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Edited by

J. R. PATADIA

(Issued: March, 2009)

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
(In the memory of late Professor R. P. Agarwal)

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Ratan Prakash Agarwal
(June 15, 1925 - February 9, 2008)

OBITUARY
Professor Ratan Prakash Agarwal
(1925 - 2008)

India has produced many mathematicians who worked in the fields of Number theory, Partition theory, Combinatorics and Special functions and made outstanding contributions. Brilliance with numbers is a curious thing. The number theorist of the 20th century, S. Ramanujan is rightly called the gift of India. In the years past, people with a gift for numbers overcame vast odds to find an outlet for their genius. S. S. Pillai and T. Vijayraghvan, the great mathematical wizards of the 20th century are undoubtedly the gift of number theorists of India to the world. India lost one of her brightest jewels in the fields of Special Functions, Number theory and Partition theory with the passing away on February 9, 2008, of Professor R. P. Agarwal, formerly Professor and Head of the Department of Mathematics and Astronomy, University of Lucknow and Vice Chancellor of Universities of Rajasthan and Lucknow. A true visionary who became an institution in his life time and whose work on mock theta and related functions, are comparable to well known mathematicians of the world which is quite evident from the recognitions and appreciations he received from the world famous mathematicians.

Between 1994 and 2003, Indian researchers accounted for 2.21 percent of world's publication of research in mathematics. India had been the largest contributor in the field of special functions and number theory in mathematical publications output in the world. Indian researchers have done well in these specific areas and the major reason is because some 'dedicated' and 'monastic' mathematicians like Prof. R. P. Agarwal who worked with the raw material to produce very different results. One of Prof. Agarwal's achievements is formation of a school of research in Ramanujan mathematics, q-series, q-transforms, mock theta functions, combinatorics and continued fractions. He was one of the most successful guides. In his school one can find research teams where juniors can freely contradict their seniors in ways that are essential to the scientific temper; dissent is, after all, the beginning of discovery in his opinion.

Professor Ratan Prakash Agarwal was born on June 15, 1925 in Basti, U.P. He was Ph.D from Lucknow University(1950) and also from London University(1953) under the supervision of W. N. Baily of Bedford College. He was on the faculty of Lucknow University from 1946 to 1985 save for a break of three years at University of Gorakhpur where he was Head of the Department of Mathematics (1963-66). He was a Fulbright Visiting Research Professor at West Virginia University, USA during 1967-68. He was invited as Visiting Professor by Pennsylvania State University, USA in 1979.

On return from University of Gorakhpur to University of Lucknow, Prof Agarwal started collecting a band of enthusiastic workers around him. He encouraged the teachers in the universities of Gorakhpur and Lucknow to devote them to research. He organized study groups for teachers, seminars and workshops and participated vigorously in them. More than 23 persons including Prof. W. H. Abdi, Prof Arun Verma, Prof. R. Y. Denis and Prof. M. A. Khan obtained doctorate degrees working under him in different classifications of analysis. He was a rare supervisor who has not bothered to have a joint paper with his Ph.D and D. Sc. students. In course of a few years he published a number of papers which had unmistakable marks of originality in them, and his great power as a mathematician began to be appreciated. As years rolled by, young people animated by the spirit of quest for knowledge gathered round this man of creative ability and soon Lucknow became an active centre for fundamental research in various disciplines of mathematical analysis (in a broad sense). His intention was to provide forum for experts to communicate recent results and exchange ideas in analysis that may have some connections with the topics of his special interest.

The early development of special function theory during the forties and later two decades or so in India was primarily influenced by the English school of researchers such as G. H. Hardy, E. C. Titchmarsh, T. M. MacRobert, F. H. Jackson, G. N. Watson, W. N. Bailey, A. Erdelyi, L. J. Slater and others. In the early fifties there were a few American number - theorists too, such as H. Rademacher, N. J. Fine , L. Carlitz etc. working on certain aspects of Ramanujan's work which had a great impact on the trends of the researches in our country, particularly in number theory and special functions. It was in the early eighties that there was a resurgence of a new interest in the study of q-series and Ramanujan's mathematics and all respected names in their own right in the fields of number theory and special functions including Professors R. P. Bambah, S. Chowla, K. G. Ramanathan, K. Venkatachaliengar and R. P. Agarwal in India, George Andrews, Richerd Askey, Bruce Berndt, M. E. H.Ismail and Mizan Rehman in U.S.A and Canada were tied up in a thread to explore the possibility of giving a new direction to the subject by attempting to express themselves in their distinct creative way. Ramanujan's works have been more thoroughly studied and better understood now than before.

Looking back on his achievements, one can not but be struck by his great originality and his undoubted command over techniques of classical analysis. Reviewing his work, one cannot fail to be impressed by its uniformly high quality. His approach to special functions was strongly influenced by Prof. W. N. Bailey of England whom he admired not only as a mathematician or as his research guide but as a truly great individual. Many a time Prof. Agarwal used to talk about the great boost that the intellectual companionship of a kind provided by Prof. Bailey gave during fifties to his mathematical research and at the same time revolutionized his thinking of mathematics.

Integral transform calculus was developed in India by R. S. Verma, Brij Mohan, B. R. Bhonsle, K. M. Saksena, S. K. Bose and scores of their co-researchers who found certain very important generalizations of some of the classical integral transforms. Out of the above generalizations the most widely studied was the generalization of the Hankel transformation given by Prof. R. P. Agarwal, who used E. M. Wright's generalized Bessel function as the kernel, because it was only one for which the inverse transformation also contained a generalized Bessel function of the same kind, which made it more amenable from application point of view. Work on transforms and self-reciprocal functions were the favourite topic throughout the country till the early fifties. In 1961, he generalized the MacRobert's E-function (Proc. Glasgow Math. Asso.,Vol.5). Some of his results are included in his book 'Generalized Hypergeometric Series', Asia Publishing House, 1963. Later in 1965, he had written the path breaking paper on Meijer's G-function of two variables defined by means of double Barnes contour integrals which appeared in the volume 31 of the Proc. Nat. Inst. Sci.(India). He became an overnight sensation in the scientific community giving (proper) birth to the theory of Fox H-function of two and more variables and scores of other analytical generalizations based on this paper. This paper became the most quoted and referred paper in the theory of special functions. And, India is one of the growth hubs for these special functions.

In an article published in the Proceedings of the International Conference on 'The works of Srinivasa Ramanujan' at Mysore in July 2000, he presented that facet of Ramanujan's work which has remained elusive to the research workers even after more than three quarters of a century, now. He was referring to Ramanujan's last monumental and ingenious gift to the mathematical world-the mock theta functions. Ramanujan announced the discovery of these functions about three months prior to his death in April,1920, in a letter to hardy, although the world of mathematics knew about them for the first time through the Presidential address of G. N. Watson to the London Mathematical Society in 1935. In 1991, Andrews and Hickerson announced the existence of eleven more identities given in the "Lost" Notebook of Ramanujan, involving seven new functions which they labeled as mock theta functions of sixth order. Ramanujan did not say what he meant by an order. In an attempt at finding a satisfactory explanation to the questions of the definition of the order of the mock theta functions and the unexplained comments of Watson on the inadequate transformation theory for the fifth and seventh order functions, Prof. Agarwal shown that the structure of twenty seven mock theta functions can be unified by basic hypergeometric series for particular values of the parameters and the variables. Developing transformations for such basic unified series can be a suitable tool for obtaining identities involving the mock theta in a unified manner. The works of Andrews and Agarwal lend some credence to this belief.

A workshop on 'Special functions and Differential equations' held at the Institute of Mathematical Sciences in Chennai from Jan.13 to Jan.24, 1997 was inaugurated by Prof. T. H. Koornwinder, Vice Chair of the SIAM Activity group on Orthogonal Polynomials and Special Functions. During the opening ceremony of the workshop, Prof. Agarwal gave a survey of special functions in India during the last century.

He retired in 1985 from University of Lucknow. He was the Vice Chancellor of Rajasthan University, Jaipur, from 1985 to 1988. On the request of Patil, the Chancellor (Governor of Rajasthan), he has to fly an early morning to Jaipur from Lucknow to immediately take up this job in these days of student unrest and political uncertainties. He took it as a challenge. He was also the Pro Vice Chancellor and Vice Chancellor of Lucknow University.

The researcher in him urged him strongly to break the bonds of officialdom, and during his Vice Chancellorship, he not only worked with his research scholars but also organized seminars and conferences including an 'Instructional conference on special functions' at Mount Abu, a well known hill station of Rajasthan in 1987. Prof. Agarwal, Prof. S. Bhargava, Prof M. S. Raghunathan and Prof. R. Balsubramanian lectured the participants from all over India. This was the time when Ramanujan centenary conferences were held at several places in India including Pune, Jaipur and Chennai. A series of lectures on the Ramanujan's work delivered by him at some of these conferences has been a valuable contribution to the subject. The book 'Srinivasa Ramanujan 1887-1920-A tribute', published by Macmillan India in 1987, contains contributions by such eminent mathematicians as G. E. Andrews, R. Askey, M. V. Subba Rao, B. Berndt, S. Bhargava et. al. and the chapter contributed by Agarwal is a remarkable review covering works of his own and others in the field. He motivated his friend Prof. W. H. Abdi to write a book on the life and work of S. Ramanujan and the outcome was the memorable book "Triumphs and toils of Srinivasa Ramanujan" which also has a chapter contributed by Prof. Agarwal. He loved lecturing, and was a lucid, effective, and brilliant lecturer, especially on mock theta functions and on mathematical topics related to Ramanujan mathematics which were of immediate interest to him.

In person Prof. Agarwal was most unassuming, charming and liberal minded, graced with dignity and a delightful humor. To his students and colleagues, he was a friend, philosopher and guide. His gentle manners and courtesy could serve as a model to any one.

Prof. R. P. Agarwal and Prof. H. C. Khare were coordinators of a tele film project sponsored by the University Grant Commission in which a number of lectures on different courses of undergraduate classes were recorded by a number of well established teachers from all over India. These lectures are regularly telecast on television by Door Darshan on national network.

The distinctions conferred on Prof. Agarwal by the scientific community have been numerous and of a varied nature. He was fellow of the Indian National Science Academy. He was for many years a member of the Indian Mathematical Society (IMS), of the Indian Science Congress Association (ISCA) and of the Bharat Ganit Parisad (BGP). He was the President of the section of Mathematics, ISCA for the year 1982-83. He was founder member and Patron of the Society for Special functions and their Applications (SSFA). From 1998 onward, he functioned as member of the Council of IMS, first as General Secretary, then as President (1984-85), and later as Editor of the journal of the I.M.S. He was ever in favor of establishing a high standard for the society's publications, and gave his unreserved support for many proposals aimed at bringing it out.

He received Ramanujan birth centenary award for outstanding contribution to Mathematics from the ISCA in 1991. He was also a recipient of Distinguished Service Award of the Mathematical Association of India in 1985. He delivered S. M. Shah memorial award lecture of IMS in 1998 and Ramanujan memorial award lecture in 1990. An International conference of SSFA was held at Lucknow to celebrate 76th birthday of Prof. Agarwal on February 2-4, 2001 which was attended by many foreign mathematicians including Prof S. Kanemitsu from Japan. His lecture on 'An attempt towards presenting an unified theory for mock theta functions' was the main attraction of this conference. His another important lecture on the 'Recent developments in the theory of hypergeometric series' is contributed to the first publication of SSFA, the book 'Selected topics in Special Functions', Allied Publishers Ltd., New Delhi, 2001.

Prof. Agarwal was a very active mathematical administrator. His concerted efforts in the development of the I.M.S and other societies have been remarkably successful. When the venue of the conference of I.M.S. for 2001 could not be materialized due to some problems, Prof. Agarwal, General Secretary of the I.M.S. at that time, contacted me and asked to organize this event within 2 months time. Incidentally I was Chairman of the department of Mathematics at A. M. U., Aligarh. With his leadership and the active involvement and encouragement of Professors H. C. Khare, N. K. Thakare, V. M. Shah and other office bearers of I.M.S., we succeeded in holding the conference at Aligarh. The conference was inaugurated by Mr. Hamid Ansari, Vice Chancellor of A.M.U. (presently, Vice President of India) and Prof R. Balsubranian, Director, Institute of Mathematical Sciences delivered the Ramanujan memorial award lecture.

The valuable work done by him during the years 1989-1998 found a very encouraged reception at the hands of mathematicians and one may make a specific reference to his work which has been incorporated in three volumes entitled 'Resonance of Ramanujan's mathematics'(1998). Dr. S. N. Singh came from Jaunpur and helped him to finish this task. The book was released in 1998 by the Governor of Uttar Pradesh.

He deeply cared, loved and gave everyone around him a sense of belonging. Whenever I met him, I always found him smiling and cheerful. It was always so refreshing and inspiring to spend a few minutes in his company. During evenings, I generally found him exchanging jokes and humor with his best friend Prof. W. H. Abdi. After the sudden death of Prof. Abdi, we missed all those memorable moments. It was not surprising that he longed for such sittings during his later periods of acute mental depression.

He was married to Dr. Sushila, a teacher of Psychology in the University of Lucknow and had a happy married life. When in 2001, Mrs. Agarwal fell ill and became immovable, it seemed like a terrible tragedy. During the long illness of his wife he cared like a devoted and loving husband. Getting that diagnosis, Prof. Agarwal could not take up any assignment outside Lucknow leaving her alone on the mercy of servants. Everyone realized she could never go back to being the person she had been. It was 2000 when he attended his last conference of the SSFA at Jodhpur. When members of Executive Councils of I. M. S. and S. S. F. A. insisted on his presence in meetings, he voluntarily undertook task of arranging such meetings in Lucknow.

His frustrations due to sad demise of his wife and then his friend Prof. H. C. Khare led to physical deterioration and a few months before his death he fractured his leg. He survived this but slowly lost his memory and became bedridden. If your memory is faulty, it is as if your life is thrown off balance. He was looked after by his servants, adopted daughter Abha and

Dr. Ahmad Ali. However, his condition got worse and he passed away on February 9, 2008.

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PROFESSOR R. P. AGARWAL (15th JUNE 1925 - 9th FEBRUARY 2008) A RATAN (JEWEL) IN MATHEMATICS

N. K. THAKARE

While working on orthogonal polynomials, I came across an interesting paper of Prof. R.P. Agarwal on Bessel Polynomials. That is how, I came to know Professor Ratan Prakash Agarwal. His researches on topics such as Basic Hypergeometric Functions, Mock-theta Functions, Transform Analysis, Number Theory and Ramanujan's Mathematics were deep, penetrating and much discussed amongst the researchers then. His book on Generalized Hypergeometric Functions was also well received.

He and the Japanese Scientist Izuru Fujiwara were the external examiners of my Ph. D. Thesis submitted to Shivaji University, Kolhapur. The report of the external referees on my thesis was so highly superlative and praiseworthy that I became an ' Instant Hero' in the academic circles of Shivaji University. I was then on the faculty of mathematics of Shivaji University in late nineteen sixties and early ninteen seventies. Professor Agarwal came all the way from Lucknow to Kolhapur to conduct my viva-voce as it was mandatory. The way he posed questions to me during the viva-voce revealed an inquiring mind hidden in that unassuming gentleman. Through his questions he enriched my knowledge of classical analysis. Since then he virtually became my 'God Father' in the real sense of the term. He was fatherly figure not only to his adopted daughter Abha but to many researchers including his own students W. H. Abdi, Arun Verma, Remy Denis, S. N. Singh, Ahmed Ali and others. He was perhaps a rare doctoral supervisor who took considered care in not having joint publication with any of his 23 Ph.D students and one D.Sc. student.

Professor Agarwal moved to Rajasthan University, Jaipur as the Vice-Chancellor in 1985. But he never neglected his researchers while executing the onerous duties of Vice-Chancellorship. On the contrary he organized many conferences, workshops during those days.

In the last week of December 1987, we organized Ramanujan Birth Centenary Symposium on Analysis at Pune with the active involvement of Defence Research and Development Organization's Institute of Armament Technology. Professor Agarwal inaugurated the said Centenary Symposium which was attended and graced by illuminaries such as Professors Venkatachaliengar, G. E. Andrews, Richard Askey, Mourad Ismail, Mizan Rahman, M. A. Al-Bassum, A. M. Mathai, H. P. Dikshit, H. M. Srivastava, Remy Denis and many eminent researchers. It was a very successful event. Professor Agarwal was the beacon light and guiding source for that event.

In January 1988, the Platinum Jubilee Session of the Indian Science Congress Association (ISCA) was organized at Pune. Prof. R. P. Agarwal delivered the first Ramanujan Lecture at the Platinum Jubilee Session. Professor P. C. Vaidya was the Sectional President and I was the Local Secretary for the Section of Mathematics at the said Session. Professor Agarwal took active interest in day to day deliberations of this Session. This was his second visit to Pune hardly after a gap of one week.

He was also honoured by ISCA with the Ramanujan Birth Centenary Award for the year 1991 for his outstanding contribution to mathematics.

Professor Agarwal's third visit to Pune was in December 1988 when we organized the 54th Annual Conference of the Indian Mathematical Society under the Presidentship of Professor M. K. Singal. The record attendance of 448 delegates at this Conference is yet to be surpassed. This was the result of concerted efforts Professor Agarwal took with the active co-operation of Professors H. C. Khare, J. N. Kapur, M. K. Singal and V. M. Shah in pulling out the I. M. S. from its slump status to proactive status. Under the leadership of Professor Agarwal the constitution of I. M. S. was modified, aptly updated and was tailored to suit the growth and development of I. M. S.

In August 1990, the University of Pune was bifurcated into two universities and the onerous burden of establishing and raising the new University – North Maharashtra University – at Jalgaon fell on my shoulders. I was then constantly guided by Professor Agarwal in imparting my functions as the First Vice-Chancellor. In October 1990, I received a call from Prof. Agarwal asking me to send willingness certificate to accept the Office of President of the Section of Mathematics of ISCA, if elected. It was complied with obediently. Because of the God- Fatherly affections bestowed on me by Professor

Agarwal, I became the Sectional President of Mathematics at the 79th Annual Session of ISCA held in January 1992 at Vadodara. With the blessings of Professors Agarwal and H. C. Khare the luck smiled on me again and I presided over the proceedings of the 61st Annual Conference of I. M. S. held at Aligarh Muslim University in December 1995. This Conference was inaugurated by the then Hon. Vice-Chancellor Dr. Md. Hamid Ansari who is now the Vice- President of India.

Professor R. P. Agarwal was founder member and patron of the 'Society for Special Functions and Its Applications' (SSFA). Other founder members included M. A. Pathan, S. Bhargara, A. K. Agarwal, R. Y. Denis, S. N. Singh, Ahmed Ali besides myself. I was the President of SSFA for six years since its inception. The journal of SSFA started at the instance of professor Agarwal is considered a valuable activity. Substantial financial support was provided by Professor Agarwal for this purpose.

Because of Professors Agarwal and Khare, a village boy and first generation learner like me whose father was hardly literate and whose mother did not know how to count beyond twenty could preside over annual deliberations of India's topmost Mathematical/Scientific societies. The readers may be aware that I.M.S. is the oldest scientific society in India.

Professor Ratan Prakash Agarwal was really a jewel – as suggested by his first name Ratan - in Mathematics. I pay my humble tribute to this great mathematician and a large hearted human being. His monumental three volume work on 'Resonance of Ramanujan's Mathematics' shall keep his memory alive for generations to come.

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THE IMPORTANCE OF MATHEMATICS

A. K. AGARWAL

Hon'ble Chief Guest Prof. C. S. Seshadri, Prof. R. G. Harshe-Hon'ble Vice- Chancellor of the University of Allahabad, Prof. Ramji Lal- Local Secretary of the 74th Annual Conference of the Indian Mathematical Society, other dignitories sitting on the dias, Fellow mathematicians, Ladies and Gentlemen! At the outset I would like to thank the members of the Indian Mathematical Society for electing me to the high office of the President for the year 2008-09. I am indeed greatly humbled by my election and assure you all that it would be my constant endeavor during my tenure and thereafter to serve the Society to the best of my abilities.

The Indian Mathematical Society has completed 101 years. 101 is an auspicious number, celebrating this occasion in the holy city of Prayag (now Allahabad) is doubly auspicious. Lord Brahama called this city "Tirth Raj" (King of all pilgrimage centres). I call this great city "Desh Raj" (King of the country) in the sense that this city has given the maximum number of Prime Ministers to the country. While three Prime Ministers, namely, Pt. Jawahar Lal Nehru, Mrs. Indira Gandhi and Shri V. P. Singh were born here, two other Shri Lal Bahadur Shastri and Shri Chandershekhar were closely associated with this city. This city has the unique feature: on the one hand it is the confluence of three holy rivers- Ganga, Yamuna and Saraswati and the confluence of history, politics, culture and literature on the other. The most popular Hindi poet Dr. Harivansh Rai Bachchan of 'Madhushala fame' and his son Mr. Amitabh Bachchan who is described as "One-man Industry" were born here. While in the poetry sessions, the audience wished to hear Dr. Harivansh Rai Bachchan alone, in a poll survey on BBC, Amitabh Bachchan was named the star of the millennium beating

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This is the text of the Presidential address(General) delivered at the 74^{th} Annual Conference of the Indian Mathematical Society held at the University of Allahabad, Allahabad - 211 002, India, during December 27-30, 2008.

top Hollywood legends like Charlie Chaplin. Another noted literary figure Smt. Mahadevi Verma made this city her “Karm Bhoomi”. Munshi Prem Chand - the ‘badshah’ of Hindi novels was also associated with this city. Before independence, many important events of our freedom struggle took place in Allahabad. Chander Shekhar Azad-the undaunted revolutionery, sacrificed his life for the freedom of his motherland while fighting bravely against the British Police in Alfred Park. I salute all these national heros of this place. We are happy to be here in this great city to participate in the 74^{th} Annual Conference of the Indian Mathematical Society. For this I express my sincere gratitude to Prof. R. G. Hershe, Hon’ble Vice-Chancellor of the University of Allahabad and the members of the Local Organizing Committee led by Prof. Ramji Lal and of course, Prof. Satya Deo Tripathy who was instrumental in holding this conference here at Allahabad.

On this occasion, we all feel with regret the absence of one of the stalwarts of the Indian Mathematical Society, Prof. R. P. Agarwal who passed away earlier this year in the month of February. An excellent academician and an able administrator, Prof. Agarwal served the Society for many years in various capacities such as President, General Secretary, Administrative Secretary, Editor of the Journal and Editor of the Math. Student. He had also delivered the Ramanujan Memorial Award Lecture of the Indian Mathematical Society. He will remain an ideal for generations to come.

I am fully aware of the challenges and difficulties which mathematicians face to convince the common man of the importance of mathematics - a subject commonly called ‘a dry subject’. Many people ask: Why should we spend a considerable portion of our life at x, y, z? Why should we bother our heads about $f(x)$? What is our concern whether $lim\ x \to \infty\ f(x)$ is finite or infinite? Even well educated people seem to be proud that mathematics was their horror subject in school and that they wanted to drop it altogether from their curriculum at the first available opportunity. They argue that with the advent of super fast computers, the study of mathematics will not be of much use as the differential equations can be solved numerically on computers and integrals can be evaluated on computers. Sometimes they create divide between pure mathematics and applied mathematics. They also allege that there is no scope for mathematics students as compared to other science subjects. In this general address I would like to touch upon the public awareness of mathematics and would also like to explode the myth that mathematics is a dry, hard and useless subject.

To me it appears to be a paradox. While mathematics is everywhere in our life, it is quite difficult for the mathematical community to convince others of its importance. In my opinion the reason for this is: unlike the other sciences, the applications of mathematics are not very visible for the public. The applications of the medical sciences are visible to everybody. When we travel by super fast trains or planes or use technical gadgets like TV, mobile phone, microwave, washing machine, fridge, we all praise modern technology. Very few people realize that at the back of all these modern technologies, mathematics is playing a very important role. For example, this is the era of information technology. Enormous amount of data can be communicated from one part of the world to the other in no time. Many of us know this technology uses digital communication in which satellites play a key role. But not many people know that at the back of this technology the driving force is the coding theory. This is an area in which finite fields which are essentially parts of abstract algebra are studied. Furthermore, cryptography is used for protecting information transmitted through public communication networks, such as those using telephone lines, microwaves or satellites. And in cryptography large prime numbers and Euler's theorem which is a generalization of Fermat's little theorem from number theory are used.

Today, the most powerful forces of the society are the financial interests. And mathematics is there in descriptions and calculations of what happens on the financial markets. During last four decades finite element method has become an important tool to solve industrial problems, related to both linear and non-linear design studies arising in high-tech engineering research. Mathematics guides all industrial activities, be they those by which processes are adjusted or estimates framed or commodities bought and sold or accounts kept.

We all know the applications of geometry and trigonometry in surveying, navigation and construction work. The surveyor who measures the land purchased; the architect in designing a mansion to be built on it; the builder when laying out the foundations; the masons in cutting the stones; and the various artisans who put up the fittings; are all guided by geometrical truths. Railway making is regulated from beginning to end by geometry; alike in the preparation of plans and sections; in staking out the line; in the mensuration of cuttings and embankments; in the designing and building of bridges, culverts, viaducts, tunnels, stations; similarly with the harbors, docks, piers and various engineering and architectural works that fringe the coasts and overspread the country, as well as the mines that run underneath it. And now-a-days even the farmer for the correct laying out of his drains, has recourse to the level, that is, to geometrical principles. Because of its importance Plato loved geometry to such a degree that the inscription over the entrance to his school ran "Let none ignorant of geometry enter my door". We also know that trigonometry is essential in most branches of physics.

From the Newton's time phrases like "mathematical reasoning" and "mathematical certainty" have been used in all sciences. Mathematical reasoning means: exact reasoning and correct reasoning. So mathematics is a synonym of "exactness". The engineer who miscalculates the strength of materials builds a bridge that breaks down. The manufacturer who uses a bad machine can not compete with another whose machine wastes less in friction and inertia. It is this exactness of mathematics which enables us to calculate accurately distances billions upon billions of miles in lengths, as the distances of stars, etc.; and it also enables us to measure magnitudes about one billionth part of a cubic inch in volume, like the size of a molecule or atom. From finite quantities it leads us on to the region of infinite. Different solar systems and different planets of the same solar system are governed by mathematical laws.

Today, in all sciences whether they are pure sciences (like physics and chemistry), life sciences (like zoology and botany), social sciences (like psychology and sociology) or applied sciences (like medical and engineering) mathematical models are used. Examples:

1. To measure the altitude, the barometric equation

$$h = (30T + 8000)l_n(P_0/P)$$

is used. Here h denotes height in meters above the sea level, the air temperature T in degrees Celsius, the atmospheric pressure P_0 in centimeters of mercury at sea level, and the atmospheric pressure P in centimeters of mercury at height h.

2. When the glucose solution is infused into the blood stream and the medical technician wants to determine the concentration y of glucose in the blood t minutes after the beginning of the infusion, he uses the mathematical model

$$y = A + (B - A)e^{-kt},$$

where B is the concentration of glucose in the blood at $t = 0$. A represents the concentration of glucose in the blood after a long period of time indicating that the concentration has stabilized.

3. Psychologists use Fechner's model to measure sensation. If S denotes the intensity of sensation and P the intensity of the physical stimulus causing it, then the Fechner's law states that S varies linearly as the natural logarithm of P, so that

$$S = A + B\ l_n P,$$

where A and B are suitable constants.

These applications of mathematics are not visible to the man on street. So, if someone says that mathematics is a dry and useless subject, I would say that he is not a fit person to make comments on mathematics. Like the solar light, mathematics appears quite colourless to the unthinking multitude, while it is in reality composed of colours of the rainbow. As to enjoy the beauty of the seven colours of the solar light one needs a prism, similarly to appreciate the importance of mathematics one needs good understanding. Now I would like to respond to some of the questions which are often raised by non-mathematicians and which I mentioned in the beginning of my address.

1. The apprehension that computers will make the study of mathematics superfluous is not true. As food processors can not replace cooks in the kitchen, similarly computers can not substitute for mathematicians. Computers are not a threat but a great help for mathematicians. The softwares like Latex, Macaulay2, Macsyma, Mathematica and Matlab are extremely useful and considering the fact that the pace of our lives is accelerating day by day we must delegate as much as we can do to machines.

2. There is no divide between the so-called pure mathematics and applied mathematics. Those who see a separation between the two, should know that in research both work side-by-side. Terence Tao (Fields Medalist) explained this point as follows:

"Pure mathematics and applied mathematics are both about applications, but with a very different time frame. A piece of applied mathematics will employ mature ideas from pure mathematics in order to solve an applied problem today; a piece of pure mathematics will create a new idea or insight that, if the insight is good one, is quite likely to lead to an application perhaps ten or twenty years in the future. For instance, a theoretical result on the stability of a PDE may lend insight as to what components of the dynamics are the most important and how to control them, which may eventually inspire an efficient numerical scheme for solving that PDE computationally."

3. The allegations that there is no scope in mathematics and the future prospects of mathematics students are not bright are not correct. One can explore well the full scope of mathematics only if he has the right approach. Many students who take up a computer degree or diploma after their M.Sc. in mathematics are hired as professionals by IT companies. M.Sc. in mathematics is also taken up by IAS, IFS, IPS and PCS aspirants as it is a scoring subject. Even the banking sector prefers mathematics graduates over other candidates and scope of promotion is also better. The shortage of mathematics teachers abroad too seems an opportunity candidates can grab once they complete M.Sc. in the subject with reasonable good marks. Mathematics research centres in India and abroad are looking for bright mathematics graduates who have research aptitude and who desire to make their own significant contributions to the subject. At many places mathematics is the most preferred subject. Mathematics is the tallest building on campus both in Princeton and in Moscow.

Those who allege that mathematics is a hard and dry subject without any scope should know that mathematics is like the ocean - rough, boisterous and fearful on the surface, but having precious pearls, and gems of the purest ray serene at the bottom. Underlying the importance of mathematics described as 'Queen of Sciences' by Gauss, Isaac Todhunter in his essays on education says that of all the subjects required for passing university examination mathematics furnishes the most reliable test of a man's working powers. According to Roger Bacon, "mathematics is the gateway and the key to other sciences."

I conclude by wishing the conference a grand success so that the organizers may feel fully contended. I am sure that after attending the various academic programs of the conference the delegates will feel enriched in their knowledge of the subject.

Thank you.

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RAMANUJAN'S CONTRIBUTIONS AND SOME RECENT ADVANCES IN THE THEORY OF PARTITIONS

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1. **Introduction**

The works of Srinivasa Ramanujan - the legendary Indian mathematician of the twentieth century, made a profound impact on many areas of modern number theoretic research. For example, partitions, continued fractions, definite integrals and mock theta functions. This lecture is devoted to his crowning achievements in the theory of partitions: asymptotic properties, congruence properties and partition identities. We shall also talk about some of the advances that have occurred in these areas after Ramanujan.

The theory of partitions is an important branch of additive number theory. The concept of partition of non-negative integers also belongs to combinatorics. Partitions first appeared in a letter written by Leibnitz in 1669 to John Bernoulli, asking him if he had investigated the number of ways in which a given number can be expressed as a sum of two or more integers. The real development started with Euler (1674). It was he who first discovered the important properties of the partition function and presented them in his book " Introduction in Analysin Infinitorum". The theory has been further developed by many of the other great mathematicians - prominent among them are Gauss, Jacobi, Cayley, Sylvester, Hardy, Ramanujan, Schur, MacMahon, Gupta, Gordon, Andrews and Stanley. The celebrated joint work of Ramanujan with Hardy indeed revolutionized the study of partitions. Because of its great many applications in different areas like

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probability, statistical mechanism and particle physics, the theory of partitions has become one of the most hot research areas of the theory of numbers today.

A partition of a positive integer n is a finite non- increasing sequence of positive integers $a_1 \geq a_2 \geq \cdots \geq a_r$ such that $\sum_{i=1}^{r} a_i = n$. The a_i are called the parts or summands of the partition. We denote by $p(n)$ the number of partitions of n.

Remark 1. we observe that 0 has one partition, the empty partition, and that the empty partition has no part. We set $p(0) = 1$.

Remark 2. It is conventional to abbreviate repeated parts by the use of exponents. For example, the partitions of 4 are written as 4, 31, 2^2, 21^2, 1^4.

Remark 3. In the definition of partitions the order does not matter. 4+3 and 3+4 are the same partition of 7. Thus a partition is an unordered collection of parts. An ordered collection is called a Composition. Thus 4+3 and 3+4 are two different compositions of 7.

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where $| q | < 1$ and $(q;q)_n$ is a rising q-factorial defined by

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})},$$

for any constant a. If n is a positive integer,then obviously

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

and

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots .$$

Plane partitions

A plane partition π of n is an array

$$\begin{array}{|cccc} n_{1,1} & n_{1,2} & n_{1,3} & \cdots \\ n_{2,1} & n_{2,2} & n_{2,3} & \cdots \\ . & . & . & \\ . & . & . & \\ . & . & . & \end{array}$$

of positive integers which is non increasing along each row and column and such that $\Sigma n_{i,j} = n$. The entries $n_{i,j}$ are called the parts of π. A plane partition is called symmetric if $n_{i,j} = n_{j,i}$, for all i and j. If the entries of π are strictly decreasing in each column, we say that π is column strict. And if the elements of π are strictly decreasing in each row, we call such a partition row strict. If π is both row strict and column strict, we say that π is row and column strict. Column strict plane partitions are equivalent to Young tableaux which are used in invariant theory. Plane partitions have applications in representation theory of the symmetric group, algebraic geometry and in many combinatorial problems. Plane partitions with atmost k rows are called k-line partitions. We denote by $t_k(n)$ the number of k-line partitions of n. Plane partition is a very active area of research. An extensive and readable account of work done in this area is given in [55,56]. However, in this lecture we shall only briefly touch the Ramanujan type congruence properties of $t_k(n)$.

F- partitions

A generalized Frobenius partition (or an F- partition) of n is a two rowed array of integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$, $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$,
such that

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

Remark. These partitions are attributed to Frobenius because he was the first to study them in his work on group representation theory under the additional assumptions: $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$.
$\phi_k(n)$ will denote the number of F- partitions of n such that any entry

appears atmost k times on a row. An F- partition is said to be a k- coloured F- partition if in each row the parts are distinct and taken from k-copies of the non-negative integers ordered as follows:

$$0_1 < 0_2 < \cdots < 0_k < 1_1 < 1_2 < \cdots < 1_k < 2_1 < \cdots < 2_k < \cdots .$$

The notation $c\phi_k(n)$ is used to denote the number of k- coloured partitions of n. For a detailed study of F- partitions the reader is referred to [26]. In this lecture we will briefly mention some Ramanujan type congruence properties of $\phi_k(n)$ and $c\phi_k(n)$.

Partitions with "$n + t$ copies of n"

A partition with "$n+t$ copies of n," $t \geq 0$, (also called an $(n+t)$- colour partition) is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$-different colours denoted by subscripts: $n_1, n_2, \cdots, n_{n+t}$. In the part n_i, n can be zero if and only if $i \geq 1$. But in no partition are zeros permitted to repeat.
Thus, for example, the partitions of 2 with "$n + 1$ copies of n" are

$$2_1,\ 2_1 + 0_1,\ 1_1 + 1_1,\ 1_1 + 1_1 + 0_1$$

$$2_2,\ 2_2 + 0_1,\ 1_2 + 1_1,\ 1_2 + 1_1 + 0_1$$

$$2_3,\ 2_3 + 0_1,\ 1_2 + 1_2,\ 1_2 + 1_2 + 0_1.$$

The weighted difference of two elements m_i and n_j, $m \geq n$ in a partition with "$n + t$ copies of n" is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$. Partitions with "$n + t$ copies of n" were used by Agarwal and Andrews [17] and Agarwal [1, 2, 3, 4, 5, 6, 7, 9, 13, 15] to obtain new Rogers - Ramanujan type identities. Agarwal and Bressoud [20] and Agarwal [8,15] connected these partitions with lattice paths. Further properties of these partitions were found by Agarwal and Balasubramanian [19]. Agarwal in [10] also defined n- colour compositions. Several combinatorial properties of n- colour compositions were found in [10, 12, 48, 49]. Anand and Agarwal [23] used n- colour partitions in studying the properties of several restricted plane partition functions. Agarwal [14] and Agarwal and Rana [21] have also used $(n+t)$- colour partitions in interpreting some mock theta functions of Ramanujan combinatorially. In fact our endeavor is to develop a complete theory for n- colour partitions parallel to the theory of classical partitions of Euler.

2. Hardy - Ramanujan - Rademacher exact formula for $p(n)$

As one many perceive, $p(n)$ grows astronomically with n. Even if a person has perfect powers of concentration and writes one partition per second, it will take him about 1,26,000 years to write all 3,972,999,029,388 partitions of 200. Ramanujan asked himself a basic question: " Can we find $p(n)$, without enumerating all the partitions of n?" This question was first answered by Hardy and Ramanujan in 1918 in their epock - making paper [39]. Their asymptotic formula for $p(n)$ can be stated as

$$p(n) = \frac{(12)^{1/2}}{(24n-1)u_n} \sum_{k=1}^{v} A_k(n)(u_n - k)exp(u_n/k) + O(n^{-1}), \tag{2.1}$$

where

$$u_n = \frac{\pi\sqrt{24n-1}}{6}, v = O\sqrt{n}$$

and

$$A_k(n) = \sum_{0 \le h < k, (h,k)=1} \omega_{h,k}\, e^{-2nh\pi i/k},$$

in which $\omega_{h,k}$ a certain $24th$ root of unity. The first few terms of (2.1) give a value the integral part of which is $p(n)$ itself. However, D.H. Lehmer [44] found that the Hardy- Ramanujan series (2.1) was divergent. But H. Rademacher [50,51] proved that if in (2.1) $(u_n - k)exp(u_n/k)$ is relpaced by $(u_n - k)exp(u_n/k) + (u_n + k)exp(-u_n/k)$, then we get a convergent series for $p(n)$. The new famous Hardy - Ramanujan - Rademacher expansion for $p(n)$ is the following:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \left[\frac{d}{dx} \frac{sinh \frac{\pi}{k}[\frac{2}{3}(x - \frac{1}{24})]^{1/2}}{(x - 1/24)^{1/2}} \right]_{x=n}, \tag{2.2}$$

where $A_k(n)$ are as defined earlier. Formula (2.2) is one of the most remarkable results in mathematics. It shows an interaction between an arithmetic function $p(n)$ and some techniques of calculus. It is not only a theoretical formula for $p(n)$ but also a formula which admits relatively rapid computation. For example, if we compute the first eight terms of the series for n=200, we find the result is 3,972,999,029,388.004 which is the correct value of $p(200)$ within 0.004. The 'Circle method' developed for proving formula for $p(n)$ has been useful in later developments of modular function theory. To know more about this subject the reader is referred to Rademacher [52] and Sections P68 and P72 of LeVeque [46].

3. Ramanujan's Congruence properties of $p(n)$

By looking carefully at MacMahon's table of $p(n)$ from n=1 to 200, Ramanujan was led to conjecture the following congruences:

$$p(5m+4) \equiv 0 (mod\ 5), \tag{3.1}$$

$$p(7m+5) \equiv 0 (mod\ 7), \tag{3.2}$$

$$p(11m+6) \equiv 0 (mod\ 11). \tag{3.3}$$

For elementary proofs of (3.1) and (3.2) see Ramanujan [53]. An elementary proof of (3.3) was given by Winquist [60]. There are also congruences to moduli 5^2, 7^2 and 11^2, such as

$$p(25m+24) \equiv 0 (mod\ 5^2).$$

All these congruences are included in Ramanujan's famous conjecture:If

$$\delta = 5^a 7^b 11^c \text{ and } 24n \equiv 1 (mod\ \delta) \text{ then } p(n) \equiv 0 (mod\ \delta). \tag{3.4}$$

Ramanujan proved this conjecture for 5^2, 7^2, 11^2 while Krečmar [41] proved it for 5^3, and G.N. Watson [59] proved it for general 5^a. Ramanujan's conjecture held good till 1934, when S. Chowla [33], using H. Gupta's table of $p(n)$ for $n \leq 300$, found that the conjecture failed for n=243, since p(243) = 133978259344888 is not divisible by 7^3 and $24.243 \equiv 1 (mod\ 7^3)$. Watson [59] modified the conjecture and proved: If $24n \equiv 1 (mod\ 7^b)$ then $p(n) \equiv 0 (mod\ 7^{[(b+2)/2]}$. The truth of Ramanujan's conjecture (3.4) was verified by Lehmer [43] for the first values of n associated with the moduli 11^3 and 11^4. Lehner [45] proved the conjecture for 11^3. Finally, in 1967, A.O.L. Atkin [28] settled the problem by proving (3.4) for general 11^c. The full truth with regard to the conjecture can now be stated as follows:

Theorem 3.1. If $\delta = 5^a 7^b 11^c$ and $24n \equiv 1 (mod\ \delta)$, then

$$p(n) \equiv 0 (mod\ 5^a\ 7^{[(b+2)/2]}\ 11^c). \tag{3.5}$$

In order to obtain the combinatorial interpretations of the Ramanujan's congruences (3.1) - (3.3), Dyson [34] defined the " rank of a partition" as the largest part minus the number of parts. He conjectured the following combinatorial interpretations of the congruences (3.1) and (3.2), respectively.

Theorem 3.2. If we write $\pi \sim \pi'$ when the ranks r and r' of π and π' are congruent (mod 5), then the equivalence classes of partitions of $5n+4$ induced by $\sim$ are equinumerous.

Theorem 3.3. If we write $\pi \sim \pi'$ when the ranks r and r' of π and π' are congruent (*mod* 7), then the equivalence classes of partitions of $7n+5$ induced by $\sim$ are equinumerous.

Atkin and Swinnerton [29] proved Theorems 3.2 and 3.3. Dyson observed that the rank did not separate the partitions of $11n+6$ into 11 equal classes. He then conjectured that there must be some other partition statistics (which he called the "crank") that would provide a combinatorial interpretation of Ramanujan's third congruence (3.3).

Andrews and Garvan [27] defined the crank of a partition as follows: For a partition π, let $l(\pi)$ denote the largest part of π, $\omega(\pi)$ denote the number of ones in π and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The "crank" $c(\pi)$ is defined by

$$c(\pi) = \begin{cases} l(\pi) & \text{if } \omega(\pi) = 0 \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

They gave the following combinatorial interpretation of Ramanujan's third congruence (3.3):

Theorem 3.4. If we write $\pi \sim \pi'$ when the cranks $c(\pi)$ and $c(\pi')$ are congruent (*mod* 11), then the equivalence classes of partitions of $11n+6$ induced by $\sim$ are equinumerous.

Garvan [36] gave other combinatorial interpretations of Ramanujan's congruences by using vector partitions. He defined a vector partition $\vec{\pi}$ of n as a 3-tuple (π_1, π_2, π_3), where π_1 is a partition into distinct parts and π_2 and π_3 are ordinary partitions such that the sum of the parts of the individual components of $\vec{\pi}$ is n. The rank of $\vec{\pi}$ is defined as the number of parts of π_2 minus the number of parts of π_3. Let $N_v(m,t,n)$ denote the number of vector partitions of n in which the rank is congruent to m *modulo* t. Garvan's combinatorial interpretations of (3.1)-(3.3) are

$$N_v(0,5,5n+4) = N_v(1,5,5n+4) = \cdots = N_v(4,5,5n+4) = \frac{p(5n+4)}{5}, \quad (3.6)$$

$$N_v(0,7,7n+5) = N_v(1,7,7n+5) = \cdots = N_v(6,7,7n+5) = \frac{p(7n+5)}{7}, \quad (3.7)$$

$$N_v(0,11,11n+6) = \cdots = N_v(10,11,11n+6) = \frac{p(11n+6)}{11}. \quad (3.8)$$

The study of Ramanujan type congruences is an active line of research. Agarwal and subbarao [22] obtained an infinite class of Ramanujan type congruences for perfect partitions. A perfect partition of a number n is one which contains just one partition of every number less than n when repeated parts are regarded as indistinguishable. The number of perfect partitions of n is denoted by $per(n)$.
Example. per(7)=4 since there are four perfect partitions of 7, viz., 41^3, 421, $2^3 1$, 1^7

Agarwal and Subbarao proved the following:

Theorem 3.5. (Agarwal and Subbarao) For $n \geq 1$, $k \geq 2$ and q any prime,

$$per(nq^k - 1) \equiv 0(mod\ 2^{k-1}). \tag{3.9}$$

Theorem 3.5 yields infinitely many Ramanujan type congruences for perfect partitions. For example, when $k = 3$, $q = 2$ with n replaced by $n + 1$, (3.9) reduces to

$$per(8n + 7) \equiv 0(mod\ 4),$$

which is very much analogous in structure to Ramanujan's congruences (3.1) - (3.3).
Cheema and Gordon [32] have obtained the following congruences for two- and three- line partitions:

Theorem 3.6. (Cheema and Gordon).

$$t_2(\nu) \equiv 0(mod\ 5),\ if\ \nu \equiv 3\ or\ 4(mod\ 5) \tag{3.10}$$

and

$$t_3(3\nu + 2) \equiv 0(mod\ 3). \tag{3.11}$$

More congruences of this type were found by Gandhi [35]. His results are given in the following theorem

Theorem 3.7. (Gandhi).

$$t_2(2\nu) \equiv t_2(2\nu + 1)(mod\ 2), \tag{3.12}$$

$$t_3(3\nu) \equiv t_3(3\nu + 1)(mod\ 3), \tag{3.13}$$

$$t_4(4\nu) \equiv t_4(4\nu + 1) \equiv t_4(4\nu + 2)(mod\ 2), \tag{3.14}$$

$$t_4(4\nu + 3) \equiv 0(mod\ 2), \tag{3.15}$$

$$t_5(5\nu+1) \equiv t_5(5n+3)(mod\ 5), \tag{3.16}$$

and

$$t_5(5n+2) \equiv t_5(5n+4)(mod\ 5). \tag{3.17}$$

Agarwal [11] defined the function $P_k(\nu)$ as the number of n- colour partition of ν with subscripts $\leq k$. He then established a bijection between the k- line partitions enumerated by $t_k(\nu)$ and the n- colour partitions enumerated by $P_k(\nu)$. This enabled him to translate the congruences (3.10) - (3.17) into n- colour partition congruences simply by replacing $t_k(\nu)$ by $P_k(\nu)$. In other words the congruences (3.10) - (3.17) can be restated in the following form:

Theorem 3.8. (Agarwal). We have

$$P_2(\nu) \equiv 0(mod\ 5),\ if\ \nu \equiv 3\ or\ 4(mod\ 5) \tag{3.18}$$

$$P_3(3\nu+2) \equiv 0(mod\ 3), \tag{3.19}$$

$$P_2(2\nu) \equiv P_2(2\nu+1)(mod\ 2), \tag{3.20}$$

$$P_3(3\nu) \equiv P_3(3\nu+1)(mod\ 3), \tag{3.21}$$

$$P_4(4\nu) \equiv P_4(4\nu+1) \equiv P_4(4\nu+2)(mod\ 2), \tag{3.22}$$

$$P_4(4\nu+3) \equiv 0(mod\ 2). \tag{3.23}$$

$$P_5(5\nu+1) \equiv P_5(5\nu+3)(mod\ 5), \tag{3.24}$$

$$P_5(5\nu+2) \equiv P_5(5\nu+4)(mod\ 5). \tag{3.25}$$

Analogues to Ramanujan's congruences for $p(n)$, congruences involving coloured and uncoloured F- partitions are also found in the literature. The following two congruences are due to Andrews [26]:

$$\phi_2(5n+3) \equiv c\phi_2(5n+3) \equiv 0(mod\ 5), \tag{3.26}$$

and

$$c\phi_p(n) \equiv 0(mod\ p^2), \tag{3.27}$$

where p is a prime and $p|\ n$.

Kolitsch [42] generalized (3.27) and proved the following congruence

$$\sum_{d|(m,n)} \mu(d)\ c\phi_{\frac{m}{d}}(\frac{n}{d}) \equiv 0(mod\ m^2). \tag{3.28}$$

Obviously, (3.28) reduces to (3.27) when $m = p$, a prime.

The following congruence between coloured and uncoloured F- partitions is found in [36]:
For p prime,

$$\phi_{p-1}(n) \equiv c\phi_{p-1}(n)(mod\ p). \tag{3.29}$$

We thus see how Ramanujan's congruences inspired several other mathematicians to pursue research in this area. In fact he himself proved many other identities connected with the congruence properties of partitions, such as

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}. \tag{3.30}$$

This result was considered to be the representative of the best of Ramanujan's work by G.H. Hardy.

4. **Rogers - Ramanujan Identities**

The following two "sum- product" identities are known as Rogers - Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}(1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1}, \tag{4.1}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty}(1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{4.2}$$

They were first discovered by Rogers in 1894 **(L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25(1894), 318-343.)** and were rediscovered by Ramanujan in 1913. He had no proof. In 1917 when he was looking through the old volumes of the Proceedings of the London Mathematical Society, came accidentally across Roger's paper. A correspondence followed between Ramanujan and Rogers. Ramanujan published a paper in 1919 **(S. Ramanujan, Proof of certain identities in combinatory analysis, Proceedings of the Cambridge Philosophical Society, XIX, (1919), 214-216)** which contains two proofs (one by Ramanujan and the other by Rogers) and a note by Hardy. After the publication of this paper these identities are known as Rogers - Ramanujan identities. MacMahon [47] gave the following combinatorial interpretations of (4.1) and (4.2), respectively:

Theorem 4.1. The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm\, 1(mod\ 5)$.

Theorem 4.2. The number of partitions n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm\, 2(mod\ 5)$.

The following generalization of Theorems 4.1 and 4.2 is due to Gordon [37]:

Theorem 4.3. (B. Gordon). For $k \geq 2$ and $1 \leq i \leq k,$, let $B_{k,i}(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$, where $b_j - b_{j+k-1} \geq 2$, and atmost $i-1$ of the b_j equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm\, i(2k+1)$. Then

$$A_{k,i}(n) = B_{k,i}(n) \ for \ all \ n.$$

Obviously, Theorem 4.1 is the particular case $k = i = 2$ of Theorem 4.3 and Theorem 4.2 is the particular case $k = i + 1 = 2$.

Andrews [24] gave the following analytic counterpart of Theorem 4.3:

Theorem 4.4. (G.E. Andrews). For $1 \leq i \leq k, k, \geq 2, |\ q\ | < 1$.

$$\sum_{n_1,n_2,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+N_i+N_{i+1}+\ldots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_{k-1}}} = \prod_{n\not\equiv 0,\pm i(mod\ 2k+1)} (1-q^n)^{-1}, \tag{4.3}$$

where $N_j = n_j + n_{j+1} + \ldots + n_{k-1.}$

It can be easily seen that Identities (4.1) and (4.2) are the particular cases $k = i = 2$ and $k = i + 1 = 2$ of Theorem 4.4.

Partition theoretic interpretations of many more q- series identities have been given by several mathematicians. See, for instance, Gordon [38], Connor [31], Hirschhorn [40], Agarwal and Andrews [16], Subbarao [57], Subbarao and Agarwal [58]. In all these results only ordinary partitions were used. Analogous to MacMahon's combinatorial interpretations (Theorem 4.1 and 4.2) of the Rogers - Ramanujan identities (4.1) and (4.2), Andrews [25] using n- coloured partitions conjectured and Agarwal [1] proved the following theorems:

Theorem 4.5. The number of partitions with "n copies of n" of ν such that each pair of summands m_i,r_j has positive weighted difference equals the numbers of ordinary partitions of n into parts $\not\equiv 0, \pm 4(mod\ 10)$.

Example. For $\nu = 6$, we have 8 relevant partitions of each type, viz., 6_1, 6_2, 6_3, 6_4, 6_5, 6_6, $5_1 + 1_1$, $5_2 + 1_1$ of the first type and 51, 3^2, 321, 31^3, 2^3, 2^21^2, 21^4, 1^6 of the second type.

Theorem 4.6. The number of partitions with "n copies of n" of ν such that each pair of summands m_i, r_j has nonnegative weighted difference equals the number of ordinary partitions of n into parts $\not\equiv 0, \pm 6(mod\ 14)$.

Theorems 4.5 and 4.6 are the combinatorial interpretations of the following q-series identities from [54]:

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q:q)_n(q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty}(1-q^{10n})(1-q^{10n-4})(1-q^{10n-6}), (I(46), p.156) \tag{4.4}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q:q)_n(q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty}(1-q^{14n})(1-q^{14n-6})(1-q^{14n-8}). \ (I(61), p.158) \tag{4.5}$$

The original impetus for Theorems 4.5 and 4.6 came from an attempt to understand the left side of the following identity which arose in Baxter's solution of the hard hexagon model [30, Chap. 14]:

For $\sigma_1 = 0$, $\sigma_i = 0$ or 1, $\sigma_i + \sigma_{i+1} \leq 1$,

$$\sum_{\sigma_2,\cdots,\sigma_m,\cdots} q^{(\sigma_2-\sigma_1\sigma_3).1+(\sigma_3-\sigma_2\sigma_4).2+(\sigma_4-\sigma_3\sigma_5).3+\cdots} = \prod_{n\not\equiv 0,\pm 4(mod\ 10)} (1-q^n)^{-1}. \tag{4.6}$$

Under the conditions $\sigma_i = 0$ or 1 and $\sigma_i + \sigma_{i+1} \leq 1$, each value in the parenthesis is 1 or 0 or -1, so the exponents look like collections of

1

2, $1 - 2 + 3$

3, $2 - 3 + 4$, $1 - 2 + 3 - 4 + 5$

4, $3 - 4 + 5$, $2 - 3 + 4 - 5 + 6$, $1 - 2 + 3 - 4 + 5 - 6 + 7$ etc.

We call these collections as blocks and denote them by $1_1, 2_1, 2_2, \cdots$. It was found by Andrews and Baxter that the partitions which we get from the above sum are such that the minimum difference between the blocks is 3. Now any block m_i is : $(m - i + 1) - \cdots \pm (m + i - 1)$, and any block n_j is : $(n - j + 1) - \cdots \pm (n + j - 1)$.

Now $m_i \geq n_j + 3$ means $m - i + 1 \geq n + j - 1 + 3$ or $m - n - i - j > 0$ that is, the weighted difference $((m_i - n_j)) > 0$.

Analogous to Gordon's Theorem (Th. 4.3) which generalizes MacMahon's combinatorial interpretations of the Rogers- Ramanujan identities, Agarwal and Andrews [17] proved the following generalization of Theorem 4.5 and 4.6:

Theorem 4.7 (Agarwal and Andrews). For $0 \leq t \leq k - 1$, $k \geq 2$, let $A_t(k, \nu)$ denote the number of partitions of ν with "$n + t$ copies of n" such that if the weighted difference of any pair of summands m_i, r_j is non-positive, then it is even and $\geq -2 \min(i-1, j-1, k-3)$. And if $t \geq 1$, then for some i, i_{i+t} is a part. Let $B_t(k, \nu)$ denote the number of partitions of n into parts $\not\equiv 0, \pm 2(k-t) (mod\ 4k+2)$. Then $A_t(k, \nu) = B_t(k, \nu),\ for\ all\ \nu$.

Obviously, Theorems 4.5 and 4.6 are special cases $k = t + 2 = 2$ and $k = t + 3 = 3$, respectively, of Theorem 4.7. As Andrews' Theorem (Th. 4.4) provides an analytic counterpart for Gordon's combinatorial theorem (Th. 4.3), similarly, the following Theorem of Agarwal, Andrews and Bressoud [18] provides an analytic counterpart for Theorem (4.7):

Theorem 4.8 (Agarwal, Andrews and Bressoud). Given $0 \leq t \leq k - 1$, $k \geq 2$, let $r = [k/2]$ and $\chi(A)$ is 1 if A is true, 0 otherwise. If

$k-r-1 \le t \le k-1$, then

$$\sum_{m_1 \ge \ldots \ge m_r \ge 0} \frac{q^{m_1^2+\ldots+m_r^2+a_1m_1+\ldots+a_rm_r+\binom{m_r}{2}\chi(k \text{ is even})}}{(q;q)_{m_1-m_2}\ldots(q;q)_{m_{r-1}-m_r}(q;q)_{m_r}(q;q^2)_{m_r+1}}$$

$$= \prod_{n \not\equiv 0, \pm 2(k-t)(mod\ 4k+2)} (1-q^n)^{-1}, \tag{4.7}$$

where $a_i = 1+\chi(i \ge k-t)$. If $0 \le t \le r$, then

$$\sum_{m_1 \ge \ldots \ge m_r \ge 0} \frac{q^{m_1^2+\ldots+m_r^2-m_1-\ldots-m_t+\binom{m_r}{2}\chi(k \text{ is even})(1-q^{m_t})}}{(q;q)_{m_1-m_2}\ldots(q;q)_{m_{r-1}-m_r}(q;q)_{m_r}(q;q^2)_{m_r}}$$

$$= \prod_{n \not\equiv 0, \pm 2(k-t)(mod\ 4k+2)} (1-q^n)^{-1}, \tag{4.8}$$

where $q^{-m_1-\ldots-m_t}(1-q^{m_t})$ is defined to be one if t is zero.

Identities (4.4) and (4.5) are special cases $k=t+2=2$ and $k=t+3=3$, respectively of Theorem 4.8.

For more Rogers- Ramanujan type identities involving coloured partitions, lattice paths and the Frobenius partitions the reader is referred to Agarwal [2, 3, 4, 5, 6, 7, 8, 9, 12, 15]. The Rogers- Ramanujan Identities are considered among the most beautiful formulae in Mathematics. Their study continues to be a very active area of research even today. They have applications in different areas. G. Andrews and R. Askey have demonstrated a very intimate connection between the Rogers- Ramanujan identities and certain families of orthogonal polynomails. J. Lepowsky, A. Feingold, S. Milne and R. Wilson have found a connection with Lie algebras. F. Dyson has found applications of Rogers- Ramanujan identities in particle physics and R. J. Baxter in statistical mechanism.

5. **Concluding Remarks**

We have only touched Ramanujan's contribution to theory of partitions which represent only a small part of his contributions to mathematics. But they are more than enough to prove the profoundness and invincible originality of his work. One of the most remarkable features of his work is: it remains youthful in this modern world of computers even after more than eight decades of his death. We have seen in the preceding sections how his exact formula for $p(n)$, his congruences and identities have inspired numerous mathematicians around the globe and continue to excite researchers today. We hope that they will continue to inspire generations to come.

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ABSTRACT OF THE PAPERS SUBMITTED FOR PRESENTATION AT THE 74^{th} ANNUAL CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY, HELD AT THE UNIVERSITY OF ALLAHABAD, ALLAHABAD-211002 DURING DECEMBER 27-30, 2009.

A: Combinatorics, Graph Theory and Discrete Mathematics:

A1: Forbidden-minor characterization for the class of graphic element splitting matroids, *Kiran Dalvi, Y. M. Borse and M. M. Shikare (Pune : ymborse@math.unipune.ernet.in).*

This paper is based on the element splitting operation for binary matroids that was introduced by Shikare and Azadi as a natural generalization of the corresponding operation in graphs. In this paper, we consider the problem of determining precisely which graph matroids M have the property that the element splitting operation, by every pair of elements on M, yields a graphic matroid. This problem is solved by proving that there is exactly one minor-minimal matroid that does not have this property.

A2: Zero-divisor graphs of lattices, *Meenakshi Wasadikar and Pradnya Suravase (Aurangabad : wasadikar@yahoo.com).*

The concept of zero divisor graph is studied for graphs derived from a lattice with 0. It is shown that girth of this graph is less than or equal to 4. Also some results for these graphs are obtained.

A3: On self-complimentary chordal comparability graphs, *Merajuddin and S. A. K. Kirmani (Aligarh : meraj1957@rediffmail.com, ajazkirmani@rediffmail.com).*

A graph G is self-complimentary (sc) if it is isomorphic to its complement. G is perfect if for all induced subgraphs H of G, the chromatic number of H (denoted by $\chi(X)$) equal the number of vertices in the largest clique of H (denoted by $\omega(H)$). A sc graph which is also perfect is known as a sc perfect graph. Sc comparability graphs and sc chordal graphs are the subclasses of sc perfect graphs. In this paper, we study a sc chordal graph and obtain a condition on a sc chordal graph to be a sc comparability graph using the concept of bipartite transformation of a sc chordal graph.

B: Algebra, Number Theory and Lattice Theory:

B1: Group rings without Lie regular units, *R. K. Sharma and Pooja Yadav (New Delhi : rksharma@maths.iitd.ernet.in, iitd.pooja@gmail.com).*

There are group algebras in which the Lie regular units generate the whole unit group but it is not necessary that the unit group always have Lie regular units. Here we study group algebras which do not have Lie regular units.

B2: On generalized derivations in prime and semiprime rings, *Asma Ali (Aligarh : asma_ali2@rediffmail.com).*

Let R be a prime ring and θ and ϕ be endomorphisms of R. An additive mapping $F : R \to R$ is called a generalized (θ, ϕ)-derivation on R if there exists a (θ, ϕ)-derivation $d : R \to R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$. Let S be a non empty subset of R. In the present paper, for various choices of S, we study the commutativity of a semiprime (prime) ring R admitting a generalized (θ, ϕ)-derivation F satisfying any one of the following properties:

1. $F(x)F(y) - xy \in Z(R)$, 2. $F(x)F(y) + xy \in Z(R)$
3. $F(x)F(y) - yx \in Z(R)$, 4. $F(x)F(y) + yx \in Z(R)$
5. $F[x, y] - [x, y] \in Z(R)$, 6. $F[x, y] + [x, y] \in Z(R)$
7. $F(x \circ y) - x \circ y \in Z(R)$ 8. $F(x \circ y) + x \circ y \in Z(R)$

B3: Generalized derivations as homomorphisms or as anti-homomorphisms in rings, *Rekha Rani (Aligarh : linerekha2@yahoo.co.in).*

Let R be a prime ring and θ and ϕ be endomorphisms of R. An additive mapping $F : R \to R$ is called a generalized $(\theta, \phi)-$ derivation on R if there exists a $(\theta, \phi)-$derivation $d : R \to R$ such that $F(xy) = F(x)\theta(y)+\phi(x)d(y)$ for all $x, y \in R$. Let U be a Lie ideal of R. In the present paper, we prove that if F is a generalized $(\theta, \phi)-$ derivation on U which acts as a homomorphism or as an anti-homomorphism on U, then either $d = 0$ or $U \subseteq Z(R)$.

B4: Some observation in the category of G-sets, *Gulam Muhiuddin (Lucknow : chishtygm@gmail.com).*

In this paper, we study the behaviour of monomorphisms, epimorphisms, co-retractions and retractions in the category of G-sets. Further, it is obtained that the category of G-sets is balanced.

B5: Some generalized results of ideals of a regular ring, *Ajay Kumar Yadav (Department of Mathematics, J. N. V. Sukhasan, Madhepura, Bihar).*

In this paper, we have proved some generalized results on regularity of a ring in case of ideals. It has been pointed out that behaviour pattern of ideals indicate that Von-Neumann regularity may be visualized as a generalization of the classical semi-simplicity concept of a ring. We observe that the ideals in a Von-Neumann regular ring are idemopotent. The impact of regularity of a ring is most strikingly felt on its principal ideals. It has been proved that the right principal ideals (left principal ideals) generated by an element a in a ring R without unity can also be written in the form aR (Ra) provided R is regular. Also a ring R with or without unity is regular if and only if each right (left) principal ideal of R is generated by an idempotent.

B6: On Wnil-Injectivity and its generalizations, *V. A. Hiremath (Dharwad : shanbhag-poonam@yahoo.com).*

In this paper, we introduce, nil π-regular rings which is a proper generalization of nil regular rings (n-regular rings) introduced by J. C. Wei and J. H. Chen and study their relations with wnil-injective rings. Also we introduce and study generalized wnil-injectivity and its properties.

B7: Some generalized results on regular near-rings, *M. K. Manoranjan (Department of Mathematics, B. N. M. V. college, Sahugarh, Madhepura, Bihar).*

In this paper, we have introduced the concept of a regular near ring with identity by imposing regularity condition over a near ring and have developed some generalized results of such rings.

B8: On forbidden configuration of 0-distributive lattices, *Vinayak V. Joshi (Pune : vinayakjoshi111@yahoo.com).*

In this paper, we obtain the forbidden configuration for 0-distributive lattices.

B9: On n-normal posets, *R. Halas, V. Joshi and V. S. Kharat (Pune : vsk@math.unipune.ernet.in, vinayakjoshi111@yahoo.com).*

A poset $(Q;)$ is called n-normal if every of its prime ideal contains at most n-minimal prime ideals. In the paper some properties and characterizations of n-normal ideal-distributive posets are presented.

B10: Some characterizations of upper semimodular lattices and posets, *R. S. Shewale and V. S. Kharat (Pune : rss@math.unipune.ernet.in, vsk@math.unipune.ernet.in).*

Semimodular lattices are extensively studied in the literature by many authors. Number of properties of Upper Semimodular lattices are generalized to posets in many authors. Upper Semimodular lattices are studied and characterized in many ways. Different conditions due to Mac Lane are well-known for characterizing upper semimodular lattices. In this paper we have introduced two new conditions that characterize upper semimodularity in lattice. Their relations with the existing conditions due to Mac Lane are established. Also, all these conditions for lattices are generalized for posets and relations amongst them are studied. We also give a forbidden structure as a characterization for finite upper semimodular posets. Several counter examples are constructed.

B-11: Lie ideal and jordan left derivation on semiprime rings, *B. Prajapati, R. K. Sharma and S. K. Shah (New Delhi : rksharma@maths.iitd.ac.in).*

Let R be a semiprime ring of char $= 2$ in which $u^2 = 0$ implies $u = 0$. Let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. In this presentation we show that if $' : R \longrightarrow R$ is an additive mapping satisfying $(u^2)' = 2uu'$ for all $u \in U$, then $(uv)' = uv' + vu'$ for all $u, v \in U$.

B-12: Dualization of cyclic purity, *V. A. Hiremath and Seema S. Gramopadhye (Dharwad : va_hiremath@rediffmail.com.)*

Cocylic copurity, the dualization of cyclic purity is studied. Examples are given to show that the concepts of Cocylic Copurity and cyclic purity are independent. We have also compared Cocyclic copurity with Cyclically purity given by Divani-Aazar, et al.

B13: Some fixed point theorems in Posets and Pseudo ordered sets, *Shiju George and S. Parameshwara Bhatta (Karnataka : shijugeorgem@rediffmail.com and s_p_bhatta@yahoo.co.in).*

In this paper, it is shown that the finite acyclic pseudo ordered sets having the weak fixed point property are precisely the generalized crowns. Also a generlized notion of the fixed point property, namely the n-fixed point property, for posets is discussed. Then n-fixed point property is proved to be equivalent to the fixed point property in lattices. Further, it is shown that a poset of finite width has the n-fixed point property for some natural number n if and only if every maximal chain in it is a complete lattice.

B14: Frobenius partition theoretic interpretation of a fifth order mock theta function, *M. Rana and A.K. Agarwal (Chandigarh : meenakshiaaru@gmail.com and aka@pu.ac.in).*

Recently we gave two combinatorial interpretations of $F_1(q)-$ a fifth order mock theta function by using $(n+2)-$ colour partitions and lattice paths. In this paper we extend the main result by using generalized Frobenius partitions. This results in a three-way combinatorial identity.

B15: A note on modular pairs and orthomodularity in posets, *Ramananda H. S. and S. Parameshwara Bhatta (Karnataka : ramanandahs@gmail, s_p_bhatta@yahoo.co.in).*

A negative solution to an open problem posed on modular pairs in orthocomplemented posets is obtained by constructing a counterexample. Some relations between jon-irreducible elements and modular pairs are discussed. It is also observed that orthomodular posets of finite length are strong and balanced.

B16: Projective modules and completion of an unimodular row to an elementary matrix, *Shiv Datt Kumar, Ratnesh Kumar Mishra and Raja Sridharan (Allahabad : sdt@mnnit.ac.in, sraja@math.tifr.res.in).*

In 1955, Serre asked whether finitely generated projective modules over $K[X_1, \cdots, X_n]$(K is a field) are free? This question of Serre (usually called Serre's conjecture), had profound impact on the development of Algebraic K-Theory. In fact an attempt to settle Serre's question, it led to the development of elementary Algebraic K-Theory. The theorem of Horrocks is the crucial ingredient in the Quillen-Suslin proof of Serre's conjecture.

B17: A new class of lattice paths and partitions with n copies of n, *S. Anand and A. K. Agarwal (Chandigarh : shivani.bedi.maths@gmail.com).*

In this paper, we consider a new class of lattice paths. Using these lattice paths, we interpret combinatorially three q-series which lead to several new combinatorial identities involving partitions with $n+t$ copies of n and these lattice paths. We also obtain a lattice path representation for partitions with $n+t$ copies of n and use this representation in providing graphical representation for partitions with n copies of n as well as for n-color compositions.

C: Real and Complex Analysis(Including Special Functions, Summability and Transforms):

C1: Nonnegative Moore-Penrose inverse and reflexive generalised inverses with application, *Sachindranath Jayaraman (Chennai : sachindranath@cmi.ac.in).*

We investigate non-negative of the Moore-Penrose inverse and of reflexive generalized inverses in connection with the notion of non-ngetive rank factorization in infinite dimensions.

C2: On certain transformations for poly-basic hypergeometric series, *Roselin and N. A. Herbert (Allahabad : roselinmaths@gmail).*

Bailey's transformation and terminating quadratic, cubic and quartic series are used to derive numerous transformation formulae for poly-basic hypergeometric series in a unified manner.

C3: On monomials satisfying certain functional equations, *P. M. Patil and A. P. Singh (Department of Mathematics, Goverment Polytechnic, Kolhapur, Maharashtra and Department of Mathematics, University of Jammu, Jammu).*

Nevanlinna theory of meromorphic functions has been used to study the properties regarding the growth and the number of zeros of monomials satisfying certain equations.

C4: Multiplicity and relative defects of homogeneous differential polynomials, *Harina P. Waghamore (Department of Mathematics, Central college campus, Bangalore University, Bangalore).*

We extend the concept of relative defects of meromorphic functions to differential monomials and find various relations involving these deficient values with the Nevanlinna deficient values.

C5: Trigonometric approximation of functions in L_p- Norm, *M. L. Mittal and Vishnu Narayan Mishra (Department of Mathematics, Indian Institute of Technology, Roorkee - 247667, Uttarakhand and Department of Mathematics, (ASHD), S. V. National institute of Technology, Surat - 395007, Gujarat).*

In this paper, we extend two theorems of Leindler [*J. Math. Anal. Appl. 302 (2005) 129-136*], where he has taken less stringent conditions on the generating sequence $\{p_n\}$ given by Chandra [*J. Math. Anal. Appl. 275, 2002, 13-26*,] to more general classes of triangular matrix methods. Our Theorems also generalize theorem 4 partially and (ii) part of theorem 5 of Mittal et. al. [*J. Math. Anal. Appl. 326 (2007) 667-676*] by dropping the monotonicity on the elements of matrix rows.

C6: A subclass of p-valently harmonic meromorphic functions defined by an operator, *Amit Kumar Yadav and Poonam Sharma (Lucknow : poonambaba@yahoo.com, yadav_kumar_amit@yahoo.co.in).*

In this paper, we study a subclass $\sum_H(\mu, p, \alpha, \lambda, k)$ of p-valently harmonic meromorphic functions which are orientation preserving in the exterior of the unit disk. We give sufficient condition for the functions belonging to $\sum_H(\mu, p, \alpha, \lambda, k)$ class. We also prove necessary condition when the coefficient are negative belonging to its subclass $\sum_{\overline{H}}(\mu, p, \alpha, \lambda, k)$. This leads to extreme points, distortion bounds, convolution conditions and convex combinations for the functions belonging to its subclass $\sum_{\overline{H}}(\mu, p, \alpha, \lambda, k)$.

C7: An evaluation of an analytic function due to an unified integral and its generating functions through Lie-group methods, *Hemant Kumar and S. P. Singh Yadav (Kanpur : palhemant2007@rediffmail.com, satyarth_pal_singh_yadav@yahoo.co.in).*

In the present paper, we evaluate an analytic function by defining a unified integral and then, obtain its some generating functions on introducing some of Lie-groups and a basic function. Finally, on making their applications, we obtain some generating relations involving H-function of two variables.

C8: Order of magnitude of double Fourier coefficients of functions of bounded p-variation, *B. L. Ghodadra (Baroda : bhikhu_ghodadra@yahoo.com).*

In this paper, the notion of bounded p-variation, for a complex-valued function of two variables, is defined. If $f : \Re^2 \to C$ is $2\pi-$ periodic in both variables and is of bounded p-variation over the torus $T^2 = [0, 2\pi]^2$, then it is shown that its Fourier coefficient $\hat{f}(n) = 0(1/|n|^{1/p})$.

D: Functional Analysis:

D1: Implicit iteration for a finite family of accretive operators, *Niyati Gurudwan and B. K. Sharma (School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur, Chhattisgarh).*

In this paper, the strong convergence of the implicit iteration process (introduced by Xu and Ori) is proved for a finite family of accretive operators using resolvent in Banach space.

D2: A Hilbert space of Dirichlet series, *B. L. Srivastava and Abhay Raj Singh (Department of Mathematics, D. B. S. (P. G.) Collage, Kanpur, Uttar Pradesh).*

In this paper, we stduy a Hilbert space Ω_u^2 of functions analytic in half plane which are represented by Dirichlet series $\sum_{n=1}^{\infty} a_n e^{s\lambda n}$, where $\{a_n\}$ is a complex sequence in general and $s = \sigma + it$ with $\sigma, t \in \Re$ and $\{\lambda_n\}$ is a fixed sequence of positive real numbers. The norm in this space is given by weighted l_2 norm of the Dirichlet coefficients $a_n's$. We find orthonormal basis in Ω_u^2 and its projection decomposition form. It has also been established that Ω_u^2 is isometric isomorphic to the energy space l_2. Characterisation of diagonal operators in this space has also been studied.

D3: Fixed point of asymptotic contractions, *S. B. Singh and A. K. Singh (Department of Mathematics, A. M. Collage, Gaya).*

In this paper, simple and elementary proofs of the unique fixed point theorems are given. Moreover, weakly asymptotic contractions are introduced and it is proved that a weakly asymptotic contraction on a complete metric space has a unique fixed point.

D4: Polynomial compactness of non-surjective composition operators on l^2, *Kamlesh Kumar Rai (Department of Mathematics, Dera Natung Government Collage, Itanagar - 791113, Arunachal Pradesh).*

Let l^2 be the Hilbert space of all square summable sequences of complex numbers under the standard inner product. Let C_ϕ be a non-surjective composition operator on l^2 induced by a function ϕ on N into intself. We characterize the polynomial compactness of non-surjective composition operators on l^2. Polynomial compactness of surjective operators on l^2 has already been characterized.

D5: Hilbert spaces versus Hilbert-Schmidt spaces, *M. A. Sofi (Srinagar : aminosfi@rediffmail.com).*

A Banach space X is said to be Hilbert-Schmidt space if every Hilbert space map factoring over X is a Hilbert-Schmidt map. Grothendieck's theorem informs us that $L_1(\mu)$ and $L_\infty(\mu)$ are Hilbert-Schmidt spaces. Dually, it turns out that a bounded linear map between Hilbert-Schmidt spaces and factoring over a Hilbert space is a 2-summing map - a class of bounded linear maps which coincide with the Hilbert-Schmidt maps when acting between Hilbert spaces. This duality phenomenon is archetypal of many such results that one encounters in the study of Hilbert-Schmidt spaces vis-a-vis the more familar Hilbert spaces. In this connection, we recall the well known characterization of Hilbert spaces X in terms of the Hahn-Banach property
(*) of operators on its subspaces.
(*) For every Banach space Y, a subspace Z of X and a bounded linear map $T : Z \to Y$, there exist a bounded linear map $S : X \to Y$ which extends T. The corresponding result for Hilbert-Schmidt spaces states that every bounded linear map $T : X \to l_2$ extends to a bounded linear map on any superspace containing X.

Our main aim in this paper is to study this duality phenomenon in the context of operator ideals involving nuclear and summing operators which can be used to yield some striking characterizations of these classes of spaces, reflecting the extreme behaviour which is manifested in the geometry of these two classes of Banach spaces.

D6: Fixed point theorem in 2 Non-Archimedean Menger PM-space using weak compatibility, *Bijendra Singh, Arihant Jain and Jaya Kushwah (Ujjain : samagri20@rediffmail.com).*

In the year 1947, Istratrescu and Crivat have initiated the concept of non-Archimedean Menger space. Afterward, in 1978, Istratrescu proved the fixed point theorems for some classes of contraction mappings on non-Archimedean probabilistic metric space. This has been the extension of the corresponding results of Sehgal and Bharucha - Reid [1972] on a Menger space. In the present paper, we introduce the notion of weak-compatible self maps in a 2 non-archimedean Menger PM-space and prove a common fixed point theorem which generalizes the results of Cho, Ha and Chang [1997]. We also construct an example for weak compatible self maps.

D7: Reduction of an operator equation into an equivalent Bifurcation equation through Schauder's fixed point theorem, *Pallav Kumar Baruah, B. V. K. Bharadwaj and M. Venkatesulu*

(Prasanthinilayam : baruahpk@sssu.edu.in, bvkbharadwaj@sssu.edu.in and Krishanankoil : venkatesulum2000@yahoo.co.in).

In this paper, we deal with the Nonlinear Coupled Ordinary Differential Equations (Nonlinear CODE). A Multipoint Boundary Value Problem (MBVP) associated with these Nonlinear equations is defined as an operator equation. This equation (infinite dimensional) is reduced to an Equivalent Bifurcation Equation (fiite dimensional) using Schauder's Fixed Point Theorem. This Bifurcation Equation being on a finite dimensional space is easy to solve.

D8: A fixed point theorem in Menger space, *Tamanna and Renu Chugh (Rohatak : tamannakadian@yahoo.co.in and Chughrenu@yahoo.com).*

The aim of this paper is to prove a common fixed point theorem in Menger space by studying the relationship between the continuity and reciprocal continuity. We also give an example to illustrate our main theorem.

D9: Coincidence and fixed points of nonself maps using generalized *T*-weak commutativity, *Praveen Kumar Shrivastava and N. P. S. Bawa (Balaghat and Rewa : pspraveenshrivastava@yahoo.co.in)*

The purpose of this paper is to introduce the concept of sequential T-weak commutativity and generalized T-weak commutativity, which is an extesion of T-weak commutativity defined by Kamran for hybrid pair of mappings. We prove that this concept is equivalent to generalized compatibility of type (N) is more general than (IT)-commutativity introduced by Singh and Mishra. Using generalized T-weak commutativity we also extend a result of Pathak and Mishra who have indicated that a result of Chang admits a counter example and presented a corrected version of the result. We have also shown that the corrected version of the result of Chang is also derivable from our results. We thus extend and generalize many known results in this way.

D10: A study of locally convex nuclear space, *Kirti Prakash (Patna : kirti_prakashl@rediffmail.com).*

In the present paper, a new characterization of a metric locally convex nuclear space has been established as a theorem:

A metric locally convex space E is nuclear iff $L(C_0, E) \equiv \pi_1(C_0, E)$.

In order to prove the above mentioned theorem we have come to the conclusion that A metric locally convex space E is nuclear iff $L(l_q, E) \equiv$

$\pi_p L(l_q, E), 1 \leq p \leq \infty, 1/p + 1/q = 1 \cdots (I)$. Since, for $p = 1$, l_q denote the classical Banach Space C_0, putting $p = 1$ in (I), we get $L(C_0, E) \equiv \pi_1(C_0, E)$ Thus, the theorem is established.

D11: On a class of Bicomplex sequences, *Mamata Nigam (Agra : nigam.mamta@gmail.com).*

In this paper, we have investigated certain properties of a particular class B of bicomplex sequences associated with the bicomplex functions which are holomorphic in the bicomplex space $C2$ with a functional analytic view point. B is a subclass of the class B which has been defined and studied by Srivastava and Srivastava in 2007. B has been provided with a modified Gelfand algebraic structure and it has been proved that B is an algebra ideal which is not a maximal ideal of B. Invertible and quasi invertible elements in B have been studied. A characterization of zero divisors is given and a sufficient condition for an element to be topological zero divisor has been derived. Algebra homomorphism between B and B has been investigated.

D12: Fekete-Szego probelm for a class defined by the integral operator, *P. Gochhayat (Orissa : pb_gochhayat@yahoo.com).*

Let A denote the family of functions analytic in the open disk U and also let A_0 be the class of functions f in A satisfying the normalization condition $f(0) = f'(0) - 1 = 0$ and given by the power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. An analogous to Ruscheweyh derivativies, Noore defined an integral operator $I_n : A_0 \longrightarrow A_0$ $(n \in N_0 = \{0, 1, 2, 3, \cdots\})$ as follows: $I_n f(z) = \frac{n+2}{z^2} \int_0^z t^{n-1} I_{n-1}(f(t))dt$. Let $f_n(z) = z/(1-z)^{n+1}$ and let $f_n^{(-1)}$ be defined such that $f_n(z) * f_n^{(-1)}(z) = \frac{z}{z-1}$. Then $I_n f(z) = f_n^{(-1)}(z) * f(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} * f(z)$. The operator $I_n f$ defined above is called n^{th} order Noor integral operator of f. By making use of this integral operator I_n due to Noor, a new subclass of analytic univalent functions related to conical domain and denoted by $I_{n,k}(0 \leq k < \infty)$ is introduced, i. e. the function $f \in A_0$ is said to be in the class $I_{n,k}$ if $I_n f$ is k-parabolic starlike. For this class the sharp coefficient bounds for the nonlinear functional $|\mu a_2^2 - a_3|$ is found. The result obtained here also gives the Fekete-Szegö inequality for k-uniformly conves functions and k-parabolic starlike functions.

D13: A Stability result for set valued operators, *Bhagwati Prasad (Noida : bhagwati.prasad@jiit.ac.in, b_prasad10@yahoo.com).*

In this paper, we discuss the stability of Picard iterative procedures for set valued operators in b-metric spaces. Several results are also obtained as special cases.

D14: Fixed points in dislocated quasi-metric space, *C. T. Aage and J. N. Salunke (Jalgaon : caage17@gmail.com).*

In this paper, we generalize some fixed point theorems and also obtain new results in Dislocated quasi-metric spaces.

D15: Common fixed point theorems for compatible mappings of type (B) in non-Archimedean Menger PM-space, *Bijendra Singh, Chanchal Chouksey and Arihant Jain (Ujjain : arihant2412@gmail.com).*

There have been a number of generalizations of metric space. One such generalization is Menger space. Istratrescu and Crivat [1974] have established the notion of non-Archimedean Menger space. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istratrescu [1978]. This has been the extension of the corresponding results of Sehgal and Bharucha Reid [1972] on a Menger space. Cho et. al. [1997] proved a common fixed point theorem for compatible mappings in non-Archimedean Menger PM-space.

In this paper, we generalize the result of Cho et. al. [1997] by introducing the notion of compatible maps of type (B)and weak compatible self maps.

D16: A fixed point theorem in non-Archimedean Menger PM-space, *Chhatrapal Singh, Bijendra Singh and Arihant Jain (Ujjain : arihant2412@gmail.com).*

The notion of non-Archimedean Menger space has been established by Istratescu and Crivat [1974]. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istratescu [1978]. This has been the extension of the corresponding results of Sehgal and Bharucha - Reid [1972] on a Menger space.

The object of this paper is to estabish a fixed point theorem for six self maps and an example using the concept of weak-compatible self maps in a non-Archimedean Menger PM-space. Our result generalizes the result of Cho, et al [1997].

D17: Fixed point theorem using weak-compatibility in Fuzzy metric space, *Bijendra Singh and Arihant Jain (Ujjain : arihant2412@gmail.com).*

The concept of Fuzzy sets was initially investigated by Zadeh [1965] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy matric space defined by Kramosil and Michalek [1975] and modified by George and Veeramani [1994]. Recently, Grebiec [1998] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [2000] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Recently, Jungck and Rhoades [1998] introduced the concept of weak compatible maps. Using this concept, Jain et al. [2005] proved the fixed point theorem in Fuzzy metric space.

In this paper, a fixed point theorem for six self maps has been established using the concept of compatibality and weak compatibility of pair of self maps, which results in the main theorem established by Cho [2006]. We also construct a example for weak compatible maps.

E: Differntial Equations, Integral Equations and Functional Equations:

E1: On Existence results of second order impulsive integrodifferential inclusions, *S. D. Kendre and M. B. Dhakne (Pune : ksubhash@math.unipune.ernet.in and mbdhakne@yahoo.com).*

This paper is concerned with the existence of solution for initial value problem of second order impulsive integrodifferential inclusions in a real separable Banach spaces by using semigroup theory and suitable fixed point theorems when the multivalued map has convex and nonconvex values.

E2: Mathematical modeling of combat: Security force versus insurgents under decapitation warfare, *Lambodara Sahu and N. S. Nagarjuna (Pune : lsahucme@gmail.com).*

There has been a sharp increase in incidences of insurgency like Naxalites activity and terrorism in the recent times, resulting enhancement in the employment of security forces to combat such activity. The security forces find themselves engaged in a war like situation where identification of insurgents and their ultimate captive/ destruction pose a major problem. The task is made more difficult for the security forces because of insurgents have freedom of movement and take active steps to defeat the search.

Beside regular combat, security forces as a part of their strategy engage in decapitation warfare like killing or arresting the key leaders of insurgents, capturing their central base, initiating undermining operations such as precision strike, missile attack etc. against insurgents. Insurgents are also involved in the decapitation warfare like assassination or abduction of key Govt. officials, ministers and high profile politicians.

Concepts derived from Lanchesters and Lanchester-type combat models are used to project the effectiveness of forces of regular combat under decapitation warfare. A conceptual model is presented to describe the effects of decapitation strategy on the regular battlefield. With extensive coverage of operational factors such as robustness of forces, undermining effects, breakpoints, attrition rates, total force level, the model is suitable to analyze complex scenario of a military operation consisting of decapitation strategy. An illustrative example is provided to demonstrate the application of the model.

E3: Existence and uniqueness of mild and strong solutions of nonlinear mixed Volterra-Fredholm integrodifferential equations with nonlocal condition in Banach spaces, *H. L. Tidke and M. B. Dhakne (Aurangabad : tharibhau@gmail.com, mbdhakane@yahoo.com).*

In this paper, we prove the existence and uniqueness of mild and strong solutions of a nonlinear mixed Volterra-Fredholm integrodifferential equation type with nonlocal condition. Our analysis is based on semigroup theory and Banach fixed point theorem.

E4: On a non-linear dynamic integrodifferential equation on time scale, *Deepak B. Pachpatte (Aurangabad : pachpatte@gmail.com).*

The main objective of the present paper is to study some basic qualitative properties of solutions of certain non-linear integrodifferential equation on time scales. The tools employed in the analysis are based on the applications of the Banach fixed point theorem and certain inequalities with explicit estimates on time scale.

E5: On system of six order simultaneous differencial equations, *Ashwani Kumar Sinha, Alok Kumar and Md Arshaduzzaman (Department of Mathematics, M. M. Mahil Collage, Ara, Bihar and Department of Mathematics, Maharaja Collage, Ara, Bihar and Mumtaj Mohalla, Naugachia, Bhagalpur, Bihar).*

In this paper, some theorems regarding Wronskian of the solutions associated with a system of simultaneous differencial equations have been proved.

E6: Some properties of certain family of integral operators, *P. Sahoo and S. Singh (Department of Mathematics, Banaras Hindu University, Banaras).*

In this paper, we discussed some properties of the class of one parameter family of integral operators $(P^{\alpha}f)(z)$.

F: Geometry:

F1: Some theorems on almost Kaehlerian spaces with recurrent and parallel Bochner curvature tensors, *U. S. Negi and Aparna Rawat (Department of Mathematics, H. N. B. Garhwal University, Campus Badshahi Thaul, Tehri Garhwal - 249199).*

Mishra (1968) has defined and studied the recurrent Hermite spaces. Prasad (1973) has defined and studied certain properties of recurrent and Ricci-recurrent almost Hermitian spaces. In the present paper, we have defined almost Kaehlerian spaces with recurrent and parallel Bochner curvature tensors and several theorems have been established. The conditions that an almost Tachibana space be a First order and First kind as well as the Second order and First kind with recurrent and parallel Bochner curvature tensor respectively have been obtained.

F2: On Racci curvature of certain submanifolds in generalized complex space forms, *Pawan Kumar Rao and S. S. Shukla (Allahabad : babapawanrao@rediffmail.com).*

In this article, we establish inequalities between the Ricci curvature and the squared mean curvature and also between k-Ricci curvature and the scalar curvature for slant, semi-slant and bi-slant submanifolds in a generalized complex space form.

F3: On D-homothetic deeormation of trans-Sasakian structure, *A. A. Shaikh (Burdwan : aask2003@yahoo.co.in).*

In the classification of Gray and Hervella [*Ann. Mat. Pura.Appl. 1980*] of almost Hermitian manifolds there appear two classes which contains locally conformal Kahler manifolds and locally conformal almost Kahler manifolds. Also these classes are preserved under conformal diffeomorphism. An almost contact metric structure is trans-Sasakian [*Oubina, Publi.Math. Debrecen, 1985*] if the almost Hermitian structure of the manifold is locally conformal Kahler.

Again, Chinea and Gonzalez have introduced [*Ann.Mat. Pura. Appl., 1990*] two subclasses of trans-Sasakian structures, which contain the Kenmotsu and Sasakian structures. The class of trans-Sasakian structures contains the class of generalized quasi-Sasakian structures. The object of the present paper is to study D-homothetic deformation of trans-Sasakian structures. Among others it is shown that the Ricci operator Q does not commute with the structure tensor ϕ and the operator $Q\phi - \phi Q$ is conformal under a D-homothetic deformation. Some non-trivial examples of trans-Sasakian (non-Sasakian) manifolds with global vector fields are obtained.

F4: On the existence of locally ϕ-recurrent LP-Sasakian Manifolds, *A. A. Shaikh and Kanak Kanti Baishya (Burdwan : aask2003@yahoo.co.in, aask@epatra.com).*

In 1989 K. Matsumoto [*Bull. Yagamata Univ. Nat. Sci. 1989*] introduced the notion of LP-Sasakian manifolds. Then I. Mihai and R. Rosca [*Classical Analysis, World Scientific, 1992*] introduced the same notion independently and obtained many interesting results. As a weaker version of local symmetry, T. Takahashi [*Tohoku Math. J., 1977*] introduced the notion of local ϕ-symmetry on a Sasakian manifold. In the context of Lorentzian geometry, the notions of weakly and strongly local ϕ-symmetry were introduced and studied by Shaikh and Baishya [*Yokohama Math. J., 2006*] with several examples. Generalizing these notions, in the present paper we introduce the notions of weakly locally ϕ-recurrent LP-Sasakian manifold and strongly locally ϕ-recurrent LP-Sasakian manifold. The class of weakly locally ϕ-recurrent LP-Sasakian manifolds contains the class of strongly locally-recurrent LP-Sasakian manifolds. Among other interesting results we have obtained various necessary and sufficient conditions for an LP-Sasakian manifold to be of weakly locally ϕ-recurrent and the existence of such a manifold is ensured by several non-trivial examples in both odd and even dimensions. Also it is shown that a strongly locall ϕ-recurrent LP-Sasakian manifold is always η-Einstein and its scalar curvature is constant.

F5: On weakly conharmonically symmetric manifolds, *Shyamal Kumar Hui and A. A. Shaikh (Burdwan : aask2003@yahoo.co.in).*

The study of Riemann symmetric manifold beagn with the work of E. Cartan (1926). Generalizing the notion of locally symmetric manifolds,

A. G. Walkar [*Proc. London Math. Soc. 1950*] introduced the notion of recurrent manifolds and later studied by various authors. Extending the notion of recurrent manifolds. L. Tamassy and T. Q. Binh [*Coll. Math. Soc. J. Bolyai, 1989*] introduced the notion of weakly symmetric manifolds. As a special subgroup of the conformal transformation group, Ishii [*Tensor N. S., 1957*] introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function.

The object of the present paper is to introduce a type of non-flat Riemannian manifold called weakly conharmonically symmetric manifold and to study its several geometric properties along with the existence of such a manifold by several non-trivial examples.

F6: On weakly symmetric Riemannian manifolds, *Sanjib Kumar Jana and A. A. Shaikh (Burdwan : aask2003@yahoo.co.in).*

The study of Riemann symmetric manifolds began with the work of E. Cartan (1926). During the last five decades the notation of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as recurrent manifold by A. G. Walkar [*Proc. London Math. Soc., 1950*], semisymmetric manifold by Z. I. Szabo [*J. Diff. Geom., 1982*], pseudosymmetric manifold in the sense of R. Deszcz [*Bu.. Belg. Math. Soc., 1992*], pseudosymmetric manifold in the sense of M. C. Chaki [*AL. I. CUZA,1987*] and weakly symmetric manifold by L. Ta massy and T. Q. Binh [*Coll. Math. Soc. J. Bolyai, 1989*].

The object of the present paper is to study weakly symmetric Riemannian manifolds. Among others it is shown that a conformally flat weakly symmetric Riemannian manifold is of hyper quasi-constant curvature which generalizes the notion of quasi-constant curvature introduced by Chen and Yano [*Tensor N. S., 1972*] and also such a manifold is a quasi-Einstein manifold by Chaki and Maity [*Publ. Math. Debrecen, 1997*]. Finally, several examples of weakly symmetric manifolds of both zero and non-zero scalar curvature are obtained.

F7: Hypersurfaces of conformally and h-conformally related Finsler spaces, *M. K. Gupta and P. N. Pandey (Allahabad : manishkg_1982@yahoo.com and pnpiaps@rediffmail.com).*

In this paper, certain geometrical properties of the hypersurfaces of two con-formally and h-conformally related Finsler spaces have been discussed.

F8: Kaehler manifold with a special type of semi-symmetric non-metric connection, *B. B. Chaturvedi and P. N. Pandey (Allahabad : manishkg_1982@yahoo.com and pnpiaps@rediffmail.com).*

In the present paper, we have obtained certain results for a Kaehler manifold equipped with the semi-symmetric non-metric connection and a special type of semi-symmetric non-metric connection. We have obtained the expressions for the curvature tensor, Ricci tensor and proved certain results related to them. We have also discussed the properties of torsion tensor as a second order parallel tensor.

F9: On P-Sasakian manifolds satisfying certain conditions on the quasi conformal and quasi concircular curvature tensors, *Rajesh Kumar, B. Prasad and S. K. Verma (Mizoram : rajesh_mzu@yahoo.com).*

T. Adati introduced the locally symmetric and semi-symmetric P-Sasakian manifold and studied a Riemannian manifold M is locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where ∇ is the Levi-Civita connection of the Riemannian manifold. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to semi-symmetric if its curvature tensor R satisfies $R(X,Y).R = 0, X, Y \in TM$ where $R(X,Y)$ acts on R as a derivation.

In this paper, we study several derivations on P-Sasakian manifold satisfying certain conditions on the quasi concircular curvature tensor (B. Prasad and A. Maurya)

$$Z(X,Y)U = aR(X,Y)U + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y,U)X - g(X,U)Y\},$$

where a and b are constant, and quasi conformal curvature tensor (M. C. Chaki and Ghosh, 1997).

$$C(X,Y)U = aR(X,Y)U + b\{S(Y,U)X - S(X,U)Y + g(Y,U)QX - g(X,U)QY\} - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y,U)X - g(X,U)Y\},$$

where a and b are constant.

We classify necessary and sufficient conditions for P-Sasakian manifolds, which satisfy the conditions
$Z(\xi,X).Z = 0$, $Z(\xi,X).R = 0$, $R(\xi,X).Z = 0$, $Z(\xi,X).S = 0$ and $Z(\xi,X).C = 0$.

G: Topology:

G1: Generalized nearness spaces and their completions, *Mona Khare and Surabhi Tiwari (Allahabad : au.surabhi@gmail.com).*

Ivanova and Ivanov [*Ixu. Akad. Nauk. SSSR Ser. Mat., 1959*] axiomatized contiguity spaces and employed them in obtaining the T_1- compactifications of a given T_1-space. In 1974, Herrlich axiomatized the nearness of collections of subsets of a nonempty set X and named it nearness space. Axiomatizing the notion of distance between a set and a point, Lowen et. al. [*Top. Appl, 1994*] introduced approach spaces, and gave various equivalent forms of it. The topological category AP of approach spaces and contractions contains the category TOP of topological spaces and continuous maps bireffectively and bicoreflectively, $pMET^{\infty}$ of extended metric spaces and non-expansive maps bicoreffectively, and $pqMET^{\infty}$ of quasi metric spaces and nonexpansive maps bicoreffectively. The notion of 'distance' in AP is closely related to the notion of nearness; further proximity and nearness concepts arise naturally in the context of approach spaces. In the present paper, we generalize nearness and contiguity spaces, and obtain some categorical results for them. The completion of generalized nearness spaces is also constructed.

G2: Paraopen sets and maps in topological spaces, *S. S. Benchalli and B. S. Ittanagi (Dharwad : benchalli-math@yahoo.com and dr.basavarajsit@gmail.com).*

In this paper, we introduce and study the concept of a new class of sets called paraopen set and paraclosed sets in topological spaces. During this process some of their properties are obtained. Also we introduce and study a new class of maps called paracontinuous, $*$ - paracontinuous, parairresolute, minimal paracontinuous and maximal paracontinuous maps and study their basic properties in topological spaces.

G3: On generalized minimal closed sets, *Sharmistha Bhattacharya (Tripura : halder-731@rediffmail.com).*

The aim of this paper is to introduce two new concept of generalized closed sets, that is, generalized minimal closed sets and generalized $*$ - minimal closed sets. Also the dual concept, that is, generalized maximal open set and generalized $*$ - maximal open sets are introduced. Though the generalized open sets do not form a topological space, but the generalized maximal open sets forms a conditional pseudo topological space. It can also be shown that, a subset A of X is a generalized minimal closed set if and only if the minimal open sets containing A are also closed sets. It is to be noted that this set is an independent concept of open sets and closed sets but is a stronger form of generalized closed sets. Again generalized $*$ - closed set is similar to dense set but generalized $*$ - minimal closed set is an independent concept. A set is both generalized minimal closed and generalized $*$ - minimal closed set if and only if the minimal opens set containing the set is same as the closure of the set.

G4: Cohomology algebra of the orbit space of some free actions on spaces of cohomology type $(a,\ b)$, *Hemant Kumar Singh and Tej Bahadur Singh (Delhi : hksinghdu@gmail.com and tej_b_singh@yahoo.co.in).*

Let X be a finitistic space with non-trivial cohomology groups $H^{im}(X;Z) \equiv Z$ with generators v_i, where $i = 0, 1, 2, 3$. We say that X has cohomology type (a, b) if $v_1^2 = av_2$ and $v_1v_2 = bv_3$. In this note, we determine the mod 2 cohomology ring of the orbit space X/G of a free action of $G = Z_2$ on X, where both a and b are even. In this case, we observed that there is no equivariant map $S^m \to X$ for $m > 3n$, where S^m has the antipodal action. Moreover, it is shown that G can not act freely on space X which is of cohomology type $(a,\ b)$ where a is odd and b is even. We also obtain the mod 2 cohomology ring of the orbit space X/G of free action of $G = S^1$ on the space X of type $(0,\ b)$.

G5: m - n proximity space, *Md. Arshaduzzaman and Ashwani Kumar Sinha (Mumtaz Mohalla, Naugachi, Bhagalpur, Bihar and Department of Mathematics, M. M. Mahila Collage, Ara, Bihar).*

The paper is addressed to an introduction of an $m - n$ proximity space, with its m, n ordoform characterization and n-ary proximal closure and interior generalizing classical spatial structures.

G6: Generalized minimal closed sets in bitopological spaces, *Suwarnlatha N. Banasode and S. S. Benchalli (Belgum : suwarn_nam@yahoo.co.in and benchalli_math@yahoo.com).*

In this paper we introduce and investigate generalized minimal closed sets in bitopological spaces. Let $i, j \in \{1, 2\}$ be fixed integers. In a bitopological space (X, τ_1, τ_2), a subset A of X be $(\tau_1, \tau_2)-$ generalized minimal closed (briefly$(\tau_1, \tau_2) - g - m_i$ closed) set if $\tau_j - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-minimal open set in $(X; \tau_1, \tau_2)$.

Some basic properties of such spaces are obtained.

G7: Disjoint neighbourhood sets in graphs, *B. Chaluvaraju (Bangalore : bchaluvaraju@yahoo.co.in).*

A set $S \subseteq V$ is a neighbourhood of set G, if $G = U_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v. The neighbourhood number $\eta(G)$ of G is the minimum cardinality of a neighbourhood set of G. In this paper, we extend the concept of neighbourhood set to disjoint neighbourhood set and its graph theoretical relationships are explored.

G8: On α and α^*- separation axioms in intuitionistic fuzzy topological spaces, *Rekha Srivastava and Amit Kumar Singh (Banaras : rekhasri@bhu.ac.in and amitkitbhu@gmail.com).*

Atanssova [*Fuzzy and Systems 20 (1986) 87-96*] gave the notion of intuitionistic fuzzy sets and using it Mondal and Samanta [*Fuzzy sets and systems 131 (2002) 323-336*] introduced intuitionis-tic fuzzy topological spaces. In this paper we have defined gradation of separation axioms $ST_0, RT_0, , T_0, T_1$ and T_2 in these spaces. Using these concepts, we introduce $\alpha - RT_0, \alpha - T_i(\alpha \in I_0)$ and $\alpha^* - RT_0, \alpha^* - T_i(\alpha \in I_0), i = 0, 1, 2$ intuitionistic fuzzy topological spaces and studied them in detail. Several results have been proved. In particular we mention that all the properties satisfy the hereditary, productive and projective properties.

G9: On Self-complimentary chordal comparability graphs, *Merajuddin and S. A. Kirmani (Aligarh : meraj1957@rediffmail.com and ajazkirmani@rediffmail.com).*

A graph G is self-complementary (sc) if it is isomorphic to its complement. G is perfect if for all induced subgraphs H of G, the chromatic number of H (denoted by $\chi(H)$) equal the number of vertices in the largest clique in H (denoted by $\omega(H)$). A sc graph which is also perfect is known as a sc perfect graph. Sc comparability graphs and sc chordal graphs are the subclasses of sc perfect graphs. In this paper we study a sc chordal graph and obtain a condition on a sc chordal graph to be a comparability graph using the concept of bipartite transformation of sc chordal graph.

G10: Some observation in the category of *G*-sets, *Gulam Muhiuddin (Lucknow : chishtygm@gmail.com).*

In this paper, we study the behaviour of monomorphisms, epimorphisms, co-retractions and retractions in the category of G-sets. Further, it is obtained that the category of G-sets is balanced.

H: Measure Theory, Probability Theory and Stochastic Processes and Information Theory:

H1: Generators, Isomorphism and Conjugation, *Mona Khare and Shraddha Roy (Department of Mathematics, University of Allahabad, Allahabad).*

Entropy is a measure of randomness or disorder. The importance of entropy in ergodic theory arises from its usefulness in connection with the isomorphism problem (see Pykacz [*Inter. J. of Theor. Phy., 1994*]) for measure preserving transformations. In ergodic theory as well as in information theory (see Abramson [*Information Theory and Coding, 1963*]) the entropies of dynamical systems are important because they are invariant under isomorphisms and has been applied in many structures. The major theorem concerning ergodic-theoretic entropy is due to Kolmogorov and Sina j, asserts that the entropy of a dynamical system can be computed by finding its entropy with respect to a generator. In the present paper we intend to study these notions and related theory of dynamical systems in the context of quantum structures.

I: Numerical Analysis, Approximation Theory and Computer Science:

I1: A $(Q,\ R)$ Model for Fuzzufied deterioration under cobweb phenomenon and permissible delay in payment, *S. S. Mishra and P. P. Mishra (Faizabad : sant_x2003@yahoo.co.in).*

There always exists a significant time leg between production and consumption of the inventory items and in this leg procedures face the risk of decision making not only about levels of production and consumption but also deterioration of the items that take place during the leg. In order to provide a scientific foundation to aforesaid decision making about optimal quantity and price of the items, a $(Q,\ R)$ model is here studied for the fuzzified deterioration occurring in time leg between production and consumption of items that deals with the traditional cobweb phenomenon and an attractive policy of permissible delay in the payment. A computational algorithm has been developed to solve the problem in order to attain the optimal quantity and its price for the model. The paper also presents the sensitivity analysis and comparative study of aforesaid model under crisp and fuzzy environments with the help of illustrative examples to simply gain the better perspectives of the model from the application point of view.

I2: Almost quintic splines and Cauchy problem, *Kulbhushan Singh, Riyaz Ahmed and Akhilesh Kumar Mishra (Lucknow : akmishra_77@rediffmail.com).*

A modified (0, 1, 4) interpolation problem has been solved using almost quintic spline interpolants of class $S^*_{n,5}(2)$. Theorems of unique existence and convergence have also been proved. Finally, these spline functions have been used to find the approximate solutions of cauchy's initial value problem $Y'' = f(x, y, y'), x \in [0, 1], y(0) = y_0$ and $y'(0) = y'_0$.

I3: A note on the relative reward pheromone update strategy for aco algorithms, *Raghavendra G. S. and Prasanna Kumar N. (Goa : gsr@bits-goa.ac.in).*

The foraging behaviour of ants has fascinated many researchers. This has lead to the evolution of new class of algorithms, which are distributive in nature and solved by group of cooperating agents. Recent stimulation in this direction was made using colony of cooperating ants by M. Dorigo. These classes of algorithms can be used as optimization tool to solve NP hard class problems. These classes of algorithms have been implemented successfully for commercial purposes. An analogy with the way ant colonies function has suggested the definition of a new computational paradigm, which we call Ant System. M Dorgio proposed this as a viable new approach to stochastic combinatorial optimization and subsequent papers on ACO have tried to improve the performance of the algorithm. In this paper we propose new way to update the pheromone trials, which is based on competition among the ants. We also propose an alternate strategy called "Hybrid Global-Best" to update globally best-path, which has significant impact on the performance of the algorithm and we report the result, which is best compared to all the algorithms available in the literature.

I4: Results on Mealy-type fuzzy finite state machines, *S. R. Chaudhary and S. A. Morye (Kolhapur : shrikant-chaudhari@yahoo.com and sam5631@rediffimail.com).*

In this paper, we have studied homomorphism and covering of Melay-type fuzzy finite state machines. We have established relationship between homomorphism and covering. We have proved (1) Full direct (Cascade, Wreath, Cartesian) product and sum of Melay-type fuzzy finite machines are associate as well as commutative, (2) Full direct product of Melay-type fuzzy finite state machines is distributive over their sum, direct sum, wreath product and cascade product, (3) Cascade product of Melay-type fuzzy finite state machines is distributive over their sum and direct sum, (4) Full direct (Cascade and wreath) product of Melay-type fuzzy finite state machines are left distributive over their sum and direct sum and (5) Factor fuzzy automata FFA (M) covers M.

I5: On Fuzzy rough oscillatory region and its applications in decision making, *Sharmishta Bhattacharya and Susmita Roy (Tripura : halder_731@rediffimail.com and rsusmita@yahoo.com).*

The aim of this paper is to introduce the concept of fuzzy-rough oscillatory range. Also its application in the field of data mining is shown. In this paper, we are considering the linguistic dataset, which can be converted into a numerical value lying between [0, 1]. But the decision value of the object are crisp, that is, $\{0,\ 1\}$. Using this concept we are finding the pattern of the attributes of an object. With the help of this pattern we can make the decision of an unknown object.

I6: Fuzzy clustering with an application on customer relationship management (CRM), *Manoj Kumar Jain, Sandeep Tiwari and A. K. Dalela (Ujjain : Manoj_kumarjain@gmail.com, skt_tiwari75@yahoo.co.in and gdalela@gmail.com).*

Unsupervised clustering methods are examined to develop a customer relationship management (CRM) system and fuzzy c-means clustering is used to assign customer to the different clusters of customer activity. The results are compared with the results of hard k-means clustering according to performance classification. This application shows that fuzzy clustering methods can be an important supportive tool for the customer relations. Clustering techniques are rich and diversified. They have been continuously developing for over a half century following a number of tends, depending upon the emerging optimization techniques, main methodology and application area. Area of application of fuzzy cluster analysis include for example data analysis, pattern recognition, and image segmentation.

I7: Behaviour of 4 by 4 Wilkinson matrix with projection method, *Arun Kumar Awasthi and Vinay Saxena (Bahraich : arun.omee@gmail.com and vinay_saxena@yahoo.com).*

In this paper, we study the behaviour of almost singular 4 by 4 Wilkinson matrix when treated with projection method. The computations have been done in double precision and various norms in order to check the different accuracy. It is investigated that projection method deals with 4 by 4 Wilkson matrix and converges to the true result in finite step.

J: Operations Research:

J1: Numerical Solution of transient finite Markovian Models, *Mohit, D. V. Gupta and P. K. Sharma (Dehradun : mohit2692003@yahoo.co.in).*

Much of the literature on queueing models is confined to steady state models. But in many practical applications, transient analysis becomes singnificant. The transient derivation of even Markovian queueing models is quite a complicated procedure. Few approaches like spectral analysis, generating functions and complex combinatorial methods have been applied but each of these have its own limitations and are capable of analyzing the simple queueing models only. Here in, a transient single server queueing model with finite birth-death process is considered. The transient distribution of the number of customers in the system and the expected length of the system for a finite birth and death process are derived by solving the system of differential-difference equations using Laplace-Transforms and finding the inversion through the properties of tridiagonal symmetric matrices. The probability distribution of the number of customers and expected length of the system are reported for various values of parameters. Some interesting conclusions are observed. The queueing model with balking and reneging is also considered using generalized birth-death model. The results obtained are compared with the results obtained by previous methods.

K: Solid Mechanics, Fluid Mechanics, Geophysics and Relativity:

K1: Creeping flow past a swarm of porous spherical particles with Mehta-Morse Boundary Condition, *Satya Deo and Panjak Shukla (Allahabad : pankajshukla_au@rediffmail.com).*

This paper concerns the problem of creeping flow through a swarm of porous spherical particles. As boundary conditions, continuity of velocity, continuity of normal and shear stresses at the porous and fluid interface are employed. On the hypothetical cell surface, uniform velocity and Mehta-Morse boundary conditions are used. The drag force experienced by each porous spherical particle in a cell is evaluated. The earlier results reported for the drag force experienced by a porous sphere in unbounded medium by Qin and Kaloni and flow past a soid sphere in a Mehta-Morse cell model, have been then deduced. Representative results are presented in graphical form and discussed.

K2: Group invariant solution for motion of free particle, *V. G. Gupta and Kapil Pal (Jaipur : guptavguor@rediffmail.com).*

The formulation of Klein-Gordon (KG) equation was as early as days of quantum mechanics, it was connected with the general theory of relativity and was put into the form which in close analogy to Schrodinger equation is known as Schrodinger relativistic wave equation. In the present paper we have obtained the most general solution of KG equation for the motion of a free particle of mass m using the general prolongation formula for their symmetry.

K3: Stokes flow of micropolar fluid past a porous sphere with non-zero boundary condition for microrotations, *Bali Ram Gupta and Satya Deo (Allahabad : gupta_baliram@yahoo.co.in and satyadeo@allduniv.ac.in).*

The Stokes flow problem is considered for micropolar fluid past a porous sphere assuming the flow at distant points is uniform and parallel to the axis of symmetry. A non-homogeneous boundary condition for the microrotation vector i.e. the microrotation on the boundary of the sphere is assumed to be proportional to the rotation rate of the velocity field on the boundary, is used. The stream functions are determined by matching the solutions of Stokes equation for flow outside the sphere with that of the Brinkman equation for the flow inside the porous sphere. The drag force experienced by a porous sphere is evaluated and its variation is studied with respect to the material parameters. Some well-known results are then deduced from the present analysis.

K4: Analytical solution of couple stress fluid flowing through a porous media, *Ashish Tiwari, Pramod Kumar Yadav and Satya Deo*

(Allahabad : ast79.ashish@gmail.com, pramodky@rediffmail.com and satyadeo@allduniv.ac.in).

The present paper concerns the analytical solution of the couple stress fluid flow through porous media, in absence of body forces. Solutions obtained are valid for flows for which vorticity distribution is proportional to stream function perturbed by a uniform stream. Solutions are obtained for both steady and unsteady flows. Limiting cases of solutions agree with previously established results. Effect of permeability is discussed on stream function.

K5: Effect of heat transfer on MHD fluctuating flow of viscous fluid with mass transfer through a porous medium bounded by a porous plate, *Vineet Kumar Sharma and N. K. Varshney (Aligarh : vnt000@rediffmail.com).*

In this paper, a theoretical analysis of two dimensional flow of viscous and incompressible electrically conducting fluid with heat and mass transfer through a porous medium bounded by a porous plate has been presented in the presence of transverse magnetic field. The suction velocity perpendicular to the plate fluctuates in magnitude but not in the direction about a nonzero constant mean and the free stream velocity fluctuates in time about nonzero constant mean.

The effects of magnetic field (M), premeability of the porous medium (k) and Prandtl number (Pr) on the velocity and temperature of fluid are discussed with the help of Graphs.

K6: Effects of couple stress on the growth rate of Rayleigh Taylor Instability at the interface in a finite thickness couple stress fluid, *G. Chandrashekar and N. Rudraiah (Bangalore : chandrashekarg.4@gmail.com and rudraiahn@hotmail.com).*

In this paper, the effects of couple stress on the control of Rayleigh-Taylor Instability (RTI) at the interface between dense fluid accelerated by lighter fluid as studied using linear stability analysis. A simple theory based on fully developed approximations is used to derive the growth rate of Rayleigh-Taylor Instability (RTI). The cutoff and maximum wave numbers and the corresponding maximum frequency are obtained, it is shown that the effect of couple stress parameter redues the growth rate considerably compare to the classical growth rate in the absence of couple-stress.

K7: Stability of viscous couple stress parallel flow in a channel,
B. M. Shankar and N. Rudraiah (Bangalore : rudraiahn@hotmail.com and shankarmahadev07@gmail.com).

In this paper, we study linear instability of parallel couple stress viscous fluid (CSVF) in a channel and estimate the eigenvalues c belonging to the solutions of Orr-Sommerfeld equation for couple stress fluid using both moment and energy methods. It is shown that CSVF causes inhomogenety in the material science processes which is shown that these inhomogeneties can be controlled by understanding the nature of eigenvalues in the presence of a shear due to horizontal basic velocity. The eigenvalue, ci, for CSVF is determined in terms of couple stress parameter and shown that a sufficient condition for stability is

$$\alpha R_e q < \frac{\left[\left(3\alpha^2 + \frac{h^2\mu}{\lambda}\right) I_2^2 + \alpha^2\left(\frac{2h^2\mu}{\lambda} + 3\alpha^2\right) I_1^2 + \alpha^4\left(\frac{h^2\mu}{\lambda} + \alpha^2\right) I_0^2 + I_3^2\right]}{I_0 I_1},$$

where $I_n^2 = \int_{-1}^{1} |D^n v|^2 dy$ and $q = \max_{-1\leq v\leq 1} |D_{u_b}|$.

K8: Effects of external constraints of electric field and magnetic field on the stability of parallel chiral fluid flow in a channel,
Chinnappa, A. and N. Rudraiah (Bangalore : chinnappa.03@gmail.com and rudraiahn@hotmail.com).

This paper deals with the study of linear stability of an incompressible, inviscid poorly conducting chiral fluid in a rectangular channel in the presence of an impressed transverse magnetic field and aligned electric field with a stratified distribution charge density together with the velocity shear. A necessary condition for instability is found using moment and energy methods. In ordinary inviscid fluid Rayleigh had shown that a necessary condition for instability is the existence of point of inflexion in the basic velocity profile. In contrast to this Rayleigh criterion we have shown that in chiral fluid even the existence of the point of inflexion makes the system stable for the proper choice of electric number and chirality parameter. In particular, we found that the effects of electromagnetic field are analogous to the effect of viscosity in ordinary fluids as far as the behavior of solution around the critical levels.

K9: The dispersion of aerosols in atmospheric turbulent fluid flow using Saffman dusty fluid model, *Devarajun, N. and N. Rudraiah (Bangalore : devmaths@bub.ernet.in and rudraiahn@hotmail.com).*

This paper describes the use of Taylors analysis to study the dispersion of aerosols and the mixture of deformable agglomeration and coalescence of aerosols in the atmosphere. A proper theory is developed assuming this mixture behaves as a sparsely packed atmospheric turbulent fluid saturated porous medium incorporating the Brinkman viscosity effect to take care of atmospheric boundary layer and using Saffmans dusty fluid model to take care of elastic deformation of agglomeration, the solid phase of aerosol. Analytical solutions for velocity are obtained using a regular perturbation technique with $\epsilon = R_2R_3\{\frac{K\mu t_0}{\rho^3\mu_\alpha}\}$ as perturbation parameter. Concentration distribution is determined using the advection of concentration by the atmospheric turbulent fluid in the presence of an irreversible first order chemical reaction to account for removal mechanism and a source flux in the boundary condition. It is shown that the aerosols are dispersed relative to a plane moving with the mean speed of atmospheric turbulent fluid as well as the mean speed of agglomeration of aerosol with a relative diffusion coefficient D_r^β, called Taylor dispersion coefficient. This D_r^β is numerically computed and the results reveal that D_r^β increases with an increase in R_4, but decreases with an increase in σ and reaction rate parameter, where R_4 has the dimensions of the reciprocal of Reynolds number and σ is the porous parameter.

K10: Double diffusive convection in a couple stress fluid saturated porous layer, *Suresh Kumar S. and I. S. Shivkumara (Karnataka : sitkumar@gmail.com).*

The linear and weakly nonlinear stability of double diffusive convection in a couple stress fluid saturated porous layer is investigated. In the case of linear theory, conditions for the occurrence of both steady and oscillatory convection are obtained. It is established that oscillatory convection always occurs at a lower value of the Rayleigh number than that of steady convection. It is noted that an increase in the value of couple stress parameter is to delay the onset of convection. Non-linear analysis is carried out by constructing a system of non-linear autonomous ordinary differential equations using a truncated representation of Fourier series. Moreover, modified perturbation theory with the help of self-adjoint operator technique is used to investigate the stability of bifurcating equilibrium solution. It is found that sub-critical instability is always possible. Besides, heat and mass transport are calculated in terms of Nusselt numbers. In addition, the transient behaviour of Nusselt numbers is analysed by solving the non-linear system of ordinary differential equations numerically using the Runge-Kutta method.

K11: Bianchi Type - *I, III, V, VIO* and Kantowski-Sachs models in scalar-Tensor Theories with dynamic cosomological constant, *T. Singh and R. Chaubey (Varanasi : drtrilokisingh@yahoo.co.in).*

The effect of a time-dependent cosmological constant is considered in a family of scalar-tensor theories. The Bianchi type *I, III, V, VIO* and Kantowski-Sachs models for vacuum and perfect fluid matter are found. The gravitational constant decreases with time so that these models satisfy the Dirac hypothesis. The "cosmological contant" also decreases with time, therefore it can have a very small value at the present time.

K12: A fractional calculus approach to the study of stokes first problem, *Amitkumar D. Patel, Ajay K. Shukla and J. Banerjee (Surat : ajayshukla2@rediffmail.com).*

In the present paper, the analytical solution of stokes first problem is discussed. The governing equation is transformed into fractional partial derivative and is solved by using the integral transforms technique.

K13: An analytical study of finite amplitude convection in a two-component fluid saturated porous layer with soret and dufour effects, *S. N. Gaikwad (Gulbarga : sngaikwadmaths@yahoo.co.in).*

The linear and non-linear stability of convection of a two-component fluid known as thermohaline convection is considered in a horizontal porous layer heated and salted from below. The analysis is based on the Boussinesq-Darcy equations for two-dimensional convection under the assumption that the amplitudes of convection are small. The linear theory is based on the Fourier analysis and the critical Rayleigh numbers for both marginal and overstable motions are determined. It is found that a vertical solute gradient sets up overstable motions and a physical reason for this is given. The finite amplitude study is based on a truncated representation of Fourier series and the critical Rayleigh number is determined. The effects of Prandtl number, ratio of diffusitives, Soret parameter, DuFour parameter, Solute Rayleigh number and the permeability parameter on convection are studied. Nusselt number N_u and its analog N_{u^s} for solute are calculated.

K14: Perturbed solution of viscous instability phenomenon in double immiscible phase flow with mean pressure, *M. S. Joshi, M.*

N. Mehta and Twinkle Patel (Surat : mitesh_joshi2003@yahoo.com, mnm@ashd.svnit.ac.in and twinklerpapel@yahoo.com).

The non-linear partial differential equation of viscous Instability in double immiscible phase flow with mean pressure has been solved by using perturbation technique with appropriate boundary conditions. The solution obtained is in form of error functions.

K15: Accelerating cosmological models with bulk viscosity in scaler tensor theories, *G. P. Singh (Nagpur : gpsingh@math.vnit.ac.in).*

This paper deals with the study of homogeneous cosmological models of the universe, filled with dissipative cosmic fluid, in the framework of scalar-tensor theories. Exact solutions of the Einsteins field equations in the presence of quintessence have been obtained. The dynamical behaviour of solutions has also been discussed.

K16: Anisotropic bulk viscous cosmological models with variable G and A, *A. Y. Kale and G. P. Singh (Nagpur : gpsingh@mth.vnit.ac.in and ashwini_kale@rediffmail.com).*

This paper deals with Bianchi - *I*, Kantowski Sachs and Bianchi - *III* anisotropic cosmological models of the universe, filled with a bulk viscous cosmic fluid, in the presence of variable gravitational and cosmological constants. A new set of exact solutions of Einsteins field equation have been obtained in both truncated and full causal theories. Physical behaviour of the models has also been discussed.

K17: Some type D accelerating fluid distributions of embedding class one, *Mukesh Kumar and Y. K. Gupta (Uttarakhand : mukeshkumar12@rediffmail.com).*

Embedding class one perfect fluid distributions are of Petrov type 0 or Petrov type D. All the solutions of former class are known while all the Zeldowich fluids and the acceleration free solutions of later class have also been derived. In the present article a class of Petrov type D accelerating fluid distributions is found starting with $G_3(2, S)$ metric and analyzed them physically.

K18: Freely propagation of converging spherical shock waves in a rotating dusty gas, *R. P. Yadav, Dal Chand and Manoj Yadav (Bilaspur and Kotdwar, Uttaranchal : rpyadav93pphysics@yahoo.co.in).*

The aim of this paper is to study the freely propagation of converging spherical shock waves in a rotating dusty gas using Chester-Chisnell-Whitham method. The variation of flow variables with distance in perturbed medium are obtained. The effect of concentration of dust particles (k_p) and the ratio of the density of the solid particles to the density of gas (G) on flow variables are also estimated. It is found that the rotation and the presence of small solid particles in the medium have significant effects on the variation of flow variables. The results obtained here are compared with earlier results.

L: Electromagnetic Theory, Magneto-Hydrodynamics, Astronomy and Astrophysics:

L1: Thermal instability of ferrofluids permeated with suspended particles, *A. K. Aggarwal (Noida : amrish_juit@yahoo.com and amrish.aggarwal@jiit.ac.in).*

A layer of incompressible, electrically non-conducting ferromagnetic fluid permeated with suspended particles heated from below in the presence of rotation is considered. For a fluid layer between two free boundaries, an exact solution is obtained using a linearized stability theory and normal mode analysis. For the case of stationary convection, it is found that suspended particles have a destablizing effect, whereas rotation has a stabilizing effect on the onset of instability.

L2: Transformations of the Equation of Equilibrium for isothermal plane-symmetric configuration, *J. P. Sharma, D. P. Jayapandian, Shalini and Sanish Thomas (Allahabad : shalini_vns19@yahoo.com and sanish_t_1@yahoo.co.in).*

We have discussed a few transformations which connect solutions of the isothermal equation (analogue of the Lane-Emden equation (LEE) for plane symmetric configuration in $(u_\rho; v_\rho)-, (u_p; v_p)-, (u_\psi; v_\psi)-$planes and $(y_\rho; z_\rho)-, (y_p; z_p)-, (y_\psi; z_\psi)-$planes. Approximate analytical solutions for $(u_\psi; v_\psi)-$planes are tabulated following the technique of Pade's approximants. Physical aspects of the problem and interpolations of the results have also been included.

L3: Effect of Magnetic field on the flow of a dusty fluid in open channels, *R. Khare and Nitu Singh (Department of Mathematics and Statastics, Allahabad Agricultural Institute, deemed University, Allahabad).*

In this paper, we have studied the flow of a dusty incompressible viscous fluid through an open channel in presence of a transverse magnetic field of constant magnitude using Saffman's modell (1962) of dusty fluid. After deriving the expressions for the velocities of fluid and particle phase and mean velocity, results have been drawn and discussed using graphs and other methematical techniques. It has been found that the applied magnetic field may control such flows very effectively as required in different fields.

L4: MHD Free and forced convection with Mass transfer from an infinite vertical porous plate, *N. Ahmed (Guwahati : saheel_nazib@yahoo.com).*

An attempt has been made to study the effect of the uniform transverse magnetic field on a steady free and forced convective flow with mass transfer of an incompressible electrically conducting viscous fluid past an infinite vertical porous plate with constant suction taking into account the induced magnetic field and viscous and magnetic dissipations of energy. An analytical solution of the problem is obtained. The expression for the velocity field, temperature field, concentration field, induced magnetic field skin-friction at the plate, rate of heat of transfer from the plate to the fluid and current density of the fluid flow are derived in non-dimensional form. Numerical vlaues of the skin-friction, Nusselt number and the induced magnetic field are demonstrated in graphs and tables and discussed. It is seen that the flow characteristics and beat transfer may be controlled by applying the transverse magnetic field.

L5: The effect of chemical reaction and magnetic field on transient free convective flow over a vertical plate in slip-flow regime, *Sahin Ahmed and N. Ahmed (Guwahati : sahin_glp@yahoo.co.in and sahee_nazib@yahoo.com).*

The paper considers the twin effects of chemical reaction as well as a uniform externally applied magnetic field on the transient free convection and mass transfer flow of an electrically conducting, viscous, incompressible fluid, past an infinite vertical porous plate in slip-flow regime. The plate is subjected to a variable suction velocity when the heat and mass transfer oscillates in time. Adopting a perturbative series expansion about a small parameter ϵ, expressions are obtained describing the velocity, temperature, skin-friction coefficient and the rate of mass transfer. The effects of governing parameters on the flow variables are discussed quantitatively with the aid of graphs for the transient velocity and concentration as well as fluctuating parts of skin-friction. Our results show that the chemical reaction decreases the fluid velocity as well as species concentration, but this behaviour of chemical reaction rises the phase of skin-friction. It is observed that, the rarefaction parameter increases both the transient velocity as well as phase tan, whereas this behaviour of h is opposite to the amplitude $|F|$. All numerical calculations are done with respect to air ($P_r = 0.71$) at 20^0C.

M: Bio-Mathematics:

M1: Effect of Toxicants on Population, *Anjali Sisodiya, Bijendra Singh and B. K. Joshi (Ujjain : sisodiyaanjali@yahoo.co.in).*

A single species model in a polluted closed environemnt is analysed for a constant influx of toxicant. We have assumed that the intake of the toxicant by the population is different than the depletion rate of toxicants in the environment. The analysis of Freedman and Shukla [1991] follows as a corollary.

M2: Mean field and discrete spatial equations representation of a three species food web, *Randhir Singh Baghel and Joydip Dhar (Gwalior : randhirsng@gmail.com and jdhar@iiitm.ac.in).*

The chaos and pattern formation is a very common behavior in many three species food chain models. Most of the models they assume that the populations are homogeneously mixed, so that the probability that any two individuals interact is uniform and space can be ignored. In this paper we propose a food wave model in which two predator classes depend on a single prey and moreover predators are competing for their survival. Treating both space and time as discrete variables we derive a set of coupled equation that describe the evolution of the populations at each site of the spatial domain. The evolution equations for the predators involve averages over the local density of preys, whereas the equation for the prey involve double averages, where the local density of both prey and predators appear.

M3: An Extension of sirs model with mass incidence, *Neetu Trivedi, Rachana Khandelwal and Ritu Ojha (Indore : rakhikhandelwal@gmail.com).*

An infectious disease model with constant immigration and mass incidence has been extended. Two equilibria P_0 and P_1 are obtained. It has been observed that when the contact number σ is less than or equal to I/π

only the point P_0 will exist whereas for greater than $1/\pi$ both P_0 and P_1 will exist. The result of Mena Lorca follows as a special case.

ABSTRACT OF THE PAPERS FOR IMS PRIZES

IMS-1: Fuzzy programming applied to multiobjective quadratic fractional program, *K. C. Lachhwani (Bikaner : kailashclachhwani@gmail.com).*

This paper deals with the solution methodology for multiobjective quadratic fractional program. Fuzzy programming approch is used to develop the methodology for multiobjective functions which works for the minimization of the perpendicular distance between parallel hyper plates at the optimum points for the objective functions. Suitable membership function is defined as the supremum perpendicular distance and a compromise optimum solution is obtained as a result of minimization of supremum perpendicular distance. An example is also given to support our model.

IMS-2: An extesion of Boas-Buck theory, *S. K. Meher (Surat : srikant_math@rediffmail.com).*

In the present paper, we generalized the Boas-Buck theory for two variables. Some properties and recurrence relation have also been obtained.

IMS-3: On certain mirrors in the Bicomplex space, *Usha (Agra : usha_ibs@yahoo.com).*

Various types of conjugations and modulii have been defined and studied on the bicomplex space by contemporary workers, including Rochon & Shapiro and Jaishree.

In this paper, we have studied the geometrical aspect of certain transformations including the bicomplex conjugations. These transformations have beem found to behave like mirrors in some sense. We have investigated such mirrors and their individual and cumulative effects. The idea of the dimesion of mirror has been conceived and developed. These mirrors have been shown to possess certain common properties and certain individual characteristic properties as well. Family of mirrors has been provided with an Abelian group structure. Certain subgroups of this group have been listed.

IMS-4: Intorduction of a circular number line, *Dharmendra Kumar Yadav (Delhi : dkyadav1978@yahoo.co.in).*

The present paper introduces a circular number line, the superset of imaginary number line previously given by Yadav and imaginary circular plane, the superset of circular complex plane given by Yadav. It also introduces the new concept of imaginary circle and imaginary spheres. Taking different values of n, the natural numbers, we find that it takes all the values of the imaginary number line. Giving the geometrical meaning as the sum of arithmatical distances, we find that the values of in lie on a circle and the imaginary number line lies on this circle, which gives the concept of circular number line. This circular number line is not a straight line but a circle of imaginary radius. At last some axioms of elliptical geometry and Euclidean geometry have been observed true in this paper. These axioms have been observed on the imaginary sphere and imaginary circle. The circular number line, imaginary sphere and imaginary circle will play major role in explaining the concept of Elliptical geometry and Hyeprbolic geometry as well as they will be very helpful in explaining the universe geometrically. In fact author has given D-theory of universe by combining different theories of universe.

IMS-5: An odd model of HIV infection in vivo with RT Inhibitor, *P. K. Srivastava (Kanpur : pksri@iitk.ac.in).*

A mathematical model for the effect of Reverse Transcription(RT) Inhibitor on the in vivo dynamics of HIV is proposed and analysed. Further, with the help of numerical simulations, the relation between efficacy of administered drug, the total number of virus particles emitted from the infected cell and the latency period is also discussed.

ABSTRACT OF PAPERS FOR AMU PRIZE

AMU-1: On Jordan triple $(\alpha, \beta)^*$−derivations in Banach *−Algebras, *Shakir Ali (Aligarh : shakir50@rediffmail.com).*

Let A be a Banach*−Algebra over a field F. A linear mapping $d : A \longrightarrow A$ is called a *-derivation (resp. Jordan *-derivation) if $d(xy) = d(x)y^* + xd(y)$(resp. $d(x^2) = d(x)x^* + xd(x)$) hold for $x, y \in A$. The study of Jordan *-derivations has been motivated by the problem of representability of quasiquadratic functionals by sesquilinear ones. It turns out that the question of whether each quasi-quadratic functional is generated by some sesquilinear functional is intimately connected with the structure of Jordan

*-derivations (viz., [*Bull. Austral. Math. Soc. 37(1988)*] and [*Proc. Amer. Math. Soc. 91(1987), 133-136*], where further refernce can be found).

Let α and β be homomorphisms of A. A linear mapping $d : A \longrightarrow A$ is called a $(\alpha, \beta)^*-$derivation (resp. Jordan $(\alpha, \beta)^*-$derivation) if $d(xy) = d(x)y\alpha(y^*) + \beta(x)d(y)$ (resp. $d(x^2) = d(x)\alpha(x^*) + \beta(x)d(x)$) hold for all $x, y \in A$. A linear mapping d of a *-algebra A into itself is called a Jordan triple $^*-$derivations if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ hold for all $x, y \in A$. A linear mapping d of a *-algebra A into itself is called d Jordan triple $(\alpha, \beta)^*-$derivation if $d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ holds for $x, y \in A$. Note that for I_A, the identity map on A, Jordan triple $(I_A, I_A)^*-$derivation is called simply a Jordan triple $^*-$derivation. Clearly, every Jordan $(\alpha, \beta)^*$ -derivation on a 2-torsion free *-algebra is a Jordan triple $(\alpha, \beta)^*-$derivation. However, the converse need not be true in general.

In this talk we will give some recent progress about Jordan $(\alpha, \beta)^*$ -derivation of Banach*-algebra and establish set of conditions under which every Jordan triple $(\alpha, \beta)^*-$derivation on A is a Jordan $(\alpha, \beta)^*-$derivation.

ABSTRACT OF PAPERS FOR V. M. SHAH PRIZE

VMS-1: An integral transform of two variables and its properties, *I. A. Salehbhai (Surat : ibrahimmaths@gmail.com).*

In the present paper, we introduced an integral transform of two variables in which Laguerre type Polynomials are considered as kernel of transforms. We obtain some basic operational properties like convolution, inverse transform of derivatives. At the end of the paper, we examined the MAPLE implementation to obtain aforesaid transform.

VMS-2: A note on Bicomplex Dirichlet series, *Jogendra Kumar (Agra: jogendra_j@rediffmail.com).*

In this paper, we have introduced the Bicomplex version of Dirichlet series which we call as Bicomplex Dirichlet series. We have investigated the region of convergence and region of uniform convergence of general Bicomplex Dirichlet series in some sense. Ordinary Bicomplex Dirichlet series have also been studied.

VMS-3: Coefficient inequalities of meromorphic multivalent starlike and convex functions, *Nidhi Jain and I. B. Bapna (Bhilwara : nidhijain@yahoo.com and bapnain@yahoo.com).*

This paper envisages some coefficient properties of function $f(z)$ belonging to the class $\psi_r(p)$ of meromorphic multivalent functions in the punctured disk D_r. Motivated by several earlier works, we consider two subclasses $S_r^*(\alpha)$ and $C_r(\alpha)$ of $\psi_r(p)$. Further we discuss the starlikeness and the convexity of functions $f(z)$ in $\psi_r(p)$. These results are more important as they generalize the results of Shigeyoshi Owa, Nicole N. Pascu, M. L. Mogra, T. R. Reddy and O. P. Juneja et al.

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ON SOME LARGE SUBSETS OF EUCLIDEAN SPACES INVOLVED IN GEOMETRY, DYNAMICS AND DIOPHANTINE APPROXIMATION

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1. **Introduction**

Which subsets S of $\mathbb{R}^d$ (euclidean spaces) can be considered as being "large"? Apart from the whole space (of course), complements of countable sets, complement of a linear subspace or more generally complement of an algebraic variety (the set of solutions of a polynomial equation) come to mind naturally as large subsets. Countable intersections of such subsets may also be considered as being large subsets.

In Analysis one finds a role for the sets whose complement has 0 Lebesgue measure (so called sets of full measure or conull sets) and dense $\mathcal{G}_\delta$ subsets as large sets.

Here we discuss some classes $\mathcal{C}$ different from these, and yet large in a certain way. Like the classes of sets recalled above these classes also have the following properties:

i) For $S_i \in \mathcal{C}$, $i = 1, 2, \ldots, \cap S_i \in \mathcal{C}$.

ii) If $S \in \mathcal{C}$ and f is a local diffeomorphism of $\mathbb{R}^d$ then $f^{-1}(S) \in \mathcal{C}$. (Here and in the sequel $f^{-1}(S)$ denotes the set $\{t \mid f(t) \in S\}$.)

The conditions in particular mean that for S in $\mathcal{C}$ the complement of S in $\mathbb{R}^d$ does not contain a "copy" of S under a local diffeomorphism of $\mathbb{R}^d$.

The new classes to be discussed here arise from certain games. Consider first the following game. We shall produce a real number, say α, by writing

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$0.a_1a_2\cdots$ (a decimal expansion where a_i are digits between 0 and 9) where you get to choose $a_1, a_3, \ldots$, and I get to choose the even indexed entries. When the sequence is fully written out we get the number; (though the process is infinite, it is understood that each of us chooses our entries by some rule, making it possible to describe the number in real time!) What properties can you ensure for the resulting number α, with my working against your objective? You can not ensure the number to be a rational number. In fact, given any countable set $\{x_n\}$ in $\mathbb{R}$ I can prevent the resulting number from being any of the x_ns, by picking the $2n$th entry at my disposal to be different from the $2n$th digit in the expansion of x_n.

You can not ensure it to be a *normal* number. (We recall that a number $\alpha = 0.a_1a_2\cdots$ as above is said to be *normal* if given any finite block $b_1b_2\cdots b_k$, with each b_i a digit between 0 and 9, it occurs somewhere in the decimal expansion, as $a_{j+1} = b_1, a_{j+2} = b_2, \ldots, a_{j+k} = b_k$ for some $j \in \mathbb{N}$.) I can in fact make sure that in the number α we are writing the digit 5 never follows 9 (for example), so the number will not be normal. Since the set of normal numbers has full Lebesgue measure, and is a dense $\mathcal{G}_\delta$ set, and thus a large set in the conventional sense, the fact that you can not ensure the number to be in it is especially worthwhile to note.

We call a set S a *winning set* for the game if you can ensure α to be from S, no matter how I choose the entries at my disposal. Which sets are winning sets? We shall actually discuss such a question in the following sections for a broader class of games; this specific case is subsumed in the game as in the last section (and also analogous to the one in the next section) and will not be discussed separately.

2. W. M. Schmidt's (α, β) game

W.M. Schmidt introduced, in [15], the following game involving a similar idea as in the game in § 1. The game can be described as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be two players and let $\alpha, \beta \in (0, 1)$ be given. The sample play in the game proceeds as follows:

$\mathcal{B}$ chooses a closed ball B_0 of radius $r > 0$; then

$\mathcal{A}$ chooses a closed ball A_1 contained in B_0 with radius $r\alpha$; then

$\mathcal{B}$ chooses a closed ball B_1 contained in A_1 with radius $r\alpha\beta$; then

...

$\mathcal{A}$ chooses a closed ball A_k inside B_{k-1} with radius $r\alpha(\alpha\beta)^{k-1}$; next

$\mathcal{B}$ chooses a closed ball B_k inside A_k with radius $r(\alpha\beta)^k$; and so on.

Then $\cap A_i = \cap B_i$ and it consists of single point p. We call S an (α, β)-*winning set* if $\mathcal{A}$ can ensure that $p \in S$. In [15] Schmidt discusses a variety of properties of the (α, β)-winning sets.

It would be convenient to have two more notions of winning sets: we say that a set S is an α-*winning set* for $\alpha \in (0, 1)$ if it is (α, β)-winning for all $\beta \in (0, 1)$; also we call a set (simply) a *winning set* if there exists $\alpha \in (0, 1)$ such that S is α-winning.

Here are some basic properties of the winning sets that go back to the paper of Schmidt [15] (see also [6]).

- If $1 - 2\alpha + \alpha\beta \leq 0$ then only the whole space is an (α, β)-winning set.
- If $1 - 2\alpha + \alpha\beta > 0$ then the complements of finite sets are winning sets; for $\alpha \leq \frac{1}{2}$ the complement of any countable set is α-winning.
- If $\{S_i\}$ is a sequence of α-winning sets for some $\alpha \in (0, 1)$ then $\cap_{i=1}^{\infty} S_i$ is an α-winning set.
- If S is an α-winning set for some $\alpha \in (0, 1)$ and f is a local diffeomorphism of $\mathbb{R}^d$ then $f^{-1}(S)$ is a α'-winning set for any $\alpha' \in (0, \alpha)$.
- If S is a winning set and $\{f_i\}$ is a sequence of local diffeomorphisms of $\mathbb{R}^d$ then $\cap_{i=1}^{\infty} f_i^{-1}(S)$ is a winning set.
- If S is a winning set in $\mathbb{R}^d$ then the Hausdorff dimension of S is d (which is the maximum possible for a subset of $\mathbb{R}^d$).

Schmidt also gave in [15] a crucial example of a winning subset in $\mathbb{R}$ related to Diophantine approximation. An irrational number $\alpha \in \mathbb{R}$ is said to be *badly approximable* if there exists a $\delta > 0$ such that

$$|\alpha - \frac{p}{q}| > \frac{\delta}{q^2} \text{ for all rationals } \frac{p}{q}.$$

It is well known that a number is badly approximable if and only if the partial quotients in its continued expansion are bounded (see [3]).

Theorem 2.1. (Schmidt) *The set of badly approximable numbers is a winning set in* $\mathbb{R}$*; in fact it is* (α, β)*-winning for all* $\alpha, \beta \in (0, 1)$ *such that* $1 - 2\alpha + \alpha\beta > 0$*, and hence also* α*-winning for all* $\alpha \in (0, \frac{1}{2}]$*.*

There are similarly notions of *badly approximable vectors* in $\mathbb{R}^d$ and *badly approximable systems of linear forms* $L_1, L_2, \ldots, L_k$ in d variables, where $1 \leq k \leq d-1$, and Schmidt generalised the above theorem (in certain later papers) to these being winning sets in the respective vector spaces (see [16] for details). I generalized Schmidt's theorem, in [4], to the following; while the generalization is straightforward, its significance lies in the applications in hyperbolic geometry and dynamics, discussed in Section 4.

Theorem 2.2. (Dani) *Let* $\{v_i\}$ *be a sequence of vectors in* $\mathbb{R}^d$ *and* $\{r_i\}$ *a sequence of positive numbers. Suppose that there exists a* $c > 0$ *such that for all* i, j *we have* $\|v_i - v_j\| \geq c\sqrt{r_i r_j}$. *Then the set*

$$\{v \in \mathbb{R}^d \mid \exists\ \delta > 0 \text{ such that } \|v - v_i\| > \delta r_i \quad \forall\ i\}$$

is an (α, β)*-winning set in* $\mathbb{R}^d$ *for all* $\alpha, \beta \in (0, 1)$ *such that* $1 - 2\alpha + \alpha\beta > 0$.

The case of badly approximable numbers $\mathbb{R}$ falls out as a special case if we choose $\{x_i\}$ to be an enumeration of the rationals, and if $x_i = \frac{p}{q}$, where p and q are coprime integers and $q \neq 0$ ($q = 1$ if $p = 0$), choose $r_i = \frac{1}{q^2}$.

3. Winning sets and Dynamics: the case of tori

In this section we discuss a simple case of occurrence of winning sets in the study of Dynamics.

Let $\mathbf{T}^2 = \{(z, w) \mid z, w \in \mathbb{C}, |z| = |w| = 1\}$, the two-torus. Any integral matrix $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ with $\det A = \pm 1$ defines an automorphism T_A of $\mathbf{T}^2$, by $(z, w) \mapsto (z^m w^p, z^n w^q)$ for all $(z, w) \in \mathbf{T}^2$. In fact all continuous automorphisms of $\mathbf{T}^2$ arise in this way.

Let $O_A(x) = \{x, T_A x, T_A^2 x, \ldots, T_A^k x, \ldots\}$; it is called the T_A-*orbit* of x. When $|m + q| > 2$ the action of T_A on $\mathbf{T}^2$ is ergodic (see [12] or [17] for details) and consequently for *almost all* $x \in \mathbf{T}^2$, $O_A(x)$ is dense in $\mathbf{T}^2$; that is, the set of $x \in \mathbf{T}^2$ for which $O_A(x)$ is not dense in $\mathbf{T}^2$ is a set of measure zero. There are indeed orbits which are not dense in $\mathbf{T}^2$. E.g. if z and w are roots of unity then the orbit of $x = (z, w)$ is in fact finite. There are also a variety of infinite orbits which are not dense. There is however no good description of such points, or any way to tell whether for a given point the orbit is dense or not.

Here we shall be concerned with a class of the exceptional orbits, whose closure does not contain any (z, w) with z and w roots of unity, namely not containing any element of finite order in $\mathbf{T}^2$; in particular such an orbit would not be dense. Let $E(T_A)$ be the set of all $v = (v_1, v_2) \in \mathbb{R}^2$ such that for $x = (e^{2\pi i v_1}, e^{2\pi i v_2})$ the closure $\overline{O_A(x)}$ of $O_A(x)$ does not contain any (z, w) of finite order.

Theorem 3.1. (Dani) *For every A as above (for which $|m + q| > 2$) the set $E(T_A)$ is an α-winning set for all $\alpha \in (0, \frac{1}{2}]$. In particular, there exists $x \in \mathbf{T}^2$ such that $\overline{O_A(x)}$ does not contain any element of finite order, for any A as above.*

The second statement follows from the first together with Schmidt's Theorem on countable intersections of winning sets recalled earlier.

Analogous statement is proved in [5] for tori in all dimensions, with the condition $|m + q| > 2$ as above replaced by the automorphism A being diagonalisable. The theorem was generalised in [1] for certain automorphisms of compact nilmanifolds.

4. **Winning sets and Hyperbolic Geometry**

Let $\mathbf{H}^{d+1} = \{(v, y) \mid v \in \mathbb{R}^d, y > 0\}$, the hyperbolic space of dimension $d+1$. For $d = 1$ it is the upper half plane, known also as the Poincaré upper half plane, consisting of complex numbers with positive imaginary part. $\mathbf{H}^{d+1}$ is equipped with the Riemannian metric given by $\frac{1}{y^2}(\sum dv_i^2 + dy^2)$ with respect which it is a manifold of constant curvature -1 (see [2] for a brief and easy exposition of what is involved here). For $p \in \mathbf{H}^{d+1}$ and a unit tangent direction ξ at p the (unit speed) geodesic $\{\gamma(t)\}_{t \in \mathbb{R}}$ in $\mathbf{H}^{d+1}$ with $\gamma(0) = p$ and ξ as its tangent at $t = 0$ is a (Euclidean) semicircle in $\mathbb{R}^{d+1}$ orthogonal to $\mathbb{R}^d$ (the latter is embedded in $\mathbb{R}^{d+1}$ as $\{(v, 0) \mid v \in \mathbb{R}^d\}$). To the positive segment $\{\gamma(t)\}_{t \geq 0}$ there corresponds a unique *endpoint* $\gamma(\infty)$ in $\mathbb{R}^d$, namely the endpoint of the semicircle as above, in the segment corresponding to the positive part of the geodesic. Conversely, given $p \in \mathbf{H}^{d+1}$ and $v \in \mathbb{R}^d$ there exists a unique unit tangent direction ξ at p such that v is the endpoint of the (positive) geodesic $\{\gamma(t)\}_{t \geq 0}$ such that $\gamma(0) = p$ and ξ is its tangent at 0.

Now let M be a manifold of dimension $d + 1$ with constant negative curvature, say -1. Then M has the form $M = \Gamma \backslash \mathbf{H}^{d+1}$, where Γ is a group of isometries of $\mathbf{H}^{d+1}$. The quotient map $\eta : \mathbf{H}^{d+1} \to M$ is a covering map, with $\mathbf{H}^{d+1}$ as the universal covering space. The action of Γ on $\mathbf{H}^{d+1}$ extends to an action on $\mathbb{R}^d$ as above; the latter may be viewed as the "boundary" of $\mathbf{H}^{d+1}$. The geodesics in M consist of the images of geodesics in $\mathbf{H}^{d+1}$. Given $q \in M$ and a unit tangent direction ζ at q and $p \in \mathbf{H}^{d+1}$ such that $q = \eta(p)$, the geodesic in M corresponding to q and ζ has a unique canonical lift in $\mathbf{H}^{d+1}$ starting at p which is a geodesic (together with time orientation).

Now suppose furthermore that M has finite Riemannian volume. Such a manifold may be compact or noncompact; in the latter case, which is what interests us below, there are finitely many "ends" in the form of tapering off pieces, referred to as "cusps" - given any $\epsilon > 0$ there exists a compact subset K of M such that the complement of M consists of finitely many subsets, each of which is within (hyperbolic) distance ϵ from a geodesic ray (positive time geodesic). It is well-known (see [2] for instance) that for a manifold M as above, for almost all pairs (q, ζ) with $q \in M$ and ζ a unit tangent direction at q, the geodesic $\{\gamma(t)\}_{t\geq 0}$ is dense in M, and moreover the pairs $\{(\gamma(t), \gamma'(t))\}$, where $\gamma'(t)$ denotes the tangent to γ at t, form a dense subset of the unit tangent bundle of M. There are however geodesics which are not dense and exhibit a variety of different behaviour. One of the interesting issues in this respect is to understand the set of geodesics with compact closure, when the manifold itself is not compact, but has finite volume. The following result was deduced in this respect, using Theorem 2.2.

Corollary 4.1. (Dani) *Let M be a Riemannian manifold of dimension $d+1$ with constant negative curvature and finite Riemannian volume. Let $q \in M$. Let Φ be the set of geodesics $\{\gamma(t)\}_{t\geq 0}$ on M such that $\gamma(0) = q$ and the closure of $\{\gamma(t)\}_{t\geq 0}$ in M is compact. Then the set of v in $\mathbb{R}^d$ such that v is an endpoint of a lift of some $\gamma \in \Phi$ in $\mathbf{H}^{d+1}$ is a winning set in $\mathbb{R}^d$.*

In view of the correspondence between the directions and the endpoints of the corresponding geodesic this can also be thought of as a result on the multitude of directions at any point $q \in M$, for which the corresponding geodesic is bounded (has compact closure) in M.

5. Large sets in Dynamics on homogeneous spaces

Let $G = SL(n+1, \mathbb{R})$, the group of all square matrices of order $n+1$ with determinant 1, under multiplication, equipped with the topology as a subset of the vector space of all square matrices of order $n+1$. Let $\Gamma = SL(n+1, \mathbb{Z})$ be the subgroup consisting of all matrices in G with integral entries. We form the quotient space G/Γ consisting of cosets of the form $g\Gamma$, $g \in G$, and equipped with the quotient topology. Then G/Γ is a manifold of dimension $(n+1)^2 - 1$. Any $g \in G$ defines a *translation* of G/Γ, given by $x\Gamma \mapsto gx\Gamma$

for all $x \in G$. It is well-known that G/Γ admits a finite measure invariant under the action of every translation (see [2]).

For $t \in \mathbb{R}$ let $d_t = \operatorname{diag}(e^t, e^t, \ldots, e^t, e^{-nt})$. The $\{d_t\}$-action on G/Γ defines a one-parameter flow on G/Γ. This action is ergodic and in particular for almost all $x \in G/\Gamma$ the orbit under the action, viz. $\{d_t x \mid t \in \mathbb{R}\}$, is dense in G/Γ. There is however no detailed information on which orbits are dense. Using Schmidt's theorem on the set of badly approximable systems of linear forms being a winning set the following was deduced with regard to orbits which have compact closure, and in particular not dense.

Theorem 5.1. (Dani) *The set of all x in G/Γ such that $\{d_t x \mid t \in \mathbb{R}\}$ has compact closure in G/Γ intersects every nonempty open subset of G/Γ in a set of Hausdorff dimension equal to* $\dim G/\Gamma = (n+1)^2 - 1$.

It was conjectured by Margulis and proved by Kleinbock and Margulis that the corresponding assertion holds for G any connected Lie group, Γ any lattice in G and the action on G/Γ by any diagonalisable one-parameter subgroup of G. The theme was pursued further by Kleinbock (see [13] for more details).

Theorem 5.1 as well as the results of Kleinbock and Margulis deal with largeness only in terms of the Hausdorff dimension. One may ask whether the sets are winning sets for certain games, implying their largeness. This remains unexplored.

6. Values of quadratic forms

Let $Q(x, y) = (\alpha x + y)(\beta x + y)$, where α, β are irrational. Then $Q(\mathbb{Z}^2)$, namely the set of values of Q at integral pairs, is not dense in $\mathbb{R}$ if and only if α and β are badly approximable (see [8]). Recall that the set of badly approximable numbers is a winning set, which brings to mind a connection of winning sets with the topic of values of quadratic forms. We describe below a similar role of large exceptional sets for values of quadratic forms in three variables, via the use of the theorem of Kleinbock and Margulis mentioned above.

Let Q be a non-degenerate indefinite quadratic form in $n \geq 3$ variables, namely $Q(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j$, where (a_{ij}) is a nonsingular matrix which is symmetric and indefinite (neither the matrix nor its negative is positive definite). There was a long-standing conjecture due to Oppenheim that $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$, unless Q is a multiple of a rational form (in the latter case of course the set of values will be discrete). The conjecture was proved by G.A. Margulis in mid nineteen-eighties (see [2], [7]).

Now let Q be a non-degenerate indefinite quadratic form in 3 variables, which is not a multiple of a rational form. Let L be a linear form on $\mathbb{R}^3$. We ask whether, given $\epsilon > 0$ there exists a simultaneous solution x with integer coordinates for the inequalities

$$|Q(x) - a| < \epsilon \text{ and } |L(x) - b| < \epsilon.$$

(This may really be viewed as a question about values of Q; being assured that there are integral solutions x to the first inequality, we would like to know whether a solution can be found near a specified affine plane, namely the one given by $\{v \mid L(v) = b\}$).

The answer is in the affirmative if the cone $C = \{v \in \mathbb{R}^3 \mid Q(v) = 0\}$ and the plane $P = \{v \in \mathbb{R}^3 \mid L(v) = 0\}$ are tangential to each other. This follows from a theorem that I proved jointly with Margulis (see [9], [8]).

If the cone C and the plane P as above intersect only at 0 then one can see easily that the inequalities do not admit integer-tuple solutions for sufficiently small $\epsilon > 0$. Thus we are left with the case when C and P intersect transversally, in two lines. In this case the answer crucially depends on the pair of lines of intersection. For almost all pairs of lines on C one can assert existence of common solutions; note that the set of lines on C can be parametrised by points on a circle, and hence the pairs of lines by points on a torus, and "almost all" may be understood in terms of the usual measure on the torus. The set of pairs of lines on C for which the inequalities do not admit simultaneous solutions also turns out to be large, at least as far as its Hausdorff dimension is concerned, it being 2, the maximum possible. This is proved using the result of Kleinbock and Margulis with $G = SL(3, \mathbb{R})$, $\Gamma = SL(3, \mathbb{Z})$ and the one-parameter subgroup $\mathrm{diag}(e^t, 1, e^{-t})$, $t \in \mathbb{R}$ (see [8]). In this case also it remains unanswered whether the set of pairs of lines as above form a winning set for an appropriate game.

7. Badly approximable numbers in Cantor-like sets

In this concluding section we return to the set of badly approximable numbers and analogous sets in Euclidean spaces.

Let $\{v_i\}$ be a sequence of vectors in $\mathbb{R}^d$ and $\{r_i\}$ a sequence of positive numbers such that for some $c > 0$ we have $\|v_i - v_j\| \geq c\sqrt{r_i r_j}$ for all $i, j,\ i \neq j$. Then by Theorem 2.2 the set $\{\, v \in \mathbb{R}^d \mid \exists\, \delta > 0$ such that $\|v -$

$v_i\| > \delta r_i \ \forall i\}$ is a winning set, and hence a large set in the sense as discussed earlier. What can we say about the intersection of these sets with thin subsets like the Cantor set, and other compact totally disconnected sets?

Intersection of badly approximable numbers, and more generally also of badly approximable vectors in higher dimension, with compact sets which are supports measures satisfying certain conditions, called "friendly measures", is considered in papers of Kleinbock and Weiss [14] and L. Fishman [11]; the class of sets includes self-similar (fractal) sets, e.g. the Sierpinski carpet.

We recently addressed the issue (joint work with Hemangi Shah) via a different approach [10]. For this we introduce the following modification of Schmidt's game; the main point about the modification is that it reduces the dependence on the metric.

Let C be a compact subset of $\mathbb{R}^d$, $d \geq 1$. Suppose that there exists a sequence $\{\mathcal{C}_n\}$, where each $\mathcal{C}_n$ is a covering of C by compact subsets such that

i) $\mathcal{C}_0 = \{C\}$ and each $X \in \mathcal{C}_n$ is contained in a $Y \in \mathcal{C}_{n-1}$;

ii) there exist $a, b > 0$ and $\theta \in (0,1)$ such that for any $X \in \mathcal{C}_n$, $a\theta^n \leq \operatorname{diam} X \leq b\theta^n$.

Then we can define on C a game like Schmidt's (α, β)-game where $\mathcal{A}$ and $\mathcal{B}$ choose alternately sets from $\mathcal{C}_n$, $n = 1, 2, \ldots$, contained in the preceding one: $\mathcal{A}$ chooses A_1 from $\mathcal{C}_1$, then $\mathcal{B}$ chooses B_1 from $\mathcal{C}_2$ contained in A_1 and so on (the role of B_0 is played by C itself). We have correspondingly the notion of a winning set (with respect to the sequence of coverings). It may be observed that (upto some minor modifications) this game subsumes the (α, β) game of Schmidt discussed in § 2 and also the number game discussed in § 1.

We note that for any $r \in \mathbb{N}$, $\{\mathcal{C}_{rn}\}$ also has the properties (i) and (ii) as above. We say that $\{\mathcal{C}_n\}$ is σ-fine for a $\sigma > 0$ if $X \in \mathcal{C}_n$ and $Y \in \mathcal{C}_{n-1}$ is an element containing X then $\operatorname{diam} X < \sigma \operatorname{diam} Y$. We note that given a sequence of coverings $\{\mathcal{C}_n\}$ satisfying conditions (i) and (ii) there exists $r \in \mathbb{N}$ such that $\{\mathcal{C}_{rn}\}$ is $\frac{1}{2}$-fine.

Theorem 7.1. (Dani and Shah) *Let C be a compact subset of $\mathbb{R}^d$, $d \geq 1$, and let $\{\mathcal{C}_n\}$ be a sequence of coverings by compact sets satisfying conditions (i) and (ii), which furthermore is $\frac{1}{2}$-fine. Let $\{v_i\}$ be a sequence in $\mathbb{R}^d$ and $\{r_i\}$ be positive numbers such that for some $c > 0$ we have $\|v_i - v_j\| \geq c\sqrt{r_i r_j}$*

for all $i, j, i \neq j$. Then $\{v \in C \mid \exists\ \delta > 0$ such that $\|v - v_i\| > \delta r_i \ \ \forall\ i\}$ is a winning set in C for the game associated to $\{\mathcal{C}_n\}$. In particular these sets are nonempty.

The conclusion applies in particular to the set of badly approximable numbers in $\mathbb{R}$ and the set of endpoints of lifts of bounded geodesics of manifolds of constant negative curvature discussed in § 4.

It may be noted that an arbitrary compact subset of $\mathbb{R}$, even an uncountable one, may not contain a badly approximable number; the complement of the set of badly approximable numbers has positive Lebesgue measure and hence contains compact subsets of positive measure. Thus a suitable additional condition on the compact set is necessary for the conclusion as in the above theorem to hold. Conversely we see that the compact sets in the complement of the set of badly approximable numbers do not possess the regularity properties as in conditions (i) and (ii).

The results of Fishman apply also to sets of badly approximable vectors in $\mathbb{R}^d$, $d \geq 2$, which can not be described in terms of the sequences $\{x_i\}$ and $\{r_i\}$ as above. Work on proving a generalisation of the above theorem to cover this case is in progress.

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TRANSCENDENTAL NUMBERS AND ZETA FUNCTIONS

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1. Introduction

The concept of "number" has formed the basis of civilization since time immemorial. Looking back from our vantage point of the digital age, we can agree with Pythagoras that "all is number". The study of numbers and their properties is the mathematical equivalent of the study of atoms and their structure. It is in fact more than that. The famous physicist and Nobel Laureate Eugene Wigner spoke of the "unreasonable effectiveness of mathematics in the natural sciences" to refer to the miraculous power of abstract mathematics to describe the physical universe.

Numbers can be divided into two groups: algebraic and transcendental. Algebraic numbers are those that satisfy a non-trivial polynomial equation with integer coefficients. Transcendental numbers are those that do not. Numbers such as $\sqrt{2}, \sqrt{-1}$ are algebraic, whereas, numbers like π and e are transcendental. To prove that a given number is transcendental can be quite difficult. It is fair to say that our knowledge of the universe of transcendental numbers is still in its infancy.

A dominant theme that has emerged in the recent past is the theory of special values of zeta and L-functions. In this article, we will touch only the hem of the rich tapestry that weaves transcendental numbers and values of L-functions in an exquisite way. This idea can be traced back to Euler and his work.

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In 1735, Euler discovered experimentally that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}. \tag{1}$$

He gave a "rigorous" proof much later, in 1742. Here is a sketch of Euler's proof [8]. The polynomial

$$(1 - x/r_1)(1 - x/r_2)...(1 - x/r_n)$$

has roots equal to $r_1, r_2, ..., r_n$. When we expand the polynomial, the coefficient of x is

$$-(1/r_1 + 1/r_2 + \cdots + 1/r_n).$$

Using this observation, Euler proceeded "by analogy." Supposing that $\sin \pi x$ "behaves" like a polynomial and noting that its roots are at $x = 0, \pm 1, \pm 2, ...,$ Euler puts

$$f(x) = \frac{\sin \pi x}{\pi x}.$$

By l'Hôpital's rule, $f(0) = 1$. Now $f(x)$ has roots at $x = \pm 1, \pm 2, ...$ and so

$$f(x) = (1 - x)(1 + x)(1 - x/2)(1 + x/2)(1 - x/3)(1 + x/3) \cdots .$$

That is,

$$f(x) = (1 - x^2)(1 - x^2/4)(1 - x^2/9) \cdots .$$

The coefficient of x^2 on the right hand side is

$$-\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \right).$$

By Taylor's expansion,

$$\sin \pi x = \pi x - (\pi x)^3/3! + \cdots$$

so that comparing the coefficients, gives us formula (1).

The main question is whether all of this can be justified. Euler certainly didn't have a completely rigorous proof of his argument. To make the above discussion rigorous, one needs Hadamard's theory of factorization of entire funtions, a theory developed much later in 1892, in Jacques Hadamard's doctoral thesis.

The next question is whether Euler's result can be generalized. For example, can we evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Euler had difficulty with the first question but managed to show, using a similar argument, that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

and more generally that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \in \pi^{2k}\mathbb{Q}.$$

It is not hard to see Euler's proof can be modified to deduce the above results. Indeed, if $i = \sqrt{-1}$, then observing that

$$f(ix) = (1 + x^2)(1 + x^2/4) \cdots$$

we see that

$$f(x)f(ix) = (1 - x^4)(1 - x^4/2^4)(1 - x^4/3^4) \cdots$$

But the Taylor expansion of $f(x)f(ix)$ is

$$\left(1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} - \cdots\right)\left(1 + \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + \cdots\right).$$

Computing the coefficient of x^4 yields

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Continuing in this way, it is not difficult to see how Euler arrived at the assertion that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \in \pi^{2k}\mathbb{Q}.$$

Euler's work is the beginning of a modern theme in number theory, namely the transcendence of special values of L-series.

As explained at the outset, a complex number α is called *algebraic* if it is the root of a monic polynomial with rational coefficients. It is a well-known fact of algebra that the set of all algebraic numbers forms a field, usually denoted by $\overline{\mathbb{Q}}$. The elements of $\mathbb{C}\backslash\overline{\mathbb{Q}}$ are called *transcendental* numbers. For example, $\sqrt{2}$ is algebraic since it is the root of $x^2 - 2$. So is $5^{1/3}/3$ since it is the root of $x^3 - 5/27$. On the other hand, numbers such as π and e are transcendental and this is due to Lindemann and Hermite, respectively. Once a number is suspected to be transcendental, it is often difficult to show that it is so. For example, it is unknown if

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is transcendental, though it is conjectured to be so. In 1978, Roger Apéry [1] surprised the world by proving that it is irrational. There are other familiar numbers about whose arithmetic nature nothing is known. For instance, Euler's constant γ defined as

$$\gamma := \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n} - \log x \right)$$

is conjectured to be transcendental. We do not even know if it is irrational. More will be said about such questions later in this article.

Historically, it took mathematicians quite a long time to show that transcendental numbers exist. This was the question back in 1844. In that year, Liouville showed that if $\alpha \in \mathbb{C} \backslash \mathbb{Q}$ and

$$\inf\{q^n |\alpha - p/q| : p/q \in \mathbb{Q}\} = 0$$

for all natural numbers n, then α is transcendental. Using this criterion, he gave an explicit construction of certain transcendental numbers. For instance, he showed that

$$\sum_{n=1}^{\infty} 1/2^{n!}$$

is transcendental.

In retrospect, it is easy to see that a simple countability argument establishes the existence of transcendental numbers, but this argument, due to Cantor, came much later, in 1874, when he introduced notions of countability and uncountability. Incidentally, Cantor was born on March 3, 1845, one year after the publication of Liouville's paper.

In 1873, Hermite showed that e is transcendental and in 1882, Lindemann, using Hermite's technique, showed that π is transcendental. Lindemann's theorem, was important for resolving the problem of "squaring the circle." This problem, arising in ancient times, is to construct a square, using only a straightedge and compass, with area equal to that of a circle of radius 1. In other words, one must construct a line segment of length, $\sqrt{\pi}$.

One can show that if $\alpha > 0$ and we can construct a line segment of length α using only a straightedge and compass, then α must be algebraic. In fact, the field $\mathbb{Q}(\alpha)$ generated by α must have degree equal to a power of 2. More precisely, there is a tower of successive quadratic extensions

$$\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = \mathbb{Q}(\alpha)$$

with $[K_{i+1} : K_i] = 2$. In fact, the converse is also true and this gives a characterization of constructible numbers. From this result, we deduce that if $\sqrt{\pi}$ is constructible, then it must be algebraic. Consequently, π is algebraic, contrary to Lindemann's theorem. Thus, the problem of "squaring the circle" is impossible.

Let us observe that the ancient problem of doubling the cube, that is, the problem of constructing a cube whose volume equals that of a cube of length 1 is equivalent to constructing $2^{1/3}$. Since this number generates a field of degree 3 over $\mathbb{Q}$, it cannot be contained in a tower of fields as described above. Therefore, it is not constructible. A similar form of reasoning dismisses the problem of trisecting an angle using straightedge and compass. Indeed, the problem of trisecting $\pi/3$ is equivalent to constructing $\cos\pi/9$ and one can show that this number is the root of

$$4x^3 - 3x - 1/2.$$

This polynomial is irreducible over $\mathbb{Q}$ and so, $\cos\pi/9$ generates a cubic extension of the rationals.

These two problems of doubling the cube and trisecting an angle, were first treated in this algebraic setting by P. Wantzel in 1837 [21].

Shortly after Lindemann's paper appeared, Hermite obtained the following extension. For any algebraic $\alpha \neq 0$, e^α is transcendental. As a corollary, we deduce that for any algebraic number $\alpha \neq 0, 1$, $\log\alpha$ is transcendental. Since $e^{\pi i} = -1$, this shows that π is transcendental.

In his 1882 paper, Lindemann stated without proof that if $\alpha_1, ..., \alpha_n$ are distinct algebraic numbers, then

$$e^{\alpha_1}, ..., e^{\alpha_n}$$

are linearly independent over $\overline{\mathbb{Q}}$ (see page 872 of [14]). In 1885, Weierstrass published a proof of this by showing the equivalent result that if $\alpha_1, ..., \alpha_n$ are linearly independent over $\mathbb{Q}$, then

$$e^{\alpha_1}, ..., e^{\alpha_n}$$

are algebraically independent over $\overline{\mathbb{Q}}$. In other words, there is no non-trivial polynomial

$$P(x_1, ..., x_n) \in \mathbb{Q}[x_1, ..., x_n]$$

such that

$$P(e^{\alpha_1}, ..., e^{\alpha_n}) = 0.$$

This is usually referred to as the Lindemann-Weierstrass theorem.

All of these results comprise the first phase in the development of transcendental number theory. The second phase begins with Hilbert's problems. In 1900, at the International Congress of Mathematicians, Hilbert posed 23 problems for the 20th century. His 7th problem asked if a number like $2^{\sqrt{2}}$ is transcendental. More generally, he asked if α is algebraic $\neq 0, 1$ and β is an algebraic irrational, then is α^β transcendental? If the answer is yes, then we can deduce that numbers like e^π are transcendental by taking $\alpha = -1 = e^{\pi i}$ and $\beta = -i$. This specific case was proved by Gelfond in 1929. Extending these techniques, Gelfond and Schneider, independently, in 1934, proved the following:

Theorem 1. (Gelfond-Schneider, 1934) *If $\alpha, \beta \in \overline{\mathbb{Q}}$, with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, then α^β is transcendental.*

In 1966, Baker [2] derived a generalization of this theorem. He showed that if $\alpha_1, ..., \alpha_n, \beta_0, \beta_1, ...\beta_n \in \overline{\mathbb{Q}}$ with $\alpha_1 \cdots \alpha_n \beta_0 \neq 0$, then

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

is transcendental. He proved this by showing that the linear form

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. In 1970, Alan Baker was awarded the Fields medal at the ICM in France for this work.

For later reference, we state Baker's theorem precisely.

Theorem 2. *If $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Q}} \backslash \{0\}$ and $\beta_1, ..., \beta_n \in \overline{\mathbb{Q}}$, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. The former case arises only if $\log \alpha_1, ...,$ $\log \alpha_n$ are linearly dependent over $\mathbb{Q}$ or if $\beta_1, ..., \beta_n$ are linearly dependent over $\mathbb{Q}$.

Proof. This is the content of Theorems 2.1 and 2.2 of [2]. Let us note that here and later, we interpret log as the principal value of the logarithm with the argument lying in the interval $(-\pi, \pi]$. $\square$

2. **The Riemann zeta function and its' special values**

Interesting transcendental numbers arise as special values of L-series. The prototypical case is that of the Riemann zeta function. This function, originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$, can be extended analytically to the complex plane and shown to satisfy the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s),$$

where

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt,$$

for $\Re(s) > 0$ and extended as a meromorphic function to the entire complex plane via the familiar functional equation $\Gamma(s+1) = s\Gamma(s)$.

The famous theorem of Euler described in the introduction is that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$ so that $\zeta(2k)$ is transcendental for all natural numbers $k \geq 1$. A natural question that arises is: what is the nature of $\zeta(3), \zeta(5), ...$? It is conjectured that the numbers $\pi, \zeta(3), \zeta(5), ...$ are algebraically independent numbers. In particular, $\zeta(3), \zeta(5), ...$ are all conjectured to be transcendental.

As mentioned earlier, Roger Apéry showed in 1977 that $\zeta(3)$ is irrational. His proof, involving complicated recurrences, has received considerable attention and simplification. Most recently, Rivoal [19] was able to extend his technique to show that infinitely many $\zeta(2k+1)$ for $k \geq 1$ are irrational.

3. Apéry's theorem

Here is a brief sketch of Apéry's proof. His starting point was the remarkable formula

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Consider the recurrence

$$n^3u_n + (n-1)^3u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}.$$

Let a_n be the sequence satisfying this recurrence with the initial conditions $a_0 = 0$ and $a_1 = 6$. Let b_n be the sequence satisfying this recurrence with the initial conditions $b_1 = 1, b_1 = 5$. Apéry showed that all the b_n are integers, which is rather surprising since in the recurrence, we are dividing by n^3 to get the expression for u_n. Even more remarkable is that the a_n's are rational numbers such that $2[1, 2, ..., n]^3 a_n$ is an integer for all n. (Here

$[1, 2, ..., n]$ denotes the least common multiple of the numbers, $1, 2, ..., n$.) With these sequences in place, Apéry shows that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \zeta(3).$$

More precisely,

$$|\zeta(3) - \frac{a_n}{b_n}| \leq \frac{6}{b_n^2}.$$

He now invokes an elementary lemma that is easy to prove: if there are infinitely many rational numbers p_n/q_n such that

$$|\theta - p_n/q_n| \leq 1/q_n^{1+\delta},$$

for some $\delta > 0$, then θ is irrational. In our case, the denominator of a_n/b_n is easily estimated and the irrationality of $\zeta(3)$ follows from the elementary lemma.

Indeed, if $\zeta(3)$ was rational and equal to A/B (say) with A, B positive integers, then

$$B|\zeta(3)b_n - a_n| = |Ab_n - Ba_n| \leq 6B/b_n.$$

Multiplying through by $2[1, 2, ..., n]^3$ clears all denominators. By the prime number theorem, this is asymptotic to

$$2e^{3n+o(n)}.$$

On the other hand, $b_n = \alpha^{n+o(n)}$ where $\alpha = (1 + \sqrt{2})^4 = 17 + 12\sqrt{2}$, by a routine calculation. Since $e^3 < (1 + \sqrt{2})^4$, we deduce that $a_n = Ab_n$ for n sufficiently large and this is quickly checked to be a contradiction.

Attempts to generalize this argument to other odd values of the zeta function have not succeeded. In this direction, Rivoal showed in 2000, that infinitely many of the numbers $\zeta(2k+1)$ are irrational. In 2001, Rivoal and Zudilin [20] showed that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. In 2003, Ball and Rivoal [4] showed that the $\mathbb{Q}$-vector space spanned by $\zeta(3), \zeta(5), ...$ has infinite dimension.

Related to this is a curious formula of Ramanujan. In his notebooks, Ramanujan wrote

$$\zeta(3) + 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)} = \frac{7\pi^3}{180}.$$

Presumably, Ramanujan had a proof but the first rigorous proof was given by Grosswald [9] in 1970. Hence, at least one of the two terms on the left hand side is transcendental!

4. Multiple zeta values

To understand the arithmetic nature of special values of the Riemann zeta function, it has become increasingly clear that multiple zeta values (MZV's for short) must be studied. These are defined as follows:

$$\zeta(a_1, ..., a_k) = \sum_{n_1 > n_2 > \cdots n_k > 0} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}},$$

where $a_1, a_2, ..., a_k$ are positive integers with the proviso that $a_1 \neq 1$. The last condition is imposed to ensure convergence of the series.

There are several advantages to introducing these multiple zeta functions. First, they have an algebraic structure which we describe. It is easy to see that

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

Indeed, the left hand side can be decomposed as

$$\sum_{n_1, n_2} \frac{1}{n_1^{s_1} n_2^{s_2}} = \sum_{n_1 > n_2} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2 > n_1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_1 = n_2} \frac{1}{n_1^{s_1} n_2^{s_2}}$$

from which the identity becomes evident. In a similar way, one can show that $\zeta(s_1)\zeta(s_2, ..., s_t)$ is again an integral linear combination of multiple zeta values (MZV's). More generally, the product of any two MZV's is an integral linear combination of MZV's. These identities lead to new relations, like $\zeta(2, 1) = \zeta(3)$, an identity which appears in Apéry's proof of the irrationality of $\zeta(3)$.

If we let A_r be the $\mathbb{Q}$-vector space spanned by

$$\zeta(s_1, s_2, ..., s_k)$$

with $s_1 + s_2 + \cdots + s_k = r$, then the product formula for MZV's shows that

$$A_r A_s \subseteq A_{r+s}.$$

In this way, we obtain a graded algebra of MZV's. Let d_r be the dimension of A_r as a vector space over $\mathbb{Q}$. For convenience, we set $d_0 = 1$ and $d_1 = 0$. Clearly, $d_2 = 1$ since A_2 is spanned by $\pi^2/6$. Zagier [13] has made the following conjecture: $d_r = d_{r-2} + d_{r-3}$, for $r \geq 3$. In other words, d_r satisfies a Fibonacci-type recurrence relation. Consequently, d_r is expected to have exponential growth. Given this prediction, it is rather remarkable that not a single value of r is known for which $d_r \geq 2$!

In view of the identity, $\zeta(2, 1) = \zeta(3)$, we see that $d_3 = 1$. What about d_4? A_4 is spanned by $\zeta(4), \zeta(3, 1), \zeta(2, 2), \zeta(2, 1, 1)$. What are these numbers? Zagier's conjecture predicts that $d_4 = d_2 + d_1 = 1 + 0 = 1$. Is this true?

Let's adapt Euler's technique to evaluate $\zeta(2,2)$. As noted in the introduction,

$$(1 - x/r_1)(1 - x/r_2)\cdots(1 - x/r_n)$$

has roots equal to $r_1, r_2, ..., r_n$. When we expand the polynomial, the coeffcient of x is

$$-(1/r_1 + 1/r_2 + \cdots + 1/r_n).$$

The coefficient of x^2 is

$$\sum_{i<j} 1/r_i r_j.$$

With this observation, we see from the product expansion

$$f(x) = \frac{\sin \pi x}{\pi x} = (1 - x^2)(1 - x^2/4)(1 - x^2/9)\cdots$$

that the coefficient of x^4 is precisely $\zeta(2,2)$. An easy computation shows that

$$\zeta(2,2) = \pi^4/5!.$$

It is now clear that this method can be used to evaluate $\zeta(2,2,...,2) = \zeta(\{2\}^m)$ (say). By comparing the coefficient of x^{2m} in our expansion of $f(x)$, we obtain that

$$\zeta(\{2\}^m) = \frac{\pi^{2m}}{(2m+1)!}.$$

We could have also evaluated $\zeta(2,2)$ using the identity

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4),$$

but we had opted to the method above to indicate its generalization which allows us to also evaluate $\zeta(2,2,...,2)$. What about $\zeta(3,1)$? This is a bit more difficult and will not come out of our earlier work. In 1998, Borwein, Bradley, Broadhurst and Lisonek [5] showed that $\zeta(3,1) = 2\pi^4/6!$. What about $\zeta(2,1,1)$? With some work, one can show that this is equal to $\zeta(4)$. Thus, we conclude that $d_4 = 1$ as predicted by Zagier.

What about d_5? With more work, we can show that

$$\zeta(2,1,1,1) = \zeta(5); \quad \zeta(3,1,1) = \zeta(4,1) = 2\zeta(5) - \zeta(2)\zeta(3).$$

$$\zeta(2,1,1) = \zeta(2,3) = 9\zeta(5)/2 - 2\zeta(2)\zeta(3).$$

$$\zeta(2,2,1) = \zeta(3,2) = 3\zeta(2)\zeta(3) - 11\zeta(5)/2.$$

This proves that $d_5 \leq 2$. Zagier conjectures that $d_5 = 2$. In other words, $d_5 = 2$ if and only if $\zeta(2)\zeta(3)/\zeta(5)$ is irrational.

Can we prove Zagier's conjecture? To this date, not a single example is known for which $d_n \geq 2$. If we write

$$(1 - x^2 - x^3)^{-1} = \sum_{n=1}^{\infty} D_n x^n,$$

then it is easy to see that Zagier's conjecture is equivalent to the assertion that $d_n = D_n$ for all $n \geq 1$. Deligne and Goncharov [10] and (independently) Terasoma [22] showed that $d_n \leq D_n$.

5. **Dirichlet L-functions**

The Riemann zeta function is only a tiny fragment of a galaxy of L-series whose special values are of deep interest. The success of Euler's explicit evaluation of $\zeta(2k)$ can be extended to a class of series known as Dirichlet L-functions. These functions are defined as follows. Let q be a natural number and $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$ be a homomorphism of the group of coprime residue classes mod q. For each natural number $n \equiv a (\bmod\, q)$ with a coprime to q, $\chi(n)$ is defined to be $\chi(a)$. If n is not coprime to q, we set $\chi(n)$ to be zero. With this definition, we set

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

If χ is the trivial character, then $L(s, \chi)$ is easily seen to be

$$\zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

so that this is (upto a known factor) essentially the Riemann zeta function. If χ is not the trivial character, then it turns out that one can evaluate $L(k, \chi)$ explicitly in certain cases. To elaborate further, we say a character χ is even if $\chi(-1) = 1$ and odd if $\chi(-1) = -1$. For $k = 1$, $L(1, \chi)$ has been studied extensively and summarized in the celebrated formulas of Dirichlet. For $k \geq 2$, it turns out that $L(k, \chi) \in \pi^k \mathbb{Q}$ if k and χ are both even or both odd. When k and χ are of opposite parity, the arithmetic nature is a complete mystery, as enigmatic as the values of the Riemann zeta function at odd arguments. The simplest case of this mystery is to determine if

$$G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

is irrational. If χ is the non-trivial character (mod 4), then χ is odd and $G = L(2, \chi)$. Attempts to generalize Apéry's argument to show the irrationality of G have failed. However, Rivoal and Zudilin [20] have shown that at least one of $L(2k, \chi)$ with $1 \le k \le 10$ is irrational.

6. **Chowla's conjecture**

Inspired by the nature of Dirichlet's L-functions and the mystery surrounding their special values, Sarvadaman Chowla [6] considered Dirichlet series of the form

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where f is an algebraic-valued periodic function, with period q. It is not hard to see that the sum converges at $s = 1$ if and only if

$$\sum_{a=1}^{q} f(a) = 0.$$

In [6], Chowla asked if there exists a rational-valued function f, not identically zero, with prime period, such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0. \tag{2}$$

In [3], Baker, Birch and Wirsing answered this question using Baker's theory of linear forms in logarithms. In the general case when q is not necessarily prime, and f is algebraic-valued, they showed that the sum (6) in question can be written as a $\overline{\mathbb{Q}}$-linear form in logarithms of algebraic numbers. They considered functions f satisfying $f(a) = 0$ whenever $1 < (a, q) < q$. In the case that the q-th cyclotomic polynomial is irreducible over the field generated by the values of f, they showed that the sum is nonzero. By Baker's theorem, Theorem 2, we deduce the sum is transcendental. Several interesting corollaries can be deduced from this work, as noted in [16]. As indicated there, Chowla's question can be connected with the theory of the Hurwitz zeta function in the following way. The Hurwitz zeta function, defined for $0 < a \le 1$, as

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}$$

has an analytic continuation to the entire complex plane with a simple pole at $s = 1$. Its' Laurent expansion at $s = 1$ is given by

$$\zeta(s, a) = \frac{1}{s - 1} - \psi(a) + O(s - 1),$$

where $\psi(a)$ is the digamma function, $\Gamma'(a)/\Gamma(a)$. As proved in [16], we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\frac{1}{q}\sum_{a=1}^{q} f(a)\psi(a/q).$$

By choosing appropriate test functions f, one can show that there is at most one a/q such that $\psi(a/q)$ is algebraic.

7. Generalized Euler constants

Following Lehmer [15], we can define generalized Euler's constants $\gamma(a,q)$ by

$$\gamma(a,q) = \lim_{x\to\infty}\left(\sum_{\substack{n\le x\\ n\equiv a(\mathrm{mod}\ q)}} \frac{1}{n} - \frac{\log x}{q}\right).$$

Then, it is not hard to show that

$$q\gamma(a,q) = \gamma - \sum_{b=1}^{q-1} e^{-2\pi iba/q}\log(1-e^{2\pi ib/q}).$$

Apart from the γ term on the right hand side, this expresses the generalized Euler constants as a $\mathbb{Q}$-linear form of logarithms of algebraic numbers. Thus, Baker's theorem can be applied to study them. In addition, we can relate these constants to Chowla's question. Indeed, as noted in [16], we have:

Lemma 3. *If f is as above and*

$$\sum_{a=1}^{q} f(a) = 0,$$

then

$$\sum_{n=1}^{\infty}\frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(a,q).$$

This result allows us to state that all of the generalized Euler constants are transcendental with at most one possible exception.

8. The Chowla-Milnor conjecture

In [7], Paromita and Sarvadaman Chowla, considered the question of non-vanishing of $L(s,f)$ at $s=2$ for rational valued functions f. In the case q is prime, they conjectured that $L(2,f)\neq 0$ unless

$$f(1) = f(2) = \cdots f(q-1) = \frac{f(q)}{1-q^2}.$$

This can be formulated equivalently as the following conjecture on special values of the Hurwitz zeta function, as noted by Milnor [12]: the numbers

$$\zeta(2,1/q), \zeta(2,2/q), ..., \zeta(2,(q-1)/q)$$

are linearly independent over $\mathbb{Q}$. This suggested to Milnor the more general conjecture: for any natural number q, and $k \geq 2$, the numbers

$$\zeta(k, a/q), \quad 1 \leq a < q, \quad (a,q) = 1$$

are linearly independent over $\mathbb{Q}$. We refer to this as the Chowla-Milnor conjecture.

The difficulty of this conjecture is partly seen by the following theorem:

Theorem 4. (S. Gun, M. Ram Murty and P. Rath, 2008) *The Chowla-Milnor conjecture for the single modulus $q = 4$ is equivalent to the irrationality of $\zeta(2k+1)/\pi^{2k+1}$ for all natural numbers $k \geq 1$.*

Proof. See [11]. □

There are other implications of the Chowla-Milnor conjecture to many classical questions related to special values of Dirichlet L-functions and these are expanded upon in [11].

9. Concluding remarks

We hope that the reader is convinced that this is only a shadow of a richer theory that is yet to unfold. We refer the reader to [11] for yet more connections between transcendental numbers and zeta functions that emerge from Chowla's conjectures and their generalizations.

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ENTIRE FUNCTION EXTENSIONS OF CONTINUOUS FUNCTIONS : A BRIEF SURVEY

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ABSTRACT. The main topic of my today's talk is provide a brief review of the linkage between the extension of continuous functions from various domains to the whole complex as an entire function and their growth characteristics in terms of various error estimates.

1. Introduction

We are all aware of the measures of growth of an entire function such as the order, type, lower order and lower type etc. Hence if $f(z)$ is an entire function of one complex variable z and $M(r) = \max_{z=r} |f(z)|$ denotes its maximum modulus then its order is given by the formula

$$\rho = \lim_{r\to\infty} \sup \frac{\ln\ln M(r)}{\ln r}. \tag{1}$$

The above formula being accepted as yardstick for comparison of growth of various entire functions in view of the simplest transcendental entire function e^z having order equal to 1. If $0 < \rho < \infty$ then $f(z)$ is said to be of finite non- zero order and growth of such functions can be further classified by defining their type as

$$\tau = \lim_{r\to\infty} \sup \frac{\ln M(r)}{r^\rho}\ \ ,\ \ 0 \le \tau \le \infty\,. \tag{2}$$

2. Order and Type Formulae

This is the text of the Twenty second P. L. Bhatnagar Memorial Award Lecture delivered at the 74$^{\text{th}}$ Annual Conference of the Indian Mathematical Society held at the University of Allahabad, Allahabad - 211 002, India, during December 27-30, 2008.

If the power series representation of $f(z)$ is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then the characterizations of order and type and other growth parameters are available in terms of the sequence $\{a_n\}$. Thus we have

$$\rho = \lim_{n\to\infty} \sup \frac{n \ln n}{\ln |a_n|^{-1}} \tag{3}$$

and

$$e\rho\tau = \lim_{n\to\infty} \sup n|a_n|^{\rho/n}. \tag{4}$$

Many other classifications of entire functions based on their growth were subsequently introduced ,such as functions of fast growth ($\rho = \infty$) or slow growth ($\rho = 0$) , giving rise to many coefficients characterizations like above. To generalize the above ideas and to broaden the base of comparison functions, M. N. Seremeta [20] introduced the concept of generalized orders. Thus we have:

3. L^0 and Λ Classes

Let $\psi : [a, \infty) \to R$ be a real valued function such that
(i) $\psi(x) > 0$,
(ii) $\psi(x)$ is differentiable $\forall\, x \in [a, \infty)$,
(iii) $\psi(x)$ is strictly increasing and
(iv) $\psi(x) \to \infty$ as $x \to \infty$.

Further for every real valued function $\gamma(x)$ such that $\gamma(x) \to 0$ as $x \to \infty$, ψ satisfies

$$\lim_{x\to\infty} \frac{\psi[\{1 + \gamma(x)\}]}{\psi(x)} = 1\,. \tag{5}$$

Then ψ is said to belong to the class L^0. The function $\psi(x)$ is said to belong to the class Λ if $\psi(x) \in L^0$ and in place of (5), satisfies stronger condition

$$\lim_{x\to\infty} \frac{\psi(cx)}{\psi(x)} = 1\ , \tag{6}$$

for all c , $0 < c < \infty$. Functions ψ satisfying (6) are also called slowly increasing functions.

4. Generalized Order and Generalized Type

Using the generalized functions of class L^0 and Λ , Seremeta [20] obtained the following characteristics:

Let $\alpha(t) \in \Lambda$, $\beta(t) \in L^0$. Set $F(t,c) = \beta^{-1}[c\alpha(t)]$. If $dF(t,c)/d\ln t = O(1)$ as $t \to \infty$ for all c , $0 < c < \infty$, then for the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$,

$$\lim_{r\to\infty} \sup \frac{\alpha[\ln M(r,f)]}{\beta(\ln r)} = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta[\ln |c_n|^{-1/n}]}. \tag{7}$$

Again let $\alpha(t) \in L^0$, $\beta(t) \in L^0$ and $\gamma(t) \in L^0$. Let ρ , $0 < \rho < \infty$. ,be a fixed number. Set $F(t,\sigma,\rho) = \gamma^{-1}\left\{[\beta^{-1}\{\sigma\alpha(t)\}]^{1/\rho}\right\}$. Suppose that for all σ , $0 < \sigma < \infty$, $F(t,o,\rho)$ satisfies:

(a) If $\gamma(t) \in \Lambda$ and $\alpha(t) \in \Lambda$, then $d\ln F(t,\sigma,\rho)/d\ln t = O(1)$ as $t \to \infty$.

(b) If $\gamma(t) \in L^0 - \Lambda$ or $\alpha(t) \in L^0 - \Lambda$, then $\lim_{t\to\infty} d\ln F(t,\sigma,\rho)/d\ln t = 1/\rho$.

Then we have

$$\lim_{r\to\infty} \sup \frac{\alpha[\ln M(r,f)]}{\beta[\{\gamma(r)\}^\rho]} = \lim_{n\to\infty} \sup \frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}|c_n|^{-1/n})]^\rho\}}. \tag{8}$$

Later, S. M. Shah [21] called the left hand quantity in (7) as the generalized order $\rho(\alpha,\beta,f)$

5. **Generalized Lower Order**

J. M. Whittakar had introduced the concept of lower order for the study of growth of entire functions .On the same pattern, Shah [21] introduced the generalized lower order $\lambda(\alpha,\beta,f)$ as

$$\lambda(\alpha,\beta,f) = \lim_{r\to\infty} \inf \frac{\alpha[\ln M(r,f)]}{\beta(\ln r)} . \tag{9}$$

Further, Shah proved the following result [21. Theorem2]:

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function. Set $F(t) = \beta^{-1}[\alpha(t)]$. Let for some function $\psi(t)$ tending to ∞ (however slowly) as $t \to \infty$,

(i) $\frac{\beta\{t\psi(t)\}}{\beta(e^t)} \to 0$ as $t \to \infty$

(ii) $\frac{dF(t)}{d(\ln t)} = O(1)$ as $t \to \infty$

(iii) $|c_n/c_{n+1}|$ is ultimately a non decreasing function of n.

Then

$$\lambda(\alpha,\beta,f) = \lim_{n\to\infty} \inf \frac{\alpha(n)}{\beta[\ln |c_n|^{-1/n}]}. \tag{10}$$

One can return to classical formulae for order and type etc. by choosing $\alpha(x) = \ln x$, $\beta(x)$ and $\alpha(x) = \beta(x) = \gamma(x) = x$ respectively.

There are other growth characteristics available for entire transcendental functions but we will not go into those details.

6. **Approximation of Entire Function**

The process was started by the classical result of Bernstein given in his classical book [2]. Thus if $f(x) : [-1,1] \to R$ is continuous and $E_n(f) = \inf_{p\in\pi_n} ||f-p||_{L^\infty[-1,1]}$, $n = 0, 1, 2...$ denote the minimum error in the Chebyshev approximation of $f(x)$ over the set π_n of real polynomials of degree at most n, then it was proved by Bernstein that

$$\lim_{n\to\infty} E_n^{1/n}(f) = 0 \tag{11}$$

if and only if $f(x)$ has an entire function extension $f(z)$.

This result is parallel to the classical property of entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where the condition that

$$\lim_{n\to\infty} |a_n|^{1/n} = 0 \;\Leftrightarrow\; f \text{ is entire}$$

is well known.The above result of Bernstein was not exploited enough by the classical analysts till R. S. Varga [28] gave the characterizations of order and type of extended entire function $f(z)$ in terms of the sequence $\{E_n\}$ i.e., $f(z)$ is of order $\rho \Leftrightarrow \rho = \lim_{n\to\infty} \sup \frac{n \ln n}{\ln[E_n(f)]^{-1}}$.
Further,

$f(z)$ is of order ρ and type $\tau \Leftrightarrow e\rho\tau 2^\rho = \lim_{n\to\infty} \sup n(E_n)^{\rho/n}$.

A. R. Reddy and other workers generalized these results for other class of functions such as entire function of fast growth (introduced by D. Sato), entire functions of (p, q) growth etc. Using the idea of generalized growth functions introduced earlier, S. M. Shah [20] gave the characterization for the lower order under the monotonicity condition of the function

$$\psi(n) = E_n(f)/E_{n+1}(f).$$

Again the case of the entire function of (generalized) slow growth was considered separately by Kapoor and Nautiyal [14]. Different workers studied the approximation of continuous functions on different domains instead of intervals.

P. A. McCoy in a series of papers studied polynomial approximation and growth of generalized axi-symmetric potentials. The regular solutions of the generalized axi symmetric potential equation were related to an analytic function by the use of transform operators. The results obtained are similar to those of Bernstein's result. The generalized axi -symmetric potentials are the solution of the potential equation

$$\frac{\partial^2 F^\alpha}{\partial x^2} + \frac{\partial^2 F^\alpha}{\partial y^2} + \frac{2\alpha+1}{y}\ \frac{\partial F^\alpha}{\partial y} \ = \ 0, \ \ \alpha > \ -1/2.$$

Considering the approximation of the GASP function F^α through the set of all real harmonic polynomials of degree at most n, H_n^α , and defining the error in approximation as

$$E_n(F^\alpha) \ = \ \inf\{ \,||\ F^\alpha - P^\alpha||, \ P^\alpha \in \ H_n^\alpha\}, \ n = 0, 1, 2, ..$$

McCoy obtained the necessary and sufficient condition for analytic continuation of F^α as an entire function. Hence we have [18, p 54] :

Theorem 1: *Let the real GASP F^α be regular in the hyper sphere Σ^α and continuous on the closure of Σ^α .Then a necessary and sufficient condition for F^α to have an analytic continuation as an entire function is that*

$$\lim_{n\to\infty} \ E_n^{1/n}(F^\alpha) = 0.$$

From the above, similar to Varga's results, coefficient characterizations of order and type of the entire function extension were also obtained. Other workers extended these results In [22], we obtained the characterizations of lower order and lower types of F^α in terms of the errors $E_n(F^\alpha)$. In the subsequent paper [23], the characterizations of order and type etc. were obtained for the analytic extension in the hyper sphere Σ^α . Due to similarity in the expressions, we are not giving them here.

7. Approximation In The L_p^2- Norm

Let E denote the closed and bounded Jordan region in the complex plane with transfinite diameter $d > 0$. Let $w(z)$ be a positive continuous function on E and let $\mathrm{H}_2(E)$ denote the Hilbert space of analytic functions in D with inner product

$$(f, g) = \iint_E w(z) f(z)\, \overline{g(z)}\, dxdy \quad , \ f, g \in \mathrm{H}_2(E) \tag{12}$$

So that for any $f \in \mathrm{H}_2(E)$, we have

$$||f|| = \left[\iint_E w(z)\, \overline{|f(z)|^2}\, dxdy\right]^{1/2} \ < \infty \tag{13}$$

If $\mathrm{A}(E) = \{p_{n-1}(z)\}_{n=1}^{\infty}$, $p_n(z)$ being a polynomial of degree n, is a complete orthonormal sequence in $\mathrm{H}_2(E)$, then we put for any $f \in \mathrm{H}_2(E)$

and $n = 0, 1, 2...$

$$\begin{aligned} \Delta_n(f) &\equiv \Delta_n(f; E) \\ &= \inf_{c_i} \left[\iint_E w(z) |f(z) - \textstyle\sum_0^n c_k p_k(z)|^2 \, dxdy \right]^{1/2} \end{aligned} \tag{14}$$

$$a_n(f) \equiv a_n(f; E) = \iint_E w(z) f(z) \, \overline{p_n(z)} \, dxdy. \tag{15}$$

Then $\Delta_n(f; E)$ is called the minimum error of f in L^2- norm with respect to the system $\mathrm{A}(E)$ and a_n is called the n^{th} Fourier coefficient of f with respect to the system $\mathrm{A}(E)$. It is well known [9] that if U denotes the unit disk and $|w(z)| \equiv 1$ then $\mathrm{A}(U) \equiv \left\{ \sqrt{n/\pi} \; z^{n-1} \right\}_{n=1}^{\infty}$ forms a complete orthonormal sequence in $\mathrm{H}_2(U)$ and that if $f(z) = \sum_{n=0}^{\infty} b_n z^n$, $|z| < 1$, is in $\mathrm{H}_2(t)$ then

$$b_n = \sqrt{\frac{n+1}{\pi}} \; a_n(f; U). \tag{16}$$

Hence if f can be extended to an entire function, its growth can be characterized in terms of the coefficient $\{b_n\}$. In [19], S. R. H. Rizvi and O. P. Juneja obtained various resuls in this direction.

8. Approximation of Entire Harmonic Functions

Like GASP discussed above, the well known harmonic functions which are solutions of the Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0 \tag{17}$$

can also be used to approximate entire functions. A harmonic function $u(x_1, x_2, x_3)$, regular about the origin can be expanded as

$$u \equiv u(x_1, x_2, x_3) = \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} [a_{nm}^1 \cos m\varphi + a_{nm}^2 \sin m\varphi] \, P_n^m(\cos\theta) \, , \tag{18}$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\varphi$, $x_3 = r\sin\theta\sin\varphi$ and $P_n^m(t)$ are associated Legendre's functions of first kind of degree m and order n. The harmonic function u is said to be regular in the disc

$$D_R = \left\{ (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < R^2 \right\} \; , 0 < R < \infty$$

if the series in (18) converges uniformly on compact subsets of D_R. A harmonic function is called entire if it is regular in D_∞. We can define

$$M(r, u) = \sup_{x_1^2 + x_2^2 + x_3^2 < r^2} |u(x_1, x_2, x_3)|.$$

The growth parameters of u can now be defined as earlier. A. J. Fryant [5] obtained these in terms of coefficients a^i_{nm} , $i = 1, 2$.

Let H_R , $0 < R < \infty$ denote the class of all harmonic functions regular in D_R and continuous in the closure $\overline{D_R}$. Let

$$E_n(H, R) = \inf_{g \in \pi_n} ||H - g||_R$$

where the infimum is taken over the class π_n of all harmonic polynomials of degree at most n and

$$||H - g||_R = \max_{(x_1,x_2,x_3) \in \overline{D_R}} |H(x_1, x_2, x_3) - g(x_1, x_2, x_3)|.$$

Various workers obtained the growth parameters in terms of the error $E_n(H, R)$ such as in [16] and [17].

9. **Approximation and Interpolation**

Let $z_1, z_2, ..., z_n, ...$,where $z_n's$ need not be all distinct, be a bounded sequence of complex numbers and let f be analytic at the points $z_n, n = 1, 2, 3..$ Set

$$\omega_0(z) = 1$$

$$\omega_n(z) = (z - z_1)(z - z_2)...(z - z_n) \ , \ n = 1, 2... \ . \tag{19}$$

A series of the form

$$\sum_{k=0}^{\infty} c_k \, \omega_k(z) \ , \tag{20}$$

where $p_n(z) = \sum_{k=0}^{n} c_k \, \omega_k(z)$ is a polynomial of degree n which interpolate f in the points $\{ \ z_1, z_2, ..., z_{n+1}, \ .... \ \}$ is called interpolation series or Newton series of f. Walsh [29], showed that if the points $\{z_n\}$ are chosen interior to a domain D within which f is analytic, it is not necessary that the sequence $\{p_n(z)\}$ converges to $f(z)$ for every $z \in D$. However, if f is entire, the series (20) converges uniformly to f on every compact subset of the complex plane and the coefficients c_n are given by

$$c_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{\omega_{n+1}(t)} dt \ , \ n = 0, 1, 2... \tag{21}$$

where Γ is any contour which contains in its interior all the points $\{z_n\}$. The Newton series of an entire function reduce to its Taylor series about $z = 0$ if $z_n = 0$ for $n = 0, 1, 2...$.

Let $\{z_n\}$ be an extremal sequence in a compact set E in the complex plane i.e.

$$\lim_{n\to\infty} \sup \left\{ \sup_{z\in E} |(z-z_1)(z-z_2)...(z-z_n)|^{1/n} \right\} = d. \tag{22}$$

Let f be an entire function. Then

$$f(z) = \sum_{k=0}^{\infty} c_k\, \omega_k(z) \ , \tag{23}$$

where $\{\omega_k\}$ and $\{c_k\}$ are given by (19) and (21) respectively. We define

$$\mu_{n,1}(f) = \inf_{g\in\pi_n} \sup_{z\in E} |f(z)-g(z)| \ = \inf_{g\in\pi_n} ||f-g|| \ , \tag{24}$$

$$\mu_{n,2}(f) = \ ||f-p_n|| \ , \tag{25}$$

$$\mu_{n,3}(f) = \ ||p_n - p_{n-1}|| \ , \tag{26}$$

where $p_n(z) = \sum_{k=0}^{n} c_k\, \omega_k(z)$.

The characterization of various growth parameters of entire function $f(z)$ were obtained in [15]. Again the expressions are of similar nature and we omit the statements.

A. B. Batyrev [1], T. Winiarski [30,31] and A. Giroux [8] extended these results for bounded closed sets and obtained similar results.

10. Approximation of Entire Function over Jordan Domains

A. Girous [8] considered the norm

$$||f||_{L^p(D)} = \left(\frac{1}{A} \iint_D |f(z)|^p dx\, dy \right)^{1/p} < \ \infty\, , \tag{27}$$

where A is area of the domain D enclosed by the Jordan arc C with interior G. Further we put $E_n^p = E_n^p(f, D) = \min_{\pi_n} ||f - \pi_n||_{L^p(D)}$, where π_n is an arbitrary polynomial of degree at most n. Giroux proved that a function $f \in L^p(D)$ is the restriction to D of an entire function if and only if $\lim_{n\to\infty} E_n^p(f) = 0$. In this case f is an entire function of order ρ if and only if

$$\rho = \lim_{n\to\infty} \sup \frac{n \ln n}{\ln (E_n^p)^{-1}} \tag{28}$$

and of finite type σ if and only if

$$e\rho\sigma d^\rho = \lim_{n\to\infty} \sup n (E_n^p)^{\rho/n} \tag{29}$$

where d is the transfinite diameter of the domain D.

Winiarski [30] and Adam Janik [11] extended these properties for entire functions of several complex variables. Recently, Vakarchuk [25, 26] has studied the approximation of functions on Banach spaces. We now give some of his results and our recent work in this direction. M. I. Gvaradze [10] introduced a class of Banach spaces in the following manner.

For $0 < p < q \leq \infty$, $0 < \lambda \leq \infty$, let $\beta(p,q,\lambda)$ denote the space of functions $f(z)$ which are analytic in the disk $|z| < 1$ and for which

$$||f||_{p,q,\lambda} = \left\{ \int_0^1 (1-r)^{\lambda(1/p-1/q)-1} M_q^{\lambda}(r,f)\, dr \right\}^{1/\lambda} < \infty,$$

where $0 < \lambda < \infty$,

$$||f||_{p,q,\infty} = \sup \left\{ (1-r)^{(1/p-1/q)} M_q(r,f) \; ; \; 0 < r < 1 \right\} < \infty, \tag{30}$$

where $M_q(r,f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q \, d\theta \right\}^{1/q}$.
For all $p > 0$ and q , $\lambda \geq 1$, $\beta(p,q,\lambda)$ is a Banach space.

Like in the case of ordinary approximation, efforts were made by Vakarchuk to study the growth of an analytic function $f(z)$ vis--vis the approximation errors introduced as follow:

Let $f(z) \in \beta(p,q,\lambda)$ and let P_k be the set of all polynomials of degree not greater than k. The best approximation of f by members of P_k is denoted by

$$E\left(f, P_k, \beta(p,q,\lambda)\right) = \inf\left\{ ||f - p_k||_{p,q,\lambda} \; : \; p_k \in P_k \right\}. \tag{31}$$

On the pattern of Bernstein's result, Vakarchuk obtained following results in $\beta(p,2,\lambda)$ space.

Theorem 2. *Suppose that* $f(z) \in \beta(p,2,\lambda)$ *where* $0 < p < 2$ *and* $\lambda \geq 1$. *The condition*

$$\lim_{k\to\infty} \left\{ E\left(f, P_k, \beta(p,2,\lambda)\right) B^{-1/\lambda}\left((k+1)\lambda + 1; \lambda(1/p - 1/2)\right) \right\}^{1/k} = 0 \tag{32}$$

is necessary and sufficient for the function $f(z)$ *to be entire.*

Theorem 3. *Suppose that* $f(z) \in \beta(p,2,\lambda)$ *where* $0 < p < 2$ *and* $\lambda \geq 1$, *is an entire function of finite order* $\rho \in (0,\infty)$ *if and only if*

$$\rho = \limsup_{k\to\infty} \frac{(k \ln k)}{\left[\ln\left\{ E^{-1}\left(f, P_k, \beta(p,2,\lambda)\right) B^{1/\lambda}\left((k+1)\lambda + 1\, ; \, \lambda(1/p - 1/2)\right)\right\}\right]} \tag{33}$$

Theorem 4 . *Suppose that* $f(z) \in \beta(p,2,\lambda)$ *where* $0 < p < 2$ *and* $\lambda \geq 1$, *is an entire function of finite order* $\rho \in (0, \infty)$ *and normal type* $\sigma \in (0, \infty)$ *if and only if*

$$\lim_{k\to\infty} \sup \frac{1}{(e\rho)} k \left\{ \frac{E\left(f, P_k, \beta(p, 2, \lambda)\right)}{B^{1/\lambda}\left((k+1)\lambda + 1\,;\, \lambda(1/p - 1/2)\right)} \right\}^{\rho/k} = \sigma. \qquad (34)$$

11. Characterization of Order in $\beta(p, q, \lambda)$ Space

Theorem 5. *A function* $f(z) \in \beta(p, q, \lambda)$ *is entire if and only if*

$$\lim_{k\to\infty} \left\{ E\left(f, P_k, \beta(p, q, \lambda)\right) B^{-1/\lambda}\left((k+1)\lambda + 1\,;\, \lambda(1/p - 1/q)\right) \right\}^{1/k} = 0 \qquad (35)$$

Theorem 6. *Suppose that* $f(z) \in \beta(p, q, \lambda)$ *where* $0 < p < q \le \infty$ *and* $\lambda \ge 1$. *Then* $f(z)$ *is an entire function of finite order* $\rho \in (0, \infty)$ *if and only if*

$$\rho = \limsup_{k\to\infty} \frac{(k \ln k)}{\left[\ln\left\{E^{-1}\left(f, P_k, \beta(p, q, \lambda)\right) B^{1/\lambda}\left((k+1)\lambda + 1\,;\, \lambda(1/p - 1/q)\right)\right\}\right]} \qquad (36)$$

Theorem 7. *Suppose that* $f(z) \in \beta(p, q, \lambda)$ *where* $0 < p < q \le \infty$ *and* $\lambda \ge 1$. *Then* $f(z)$ *an entire function of finite order* $\rho \in (0, \infty)$ *and normal type* $\sigma \in (0, \infty)$ *if and only if*

$$\lim_{k\to\infty} \sup \frac{1}{(e\rho)} k \left\{ \frac{E\left(f, P_k, \beta(p, q, \lambda)\right)}{B^{1/\lambda}\left((k+1)\lambda + 1\,;\, \lambda(1/p - 1/q)\right)} \right\}^{\rho/k} = \sigma. \qquad (37)$$

12. Hardy and Bergman Spaces

We observe that above results (3 theorems each) are parallel to those obtained by Bernstein and Varga. Vakarchuk [26] introduced the Hardy class and Bergman spaces. Thus H_q, $q > 0$ denote the Hardy class of functions satisfying the condition

$$||f||_{H_q} = \lim_{r\to 1^-} M_q(r, f) \ < \ \infty,$$

and $H_q^{'}$ $q > 0$ denote the Bergman class of analytic functions satisfying the condition

$$||f||_{H_q'} = \left(\frac{1}{\pi} \iint_{|z|<1} |f(z)|^q dx\, dy \right)^{1/q} \ < \ \infty\ , \qquad (38)$$

Also we put

$$||f||_{H_q'} = ||f||_{H_q} = \sup\left\{|f(z)|\ ,\ |z| < 1\right\}\ .$$

Then H_q and $H_q^{'}$ are Banach spaces for $q \ge 1$. Results parallel to those stated above were obtained for functions belonging to Hardy class. Vakarchuk

[26] obtained the approximation formulae for entire functions of slow growth also.Thus we have

Theorem 8. *For the function* $f(z) \in \beta(p, q, \lambda)$ *where* $0 < p < q \leq \infty$ *and* $\lambda \geq 1$, *to be*
(i) an entire function of logarithmic order $\rho_1 \in (1, \infty)$, *it is necessary and sufficient that*

$$\lim_{k\to\infty} \sup \frac{\ln k}{\ln\left[k^{-1} \ln E^{-1}\left(f, P_k, \beta(p, q, \lambda)\right)\right]} = \rho_1 - 1 \ , \tag{39}$$

(ii) an entire function of logarithmic order $\rho_1 \in (1, \infty)$ *and logarithmic type* $\tau_1 \in (0, \infty)$, *it is necessary and sufficient that*

$$\lim_{k\to\infty} \sup \frac{(\rho_1 - 1)^{\rho_1 - 1}}{(\rho_1)^{\rho_1}} \frac{k^{\rho_1}}{\left[\ln E^{-1}\left(f, P_k, \beta(p, q, \lambda)\right)\right]^{\rho_1 - 1}} = \tau_1 \ . \tag{40}$$

Theorem 9. *For the function* $f(z) \in H_q$, $q \geq 1$, *to be*
(i) an entire function of logarithmic order $\rho_1 \in (1, \infty)$, *it is necessary and sufficient that*

$$\lim_{k\to\infty} \sup \frac{\ln k}{\ln\left[k^{-1} \ln E^{-1}\left(f, P_k, H_q\right)\right]} = \rho_1 - 1 \ , \tag{41}$$

(ii) an entire function of logarithmic order $\rho_1 \in (1, \infty)$ *and logarithmic type* $\tau_1 \in (0, \infty)$, *it is necessary and sufficient that*

$$\lim_{k\to\infty} \sup \frac{(\rho_1 - 1)^{\rho_1 - 1}}{(\rho_1)^{\rho_1}} \frac{k^{\rho_1}}{\left[\ln E^{-1}\left(f, P_k, H_q\right)\right]^{\rho_1 - 1}} = \tau_1 \ . \tag{42}$$

In [27], Vakarchuk and Zhir considered the generalized order on the lines of Seremeta and Shah as reported and obtained the following results.

Theorem 10 . *Suppose that* $\gamma(\alpha, \beta)$ *is a finite positive number. Then a function* $f(z) \in \beta(p, q, \lambda)$ *to be an entire function of generalized growth order* $\gamma(\alpha, \beta)$, *it is necessary and sufficient that*

$$\lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta\left[-n^{-1} \ln E_n\left(f, \beta(p, q, \lambda)\right)\right]} = \gamma(\alpha, \beta) \ . \tag{43}$$

Theorem 11. *Suppose that* $\delta(\alpha, \beta)$ *is a finite positive number. Then a function* $f(z) \in H_q$ *to be an entire function of generalized growth order* $\delta(\alpha, \beta)$, *it is necessary and sufficient that*

$$\lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta\left[-n^{-1} \ln E_n\left(f, H_q\right)\right]} = \gamma(\alpha, \beta) \ . \tag{44}$$

The above two theorems were obtained under the assumptions of results of Seremeta stated above. These results leave some gaps and we tried to fill these and obtained the following results. The next result gives a characterization of generalized type [6, 7]

Theorem 12. *For the function* $f(z) \in \beta(p,q,\lambda)$, *we have*

$$\lim_{r\to\infty}\sup \frac{\alpha[\ln M(r,f)]}{\beta[\{\gamma(r)\}^{\rho}]} = \lim_{n\to\infty}\sup \frac{\alpha(n/\rho)}{\beta\left\{\gamma\left[e^{1/\rho}\left(E_n\left(\beta(p,q,\lambda)\ ;\ f\right)\right)^{-1/n}\right]^{\rho}\right\}} . \quad (45)$$

provided for $F(z) = \sum_n a_n z^n,\ |a_n/a_{n+1}| \to d\ , 0 < d < \infty$.

When we consider the functions of slow growth, i.e when in the Seremeta's definition, then certain assumptions are no longer valid and above results do not hold.For such cases, we have the following results:

Theorem 13. *For the entire function* $F(z) = \sum_n a_n z^n$, *of generalized order* $\rho_1 \in (1,\infty)$, *the generalized type* τ *is given by*

$$\tau = \lim_{r\to\infty}\sup \frac{\alpha[\ln M(r,f)]}{\{\alpha(\ln r)\}^{\rho}} = \lim_{n\to\infty}\sup \frac{\alpha(n/\rho)}{\left\{\alpha\left[\frac{\rho}{\rho-1}\ln|a_n|^{-1/n}\right]\right\}^{\rho-1}} . \quad (46)$$

The above result is utilized to obtain approximation criterion for functions belonging to Bergman spaces. Hence we have

Theorem 14. *A necessary and sufficient condition for the entire function* $F(z) = \sum_n a_n z^n$, *of generalized order* $\rho_1 \in (1,\infty)$, *and belonging to the class* $\beta(p,q,\lambda)$, *to be of generalized type* τ *is that*

$$\tau = \lim_{n\to\infty}\sup \frac{\alpha(n/\rho)}{\left\{\alpha\left[\frac{\rho}{\rho-1}\ln\left(\left(\beta(p,q,\lambda)\ ;\ f\right)\right)^{-1/n}\right]\right\}^{\rho-1}} . \quad (47)$$

A similar result holds for the function $f(z) \in H_q$.

13. Several Complex Variables

T. Winiarski [31] and A. Janik [13] considered the approximation of functions of several complex variables.Janik considered the 'order systems' of entire functions of several complex variables.Using the definitive characterizations of order and type introduced by Bose and Sharma [4], we have obtained the following results, which have appeared recently [7].

Let H_q , $q > 0$ denote the space of functions $f(z_1,z_2)$ analytic in the unit bi-disk $U = \{z_1,z_2 \in C\ ;\ |z_1| < 1,\ |z_2| < 1\}$, such that

$$||f||_{H_q} = \lim_{r_1,r_2\to 1-0} M_q(f;r_1,r_2) < \infty ,$$

$$M_q(f;r_1,r_2)=\left\{\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f\left(r_1e^{i\theta_1},r_2e^{i\theta_2}\right)|^q\ d\theta_1d\theta_2\right\}^{1/q}$$

and let $H_q^{'}$ denote the space of functions $f(z_1,z_2)$ analytic in U and satisfying the condition

$$||f||_{H_q^{'}}=\left\{\frac{1}{\pi^2}\int_{|z_1|<1}\int_{|z_2|<1}|f(z_1,z_2)|^q dx_1dy_1dx_2dy_2\right\}\ <\ \infty\,.$$

We set $||f||_{H_\infty^{'}}=||f||_{H_\infty}=\sup\left\{|f(z)|\ ,\ z\in U\right\}$.

Then H_q and $H_q^{'}$ are Banach spaces for $q\geq 1$. In analogy with space of functions of one complex variable, we call H_q and $H_q^{'}$ the Hardy and Bergman spaces respectively.The function $f(z_1,z_2)$ analytic in the unit bidisk $U=\{z_1,z_2\in C\ ;\ |z_1|\ <\ 1\ ,\ |z_2|\ <1\}$ is said to belong to the space $\beta(p,q,\lambda)$, where $0<p<q\leq\infty$ and $0<\lambda\leq\infty$ if

$$||f||_{p,q,\lambda}=\left\{\int_0^1\int_0^1(1-r)^{\lambda(1/p-1/q)-1}M_q^{\lambda}(f,r_1,r_2)\,dr_1dr_2\right\}^{1/\lambda}<\ \infty\ ,$$

$$||f||_{p,q,\infty}=\sup\left\{(1-r)^{(1/p-1/q)-1}M_q(f;r_1,r_2)\right\}\ ,$$

where $(1-r)=(1-r_1)(1-r_2)$, $0<r_1\,,r_2<\infty$. The space $\beta(p,q,\lambda)$ is a Banach space for $p>0$ and $q,\lambda\ \geq\ 1$, otherwise it is a Frachet space. Further we have $H_q\subseteq H_q^{'}=B(q/2,q,q)$, $1\leq q<\infty$.

Let X be any of the above Banach spaces and let $E_{m,n}(f,X)$ be the error in the best approximation of a function $f\in X$ by the elements of the space P that consists of algebraic polynomials of degree $\leq\ m+n$ in two complex variables,

$$E_{m,n}(f,X)=\inf\left\{||f-p||_x\,,p\in P\right\}.$$

We have following results:

Theorem 15. *Let* $f(z_1,z_2)=\sum_{m,n=0}^{\infty}a_{mn}z_1^mz_2^n$ *, then the entire function* $f\in B(p,q,\lambda)$ *is of finite order* ρ *if and only if*

$$\rho=\lim_{m,n\to\infty}\sup\frac{\ln\left(m^mn^n\right)}{-\ln E_{m,n}\left(f,B(p,q,\lambda)\right)}. \tag{48}$$

We have similar result for the space H_q :

Theorem 16. *Let* $f(z_1,z_2)=\sum_{m,n=0}^{\infty}a_{mn}z_1^mz_2^n$ *, then the entire function* $f\in H_q$ *is of finite order* ρ *if and only if*

$$\rho=\lim_{m,n\to\infty}\sup\frac{\ln\left(m^mn^n\right)}{-\ln E_{m,n}\left(f,H_q\right)}. \tag{49}$$

The characterization of functions in terms of the type is similarly given as

Theorem 17. *Let* $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ *, then the entire function* $f \in B(p,q,\lambda)$ *is of finite order* ρ *and finite type* τ *if and only if*

$$\tau = \frac{1}{e\rho} \lim_{m,n\to\infty} \sup \left\{ m^m n^n E_{m,n}^p \left(f, B(p,q,\lambda)\right) \right\}^{1/(m+n)}. \tag{50}$$

Theorem 18. *Let* $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ *, then the entire function* $f \in H_q$ *is of finite order* ρ *and finite type* τ *if and only if*

$$\tau = \frac{1}{e\rho} \lim_{m,n\to\infty} \sup \left\{ m^m n^n E_{m,n}^p \left(f, H_q\right) \right\}^{1/(m+n)}. \tag{51}$$

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COLORING AND DOMINATION IN GRAPHS

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Abstract. Graph coloring and domination are two major areas in graph theory that have been well studied. In recent years several graph theoretic parameters which combine the concepts of domination and coloring have been investigated by several authors. In this paper we present a brief survey of four such concepts, namely, fall colorings, irredundant colorings, dominator chromatic number and dom-coloring number. We also list several interesting open problems on these topics for further investigation.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

Graph coloring and domination in graphs are two major areas within graph theory which have been well studied. Graph coloring deals with the fundamental problem of partitioning a set of objects into classes according to certain rules. The fundamental parameter in the theory of graph colorings is the chromatic number $\chi(G)$, which is the least number of colors required to color the vertices of a graph in such a way that no two adjacent vertices receive the same color. The chromatic number is a well studied parameter

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whose history dates back to the famous four color problem and the early work of Kempe [13] in 1899 and Heawood [12] in 1890.

Another fastest growing area within graph theory is the study of domination and related subset problems such as independence, irredundance and matching. A comprehensive treatment of fundamentals of domination in graphs is given in Haynes et al. [9]. Several advanced topics in domination are given in the book edited by Haynes et al. [10].

A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set in G is called the domination number of G and it is denoted by $\gamma(G)$. The upper domination number $\Gamma(G)$ is the maximum cardinality of a minimal dominating set in G. The open neighborhood $N(v)$ of a vertex v is the set of all vertices adjacent to v. The closed neighborhood $N[v]$ of v is the set $N(v) \cup \{v\}$. The open neighborhood $N(S)$ of a subset S of V is defined by $N(S) = \bigcup\limits_{v \in S} N(v)$ and the closed neighborhood $N[S]$ of S is the set $N(S) \cup S$. Let $S \subseteq V$ and $u \in S$. We say that u has a private neighbor with respect to S if $N[u] - N[S - \{u\}] \neq \emptyset$. We observe that u is its own private neighbor if u is not adjacent to any vertex in S. A set S is called irredundant if every vertex in S has a private neighbor with respect to S. The irredundance number $ir(G)$ and the upper irredundance number $IR(G)$ are defined by

$ir(G) = min\{|S| : S$ is a maximal irredundant set in $G\}$ and

$IR(G) = max\{|S| : S$ is a maximal irredundant set in $G\}$.

A subset S of V is called an independent set if no two vertices in S are adjacent. The independence number $\beta_0(G)$ and the independent domination number $i(G)$ are defined by

$$\beta_0(G) = \max\{|S| : S \text{ is an independent set in } G\} \quad \text{and}$$

$$i(G) = \min\{|S| : S \text{ is a maximal independent set in } G\}.$$

We observe that the maximality condition for an independent set is the definition of dominating set and the minimally condition for a dominating set is the definition of irredundant set. Hence the six parameters concerning domination, irredundance and independence satisfy the following inequality chain which was first observed by Cockayne et al. [3].

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

This domination chain has been the focus of more than 200 research papers. A domatic partition of G is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each V_i is a dominating set of G. The maximum order of a domatic partition of G is called the domatic number of G and it is denoted by $d(G)$. A partition of the vertex set $V(G)$ into independent dominating sets is called an *idomatic partition* of G. The maximum order of an idomatic partition of G is called the *idomatic number* $id(G)$.

In recent years several graph theoretic parameters which combine the concepts of domination and coloring have been investigated by several authors and in this paper we present a brief survey of results on four such parameters.

2. **Fall Colorings**

Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a k-coloring of a graph G. A vertex $v \in V_i$ is called a *colorful vertex* if v is adjacent to at least one vertex in each color class V_j, $i \neq j$. A color class V_i is called colorful if it contains at least one colorful vertex. The k-coloring π is called colorful if every color class is colorful. A colorful k-coloring $\pi = \{V_1, V_2, \ldots, V_k\}$ is called a *fall coloring* if every vertex $v \in V$ is colorful. The concept of fall coloring was introduced by Dunbar et al. [5]. We observe that not every graph admits a fall coloring. For example the cycle C_5 has chromatic number 3, but does not admit a fall k-coloring for any k. If a graph G has a fall coloring, we define the fall chromatic number (fall achromatic number) of G, denoted by $\chi_f(G)$ ($\psi_f(G)$) to be the minimum (maximum) order of a fall coloring of G.

It is easy to see that if G is a bipartite graph without isolated vertices then any 2-coloring of G is a fall 2-coloring. Further the cycle C_n has a fall 3-coloring if and only if $n \equiv 0 \pmod 3$.

There is a very close connection between fall colorings of a graph and the existence of disjoint independent dominating sets. In fact for every graph G, $\psi_f(G) = id(G)$. In [4] the following classes of graphs are shown to have fall colorings:

(1) Complete graphs K_n have (only) fall n-colorings.
(2) Connected bipartite graphs have fall 2-colorings.
(3) Complete k-partite graphs have (only) fall k-colorings.
(4) Cycles of the form C_{2n} have fall 2-colorings and cycles of the form C_{3n} have fall 3-colorings.

(5) Uniquely k-colorable graphs, including maximal outerplanar graphs, have fall k-colorings.
(6) All k-trees have fall $(k+1)$-colorings.
(7) Graphs of the form $K_{m,m}$ − 1-factor have (only) fall 2- and m-colorings.
(8) Domatic-critical graphs, i.e. graphs for which $d(G) = k$, but for which $d(G-e) < k$ for every edge $e \in E$, have fall k-colorings.
(9) The complements of graphs G of order n, which have no triangles and have a 1-factor, have fall $n/2$-colorings.
(10) Regular graphs for which $d(G) = k = \delta(G)+1$ have fall k-colorings.
(11) The complement of the Petersen graph has a fall 5-coloring, although the Petersen graph has no fall k-coloring for any k.
(12) The join $G+H$ of two graphs, each of which has a fall coloring, say of orders k and l, has a fall $(k+l)$-coloring.

The following are some of the theorems that have been established in [5] regarding fall colorings of Cartesian product of graphs.

Theorem 2.1. *For any positive integers m and n, the Cartesian product $P_m \square P_n$ has a fall 2-coloring, but does not have a fall k-coloring for any integer $k \geq 3$.*

Proposition 2.2. *The graph $C_m \square P_n$ has a fall 2-coloring if and only if $m \geq 4$ is even.*

Proposition 2.3. *The graph $C_m \square P_n$ has a fall 3-coloring for all $n \geq 1$.*

Proposition 2.4. *The graphs $C_{3k+1} \square P_n$ and $C_{3k+2} \square P_n$ do not have a fall 3-coloring for any value of $k \geq 1$.*

Proposition 2.5. *The graph $C_m \square C_n$ has a fall 2-coloring if and only if both $m \geq 4$ and $n \geq 4$ are even.*

Theorem 2.6. *The graph $C_m \square C_n$ has a fall 3-coloring if and only if either $m \equiv 0 \pmod 3$ or $n \equiv 0 \pmod 3$.*

Proposition 2.7. *The graphs $C_{4m} \square P_{2n}$ and $C_{4m} \square C_{2n}$ have a fall 4-coloring for all $m \geq 1$ and $n \geq 2$.*

Proposition 2.8. *The graph $C_m \square C_n$ has a fall 5-coloring if and only if $m \equiv 0 \pmod 5$ and $n \equiv 0 \pmod 5$.*

Theorem 2.9. *If a graph G has a fall s-coloring and a graph H has a fall r-coloring, for $s \geq r$, then the Cartesian product $G \square H$ has a fall s-coloring.*

For further results on fall colorings of categorical products and complexity issues concerning the fall chromatic number we refer to [5]. The following are some of the interesting open problems suggested in [5].

Problem 2.10. *Can the difference between $\chi(G)$ and $\chi_f(G)$ be arbitrarily large?*

Problem 2.11. *What is the smallest graph with $\delta = k$ which does not have a fall coloring? For $\delta = 1$ the smallest graph with no fall coloring is the graph K_3 with a pendant vertex. For $\delta = 2$ the smallest graph with no fall coloring is C_5. For $\delta = 3$ the smallest graphs with no fall colorings are the wheel of order 6 and the graph which is the complement of the graph $K_3 \cup P_3$. Finally, it can be shown that the complement of the cycle C_7 is the unique smallest graph with $\delta = 4$ having no fall coloring.*

Problem 2.12. *What fall colorings do the n-cubes have? We note that the 3-cube Q_3 has a fall 2-coloring and a fall 4-coloring, but does not have a fall 3-coloring.*

3. Irredundant Colorings

As pointed out in the introduction, the concepts of independence, domination and irredundance form the basic building blocks in the study of domination in graphs. Chromatic number of a graph is simply the minimum order of a partition of V into independent sets and the domatic number of a graph is the maximum order of a partition of V into dominating sets. Along similar lines Haynes et al. [11] introduced the concept of irredundant colorings.

An *irredundant coloring* is a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of V into nonempty irredundant sets. The *irratic number*, denoted $\chi_{ir}(G)$, equals the minimum order of an irredundant coloring of G.

Since every independent set $S \subseteq V$ is an irredundant set, every proper coloring of a graph is also an irredundant coloring. Hence it follows that for any graph G, $\chi_{ir}(G) \leq \chi(G)$. The following are some simple observations about the irratic number $\chi_{ir}(G)$.

Proposition 3.1. *A graph G has $\chi_{ir}(G) = 1$ if and only if $G = \overline{K_n}$.*

Proposition 3.2. *A graph G has $\chi_{ir}(G) = n$ if and only if $G = K_n$.*

Proposition 3.3. *If $\chi(G) = 2$, then $\chi_{ir}(G) = 2$.*

Proposition 3.4. *If G is a complete t-partite graph, then $\chi_{ir}(G) = t$.*

If the collection of all edges between two sets of vertices, say A and B, define a bijection (one-to-one and onto) between A and B, then we call such a perfect matching a *bijective matching*.

Proposition 3.5. *Let G and H be two graphs of the same order. The graph formed from G and H by adding edges of a bijective matching between $V(G)$ and $V(H)$ is bi-irratic.*

It follows from Proposition 3.5 that the irratic number of a subgraph H of G can be arbitrarily larger than the irratic number of G. For example, $\chi_{ir}(K_n) = n$ and $\chi_{ir}(K_n \Box K_2) = 2$.

Theorem 3.6. *If G is a connected, non-bipartite graph with minimum degree at least two, then G is bi-irratic if and only if G has a bijective matching.*

Corollary 3.7. *A connected non-bipartite graph G with minimum degree at least two is bi-irratic if and only if its vertices can be colored red and blue such that each blue vertex is adjacent to exactly one red vertex, and each red vertex is adjacent to exactly one blue vertex.*

A dominating set S of a graph G is called a perfect dominating set if every vertex in $V - S$ is adjacent to exactly one vertex in S.

Corollary 3.8. *A connected non-bipartite graph G with minimum degree at least two is bi-irratic if and only if its vertices can be partitioned into two perfect dominating sets.*

Corollary 3.9. *For any maximal planar graph G, $3 \leq \chi_{ir}(G) \leq 4$.*

Corollary 3.10. *For any maximal outerplanar graph G, $\chi_{ir}(G) = 3$.*

It is possible to adapt several well-known results from the literature on graph colorings and the chromatic number to obtain bounds for the irratic number. It is easy to see that the chromatic number has the following bounds.

Observation 3.11. *For any graph G, $n/\beta_0(G) \leq \chi(G) \leq n - \beta_0(G) + 1$.*

The corresponding result for irredundant colorings is the following.

Proposition 3.12. *For any graph G, $n/IR(G) \leq \chi_{ir}(G) \leq n - IR(G) + 1$.*

Proposition 3.13. *For any graph G which is triangle-free and has minimum degree at least two,*

(1) $\chi_{ir}(G) \leq \lceil \frac{n}{2} \rceil$.
(2) $\chi_{ir}(G) \leq \gamma'(G) + 1$.
(3) $\chi_{ir}(G) \leq \alpha_0(G)/2 + 1$, *where* $\alpha_0(G)$ *is the vertex covering number of* G.

Using the same idea as that used to prove that $\chi_{ir}(G) \leq \lceil \frac{n}{2} \rceil$, one can show that in a graph having girth $g(G) \geq 5$ and minimum degree $\delta(G) \geq 3$, every set of three vertices is an irredundant set. Therefore,

Proposition 3.14. *For any graph G having girth $g(G) \geq 5$ and minimum degree $\delta(G) \geq 3$, $\chi_{ir}(G) \leq \lceil \frac{n}{3} \rceil$.*

Proposition 3.15. *For any cycle C_{2n+1}, $\chi_{ir}(C_{2n+1}) = 3$.*

Proposition 3.16. *For any cubic graph G, $G \neq K_4$, $2 \leq \chi_{ir}(G) \leq \chi(G) \leq 3$.*

For details of several interesting questions regarding complexity issues for irratic numbers we refer to [11]. Also in [11] the following questions which arise naturally from some well known theorems on graph coloring are posed.

Question 3.17. *Do there exist graphs of arbitrarily large irratic number and arbitrarily large girth? It is known that there are graphs with arbitrarily large chromatic number and arbitrarily large girth.*

Question 3.18. *Do there exist triangle-free graphs having arbitrarily large irratic number? The same is true for the chromatic number.*

Question 3.19. *We have seen that for complete multipartite graphs G, $\chi_{ir}(G) = \chi(G)$. For which graphs G is $\chi_{ir}(G) = \chi(G)$?*

It would be interesting to study the *open irratic number* of a graph G, denoted $\chi_{oir}(G)$, which equals the minimum order of a partition of $V(G)$ into open irredundant sets. Since the partition into singleton sets is an open irredundant partition in any graph without isolated vertices, all graphs without isolated vertices have open irredundant colorings. By definition, therefore, $\chi_{ir}(G) \leq \chi_{oir}(G)$.

Question 3.20. *What can you say about open irredundant colorings of graphs?*

We have obtained further results on irredundant colorings and open irredundant colorings, which will be reported in a subsequent paper.

4. Dominator Colorings in Graphs

The concept of dominator coloring was introduced by Gera et al. [6].

A *dominator coloring* of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The *dominator chromatic number* $\chi_d(G)$ is the minimum number of color classes required in a dominator coloring of a graph G. Given a graph G and an integer k, the problem of deciding whether G admits a dominator coloring using k colors is NP-complete on arbitrary graphs [6].

The dominator chromatic number of standard graphs are given in the following propositions.

Proposition 4.1.

(1) For the star $K_{1,n}, \chi_d(K_{1,n}) = 2$.

(2) For the path P_n of order $n \geq 2$,

$$\chi_d(P_n) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil & \text{if } n = 2, 3, 4, 5, 7 \\ 2 + \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

(3) For the complete graph K_n, $\chi_d(K_n) = n$.

Proposition 4.2.

(1) For the cycle C_n,

$$\chi_d(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n = 4 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n = 5 \\ \lceil \frac{n}{3} \rceil + 2 & \text{otherwise.} \end{cases}$$

(2) For the multistar $K_n(a_1, a_2, \ldots, a_n)$, $\chi_d(K_n(a_1, a_2, \ldots, a_n)) = n+1$.

(3) For the wheel graph $W_{1,n}$,

$$\chi_d(W_{1,n}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

(4) For the complete k-partite graph $K_{a_1,a_2,\ldots,a_n}, \chi_d(K_{a_1,a_2,\ldots,a_n}) = k$.

Let G be a connected graph of order $n \geq 2$. Then at least two different colors are needed in a dominator coloring since there are at least two vertices adjacent to each other. Thus $2 \leq \chi_d(G) \leq n$.

Proposition 4.3. *Let G be a connected graph of order n. Then $\chi_d(G) = 2$ if and only if G is isomorphic to a complete bipartite graph.*

Proposition 4.4. *Let G be a connected graph of order n. Then $\chi_d(G) = n$ if and only if G is complete.*

Theorem 4.5. *Let G be a connected graph. Then* $\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$.

Question 4.6. *For which graphs does $\chi_d(G) = \chi(G)$?*

Question 4.7. *For which graphs does $\chi_d(G) = \gamma(G)$?*

Question 4.8. *For which graphs does* $\chi_d(G) = \gamma(G) + \chi(G)$?

Question 4.9. *For which graphs does* $\chi_d(G) = \gamma(G) = \chi(G)$?

Question 4.10. *What is the complexity of the dominator coloring problem on particular classes of graphs?*

We have obtained further results on dominator colorings and edge version of dominator colorings, which will be reported in a subsequent paper.

5. Dom-color Number

Walikar et al. [14] have proved the following theorem.

Theorem 5.1. *In a k-chromatic graph G, any k-coloring of G yields another k-coloring of G containing a color class which is a dominating set of G.*

Hence it is natural to consider the maximum number of color classes which are dominating sets of G, where the maximum is taken over all k-colorings of G, which we call the dom-color number of G.

Some basic results on dom-color number are given in [1].

Observation 5.2. *For all graphs G, $1 \leq d_\chi(G) \leq \chi(G)$.*

Observation 5.3. *Let G be a graph of order n. Then $\chi(G) = n$ if and only if $d_\chi(G) = n$.*

Proposition 5.4. *If G is uniquely χ-colorable, then $d_\chi(G) = \chi(G)$.*

Remark 5.5. *The converse of Proposition 5.4 is not true. For the graph G given in Figure 1, $d_\chi(G) = \chi(G) = 3$ and G is not uniquly colorable.*

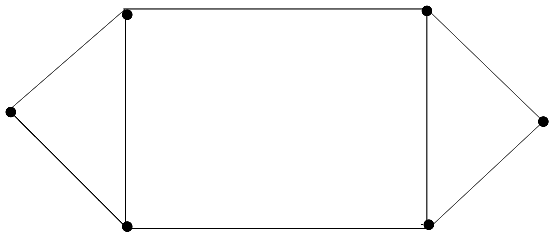

Figure 1

Observation 5.6. *Let G be a graph with no isolated vertex. If $\chi(G) = 2$, then $d_\chi(G) = 2$.*

Proposition 5.7. *For $n \geq 3$,* $d_\chi(C_n) = \begin{cases} 3 & \text{if } n \equiv 3 \pmod 6 \\ 2 & \text{otherwise.} \end{cases}$

Problem 5.8. *Characterize graphs for which $d_\chi = 1$.*

Problem 5.9. *Characterize graphs for which $d_\chi = \chi(G)$.*

We have obtained several new results on dom-color number including new bounds and complexity results, which will be presented in a subsequent paper.

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THE RELATION BETWEEN TWO COMBINATORIAL GROUP INVARIANTS : HISTORY AND RAMIFICATIONS

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1. **Introduction**

In this expository article, we shall be discussing about two combinatorial group invariants (and certain generalizations of them) related to some extremal problems which lie at the juncture of combinatorial number theory and combinatorial group theory.

Given any sequence of n integers $a_1, a_2, \ldots, a_n$, considering the sums $s_1 = a_1, s_2 = a_1 + a_2, \ldots, s_n = a_1 + \cdots + a_n$, by the pigeonhole principle, it follows that either an s_i is $0 \pmod n$ or at least two of the s_i's are equal modulo n. Thus, given any sequence of n integers, the sum of a non-empty subsequence is $0 \pmod n$.

In general, for a finite abelian group G (written additively), the Davenport constant $D(G)$ is defined to be the smallest natural number k such that any sequence of k elements in G has a non-empty subsequence whose sum is zero (the identity element of the group) and by the above argument, the value of this constant doesn't exceed the cardinality of the group G. When $G = \mathbb{Z}/n\mathbb{Z}$, observing that the sequence

$$(\underbrace{1, 1, \ldots, 1}_{(n-1) \text{ times}})$$

does not have any non-empty subsequence whose sum is zero, one has the equality $D(\mathbb{Z}/n\mathbb{Z}) = n$.

As was observed in some of the early papers on the subject (see [30], for instance), the problem of finding $D(G)$ had been proposed by H. Davenport

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in the 'Midwestern Conference on Group Theory and Number Theory' at Ohio State University held in April 1966, in the connection that if G is the class group of an algebraic number field K, then $D(G)$ is the maximal number of prime ideals (counting multiplicity) in the decomposition of an irreducible integer in K. Being attributed to Davenport, it is now known as the *Davenport constant*; however, K. Rogers [32] had studied it in 1962 in connection with the same question (attributing the question to Sudler) and this reference was somehow missed out by most of the authors in this area.

Later on, the study of the Davenport constant went on finding many more applications. One of the most important applications of this can be seen in the proof of the infinitude of Carmichael numbers by Alford, Granville and Pomerance [9]. It was also useful in another paper on Number Theory written by Brüdern and Godinho [14].

Just about a couple of years before the publication of the above mentioned paper of Rogers [32], a related result was obtained by Erdős, Ginzburg and Ziv (known as the EGZ theorem) [17], which continues to play a central role in the development of this area of combinatorics.

Theorem 1. (EGZ theorem) *For any positive integer n, any sequence of $2n-1$ integers has a subsequence of n elements whose sum is* $0 \pmod{n}$.

For an abelian group G of cardinality n, writing $E(G)$ for the smallest natural number k such that any sequence of k elements in G has a subsequence of length n whose sum is zero, we observe that the EGZ theorem implies that $E(\mathbb{Z}/n\mathbb{Z}) \leq 2n-1$. Since the sequence

$$(\underbrace{0, 0, \ldots, 0}_{(n-1)\text{ times}},\ \underbrace{1, 1, \ldots, 1}_{(n-1)\text{ times}})$$

does not have any subsequence of length n whose sum is zero, one has the equality $E(\mathbb{Z}/n\mathbb{Z}) = 2n-1$.

In the present article, we shall discuss about some recent development around the relation between the constants $D(G)$ and $E(G)$, which was established by Gao. Along with the weighted version of Harborth's constant, weighted versions of these constants and the corresponding generalization of the above mentioned relation of Gao were also touched upon in a recent article [2]; here we take the opportunity to elaborate just on the development related to the generalization of the relation of Gao and supply sketches of proofs of many results.

For various proofs of the EGZ theorem, apart from the original paper [17], one may look into the references [1], [11] and [29]. For expository accounts of various developments on these zero-sum problems and related references, one may look into [16],[21] and [22].

2. Some early results and the relation of Gao

For a finite abelian group G, it is known that $D(G) = |G|$ holds if and only if G is the cyclic group of order n. Exact values for $D(G)$ were obtained by Olson [30] [31] for all finite abelian groups of rank 2 and for all p-groups; more precisely, he proved that in these cases

$$D(G) = 1 + \sum_{i=1}^{r}(n_i - 1), \tag{1}$$

where $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, with $n_1|n_2|\cdots|n_r$.

The constants $D(G)$ and $E(G)$ were being studied independently before Gao [18] (see also [22], Proposition 5.7.9) established the following:

Theorem 2. *For a finite abelian group of order n, we have*

$$E(G) = D(G) + n - 1. \tag{2}$$

Remark 1. *Since we have already observed that $D(G) \leq n$, from the above theorem it follows that $E(G) \leq 2n - 1$ and hence, in particular, the EGZ theorem follows from the above theorem. It should be also remarked that from the statement of the EGZ theorem, namely from $E(\mathbb{Z}/n\mathbb{Z}) \leq 2n - 1$, by induction on the rank it follows that for a finite abelian group G of order n, $E(G) \leq 2n - 1$.*

It is easy to observe that by appending a sequence of $n - 1$ zeroes to a sequence of length $D(G) - 1$ with no zero sum (which exists by the definition of $D(G)$), the resulting sequence of length $D(G) + n - 2$ will have no subsequence of length n which sums up to zero and hence

$$E(G) \geq D(G) + n - 1. \tag{3}$$

For the proof of Theorem 2 we shall need the following result of Scherk [33] (see also [26]).

Lemma 1. *Let A and B be two subsets of a finite abelian group G. Suppose $0 \in A \cap B$ and suppose that the only solution of $a + b = 0,\ a \in A, b \in B$ is $a = b = 0$. Then,*

$$|A + B| \geq |A| + |B| - 1.$$

Proof of Lemma 1. We shall closely follow the proof of Scherk [33].

Since the result is obvious for $|B| = 1$, we assume that $|B| = n > 1$ and that the result is true for $|B| \leq n - 1$.

Choose $b\ (\neq 0) \in B$. By our assumption, $0 \notin A + b$. Since $0 \in A$ and $|A + b| = |A|$, $\exists t \in A$ such that $t + b \notin A$.

Let

$$B_1 \stackrel{\text{def}}{=} \{g \in B \mid t + g \notin A\}.$$

We observe that $b \in B_1$ and hence B_1 is non-empty. Also, since $t \in A$, $0 \notin B_1$, which shows that $B_1 \subsetneq B$.

Let

$$\begin{aligned} A_1 &\stackrel{\text{def}}{=} t + B_1 \subset A + B, \\ A_2 &\stackrel{\text{def}}{=} A \cup A_1, \\ B_2 &\stackrel{\text{def}}{=} B \setminus B_1. \end{aligned}$$

Now, $A_1 \cap A = \emptyset$ and $|A_1| = |B_1|$ and therefore,

$$|A_2| + |B_2| = |A| + |A_1| + |B_2| = |A| + |B_1| + |B_2| = |A| + |B|. \quad (4)$$

We claim that

$$A_2 + B_2 \subset A + B. \quad (5)$$

Let $a_2 \in A_2, b_2 \in B_2$. If $a_2 \in A$, then $a_2 + b_2 \in A + B_2 \subset A + B$. If $a_2 \in A_1$, then $a_2 + b_2 = (t + b') + b_2$, for some $b' \in B_1$. Now, $(t + b') + b_2 = (t + b_2) + b' \in A + B_1 \subset A + B$. This establishes the claim (5).

We observe that $0 \in A \subset A_2$ and since $0 \notin B_1$, $0 \in B_2$. Now, if possible let $a_2 + b_2 = 0$ with $a_2 \in A_2, b_2 \in B_2$. If $a_2 \in A_1$, then as before, $a_2 + b_2 = (t + b_2) + b'$, where $(t + b_2) \in A$ and $b' \in B_1 \subset B$ and hence $0 = a_2 + b_2 = (t + b_2) + b'$ would imply $b' = 0$, contradicting the fact that $0 \notin B_1$. Therefore, $a_2 \in A$. Since $B_2 \subset B$, we have $a_2 = b_2 = 0$. Since $B_1 \subsetneq B$, by the induction hypothesis,

$$|A_2 + B_2| \geq |A_2| + |B_2| - 1,$$

and therefore from (4) and (5),

$$|A + B| \geq |A_2 + B_2| \geq |A_2| + |B_2| - 1 = |A| + |B| - 1.$$

Given a sequence $S = (x_1, \ldots, x_k)$ of elements of an abelian group G, for any $1 \leq l \leq k$, we shall use the notation $\sum_l(S)$ to denote the set of all l-sums of S (by an l-sum one means a sum $x_{i_1} + \cdots + x_{i_l}$ of a subsequence of length l of S) and $\sum_{\leq l}(S)$ to denote $\cup_{1 \leq t \leq l} \sum_t(S)$.

We shall simply write $\sum(S)$ in place of $\sum_{\leq k}(S)$, the set of sums of all the subsequences of S. We shall write $\sum S$ to denote the sum of all elements of S and $|S|$ to denote the length k of S.

From Lemma 1, one can easily deduce the following:

Lemma 2. *Let S be a sequence of n elements in an abelian group G of order n and let h be the maximum number that any element of G occurs in S. Then*

$$0 \in \sum_{\leq h}(S).$$

Proof of Theorem 2. We give a sketch of the original proof of Gao [18].

In view of (3), one has only to establish that

$$E(G) \leq D(G) + n - 1. \tag{6}$$

Let A be any sequence of $m = D(G)+n-1$ elements of G and we have to show that $0 \in \sum_n(A)$. We can assume that 0 is the most repeated element in the sequence and arrange A to be of the form

$$A = (a_1, a_2, \ldots, a_{m-h},\ \underbrace{0, 0, \ldots, 0}_{h \text{ times}}),$$

with all $a_i \neq 0$.

Clearly, we can assume that $h \leq n$.

Let W be a subsequence of $(a_1, a_2, \ldots, a_{m-h})$, maximal subject to the condition that $\sum W = 0$.

From the maximality of W and the definition of $D(G)$, it is easy to observe that $m-h-|W| < D(G)$, so that $|W| > m-h-D(G) = n-h-1$.

If $|W| \leq n$, then appending $n - |W| \leq h$ zeros, we see that $0 \in \sum_n(A)$.

If $|W| \geq n+1$, repeatedly applying Lemma 2 to W, we find a system of disjoint subsequences $W_1, W_2, \ldots, W_v$ satisfying the conditions $\sum W_i = 0$ and $|W_i| \leq h$, for each i such that

$$|W - W_1 - \cdots W_v| \leq n-1$$

and

$$|W - W_1 - \cdots W_{v-1}| \geq n.$$

Therefore,

$$n-1 \geq |W - W_1 - \cdots W_v| \geq n - |W_v| \geq n-h.$$

Writing $W_0 = W - W_1 - \cdots W_v$, we have $n-1 \geq |W_0| \geq n-h$ and as before, appending $n - |W_0| \leq h$ zeros, $0 \in \sum_n(A)$. This completes the proof.

3. General upper bounds for $D(G)$ and some generalizations of the EGZ theorem

As for general upper bound for $D(G)$, Emde Boas and Kruyswijk [15], Baker and Schmidt [12] and Meshulam [28] gave upper bounds for Davenport constant which involves the exponent of the group and the cardinality of the group G. The best known bound is due to Emde Boas and Kruyswijk [15]:

$$D(G) \leq m\left(1 + \log\frac{|G|}{m}\right),$$

where m is the exponent of G. This was again proved by Alford, Granville and Pomerance [9].

With notations as in Equation (1), it is known that

$$D(G) > M(G) = 1 + \sum_{i=1}^{r}(n_i - 1),$$

for infinitely many finite abelian groups of rank $d > 3$ (see [23], for instance). However, for rank $d = 3$ and groups of the form $(\mathbb{Z}/n\mathbb{Z})^d$, one expects ([19] and [20]) that $D(G) = M(G)$.

The following generalization of the EGZ theorem is due to Hamidoune [25].

Theorem 3. *Let G be an abelian group of order n and k a positive integer. Let $(w_1, w_2, ..., w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $S : (x_1, x_2, ..., x_{k+n-1})$ of elements of G, if x_1 is the most repeated element in the sequence, we have*

$$\sum_1^k w_i x_{\sigma(i)} = \left(\sum_1^k w_i\right) x_1,$$

for some permutation σ of $\{1, 2, \ldots, k+n-1\}$.

Remark 2. *Taking $G = \mathbb{Z}/n\mathbb{Z}$, $k = n$ and $w_i = 1$, for all i, in the above theorem, we get the EGZ theorem.*

We now state another theorem due to Hamidoune [25]; this is a generalization of Theorem 2 of Gao.

Theorem 4. *Let G be an abelian group of order n and d the Davenport constant of G. Let $k \geq n-1$ be a natural number. Given a sequence $S : (x_1, x_2, ..., x_{d+k})$ of elements of G, where x_1 is the most repeated element in the sequence, there is a k-subset $K \in \{2, \ldots, d+k\}$ such that*

$$\sum_{i \in K} x_i = kx_1.$$

Remark 3. *Taking $k = n-1$ in the above theorem, there is an $(n-1)$ -subset $L \subset \{2, \ldots, d+n-1\}$ such that $\sum_{i \in L} x_i = -x_1$ and hence Theorem 2 of Gao follows.*

To mention one more generalization of the EGZ theorem, the following interesting result of Bollobás and Leader [13] clearly implies the EGZ theorem by taking $r = n-1$.

Theorem 5. *Let G be an abelian group of order n and r be a positive integer. Let A denote the sequence $(a_1, a_2, \ldots, a_{n+r})$ of $n+r$ elements of G. Then if 0 is not an n-sum, the number of distinct n-sums of A is at least $r+1$.*

We would like to remark that a simple proof of the above theorem has been given by Yu [38] using the result of Scherk (Lemma 1), which was used in proving Theorem 2 in the previous section.

4. **Generalizations with weights**

Generalizations of the constants $E(G)$ and $D(G)$ with weights were considered in [5] and [8] for the particular group $\mathbb{Z}/n\mathbb{Z}$. Later in [4], the following generalizations of both $E(G)$ and $D(G)$ for an arbitrary finite abelian group G of order n have been introduced. For a finite abelian group G and any non-empty $A \subseteq \mathbb{Z}$, the *Davenport constant of G with weight A*, denoted by $D_A(G)$, is defined to be the least natural number k such that for any sequence $(x_1, \ldots, x_k)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \ldots, x_{j_l})$ and $a_1, \ldots a_l \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$. Clearly, if G is of order n, one may consider A to be a non-empty subset of $\{0, 1, \ldots, n-1\}$ and we avoid the trivial case $0 \in A$.

Similarly, for any such weight set A, for a finite abelian group G of order n, the constant $E_A(G)$ is defined to be the least $t \in \mathbb{N}$ such that for any sequence $(x_1, \ldots, x_t)$ of t elements with $x_i \in G$, there exist indices $j_1, \ldots, j_n \in \mathbb{N}, 1 \le j_1 < \cdots < j_n \le t$, and $\vartheta_1, \ldots, \vartheta_n \in A$ with $\sum_{i=1}^{n} \vartheta_i x_{j_i} = 0$.

The case $A = \{1\}$ corresponds to $D(G)$ and $E(G)$.

After the introduction of $D_A(G)$ for the particular group $\mathbb{Z}/n\mathbb{Z}$, the generalized constant $D_A(G)$ for an arbitrary abelian group G was also considered independently in [35], where in the spirit of Olson [30], an upper bound for $D_A(G)$ was obtained when G is a finite abelian p-group, by using some results from [36].

We simply write $E_A(n)$ and $D_A(n)$ to denote respectively $E_A(\mathbb{Z}/n\mathbb{Z})$ and $D_A(\mathbb{Z}/n\mathbb{Z})$ and consider some special cases.

(I) The case $A = \{1, -1\}$. In one of the early papers [5] on this theme, considering the case $A = \{1, -1\}$, it was shown that

$$E_A(n) = n + [\log_2 n]. \tag{7}$$

While by the pigeonhole principle (see [5]) it follows that $D_A(n) \le [\log_2 n] + 1$, considering the sequence $(1, 2, \ldots, 2^r)$, where r is defined by $2^{r+1} \le n < 2^{r+2}$, we have $D_A(n) \ge [\log_2 n] + 1$.

Thus, in this case

$$D_A(n) = [\log_2 n] + 1. \tag{8}$$

(II) The case $A = (\mathbb{Z}/n\mathbb{Z})^*$. For the case $A = (\mathbb{Z}/n\mathbb{Z})^* = \{a : (a, n) = 1\}$, settling a conjecture from [5], it was proved in [27] that

$$E_A(n) = n + \Omega(n), \tag{9}$$

where $\Omega(n)$ denotes the number of prime factors of n, multiplicity included.

Later, the result (9) was also established in [24].

Writing $n = p_1 \cdots p_s$ as product of $s = \Omega(n)$ not necessarily distinct primes, the sequence $(1, p_1, p_1p_2, \ldots, p_1p_2 \cdots p_{s-1})$ gives the lower bound $D_A(n) \ge 1 + \Omega(n)$. Therefore, from (9) and the trivial inequality

$$E_A(n) \ge D_A(n) + n - 1,$$

we have

$$n + \Omega(n) = E_A(n) \ge D_A(n) + n - 1 \ge n + \Omega(n)$$

and hence

$$D_A(n) = 1 + \Omega(n). \tag{10}$$

(III) The case when $n = p$, a prime, and $A = \{1, 2, \ldots, r\}$, where r is an integer such that $1 < r < p$. In this case, it has been proved in [8] that

$$D_A(p) = \left\lceil \frac{p}{r} \right\rceil, \ E_A(p) = p - 1 + D_A(p). \tag{11}$$

(IV) The case where A is the set of quadratic residues $\pmod p$. For a prime p, if A consists of all the squares in $(\mathbb{Z}/p\mathbb{Z})^*$, then it has been proved in [8] that

$$D_A(p) = 3, \ E_A(p) = p + 2. \tag{12}$$

Proof of (12). Given a sequence $S = (s_1, \ldots, s_{p+2})$ of elements of $\mathbb{Z}/p\mathbb{Z}$, if we consider the following system of equations over $\mathbb{F}_p$:

$$\sum_{i=1}^{p+2} s_i x_i^2 = 0, \ \sum_{i=1}^{p+2} x_i^{p-1} = 0,$$

then by Chevalley - Warning Theorem (see [34] or [1], for instance), there is a nontrivial solution $(y_1, \ldots, y_{p+2})$ of the above system. Writing $I = \{i : y_i \neq 0\}$, from the first equation it follows that $\sum_{i\in I} a_i s_i = 0$ where a_i's belong to the set of squares in $(\mathbb{Z}/p\mathbb{Z})^*$. By Fermat's little theorem, from the second equation we have $|I| = p$. This establishes that

$$E_A(p) \leq p + 2.$$

Now, considering a sequence $v_1, -v_2$, where v_1 is a quadratic residue and v_2 a quadratic non-residue $\pmod p$, for any two elements $a_1, a_2 \in A$, $a_1 v_1 + a_2(-v_2) = 0$ is not possible and hence

$$D_A(p) \geq 3.$$

Therefore,

$$p + 2 \geq E_A(p) \geq D_A(p) + p - 1 \geq p + 2,$$

and from this the results in (12) follow.

5. Relation between the generalized constants $D_A(G)$ and $E_A(G)$ and related results. Results in the special cases observed in the pervious section, lead (see [4], [8]) to the expectation that for any set $A \subset \{1, \ldots, n-1\}$ of weights, the equality $E_A(n) = D_A(n) + n - 1$ holds.

When $n = p$, a prime, the method employed in [8] while determining the value of $E_A(p)$ in the case $A = \{1, 2, \ldots, r\}$, for $1 < r < p$ (see the special case III in the previous section), gives the result $E_A(p) = D_A(p) + p - 1$ for any nonempty $A \subseteq \{1, 2, \ldots, p-1\}$; another proof of this result was recently given in [7] by using the theory of permanents. It should be mentioned that the idea of using permanents to tackle this type of problems was initiated by Alon (see [10] for instance, see also [1]).

For a general finite abelian group G of order n, the following conditional result was proved by Adhikari and Chen [4].

Theorem 6. *Let G be a finite abelian group of order n, and $A = \{a_1, a_2, \ldots, a_r\}$ be a finite subset of $\mathbb{Z}$ with $|A| = r \geq 2$ and*

$$\gcd(a_2 - a_1, a_3 - a_1, \ldots, a_r - a_1, n) = 1.$$

Then we have $E_A(G) = D_A(G) + n - 1$.

Remark 4. *One observes that the above result does not include the result (2) of Gao which corresponds to the case $|A| = 1$.*

Now we state the following result which has been proved in [3].

Theorem 7. *Let $W \subseteq (\mathbb{Z}/p\mathbb{Z})^*$ such that $|W| \geq \frac{p+2}{3}$, and suppose there exists α in $(\mathbb{Z}/p\mathbb{Z})^*$ such that $\frac{a}{b} \neq \alpha$ for any a, b in W. Then*

$$D_W(p) = 3.$$

Remark 5. *If $p \geq 7$, then choosing one element each from the $\frac{p-1}{2}$ pairs $(a, -a)$ of non-zero elements of $\mathbb{Z}/p\mathbb{Z}$, we see that there are $2^{(p-1)/2}$ subsets W with $\frac{p-1}{2} \geq \frac{p+2}{3}$ elements satisfying the condition that $\frac{a}{b} \neq -1$ for all a, b in W. Hence Theorem 7 provides the exact value of $D_W(p)$ for many subsets W of $(\mathbb{Z}/p\mathbb{Z})^*$.*

Taking W to be the subset $\{2, 4, \cdots, p-1\}$, since $\gcd(2, p) = 1$, Theorem 6 can be applied to obtain $E_W(p) = p + 2$.

Remark 6. *If $p \geq 7$, then taking W to be the set of quadratic residues $\pmod p$, $|W| \geq \frac{p+2}{3}$ and taking any quadratic non-residue α, we have $\frac{a}{b} \neq \alpha$ for any a, b in W. Hence Theorem 7 can be applied in this case to obtain the result $D_W(p) = 3$.*

In [6], considering the problem of determining $E_A(n)$ and $D_A(n)$, where A is the set of squares in the group of units in the cyclic group $\mathbb{Z}/n\mathbb{Z}$ for a general integer n, the generalization of (12) has been obtained in some cases. For instance, the following result has been obtained.

Theorem 8. *Let n be a square-free integer, coprime with 6 or $n = p^r$, where $p > 3$ is a prime. Then*

$$\begin{aligned} D_R(n) &= 2\Omega(n) + 1, \\ E_R(n) &= n + 2\Omega(n). \end{aligned}$$

where $R = R_n = \{x^2 : x \in (\mathbb{Z}/n\mathbb{Z})^*\}$.

In [6], it has also been observed that the following generalization of (11) holds.

Theorem 9. *Let n, r be positive integers, $1 \leq r < n$ and consider the subset $A = \{1, \ldots, r\}$ of $\mathbb{Z}/n\mathbb{Z}$. Then,*

$$\begin{array}{rrcl} \text{(i)} & D_A(n) & = & \left\lceil \frac{n}{r} \right\rceil, \\ \text{(ii)} & E_A(n) & = & n - 1 + D_A(n). \end{array}$$

Remark 7. *As we have observed, for $r = 1$, the result (i) of the above theorem is trivial and in this case, Part (ii) follows from the EGZ theorem. For $r > 1$, that Part (i) of the above theorem can be obtained following the method of* [8] *has been observed independently in* [37]*; Part (ii) follows by invoking Theorem 6 in this case.*

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ORDERS ON MANIFOLDS AND SURGERY

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Abstract. There is a well-studied natural order on manifolds given by the existence of degree-one maps, which is especially important in the case of 3-dimensional manifolds. This article introduces and explains our result giving a surgery description of this order.

1. **Classification of surfaces**

Much of the study of topology is guided by a classical result – the classification of surfaces. We begin by recalling this classification. We first give a (temporary) working definition of surfaces.

Consider a smooth, proper function $f : \mathbb{R}^3 \to \mathbb{R}$ with gradient $\nabla f(x) \neq 0$ whenever $f(x) = 0$. The set of points

$$\Sigma = \{x \in \mathbb{R}^3 : f(x) = 0\}$$

is a surface (without boundary). We shall define a surface to be a set of this form. For example, taking $f(x, y, z) = x^2 + y^2 + z^2 - 1$ gives a sphere. We recall that the function f is *proper* if as $|x| \to \infty$, $|f(x)| \to \infty$.

A surface is said to be *closed* if it is compact (and has no boundary). It is said to be *oriented* if we can define a notion of right-handed and left-handed on the surface. Surfaces given by a function $f : \mathbb{R}^3 \to \mathbb{R}$ as above are automatically oriented.

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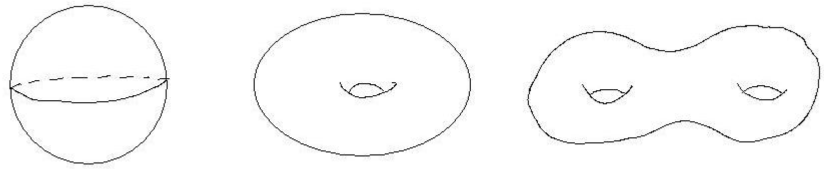

FIGURE 1. The first three surfaces

We wish to classify surfaces from the point of view of *topology*, i.e., where we take into account only global properties and not distances. More precisely, we say that two surfaces Σ and Σ' are homeomorphic if there is a continuous function $\Phi : \Sigma \to \Sigma'$ that is bijective and so that Φ^{-1} is continuous. Thus, we have a one-to-one correspondence between points of Σ and Σ' which is permitted to distort distance but not to *tear* the surface. For example, a sphere is homeomorphic to an ellipsoid.

Three closed, oriented surfaces are given in figure 1. These are the *sphere*, the *torus* and the *two-holed torus*. It is clear that we can extend this picture to a sequence of surfaces, namely the *n-holed tori*. We shall call the n-holed torus the *surface of genus n*, and denote this by Σ_n. In particular the sphere is the surface of genus 0.

Theorem 1.1 (Classification of surfaces)**.** *Suppose Σ is a closed, oriented surface. Then there is a unique integer n so that Σ is homeomorphic to Σ_n.*

2. **Ordering surfaces**

It is clear that in some sense a surface of larger genus is more complex than one of lower genus. Our first step is to understand this order in topological terms, i.e., without reference to the classification of surfaces. This will allow us to generalise the order to higher dimensions.

We shall first make a (temporary) definition, which may seem peculiar to the reader at first.

Definition 2.1. A surface M is said to *dominate* a surface N if there is a map $f : M \to N$ such that there is a disc D in N with $f^{-1}(D)$ a disc and $f|_{f^{-1}(D)} : f^{-1}(D) \to D$ one-to-one and onto (hence a homeomorphism).

The basic example that motivates this definition is the *pinch map*, which is illustrated in figure 2. For instance, if M is a figure of genus 2 and N is the torus, we consider a curve α that separates M into two components, say

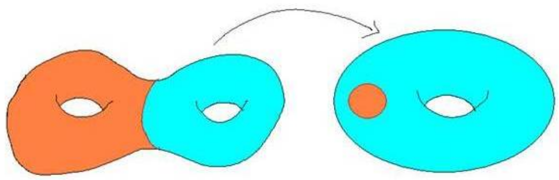

FIGURE 2. A pinch map

Σ and Σ', each of which has one of the holes giving the genus. We consider the quotient space of M obtained by identifying the space Σ to a single point. This is clearly homeomorphic to the torus N, so we have a *quotient* map $f : M \to N$.

For a disc $D \subset N$ that is disjoint from the image $f(\Sigma)$ (which is a single point), by construction $f^{-1}(D)$ is a disc in Σ' which is mapped homeomorphically to D. Hence M dominates N.

It is easy to generalise this to surface Σ_m and Σ_n, with genus m and n, $m \geq n$. Hence we have the following proposition.

Proposition 2.2. *If $m \geq n$, then Σ_m dominates Σ_n.*

Thus, we know that a surface of higher genus dominates one of lower genus – this amounted to a simple construction. The crux of the matter is to show that the opposite does not happen. This is a consequence of a theorem of Hopf.

We shall give an intuitive proof of the first case of Hopf's theorem, namely that the sphere *does not* dominate the torus, with the extra assumption that the maps we consider is smooth.

Proposition 2.3 (Hopf)**.** *The sphere does not dominate the torus.*

Proof. Suppose there is a map $f : S^2 \to T$ from the sphere to the torus satifying the conditions of Definition 2.1 and let D be as in the definition. Assume further that f is smooth.

The proof is illustrated in figure 3. Consider two curves α and β in the torus that intersect *transverally* in a single point P (i.e., the curves cross at P), so that P lies in the disc D. The inverse images $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ can be assumed to be unions of circles (see [4]).

As $P \in D$, $f^{-1}(P)$ is a single point P'. Further, there are (unique) components α' and β' of $f^{-1}(\alpha)$ and $f^{-1}(\beta)$, respectively, that contain the

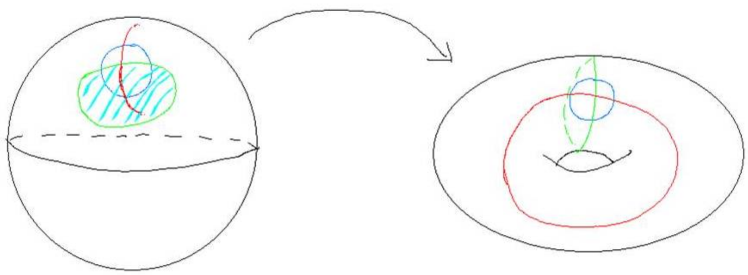

FIGURE 3. The sphere does not dominate the torus

point P'. As α and β do not intersect outside P, it follows that $\alpha' \cap \beta' = \{P'\}$.

However, by the Jordan curve theorem, the curve α' separates the sphere S^2 into two components. If γ' is a small subarc of β' containing P', by transversality of the intersection of α and β, hence α' and β', the endpoints of γ' are in different components of $S^2 \setminus \alpha'$. But as $\alpha' \cap \beta' = \{P'\}$, the arc $\beta' - int(\gamma')$ joins these two endpoints in $S^2 \setminus \alpha'$, giving a contradiction. $\square$

For readers familiar with algebraic topology, this construction is essentially a transaltion of Hopf's result concerning the cup product on cohomology. The details of Hopf's theorems are in Section 10.

Our goal is to understand the analogous relation in higher dimensions. This is defined in terms of *degree-one* maps between manifolds. In the next two sections we recall the concepts of (smooth) manifolds and the degree of maps.

3. Manifolds and smooth structures

The natural generalisation of surfaces to higher dimensions are *manifolds*. Observe that a small piece of a surface looks like a disc. Similarly, manifolds are spaces that look like Euclidean space. For the purposes of this article, the reader can instead use the definition of manifolds in [4] or [2]. For a treatment along the more general lines taken here, we refer to [5].

Definition 3.1. A topological n-manifold is a metric space X so that there is an open cover $\{U_i\}$ and homeomorphisms $\varphi : U_i \to V_i \subset \mathbb{R}^n$, with V_i open in $\mathbb{R}^n$.

The maps $\varphi_i : U_i \to V_i$ are called *charts*, and the collection of charts is called an *atlas*. Indeed an atlas of the earth is such an atlas – the separate charts (maps) map a region of the earth into a set in the plane and they together cover the earth.

It is very useful to be able to apply the methods of calculus to study manifolds. To do this, we need to first be able to determine when a function is differentiable, or better still smooth (infinitely differentiable).

Consider a topological n-manifold M and fix a collection $\varphi_i : U_i \to V_i$ of charts (i.e., an atlas). Let $f : M \to \mathbb{R}$ be a continuous function and let $p \in M$ be a point. As $\{U_i\}$ forms an open cover, $p \in U_i$ for some i. The function $f \circ \varphi_i^{-1} : V_i \to \mathbb{R}$ is then a function on an open set in $\mathbb{R}^n$. It is natural to define f to be smooth at the point p if and only if $f \circ \varphi_i^{-1} : V_i \to \mathbb{R}$ is smooth at $\varphi_i(p)$.

However, the above condition for smoothness is not well defined. This is because if U_j also contains p, then $f \circ \varphi_i^{-1} : V_i \to \mathbb{R}$ may be smooth at $\varphi_i(p)$ but $f \circ \varphi_j^{-1} : V_i \to \mathbb{R}$ may not be smooth at $\varphi_j(p)$. To be able to define smooth functions on M, we need to avoid this situation, i.e., we need to ensure that $f \circ \varphi_j^{-1} : V_i \to \mathbb{R}$ is smooth if and only if $f \circ \varphi_i^{-1} : V_i \to \mathbb{R}$ is smooth on the images of points in $U_i \cap U_j$.

We can write $f \circ \varphi_j^{-1} : V_i \to \mathbb{R}$ as $(f \circ \varphi_i^{-1}) \circ (\varphi_i \circ \varphi_j^{-1})$ on $\varphi_j(U_i \cap U_j)$. By the chain rule, this is smooth on $U_i \cap U_j$ if $f \circ \varphi_i^{-1} : V_i \to \mathbb{R}$ is smooth and $\varphi_i \circ \varphi_j^{-1}$ is smooth. This leads us to make the following definition.

Definition 3.2. A *smooth manifold* of dimension n is a topological manifold M equipped with an atlas $\{\varphi_i : U_i \to V_i\}$ so that if $U_i \cap U_j \neq \phi$, then $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to V_i$ is smooth.

The functions $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to V_i$ are called *transition functions*. Observe that these are invertible, with the inverse also a transition function. It follows that the total derivative (i.e., the Jacobian matrix) is invertible at all points of the domain of definition.

We shall need the notion of *orientation* of a manifold, i.e. a notion of left-handedness and right-handedness. A linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserves orientations if it determinant satisfies $det(L) > 0$. For a general smooth function, we consider instead the determinant of the total derivative. For smooth manifolds, we have the following definition.

Definition 3.3. An oriented smooth manifold M is a smooth manifold with an atlas $\{\varphi_i : U_i \longrightarrow V_i\}$ so that if $p \in U_i \cap U_j \neq \phi$, then $det(D(\varphi_i \circ \varphi_j^{-1})(\varphi_j(p))) > 0$.

We leave it as an exercise to the reader to check that for a smooth manifold M, we have well defined notions of when functions $f : M \to \mathbb{R}$ are smooth, and also when functions $\Phi : M \to N$ between smooth manifolds are smooth.

Definition 3.4. We say that $\Phi : M \to N$ is a *diffeomorphism* if it is a homeomorphism and Φ and Φ^{-1} are both smooth.

4. Degree of a map

The notion of degree, due to Brouwer, is a natural extension of the familiar case of a complex polynomial. We shall begin by examining this case and put it in a more general setting.

4.1. **The degree of a complex polynomial.** Consider a complex polynomial in one variable

$$f(z) = a_n z^n + a_{n-1} z^{n-1} \cdots + a_0$$

The degree of f is the largest k for which $a_k \neq 0$. The fundamental theorem of algebra allows one to interpret the degree in another way, in terms of the map $f : \mathbb{C} \to \mathbb{C}$ rather than the formula for f. Namely, if $z_0 \in \mathbb{C}$ is a complex number, the number of points in $f^{-1}(z_0)$ counted with *multiplicity* (i.e., the number of roots of $f(z) = z_0$), is the degree of the polynomial f.

The Brouwer degree is a generalisation of this observation. However, in the case of smooth maps, we cannot in general define a notion of multiplicity. Instead we ensure that the point z_0 is chosen so that $f(z) = z_0$ has no multiple roots.

A root a of $f(z) = z_0$ is a multiple root if and only if $f'(a) = 0$. We say z is a *critical point* if $f'(z) = 0$, otherwise we say that z is a *regular point*. A *critical value* is the image $f(z)$ of a critical point z.

A *regular value* is a complex number z_0 that is not the image of a critical value. Thus, if z_0 is a regular value, then all roots of $f(z) = z_0$ are simple. It follows that we have the following description of the degree.

Proposition 4.1. *The degree of a complex polynomial $f(z)$ is the number of points in $f^{-1}(z_0)$ for z_0 a regular value.*

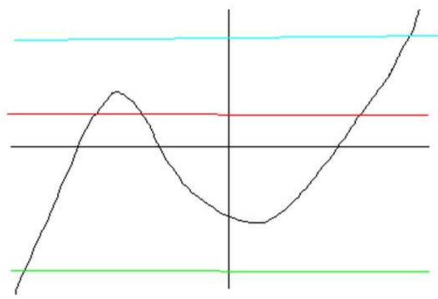

FIGURE 4. The graph of $f(x) = x^3 - 3x$

This description does not make essential use of the fact that $f(z)$ is a polynomial, and hence is suitable for generalisations. However, for this to be useful, we need to know that there are regular values. We next observe that there are regular values in the case of complex polynomials.

Proposition 4.2. *The set of critical values for a complex polynomial $f(z)$ is finite.*

Proof. The set of critical points is the set of z for which $f'(z) = 0$. As f is a polynomial, this is either finite or $f'(z) = 0$ for all z.

In the first case, as the set of critical values is the image of a finite set, it is finite. On the other hand, if f' is zero everywhere, then $f(z)$ is a constant polynomial $f(z) = a_0$. Hence the image of f is a single point a_0, which is thus the only critical value. $\square$

We emphasise that if the inverse image $f^{-1}(z)$ of a point z is empty, then z is a *regular* value (a so called vacuous implication).

4.2. **A real example.** The above examples suggest that we define the degree of a function as the cardinality of the inverse image of a regular value. However, the case of complex polynomials, or more generally complex analytical functions, is special. To understand the additional subtleties in the real case, we consider an example.

Consider the function $f(x) = x^3 - 3x$. The critical values are points where $0 = f'(x) = 3x^2 - 3$, i.e., $x = \pm 1$. The critical values are the images of these points, namely $f(\{\pm 1\}) = \{-2, 2\}$. Hence the set of regular values has three components, $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

If y lies in $(-\infty, 2)$ or $(2, \infty)$, then $f^{-1}(y)$ consists of a single point (see figure 4). However, if y lies in $(-2, 2)$, then $f^{-1}(y)$ consists of three points. Hence the cardinality of $f^{-1}(y)$ depends on y, so cannot be taken as the definition of the degree.

However, we can associate signs to points in $f^{-1}(y)$. Namely, if $f(x) = y$ for a regular value y, then $f'(x) \neq 0$. We assign a positive sign to x if $f'(x) > 0$ and a negative sign if $f'(x) < 0$.

If we count the number of points with sign, then we see that this is independent of y for y a regular value. Thus, we can take this as the definition of the degree.

4.3. **The degree of maps between manifolds.** Now let M and N be compact, oriented n-manifolds with atlases $\{\varphi_i : U_i \to \varphi_i(U_i)\}$ and $\{\psi_j : W_j \to \psi_j(W_j)\}$ respectively. Consider a smooth function $f : M \to N$. Let $p \in U_i$ and $f(p) \in W_j$. As f is continuous, by replacing U_i by a smaller open set we can assume that $f(U_i) \subset W_j$. To apply the methods of real analysis to f, we consider the function between open sets of $\mathbb{R}^n$ given by $g = \psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(U_j)$ and let $q = \varphi_i(p)$. We say p is a regular point if the total derivative $Dg(q)$ is invertible, otherwise we say that p is a critical point.

For a regular point p, we assign a positive or negative sign according as $det(Dg(q))$ (with q related to p as above) is positive or negative. We note that the definition of orientability implies that this depends only on f and p, and not on the choice of charts. Let $\varepsilon(p; f)$ be $+1$ if the sign of p is positive and -1 if the sign of q is negative.

As before, critical values are the images of critical points. We define the degree as the number of inverse images of a regular point, counted with sign. More explicitly, we define

$$deg_y(f) = \sum_{\{p \in M : f(p) = y\}} \varepsilon(p; f)$$

for a regular value $y \in N$

For this definition to be useful, there must be regular values p for the function f. This is guaranteed by the following result.

Theorem 4.3 (Sard's theorem)**.** *The set of critical values of a smooth function $f : M \to N$ has measure zero.*

This means in particular that the set of regular values is non-empty. Furthermore, the set of common regular values for any finite or countable collection of functions is non-empty. The definition of degree is based on the following lemma.

Lemma 4.4. *For a regular value $y \in N$, the set $f^{-1}(y)$ is finite. Furthermore, the number of elements counted with sign is independent of y.*

The first part of this lemma is an exercise in application of the inverse function theorem. The second part is non-trivial. We refer to [4] for the proofs.

We can now define the degree of the map f.

Definition 4.5. The degree of $f : M \to N$ is

$$deg(f) = deg_y(f)$$

for $y \in N$ a regular value.

By Theorem 4.3, there is at least one regular value y to apply the above definition. By Lemma 4.4, the $deg_y(f)$ is independent of $y \in N$ and so $deg(f)$ is well-defined.

4.4. **Multiplicativity of the degree.** An important property of the degree is that it is *multiplicative* under composition of maps. More precisely, we have the following result.

Theorem 4.6. *Let $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ be smooth maps between compact, connected, n-dimensional smooth manifolds. Then*

$$deg(f_2 \circ f_1) = deg(f_1) \cdot deg(f_2)$$

Proof. Let $x_1 \in M_1$ be a point. Let $x_2 = f_1(x_1)$ and $x_3 = f_2(x_2) = f_2 \circ f_1(x_1)$. As M_1, M_2 and M_3 are manifolds there are open sets U_i in M_i containing x_i and charts $\varphi_i : U_i \to V_i$ with V_i open subsets of $\mathbb{R}^n$. As the functions f_1 and f_2 are continuous, we can assume that $f_1(U_1) \subset U_2$ and $f_2(U_2) \subset U_3$.

We consider the functions $g_1 = \varphi_2 \circ f_1 \circ \varphi_1^{-1} : V_1 \to V_2$ and $g_2 = \varphi_3 \circ f_1 \circ \varphi_2^{-1} : V_2 \to V_3$. Recall that for $i = 1, 2$, x_i is a regular point for f_i if and only if $z_i := \varphi_i(x_i)$ is a regular point for g_i. Furthermore, if x_i is a regular point, then $\varepsilon(x_i, f_i) = det(Dg_i(z_i))$.

Lemma 4.7. *If x_1 is a regular point for $f_2 \circ f_1$, then x_1 is a regular point for f_1 and x_2 is a regular point for f_2. Moreover $\varepsilon(x_1; f_2 \circ f_1) = \varepsilon(x_1; f_1) \cdot \varepsilon(x_2; f_2)$*

Proof. Using the charts φ_1 and φ_3 for x_1 and $x_3 = f_2 \circ f_1(x_1)$, we see that x_1 is a regular point for $f_2 \circ f_1$ if and only if z_1 is a regular point for $g_2 \circ g_1$, i.e. $D(g_2 \circ g_1)(z_1)$ is non-singular. By the chain rule,

$$D(g_2 \circ g_1)(z_1) = D(g_2)(z_2) \circ Dg_1(z_1)$$

It follows that both terms on the right hand side are non-singular, giving the first claim. The second claim follows from the same equation using the multiplicativity of the determinant. □

We now complete the proof of Theorem 4.6. Let y be a regular value for $f_2 \circ f_1$. By Lemma 4.7, if $f_2 \circ f_1(x_1) = y$, then x_1 is a regular point for f_1 and $f_1(x_1)$ is a regular point for f_2. Hence, we have the following.

$$\begin{aligned}
deg(f_2 \circ f_1) &= \sum_{\{x_1 \in M_1 : f_2 \circ f_1(x_1) = y\}} \varepsilon(x_1; f_2 \circ f_1) \\
&= \sum_{\{x_1 \in M_1 : f_2 \circ f_1(x_1) = y\}} \varepsilon(x_1; f_1)\varepsilon(f_1(x_1); f_2) \\
&= \sum_{\{x_2 \in M_2 : f_2(x_2) = y\}} \sum_{\{x_1 \in M_1 : f_1(x_1) = x_2\}} \varepsilon(x_1; f_1)\varepsilon(x_2; f_2) \\
&= \sum_{\{x_2 \in M_2 : f_2(x_2) = y\}} \varepsilon(x_2; f_2) \left(\sum_{\{x_1 \in M_1 : f(x_1) = x_2\}} \varepsilon(x_1; f_1) \right) \\
&= \sum_{\{x_2 \in M_2 : f_2(x_2) = y\}} \varepsilon(x_2; f_2) deg(f_1) \\
&= deg(f_1) \sum_{\{x_2 \in M_2 : f_2(x_2) = y\}} \varepsilon(x_2; f_2) \\
&= deg(f_1) \cdot deg(f_2)
\end{aligned}$$

□

5. **Degree-one maps and dominance**

We can define an ordering on (smooth) manifolds in terms of the existence of maps of degree one. Let M and N be smooth, compact, oriented n-dimensional manifolds.

Definition 5.1. We say that M dominates N if there is a smooth map $f : M \to N$ such that f has degree one.

We observe that this is indeed transitive, i.e., an order.

Proposition 5.2. *If M_1 dominates M_2 and M_2 dominates M_3, then M_1 dominates M_3.*

Proof. If M_1 dominates M_2 and M_2 dominates M_3, then there are maps $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ of degree one. By Theorem 4.6, the map $f_2 \circ f_1$ has degree one. Hence M_1 dominates M_3. □

The definition of dominance in Definition 5.1 agrees with the earlier definition given in Definition 2.1 except that we may need to reverse the orientation of one of the manifolds. We shall see the easier half of this below.

Proposition 5.3. *If M dominates N in terms of Definition 2.1, there is a degree one map from M to N after possibly reversing the orientation of M.*

Proof. By hypothesis there is a map $f : M \to N$ and a disc D in N with $f^{-1}(D)$ a disc and $f|_{f^{-1}(D)} : f^{-1}(D) \to D$ one-to-one and onto. By perturbing f we may assume that f is smooth. By Sard's theorem, there is a regular value y in the disc D. The inverse image of y consists of a single point x. Hence the degree of f is $\varepsilon(x; f) = \pm 1$. By reversing the orientation of M if necessary, we can ensure that $deg(f) = 1$. □

While dominance is transitive, it is not true that it is a partial order, i.e., if M dominates N and N dominates M, it need not be true that $M = N$.

6. Degree-one maps of 3-manifolds

Motivated by the relation of degree-one maps to the Poincaré conjecture, which we see below, Haken initiated the intensive study of degree-one maps of 3-manifolds. Recall that the Poincaré conjecture, now a theorem of Perelman, says that any *simply-connected*, smooth 3-manifold is homeomorphic to S^3.

Proposition 6.1. *A compact, smooth 3-dimensional manifold M is simply-connected if and only if there is a degree one smooth map $f : S^3 \to M$.*

Thus, the Poincaré conjecture says that if S^3 dominates M, then S^3 is homeomorphic to M. Thus, a natural approach to the Poincaré conjecture, attempted by Haken, was to study the structure of mas of degree-one.

Haken was able to show that a map $f : S^3 \to M$ of degree one can be deformed to a map of a special type, a so called 1-pinch. This is essentially a higher dimensional version of the maps of degree one we considered between surfaces in Section 2. This was generalised to the case of smooth maps of degree one between arbitrary smooth manifolds by Rong and Wang.

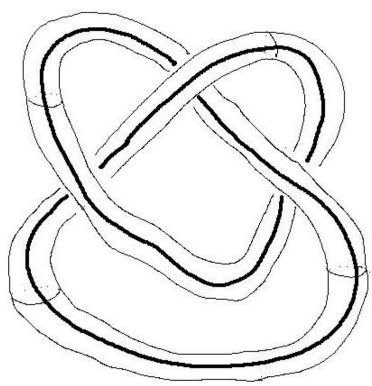

FIGURE 5. A knot and its regular neighbourhood

The existence of a nice description, as well as relations to geometry and other considerations have led to an extensive study of degree-one maps of 3-manifolds.

7. Surgery, Knots and 3-manifolds

A *knot* K is a smooth, 3-dimensional manifold M is a smooth embedding of a circle in M. It is easy to see that a neighbourhood $nbd(K)$ of such a knot K is homeomorphic to $D^2 \times S^1$(see figure 5). More formally, this is a consequence of the *tubular neighbourhood theorem* (see [4]).

The *knot exterior* $\widehat{M}_K$ of the knot K is the manifold $\widehat{M}_K = M \backslash int(nbd(K))$ obtained from M by deleting the interior of $nbd(K)$. The boundary of $nbd(K)$ is also the boundary of $\widehat{M}_k$ regarded as a *manifold with boundary.* In particular the boundary of $\widehat{M}_K$ is homeomorphic to $S^1 \times S^1 = \partial(D^2 \times S^1)$. For a formal definition of manifolds with boundary we refer to [4] and [5].

Surgery of M about K is the process of gluing to the knot exterior $\widehat{M}_K$ a solid torus $D^2 \times S^1$ using some homeomorphism $\varphi : \partial(D^2 \times S^1) \to \partial M$. Thus, surgery is the process of deleting a solid torus from M and gluing back a solid torus, but in general in a different way.

We can more generally delete several disjoint solid tori and glue back solid tori. This is the process of surgery about a *link* L – a finite collection of disjoint embeddings of the circle in M. The importance of this construction lies in the following fundamental result of Lickorish and Wallace.

Theorem 7.1 (Lickorish-Wallace)**.** *Suppose M and N are compact, smooth, oriented 3-dimensional manifolds. Then M can be obtained from N by surgery about some link L in N.*

In particular, taking $N = S^3$ we see that all compact 3-manifolds can be obtained by surgery about some link in the 3-sphere. Thus, surgery gives a universal construction of 3-manifolds.

8. **Degree-one maps and surgery**

We now state our result (from [1]) giving a description of dominance in terms of surgery. An *unknot* is an embedding of the circle in a 3-dimensional manifold M (i.e., a knot) which is the boundary of the embedding of a disc in M. Note that each component of a link is a knot, which may or may not be an unknot.

Theorem 8.1. *Suppose M and N are compact, smooth, oriented 3-dimensional manifolds. Then there is a degree one map $f : M \to N$ if and only if M can be obtained from N by surgery about some link L in N each of whose components is an unknot.*

Thus, our result gives a natural description in terms of surgery of manifolds dominating a fixed manifold. Intuitively, surgeries both create and destroy topological complexity. What we have shown is that, in a precise sense, if a link has unknots as its components, then surgery about such a link only creates topological complexity.

9. **Dominance and smooth four-manifolds**

Topology in different dimensions behaves very differently. Somewhat surprisingly, topology in dimensions 5 and above can be reduced to algebra, while this is not the case in dimensions 3 and for smooth manifolds in dimension 4. The work of Thurston and Perelman shows that topology in dimension 3 instead reduces to geometry.

Smooth 4-dimensional manifolds still remain very mysterious, with not even a conjectural theory. Part of our motivation in proving Theorem 8.1 was its relation to smooth 4-manifolds via so called *topological field theories*. The relation is through an order that is a strengthening of dominance.

Suppose we are given a smooth embedding of a 3-dimensional manifold M into $N \times (0, 1)$, which is the interior of $W = N \times [0, 1]$. Then M separates W into two components. Let us further assume that the two boundary component $N \times \{0\}$ and $N \times \{1\}$ of W are in different components of $W \setminus M$. Then we can deduce that the projection of the embedding of M to N has degree ± 1. We assume that the orientation of M is so that this projection has degree one.

Definition 9.1. We say that M strongly dominates N if there is a smooth embedding of M in $N \times (0, 1)$ separating the boundary components of $N \times [0, 1]$ with M having the appropriate orientation so that the projection has degree one.

By the above, strong dominance implies dominance. We showed that Theorem 8.1 can be interpreted in terms of a weakened version of strong dominance, where we replace W by $N \times [0, 1]$ blown up positively and negatively. For details, we refer to [1].

10. Appendix: Hopf's results and consequences

For readers familiar with Algebraic Topology, we sketch the fundamental results of Hopf concerning the existence of degree-one maps in this section.

Theorem 10.1 (Hopf)**.** *Suppose $f : M \to N$ is a map of degree one between smooth manifolds. Then $f_* : \pi_1(M) \to \pi_1(N)$ is a surjection.*

Proof. Suppose $f_* : \pi_1(M) \to \pi_1(N)$ is not a surjection, then there is a connected cover $p : \bar{N} \to N$ of N with $p_*(\pi_1(N)) = f_*(\pi_1(M))$ and a lift of f to $\bar{f} : M \to \bar{N}$. Assume first that $\bar{N}$ is compact. As $\bar{N}$ is a non-trivial cover, it is easy to see that $deg(p) > 1$.

But as $\bar{f}$ is a lift of f, $f = p \circ \bar{f}$. Hence, by Theorem 4.6,

$$1 = deg(f) = deg(\bar{f})deg(p)$$

which is impossible as $deg(p) > 1$.

In case $\bar{N}$ is non-compact, a similar argument shows that $deg(f) = 0$, giving a contradiction. Alternatively, one can use the definition of degree one maps in terms of homology below to get a simple proof (but which uses more technology). □

We now turn to Hopf's theorems regarding induced maps in (co)homology. Let $f : M \to N$ be a continuous map between closed, oriented n-dimensional manifolds. Recall that for a closed orientable n-dimensional manifold M, $H_n(M) = \mathbb{Z}$ (we use $\mathbb{Z}$-coefficients throughout). An orientation is a choice of generator $[M] \in H_n(M)$. The generator $[M]$ is called the fundamental class. We refer to [3] for results concerning the fundamental class, local homology and Poincaré duality.

The map $f : M \to N$ induces a map $f_* : H_n(M) \to H_n(N)$. It follows that for some $d \in \mathbb{Z}$, $f_*([M]) = d[N]$. We define the degree of the map f to be $deg(f) = d$.

This definition coincides with our previous definition, as can be seen by using local homology.

Theorem 10.2 (Hopf)**.** *Let $f : M \to N$ be a map of degree one between n-dimensional compact, oriented manifolds. Then $f^* : H^*(N) \to H^*(M)$ is injective.*

Proof. By hypothesis, $f_*([M]) = [N]$. Let $\alpha \in H^k(N)$ be non-zero. By Poincaré duality, its image in homology $[N] \cap \alpha \neq 0$. By naturality of the cup and cap products, as $f_*([M]) = [N]$,

$$f_*([M] \cap f^*(\alpha)) = f_*([M]) \cap \alpha = N \cap \alpha \neq 0.$$

But if $f^*(\alpha)$ is zero, then so is $f_*([M] \cap f^*(\alpha))$. Hence $f^*(\alpha) \neq 0$ whenever $\alpha \neq 0$, i.e., f^* is injective. □

Our proof in Section 2 that the sphere does not dominate the torus is a geometric translation of the result of Hopf.

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RANDOM FIXED POINT OF THREE-STEP RANDOM ITERATIVE PROCESS WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE RANDOM OPERATOR

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Abstract. In this paper, we study convergence of random fixed point of three-step random iterative process with errors for asymptotically quasi-nonexpansive random operator in uniformly convex Banach space. Our results extend and improve the corresponding results of Beg and Abbas [5] and many others.

1. Intorduction

Random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is needed for the study of various classes of random equations. The study of random fixed point theory was initiated by the Prague school of Probabilities in the 1950s [11, 12, 22]. Common random fixed point theorems are stochastic generalization of classical common fixed point theorems. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see e.g. [17]) and references mentioned therein. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [9] attracted the attention of several mathematicians and gave wings to the theory. Itoh [14] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become the full fledged research area and various ideas

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associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [4, 5, 8, 13, 20]). Papageorgiou [15, 16], Beg [2, 3] studied common random fixed points and random coincidence points of a pair of compatible random operators and proved fixed point theorems for contractive random operators in Polish spaces. Recently, Beg and Shahzad [7], Choudhury [10] and Badshah and Sayyed [1] used different iteration processes to obtain random fixed points. More recently, Beg and Abbas [6] studied common random fixed points of two asymptotically nonexpansive random operators through strong as well as weak convergence of sequence of measurable functions in the setup of uniformly convex Banach spaces. Also they construct different random iterative algorithms for asymptotically quasi-nonexpansive random operators on an arbitrary Banach space and established their convergence to random fixed point of the operators. In this paper, we study the convergence of three-step random iterative processes with errors for asymptotically quasi-nonexpansive random operator in uniformly convex Banach space. Our result extend the corresponding result of Beg and Abbas [5] and many others.

2. **Prliminaries**

Let (Ω, Σ) be a measurable space (Σ-sigma algebra) and let F be a nonempty subset of a Banach space X. We will denote by $C(X)$ the family of all compact subsets of X with Hausdorff metric H induced by the metric of X. A mapping $\xi\colon \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$, for each open subset U of X. The mapping $T\colon \Omega \times X \to X$ is a random map if and only if for each fixed $x \in X$, the mapping $T(.,x)\colon \Omega \to X$ is measurable and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega, .)\colon X \to X$ is continuous. A measurable mapping $\xi\colon \Omega \to X$ is a random fixed point of a random map $T\colon \Omega \times X \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$, for each $\omega \in \Omega$. We denote the set of random fixed points of a random map T by $RF(T)$.

Let $B(x_0, r)$ denote the spherical ball centered at x_0 with radius r, defined as the set $\{x \in X : \|x - x_0\| \leq r\}$.

We denote the nth iterate $T(\omega, T(\omega, T(\omega, \ldots, T(\omega, x) \ldots,)))$ of T by $T^n(\omega, x)$. The letter I denotes the random mapping $I\colon \Omega \times X \to X$ defined by $I(\omega, x) = x$ and $T^0 = I$.

Definition 2.1: Let F be a nonempty separable subset of a Banach space X and $T\colon \Omega \times F \to F$ be a random map. The map T is said to be the following:

(**a**) A *nonexpansive random operator* if for $x, y \in F$, one has

$$\|T(\omega, x) - T(\omega, y)\| \leq \|x - y\|, \quad \text{for each } \omega \in \Omega. \tag{1}$$

(**b**) *Asymptotically nonexpansive random operator* if there exists a sequence of measurable mapping $k_n\colon \Omega \to [1, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 1$, such that for $x, y \in F$

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq k_n(\omega)\, \|x - y\|, \quad \text{for } \omega \in \Omega. \tag{2}$$

(**c**) *Asymptotically quasi-nonexpansive random operator* if for each $\omega \in \Omega$, $G(\omega) = \{x \in F : x = T(\omega, x)\} \neq \phi$ and there exists a sequence of measurable mapping $k_n\colon \Omega \to [0, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 0$, such that for $x \in F$ and $y \in G(\omega)$, the following inequality holds:

$$\|T^n(\omega, x) - y\| \leq (1 + k_n(\omega))\, \|x - y\|, \quad \text{for each } \omega \in \Omega. \tag{3}$$

(**d**) Uniformly $(L - \alpha)$-*Lipschitzian random operator* if for $x, y \in F$

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq L\, \|x - y\|^{\alpha}, \quad \text{for each } \omega \in \Omega. \tag{4}$$

where $n = 1, 2, \ldots$, L is a positive constant, and $\alpha > 0$.
(**e**) *Quasi-nonexpansive random operator* if for each $\omega \in \Omega$,
$G(\omega) = \{x \in F : x = T(\omega, x)\} \neq \phi$ and for any $x \in F$ and $y \in G(\omega)$,

$$\|T(\omega, x) - y\| \leq \|x - y\|, \quad \text{for each } \omega \in \Omega. \tag{5}$$

(**f**) A *semicompact random operator* if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to F, with $\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$, for every $\omega \in \Omega$, one has a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and a measurable mapping $\xi\colon \Omega \to F$ such that $\{\xi_{n_k}\}$ converges pointwisely to ξ as $k \to \infty$.

Definition 2.2 (three step random iterative process, cf. [5]): Let $T\colon \Omega \times F \to F$ is a random operator, where F is a nonempty convex subset of a separable Banach space X. Let $\xi_0\colon \Omega \to F$ be a measurable mapping from Ω to F. Define sequence of functions $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$, as given below:

$$\begin{aligned}
\zeta_n(\omega) &= \alpha''_n T^n(\omega, \xi_n(\omega)) + \beta''_n \xi_n(\omega),\\
\eta_n(\omega) &= \alpha'_n T^n(\omega, \zeta_n(\omega)) + \beta'_n \xi_n(\omega), \qquad (6)\\
\xi_{n+1}(\omega) &= \alpha_n T^n(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega),
\end{aligned}$$

for each $\omega \in \Omega$, $n = 0, 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$ and $\{\beta''_n\}$ are sequences of real numbers in $[0, 1]$. Obviously $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ are sequences of measurable functions from Ω to F.

Motivated by above we define the following random iterative scheme:

Definition 2.3: Let $T\colon \Omega \times F \to F$ be a random operator, where F be a nonempty convex subset of a separable Banach space X. Let Let $\xi_0\colon \Omega \to F$ be a measurable mapping from Ω to F, let $\{f_n\}$, $\{f'_n\}$, $\{f''_n\}$ be bounded sequences of measurable functions from Ω to F. Define sequences of functions $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$, as given below:

$$\begin{aligned} \zeta_n(\omega) &= \alpha''_n T^n(\omega, \xi_n(\omega)) + \beta''_n \xi_n(\omega) + \gamma''_n f''_n(\omega), \\ \eta_n(\omega) &= \alpha'_n T^n(\omega, \zeta_n(\omega)) + \beta'_n \xi_n(\omega) + \gamma'_n f'_n(\omega), \\ \xi_{n+1}(\omega) &= \alpha_n T^n(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega), \end{aligned} \tag{7}$$

for each $\omega \in \Omega$, $n = 0, 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are sequences of real numbers in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

Remark 2.4: If we take $\gamma_n = \gamma'_n = \gamma''_n = 0$, then (7) reduces to (6).

The purpose of this paper is to establish some convergence results of the three-step random iterative process with errors given in (7) for asymptotically quasi-nonexpansive random operator.
In the sequel, we will need the following lemmas.

Lemma 2.5 (Tan and Xu [21]): Let $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \quad \forall n \in N.$$

If

$$\sum_{n=1}^{\infty} r_n < \infty, \quad \sum_{n=1}^{\infty} \beta_n < \infty.$$

Then

(i) $\lim_{n\to\infty} a_n$ exists.

(ii) If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (Schu [19]): Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying

$$\limsup_{n\to\infty} \|x_n\| \leq r,$$

$$\limsup_{n\to\infty} \|y_n\| \leq r,$$

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

3. **Main Results**

In this section, we investigate the convergence of three-step random iterative process with errors for asymptotically quasi-nonexpansive random operator to obtain the random solution of the random fixed point. This iterative process extend the three-step random iterative process [5] for asymptotically quasi-nonexpansive random operator. In order to prove our main results, we need the following lemmas.

Lemma 3.1: Let X be a uniformly convex separable Banach space, and let F be a nonempty closed and convex subset of X. Let $T\colon \Omega \times F \to F$ be asymptotically quasi-nonexpansive random operator from $\Omega \times F$ to F with sequence of measurable mapping $k_n\colon \Omega \to [0,\infty)$ satisfying

$$\sum_{n=1}^{\infty} k_n(\omega) < \infty,$$

for each

$$\omega \in \Omega.$$

Let $\{\xi_n\}$ be the sequence as defined by (7) with

$$\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \gamma_n' < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n'' < \infty.$$

Then

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$$

exists for all $\xi(\omega) \in RF(T)$ and for each $\omega \in \Omega$.

Proof: The existence of random fixed point of T follows from Bharucha-Reid's stochastic analogue (see [9]) of well-known Schauder's fixed point theorem. Let $\xi\colon \Omega \to F$ be the random fixed point of T. Since $\{f_n\}$, $\{f_n'\}$

and $\{f''_n\}$ are bounded sequences of measurable functions from Ω to F, we can put

$$M(\omega) = \sup_{n\geq 1} \|f_n(\omega) - \xi(\omega)\| \vee \sup_{n\geq 1} \|f'_n(\omega) - \xi(\omega)\| \vee \sup_{n\geq 1} \|f''_n(\omega) - \xi(\omega)\| .$$

Then $M(\omega)$ is a finite number for each $\omega \in \Omega$. For $n \geq 1$, we have

$$\begin{aligned}
& \|\xi_{n+1}(\omega) - \xi(\omega)\| \\
= \ & \|\alpha_n T^n(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega) - \xi(\omega)\| \\
\leq \ & \alpha_n \|T^n(\omega, \eta_n(\omega)) - \xi(\omega)\| + \beta_n \|\xi_n(\omega) - \xi(\omega)\| \\
& +\gamma_n \|f_n(\omega) - \xi(\omega)\| \\
\leq \ & \alpha_n(1 + k_n(\omega)) \|\eta_n(\omega) - \xi(\omega)\| + \beta_n \|\xi_n(\omega) - \xi(\omega)\| \\
& +\gamma_n \|f_n(\omega) - \xi(\omega)\| \\
\leq \ & \alpha_n(1 + k_n(\omega)) \|\eta_n(\omega) - \xi(\omega)\| + \beta_n \|\xi_n(\omega) - \xi(\omega)\| \\
& +\gamma_n M(\omega)
\end{aligned} \tag{8}$$

Similarly, we have

$$\begin{aligned}
\|\eta_n(\omega) - \xi(\omega)\| \ \leq \ & \alpha'_n(1 + k_n(\omega)) \|\zeta_n(\omega) - \xi(\omega)\| \\
& +\beta'_n \|\xi_n(\omega) - \xi(\omega)\| + \gamma'_n \|f'_n(\omega) - \xi(\omega)\| \\
\leq \ & \alpha'_n(1 + k_n(\omega)) \|\zeta_n(\omega) - \xi(\omega)\| \\
& +\beta'_n \|\xi_n(\omega) - \xi(\omega)\| + \gamma'_n M(\omega)
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\|\zeta_n(\omega) - \xi(\omega)\| \ \leq \ & \alpha''_n(1 + k_n(\omega)) \|\xi_n(\omega) - \xi(\omega)\| \\
& +\beta''_n \|\xi_n(\omega) - \xi(\omega)\| + \gamma''_n \|f''_n(\omega) - \xi(\omega)\| \\
\leq \ & \alpha''_n(1 + k_n(\omega)) \|\xi_n(\omega) - \xi(\omega)\| \\
& +\beta''_n \|\xi_n(\omega) - \xi(\omega)\| + \gamma''_n M(\omega)
\end{aligned} \tag{10}$$

Substituting (10) in (9), we get

$$\begin{aligned}
\|\eta_n(\omega) - \xi(\omega)\| \ \leq \ & \alpha'_n \alpha''_n (1 + k_n(\omega))^2 \|\xi_n(\omega) - \xi(\omega)\| \\
& +\alpha'_n \beta''_n (1 + k_n(\omega)) \|\xi_n(\omega) - \xi(\omega)\| \\
& +\alpha'_n \gamma''_n (1 + k_n(\omega)) M(\omega) + \beta'_n \|\xi_n(\omega) - \xi(\omega)\| \\
& +\gamma'_n M(\omega)
\end{aligned}$$

$$\begin{aligned}
&= (1-\beta_n'-\gamma_n')\alpha_n''(1+k_n(\omega))^2\|\xi_n(\omega)-\xi(\omega)\|+\beta_n'\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +(1-\beta_n'-\gamma_n')\beta_n''\|\xi_n(\omega)-\xi(\omega)\|+m_n(\omega) \\
&\le (1-\beta_n')\alpha_n''(1+k_n(\omega))^2\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +\beta_n'(1+k_n(\omega))^2\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +(1-\beta_n')\beta_n''(1+k_n(\omega))^2\|\xi_n(\omega)-\xi(\omega)\|+m_n(\omega) \\
&\le (1-\beta_n')(1+k_n(\omega))^2\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +\beta_n'(1+k_n(\omega))^2\|\xi_n(\omega)-\xi(\omega)\|+m_n(\omega) \\
&= (1+k_n(\omega))^2\|\xi_n(\omega)-\xi(\omega)\|+m_n(\omega) \qquad (11)
\end{aligned}$$

where $m_n(\omega)=\alpha_n'\gamma_n''(1+k_n(\omega))M(\omega)+\gamma_n'M(\omega)$.
Note that $\sum_{n=1}^{\infty} m_n(\omega)<\infty$. Substituting (11) in (8), we get

$$\begin{aligned}
&\|\xi_{n+1}(\omega)-\xi(\omega)\| \\
&\le \alpha_n(1+k_n(\omega))^3\|\xi_n(\omega)-\xi(\omega)\|+\alpha_n(1+k_n(\omega))m_n(\omega) \\
&\quad +\beta_n\|\xi_n(\omega)-\xi(\omega)\|+\gamma_nM(\omega) \\
&\le (\alpha_n+\beta_n)(1+k_n(\omega))^3\|\xi_n(\omega)-\xi(\omega)\|+A_n(\omega) \\
&\le (1+k_n(\omega))^3\|\xi_n(\omega)-\xi(\omega)\|+A_n(\omega)
\end{aligned}$$

where $A_n(\omega)=\alpha_n(1+k_n(\omega))m_n(\omega)+\gamma_nM(\omega)$ with $\sum_{n=1}^{\infty} A_n(\omega)<\infty$.

Since $\sum_{n=1}^{\infty} k_n(\omega)<\infty$, $\sum_{n=1}^{\infty} A_n(\omega)<\infty$, it follows from Lemma 2.5 [21] that $\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|$ exists for all $\omega\in\Omega$. This completes the proof.

Lemma 3.2: Let X be a uniformly convex separable Banach space, and let F be a nonempty closed and convex subset of X. Let $T\colon\Omega\times F\to F$ be asymptotically quasi-nonexpansive random operator from $\Omega\times F$ to F with sequence of measurable mapping $k_n\colon\Omega\to[0,\infty)$ satisfying $\sum_{n=1}^{\infty} k_n(\omega)<\infty$, for each $\omega\in\Omega$. Let $\{\xi_n\}$ be the sequence as defined by (7) with the following restrictions:

(i) $\alpha_n+\beta_n+\gamma_n=\alpha_n'+\beta_n'+\gamma_n'=\alpha_n''+\beta_n''+\gamma_n''=1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n < 1 - \alpha$, for some $\alpha \in (0,1)$, for all $n \geq n_0$, $\exists n_0 \in N$.

Then

$$\begin{aligned}\lim_{n\to\infty} \|T^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| &= \lim_{n\to\infty} \|T^n(\omega, \zeta_n(\omega)) - \xi_n(\omega)\| \\ &= \lim_{n\to\infty} \|T^n(\omega, \xi_n(\omega)) - \xi_n(\omega)\| \\ &= 0.\end{aligned}$$

Proof: For any $\xi(\omega) \in RF(T)$, it follows from Lemma 3.1, we have

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$$

exists for all $\omega \in \Omega$. Let

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = a$$

for some $a \geq 0$. From (11), we have

$$\|\eta_n(\omega) - \xi(\omega)\| \leq (1 + k_n(\omega))^2 \|\xi_n(\omega) - \xi(\omega)\| + m_n(\omega).$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| &\leq \limsup_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| \\ &= \lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| \\ &= a.\end{aligned}$$

Note that

$$\begin{aligned}\limsup_{n\to\infty} \|T^n(\omega, \eta_n(\omega)) - \xi(\omega)\| &\leq \limsup_{n\to\infty} (1 + k_n(\omega)) \|\eta_n(\omega) - \xi(\omega)\| \\ &\leq a.\end{aligned}$$

Next consider

$$\begin{aligned}\|T^n(\omega, \eta_n(\omega)) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\| &\leq \|T^n(\omega, \eta_n(\omega)) - \xi(\omega)\| \\ &\quad + \gamma_n \|f_n(\omega) - \xi_n(\omega)\|.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \|T^n(\omega, \eta_n(\omega)) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\| \leq a.$$

Also,

$$\begin{aligned} &\|\xi_n(\omega) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\| \\ &\leq \|\xi_n(\omega) - \xi(\omega)\| + \gamma_n \|f_n(\omega) - \xi_n(\omega)\| \end{aligned}$$

gives that

$$\limsup_{n\to\infty} \|\xi_n(\omega) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\| \leq a.$$

Moreover, we note that

$$\begin{aligned} a &= \lim_{n\to\infty} \|\xi_{n+1}(\omega) - \xi(\omega)\| \\ &= \lim_{n\to\infty} \|\alpha_n T^n(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega) - \xi(\omega)\| \\ &= \lim_{n\to\infty} \|\alpha_n T^n(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega) - (1-\alpha_n)\xi(\omega) - \alpha_n \xi(\omega)\| \\ &= \lim_{n\to\infty} \|\alpha_n [T^n(\omega, \eta_n(\omega)) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))] \\ &\qquad + (1-\alpha_n)[\xi_n(\omega) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))]\| \end{aligned}$$

By Lemma 2.6 (Schu [19]), we have

$$\lim_{n\to\infty} \|T^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| = 0.$$

Next we prove that

$$\lim_{n\to\infty} \|T^n(\omega, \zeta_n(\omega)) - \xi_n(\omega)\| = 0.$$

For each $n \geq 1$, we have

$$\begin{aligned} \|\xi_n(\omega) - \xi(\omega)\| &\leq \|T^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| + \|T^n(\omega, \eta_n(\omega)) - \xi(\omega)\| \\ &\leq \|T^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| + (1 + k_n(\omega)) \|\eta_n(\omega) - \xi(\omega)\|. \end{aligned}$$

Since

$$\lim_{n\to\infty} \|T^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| = 0 = \lim_{n\to\infty} k_n(\omega),$$

we obtain that

$$a = \lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| \leq \liminf_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\|.$$

It follows that

$$a \leq \liminf_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| \leq \limsup_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| \leq a.$$

This implies that

$$\lim_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| = a.$$

On the other hand, we note that

$$\begin{aligned}
\|\zeta_n(\omega)-\xi(\omega)\| &= \|\alpha''_n T^n(\omega,\xi_n(\omega))+\beta''_n\xi_n(\omega)+\gamma''_n f''_n(\omega)-\xi(\omega)\| \\
&\le \alpha''_n\|T^n(\omega,\xi_n(\omega))-\xi(\omega)\|+\beta''_n\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +\gamma''_n\|f''_n(\omega)-\xi(\omega)\| \\
&\le \alpha''_n(1+k_n(\omega))\|\xi_n(\omega)-\xi(\omega)\|+\beta''_n\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +\gamma''_n\|f''_n(\omega)-\xi(\omega)\| \\
&\le \alpha''_n(1+k_n(\omega))\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +(1-\alpha''_n)(1+k_n(\omega))\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +\gamma''_n\|f''_n(\omega)-\xi(\omega)\| \\
&\le (1+k_n(\omega))\|\xi_n(\omega)-\xi(\omega)\|+\gamma''_n\|f''_n(\omega)-\xi(\omega)\|.
\end{aligned}$$

By boundedness of $\{f''_n(\omega)\}$ and

$$\lim_{n\to\infty} k_n(\omega)=0=\lim_{n\to\infty}\gamma''_n,$$

we have

$$\limsup_{n\to\infty}\|\zeta_n(\omega)-\xi(\omega)\|\le\limsup_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|\le a,$$

and

$$\limsup_{n\to\infty}\|T^n(\omega,\zeta_n(\omega))-\xi(\omega)\|\le\limsup_{n\to\infty}(1+k_n(\omega))\|\zeta_n(\omega)-\xi(\omega)\|\le a.$$

Next, we consider

$$\begin{aligned}
\|T^n(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma'_n(f'_n(\omega)-\xi_n(\omega))\| &\le \|T^n(\omega,\zeta_n(\omega))-\xi(\omega)\| \\
&\quad +\gamma'_n\|f'_n(\omega)-\xi_n(\omega)\|.
\end{aligned}$$

Taking $\limsup_{n\to\infty}$ in both sides, we have

$$\limsup_{n\to\infty}\|T^n(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma'_n(f'_n(\omega)-\xi_n(\omega))\|\le a.$$

Also,

$$\begin{aligned}
&\|\xi_n(\omega)-\xi(\omega)+\gamma'_n(f'_n(\omega)-\xi_n(\omega))\| \\
\le\ &\|\xi_n(\omega)-\xi(\omega)\|+\gamma'_n\|f'_n(\omega)-\xi_n(\omega)\|
\end{aligned}$$

gives that

$$\limsup_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)+\gamma'_n(f'_n(\omega)-\xi_n(\omega))\|\le a.$$

Since $\lim_{n\to\infty}\|\eta_n(\omega)-\xi(\omega)\|=a$, we obtain

$$\begin{aligned} a &= \lim_{n\to\infty}\|\eta_n(\omega)-\xi(\omega)\| \\ &= \lim_{n\to\infty}\|\alpha'_n T^n(\omega,\zeta_n(\omega))+\beta'_n\xi_n(\omega)+\gamma'_n f'_n(\omega)-\xi(\omega)\| \\ &= \lim_{n\to\infty}\|\alpha'_n[T^n(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma'_n(f'_n(\omega)-\xi_n(\omega))] \\ &\qquad +(1-\alpha'_n)[\xi_n(\omega)-\xi(\omega)+\gamma'_n(f'_n(\omega)-\xi_n(\omega))]\| \end{aligned}$$

By Lemma 2.6 (Schu [19]), we have

$$\lim_{n\to\infty}\|T^n(\omega,\zeta_n(\omega))-\xi_n(\omega)\|=0.$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty}\|T^n(\omega,\xi_n(\omega))-\xi_n(\omega)\|=0,$$

for all $\omega\in\Omega$. This completes the proof.

Theorem 3.3: Let X be a uniformly convex separable Banach space, and let F be a nonempty closed and convex subset of X. Let $T\colon\Omega\times F\to F$ be uniformly L-Lipschitzian completely continuous asymptotically quasi-nonexpansive random operator from $\Omega\times F$ to F with sequence of measurable mapping $k_n\colon\Omega\to[0,\infty)$ satisfying $\sum_{n=1}^{\infty}k_n(\omega)<\infty$, for each $\omega\in\Omega$. Let $\{\xi_n\}$ be the sequence as defined by (7) with the following restrictions:

(i) $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=\alpha''_n+\beta''_n+\gamma''_n=1$.

(ii) $\sum_{n=1}^{\infty}\gamma_n<\infty,\ \sum_{n=1}^{\infty}\gamma'_n<\infty,\ \sum_{n=1}^{\infty}\gamma''_n<\infty$.

(iii) $0<\alpha\le\alpha_n,\alpha'_n,\alpha''_n<1-\alpha$, for some $\alpha\in(0,1)$, for all $n\ge n_0$, $\exists n_0\in N$.

Then sequences $\{\xi_n\}$ converges to some random fixed point of T.

Proof: It follows from Lemma 3.2, that

$$\lim_{n\to\infty}\|T^n(\omega,\eta_n(\omega))-\xi_n(\omega)\|=\lim_{n\to\infty}\|T^n(\omega,\zeta_n(\omega))-\xi_n(\omega)\|=$$
$$\lim_{n\to\infty}\|T^n(\omega,\xi_n(\omega))-\xi_n(\omega)\|=0,$$

and this implies that

$$\begin{aligned}\|\xi_{n+1}(\omega)-\xi_n(\omega)\| &\le \alpha_n \|T^n(\omega,\eta_n(\omega))-\xi_n(\omega)\|+\gamma_n\|f_n(\omega)-\xi_n(\omega)\| \\ &\to 0\end{aligned}$$

as $n\to\infty$ and for each $\omega\in\Omega$.
We now to show that $\lim_{n\to\infty}\|\xi_n(\omega)-T(\omega,\xi_n(\omega))\|=0$.
Note that

$$\begin{aligned} & \left\|\xi_n(\omega)-T^{n-1}(\omega,\xi_n(\omega))\right\| \\ \le\ & \|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\left\|\xi_{n-1}(\omega)-T^{n-1}(\omega,\xi_{n-1}(\omega))\right\| \\ & +\left\|T^{n-1}(\omega,\xi_{n-1}(\omega))-T^{n-1}(\omega,\xi_n(\omega))\right\| \\ \le\ & \|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\left\|\xi_{n-1}(\omega)-T^{n-1}(\omega,\xi_{n-1}(\omega))\right\| \\ & +L\|\xi_{n-1}(\omega)-\xi_n(\omega)\| \\ & \to 0\end{aligned}$$

as $n\to\infty$ and for each $\omega\in\Omega$.
Thus by the above inequality, we have

$$\begin{aligned} & \|\xi_n(\omega)-T(\omega,\xi_n(\omega))\| \\ \le\ & \|\xi_n(\omega)-T^n(\omega,\xi_n(\omega))\|+\|T^n(\omega,\xi_n(\omega))-T(\omega,\xi_n(\omega))\| \\ \le\ & \|\xi_n(\omega)-T^n(\omega,\xi_n(\omega))\|+L\left\|T^{n-1}(\omega,\xi_n(\omega))-\xi_n(\omega)\right\| \\ & \to 0\end{aligned}$$

as $n\to\infty$ and for each $\omega\in\Omega$.
It implies that

$$\lim_{n\to\infty}\|\xi_n(\omega)-T(\omega,\xi_n(\omega))\|=0. \qquad (*)$$

By Lemma 3.1 $\{\xi_n\}$ is a bounded sequence. It follows by our assumption that T is completely continuous there exists a subsequence $\{T\xi_{n_k}\}$ of $\{T\xi_n\}$ and a measurable mapping $p\colon\Omega\to F$ such that for each $\omega\in\Omega$, $T\xi_{n_k}\to p\in RF(T)$ as $n_k\to\infty$. Moreover, by $(*)$, we have $\|T(\omega,\xi_{n_k}(\omega))-\xi_{n_k}(\omega)\|\to 0$ which implies that $\xi_{n_k}(\omega)\to p(\omega)$ as $n_k\to\infty$. By $(*)$ again, we have

$$\|p(\omega)-T(\omega,p(\omega))\|=\lim_{n_k\to\infty}\|\xi_{n_k}(\omega)-T(\omega,\xi_{n_k}(\omega))\|=0.$$

It shows that p is a random fixed point of T i.e. $p\in RF(T)$. Furthermore, since $\lim_{n\to\infty}\|\xi_n(\omega)-p(\omega)\|$ exists. Therefore $\lim_{n\to\infty}\|\xi_n(\omega)-p(\omega)\|=0$, that is, $\{\xi_n\}$ converges to some random fixed point of T. This completes the proof.

Corollary 3.4: Let X be a uniformly convex separable Banach space, and let F be a nonempty closed and convex subset of X. Let $T\colon \Omega \times F \to F$ be uniformly L-Lipschitzian completely continuous asymptotically quasi-nonexpansive random operator from $\Omega \times F$ to F with sequence of measurable mapping $k_n\colon \Omega \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} k_n(\omega) < \infty$, for each $\omega \in \Omega$. Let $\{\xi_n\}$ be the sequence as defined by (7) with the following restrictions:

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n < 1 - \alpha$, for some $\alpha \in (0,1)$, for all $n \geq n_0$, $\exists n_0 \in N$.

Then sequences $\{\eta_n\}$ and $\{\zeta_n\}$ converges to some random fixed point of T.

Proof: By Theorem 3.3, we have the sequence $\{\xi_n\}$ converges to a random fixed point p of T, that is, $p \in RF(T)$, so that it is also a bounded sequence. Thus by Lemma 3.2 and Theorem 3.3, we obtain that

$$\|\eta_n(\omega) - \xi_n(\omega)\| \leq \alpha'_n \|T^n(\omega, \zeta_n(\omega)) - \xi_n(\omega)\| + \gamma'_n \|f'_n(\omega) - \xi_n(\omega)\| \to 0,$$

as $n \to \infty$ and for each $\omega \in \Omega$,
and

$$\|\zeta_n(\omega) - \xi_n(\omega)\| \leq \alpha''_n \|T^n(\omega, \xi_n(\omega)) - \xi_n(\omega)\| + \gamma''_n \|f''_n(\omega) - \xi_n(\omega)\| \to 0,$$

as $n \to \infty$ and for each $\omega \in \Omega$.

$$\therefore \lim_{n\to\infty} \eta_n(\omega) = p(\omega) = \lim_{n\to\infty} \zeta_n(\omega).$$

This completes the proof.

Remark 3.5: Our results extend and improve the corresponding results of Beg and Abbas [5] to the case of three-step random iterative process with errors for more general class of asymptotically nonexpansive random operator and also we will extend the random iterative scheme (6) into (7) for asymptotically quasi-nonexpansive random operator.

Remark 3.6: (i) If we take $\gamma_n = \gamma'_n = \gamma''_n = 0$, $\alpha''_n = 0$ and replace T^n by T in Theorem 3.3, then we obtain Ishikawa random scheme for asymptotically quasi-nonexpansive random operator.

(ii) If we take $\gamma_n = \gamma'_n = \gamma''_n = 0$ and $\alpha''_n = 0$ in Theorem 3.3, then we obtain modified Ishikawa random scheme for asymptotically quasi-nonexpansive random operator.

(iii) If we take $\alpha''_n = 0$ in Theorem 3.3, then we obtain modified Ishikawa random scheme with errors for asymptotically quasi-nonexpansive random operator.

Remark 3.7: (i) If we take $\gamma_n = \gamma'_n = \gamma''_n = 0$, $\alpha'_n = \alpha''_n = 0$ and replace T^n by T in Theorem 3.3, then we obtain Mann random scheme for asymptotically quasi-nonexpansive random operator.

(ii) If we take $\gamma_n = \gamma'_n = \gamma''_n = 0$ and $\alpha'_n = \alpha''_n = 0$ in Theorem 3.3, then we obtain modified Mann random scheme for asymptotically quasi-nonexpansive random operator.

(iii) If we take $\alpha'_n = \alpha''_n = 0$ in Theorem 3.3, then we obtain modified Mann random scheme with errors for asymptotically quasi-nonexpansive random operator.

Remark 3.8: If we take $\gamma''_n = 0$ and $\alpha''_n = 0$ in Theorem 3.3, then Theorem 3.3 is a stochastic version of the result due to Qihou [18] for $\alpha = 1$.

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ON CERTAIN NEW SUMMATION FORMULAE FOR FOX H-FUNCTION

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Abstract. In the present paper, we establish four summation formulae for the well-known H-function which are believed to be new. Corresponding summation formulae for the familiar G-function follow as their special cases.

1. Introduction

The H-function occurring in this paper will be defined and represented in the following manner [1, p.10, eq.(2.1.1)]:

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi\omega}\int_L \theta(\xi) z^{\xi}\, d\xi, \ (\omega=\sqrt{-1}) \qquad (1)$$

where

$$\theta(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\ \prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=M+1}^{Q}\Gamma(1-b_j+\beta_j\xi)\ \prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (2)$$

M, N, P and Q are non-negative integers satisfying $0 \le N \le P, 1 \le M \le Q$; $\alpha_j (j=1,2,...P)$ and $\beta_j (j=1,2,...Q)$ are assumed to be positive quantities for standardization purposes and an empty product in (2), if it occurs , is taken as unity. Further, the contour L runs from $-\omega\infty$ to $+\omega\infty$ such that the poles of $\Gamma(b_h-\beta_h\xi)$, $h=1,...,M$, lie to the right of L and the poles of $\Gamma(1-a_j+\alpha_j\xi)$, $j=1,...,N$, lie to the left of L(see also [2]).

The contour integral given by (1) is absolutely convergent when the following conditions are satisfied [1,p.13,eq.(2.2.11)]:

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$$A = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P} \alpha_j + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q} \beta_j > 0$$

and $|argz| < \frac{1}{2} A\pi$

It may not be out of place to mention here that various other sets of conditions of convergence of the defining integral (1) can also be given, but we have restricted ourselves in the present paper only to the conditions mentioned above.

The H-function is presently becoming a function of choice. Thus, recently Kilbas and Saigo have written a useful monograph on H-transforms [3]. Again, a number of researchers are trying to locate special cases of Fox H-function that are solutions of boundary value problems occurring in diverse fields of engineering, physics and statistics.

2. Summation Formulae

$$H_{P+r,Q+r}^{M,N+r}\left[z^{\sigma} \left| \begin{matrix} (0,\sigma)_{1,r}, (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q}, (1,\sigma)_{1,r} \end{matrix} \right. \right] = \frac{\sigma^r}{\alpha_1^r} \left\{ A_1^r H_{P,Q}^{M,N}\left[z^{\sigma} \left| \begin{matrix} (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right. \right] \right. \quad (3)$$

$$\left. + \sum_{k=1}^{r} \sum_{r_1,\cdots,r_k=0}^{r_1+\cdots+r_k \le r-k} A_1^{r_1} \cdots A_k^{r_k} A_{k+1}^{r-(r_1+\cdots+r_k)-k} H_{P,Q}^{M,N}\left[z^{\sigma} \left| \begin{matrix} (a_1-k,\alpha_1), (a_j,\alpha_j)_{2,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right. \right] \right\}$$

where A_i stands for $a_1 - i$; $i = 1, 2, \cdots, r$.

Proof: We shall prove (3) by the method of mathematical induction. To do so, we first prove the above formula for $r = 1$ i.e.

$$H_{P+1,Q+1}^{M,N+1}\left[z^{\sigma} \left| \begin{matrix} (0,\sigma), (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q}, (1,\sigma) \end{matrix} \right. \right] \quad (4)$$

$$= \frac{\sigma}{\alpha_1} \left\{ A_1 H_{P,Q}^{M,N}\left[z^{\sigma} \left| \begin{matrix} (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right. \right] + H_{P,Q}^{M,N}\left[z^{\sigma} \left| \begin{matrix} (a_1-1,\alpha_1), (a_j,\alpha_j)_{2,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right. \right] \right\}$$

To establish (4), we first write the H-functions occurring in its right hand side in the form of a contour integral given by (1). Further, on using the following simple property of the gamma function:

$$\alpha_1 s \Gamma(1 - a_1 + \alpha_1 s) = (a_1 - 1)\Gamma(1 - a_1 + \alpha_1 s) + \Gamma(2 - a_1 + \alpha_1 s) \quad (5)$$

We arrive at the l.h.s. of (4) after a little simplification. Now, we assume this result to be true for r and prove it for $r+1$.
The r.h.s. of (3) for $r = r+1$ is

$$\frac{\sigma^{r+1}}{\alpha_1^{r+1}}\left\{A_1^{r+1}H_{P,Q}^{M,N}\left[\; z^{\sigma}\;\middle|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\(b_j,\beta_j)_{1,Q}\end{matrix}\right]+\sum_{k=1}^{r+1}\sum_{r_1,\cdots,r_k=0}^{r_1+\cdots+r_k\leq(r+1)-k}A_1^{r_1}\cdots A_k^{r_k}\right.$$

$$\left.\times A_{k+1}^{r+1-(r_1+\cdots+r_k)-k}H_{P,Q}^{M,N}\left[\; z^{\sigma}\;\middle|\begin{matrix}(a_1-k,\alpha_1),(a_j,\alpha_j)_{2,P}\\ \\(b_j,\beta_j)_{1,Q}\end{matrix}\right]\right\}$$

Expressing the various H-functions occurring in the above expression in terms of their corresponding contour integrals and using the property of the gamma function given by (5) in the slightly altered form repeatedly, we arrive at the following result after a little simplification:

$$H_{P+r+1,Q+r+1}^{M,N+r+1}\left[\; z^{\sigma}\;\middle|\begin{matrix}(0,\sigma)_{1,r+1},(a_j,\alpha_j)_{1,P}\\ \\(b_j,\beta_j)_{1,Q},(1,\sigma)_{1,r+1}\end{matrix}\right]$$

This proves the summation formula for $r+1$.

Thus the result (3) follows for all positive integral values of r by mathematical induction. The following three summation formulae can also be established on similar lines:

$$H_{P+r,Q+r}^{M,N+r}\left[\; z^{\sigma}\;\middle|\begin{matrix}(0,\sigma)_{1,r},(a_j,\alpha_j)_{1,P}\\ \\(b_j,\beta_j)_{1,Q},(1,\sigma)_{1,r}\end{matrix}\right]=\frac{\sigma^r}{\alpha_P^r}\left\{A_1^r H_{P,Q}^{M,N}\left[\; z^{\sigma}\;\middle|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\(b_j,\beta_j)_{1,Q}\end{matrix}\right]\right. \tag{6}$$

$$+\sum_{k=1}^{r}\sum_{r_1,\cdots,r_k=0}^{r_1+\cdots+r_k\leq r-k}(-1)^k A_1^{r_1}\cdots A_k^{r_k}A_{k+1}^{r-(r_1+\cdots+r_k)-k}$$

$$\left.\times H_{P,Q}^{M,N}\left[\; z^{\sigma}\;\middle|\begin{matrix}(a_j,\alpha_j)_{1,P-1},(a_P-k,\alpha_P)\\ \\(b_j,\beta_j)_{1,Q}\end{matrix}\right]\right\}$$

where A_i stands for $a_P - i$; $i = 1, 2, \cdots, r$.

$$H_{P+r,Q+r}^{M,N+r}\left[\; z^{\sigma}\;\middle|\begin{matrix}(0,\sigma)_{1,r},(a_j,\alpha_j)_{1,P}\\ \\(b_j,\beta_j)_{1,Q},(1,\sigma)_{1,r}\end{matrix}\right]=\frac{\sigma^r}{\beta_1^r}\left\{B_0^r H_{P,Q}^{M,N}\left[\; z^{\sigma}\;\middle|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\(b_j,\beta_j)_{1,Q}\end{matrix}\right]\right. \tag{7}$$

$$+\sum_{k=1}^{r}\sum_{r_1,\cdots,r_k=0}^{r_1+\cdots+r_k\le r-k}(-1)^k B_0^{r_1}\cdots B_{k-1}^{r_k}B_k^{r-(r_1+\cdots+r_k)-k}$$

$$\times H_{P,Q}^{M,N}\left[z^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_1+k,\beta_1),(b_j,\beta_j)_{2,Q}\end{matrix}\right.\right]\Bigg\}$$

where B_i stands for b_1+i; $i=0,1,2,\cdots,r$.

$$H_{P+r,Q+r}^{M,N+r}\left[z^{\sigma}\left|\begin{matrix}(0,\sigma)_{1,r},(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q},(1,\sigma)_{1,r}\end{matrix}\right.\right]=\frac{\sigma^r}{\beta_Q^r}\Bigg\{B_0^r H_{P,Q}^{M,N}\left[z^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right.\right] \tag{8}$$

$$+\sum_{k=1}^{r}\sum_{r_1,\cdots,r_k=0}^{r_1+\cdots+r_k\le r-k}B_0^{r_1}\cdots B_{k-1}^{r_k}B_k^{r-(r_1+\cdots+r_k)-k}$$

$$\times H_{P,Q}^{M,N}\left[z^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q-1},(b_Q+k,\beta_Q)\end{matrix}\right.\right]\Bigg\}$$

where B_i stands for b_Q+i; $i=0,1,2,\cdots,r$.

3. Special Cases

If we put $\alpha_i=\beta_j=1\ \forall\, i=1,2,\cdots,P,\ j=1,2,\cdots,Q$ in the result (3), we obtain the following summation formula for Meijer G-function:

$$G_{P+r,Q+r}^{M,N+r}\left[z^{\sigma}\left|\begin{matrix}\underbrace{0,\cdots,0}_{r\text{ times}},(a_j)_{1,P}\\ \\ (b_j)_{1,Q},\underbrace{1,\cdots,1}_{r\text{ times}}\end{matrix}\right.\right]=\frac{\sigma^r}{\alpha_1^r}\Bigg\{A_1^r G_{P,Q}^{M,N}\left[z^{\sigma}\left|\begin{matrix}(a_j)_{1,P}\\ \\ (b_j)_{1,Q}\end{matrix}\right.\right] \tag{9}$$

$$\sum_{k=1}^{r}\sum_{r_1,\cdots,r_k=0}^{r_1+\cdots+r_k\le r-k}A_1^{r_1}\cdots A_k^{r_k}A_{k+1}^{r-(r_1+\cdots+r_k)-k}G_{P,Q}^{M,N}\left[z^{\sigma}\left|\begin{matrix}(a_1-k),(a_j)_{2,P}\\ \\ (b_j)_{1,Q}\end{matrix}\right.\right]\Bigg\}$$

Finally, if we take $\alpha_i=\beta_j=1\ \forall\, i=1,2,\cdots,P,\ j=1,2,\cdots,Q$ in the three alternate summation formulae for the H-function given earlier, we shall easily obtain the corresponding results for Meijer G-function. We omit the details.

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SOME NEW PROPERTIES OF PRIME POWERS

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ABSTRACT. The aim of this paper is to establish new properties of prime powers.

1. Introduction

In 1960 E.Cohen [1] introduced the idea of a positive integer m being semiprime to a positive integer n. He defined $(m, n)^*$ to be the greatest divisor of m which is a unitary divisor of n. A divisor $d > 0$ of the positive integer n is called unitary if $n = d\delta$ and $(d, \delta) = 1$. In case $(m, n)^* = 1,\ m$ is said to be semiprime to n. It is obvious that a number n is a prime power if and only if every number less than n is semi prime to n.

In this paper, the prime number theorem and Dirichlet's theorem on primes in arithmetical progression are extended to prime powers. This paper also contains asymptotic relations involving sums and products of prime powers.

Notation: In this paper $p_1, p_2, \ldots$ represent primes and $q_1, q_2, \ldots$ represent prime powers and α, β, k, m, n, s, x denote positive integers. p_k denotes the k^{th} prime and q_k denotes the k^{th} prime power.

2. Main Theorems

Theorem 2.1. *In any arithmetical progression $h + mk$, where $(h, k) = 1$ and m runs through all the positive integers, there are infinitely many prime powers which are not primes.*

Theorem 2.2. *If $\Pi^*(x)$ denotes the number of prime powers less than or equal to x, then $\Pi^*(x) \sim x/logx$.*

Note: This theorem is an extension of the well known Prime Number Theorem to prime powers.
In the statement of the theorems 2.3 to 2.5, x could be any real number and the sums and product are over all those prime powers q that do not exceed x.

Theorem 2.3.

$$\sum_{q \le x} 1/q = \log\log\ x + B + O(1/log\ x),$$

where B is a constant.

Theorem 2.4.

$$\sum_{q \le x} \log\ q/q = \log\ x + O(1).$$

Theorem 2.5.

$$\prod_{q \le x} (1 - 1/q) = e^{-c}/\ \log\ x + O(1/log^2\ x),$$

where c is a constant.

Remark: Every positive integer can be expressed as the product of prime powers which is unique apart from the order of the prime powers if and only if the number of prime powers in the representation is equal to the number of distinct prime factors.

Proof of Theorem 2.1 In the arithmetical progression $h, h+k, h+2k, \ldots h{+}mk, \ldots$ with $(h, k) = 1$, there are infinitely many primes by Dirichlet's theorem. Let these primes be $h+n_1k, h+n_2k, h+n_3k, \ldots h+n_rk, \ldots$ Write $h + n_1k = p$.
By Euler's theorem, $h^{\phi(k)} \equiv 1(\text{mod } k)$.
Therefore, $h^{\phi(k)+1} \equiv h(\text{mod } k)$.
Now,

$$p \equiv h(\text{mod } k). \tag{1}$$

$$\therefore\quad p^{s\phi(k)+1} \equiv h^{s\phi(k)+1}(\text{mod } k). \tag{2}$$

From (1) and (2) we have

$$p^{s\phi(k)+1} \equiv h(\text{mod } k).$$

Therefore, $p^{s\phi(k)+1}$ belongs to the arithmetical progression and is a prime power and is not a prime. As s runs through all the positive integers we have in the given arithmetical progression infinitely many prime powers, which are not primes. This completes the proof of theorem 2.1.

Proof of Theorem 2.2

$$\Pi^*(x) = \sum_{p^k \le x} 1.$$

$$\therefore \quad \Pi^*(x) = \sum_{p \le x} 1 + \sum_{p \le x^{1/2}} 1 \cdots + \sum_{p \le x^{1/m}} 1.$$

where $m = [\log x/\log 2]$.

$$\therefore \quad \Pi^*(x) = \Pi(x) + \Pi(x^{1/2}) + \Pi(x^{1/3}) + \cdots + \Pi(x^{1/m}).$$

We know that $\Pi(x) < C(x/\log x)$, for some constant C and for all $x \ge 2$.

$$\therefore \quad \sum_{r=2}^{m} \Pi(x^{1/r}) < \sum_{r=2}^{m} r x^{1/r}/\log x.$$

It can be easily seen that the function $yx^{1/y}/\log x$ of the independent variable y is monotonically decreasing in $[2, \log x]$ and monotonically increasing in $(\log x, \infty)$. Hence, the terms $rx^{1/r}/\log x$ are monotonically decreasing for $r = 2, 3, \ldots [\log x]$, and they are monotonically increasing for $r = [\log x] + 1$ to $[\log x/\log 2]$.

Now,

$$yx^{1/y}/\log x = 2\ x^{1/2}/\log x, \quad \text{for } y = 2.$$

And,

$$yx^{1/y}/\log x = 2/\log 2, \tag{A}$$

for

$$y = \log x/\log 2. \tag{B}$$

The result of (A) and (B) is proved as follows :

$y = [\log x/\log 2]$

$\Longrightarrow \log x = y \log 2 = [\log x/\log 2] \log 2.$

$\Longrightarrow \log x = \log x$

$\Longrightarrow x = e^{\log x}$

Putting the above value of x and the value of y from (B) in the R.H.S. of (A) we get

$(\log x/\log 2)((e^{\log x})^{\log 2/\log x}/\log x)$

$= (\log x/\log 2)(e^{\log 2}/\log x)$

$= e^{\log 2}/\log 2 = 2/\log 2 =$ R.H.S. of (A). (Proved).

It can easily be verified that $(2x^{1/2}/\log x) > 2/\log 2$ if $x > e^2$. Hence each term in

$$\sum_{r=2}^{m} r.x^{1/r}/\log x$$

is less than or equal to the first term in the sum. The number of terms in this summation is equal to $[\log x/\log 2] - 1$.
Also, $[\log x/\log 2] - 1 < \log x/\log 2$.

$$\begin{aligned}
\therefore \sum_{r=2}^{m} \Pi x^{1/r} &< \Pi x^{1/2}\log x/\log 2 \\
&< (2Cx^{1/2}/\log x)(\log x/\log 2). \\
\therefore \sum_{r=2}^{m} \Pi x^{1/r} &= \mathrm{O}(x^{1/2})\log x/\log 2 \\
&= \mathrm{o}(x/\log x).
\end{aligned}$$

Hence, $\Pi^*(x) = \Pi(x) + \mathrm{o}(x/\log x)$.
So, $\Pi(x) \sim x/\log x$.
Hence, $\Pi^*(x) = \Pi(x) + \mathrm{o}(\Pi(x))$.
Therefore, $\Pi^*(x) \sim \Pi(x)$.
That is $\Pi^*(x) \sim x/\log x$.
Therefore, the proof of theorem 2.2 is complete.

Remark: We know that$p_n \sim n\log n$, where p_n is the n^{th} prime. This has been proved by using the result $\Pi(x) \sim x/\log x$. Since $\Pi^*(x) \sim x/\log x$, in a similar manner it can be deduced that $q_n \sim n\log n$, where q_n is the n^{th} prime power.

Proof of Theorem 2.3

$$\sum_{q\le x} 1/q = \sum_{p^k\le x} 1/p^k$$

where, $1 \le k \le m = [\log x/\log p]$

$$= \sum_{p\le x} 1/p \; + \sum_{p^k\le x, 2\le k\le m} 1/p^k \tag{3}$$

It is well known that,

$$\sum_{p\le x} 1/p = \log\log x + B_1 + \mathrm{O}(1/\log x) \tag{4}$$

where B_1 is a constant.

Now $1/p(p-1) = p^{-2} + p^{-3} + p^{-4} + \ldots \text{to } \infty$.
It is well known that

$$\sum_p 1/p(p-1)$$

is convergent. Let its sum be equal to a constant B_2. Write

$$\begin{aligned} S_1 &= \sum_{p \le x^{1/2}} (p^{-2} + p^{-3} + p^{-4} + \ldots p^{-m}), \\ S_2 &= \sum_{p \le x^{1/2}} (p^{-(m+1)} + p^{-(m+2)} + \ldots \infty), \\ S_3 &= \sum_{x^{1/2} < p \le x} (p^{-2} + p^{-3} + p^{-4} + \ldots \infty), \\ S_4 &= \sum_{p > x} (p^{-2} + p^{-3} + p^{-4} + \ldots \infty), \end{aligned}$$

We have

$$\begin{aligned} S_2 &= \sum_{p \le x^{1/2}} p/p^{m+1}(p-1) \\ &\le \sum_{p \le x^{1/2}} 1/p^{m+1}. \end{aligned}$$

Since, $m = [\log x/\log p]$,
we know that $p^{m+1} > x^{1/2}$. Hence, we have $1/p^{m+1} < 1/x^{1/2}$.
$\therefore \quad S_2 < 2\Pi(x^{1/2})1/x^{1/2}$
and $\Pi(x^{1/2}) = \mathrm{O}(x^{1/2}\log x^{1/2})$.
Hence,

$$S_2 = \mathrm{O}(1/\log x) \tag{5}$$

Now

$$\begin{aligned} S_3 &= \sum_{x^{1/2} < p \le x} (p^{-2} + p^{-3} + p^{-4} + \ldots \infty), \\ &= \sum_{x^{1/2} < p \le x} 1/p(p-1) \\ &< \Pi(x)/x^{1/2}(x^{1/2} - 1). \end{aligned}$$

Hence,

$$\begin{aligned} S_3 &= \mathrm{O}(x/\log x/x^{1/2}(x^{1/2}-1)) \\ &= \mathrm{O}(1/\log x). \end{aligned} \tag{6}$$

Again,

$$\begin{aligned} S_4 &= \sum_{p>x} 1/p(p-1) \\ &< \sum_{n=[x]-1}^{\infty} 1/n(n+1) \\ &= \mathrm{O}(1/\ x). \end{aligned} \quad (7)$$

We see that

$$\sum_{p} 1/p(p-1) = S_1 + S_2 + S_3 + S_4.$$

Hence by using (5), (6) and (7) we have

$$B_2 = S_1 + \mathrm{O}(1/\log x) \quad (8)$$

But

$$S_1 = \sum_{p^k \le x,\ 2\le k\le m} 1/p^k.$$

Hence, from (8) we have

$$S_1 = B_2 + \mathrm{O}(1/\log x).$$

Therefore, from (3)and (4), we have

$$\sum_{q\le x} 1/q = \log\log x + B_1 + \mathrm{O}(1/\log x) + B_2 + \mathrm{O}(1/\log x)$$

$$= \log\log x + B + \mathrm{O}(1/\log x)$$

where B is a constant equal to $B_1 + B_2$.
Hence, theorem 2.3 is proved.

Proof of Theorem 2.4

$$\sum_{q\le x} \log q/q = \sum_{p^m\le x} m \log p/p^m$$

$$= \sum_{p\le x} \log p \sum_{m=1}^{\log x/\log p} mp^{-m}.$$

It is obvious that

$$p^{-1} \le \sum_{m=1}^{\log x/\log p} mp^{-m} \le \sum_{m=1}^{\infty} mp^{-m}.$$

Therefore,

$$p^{-1}\log p \ \le \log p \sum_{m=1}^{\log x/\log p} mp^{-m} \le \log p \sum_{m=1}^{\infty} mp^{-m}$$
$$= (\log p)(p/(p-1)^2)$$
$$= \log p/p + (\log p/p)((2p-1)/(p-1)^2).$$

Therefore,

$$\sum_{p\le x}\log p/p \le \sum_{q\le x}\log q/q$$
$$< \sum_{p\le x}\log p/p + \sum_{p\le x}(\log p/p)((2p-1)/(p-1)^2) \tag{9}$$

It is well known that

$$\sum_{p\le x}\log p/p = \log x + \mathrm{O}(1) \tag{10}$$

We know that $\log p/p < 2/p^{1/2}$.

$$\therefore \ \sum_{p\le x}(\log p/p)((2p-1)/(p-1)^2) < \sum_{p\le x}(2/p^{1/2})((2p-1)/(p-1)^2) = \mathrm{O}(1)$$

Therefore, from (9) and (10), we have

$$\sum_{q\le x}(\log q/q) \le \sum_{p\le x}(\log p/p) + \mathrm{O}(1) = \log x + \mathrm{O}(1).$$

This completes the proof of theorem 2.4.

Proof of Theorem 2.5

Let

$$Q(x) = \prod_{q\le x}(1-q^{-1}).$$

Hence,

$$\log Q(x) = \sum_{k=1}^{\prod(x)} \sum_{m=1}^{[\log x/\log p]} \log(1-p_k^{-m})$$

Now,

$$-\log(1-p_k^{-m}) - p_k^{-m} = 1/2p_k^{-2m} + 1/3p_k^{-3m} + \cdots\infty.$$

The series on the right converges and its sum is $< 1/2p_k^m(p_k^m-1)$.
Therefore,

$$0 < -\log(1-p_k^{-m}) - p_k^{-m} < 1/2p_k^m(p_k^m-1).$$

Now the terms of the series

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} 1/p_k^m(p_k^m - 1)$$

are all different and they are terms of the convergent series

$$\sum_{n=1}^{\infty} 1/n(n-1).$$

Hence the series is convergent.

$$\therefore \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}[-\log(1-p_k^{-m}) - p_k^{-m}]$$

is a convergent series and let its sum be denoted by A.
For convenience, let us write $-\log(1-p_k^{-m}) - p_k^{-m} = f(k,m)$

$$\therefore \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} f(k,m) = A$$

Now,

$$A = \sum f(k,m) \quad [k, m \text{ run through those values for which } p_k^m \leq x]$$

$$+\sum f(k,m) \quad [k, m \text{ run through those values for which } p_k^m > x]$$

Therefore,

$$A - \sum_{p_k^m \leq x} f(k,m) = \sum_{p_k^m > x} f(k,m) < \sum_{p_k^m > x} 1/p_k^m(p_k^m - 1)$$

$$< \sum_{n \geq x} 1/n(n-1)$$

$$= \mathrm{O}(1/x).$$

Hence,

$$A - \sum_{p_k^m \leq x} f(k,m) = \mathrm{O}(1/x).$$

That is,

$$\sum_{p_k^m \leq x} f(k,m) = A + \mathrm{O}(1/x).$$

Therefore,

$$\sum_{p_k^m \leq x} -[\log(1-p_k^{-m}) + p_k^{-m}] = A + \mathrm{O}(1/x).$$

Hence

$$-\sum_{p_k^m \leq x} [\log(1-p_k^{-m})] = \sum_{p_k^m \leq x} p_k^{-m} + A + \mathrm{O}(1/x).$$

By theorem 2.3, we have

$$\sum_{p_k^m \le x} p_k^{-m} = \log\log x + B + \mathrm{O}(1/\log x)$$

Therefore,

$$-\sum_{p_k^m \le x} [\log(1 - p_k^{-m})] = \log\log x + B + \mathrm{O}(1/\log x) + A + \mathrm{O}(1/x).$$

$$= \log\log x + C + \mathrm{O}(1/\log x).$$

where C is the constant $A + B$.
Therefore,

$$\log \prod_{p_k^m \le x} (1 - p_k^{-m}) = -\log\log x - C + \mathrm{O}(1/\log x)$$

That is,

$$\prod_{p_k^m \le x} (1 - p_k^{-m}) = e^{-\log\log x} e^{-C} e^{\mathrm{O}(1/\log x)}$$

$$= (e^{-C}/\log x) e^{\mathrm{O}(1/\log x)}$$

We have, if $0 < u < 1$,

$$e^u = (1 + \mathrm{O}(u)).$$

Therefore,

$$\prod_{p_k^m \le x} (1 - p_k^{-m}) = (e^{-C} e^{\mathrm{O}(1/\log x)})(1 + \mathrm{O}(1/\log x))$$

$$= (e^{-C}/\log x) + \mathrm{O}(1/\log^2 x).$$

This completes the proof of theorem 2.5.

Remark: The corresponding result for primes is well known and is stated below:

$$\prod_{p \le x} (1 - 1/p) = e^{-A}/\log x + \mathrm{O}(1/\log^2 x).$$

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THE NOWHERE-DIFFERENTIABLE CONTINUOUS FUNCTIONS OF WEIERSTRASS AND VAN DER WAERDEN

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Abstract. In this article is presented an exposition of the nowhere-differentiable continuous functions of Weierstrass and Van der Waerden with what is felt well-motivated and desirable versions of the proofs of nowhere-differentiability.

1. Introduction

Discovering a function $f : \mathbb{R} \to \mathbb{R}$, *continuous* but nowhere differentiable, Weierstrass established the existence of such a function not known before; later, a basically simpler alternative g to f was found by Van der Waerden. Well-motivated versions of the proofs of nowhere-differentiability of f and g, felt desirable, are presented here. The proofs consist in showing that, for every point P on the graph of f (resp. g) in $\mathbb{R}^2$, there is a sequence (PP_n) of chords of the graph such that (P_n) converges to P but the sequence (t_n) of slopes of those chords does not converge. Very much in this picture is the *linear* mapping $h \longmapsto \hat{h}$ of each function $h : \mathbb{R} \to \mathbb{R}$ into the function

$$(x, x^{'}) \longmapsto \hat{h}(x, x^{'}) = \frac{h(x^{'}) - h(x)}{x^{'} - x}, \ \ x \in \mathbb{R}, \ \ x^{'} \in \mathbb{R} \setminus \{x\}$$

whose values are the slopes of chords of the graph of h in $\mathbb{R}^2$.

2. For each integer $k \geq 0$, let f_k be the C^{∞}-function

$$x \longmapsto f_k(x) = a^k \cos b^k x \quad (0 < a < 1 < b < +\infty)$$

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on $\mathbb{R}$. The series $\sum_{k=0}^{\infty} f_k^{(p)}$ converges uniformly on $\mathbb{R}$ as

$$|f_k^{(p)}(x)| \leq (ab^p)^k, \quad x \in \mathbb{R}$$

for $ab^p < 1$. The case $p = 0$ of this fact shows that the function

$$x \longmapsto f(x) = \sum_{k=0}^{\infty} f_k(x) \quad \text{[Weierstrass]}$$

is well-defined on $\mathbb{R}$ and is continuous; the case $p = 1$ shows that, for $b < 1/a$, f is differentiable on $\mathbb{R} : f^{'}(\mathbb{R}) \subset \mathbb{R}$! The nowhere-differentiability of f for an odd integer $b > (1 + \frac{3\pi}{2})/a$ is the pioneering result of Weierstrass extended to all $b \geq 1/a$ by Hardy [**1**]. We recast the proof of Weierstrass's result.

Let $x \in \mathbb{R}$ be given. For $x^{'} \in \mathbb{R} \setminus \{x\}$ we can write

$$\hat{f}(x, x^{'}) = \hat{s}_n(x, x^{'}) + \hat{r}_n(x, x^{'}), \quad s_n = \sum_{k=0}^{n-1} f_k, \quad r_n = \sum_{k=n}^{\infty} f_k.$$

For $b > 1/a$, the mean value theorem at once yields

$$|\hat{s}_n(x, x^{'})| \leq \sum_{k=0}^{n-1} |\hat{f}_k(x, x^{'})| \leq \sum_{k=0}^{n-1} (ab)^k < \frac{(ab)^n}{ab - 1}.$$

For an estimate of $\hat{r}_n(x, x^{'})$, which is intractable for arbitrary $x^{'}$, we seek a properly restricted $x^{'} = x_n$: if b is odd and

$$b^n x_n = q_n \pi \qquad (q_n \in \mathbb{Z}),$$

$b^k x_n$ $(k \geq n)$ is an odd multiple of $q_n \pi$ and so

$$(-1)^{q_n}[f_k(x_n) - f_k(x)] = a^k - f_k(x_n - x) \geq 0, \quad k \geq n;$$

if, to secure $f_n(x_n - x) \leq 0$, we require also that

$$b^n(x_n - x) \in ((2\epsilon_n - 1)\frac{\pi}{2}, (2\epsilon_n + 1)\frac{\pi}{2}] \ \ (\epsilon_n = \pm 1),$$

x_n gets well defined, $b^n x_n$ being the unique integral multiple $q_n \pi$ of π contained in the interval

$$(b^n x + (2\epsilon_n - 1)\frac{\pi}{2}, b^n x + (2\epsilon_n + 1)\frac{\pi}{2}]$$

of length π, and we have the desired x_n for odd $b > 1$ with

$$(-1)^{q_n}[r_n(x_n) - r_n(x)] \geq a^n, \quad \frac{\pi/2}{b^n} \leq \epsilon_n(x_n - x) \leq \frac{3\pi/2}{b^n},$$

$$|\hat{r}_n(x, x_n)| \geq \frac{(ab)^n}{3\pi/2}.$$

For odd $b > 1/a$, the estimates of $x_n - x$, $\hat{r}_n(x, x_n)$ and $\hat{s}_n(x, x_n)$ hold simultaneously and imply

$$x_n \neq x \quad (n \geq 0), \quad x_n \to x \quad (n \to \infty),$$

$$|\hat{f}(x, x_n)| > (\frac{1}{3\pi/2} - \frac{1}{ab-1})(ab)^n.$$

If $f'(x)$ exists, then the sequence $(\hat{f}(x, x_n)$ converges to $f'(x)$; but $(\hat{f}(x, x_n))$ is unbounded, and so cannot converge, for odd $b > (1 + \frac{3\pi}{2})/a$.

3. For each integer $k \geq 0$, define the function g_k on $\mathbb{R}$ by

$$g_k(x) = |x|, \quad x \in [-\omega_k, \omega_k)$$

and $2\omega_k$-periodicity; to meet the requirements

(a) $\sum_0^\infty \omega_k < +\infty$, (b) $\omega_k : \omega_n \in \mathbb{N}$ $(k \leq n)$, (c) $\frac{1}{2}\,\omega_n : 2\omega_k \in \mathbb{N}$ $(k > n)$

arising later, the period of g_k is restricted to its simplest choice

$$2\omega_k = \frac{1}{4^k}$$

decided by (c) which covers (b) obviously and (a) by $\omega_k \leq \frac{1}{4}\omega_{k-1}$. Linear in $[-\omega_k, 0)$ with slope -1 and in $[0, \omega_k)$ with slope $+1$, g_k is linear, by its $2\omega_k$-periodicity, in the interval

$$[q\omega_k, (q+1)\omega_k), \quad q \in \mathbb{Z}$$

with slope $(-1)^q$. The uniform convergence of $\sum_0^\infty g_k$ on $\mathbb{R}$ by

$$0 \leq g_k(x) \leq \omega_k, \quad x \in \mathbb{R}$$

with (a) shows that the function

$$x \longmapsto g(x) = \sum_0^\infty g_k(x) \quad \text{[Van der Waerden]}$$

is well defined on $\mathbb{R}$, and, by the continuity of the g_k, continuous. We prove that g is differentiable nowhere (see [**2**]).

Let $x \in \mathbb{R}$ be given. For each integer $n \geq 0$, the intervals

$$[a\omega_n, b\omega_n), \quad a \leq x/\omega_n < b \ \ (a, b \in \mathbb{Z})$$

enclose x, have the intersection

$$I_n = [q_n\omega_n, (q_n+1)\omega_n), \quad q_n = [x/\omega_n]$$

and by (b), include I_k $(k \leq n)$, so that x and *each* $x^{'} \in I_n \setminus \{x\}$ are points of I_k in which g_k is linear with slope $(-1)^{q_k}$:

$$\hat{g}_k(x, x^{'}) = (-1)^{q_k}, \quad k \leq n.$$

Thanks to the $2\omega_k$-periodicity of g_k, there exists $x^{'} = x_n$ with

$$\hat{g}_k(x, x_n) = 0, \quad (k > n).$$

In fact, this means $g_k(x_n) = g_k(x)$ and is therefore ensured, by (c), if x_n can be chosen such that

$$x_n \in I_n, \quad (\omega_n >) \ |x_n - x| = \frac{1}{2}\omega_n.$$

These conditions actually *define* x_n as the image of x under the translation that maps the half $[u_n, v_n)$ of I_n containing x onto the other half $[u_n^{'}, v_n^{'})$, and we have

$$x_n \neq x \ \ (n \geq 0), \quad x_n \to x \ \ (n \to \infty),$$
$$\hat{g}(x, x_n) = \sum_{k=0}^{n} (-1)^{q_k}.$$

If $g^{'}(x)$ exists, then $(\hat{g}(x, x_n))$ converges to $g^{'}(x)$; but $(\hat{g}(x, x_n))$ cannot converge as $|\hat{g}(x, x_n) - \hat{g}(x, x_{n-1})| = 1$.

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ON THE EXISTENCE OF A UNIQUE ALMOST PERIODIC SOLUTION OF A NON-LINEAR ABSTRACT DIFFERENTIAL EQUATION

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Abstract. Suppose B is an operator on a Banach space X into itself, R is the real line, and $f : X \times R \longrightarrow X$is a function. We demonstrate the existence of a unique almost periodic solution of the differential equation $u'(t) = Bu(t) + f(u(t), t)$, with B and f satisfying certain conditions.

1. Introduction

Suppose X is a Banach space and R is the real line. A continuous function $g : R \longrightarrow X$ is said to be (Bochner or strongly) almost periodic if, given $\epsilon > 0$, there is a positive real number $r = r(\epsilon)$ such that any interval in R of length r contains at least one point τ for which

$$\sup_{t\in R} \|g(t+\tau) - g(t)\| \leq \epsilon. \quad (1.1)$$

A continuous function $f(x,t), (x,t) \in X \times R \longrightarrow X$ is said to be almost periodic in t uniformly for x in compact subsets K of X if, given $\epsilon > 0$, there exists a positive real number $r = r(\epsilon, K)$ such that any interval in R of length r contains at least one point τ for which

$$\sup_{t\in R} \|f(x, t+\tau) - f(x,t)\| \leq \epsilon \quad \text{for all} \quad x \in K. \quad (1.2)$$

Assume that a function $f(x,t), (x,t) \in X \times R \longrightarrow X$ is continuous in t for any fixed $x \in X$. We say that the function $f(x,t)$ satisfies a Lipschitz

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condition if

$$\|f(x_1,t)-f(x_2,t)\| \le N\,\|x_1-x_2\| \quad \text{for all } x_1,x_2\in X \text{ and all } t\in R, \tag{1.3}$$

where $N>0$ is called a Lipschitz constant. We note that, as a consequence of Lipschitz condition, the function $f(\varphi(t),t), t\in R \longrightarrow X$ is continuous if $\varphi: R\longrightarrow X$ is continuous.
Our result is as follows.

Theorem 1. *Suppose that a continuous function $f(x,t),(x,t)\in X\times R\longrightarrow X$ is almost periodic in t uniformly for x in compact subsets K of X, and satisfies a Lipschitz condition with a sufficiently small Lipschitz constant N. Further, suppose that B is a bounded linear operator on X into itself such that*

$$\|\exp(tB)\| \le M\exp(\beta t) \text{ for some } M>0, \text{ some } \beta<0 \text{ and all } t\ge 0. \tag{1.4}$$

Then the differential equation

$$u'(t)=Bu(t)+f(u(t),t) \quad \text{on } R \tag{1.5}$$

has a unique almost periodic solution.

2. Statement of a lemma

Lemma 1. *If $h: R\to X$ is an almost periodic function, then the differential equation*

$$u'(t)=Bu(t)+h(t) \quad \text{on } R \tag{2.1}$$

has a unique almost periodic solution, where B is a bounded linear operator on X into itself, satisfying the condition (1.4).

Proof. **(a) Uniqueness:** If u_1 and u_2 are two bounded solutions of the differential equation (2.1), then $v=u_1-u_2$ is a bounded solution of the differential equation

$$v'(t)=Bv(t) \quad \text{on } R. \tag{2.2}$$

So, for any $t_0\in R$, we have the representation

$$v(t)=\exp((t-t_0)B)v(t_0) \quad \text{on } R. \tag{2.3}$$

Now take a sequence $\{t_n\}_{n\ge1}$ with $t_n\longrightarrow -\infty$ as $n\longrightarrow\infty$. Let

$$\|v(t)\|\le C_1 \quad \text{for all } t\in R. \tag{2.4}$$

Then, for an arbitrary but fixed $t'\in R$, we have $t'\ge t_n$ for some sufficiently large n, and so, by (1.4),

$$\|v(t')\| \le \|\exp((t'-t_n)B)\|.\|v(t_n)\| \tag{2.5}$$

$$\leq M \exp(\beta(t' - t_n)).C_1 \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, $v(t') = \theta =$ the null element of X, and hence $u_1(t') = u_2(t')$.

(b) Existence: We define a function u_k on R into X by

$$u_k(t) = \int_k^{k+1} \exp(sB)h(t-s)ds \text{ for } k = 0, 1, 2, \cdots . \tag{2.6}$$

We set

$$C_2 = \sup_{s \in R} \|h(s)\|. \tag{2.7}$$

Then we have, by (1.4),

$$\begin{aligned} \|u_k(t)\| &\leq \int_k^{k+1} \|\exp(sB)\|.\|h(t-s)\|ds \qquad (2.8) \\ &\leq \int_k^{k+1} M \exp(\beta s)\|h(t-s)\|ds \\ &\leq M \exp(\beta k) \int_{t-k-1}^{t-k} \|h(\zeta)\|d\zeta \\ &\leq C_2 M \exp(\beta k) \quad \text{on} \quad R. \end{aligned}$$

Since $\beta < 0$, we obtain

$$\sum_{k=0}^{\infty} \exp(\beta k) = 1/[1 - \exp(\beta)].$$

Therefore, the series $\sum_{k=0}^{\infty} u_k(t)$ converges uniformly on R. We write

$$u(t) = \sum_{k=0}^{\infty} u_k(t) \quad \text{on } R. \tag{2.9}$$

Given $\epsilon > 0$, if τ is an $(\epsilon/M)-$ almost period of h, we have

$$\begin{aligned} \|u_k(t+\tau) - u_k(t)\| &= \|\int_k^{k+1} \exp(sB)[h(t+\tau-s) - h(t-s)]ds\| \qquad (2.10) \\ &\leq \int_k^{k+1} M \ \exp(\beta s)\|h(t+\tau-s) - h(t-s)\| \ ds \\ &\leq M \ \exp(\beta k) \int_{t-k-1}^{t-k} \|h(\tau+\zeta) - h(\zeta)\| \ d\zeta \\ &\leq M \ \exp(\beta k). \ \epsilon/M \leq \epsilon \ \text{ on } \ R. \end{aligned}$$

Similarly, we can show that u_k is continuous on R (by the uniform continuity of h on R). So u_k is almost periodic from R to X for $k = 0, 1, 2, \cdots$. Hence, u is almost periodic from R to X.

Further, by setting $\zeta = t - s$ in (2.6), we obtain

$$u_k(t) = \int_{t-k-1}^{t-k} \exp((t-\zeta)B)h(\zeta)d\zeta \quad \text{on} \quad R, \tag{2.11}$$

and

$$u_k'(t) = Bu_k(t) + [\ \exp\ (kB)h(t-k) - \ \exp((k+1)B)h(t-k-1)]. \tag{2.12}$$

Then, by (1.4), (2.7) and (2.8), we have

$$\begin{aligned} \|u_k'(t)\| &\le \|B\|.\|u_k(t)\| + \|\ \exp\ (kB)\|.\|h(t-k)\|\ + \\ &\quad \|\exp((k+1)B)\|.\|h(t-k-1)\| \\ &\le \|B\|.C_2\ M\ \exp(\beta k) + M\ \exp\ (\beta k).C_2 + M\ \exp(\beta(k+1)).C_2 \\ &\le (\|\ B\| + 2)C_2\ M\ \exp\ (\beta k) \quad \text{on} \quad R. \end{aligned} \tag{2.13}$$

Consequently, the series $\sum_{k=0}^{\infty} u_k'(t)$ converges uniformly on R, and we have

$$u'(t) = \sum_{k=0}^{\infty} u_k'(t) \quad \text{on } R. \tag{2.14}$$

Furthermore, from (2.12), we obtain

$$s_n'(t) = Bs_n(t) + h(t) - \exp((n+1)B)h(t-n-1) \quad \text{on } R, \tag{2.15}$$

where

$$s_n(t) = \sum_{k=0}^{n} u_k(t). \tag{2.16}$$

Now, for all $t \in R$, we have

$$\|\exp((n+1)B)h(t-n-1)\| \le \|\exp((n+1)B)\|.\|h(t-n-1)\| \tag{2.17}$$

$$\le M\ \exp(\beta(n+1)).C_2 \to 0 \quad \text{as} \quad n \to \infty.$$

So, from (2.15), it follows that $\quad u'(t) = Bu(t) + h(t)$ on R. $\square$

3. **Proof of Theorem 1**

Let $AP(X) = \{\varphi : \varphi$ is an almost periodic function from R to $X\}$, and let

$$\|\varphi\| = \|\varphi\|_{AP(X)} = \sup_{t\in R} \|\varphi(t)\|. \tag{3.1}$$

Then $AP(X)$ with this uniform norm is a Banach space.

Further, for each $\varphi \in AP(X)$, the composite function $f(\varphi(t), t)$ is almost periodic from R to X (see the Appendix of Zaidman [3]). Now consider the mapping $T : AP(X) \to AP(X)$ given by (our Lemma 1)

$$T(\varphi) = u, \tag{3.2}$$

where u is the unique almost periodic solution of the differential equation

$$u'(t) = Bu(t) + f(\varphi(t), t) \quad \text{on} \quad R. \tag{3.3}$$

We proceed to show that T is a contraction on $AP(X)$. For $\varphi_1, \varphi_2 \in AP(X)$, let

$$u^i = T(\varphi_i), \quad i = 1, 2.$$

Then, by (2.11), we have

$$\begin{aligned}
\|u^1(t) - u^2(t)\| &= \|\sum_{k=0}^{\infty}\int_{t-k-1}^{t-k} \exp((t-\zeta)B)[f(\varphi_1(\zeta),\zeta) - f(\varphi_2(\zeta),\zeta)]d\zeta\| \quad (3.4)\\
&\leq \sum_{k=0}^{\infty}\int_{t-k-1}^{t-k} \|\exp((t-\zeta)B)\|.\|f(\varphi_1(\zeta),\zeta) - f(\varphi_2(\zeta),\zeta)\|d\zeta\\
&\leq \sum_{k=0}^{\infty}\int_{t-k-1}^{t-k} M \ \exp\ (\beta(t-\zeta)).N\|\varphi_1(\zeta) - \varphi_2(\zeta)\|d\zeta\\
&\leq MN\|\varphi_1 - \varphi_2\|[\sum_{k=0}^{\infty}\int_{t-k-1}^{t-k} \ exp\ (\beta(t-\zeta))d\zeta]\\
&= MN\|\varphi_1 - \varphi_2\|\{\sum_{k=0}^{\infty}(1/|\beta|)[\ \exp\ (\beta(t-\zeta))]_{t-k-1}^{t-k}\}\\
&= (MN\|\varphi_1 - \varphi_2\|/|\beta|).\{\sum_{k=0}^{\infty}[\ \exp\ (\beta k) - \exp(\beta(k+1))]\}\\
&= MN\|\varphi_1 - \varphi_2\|/|\beta| \quad \text{on} \quad R.
\end{aligned}$$

Consequently,

$$\|T(\varphi_1) - T(\varphi_2)\| \leq (MN/|\beta|).\|\varphi_1 - \varphi_2\|,$$

which is a contraction mapping if $N < |\beta|/M$.

Let $u^* \in AP(X)$ be a (unique) fixed point of T. Then u^* is an almost periodic solution of the differential equation

$$u'(t) = Bu(t) + f(u(t), t) \quad \text{on} \quad R,$$

completing the proof of the theorem. □

Note: If a function f satisfies a Lipschitz condition, then $f/C(C > 0$, a constant) also satisfies a Lipschitz condition. So choosing a sufficiently large C, we can make

$$N/C < |\beta|/M.$$

Now we may replace f by f/C in the Theorem 1 of the paper.

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EXTENDING THE LAPLACE TRANSFORM

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ABSTRACT. A space $\mathcal{P}$ is constructed algebraically with the aid of convolution. The Laplace transform is defined on elements of $\mathcal{P}$ in a natural way. An operational calculus is developed. The space $\mathcal{P}$ is shown to be isomorphic to the space of transformable distributions. Some illustrative examples are given.

1. **Introduction**

Engineers, physicists, and scientists find distribution theory useful for solving many physical problems. However, constructing spaces of distributions sometimes requires knowledge of functional analysis. A familiarity with some notions as topology or convergence structures, dual spaces, and linear functionals is needed [1, 2, 3, 4, 5].

In this note, we present a construction of a space $\mathcal{P}$ for which the classical Laplace transform can be extended. The space $\mathcal{P}$ contains the Dirac delta function and all of its derivatives. In fact, the space $\mathcal{P}$ is isomorphic to the space of transformable distributions [4].

Unlike the construction of Schwartz distributions, construction of the space $\mathcal{P}$ only requires the knowledge of elementary calculus and classical Laplace transform theory as presented in the text by R. V. Churchill [6], and therefore, may be of interest to university professors, engineers, and scientists, as well as mathematicians.

This paper is organized as follows. In Section 2, the space $\mathcal{P}$ is constructed. Operations which are useful in applications are defined for elements of $\mathcal{P}$. For example, convolution, differentiation, multiplication by a

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polynomial, and translation are defined for elements of $\mathcal{P}$. In Section 3, the Laplace transform is defined for each element of $\mathcal{P}$. Then, an operational calculus is developed. Finally, the space $\mathcal{P}$ is shown to be isomorphic to the space of transformable distributions. In the last section, Section 4, some examples are presented.

2. **The Space $\mathcal{P}$**

In this section we construct the space $\mathcal{P}$. Our construction is similar to the one Yosida [7] uses to construct a simplified version of Mikusiński operators [8]. The functions used in our construction differ in two ways from the functions used by Yosida. We use piecewise continuous functions while Yosida uses continuous functions. This is only a minor difference. Also, since our goal is to extend the Laplace transform, we require the functions to have at most exponential growth.

Let $\mathcal{A}$ denote the set of all piecewise continuous functions on $(-\infty, \infty)$ which vanish on $(-\infty, 0)$ and satisfy $|f(t)| \leq Me^{\alpha t}$, for all t (for some $\alpha \geq 0$ and $M \geq 0$).

The convolution of two functions $f, g \in \mathcal{A}$ is defined as

$$(f * g)(t) := \int_0^t f(t-x)g(x)dx \tag{1}$$

Let $f, g, h \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. Then, the convolution product satisfies the following properties.

$$f * g \in \mathcal{A} \tag{2}$$

$$f * g = g * f \tag{3}$$

$$\alpha(f * g) = (\alpha f) * g = f * (\alpha g) \tag{4}$$

$$f * (g * h) = (f * g) * h \tag{5}$$

$$f * (g + h) = f * g + f * h \tag{6}$$

We denote by H the Heaviside function. This is the function from $\mathcal{A}$ such that $H(t) = 1$ for $t \geq 0$. We note that $(H * f)(t) = \int_0^t f(x)dx$ for every $f \in \mathcal{A}$. This implies that the relation $H * f = 0$ yields $f = 0$.

For each $n \in \mathbb{N}$, we denote by H^{*n} the function $H * \ldots * H$ where H is repeated n times.

Let $f, g \in \mathcal{A}$ and $m, n \in \mathbb{N}$. Then, $\frac{f}{H^{*n}}$ and $\frac{g}{H^{*m}}$ are said to be equivalent if $H^{*m} * f = H^{*n} * g$. This is an equivalence relation. In the case $\frac{f}{H^{*n}}$ and $\frac{g}{H^{*m}}$ are equivalent, we write $\frac{f}{H^{*n}} = \frac{g}{H^{*m}}$.

Let $\mathcal{P}$ denote the set of equivalence classes. A typical element of $\mathcal{P}$ will be denoted by $p = \frac{f}{H^{*n}}$.

Addition, multiplication, and scalar multiplication are defined in the natural way, and $\mathcal{P}$ with these operations is an algebra with identity $\delta = \frac{H}{H}$.

$$\frac{f}{H^{*n}} + \frac{g}{H^{*m}} = \frac{H^{*m} * f + H^{*n} * g}{H^{*(n+m)}} \tag{7}$$

$$\frac{f}{H^{*n}} * \frac{g}{H^{*m}} = \frac{f * g}{H^{*(n+m)}} \tag{8}$$

$$\alpha \frac{f}{H^{*n}} = \frac{\alpha f}{H^{*n}}, \text{ where } \alpha \in \mathbb{C} \tag{9}$$

It is straightforward to show that the above operations are well-defined (see [7]). The space $\mathcal{A}$ may be identified with a subspace of $\mathcal{P}$ by identifying $f \in \mathcal{A}$ with $\frac{f*H}{H} \in \mathcal{P}$. Thoughout this note, we will use this identification.

We now define the generalized derivative.

Let $p = \frac{f}{H^{*n}} \in \mathcal{P}$. Then,

$$Dp := \frac{f}{H^{*(n+1)}} \tag{10}$$

It is clear that this is a well-defined operation. That is, if $\frac{f}{H^{*n}} = \frac{g}{H^{*m}}$, then $\frac{f}{H^{*(n+1)}} = \frac{g}{H^{*(m+1)}}$.

For $a \geq 0$, we denote by H_a the function $t \mapsto H(t-a)$. Also, δ_a will denote $\frac{H_a}{H} \in \mathcal{P}$, with $\delta_0 = \delta$.

Theorem 1. *Let $a \geq 0$. If f is a differentiable function such that $f, f' \in \mathcal{A}$ and f' is continuous, then*

$$D(fH_a) = f'H_a + f(a)\delta_a \tag{11}$$

Proof.

$$D(fH_a) = \frac{fH_a * H}{H^{*2}} = \frac{fH_a}{H} \tag{12}$$

$(H * f'H_a)(t) = \int_0^t f'(x)H_a(x)dx = f(t)H_a(t) - f(a)H_a(t), \quad t \in \mathbb{R}.$
This yields

$$\frac{fH_a}{H} = \frac{f'H_a * H}{H} + f(a)\frac{H_a}{H} \tag{13}$$

The conclusion follows by (12) and (13). □

Suppose that f is n-times differentiable such that $f^{(k)} \in \mathcal{A}\,(k = 0, 1, 2, \ldots, n)$ and $f^{(n)}$ is continuous. By repeated use of Theorem 1, we obtain the formula

$$D^n(fH_a) = f^{(n)}H_a + \sum_{k=0}^{n-1} f^{(k)}(a)D^{n-(k+1)}\delta_a \tag{14}$$

$$(n = 1, 2, \ldots)$$

Our next goal is to define multiplication by "t" . We denote by T the function defined by $T(t) = tH(t)$, $t \in \mathbb{R}$.

Let $p = \frac{f}{H^{*n}} \in \mathcal{P}$, where $n \geq 2$. We define $T \bullet p$ as follows.

$$T \bullet p := \frac{-nf}{H^{*(n-1)}} + \frac{Tf}{H^{*n}} \tag{15}$$

We note that every $p \in \mathcal{P}$ can be written in the form $\frac{f}{H^{*n}}$, where $n \geq 2$. Let $p = \frac{f}{H^{*n}} \in \mathcal{P}$. Then, p can be written in the form $p = \frac{f*H}{H^{*(n+1)}}$, where $n + 1 \geq 2$.

The Laplace transform will be used to show that $T \bullet p$ is well-defined. We assume knowledge of the elementary theory of the Laplace transform as presented in [6].

Let $\mathbb{C}$ denote the set of complex numbers. A typical element of $\mathbb{C}$ will be denoted by s. For $f \in \mathcal{A}$ with $|f(t)| \leq Me^{\alpha t}$, the (classical) Laplace transform of f, denoted $\mathcal{L}[f]$, is given by

$$\mathcal{L}[f](s) = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{16}$$

for Re $s > \alpha$.

We now show that $T \bullet p$ is well-defined.

Proposition. *Let* $\frac{f}{H^{*n}} = \frac{g}{H^{*m}}$*, where* $m, n \geq 2$*. Then,*

$$\frac{-nf}{H^{*(n-1)}} + \frac{Tf}{H^{*n}} = \frac{-mg}{H^{*(m-1)}} + \frac{Tg}{H^{*m}} \tag{17}$$

Proof. Since $\frac{f}{H^{*n}} = \frac{g}{H^{*m}}$, we have

$$H^{*m} * f = H^{*n} * g \tag{18}$$

After applying the Laplace transform to both sides of (18), we obtain

$$s^n F(s) = s^m G(s) \tag{19}$$

Now, differentiating both sides of (19) gives

$$\frac{1}{s^m}\left(\frac{-nF(s)}{s} - \frac{d}{ds}F(s)\right) = \frac{1}{s^n}\left(\frac{-mG(s)}{s} - \frac{d}{ds}G(s)\right) \tag{20}$$

which implies

$$H^{*m} * (-nf * H + Tf) = H^{*n} * (-mg * H + Tg) \tag{21}$$

Thus,

$$\frac{-nf}{H^{*(n-1)}} + \frac{Tf}{H^{*n}} = \frac{-mg}{H^{*(m-1)}} + \frac{Tg}{H^{*m}} \tag{22}$$

□

If $f \in \mathcal{A}$, then $T \bullet f$ is ordinary multiplication. That is, $T \bullet f = Tf$ or

$$T \bullet \frac{f * H^{*_2}}{H^{*_2}} = \frac{Tf * H}{H} \tag{23}$$

This follows by using the Laplace transform as in the previous proposition.

Let $\alpha \geq 0$. The shift operator τ_α on $\mathcal{A}$ is given by $(\tau_\alpha f)(t) = f(t - \alpha)$ for $t \in \mathbb{R}$. The shift operator can be extended to $\mathcal{P}$.

Let $p = \frac{f}{H^{*_n}} \in \mathcal{P}$, then

$$\tau_\alpha p := \frac{\tau_\alpha f}{H^{*_n}} \tag{24}$$

By using the fact that $\tau_\alpha(f * g) = (\tau_\alpha f) * g = f * (\tau_\alpha g)$, it is easy to show that the above is a well-defined operation on $\mathcal{P}$.

3. Extending the Laplace Transform

We now extend the Laplace transform to the space $\mathcal{P}$. Let $p = \frac{f}{H^{*_n}} \in \mathcal{P}$. The Laplace transform of p, denoted $\mathcal{L}[p]$, is defined as

$$\mathcal{L}[p](s) := P(s) := s^n F(s) \tag{25}$$

By using the Convolution Theorem for Laplace transforms, i.e. $\mathcal{L}[f*g] = \mathcal{L}[f]\mathcal{L}[g]$, we see that the Laplace transform of each element of $\mathcal{P}$ is well-defined. Moreover, for each $p \in \mathcal{P}$ there exists a half-plane in which $P(s)$ is an analytic function.

By using the classical Laplace transform we obtain a uniqueness theorem.

Theorem 2. *Let $p, q \in \mathcal{P}$ such that $P(s) = Q(s)$, for Re $s > \alpha \geq 0$. Then $p = q$.*

Proof. Let $p = \frac{f}{H^{*n}}$ and $q = \frac{g}{H^{*m}}$ such that $P(s) = Q(s)$, for Re $s > \alpha \geq 0$. Thus, $s^n F(s) = s^m G(s)$, for Re$s > \alpha$. Hence, $\mathcal{L}[H^{*_m} * f](s) = \frac{1}{s^m} F(s) = \frac{1}{s^n} G(s) = \mathcal{L}[H^{*_n} * g](s)$, Re $s > \alpha$. Thus, by the Uniqueness Theorem for the classical Laplace transform we obtain,

$$H^{*_m} * f = H^{*_n} * g,$$

which yields

$$p = \tfrac{f}{H^{*n}} = \tfrac{g}{H^{*m}} = q. \qquad \square$$

As the next theorem shows, many of the standard properties concerning the classical Laplace transform are preserved. Since the proof follows directly from the definitions and the properties of the classical Laplace transform, it is omitted.

Theorem 3. *Let $p, q \in \mathcal{P}$ and $\alpha \in \mathbb{C}$. Then,*

$$\mathcal{L}[p+q] = \mathcal{L}[p] + \mathcal{L}[q] \tag{26}$$

$$\mathcal{L}[\alpha p] = \alpha P(s) \tag{27}$$

$$\mathcal{L}[D^n p] = s^n P(s) \qquad (n = 0, 1, 2, \ldots) \tag{28}$$

$$\mathcal{L}[p * q] = P(s)Q(s) \tag{29}$$

$$\mathcal{L}[T \bullet p] = -\frac{d}{ds}P(s) \tag{30}$$

$$\mathcal{L}[\tau_\alpha p] = e^{-as}P(s) \tag{31}$$

$$\mathcal{L}[\delta_a] = e^{-as} \tag{32}$$

We now give a characterization for the range of the Laplace transform. This characterization shows that the space $\mathcal{P}$ is isomorphic to the space of transformable distributions [4].

Theorem 4. *Let $p \in \mathcal{P}$. Then, there exist $\alpha \geq 0$ and a polynomial φ such that $P(s)$ is an analytic function in the half plane Re $s > \alpha$ and $|P(s)| < \varphi(|s|)$ for Re $s > \alpha$. Conversely, let $\alpha \geq 0$ and φ be a polynomial. If g is an analytic function in Re $s > \alpha$ such that $|g(s)| < \varphi(|s|)$ for Re $s > \alpha$, then there exists a unique $p \in \mathcal{P}$ such that $P(s) = g(s)$ for Re $s > \alpha$.*

Proof. Let $p = \frac{f}{H^{*n}} \in \mathcal{P}$. We may assume that f is differentiable and that f' is piecewise continuous. If not, $f * H$ satisfies these conditions and $p = \frac{f*H}{H^{*(n+1)}}$.

Thus, $\lim_{|s|\to\infty} F(s) = 0$ in some half-plane [6]. Therefore,$|F(s)| < 1$, Re $s > \alpha$ (for some $\alpha \geq 0$).

So, $P(s) = s^n F(s)$ is analytic in the half-plane Re $s > \alpha$ and

$$|P(s)| = |s^n F(s)| < |s|^n, \text{ Re } s > \alpha.$$

For the other direction, suppose that g is analytic in some half-plane Re $s > \alpha \geq 0$ and $|g(s)| < \varphi(|s|)$ for Re $s > \alpha$, for some polynomial φ.

Let $f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\frac{g(s)}{s^{m+2}}ds$, where $\gamma > \alpha$ and m is the degree of the polynomial φ.

Then, $f \in \mathcal{A}$ and $\mathcal{L}[f](s) = \frac{g(s)}{s^{m+2}}$, Re $s > \alpha$ (see [6]).

Now, put $p = \frac{f}{H^{*m+2}}$. Then, $p \in \mathcal{P}$ and

$$P(s) = s^{m+2}F(s) = g(s), \text{Re } s > \alpha.$$

The uniqueness follows from Theorem 2. □

The proof of the previous theorem leads to an inversion formula.

Let $p \in \mathcal{P}$. Then there exist $n \in \mathbb{N}, \alpha \geq 0$, and $M > 0$ such that $|P(s)| < M(1+|s|)^n$ for Re $s > \alpha$. Moreover, $p = \frac{f}{H^{*(n+2)}}$, where

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\frac{P(s)}{s^{n+2}}ds, \quad (\gamma > \alpha) \tag{33}$$

4. Some illustrative examples

Example 1. *Find all $p \in \mathcal{P}$ such that $tD^2p + (3-t)Dp - p = 0$.*
(i.e. $T \bullet D^2p + 3Dp - T \bullet Dp - p = 0$)

After applying the Laplace transform and Theorem 3, we obtain

$$-\frac{d}{ds}\left(s^2P(s)\right) + 3sP(s) + \frac{d}{ds}(sP(s)) - P(s) = 0 \tag{34}$$

Differentiating and collecting terms yields

$$(s - s^2)\frac{d}{ds}P(s) + sP(s) = 0 \tag{35}$$

Solving this first order linear equation yields

$$P(s) = c(1-s), \, c \in \mathbb{C} \tag{36}$$

Thus,

$$p = c(\delta - D\delta) \tag{37}$$

Example 2. *It is found experimentally that a 6 lb weight stretches a certain spring 6 inches. The weight is pulled 4 inches below the equilibrium position and released. At the instant $t = 2\pi$ seconds the mass is struck with a hammer, providing an impulse of 8 lbs. Determine the motion of the mass. That is, we need to solve the initial value problem*

$$y'' + 64y = 8\delta_{2\pi}, \quad y(0) = \frac{1}{3}, \quad y'(0) = 0 \tag{38}$$

By using formula (14), with $n = 2$ and $a = 0$, we obtain

$$D^2y + 64y = \frac{1}{3}D\delta + 8\delta_{2\pi} \tag{39}$$

Applying the Laplace transform gives

$$s^2Y(s) + 64Y(s) = \frac{1}{3}s + 8e^{-2\pi s} \tag{40}$$

Thus,

$$Y(s) = \frac{1}{3}\frac{s}{s^2+64} + e^{-2\pi s}\frac{8}{s^2+64} \tag{41}$$

The solution is obtained by using Theorem 3 and a standard table of Laplace transforms [9].

$$y(t) = \frac{1}{3}\cos 8t + H_{2\pi}(t)\sin 8(t-2\pi) = \frac{1}{3}\cos 8t + H_{2\pi}(t)\sin 8t \tag{42}$$

Example 3. *Consider the RLC circuit with $R = 11$ ohms, $L = 1$ henry, $C = .1$ farads, and a battery supplying $E = 180$ volts. Initially, there is no current in the circuit and no charge on the capacitor. At time $t = 0$ the switch is closed and left closed for one second. At time $t = 1$ it is opened and left open. Find the resulting current in the circuit at any time.*

Using Kirchhoff's law

$$i' + 11i + 10q = 180(1 - H_1) \tag{43}$$

where i is the current, q is the charge on the capacitor, and $q' = i$. Taking the generalized derivative of both sides of (43) and using Theorem 1 yields

$$Di' + 11Di + 10i = 180D(1 - H_1) \tag{44}$$

After applying the Laplace transform and using Theorem 3 we obtain

$$s(sI(s) - i(0)) + 11sI(s) + 10I(s) = 180s(\frac{1}{s} - \frac{e^{-s}}{s}) \tag{45}$$

Noting that $i(0) = 0$ and solving for I yields

$$I(s) = 20\left(\frac{1}{s+1} - \frac{1}{s+10}\right)(1 - e^{-s}) \tag{46}$$

The solution is obtained by using Theorem 3 and a standard table of Laplace transforms [9].

$$i(t) = 20(e^{-t} - e^{-10t}) - 20H_1(t)\left[e^{-(t-1)} - e^{-10(t-1)}\right] \tag{47}$$

Example 4. *([10], p. 323) Consider a semi-infinite string fixed at the end $x = 0$. The string is initially at rest. Let there be an external force $f(x,t) = -f_0\delta_{x/v}$ acting on the string. This is a concentrated force f_0 acting at the point $x = vt$.*

The motion of the string is governed by

$$u_{tt} = c^2 u_{xx} - f_0\delta_{x/v} \tag{48}$$

$$u(x,0) = 0 \tag{49}$$

$$u_t(x,0) = 0 \tag{50}$$

$$u(0,t) = 0 \tag{51}$$

Let $U(x, s)$ be the Laplace transform of $u(x, t)$. Transforming the wave equation and using the initial conditions, we obtain

$$U_{xx} - \frac{s^2}{c^2}U = \frac{f_0}{c^2}e^{-xs/v} \tag{52}$$

The solution of this equation is

$$U(x,s) = Ae^{sx/c} + Be^{-sx/c} + \begin{cases} \dfrac{f_0 v^2 e^{-sx/v}}{(c^2-v^2)s^2} & \text{for } v \neq c \\ \dfrac{-f_0 x e^{-sx/v}}{2cs} & \text{for } v = c \end{cases} \tag{53}$$

Theorem 4 implies that $A = 0$.
Application of the condition $U(0, s) = 0$ yields

$$B = \begin{cases} \dfrac{-f_0 v^2}{(c^2-v^2)s^2} & \text{for } v \neq c \\ 0 & \text{for } v = c \end{cases} \tag{54}$$

Hence the transform is given by

$$U(x,s) = \begin{cases} \dfrac{f_0 v^2 (e^{-sx/v} - e^{-sx/c})}{(c^2-v^2)s^2} & \text{for } v \neq c \\ \dfrac{-f_0 x e^{-sx/c}}{2cs} & \text{for } v = c \end{cases} \tag{55}$$

The solution is obtained by using Theorem 3 and a standard table of Laplace transforms [9].

$$u(x,t) = \begin{cases} \dfrac{f_0 v^2}{(c^2-v^2)}\left[H_{x/v}(t)(t-\tfrac{x}{v}) - H_{x/c}(t)(t-\tfrac{x}{c})\right] & \text{for } v \neq c \\ \dfrac{-f_0 x}{2c}H_{x/c}(t) & \text{for } v = c \end{cases} \tag{56}$$

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HANKEL TRANSFORM ON A GEVREY SPACE

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ABSTRACT. In this paper we give a completion for the distributional theory of Hankel transformation developed in [6]and [8]. The space H_ω generalizing the Altenburg space H is defined. Some properties of this space are studied. It is shown that the Hankel transformation h_μ is an automorphism of H_ω. The generalized Hankel transform of Gevrey spaces is defined and it is found that Hankel transform is an automorphism of H'_ω. Product on H_ω, Hankel translation and Hankel convolution on H_ω are investigated.

1. Introduction

The space $L^p_\mu, 1 \leq p < \infty$, consists of all those measurable functions ϕ on $I = (0, \infty)$. Such that

$$\|\phi\|_{L^p_\mu} = (\int_0^\infty |\phi(x)|^p d\mu(x))^{\frac{1}{p}} < \infty \tag{1}$$

where as $L^\infty_\mu(I)$ denotes the space of those functions ϕ for which

$$\|\phi\|_{L^\infty_\mu} = \underset{x\varepsilon I}{ess.l.u.b}\ |\phi(x)| < \infty. \tag{2}$$

Note that

$$d\mu(y) = \frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy. \tag{3}$$

The Hankel transformation of $\phi\varepsilon L^1_\mu(I)$, is defined by

$$(h_\mu\phi)(x) = \int_0^\infty j_\mu(xy)\phi(y)d\mu(y), x\varepsilon I \tag{4}$$

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where $j_\mu(x) = x^{-\mu}J_\mu(x)$ and $J_\mu(x)$ represents the Bessel function of the first kind and order μ. We shall assume that throughout this paper $\mu \geq -1/2$. Since $z^{-\mu}J_\mu(z)$ is bounded on I, the Hankel transform $h_\mu(\phi)$ is bounded on I. The inversion for (4) is given by

$$\phi(y) = \int_0^\infty j_\mu(xy)(h_\mu\phi)(x)d\mu(x), y\varepsilon I. \tag{5}$$

G. Altenburg [1] introduced the space H that consist of all those complex valued and smooth function ϕ defined on I, such that for every $m, n\varepsilon \mathbf{N_0}$

$$\gamma_{m,n}(\phi) = \sup_{x\varepsilon I}(1+x^2)^m|(x^{-1}D)^n\phi(x)| < \infty. \tag{6}$$

When H is endowed with the topology associated with the family $\{\gamma_{m,n}\}_{m,n\varepsilon\mathbf{N_0}}$ of semi norms, H is a Fréchet space and the Hankel transformation h_μ is an automorphism of H, [1]. The Hankel transform is defined on H' , the dual space of H, as the transpose of h_μ on H and is defined by h'_μ. That is $f\varepsilon H'$, then the Hankel transform $h'_\mu f$ of f is the element of H' defined by

$$< h'_\mu f, \phi > = < f, h_\mu\phi >, \quad \phi\varepsilon H.$$

If $f\varepsilon L^p_\mu$ then f defines an element of H' through

$$< f, g > = \int_0^\infty f(x)g(x)d\mu(x), \quad g\varepsilon H. \tag{7}$$

Thus, L^p_μ can be seen as a subspace of H'. The convolution associated to the h_μ transformation is defined as follows. The Hankel convolution $f\sharp_\mu g$ of order μ of the measurable functions $f, g\varepsilon L^1_\mu(I)$ is given through

$$(f\sharp_\mu g)(x) = \int_0^\infty f(y)({}_\mu\tau_x g)(y)d\mu(y), \tag{8}$$

where the Hankel translation operator ${}_\mu\tau_x g$ of g is defined by

$$({}_\mu\tau_x g)(x) = \int_0^\infty g(z)D_\mu(x,y,z)d\mu(z), \tag{9}$$

provided that the above integrals exist. Here D_μ is the following function

$$D_\mu(x,y,z) = \int_0^\infty j_\mu(xt)j_\mu(yt)j_\mu(zt)d\mu(t), \tag{10}$$

and we have the following basic formula

$$\int_0^\infty j_\mu(zt)D_\mu(x,y,z)d\mu(z) = j_\mu(xt)j_\mu(yt). \tag{11}$$

Lemma 1.1. *Let f and g be functions on $L^1_\mu(I)$, then*

(i) $(h_\mu({}_\mu\tau_x f))(y) = j_\mu(xy)(h_\mu f)(y)$,

(ii) $(h_\mu(f\sharp_\mu g))(y) = (h_\mu f)(y)(h_\mu g)(y)$.

Proof. (i) Since $({}_\mu\tau_x f)(y) = \int_0^\infty f(z)D_\mu(x,y,z)d\mu(z)$.
We therefore $(h_\mu({}_\mu\tau_x f))(y) = \int_0^\infty j_\mu(yt)({}_\mu\tau_x f)(t)d\mu(t)$

$$= \int_0^\infty j_\mu(yt)d\mu(t)\int_0^\infty f(z)D_\mu(x,t,z)d\mu(z)$$
$$= \int_0^\infty f(z)d\mu(z)\int_0^\infty j_\mu(yt)D_\mu(x,t,z)d\mu(t).$$

Using (11), we have

$$\begin{aligned}(h_\mu({}_\mu\tau_x f))(y) &= \int_0^\infty f(z)j_\mu(xy)j_\mu(zy)d\mu(z)\\ &= j_\mu(xy)\int_0^\infty j_\mu(zy)f(z)d\mu(z)\\ &= j_\mu(xy)(h_\mu f)(y). \qquad (12)\end{aligned}$$

(ii) See [[5], p. 339]. □

2. **The space** H_ω

As G. Bj*örck* [4], we consider continuous, increasing and nonnegative functions ω defined on I, such that $\omega(0) = 0, \omega(x) > 0$ and it satisfying the following three properties

$$(i)\ \ \omega(x+y) \leq \omega(x) + \omega(y), x, y\varepsilon I \qquad (13)$$

$$(ii)\ \ \int_0^\infty \frac{\omega(x)}{1+x^2}dx < \infty \ \ and \qquad (14)$$

$$(iii)\ \ \omega(x) \geq a + blog(1+x),\ \text{for some } a\varepsilon R \text{ and } b > 0. \qquad (15)$$

The class of all such ω functions is denoted by M. Note that if ω is extended to $\mathbf{R}$ as an even function then ω satisfies the subadditivity property (i) for every $x, y\varepsilon\mathbf{R}$.

A. Beurling [3] developed the foundations of a general theory of distributions that extends the Schwartz theory. Some aspects of that theory were presented and completed by Björck [4]. We now collect some definitions and properties for the purpose of the present paper.
A function $\phi\varepsilon L^1_\mu(I)$ is in H_ω when ϕ is smooth function and every $m, n\varepsilon\mathbf{N_0}$

$$\alpha_{m,n}(\phi) = \sup_{x\varepsilon I} e^{m\omega(x)}|(x^{-1}D)^n\phi(x)| < \infty. \qquad (16)$$

On H_ω we consider the topology generated by the family $\{\alpha_{m,n}\}_{m,n\varepsilon\mathbf{N_0}}$ of seminorms. H_ω is clearly a linear space. In [2], H_ω is a Fréchet space. For

$\omega(x) = log(1+x^2)$ it reduces to H and for $\omega(x) = x^\alpha, (0 < \alpha < 1), H_\omega$ is a Gevrey space. From definitions (6), (16) and the inequality $(1+x^2) \leq e^{\omega(x)}$ it follows that $H_\omega \subseteq H$. It is clear that $D(I) \subset H_\omega(I) \subset E(I)$. Since $D(I)$ is a dense subspace of $E(I)$, then $H_\omega(I)$ is dense in $E(I)$. Hence $E'(I) \subset H'_\omega(I)$ the dual of $H_\omega(I)$. Since $H_\omega \subset H$ the following properties given in [2, 5] hold in the present case also when $\omega \varepsilon M$. The Bessel operator defined by

$$\begin{aligned} \Delta_x &= x^{-2\mu-1} D x^{2\mu+1} D \\ &= \frac{d^2}{dx^2} + \frac{(2\mu+1)}{x}\frac{d}{dx}. \end{aligned}$$

By an application of Bessels equation for t fixed, we have

$$\Delta_x j_\mu(xt) = -t^2 j_\mu(xt).$$

Theorem 2.1. *(i) The Hankel transformation h_μ is an automorphism of H_ω. (ii) The generalized Hankel transformation h'_μ is an automorphism of H'_ω.*

Proof. Here we prove (i) and part (ii) can be proved in a similar way.
Since Hankel integral transformation H_μ is defined by

$$(H_\mu \phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\phi(y)dy, x\varepsilon(I) \tag{17}$$

we have

$$\begin{aligned} (h_\mu\phi)(x) &= \int_0^\infty j_\mu(xy)\phi(y)d\mu(y) \\ &= \frac{1}{2^\mu\Gamma(\mu+1)}\int_0^\infty (xy)^{-\mu} J_\mu(xy)\phi(y) y^{2\mu+1} dy \\ &= \frac{1}{2^\mu\Gamma(\mu+1)}\int_0^\infty (xy)^{1/2} J_\mu(xy)(xy)^{-\mu-1/2}\phi(y) y^{2\mu+1} dy \\ &= \frac{1}{2^\mu\Gamma(\mu+1)} x^{-\mu-1/2}\int_0^\infty (xy)^{1/2} J_\mu(xy)(y^{\mu+1/2}\phi)(y)dy \\ &= \frac{1}{2^\mu\Gamma(\mu+1)} x^{-\mu-1/2} H_\mu(y^{\mu+1/2}\phi)(x). \end{aligned} \tag{18}$$

Using relation (18), we have

$$\begin{aligned} & (1+x^2)^m (x^{-1}D)^n (h_\mu\phi)(x) \\ = & \frac{1}{2^\mu\Gamma(\mu+1)}(1+x^2)^m (x^{-1}D)^n x^{-\mu-1/2} H_\mu(y^{\mu+1/2}\phi). \end{aligned}$$

By the technique of Zemanian [[9], p 141], we can write
$(1+x^2)^m|(x^{-1}D)^n(h_\mu\phi)(x)|$

$$= \frac{1}{2^\mu\Gamma(\mu+1)}|\int_0^\infty (1+y^2)^{2\mu+m+2n+1}(y^{-1}D)^m y^{-\mu-1/2} \times (y^{\mu+1}\phi)(y)[(xy)^{-\mu-n}J_{\mu+m+n}(xy)]dy|.$$

From (ii) it follows that to every $\varepsilon > 0$, there exists a constant $c(\epsilon)$ such that $\omega(\xi) \leq \epsilon\xi + c(\epsilon)$, so that

$$e^{\nu\omega(\xi)} < e^{\nu c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!}\xi^m \leq e^{\nu c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!}(1+\xi^2)^m.$$

Now, for any choice ν and n, we have

$$\begin{aligned} \alpha_{\nu,n}(h_\mu\phi) &= \sup_{\xi\varepsilon I} e^{\nu\omega(\xi)}|(\xi^{-1}D)^n(h_\mu\phi)(\xi)| \\ &\leq \sup_{\xi\varepsilon I} e^{\nu c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!}(1+\xi^2)^m|(\xi^{-1}D)^n(h_\mu\phi)(\xi)| \\ &\leq \sup_{\xi\varepsilon I} e^{\nu c(\epsilon)}|\sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!}\frac{1}{2^\mu\Gamma(\mu+1)} \times \\ &\quad \int_0^\infty (1+y^2)^{2\mu+m+2n+1}(y^{-1}D)^m\phi(y)[(\xi y)^{-\mu-n}J_{\mu+m+n}(\xi y)dy]|. \end{aligned}$$

Since $\frac{1}{2^\mu\Gamma(\mu+1)}(\xi y)^{-\mu-n}J_{\mu+m+n}(\xi y)$ is bounded by Q_μ, for $\mu \geq -1/2$, so that

$$\begin{aligned} &\alpha_{\nu,n}(h_\mu\phi) \\ &\leq Q_\mu e^{\nu c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!} \sup_{y\varepsilon I}(1+y^2)^{2\mu+m+2n+2}|(y^{-1}D)^m\phi(y)| \int_0^\infty \frac{1}{1+y^2}dy \\ &\leq Q_\mu e^{\nu c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!}\gamma_{2(\mu+m/2+n+1),m}(\phi) \; < \; \infty. \end{aligned}$$

as infinite series can be made convergent for $m \geq 1$, by choosing

$$\epsilon < \nu^{-1}(\gamma_{2(\mu+m/2+n+1),m}(\phi))^{-1/m}.$$

This proves that $(h_\mu\phi)$ is also in H_ω and that h_μ is continuous linear mapping from $H_\omega$into itself. Since $H_\omega \subset L^1_\mu(I)$ for $\mu \geq -1/2$, we can apply inversion theorem and also the fact that $h_\mu^{-1} = h_\mu$ to this case and conclude that h_μ is an automorphism on H_ω. □

Now the generalized Hankel transformation h'_μ on H'_ω is defined as the adjoint of h_μ on H_ω . More specifically, for any $\phi\varepsilon H_\omega$ and $\psi\varepsilon H'_\omega$, we have

$$< h'_\mu\psi, \phi > = < \psi, h_\mu\phi > .$$

3. Product, Hankel Translation and Hankel Convolution on H_ω

We shall denote by Λ_m the space of all C^∞ function $\phi(x),\ x\ \varepsilon\ I$ such that $m\ \varepsilon\ \mathbf{N_0}$ and there exists a $\lambda = \lambda(m)\ \varepsilon\ \mathbf{N_0}$ for which

$$e^{-\lambda\omega(x)}|(x^{-1}D)^m\phi(x)| < \infty. \tag{19}$$

Here Λ_m is the space of multipliers for H_ω.

Theorem 3.1. *If $\phi,\ \psi\varepsilon H_\omega(I)$, then $fg\varepsilon H_\omega(I)$.*

Proof. For $k, n\varepsilon\mathbf{N_0}$, we have by definition (16)

$$\alpha_{k,n}(\phi\psi)(x) = \sup_{x\varepsilon I} e^{k\omega(x)}|(x^{-1}D)^n\phi(x)\psi(x)|.$$

Using Leibnitz theorem, we have

$$\begin{aligned}
&\alpha_{k,n}(\phi\psi)(x)\\
= &\sup_{x\varepsilon I} e^{k\omega(x)}|\sum_{\nu=0}^{n}\binom{n}{\nu}(x^{-1}D)^{n-\nu}\phi(x)(x^{-1}D)^\nu\psi(x)|\\
= &\sum_{\nu=0}^{n}\binom{n}{\nu}\sup_{x\varepsilon I} e^{k\omega(x)}|(x^{-1}D)^{n-\nu}\phi(x)||\sup_{x\varepsilon I}(x^{-1}D)^\nu\psi(x)|\\
= &\sum_{\nu=0}^{n}\binom{n}{\nu}\alpha_{k,n-\nu}\phi(x)\alpha_{0,\nu}\psi(x) < \infty.
\end{aligned}$$

Hence $\phi\psi\varepsilon H_\omega(I)$. This completes the proof of theorem. □

Theorem 3.2. *The mapping $\phi \mapsto {}_\mu\tau_x\phi$ is continuous from H_ω into itself.*

Proof. Let $\phi\varepsilon H_\omega(I)$. Then $h_\mu\phi\varepsilon H_\omega(I)$. By Lemma 1.1(i) we have

$$h_\mu({}_\mu\tau_x\phi)(y) = j_\mu(xy)(h_\mu\phi)(y).$$

Now we show that

$$j_\mu(xy)\varepsilon\Lambda_m.$$

We have

$$\begin{aligned}
(y^{-1}D)^m(j_\mu(xy)) &= (y^{-1}D)^m((xy)^{-\mu}J_\mu(xy))\\
&= (-1)^m x^{2m}(xy)^{-\mu-m}J_{\mu+m}(xy).
\end{aligned}$$

$$|(y^{-1}D)^m j_\mu(xy)| \le Q_\mu x_0^{2m},$$

so that there exists $\lambda > 0$ such that

$$|e^{-\lambda\omega(y)}(y^{-1}D)^m j_\mu(xy)| < \infty, \quad \text{for every } \ x \ \varepsilon \ I.$$

Here $j_\mu(xy)\varepsilon\Lambda_m$ for fixed $x\varepsilon I$. But $(h_\mu\phi)(y)\varepsilon H_\omega$, then $j_\mu(xy)h_\mu\phi(y)\varepsilon H_\omega$. Since h_μ is an automorphism of H_ω, we have ${}_\mu\tau_x\phi\varepsilon H_\omega$ and the mapping $\phi \mapsto {}_\mu\tau_x\phi$ is continuous from $H_\omega$into itself. □

Theorem 3.3. *If* $f, g\varepsilon H_\omega(I)$, *then* $f\sharp_\mu g\varepsilon H_\omega(I)$.

Proof. By definition (16), we have

$$\alpha_{k,n}(h_\mu(\phi\sharp_\mu\psi))(x) = \sup_{x\varepsilon I} e^{k\omega(x)}|(x^{-1}D)^n h_\mu(\phi\sharp_\mu\psi)|$$

Using Lemma 1.1 (ii)

$$\begin{aligned}\alpha_{k,n}(h_\mu(\phi\sharp_\mu\psi))(x) \ &= \ \sup_{x\varepsilon I} e^{k\omega(x)}|(x^{-1}D)^n(h_\mu\phi)(x)(h_\mu\psi)(x)\\ = \quad & \sup_{x\varepsilon I} e^{k\omega(x)}\sum_{\nu=0}^{n}\binom{n}{\nu}|(x^{-1}D)^{n-\nu}(h_\mu\phi)(x)(x^{-1}D)^{\nu}(h_\mu\psi)(x)|\\ = \quad & \sum_{\nu=0}^{n}\binom{n}{\nu}\sup_{x\varepsilon I} e^{k\omega(x)}|(x^{-1}D)^{n-\nu}(h_\mu\phi)(x)|\sup_{x\varepsilon I}|(x^{-1}D)^{\nu}(h_\mu\psi)(x)|\\ = \quad & \sum_{\nu=0}^{n}\binom{n}{\nu}\alpha_{k,n-\nu}(h_\mu\phi)(x)\alpha_{0,\nu}(h_\mu\psi(x) < \infty.\end{aligned}$$

Hence $h_\mu(\phi\sharp_\mu\psi)\varepsilon H_\omega(I)$. Since h_μ is an automorphism of $H_\omega(I)$, we have $\phi\sharp_\mu\psi\varepsilon H_\omega(I)$. □

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METHOD OF INFINITE ASCENT APPLIED ON $mA^6 + nB^3 = C^2$

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ABSTRACT. In this paper we will use the Method of Infinite Ascent to find infinite number of integral solutions for the Diophantine equation $mA^6 + nB^3 = C^2$ with m, n ,A, B and C as integers and A, B, C pair-wise co-prime. In addition to this, many fundamental results relating to this equation will be obtained.

1. Introduction

In the VII^{th} Joint International Meeting of the American Mathematical Society and the Mexican Society of Mathematics [1] the author of this paper presented a talk titled " Method of Infinite Ascent applied on $A^6 + nB^3 = C^2$" in which this Diophantine equation has been shown to have infinitely many co-prime integral solutions for (A, B, C) for a class of integral values of n. The present paper is a generalization of that result.

Sometimes a Diophantine equation possessing an infinite number of integral solutions does not exhibit this infinitude characteristics as seen in its original form. Putting into a slightly modified form – which we need to discover - this equation becomes regenerative, so that any set of solution for the equation will lead to the next set of solution for the same; the first set leading to the second, the second set leading to the third and so on without end. This is a regeneration technique which we wish to call the *Method of Infinite Ascent(MIA)*, explicitly showing on how to generate the endless set of integral solutions for the Diophantine equation. We will tell more on MIA latter; but for the present, let us concentrate on the following Diophantine

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equation.

$$mA^6 + nB^3 = C^2 \tag{1.1}$$

2. **Diophantine Equation** $mA^6 + nB^3 = C^2$

Equation (1.1) has infinite number of integral solutions for $(A,\ B,\ C)$ with $A,\ B$ and C pair-wise co-prime for each pair of $(m,\ n)$ values, a result which would be clear if we take the results of *Theorem 2.1, Corollary 2.1, Corollary 2.2, Corollary 2.3* and *Theorem 3.1.* The integers m and n are taken to be co-prime and they are related by the equation $n = mp^6 - q^2$.

Theorem 2.1. *For any integer m, p and q,*

$$m(2pq)^6 + (mp^6 - q^2)(9mp^6 - q^2)^3 = (27m^2p^{12} - 18mp^6q^2 - q^4)^2 \tag{2.1}$$

Proof. L.H.S. of (2.1) = $m(2pq)^6 + (mp^6 - q^2)(9mp^6 - q^2)^3$

$$\begin{aligned}
&= 64mp^6q^6 + mp^6(9mp^6 - q^2)^3 - q^2(9mp^6 - q^2)^3 \\
&= 64mp^6q^6 + mp^6(729m^3p^{18} - 243m^2p^{12}q^2 + 27mp^6q^4 - q^6) \\
&\quad -q^2(729m^3p^{18} - 243m^2p^{12}q^2 + 27mp^6q^4 - q^6) \\
&= 64mp^6q^6 + 729m^4p^{24} - 243m^3p^{18}q^2 + 27m^2p^{12}q^4 - mp^6q^6 \\
&\quad -729m^3p^{18}q^2 + 243m^2p^{12}q^4 - 27mp^6q^6 + q^8 \\
&= 729m^4p^{24} + (-243 - 729)m^3p^{18}q^2 + (27 + 243)m^2p^{12}q^4 \\
&\quad +(64 - 1 - 27)mp^6q^6 + q^8 \\
&= 729m^4p^{24} - 972m^3p^{18}q^2 + 270m^2p^{12}q^4 + 36mp^6q^6 + q^8 \\
&= (27m^2p^{12} - 18mp^6q^2 - q^4)^2 = \text{R.H.S. of (2.1)}
\end{aligned}$$

Hence, *Theorem 2.1* is proved. □

Corollary 2.1. *For any integer m and q,*

$$m(2q)^6 + (q^2 - m)(q^2 - 9m)^3 = (q^4 + 18mq^2 - 27m^2)^2 \tag{2.2}$$

with $q^2 \neq m$ or $9m$.

Proof. Taking $p = 1$ in (2.1) and rearranging, *Corollary 2.1* is proved. □

Corollary 2.2. *For any integer q,*

$$(2q)^6 + (q^2 - 1)(q^2 - 9)^3 = (q^4 + 18q^2 - 27)^2 \tag{2.3}$$

with $q^2 \neq 1$ or 9.

Proof. Taking $p = m = 1$ in (2.1) and rearranging, *Corollary 2.2* is proved. □

Corollary 2.3. *For any integer k,*

$$2^{k+6} + (2^k - 1)(9.2^k - 1)^3 = (27.2^{2k} - 18.2^k - 1)^2 \tag{2.4}$$

Proof. Taking $p = q = 1$ and $m = 2^k$ in (2.1) and rearranging, *Corollary 2.3* is proved. □

3. Method of Infinite Ascent Applied on $mA^6 + nB^3 = C^2$

For applying the Method of Infinite Ascent to the equation

$$mA^6 + nB^3 = C^2 \tag{3.1}$$

we will require the result of *Theorem 3.1*.

Theorem 3.1. *If (A_t, B_t, C_t) is a solution of the Diophantine equation $mA^6+nB^3 = C^2$ with m, n, A, B and C as integers then, $(A_{t+1}, B_{t+1}, C_{t+1})$ is also a solution of the same equation such that*

$$(A_{t+1}, B_{t+1}, C_{t+1}) = \{(2A_tC_t), -B_t(9mA_t^6 - C_t^2), (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4)\} \tag{3.2}$$

and if mA_t , nB_t and C_t are pair-wise co-prime where nB_t is an odd integer and 3 is not a factor of C_t ,then mA_{t+1}, nB_{t+1} and C_{t+1} are also pair-wise co-prime where nB_{t+1} is an odd integer and 3 is not a factor of C_{t+1}; in addition to this, mA_{t+1} will be always an even integer and C_{t+1}, always an odd integer.

Proof. In three parts we will prove the statement of *Theorem 3.1* completely.

I. Since (A_t, B_t, C_t) is a solution of (3.1), we have

$$mA_t^6 + nB_t^3 = C_t^2 \tag{3.3}$$

Or,

$$mA_t^6 - C_t^2 = -nB_t^3 \tag{3.4}$$

Take $p = A_t$ and $q = C_t$ in (2.1). So,

$$m(2A_tC_t)^6 + (mA_t^6 - C_t^2)(9mA_t^6 - C_t^2)^3 = (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4)^2 \tag{3.5}$$

Replacing $(mA_t^6 - C_t^2)$ by $(-nB_t^3)$ from (3.4) in (3.5) we get

$$m(2A_tC_t)^6 - nB_t^3(9mA_t^6 - C_t^2)^3 = (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4)^2 \tag{3.6}$$

Comparing (3.1) to (3.6) we get the next solution for (3.1) as

$$(A_{t+1}, B_{t+1}, C_{t+1})$$
$$= ((2A_tC_t), -B_t(9mA_t^6 - C_t^2), (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4)) \tag{3.7}$$

II. Now from (3.7)

$$mA_{t+1} = m.2A_tC_t = 2mA_tC_t \tag{3.8}$$

$$nB_{t+1} = -nB_t(9mA_t^6 - C_t^2) \tag{3.9}$$

and

$$C_{t+1} = (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4) \tag{3.10}$$

From (3.8) it is obvious that mA_{t+1} is always an even integer.

It is given that nB_t is an odd integer. Since mA_t, nB_t and C_t are pairwise co-prime, then out of mA_t and C_t, one should be odd and the other even. In either case, $(9mA_t^6 - C_t^2)$ is odd and hence, from (3.9) we see that nB_{t+1} is odd.

Again, from (3.10), C_{t+1} is odd because, out of mA_t and C_t , one is odd and the other is even.In the right hand side of equation (3.10), two terms: $27m^2A_t^{12}$ and $18mA_t^6C_t^2$ are divisible by 3 but the third term C_t^4 is not divisible by 3. So, 3 can not be a factor of C_{t+1}.

III. Since, $(A_{t+1}, B_{t+1}, C_{t+1})$ is a solution for $(A\ ,\ B\ ,\ C)$ of the equation (3.1), we get,

$$mA_{t+1}^6 + nB_{t+1}^3 = C_{t+1}^2 \tag{3.11}$$

We proved mA_{t+1} to be an even integer and nB_{t+1} as an odd integer. So, any common factor of these two terms should be an odd integer > 1. Let us suppose mA_{t+1} and nB_{t+1} are not co-prime. Then there is a positive odd integer $p > 1$ which divides mA_{t+1} and nB_{t+1}.

From (3.8), we get that p is a factor of $2mA_tC_t$ .Since p is odd, mA_t and C_t are co-prime, p will be a factor of either mA_t or C_t.

Case 1. Let p is a factor of mA_t. p is not a factor of nB_t because the given condition is that mA_t and nB_t are co-primes. Similarly, p can not be a factor of $(9mA_t^6 - C_t^2)$ as mA_t and C_t are co-primes, and 3 is not a factor of C_t. Hence, p is not a factor of $nB_t(9mA_t^6 - C_t^2)$. From (3.9), we conclude that p is not a factor of nB_{t+1}.

Case 2. Let p is a factor of C_t. p is not a factor of nB_t because the given condition is that C_t and nB_t are co-primes. Similarly, p can not be a factor of $(9mA_t^6 - C_t^2)$ as mA_t and C_t are co-primes and 3 is not contained in C_t. Hence, p is not a factor of $nB_t(9mA_t^6 - C_t^2)$. From (3.9), we conclude that p is not a factor of nB_{t+1}.

Combining the results of *case1* and *case 2*, we conclude that there is no common factor between mA_{t+1} and nB_{t+1}. From (3.11), we observe that the factors of C_{t+1} are different from the factors of mA_{t+1} and nB_{t+1} because these latter two terms are co-prime. Hence, it is proved that mA_{t+1}, nB_{t+1} and C_{t+1} are pair-wise co-prime.
This completes the proof of *Theorem 3.1*. □

4. **Comments**

Comment 4.1: The Diophantine equation (1.1) looks innocent about the number of its integral solutions. But, in form (2.1), it exhibits its deep inner nature of having infinite number of integral solutions for all integral values of m and q characterized by (2.2) or (2.3). The result of (2.4) is interesting and possibly be used to get some deeper results. It is notable that in (2.1), if we take $(m, p, q) = (2, 1, 1)$, then we get one solution to the Diophantine equation relating to the *Fermat-Catalan Conjecture* [2] as $2^7 + 17^3 = 71^2$. Allowing the algebraic numbers and taking $(m, p, q) = (1, 1, \sqrt{2})$ in (2.1) leads to another solution

$$2^9 - 7^3 = 13^2 \quad \text{Or,} \quad 13^2 + 7^3 = 2^9 \tag{4.1}$$

Writing (4.1) as

$$8.2^6 + (-7)^3 = 13^2 \tag{4.2}$$

and comparing (4.2) with (1.1) will lead to one solution of (1.1) as
$(m, n, A_1, B_1, C_1) = (8, 1, 2, -7, 13)$.
With $(m, n, A_1, B_1, C_1) = (8, 1, 2, -7, 13)$ as the first solution of (1.1), we will get its second solution as $(m, n, A_2, B_2, C_2) = (8, 1, 52, 31073, 5491823)$ by applying (3.2). This list of solutions of (1.1) will go up without end with B taking either positive or negative integral values.

Comment 4.2: Many people have investigated on the solution of the Diophantine equation $x^2 + C = y^n$ with $x \geq 1, y \geq 1, n \geq 3$ and C is any integer, positive or negative. Lebesgue [7] proved that there are no solutions for $C = 1$. Ljunggren [8] solved this equation for $C = 2$. Nagell [11, 12] solved it for $C = 3$, 4 and 5. Chao[4] proved that it has no solution for C = -1. Cohn's paper [5] gives solution of the said equation for 77 values of $C \leq 100$. For C =74 and C = 86 Mignotte and de Weger [10] gave the solution for this equation. All the remaining cases for $0 < C < 100$ have been solved by Y. Bugeaud, M. Mignotte and S. Siksek [13].

Other people like Le [6], Arif and Muriefah [3], and Luca [9] have considered a different form of the equation $x^2 + C = y^n$, when C is no longer a fixed integer but the power of one or two fixed primes. If we allow C to be the product of the power of 2 and an integral sixth power, then we have the following result of *Theorem 4.1*.

Theorem 4.1. *For any positive integer q, the Diophantine equation*

$$2^{6q-3}.A^6 + B^3 = C^2 \tag{4.3}$$

has infinitely many co-prime integral solutions for (A, B, C).

Proof. We will prove *Theorem 4.1* in three steps. Firstly, we have to establish that equation (4.3) has infinitely many co-prime integral solutions for $(A,\ B,\ C)$ when $q = 1$. Secondly, we will see how to use these co-prime solutions of first step to find the initial co-prime solutions for $(A,\ B,\ C)$ of equation (4.3) for other values of $q > 1$. Next, we will show that the conditions of generating infinite number of co-prime integral solutions, as proposed by *Theorem 3.1*, are applicable to (4.3) for each value of q.

Step I. Putting $q = 1$ in (4.3) we get,

$$2^3.A^6 + B^3 = C^2 \tag{4.4}$$

We will denote the i^{th} solution for $(A,\ B,\ C)$ of equation (4.4) as (A_i, B_i, C_i), where i takes positive integral values. Using the result of (4.2), we can get the starting solution for $(A,\ B,\ C)$ of equation (4.4) as $(A_1, B_1, C_1) = (2, -7, 13)$. Comparing (4.4) with (3.1) we get the values of m and n to be 2^3 and 1 respectively. Now, the conditions of generating infinite number of co-prime integral solutions, as proposed by *Theorem 3.1*, are applicable for equation (4.4) because, the three terms: mA_1, nB_1 and C_1, taking values $2^2.2, -7$ and 13 respectively, are pair-wise co-prime; nB_1 is an odd integer and 3 is not a factor of C_1.

Thus, the repeated use of *Theorem 3.1* will enable us to generate an infinite number of co-prime integral solutions for $(A,\ B,\ C)$.

Using the equation (3.2) we calculate the second solution of (4.4) as

$$(A_2, B_2, C_2) = (2.2.13, 7(9.2^3.2^6 - 13^2), (27.2^6.2^{12} - 18.2^3.2^6.13^2 - 13^4))$$

$$= (2^2.13, 31073, 5491823).$$

In general, using equation (3.2), we calculate the k^{th} solution of (4.4) as $(A_k, B_k, C_k) = (2^k.A'_k, B_k, C_k)$, where the integer k is greater than 1, $A_k = 2^k.A'_k$ and all three terms: A'_k, B_k and C_k are odd.

Now, the repeated use of equation (3.2) will enable us to find any number of co-prime integral solutions for (A, B, C) of equation (4..4).

Step II. The first solution for (A, B, C) of equation (4.4) is (2, -7, 13). Using these values for $(A,\ B,\ C)$ in (4.4) we get,

$$2^3.2^6 - 7^3 = 13^2 \quad \text{Or,} \quad 2^9.1^6 + (-7)^3 = 13^2 \tag{4.5}$$

The second solution for (A, B, C) of equation (4.4) is $(2^2.13, 31073, 5491823)$. Using these values for (A, B, C) in (4.4) we get,

$$2^3.2^{12}.13^6 + 31073^3 = 5491823^2 \quad \text{Or,} \quad 2^{15}.13^6 + 31073^3 = 5491823^2 \tag{4.6}$$

The k^{th} solution for (A, B, C) of equation (4.4) is $(2^k.A'_k, B_k, C_k)$. Using these values for (A, B, C) in (4.4) we get,

$$2^{6k+3}.A'^6_k + B^3_k = C^2_k \tag{4.7}$$

When $q = 2$, from (4.5) we get the starting solution for (A, B, C) of equation (4.3) as (1, -7, 13).

When $q = 3$, from (4.6) we get the starting solution for (A, B, C) of equation (4.3) as (13, 31073, 5491823).

When $q = k$, from (4.7) we get the starting solution for (A, B, C) of equation (4.3) as (A'_k, B_k, C_k).

Step III. In *step I*, we have already proved the validity of the statement of *Theorem 4.1* for $q = 1$. Putting $q = 2$ in (4.3) we get

$$2^9.A^6 + B^3 = C^2 \tag{4.8}$$

Now, for each integral value of $q > 1$, there is a starting solution for $(A\ ,\ B,\ C)$ for equation (4.3) as we showed in *Step II.* Since the values of B and C in these starting solutions are the same values which are generated by the subsequent solutions of equation (4.4), they should be co-prime; B and C are odd integers; and 3 is not a factor of C. Hence, for any integer $q > 1$, the statement of *Theorem 4.1* should also be valid, because the conditions of generating infinite number of co-prime integral solutions, as proposed by *Theorem 3.1,* are satisfied. This completes the proof of *Theorem 4.1.* □

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DUALIZATION OF CYCLIC PURITY

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Abstract. Cocyclic copurity, the dualization of cyclic purity is studied. Examples are given to show that the concepts of Cocyclic Copurity and cyclic purity are independent. We have also compared Cocyclic copurity with Cyclically purity given by Divani-Aazar. et. al. [4].

1. Introduction

The notion of purity has an important role in module theory and in model theory. In model theory, the notion of pure exact sequence is more useful than split exact sequences. There are several variants of this notion. R.Wisbauer [20] generalized the notion of purity for a class P of R-modules. He defines a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules to be P-pure, if every member of P is projective with respect to this sequence. Cohn's [3] purity is precisely P-purity for the class P of all finitely presented R-modules. Cohn's purity is called as purity.

Two more types of P-purity are of interest. One introduced by Simmons [15], called cyclic purity, in which P is the class of all cyclic R-modules. In the Section 3, we give examples to show that the cyclic purity and Cohn's purity are independent. Another type of P-purity considered by Divaani-Aazar et. al. [4], called Cyclically purity, in which P is the class of R-modules which are isomorphic to R^n/G where n is any natural number and G any cyclic submodule of R^n. Actually this purity is a generalization of Cohn's purity. Divaani[4], proved that this purity is precisely the intersection purity which was introduced by Stenstrom. In Section 4, we give

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examples to show that the concepts of cyclic pure and cyclically pure are independent.

The First author dualized the Cohn's purity, by introducing Copurity[7]. In this paper we study the cocyclic copurity as the dualization of cyclic purity.

2. **Definitions and Notations**

In this paper, by a ring R we mean an associative ring with unity and by an R-module we mean a unitary right R-module.Consider a short exact sequence, $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules. We call an R-module M to be ϵ-injective (resp. ϵ-projective) if M is injective (resp. projective) with respect to the short exact sequence ϵ.

Definition 2.1. *An exact sequence $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is said to be cyclic pure (c-pure in short) if every cyclic R-module is ϵ-projective.*

Definition 2.2. *An exact sequence $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is said to be cyclically pure (CP in short) if every R-module which is isomorphic to R^n/G where n any natural number and G cyclic submodule of R^n, is ϵ-projective.*

Definition 2.3. *An R-module M is said to be cocylic, if it can be embedded in the injective hull of a simple R-module. M is said to be subdirectly irreducible if the intersection of any family of nonzero submodules of M is nonzero. It is easy to prove that the concepts of cocyclic module and subdirectly irreducible module are equivalent.*

Definition 2.4. *An R-module M is said to be finitely embedded (f.e) [18] (later called by J.P.Jans [8] as cofinitely generated) if, $E(M) = E(S_1) \oplus E(S_2) \oplus \ldots \oplus E(S_n)$ for some simple R-modules $S_1, \ldots, S_n$ (here $E(A)$ denotes the injective hull of an R-module A).*

Definition 2.5. *An R-module M is said to be finitely cogenerated [1, p.124], if the following condition is satisfied. If $\Im$ is any family of submodules of M such that $\cap\Im = (0)$ then there is finite subfamily F of $\Im$ such that $\cap F = (0)$.*

It was shown that the concepts of finitely embedded and finitely cogenerated are same c.f [8].

Definition 2.6. *An R-module M is said to be cofree [6], if M is embeddable in a direct product of the injective hulls of a family of simple R-modules.*

Definition 2.7. *An R-module M is cofinitely related if there exists a short exact sequence, $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$ where N is cofree and N and K are cofinitely generated.*

Definition 2.8. *An exact sequence $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is said to be copure if every cofinitely related R-module is ϵ-injective.*

Definition 2.9. *A submodule A of an R-module B is said to be $\cap$-pure in B [16] if $AI = BI \cap A$, for every left ideal I of R.*

3. C-purity versus Cohn's Purity

Remark 3.1. *In general purity does not imply c-purity.*

Example Let $R = \prod_{\alpha\in\Lambda} R_\alpha$ and $S = \oplus_{\alpha\in\Lambda} R_\alpha$ where $\{R_\alpha\}$ is any infinite family of fields. Clearly S in an essential ideal of R. If S is cyclic pure in R then R/S is projective with respect to the canonical short exact sequence $0 \longrightarrow S \longrightarrow R \longrightarrow R/S \longrightarrow 0$. Then S is a direct summand of R. Since, S is an essential ideal of R, this implies that $R = S$, which is impossible. So, S is not cyclic pure in R. Since, R is a von-Neumann regular ring by [17, Proposition 11.1], S is pure in R.

Remark 3.2. *In general cyclic purity does not imply purity.*

Example: Let $R = k[X,Y]$ be a polynomial ring in two variables X, Y over a field k. Here, the ideal (X,Y) of R is torsion-free but not flat as an R-module. ((cf. [1,Chapter I, Exercise 2.3])). Let $(X,Y) = F/K$ for a free module F and a submodule K. Since, $(X,Y) = F/K$ is not flat, K is not pure in F. Since, R is a commutative integral domain, by [15, Remark C], K is c-pure in F.

4. C-purity versus CP

Remark 4.1. *In general CP does not imply c-purity.*

Example: Since, CP is a generalization of purity, a pure exact sequence is always CP-exact. Hence, the example given after the Remark 3.1, serves our purpose.

Remark 4.2. *In general c-purity does not imply CP.*

Example: Consider the example given after Remark 3.2. In that example, if K is CP in F, then by [4, Proposition 2.2], $KI = K \cap FI$, for every ideal I of R. Then by [20, 36.6], (X, Y) is flat, which is a contradiction. Hence K is not a CP submodule of F.

5. **Cocyclic Copurity**

In this section, we study the cocyclic copurity, which is given by Choudhary and Tiwari [2]. But we study the concept as the dualization of c-purity.

Definition 5.1. *A short exact sequence,* $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *of R-modules is said to be cocyclically copure [2, p.1568] (in short ccp) if every cocyclic R-module is* ϵ*-injective.*

Definition 5.2. *A submodule* A *of an R-module* B *is said to be a ccp submodule of* B *if the canonical short exact sequence* $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ *of R-modules is ccp.*

Choudhary and Tiwari [2, Proposition 4.2] have proved the following characterization for ccp submodule.

Proposition 5.3. *For a submodule* A *of an R-module* B*, the following conditions are equivalent.*
i) *A is ccp in B.*
ii) *For each* $0 \neq a \in A$ *and a submodule* C *of* A *maximal with respect to the property not containing* a*, there exists a submodule* D *of* B *containing* C *maximal with the following properties* $a \notin D$, $C = D \cap A$ *and* $B = D + A$.

Proposition 5.4. *Let* A, B, C *be R-modules such that* A *is a submodule of* B *and* B *is a submodule of* C,
i) *If A is ccp in B and B is ccp in C then A is ccp in C.*
ii) *If A is ccp in C then A is ccp in B.*
iii) *If B is ccp in C then B/A is ccp in C/A.*
iv) *If A is ccp in C and B/A is ccp in C/A then B is ccp in C.*

Proof. The proofs of (i) ,(ii),(iii) are straightforward. We prove (iv).

Let M be a cocyclic R-module and $f \in Hom(B, M)$. Let $f_1 = f/A$ be the restriction map. Since, A is ccp in C, there exists $g_1 \in Hom(C, M)$ such that $g_1/A = f_1$. Let $k = g_1/B$. Since, $h = f - k$ vanishes on A, it induces a homomorphism $h^* : B/A \longrightarrow M$ defined by $h^*(\bar{b}) = g_1(b) - f(b)$. Since, B/A

is ccp in C/A, there exists $g_2 \in Hom(C/A, M)$, such that $g_2/(B/A) = h^*$. Now define a map, $g : C \longrightarrow M$ by, $g(c) = g_1(c) - (g_2 \circ \eta(c))$. Clearly, g is well-defined homomorphism. Now, if $b \in B$, $g(b) = g_1(b) - g_2 \circ \eta(b) = g_1(b) - g_2(\bar{b}) = g_1(b) - f_2(\bar{b}) = g_1(b) - (g_1(b) - f(b)) = f(b)$. Hence, $g/B = f$. Hence, the result. □

Remark 5.5. *By the Proposition 5.4, it is clear that the family of all ccp short exact sequences of R-modules forms a Proper Class in the sense of MacLane [10].*

Proposition 5.6. ***i)*** *If A is a cocyclic, ccp submodule of an R-module B then A is a direct summand of B.*
ii) *A submodule B of an R-module C is ccp in C if and only if for every submodule A of B such that B/A is cocyclic, B/A is a direct summand of C/A.*

Proof. **i)** Since, by hypothesis, A is cocyclic, ccp submodule of B the identity map I_A of A extends to an R-homomorphism from B to A. This implies that A is a direct summand of B.
ii) Only if: Let A be a submodule of B such that B/A is cocyclic. Since, by hypothesis, B is ccp in C, it follows by Proposition 5.4(iii), that B/A is ccp in C/A. It now follows by (i) above that B/A is direct summand of C/A.
If: Let $f : B \longrightarrow M$ be any R-homomorphism from B into a cocyclic R-module M. Let $A = Ker f$. Then f induces a monomorphism $f^* : B/A \longrightarrow M$ from B/A into M. Since, M is cocyclic so is B/A. Then, by hypothesis, B/A is a direct summand of C/A. So, there exists an R-homomorphism $\theta : C/A \longrightarrow B/A$ which is identity on B/A. Let $\eta : C \longrightarrow C/A$ be the canonical epimorphism and let $g = f^* \circ \theta \circ \eta : C \longrightarrow M$. We prove that g extends f. For $b \in B$, we have $g(b) = (f^* \circ \theta \circ \eta)(b) = (f^*(\theta(\eta(b))) = (f^*(\theta(b + A)) = f^*(b + A) = f(b)$. Thus g extends f. This proves that B is a ccp submodule of C. □

Proposition 5.7. *For a ring R the following conditions are equivalent.*
i) *R is right V-ring.*
ii) *Every short exact sequence of R-modules is copure.*
iii) *Every short exact sequence of R-modules is ccp.*
iv) *Every right ideal of R is a ccp submodule of R.*

Proof. $(i) \Rightarrow (ii)$: Follows by $(i) \Rightarrow (iii)$ of [7, Proposition 5]
$(ii) \Rightarrow (iii)$: Follows by the remark after the proof of the Proposition 5 of

[7].

$(iii) \Rightarrow (iv)$: Obvious.

$(iv) \Rightarrow (i)$: By [7, Proposition 4(i)], we need only prove that every cocyclic R-module is injective. Let M be any cocyclic R-module and let $f : I \longrightarrow M$ be any R-homomorphism from a right ideal I of R into M. Since, by hypothesis (iv), I is ccp in R, f extends to an R-homomorphism from R into M. This proves the injectivity of M by the Baer's criterion of injectivity. This proves that R is a right V-ring. □

Proposition 5.8. *If A is a ccp submodule of an R-module B then, $AI = BI \cap A$, for every right ideal I of R.*

Proof. Let I be any right ideal of R. Clearly $AI \subseteq BI \cap A$. For the reverse inclusion, let $x \in BI \cap A$. Suppose $x \notin AI$. Let K be a submodule of A maximal with the property that $AI \subseteq K$ and $x \notin K$ (we note that since I is a right ideal of R, AI is a submodule of A). Then A/K is subdirectly irreducible and hence cocyclic. Let $\eta : A \longrightarrow A/K$ be the canonical epimorphism. Since, by hypothesis, A is ccp submodule of B there exists an R-homomorphism $f : B \longrightarrow A/K$ such that $f/A = \eta$. Since $x \in A$, $\eta(x) = f(x) \in f(BI) \subseteq f(B)I \subseteq (A/K)I = (\bar{o})$ (because $AI \subseteq K$) which implies that $\eta(x) = \bar{o}$ which is impossible since $x \notin K$. Hence $x \in AI$. This proves that $BI \cap A \subseteq AI$. This completes the proof of the proposition. □

Corollary 5.9. *If a right ideal I is ccp submodule of R, then I is idempotent.*

Corollary 5.10. *If R is a commutative ring then every ccp submodule of an R-module M is an $\cap$-pure submodule of M.*

Corollary 5.11. *If R is a commutative ring and if $0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$ be ccp with F flat, then B is flat.*

Proof. Let $0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$ be ccp then by above proposition $AI = FI \cap A$ for every ideal I of R. Hence by [20, 36.6] B is flat. □

Fuchs in [5, p.121, Ex 1] has mentioned an equivalent condition for the pure subgroup. This condition is partially generalized in the following proposition for ccp submodules over commutative rings.

Proposition 5.12. *If R is a commutative ring and A is a ccp submodule of an R-module B then, $(A : r) = A + (0 : r)$ where, for an R-submodule C of B, $(C : r) = \{b \in B / br \in C\}$.*

Proof. $A + (0 : r) \subseteq (A : r)$ is obvious. Let, $x \in (A : r)$ then $xr \in A$. Consider the ideal $rR = I$ of R. Now $xr \in BI$. By above Proposition 5.8, $xr \in AI$. So, $xr = \sum_{i=1}^{n} a_i r r_i$ for some $a_i \in A$ and $r_i \in R$ for $1 \leq i \leq n$. Then $xr = a_0 r$ where, $a_0 = \sum_{i=1}^{n} a_i r_i \in A$. Then, $xr - a_0 r = o \Longrightarrow (x - a_0)r = o \Rightarrow x - a_0 \in (0 : r) \Rightarrow x \in A + (0 : r)$. This implies, $(A : r) \subseteq A + (0 : r)$. Hence the proof. □

We do not know whether in general copurity implies ccp. However, we have examples to show that the concepts cofinitely related and cocyclic are different.

Remark 5.13. *A cocyclic R-module need not be cofinitely related.*

Example: Let $R = Z_{(p)} \oplus Z(p^\infty)$ be the direct sum of the abelian group $Z_{(p)}$ of p-adic integers and the $p-Prüfer$ group $Z(p^\infty)$ for a fixed prime p. Then R becomes a ring with multiplication defined by $(\lambda, x)(\mu, y) = (\lambda\mu, \lambda y + \mu x)$ for $(\lambda, x), (\mu, y) \in R$. Then, by [13, p.378-379]), (i)R is a commutative ring with identity (ii) R is self-injective and (iii) $Z(p)$ is the socle of R and is essential in R. So, R is cocyclic as an R-module. Clearly, $Z(p^\infty)$ is an ideal of R and the factor module $R/Z(p^\infty)$ is not cofinitely generated as an R-module. It now follows, by [6, Proposition 15], that $Z(p^\infty)$ is not cofinitely related as an R-module. Clearly, $Z(p^\infty)$ is cocyclic as an R-module.

Remark 5.14. *A cofinitely related R-module need not be cocyclic.*

Example: Consider a ring R. Let S_1 and S_2 be any two simple R-modules. Clearly, $M = E(S_1) \oplus E(S_2)$ is cofinitely related. Since M cannot be subdirectly irreducible it cannot be cocyclic.

We have the following Proposition, to compare copurity and ccp.

Proposition 5.15. *If R is a right co-Noetherian ring, every copure short exact sequence of R-modules is ccp.*

Proof. Since, by hypothesis, R is right co-Noetherian, it follows by [6, proposition 17], every cofinitely generated (and hence cocyclic) R-module is cofinitely related. Hence the proposition follows. □

6. **CCP versus Purity**

Remark 6.1. *In general a pure submodule of an R-module need not be ccp submodule.*

Example: In [7, Remark 13], the first author gave an example for a pure submodule which is not copure. We prove that the same example serves our purpose.

Let V be a countably infinite dimensional vector space over the field Q of rational numbers and R be the ring of linear operators of V. Then R is von Neumann regular ring by [11, Theorem 7.3]. Let I be the two-sided ideal of R of all linear operators of V of finite rank. Then, by [11, Theorem 7.14], I is the only proper ideal of R. Then, by [14, Theorem 1], (0) is the only cyclic, injective module annihilated by the maximal two-sided ideal I of R. Let J be a maximal right ideal of R containing I. Since, I annihilates the simple R-module $S = R/J$ this implies S is non-injective R-module and hence $S \neq E(S)$. Let, $0 \neq x \in E(S) - S$. Let T be a submodule of $E(S)$ maximal with respect to the property that $S \subseteq T$ and $x \notin T$. Then, T is a cocyclic, essential submodule of $E(S)$ and $T \neq E(S)$. Hence, T cannot be a direct summand of $E(S)$. By Proposition 5.5(i), T is not a ccp submodule of $E(S)$. Since, R is a von Neumann regular ring, it implies, by [17, Propositions 11.1], that every short exact sequence of R-modules is pure. In particular T is a pure submodule of $E(S)$.

Proposition 6.2. *Over a commutative Noetherian (co-Noetherian) ring a pure submodule of an R-module is ccp submodule.*

Proof. Follows from [7, Proposition 12] and Proposition 5.15. □

Remark 6.3. *In general a ccp submodule of an R-module need not be pure submodule.*

Example: In [7, Remark 19] the first author gave an example for a copure submodule of an R-module, which is not pure. In that example the ring R is a right V-ring and by above Proposition 5.7, the same example serves our purpose.

7. CCP versus C-purity

Remark 7.1. *In general ccp does not imply cyclic purity.*

Example: In the above example after Remark 3.1, the ring R is a commutative von Neumann regular ring and hence V-ring. So, by Proposition 5.7, S is ccp in R. But it is proved that S is not cyclic pure in R.

Remark 7.2. *In general cyclic purity does not imply ccp.*

Example: In the above example after Remark 3.2, K is proved to be cyclic pure in F. But by the Corollary 5.11 above, K is not ccp in F, since $F/K = (X, Y)$ is not flat.

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74^{th} IMS CONFERENCE: A BRIEF REPORT

RAMJI LAL

The 74^{th} Annual Conference of the Indian Mathematical Society was held at the University of Allahabad, Allahabad during the period December 27 - 30, 2008 under the Presidentship of Prof. A. K. Agarwal, Centre for Advanced Study in Mathematics, Panjab University, Sector 14, Chandigarh 160 014. The Conference was attended by approximately 350 delegates from all over the country.

Prof. C. S. Seshadri, FRS, Director of the Chennai Mathematical Institute, Chennai, inaugurated the conference. The Inaugural function was held in the forenoon of December 27, 2008 in the historic Senate Hall of the University of Allahabad. The Vice Chancellor of the University Prof. R. G. Harshe presided over the inaugural function. Prof. Ramji Lal, Head, Department of Mathematics, University of Allahabad and the Local Secretary of the Conference offered a warm welcome to the delegates. The General Secretary of the Society Prof. V. M. Shah spoke about the Indian Mathematical Society and expressed his sincere and profuse thanks, on behalf of the Society, to the host for organizing the Conference. The Academic Secretary Prof. Satya Deo reported about the academic programmes of the Conference. Prof. V. M. Shah reported about Prof. P. L. Bhatnagar Memorial Prize for 2008. The Prize was awarded to the top scorers of the Indian Team at the International Mathematics Olympiad Anirudh Gurjale (Hyderabad) and Utkarsh Tripathi (Kanpur). Prof. A. K. Agarwal delivered his Presidential address (General) on " THE IMPORTANCE OF MATHEMATICS ". The function ended with a vote of thanks proposed by Prof. D. B. Chaudhuri of the Department of Mathematics, University of Allahabad.

The Academic Sessions of the conference began with the technical address of the President of the Society on "RAMANUJAN'S CONTRIBUTIONS AND SOME RECENT ADVANCES IN THE THEORY OF PARTITIONS" in the Senate Hall itself which was presided by Prof. V. M. Shah, senior most past president of the Society present in the Conference. The rest of the academic programmes were held in the Banerjii Hall of the Department of Mathematics and Ghosh Auditorium of Applied Physics Department.

Three plenary Lectures were delivered during the Conference - by Prof. C. S. Seshadri, Prof. S. G. Dani and Prof. M. Ram Murty, Also, three award lectures and Six invited Lectures were delivered by eminent mathematicians during the conference. Besides these, there were Six symposia on various topics of mathematics and computer science - including one dedicated to the Memory of late Prof. T. Pati. There was a paper presentation competition for various IMS and other prizes. In all about 120 papers were presented during the conference on various disciplines of mathematics and computer science. Half an hour condolence session was arranged on 30th December for late Prof. R. P. Agarwal, former President of the Society, before General body meeting and the valedictory function was held after lunch in which the IMS and other prizes were awarded by the president of the Society to the winners of the paper presentation competition. It was also announced that the next annual conference of the Society will be held in Kalaslingam University, near Madurai, Tamil Nadu.

Apart from academic sessions, a Cultural Evening was also arranged on December 27, 2008.

The conference was supported by the NBHM, CSIR, University of Allahabad, INSA, DST, some local Institutions and Union Bank of India.

PLENARY LECTURES

The details of the Plenary Lectures delivered is as follows:

1. **Professor C. S. Seshadri** (FRS and Director, Chennai Mathematical Institute) delivered a Plenary Lecture on *"Moduli and Monodromy"*.
   Chairperson: H. P. Dikshit.
2. **Professor S. G. Dani** (TIFR, Mumbai; Chairman, NBHM) delivered a Plenary Lecture on *"Some large subsets of Euclidean spaces involved in Geometry, Dynamics and diophantine approximation"*.
   Chairperson: Prof. Satya Deo.
3. **Professor M. Ram Murty**(FRSC, Queens University, Canada) delivered a Plenary Lecture dedicated to the memory of late Prof.

R. P. Agarwal on *"Transcendental Numbers and zeta functions"*.
Chairperson: Prof. N. K. Thakare.

MEMORIAL AWARD LECTURES

The details of the Memorial Award Lectures delivered is as follows:

1. 22^ed **P. L. Bhatnagar Memorial Award Lecture** was delivered by Prof. G. S. Srivastava, Professor of Mathematics, Indian Institute of Technology Roorkee, Roorkee - 247 667 (Uttarakhand), India. He spoke on *"ENTIRE FUNCTIONS EXTENSIONS OF CONTINUOUS FUNCTIONS - A BRIEF SURVEY"*.
   Chairperson: Prof. Manjul Gupta.
2. 19^th **V. Ramaswamy Aiyer Memorial Award Lecture** was delivered by Professor S. Arumugam, Kalasalingam University, Anand Nagar, Krishnankoil-626 190 (Tamil Nadu), India. He spoke on *"COLORING AND DOMINATION IN GRAPHS "*.
3. 19^th **Hansa Raj Gupta Memorial Award Lecture** was delivered by by Professor S. D. Adhikari, Harish-Chandra Research Institute, Allahabad - 211 009 (UP), India. He spoke on *"THE RELATION BETWEEN TWO COMBINATORIAL GROUP INVARIANTS : HISTORY AND RAMIFICATIONS "*.
   Chairperson: Prof. A. K. Agarwal.

SPECIAL SESSION OF PAPER PRESENTATION COMPETITION FOR VARIOUS PRIZES

During the special session of paper presentation competition, research papers were presented for the award of Six IMS Prizes (areas : Algebra, Geometry, Topology, Functional Analysis, Discrete Mathematics, Number Theory, Operations Research, Fluid Dynamics, Biomathematics and Computer science), AMU Prize (areas : Algebra, Functional Analysis, Differential geometry) and V. M. Shah Prize (in Analysis). Prof. Satya Deo (Chairperson), Prof. Peeyush Chandra, Prof. G. S. Srivastava and Prof. B. N. Waphare were the judges. The result was as follows:

1. IMS Prize (Algebra) : No entry.
2. IMS Prize (Geometry) : No entry.
3. IMS Prize (Topology) : No entry.
4. IMS Prize (Functional Analysis) : No Prize awarded.
5. IMS Prize (Descrete Mathematics): No entry.
6. IMS Prize (Number Theory) : No entry.

7. IMS Prize (Operations Research) : No entry.
8. IMS Prize (Fluid Dynamics) : No entry.
9. IMS Prize (Biomathematics) : P. K. Srivastava (Kanpur).
10. IMS Prize (Computer Science) : No entry.
11. AMU Prize (Algebra) : Sakir Ali (Aligarh).
12. VMS Prize (Analysis) : Ibrahim Salebhai (Surat).

INVITED LECTURES

The details of the half an hour invited Lectures delivered are as follows:

1. **Dipendra Prasad**(TIFR, Mumbai):
   "On self-dual representations of p-adic groups, and the Langlands Functoriality".
2. **N. S. N. Sastry**(ISI, Bangalore):
   "Codes, permutation groups and finite geometries".
3. **Ravindra Shukla**(University of Allahabad, Allahabad):
   "Structure of some codes related to $Sp(4, q)$, *q even".*
4. **A. Ojha**(PDPM IIIT, Jabalpur):
   "Constrained Curve Drawing Problems in CAD".
5. **H. K. Mukherjee**(NEHU, Shillong).
6. **B. N. Waphare**(University of Pune, Pune):
   "Birkhoff's problem concerning posets".
7. **Milind Kulkarni**(P G Center, Goa):
   "Value distribution of Differential Polynomials".

SYMPOSIA

Details of symposia organized is as under:

1. On *"Fourier Analysis and Allied Topics".*
   Organizer: H. P. Dikshit (Director General, School of Good Governance and Policy Analysis, Bhopal), Dedicated to the Memory of late Prof. T. Pati.
   Speakers:
   - U. B. Tewari (I.I.T. Kanpur, Kanpur):
     "The Fourier Algebra on Locally Compact Groups and Completely Bounded Multiplers".
   - R. S. Pathak (BHU, Varanasi):
     "Wavelet Transforms".
   - P. N. Rathie (Brazil):
     "Some aspects of one-dimensional stable Levy and related Distributions".

- H. P. Dikshit (Bhopal):
  "Lebesgue Constants and Norlund Means".

2. On *"Modular forms and L-functions"*.
   Organizer: Ramakrishnan Balakrishnan (HRI, Allahabad).
   Speakers:
   - M. Ram Murty (Kingston):
     "Transcendence of certain Petersson inner products".
   - V. Kumar Murty (Toronto):
     "The field of Fourier coefficients of a modular form".
   - Dipendra Prasad (TIFR, Mumbai):
     "Central critical L-values, and representations".
   - S. Kanemitsu (Japan):
     "Dedekind sums and Riemann's posthumous fragemets".
3. On *"Operator Theory"*.
   Organizer: Raj K. Singh (Lucknow).
   Speakers:
   - S. C. Arora (Delhi University):
     "Slant Toeplitz Operators and their generalizations".
   - B. P. Guggal (UK):
     "Properties of linear operators through Invariant subspaces".
   - O. P. Singh (IT, BHU, Varanasi):
     "Applications of Wavelets in Science and Technology".
   - Harish-Chandra (BHU, Varanasi):
     "Invariant subspaces of Composition Operators".
4. On *"Algebra"*.
   Organizer: Dinesh Khurana (IISER, Mohali, Chandigarh).
   Speakers:
   - Dipendra Prasad (TIFR, Mumbai):
     "On self-dual representation of groups".
   - J. K. Verma (IIT Powai, Mumbai):
     "On the Chern number of an ideal".
   - S. K. Khanduja (Punjab University, Chandigarh):
     "On Dedikind Criterion and simple extensions of Valuation Rings".
   - Chanchal Kumar (IISER, Mohali, Chandigarh):
     "Cellulor Resolutions of Generalized Permutohedorn Ideals".

- Manoj Yadav (HRI, Allahabad):
  "On finite p-groups whose all automorphisms are class-preserving".
- Selby Jose (Mumbai):
  "Unimodular Rows and Orthogonal Transformations".

5. On *"Fuzzy Sets and Systems".*
   Organizer: A. K. Srivastava (BHU, Varanasi).
   Speakers:
   - M. K. Chakraborty (University of Calcutta, Kolkata):
     "Rough sets and Fuzzy sets: A Unified Categorical approach".
   - Wagish Shukla (IIT Delhi, Delhi):
     "Fuzzy sets and Quantum computations".
   - Mohua Banerjee (IIT Kanpur, Kanpur):
     "Rough sets: Algebraic aspects".
   - M. N. Mukherjee (University of Calcutta, Kolkata):
     "Fuzzy Grills: Some Applications".
   - A. K. Srivastava (BHU, Varanasi):
     "Sierpinski Objects, their associated Rough sets and Fuzzy sets: A unified Categorical approach Epireflective and Coreflective Subcategories, and injective objects in Fuzzy Topology".
6. On *"Mathematical Ecology".*
   Organizer: Bindhya Rai (Allahabad University, Allahabad).
   Speakers:
   - Peeyush Chandra (IIT Kanpur, Kanpur):
     "Mathematical Ecology with an example in Marine ecology".
   - Suneeta Gakkhar (IIT Roorkee, Roorkee):
     "The complex dynamic behaviors in Food Webs".
   - O. P. Mishra (Jiwaji University, Gwalior):
     "Transmission and control of Japanese Encephalitis: A mathematical model".
   - Manju Agarwal (Lucknow University, Lucknow):
     "A note on Prey-predator model of stage structure".
   - P. N. Pandey (Allahabad University, Allahabad):
     "Starfish Predation of a Growing Coral reef community".

P. L. BHATNAGAR MEMORIAL PRIZE FOR 2008

At the 49thh International Mathematical Olympiad, held during July 10 - 22, 2008 at Madrid, Spain. The Indian team secured the 31st position among 97 countries, winning five Bronze medals and one Honourable Mention. **Anirudh Gurjale**(Hyderabad) and **Utkarsh Tripathi**(Kanpur) were the top scorers with 19 points each. They were awarded the P. L. Bhatnagar Memorial Prize (carrying a Certificate and a cash award of Rs.1000/- each) for 2008 in the inaugural session.

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REPORT ON IMS SPONSORED LECTURES

The details of the IMS Sponsored Lectures/Popular Talks held are as under:

1.	Speaker	:	**Prof A. R. Rao** (Aged 100 years!) Community Science Center, Ahmedabad.
	Title of the Popular Talk	:	**Some non-routine Problems**.
	Date	:	Monday, September 08, 2008.
	Venue	:	Department of Mathematics, Gujarat University, Ahmedabad - 380 009.
	Organizer	:	Prof. N. R. Ladhawala.
2.	Speaker	:	**Prof Subhash J. Bhatt**. Department of Mathematics, Sardar Patel University, V. V. Nagar - 388 120.
	Title of the Popular Talk	:	**Harmonic Analysis on Groups with weights**.
	Date	:	January 29, 2008.
	Venue	:	Department of Mathematics, Gujarat University, Ahmedabad.
	Organizer	:	Prof. N. R. Ladhawala.
3.	Speaker	:	**Prof V. Kumar Murty**. Professor of Mathematics, University of Toronto, Canada.
	Title of the Lecture	:	**Artin L-functions near $s = 1$**.
	Date	:	January 14, 2008.
	Venue	:	H R I, Allahabad.
	Organizer	:	Prof. Satya Deo.

Abstract: The concept of L-functions is central to many problems in number theory. These functions are usually defined as a Dirichlet series convergent in some half plane by using local data. They are expected to have analytic continuation in a larger domain, and in that domain, they seem to exhibit "knowledge" of global phenomenon. This mysterious behavior is what makes L-functions so intriguing and useful.

Artin L-functions are defined using finite dimensional complex representations of absolute Galois group of a global field. Their behavior near $s = 1$, the first point at which one may need analytic continuation, is interesting both from an arithmetic and analytic point of view.

In this talk we discuss some recent results on the growth of Artin L-functions near $s = 1$. In particular, we report on some recent progress on Kummer's conjecture on the growth of the first factor of the class number of a cyclotomic field.

4.	Speaker	:	**Prof Dr. Kyo Nishiyama**. Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 605-8502., Japan.
	Title of the Lecture	:	**Representation Theory & Recent Advances**.
	Date	:	September 25, 2007; 3.00 p.m.
	Venue	:	Department of Mathematics, Central College Campus, Bangalore University, Bangalore.
	Organizer	:	Dr. M. S. Mahadeva Naika.
5.	Speaker	:	**Prof S. G. Dani**, Chairman, NBHM. School of Mathematics, T. I. F. R., Mumbai - 400 005.
	Title of the Lecture	:	**The Number Line and the Modular Group**.
	Date	:	Monday, September 22, 2008; 12.00 noon.
	Venue	:	Department of Mathematics, Panjab University, Chandigarh.
	Organizer	:	Prof. Madhu Raka.
6.	Speaker	:	**Prof K. Parthasarathy**. Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai - 600 005.

Title of the Lecture : **The Sphere - some explorations**.
Date : Friday, October 26, 2007; 11.30 a.m.
Venue : Department of Mathematics, Faculty of Science, University of Mysore, Mysore.
Organizer : Prof. Madhu Raka.

Abstract: In an interesting way some elementary properties of the unit sphere $S^n, n \geq 1$, were discussed from different perspectives - algebraic, analytic, geometric and topological. Some of the topics mentioned for discussion are

- Group structure on S^1 and on S^3 (as real quaternions of norm 1 or as special unitary group $SU(2)$); the absence of a (Lie) group structure on $S^n, n \neq 1, 3$.
- Realization of S^{n-1} as a homogeneous space

$$SO(n)/SO(n-1)o(n)/O(n-1)$$

via the natural action of $O(n)$ on S^{n-1}.
- Surface measure on S^n and spherical harmonics.
- Geodesics on S^2 and non-Euclidean geometry.
- Borsuk - Ulam theorem and the pancake theorem.
- $S^n, n > 1$, as a compact simply connected manifold.
- Poincare conjecture.

7. Speaker : **Prof A. R. Aithal**.
University Department of Mathematics, of Mumbai, Mumbai - 400 098.
Title of the Lecture : **Shape Calculus**.
Date : Tuesday, October 30, 2007 ; 11.30 a.m.
Venue : Department of Mathematics, Faculty of Science, University of Mysore, Mysore.
Organizer : Prof. Soner N. D.

Abstract: In a very lucid way, the speaker introduced the basic and elementary concepts of Riemannian manifold (M, g) with the unique solution of the Dirichlet Boundary Value Problem. He started with definitions and then proceeded to give basic results of the theory indicating on the way the different directions where researches are going on and suggesting several references.

8.	Speaker	:	**Prof Remy Y. Denis**. Department of Mathematics, Gorakhpur University, Gorakhpur - 273 009.
	Title of the Lecture	:	**Hypergeometric Functions and Continued Fractions of Ramanujan**.
	Date	:	Friday, October 03, 2008; 11.45 p.m.
	Venue	:	Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Vadodara - 390 002.
	Organizer	:	Dr. J. R. Patadia.
9.	Speaker	:	**Prof Ramesh R. Tikekar**. Department of Mathematics, S. P. University, Vallabh Vidyanagar - 288 120.
	Title of the Popular Talk	:	**Geometry and Universe**.
	Date	:	Thursday, September 25, 2008; 11.45 p.m.
	Venue	:	Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Vadodara - 390 002.
	Organizer	:	Dr. J. R. Patadia.
10.	Speaker	:	**Dr. A. K. Desai**. Department of Mathematics, Gujarat University, Ahmedabad - 380 009.
	Title of the Popular Talk	:	**Finite Geometries**.
	Date	:	Monday, February 4, 2008 ; 11.45 a.m.
	Venue	:	Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Vadodara - 390 002.
	Organizer	:	Dr. J. R. Patadia.
11.	Speaker	:	**Dr. D. J. Karia**. Department of Mathematics, S. P. University, Vallabh Vidyanagar - 388 120.

	Title of the Popular Talk	:	**The Banach Algebra of Functions with limits**.
	Date	:	Thursday, December 20, 2007; 12.45 p.m.
	Venue	:	Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Vadodara - 390 002.
	Organizer	:	Dr. J. R. Patadia.
12.	Speaker	:	**Prof Poornima Raina**. Department of Mathematics, University of Mumbai, Mumbai.
	Title of the Popular Talk	:	**Random Walk**.
	Date	:	Monday, February 05, 2007; 12.30 p.m.
	Venue	:	Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Vadodara - 390 002.
	Organizer	:	Dr. J. R. Patadia.
13.	Speaker	:	**Prof P. Shankaran**. Institute of Mathematical Sciences, Taramani P. O., Chennai.
	Title of the Popular Talk	:	**Poincare Conjecture**.
	Date	:	Friday, January 05, 2007; 12.15 p.m.
	Venue	:	Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Vadodara - 390 002.
	Organizer	:	Dr. J. R. Patadia.

Abstract: In the year 1904 the French mathematician Henri Poincare asked the question whether every compact connected three-dimensional manifold without boundary which is simply connected is homeomorphic to the three-dimensional sphere S^3. The assertion that it is so has been known as the Poincare conjecture. Topologists have been fascinated by this problem and had made many attempts to solve it. Their efforts lead to great developments in geometric topology enriching our understanding, in particular, of the three-dimensional topology.

The Poincare conjecture can be generalized to higher dimensions; it was solved in 1960 for dimensions at least five by Stephen Smale which won him the Fields Medal in the year 1966. The solution for dimension four had to wait another 22 years, which was obtained by Michael Freedman in 1982 for which he received the Fields Medal in the year 1986.

In the year 2003, Grisha Perelman posted in the ArXiV {http:// xxx.imsc.res.in} his proof of the Poincare conjecture using earlier work of Richard Hamilton. It was declared in the International Congress of Mathematicians, 2006, held at Madrid, that Poincare conjecture had been proved. Although Perelman was awarded the Fields Medal he refused to accept it.

In his talk, the speaker discussed the classification problem for compact connected surfaces (= two-dimensional manifolds) and the statement of Poincare conjecture was explained. Poincare's example of a homology three-manifold which is not homeomorphic to the three-sphere was also recalled. A generalization due to W. P. Thurston who was awarded the Fields Medal in 1983, known as the geometrization conjecture was also stated.

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FORM IV

(See Rule 8)

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I , V. M. Shah, hereby declare that the particulars given above are true to the best of my knowledge and belief.

V. M. SHAH

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Back volumes of both the periodicals, except for a few numbers out of stock, are available. The following publications of the Society are also available: (1). Memoir on cubic transformation associated with a desmic system, by R. Vaidyanathswamy, pp. 92, Rs. 250/- (or, $ 10/-), and (2). Tables of partitions, by Hansaraj Gupta, pp. 81, Rs. 350/- (or, $ 15/-).

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