path. By Theorem 1.1, $P(G)$ is planar. Also $\Delta(P(G)) = 3$ with every block as $K_2$. Again, by Theorem 1.1, $P^2(G)$ is planar, a contradiction .
**Subcase 1.3.** $\Delta(G) = 3$. Suppose $G$ contains only nonconsecutive points of degree 3. By Theorem 1.1, $P(G)$ is planar. Also, $\Delta(P(G)) = 4$ with every block, a cycle or a $K_2$. Once again, by Theorem 1.1, $P^2(G)$ is planar, a contradiction. Hence the points must be consecutive.

Suppose $G$ has exactly four points of degree 3 which are consecutive. Then, by Theorem 1.1, $P(G)$ is planar. Further, $P(G)$ contains two line disjoint subgraphs isomorphic to $Q_i$; $i = 4$, a contradiction to the hypothesis of Theorem 2.1.
**(a)** Suppose $G$ has exactly one point of degree 3. By Theorem 1.1, $P(G)$ is planar. Also, $\Delta(P(G)) = 4$ and each block is a cycle or a $K_2$. By Theorem 1.1, $P^2(G)$ is planar, a contradiction.
**(b)** Suppose $G$ has exactly two or three consecutive points of degree 3. Then $P(G)$ is planar and has a subgraph isomorphic to $Q_i$; $i = 4$ or has exactly one line-disjoint subgraph isomorphic to $Q_i$; $i = 4$. Hence $P(G)$ satisfies the hypothesis of Theorem 2.1.

Now, assume $G$ to be the graph mentioned in $(b)$ with three consecutive points of degree 3. Let $v_1$, $v_2$, $v_3$ be the three points of degree 3 such that $v_2$ is adjacent to $v_1$ and $v_3$. Let $v_4$ be a point of degree 3 adjacent to either $v_1$ or $v_3$. Then, by Theorem 1.1, $P(G)$ is planar. Also, $P(G)$ contains two line-disjoint subgraphs isomorphic to $Q_i$; $i = 4$. Thus, $P(G)$ fails to hold the hypothesis of Theorem 2.1, a contradiction.
**(c)** Assume $G$ to be a graph as mentioned in $(b)$ with three consecutive points of degree 3. Let $v_4$ be the point of degree 3 adjacent to $v_2$. Then $P(G)$ is planar and contains only one line-disjoint subgraph isomorphic to $Q_i$; $i = 4$. Thus, $P(G)$

satisfies the hypothesis of Theorem 2.1.

Now assume $G$ to be a graph as mentioned in $(c)$. Let $v_5$ be a point of degree 3 adjacent to any of the points of degree 3. Then, $P(G)$ is planar and contains two line-disjoint subgraphs isomorphic to $Q_i$; $i = 4$. This leads to a contradiction.

Further, $G$ cannot contain two subgraphs isomorphic to the graphs mentioned in $(b)$ or $(c)$.

Finally, from the above discussion the graphs mentioned in $(b)$ and $(c)$ are isomorphic to $G_{11}$ and $G_{22}$ respectively.

**Subcase 1.4.** $\Delta(G) = 4$. Suppose $G$ contains two points of degree 4. Then $P(G)$ is planar and contains two line-disjoint subgraphs isomorphic to $Q_i$, $Q_j$; $i, j = 1, 4$, respectively, a contradiction to the hypothesis of Theorem 2.1. Hence $G$ must contain only one point of degree 4.

**Case 2.** Suppose $G$ is not a tree.

We consider the following subcases:

**Subcase 2.1.** Suppose $G$ is a block. Then, $G$ is either a cycle or a $K_2$.

**2.1.1.** Suppose $G$ is $K_2$. Then, this is discussed in subcase 1.1 of this theorem.

**2.1.2.** Suppose $G$ is a cycle. Assume $G$ to be a cycle of length $n$; $n \geq 7$.

**2.1.2.1.** Now, assume $G$ to be a cycle of length 7. Then, by Theorem 1.3, $P(G)$ is a wheel. Clearly, $P(G)$ is planar and contains two line-disjoint subgraphs $Q_i$, $Q_j$; $i, j = 1, 2$, respectively. Thus $\xi'(P^2(G)) \geq 2$.

**2.1.2.2.** Assume $G$ to be a cycle of length $n$; $n \geq 8$. Then, $P(G)$ is planar and contains two line-disjoint subgraphs isomorphic to $Q_i$, $Q_j$; $i, j = 1, 4$, respectively. This leads to a contradiction as $P(G)$ fails to satisfy the hypothesis of Theorem 2.1.

**2.1.2.3.** Assume $G$ to be a cycle of length $n$; $3 \leq n \leq 6$. Then, by Theorem 1.3, $P(G)$ is a wheel. Clearly, $P(G)$ is planar and contains only one subgraph $Q_i$; $i = 1$. $\xi'(P^2(G) = 1$. Hence the cycle must be of length $3 \leq n \leq 6$.

**2.1.3.** Suppose $G$ is not a block. Then, $G$ may contain two line-disjoint cycle of length $n$ or $C_n \bullet C_n$; $3 \leq n \leq 6$ as subgraphs. Clearly, by 2.1.2.3 it follows that $P^2(G)$ contains two point-disjoint subgraphs homeomorphic to $K_5$, a contradiction. Hence $G$ must be unicyclic.

**2.1.3.1.** Suppose $G$ is unicyclic with at least one point of degree 4 either on the cycle or other than the cycle.

If the point of degree 4 is on the cycle $C_3$. Then $P(G)$ is planar and contains two points of degree 5 which are adjacent. Also, Remark 1.1 implies $L(P(G))$ is a subgraph of $P^2(G)$. $L(P(G))$ contains two line-disjoint subgraphs isomorphic to $K_5$ and $K_4$. The point $u$ of $P^2(G)$ which corresponds to the block $B$ of the graph $P(G)$ containing the two adjacent points of degree 5 is in $P^2(G)$, adjacent to every point of $L(B)$. This produces two point-disjoint subgraphs isomorphic to $K_5$ in $P^2(G)$, a contradiction.

If the point of degree 4 is on the cycle $C_n$; $n = 4, 5, 6$. Then $P(G)$ has two line-disjoint subgraphs $Q_i, Q_j$; $i, j = 1, 4$, respectively. This is a contradiction to the hypothesis of Theorem 2.1.

If the point of degree 4 is on other than the cycle. Then $P(G)$ is planar and contains two line-disjoint subgraphs $Q_i$, $Q_j$; $i, j = 1, 4$, respectively once again a contradiction to the hypothesis of Theorem 2.1. Hence no point of the unicyclic graph is of degree 4.

Now assume $G$ is unicyclic with $C_n$; $3 \le n \le 6$ and contains either $G_{11}$ or $G_{22}$ such that $G_{11}$ or $G_{22}$ may or may not share two adjacent lines from the cycle. In either possibilities $P(G)$ contains two line-disjoint subgraphs to $Q_i, Q_j$; $i, j = 1, 4$, respectively. Hence $\xi'(P^2(G)) > 1$, a contradiction.

Conversely, suppose $G$ satisfies the condition of Theorem 2.2. Then $P(G)$ satisfies the hypothesis of Theorem 2.1, and hence $\xi'(P^2(G)) = 1$.

**Theorem 2.3.** $\xi'(P^3(G)) = 1$ *if and only if* $G$ *is* $P_5$ *or* $P_6$.

**Proof:** Suppose $\xi'(P^3(G)) = 1$ for a connected graph. Then $P^2(G)$ is planar and satisfies the hypothesis of Theorem 2.1. Since $P^2(G)$ is planar, $P(G)$ must be planar. Consequently, $\Delta(G) = 4$ and every block of $G$ is a cycle or a $K_2$. Clearly, $G$ cannot be the graphs mentioned in Theorem 2.2. Hence G must be a path.

Assume $G \neq P_5$ or $P_6$. Then immediately $G$ is either $P_n$; $n \le 4$ or $P_n$; $n \ge 7$.

If $G = P_n$; $n \le 4$, then $\xi'(P^3(G)) = 0$, a contradiction.

If $G = P_n$; $n \ge 7$, then $P^2(G)$ is planar and contains at least two line-disjoint subgraphs isomorphic to $Q_i; i = 4$, a contradiction to the hypothesis of Theorem 2.1.

If $G = P_5$ or $P_6$ , then $P(G)$ is planar. Moreover, each block of $P(G)$ is $K_2$ with $\Delta(P(G)) = 3$ and contains $G_{11}$ as a subgraph. As in Theorem 2.2, $P^2(G)$ contains exactly one subgraph isomorphic $Q_i$; $i = 4$ or one line-disjoint subgraph isomorphic to $Q_i$; $i = 4$. Hence $\xi'(P^3(G)) = 1$.

**Theorem 2.4.** *For* $n \ge 4$, *there is no graph whose* $n^{th}$ *plick graph has point coarseness* 1.

**Proof:** Assume $\xi'(P^4(G)) = 1$. Then $P^3(G)$ is planar and must satisfy the hypothesis of Theorem 2.1. Consequently, $P(G)$ is planar and hence by Theorem 1.1, $\Delta(G) \le 4$ and every block of $G$ is a cycle or $K_2$. By Theorems 2.2 and 2.3, $G$ cannot be unicyclic or a tree or a path. Thus there does not exist any graph whose $n^{th}$; $n \ge 4$ plick graph have point-coarseness 1.

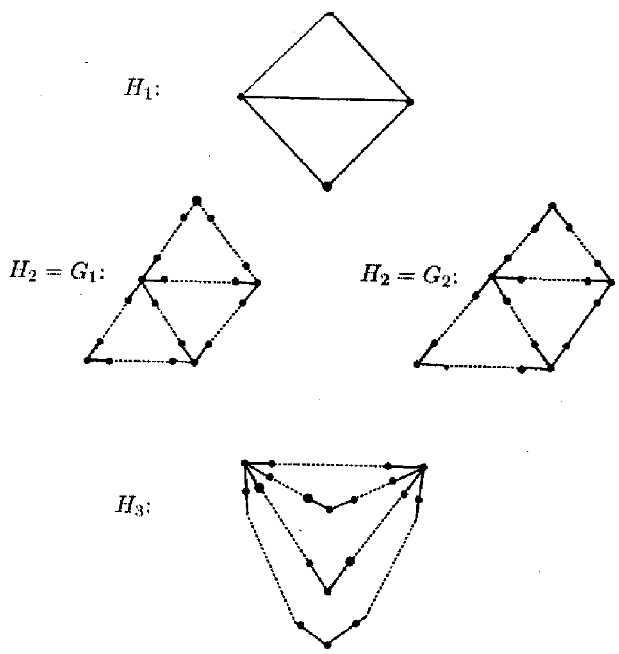


Figure 1

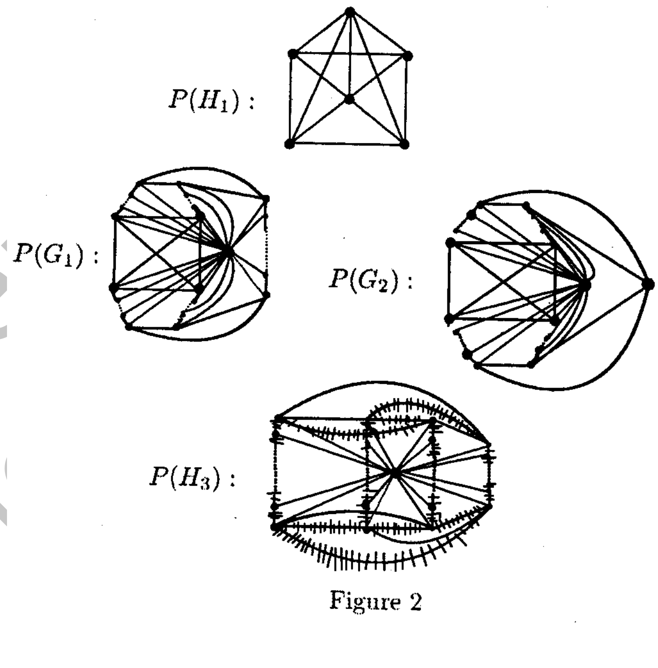


Figure 2

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# AN INTRODUCTION TO TRIGONOMETRY (A NEW OUTLOOK)

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## HISTORICAL INTRODUCTION

THE LEGENDARY FIGURE SRINIVASA RAMANUJAN when he was studying in 10th standard (or even earlier) rediscovered the formulae

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})\text{and } \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

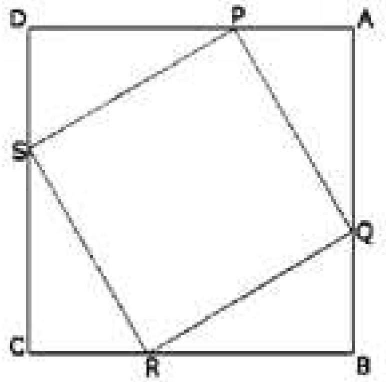
originally due to L.EULER. When Ramanujan came to know that his was only a rediscovery he felt dejected and never published his proof of these formulae. In this short monograph we give proofs of this formulae by a method which was probably the same as that of Ramanujan. The proof hinges very much on PYTHAGORAS' theorem: *In any right angled triangle the square on the hypotenuse is equal in area to the sum of the areas of squares on the other two sides.* We may remark that this theorem was known to Indian's long long ago (was around 5th or 6th century B.C. being recorded in authoritative books like SULVASUTRAS).

## 1. PYTHAGORAS' THEOREM

The purpose of this note is to give a self contained proof of Euler's equation. We give two proofs of this Theorem one based on $(a+b)^2 = a^2 + 2ab + b^2$ and another on $(a-b)^2 = a^2 - 2ab + b^2$ (both these identities are valid for any two complex numbers $a, b$).

**FIRST PROOF** (using $(a+b)^2 = a^2 + 2ab + b^2$). Consider the triangle $\Delta$ SRC, where $\hat{C}$ is a right angle. Draw the square PQRS. Draw $\Delta$ QBR in such a way that $CR = QB$ and $SC = RB$. Similarly draw the triangles $\Delta$ PAQ and $\Delta$PDS. All the four triangles are congruent (since three sides of $\Delta$ PAQ and

$\Delta$PDS. All the four triangles are congruent (since three sides of one equal three sides of the other). Each of these triangles is equal in area to that of $\Delta$ SCR, which is plainly equal to $\frac{1}{2}$ CR.SC. The four triangles together make up $4\frac{1}{2}CR \times SC = 2CR \times SC$. Also they are equal to $(CD)^2 - (SR)^2$. Thus (since $CD = DS + SC = (CR + SC)$),




$$2CR \times SC = (CR + SC)^2 - (SR)^2.$$

Hence

$$\begin{aligned}(SR)^2 &= (CR)^2 + 2CR \times SC + (SC)^2 - 2CR \times SC \\ &= (CR)^2 + (SC)^2.\end{aligned}$$

**SECOND PROOF**(using $(a-b)^2 = a^2 - 2ab + b^2$). Consider the triangle $\Delta ABP$ right angled at $\hat{P}$. Construct a triangle $\Delta QBC(\hat{Q}$ = a right angle) such that $BQ = AP$ and $CQ = BP$. Plainly $PQ = BP - AP$ (since $\Delta ABP$ is congruent to $\Delta QBC$). Certainly $AB = BC$. Do the same thing for triangle $\Delta RDC$ and $\Delta ADS$. The four small triangles make up an area of $4.\frac{1}{2}AP \times BP = 2AP \times BP$. Hence

$$\begin{aligned}(AB)^2 &= (BP - AP)^2 + 2AP \times BP \\ &= (BP)^2 + (AP)^2,\end{aligned}$$

using $(a - b)^2 = a^2 - 2ab + b^2$.

**REMARK** We have used the fact that the area of a right angled triangle is equal to half the product of the sides containing the right angle. We will amplify this remark again in section III.

## 2. SIMILAR TRIANGLES THEOREM

In this section we prove that PYTHAGORAS' theorem implies that if two triangles are similar (in other words if three angles of one are equal to three angles of the other) then their corresponding sides are proportional. In order to prove this we can not do any thing better than reproducing from the reference [3]. We deduce the general case from the following theorem as a corollary.

**Theorem 2.1.** *If two right angled triangles are similar then their corresponding sides are proportional.*

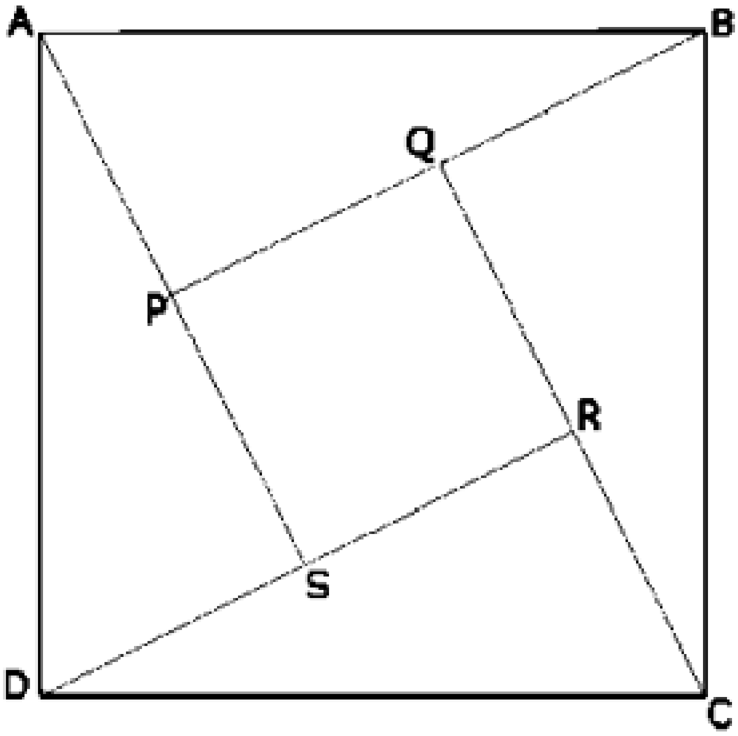



FIGURE 1



To prove the corollary, divide each of the similar triangles $PQR$ and $XYZ$ into right angled triangles as shown in figure 3 and apply theorem 1 twice

There is a well-known method of proving Pythagoras' Theorem using theorem 1, but it is less well known that deduction can be made in the opposite direction namely: we can prove theorem 1 using Pythagoras' theorem. The following proof may be new.

Let ABC and DEF be similar right angled triangles. If they are congruent then there is nothing to prove, so assume without loss of generality that DE is less than AB. Superimpose $\Delta DEF$ on $\Delta ABC$ in such a way that $D$ coincides with $A$ and $DE, DF$ fall respectively on $AB, AC$ as in figure 4. Draw EG perpendicular to BC. Write $AB = c, BC = a, CA = b, AE = \lambda c, AF = \mu b$, where $0 < \lambda < 1$ and $\mu$ is positive. It suffices to show that $\lambda = \mu$. since $E$ lies between $A$ and $B, F$ cannot lie outside $AC$, for otherwise the parallel lines $EF$ and $BC$ would intersect. Hence $0 < \mu < 1$.

Also write $BG = x, GC = EF = y$. Applying Pythagoras' theorem we have

$$x^2 = (c - \lambda c)^2 - (b - \mu b)^2,\ y^2 = \lambda^2 c^2 - \mu^2 b^2 \text{and } a^2 = c^2 - b^2. \quad (2.1)$$

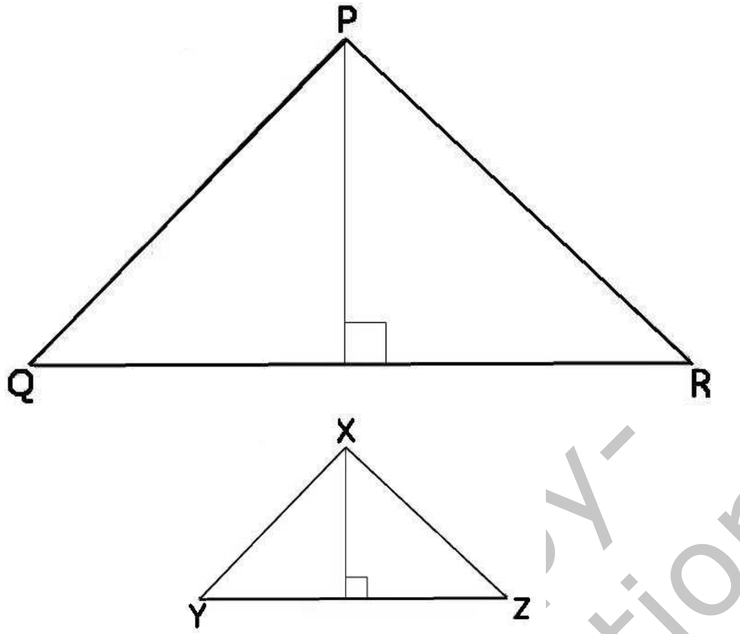



FIGURE 2

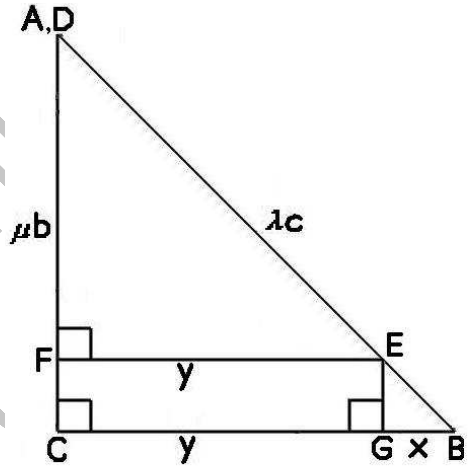



FIGURE 3

Now $x = a - y$, so $x^2 = a^2 + y^2 - 2ay$. Hence $2ay = a^2 + y^2 - x^2$.

Substituting from (1) we obtain

$$2ay = 2a\lambda c^2 - 2\mu b^2$$

Hence

$$a^2y^2 = (\lambda c^2 - \mu b^2)^2.$$

Substituting again for $a^2$ and $y^2$ from (1) we obtain after simplification $(\lambda - \mu)^2 b^2 c^2 = 0$ and hence $\lambda = \mu$ as required.

**REMARK** The proof of the theorem on similar triangles reproduced here is a simplified version (due to the referee) of the paper by K.Ramachandra.

## 3. ADDITION THEOREM FOR SINES AND COSINES

From the results of chapter II it is clear that the ratios

$$\sin\theta = \frac{\textit{opposite side}}{\textit{hypotenuse}} \text{ and } \cos\theta = \frac{\textit{adjacent side}}{\textit{hypotenuse}}$$

are absolute quantities which do not depend on the right angled triangle. The same is true of

$$\tan\theta = \frac{\textit{opposite side}}{\textit{adjacent side}},\ cot\theta = (\tan\theta)^{-1}$$

and other ratios $\text{cosec}\ \theta = (\sin\theta)^{-1}, \sec\theta = (\cos\theta)^{-1}$.

It is very surprising that these ratios have an addition theorem.

**ADDITION THEOREM** *We have*

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

*and*

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

*where $\alpha, \beta$ and $\alpha+\beta$ are positive angles (all less than a right angle).*

**Corollary 3.1.** *Let $i = \sqrt{-1}$. Then*

$$\begin{aligned} & \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\ = \ & \cos\beta(\cos\alpha + i\sin\alpha) + \sin\beta(i\cos\alpha - \sin\alpha) \\ = \ & \cos\beta(\cos\alpha + i\sin\alpha) + i\ \sin\beta(\cos\alpha + i\sin\alpha) \\ = \ & (\cos\alpha + i\sin\alpha)\ (\cos\beta + i\sin\beta) \end{aligned}$$

**Corollary 3.2.** *Let $\theta$ be any angle. Then for all positive integers $n$*

$$\cos\theta + i\sin\theta = (\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})^n$$

*(Note: RH makes sense whatever $\theta$ (positive angle) provided $n$ is large). Also if we take $cos^2\theta + \sin^2\theta = 1$ to be valid for negative $\theta$ and interpret $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ we will have $\cos^2\theta + \sin^2\theta = 1 = (\cos\theta + i\sin\theta)(\cos(-\theta) + i\sin(-\theta))$.*

Hence we can uphold corollary 2 to negative integers $n$ also.

**PROOF OF THE MAIN THEOREM** (Proof of addition theorem for $\sin\theta$ due to I.M.Gelfand)

We begin with

**LEMMA 1.** *Area of any triangle is equal to half the product of any two sides multiplied by the sine of the included angle.*

**PROOF:** Suppose the triangle is $\Delta ABC$. Draw AD perpendicular to BC. Area of $\Delta ABC$ is equal to the sum of the areas of $\Delta ABD$ and $\Delta ADC$.

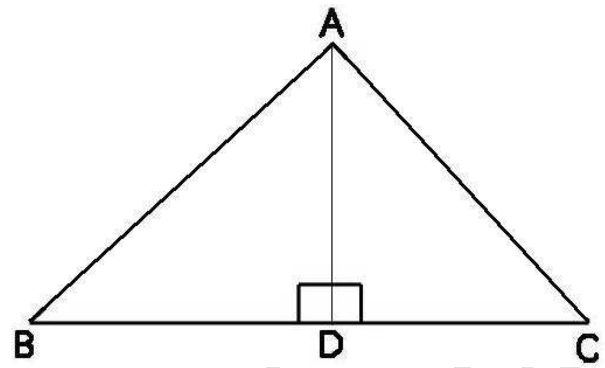



FIGURE 4

$$= \frac{1}{2}BD \times AD + \frac{1}{2}DC \times AD$$
$$= \frac{1}{2}AD(BD + DC) = \frac{1}{2}AD \times BC.$$

But $AD = AB\times$ sine of the angle $A\hat{B}D$. Hence area of $\Delta ABC = \frac{1}{2}AB \times BC\times$ sine of the angle $A\hat{B}D$ and this proves the lemma.

**REMARK.** We have used the fact that the area of a right-angled triangle (say $\Delta ACD$) is $\frac{1}{2}DC \times AD$ which is half of the area of the rectangle APCD (see the figure 6)

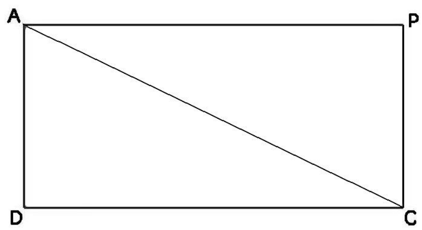



FIGURE 5

Consider the figure 7, where angle $B\hat{A}D = \alpha$ and $D\hat{A}C = \beta$, and $AD$ is perpendicular to BC. Select B such that angle $B\hat{A}D = \alpha$ and $C$ such that $D\hat{A}C = \beta$. Plain by $B\hat{A}D = \alpha + \beta$. Area of $\Delta BAC = \frac{1}{2}AB \times AC\ sin(\alpha + \beta)$. Also area of $\Delta BAC =$ the sum of the areas $\Delta BAD$ and $\Delta DAC$.

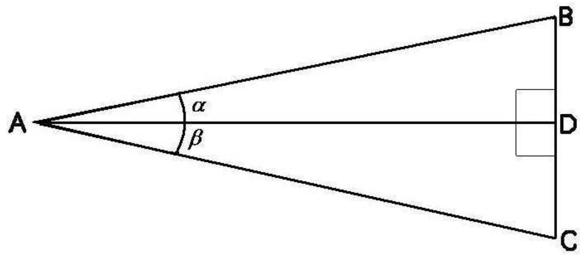



FIGURE 6

$$= \frac{1}{2}AD \times BD + \frac{1}{2}AD \times DC.$$

Hence $AB \times AC \sin(\alpha + \beta) = AD \times BD + AD \times DC$. But $AD = AC\cos\beta = AB\cos\alpha$ and $BD = AB\sin\alpha$ and also $DC = AC\sin\beta$.

There follows

$$AB \times AC\sin(\alpha + \beta) = AC\cos\beta \times AB\sin\alpha + AB\cos\alpha \times AC\sin\beta.$$

Canceling $AB \times AC$ throughout we obtain

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Next (using $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ which follows from a consideration of right angled isosceles triangle)

$$\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{4} + \frac{\pi}{4} - \theta\right) \\
&= \sin\frac{\pi}{4}\cos\left(\frac{\pi}{4} - \theta\right) + \cos\frac{\pi}{4}\sin\left(\frac{\pi}{4} - \theta\right) \\
&= \sin\frac{\pi}{4}\sin\left(\frac{\pi}{4} + \theta\right) + \cos\frac{\pi}{4}\sin(\frac{\pi}{4} - \theta) \\
&= \sin\frac{\pi}{4}(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta) + \cos\frac{\pi}{4}\sin(\frac{\pi}{4} - \theta) \\
&= \frac{1}{2}\cos\theta + \frac{1}{2}\sin\theta + \frac{1}{\sqrt{2}}\sin(\frac{\pi}{4} - \theta)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{\sqrt{2}}\sin\ \left(\frac{\pi}{4} - \theta\right) &= \frac{1}{2}\cos\theta - \frac{1}{2}\sin\theta \\
i.e\ \sin\left(\frac{\pi}{4} - \theta\right) &= \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta).
\end{aligned}$$

From this we obtain (by using $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ and $\sin\theta = \cos(\frac{\pi}{2} - \theta)$

$$\begin{aligned}
\cos(\theta+\phi) &= \sin\left(\frac{\pi}{4} - \theta + \frac{\pi}{4} - \phi\right) \\
&= \sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} - \theta) + \cos(\frac{\pi}{4} + \theta)\sin(\frac{\pi}{4} - \phi) \\
&= \sin(\frac{\pi}{4} - \theta)\sin(\frac{\pi}{4} + \phi) + \sin(\frac{\pi}{4} + \theta)\sin(\frac{\pi}{4} - \phi) \\
&= \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)\frac{1}{\sqrt{2}}(\cos\phi + \sin\phi) \\
&\quad + \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)\frac{1}{\sqrt{2}}(\cos\phi - \sin\phi) \\
&= \frac{1}{2}(\cos\theta\cos\phi + \cos\theta\sin\phi - \sin\theta\cos\phi - \sin\theta\sin\phi) \\
&+ \frac{1}{2}(\sin\theta\cos\phi - \sin\theta\sin\phi + \cos\theta\cos\phi - \cos\theta\sin\phi) \\
&= \cos\theta\cos\phi - \sin\theta\sin\phi
\end{aligned}$$

which is the addition theorem for $\cos\theta$ namely $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\phi\sin\phi$. i.e. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ with a change of notation. This proves our addition theorem completely. It must be remembered that $0 \leq \alpha \leq \frac{\pi}{2}$ and $0 \leq \beta \leq \frac{\pi}{2}$ $0 \leq \alpha + \beta \leq \frac{\pi}{2}$. But we extend it to other (real) values of $\alpha$ and $\beta$ by using

$$(\cos\theta + i\sin\theta) = \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n$$

valid for all integers $n \neq 0$.



## 4. RADIAN MEASURE AND CALCULATION OF TRIGONOMETRIC RATIOS

Although we start with sexagesimal measure such as $30^o, 45^o, 180^o$ and so on, we find it convenient to designate by $2\pi$ the circumference of a circle of unit radius. We call the angle around a point a $2\pi$ radians. We divide the circumference into very small equal parts say $k$ parts and each part is of length $\frac{2\pi}{k}$. The angle subtended at the center by each part is $\frac{2\pi}{k}$ radians. By choosing $k$ large, we can define (by rule of three) the angle $\theta (0 < \theta \leq 2\pi)$ subtended at the center by arc of length $\theta$ will be $\theta$ radians. Of course we can calculate $\pi$ by using

$$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

and putting $x = 1$ and $\tan^{-1} 1 = \frac{\pi}{4}$ (radian measure). These things will be worked out in section VI. For the calculation of trigonometric ratios we start with

**LEMMA1.** *For* $0 < \theta \leq \frac{\pi}{2}$, *we have*

$$\cos\ \theta = 1 + O(\theta^2)$$

**NOTE** From now on we employ the notation $O(\dots)$ to mean "less than a constant times ...". Thus stated in other words the lemma reads

$$|\cos\theta - 1| \leq C\theta^2$$

where $C > 0$ is an absolute constant. From now on, all angles will be in radian measure.

**PROOF:** $|\cos\theta - 1| = 1 - \cos\theta = \frac{1-\cos^2\theta}{1+\cos\theta} \leq (\sin\theta)^2 \leq \theta^2$ and we prove a more precise result regarding $(\sin\theta)$ in the next lemma.

**Lemma 4.1.** $|\sin\theta - \theta| = O(\theta^3)$,

**PROOF:** In the figure 8, $AB = AC = AF = 1, B\hat{A}C = \theta =$ arc BFC, and AGF is perpendicular to BC. DFE is tangent to arc BFC touching it at F. Also F is the middle point of arc BFC. We recall that the area of any triangle is half the product of any two sides multiplied by the sine of the included angle. Hence by dividing the arc $\theta$ into small bits area of sector ABFC=$\theta$. Area of $\Delta ABC$ is $\frac{1}{2}\ \sin\theta$.

Again $BD = GF = AF - AG = 1 - \cos\frac{\theta}{2}, BC = 2\sin\frac{\theta}{2}$. Hence

$$0 < \frac{1}{2}\theta - \frac{1}{2}\sin\theta \leq (2\sin\frac{\theta}{2})(1-\cos\frac{\theta}{2}) \leq (2\sin\frac{\theta}{2})(1-\cos^2\frac{\theta}{2})(1+\cos\frac{\theta}{2})^{-1}$$

$$\leq \theta(\sin\frac{\theta}{2})^2 \leq \frac{1}{4}\theta^3.$$

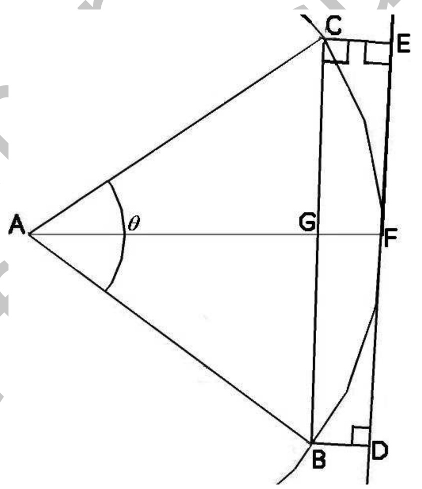



Figure 7

Thus $0 < \theta - \sin\theta \leq \frac{1}{2}\theta^3$. This proves the lemma. Our next step is

**Theorem 4.1.** *If $\theta$ is in radian measure then*

$$(\cos\theta + i\sin\theta) = (1 + \frac{i\theta}{n})^n + O(\frac{1}{n^2}).$$

**PROOF:** We begin with the identity

$$\begin{aligned} & A^n - B^n \\ = \;& (A-B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + B^{n-1}) \\ \leq \;& |A-B|(nJ^n) \end{aligned}$$

where $J = \max(|A|, |B|)$. We observe

$$\cos\theta + i\sin\theta = \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n \text{ and also}$$

$$\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)n - \left(1 + \frac{i\theta}{n}\right)^n = A^n - B^n,$$

$$\left(\text{where A} = \cos\tfrac{\theta}{\text{n}} + \text{isin} = \tfrac{\theta}{\text{n}} \text{ and B} = 1 + \tfrac{\text{i}\theta}{\text{n}}\right),$$

$$|A^n - B^n| \leq |A-B|nJ^n \text{ where J} = \max(|\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}|, |1 + \frac{i\theta}{n}|) \leq 1 + \frac{C_o}{n}$$

(where $C_o$ is a positive and constant, and so $nJ^n \leq n(1+\frac{C_o}{n})^n \leq Dn$, where $D$ is a positive constant. The inequality $(1+\frac{C_o}{n})^n \leq D$ for all large $n$ will be proved in the next section).

$$\begin{aligned} |A-B| &= |\cos\frac{\theta}{n} - 1 + i(\sin\frac{\theta}{n} - \frac{\theta}{n})| \\ &\leq |\cos\frac{\theta}{n} - 1| + |\sin\frac{\theta}{n} - \frac{\theta}{n}| \\ &= O(\frac{1}{n^2} + \frac{1}{n^3}) \\ &= O(\frac{1}{n^2}), \text{by using lemmas 1 and 2.} \end{aligned}$$

Thus

$$\begin{aligned} & \cos\theta + isin\theta - (1 + \frac{i\theta}{n})^n \\ = \;& (\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})^n - (1 + \frac{i\theta}{n})^n \\ = \;& O(\frac{D}{n} .n .\frac{1}{n^2}) = O(\frac{1}{n^2}) \end{aligned}$$

and this proves the theorem.

**Lemma 4.2.** *We have the inequality*

$$\sum_{n=0}^{\infty} \frac{C^n}{n!} \leq \sum_{0 \leq n \leq 100C^3} \frac{C^n}{n!} + 4$$

*where $C$ is any positive constant.*

**PROOF:** Put $K = 100C^3$ (without loss of generality we can assume that $C$ is a positive integer). We can ignore $\sum_{0 \le n \le |<} \frac{C^n}{n!}$. Now

$$\begin{aligned}
\sum_{n \ge K+1} \frac{C^n}{n!} &\le \sum_{n \ge K+1} \frac{C.C\ldots,C \ to\ n \text{ terms}}{100C^3.100C^3 \ldots \ to\ n \text{ terms}} \\
&\le \sum_{n \ge K+1} \frac{C^2.C^2.\ldots.C^2 \ to[\frac{n}{2}]+1 \text{ terms}}{100C^3.100C^3\ldots to[\frac{n}{2}]+1 \text{ terms}} \\
&\le \sum_{n \ge K+1} \frac{1}{100C.100C\ldots[\frac{n}{2}]+1 \text{ terms}} \\
&\le \sum_{n=0}^{\infty} 2.2^{-n} = 4. \text{ since 100C} \ge 4 \text{ and} \\
&4.4.\ldots \text{to } [\frac{\text{n}}{2}] + 1 \text{ terms } \ge 2.2^{-\text{n}}.
\end{aligned}$$

Letting $n \to \infty$ we have

**Corollary 4.1.** . $\lim_{n\to\infty} \left(1 + \frac{i\theta}{n}\right)^n$ *exists and is equal to* $\cos\theta + i\sin\theta, \theta$ *being in radian measure.*

**REMARK** The notion of a limit(and consequently the concept of convergence of a sequence (such as infinite series, infinite products and infinite continued fractions) depends on the notion of a distance of a real or a complex number from the origin). This necessitates the notion of the real line or the complex plane as the case may be. The distance of real number $a$ being $|a|$ and that of a complex number $x + iy$ being the positive square root $\sqrt{x^2+y^2}$.

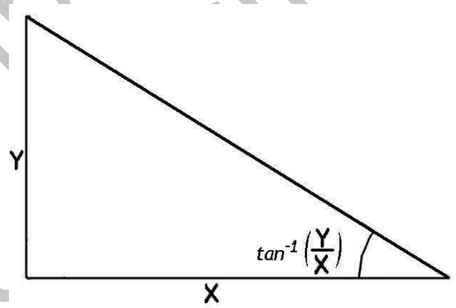



FIGURE 8

Also

$$\begin{aligned}
x + iy &= \sqrt{x^2+y^2}\ e^{itan^{-1}\frac{y}{x} + 2\pi ik} \\
&= \sqrt{x^2+y^2}\ \left\{\frac{x}{\sqrt{x^2+y^2}} + \frac{iy}{\sqrt{x^2+y^2}}\right\}.
\end{aligned}$$

Hence

$$\log(x+iy) = \log\sqrt{x^2+y^2} + \mathrm{i}\tan^{-1}\ \frac{y}{x} + 2\pi ik$$
$$k = 0, \pm 1, \pm 2, \ldots$$

.

These are in some ways a natural way of introducing the distance function (also called Archimedean valuation). There are other ways (called non-Archimedean valuations). For example

$$|2^n|_2 = 2^{-n} \ \mathit{and}\ |6^n|_2 = 2^{-n}.$$

You will surely raise your eyebrows if I say that the distance of $2^n$ from the origin is $2^{-n}$ and so $2^n$ tends to zero as $n \to \infty$. But you need not. There is a rich subject called "p-adic analysis". Every rational number has a p-adic distance called p-adic valuation" associated with every prime $p$. For example if $(p, 6) = 1$ then $|6|_p = 1$. Also we can do differentiation, integration theory of analytic functions and so on. In this theory, the radius of converge of $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ is not infinity but $p^{-\delta}, \delta = \frac{1}{p-1}$, !!!.

This is certainly not perverted intelligence. The extra-ordinary results of A.Baker have been extended to p-adic valuations. These give very rich dividends to ordinary Diophantine equations and more general diophantine problems. The best Indian expert on p-adic analysis is Professor T.N. Shorey. He has worked on p-adic transcendence and application of Baker's work to diophantine questions. From now on we use only the ordinary distances of real and complex numbers from the origin.

In the next section we will express $\lim_{n\to\infty}(1+\frac{i\theta}{n})^n$ as an infinite power series in powers of $i\theta$. Separating real and imaginary parts we obtain series for the $\sin\theta$ and $\cos\theta$ in powers of $\theta$.



## 5. EXPONENTIAL SERIES

The object of this section is to prove the following theorem.

**Theorem 5.1.** *For any fixed complex number $z = x + iy$ we have*

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.$$

**PROOF:** For proving this theorem we need the expansion

$$(1+\frac{z}{n})^n = 1 + \frac{n}{1!}(\frac{z}{n}) + \frac{n(n-1)}{2!}(\frac{z}{n})^2 + \frac{n(n-1)(n-2)}{3!}(\frac{z}{n})^3 + \ldots$$

to $n+1$ terms, where $n$ is a positive integer (this could have been proved by Ramanujan).

**MOTIVATION** We can start with

$$(1-x)^{-1} = 1 + x + x^2 + \ldots.$$

differentiate both sides with respect to $x$ n times. We get successively (for LHS)

$$1!(1-x)^2, 2!(1-x)^{-3}, 3!(1-x)^{-4}, \dots \textit{ to n terms}$$

RHS will be

$$\begin{aligned} &1.x^0 + 2.x^1 + 3x^2 + 4x^3 + \dots \\ &2.1.x^0 + 2.3x^1 + 3.4x^2 + \dots \end{aligned}$$

We can guess what happens at the $r^{th}$ stage and we get an expansion for

$$(1-x)^{-r}.$$

Here we can replace $r$ by $-r$ and $x$ by $-x$ and thus we can get a formula for

$$(1+x)^n.$$

From this we can get the expansion for $(1+\frac{z}{n})^n$, (which is the well known Binomial theorem for a positive integral index) stated earlier.

We need a lemma

**Lemma 5.1.**

$$\begin{aligned} &\left|\frac{n(n-1)}{n^2} - 1\right| \leq \frac{2^2}{n},\ \left|\frac{n(n-1)(n-2)}{n^3} - 1\right| \leq \frac{2^3}{n}, \\ &\left|\frac{n(n-1)(n-2)(n-3)}{n^4} - 1\right| \leq \frac{2^4}{n}, \dots \end{aligned}$$

**PROOF:** We have for $0 < r < n$

$$\begin{aligned} &\left|\frac{n(n-1)(n-2)\dots(n-r)}{n^{r+1}} - 1\right| \\ &\leq \left|\frac{(n-1)(n-2)\dots(n-r)}{n^r} - 1\right| \end{aligned}$$

(Note that the first term is positive and less than 1 and so the quantity in question is)

$$\begin{aligned} &\leq \left|\left(\frac{n-r}{n}\right)^r - 1\right| = n^{-r}(n^r - (n-r)^r) \\ &= n^{-r}(n-(n-r))\left(\sum_{j=0}^{r-1} n^j (n-r)^{r-1-j}\right) \\ &\leq n^{-r}.r.r\ n^{r-1} = \frac{r^2}{n} = \frac{r^2}{2}.2.\frac{1}{n} \leq \frac{2^{r+1}}{n} \\ &\text{since } 2^{\mathrm{r}} \geq 1 + \mathrm{r} + \frac{\mathrm{r}(\mathrm{r}-1)}{2} = 1 + \frac{\mathrm{r}^2+\mathrm{r}}{2} \geq \frac{\mathrm{r}^2}{2}. \end{aligned}$$

This proves the lemma.

Now

$$\begin{aligned} &\left|(1+\frac{z}{n})^n - (1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots)\right| \\ &\leq \left|(1+\frac{|z|}{n})^n - \sum_{m=0}^{n} \frac{|z|^m}{m!}\right| + \sum_{m>n} \frac{|z|^m}{m!} \end{aligned}$$

$$\leq (\frac{|z|^2}{2!}\ \frac{2^2}{n} + \frac{|z|^3}{3!}\ \frac{2^3}{n} + \frac{|z|^4}{4!}\ \frac{2^4}{n} + \cdots + \frac{|z|^n}{n!}\ \frac{2^n}{n})$$
$$+ \sum_{m>n} \frac{|z|}{m!}^m = P + Q \ say.$$

It is clear that

$$Q \leq \frac{|z|}{n!}^n \sum_{m=n+1}^{\infty} \frac{|z|^{m-n} n!}{m!}$$
$$= \frac{|z|}{n!}^n \sum_{r=1}^{\infty} \frac{|z|^r n!}{(n-r)!} = \frac{|z|}{n!}^n \sum_{r=1}^{\infty} \frac{|z|}{r!}^r$$

since for $r \geq 1$ we have $\frac{n!}{(n+r)!} \leq \frac{1}{n!}(\frac{(n+r)!r!}{n!}$ being a binomial coefficient in $(1+x)^{n+r}$ and hence an integer). Hence if $|z| \leq C$(we have for some $C > 2$)

$$Q \ \leq \ \frac{C^n}{n!} \sum_{r=1}^{\infty} \frac{C^r}{r!} \ \leq \frac{C^n}{n!} \ times\ a\ constant.$$

Hence as $n \to \infty,\ Q \to 0$ as is evident from

$$\frac{C^n}{n!} \ \leq \ \frac{C^n}{[\frac{n}{2}] \dots [n]} \leq \left( \frac{C^2}{\frac{n}{2} - 1} \right)^{\frac{n}{2}}$$

valid for $n \geq 100C^2$.

Again

$$P \ \leq \ \frac{1}{n} \left\{ \frac{|2z|^2}{2!} + \frac{|2z|^3}{3!} + \dots \right\}$$
$$\leq \ \frac{1}{n} \sum_{m=1}^{\infty} \frac{|2z|^m}{m!}$$
$$\leq \frac{1}{n} \sum_{m=1}^{\infty} \frac{|2C|^m}{m!} \text{ for } |z| \leq C$$

It is not hard to prove that the last infinite sum is bounded by a constant depending on $C$ (see the remark before section V).

Hence $P \to 0, Q \to 0$ *as* $n \to \infty$, and this proves the theorem.

**REMARK** We do not go in detail to the theory of infinite series. For our purposes an infinite series of complex terms $\sum_{n=1}^{\infty} a_n$ represents a complex number if it is "convergent". For our purposes a series is said to be convergent if the tail portion

$$\sum_{n=m}^{\infty} a_n$$

tends to zero as $m \to \infty$. In our case

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is convergent. (In fact it can be differentiated term by term any number of times). Call this $e(z)$.

$$e(z_1)e(z_2) = \lim_{n\to\infty} \left\{(1+\frac{z_1}{n})^n(1+\frac{z_2}{n})^n\right\}$$

and the limit can be seem to be

$$\lim_{n\to\infty} \left\{1+\frac{z_1+z_2}{n}\right\}^n = e(z_1+z_2)$$

and so

$$(e(z))^n = e(nz), (e(1))^z = \lim_{n\to\infty} \{(1+\frac{1}{n})^{zn}\}$$

$$= \lim_{n\to\infty} \{(1+\frac{z}{n})^n. \text{ Hence } e(z) = e(1)^z,\ e(1)$$

is usually denoted by $e$.

Thus we have the following theorem.

**Theorem 5.2.** *We have* $\cos\theta + i\sin\theta = \lim_{n\to\infty}(1+\frac{i\theta}{n})^n = e^{i\theta} = 1 + i\theta + \frac{(i\theta^2)}{2!} + \frac{(i\theta)^3}{3!} + \ldots$ *whereby*

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} - + \ldots$$

*and*

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \ldots$$

**REMARK** If $a$ and $z$ are complex number we have to interpret $a^z$ as $\exp\{z\log a\}$. Since $\log a_1 = \log a_2$ will happen with $a_1 = a_2 e^{2\pi ik}$ for any integer $k$, we have to specify the logarithm. In case $a$ is a positive real number $\log a$ is in uniquely defined (in practice) as the unique real solution of $e^b = a$. But even in this case $\log a$ is in general any of the numbers $b + 2ki\pi (k = 0, \pm 1, \pm 2, \ldots,)$. If $a(\neq 0)$ is complex we write $\frac{a}{|a|} = e^{i\theta}$ (note the LHS has absolute value 1. If $A + iB$ is of absolute value 1 so is its square $A^2 + B^2 = 1$, where $A$ and $B$ are real.

## 6. LOGARITHMIC SERIES

We have seen that

$$\lim_{n\to\infty}(1+\frac{z}{n})^n = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z.$$

Let $1 - x(|x| < 1)$ be the series on the RHS. It is possible to invert and find an expression for $z$ in terms of $x$. Put $(1+\frac{z}{n})^n = 1 - x$. There follows

$$\lim_{n\to\infty} n((1-x)^{\frac{1}{n}} - 1) = z.$$

If we use binomial theorem for the (non-integral) index $\frac{1}{n}$, we have

$$\begin{aligned} z &= \lim_{n\to\infty} n\left(-x.\frac{1}{n} + \frac{(-x)^2}{2!}.\frac{1}{n}\ (\frac{1}{n}-1) + \frac{-x^3}{3!}.\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2) + \ldots\right) \\ &= \lim_{n\to\infty}(-x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots + Q_n) \end{aligned}$$

where

$$Q_n = \{-(1-\frac{1}{1.n})\frac{x^2}{2} + \frac{x^2}{2} - (1-\frac{1}{1.n})(1-\frac{1}{2.n})\frac{x^3}{3} + \frac{x^3}{3}$$
$$-(1-\frac{1}{1.n})(1-\frac{1}{2.n})(1-\frac{1}{3.n})\frac{x^4}{4} + \frac{x^4}{4} + \dots\}$$

Hence

$$\begin{aligned}
|Q_n| &\leq \frac{|x|^2}{2}(1-(1-\frac{1}{1.n})) + \frac{|x|^3}{3}(1-(1-\frac{1}{1.n})(1-\frac{1}{2.n})) + \dots \\
&= \sum_{r=2}^{\infty}\left\{\frac{|x|^r}{r}((1-(1-\frac{1}{1.n})(1-\frac{1}{2.n})\dots(1-\frac{1}{(r-1)n}))\right\} \\
&\leq \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left|(1+\frac{1}{1.n})(1+\frac{1}{2.n})\dots(1+\frac{1}{(r-1)n}) - 1\right| \\
&\quad \Big(\text{(since } (1+\frac{1}{1.\mathrm{n}})(1+\frac{1}{2.\mathrm{n}})\dots\ (1+\frac{1}{(\mathrm{r}-1)\mathrm{n}}) \\
&+ \quad (1-\frac{1}{1.n})(1-\frac{1}{2.n})\dots\ (1-\frac{1}{(r-1)n}) \geq 2\Big) \\
&\leq \sum_{r=2}^{\infty}\frac{x^r}{r}\left\{\frac{1}{n}\sum_{j\leq r}\frac{1}{j} + \frac{1}{n^2}(\sum_{j\leq r}\frac{1}{j})^2 + \dots \text{to r terms}\right\} \\
&\leq \sum_{r=2}^{\infty}\frac{x^r}{r}\left\{\frac{1}{n}(\log r + O(1)) + \frac{1}{n^2}(\log r + O(1))^2 + \dots \text{to r terms}\right\} \\
&= \sum_1 + \sum_2
\end{aligned}$$

where $\sum_1$ is over those $r$ with $n > (10\log r + O(1))^2$ i.e. $(\log r + O(1)) \leq 5\sqrt{n}$ and $\sum_2$ those with the remaining $r$ namely $r > N, N = Exp(\frac{n}{10} + O(1))$.

$$\begin{aligned}
\sum_1 &\leq \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left\{\frac{1}{n}5\sqrt{n} + \frac{1}{n^2}(5\sqrt{n})^2 + \text{ to r terms}\right\} \\
&\leq \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left\{\frac{5}{\sqrt{n}} + (\frac{5}{\sqrt{n}})^2 + (\frac{5}{\sqrt{n}})^3 + \ \dots\ to\ \infty\right\} \\
&= \sum_{r=2}^{\infty}\frac{|x|^2}{r}\left\{\frac{\frac{5}{\sqrt{n}}}{1-\frac{5}{\sqrt{n}}}\right\} = \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left\{\frac{5}{\sqrt{n}-5}\right\} \\
&\leq \sum_{r=2}^{\infty}\frac{|x|^r}{r}.\frac{10}{\sqrt{n}}\ if\ n\ \sqrt{n} - 5 \geq \frac{\sqrt{n}}{2}\ i.e.\ if\ n \geq 100. \\
&\leq \frac{|x|^2}{1-|x|}.\frac{10}{\sqrt{n}}.
\end{aligned}$$

Next

$$\sum_2 \leq \frac{1}{N}\sum_{r\geq N}|x|^r(1+\frac{1}{n})^{r-1}\text{see the third step of inequality for}|\mathrm{Q_n}|$$

$$\leq \frac{1}{N} \sum_{r\geq 0} |x|^r (1+\frac{1}{n})^{r-1}$$
$$\leq \frac{n}{N}(1+\frac{1}{n})^{-1} \left\{1-|x|(1+\frac{1}{n})\right\}^{-1}$$
$$= \frac{2n}{N} \left\{1-|x|(1+\frac{1}{n})\right\}^{-1}, \text{ provided } |x|(1+\frac{1}{\text{n}}) < 1,$$

$$\leq N^{-\frac{1}{2}} \left\{1-|x|(1+\frac{1}{n})\right\}^{-1} \text{ since N} = \exp(\frac{\text{n}}{10} + \text{O}(1)).$$

Thus we have proved the following

**Theorem 6.1.** *Let* $|x| < 1$ *and*

$$n((1-x)^{\frac{1}{n}} - 1) + \sum_{k=1}^{\infty} \frac{x^k}{k} = E(x).$$

*Then*

$$|E(x)| \leq \frac{10|x|^2}{1-|x|}\ n^{-\frac{1}{2}} + N^{-\frac{1}{2}} \left\{1-|x|(1+\frac{1}{n})\right\}^{-1}$$

*where* $n \geq 100$ *and* $N = Exp(\frac{n}{10} + O(1))$.

**Corollary 6.1.** *Letting* $n \to \infty$ *we have for* $|x| < 1$, *the equality* $\lim_{n\to\infty} n((1-x)^{\frac{1}{n}} - 1) = -\sum_{k=1}^{\infty} \frac{x^k}{k} = \log(1-x)$.

An alternative approach is to assume an expansion of the following type (in this approach binomial theorem for a non-integral positive index is not necessary. We need "multinomial theorem" for a positive integral index). We start with

$$(1+\frac{z}{n})^n = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \frac{1}{1-x}\ (|x| < 1)$$

$$e^z = \frac{1}{1-x} = 1 + x + x^2 + \ldots |x| < 1$$

Assume that $z = a_0 + a_1 x + a_2 x^2 + \ldots$. There follows

$$\frac{1}{1-x} = 1 + \frac{(a_0 + a_1 x + a_2 x^2 + \ldots)}{1!} + \frac{(\ldots)^2}{2!} + \frac{(\ldots)^3}{3} + \ldots.$$

Equating constant terms we get $1 = e^{a_0}$ and so $a_0 = 0$. Equating coefficients of $x$ we have $1 = a_1$. Equating coefficients of $x^2$ we have $1 = \frac{a_2}{1!} + \frac{a_1^2}{2!} i.e.\ a_2 = \frac{1}{2}$ and so on. By induction we may complete the proof that

$$\log(\frac{1}{1-x}) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

But we will not pursue this proof and do error estimations.

## 7. (TRIGONOMETRIC FUNCTIONS AND THEIR INVERSES)

We start with the diagram of triangle ABC

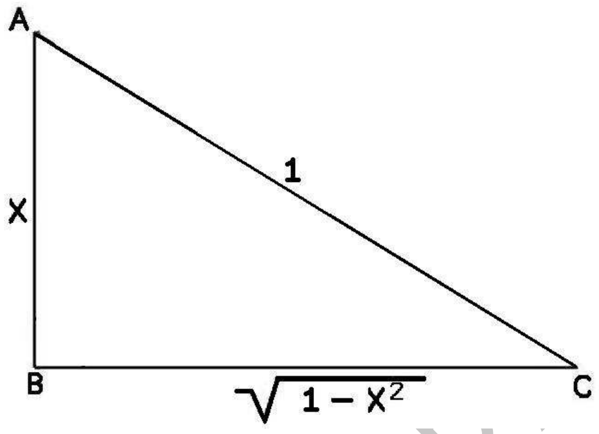



FIGURE 9

where angle B (denoted by $\hat{B}$) $= \frac{\pi}{2}$. It is clear that $\sin^{-1} x = \hat{C}, \cos^{-1} x + \sin^{-1} x = \hat{A} + \hat{C} = \frac{\pi}{2}, \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \hat{C},\ cot^{-1} \frac{x}{\sqrt{1-x^2}} = \hat{A}$ and so $\tan^{-1} \frac{x}{\sqrt{1-x^2}} + \cot^{-1} \frac{x}{\sqrt{1-x^2}} = \hat{C} + \hat{A} = \frac{\pi}{2}$ and so on. These require the condition $0 < x < 1$ (but relaxable by "Analytic continuation", a term which we do not explain).

From

$$\cos\theta + i\sin\theta = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

Follows (on equating real and imaginary points),

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \dots \text{and}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \dots$$

The usual (nice) series for $tan^{-1}x$ can be obtained as follows.

We have

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Put $z = x + iy$ and we get

$$-\log(1-x-iy) = \sum_{n=1}^{\infty} \frac{(x+iy)^n}{n}.$$

Specializing this to $x = 0$ we obtain

$$-\log(1-iy) = \sum_{n=1}^{\infty} \frac{(iy)^n}{n}.$$

Here LHS $= -\log|1-iy| - \mathrm{i}\ \tan^{-1}(-y) = -\log|1-iy| + \mathrm{i}\ \tan^{-1}$y Thus

$$-\log(1-iy) = \sum_{n=1}^{\infty} \frac{(iy)^n}{n} = -\log|1-iy| + i\ tan^{-1}y$$

Equating imaginary points in the last two formulae we get

$$tan^{-1}y = y - \frac{y^3}{3} + \frac{y^5}{5} - + \dots, \text{where}|y| < 1.$$

But $tan^{-1}1 = \frac{\pi}{4}$ and we recover

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \dots.$$

by justifying the limit operation $y \to 1$.

So far we have not employed calculus. But we now use calculus.

$$\frac{d}{dy}tan^{-1}y = \frac{1}{1+y^2} \text{ and so } \tan^{-1}y = \int_0^y \frac{\mathrm{d}}{\mathrm{d}y}(\tan^{-1}y)dy \text{ and so for } |y| < 1$$

$$= \int_0^y (1 - y^2 + y^4 - \dots)dy = y - \frac{y^3}{3} + \frac{y^5}{5} - + \dots$$

Again $\frac{d}{dy}\ \sin^{-1}y = \frac{1}{\sqrt{1-y^2}}$ and so $\sin^{-1}y = \int_0^y \frac{dy}{\sqrt{1-y^2}}$

$$\begin{aligned} = \quad & \int_0^y (1 + (-\frac{1}{2})\frac{(-y^2)}{1!} + (-\frac{1}{2})(-\frac{1}{2}-1)\frac{(-y^2)^2}{2!} \\ & + (-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\frac{(-y^2)^3}{3} + \dots\ dy \\ = \quad & y + \frac{1}{2}\frac{y^3}{3} + \frac{1.3}{2.4}\ \frac{y^2}{5} + \frac{1.3.5}{2.4.6}\ \frac{y^7}{7} + \dots. \end{aligned}$$

Also $cos^{-1}y = \frac{\pi}{2} - sin^{-1}y = \frac{\pi}{2} - (y + \frac{1}{2}\frac{y^3}{3} + \frac{1.3}{2.4}\ \frac{y^5}{5} + \dots)$.

We end this article by proposing a new method for a nice expansion of $(F(x))^k$ where $k$ is any positive integer constant. We limit ourselves to $(tan^{-1}x)^2$ and $(sin^{-1}x)^2$. [Before proceeding further we note (p.203 of part II of the excellent books by S.L.Loney (parts I and II)) the following result.

$$\frac{1}{6}(sin^{-1}x)^3 = \frac{1}{2}\frac{x^3}{3} + (\frac{1}{1^3} + \frac{1}{3^3})\ \frac{1.3}{2.4}\ \frac{x^5}{5} + (\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3})\ \frac{1.3.5}{2.4.6}\ \frac{x^7}{7} + \dots$$

We hope that we can obtain this by our method].

We now proceed with our method.

Let

$$(tan^{-1}x)^2 = a_0 + a_1x + a_2x^2 + \dots. \tag{7.1}$$

Differentiating both sides with respect to $x$ and multiplying both sides by $1 + x^2$ we get

$$2tan^{-1}x \quad = \quad (1 + x^2)(a_1 + 2a_2x + 3a_3x^2 + \dots)$$

$$= \quad 2(x - \frac{x^3}{3} + \frac{x^5}{5} - + \dots)$$

Equating coefficients of like powers of $x$, we can obtain $a_0, a_1, a_2, \dots$. Certainly $a_0 = 0$. But in this special case we can get $a_0, a_1, a_2, \dots$ by direct squaring in

$$\left(x - \frac{x^3}{3} + \frac{x^5}{5} - + \dots\right)^2.$$

We now turn to

$$(sin^{-1}x)^2 = a_0 + a_1 x + a_2 x^2 + \dots \tag{1}$$

differentiating we get

$$\frac{2sin^{-1}x}{\sqrt{1-x^2}} = a_1 + 2a_2 x + 3a_3 x^2 + \dots \tag{2}$$

One more differentiation gives

$$\frac{2}{1-x^2} + 2(sin^{-1}x)(-\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)) = 2.1.a_2 + 3.2.a_3 x + 4.3.a_4 x^2 + \dots \tag{3}$$

Multiplying throughout by $1 - x^2$, we obtain

$$2 + \frac{2(sin^{-1}x)}{\sqrt{1-x^2}} = (1-x^2)(2.1.a_2 + 3.2.a_3 x + 4.3.a_4 x^2 + \dots$$

Here we substitute for LHS (using (2))

$$2 + a_1 + 2a_2 x + 3a_3 x^2 + \dots$$
$$= (1-x^2)(2.1.a_2 + 3.2.a_3 x + 4.3.a_4 x^2 + \dots). \tag{4}$$

In (4) we equate coefficients of like powers of $x$ and we get $a_1, a_2, a_3, \dots$ (trivially $a_0 = 0$ and $a_1 = 0$).

We leave it as an exercise for the reader to complete the expansion of $(sin^{-1}x)^2$.

**ONE FINAL REMARK** JONATHAN M BORWEIN and MARC CHAMBERLAND have proved the following surprising result.

**Theorem 7.1.** *(1) for $|x| \leq 2$ and $N = 1, 2, 3, \dots$ we have*

$$\frac{1}{(2N)!}\left(\sin^{-1}\frac{x}{2}\right)^{2N} = \sum_{R=1}^{\infty} \frac{H_N(k)}{\begin{pmatrix} 2k \\ k \end{pmatrix} k^2} x^{2k}$$

*where $H_1(k) = \frac{1}{4}$ and*

$$H_{N+1}(k) = \frac{1}{4}\sum_{n_1=1}^{k-1} \frac{1}{(2n_1)^2} \sum_{n_2=1}^{n_1-1} \frac{1}{(2n_2)^2} \cdots \sum_{n_N=1}^{n_{N-1}-1} \frac{1}{(2n_N)^2}.$$

*(2) For $|x| \leq 2$ and $N = 0, 1, 2, \dots$ we have*

$$\frac{1}{(2N+1)!}\,(\sin^{-1}\ \frac{x}{2})^{2N+1} = \sum_{k=0}^{\infty} \frac{G_N(k)\begin{pmatrix} 2k \\ k \end{pmatrix}}{2(2k+1)4^{2k}}\ x^{2k+1}$$

*where* $G_0(k) = 1$ *and*

$$G_N(k) = \sum_{n_1=0}^{k-1} \frac{1}{(2n_1+1)^2} \sum_{n_2=0}^{n_1-1} \frac{1}{(2n_2+1)^2} \cdots \sum_{n_N=0}^{n_{N-1}-1} \frac{1}{(2n_N+1)^2}$$

the convention is that the sum is zero if the starting index exceeds the finishing index.

The beautiful result

$$\sum_{k=1}^{\infty} \frac{4^k}{\binom{2k}{k} k^3} = \pi^2 \ \log 2 - \frac{7}{8}\ \zeta(3)$$

is deduced in their paper as a special case.[-0.2 cm]

## References


[1] S.L.Loney, *Plane Trigonometry volume I,* Essential books 4393/4, 1st floor, Tulsidas Street, Ansari Road, Daryaganj, New Delhi - 110 002.

[2] S.L.Loney, *Plane Trigonometry Volume II,* Publishers same as in [1].

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# POROSITY OF THE TWO RATIO CANTOR SET

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Abstract. In this note we prove that the two ratio Cantor set is uniformly very porous in the set of real numbers.

## 1. INTRODUCTION

Let $r_1$ and $r_2$ be positive real numbers such that $r_1 + r_2 < 1$. Put

$E_0 = [0, 1]$,

$E_1 = [0, r_1] \cup [1 - r_2, 1]$,

$E_2 = [0, r_1^2] \cup [r_1(1 - r_2), r_1] \cup [1 - r_2, 1 - r_2(1 - r_2)] \cup [1 - r_1 r_2, 1]$,

and so on. For any positive integer $k, E_k$ is a union of $2^k$ disjoint closed intervals. If $[a, b]$ is a typical interval in $E_k$ then the intervals in $E_{k+1}$ are given by $[r_1 a, r_1 b]$ and $[r_1 a, r_1 b]$. Further if $c$ be such that $r_1 < c < 1 - r_2$ then all intervals $[r_1 a, r_1 b]$ lie to the left of $c$ and all intervals $[1 - r_2 b, 1 - r_2 a]$ lie to the right of $c$. Clearly $E_{k+1} \subset E_k$. Also note that if $J$ is any interval in $E_{k+1}$, then there is a unique interval $I$ in $E_k$ such that $J \subseteq I$ and moreover

(1.1) $$length(J) = r_1\, length(I) \text{ or } length(J) = r_2\, length(I)$$

We set
$$C(r_1, r_2) = \bigcap_{k=0}^{\infty} E_k.$$

The set $C(r_1, r_2)$ has been called by Coppel [1] the two ratio Cantor set. If $r_1 = r_2 = 1$ then $C(r_1, r_2)$ coincides with the classical Cantor set $C$. In this note we like to discuss the porosity behaviour of $C(r_1, r_2)$.

## 2. SOME BASIC FACTS ABOUT POROSITY

We recall the following basic facts about porosity from [2] and [3] . Let $M \subset \mathbb{R}$ and let $y$ be a real number. Let $\nu(y, r, M) = \sup\{t > 0; \text{thereisa}\, z \in (y - r, y + r)\, \text{suchthat}\ (z - t, z + t) \subset (y - r, y + r)\, \text{and}\ M \cap (z - t, z + t) = \phi\}$

Note that if $M$ is dense then $\nu(y, r, M) = -\infty$.

Let
$$\overline{P}(y, M) = \limsup_{r \to 0^+} \frac{\nu(y, r, M)}{r},$$

$$\underline{P}(y, M) = \liminf_{r \to 0^+} \frac{\nu(y, r, M)}{r},$$

and if $\overline{p}(y, M) = \underline{p}(y, M)$ then we set

$$p(y, M) = \underline{p}(y, M) = \overline{p}(y, M) = \lim_{r \to 0^+} \frac{\nu(y, r, M)}{r}$$

The set $M$ is said to be porous at $y$ if $p(y, M) > 0$ and $\sigma$-porous at $y$ provided $M = \bigcup_{n=1}^{\infty} M_n$ and each of the sets $M_n$ is porous at $y$. $M$ is called a porous or $\sigma$-porous set if it is so at each $y \in \mathbb{R}$.

A porous set is nowhere dense while a $\sigma$-porous set is a set of first category. Very simple examples of porous sets are the set of natural numbers and the set of all integers as also the classical Cantor set. The set of all rational numbers is an example of a $\sigma$-porous set which is not porous. However not all nowhere dense sets are porous and there are examples of sets which are nowhere dense but not porous.

The set $M$ is said to be *very porous* at $y$ if $\underline{p}(y, M) > 0$ and $M$ is said to be *uniformly very porous* in $Y_0 \subset \mathbb{R}$ if there is a $c > 0$ such that for each $y \in y_0$ we have $\underline{p}(y, M) \geq c$.

A very porous set or a uniformly very porous set is evidently porous but the converse is not generally true.

As already mentioned, for the classical Cantor set $C$, it is known [3, p 318] that $C$ is porous at each point of $C$. We prove here a stronger result for $C(r_1, r_2)$ in respect of porosity. Incidentally, on taking $r_1 = r_2 = \frac{1}{3}$ our theorem extends the theorem on porosity of $C$ [3].

## 3. THEOREM

**Theorem 3.1.** . *The set $C(r_1, r_2)$ is uniformly very porous in $\mathbb{R}$.*

**Proof :** If $x \notin C(r_1, r_2)$, then clearly $\underline{p}[x, C(r_1, r_2)] = 1$. So let $x \in C(r_1, r_2)$. Then $x \in E_k$ for all non-negative integers $k$. Let us take $I_0 = [0, 1]$ and $t_0 = length(I_0) = 1$. Since $E_k$ is the union of disjoint closed intervals, there exists such an interval $I_k$ from $E_k$ that $x \in I_k$ and this is true for each $k$. Thus $\{I_k\}$ is a strictly decreasing sequence of closed intervals with $length(I_k) = t_k$ (say) tending to zero as $k \to \infty$ . Let $r > 0$. Without any loss of generality we choose $r < 1$. Select a positive integer $m$ such that $t_{m+1} < r \leq t_m$. From (1) it follows that $t_{m+1} = r_1 t_m$ or $t_{m+1} = r_2 t_m$

Now since $x \in I_{m+1}$ and $r > t_{m+l} = length(I_{m+l})$, we must have $(x - r, x + r) \supset I_{m+1}$. The interval $I_{m+1}$ contains an interval of length $(1 - r_l - r_2)t_{m+l}$ which does not contain any point of $C(r_1, r_2)$. Choose $t = \frac{t_{m+1}}{2}(1 - r_l - r_2)$ . Then

$$\frac{1}{r}\nu[x, r, C(r_1, r_2)] \geq \frac{1}{r} \geq \frac{1}{t_m} = \frac{(1 - r_l - r_2)t_{m+1}}{2t_{m+1}}$$
$$\geq \frac{(1 - r_l - r_2)}{2}\min(r_1, r_2) = c \quad \text{say.}$$

Thus $c$ is positive and

$$\underline{p}\,[x, C(r_1, r_2)] = \liminf_{r \to 0^+} \frac{\nu[x, r, C(r_1, r_2)]}{r} \geq c.$$

This shows that $C(r_1, r_2)$ is uniformly very porous in $R$ and thus the theorem is proved.

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# ON FATOU COMPONENTS OF CERTAIN COMPOSITIONS OF ENTIRE FUNCTIONS

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Abstract. Let $f$, $g$ and $h$ be transcendental entire functions. Then $f$ and $g$ are said to be permutable if $f \circ g = g \circ f$. Further $g$ and $h$ are said to be semiconjugated by $f$ if $g \circ f = f \circ h$. In this paper we consider the functions $f$ and $g$ which are permutable entire functions and further one of the factors say $f$ semiconjugates with the other factor $g$ and with another entire function $h$. Thus we are interested in transcendental entire functions $f, g, h$ which satisfy

$$f \circ g = g \circ f = f \circ h$$

and obtain results on the Fatou sets and Julia sets of such, and related entire functions.

**Keywords and phrases :** Entire functions, Semiconjugation, Fatou set, Julia set

**2000 Mathematics Subject Classification :** 30D05, 37F10



## 1. INTRODUCTION

Let $f$ be a transcendental entire function or a rational function of degree atleast two. For $n \in \mathbb{N}$, $f^n$ will denote the $n^{th}$ iterate of $f$. Thus, $f^1 = f, f^n = f(f^{n-1})$, $n = 2, 3, \ldots$. The Fatou set $F(f)$ is the set of all $z$ where the family $\{f^n\}$ of iterates of $f$ is normal in some neighbourhood of $z$. The Julia set of $f$ denoted by $J(f)$, is the complement of the Fatou set of $f$. By definition $F(f)$ is open and consequently $J(f)$ is closed.

For any $p \in \mathbb{N}$, $F(f^p) = F(f)$ and $J(f^p) = J(f)$. Further, $J(f)$ is completely invariant under $f$, i.e., $z \in J(f)$ implies $f(z) \in J(f)$ and also $w \in J(f)$ for any $w$ which satisfies $f(w) = z$. $F(f)$ is also completely invariant. $J(f) \neq \phi$ and is infact infinite set and is dense in itself. For a transcendental entire function $f$, the interior of $J(f)$ viz., $(J(f))^o = \phi$ or else $J(f) = \mathbb{C}$ and for rational function ( of degree $\geq 2$ ), $(J(f))^o = \phi$ or else $J(f) = \hat{\mathbb{C}}$,where $(J(f))^o$ denotes the interior of $J(f)$. Note that $J(f) = \hat{\mathbb{C}}$ for rational $f$ and $J(f) = \mathbb{C}$ for transcendental entire function are indeed possible. Infact Lattés showed that $J(\frac{(z^2+1)^2}{4z(z^2-1)}) = \hat{\mathbb{C}}$

(see [2, 73] for more details) and Misiurewicz [9] showed that $J(e^z) = \mathbb{C}$.

If $U_n \cap U_m = \phi$ for $n \neq m$ where $U_n$ denotes the component of $F(f)$ which contains $f^n(U)$ then $U$ is called a wandering domain. If $U_n = U$ for some $n$, then $U$ is called a periodic domain ( of period $n$ if $U_n = U$ and $U_i \neq U$ for $i = 1, 2, ..., n-1$ ). For more details on the subject we refer the reader, for instance to [2], [5],[10].

If $f$ and $g$ are transcendental entire functions then so are $f \circ g$ and $g \circ f$. Thus one would be interested in studying the dynamics of these two composite functions along with its factors. It is interesting to note that the dynamics of $f \circ g$ can help in studying the dynamics of $g \circ f$. For instance Bergweiler and Wang [4] ( see also Poon and Yang [13]) have shown that $f \circ g$ has no wandering domain if and only if $g \circ f$ has no wandering domain. Begweiler and Wang [4] have also shown that if $f$ and $g$ are non-linear entire functions then $z \in J(f \circ g)$ if and only if $g(z) \in J(g \circ f)$.

If $f$ and $g$ are rational functions then so are $f \circ g$ and $g \circ f$. If further $f$ and $g$ are of degree at least two, and $f \circ g = g \circ f$ then it is well known that ( see [2, 72]) $J(f) = J(g)$. It is still an open question whether this is true if both $f$ and $g$ are transcendental entire functions. The reason why the proof of rational function does not carry over to transcendental entire functions is that the rational functions satisfy Lipschitz condition whereas transcendental entire functions do not. Thus a different approach would be needed. Of course several partial results in affirmation for this result have been obtained, for instance [3] if $f$ and $g$ both do not have wandering domain and if $f \circ g = g \circ f$ then $J(f) = J(g)$. The case with both $f$ and $g$ having wandering domain is still open. For more results, see for instance [3], [7], [8],[11],[12], [14].

A more general concept than permutation is the concept of semi-conjugation. Following [3], let $g$ and $h$ be transcendental entire functions and let $f$ be continuous and open map on the complex plane. Let $(g \circ f)(z) = (f \circ h)(z)$ for every $z \in \mathbb{C}$. Then $g$ and $h$ are said to be semiconjugated by $f$. In particular if $h = g$ and $f$ is entire function, then $f$ and $g$ are permutable entire functions. In this paper we shall obtain some results on the Fatou sets and Julia sets of semiconjugated and permutable entire functions. We shall work when $f$ and $g$ are permutable entire functions and further one of the factors say $f$ semiconjugates with other entire functions $g$ and $h$. Thus we shall be interested in transcendental entire functions $f, g, h$ which satisfy

$$f \circ g = g \circ f = f \circ h \tag{1.1}$$

Note that if $h = g$ then (1.1) reduces to $f \circ g = g \circ f$. We also note that there do exist distinct entire functions $f, g, h$ which satisfy condition (1.1). For instance, clearly the functions $f(z) = 2\pi i + e_3(z), g(z) = 2\pi i + e_2(z)$, and $h(z) = e_2(z)$, where $e_1(z) = e^z, e_n(z) = e_1(e_{n-1}(z)), n = 2, 3, \ldots,$ satisfy (1.1).

## 2. MAIN RESULTS

We begin with an extension of Lemma 4.3 of Baker [1].

**Theorem 2.1.** *Let $f, g, h$ be transcendental entire functions such that $g \circ f = f \circ g = f \circ h$.*

*i) If $\alpha \in F(f)$ and there is a sequence $f^{n_k}$ with $n_k \to \infty$ which has a finite limit in the component of $F(f)$ which contains $\alpha$, then $h(\alpha) \in F(f)$.*

*ii) If $\infty$ is never a limit function of a subsequence of $f^n$ in a component of $F(f)$ then $h(F(f)) \subset F(f)$.*

For the proof of the theorem, we shall need the following Lemma.

**Lemma 2.1.** *([10, 75]). If $f$ is transcendental entire ( or rational function of degree atleast two), $\alpha \in J(f)$ and $V$ is any neighbourhood of $\alpha$. Then for every compact set $K$ containing no exceptional points, $f^n(V) \supset K$ for sufficiently large $n$.*

**Proof of Theorem 2.1** By hypothesis, $\alpha \in F(f)$ and $f^{n_k}$ converges to say, $\beta(\neq \infty)$ in the component containing $\alpha$. As $g$ is continuous, given $\epsilon > 0$ choose $\delta > 0$ such that $|g(z) - g(\beta)| < \epsilon$ whenever $|z - \beta| < \delta$. Also we can choose a small neighbourhood $U$ of $\alpha$ such that $\overline{U} \subset F(f)$ and also $f^{n_k}(U)$ contains $\beta$ and $diam(f^{n_k}(U)) < \delta$ for all $n_k \geq n_o$.

Suppose $h(\alpha) \in J(f)$. Let $K$ be any compact set which does not contain exceptional point and such that $K \cap V = \phi$ where $V = (|w - g(\beta)| < \epsilon)$. Then by Lemma 2.1, for sufficiently large $n_k$, $f^{n_k}(h(U)) \supset K$ and so if $\eta \in K$ then there exists $\xi \in h(U)$ and consequently $t \in U$ such that $f^{n_k}(h(t)) = f^{n_k}(\xi) = \eta$. On the other hand $f^{n_k}(h(t)) = g(f^{n_k}(t)) = g(w) \in V$ where $w = f^{n_k}(t) \in B(\beta, \delta)$. Thus $\eta \in V$, contradicting the disjointness of K and V.

The proof of (ii) is immediate.

C. C. Yang and L. Liao [8] have proved the following:

**Theorem A.** *Let $f$ and $h$ be two permutable transcendental entire functions. If there exists a non constant polynomial $p$ and an entire function $k$ such that $p(h(z)) = k(f(z))$ then $F(h) \subset F(f)$ and $f(F(h)) \subset F(h)$.*

We extend this theorem for three functions $f, g, h$ as follows.

**Theorem 2.2.** *Let $f, g, h$ be transcendental entire functions such that $f \circ g = g \circ f = f \circ h$. Suppose there exists a non constant polynomial $p(z)$ and a non constant entire function $k(z)$ such that $p(f(z)) = k(g(z))$. Then $h(F(f)) \subset F(f)$ and $J(h) \subset J(f)$.*

**Lemma 2.2.** *([8]) Let $f$ and $g$ be two entire functions. If $g(F(f)) \subset F(f)$ then $F(f) \subset F(g)$.*

**Proof of Theorem 2.2** Let $\alpha \in F(f)$. By Theorem 2.1 and Lemma 2.2, it is sufficient to consider the case that $f^n \to \infty$ in $U$ where $U$ is a small neighbourhood of $\alpha$ such that $\overline{U} \subset F(f)$.

Let $M = \max_{|w|=1} |k(w)|$. Since $p$ is a nonconstant polynomial, there exists a constant $P_o$ such that $|p(z)| > M+1$ for all $|z| > P_o$.

Now $f^n \to \infty$ in $U$ and hence there exists $n_o \in \mathbb{N}$ such that $|f^n(z)| > P_o$ for all $n \geq n_o$ and for all $z \in U$. So $|p(f^n(z))| > M+1$ for all $z \in U$ and $n \geq n_o$.

Now suppose $h(\alpha) \in J(f)$. Then $h(U)$ is an open set containing a point of Julia set, viz. $h(\alpha)$ . Hence by Lemma 2.1, for $n$ sufficiently large say $n \geq m_o(\geq n_o)$, $f^n(h(U)) \supset (|z| < 1)$.

We assume $|z| \leq 1$ does not contain Fatou exceptional point of $f$. For otherwise, we work with $|z| < \delta$ instead of $|z| < 1$ where $\delta > 0$ is chosen so small that $|z| \leq \delta$ does not have Fatou exceptional point of $f$, and we take $M = \max_{|w|=\delta} |f(z)|$ in this case. And if 0 is Fatou exceptional point, then we choose the disk $|z-a| < \delta$ where $a \neq 0$ and $\delta > 0$ is suitabley small and $M = \max_{|z-a|<\delta} |f(z)|$.

Thus there exists a point $t \in U$ and $\xi \in h(U)$ such that $h(t) = \xi$ and $|f^n(\xi)| < 1$. Thus $1 > |f^n(\xi)| = |f^n(h(t))| = |g(f^n(t))| = |g(\eta)|$ where $\eta = f^n(t) \in f^n(U)$. Thus $|g(\eta)| < 1$ and so $|k(g(\eta))| \leq M < M+1$, and so by hypothesis $|p(f(\eta))| < M+1$. Thus $|f(\eta)| \leq P_o$. Thus $|f^{n+1}(t)| \leq P_o$ where $n \geq m_o$. This contradiction shows that $h(\alpha) \in F(f)$. Thus $h(F(f)) \subset F(f)$ and so by Lemma 2.2, $F(f) \subset F(h)$ and consequently $J(h) \subset J(f)$.

As an example, let $f(z) = e_3(z) + 2\pi i$, $g(z) = e_2(z) + 2\pi i$, $h(z) = e_2(z)$, $k(z) = e_1(2z) + 4e_1(z) - 4\pi$ for all $z \in \mathbb{C}$. Then all the conditions of Theorem 2.2 are satisfied and so $J(h) \subset J(f)$ i.e., $J(e_2(z)) \subset J(e_3(z) + 2\pi i)$. Also since it is well known that $J(f) = J(f^p)$ for all $p \geq 1$, and $J(e^z) = \mathbb{C}$ it follows that $J(e^z) = J(e_2(z))$ and consequently $J(e_3(z) + 2\pi i) = \mathbb{C}$.

Our next theorem deals with transcendental entire functions $g, h, k$ which satisfy $g \circ h = h \circ k$ and $k \circ h = h \circ g$. Note that if further $g$ and $h$ permute then so will $k$ and $h$.

**Theorem 2.3.** *Let $g, h, k$ be transcendental entire function such that $k \circ h = h \circ g$ and $g \circ h = h \circ k$. If $\alpha \in F(h)$ and there is a sequence $h^{n_k}$ with $n_k \to \infty$ which has a finite limit $\beta$ in the component of $F(h)$ which contains $\alpha$, then $g(\alpha) \in F(h)$.*

**Proof:** Given $\epsilon > 0$, choose $\delta > 0$ such that $|k(z) - k(\beta)| < \epsilon$, and $|g(z) - g(\beta)| < \epsilon$ for all $|z - \beta| < \delta$. As in Theorem 2.1, let $U$ be a neighbourhood of $\alpha$ such that

$\overline{U} \subset F(h)$ and $|h^{n_k}(z) - \beta| < \delta$, for all $n_k \geq n_o$, and for all $z \in U$. Let $M$ be any compact set which has no Fatou exceptional point of $h$ and such that $M \cap \{(|z - k(\beta)| < \epsilon) \cup (|w - g(\beta)| < \epsilon)\} = \phi$.

We shall prove the theorem by contradiction. Suppose $g(\alpha) \in J(h)$. By Lemma 2.1, for all sufficiently large $n$ $(> n_o)$, $h^n(g(U)) \supset M$.

Let $m \in M$, then there exists $\xi \in g(U)$ and some $n$, $(> n_o)$, such that $h^{n_k+1}(\xi) = m$. We first take the case when $n_k$ is even. We have $h(\eta) = m$ where $\eta = h_k^n(\xi)$. Thus there exists $t \in U$ such that $g(h_k^n(t)) = h_k^n(g(t)) = \eta$. And so $h^n(g(t)) = \eta$.

Also $h^{n_k+1}(U) \subset (|z - \beta| < \delta)$. Hence by continuity of $k$, $|k(h(\mu)) - k(\beta)| < \epsilon$, where $\mu = h_k^n(t)$. And so $|h(g(\mu)) - k(\beta)| < \epsilon$.

But $h(g(\mu)) = h(g(h^{n_k}(t))) = h(h^{n_k}(g(t))) = h^{n_k+1}(g(t)) \in M$.

Thus $|h^{n_k+1}(t) - k(\beta)| < \epsilon$. This contradicts that $M \cap (|z - k(\beta)| < \epsilon) = \phi$.

Our next theorem deals with the image of wandering domain.

**Theorem 2.4.** *Let $g$ and $h$ be transcendental entire functions such that $g \circ h = h \circ g$. Let $W$ be a wandering domain of $h$ with $h^n(W) \to \infty$ as $n \to \infty$. If $h \circ h = f \circ g$ for some entire function $f$, then $g(W) \subset F(h)$.*

**Proof:** As in Theorem 2.2, without any loss of generality, we assume $|z| \leq 1$ does not contain any Fatou exceptional point of $g$. Let $D$ be the unit disk. Let $M = \max_{|z|=1} |f(z)|$.

Let $\alpha \in W$. We choose a small neighbourhood of $\alpha$ say $U$ such that $U \subset W$ and $h^n(U) \to \infty$ as $n \to \infty$. Also by hypothesis, there exits $m_o$ such that $|h^n(w)| > M + 1$ for all $n \geq m_o$ and for all $w \in W$.

Suppose $g(\alpha) \in J(h)$. Now $V := g(U)$ is an open set containing the point $g(\alpha)$ of the Julia set of $h$. Hence if $z_o$ is any point in $D$ then there exists $\xi \in V$ such that $h^n(\xi) = z_o$, where $n$ is taken to be sufficiently large $(\geq m_o)$. Also there exists $t \in U$ such that $g(t) = \xi$. Now $\eta := h^n(t) \in h^n(U)$ and so

$$|h^n(t)| > M + 1,$$
$$|h^{n+1}(t)| > M + 1,$$
$$|h^{n+2}(t)| > M + 1,$$

Thus

$$|h^2(\eta)| > M + 1.$$

On the other hand

$$|f(z_o)| \leq M.$$

But $f(z_o) = f(h^n(\xi)) = f(h^n(g(t)) = f(g(h^n(t))) = f(g(\eta)) = (h^2(\eta)$.

Thus

$$|h^2(\eta)| \leq M.$$

This contradiction proves the theorem.

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# CONVERGENCE OF DEFICIENT CUBIC SPLINE INTERPOLATION

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Abstract. In the present paper we have studied the existence ,uniqueness and convergence properties of a Hermite-Birkhoff interpolation problem by deficient cubic splines for a general interpolatory mean averaging condition and discussed convergence properties for such a spline inpterpolant.

**Keywords and Phrases :** Deficient cubic spline interpolation, mean averaging condition,convergence properties.

**1980 AMS Subject classification code.** Primary 41A05, 65D07.



## 1. INTRODUCTION

The most popular choice for reasonably efficient approximating functions still continues to be in favour of cubic splines (See de Boor [4]) . The convergence properties of periodic deficient cubic splines which interpolate a given function at two interior points of a given mesh interval have been studied by Meir and Sharma [7] . Dikshit and Powar [5] have studied deficient cubic splines having two interpolatory conditions one of which is the interpolation at the mesh points and the other is matching of the integral means. Instead of supplicating the additional degrees of freedom by prescribing values at two intermediate points for deficient cubic splines (or called splines with multiple knots or $C^m$ smooth splines). Rana [8] studied Hermite - Birkhoff type problem with two interpolatory conditions by matching the function and derivative at intermediate points of the dividing intervals and obtained sharp convergence for such a spline interpolant when $f \in C^4$. Kopotun [6] has shown the equivalance of moduli of smoothness and applications of univariate splines.Brakhage and Lamby [2] have shown the application of $B$ spline techniques to the modelling of airplane wings and numerical grid generation. Following Schoenberg [11] and deBoor [3] ,the problem of deficient cubic

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spline has been studied by Rana and Purohit [9] . A study of data visualization using rational spline interpolation has been made by Sarfraz and Hussain [10] .In the present paper, we shall study a problem of Hermite Birkhoff interpolation by deficient cubic splines for a more general interpolatory condition which in particular includes the results proved in [9] and others.

## 2. EXISTENCE AND UNIQUENESS

Let a mesh on $[0, 1]$ be given by

$$\Delta : 0 = x_0 < x_1 < \cdots < x_n = 1$$

with equidistant mesh points $x_i$, so that $p = x_i - x_{i-1} = 1/n$. For a positive integer $m; \pi_m$ denotes the set of all algebraic polynomials of degree not greater than $m$. For a function $s$ defined over $\Delta$ we denote the restriction of $s$ over $[x_{i-1}, x_i]$ by $s_i$. The class of all deficient cubic splines over $\Delta$ is given by

$$S(3, \Delta) = \{s : s \in C^1[0, 1];\ s_i \in \pi_3, i = 1, 2, \ldots, n\}.$$

Considering a periodic non-negative increasing function $g$, with period $p = \frac{1}{n}$, which satisfies the condition :

$$\int_0^p dg = K, (\textit{some positive constant}), \tag{2.1}$$

we set,

$$p^{r+t} A(r, t) = \int_0^p x^r (p - x)^t dg, \quad r, t = 0, 1, 2, 3, 4.$$

and observe that as a consequence of (2.1) we have,

$$p^{r+t} A(r, t) = \int_{x_{i-1}}^{x_i} (x - x_{i-1})^r (x_i - x)^t dg, \quad \text{for } i = 1, 2, \ldots, n. \tag{2.2}$$

Writing $\eta_i = x_{i-1} + \theta p,\ 0 \le \theta \le 1$, we shall prove the following :

**Theorem 2.1.** *Suppose $f'$ exist at $\eta_i$ and let $f$ be 1 - periodic locally integrable function with respect to a positive measure $dg$ satisfying (2.1). Suppose further that*

$$\int_0^p (3\theta p - 2x) dg > 0 \quad \textit{and} \quad \int_0^p (2x + p - 3\theta p) dg > 0$$

*Then there exists a unique* 1 - *periodic spline* $s \in S(3, \Delta)$ *satisfying the following conditions,*

$$\int_{x_{i-1}}^{x_i} (s - f)dg = 0, \ i = 1, 2, \ldots, n \tag{2.3}$$

$$s'(\eta_i) = f'(\eta_i), \ i = 1, 2, \ldots, n \tag{2.4}$$

*when*

$$(3 - \sqrt{3}) < 6\theta < (3 + \sqrt{3}).$$

**Remark 2.1.** *Assuming that* $g$ *has a single jump of* 1 *in* $[0, p]$ *at the point* $\eta_i$ *, the condition (2.3) reduces to matching of* $f$ *and* $s$ *at the points* $\eta_k, k = 1, 2, \ldots, n$*, due to periodicity of* $g$*.*

**Proof of theorem 2.1** Writing $s'(x_i) = m_i, \ i = 0, 1, \ldots, n$ , we now proceed to obtain the representation of $s$ in terms of $m_i's$. Since $s'$ is quadratic in $x_{i-1} \leq x \leq x_i$, we have for $x \in [x_{i-1}, x_i]$ ,

$$p^2 s' = m_i(x - x_{i-1})p + m_{i-1}(x_i - x)p + c_i(x_i - x)(x - x_{i-1}) \tag{2.5}$$

where $c_i's$ are appropriate constants to be determined by interpolating condition (2.4). Thus,

$$f'(\eta_i) = \theta m_i + (1 - \theta)m_{i-1} + \theta(1 - \theta)c_i. \tag{2.6}$$

This determines $c_i's$ and now integrating (2.5), we have

$$\begin{aligned} 6\theta(1 - \theta)p^2 s &= 3\theta(1 - \theta)p[(x - x_{i-1})^2 m_i - (x_i - x)^2 m_{i-1}] \\ &+ [f'(\eta_i) - \theta m_i - (1 - \theta)m_{i-1}][3(x_i - x) + x - x_{i-1}] \\ &(x - x_{i-1})^2 + 6p^2 d_i. \end{aligned} \tag{2.7}$$

We now use interpolatory condition (2.3) and relation (2.2) to get,

$$\begin{aligned} &3\theta(1 - \theta)p[A(2, 0)m_i - A(0, 2)m_{i-1}] + p[f_i'(\eta_i) - \theta m_i - (1 - \theta)m_{i-1}] \\ &[3A(2, 1) + A(3, 0)] + 6d_iK = 6\theta(1 - \theta)\int_{x_{i-1}}^{x_i} fdg = 6\theta(1 - \theta)F_i \ (say). \end{aligned} \tag{2.8}$$

Thus, using the continuity requirement for $s$, we have

$$\begin{aligned} &[3\theta(1 - \theta)pA(0, 2) - (1 - \theta)pK + (1 - \theta)(3A(2, 1) + A(3, 0))p]m_{i-1} \\ &+ [6\theta(1 - \theta)pK - \theta pK - 3\theta(1 - \theta)(A(2, 0) \\ &+ A(0, 2))p + (2\theta - 1)(3A(2, 1) + A(3, 0))p]m_i \\ &+ \theta[3(1 - \theta)A(2, 0) - 3A(2, 1) - A(3, 0)]pm_{i+1} = F_i^*, \end{aligned} \tag{2.9}$$

where $F_i^* = 6\theta(1-\theta)(F_{i+1}-F_i)-(3A(2, 1)+A(3, 0))p(f'(\eta_{i+1})-f'(\eta_i))-pKf'(\eta_i)$.

We now show that system of equation (2.9) has a unique solution under the assumptions of our theorem 2.1. Clearly the coefficients of $m_{i-1}$ and $m_{i+1}$ in (2.9) are non positive. Now the excess of the absolute value of the coefficient of $m_i$ over the sum of absolute value of the coefficients of $m_{i-1}$ and $m_{i+1}$ is given by

$$T(p,\theta) = (6\theta - 6\theta^2 - 1)pK > 0. \tag{2.10}$$

Therefore, the coefficient matrix of the system of equations (2.9) is diagonally dominant and hence invertible.Thus,we can determine all $m_i'$s uniquely which completes the proof of theorem 2.1.

**Remark 2.2.** *The proof of the above theorem is also valid for the following:*

**Theorem 2.2.** *Suppose $f'$ exist at $\eta_i$ and locally integrable function with respect to a positive measure $dg$ satisfying (2.1) and boundary conditions $s'(x_0) = f'(x_0)$ and $s'(x_n) = f'(x_n)$. Suppose further that*

$$\int_0^p (3\theta p - 2x)dg > 0 \quad \textit{and} \quad \int_0^p (2x + p - 3\theta p)dg > 0$$

*Then there exists a unique spline $s \in S(3, \Delta)$ satisfying the above conditions (2.3) and (2.4) when*

$$(3 - \sqrt{3}) < 6\theta < (3 + \sqrt{3}).$$

## 3. ERROR ESTIMATES

In this section, we obtain the bounds of the function $e = s - f$ , where $s$ is the deficient cubic spline interpolant of $f$ in $S(3, \triangle)$ of theorem 2.1 and let $f \in C^3[0, 1]$. In what follows, we shall use the notation $f_i$ for $f(x_i)$, $f_j'$ for $f'(x_j)$ and $w(f, p)$ for modulus of continuity of $f$. In fact, we shall prove the following,

**Theorem 3.1.** *Let $f$ be 1 - periodic function in the class $C^3[0, 1]$ and $s$ be its spline interpolant in $S(3, \triangle)$ of theorem 2.1. Then for $r = 0, 1$, we have*

$$||(s - f)^r|| \le p^{3-r}[5pH(x, p)T(p) + 1/4]w(f''', p), \tag{3.1}$$

*where $H(x, p)$ is some positive function of $x$ and $p$.*

**Proof of theorem 3.1:** Let us write the equation (2.9) as

$$AM = F, \tag{3.2}$$

where $A$ is the coefficient matrix and $M$ and $F$ are single column matrices $(m_i)$ and $(F_i)$ respectively. It may be seen easily [cf.[1, p. 21] ] that the row -max norm

$$||A^{-1}|| \le T^*(p) \tag{3.3}$$

where $T^*(p) = \{min_\theta T(p, \theta)\}^{-1}$ .

Replacing $m_i$ by $e'_i$ in (3.2), it follows that

$$A(e'_i) = (f^*_i) - A(f'_i) = (D_i)\ (say), \tag{3.4}$$

In order to get the derivative of the error function, we estimate $(D_i)$.
Considering $f \in C^3[0,1]$ and using the results that

$$f = f_j + (x - x_j)f'_j + (x - x_j)^2 f''_j/2 + (x - x_j)^3 f'''(\alpha_j)/6$$

and

$$f' = f'_j + (x - x_j)f''_j + (x - x_j)^2 f'''(\beta_j)/2$$

where $\alpha_j$ and $\beta_j$ lie in appropriate intervals, we get the following,

$$\begin{aligned} 2(D_i)/p^3 = {} & 2\theta(1-\theta)A(3,0)f'''(z_{i+1}) + 2\theta(1-\theta)A(0,3)f'''(z_i) \\ & + [(1-\theta)^2(3A(2,1) + A(3,0)) - K(1-\theta)^2]f'''(\theta_i) \\ & - \theta^2[3A(2,1) + A(3,0)]f'''(\theta_{i+1}) \\ & - \theta[3(1-\theta)A(2,0) - 3A(2,1) - A(3,0)]f'''(\delta_{i+1}) \\ & - (1-\theta)[3A(2,1) + A(3,0) + 3\theta A(0,2) - K]f'''(\delta_{i+1}), \end{aligned} \tag{3.5}$$

where $k_i \in [x_{i-1}, x_i]$, for $k = z, \theta, \delta$. Rearranging the terms suitably, we get

$$\begin{aligned} 2(D_i)/p^3 = {} & \theta^2[3A(2,1) + A(3,0)][f'''(\theta_i) - f'''(\theta_{i+1})] \\ & + (1-\theta)[3A(2,1) + A(3,0)][f'''(\theta_i) - f'''(\delta_{i-1})] \\ & + \theta[3A(2,1) + A(3,0)][f'''(\delta_{i+1}) - f'''(\theta_i)] \\ & + 2\theta(1-\theta)A(3,0)[f'''(z_{i+1}) - f'''(z_i)] \\ & + 3\theta(1-\theta)A(2,0)[f'''(z_i) - f'''(\delta_{i+1}) + (f'''(z_i) - f'''(\delta_{i-1}))] \\ & + 6\theta(1-\theta)A(1,0)[f'''(\delta_{i-1}) - f'''(z_i)] \\ & + 2\theta(1-\theta)K[f'''(z_i) - f'''(\delta_{i-1})] \\ & + (1-\theta)^2 K[f'''(\theta_i) - f'''(\delta_{i-1})]; \end{aligned} \tag{3.6}$$

Thus, the right hand side $(D_i)$ of equation (3.4) is estimated as

$$|D_i| \le H(x,p)p^3 w(f''', p), \tag{3.7}$$

where

$$H(x,p) = [24A(2,1) + 8A(3,0) + 12A(2,1) + 12A(1,0) + 3K]/4$$

Now using (3.7) and (3.3) in (3.4), we have

$$||(e'_i)|| \le H(x,p)T^*(p)p^3 w(f''', p) \tag{3.8}$$

where $T^*(p)$ is given by (3.3). We now substitute the value of $c_i$ from (2.6) in (2.5) and replace $s'$ by $e', m_i$ by $e'_i$ in (2.5) and arrange the terms suitably after applying the Taylors expansion to get

$$||e'|| \leq 5||e'_i|| + (p^2/4)w(f''', p), \tag{3.9}$$

Now, we have the following when we appeal to (3.8) in (3.9),

$$||e'|| \leq p^2[5pH(x,p)T^*(p) + (1/4)]w(f''', p) \tag{3.10}$$

which proves theorem 3.1 for $r = 1$.The other inequality of theorem 3.1 follows by the standard reasoning used elsewheve.

**Remark 3.1.** *Theorem 3.1 can be proved in the case when $f \in C^1[0,1]$ or $C^2[0,1]$ by the same reasoning but in that case the rate of convergence is comparatively less than that of theorem 3.1.Moreover,one could carry out a proof of Theorem 2.1 for a interpolatory condition (2.4) replaced by mean averaging condition.*

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# SOME EXPONENTIAL DIOPHANTINE EQUATIONS

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Abstract. This paper gives the complete solution in rational numbers of the hitherto unsolved diophantine equation $x^x = y^y$. It also gives integer solutions of certain other exponential diophantine equations such as $x_1^{x_1} x_2^{x_2} = y_1^{y_1} y_2^{y_2}$ and $x_1^{x_1} x_2^{x_2} x_3^{x_3} = y_1^{y_1} y_2^{y_2} y_3^{y_3}$ for which no solutions have been published till now.



## 1. INTRODUCTION

The diophantine equations

(1.1) $$x^x y^y = z^z$$

and

(1.2) $$\prod_{i=1}^{m} x_i^{x_i} = y^y$$

were considered by Chao Ko who gave parametric solutions of these equations (as stated in [1, p. 266] and [2, p. 111]). Another related exponential equation is

(1.3) $$x^y = y^x$$

for which the complete solution in rational numbers is given in [2, p. 109]. Till now, however, the diophantine equation

(1.4) $$x^x = y^y$$

has not been considered. Similarly, no solutions seem to have been published for diophantine equations of the type

(1.5) $$\prod_{i=1}^{m} x_i^{x_i} = \prod_{j=1}^{n} y_j^{y_j}, \quad m \geq 2, \quad n \geq 2.$$

In this paper we first give the complete solution in rational numbers of the equation (1.4). As an example, a numerical solution of (1.4) is given by

$$x = 1/2, \quad y = 1/4.$$

Next we obtain parametric solutions of equation (1.5) for various values of $m \geq 2$ and $n \geq 2$. As a numerical example, a solution of the equation

$$x_1^{x_1} x_2^{x_2} = y_1^{y_1} y_2^{y_2} \tag{1.6}$$

is given by

$$x_1 = 2^{19}3^{12}, \quad x_2 = 2^{15}3^{14}, \quad y_1 = 2^{18}3^{13}, \quad y_2 = 2^{14}3^{12} \tag{1.7}$$

while a solution of the equation

$$x_1^{x_1} x_2^{x_2} x_3^{x_3} = y_1^{y_1} y_2^{y_2} y_3^{y_3} \tag{1.8}$$

is given by

$$\begin{array}{rclrclrcl} x_1 &=& 2^{69}3^{48}, & x_2 &=& 2^{65}3^{50}, & x_3 &=& 2^{63}3^{48}, \\ y_1 &=& 2^{68}3^{49}, & y_2 &=& 2^{62}3^{48}, & y_3 &=& 2^{65}3^{48}. \end{array} \tag{1.9}$$

## 2. THE EXPONENTIAL EQUATION $x^x = y^y$

We first note that if $x$ and $y$ are assumed to be positive integers, and $x > y$, then $x^x > y^y$. Similarly, if $y > x$ then $y^y > x^x$. Thus equation (1.4) has no non-trivial solutions in positive integers. To solve equation (1.4) in positive rational numbers, we assume without loss of generality that $y < x$, and we substitute $y = hx$ where $0 < h < 1$. Now (1.4) gives

$$x^x = (hx)^{hx} = ((hx)^h)^x$$

and hence,

$$\begin{array}{rrcl} & x &=& (hx)^h \\ \text{or,} & x &=& h^{h/(1-h)}. \end{array}$$

We write $h/(1-h) = r$, so that $h = r/(1+r)$, and

$$x = \left(\frac{r}{r+1}\right)^r \tag{2.1}$$

where $r > 0$ since $h < 1$. We will now show that for $x$ to be a rational number, $r$ must be an integer. If $r = p/q$ is a rational number with $p$, $q$ being positive coprime integers, then

$$x = \left(\frac{p}{p+q}\right)^{p/q}. \tag{2.2}$$

Since $\gcd(p,\, q) = 1$, we must also have $\gcd(p,\, p+q) = 1$, and it now follows from (2.2) that for $x$ to be rational, both $p$ and $p+q$ must be perfect $q^{\text{th}}$ powers. Thus, there must exist positive integers $s$ and $t$ such that $p + q = s^q$ and $p = t^q$. Thus, $q = s^q - t^q$. Since $q > 0$, we must have $s > t$, and hence $s \geq t + 1$. If $q > 1$, it follows that $q \geq (t+1)^q - t^q$, or, $q \geq qt^{q-1} + \cdots + 1$, which is a contradiction. Thus, we must have $q = 1$, and it follows that all solutions of (1.4) with $x > y$ are given by

$$x = \left(\frac{r}{r+1}\right)^r, \quad y = \left(\frac{r}{r+1}\right)^{r+1} \tag{2.3}$$

where $r$ is a positive integer. As numerical examples, when $r = 1$, we get the solution $x = 1/2,\ y = 1/4$, and when $r = 2$, we get the solution $x = 4/9,\ y = 8/27$.

## 3. THE EXPONENTIAL EQUATION $\prod_{i=1}^m x_i^{x_i} = \prod_{j=1}^n y_j^{y_j}$.

**3.1** We first prove some general results concerning this equation. We begin by proving a preliminary lemma which we shall use repeatedly to obtain integer solutions of exponential equations of this type.

**Lemma 1:** If $p_1,\ q_1,\ q_2$ are integers defined by

$$\begin{aligned} p_1 &= 2^{2r+s}, \\ p_2 &= 2^s(2^r-1)^2, \\ q_1 &= 2^{r+s+1}(2^r-1) \end{aligned} \tag{3.1}$$

where $r$ and $s$ are arbitrary nonnegative integers such that $s < 2^{r+1}(2^r - r - 1)$, then $p_1 + p_2 - q_1 = 2^s > 0$ while

$$\frac{q_1^{q_1}}{p_1^{p_1} p_2^{p_2}} = 2^{2^s\{2^{r+1}(2^r-r-1)-s\}}(2^r-1)^{2^{s+1}(2^r-1)} \tag{3.2}$$

is an integer.

**Proof:** Straightforward computation proves the lemma.

To obtain integer solutions of the exponential equation (1.5), we assume that $g$ is the gcd of the integers $x_i,\ y_j,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$, and so we may write

$$\begin{aligned} x_i &= p_i g, \quad i = 1, 2, \ldots, m, \\ y_j &= q_j g, \quad j = 1, 2, \ldots, n \end{aligned} \tag{3.3}$$

where $p_i, i = 1, 2, \ldots, m,\ q_j,\ j = 1, 2, \ldots, n$, are integers. On substituting the above values of $x_i,\ y_j$ in (1.5), we get

$$\left\{ g^{\sum_{i=1}^m p_i} \prod_{i=1}^m p_i^{p_i} \right\}^g = \left\{ g^{\sum_{j=1}^n q_j} \prod_{j=1}^n q_j^{q_j} \right\}^g$$

$$\text{or,} \quad g^{\sum_{i=1}^m p_i - \sum_{j=1}^n q_j} \prod_{i=1}^m p_i^{p_i} = \prod_{j=1}^n q_j^{q_j}. \tag{3.4}$$

If the integers $p_i, i = 1, 2, \ldots, m,\ q_j,\ j = 1, 2, \ldots, n$, satisfy the relation

$$\sum_{i=1}^{m} p_i - \sum_{j=1}^{n} q_j = 1 \tag{3.5}$$

and, in addition, $\prod_{j=1}^{n} q_j^{q_j} / \prod_{i=1}^{m} p_i^{p_i}$ is an integer, then it follows from (3.4) that a solution in integers of the exponential equation (1.5) is given by (3.3) where

$$g = \frac{\prod_{j=1}^{n} q_j^{q_j}}{\prod_{i=1}^{m} p_i^{p_i}}. \tag{3.6}$$

Thus, to obtain integer solutions of any exponential equation of type (1.5), it suffices to find integers $p_i, i = 1, 2, \ldots, m,\ q_j,\ j = 1, 2, \ldots, n$, satisfying the relation (3.5) and such that $\prod_{j=1}^{n} q_j^{q_j} / \prod_{i=1}^{m} p_i^{p_i}$ is an integer.

**3.2** We will now obtain integer solutions of the exponential equation

$$x_1^{x_1} x_2^{x_2} = y_1^{y_1} y_2^{y_2}. \tag{3.7}$$

We choose integers $p_1,\ p_2,\ q_1$ as defined by (3.1) and take $q_2 = 2^s - 1$ when condition (3.5) is satisfied and, in view of Lemma 1, it readily follows that $q_1^{q_1} q_2^{q_2} / (p_1^{p_1} p_2^{p_2})$ is an integer. Hence, as indicated above, we can readily solve equation (3.7) to obtain a parametric solution which is given by

$$\begin{aligned} x_1 &= 2^{2r+s} g, \\ x_2 &= 2^s (2^r - 1)^2 g, \\ y_1 &= 2^{r+s+1} (2^r - 1) g, \\ y_2 &= (2^s - 1) g \end{aligned} \tag{3.8}$$

where

$$g = 2^{2^s \{2^{r+1}(2^r - r - 1) - s\}} (2^r - 1)^{2^{s+1}(2^r - 1)} (2^s - 1)^{2^s - 1}. \tag{3.9}$$

In this solution, $r$ and $s$ are arbitrary positive integers such that $s < 2^{r+1}(2^r - r - 1)$. As a numerical example, when $r = 2,\ s = 1$, we get the solution (1.7) of equation (3.7).

**3.3** Next we will obtain integer solutions of the exponential equation

$$x_1^{x_1} x_2^{x_2} x_3^{x_3} = y_1^{y_1} y_2^{y_2} y_3^{y_3}. \tag{3.10}$$

As in the previous section, we choose integers $p_1,\ p_2,\ q_1$ as defined by (3.1) with $s < 2^{r+1}(2^r - r - 1)$, and take $p_3 = 2^t,\ q_2 = q,\ q_3 = 2^s + 2^t - q - 1$ where $q,\ t$ are arbitrary positive integers such that $q < 2^s + 2^t - 1$ and $t < \min\{s,\ 2^{r+1}(2^r - r - 1) - s\}$. Then $p_1 + p_2 + p_3 - q_1 - q_2 - q_3 = 1$, and following the general procedure outlined in subsection **3.1**, we find that a parametric solution of (3.10) is given by

$$\begin{aligned} x_1 &= 2^{2r+s}g, \\ x_2 &= 2^s(2^r-1)^2 g, \\ x_3 &= 2^t g, \\ y_1 &= 2^{r+s+1}(2^r-1)g, \\ y_2 &= qg, \\ y_3 &= (2^s+2^t-q-1)g \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} g = {} & 2^{2^s\{2^{r+1}(2^r-r-1)-s\}-t2^t}(2^r-1)^{2^{s+1}(2^r-1)} \\ & q^q(2^s+2^t-q-1)^{(2^s+2^t-q-1)}. \end{aligned} \tag{3.12}$$

As a numerical example, when $q=1, r=2, s=3, t=1$, we get the solution (1.9) of equation (3.10).

**3.4** We can similarly obtain solutions of equation (1.5) for various other values of $m$ and $n$. For instance, to obtain a solution of the equation $x_1^{x_1}x_2^{x_2}=y_1^{y_1}$, we may take $p_1$, $p_2$, $q_1$ as defined by (3.1) where we take $s=0$ so that we have $p_1+p_2-q_1=1$. This leads to the solutions of this equation given by Chao Ko and mentioned in [2, p. 111].

As a final example, we obtain a parametric solution of the equation

$$x_1^{x_1}x_2^{x_2}=\prod_{j=1}^{n} y_i^{y_i}, \quad n \geq 3. \tag{3.13}$$

We choose $p_1$, $p_2$, $q_1$ as defined by (3.1), and take $q_2, q_3, \ldots, q_n$ to be arbitrary integers such that $\sum_{j=2}^{n} q_j = 2^s-1$, when condition (3.5) is satisfied while, in view of Lemma 1, $\prod_{j=1}^{n} q_j^{q_j}/(p_1^{p_1}p_2^{p_2})$ is an integer. We thus get a parametric solution of (3.13) which may be written as

$$\begin{gathered} x_1=2^{2r+s}g, \quad x_2=2^s(2^r-1)^2g, \\ y_1=2^{r+s+1}(2^r-1)g, \quad y_j=q_jg,\ j=1,\,2,\,\ldots,\,n \end{gathered} \tag{3.14}$$

where

$$g=2^{2^s\{2^{r+1}(2^r-r-1)-s\}}(2^r-1)^{2^{s+1}(2^r-1)}\prod_{j=2}^{n} q_j^{q_j}. \tag{3.15}$$

In this solution, $r$ and $s$ are arbitrary integers such that $s<2^{r+1}(2^r-r-1)$, while $q_j,\ j=2,\,3,\,\ldots,\,n$ are arbitrary integers such that $\sum_{j=2}^{n} q_j=2^s-1$. As a numerical example, taking $r=2$, $s=3$, $q_2=1$, $q_3=2$, $q_4=4$, we get a solution of the equation

$$x_1^{x_1}x_2^{x_2}=y_1^{y_1}y_2^{y_2}y_3^{y_3}y_4^{y_4}, \tag{3.16}$$

which is as follows:

$$
\begin{aligned}
x_1 &= 2^{57}3^{48}, & x_2 &= 2^{53}3^{50}, & y_1 &= 2^{56}3^{49}, \\
y_2 &= 2^{50}3^{48}, & y_3 &= 2^{51}3^{48}, & y_4 &= 2^{52}3^{48}.
\end{aligned} \tag{3.17}
$$

Finally, we may mention that while we have obtained parametric solutions of equations of type (1.5) for various values of $m$ and $n$, we have not obtained the complete solution in integers of these equations. It would be interesting to find new integer solutions of equation (1.1) different from the solutions published by Chao Ko.

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# METHOD OF INFINITE ASCENT APPLIED ON $\mathbf{A^4 \pm nB^2 = C^3}$

**SUSIL KUMAR JENA***

(Received : 25-11-2009)

Abstract. In the Diophantine equations $A^4 \pm nB^2 = C^3$, it is always possible to generate infinite number of integral solutions for $(A, B, C)$ for any positive integral value of $n$. In this paper, we will show how the Method of Infinite Ascent helps us to generate these solutions. Again, we will see the conditions when $A, B$ and $C$ would be pair-wise co-prime.

**2000 Mathematics Subject Classification:** 11D41.



## 1. INTRODUCTION

A technique called the Method of Infinite Ascent (MIA) can be used in getting many peculiar but beautiful results in number theory. We have already presented a paper [13] in an International Mathematics Conference in Zacatecas, Mexico during May 23-26, 2007 with the title 'Method of Infinite Ascent applied on $A^6 \pm nB^3 = C^2$ highlighting the use of this method. A more generalized paper [14] appears in The Mathematics Student which uses MIA. There are many Diophantine equations which can be expressed in certain forms so that they will generate infinite number of integral solutions for their parameters. Hence, the Method of Infinite Ascent is an insight to look beyond the limits of finiteness.To appreciate its use let us examine the following Diophantine equations.

$$A^4 \pm nB^2 = C^3 \tag{1.1}$$

## 2. ON EXAMINING THE EQUATIONS $A^4 \pm nB^2 = C^3$

The results of Theorem 2.1 and Theorem 2.2 suggest that each of the two Diophantine equations of (1.1) has infinite number of co-prime integral solutions for $(A, B, C)$ for any positive integer $n$. Before stating these theorems let us state

* Dedicated to the immortal soul of Professor Venkatachaliengar.



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and prove the following Lemmas which would be helpful in establishing these theorems.

**Lemma 2.1.** *For any two positive integers p and q,*

$$(p+q)^3 = p(p-3q)^2 + q(3p-q)^2 \tag{2.1}$$

**Proof :**

$$\begin{aligned}\text{R.H.S. of (2.1)} &= p(p-3q)^2 + q(3p-q)^2\\ &= p(p^2-6pq+9q^2) + q(9p^2-6pq+q^2)\\ &= p^3-6p^2q+9pq^2+9p^2q-6pq^2+q^3\\ &= p^3+3p^2q+3pq^2+q^3\\ &= (p+q)^3\\ &= \text{L.H.S. of (2.1) (Proved)}\end{aligned}$$

**Lemma 2.2.** *For any two positive integers p and q,*

$$(p-q)^3 = p(p+3q)^2 - q(3p+q)^2 \tag{2.2}$$

**Proof :**

$$\begin{aligned}\text{R.H.S. of (2.2)} &= p(p+3q)^2 - q(3p+q)^2\\ &= p(p^2+6pq+9q^2) - q(9p^2+6pq+q^2)\\ &= p^3+6p^2q+9pq^2-9p^2q-6pq^2-q^3\\ &= p^3-3p^2q+3pq^2-q^3\\ &= (p-q)^3\\ &= \text{L.H.S. of (2.2) (Proved)}\end{aligned}$$

**Lemma 2.3.** *For any two positive integers m and n,*

$$(m^2-3n)^4 + 3.16m^2n(m^2+3n)^2 = (m^4+18m^2n+9n^2)^2 \tag{2.3}$$

**Proof :**

$$\begin{aligned}\text{L.H.S. of (2.3)} &= (m^2-3n)^4 + 3.16m^2n(m^2+3n)^2\\ &= (m^2-3n)^4 + 48m^2n(m^4+6m^2n+9n^2)\\ &= m^8-12m^6n+54m^4n^2-108m^2n^3+81n^4+48m^6n+288m^4n^2\\ &\quad +432m^2n^3\\ &= m^8+36m^6n+342m^4n^2+324m^2n^3+81n^4\\ &= m^8+324m^4n^2+81n^4+36m^6n+324m^2n^3+18m^4n^2\end{aligned}$$

$$
\begin{aligned}
&= (m^4)^2 + (18m^2n)^2 + (9n^2)^2 + 2(m^4)(18m^2n) + 2(18m^2n)(9n^2) \\
&\quad + 2(m^4)(9n^2) \\
&= (m^4 + 18m^2n + 9n^2)^2 \\
&= \text{R.H.S. of (2.3) (Proved)}
\end{aligned}
$$

**Lemma 2.4.** *For any two positive integers $m$ and $n$,*

$$(m^2 + 3n)^4 - 3.16m^2n(m^2 - 3n)^2 = (m^4 - 18m^2n + 9n^2)^2 \tag{2.4}$$

**Proof :**

$$
\begin{aligned}
\text{L.H.S. of (2.4)} &= (m^2 + 3n)^4 - 3.16m^2n(m^2 - 3n)^2 \\
&= (m^2 + 3n)^4 - 48m^2n(m^4 - 6m^2n + 9n^2) \\
&= m^8 + 12m^6n + 54m^4n^2 + 108m^2n^3 + 81n^4 - 48m^6n \\
&\quad + 288m^4n^2 - 432m^2n^3 \\
&= m^8 - 36m^6n + 342m^4n^2 - 324m^2n^3 + 81n^4 \\
&= m^8 + 324m^4n^2 + 81n^4 - 36m^6n - 324m^2n^3 + 18m^4n^2 \\
&= (m^4)^2 + (-18m^2n)^2 + (9n^2)^2 + 2(m^4)(-18m^2n) \\
&\quad + 2(-18m^2n)(9n^2) + 2(m^4)(9n^2) \\
&= (m^4 - 18m^2n + 9n^2)^2 \\
&= \text{R.H.S. of (2.4) (Proved)}
\end{aligned}
$$

With the results of these four Lemmas, we can state and prove the following two theorems.

**Theorem 2.1.** *For any positive integer $n$, the Diophantine equation: $A^4 + nB^2 = C^3$ has infinitely many co-prime integral solutions for $A, B$ and $C$ such that*

$$
\begin{aligned}
(A, B, C) = [&\{(m^2 + 3n)(m^4 - 18m^2n + 9n^2)\}, 4m(m^2 - 3n)\left\{3(m^2 + 3n)^4\right. \\
&\left.-16m^2n(m^2 - 3n)^2\right\}, \{(m^2 + 3n)^4 + 16m^2n(m^2 - 3n)^2\}],
\end{aligned}
$$

*where $m$ takes any integral value not divisible by 3 so that $m$ and $n$ are co-prime.*

**Proof :** The statement of Theorem 2.1 tells that if

$$A^4 + nB^2 = C^3 \tag{2.5}$$

then,

$$
\begin{aligned}
&\{(m^2 + 3n)(m^4 - 18m^2n + 9n^2)\}^4 \\
&+ n\left[4m(m^2 - 3n)\left\{3(m^2 + 3n)^4\right.\right. \\
&\left.\left.-16m^2n(m^2 - 3n)^2\right\}\right]^2 \\
&= \{(m^2 + 3n)^4 + 16m^2n(m^2 - 3n)^2\}^3
\end{aligned} \tag{2.6}
$$

From (2.5) and (2.6) we get,

$$C^3 = \{(m^2+3n)^4 + 16m^2n(m^2-3n)^2\}^3 \tag{2.7}$$

In (2.7), take

$$p = (m^2+3n)^4 \quad \text{and} \quad q = 16m^2n(m^2-3n)^2 \tag{2.8}$$

Hence, from (2.7), (2.8), (2.1) and Lemma 2.4 we get,

$$\begin{aligned}
C^3 &= \{(m^2+3n)^4 + 16m^2n(m^2-3n)^2\}^3 \\
&= (m^2+3n)^4\{(m^2+3n)^4 - 3.16m^2n(m^2-3n)^2\}^2 \\
&\quad + 16m^2n(m^2-3n)^2\{3(m^2+3n)^4 - 16m^2n(m^2-3n)^2\}^2 \\
&= (m^2+3n)^4\{(m^4-18m^2n+9n^2)^2\}^2 + 16m^2n(m^2-3n)^2\{3(m^2+3n)^4 \\
&\quad - 16m^2n(m^2-3n)^2\}^2 \\
&= \{(m^2+3n)(m^4-18m^2n+9n^2)\}^4 + n(4m)^2(m^2-3n)^2\{3(m^2+3n)^4 \\
&\quad - 16m^2n(m^2-3n)^2\}^2 \\
&= \{(m^2+3n)(m^4-18m^2n+9n^2)\}^4 \\
&\quad + n\cdot[4m(m^2-3n)\{3(m^2+3n)^4 - 16m^2n(m^2-3n)^2\}]^2 \\
&= A^4 + nB^2 \quad \text{[From (2.5) and (2.6)]}
\end{aligned}$$

Hence, $A^4 + nB^2 = C^3$. (Proved).

**Theorem 2.2.** *For any positive integer $n$, the Diophantine equation:$A^4 - nB^2 = C^3$ has infinitely many co-prime integral solutions for $A, B$ and $C$ such that*

$$\begin{aligned}
(A,B,C) = [&\{(m^2-3n)(m^4+18m^2n+9n^2)\}, 4m(m^2+3n)\left\{3(m^2-3n)^4\right. \\
&\left.+16m^2n(m^2+3n)^2\right\}, \{(m^2-3n)^4 - 16m^2n(m^2+3n)^2\}],
\end{aligned}$$

*where $m$ takes any integral value not divisible by 3 so that $m$ and $n$ are co-prime.*

**Proof :** The statement of Theorem 2.2 tells that if

$$A^4 - nB^2 = C^3 \tag{2.9}$$

then,

$$\begin{aligned}
&\{(m^2-3n)(m^4+18m^2n+9n^2)\}^4 \\
&- n[4m(m^2+3n)\{3(m^2-3n)^4 + 16m^2n(m^2+3n)^2\}]^2 \\
&= \{(m^2-3n)^4 - 16m^2n(m^2+3n)^2\}^3
\end{aligned} \tag{2.10}$$

From (2.9) and (2.10) we get,

$$C^3 = \{(m^2-3n)^4 - 16m^2n(m^2+3n)^2\}^3 \tag{2.11}$$

In (2.11), we take

$$p = (m^2 - 3n)^4 \text{ and } q = 16m^2n(m^2 + 3n)^2 \tag{2.12}$$

Hence, from (2.11), (2.12), (2.2) and Lemma 2.3 we get,

$$\begin{aligned}
C^3 &= \{(m^2 - 3n)^4 - 16m^2n(m^2 + 3n)^2\}^3 \\
&= (m^2 - 3n)^4\{(m^2 - 3n)^4 + 3.16m^2n(m^2 + 3n)^2\}^2 \\
&\quad - 16m^2n(m^2 + 3n)^2\{3(m^2 - 3n)^4 + 16m^2n(m^2 + 3n)^2\}^2 \\
&= (m^2 - 3n)^4\{(m^4 + 18m^2n + 9n^2)^2\}^2 \\
&\quad - n(4m)^2(m^2 + 3n)^2\{3(m^2 - 3n)^4 + 16m^2n(m^2 + 3n)^2\}^2 \\
&= \{(m^2 - 3n)(m^4 + 18m^2n + 9n^2)\}^4 \\
&\quad - n(4m)^2(m^2 + 3n)^2\{3(m^2 - 3n)^4 + 16m^2n(m^2 + 3n)^2\}^2 \\
&= \{(m^2 - 3n)(m^4 + 18m^2n + 9n^2)\}^4 \\
&\quad - n.[4m(m^2 + 3n)\{3(m^2 - 3n)^4 + 16m^2n(m^2 + 3n)^2\}]^2 \\
&= A^4 - nB^2 \text{ [From (2.9) and (2.10)]}
\end{aligned}$$

Hence, $A^4 - nB^2 = C^3$. (Proved).

## 3. COMMENT

With Theorem 2.1 and Theorem 2.2 we see that the Diophantine equations: $A^4 \pm nB^2 = C^3$ would have infinitely many non-zero integral solutions for $(A, B, C)$ for any positive integer $n$. Again, $A, B$ and $C$ will be pair-wise co-prime if we choose those integral values of $m$ not divisible by 3, such that $m$ and $n$ are also co-prime.

In 1913, Ramanujan [11] asked if the Diophantine equation $2^n - 7 = x^2$, sometimes called the Ramanujan-Nagell equation has any solution for $(n, x)$ other than (3,1), (4, 3), (5, 5), (7, 11) and (15, 181). Nagell [1] and Skolem et al. [2] showed there are no solutions past $2^{15}$, thus settled Ramanujan's question in negative. Relating to this equation there are other papers available in the literature due to Nagell [3], Mignotte [4], Johnson [5], Bundschuh [6], Cohen [7] and Ramasamy [8]. People like Le [9], Arif and Muriefah [10], and Luca [12] have investigated on the solution of the Diophantine equation

$$x^2 + C = y^n \tag{3.1}$$

with $x \geq 1, y \geq 1, n \geq 3$ and $C$ is the power of one or two fixed primes. The results of Theorem 2.1 and Theorem 2.2 add to our understanding of a similar but essentially different Diophantine equation.

**ACKNOWLEDGEMENT** : I am very much thankful to the heavenly spirit of *Martin Gardner* for igniting in me the passion for numbers through his regular articles in the *Mathematical Games* column of the *Scientific American.* The first conviction that I could be of some help to this field of Number Theory came from the *American Mathematical Society.* Probably, this would be the right time and place to document my candid gratitude to them. My sincere thanks would also be for the anonymous referee for reviewing this paper.

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# PLATINUM JUBILEE $75^{TH}$ IMS CONFERENCE : A BRIEF REPORT

S. ARUMUGAM

The Platinum Jubilee $75^{th}$ Annual Conference of the Indian Mathematical Society was held at the Kalasalingam University, Anand Nagar, Krishnankoil 626 190, (Via) Srivilliputtur, Dist. Virudunagar, Tamilnadu during the period December 27 - 30, 2009 under the Presidentship of Prof. Peeyush Chandra, Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208 016. The Conference was attended by more than 300 delegates from all over the country.

Prof. K. Thulasiraman, Professor and Hitachi Chair, Department of Computer Science, University of Oklahoma, USA inaugurated the conference. The Inaugural function was held in the forenoon of December 27, 2009 in the TIFAC-CORE Seminar Hall at Kalasalingam University. Prof. Peeyush Chandra, President of the Indian Mathematical Society presided over the inaugural function.

Dr. C. Thangaraj, Vice-Chancellor, Kalasalingam University delivered the welcome address. The General Secretary of the Society Prof. V. M. Shah spoke about the Indian Mathematical Society and expressed his sincere and profuse thanks, on behalf of the Society, to the host for organizing the Conference. The Academic Secretary Prof. Satya Deo reported about the academic programmes of the Conference. Prof. V. M. Shah reported about Prof. P. L. Bhatnagar Memorial Prize for 2009. The Prize was awarded to the top scorers of the Indian Team at the International Mathematics Olympiad Mr. Sameer N. Wagh, Pune. Prof. Peeyush Chandra delivered his Presidential address (General) on MATHEMATICS EDUCATION IN INDIA : SOME CONCERNS & OBSERVATIONS. The function ended with a vote of thanks proposed by Prof. S. Arumugam, Director, $n$-CARDMATH, Kalasalingam University and the Local Secretary of the Conference.

The Academic Sessions of the conference began with the technical address by the President of the Society on "MATHEMATICAL MODELLING OF HIV DYNAMICS : IN VIVO in the TIFAC-CORE Seminar Hall, which was presided by Prof. V. M. Shah, the seniormost past president of the Society present in the Conference. The rest of the academic programmes were held in the TIFAC-CORE

Seminar Hall, TIFAC-CORE Senate Hall and n-CARDMATH Conference Hall.

In the afternoon of the first day of the conference the Department of Posts (Philately Division), Government of India, released a **Postal Stamp** in commemoration of the completion of 100 years of the Indian Mathematical Society. The stamp was released by Shri. K. Ramachandiran N, Postmaster General, Southern Region, Madurai and was received by "Kalvivallal" Thiru T. Kalasalingam, Chancellor, Kalasalingam University. This is a historic event in the annals of the Indian Mathematical Society.

One plenary Lecture was delivered during the Conference - by Prof. R. Balasubramaniam, (Director, IMSc, Chennai). Another one by Prof. M. S. Narasimhan (TIFR, Mumbai) could not be delivered due to unavoidable circumstances. Also, Four Award Lectures and Eight invited Lectures were delivered by eminent mathematicians during the conference. Besides these, there were Six symposia on various topics of mathematics and computer science. There was a **Paper Presentation Competition** for various IMS and other prizes. In all about 120 papers were presented during the conference on various disciplines of mathematics and computer science. During the valedictory function, the IMS and other prizes were awarded by the president of the Society to the winners of the Paper Presentation Competition. It was also announced that the next annual conference of the Society will be held at the Sardar Vallabhbhai National Institute of Technology (SVNIT), SURAT- 395 007, (Gujarat State).

Apart from academic sessions, a Cultural Evening was also arranged on December 29, 2009.

The conference was supported by the DST, NBHM, CSIR, DRDO and Kalasalingam University, Krishnankoil.

**PLENARY LECTURES:**

The details of the Plenary Lectures delivered is as follows:

1. **R. Balasubramaniam** (Director, Institute of Mathematical Sciences, Chennai) delivered a Plenary Lecture on *"Additive Combinatorics"*.
   Chairperson: Satya Deo.

**MEMORIAL AWARD LECTURES:**

The details of the Memorial Award Lectures delivered are as follows:

1. THE TWENTYTHIRD P. L. BHATNAGAR MEMORIAL AWARD LECTUR : The $23^{rd}$ P. L. Bhatnagar Memorial Award Lecture was delivered by **Girija Jayaraman** (IIT Delhi, Delhi). She spoke on *"Mathematical Challenges in Modeling the Cardiovascular Flows and Associated Clinical Procedures"*.
   Chairperson: Peeyush Chandra.

2. THE TWENTIETH V. RAMASWAMY AIYER MEMORIAL AWARD LECTURE :

   The $20^{th}$ V. Ramaswamy Aiyer Memorial Award Lecture was delivered by **M. S. Chaudhary,** Shivaji University, Kolhapur. He spoke on *" Transform Analysis on some distribution spaces".*
   Chairperson: N.K. Thakare.
3. THE TWENTIETH SRINIVASA RAMANUJAN MEMORIAL AWARD LECTURE :

   The $20^{th}$ Srinivasa Ramanujan Memorial Award Lecture was delivered by **Ravindra Kulkarni** (IIT Bombay, Powai, Mumbai). He spoke on *"Spaces of Geometric Objects and Parametrization Problems".*
   Chairperson: A .K. Agarwal.
4. THE TWENTIETH HANSA RAJ GUPTA MEMORIAL AWARD LECTURE:

   The $20^{th}$ Hansraj Gupta Memorial Award Lecture was delivered by **A. M. Mathai** (Director, Centre for Mathematical Sciences, Pala). He spoke on *"Random Geometrical Configurations and Random Volumes".*
   Chairperson: S. Arumugam

**SPECIAL SESSION OF PAPER PRESENTATION COMPETITION FOR VARIOUS PRIZES.**

During the special session of Paper Presentation Competition, research papers were presented for the award of Six IMS Prizes (areas : Algebra, Geometry, Topology, Functional Analysis, Discrete Mathematics, Number Theory, Operations Research, Fluid Dynamics, Biomathematics and Computer science), AMU Prize (areas: Algebra, Functional Analysis, Differential geometry) and V. M. Shah Prize (in Analysis). Satya Deo (Chairperson) (N. K. Thakare was chair in lieu of Satya Deo), A. K. Agarwal, Geetha Rao, J. R. Patadia and B. N. Waphare were the judges.

01. IMS Prize (Descrete Mathematics) : **S. Francis Raj** (Bharathidashan University, Tiruchirapalli).
02. IMS Prize (Number Theory) : **Roselin** (Allahabad Agriculture Institute - Deemed University, Allahabad).
03. IMS Prize (Fluid Dynamics) : **K.P. Vishwanath** (Jawaharlal Nehru Center for ASR, Bangalore).
04. AMU Prize (Algebra) : No entry.
05. VMS Prize (Analysis) : No prize awarded.

**INVITED SPEAKERS.**

The details of the half an hour invited Lectures delivered are as follows.

1. **Bimal Roy** (ISI Kolkata).

Title: *" Key Pre-distribution : A Combinatorial Approach."*

2. **Punita Batra** (HRI, Allahabad).

Title: *"Highest weight representations of pre-exp-polynomial Lie algebras."*

3. **B. K. Dass** (University of Delhi).

Title: *"A New Construction of Burst Correcting Code."*

4. **S. Sundar** (IIT, Madras, Chennai).

Title: " *Navier-Stokes-Brinkman system for interaction of viscous waves with a submerged porous structure.*"

5. **D. Wulbert** (University of California,San Diego, USA).

Title: "*Mathematics Research and Teaching in California and India.*"

**SYMPOSIA :**

Details of symposia organized is as under :

1. On **"Contributions of the Medieval Indian Mathematician Narayana Pandita".**
   Organizer : **R. Sridharan** (CMI, Chennai).
   Speakers :
   - **M S Sriram** ( Univ. of Madras, Guindy Campus) : *"Progressions in Ganitakaumudi".*
   - **K. Ramasubramanian** ( Cell for Indian Science and Technology in Sanskrit, IIT Powai, Mumbai): *"Ksetravyavahara in Ganitakaumudi".*
   - **M. D. Srinivasa** (Center for Policy Studies, 6, Balaiah Avenue, Luz, Mylapore, Chennai) : *"Vargaprakriti (Brahmagupta to Narayana) ".*
   - **Raja Sridharan** (TIFR, Mumbai) : *"Combinatorics".*
   - **R. Sridharan** (CMI, Chennai) : *"Sarvatobhadra of Varahamihir".*
2. On **"Partial Differential Equations, Scientific Computing & Applications"**
   Organiser : **B. V. Rathish Kumar** (IIT Kanpur, Kanpur).
   Speakers :
   - **V. Raghavendra** (IIT Kanpur) : *"Glimpses of Theoretical elements used in Numerical Methods for PDEs".*
   - **Amiya Pani** (IIT Bombay): *"A Bird's Eye view on Finite Element Analysis of PDEs".*
   - **Rajen Sinha** (IIT Guwahati) : *"Finite Element Computations & Applications".*
   - **Pravir Dutt** (IIT Kanpur) : *"A look at the basics of Spectral Element Method for PDEs ".*

- **B. V. Rathish Kumar** (IIT Kanpur): *"A Review on Spectral Element Computation of Viscous Fluid Flows".*

3. On **"Optimization Theory and Algorithms".**
   Organizer: **C. S. Lalitha** (University of Delhi).
   Speakers :
   - **T. Parthasarathy** (CMI, Chennai) : *"On $P_2$ prime and P2 property in Semidefinite Linear Complementarity problems".*
   - **J. Jeyaraman** (IIT Madras) : *"The P2-Property of Linear Transformations on Euclidean Jordan Algebras".*
   - **C. S. Lalitha** (University of Delhi) : *"Optimality Criteria in terms of Weak Clarke Epiderivative for Set-valued Optimization".*
   - **A. Chandrashekaran** (IIT Madras) : *"On the SDLCP problem".*
4. On **"Recent Trends in Discrete Mathematics".**
   Organizer :**Ambat Vijayakumar** (Cochin University of Science and Technology, Cochin).
   Speakers :
   - **Aparna Lakshmanan** (St Xaviers College for Women, Kerala) : *" On Vizing's Conjecture".*
   - **S. M. Hegde** (NIT, Karnatak) : *"An insight in to Graph Labelling problems".*
   - **Rajendra Pawale** (University of Mumbai) : *"Diophantine equations associated with quasi-symmetric designs ".*
   - **L. Sunil Chandran** (IISc, Bangalore): *"Geometric Representations of Graph ".*
   - **T. Tamizh Chelvam** (Manonmaniam Sundaranar University, Tirunelveli) : *"Non-commuting Graph of a group".*
5. On**"Advances in Functional Analysis".**
   Organizer : **M.A.Sofi** (University of Kashmir).
   Speakers :
   - **S. H. Kulkarni** (IIT Mardas): *"Approximation numbers of bounded linear operators in normed linear spaces".*
   - **Daniel Wulbert** (University of California, San Diego) : *"Liapunov convexity theorem without compactness ".*
   - **Indumathi** (University of Pondicherry) : *"Best Approximation from subspaces and their closed unit balls".*
   - **Geetha S. Rao** (Ramanujan Institute of Advanced Study in Mathematics, Chennai) : *"The Art of Frame Theory".*
   - **P. N. Pandey** (Allahabad University, Allahabad) : *"Starfish Predation of a Growing Coral reef community".*

6. On **"Dynamical Systems".**
   Organizer : **T. K. Das** (The M. S. University of Baroda, Vdodara).
   - **V. Kannan** (University of Hyderabad) : *"Conjugacy classes in the group of homeomorphisms".*
   - **A. P. Singh** (University of Jammu) : *"Escaping sets of entire and meromorphic functions".*
   - **Raju K. George** (IIST, Trivandrum) : *"Application of Differential Equations to Control Theory".*
   - **Gyan Chandra Yadav** (Allahabad University, Allahabad) : *"Dynamics of wavelet sets".*
   - **Tarun K. Das** (The M. S. University of Baroda) : *"Chaotic Dynamical Systems".*

**P. L. BHATNAGAR MEMORIAL PRIZE FOR 2009.**

At the $50^{th}$ International Mathematical Olympiad, held in Bremen, Germany from 10 to 22 July 2009, the Indian team secured the $28^{th}$ position among 104 countries, winning three Silver medals, two Bronze medals and one Honorable Mention. **Mr. Sameer Wagh** (Pune) was the top scorer with 25 points. He was awarded the P. L. Bhatnagar Memorial Prize (carrying a Certificate and a cash award of Rs.1000/- each) for 2009 in the inaugural session. "Kalvivallal" Thiru T. Kalasalingam, Chancellor, Kalasalingam University gave the cash award and certificate to Mr. Sameer Wagh.

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# REPORTS ON IMS SPONSORED LECTURES

The details of the IMS Sponsored Lectures / Popular Talks held is as under :

| | | | |
|---|---|---|---|
| 1. | Speaker | : | **Prof M. Ram Murty, FRSC.** |
| | | | Queen's University, Canada. |
| | Title of the Lecture | : | Ramanujan and the Zeta Function. |
| | Day & Date | : | October 27, 2009. |
| | Venue | : | H R I , Allahabad. |
| | Organizer | : | Prof. Satya Deo. |

**Abstract :** In his notebooks, Ramanujan derived beautiful formulas for the special values of the Riemann zeta function at odd arguments. The notebooks were published in 1957 by TIFR and several authors have given proofs and generalizations of the formulas since that time. However, the conceptual meaning of some of the other terms that occur in these formulas was never discussed. In this talk, we will discuss this and show that the other terms are special values of what are now called Eichler integrals. We will also discuss the transcendental nature of these integrals and report on some recent joint work with Sanoli Gun and Purusottam Rath.

| | | | |
|---|---|---|---|
| 2. | Speaker | : | Prof S. D. Adikari. . |
| | | | H. R. I., Allahabad. |
| | Title of the Lecture | : | Visibility of Lattice points. |
| | Day & Date | : | December 18, 2009 ; 3.30 p.m. |
| | Venue | : | Department of Mathematics, |
| | | | Panjab University, Chandigarh. |
| | Organizer | : | Dr. A. K. Bhandari. |

**Abstract :** If $d \geq 2$, we say $\mathbf{a} \in Z^d$ is visible from $\mathbf{b} \in Z^d$ if there is no element of $Z^d$ on the straight line segment in between $\mathbf{a}$ and $\mathbf{b}$. After an introductory discussion on some of the early results, the particular problem of determining the size of the smallest $B \subset Z^d$ such that the set of integerlattice points inside a rectangular box $A$ is visible from $B$. Some recent results obtained in collaboration with Andrew Granville after describing some earlier results obtained jointly with R. Balasubramanian and Yong-Gao Chen were discussed.

| | | | |
|---|---|---|---|
| 3. | Speaker | : | Prof. Riddhi Shah.<br>J. N. U., Delhi |
| | Title of the Lecture | : | Distal and Ergodic Actions of Groups. |
| | Date | : | January 25, 2010 |
| | Venue | : | Department of Mathematics,<br>Gujarat University, Ahmedabad - 380 009. |
| | Organizer | : | Prof. N. R. Ladhawala. |

## FORM IV

(See Rule 8)

| | | |
|---|---|---|
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| 2. | Periodicity of publication. | QUARTERLY. |
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I, V. M. Shah, hereby declare that the particulars given above are true to the best of my knowledge and belief.

V. M. SHAH
Dated: $23^{rd}$ March, 2010. Signature of the Publisher

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Back volumes of both the periodicals, except for a few numbers out of stock, are available.

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