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**THE  
MATHEMATICS  
STUDENT**

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Edited by

**J. R. PATADIA**

(Issued: October, 2008)

**PUBLISHED BY  
THE INDIAN MATHEMATICAL SOCIETY**  
(In the memory of late Professor M. K. Singal)

# THE MATHEMATICS STUDENT

Edited by

J. R. PATADIA

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\topmargin=1.5 cm
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**Mahendra Kumar Singal**  
(November 10, 1932 - July 16, 2006)

#### **OBITUARY**

#### **Professor Mahendra Kumar Singal (1932-2006)**

Where do you begin to describe a man like Dr M K Singal? Do you talk about his life, his strong yet kind character, his indefatigable enthusiasm for work and his passion for Mathematics, or about the people whose lives he touched every day?

Born on November 10, 1932, Professor M K Singal passed his Matriculation Examination at the early age of 12 years. He got one of the five All India Entrance Scholarships. At the B.A. (Mathematics Honours) Examination, he was awarded the Bholanath Gold Medal and the Rabikanta Debi Prize.

In his B.A. Examination, he topped in seven of the eight papers but by some incomprehensible happenings somewhere, he got only 80 marks instead of a 100 in one paper. Frustrated, he left Mathematics and joined Law instead. In the BCL Examination, it was compulsory to clear the Marriage Law paper in order to pass. Unfortunately he had a bad boil in his right hand and could not write this paper. He still got the highest marks with a zero counted in this paper. There was no provision for a re-examination in Delhi University at that time but on a request from his father on the plea of still highest marks, the Vice Chancellor Sir Maurice Gwayr allowed a re-examination. This created history. After obtaining an L.L.B., he came back to Mathematics. He had realized that Law was not a career for him as that often involved working in favour of the client by hook or by crook. He opted for an honest career.

On the performance of his M. A. (Mathematics) Examination, he was awarded a scholarship to carry out research. However, at the request of the Principal of Ramjas College, Delhi University, he forsook the research

scholarship and took up a Lectureship at Ramjas College to which he was invited without application and interview.

He appeared for the IAS Examination but was rejected in personal interview. At that time, one had to clear both the theory and the personal interview separately. Yet he had obtained the 5th position in aggregate marks among all the students that appeared that year. That started a public debate. Should such a brilliant student be rejected just because, may be, some of the interviewers were biased? The matter reached the Parliament and an act was passed whereby only the aggregate marks were to be considered. Fortunately for the Mathematics Community, the rule was not applied retrospectively and he became a mathematician rather than an IAS Officer.

He pursued research along with his heavy work-load of teaching 32 periods a week. He obtained his Ph.D. degree in Differential Geometry. His research interests were in the area of Differential Geometry, General Topology and Mathematics Education. He supervised the Ph.D. research work of more than a dozen students and published 75 research papers.

He became a Reader in Mathematics at Delhi University in 1964. In 1969, he was invited to set up the Mathematics Department at the newly formed Meerut University. He joined Meerut University as Founder-Professor and Head of the Mathematics Department and also served tenure at the same University as the Vice-Chancellor.

Professor Singal was a Fellow of the National Academy of Sciences (India) and also that of the Institute of Mathematics and its Applications, UK. He travelled and lectured in dozens of countries. At the home front also, he lectured in a large number of cities, Staff Academic Colleges, as well as in most of the colleges and a large number of schools in Delhi. He was also a reputed author. Of the odd fifty books authored by him, *Algebra*, *Matrices*, *A first Course in Real Analysis*, *Topics in Analysis I and II*, *Mathematics for the Physical Sciences*, *Mathematics for the Life Sciences*, *A Complete Course for ISC Mathematics*, *Elements of Algebra*, *Elements of Calculus*, and *Olympiad Mathematics* are the current popular titles.

Professor Singal was connected with the Indian Science Congress Association (ISCA) in various capacities: as President of the Mathematics Section, General Secretary (Outstation), Platinum Jubilee Lecturer, and Member of the Executive Committee/Council continuously since 1979. He received the Distinguished Service Award of ISCA at the hands of the Late Shri Rajiv Gandhi and the Srinivasa Ramanujan Birth Centenary Award at the hands of the Late Shri P V Narsimha Rao, both the then-Prime-Ministers of India, for his services to the cause of scientific enquiry in general, and Mathematics in particular.

He produced six video-lectures for the undergraduate Mathematics students and also a video film PARABOLA for the Country-wide Classroom of



the UGC. As Director of the UGC sponsored Project *University Leadership Project*, he developed the first Objective Type Questions Bank at Meerut. He also established a Mathematics Museum containing photographs of 200 eminent mathematicians, 350 mathematical charts, 75 unique polyhedron models, and a large number of books, games and puzzles. This museum is unique in the whole of the country.

Professor Singal desired to put India on the world map of Mathematics. He undertook, along with Late Prof P L Bhatnagar, to bring about the culture of Mathematical Olympiads in the country. His efforts bore fruit. Today India is a visible force in International Mathematical Olympiad (IMO). It was the Math Olympiad for him was not just about the IMO, but also about training lacs and lacs of students in the culture of Mathematics. He established the Indian Mathematical Olympiads Foundation that conducts Mathematical Olympiads every year at school level to nurture the mathematical talent of young students. About 15,000 students spread over 200 schools participate in this competition. Math Olympiad is a domestic term in schools today. He also produced a book *Olympiad Mathematics* for students of secondary classes. He was in the process of finalizing a similar book for students of the middle schools when God decided he was needed elsewhere.

Professor Singal's association with Indian Mathematical Society (IMS) has been a great service to the Mathematical Community of India. He took reigns of the Society as Secretary at a time when the Society and its publications seemed to be losing steam. With his efforts, the Society turned a decisive corner. He collected and persuaded a number of friends to hold various offices in IMS, work diligently with dedication and devotion, and make IMS a vibrant Society. By the time he relinquished the office of Secretary after three years, IMS had already found a sound footing. On this occasion a friend appropriately paid tributes to his work by writing, *no secretary ever found the Society in a worse shape nor left it in a better shape*. He continued to hold other offices such as President, Treasurer, Editor, and Administrative Secretary in the Society and to be the guiding force behind the Society. The cause of IMS was very close to his heart and he worked tirelessly for the Society till his last day.

Another remarkable service Dr. Singal rendered to the cause of Mathematics was his observance on April 13 at Delhi of *A Date with Mathematicians*. It was his dedication and tenacity that has been responsible for the continuance of this activity year-after-year at the same day (April 13, the *Baisakhi* day) at the same place (Physics Lecture Theatre of the University of Delhi) for a long period of 29 years. This activity is oriented towards the University and College Teachers of Delhi. The event comprises of three Lectures by eminent scientists, conferring a Distinguished Service Award on an eminent scientist to recognize his significant contribution to Science and

Society, and felicitating one or two Senior Teachers in line with our age-old golden tradition *Guroor Brahma* . . .

Prof Singal was not only a Mathematician extra-ordinaire; he equally excelled in many subjects like Physics, Chemistry, History, Geography, Philosophy, and Literature. He had written not only on Mathematics but he also wrote text-books on History, Hindi and Geography. He had travelled widely in India and knew about the history and culture of many places of historical value.

He was a great visionary and the expanse of his dreams had a surprising depth and the potential of being realized. He served the cause of Mathematics in a myriad ways. Besides all his mathematical work, he also found time to force Science Academies, Indian Science Congress Association (ISCA), and the Government (through his discussions with the various Prime ministers every year at the ISCA Sessions in January) to accord due respect to mathematicians.

Among his personal assets, he had an excellent memory, a keen observation power, surprising foresightedness and a great tenacity and capacity for hard work. There was a rare 100-page Law book which his father was able to borrow for one night. He memorized the book during the night and put it in black and white the next day. He remembered the names of all his students and most of the time could tell which year a particular student was with him and who all were his class-fellows. He was a lover of nature and enjoyed music immensely. He helped all those who were around him.

He was no ordinary person; he was an Institution unto his numerous friends and followers. He spent ten life-spans in serving the cause of Mathematics in a single life. He left an indelible impact on the Mathematical Scene in India. As they say, he passed away on July 16, 2006. But did he really? In line with *Yam's* answer to *Nachiketa's* query about the abode of a person on leaving the world, Prof Singal lives in the numerous books, papers and articles written by him. He lives in the various activities started by him and taken up by his students, family and friends. He lives in the *Mahendra Kumar Singal Mathematics Trust* formed by his followers to carry on in his footsteps. The motto of the Trust is *Charaiveti-Charaiveti* (Keep going, keep going). The journey he started continues. May God bless his soul.

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## PROFESSOR M. K. SINGAL: AN ORGANIZER PAR EXCELLENCE

SATYA DEO

I first saw Prof M. K. Singal way back in 1970 as an expert in a mathematics faculty appointment selection committee. As the youngest professor of mathematics in that panel of experts, he seemed to display an exemplary confidence and depth of mathematical knowledge. Candidates coming out of the interview would declare that only he/she will be selected who can impress the young professor from Meerut who was none other than Prof M. K. Singal. His questions in the interview were directed to test basically the potential of a candidate whether or not he would be a good mathematics teacher of a university. It was only after several years when I discovered that he himself was one of the most popular teachers of Delhi University.

After a gap of about ten years I again met him in a conference on topology where I learnt about his group of research scholars and his mathematical family- at that time all of them worked on problems of set topology around paracompactness and its various generalizations. It was not difficult to conclude that Prof Singal himself was a self-read mathematician and yet had attracted a good number of Ph.D. students working under his supervision. Prof Singal's knack for meeting eminent mathematicians in the country, finding all about their accomplishments starting from student days onward and then getting them in some conference to deliver an invited talk was simply extraordinary. Once the programme of an event has been chalked out, he would make sure that everything is done in a systematic and remarkably formal way. There will be a good introduction of the speaker by him in each lecture which will be quite elaborate and highly attractive, then he will join the audience and listen to the lecture attentively, will make sure that no one disturbs the speaker, will himself rise and move very quietly if there is some urgent message to be attended to by himself, will propose quite an impressive vote of thanks to the speaker, and finally will join in the applause to conclude the lecture. I never saw him in hurry even if it was

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the last minute of the last lecture of the last session of the last day!! All of this had the obvious affect that in the events organized by him, the speakers would definitely turn up, will reach at the right time and will give the best of what they had to say.

Annual Conferences of the Indian Mathematical Society, annual conferences of the Mathematics Section of the Indian Science Congress, "April 13 - A date with mathematicians", annual conference of the Delhi Chapter of the Indian Mathematical Olympiads are some of the regular programmes which he organized successfully year after year for more than two decades. Memorial lectures in the honour of our famous Indian Mathematician like, to name a few, Srinivas Ramanujan, Prof P. L. Bhatnagar of IISc, Bangalore, Ramaswamy Aiyer- the founder of the Indian Mathematical Society, Hansraj Gupta - the famous self-made number theorist from Panjab University, Chandigarh, Prof U. N. Singh - Pro Vice Chancellor of Delhi University and a Vice Chancellor of Allahabad University, Prof D. S. Kothari, the famous physicist and author of the Education commission of India etc. are notable examples of his superb ability to organize something and leave it for future generations of mathematicians to nourish and keep them alive. He was never tired or disappointed by small failures which may have occurred in arranging any of these academic activities throughout his life. Most of these activities were implemented by Prof Singal while he was Professor and Head of Mathematics Department at Meerut University - in fact Prof Singal and Meerut became synonyms in the Indian Mathematical world at least for a few years. His close family members, like his wife who herself was Professor as well as Head of the Department of Mathematics, and also a Vice Chancellor of Meerut University; his sister-in-law who practised mathematics teaching and research from a College of Delhi University, his own sister who also taught mathematics in a College of Delhi University etc, all cooperated him to their best of abilities in his academic activities. He has really left a mark on Indian Mathematics for organizing above mentioned activities with utmost regularity and perfection, and for this quality Professor Singal will definitely be remembered for a long time to come.

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## A DATE WITH MAHENDRA KUMAR SINGAL

A. N. MITRA

Mahendra Kumar Singal was sound Mathematician, an excellent organizer, a humane personality, and a good personal friend. His premature death has left a big void not only in his immediate family but among his wider circle of friends and admirers in which I have a claim for inclusion. It was therefore appropriate that the first Baisakhi day, 13 April 2007, after his passing away, was devoted to his memory.

It was on a Baisakhi day, 13 April 1978, that he gave effect to his novel idea of honouring his departed teachers under the grand name of “Date with MATHEMATICIANS”! In this venture he had the firm support of his distinguished sister, Prof Sashi Prabha Arya, all working under the gentle guidance (and active involvement) of a famous Mathematician, the late Prof Jagat Narain Kapoor. The celebrations have over the years had acquired a characteristic pattern: Morning and afternoon lectures by eminent mathematicians/Scientists on highly topical subjects of Mathematical and physical interest; an evening function in which distinguished personalities with a long record of service to the cause of Mathematical Sciences are honored. It is therefore in the fitness of things that celebrations on 13 April 2007 should start with a lecture in the memory of the very Founder of this series.

I may be forgiven if I recall a few personal reminiscences on this occasion. Prof Singal started this series with a memorial lectures in honour of his teacher (and mine) the late Prof J. N. Mitra the very first one that was inaugurated most appropriately by a long term colleague of Prof Mitra, viz Prof B. R. Seth, and presided over by Prof. R. C. Majumdar (Who happened to be my teacher in physics in addition to the guidance of my late father. Now Father had a large constituency of students, we were only two, all of whom received equal attention from him. This spectacle was most manifest in the afternoons at our home in Anand Parvat and subsequently

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in Daryaganj. The usual pattern was that every afternoon some 5-10 of his students gathered for interactive lessons in Mathematics, all the different topics of relevance to the mathematics syllabus. Needless to add, these lessons were given strictly for love's labour any sinecure aspects were totally unknown in those days. It was this special relationship of Father with his students that perhaps singled him out for a special place in their hearts all for the altruistic love for mathematics! Prof Singal and I were only two of them. But it had never occurred to any of his students (Except one!) to perpetuate this wonderful memory for all the years since his death in 1970, until Prof. Singal came up with his brilliant idea in 1978 which quickly expanded to include memorial lectures in honour of several other Professors of ours who kept disappearing from the scene in quick succession (Prof. R. S. Verma, Prof. Ram Behari, Prof. B. R. Seth, culminating in the memorial to Prof. D. S. Kothari who represented a synthesis of physics and mathematics). And the tremendous enthusiasm for attending the *DATE WITH MATHEMATICIANS* year by year has been there for all to witness. All kudos to Prof. Singal for this wonderful feat. And with his passing away in July 2006, he has himself joined their ranks with effect from the Baisakhi Day of 2007! His memory will outlive him by many many years.

I am thankful to Prof. Mrs Singal and Prof. S. P. Arya for this opportunity to associate myself with this volume of the Mathematics Student.

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## PROFESSOR M. K. SINGAL: A TRIBUTE

N. K. THAKARE

The sad demise of Professor M. K. Singal has left a void in the functioning of the Indian Mathematical Society (IMS). In early (nineteen) eighties the IMS was sailing through rough weather. Professor M. K. Singal was greatly instrumental for bringing out the IMS from this untoward situation. Professor J. N. Kapur, R. P. Agarwal, V. M. Shah, M. K. Singal, I. B. S. Passi and A. M. Vaidya took charge of the affairs of the IMS and the things changed dramatically. Some regularity in publishing the two periodicals namely the Journal of the Indian Mathematical Society and the Mathematics Student was achieved. The monetary condition of the IMS started improving.

Professor Singal was associated not only with the IMS, but with many more scientific societies such as Mathematical Association of India, Indian Science Congress Association (ISCA), National Academy of Sciences and what not. The Platinum Jubilee Session of ISCA was held at Pune in January 1988. The then Prime Minister of India, Mr. Rajiv Gandhi inaugurated the Platinum Jubilee Session on 3<sup>rd</sup> January, 1988. After the inaugural function, members of the Executive Council of ISCA, the Sectional Presidents and local organizing secretaries had a lunch with the honourable Prime Minister. Being members of ISCA Council both Professors M. K. Singal and S. P. Arya were also present for the lunch. By chance, Member of Parliament Mr. V. N. Patil who hailed from North Maharashtra, accompanied Mr. Gandhi. No sooner Mr. Patil spotted me, he said, "Come on, I shall introduce you to the Hon'ble Prime Minister." I said, "It would be a unique opportunity for me" and then I requested the M. P. that we would like to discuss some important issues alongwith some mathematicians present for the lunch. Thus, Professor M. K. Singal, Dr. S. P. Arya and myself had a very fruitful dialogue with the Hon'ble Prime Minister regarding the non-inclusion of Mathematics under the Department of Science and Technology, Government of India. Professor Singal forcefully put forward the point that

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Mathematics- considered as the queen of sciences - was not included in the list of subjects that came under the Department of Science and Technology for funding purposes. Mr. Rajiv Gandhi was so much convinced about the subordinate treatment being meted out to the subject of Mathematics that he asked his secretary (Mr. Chaube, if I remember it correctly) accompanying him to move the matter with utmost urgency. Both Professor Singal and Dr. Arya, after their return to Delhi, pursued the matter. Finally, around 1992 or so Mathematics was brought under the Department of Science and Technology on par with other subjects for funding and allied purposes. Since then substantial funds are being earmarked for Mathematics under DST. Thus, in a real sense of the term, Professor Singal was instrumental in rooting out the injustice that was being done to Mathematics.

Professor Singal was the President - Elect of the IMS and was to take over the charge of Presidentship from 1st April 1988. While attending the Platinum Jubilee Session of ISCA in January 1988, he asked me whether I would be able to host the 54th Annual Conference of the IMS in December 1988 at Pune. He said "I am highly impressed by your University and the way you managed day-to-day activities of the Section of Mathematics as the Local Secretary. As I am to take over the Presidentship of the IMS, I shall feel happy if this year's Annual Conference can be organized at Pune".

His offer was plain and straight. Through 1986 to January 1988, we organized several academic activities at the Department of Mathematics. I was rather reluctant to accept the begin suggestion of Professor Singal. However, I couldn't say "No" to him. We had 3-week long refresher course in April/May 1987, a workshop in Coding Theory in September/October 1986, Ramanujan Birth Centenary International Symposium on Analysis in December 1987, the Platinum Jubilee Session of ISCA in January 1988; hence my reluctance. But there was such an ardent appeal in the suggestion of Professor Singal that I had no courage to say "No" to him.

Professor Singal gave me complete freedom in organizing the 54th Annual Conference of the IMS at Pune. That was perhaps the last instant wherein the Local Organizing Secretary shouldered the responsibilities of academic programmes, as well as funding and everything connected with organizing an All India Event. The academic programme of the conference was of high order. But what stood out was the record attendance of 448 delegates for the conference. To this date this record is undisturbed and Prof. Singal



was the President of the IMS.

Under the leadership of Professor R. P. Agarwal and with the active association of Professors M. K. Singal, V. M. Shah, S. R. Sinha and D. N. Verma the old constitution of the IMS was modified to help in streamlining the IMS academically, legally and financially. An additional post of Administrative Secretary was created to strengthen the functioning of IMS. Duties and functions of the General Secretary, the Academic Secretary, the Administrative Secretary were formulated in detail by them. Up to his last breath he was tirelessly working for the IMS. The future generations of Mathematicians will always cherish his yeoman contribution in strengthening the IMS. In him we lost a great human being, a great organizer and a prolific mathematician.

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## PROFFESOR M. K. SINGAL: A LIFE DEVOTED TO MATHEMATICS

SUNDER LAL

It is difficult for me to accept the fact that Prof. M. K. Singal is no more. He was much more than a teacher to me. In fact he was my mentor in almost all activities of my life during the last 37 years.

I came in his contact in 1969 when I joined Meerut University as an M. Phil. student. Prof. Singal had just joined as founder Professor and Head of the Mathematics Department of the University, the University where the seeds of M. Phil were geminated and which at time was perhaps the only university in the country awarding M. Phil. Degree in mathematics. During these 37 years of my close association with him I have seen many shades of his personality. First and foremost he was an extremely inspiring teacher. During more than 23 years of his stewardship at Meerut he, together with his colleagues, trained, initiated and inspired a large number of would be teachers. Through COSIP, COSIST and the University Leadership Project, he made lasting qualitative changes in the teaching and curriculum of Mathematics in that part of the country. The collection of objective type questions prepared under his directorship of University Leadership Project is among the earliest question-banks in undergraduate mathematics. His impact on teaching of mathematics is not confined to just undergraduate mathematics. Through his association with NCERT and through his initiative and involvement in Mathematics Olympiads, he made great service to mathematics education at the school level and the secondary level too.

As a researcher his influence was even greater so far as the region of Meerut University is concerned. Before his joining as a professor, the land was almost barren in the matter of research specially in modern topics. In 1967, there were just 2 mathematics teachers having Ph. D. degree in the entire Agra University region which is now covered by as many as seven state universities. For teachers of this region he organized more than a dozen

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summer schools and as many conferences to introduce modern topics, for teaching and research at Meerut University. It was mainly due to his effects that the barren land of Meerut University was put on the national map as a research activity center in mathematics.

He was a great administrator. He occupied important administrative positions in The Mathematical Association of India, The Indian Science Congress Association and The Indian Mathematical Society. He influenced the working of these bodies, through his strong personality and dynamic leadership. His first encounter as administrator with IMS was in 1970 when he took over as editor of the Mathematics Student for the remaining period of the editorship of Prof. J. N. Kapur who had joined Meerut University as Vice-Chancellor. I was closely associated with him in this job and I know things were far from happy. During that period I as his assistant, had the opportunity of communicating with stalwarts like late Professors P. L. Bhatnagar, V. V. Narliker, P. Keshwa Menon, H. Gupta etc. I know how unhappy they were with the affairs of I. M. S. at that time. I am witness to the tremendous efforts that Prof Singal made to streamline things and to update its publications. Later as general secretary and as treasurer, he really worked hard and established order in the Society. It was indeed befitting that the Indian Mathematical Society honored him by electing him as its President.

Prior to his involvement with the Indian Science Congress Association, mathematics as a subject did not occupy a justified position in spite of the fact that mathematicians like Prof. B. N. Prasad and Prof. R. S. Mishra had been honoured by its General President. The activities of the Mathematics section were limited and subdued. It was largely due to his efforts as Sectional President of Mathematics, as General Secretary and then as Council member that Mathematics section became one of the most thickly attended sections of the Science Congress sometimes even surpassing the annual gatherings of IMS. There was a time when mathematicians were occupying all top positions in ISCA, Prof. R. P. Bambah as General President, Prof. Singal as General Secretary and Prof. D. K. Sinha as the Treasurer. Mathematics never had (and I doubt it can achieve in near future) such a prominent position in ISCA. In this rise of mathematics in ISCA, Prof. Singal was largely, if not solely, responsible. It was therefore befitting that Indian Science Congress Association bestowed upon him the Distinguished Service Award, which was presented to him by the then Prime Minister late Sri Rajeev Gandhi.

To the Mathematical Association of India, “*A DATE WITH MATHEMATICANS*” is his greatest contribution. A concept solely conceived and meticulously executed by him 30 years ago was an instant success. The zeal and success with which he organized it year after year is really exemplary and has made it one of the most keenly awaited events for a large number of lovers of mathematics and science.

He was a fine human being, always ready to help the needy. I have personal knowledge of several people whose life changed beyond recognition simply because of his help, guidance and support. He was a hard taskmaster and a very successful group leader. Almost all his activities were related to mathematics either teaching or research or administration. During the last two years or so he was making efforts and persuading the power that be to declare December 22, the birthday of Ramanujan the great, as Mathematics Day. He lectured, wrote articles and posted letters to editors of newspapers on this issue. It was largely due to his motivation and support that at a number of places (including Agra University) people are celebrating this day as Mathematics Day for the last several years.

Prof. Singal’s sudden demise on July 16, 2006 was a big loss to his students, friends, and a vast circle of his admirers. It is a void difficult to fill. I pay my homage and heartfelt tributes to this lover and server of mathematics. May his soul guide us all in the service of mathematics and humanity.

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## JACOBI CONVOLUTION OF DISTRIBUTIONS

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ABSTRACT. Jacobi convolution studied by Gasper is extended to distributions. Jacobi convolution of a distribution and a test function and that of two distributions are defined and their properties are investigated.

### 1. Introduction

Jacobi transform is a general finite integral transform of which Legendre, Chebyshev and Gegenbauer transforms are special cases. Let us set

$$R_n(x) = R_n^{(\alpha, \beta)}(x) = P_n^{(\alpha, \beta)}(x)/P_n^{(\alpha, \beta)}(1), \quad (1.1)$$

where  $P_n^{(\alpha, \beta)}(x)$  is the Jacobi polynomial of order  $(\alpha, \beta)$  and degree  $n$ . Now, let

$$U := \left\{ (\alpha, \beta) : \alpha \geq \beta \geq -\frac{1}{2}, \alpha > -\frac{1}{2} \right\}. \quad (1.2)$$

For  $(\alpha, \beta) \in U$ , let  $L_p^{(\alpha, \beta)}(I)$ ,  $1 \leq p \leq \infty$ , denote the class of measurable functions  $f$  on  $I = (-1, 1)$  for which

$$\|f\|_p = \left( \int_{-1}^1 |f(x)|^p d\Lambda(x) \right)^{1/p} < \infty, 1 \leq p < \infty, \quad (1.3)$$

$$\|f\|_\infty = \sup_{x \in I} |f(x)|,$$

where  $d \wedge(x) = (1-x)^\alpha(1+x)^\beta dx$ .

Then Jacobi transform of  $f \in L_1^{(\alpha, \beta)}(I)$  is defined by

$$\hat{f}(n) := \int_{-1}^1 f(x)R_n(x)d \wedge(x). \quad (1.4)$$

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This article is dedicated to the late Professor M. K. Singal.

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The inverse Jacobi transform is given by

$$f(x) = \sum_{n=0}^{\infty} h_n^{(\alpha, \beta)} \hat{f}(n) R_n^{(\alpha, \beta)}(x), \quad (1.5)$$

where

$$h_n = h_n^{(\alpha, \beta)} = \left( \int_{-1}^1 [R_n^{(\alpha, \beta)}(x)]^2 d\wedge(x) \right)^{-1} \quad (1.6)$$

$$= \frac{(2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)\Gamma(n + \alpha + 1)}{2^{\alpha + \beta + 1}\Gamma(n + 1)\Gamma(\alpha + 1)\Gamma(\alpha + 1)}. \quad (1.7)$$

Let us recall that  $R_n(x)$  satisfies the differential equation [1, p.115]:

$$\Delta_x[R_n(x)] = -n(n + \alpha + \beta + 1)R_n(x), \quad (1.8)$$

where

$$\Delta_x := (1 - x^2) \frac{d^2}{dx^2} + [(\beta - \alpha) - (\alpha + \beta + 2)x] \frac{d}{dx}. \quad (1.9)$$

For several other properties of Jacobi polynomial refer to [5].

It has been proved by Gasper [1] that if  $(\alpha, \beta) \in U$  and  $-1 < x, y, z < 1$ , then

$$R_n^{(\alpha, \beta)}(x)R_n^{(\alpha, \beta)}(y) = \int_{-1}^1 R_n^{(\alpha, \beta)}(z)K(x, y, z)d\wedge(z), \quad (1.10)$$

where  $K(x, y, z) \geq 0$  and

$$\int_{-1}^1 K(x, y, z)d\wedge(z) = 1, \quad (1.11)$$

since  $R_0(x) = 1$ . From (1.11), using inversion formula (1.5), we get

$$K(x, y, z) = \sum_{n=0}^{\infty} h_n^{(\alpha, \beta)} R_n^{(\alpha, \beta)}(x)R_n^{(\alpha, \beta)}(y)R_n^{(\alpha, \beta)}(z). \quad (1.12)$$

**Definition 1.1:** Jacobi translation of a function  $f \in L_1^{(\alpha, \beta)}(I)$ , where  $I = (-1, 1)$ , is defined by

$$\begin{aligned} (\tau_x f)(y) = f(x, y) &:= \int_{-1}^1 f(z)K(x, y, z)d\wedge(z) \\ &= (\tau_y f)(x). \end{aligned} \quad (1.13)$$

It has been proved by Gasper [1] that  $\tau_x$  is a positive operator in the sense that if  $f(x) \geq 0$ ,  $-1 \leq x \leq 1$ , then  $f(x, y) \geq 0$ ,  $-1 \leq x, y \leq 1$ . If  $f \in L_{\infty}^{(\alpha, \beta)}(I)$ , then by using (1.11), we have

$$\|\tau_x f\|_{\infty} \leq \|f\|_{\infty}. \quad (1.14)$$

**Lemma 1.2:** Let  $-1 < x < 1$ ; then  $\tau_x : L_1^{(\alpha, \beta)}(I) \rightarrow L_1^{(\alpha, \beta)}(I)$  is a continuous linear map.



**Proof:** Using (1.11) we have

$$\int_{-1}^1 |(\tau_x f)(y)| d \wedge (y) \leq \int_{-1}^1 |f(z)| \left( \int_{-1}^1 K(x, y, z) d \wedge (y) \right) d \wedge (z);$$

so that

$$\|\tau_x f\|_1 \leq \|f\|_1 \tag{1.15}$$

from which the conclusion of the lemma follows.

**Definition 1.3:** Let  $f_1, f_2 \in L_1^{(\alpha, \beta)}(I)$  and  $(\alpha, \beta) \in U$  then their convolution is defined by

$$(f_1 * f_2)(x) := \int_{-1}^1 (\tau_x f_1)(y) f_2(y) d \wedge (y) \tag{1.16}$$

$$= \int_{-1}^1 \int_{-1}^1 f_1(z) f_2(y) K(x, y, z) d \wedge (z) d \wedge (y). \tag{1.17}$$

Many properties of the Jacobi convolution are contained in the following theorem due to Gasper [1, p. 114].

**Theorem 1.4:** Let  $(\alpha, \beta) \in U$  and  $f_1, f_2, f_3 \in L_1^{(\alpha, \beta)}(I)$ . Then  $f_1 * f_2, \in L_1^{(\alpha, \beta)}(I)$  and

- (i)  $\|f_1 * f_2\|_1 \leq \|f_1\|_1 \|f_2\|_1$
- (ii) ess.  $\sup_{-1 \leq x \leq 1} |f_1 * f_2| = \|f_1 * f_2\|_\infty \leq \|f_1\|_\infty \|f_2\|_1$
- (iii)  $f_1 * f_2 = f_2 * f_1$
- (iv)  $f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$
- (v)  $(f_1 * f_2)^\wedge(n) = \hat{f}_1(n) \cdot \hat{f}_2(n)$   
 where  $n = 0, 1, 2, \dots$ .

In this paper we shall study Jacobi translation and Jacobi convolution on certain test function and distribution spaces.

## 2. The Spaces $J_{(\alpha, \beta)}(I)$ and $J'_{(\alpha, \beta)}(I)$

Let  $I$  denote the open interval  $(-1, 1)$  and let  $x$  be a real variable restricted on  $I$ . Then the test function space  $J_{(\alpha, \beta)}(I)$  consists of all the complex valued infinitely differentiable functions  $\phi(x)$  defined over  $I$  such that

$$\gamma_k(\phi) := \sup_{x \in I} |\Delta_x^k [\phi(x)]| < \infty, \tag{2.1}$$

for all  $k = 0, 1, 2, \dots$ , where operator  $\Delta_x$  is the same as defined by (1.9).

The topology of  $J_{(\alpha, \beta)}(I)$  is defined by the separating collection of seminorms  $\{\gamma_k\}_{k=0}^\infty$  [6]. A sequence  $\{\phi_i\}_{i=1}^\infty$  is said to be converge in  $J_{(\alpha, \beta)}(I)$  to the limit  $\phi$  if  $\gamma_k[\phi_i - \phi] \rightarrow 0$  as  $i \rightarrow \infty$ , for each  $k$ . A sequence  $\{\phi_i\}_{i=1}^\infty$  is said to be a Cauchy sequence in  $J_{(\alpha, \beta)}(I)$  if  $\gamma_k[\phi_j - \phi_i]$  tends to zero as  $i$  and  $j$  both tend to infinity independently of each other. It can be readily seen

that  $J_{(\alpha,\beta)}(I)$  is a locally convex, sequentially complete Hausdorff topological vector space[6]. The dual of  $J_{(\alpha,\beta)}(I)$  will be represented by  $J'_{(\alpha,\beta)}(I)$ . It can easily be seen that the space of infinitely differentiable functions having compact support in  $I$ , denoted by  $D(I)$ , is a subspace of  $J_{(\alpha,\beta)}(I)$  and  $J'_{(\alpha,\beta)}(I) \subset D'(I)$ .

Using (1.8) it can be easily seen that  $R_n(x) \in J_{(\alpha,\beta)}(I)$  for each  $n = 0, 1, 2, \dots$ . Following Pandey and Pathak [3] we define generalized Jacobi transform of  $f \in J'_{(\alpha,\beta)}(I)$  by

$$\tilde{f}(n) = \langle f(x), R_n(x) \rangle, n = 0, 1, 2, \dots \quad (2.2)$$

It can be shown, as in [3, Theorem 2], that

$$f(x) = \sum_{n=0}^{\infty} h_n \tilde{f}(n) R_n(x) \text{ in } D'(I). \quad (2.3)$$

### 3. Jacobi Translation and Convolution on $J_{(\alpha, \beta)}(I)$

In this section at first we investigate the boundedness, continuity and differentiability of the translation operator  $\tau_x$  in  $J_{(\alpha,\beta)}(I)$ . To begin with, we prove a lemma which will be used in the sequel.

**Lemma 3.1:** Suppose that  $-1 < x, y, z < 1$  and  $\phi \in J_{(\alpha,\beta)}(I)$ . Then

$$\int_{-1}^1 \Delta_z^k [R_n(z)] \phi(z) d \wedge (z) = \int_{-1}^1 R_n(z) \Delta_z^k \phi(z) d \wedge (z), \quad (3.1)$$

for  $k = 1, 2, \dots$ , where  $(\alpha, \beta) \in U$ .

**Proof:** In view of definition (1.9) we have,

$$\begin{aligned} \int_{-1}^1 \Delta_z [R_n(z)] \phi(z) d \wedge (z) &= \int_{-1}^1 (1 - z^2) \frac{d^2 R_n(z)}{dz^2} \phi(z) d \wedge (z) \\ &+ (\beta - \alpha) \int_{-1}^1 \frac{dR_n(z)}{dz} \phi(z) d \wedge (z) - (\alpha + \beta + 2) \int_{-1}^1 z \frac{dR_n(z)}{dz} \phi(z) d \wedge (z). \end{aligned} \quad (3.2)$$

Now, integrating by parts and using the fact that limit-terms vanish, we find that the right-hand side of (3.2)

$$\begin{aligned} &= \int_{-1}^1 (1 - z^2) \phi''(z) R_n(z) d \wedge (z) \\ &+ \int_{-1}^1 [(\beta - \alpha) - (\alpha + \beta + 2)z] \phi'(z) R_n(z) d \wedge (z) \\ &= \int_{-1}^1 \Delta_z [\phi(z)] R_n(z) d \wedge (z), \end{aligned}$$

where dashes denote derivatives. Now, the general result (3.1) follows by induction.

**Theorem 3.2:** Let  $-1 < x, y, z < 1$  and  $\phi \in J_{(\alpha, \beta)}(I)$ , then mapping  $\phi \rightarrow \tau_x \phi$  is bounded and continuous from  $J_{(\alpha, \beta)}(I)$  into itself.

**Proof:** Using lemma 1.2, we can show that if  $\phi \in J_{(\alpha, \beta)}(I)$ , then  $(\tau_y \phi)(x)$  is bounded in  $J_{(\alpha, \beta)}(I)$ .

In view of (2.1), we have

$$\gamma_m(\tau_y \phi) = \sup_{x \in I} |\Delta_x^m [\tau_y \phi](x)|, \quad (3.3)$$

where  $y \in [-1, 1]$  is fixed,  $m = 0, 1, 2, \dots$ . Using (1.13), we can write

$$\begin{aligned} \Delta_x^m [\tau_y \phi](x) &= \Delta_x^m \left[ \int_{-1}^1 \phi(z) K(x, y, z) d \wedge (z) \right] \\ &= \int_{-1}^1 \phi(z) \left( \sum_{n=0}^{\infty} h_n \Delta_x^m [R_n(x)] R_n(y) R_n(z) \right) d \wedge (z) \\ &= \sum_{n=0}^{\infty} h_n R_n(x) R_n(y) \int_{-1}^1 \phi(z) \Delta_z^m [R_n(z)] d \wedge (z). \end{aligned}$$

Now, applying lemma 3.1, we get

$$\begin{aligned} |\Delta_x^m [\tau_y \phi](x)| &= \left| \sum_{n=0}^{\infty} h_n R_n(x) R_n(y) \int_{-1}^1 [\Delta_z^m \phi(z)] R_n(z) d \wedge (z) \right| \\ &\leq \int_{-1}^1 |\Delta_z^m [\phi(z)]| K(x, y, z) d \wedge (z) \\ &\leq \sup_{z \in I} |\Delta_z^m [\phi(z)]|, \end{aligned}$$

by property (1.11) of  $K(x, y, z)$ . Thus, we get

$$\gamma_m(\tau_y \phi) \leq \gamma_m(\phi). \quad (3.4)$$

from which the conclusion of the theorem follows.

**Theorem 3.3:** Let  $-1 < x, y, z < 1$  and  $\phi \in J_{\alpha, \beta}(I)$ , then

$$\tau_x [\Delta^m \phi(y)] = \Delta_y^m [(\tau_x \phi)(y)], \quad m = 0, 1, 2, \dots \quad (3.5)$$

**Proof:** Using Lemma 3.1, we have

$$\begin{aligned}
\tau_x[\Delta^m \phi](y) &= \int_{-1}^1 [\Delta^m \phi](z) K(x, y, z) d \wedge (z) \\
&= \sum_{n=0}^{\infty} h_n R_n(x) R_n(y) \int_{-1}^1 (\Delta^m \phi)(z) R_n(z) d \wedge (z) \\
&= \sum_{n=0}^{\infty} h_n R_n(x) R_n(y) \int_{-1}^1 \phi(z) \Delta^m [R_n(z)] d \wedge (z) \\
&= \int_{-1}^1 \phi(z) \left( \sum_{n=0}^{\infty} h_n R_n(x) \Delta_y^m [R_n(y)] R_n(z) \right) d \wedge (z) \\
&= \Delta_y^m \left[ \int_{-1}^1 \phi(z) K(x, y, z) d \wedge (z) \right] \\
&= \Delta_y^m [\tau_x \phi](y).
\end{aligned}$$

Now, using the above properties of the translation operator  $\tau_x$  we derive some results for the Jacobi convolution.

**Theorem 3.4:** Let  $\phi, \psi \in J_{(\alpha, \beta)}(I)$ , then mapping  $(\phi, \psi) \rightarrow \phi * \psi$  is continuous from  $J_{(\alpha, \beta)}(I) \times J_{(\alpha, \beta)}(I)$  into  $J_{(\alpha, \beta)}(I)$ .

**Proof:** In view of definition (1.16) and relation (3.5), we have

$$\begin{aligned}
|\Delta_x^r [\phi * \psi](x)| &= \left| \Delta_x^r \int_{-1}^1 (\tau_x \phi)(y) \psi(y) d \wedge (y) \right| \\
&= \left| \int_{-1}^1 \Delta_x^r (\tau_y \phi)(x) \psi(y) d \wedge (y) \right| \\
&= \left| \int_{-1}^1 \tau_y (\Delta^r \phi)(x) \psi(y) d \wedge (y) \right|.
\end{aligned}$$

Now, invoking Theorem 3.2, we get

$$\begin{aligned}
\gamma_r(\phi * \psi) &\leq \gamma_r(\phi) \int_{-1}^1 |\psi(y)| d \wedge (y) \\
&\leq \gamma_r(\phi) \gamma_0(\psi) \times 2^{\alpha + \beta + 1} B(\alpha + 1, \beta + 1)
\end{aligned}$$

from which the conclusion of the theorem follows.

**Theorem 3.5:** Let  $\phi, \psi \in J_{(\alpha, \beta)}(I)$ ; then

$$\Delta^m [\phi * \psi](x) = [\Delta^m \phi * \psi](x) = [\phi * \Delta^m \psi](x),$$

where  $m = 0, 1, 2, \dots$ .

**Proof:** In view of (1.16), we have

$$\begin{aligned} \Delta_x^m[\phi^*\psi](x) &= \Delta_x^m \left[ \int_{-1}^1 (\tau_x\phi)(y)\psi(y)d \wedge (y) \right] \\ &= \int_{-1}^1 \Delta_x^m[\tau_y\phi](x)\psi(y)d \wedge (y). \end{aligned}$$

Using identity (3.5), we get

$$\begin{aligned} \Delta^m[\phi^*\psi](x) &= \int_{-1}^1 \tau_x[\Delta^m\phi](y)\psi(y)d \wedge (y) \\ &= (\Delta^m\phi^*\psi)(x). \end{aligned}$$

Similarly, we can show the second part also, and thus we get complete proof of the theorem.

#### 4. Jacobi Convolution of a Distribution and Function

Using results obtained in the previous section we investigate convolution of a distribution  $f \in J'_{(\alpha, \beta)}(I)$  and test function  $\phi \in J_{(\alpha, \beta)}(I)$ .

**Theorem 4.1:** Let  $f \in J'_{(\alpha, \beta)}(I)$  and  $\phi \in J_{(\alpha, \beta)}(I)$ .

$$\text{Define } F(x) := (f^*\phi)(x) := \langle f(y), \phi(x, y) \rangle, \quad (4.1)$$

where  $\phi(x, y) = (\tau_x\phi)(y)$ . Then

- (i)  $F(x)$  is continuous,
- (ii)  $F(x)$  is differentiable.

Moreover,  $F(x) \in J_{(\alpha, \beta)}(I)$ .

**Proof:** Let  $\{x_j\}_{j \in N_0}$  be a sequence in  $I$ , which converges to  $x$ . Then  $\{\phi(x_j, y)\}$  is a sequence in  $J_{(\alpha, \beta)}(I)$  and we have

$$F(x_j) = \langle f(y), \phi(x_j, y) \rangle. \quad (4.2)$$

First we show that  $\phi(x_j, y) \rightarrow \phi(x, y)$  in  $J_{(\alpha, \beta)}(I)$ , as  $j \rightarrow \infty$ . For any  $s \in N_0$ ,  $\sup_{y \in [-1, 1]} |\Delta_y^{(s)}\phi(x_j, y) - \Delta_y^{(s)}\phi(x, y)| \rightarrow 0$ , as  $j \rightarrow \infty$  and  $\phi(x_j, y)$  is a bounded sequence in  $J_{(\alpha, \beta)}(I)$ .

Hence, it gives that  $\phi(x_j, y)$  converges to  $\phi(x, y)$  in  $J_{(\alpha, \beta)}(I)$ . Since  $f \in J'_{(\alpha, \beta)}(I)$ , its continuity implies that

$$F(x_j) = \langle f(y), \phi(x_j, y) \rangle \rightarrow \langle f(y), \phi(x, y) \rangle = F(x).$$

Thus,  $F(x)$  is continuous.

Next, we show that  $F(x)$  is differentiable.

Let  $x$  be a fixed point in  $I$  and  $h$  be an increment in  $x$ , and we denote  $\Delta_y^{(m)}\phi(x, y)$  by  $\phi^{(m)}(x, y)$ . Then we have

$$\left| \frac{\phi^{(m)}(x+h, y) - \phi^{(m)}(x, y)}{h} - \frac{d}{dx}\phi^{(m)}(x, y) \right|$$

$$\begin{aligned}
&= \left| \frac{1}{h} \int_x^{x+h} \left( \int_x^t \frac{d^2}{du^2} [\phi^{(m)}(u, y)] du \right) dt \right| \\
&\leq \sup_{(u, y) \in I \times I} \left| \frac{d^2}{du^2} [\phi^{(m)}(u, y)] \right| \left[ \frac{1}{|h|} \int_x^{x+h} [|x-t|] dt \right] \\
&\leq \sup_{(u, y) \in I \times I} \left| \frac{d^2}{du^2} [\phi^{(m)}(u, y)] \right| \times \frac{1}{2} |h| \rightarrow 0
\end{aligned}$$

as  $h \rightarrow 0$ .

This proves that

$$\frac{\phi(x+h, y) - \phi(x, y)}{h} - \phi(x, y) \rightarrow 0$$

in  $J_{(\alpha, \beta)}(I)$  as  $h \rightarrow 0$ .

Hence,  $\frac{F(x+h) - F(x)}{h} - \frac{d}{dx} F(x)$

$$= \left\langle f(y), \frac{\phi(x+h, y) - \phi(x, y)}{h} - \frac{d}{dx} \phi(x, y) \right\rangle \rightarrow 0 \text{ as } h \rightarrow 0.$$

Thus,

$$\frac{d}{dx} [F(x)] = \left\langle f(y), \frac{d}{dx} [\phi(x, y)] \right\rangle. \quad (4.3)$$

Now, we have to show that  $F(x) \in J_{(\alpha, \beta)}(I)$ . By boundedness property of distributions there exists a constant  $Q$  and a non-negative integer  $l$  such that

$$\begin{aligned}
\left| \Delta_x^{(p)} [F(x)] \right| &= \left| \left\langle f(y), \Delta_x^{(p)} [\phi(x, y)] \right\rangle \right| \\
&\leq Q \sum_{|q| \leq l} \sup_{y \in I} \left| \Delta_y^{(q)} [\Delta_x^{(p)} (\tau_x \phi)(y)] \right|, \quad (4.4)
\end{aligned}$$

where  $p, q$  and  $l$  are in  $N_0$ .

Consider the term  $\Delta_y^{(q)} [\Delta_x^{(p)} (\tau_x \phi)(y)]$ . Using lemma 3.1, we get

$$\begin{aligned}
\Delta_y^{(q)} \left[ \Delta_x^{(p)} (\tau_x \phi)(y) \right] &= \Delta_y^{(q)} \left[ \Delta_x^{(p)} \left\{ \int_{-1}^1 \phi(z) K(x, y, z) d \wedge (z) \right\} \right] \\
&= \Delta_y^{(q)} \left[ \int_{-1}^1 \Delta_z^{(p)} [\phi(z)] K(x, y, z) d \wedge (z) \right] \\
&= \int_{-1}^1 \Delta_z^{(q)} \left[ \Delta_z^{(p)} [\phi(z)] K(x, y, z) d \wedge (z) \right] \\
&= \int_{-1}^1 \Delta_z^{(i)} [\phi(z)] K(x, y, z) d \wedge (z) \\
&\leq \sup_z \left| \Delta_z^{(i)} [\phi(z)] \right|, \quad (4.5)
\end{aligned}$$

where  $p + q = i \in N_0$ . Hence,

$$\gamma_p(F) = \sup_{x \in I} \left| \Delta_x^{(p)} [F(x)] \right| \leq Q \sum_{|q| \leq l} \sup \left| \Delta_z^{(p+q)} [\phi(z)] \right| < \infty.$$

This completes the proof.

**Theorem 4.2:** Let  $f \in J'_{(\alpha,\beta)}(I)$  and  $\phi \in J_{(\alpha,\beta)}(I)$ , then the following identity holds :

$$(f * \phi)^\wedge(n) = \tilde{f}(n) \phi^\wedge(n), \quad (4.6)$$

where  $\tilde{f}(n)$  denotes generalized Jacobi transform of  $f$ .

**Proof:** Since

$$(f * \phi)(x) = \langle f(y), (\tau_x \phi)(y) \rangle$$

is an element of  $J_{(\alpha,\beta)}(I) \subset L_1^{(\alpha,\beta)}(I)$ , its Jacobi transform exists. In view of (1.4), using property  $(\tau_x \phi)(y) = (\tau_y \phi)(x)$ , we have

$$\begin{aligned} (f * \phi)^\wedge(n) &= \int_{-1}^1 (f * \phi)(x) R_n(x) d \wedge(x) \\ &= \int_{-1}^1 \langle f(y), (\tau_x \phi)(y) \rangle R_n(x) d \wedge(x) \\ &= \left\langle f(y), \int_{-1}^1 (\tau_y \phi)(x) R_n(x) d \wedge(x) \right\rangle \\ &= \langle f(y), (\tau_y \phi)^\wedge(n) \rangle. \end{aligned} \quad (4.7)$$

Now, using (1.13),

$$\begin{aligned} (\tau_y \phi)^\wedge(n) &= \int_{-1}^1 (\tau_y \phi)(x) R_n(x) d \wedge(x) \\ &= \int_{-1}^1 \int_{-1}^1 \phi(z) K(x, y, z) R_n(x) d \wedge(z) d \wedge(x) \\ &= \int_{-1}^1 \phi(z) R_n(y) R_n(z) d \wedge(z) \\ &= R_n(y) \phi^\wedge(n). \\ \therefore (f * \phi)^\wedge(n) &= \langle f(y), R_n(y) \rangle \phi^\wedge(n) \\ &= \tilde{f}(n) \phi^\wedge(n). \end{aligned}$$

### 5. Jacobi Convolution of Two Distributions

Let  $f, g \in J'_{(\alpha,\beta)}(I)$  and  $\phi \in J_{(\alpha,\beta)}(I)$ . Then, by Theorem 4.1,  $g * \phi \in J_{(\alpha,\beta)}(I)$ .

Therefore, we can define  $f * g$  by

$$\begin{aligned} \langle (f * g)(x), \phi(x) \rangle &= \langle f(x), \langle g(y), \phi(x, y) \rangle \rangle \\ &= \langle f(x), (g * \phi)(x) \rangle. \end{aligned} \quad (5.1)$$

It can easily be shown that  $f * g$  is a linear, continuous functional on  $J_{(\alpha, \beta)}(I)$ .

**Theorem 5.1:** Let  $f, g \in J'_{(\alpha, \beta)}(I)$ ; then

$$(f * g)^\sim(n) = \tilde{f}(n) \tilde{g}(n). \quad (5.2)$$

**Proof:** Let  $\phi \in D(I)$ . Then using (2.3), we have

$$\begin{aligned} \langle (f * g)(x), \phi(x) \rangle &= \langle f(x), \langle g(y), \phi(x, y) \rangle \rangle \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n h_m \tilde{f}(n) \tilde{g}(m) \langle R_n(x), \langle R_m(y), \phi(x, y) \rangle \rangle \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n h_m \tilde{f}(n) \tilde{g}(m) \int_{-1}^1 \int_{-1}^1 R_n(x) \phi(z) \\ &\quad \times \left[ \int_{-1}^1 R_m(y) K(x, y, z) d \wedge (y) \right] d \wedge (x) d \wedge (z) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n h_m \tilde{f}(n) \tilde{g}(m) \int_{-1}^1 R_m(x) R_n(x) d \wedge (x) \int_{-1}^1 \phi(z) R_m(z) d \wedge (z) \\ &= \left\langle \sum_{n=0}^{\infty} h_n \tilde{f}(n) \tilde{g}(n) R_n(z), \phi(z) \right\rangle. \end{aligned} \quad (5.3)$$

But in view of (2.3) we have

$$\langle (f * g)(z), \phi(z) \rangle = \left\langle \sum_{n=0}^{\infty} h_n (f * g)^\sim(n) R_n(z), \phi(z) \right\rangle. \quad (5.4)$$

Hence, by uniqueness of Jacobi transform,

$$(f * g)^\sim(n) = \tilde{f}(n) \tilde{g}(n).$$

**Theorem 5.2:** Let  $f, g, h \in J'_{(\alpha, \beta)}(I)$ , then the following holds:

- (i)  $f * g = g * f$  in  $D'(I)$
- (ii)  $(f * g) * h = f * (g * h)$  in  $D'(I)$ .

**Proof:** Let  $\phi \in D(I) \subset J_{(\alpha, \beta)}(I)$ . Then from Theorem 5.1, we have

$$(f * g)^\sim(n) = \tilde{f}(n) \tilde{g}(n) = \tilde{g}(n) \tilde{f}(n) = (g * f)^\sim(n), \quad n = 0, 1, 2, \dots$$

Now, invoking (2.3) we get (i). Next, in view of (5.1), for  $\phi \in D(I)$ , we have

$$\begin{aligned} \langle [(f * g) * h](x), \phi(x) \rangle &= \langle (f * g)(x), (h * \phi)(x) \rangle \\ &= \langle f(x), [g * (h * \phi)](x) \rangle. \end{aligned}$$



But  $[g * (h * \phi)](x) = \langle g(y), \langle h(z), (\tau_x \phi)(z) \rangle \rangle = \langle (g * h)(z), (\tau_x \phi)(z) \rangle$ .

Hence,

$$\begin{aligned} \langle [(f * g) * h](x), \phi(x) \rangle &= \langle f(x), [(g * h) * \phi](x) \rangle \\ &= \langle [f * (g * h)](x), \phi(x) \rangle. \end{aligned}$$

This is the conclusion of part (ii).

**Theorem 5.3:** Let  $f, g \in J'_{(\alpha, \beta)}$  and  $\Delta_x$  be the Jacobi differential operator. Then we have

$$\Delta_x^r [f * g](x) = [\Delta^r f * g](x) = [f * \Delta^r g](x) \text{ in } D'(I), \quad (5.5)$$

for  $r = 0, 1, 2, \dots$

**Proof:** Let  $\phi \in D(I)$ . Then by (2.3)

$$\begin{aligned} \langle \Delta^m f, \phi \rangle &= \left\langle \Delta_x^m \left[ \sum_{n=0}^{\infty} h_n \tilde{f}(n) R_n(x) \right], \phi(x) \right\rangle \\ &= \sum_{n=0}^{\infty} h_n \tilde{f}(n) \langle \Delta_x^m R_n(x), \phi(x) \rangle \\ &= \left\langle \sum_{n=0}^{\infty} h_n \tilde{f}(n) R_n(x), \Delta_x^m \phi(x) \right\rangle \quad \text{by lemma 3.1} \\ &= \langle f(x), \Delta_x^m \phi(x) \rangle. \end{aligned}$$

Since  $f * g \in J'_{(\alpha, \beta)}(I)$ , we have

$$\langle \Delta_x^m (f * g)(x), \phi(x) \rangle = \langle f * g, \Delta^m \phi \rangle = \langle f, g * \Delta^m \phi \rangle.$$

But by (4.1),

$$\begin{aligned} (g * \Delta^m \phi)(x) &= \langle g(y), (\tau_x \Delta^m \phi)(y) \rangle \\ &= \langle g(y), (\Delta^m \tau_x \phi)(y) \rangle \\ &= \Delta_x^m [g * \phi](x). \end{aligned}$$

Therefore, using representation (2.3) and relation (3.1), we get

$$\begin{aligned} \langle \Delta_x^m (f * g)(x), \phi(x) \rangle &= \langle f(x), \Delta_x^m (g * \phi)(x) \rangle \\ &= \langle \Delta_x^m f(x), (g * \phi)(x) \rangle \\ &= \langle (\Delta^m f * g)(x), \phi(x) \rangle. \end{aligned} \quad (5.6)$$

The proof of the second part is similar.

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## S. CHOWLA'S CONTRIBUTIONS TO COMBINATORICS

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### Introduction

Professor Sarva Daman Chowla was one of the most prolific Number theorists of the 20<sup>th</sup> century. He was born in England on October 22, 1907. His father Gopal Singh Chowla was a Professor of Mathematics at Government College, Lahore. Young Sarva Daman was 22 when his father passed away in Paris. He had been contributing to the problem section of the Journal of the Indian Mathematical Society from a very young age and had published some research papers in the Journal even before his graduation. He got his Ph.D. in Cambridge in 1931 with Professor J. E. Littlewood. He taught at St. Stephen's College, Delhi, at BHU and at Andhra University before settling down in his father's position at Lahore in 1936. He had to leave Lahore at the time of the partition in 1947. Eventually, early in 1948, he migrated to USA and worked there at the Institute for Advanced Studies in Princeton and at the Universities of Kansas, Colorado (where this author was one of his students) and Penn State. He died in Laramie, WY in USA in 1995.

Professor Chowla published about 350 research papers on nearly all areas of Number Theory as also in Algebra, Combinatorics and other areas of Mathematics. His collected papers have been published in three volumes [1] and NBHM has presented a copy of these volumes to each University Library in India.

On the occasion of the centenary of Professor Chowla's birth and as my personal tribute to him, I would like to report in this article on his work in topics on Combinatorics.

Chowla's contributions to combinatorics are mostly in the areas of Difference Sets,  $l$ - $m$ - $n$  configurations and Balanced Incomplete Block Designs

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This paper is respectfully dedicated to the memory of late Professor M. K. Singal.

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(BIBD). His work culminated with the famous Bruck - Chowla - Ryser Theorem. He also worked on the number of Mutually Orthogonal Latin Squares of a given order.

### The a - b Problem

Professor Chowla's first contribution to Combinatorics came in 1939 [2]. He gave a new solution to the "10-21 Problem". The problem is to arrange the 21 numbers 1, 2, 3, ..., 21 around a circle in 10 different ways so that no number has the same neighbour (on either side) in two distinct arrangements.

For instance, it is easy to solve the corresponding 2-5 Problem. The two ways of arranging five numbers 1, 2, 3, 4, 5 around a circle being 1, 2, 3, 4, 5 and 1, 3, 5, 2, 4. But solving the 10-21 problem would clearly be a lot tougher. Professor Chowla just gives the 10 arrangements without telling us how he arrived at them.

The same year (1939) and in the same journal he also considers the general " $n - 2n + 1$  Problem" [3]. He tries for an inductive proof. He does not fully succeed but is able to prove that if the problem is solvable for some  $n$  and if  $2n + 1$  is a prime, then it can be solved for  $n + 1$ .

While correcting the proofs of this paper, he adds that Levi has shown that the problem is solvable for all  $n$ .

### The $l - m - n$ Configurations

Professor Chowla published two papers in 1944 [4,5] on  $l - m - n$  configurations. These are defined as follows. Suppose  $Q$  is a set with  $q$  elements. We take  $N$  subsets of  $Q$  each containing  $n$  elements. If every subset of  $Q$  containing  $m$  elements is a subset of exactly  $l$  of the  $N$  sets, then we say that the family of  $N$  sets is an  $l - m - n$  configuration on  $q$  symbols. Experts would not fail to observe that Chowla is firmly driving in the direction of Block Designs.

Chowla tells us in these papers that a 2-2-9 configuration on 37 symbols is known. Chowla gives us a 12 - 2 - 49 configuration on 197 symbols. Here  $Q = \{1, 2, 3, \dots, 197\}$  and  $S$  the set of the 49 biquadratic residues of 197. These are those elements of the field of 197 elements that are perfect fourth powers in the field.

He claims that the family of sets  $S, S+1, S+2, \dots, S+196$  (all additions are mod 197) forms a 12-2-49 configuration.

In the same way, starting with the 25 biquadratic residues of the prime 101, he gets a 6 - 2 - 25 configuration on 101 symbols.

To illustrate the 2 - 2 - 9 configuration on 37 symbols, we start with  $Q = \{0, 1, 2, 3, \dots, 36\}$  and take the set  $S$  of the nine biquadratic residues of 37, namely  $S = \{1, 7, 9, 10, 12, 16, 26, 33, 34\}$ . We then take the 37 subsets

of  $Q$ , namely  $S, S+1, S+2, \dots, S+36$  where again the addition is mod 37. It can be checked that every 2-element subset of  $Q$  is a subset of exactly two of the 37 sets  $S+i$  above. For instance,  $\{1, 10\}$  is a subset only of  $S$  and of  $S+31$ ,  $\{9, 16\}$  is a subset only of  $S$  and of  $S+20$ ,  $\{15, 33\}$  is a subset only of  $S+26$  and of  $S+36$  while  $\{0, 11\}$  is a subset only of  $S+4$  and of  $S+36$ .

At the end, Chowla remarks that "37, 101 and 197 are the only primes  $\equiv 5 \pmod{32}$  with the above property". Perhaps what he means is that the set of all biquadratic residues of any other prime  $\equiv 5 \pmod{32}$  and its translates would not give us a  $k-2-n$  configuration where  $n$  is the number of biquadratic residues of the prime. He does not give any proof of this statement but he must have had a proof. So if one starts with say the prime  $229 \equiv 5 \pmod{32}$  and the set  $S$  of its 57 biquadratic residues, then  $S$  and its translates would not furnish a  $k-2-57$  configuration on 229 symbols. In fact I have checked that they do not furnish a  $k-m-57$  configuration for any  $k$  and  $m$ . The mystery behind this remark would be revealed a little later in this paper.

### Difference Sets

Chowla now turns to Difference Sets. A Difference Set (mod  $n$ ) is a set of integers in which every non-zero difference (mod  $n$ ) of two integers occurs the same number of times. For instance,  $\{1, 2, 4\}$  is a Difference Set (mod 7). This is because every non-zero integer (mod 7) occurs in the set as a difference in exactly one way. Thus 1 is 2-1, 2 is 4-2, 3 is 4-1, 4 is 1-4, 5 is 2-4 and 6 is 1-2.

In a 1944 paper [6] he quotes a result of R. C. Bose that says that the set of quadratic residues of a prime  $p \equiv 3 \pmod{4}$  is a difference set (mod  $p$ ). Thus for  $p = 19$ , the set of quadratic residues of 19 is  $\{1, 4, 5, 6, 7, 9, 11, 16, 17\}$ . Here it is easily checked that every non-zero integer (mod 19) is a difference in exactly 4 different ways. Thus 15 is a difference in the following 4 ways: 16-1, 1-5, 5-9 and 7-11 and 9 is a difference in the following 4 ways: 16-7, 1-11, 6-16 and 7-17.

Although Chowla does not mention it in this paper, the sets  $S$  of biquadratic residues of the primes 37, 101 and 197 constructed earlier are all difference sets modulo the respective primes. Indeed this is the property that ensures that  $S$  and its translates furnish a  $k-2-n$  configuration for some  $k$  and for  $n = (p-1)/4$ . On the other hand I have checked that the set of biquadratic residues of 229 is not a difference set. It can be easily shown that if a prime  $p \equiv 3 \pmod{4}$ , the set of quadratic residues of  $p$  is precisely the set of biquadratic residues of  $p$ . Therefore it would appear that the set of all biquadratic residues of all primes  $p \equiv 3 \pmod{4}$  and many primes

$p \equiv 1 \pmod{4}$  is a difference set. This way we can get many difference sets modulo primes. In this paper, Chowla produces a difference set modulo a composite number. Let  $p$  and  $p + 2$  be twin primes and let  $m = p(p + 2)$ . There are exactly  $(p^2 - 1)/2$  numbers  $(\text{mod } m)$  that are either quadratic residues of both  $p$  and  $p + 2$  or are quadratic non-residues of both  $p$  and  $p + 2$ . These  $(p^2 - 1)/2$  numbers (between 0 and  $m$ ) together with the  $p$  multiples of  $p + 2$  that lie between 1 and  $m$  give us a total of  $(p^2 - 1)/2 + p = (m - 1)/2$  numbers. This set is a difference set  $(\text{mod } m)$ .

For example, if we take the twin primes 3 and 5, we get the difference set  $S = \{1, 2, 4, 5, 8, 10, 15\} \pmod{15}$ . Every non-zero integer  $(\text{mod } 15)$  comes here as a difference in exactly 3 ways.

### Balanced Incomplete Block Designs (BIBD)

A Balanced Incomplete Block Design with parameters  $v, k, \lambda, r, b$  is an incidence system in which there is a set  $X$  with  $v$  elements and a family  $A$  of  $b$  subsets (called blocks) of  $X$  each containing  $k$  elements such that every element of  $X$  is in exactly  $r$  different blocks and every 2-element subset of  $X$  is a subset of exactly  $\lambda$  blocks. The five parameters are not independent. They are related by  $vr = bk$  and  $\lambda(v - 1) = r(k - 1)$ .

Here is an example of a BIBD with  $v = 7 = b, r = 3 = k$  and  $\lambda = 1$ . The set  $X = \{1, 2, 3, 4, 5, 6, 7\}$ . The blocks are actually the translates of the difference set  $\{1, 2, 4\}$  of the quadratic residues of 7, that we have seen before. The blocks are:

$$\begin{array}{ll} \{1, 2, 4\} & \{5, 6, 1\} \\ \{2, 3, 5\} & \{6, 7, 2\} \\ \{3, 4, 6\} & \{7, 1, 3\} \\ \{4, 5, 7\} & \end{array}$$

A BIBD is commonly referred to as simply  $(v, k, \lambda)$  since  $b$  and  $r$  are given in terms of  $v, k, \lambda$  by  $b = \{v(v - 1)\lambda\}/\{k(k - 1)\}$  and  $r = \{\lambda(v - 1)\}/(k - 1)$ . So the example above is a  $(7, 3, 1)$  BIBD.

A BIBD is called symmetric if  $b = v$  or equivalently,  $r = k$ .

If  $X = \{x_1, x_2, \dots\}$  and the blocks are  $A_1, A_2, \dots$ , then the BIBD is often studied through its incidence matrix  $M = (m_{ij})$  defined by  $m_{ij} = 1$  if  $x_i$  belongs to  $A_j$  and 0 otherwise.  $M$  is a  $v \times b$  matrix and it satisfies  $MM^T = (r - \lambda)I + \lambda J$ , where  $I$  is the  $v \times v$  identity matrix and  $J$  is the  $v \times v$  unit matrix.

Let us now return to the difference set  $(\text{mod } 15)$ , namely  $S = \{1, 2, 4, 5, 8, 10, 15\}$ . If we take the translates of  $S$ , namely  $S + 1, S + 2, \dots, S + 15$  we get a BIBD with  $v = b = 15, r = k = 7$  and  $\lambda = 3$ . This is a Symmetric

design and as the blocks are translates of the same set, Chowla calls this a Cyclic Symmetric Block Design (csbd).

Of course, it is obvious that if  $S$  is a difference set (mod  $n$ ), then any translate  $S + i$  is also a difference set (mod  $n$ ). In one of his papers [7], Chowla discusses many properties of Difference Sets including the linear transformations under which a difference set remains a difference set.

Earlier, in 1944, Chowla proved [8] that the set of biquadratic residues of a prime  $p \equiv 1 \pmod{4}$  is a difference set (mod  $p$ ) if  $(p-1)/4$  is the square of an odd number. Note that this is satisfied by the primes 37, 101 and 197 but not by 229. It is well known in Number Theory that the square of every odd number is of the type  $8m+1$ . If  $(p-1)/4 = 8m+1$ , then  $p = 32m+5$ . This explains why Chowla is talking of primes  $\equiv 5 \pmod{32}$  and is working with primes 37, 101 and 197. The next prime  $p$  for which  $(p-1)/4$  is the square of an odd number is 677. This prime has 169 biquadratic residues. Chowla verified his theorem by very painstakingly finding all the 169 biquadratic residues of 677. Note that there is no quick fire method to find all the residues and he did this long before the advent of computers or Mathematica or any other software. I tried to find these residues myself and even though I had the help of (a non-programmable) Mathematica software, it took me 2 days simply to get the list. Chowla went on to verify that this set is a difference set in which every difference comes 42 times. So the set and its translates would furnish us with an example of a  $(677, 169, 42)$  symmetric block design.

The BI and symmetric block designs are important not only in the Design of Experiments but they also figure importantly in finite geometries. A finite projective plane is defined by means of postulates that imply that there is the same number of points (say,  $m+1$ ) on every line and the same number of lines (again, the same  $m+1$ ) pass through each point, that there are  $m^2+m+1$  points and an equal number of lines. Given two distinct lines, they intersect in exactly one point and given two distinct points they lie on exactly one line. If each line is considered as a block then we get that every projective plane is actually a symmetric block design with  $v = m^2 + m + 1 = b, r = m + 1 = k$  and  $\lambda = 1$ . Such a projective plane is said to be of order  $m$ . A Finite projective plane of order  $n$  is a symmetric  $(n^2 + n + 1, n + 1, 1)$  block design. Thus the  $(7,3,1)$  design we have seen above is the projective plane of order 2 also known as the Fano plane.

It is not true that there exist finite projective planes of every order. There are no finite projective planes of orders 6 or 10. All finite projective planes known so far are of prime power orders. This has something to do with the fact that all finite fields have prime power orders. Bruck and Ryser [9] proved in a famous theorem in 1949 that if there is a finite projective plane

of order  $n$  with  $n \equiv 1$  or  $2 \pmod{4}$  then  $n$  must be a sum of two squares. That there is no projective plane of order 6 was proved by Tarry around 1900, he had done so by a systematic and long enumeration of many cases. Bruck and Ryser's theorem gave the first simple proof of the non-existence of a projective plane of order 6 because  $6 \equiv 2 \pmod{4}$  but 6 is not a sum of two squares. That Bruck-Ryser theorem gives only necessary conditions was established when Lam showed that there is no projective plane of order 10.

Singer had proved [10] in 1938 by using Geometrical techniques that whenever  $m$  is a prime power, we can find a set  $S$  of  $m + 1$  integers  $d_0, d_1, \dots, d_m$  such that all the  $m^2 + m$  differences  $d_i - d_j (i \neq j)$  are distinct  $\pmod{m^2 + m + 1}$ . This means that  $S$  and its translates would give us a projective plane of order  $m$ . Chowla and Vijayraghavan [11] gave a direct proof in 1945 in a paper in Proc. Ind. Acad. Sc. What Singer proved for projective planes was proved by R. C. Bose [12] for Affine planes in 1938 in J. Ind. Math. Soc. In this case the direct proof was given by Chowla and Bose [13] in 1945. An Affine plane of order  $n$  is a block design of the form  $(n^2, n, 1)$ . It exists if and only if a projective plane of order  $n$  exists.

Chowla also proved [14] in 1945 that given  $k$  and  $\lambda$ , we can always find suitable integers  $v, b$  and  $r$  and a BIBD with parameters  $v, b, r, k, \lambda$ .

Nearly 25 years after his work on block designs, Chowla returned to the topic in 1969 (Abha. Math. Uni. Hamburg). He considered cyclic symmetric block designs  $(v, k, \lambda)$  where  $v \equiv 3 \pmod{4}$ . We shall presently report on this in the next section.

### Bruck Chowla Ryser Theorem

We now come to what is arguably Chowla's most cited theorem. This was an extension of Bruck-Ryser Theorem to symmetric block designs. Chowla and H. J. Ryser published their paper [15] in the Canad. J. Math. in 1950. They got necessary conditions for the existence of symmetric  $(v, k, \lambda)$  block designs. What they proved can be put in the following form (Bruck-Ryser-Chowla theorem):

If a symmetric  $(v, k, \lambda)$ - Block Design exists, then

- (i) if  $v$  is even, then  $k - \lambda$  is a square, and
- (ii) if  $v$  is odd then the Diophantine Equation  $x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2} \lambda z^2$  has integer solutions, not all zero.

Thus, for arbitrary  $v, k$  and  $\lambda$ , a symmetric  $(v, k, \lambda)$ -Block Design may not exist. For instance there is no  $(20, 10, 2)$  or  $(44, 12, 5)$  symmetric design by (i) above and there is no  $(25, 8, 2)$  symmetric design by (ii). So far no necessary and sufficient conditions for a symmetric  $(v, k, \lambda)$  design to exist are known but the existence or otherwise of all  $(v, b, k, r, \lambda)$  block designs



with  $r \leq 10$  has been settled with at most a couple of exceptions, namely  $(46,69,9,6,1)$  and  $(51,8,5,10,6,1)$ .

Let us now complete our reporting of Chowla's 1969 work [16] on csbd  $(v, k, \lambda)$  where  $v \equiv 3 \pmod{4}$ . He begins by noting that in view of (ii) above, if a  $(v, k, \lambda)$  (with  $v \equiv 3 \pmod{4}$ ) BIBD exists then the Diophantine equation

$$(k - \lambda)y^2 = x^2 + \lambda z^2$$

has integral solution in integers not all zero. He proves in this paper that in addition to this necessary condition for a BIBD to exist, for csbd to exist, it is also necessary that for every prime divisor  $p \equiv 3 \pmod{4}$  of  $v$ , the Diophantine equation (or equations)

$$(k - \lambda)y^2 = x^2 + pz^2$$

must have solutions with none of  $x, y, z$  zero. His method of proof is not combinatorial. He uses results from cyclotomy. He also shows that the additional condition is not empty because with  $v = 111, k = 11, \lambda = 1, p = 3$ , the first equation is

$$10y^2 = x^2 + z^2$$

and it has the solution  $x = 1 = y, z = 3$ . But the second equation is

$$10y^2 = x^2 + 3z^2$$

the only integral solution of which is  $x = y = z = 0$ . Therefore there is no  $(111, 11, 1)$  csbd.

Chowla states that while it is not known whether (ii) above is a sufficient condition for a  $(v, k, \lambda)$  BIBD to exist for odd  $v$  (there is a strong Marshall Hall conjecture that it is), we now know that at least for csbd to exist, it is not sufficient.

### Latin Squares

An  $n \times n$  Latin square is an array of  $n$  rows and  $n$  columns each containing objects from a set having  $n$  objects in such a way that no object appears more than once in any row or column. Thus

$$\begin{array}{cc} A & B \\ B & A \end{array}$$

is a  $2 \times 2$  Latin square. For each  $n$ , there is an  $n \times n$  Latin square and the number of  $n \times n$  Latin squares increases exponentially with  $n$ . For  $n = 3$ , there are 12 Latin Squares but for  $n = 4$ , there are 576. For  $n = 6$ , there are 812851200 Latin Squares. It is claimed that there are more  $15 \times 15$  Latin Squares than there are atoms in the universe.

Two  $n \times n$  Latin squares are said to be orthogonal if when corresponding elements are written as a pair, each pair occurs exactly once. For instance, when the two  $4 \times 4$  Latin Squares

$$\begin{array}{cccc} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{array}$$

and

$$\begin{array}{cccc} \alpha & \beta & \gamma & \delta \\ \gamma & \delta & \alpha & \beta \\ \delta & \gamma & \beta & \alpha \\ \beta & \alpha & \delta & \gamma \end{array}$$

are "super-imposed", we get the square

$$\begin{array}{cccc} A\alpha & B\beta & C\gamma & D\delta \\ B\gamma & A\delta & D\alpha & C\beta \\ C\delta & D\gamma & A\beta & B\alpha \\ D\beta & C\alpha & B\delta & A\gamma \end{array}$$

in which each pair comes exactly once. Therefore the two Latin squares with which we started (the "roman" and the "greek") are orthogonal.

Orthogonal Latin squares are important in the design of experiments. If you have 4 species  $A, B, C, D$  of a plant and 4 types  $\alpha, \beta, \gamma, \delta$  of Fertilizers, you can arrange them in the form of the final ("Greco-Roman") square above, so that in each row and each column, we have a combination of each species and each type of fertilizer so that it eliminates the effect of soil etc. in a certain row or column and any difference can only be ascribed to the difference in species or fertilizer.

It is therefore important to know of orthogonal Latin Squares of every order. It is trivial to see that there cannot be two orthogonal  $2 \times 2$  Latin Squares. Euler had conjectured that there would be no pair of  $n \times n$  Orthogonal Latin squares (OLS) if  $n = 4k + 2, k = 0, 1, 2, 3, \dots$ . Around 1900, it was proved that there are no pairs of  $6 \times 6$  OLSs. Thus Euler's conjecture was true at least for  $k = 0$  and 1.

Bose, Shrikhande and Parker proved in 1959 that there exist pairs of OLS for every  $n = 4k + 2$  for all  $k > 1$ . Their paper was published in *Canad. J. Math.* In 1960. In the same issue of *Canad. J.* and immediately following the Bose-Shrikhande-Parker paper, there was a paper [17] by Chowla, Erdos and Strauss. These three had strengthened the result of Bose-Shrikhande-Parker. They proved that if  $f(n)$  is the number of mutually orthogonal  $n \times n$  Latin Squares, then as  $n$  tends to infinity,  $f(n)$  also tends to infinity. Note

that in this notation, the Bose-Shrikhande-Parker result was that  $f(4k + 2) > 0$  for  $k > 1$ .

At present, of course it is known that except for  $n = 2$  and  $6$ , there exist orthogonal  $n \times n$  Latin squares for every  $n$ . It is also known that  $f(n) \leq n - 1$  for all  $n$  and that  $f(n) = n - 1$  whenever  $n$  is a power of a prime. We do not know if  $f(n)$  can be  $n - 1$  even when  $n$  is not the power of a prime. R.C. Bose gave the result that if  $f(n) = n - 1$ , then there is a finite projective plane of order  $n$ .

### Conclusion

Professor Sarva Daman Chowla was principally a number theorist but he devoted a good deal of his time and talent to important areas of Combinatorics and obtained lasting results.

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## THE TRANSMISSION/DISEQUILIBRIUM TEST (TDT) USED IN HUMAN GENETICS

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### 1. Introduction

Professor M. K. Singal was a great mathematician and educationist. He contributed very significantly for the cause of mathematics through various organizations as well as in his personal capacity. I had the privilege of meeting him for the first time in 1985 in a function 'A Date with Mathematicians' organized by the Mathematical Association of India (MAI), Delhi Chapter. Our friendship since then continued over time. He was instrumental in roping me as Chairman or Speaker on several occasions in the deliberations of this Association as well as of the Indian Mathematical Society, the oldest society in India of which I became a life member at his instance. Professor Singal had a knack of identifying researchers from diverse disciplines and getting them due recognition. He identified me for the Distinguished Service Award of MAI given to me in 1993. He chose me to be invited, from amongst the only two from India, to be a speaker and participant in the Second Asian Mathematics Congress of mathematicians held at Nakhon Ratchasima, Thailand in October 1995. As a past General Secretary of the Indian Science Congress Association (ISCA), he championed the cause of mathematics and related subjects in more than one way. Before his untimely sad demise, he was discussing with me how to get reintroduced a separate section of statistics in the ISCA that was earlier established in 1940 due to the efforts of Prof. P.C. Mahalanobis but was clubbed with the section of mathematical sciences in recent years. He had several other plans for the propagation of mathematics in the country. But alas, his life was cut short and left the mathematical community to accomplish his dreams. I benefited a lot by his association, encouragement and advice. I pay my humble tribute to his memory by discussing a front line research activity

in human genetics viz. the transmission/disequilibrium test abbreviated as TDT.

## 2. Emergence of Genomics and related fields

A very significant development took place at the turn of the twentieth century when the human genome was sequenced by the International Human Genome Sequencing Consortium. At a video conference televised the world over on 20 June 2000, Mr. Bill Clinton, the then President of the USA, rated this feat as that of landing on the moon and remarked that the book of life had been deciphered that would lead to prevention and treatment of common human diseases from which the mankind has been suffering so long. Prime Minister of Britain, Mr. Tony Blair expressed similar views in this conference. This gigantic effort of Human Genome Project has no doubt provided massive amount of data in the form of the sequences of DNA (deoxyribonucleic acid) letters in the human genome, there being around three billion DNA letters in one cell of our body. But it is not known which DNA letter affects which part of our body. It is like having a crazy English dictionary where all letters, words, and lines are arranged randomly in a single continuous string without spacing or punctuation. We can't make head or tail of it unless we have some means of deciphering the words, their meaning and arrangement in an alphabetical order. In a similar way, in order to know how DNA works to produce various genetic disorders and diseases like cancer and heart problems, we need to develop methods to handle the sequence data that would tell us where the genes of complex disease are located on the chromosomes. This requires advanced and sophisticated mathematical techniques that need be addressed by interested scientists (Narain, 2004). Besides, to handle massive quantities of data being created in the field of genomics, a new field of bioinformatics, existing at the interface of biological and computational sciences, has recently emerged that requires statistical tools (Narain, 2006).

Genetic disorders in humans are broadly grouped into two categories. In the first category, we have diseases whose genetic control follows simple Mendelian principles - diseases such as cystic fibrosis and Huntington disease. In the former disease for example, a single-nucleotide polymorphism in the CFTR gene profoundly affects the bearer's digestive, reproductive, and respiratory systems and causes excessive loss of salt through sweating. This group of symptoms collectively is known as cystic fibrosis. In the second category, we have complex diseases such as diabetes, schizophrenia, various types of cancer etc. where the mode of inheritance does not follow a Mendelian pattern. Many genes, each with small and supplementary effects are involved in such cases that cannot be identified individually and followed

through generations as we do in the case of Mendelian genes. For such cases, genomic techniques like restriction fragment length polymorphism (RFLP), random amplified polymorphic DNA (RAPD), amplified fragment length polymorphism (AFLP), variable number of tandem repeats (VNTR) - that consist of microsatellites (short sequences) termed as short tandem repeats (STR) or simple sequence repeats (SSR) and minisatellites (long sequences) - and single nucleotide polymorphism (SNP) have opened up new possibilities for the identification of disease genes by means of the correlation between the disease genes and the specific DNA markers (Narain, 2000). This involves methods of genetic analysis such as quantitative trait locus (QTL) and linkage disequilibrium (LD) mapping. For a review on the QTL analysis, one can refer to Narain (2003, 2005). The method of LD mapping to ascertain linkage between the disease genes and molecular markers requires no assumptions about the mode of inheritance of the disease genes. Such methods use either case-control studies or family-based controls. In the latter category, a powerful test known as transmission/disequilibrium test was developed by Spielman et al. (1993) for studying insulin-dependent diabetes mellitus (IDDM). We discuss the theory of this test along with its power, from a statistical point of view, in what follows.

### 3. Transmission Disequilibrium Test (TDT)

Let  $A - a$  denote a marker locus with allelic frequencies  $p$  for  $A$  and  $q = 1 - p$  for  $a$  that is to be evaluated in relation to a disease trait locus  $D - d$  with a recombination probability between them as  $r$ , the disease being caused by the allele  $d$  with frequency  $p_d$  and  $q_d = 1 - p_d$  being the frequency of allele  $D$  in a large random mating population. We assume that the random mating population, under consideration, is in a steady state with a constant population size i.e. in equilibrium between the effects of genetic drift and recombination. This means that the time that has passed since the disease mutant was introduced is of the same order as the effective population size.

Let  $\phi_{DD}$ ,  $\phi_{Dd}$  and  $\phi_{dd}$  denote the penetrance (the probability of being affected by the disease) of the three genotypes  $DD$ ,  $Dd$  and  $dd$  respectively with  $\phi_{DD} < \phi_{Dd} < \phi_{dd}$ , the risk for an individual with two copies of disease gene  $d$  being more than that with one copy of  $d$  which in turn being more than that with no copies of  $d$ . The population disease prevalence is then

$$\phi = q_d^2 \phi_{DD} + 2p_d q_d \phi_{Dd} + p_d^2 \phi_{dd} \quad (1)$$

For later reference we define

$$C = -q_d \phi_{DD} + (q_d - p_d) \phi_{Dd} + p_d \phi_{dd} \quad (2)$$

The TDT compares the frequencies of marker alleles,  $A$  and  $a$ , transmitted from the parents to affected offspring with those of the alleles that are not transmitted and so is based on a  $2 \times 2$  table containing frequencies for the marker alleles transmitted ( $T$ ) or not transmitted ( $NT$ ) from parents to affected offspring in a random sample of  $2N$  parents ascertained through their  $N$  affected offspring from a population in Hardy-Weinberg equilibrium as given in the following Table.

**Table 1: Observed counts for transmitted and non-transmitted marker alleles  $A$  and  $a$  among  $2N$  parents of  $N$  affected offspring.**

Non-transmitted (NT) allele	Transmitted (T) allele		Total
	$A$	$a$	
$A$	$a$	$b$	$(a+b)$
$a$	$c$	$d$	$(c+d)$
Total	$(a+c)$	$(b+d)$	$2N$

The expected values of the counts in Table 1 are conditional probabilities with which a parent transmits one marker allele and not the other, given that the offspring is affected. In order to determine them we need to consider the population genetics model of a two loci system as discussed below.

With respect to the two loci,  $A - a$  and  $D - d$ , there are ten genotypes, taking into account the two phases of linkage. There are four possible two-locus haplotypes  $AD$ ,  $Ad$ ,  $aD$  and  $ad$  with frequencies say  $p_{AD}$ ,  $p_{Ad}$ ,  $p_{aD}$  and  $p_{ad}$  respectively when the genotypes are mated at random. Then the linkage disequilibrium coefficient, between the two loci, denoted by  $D$  is defined as the deviation of the haplotype frequency from its expected frequency under equilibrium which is simply the product of the corresponding gene frequencies. For example, if we take the haplotype  $Ad$  we have

$$D = p_{Ad} - p_A p_d. \quad (3)$$

The disequilibrium coefficient can also be expressed entirely in terms of the four haplotype frequencies as

$$D = p_{Ad} p_{aD} - p_{AD} p_{ad} \quad (4)$$

(Narain, 1990). This coefficient measures allelic association that could be either due to linkage for loci on the same chromosome or just association without any linkage for loci on non-homologous chromosomes showing independent segregation at meiosis.

Due to conditioning for the genotypes at the disease locus with respective penetrances, we have to consider the probability of only relevant mating types that result in the formation of desired gametes. The total prevalence of the disease in the population being  $\phi$ , the relevant probabilities need to



be divided by  $\phi$ . From a table of such probabilities, one can determine the required conditional probabilities of transmission of gametes. For instance, for the expected value of the count  $c$ , we determine the probability that, given that the offspring is affected, the heterozygous parent  $Aa$  transmits the marker allele  $A$  and not the other allele  $a$ . This is written symbolically as  $Pr.[T : A, NT : a/Aa, \text{offspring is affected}]$  and is given by

$$\begin{aligned} E(c) &= 2NPr.[T : A, NT : a/Aa, \text{offspring is affected}] \\ &= 2N\phi^{-1}[(p_d \phi_{dd} + q_d \phi_{Dd})\{p_{Ad}p_{ad} + (1-r)p_{Ad}p_{aD} + r p_{AD}p_{ad}\} \\ &\quad (p_d \phi_{Dd} + q_d \phi_{DD})\{p_{AD}p_{aD} + (1-r)p_{AD}p_{ad} + r p_{Ad}p_{aD}\}] \\ &= 2N[pq + (q-r)D(C/\phi)] \end{aligned} \quad (5)$$

In a similar manner, we get the expectations of  $a$ ,  $b$  and  $d$  as given below:

$$E(a) = 2N[p\{p + D(C/\phi)\}] \quad (6)$$

$$E(b) = 2N[pq + (r-p)D(C/\phi)] \quad (7)$$

$$E(d) = 2N[q\{q - D(C/\phi)\}] \quad (8)$$

#### (a) Test Statistic

From the expectations given by (5) and (7), we get

$$E(c-b) = 2N[(1-2r)D(C/\phi)] \quad (9)$$

$$E(c+b) = 2N[2pq + (q-p)D(C/\phi)] \quad (10)$$

This shows that the expectation of the difference  $(c-b)$  would be zero if either  $r = 1/2$  or  $D = 0$  which indicates either no linkage or no disequilibrium. In that case the expectations of both  $c$  and  $b$  will be same and equal to half. The statistic for TDT is therefore

$$\chi^2 = \frac{(c-b)^2}{(c+b)} \quad (11)$$

which follows a chi-square distribution with one degree of freedom and therefore can be used to test whether there is an association between marker  $A$  and the trait gene  $d$ . It may be noted that  $(c+b)$  provides with an estimate of the variance of  $(c-b)$ . In statistical terms, the above  $2 \times 2$  contingency table is the result of a matched-pair design in which, over the  $2N$  parents, each case of the transmitted allele is paired with that of a non-transmitted allele. Because of the matching, the observations tend to be dependent and one has to use the test for comparing correlated proportions. This leads to McNemar's test which is the same as that given by (11). In fact, in this test we test the hypothesis of marginal homogeneity. It implies symmetry across the main diagonal so that hypotheses of marginal homogeneity and symmetry are equivalent.

In the study on insulin dependent diabetes mellitus (IDDM) reported by Spielman et al. (1993), there were 94 families of which 57 parents were heterozygous for a marker, with "1" and "X" alleles, on chromosome 11p. The total of 62 children affected by IDDM indicated 124 alleles transmitted to the children. Among them there were  $c = 78$  "1" alleles and  $b = 46$  "X" alleles. This gave, by (11), a chi-square value of 8.26 with 1 d.f. that is significant at 0.004 level, thus demonstrating that the marker is linked to the susceptible gene for IDDM.

**(b) Power of the test**

Under the alternative hypothesis that there is linkage between the marker and the disease gene, given that there is linkage disequilibrium, the chi-square statistic, given by (11), follows a non-central  $\chi^2(1, \lambda)$  distribution with 1 d.f. and non-centrality parameter  $\lambda$  given by

$$\begin{aligned}\lambda &= [E(c) - E(b)]^2 / [E(c) + E(b)] \\ &= 2N[(1 - 2r)^2 D^2] / (\phi/C)[2pq(\phi/C) + D(q - p)]\end{aligned}\quad (12)$$

The power of the test is then the probability that the deviate from  $\chi^2(1, \lambda)$  is greater than or equal to  $\chi^2(a)$ , the critical value of  $\chi^2$  to reject the null hypothesis at significance level  $a$ .

Although not required for the validity of the TDT, for power computation we need a model for the mode of inheritance (MOI) in terms of the genotypic relative risk (GRR). Taking  $DD$  (with no copy of risk allele  $d$ ) as the reference genotype, the GRR of the three genotypes  $DD$ ,  $Dd$ , and  $dd$  are defined as 1,  $\gamma_1 = \phi_{Dd}/\phi_{DD}$  and  $\gamma = \phi_{dd}/\phi_{DD}$  with  $\gamma_1 \geq 1$  and  $\gamma > 1$ . If we consider only recessive genetic effects of the disease so that  $\phi_{DD} = \phi_{Dd}$  and therefore  $\gamma_1 = 1$ , we have only one GRR parameter  $\gamma$ . The non-centrality parameter  $\lambda$  can then be expressed as

$$\lambda = 2N[(1 - 2r)^2 D^2] / \theta[2pq\theta + D(q - p)]\quad (13)$$

where

$$\theta = p_d + 1/p_d(\gamma - 1).\quad (14)$$

It may be noted that  $\theta$  being a function of  $p_d$  and  $\gamma$ , the GRR parameter that is greater than unity, is necessarily positive and would have the minimum value of  $p_d$  when  $\phi_{DD} = \phi_{Dd} = 0$ . In that case the non-centrality would be maximum indicating maximum power of the test. We therefore consider the case when  $\phi_{DD} = \phi_{Dd} = 0$  and  $\phi_{dd} = 1$ , i.e. the genotypes  $DD$  and  $Dd$  have no risk of being affected by the disease whereas the genotype  $dd$  is sure to be affected by the disease. This gives  $1/(\gamma - 1) = 0$  and  $\theta = p_d$ . The population then consists of individuals affected by the disease as having genotype  $dd$  and normal individuals unaffected by the disease can be either the homozygous genotype  $DD$  or the heterozygous genotype  $Dd$ . We thus

take an affected individual as the one whose genotype at the disease locus is known as  $dd$  as well as the two parents of this individual whose genotypes at the disease locus are not known but both must have contributed the allele  $d$  to their child. The non-centrality parameter then becomes

$$\lambda = 2N[(1 - 2r)^2 D^2] / p_d [2pqp_d + D(q - p)] \quad (15)$$

and is a function of five parameters  $N$ ,  $p$ ,  $p_d$ ,  $r$  and  $D$ .

Liu (1997) gives the power of the TDT in this simplified form for several values of  $N$ ,  $p$ ,  $p_d$ ,  $r$  and  $D$ . The power increases with increase in  $D$  but with decrease in  $r$ . It is high when  $p_d$  is lower. The frequency  $p$  has however small effect on the power. It also increases with increase in  $N$ .

It may be seen that  $\lambda$  will be strictly zero when either  $r = 1/2$  or  $D = 0$ , the values under the null hypothesis in which case the chi-square follows a central  $\chi^2$  distribution with 1 d.f.. Values of  $\lambda$ , therefore, under different values of the five parameters,  $N$ ,  $p$ ,  $p_d$ ,  $r$  and  $D$  reflect the power of the TDT. We give these values in Table 2 below for  $N = 200$  and  $D = 0.1$  when  $r$  varies between 0.45 and 0.01 for each of the two values of  $p_d$  and  $p$ .

**Table 2: Values of non-centrality parameter for different recombination probability between the marker  $A - a$  and the disease locus for two combinations of gene frequencies when  $N = 200$ .**

$r$	$p_d = 0.5, p = 0.2$	$p_d = p = 0.2$
0.45	0.36	1.61
0.30	5.82	25.01
0.20	13.09	58.06
0.10	23.27	103.22
0.04	30.78	136.52
0.01	34.92	154.90

From Table A.3 in Weir (1996), we find that for the significance value of 0.05 and 1 d.f., the power is 0.90 and 0.99 for non-centrality parameters of 10.5 and 18.4 respectively. As such when  $r$  is 0.2, the power of the test for  $p_d = 0.5$  and  $p = 0.2$  with  $\lambda = 13.09$  would be greater than 0.90 but it would even be greater than 0.99 for  $p_d = p = 0.2$  with  $\lambda = 58.06$ , indicating thereby that the power is high when the value of  $p_d$  is smaller as shown in Liu (1997). Thus we can compare the power of the test in terms of the non-centrality parameter without actually working out the power.

When the marker is at the disease gene locus itself,  $r = 0$ ,  $p = p_d$ , and  $D = p_d q_d$ , giving

$$\lambda = 2Nq_d \quad (16)$$

#### 4. Discussion

Most of the common human diseases such as diabetes, hypertension, various types of cancer, asthma and psychiatric disorders are complex in nature being mediated by a large number of genes and modified by environmental factors along with their interactions with the genotypes. Until recently, there was no way to identify and follow the corresponding individual genes with the help of Mendelian principles as is possible in the case of monogenic diseases. Recent discovery of molecular markers such as micro-satellites, RFLP and SNP spread all along the chromosomes has opened up new ways to identify the genes by determining the association between the markers and the disease phenotypes. This association depends on the linkage disequilibrium (LD) between the disease susceptibility gene and the marker that got generated when the disease gene arose by mutation from the normal type quite sometime back in the past. The conservation of LD over the generations, either because of linkage with the gene and the marker on the same chromosome or because of association when they lie on different chromosomes, produces the correlation between the marker and the phenotype. In this context the transmission/disequilibrium test (TDT), introduced by Spielman et al. (1993) on the basis of case-parents trios, is a powerful and major approach to search for genes for complex diseases in the humans. It tests directly for linkage between them when association due to LD is present. We have discussed here the theory of this method from a statistical point of view and given results on the power of the test.

Classical association studies such as case-control analyses in unrelated cases and controls suffer from the drawback that population admixture/stratification tends to mask or even reverse true genetic effects. TDT, on the other hand, controls this factor in testing for linkage and/or association between marker loci and the disease susceptibility gene.

It may be noticed that the TDT requires data on a sample of families, in each of which there is a single affected offspring with two of its parents, at least one parent being heterozygous at the marker locus - a trio - and all individuals have been genotyped at the particular marker locus. However, when the disease, under study, has a late age of onset, the parental marker genotypes may not be available at all and we are unable to adopt TDT. In this situation, one possibility is that the missing parental genotypes may be reconstructed from the genotypes of their offspring and treated as if they have been typed (Spielman and Ewens, 1996). A better alternative, however, is to use the 'sib TDT' or S-TDT type procedure discussed in Spielman and Ewens (1998) where data consist of marker genotypes of the

offspring only, both affected and unaffected, for each family included in the sample.

## 5. Summary

The theory of the transmission/disequilibrium test (TDT) used in human genetic studies has been discussed from a statistical point of view. For a sample of children affected by a disease together with their parents, ascertained at random from a population in Hardy-Weinberg equilibrium, the probability that a parent having a particular genotype at the marker locus transmits one of its allele and not the other, on the condition that the offspring is affected by the disease has been determined for the four possible cases. Using them we get a McNemar's type chi-square test from a  $2 \times 2$  table. Entries only for the heterozygous parents are used. On the null hypothesis of no linkage, in the presence of association, individual alleles should be transmitted from heterozygous parents to affected offspring randomly with equal probabilities. The test has been illustrated with data reported on insulin dependent diabetes mellitus (IDDM). The power of the test, in terms of non-centrality parameter, has been derived in general terms. In the special case of recessive genetic effects with recessive genotypes at the disease locus being sure to be affected by the disease but the dominant and heterozygous genotypes having no disease risk, the power has been numerically studied via non-centrality parameter. The limitation of TDT when parental data are not available due to late onset of disease has been discussed.

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## CHALLENGES AND FRUSTRATIONS OF BEING A MATHEMATICIAN

R. B. BAPAT

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It is a great honour and pleasure for me to have had this opportunity of addressing this distinguished gathering. I express my gratitude to the members of the Indian Mathematical Society for having elected me to the office of the President, in the centenary year of the Society. I am also thankful to the local organizers for undertaking a task of such magnitude towards a successful and productive conference.

As mathematicians, we belong to a minority. Most people are turned off by mathematics. Very few have some liking for the subject and even a smaller number choose it as a profession. In this talk I have tried to give way to some feelings about the nature of our profession and the problems that we face. We all agree that it is a pleasure to do mathematics and so I dwell mostly on things that tend to obstruct that pleasure. The views expressed here have evolved largely out of my personal experiences and perspectives. It is hoped that they might induce a discussion which might result in further inputs and exchange of ideas. The tone of the writing is definitive, but since it is meant to generate a debate, it is justified.

We are mathematicians by choice. We chose the profession because we love the subject. Reading and assimilating deep results of masters and then solving some of our own small problems brings us pleasure to which nothing else compares much. Yet we live in a world populated mostly by non-mathematicians. We must survive and thrive in their midst. This brings forth its own challenges and frustrations.

Our professional activity is divided mainly into teaching and research, except for some administrative duties. A lot has been said about mathematics education and I will confine myself to a few comments. It is a fact that most of us like to do our own thing. We enjoy teaching if it is a course of our choice and the class consists of a few eager, motivated, well-behaved

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students. That is only a dream. Often we must teach large classes of uninterested students who are there only for completing the requirements. But in spite of all this we must strive to teach, giving it our best and at the same time maintaining the standard of our subject. Compromises have no place here. We believe in teaching in a certain way and it can be fine-tuned depending upon the reactions of the students. But it should not prevent us from communicating the basic spirit of mathematics, especially the importance of logical enquiry. All aspects of mathematics, including history, biographies, motivation, definitions, lemmas, theorems, corollaries, proofs, examples, counterexamples, conjectures, construction, computation and applications can and should find a place in the classroom.

The views expressed in this talk pertain to mathematics in general and not particularly to the Indian context. But with special reference to the situation in India it must be remarked that the bifurcation of undergraduate teaching and research has not served us well. Our best researchers are concentrated in research institutes and do not teach undergraduate courses in mathematics. This is one of the reasons for a steady decline in the quality of undergraduate mathematics education in the country.

Examinations and tests are a major part of the teaching process and let me say something about them. I find that in physics, students are routinely asked questions which are not in any of their prescribed books and require some extra application of thought. In mathematics however, we are supposed to stick to routine questions, except in olympiad type exams. This is true at the high school board examinations level as well as the college level. Submitting to this requirement makes the subject dull for gifted students. Examination boards need to be persuaded to consider providing for ten percent marks for nonroutine questions, without changing the syllabus. We will then continue to have students securing 100 percent marks but that will mean something much more than in the present system.

Mathematics has been projected as a dry and difficult subject. Being poor in mathematics is considered natural and in fashion, whereas being good in mathematics is taken as being eccentric and queer. This perception has been created by all, including press and popular media. Television interviews of celebrities invariably include the "I was awful in maths, just hated it ..." bit. All this has its effect on students. The forces that are at work pulling talented students away from mathematics and towards other sciences are too powerful and the only thing that works in favour of mathematics is the enjoyment it provides to some of the students who cannot think of doing anything else. Why can't mathematics be both enjoyable and fashionable, the *in-thing*, to do?

We are repeatedly asked to provide real life examples, motivation, while teaching or while lecturing to a general audience. This is justified to some extent, but mathematics manifests its beauty only when it is stripped off



of its worldly connections. It flourishes in the abstract and then again turns messy when brought back to apply. The messy part is the one that engineers and scientists (those who use the mathematics) are to deal with. Why should a mathematician be required to constantly make this back and forth transition? To protect our interests let us make this feeling known, after debating it among ourselves.

Now I turn to the second component of our job, research. Writing our work and then managing to get it published is an important part of our profession. In a subject that is nearly two thousand years old, and in areas that are more than two hundred years old, getting a drop of something original is not easy. The situation is perhaps different in experimental sciences. But a comparison of publication record with other sciences is always made for all policy decisions. The recent debate about the inappropriate use of *citation index* in mathematics is a case in point. This leads to many ills of our profession. There is too much pressure to publish, quality suffers in the process. Refereeing is a challenging task with no apparent reward except a feeling of satisfaction towards contributing to the health of your area. It is nearly impossible to track all that is published and that results in further narrowing of one's interests.

In this context I wish to propose a scheme, which addresses the question of decreasing the number of publications to some extent. There can be free, possibly electronic, journals, run by well-established societies or academies, in which all papers are by invitation only. Papers must be refereed but the role of the referee must be limited mainly to checking for correctness, style etc. To fix ideas, suppose one such journal is called *The Free Journal of Combinatorics*. The job of the editorial board of this journal would be to identify promising mathematicians in Combinatorics, at all levels, and invite them to contribute one paper to the journal per year for the next five years. The author in turn should agree to (i) submit the best of his work to the journal and (ii) to restrict his publications in other journals to either zero or a very small number. If the journal acquires enough prestige, then being an invited author of the journal will carry lot of value and then there is no need to publish more. Also in the course of a few years the Free Journal of Combinatorics will be a reflection of the best work in the area of Combinatorics and give a fair picture of the development of the subject.

I am proposing this scheme after giving it some thought from my personal perspective. I would welcome a situation where a reputed publisher will publish one paper of mine per year for a certain number of years. That will then contain the best I have to offer. Apart from that I can stick mainly to expository writing or survey papers. In any event it is true that barring exceptions, the average professional mathematician does not have enough research to report and generate more than one or two papers per year after the initial burst of activity has subsided. In practice however,

the number increases since more publications means better salary and more grants, among other things.

The very idea of limiting the role of a referee may appear drastic. Mathematics is an art as much as it is a science. Can one imagine the painting of an accomplished painter being subjected to a refereeing process, before it is exhibited? And with all the stringent refereeing regimen in place in the present system, has it really eliminated erroneous papers or duplication of results? Let us recognize the best talent amongst us and *invite* them to write for us. There are invited papers at present but an invited paper often means a paper which the author would never care to publish otherwise.

Mathematicians, like other scientists, need support for their research and hence must write proposals for research grants. The mechanics of seeking research grants is designed by and suited to experimental scientists. A natural scientist wishes to propose a theory about an enzyme or a drug and must conduct experiments to test the claims. This requires some equipment, graduate students, site visits and these constitute the bulk of the proposal requirements. This process does not quite suit us mathematicians. We like to think about a problem, and at the same time let our thoughts wander in a random fashion. If something along the way catches our attention then we may follow that route. Thus, in reality, we cannot indeed write an honest research proposal which gives too many specifics about what we are going to achieve. If someone claims in a proposal that he or she is going to investigate bounds for the eigenvalues of a certain class of  $0 - 1$  matrices, very likely the proposer has already obtained some such bounds which will only be written up when the proposal is approved. There cannot be any other way. If I do not have any eigenvalue bounds already obtained with me, there is a chance I will not get any, even if a very big grant is awarded.

This special nature of mathematical research needs to be kept in mind by one and all, particularly the funding agencies. Past achievements can be given more weight and a sketchy proposal should not necessarily mean automatic rejection. Travel and short visiting appointments are a big attraction to mathematicians. If a mathematician can spend a few months in a place with all facilities, an intellectually stimulating atmosphere, and no teaching or administrative duties, then he can achieve wonderful results. Needs of a mathematician are meagre compared to that of an experimental scientist but they need to be addressed sensitively.

Even though outsiders may recognize one of us simply as a mathematician, within our community there are many subdivisions. One is not just a mathematician, he is a *differential geometer* or a *quantum probabilist* or a *commutative ring theorist*. Mathematics is neatly divided into areas: algebra, geometry, analysis, topology are some of the respected ones and there are many others which do not enjoy a similar standing. A research mathematician must make his or her area known and then should stick to it,

if he/she doesn't want the professional career to suffer. Contacts must be developed in that particular area, journals should be identified, one must become known to the editorial board members and then life may be easier. Except that if you get bored with the same type of problems and want to be adventurous and venture into new territory, you better be first rate and adapt quickly, otherwise getting a foothold in the new area is not easy. So if your Ph.D. thesis has been in *uniform bounded cohomology of sections of holomorphic vector bundles*, twenty years later you would at best be venturing into the *locally nonuniform* case. More specialization has created further divisions among the dwindling number of mathematicians.

In ground reality, however mathematicians are indeed divided, but these divisions are of a different kind. There are those who enjoy teaching and mentoring students, researchers who excel in what they do but are hardly comfortable or efficient in a classroom, good expositors who can make a difficult subject look simple in their writing, people who enjoy organizing conferences; they don't mind if their own area is far removed from the area of the conference, good Ph.D. guides; their number of students is in double digits in a short time, those who are interested in foundations, those who like to write proposals and have several projects to their credit simultaneously, those who like to chair departments and so on.

Can we recognize this classification and take it into account in our decision making, rather than the narrow area-wise breakup? Some of our sister areas, notably Economics, are devoid of the sort of tight division that we mathematicians have imposed upon ourselves.

Now I come to a topic where mathematicians are not really at fault, but it is made to appear that they are doing something wrong, or rather, not doing something right. And here I am referring to the task of communicating our work to others, especially to a general audience. Explaining mathematics to non-mathematicians, even educated ones, is an ordeal. Our subject does not lend itself to explanation to nonspecialist, period. However most people, including mathematicians, believe that it is a shortcoming of mathematicians that they are not able to "explain" their subject to others. Explaining to a layman (an educated one) what a *group* means, appears interesting - talk about geometrical figures, rotations, reflections and so on. But that is just the beginning. Can we get to *normal subgroups* and still retain the same clarity? And then what about *trivial torsion units in G-adapted group rings*?

A recent book review by Daniel Bliss in the Notices of the AMS (June/July 2007) brings forth this dilemma in a very interesting fashion. I recommend reading it in original, but the highlight is that the author (John Stillwell) of the book under review (*Yearning for the Impossible*, A.K.Peters Limited, 2006) tries very sincerely and very hard to explain what an *ideal* is, develops the concept very patiently and finally has an explanation. But at the end

of it one wonders whether it is worth the trouble. Why can't we be frank and say that, look, these concepts are really very abstract and cannot be explained but then it is also required that we justify why we are doing what we do. In any event whatever good nontechnical exposition of mathematics has been achieved has remained confined to a few areas, notably discreet mathematics, where more ground can be covered. Even the great Martin Gardner has had to restrict himself in terms of range of topics. But then we are not conveying a true picture of the vastness and depth of mathematics.

This difficulty is also faced by other areas of arts and science. But it is interesting to see how they get around the problem. We are made to believe that physics, chemistry or biology are easier to explain to the general audience. Nothing can be farther from the truth but this belief is implanted successfully in our minds by the way scientists in these areas present their work. Their abstract concepts are presented in a way as if they are actual realities. Atoms, quarks, dark energy, strings, black holes, are all concepts but people believe in them. In comparison mathematicians are awfully shy of presenting anything for which they do not have a refereed proof. We need to be bold. Our ongoing investigations should be presented with a passion and sense of self-belief. Our deceptively simple terminology is also a culprit. How can anyone believe that simple sounding concepts like group, ring and field can have anything deep to connote?

I have tried to present some views about our profession and as remarked in the beginning, it will be helpful if it generates more exchange of ideas. Our subject, known as the *Queen of Sciences* and one with a long history, is losing its image in the eyes of the young student, policy makers and general public. Riding on the wave of computer science does not solve the problem, since the nature of mathematics is unique and computer science is at best a glimpse into a small portion of it. We must strive to effectively communicate the unique nature of our subject, its beauty as perceived by us, and its applications to the betterment of life, to the layman, as well as scientists and policy makers in order to create a positive feeling towards our profession.

I conclude by wishing all the members and delegates a very fruitful and memorable conference.

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## RECENT DEVELOPMENTS AND OPEN PROBLEMS IN THE THEORY OF PERMANENTS

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### 1. Introduction

If  $A$  is an  $n \times n$  complex matrix, then the permanent of  $A$ , denoted  $\text{per} A$ , is defined as

$$\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$$

where  $S_n$  is the symmetric group of degree  $n$ . As an example,

$$\text{per} \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix} = 10 + 40 + 6 + 8 + 4 + 75 = 143.$$

Thus the definition of the permanent is similar to that of the determinant except for the sign associated with each term in the summation. This minor difference in the definition makes the two functions quite unlike each other. Perhaps the permanent cannot compete with its cousin, the determinant, in terms of the depth of theory and the breadth of applications, but it is safe to say that the permanent also exhibits both these characteristics in ample measure, a fact that has not received enough attention.

The permanent has a rich structure when restricted to certain classes of matrices, particularly, matrices of zeros and ones, (entrywise) nonnegative matrices and positive semidefinite matrices. Furthermore, there is a certain similarity of its properties over the class of nonnegative matrices and the class of positive semidefinite matrices, which is not yet fully understood. In this article we describe some properties of permanents over these three classes, putting emphasis on recent developments and open problems. The

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article is by no means an extensive survey of permanents, rather it is biased towards topics in which I have been interested over the years. The interested reader can however follow the references at the end that point to several sources dealing with various aspects of the theory of permanents.

## 2. Matrices of zeros and ones

Let  $A$  be an  $n \times n$  matrix. If  $\sigma \in S_n$ , then the set  $\{a_{1\sigma(1)}, \dots, a_{n\sigma(n)}\}$  is called a diagonal of  $A$  corresponding to the permutation  $\sigma$ . The product  $\prod_{i=1}^n a_{i\sigma(i)}$  is called a diagonal product. A diagonal is positive if the corresponding diagonal product is positive. Note that the permanent of a 0 – 1 matrix equals the number of positive diagonals of  $A$ .

The concept of a positive diagonal is clearly related to that of a system of distinct representatives in a family of sets, as well as to perfect matching in a bipartite graph. This is explained as follows.

If  $A$  is a 0 – 1  $n \times n$  matrix then let  $A_i = \{j : a_{ij} = 1\}, i = 1, \dots, n$ . We say that the set  $\{x_1, \dots, x_n\}$  is a system of distinct representatives of the family  $\{A_1, \dots, A_n\}$  if  $x_i \in A_i, i = 1, \dots, n$ . The well-known theorem due to P.Hall asserts that the family  $\{A_1, \dots, A_n\}$  admits a system of distinct representatives if and only if the union of any  $k$  members of the family contain at least  $k$  elements,  $k = 1, \dots, n$ .

Given a 0 – 1  $n \times n$  matrix we may naturally associate a bipartite graph  $G$  with  $A$ . The partite sets  $X$  and  $Y$  are the index sets of rows and columns respectively. There is an edge from the  $i$ -th vertex of  $X$  to the  $j$ -th vertex of  $Y$  if and only if  $a_{ij} = 1$ . Then by the König-Egervary Theorem  $G$  has a perfect matching if and only if for any  $S \subset X$ , the neighbour set of  $S$  has at least  $|S|$  elements. This statement as well as Hall's Theorem are equivalent to the next result.

**Theorem 1.** [Frobenius-König Theorem] *Let  $A$  be a 0 – 1  $n \times n$  matrix. Then  $\text{per} A$  is zero if and only if  $A$  has a zero submatrix of order  $r \times s$  such that  $r + s = n + 1$ .*

Another result equivalent to Theorem 1 is the Marriage Theorem of Halmos and Vaughan. As these equivalences show, the permanent function on the set of 0 – 1 matrices is intimately connected with several classical combinatorial problems. However, Theorem 1 also admits extensions of a different type, making it evident that the result is important in classical matrix theory as well. If  $A$  is a 0 – 1 matrix then the term rank of  $A$  is the maximum  $k$  such that  $A$  has a  $k \times k$  submatrix with positive permanent. Although the term rank appears to be a purely combinatorial concept, related to the matching number of a bipartite graph, it is also related to the classical rank as we indicate next.

We say that a matrix over a field is of zero type if each of its rows is a linear combination of the remaining rows and each of its columns is a linear combination of the remaining columns. The following result is obtained in [5].

**Theorem 2.** *Let  $A$  be an  $n \times n$  matrix over a field  $F$ . Then  $A$  is singular if and only if  $A$  has a zero type submatrix  $B$  of order  $r \times s$  such that  $r + s \geq n + \text{rank}(B)$ .*

A matrix is said to be generic if its nonzero elements are algebraically independent indeterminates. The term rank of a generic matrix coincides with its rank (over the field generated by its nonzero elements). If Theorem 2 is applied to a generic matrix then we recover Theorem 1. For related results and extensions to bimatroids, see [23]. It was conjectured by Minc in 1963 that if  $A$  is a 0 – 1 matrix with row sums  $r_1, \dots, r_n$ , then  $\text{per} A \leq \prod_{i=1}^n (r_i!)^{\frac{1}{r_i}}$ .

The conjecture was proved by Brègman in 1973 and Schrijver gave an easier proof in 1978. Recently, Soules [31, 32] has obtained extensions of the bound to nonnegative matrices, using a different proof technique.

### 3. Nonnegative matrices

The class of entrywise nonnegative matrices, which properly includes the class of 0 – 1 matrices, is another class on which the permanent is well-behaved. We will write  $A \geq 0$  to indicate that each element of  $A$  is nonnegative. Note that when  $A \geq 0$ , there are no cancelations in the expression for the permanent and it provides the sum of all the diagonal products of the matrix. In the context of graph theory it can be interpreted as the sum of the weights of all perfect matchings in the associated bipartite graph.

The  $n \times n$  matrix  $A$  is said to be doubly stochastic if  $A \geq 0$  and each row and column sum of  $A$  is 1. The set of  $n \times n$  doubly stochastic matrices is a compact, convex set, and we denote it by  $\Omega_n$ .

A permutation matrix is a matrix obtained from the identity matrix by permuting its rows and columns. Clearly a permutation matrix is doubly stochastic. A celebrated theorem of Birkhoff and von Neumann asserts that the extreme points of  $\Omega_n$  are precisely the  $n!$  permutation matrices of order  $n$ . Thus a matrix is doubly stochastic if and only if it can be expressed as a convex combination of permutation matrices. As a simple consequence of the Birkhoff-von Neumann Theorem we can conclude that the permanent of a doubly stochastic matrix must be positive. Therefore it is natural to enquire about the minimum of the permanent over  $\Omega_n$ . Van der Waerden conjectured in 1928 that the minimum of the permanent over  $\Omega_n$  equals  $\frac{n!}{n^n}$ .

and is attained uniquely at the matrix  $J_n$ , which is the  $n \times n$  matrix with each entry equal to  $\frac{1}{n}$ .

Marcus and Newman [21] obtained some important partial results towards the solution of the van der Waerden conjecture. In particular they showed that if  $A \in \Omega_n$  is a permanent minimizer, then all permanent cofactors of  $A$  must exceed or equal  $\text{per} A$ . From this result it can be seen that if  $A \in \Omega_n$  is a permanent minimizer such that each entry of  $A$  is positive, then  $A = J_n$ .

After the Marcus and Newman paper, attempts towards the solution of the van der Waerden conjecture again picked up in the late sixties and seventies and it gave a significant impetus to work in the area of combinatorial matrix theory.

The breakthrough came around 1981 when Egorychev and Falikman independently proved the van der Waerden conjecture. The main tools in the proof due to Egorychev were the Marcus-Newman result and the Alexandroff inequality which we now describe.

If  $A$  is an  $n \times n$  matrix, then let  $a_i$  denote the  $i$ -th column of  $A$ ,  $i = 1, 2, \dots, n$ . Let  $A$  be an  $n \times n$  positive matrix and consider the bilinear form

$$(1) \quad \text{per}(a_1, \dots, a_{n-2}, x, y), x, y \in R^n.$$

It turns out that the bilinear form (1) is indefinite with exactly one positive eigenvalue. As a consequence one obtains the Alexandroff inequality which asserts that if  $A$  is a nonnegative  $n \times n$  matrix and if  $x, y \in R^n$  where  $x$  has positive coordinates, then

$$(2) \quad (\text{per} A)^2 \geq \text{per}(a_1, \dots, a_{n-2}, x, x) \text{per}(a_1, \dots, a_{n-2}, y, y).$$

The Alexandroff inequality was in fact proved for the mixed discriminant, which we will discuss in the next section. The Birkhoff-von Neumann Theorem has been extensively studied and generalized. For a recent extension in the area of quantum probability, see [24].

Let  $A$  be an  $n \times n$  0-1 matrix with  $k$  ones in each row and column. The permanent of such a matrix is of special interest since it counts the number of perfect matchings in a regular bipartite graph. Note that  $\frac{1}{k}A \in \Omega_n$  and hence we do get a bound for  $\text{per} A$  by the Egorychev-Falikman proof of the van der Waerden bound. Let  $\Omega_{k,n}$  be the set of matrices in  $\Omega_n$  with each entry 0 or  $\frac{1}{k}$ . The following bound was conjectured by Schrijver and Valiant in 1980, and proved by Schrijver [27]. If  $A$  is an  $n \times n$  matrix in  $\Omega_{k,n}$  with each entry 0 or  $1/k$ , then

$$\min\{\text{per} A : A \in \Omega_{k,n}\} \geq \left(\frac{k-1}{k}\right)^{(k-1)n}.$$



For any  $k$  and  $n$ , let  $p(k, n)$  be the number of perfect matchings in any  $k$ -regular bipartite graph with  $2n$  vertices. The van der Waerden bound implies that

$$\inf_{k \in \mathbb{N}} \frac{p(k, n)}{k^n} = \frac{n!}{n^n}.$$

Note that the Schrijver-Valiant bound implies that

$$p(k, n) \geq \frac{(k-1)^{(k-1)}}{k^{(k-2)}}.$$

Moreover, Schrijver[27] has shown that

$$\inf_{n \in \mathbb{N}} p(k, n)^{1/n} = \frac{(k-1)^{(k-1)}}{k^{(k-2)}}.$$

Thus both bounds are best possible in different asymptotic directions.

Recently, Gurvits[14] has given a unified proof of the van der Waerden conjecture and the Schrijver-Valiant conjecture using hyperbolic polynomials. According to the van der Waerden bound the permanent achieves its minimum over  $\Omega_n$  at the matrix with each entry  $\frac{1}{n}$ . However finding the minimum of the permanent on certain subsets has also been considered. In this context the following folklore conjecture appears to be very notorious

**Conjecture 1:** The permanent achieves its minimum over the set of  $n \times n$  doubly stochastic matrices with zeros on the diagonal uniquely at the  $n \times n$  matrix with each diagonal entry zero and each off-diagonal entry  $\frac{1}{n-1}$ .

#### 4. Mixed discriminants

If  $A^k = (a_{ij}^k)$  are  $n \times n$  matrices,  $k = 1, 2, \dots, n$ , then their mixed discriminant, denoted by  $D(A^1, \dots, A^n)$ , is defined as

$$(3) \quad D(A^1, \dots, A^n) = \frac{1}{n!} \sum_{\sigma \in S_n} \begin{vmatrix} a_{11}^{\sigma(1)} & \dots & a_{1n}^{\sigma(n)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\sigma(1)} & \dots & a_{nn}^{\sigma(n)} \end{vmatrix},$$

where  $S_n$  denotes, as usual, the set of permutations of  $1, 2, \dots, n$ . (Throughout this section,  $A^k$  should not be confused with the  $k$ -th power of  $A$ .) Thus, if  $A = (a_{ij}), B = (b_{ij})$  are  $2 \times 2$  matrices, then

$$D(A, B) = \frac{1}{2}(a_{11}b_{22} - a_{21}b_{12} - a_{12}b_{21} + a_{22}b_{11}).$$

We now indicate that the mixed discriminant provides a generalization of both the determinant and the permanent. If  $A^k = A, k = 1, 2, \dots, n$ ,

then clearly,  $D(A^1, \dots, A^n) = \det A$ . Also, if each  $A_k$  is a diagonal matrix,

$$A^k = \begin{bmatrix} a_{11}^k & & \\ & \ddots & \\ & & a_{nn}^k \end{bmatrix},$$

then  $D(A^1, \dots, A^n)$  equals  $\frac{1}{n!} \text{per} B$  where  $B = (b_{ij}) = (a_{ii}^j)$ .

We now consider some properties of mixed discriminants of positive semidefinite matrices.

Let  $A^k, k = 1, 2, \dots, n$ , be positive semidefinite  $n \times n$  matrices and suppose  $A^k = X_k X_k^T$  for each  $k$ . Then it can be proved that

$$D(A^1, \dots, A^n) = \frac{1}{n!} \sum (\det(x_1, \dots, x_n))^2,$$

where the sum is over all choices  $\{x_1, \dots, x_n\}$  such that  $x_k$  is a column of  $X_k, k = 1, 2, \dots, n$ . As an immediate consequence we conclude that the mixed discriminant of positive semidefinite matrices is nonnegative. When each  $A^k$  is a diagonal positive semidefinite matrix then the statement merely reduces to the fact that the permanent of a nonnegative matrix is nonnegative.

It is natural to enquire about the positivity of  $D(A^1, \dots, A^n)$  when each  $A^k$  is positive semidefinite. Here one can prove the following, using Rado's generalization of Hall's theorem. Let  $A^k, k = 1, 2, \dots, n$ , be  $n \times n$  positive semidefinite matrices. Then  $D(A^1, \dots, A^n) > 0$  if and only if for any  $T \subset \{1, 2, \dots, n\}$ , the rank of  $(\sum_{i \in T} A^i)$  is at least  $|T|$ .

Let  $\mathcal{D}_n$  denote the set of all  $n$ -tuples  $\mathbf{A} = (A^1, A^2, \dots, A^n)$  of  $n \times n$  positive semidefinite matrices satisfying  $\text{trace } A^i = 1, i = 1, 2, \dots, n; \sum_{i=1}^n A^i = I$ . Then by the process of identifying a nonnegative  $n \times n$  matrix with an  $n$ -tuple of diagonal matrices described in the preceding discussion,  $\mathcal{D}_n$  can be viewed as a generalization of the class of  $n \times n$  doubly stochastic matrices. The permanent function on  $\Omega_n$ , the polytope of  $n \times n$  doubly stochastic matrices, is generalized to the mixed discriminant over  $\mathcal{D}_n$ . It can be shown [2] that if  $\mathbf{A} = (A^1, \dots, A^n) \in \mathcal{D}_n$ , then  $D(A^1, A^2, \dots, A^n) > 0$ .

It was conjectured in [2] that the mixed discriminant achieves its minimum over  $\mathcal{D}_n$  precisely at  $(A^1, \dots, A^n)$  where each  $A^i$  is the diagonal matrix

$$\begin{bmatrix} \frac{1}{n} & & \\ & \ddots & \\ & & \frac{1}{n} \end{bmatrix}.$$

The conjecture has been recently proved by Gurvits [13]. Another open problem posed in [2] is to characterize the extreme points of  $\mathcal{D}_n$ .

### 5. Positive definite matrices

When one deals with the concept of positivity in the context of matrices then one must keep in mind the two notions of positivity, the notion of a positive operator, which corresponds to a positive semidefinite matrix and the notion of an entrywise positive matrix. There are curious similarities regarding the properties of the two classes. Olga Taussky Todd proposed the problem of explaining this similarity and it is commonly known as the Taussky unification problem, see [26]. If  $A$  is an  $n \times n$  complex hermitian matrix, then  $A$  is said to be positive definite if  $x^*Ax > 0$  for any nonzero  $x$ , and positive semidefinite if  $x^*Ax \geq 0$  for any  $x$ .

The permanent exhibits interesting properties on the class of positive definite matrices as well. As an example, if  $A$  is a nonnegative matrix then  $\text{per}A$  is clearly nonnegative. It turns out that if  $A$  is positive semidefinite, then  $\text{per}A \geq 0$ . This fact is not obvious from the definition of permanent, since the sum in the definition may contain both positive as well as negative terms.

If  $A$  and  $B$  are both positive semidefinite such that  $A - B$  is also positive semidefinite, then it is known that  $\text{per}A \geq \text{per}B$ . Now if  $A$  is a positive semidefinite matrix which is also doubly stochastic, then it is not difficult to show that  $A - J_n$  is positive semidefinite. Then  $\text{per}A \geq \text{per}J_n$  and thus the van der Waerden conjecture is verified for positive semidefinite matrices. Later we will give an example of an open problem which appears difficult when the matrix is positive semidefinite but is easy for nonnegative matrices.

We now introduce some notation. If  $A, B$  are matrices of order  $m \times n$  and  $p \times q$  respectively, then the Kronecker product of  $A, B$  is denoted by  $A \otimes B$ . Thus  $A \otimes B$  is the  $mp \times nq$  matrix given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

The main property of Kronecker product is that if  $A, B, C, D$  are matrices such that  $AC$  and  $BD$  are defined, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

It can be proved using the preceding property that if  $A, B$  are positive semidefinite, then  $A \otimes B$  is positive semidefinite.

Let us assume that the elements of  $S_n$ , the permutation group of degree  $n$ , have been ordered in some way. This ordering will be assumed fixed in the subsequent discussion. Let  $A$  be an  $n \times n$  matrix. The Schur power of

$A$ , denoted by  $\pi(A)$ , is the  $n! \times n!$  matrix whose rows as well as columns are indexed by  $S_n$  and whose  $(\sigma, \tau)$ -entry is  $\prod_{i=1}^n a_{\sigma(i)\tau(i)}$  if  $\sigma, \tau \in S_n$ .

As an illustration, suppose the elements of  $S_3$  are ordered as 123, 132, 213, 231, 312, 321, and let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ . Then

$$\pi(A) = \begin{bmatrix} aek & afh & bdk & bfg & cdh & ceg \\ afh & aek & bfg & bdk & ceg & cdh \\ bdk & cdh & aek & ceg & afh & bfg \\ cdh & bdk & ceg & aek & bfg & afh \\ bfg & ceg & afh & cdh & aek & bdk \\ ceg & bfg & cdh & afh & bdk & aek \end{bmatrix}.$$

We make some simple observations about  $\pi(A)$ . The diagonal entries of  $\pi(A)$  are all equal to  $a_{11} \cdots a_{nn}$  where  $A$  is  $n \times n$ . The sum of the entries in any row or column of  $\pi(A)$  equals  $\text{per}A$ . In particular,  $\text{per}A$  is an eigenvalue of  $\pi(A)$ .

If  $A$  is  $n \times n$ , then after a permutation of the rows and an identical permutation of the columns,  $\pi(A)$  can be viewed as a principal submatrix of  $\otimes^n A = A \otimes A \otimes \cdots \otimes A$ , taken  $n$  times. If  $A$  is positive semidefinite, then  $\otimes^n A$  is positive semidefinite and hence so is  $\pi(A)$ . Since  $\text{per}A$  is an eigenvalue of  $\pi(A)$ , we immediately have a proof of the fact that if  $A$  is positive semidefinite, then  $\text{per}A \geq 0$ .

It is true that  $\det A$  is also an eigenvalue of  $\pi(A)$ . To see this, define a vector  $\epsilon$  of order  $n!$  as follows. Index the elements of  $\epsilon$  by  $S_n$ . If  $\tau \in S_n$ , then set  $\epsilon(\tau) = 1$  if  $\tau$  is even and  $-1$  if  $\tau$  is odd. Then for any  $\sigma \in S_n$ ,

$$\begin{aligned} \sum_{\tau \in S_n} \epsilon(\tau) \prod_{i=1}^n a_{\sigma(i)\tau(i)} &= \sum_{\tau \in S_n} \epsilon(\tau) \prod_{i=1}^n a_{i\tau\sigma^{-1}(i)} \\ &= \sum_{\rho \in S_n} \epsilon(\rho\sigma) \prod_{i=1}^n a_{i\rho(i)} \\ &= \epsilon(\sigma) \sum_{\rho \in S_n} \epsilon(\rho) \prod_{i=1}^n a_{i\rho(i)} \\ &= \epsilon(\sigma) \det A. \end{aligned}$$

Thus  $\pi(A)\epsilon = (\det A)\epsilon$  and hence  $\det A$  is an eigenvalue of  $\pi(A)$  with  $\epsilon$  as the corresponding eigenvector.

Thus  $\text{per}A$  and  $\det A$  are both eigenvalues of  $\pi(A)$  and this explains the similarity of certain properties of the permanent and the determinant, restricted to the class of positive semidefinite matrices.

A remarkable result due to Schur asserts that if  $A$  is positive semidefinite, then  $\det A$  is in fact the smallest eigenvalue of  $\pi(A)$ . Recall the extremal characterization of the eigenvalues of a hermitian matrix. If  $B$  is hermitian with the least eigenvalue  $\lambda_n$ , then  $\lambda_n$  is the minimum of  $x^* B x$  taken over unit vectors  $x$ . Therefore Schur's result provides a rich source of inequalities for the determinant of a positive definite matrix since we can make a judicious choice of  $x$  and get an inequality. For example, if  $x$  has all coordinates zero except one, then Schur's Theorem reduces to the well-known Hadamard Inequality, that the determinant of a positive semidefinite matrix is bounded above by the product of the main diagonal elements. Another example is the following.

Let  $A$  be an  $n \times n$  positive semidefinite matrix and let  $G$  be a subgroup of  $S_n$ . Then

$$(4) \quad \det A \leq \sum_{\sigma \in G} \prod_{i=1}^n a_{i\sigma(i)}.$$

The expression appearing on the right hand side of (4) is an example of an immanant of  $A$ . Note that in (4), if  $G$  is the subgroup consisting of the identity permutation only, then we get the Hadamard Inequality.

One of the most important outstanding open problems at present is to decide whether an analogue of Schur's result holds for the permanent. More precisely, the problem is formulated as a conjecture due to Soules as follows:

**Conjecture 2:** If  $A$  is positive semidefinite, then  $\text{per} A$  is the largest eigenvalue of  $\pi(A)$ .

The conjecture has been proved for matrices of order at most 3, see [1]. For some further ideas, see [29, 30]. We remark in passing that Conjecture 2 is easily verified if  $A$  is assumed to be entrywise nonnegative as well. For in this case  $\pi(A)$  is a nonnegative matrix and the vector of all ones is an eigenvector of  $\pi(A)$  corresponding to  $\text{per} A$ . It follows by the Perron-Frobenius theorem that  $\text{per} A$  must be the spectral radius, and hence the maximal eigenvalue, of  $\pi(A)$ .

There are several conjectures weaker than Conjecture 2 which have also received considerable attention. We discuss some of these below. The next is the so called "permanent- on-top conjecture" or the "permanental dominance conjecture" which asserts that any immanant of a positive semidefinite matrix is dominated by the permanent, see [17, 22]. More precisely, the conjecture is the following.

**Conjecture 3:** Let  $G$  be a subgroup of  $S_n$  and let  $\chi$  be a complex character on  $G$ . If  $A$  is an  $n \times n$  positive semidefinite matrix, then

$$\text{per} A \geq \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)}.$$

The Schur product of two  $n \times n$  matrices  $A$  and  $B$ , denoted  $A \circ B$ , is simply their entrywise product. If  $A$  and  $B$  are  $n \times n$  positive semidefinite matrices then  $A \circ B$  is also positive semidefinite. Oppenheim's inequality asserts that if  $A$  and  $B$  are  $n \times n$  positive semidefinite matrices then  $\det(A \circ B) \geq (\det A)b_{11} \cdots b_{nn}$ . Note that the inequality reduces to the Hadamard inequality when  $B$  is the identity matrix. The next conjecture, which can be shown to be weaker than Conjecture 2, has been proposed in [1].

**Conjecture 4:** If  $A$  and  $B$  are  $n \times n$  positive semidefinite matrices then  $\text{per}(A \circ B) \leq (\text{per} A)b_{11} \cdots b_{nn}$ .

In this context it may be mentioned that the Hadamard inequality for the permanent (which is a special case of Conjecture 4) has been proved by Marcus[20]. For the relevance of Conjecture 4 in some topics in mathematical physics, see [9]. If  $A$  is an  $n \times n$  matrix, then let  $A(i, j)$  denote the submatrix of  $A$  obtained by deleting its  $i$ -th row and  $j$ -th column. The next conjecture, which is also weaker than Conjecture 2, was proposed in [1].

**Conjecture 5:** Let  $A$  be an  $n \times n$  positive semidefinite matrix. Then  $\text{per} A$  is the largest eigenvalue of the  $n \times n$  matrix with its  $(i, j)$ -entry equal to  $a_{ij} \text{per} A(i, j)$ ,  $i, j = 1, \dots, n$ .

If  $k \leq n$ , then let  $G_{k,n}$  denote the set of all strictly increasing functions from  $\{1, \dots, k\}$  to  $\{1, \dots, n\}$ , ordered lexicographically. If  $A$  is an  $n \times n$  matrix and  $\alpha, \beta \in G_{k,n}$ , then  $A[\alpha, \beta]$  will denote the  $k \times k$  submatrix of  $A$  whose rows and columns are indexed by  $\alpha, \beta$  respectively, while the  $(n-k) \times (n-k)$  submatrix of  $A$  whose rows and columns are indexed by  $\alpha^c, \beta^c$  will be denoted  $A(\alpha, \beta)$ . Let  $\mathcal{C}_k(A)$  be  $\binom{n}{k} \times \binom{n}{k}$  matrix indexed by  $G_{k,n}$  with  $(\mathcal{C}_k(A))_{\alpha, \beta} = \text{per}(A(\alpha, \beta)) \text{per}(A[\alpha, \beta])$ . The following generalization of Conjecture 5 has recently been proposed by Pate[25] who has also obtained some partial results.

**Conjecture 6:** If  $A$  is an  $n \times n$  positive semidefinite matrix, then  $\text{per} A$  is the largest eigenvalue of  $\mathcal{C}_k(A)$ ,  $k = 1, \dots, n-1$ .

We remark that Conjecture 6 reduces to Conjecture 5 when  $k = n - 1$ .

### 6. A $q$ -analogue of the permanent

If  $\sigma \in S_n$ , then an *inversion* of  $\sigma$  is a pair  $(i, j)$  such that  $1 \leq i < j \leq n$  and  $\sigma(i) > \sigma(j)$ . As an example, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 2 & 1 & 5 \end{pmatrix}$$

in  $S_6$  has 9 inversions. If  $\sigma \in S_n$  then let  $\ell(\sigma)$  denote the number of inversions of  $\sigma$ . The identity permutation has zero inversions. The maximum number of inversions in  $S_n$  is  $\frac{n(n-1)}{2}$ , attained at the permutation  $n, n-1, \dots, 2, 1$ . Note that  $\ell(\sigma)$  is even (odd) if  $\sigma$  is an even (odd) permutation. If  $A$  is an  $n \times n$  matrix and  $q$  a real number, then we define the  $q$ -permanent of  $A$ , denoted by  $\text{per}_q(A)$  as

$$\text{per}_q(A) = \sum_{\sigma \in S_n} q^{\ell(\sigma)} \prod_{i=1}^n a_{i\sigma(i)}.$$

Observe that  $\text{per}_{-1}(A) = \det A$ ,  $\text{per}_0(A) = \prod_{i=1}^n a_{ii}$  and  $\text{per}_1(A) = \text{per } A$ . Here we have made the usual convention that  $0^0 = 1$ . The  $q$ -permanent thus provides a parametric generalization of both the determinant and the permanent. The  $q$ -permanent appears to be a function with a very rich structure but at the same time it does not lend itself to manipulations very easily.

If  $A$  is a positive semidefinite matrix and  $-1 \leq q \leq 1$ , then  $\text{per}_q(A) \geq 0$ . This result has been proved by Bożejko and Speicher [8] in connection with a problem in Mathematical Physics dealing with parametric generalizations of Brownian motion. A proof based on conditionally negative definite matrices has been given in [6].

The following monotonicity property of the  $q$ -permanent has been conjectured in [3].

**Conjecture 7:** If  $A$  is positive semidefinite, then  $\text{per}_q(A)$  as a function of  $q$  is monotonically increasing in  $[-1, 1]$ .

Note that Conjecture 7 can be motivated by the known fact that

$$\text{per } A \geq \prod_{i=1}^n a_{ii} \geq \det A$$

for a positive semidefinite matrix  $A$ . Conjecture 7 has been verified for  $n \leq 3$  and there is overwhelming numerical evidence in its favour.

A different generalization of the permanent and the determinant has been considered by Vere-Jones [33], which is as follows. If  $A$  is an  $n \times n$  matrix

and if  $\alpha$  is a real number then let

$$\det_{\alpha} A = \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^n a_{i\sigma(i)},$$

where  $\nu(\sigma)$  is the number of cycles in  $\sigma$ . Note that when  $\alpha = -1$ , then  $\det_{\alpha} A = \det A$ , while if  $\alpha = 1$ , then  $\det_{\alpha} A = \text{per} A$ . This function arises in connection with some random point processes; see [28], where the following conjecture is posed.

**Conjecture 8:** If  $A$  is positive semidefinite  $n \times n$  matrix and if  $0 \leq \alpha \leq 2$ , then  $\det_{\alpha} A \geq 0$ .

For some related stochastic processes involving the permanent, see [12, 15].

### 7. Computation of the permanent

The determinant can be evaluated efficiently using Gaussian elimination. The computation of the permanent is however much more complicated. In the last two decades many contributions in the area of computational complexity have been made towards exact or approximate computation of the permanent. A classical result of Valiant asserts that the problem of computing the permanent is  $\#P$ -complete, which basically means that there is almost no possibility of finding a polynomial time algorithm for computing the permanent. At the same time the possibility of computing the permanent within arbitrarily small relative error in polynomial time is not ruled out.

Given a  $0-1$  matrix  $A$ , form a random matrix  $B$  by assigning  $\pm$  signs independently at random to the elements of  $A$ . Then  $(\det B)^2$  is an unbiased estimator of  $\text{per} A$ . In general the variance of the estimator may be very large. Karmarkar et al [18] replaced the  $\pm 1$  entries of  $B$  by randomly choosing complex roots of unity and later Barvinok[7] used random quaternions with the aim of reducing the variance. This idea was naturally extended by employing Clifford algebras, see [11]. A polynomial-time approximation algorithm for the permanent of a nonnegative matrix is given in [16].

### 8. Symmetric function means and permanents

We conclude by recalling a conjecture posed in [4]. First we introduce some notation. Let  $x = (x_1, \dots, x_n)$  be a positive vector. We denote by  $e_{r,n}(x)$  the  $r$ -th elementary symmetric function in  $x_1, \dots, x_n$ . Thus

$$e_{r,n}(x) = \sum_{i_1 < \dots < i_r} x_{i_1} \dots x_{i_r}.$$



We set  $e_{0,n}(x) = 1$ .

Several inequalities are available in the literature for ratios of elementary symmetric functions, also known as symmetric function means. Let

$$M_{r,n}(x) = \frac{e_{r,n}(x)}{e_{r-1,n}(x)}, r = 1, 2, \dots, n.$$

A well-known result of Marcus and Lopes [19] asserts that for any two positive vectors  $x, y$ ;

$$(5) \quad M_{r,n}(x+y) \geq M_{r,n}(x) + M_{r,n}(y).$$

Let  $c, b_1, b_2, \dots$  be positive vectors in  $R^n$  which will be held fixed. For any positive vector  $x$  in  $R^n$  and for  $1 \leq r \leq n$ , define

$$(6) \quad S_{r,n}(x) = \frac{\text{per}[\underbrace{x, \dots, x}_r, b_1, \dots, b_{n-r}]}{\text{per}[\underbrace{x, \dots, x}_{r-1}, b_1, \dots, b_{n-r}, c]}$$

If  $c = b_i = \mathbf{1}$ , the vector of all ones, for all  $i$ , then

$$S_{r,n}(x) = \frac{r}{n-r+1} M_{r,n}(x)$$

and thus the function in (6) is more general than a symmetric function mean. It is thus natural to conjecture that a generalization of (5) holds; more precisely, the following was posed in [4].

**Conjecture 9:** For any positive vectors  $x, y$ ;

$$(7) \quad S_{r,n}(x+y) \geq S_{r,n}(x) + S_{r,n}(y).$$

The case  $r = 1$  of (7) is trivial. The case  $r = 2$  which is closely related to the Alexandroff inequality, is proved in [4].

For a recent survey concerning permanents we refer to [10] where further references can be found.

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**ABSTRACT OF THE PAPERS PRESENTED AT THE  
73<sup>rd</sup> ANNUAL CONFERENCE OF INDIAN  
MATHEMATICAL SOCIETY, HELD AT THE  
UNIVERSITY OF PUNE, PUNE-411007 DURING  
DECEMBER 27-30, 2007.**

**A: Combinatorics, Graph Theory and Discrete  
Mathematics:**

**A1: On imbalance sequences of oriented graphs,** *Merajuddin and  
Madhukar Sharma (Aligarh: meraj1957@rediffmail.com).*

A necessary and sufficient condition for a sequence of integers to be an irreducible imbalance sequence is obtained. We found bound for imbalance  $b_i$  of a vertex  $v_i$  of oriented graphs. Some properties of imbalance sequences of oriented graphs, arranged in lexicographic order, are investigated. In the last we report a result on an imbalance sequence for a self-converse tournament and conjecture that it is true for oriented graphs.

**A2: On n-Ray  $H_v$ -structures,** *B. N. Waphare and N. Ghadiri (Pune,  
Maharashtra: bnwaph@math.unipune.ernet.in).*

In this paper the class of n-ray  $H_v$ -groups (rings) are introduced and the fundamental relation defined on n-ray  $H_v$  groups are studied. Also the n-ray  $H_v$ -groups and  $H_v$ -rings with n-ray P-hyper operations are investigated.

**A3: Quasiindependence spaces,** *Anil Pedgaonkar (Mumbai).*

The concept of a matroid is an abstraction of the notion of linear independence. The spanning forests of a graph and the independent sets in a vector space are both examples of independent sets. The greedy algorithm of Kruskal for finding a spanning tree of maximum weight is valid in matroid setting and characterises the matroid structure. Edmonds formulated a continuous analogue of matroid known as polymatroid. In this paper, we unify the concepts of a matroid and a polymatroid to define a new space called as Quasiindependence space. Their study may shed further light on linear programming and integer programming. Matroid structure arises from Boolean valued mappings while this structure arises from mappings with values in

any compact subset of  $R$  containing 0 and satisfy certain properties. Further, quasiindependence spaces are further characterised by submodularity of rank function like matroids. However, there exists a set rank function as well as a vector rank function for a quasiindependence space. These notions coincide in the case of a matroid. An important application of the concept is that it paves the way to define the continuous counterpart of the concept of greedoid which is termed as greedy spaces.

## **B: Algebra, Number Theory and Lattice Theory:**

### **B1: Congruences for Jacobi sums of order 25,**

*Devendra D. Shirolkar (Pune: dshirolkar@gmail.com).*

Here we find congruences for Jacobi sums  $J(1,1)$  and  $J(1,2)$  of order 25 for a finite field  $F_p$ . We show that simpler congruences hold for  $J(1,1)$  and  $J(1,2)$  in the case of artiad and hyperartiad primes. The congruences obtained here are the appropriate congruences useful in giving algebraic characterisation of the Jacobi sums of order 25.

### **B2: Blind signature scheme over braid groups,** *Girraj Kumar*

*Verma (Mathura, UP: girraj\_ibs@rediffmail.com).*

A blind signature scheme is a cryptographic protocol for obtaining a signature from a signer such that the signer's view of the protocol cannot be linked to the resulting message signature pair. In this paper, we have proposed two blind signature schemes using braid groups. The security of the given scheme depends upon conjugacy search problem in braid groups.

### **B3: Applications of linear algebra to roots of unity in finite fields,** *Jagmohan Tanti (Pune: jtanti@math.bprim.org).*

Let  $e \geq 2$  be an integer and  $p \equiv 1 \pmod{e}$  be a prime. Expressions of  $e$ th roots of unity in finite fields  $F_p$  have been obtained by researchers for  $e = 3, 4, 5, 6, 8$  etc. in terms of solutions of certain diophantine systems arising from Gauss and Jacobi sums or Jacobsthal sums. Here we get expressions for 7th and 11th root so  $f$  unity in  $F_p$  in terms of the coefficients of the Jacobi sums of order 7 and 11 respectively by using elementary techniques in linear algebra. It is expected that the technique will also be useful to get the  $e$ -th roots of unity in  $F_p$  for a general prime  $p$ .

### **B4: Some result on strongly regular rings,** *M. K. Manoranjan*

*(Sahugarh-Madhepura, Bihar).*

A ring  $R$  is called strongly regular if to each element  $a$  of  $R$ , there exists at least one element  $x$  in  $R$  such that  $a = a^2x$ . It can be seen that every strongly regular ring is regular. A ring  $R$  is said to be two-sided ring if every one-sided ideal of  $R$  is a two-sided ideal of  $R$ . Evidently every division ring and

every commutative ring is two-sided ring. We prove some characterisations of strongly regular rings.

**B5: n-c pure exact sequences**, *Seema S. Gramopadhye (Dharwad, Karnataka: semaalg@yahoo.com).*

In this paper, we introduce the concept of n-cyclic purity (n-c pure), as generalisation of cyclic purity. Also, we study absolutely n-c-pure, n-c-flat and n-c-regular R-modules. We have shown that in a commutative integral domain R, Cohn's purity implies 1-c-purity, 1-absolutely c-pure R-modules are divisible and 1-c-flat R-modules are precisely torsion free R-modules.

**B6: On derivations in rings and algebras**, *Moharram A. Khan (New Delhi).*

In this talk, we first establish a result which asserts that : Take two derivations  $f$  and  $g$  of a semi-rime ring  $R$  such that at least one is non-zero and the relation  $f(x)x + xg(x) = 0$ , for all  $x$  in  $R$ . Then  $R$  has a non-central ideal. This result is a generalization of a well known result of Posner [AMS (1957)] and a partial extension of a theorem of Bresar [J. Algebra (1993)]. Secondly, this proves that both derivations  $f, g : R \rightarrow R$  with above related relation satisfied. Finally, from this purely algebraic result one can obtain a result of continuous derivations on Banach algebras also discussed.

**B7: On fuzzy ideals of modules over gamma near-rings**, *Nagaraju Dasari (AP.: nagaraju\_dasari9@rediffmail.com).*

A studied the concepts fuzzy ideal, normal fuzzy ideal, maximal fuzzy ideals of modules over gamma near-rings and obtained few important results on them.

**B8: Id based blind generalised signcryption**, *P. Kushwaha and S. Lal (Agra: prashantkushwah.ibs@gmail.com).*

Generalized signcryption is a new cryptographic primitive in which a signcryption scheme can work as an encryption scheme as well as a signature scheme. This paper presents a identity based generalised signcryption scheme and an identity base blind generalised signcryption scheme.

**B9: Remarks on IBE scheme of Wang and Cao**, *P. Sharma and S. Lal (Agra: priyam.sharma.ibs@rediffmail.com).*

In this paper we analyse and find an anamoly in the security proof of he identity based encryption (IBE) scheme full M-IBE of Wang and Cao [8], which is based on mBDHP. Here we give another proof of full M-IBE which is based on Bilinear Diffie-Hellman Problem (BDHP). We also obtain a tightness improvement using stronger assumption, namely the Bilinear Inverse Decision Diffie-Hellman Problem (BIDDHP).

**B10: Fuzzy identity based encryption scheme, R. Agarwal and S. Lal** (Agra: *agrawalrubiybs@rediffmail.com*).

This paper introduces a Fuzzy identity based encryption scheme different from that of Sahai and Waters. It is computationally more efficient and its security is comparable to that of SW-RO scheme.

**B-11: Matrix partial orders and reversal law, Saroj Malik** (Delhi: *saroj.malik@gmail.com*).

Let  $A, B$  be any  $m \times n$  matrices such that  $A$  is below  $B$  under any of the three matrix order relations, namely space pre-order and therefore minus, star and also when matrices are square under the sharp order. We give conditions on matrices that ensure the validity of the reverse order law for various  $g$ -inverses.

**B-12: Right semisimple right near-rings, Ravi Srinivasa Rao** (Vijayawada).

Near-rings considered are right near-rings. Recently, the authors have introduced and studied the right Jacobson radicals  $J_v^r, v \in \{1, 2, 3, s\}$ . In this paper semi-simple near-rings corresponding to these right Jacobson radicals are studied. Unlike the left Jacobson radicals, it is shown that the right Jacobson radicals extend a form of the Wedderburn-Artin theorem of rings involving matrix rings to d.g. near-rings, and to finite and general near-rings satisfying certain conditions. It is also shown that a right-primitive d.g. near-ring  $R$  satisfying DCC on right ideals is isomorphic to a matrix near-ring  $M_n(B)$ , where  $n = \dim R$  and the near-ring  $B$  is a right B-group of type- $v, v \in \{1, 2\}$ . Some generalizations of the Wedderburn-Artin theorem of rings involving matrix rings are also presented.

**B13: Group algebras satisfying a certain lie identity, Meena Sahai** (Lucknow).

Let  $K$  be a field of characteristic  $p = 2$  and let  $G$  be any group. A characterization of group algebras  $KG$  satisfying the Lie identity is  $[[x, y], u, v], [z, t]] = 0$  for all  $x, y, u, v, z, t \in KG$  is obtained.

**B14:  $\sigma$  elements in multiplicative lattices, Nitin S. Chavan** (Pune).

The concept of  $\sigma$ -elements in multiplicative lattices is introduced by C. Jayaram in 1998. Here in this paper we generalize this concept and then we generalize some of the results proved by C. Jayaram. We prove the following important results.

- (1) Let  $L$  be an  $\sigma$ -normal lattice and let  $u$  be a compact radical element. (i) An element  $p$  is a minimal prime over  $u$  if and only if  $p$  is a maximal-element.
- (ii) Every prime-element is a maximal-element.



(2) Let  $u$  be a radical element. Then  $L$  is a  $u$ -Baer lattice if and only if every prime  $u$ -Baer element is a prime-element.

**B15: Review of the identity of subset and of the theorem the empty set is a subset of every set**, Dharmendra Kumar Yadav (Delhi: *dkiyadav1978yahoo.co.in*).

In the present paper, the term subset in set theory has been redefined by contradicting the classical definition and the theorem ‘the empty set is a subset of every set’. This new definition removes the drawback of the classical definition of the subset and the theorem.

**B16: On generalised derivations in rings and modules**, Mohammad Ashraf (Aligarh: *mashraf80@hotmail.com*).

Let  $R$  be an associative ring and  $U$  a Lie ideal of  $R$ . Let  $\theta, \phi$  be endomorphisms of  $R$  and  $M$  be a 2-torsion free  $R$ -bimodule. An additive mapping  $d : R \rightarrow M$  is called a  $(\theta, \phi)$ -derivation (resp. Jordan  $(\theta, \phi)$ -derivation) on  $R$  if  $d(xy) = d(x)\theta(y) + \phi(x)d(y)$  (resp.  $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$ ), holds for all  $x, y \in R$ . Following [International J. Math. Game Theory and Algebra, 12 (2002), 295-300], an additive mapping  $F : R \rightarrow M$  is called a generalized  $(\theta, \phi)$ -derivation (resp. generalized Jordan  $(\theta, \phi)$ -derivation) of  $U$  if there exists a  $(\theta, \phi)$ -derivation  $d : R \rightarrow M$  such that  $F(uv) = F(u)\theta(v) + \phi(u)d(v)$  (resp.  $F(u^2) = F(u)\theta(u) + \phi(u)d(u)$ ), holds for all  $u, v \in U$ . It is obvious to see that every generalized  $(\theta, \phi)$ -derivation  $F : R \rightarrow M$  is a generalized Jordan  $(\theta, \phi)$ -derivation. However, the converse need not be true in general. There exist several results in the existing literature which deal with the converse part of this problem. It was Herstein [Proc. Amer. Math. Soc. 8(1957), 1104-10] who first investigated this problem and proved that every Jordan derivation on a 2-torsion free prime ring  $R$  is a derivation. In the present paper, we shall discuss such types of problems in the setting of generalized derivation in rings with module values.

**B17: An identity related to generalised  $(\alpha, \beta)$ -derivations**, Nademur Rehman (Aligarh: *rehman100@gmail.com*).

The concept of derivations as well as generalized inner derivations have been generalized as a function  $F : R \rightarrow R$  satisfying  $F(xy) = F(x)y + yd(x)$  for all  $x, y \in R$ , where  $d$  is derivation on  $R$ , such a function  $F$  is said to be a generalized derivation. Let  $R$  be a prime ring and  $\alpha, \beta$  be endomorphisms of  $R$ . An additive mapping  $d : R \rightarrow R$  is said to be a  $(\alpha, \beta)$ -derivation if  $d(ab) = d(a)\alpha(b) + \beta(a)d(b)$ , holds for all  $a, b \in R$ . An additive mapping  $F : R \rightarrow R$  is called a generalized  $(\alpha, \beta)$ -derivation (resp. Jordan generalized  $(\alpha, \beta)$ -derivation) on  $R$  if there exists a  $(\alpha, \beta)$ -derivation (resp. Jordan  $(\alpha, \beta)$ -derivation  $d : R \rightarrow R$  such that  $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$  (resp.

$F(a^2) = F(a)\alpha(a) + \beta(a)d(a)$ , holds for all  $x, y \in R$ . An additive mapping  $F : R \rightarrow R$  is called generalized triple  $(\alpha, \beta)$ -derivation on  $R$  if there exists a  $(\alpha, \beta)$ -derivation  $d : R \rightarrow R$  such that  $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$  for all  $x, y \in R$ . In the present paper, we prove that if  $\alpha, \beta$  are endomorphisms, then every generalized triple  $(\alpha, \beta)$ -derivation is Jordan generalized  $(\alpha, \beta)$ -derivation.

**B18: Intuitionistic  $P^n$ -fuzzy subalgebras in  $BCI(BCK)$ -algebras,** *B. N. Waphare and Alireza Gilani (Pune: bnwaph@math.unipune.ernet.in).*

After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalization of the notion of fuzzy sets. The idea of intuitionistic fuzzy set as first published by Atanassov, as a generalization of the notion of fuzzy sets. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets give both degree of membership and of non-membership. Both degrees belong to the interval  $[0,1]$ , and their sum should not exceed 1. Intuitionistic fuzzy sets have also been defined by G. Takeuti and S. Titanti. They considered intuitionistic fuzzy logic in the narrow sense and derived a set theory from logic which they called intuitionistic fuzzy set theory. Y.L.B. Jun introduced the notion of closed ideal in BCI-algebra and discussed their properties. In Section 2, we apply the concept of an intuitionistic fuzzy set to P-fuzzy sets and intuitionistic P-fuzzy ideal of a BCI-algebras and investigate some related properties. In Section 3, we considered the intuitionistic P-fuzzification of the concept of fuzzy closed  $a$ -ideals, we consider some of their properties. In Section 4, we conducted on the generalization of the notion of the concept of sub algebras in BCI-algebras.

**B19: On a property of regular rings,** *M. M. Singh (Durg, CG.).*

Von Neumann (1936) characterised that if  $R$  is a ring with unity element then  $R$  is (Von Neumann) regular iff each right (left) principal ideal in  $R$  is generated by an idempotent. It may be recalled that Waddel (1952) pointed out without proof that this equivalence can be seen to hold for rings without unity. But it can be seen that this equivalence will not hold for the general case of rings without unity. In what follows the author traces out a proof of Waddel's suggestion.

**B20: Intuitionistic fuzzy near-rings and its properties,** *Okram Ratnabala Devi (Imphal: ord2007mu@yahoo.com).*

In this paper, we introduced the notion of intuitionistic fuzzy near-rings and ideals on N-groups and obtain some related properties and homomorphism theorems. We also introduce the intuitionistic fuzzy cosets of intuitionistic fuzzy ideals.

### **C: Real and Complex Analysis (Including Special Functions, Summability and Transforms):**

**C1: Lambert  $W$ -function in statistical distributions, *P. N. Rathie (Brasilia, DF, Brazil, 70910-900).***

Lambert  $W$ -function is relatively a new special function with applications in various areas of mathematics and engineering. In this paper, some new applications of  $W$ -function are given for logarithmic series and generalized gamma (one and two sided) distributions and for optimal maintenance time for repairable systems in Reliability Analysis. Computation of  $W$ -function, with good precision, is already available in Mathematics and Maple softwares for numerical applications.

**C2: Generalisations of Calderon's reproduction formula for Hankel convolution, *B. B. Waphare and S. P. Malgonde (Pune: bbwaphare@mitpune.com).***

In recent years a lot of research is going on in transform theory and its applications. Many are working on Hankel transformation theory. Calderon's reproducing formula is studied by many author's for Hankel and Hankel type convolutions from time to time. The purpose of the present paper is to generalise the Calderon type reproducing formula for Hankel convolution by using the theory of Hankel transform.

**C3: On some classes of difference double sequence spaces, *B. Sarma and B. Tripathy (Guwahati: sarmabipu101@yahoo.co.in).***

In this article we introduce the difference double sequences defined over a semi-normed space  $(X, q)$ , semi-normed by  $q$ . We examine some topological and algebraic properties of these spaces like symmetricity, solidness, monotonicity, convergence free, nowhere denseness etc. Some inclusion results are also proved.

**C4: Difference sequence on fuzzy real numbers defined by Orlicz function, *S. Borgohain and B. Tripathy (Guwahati, Assam).***

In this article we introduce the notion of difference sequences on fuzzy real numbers defined by Orlicz functions. It is shown that these spaces are complete matric spaces and their different properties are studied. Some inclusion results are also proved.

**C5: On fuzzy real-valued bounded variation double sequences space,** *A. J. Dutta and B. C. Tripathi (Gowahati, Assam).*

In this article we introduce the notion of fuzzy real-valued bounded variation double sequence space 2bvF. It is shown that the space 2bvF is a complete metric space and have studied their different properties. It is verified that the space 2bvF is neither solid nor convergence free nor symmetric. We also prove some inclusions results.

**C6: Fractional integration of hypergeometric functions on three variables,** *Shanu Sharma (Jodhpur, Rajasthan).*

In this paper, I introduce the fractional integral representations and the definitions of the three variable hypergeometric  $G_C$  and  $G_D$  with three additional parameters  $\tau_1, \tau_2, \tau_4$ , which extends the work done earlier by several authors.

**C7: On undefined integral involving product of  $\bar{H}$  functions under more general conditions of existances,** *R. Jain and R. Nahar (Jaipur, Rajasthan: rashmiramesh@gmail.com).*

In this paper, two integrals involving product of  $\bar{H}$  function, single and multivariable general polynomials and a general sequence of functions have been established under more general conditions of the existence of the defining Mellin-Barnes type contour integral of the  $\bar{H}$ -function than those given earlier. Both the integrals are believed to be new and a very large number of new and known integrals involving simpler functions and polynomials follow as their special cases. We work out three integrals that involve Riemann zeta function, reduced Green function, Mittag Leffler function, Bessel function, generalized hypergeometric function, generalised Bessel function, Shivley's function, multivariable Jacobi polynomial and multivariable Bessel polynomial which follow as special cases of our main integrals. These integrals may find applications in science and engineering. Several basic integrals obtained earlier by Gupta and Soni, Jain, Kilbas and Saigo also follow as special cases of our findings.

**C8: On a class of multivalent functions,** *A. L. Pathak and K. K. Dixit (Kanpur, UP.: alpathak@rediffmail.com).*

In this paper, we introduce the class  $R^b(A, B, p)$  of functions regular in the disc  $D = \{z \mid |z| < 1\}$ . We determine sharp coefficient estimates, sufficient condition in terms of coefficients, distortion theorem and maximization theorem for this class.

**C9: On general class of polynomials and its integral transforms,** *I. A. Salehbhai, J. C. Prajapati and A. K. Shukla (Surat, Gujarat).*

Rodrigues formula have drawn the attention of several researchers and variety of polynomials studied time by time. Recently a general class of polynomials  $A_{qn}^{\alpha, \beta, \gamma, \delta}(x, a, k, s)$  was introduced by Shukla and Prajapati. The integral transforms have many physical and mathematical applications and their uses are still predominant in advanced study and research. Shukla and Prajapati also obtained the explicit representations and integral transforms viz. Beta transform, Laplace transform, Laguerre transform, Generalized Stieltjes transform and Whittaker transform of a general class of polynomials  $A_{qn}^{\alpha, \beta, \gamma, \delta}(x, a, k, s)$ . The aim of this paper is to obtain several integral transforms viz. Fourier transform, Hankel transform, Hermite transform, Jacobi transform, Legendre transform and Gegenbaur transform of  $A_{qn}^{\alpha, \beta, \gamma, \delta}(x, a, k, s)$ .

**C10: Approximation of single (functions) belonging to the weighted  $W(L_p, \xi(t), (p \geq 1)$  class by almost matrix summability method of its fourier series, M. L. Mittal and V. N. Mishra (Roorkee, UP.: vishnu\_narayanmishra@yahoo.co.in).**

Qureshi and Neha determined the degree of approximation of certain functions by almost  $Nö$ rlund  $(N_p)$  and almost generalized  $Nö$ rlund  $(N_p, q)$ -means. Lal [2004] has determined the degree of approximation of a function belonging to weighted  $W(L_p, \xi(t), (p \geq 1)$ - class by almost matrix summability means of its Fourier series using monotonicity on the matrix elements  $\{a_{n,k}\}$ . In this paper, we prove the results of Lal and Qureshi by dropping the monotonicity of the matrix elements  $\{a_{n,k}\}$  for the functions (signals)  $f$  weighted  $W(L_p, \xi(t), (p \geq 1)$ - class.

**C11: The distributional Laplace-Stieltjes transform, M. K. Nikam and M. S. Chaudhary (Pune: nikam.mk@yahoo.com).**

A Laplace-Stieltjes transform given by Giona and Paterno is  $m(t, z) = \int_0^\infty \frac{e^{-sx}}{1+tx} f(x) dx$ . In this paper, we develop a space of generalized functions on  $(0, \infty)$  and then we have discussed the Laplace-Stieltjes transform for this space. Also, some of operational properties and an inversion theorem are obtained.

**C12: Applications of generalised Ruschweyh derivatives to subordination of multivalent functions, K. Arab, J. M. Jahangiri and S. R. Kulkarni (Pune: kulkarni\_ferg@yahoo.com).**

Generalization of Ruschweyh derivatives are applied to multivalent functions defined by subordinations, which in special cases provide new approaches to some previously known results.

**C13: On entire function of zero order, K. Singh and A. P. Singh (Jammu).**

Let  $f$  be a transcendental entire function of zero lower order. Assume that for any  $m > 1$  and  $\epsilon > 0$ ,  $\log M(r^m, f) > m^{2+\epsilon} \log M(r, f)$  holds for arbitrary large  $r$ . Then we show that every component of  $F(f)$  is bounded. As a consequence, we prove that if  $f$  be any transcendental entire function with zero order, then every periodic component is bounded.

#### **D: Functional Analysis:**

**D1: Some common fixed point theorem for weakly compatible mappings satisfying a general contractive condition of integral type**, *B. Choudhary (Botswana: choudha2000@yahoo.com)*.

The first important result on the fixed point of contractive type mappings is the famous Banach fixed point theorem. In 2002 Branciari established a fixed point theorem for mappings on complete metric spaces satisfying contractive inequality of integral type. In this paper we prove some fixed point theorems for weakly compatible mappings on metric spaces satisfying a general contractive inequality of integral type.

**D2: Ridgelet transform for Boehmians**, *R. Roopkumar (Karaikudi, Tamilnadu: roopkumarr@rediffmail.com)*.

The ridgelet transform is defined on the space of square integrable functions as a combination of Radon transform and wavelet transform. This transform found applications in reconstruction of the image from the discrete data, image fusion and noise reduction in image processing. On the other hand the Boehmian spaces are introduced which consists of convolution quotients of sequences of functions, to generalise the Schwartz distributions and various other generalised function spaces. In this paper, the extended ridgelet transform of an  $L^2$ -Boehmian is defined as a new Boehmian. It is also proved that the extended ridgelet transform on the space of  $L^2$ -Boehmians, is linear, one-to-one, onto another suitable Boehmian space and continuous with respect to  $\delta$ -convergence as well  $\Delta$ -convergence.

**D3: Existence of fixed point for non-expansive mappings in Banach spaces**, *Dhirendra Kumar Singh and D. P. Shukla (Rewa MP.)*.

The aim of this paper is to find out fixed points for non-expansive mappings in Banach spaces.

**D4: Fixed point theorems for non-expansive mappings in modular spaces**, *D. P. Shukla (Rewa, MP.)*.

The aim of this paper is to give a fixed point theorem in former setting and with  $X_\lambda$  modular space and  $T : C \rightarrow C$  a  $\lambda$ -non-expansive mapping. Then  $T$  has a fixed point.

**D5: Common fixed point of two commuting self mappings in Banach spaces**, *Sunil Kumar and D. P. Shukla (Rewa, MP.)*.

The aim of this paper is to find the common fixed points of two commuting self mappings in Banach spaces using non-expansive mapping and this paper is to give a fixed point theorem in the former setting of the work of Dotson and Mann.

**D6: Coincidence and fixed points of non-self hybride contractions using generalised compatibility of type(N)**, *P. Shrivastava and N. P. S. Bawa (Bagalghat, MP.)*.

The aim of this paper is to introduce the concept of generalised compatibility of type (N), which is a generalisation of compatibility of type (N) introduced by us, and to prove that the concept of generalised compatibility of type (N) is more general than the concept of (IT)-commutativity recently introduced by Singh and Mishra. We thus, generalise the results of Singh and Mishra, who have indicated many errors in results and presented correct versions of their results under weaker conditions. We also claim that the condition of generalised compatibility of type (N) is the minimal condition for the existence of common fixed point of hybrid pair of nonseld mappings.

**D7: A study of generalised Aluthge transformation of composition operations**, *S. K. Latha and S. Panayappan (Coimbatore, Tamilnadu)*.

Let  $(X, \Sigma, \lambda)$  be a sigma-finite measure space and let  $T$  be a measurable transformation from  $X$  into itself. The equation  $Cf = f \circ T$  for  $f \in L^2(\lambda)$  defines a composition transformation on  $L^2(\lambda)$ . When the measure  $\lambda \circ T^{-1}$  is absolutely continuous with respect to  $\lambda$  and the Radon-Nikodym derivative is essentially bounded,  $T$  induces a composition operator  $C$  on  $L^2(\lambda)$ . In this paper various properties of generalised Aluthge Transformation of the composition operator  $C$  are studied.

**D8: Fixed point for two pairs of set valued mapping on two matrix spaces**, *S. Verma, B. Singh and A. Jain(Ujjain, MP.)*.

The existing literature of fixed point theory contains numerous results on single as well as set valued mappings. Fisher [1981] proved related common fixed point theorems for set valued mappings on metric spaces. Afterwards, Rhoades and Watson [1990] proved some results for multivalued mappings on metric spaces. Related fixed point theorems were later extended to two pairs of mappings on metric spaces by Fisher and Murthy [1997]. Fisher et al., [2000] and Chourasia et. al., [2003] made a great contribution to the field of set valued mappings. Recently, Namdeo and Fisher [2004] proved a related fixed point theorem for set valued mappings on two complete

metric spaces. In the present paper, we prove a related common fixed point theorem for two pairs of set valued mappings on two metric spaces. Our results generalises the results of Namdeo and Fisher [2004].

**D9: Semi-compatibility in non-Archimedean Menger MP- space,** *A. Jain, B. Singh and P. Agarwal (Ujjain, MP).*

The notion of non-Archimedean Menger space has been established by Istratescu and Crivat [1974]. The existence of fixed points of mappings on non-Archimedean. Menger space has been given by Istratescu [1978]. This has been the extension of the corresponding results of Sehgal and Bharucha-Reid [1972] on a Menger space. The object of this paper is to establish fixed point theorem for six self maps and an example using the concept of semi-compatible self maps in a non-Archimedean Menger MP-space. Our result generalises the result of Cho et al., [1997].

**D10: On the convergence of Noor iterative process for Zamfirescu mapping in arbitrary banach space,** *S. Dasputre and S. D. Diwan (Bhilai: samir\_nmrg@yahoo.com).*

In this paper, convergence theorem of Rafiq, regarding to approximation of fixed point of Zamfirescu mapping in arbitrary Banach space using Mann iteration process with errors in the sense of Liu is extended to Noor iteration process with errors. Our results generalise and improve the corresponding results of Berinde, Rhoades and Fafiq in arbitray Banach space.

**D11: A strong convergence of perturbed iteration using viscosity approximation,** *N. Gurudwan and B. K. Sharma (Raipur, CG.).*

In this paper, we use the perturbed iterative scheme given by Quin and  $Su : y_n = \alpha_n x_n + (1 - \alpha_n) J_{t_n} x_n x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n$  for  $n \geq 0$  for the proximal point algorithm to obtain strong convergence theorem by the viscosity approximation method in a reflexive Banach space for the accretive operator; which is a generalisation of the results of Takahashi and Qin-Su.

**D12: Composition operations of case Q,** *S. Panayappan, D. Senthilkumar and K. Thirugnanasambandam.*

In this paper, Class Q composition operations 1 on  $L_2$  space are characterised and their various properties are studied.

**D13: A Voronovskja-type theorem for positive linear operations,** *A. K. Sao (Bilaspur: awanish16@yahoo.co.in).*

In this paper we modify the positive linear operators: introduced by Firlej and Rempulska (B. Firlej and L Rempulska Fasciculi Mathematici 27 (1997), 65-79) and prove some approximation results.

**D14: Differential operator on the Orlicz spaces of entire functions,** *M. Gupta and S. Pradha (Kanpur).*



The hypercyclicity of the translation operator on the space  $E$  of entire functions equipped with the compact open topology was studied by G.D. Birkhoff around the year 1929; where as Mc Lane considered this property for the differential operator in 1952. Motivated by the work of K.C. Chan and J.H. Shapiro who proved the hypercyclicity of the translation operator on certain Hilbert subspaces of  $E$ , we consider here this property for the translation operator obtained as sum of the series of powers of differential operators, on the Banach space  $EM()$  of entire functions defined with the help of an Orlicz function  $M$  and a particular type of entire function. Besides we also study the boundedness, compactness and nuclearity of the differential operator on the space  $EM()$ .

**D15: A new finding of locally convex nuclear spaces**, *Kirti Prakash (Patna, Bihar: drnikhalkumar@rediffmail.com)*.

In this paper a new characterisation of locally convex nuclear spaces has been established as a theorem.

**D16: Common fixed point and weak\*\* commuting mapping**, *R. N. Patel and D. Patel (Bilaspur, Chattisgarh)*.

In this paper we prove some fixed point theorems on weak\*\* commuting mapping of complete metric spaces.

### **E: Differential equations, Integral equations, Integral equations and functional equations:**

**E1: On a class of univalent functions with positive coefficients**, *P. M. Patil and S. R. Kulkarni (Kolhapur)*.

In this paper we have introduced a class  $S_p^*(\alpha, \beta, \mu)$  and obtained coefficients estimates and distortion theorems for the same.

**E2: On time scale analogues of certain applicable inequalities**, *D. B. Pachpatte (Aurangabad, Maharashtra: pachpatte@gmail.com)*.

In the present paper we establish time scale analogues of certain fundamental inequalities used in the theory of differential and integral equations. The obtained inequalities can be used to study the qualitative properties of solutions of certain dynamic equations on time scale. Some immediate applications are also given.

**E3: System of diophantine simultaneous of differential equations  $\sum_{i=1}^3 x_i^k = \sum_{i=1}^3 y_i^k$  for  $i = 1, 2$  and 5 has no non-trivial solutions in integers**, *A. Sinha and A. Mishra (Ara, Bihar)*.

In this paper it is shown that the system of diophantine of differential equations  $\sum_{i=1}^3 x_i^k = \sum_{i=1}^3 y_i^k$  for  $i = 1, 2$  and 5 has no non-trivial solutions in integers. Infact it has been proved that this system of equations has no

non-trivial real solutions although non-trivial solutions do exist in the field of complex numbers.

**E4: Application of lie groups in two dimensional heat equation,**  
*V. G. Gupta (Jaipur: guptavguor@rediffmail.com).*

In Lie groups, the symmetry group of differential equation is the largest local group of transformations acting on the independent and dependent variables of the system with the property that it transform the solution of the equation to other solution. In the present paper we obtain the most general solution for the two dimensional heat conduction equation  $u_t = k_0(u_x x + u_y y)$  for the conduction of heat in a finite rod having the thermal diffusivity  $k_0$ , without source, with suitable assumptions by using the general prolongation formula by Olver in explicit form.

**E5: Application of Sumudu transforms to partial differential equations,** *V. D. Raghatare and D. K. Gadge (Shegaon, MS.).*

J. M. Tchuente and N. S. Mbare have established some applications of double Sumudu transform in population dynamics as well as partial differential equations. F. M. Belgacem and A.A. Karaballi, Shyam L. Kalla established analytic investigation of the Sumudu transform and its applications to integral production equations. S. Weerakoon and Nugegoda have established the complex inversion formula for Sumudu transform. In this paper we have obtained the solutions of some partial differential equations by using Sumudu transforms.

**E6: Application of Sumudu transforms to differential equations of electrical circuits,** *V. D. Raghatare and D. K. Gadge (Shegaon, MS: raghatate@rediffmail.com).*

J. M. Tchuente and N. S. Mbare have established some applications of double Sumudu transform in population dynamics as well as partial differential equations. F. M. Belgacem and A. A. Karaballi, Shyam L. Kalla established analytic investigation of the Sumudu transform and its applications to integral production equations. S. Weerakoon and Nugegoda have established the complex inversion formula for Sumudu transform. In this paper we have obtained the solutions of some partial differential equations by using Sumudu transforms.

**E7: On certain class of univalent functions defined by convolution operator made by hypergeometric function,** *W. G. Atshan and S. R. Kulkarni (Pune: waggasg@yahoo.com).*

In this paper, we introduce the class  $A(a, b, c, \sigma, A, B, \beta)$  defined by convolution operator. We study coefficient estimates, linear combinations, extreme points, integral representation and arithmetic mean. We also investigate some interesting properties of operator introduced here.

**E8: Quasi convolution properties of multivalent functions used by generalised Ruschewyh derivative, A. B. Rahman, S. Juma and S. R. Kulkarni (Pune: absa662004@yahoo.com).**

The aim of this paper is to attempt the study of quasi convolution properties by applying the generalised Ruschewyh derivative to subclass of multivalent functions, we have obtained coefficient and distortion bounds and other interesting results.

**E9: Food chain, M. H. Rahmani Doust and R. Rangarajan (Mysore: mh.rahmanidoust@gmail.com).**

In the present paper a system of 3 non-linear ODEs will be studied for modelling the interaction of three-species food chain where intraspecies competition exist. The first population is the prey for second; the second population is the prey for the third which is at the top of food pyramid. To do this, the techniques of linearization and first integral are employed. We discuss the implication of this work for population dynamics of three species briefly.

## **F: Geometry:**

**F1: Some theorems on almost Kaehlerian projective recurrent and symmetric spaces, U. S. Negi and A. K. Singh (Uttarkhand).**

Mishra (1968) has defined and studied the recurrent hermite spaces. Further, Prasad (1973) has defined and studied certain properties of recurrent and Ricci-recurrent almost hermitian spaces and almost Tachibana spaces. In this paper we have investigated some theorems of almost Kaehlerian projective recurrent and symmetric spaces. The condition that an almost techibana space be projective recurrent space of first order as well as second order and first kind have been investigated. Further, the conditions for an almost techibaba space to be projective symmetric space of first order and first kind as well as of the second order and first kind have been obtained.

**F2: On Sasakian conharmonic recurrent and Sasakian conharmonic symmetric spaces, A. K. Singh and U. C. Gupta (Uttarkhand).**

In this paper, we have investigated several theorems in Sasakian conharmonic recurrent and Sasakian conharmonic symmetric spaces. The necessary and sufficient condition for an  $S_n$ -space to be Sasakian conharmonic

recurrent space have been obtained. Further, the necessary and sufficient condition for an  $S_n$ -space to be Sasakian Ricci-recurrent have also been investigated. In the end, we have obtained a necessary and sufficient condition for Sasakian conharmonic symmetric space to be  $R - S_n$ -space with a non-zero recurrence vector.

**F3: On Einstein-Sasakian conharmonic recurrent spaces,**  
*U. G. Gupta and A. K. Singh(Uttarkhand).*

In this paper, we have defined and studied Einstein-Sasakian conharmonic recurrent spaces and Einstein-Sasakian spaces with recurrent Bochner curvature tensor and several theorems have been investigated. The necessary and sufficient condition an Einstein-Sasakian conharmonic recurrent space to be Sasakian recurrent have been investigated.

**F4: On decomposability of curvature tensor in recurrent conformal finsler manifold,** *C. K. Mishra and Gautam Lodhi*

M. S. Mnebelman has developed conformed geometry of generalised metric spaces. The projective tensor and curvature tensors in conformal Finsler space were discussed by R. B. Misra. M. Gamma has decomposed recurrent curvature tensor in an areal space of submetric class. The Decomposition of recurrent curvature tensor in Finsler manifold was studied by B. B. Sinha and S. P. Singh. P. N. Pandey has also studied on decomposibility of curvature tensor in a Finsler manifold. The purpose of the present paper is to be decompose the recurrent conformal curvature tensor and study the properties of conformal decomposition tensors.

**F5: On hyper darbox-lines in techibana hyperspace,** *Sandeep Kumar and A. K. Singh (Uttarkhand).*

A Darbox-line (or D-line) on the surface of a 3-dimensional Euclidean space is a curve for which the osculating sphere at each point is the tangent to the surface. The present paper is devoted to the study of hyper Darbox-lines in a Tachibana hypersurface. Some properties of hyper D-lines have also been investigated. The necessary and sufficient conditions for curvature congruences to be hyper D-lines have also been driven therein.

**F6: The study of pseudo-analytic vectors on pseudo- harmitian hyperspaces,** *K. C. Petwal and Sandeep Kumar (Uttarkhand).*

Bochner has established some crucial results concerning the study of vector fields and Ricci curvature and several important theorems have been proved. In addition, Yano has discussed a vital theory on group of transformations in generalised spaces and various technical transformations have been defined and studied. Recently, Singh and Rana have published resir useful results pertaining pseudo-analytic vectors on pseudo-Kaehlerian

manifolds and have investigated several theorems. The present paper is intended to study pseudo-analytic vectors on pseudo-Hermitian hypersurfaces wherein we shall study the pseudo-analyticity of any function along with various definitions and theorems.

### **G: Topology:**

**G1: On fuzzy strongly precontinuous maps**, *J. K. Maitra, M. Shrivastava and M. Shukla (Jabalpur, MP.)*.

In this paper, we have introduced and studied the concept of fuzzy strongly precontinuous maps on fuzzy topological spaces. We have established equivalent conditions for the map to be fuzzy strongly precontinuous map. Also we have studied some properties of fuzzy strongly precontinuous maps.

**G2: On weakly  $\alpha - I$ -open sets and a new decomposition of continuity via ideals**, *Rajesh and M. K. Gupta (Meerut, UP.)*.

Recently, several authors have studied ideal topological spaces. Hatir and Noiri introduced and investigated  $\alpha - I$ -open, semi-I-open and  $\beta - I$ -open sets and weakly pre-I-open sets in topological spaces. Dontchev introduced pre-I-open sets and obtained a decomposition of I-continuity via idealisation. In the present paper, we introduced and study the notion of weakly  $\alpha - I$ -open sets and weakly  $\alpha - I$ -continuous functions to obtain new decompositions of continuity. We also investigate some fundamental properties of these functions.

**G3: Strongly  $\alpha g^*$  closed sets in bitopological spaces**, *M. Sheik John and S. Maragathavalli (Tamilnadu)*.

In this paper, we introduce strongly  $\alpha g^*$  closed sets in bitopological spaces. Properties of these sets are investigated and we introduce two new bitopological spaces  $(i, j) - s^*T_c$  and  $(i, j) - s^*T_{g^*}$  as applications. Further, we obtain a characterization of the topological space  $(i, j) - s^*T_c$ .

**G4: Pre-induced fuzzy supra topological spaces**, *Anjan Mukherjee (Agartala, Tripura)*.

Let  $(X, T_1, T_2)$  be a bitopological space and  $\xi(T_i)(I = 1, 2)$  be the set of all pre-lower semi-continuous functions, defined from  $X$  into the closed unit interval  $I = [0, 1]$ . In this paper the concept of pre-induced fuzzy supra topology due to A. Mukherjee has been generalized to the bifuzzy supra topological setting and some of its basic properties are discussed. It is seen that the study of pre-induced bifuzzy supra topological space turns out to be the study of a special kind induced bifuzzy supra topological space. Finally some applications and the connection between some separation properties

of the bitopological space  $(X, T_1, T_2)$  and that of its pre-induced bifuzzy supra topological space  $(X, \xi(T_1), \xi(T_2))$  are shown.

**G5: Some more results on intuitionistic fuzzy soft set relations,** *A. Mukherjee and S. B. Chakraborty (Agartala, Tripura).*

The aim of this paper is to define some new operations on intuitionistic fuzzy soft relations and establish some properties on them. We also define some new concepts in intuitionistic fuzzy soft relational conditional topology.

**G6: On minimal open sets and maximal open sets in bitopological spaces,** *B. M. Ittanagi and S. S. Benchalli(Karnataka).*

In this paper a new class of sets called  $(\tau_i, \tau_j)$ -minimal open sets and  $(\tau_i, \tau_j)$ -maximal open sets have been introduced in bitopological spaces. A subset  $M$  of a bitopological space  $(X, \tau_i, \tau_j)$  is said to be a  $(\tau_i, \tau_j)$ -minimal open (resp.  $(\tau_i, \tau_j)$ -maximal open) set of any  $\tau_j$ -open set which is contained in (resp. which contains)  $M$  is either (resp.  $X$ ) or  $M$  itself. Here,  $i, j \in \{1, 2\}$  are fixed integers. Some properties of this new concept have been studied.

**G7: Fixed point result with single valued and multivalued maps,** *Akshay Sharma (Bhopal, Madhya Pradesh).*

In this paper we prove some results with JSR mapping which contains the property (EA) for hybrid pair of single and multivalued maps. Also, we will prove our result with fuzzy maps which contains the property (EA). Also furnish examples to support the result.

**G8: On fuzzy  $\theta$ -preirresolute functions,** *Rupak Sarkar and Anjan Mukherjee (Agartala, Tripura).*

In this paper, we introduce a new class of functions called fuzzy  $\theta$ -preirresolute function, which contained in the class of fuzzy quasi-preirresolute function and contains the class of fuzzy preirresolute functions. Besides giving characterizations of these functions, several interesting properties of these functions are also given. More examples and counter examples are given to illustrate the concepts introduced in this paper.

**G9: On generalised minimal regular spaces and generalised minimal normal spaces,** *S. N. Banasode and S. S. Benchalli(Belgaum).*

In this paper the notion of  $g$ -minimal regular spaces and  $g$ -minimal normal spaces are introduced and studied in topological spaces. Some basic properties of generalised minimal regular and generalised minimal normal spaces are also obtained.

**G10: PZ-Prenormal spaces,** *Debadatta Roy Chaudhuri (Mumbai).*

A. S. Mashhour et al in 1983, have defined and studied the concepts of preopen sets and precontinuity in topology. Later in 1990, Malghan

et al have introduced the concepts of almost p-regular, p-completely regular and almost p-completely regular spaces using preopen sets and pre-continuity. Recently, Mavalagi had studied some more characterisations of p-completely regular spaces and almost p-completely regular spaces by introducing the concept of prezero sets in topology. The authors have introduced a new separation concept using prezero sets, called pz-normal space which is a stronger form than the concepts of lightly normal spaces and lightly prenormal spaces. We have also studied some basic characterisations of these spaces.

## **H: Measure Theory, Probability Theory and Stochastic Processes and Information Theory:**

**H1: Existence of Caratheodory type selectors, S. K. Pandey (Satna, Madhya Pradesh).**

In this paper we have tried to investigate the existence of Caratheodory type selectors for multi valued measurable functions defined on two variables, such that it is measurable in one and continuous in other. The continuity conditions is based on Michael [11] results on continuous selections related to Polish spaces.

**H2: Composition operators of class Q, S. Panayappan, D. Senthilkumar and K. Thirugnanasambandam (Coimbatore).**

Let  $(X, \Sigma, \lambda)$  be a sigma finite measure space and  $T : X \rightarrow X$  be a non-singular measurable transformation.  $C_T$  on  $L^2(\lambda)$  induced by  $T$  is a coposition transformation given by  $C_T(f) = f \circ T$ . If  $C_T$  is bounded then we call  $C_T$  a composition operator on  $L^2(\lambda)$ . It is well known that  $T$  induces a bounded composition operator  $C_T$  on  $L^2(\lambda)$  if and only if the measure  $\lambda T^{-1}$  is absolutely continuous with respect to the measure  $\lambda$  and  $f_0$  is essentially bounded, where  $f_0$  is the Radon-Nikodym derivative of  $\lambda T^{-1}$  with respect to  $\lambda$ . A weighted composition operator is a linear transformation acting on a set of complex valued  $\Sigma$  measurable functions  $f$  of the form  $Wf = w \circ T$ , when  $w$  is a complex valued,  $\Sigma$  measurable function. In case  $w = 1, W$  becomes a composition operator, denoted by  $C_T$ . Let  $B(H)$  denote the Banach space of all bounded linear operators on a Hilbert space  $H$ . Duggal et al. introduced a new class, "Operators of class Q" and showed that this class of operators properly includes the paranormal operators. In this paper, class  $Q$  coposition operators on  $L^2$  space are characterised and their various properties are studied.

**H3: On generalised measures of fuzzy directed-divergence, Rakesh Bajaj, H. Singh and D. S. Hooda (Solan, Himachal Pradesh).**

In the present paper we have suggested a generalised fuzzy measure of Renyi's directed divergence, symmetric divergence and proved its validity. Similarly, generalised fuzzy measure of another directed divergence has been introduced with its validity. Particular cases of corresponding directed divergence and symmetric divergence have also been studied.

## **I: Numerical Analysis, Approximation Theory and Computer Science:**

**I1: A layer adaptive B-splitting collection method for singularly perturbed one-dimensional parabolic problem with a boundary turning point,** *Mohan K. Kadalbajoo and Vikas Gupta (Kanpur).*

In the present paper, we develop a numerical method for a class of singularly perturbed parabolic equations with a multiple boundary turning point on a rectangular domain. The coefficient of the first derivative with respect to  $x$  is given by the formula  $a_0(x, t)x^p$ , where  $a_0(x, t) \geq \alpha_0$  and the parameter  $p \in (0, \infty)$  takes the arbitrary value. For small values of the parameter  $\epsilon$ , the solution of this particular class of problem exhibits the parabolic boundary layer in a neighbourhood of the boundary  $x = 0$  of the domain. We use the implicit Euler method to discretize the temporal variable on uniform mesh and to discretize the spatial variable. The resulting method has been shown almost second order accurate in space and first order accurate in time for a fixed value of  $\epsilon$ . Some numerical results are given to confirm the predicted theory and comparison of numerical results made with a scheme consisting of a standard upwind finite difference operator on a piecewise uniform Shishkin mesh.

**I2: Adaptive ant algorithm - A new genetic programming approach to ACO algorithm,** *G. S. Raghavendra and N. Prasanna Kumar (Goa).*

The foraging behaviour of ants has fascinated many researchers. This has led to the evolution of a new class of algorithms, which are distributive in nature and solved by group of cooperating agents. Recent stimulation in this direction was made by using colony of cooperating ants by M Dorigo. These classes of algorithms can be used as an optimization tool to solve NP hard class problems. These classes of algorithms have been implemented successfully for commercial purposes. An analogy with the ant colonies function has suggested the definition of a new computational paradigm, which we call Ant System. M Dorigo proposed this is a viable new approach to stochastic combinatorial optimization and subsequent papers on ACO have tried to find solutions to classical problems like Travelling Salesman Problem, Quadratic Assignment Problem, Four Colour



Problem, Job-Shop Scheduling Problem etc. In this paper we discuss the salient characteristics of the Ant system and the recent advances in the approach towards solving problems using ACO. Also an attempt is made to refine the existing ACO techniques with an appropriate parameter selection.

## **J: Operations Research:**

### **J1: Review of innovation diffusion models for new product diffusion marketing, Naresh Kumar and L. P. Rai (Delhi).**

Innovation diffusion modelling describe the path of new product diffusion and development in social system. It is probably valuable to the field of technology diffusion and management for two reasons viz.

- (1) it helps to understand the trajectory of new innovation diffusion by means of new product adoption process and
- (2) the study of diffusion modelling could lead to the development of systematic and prescriptive models of adoption and diffusion of an innovation.

The importance of innovation diffusion modelling increased due to manufacturing sector, which used diffusion models for identifying the market potential of a new product. Currently the study of innovation diffusion modelling is a topic of increased economic enquiries. During innovation diffusion, there exist a large number of diversified factors and determinants, which interact and jointly contribute to the process of technological change. In this paper, different theoretical approaches of innovation diffusion modelling are systematically reviewed and analyzed. Theoretical framework of innovation diffusion and empirical development of innovation diffusion models based on evolutionary conjecture, have also been discussed.

## **K: Solid Mechanics, Fluid Mechanics, Geophysics and Relativity:**

### **K1: Non-linear parametric resonance stability of cable connected satellites system under the combined effects of the solar radiation pressure and the magnetic field of the earth, S. Shrivastava and A. Narayan (Durg, CG).**

Non-linear parametric resonance stability of the inextensible cable connected satellites system in the central gravitational field of the earth under the combined effects of the solar radiation pressure and magnetic field of the earth have been studied. The cable connecting the two satellites system is assumed to be light, flexible and inextensible in nature. The motion of each of the satellite relative to their centre of mass have been studied, while the

centre of mass of the system orbiting around the earth. It is observed that the combined effects of the solar radiation pressure and magnetic field due to earth may cause periodic changes in the amplitude of oscillation system in parametric resonance in the case of slight defection from stationary regime of oscillations. It has been observed that the system always moves like a dumbbell satellite at the stationary amplitude.

**K2: Bianchi type III cosmological models with variable  $G$  and  $\Lambda$ ,** *Pratibha Shukla, R. K. Tiwari and J. P. Singh (Rewa, MP).*

Einstein field equations with variable gravitational and cosmological constants are considered in the presence of perfect fluid for the Bianchi type III universe by assuming conservation law for the energy-momentum tensor. Exact solutions of the field equations are obtained by using the scalar of expansion proportional to the shear scalar, which leads to a relation between metric potential  $B = C^n$ , where  $n$  is a constant. The corresponding physical interpretation of the cosmological solutions are also discussed.

**K3: Static deformation of an anisotropic solid,** *N. R. Garg (Rohatak, Haryana: nrgmath@rediffmail.com).*

The plain strain static deformation analysis of a monoclinic elastic solid has been studied using eigenvalue method. We have applied Fourier transform to the governing equilibrium equations and then used the matrix notation and eigenvalue methodology to solve them. We have obtained the expressions for displacements and stresses for the monoclinic elastic medium in the Fourier transformed domain. As an application of the above theory developed, the particular case of a line-load acting inside an orthotropic elastic half space has been considered in detail and the closed form expressions for the displacements and stresses are obtained. Further, the results for the displacements for a transversely isotropic half-space as well as for an isotropic half-space have also been derived in the closed form. The results for an isotropic half-space coincide with the existing available results of Maruyama (1966) who had obtained the results by complicated methods of Green's function. Numerically, we have examined the effect of anisotropy by comparing the results for orthotropic, transversely isotropic and isotropic elastic half-spaces due to the same source of a normal line-load. It has been found that the anisotropy is affecting the deformation significantly.

**K4: Rotating anisotropic two-fluid universes coupled with magnetic field in general relativity,** *K. Manihar Singh (Imphal, Manipur).*

In the course of presentation of some interesting new solutions, rotating anisotropic two-fluid universes coupled with a magnetic field are investigated and studied, where the anisotropic pressure is generated by the presence of

two non-interacting perfect fluids which are in relative motion with respect to each other. Special discussion is made of the physical interesting class of models in which one fluid is a comoving radiative perfect fluid which is taken to model the cosmic microwave background and the second a non comoving perfect field which will model the observed material content of the universe. Besides studying their physical and dynamical properties, the effects of rotation on these models are investigated and the reactions of the magnetic and gravitational fields with respect to the rotational motion are discussed. Analysis on the rotational perturbations are also made, in the course of which the amount of anisotropy induced in the pressure distribution by a small deviation from the Friedmann metric is also investigated. The models obtained here are found to be theoretically satisfactory and thereby substantiates the possibilities of existence of such cosmological objects in the universe and may be taken as good examples of real astrophysical situations.

**K5: Quasi-static thermal stresser due to heat generation in a thin hollow cylinder**, *M. V. Khandait, S. D. Warbhe and K. C. Deshmukh (Nagpur, MS.)*.

The present paper deals with the determination of displacement and thermal stresses in a thin hollow cylinder due to heat generation within a solid. Time dependent heat flux is applied at the outer circular boundary whereas inner circular boundary is at zero heat flux. Also, initially the solid is at arbitrary temperature. The governing heat conduction equation has been solved by using transform method. The results are obtained in series form in terms of Bessel functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

**K6: Thermoelastic problems in a rectangular plate with heat generation**, *M. V. Khandait, S. D. Warbhe and K. C. Deshmukh (Nagpur, MS.)*.

The present paper deals with the determination of quasi-static thermal stresses in a finite thin rectangular plate defined as  $0 < x < a$  and  $0 < y < b$  subjected to the heat generation with the solid at a rate of  $g(x, y, t)$ . The constant temperature are prescribed at the boundaries  $x = a$  and  $y = b$  while the initial edges  $x = 0$  and  $y = 0$  are thermally insulated. The governing non-homogeneous heat conduction in a thin rectangular plate is solved with the help of integral transform technique. The results are obtained in the series forms in terms of trigonometric functions.

**K7: Linear instability of an inviscid compressible parallel shear flow**, *Hari Kishan, Geeta Devi and Naresh Kumar (Meerut, UP)*.

In the present paper, the linear instability of an inviscid compressible parallel shear flow has been discussed. Sufficient condition of stability, necessary condition of instability and upper bound of growth rate have been obtained under the assumption considered by Shivamoggi [1986].

**K8: Effective latage suction on compressible boundary layer flow due to a step change in heat flux, B. B. Singh and S. A. Wagh**  
(Raigad: *brijbhansingh@yahoo.com.*)

This paper deals with the effect of large suction in compressible boundary layer flow of a semi-infinite flat plate with heat flux. Here, perturbation methods have been used to obtain the expressions for wall temperature gradient and skin friction parameter. The numerical values of these parameters have also been computed for different values of suction parameter Prandtl number, and it has been found that the skin friction parameter increases as the suction parameter increases. Similarly, the wall temperature gradients are in decreasing order with suction parameter and Prandtl number increasing.

**K9: Non-adiabatic gravitational collapse with anisotropic core, B. S. Ratanpal and V. O. Thomas (Vadodara, Gujarat).**

The non-adiabatic gravitational collapse of a spherical distribution of matter accompanied by radial heat flux has been studied on the background of a pseudo spheroidal spacetime. The spherical distribution is divided into two regions: a core consisting of anisotropic pressure distribution and envelop consisting of isotropic pressure distribution. Various aspects of the collapse have been studied using both analytic and numerical methods.

**K10: Finite element solution of heat and mass transfer in a hydrogeometric flow of a micropolar fluid past a stretching sheet, Lokendra Kumar (Solan, HP.: *lkdma@rediffmail.com.*)**

This paper presents a finite element solution of the problem of heat and mass transfer in a hydromagnetic flow of a micropolar fluid past a stretching sheet. The governing differential equations are solved numerically by using finite element method. The effect of important parameters namely magnetic field parameter, material parameter, Eckert number and Schmidt number have been studied over velocity, microrotation, temperature and concentration function. It has been observed that the magnetic field parameter has the effect of reducing the velocity and increasing the microrotation, temperature and concentration while the micropolar parameter has the opposite effect on these functions except the temperature function. The temperature increases with increase in Eckert number and concentration decreases with increase in Schmidt number.

**K11: Partial integro-differential equations with integral boundary conditions**, *J. Dabas and D. Bahuguna (Kanpur)*.

In this paper we consider a heat conduction problem for a material with memory with nonlocal boundary conditions. We apply the method of semi-discretization in time, also known as the method of lines, to establish the existence of weak solutions.

**K12: Positive solutions for boundary value problems with one dimensional P-Laplacian**, *R. S. Torogh and M. Y. Gokhale (Pune)*.

In this paper, we prove the existence of triple positive solutions of two classes of quasi-linear multi-point boundary value problem. By using a fixed point theorem, we show the existence of at least three positive solutions.

**K13: Non-linear oscillations of extensible cable connected satellites system in circular orbit**, *Shilpa Dewangan and A. Narayan (Bhilai, Durg, CG.)*.

Non-linear oscillations of extensible cable connected satellites system in the central gravitational field of the earth under the combined effects of the solar radiation pressure and the magnetic field of the earth in circular orbit have been studied. The cable connecting the two satellites system is assumed to be light, flexible, extensible and non-conducting the nature. The motion of earth of the satellite relative to their centre of mass have been studied, while the centre of mass of the system orbiting around the earth. The presence of these forces enables the application of variation of arbitrary constant methods due to Milkin to the equations governing the non-linear oscillations of the system in non-resonance as well as resonance case. The system exhibits stable behaviour inspite of change in frequency of the system in non-resonance case as well resonance.

**L: Electromagnetic Theory, Magneto-Hydrodynamics, Astronomy and Astrophysics:****L1: Kinetic equations for non-linear plasma waves interaction in electromagnetic fields**, *R. K. Dubey (Rewa, Madhya Pradesh)*.

The transverse phonon-helicon interaction is studied in the presence of a longitudinal electrostatic field  $E_0$  and magnetostatic field  $B_0$ . The generalised coupled dispersion relation for acoustic-helicon interaction is given with the help of lattice displacement and Boltzmann's transport equation. The expression for the normalised growth rate and propagation constant are obtained for piezoelectric semi conducting plasma.

**L2: Profiles of flow variables of real gas in strong cylindrical shock waves**, *R. P. Yadav, V. Kumar, M. Singh and V. Pal (Bilaspur, UP.: rpyadav93pphysics@yahoo.com)*.

This paper deals with propagation of strong cylindrical shock waves in uniform real gas. Analytical relations for shock velocity and shock strength have been obtained for cases viz:

1. When shock moves freely and
2. When it moves under the influence of overtaking disturbances.

The dependence of shock velocity and shock strength on propagation distance, specific heat index and real gas constant is discussed with the help of tables. Using the relations, profiles of flow variables of perturbed real gas have been prepared. The results for real gases obtained here are compared for ideal gas obtained elsewhere.

**L3: Transient MHD free convection flow with mass transfer on an impulsively started semi-infinite isothermal vertical plate with temperature dependent heat source**, *Behnaz Farnam and M. Y. Gokhale (Pune)*.

Transient MHD free convection flow with mass transfer of an incompressible fluid past an impulsively started semi-infinite isothermal vertical plate with temperature dependent heat source is studied. Transient temperature, concentration and velocity profiles are plotted to show the effect of heat source. Average Nusselt number, Sherwood number and skin friction are shown graphically to get physical insight of the problem.

## **M: Bio-Mathematics:**

**M1: Philosophical flow aspects of stenosis and its effects on blood diseases**, *A. R. Haghighi and R. N. Pralhad (Pune)*.

Effects of contraction (stenosis) in the arteries have been modeled in the present paper with view to study its effects on blood diseases which are prone for heart attack. Flow parameters such as velocity, flow rates, shear stress and resistance to flow have been computed for blood diseases and compared with normal blood.

**M2: Fluctuation and stability of a delayed allelopathic phytoplankton model within fluctuating environment**, *M. Bandhopadhyay (Kolkata: mb\_math\_scc@yahoo.com)*.

In this paper we consider a two-dimensional model of two competitive phytoplankton species where one species is toxic phytoplankton and other is a non-toxic species. The logistic growth of both the species follow the

Hutchinson type growth law. Firstly, we discuss basic dynamical properties of non-delayed and delayed model system in brief, within deterministic environment. next we construct the stochastic delayed differential equation model system to study the effect of environmental driving forces on the dynamical behaviour. We calculate population fluctuation intensity for both species by fourier transform method. Numerical simulations are carried out to substantiate the analytical findings. Significant outcomes of our analytical findings and their interpretations from ecological point of view are provided in the concluding section.

**M3: A mathematical model for tumor treatment with a dose of adriamycin, Sanjeev Kumar and Alpna Mishra (Agra).**

Consider a model that represent the procedure for tumor treatment, which consists tumor cell energy and specific dose of Adriamycin. The tumor cell energy depends upon tumor cell density and Adriamycin is a specific drug that works against tumor cell energy. We modelled the problem in the form of partial differential equations for the tumor cell density, tumor cell energy and the effect of Adriamycin. The result of this work suggests that the tumor cell energy decreases by the effect of Adriamycin, if the tumor cell energy decreases then the size of tumor reduced.

**M4: Oscillatory MHD flow of blood through an artery with mild stenosis, Sanjeev Kumar and Alpna Mishra (Agra, UP).**

The purpose of this work is to study the effect of oscillatory MHD flow of blood in stenosed artery. The analytical and numerical results are obtained for oscillatory MHD blood flow, which is assumed to be a Newtonian fluid. We also assumed that the surface roughness is of cosine shaped and maximal height of roughness is very small as compared with the radius of an unstricted tube. The fluid mechanics of an MHD blood flow in a stenosed artery is studied through mathematical analysis and discussed the impact of magnetic effect on the instantaneous flow rate, which is reducing, if we increase the Hartman number.

**M5: A mathematical model for diffusion process in a binary mixture of chemical species, S. Kumar and A. Mishra (Agra, UP.: sanjeevibs@yahoo.com).**

We worked here on a mathematical model for diffusion process which is for a binary mixture of chemical species. Here an attempt is made for the applicability of Green's function for solving a one dimensional diffusion equation. We consider the initial and boundary conditions for  $t = 0$  and by using these conditions we get the required solution to this diffusion equation, which describes the rate of change of concentrations of one chemical species to other with a constant diffusion coefficient. While solving the equation

we consider  $t = 0$ , a constant quantity, so that it can be applicable in an isothermal diffusion also.

**M6: Mathematical model of unsteady-state condition of oxygen diffusion through biological floc particles, S. Kumar and A.**

*Mishra(Agra, UP.: sanjeevibs@yahoo.com).*

A model for unsteady-state condition of oxygen diffusion, through biological floc-particles is made here, to explore the mathematical foundation of reaction diffusion system. We are considering a mass balanced equation on the floc particle in conjunction with Flick's law. The governing partial differential equations are then solved using Matlab 6.0 and obtained the result for oxygen concentration. We observed that the oxygen concentration became less at middle of the floc and high at the surface of the floc.

**M7: A model of cyclic type with temporary immunity for the transmission of pertussis, Deepak Maheshwari and Bijendra Singh**

*(Indore, MP.).*

Using a model of cyclic type, the present paper deals with the transmission of infection disease Pertussis (Whooping cough). Two states of equilibrium are studied, one is the infection-free and the other is endemic. It is noted that for a basic reproduction number only infection-free equilibrium exists and is stable. For  $R > 1$  both the equilibrium exist. Stability criteria for them are also studied.

**N: History and Teaching Mathematics:**

**N1: Excitements in mathematics- a historical perspective,**

*Ambat Vijaykumar(Cochin: vijay@cusat.ac.in).*

This lecture will share the excitements in a historical perspective. Mathematics - described as the queen of all sciences (Gauss) and which is at the top of all Vedic sciences (Lagandha) is undoubtedly the most charming, universal language, filled with surprises. This talk will describe the culture, development and relevance of mathematics starting from the times of the Greeks and make a quick journey through the sixteenth century to the present, by anecdotes, quotes (by even non mathematics) and pictures of great mathematicians including the Indian contributors to the modern mathematics and a sketch of Kerala School of Mathematics (1350-1650). My expectation is that after the talk here will be one more lover of Mathematics.

**N2: Study of ttaditional statistical mathematics in India,**

*Kamini Sinha and Ashwini Kumar Sinha(Ara, Bihar).*



For many years, I have been writing and talking with rather missionary zeal, on various platforms, on the contribution of India to traditional statistical mathematics in the late medieval and early modern periods. The reason is obvious: Although it is a fact that contribution is significant in many respects, it has yet not been suitably recognized today. I would like to say something more in this direction regarding a new approach needed. Not much is gained by nearly boasting of what we have inherited; but we have to use this heritage profitably for the betterment of statistical mathematics and science is general.

## ABSTRACT OF THE PAPERS FOR IMS PRIZES

### **IMS-A1: Clique irreducibility and clique vertex of graphs,**

*Aparna Lakshmanan S. (Aluva, Kerala).*

A graph  $G$  is clique irreducible if every clique in  $G$  has an edge which does not lie in any other clique of  $G$  and is clique reducible if it is not clique irreducible. A graph  $G$  is clique vertex irreducible if every clique in  $G$  and clique vertex reducible if it is not clique vertex irreducible. The clique vertex irreducibility and clique irreducibility of graphs which are non-complee extended p-sums (NEPS) of two graphs are studied. We prove that if  $G^c$  has at least two non-trivial components then  $G$  is clique vertex reducible and if it has at least three non-trivial components then  $G$  is clique reducible. The cographs and the distance hereditary graphs which are clique vertex irreducible and clique irreducible are also recursively characterised.

**IMS-A2: On a combinatorial identity,** *Meenakshi Rana (Chandigarh, Panjab: meenakshiarru@gmail.com).*

Using Frobenius partitions we extend the main results. This leads to an infinite family of 4-way combinatorial identities. In some particular cases we get even 5-way combinatorial identities which give us four new combinatorial versions of Gollnitz-Gordon identities.

**IMS-A3: An indeterminate equation: Chidula problem and solution through discrete mathematics,** *R. Sree Parimala and K. K. Velukutty (Pollachi, Tamilnadu).*

Some indeterminate equation from classical mathematics is picked out and analysed to explore for a new version of procedure for a solution in the style of discrete mathematics. Sell exactly some chickens and ducks to purchase a lamb (Chidula Problem). The problems may be versioned in mathematical form with a generalization as solve for non negative integers  $x, y, z$  given  $l, m, p$  are integers. Classical methods are those of indeterminate equation (Diophantine equation) and Kuttakarakiya (due to Aryabhata,

Mad-have and Parameshwara). Here, Chidula square/triangle is defined and constructed in the style of Pascal triangle to generate the solutions. The feasibility Belt and the Tip of the feasibility Belt, two of the most important portions of the square/triangle pertinent to the particular problem and the generation of the solutions from the tip of the feasibility. Belt at which one solution exists are the majotinnovations of the procedure. The narrowest feasibility Belt is defined to be the feasibility ribbon. The tip of the feasibility ribbon is sufficient enough to generate all the solutions. Examples are illustrated for clarification and assessment. Concludingly, remarks are added fo r further extensions.

**IMS-A4: Extension of splitting lemma of graphs to binary matroids**, *Santosh B. Dhotre (Pune, MS: dsantosh2@yahoo.co.in).*

In discrete Math, 184 (1998), 267-271, the author has generalized splitting operation of graphs to binary matroids. In this paper we extend the splitting lemma of graphs to a class of binary matroids. We also characterise the matroids for which the splitting of dual matroid is same as the dual of splitting matroids.

**IMS-A5: Intuitionistics rough Fuzzy sets using generalised  $S$  and  $T$  norms**, *G. Ganesan (Dharmapur, Mahabubnagar, AP).*

This paper gives a new approach of rough fuzzy concepts, which are commonly used in several information systems. Here, the work is developed based on approximation of intuitionistic fuzzy sets under crisp knowledge base.

**IMS-B1: Some theorems on the explicit evaluation of singular moduli**, *K. S. Bairy (Bangalore: ksbairy@rediffmail.com).*

At scattered places in his notebook, Ramanujan recorded some theorems for calculating singular moduli and also recorded several values of singular moduli. In this paper, we establish several general theorems for the explicit evaluations of singular moduli. We also obtain some values of class invariant and singular moduli.

**IMS-B2: On prime rings with generalized left derivatives**, *Shakir Ali (Aligarh, UP.: shakir50@rediffmail.com).*

In the present paper we define the notion of generalized Jordan left derivation on a ring  $R$  and prove that every generalised Jordan left derivation on a 2-torsion free prime ring is a generalised left derivation of  $R$ . Finally, some related results have also been obtained.

**IMS-F1: On the eigenvalues of the Laplacian for left- invariant Riemannian matrices on  $S^3$** , *A. Mangasuli (Pune).*

We show that if the Ricci curvature of a left-invariant metric  $g$  on  $S^3$  is greater than that of the standard metric  $g_0$ , then the eigenvalues of  $\Delta_g$  are greater than the corresponding eigenvalues of  $\Delta_{g_0}$ .

**IMS-J1: An effective approach for order reduction of fractional programming in linear system model reduction problem, S.**

*Jain(Ajmer, Rajasthan: drjainsanjay@gmail.com).*

This paper presents a new and efficient way of implementing fractional programming to use in the linear system model reduction problem. Linear system model having higher order fractional objective function and it is reduced to lower order fractional objective function by using our proposed method. This method is simple, computationally straightforward and takes the least time. The proposed method has been illustrated by an example.

**IMS-K1: Marangoni convection in an Oldroyd-B fluid layer with throughflow, S. Saravanan(Coimbatore: sshravan@lycos.com).**

The onset of Marangoni convection in an horizontal Oldroyd-B fluid layer in the presence of a vertical throughflow is determined by linear analysis. We find an approximate solution to the corresponding eigenvalue problem by Galerkin method. The effects of viscoelastic parameters on the critical Marangoni number, wavenumber and frequency are discussed. The study also reveals the existence of a critical retardation time for which the oscillatory motion reaches its maximum strength. This study has possible implications in microgravity situations.

**ABSTRACT OF PAPERS FOR V. M. SHAH PRIZE**

**VMS-C1: On the magnitude of Vilenkin Fourier coefficient**

*B. L. Ghodadra(Varodara, Gujarat: bhikhu\_ghodadra@yahoo.com).*

On the compact, metrisable, 0-dimensional, abelian group(Vilenkin)  $G$ , the order of the magnitude of Vilenkin Fourier coefficient of  $p$ -generalised bounded fluctuation ( $p > 0$ ),  $\phi$ -generalised bounded fluctuation,  $\phi - \Lambda$ -bounded fluctuation and  $\phi - \Lambda$ -bounded fluctuation is studied.

**VMS-C2: Some further results on Lahiri**

*Abhijit Banarjee(Nadia, West Bengal: abanerjee@yahoo.co.in).*

With the aid of weighted sharing of values we prove a result on the uniqueness of meromorphic functions sharing three values. Our results will improve and supplement some earlier result of Lahiri, Yi and some examples to show that our results are the best possible. In the application part of our result we will show that some results obtained by Chen, Shen and

Lin are wrong by obtaining the actual result.

**VMS-C3: A note on the zeros and poles of the bicomplex Riemann zeta function**

*Jogendra Kumar (Khandari, Agra: jogendra\_j@rediffmail.com).*

In this paper, we have studied certain characteristics of the bicomplex Riemann zeta function. The locations of the trivial and non-trivial zeroes of the bicomplex Riemann zeta function have been depicted, geometrically, in some sense. The concept of the critical line of the complex Riemann zeta function has been generalised to the concept of critical family of planes for the bicomplex Riemann zeta function. We characterise the element of  $1 + O_2$  in terms of its idempotent components,  $O_2$  being the set of all singular elements in  $C_2$ . The complement of  $1 + O_2$  has been shown to be the certain set determined by two complex planes both punctured at point  $z = 1$ . Singular elements of  $1 + O_2$  have been characterised. The hyperbolic numbers present in  $1 + O_2$  have also been characterised. Finally, poles of bicomplex Riemann zeta function have been discussed.

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## THE PAST, PRESENT AND FUTURE OF MATHEMATICS IN CHINA AND INDIA

SHING-TUNG YAU

(Received: 24-04-2008)

Above is the title of the talk that Professor Ravindra Kulkarni asked me to give. It is a tall order as I am not a professionally trained historian, although I like to read history in my leisure. Naturally, I am more familiar with the mathematical achievements of China than of other countries. In reading through some of the writings by Western historians on mathematics, I am often surprised by the somewhat biased discussions towards the mathematical achievements of Asia.

There is no question that the ancient Greeks made numerous fundamental contributions to mathematics, as we can see many of them in use in our present day. However, many of the Greek manuscripts were partially destroyed during the early centuries, and therefore, it takes great efforts to identify the form of the original manuscripts. The Romans, the Byzantines, the Arabs and the Moors all spent much efforts to translate and circulate the classic Greek manuscripts.

Often, great ancient books incorporate a collection of many people's works and were sometimes edited by a number of authors. Commenting on the Euclid's *Elements* (believed to be written around 300 BC), the Greek mathematician Proclus of fifth century AD wrote, "Euclid, who put together the *Elements*, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors." There are copies of Euclid's *Elements* in Greek that are in the Vatican Library of the Holy See and the Bodleian Library of Oxford University. However, they are not complete and of variable quality. (See a fragment of the *Elements* from

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This is the text of Plenary Lecture(M. K. Singal Memorial) delivered at the Centenary year 73<sup>rd</sup> Annual Conference of Indian Mathematical Society, held at the University of Pune, Pune-411007 during December 27-30, 2007.

around 100 AD in Figure 1.) Many hypotheses have been put forth on the contents of the original text based on translations and originals. Similarly, the origins of ancient Chinese books of mathematics such as the *Nine Chapters on the Mathematical Art* are difficult to know for sure. Most likely, the *Nine Chapters* is an accumulation of works of many generations and was updated by many people. The recent discovery of an old book from an old tomb dated around 186 BC showed some form of the preliminary version of the *Nine Chapters*.

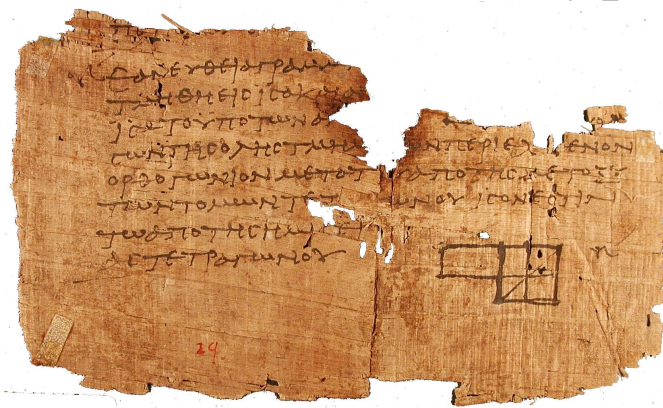


FIGURE 1. A fragment of papyrus of Euclid's *Elements* dug up at Oxyrhynchus, Egypt in 1896-1897. It has been dated to between 75 - 125 AD (from Bill Casselman's web page.)

There are several outstanding contributions in mathematics that appeared in different cultures. The most notable is the Pythagorean theorem. Pythagoras is believed to be born between 580 to 568 BC. This is not much later than the time when the Egyptian papyrus was introduced into Greece around 650 BC. It was said that he traveled to Egypt, Babylon and perhaps even India. Since there are ancient records of the Pythagorean triples in all these countries besides China, it is likely that Pythagoras learned the statement of his theorem from these countries. On the other hand, his great contribution is that he is the first one who proved the theorem. In fact, the concept of a proof based on formal logic found in Greek mathematics is rather unique as compared with other countries. It may be a mystery that countries far apart discovered the same theorem at different levels of depth. For the great desire to build a right-angled triangle, it is rather natural to imagine some primitive form of the Pythagorean theorem. Therefore, it is quite possible such a development can be made independent of each other and evolved in its own right.

A great deal of mathematical developments were necessitated both by economic needs, such as counting grain and the area of land, and also by cultural needs, for examples, the calendar, astronomy, rituals in the temple, musical instruments, and *etc.* As a result, mathematical developments can be rather unique to the culture of that country. For example, it is puzzling to me why the ancient Chinese were so much interested in the magic squares. A legend claims that interest in such objects dates back to two thousand BC and continued all the way to the Song Dynasty. On the other hand, the Muslims were very much interested in patterns - tiling of geometric figures - in the years around 1200 AD to 1600 AD. It was said that they knew the non-periodic tiling: Penrose tiling. It is quite likely that each culture depending on its needs, developed its mathematics and at the same time absorbed ideas from different countries. Moreover, the interactions between countries were likely much more frequent than what we have on records.

Interactions between ancient Greece, Egypt, Babylonia, Persia and India are not that hard to imagine. But the Silk Road between China and the West dates back to the very early days. For the merchants traveling along the Silk Road, they were usually in groups of one to three hundred people where the engineers were an important core group. The merchants needed the engineers' help to build bridges and overcome any obstacles along the road. These engineers knew mathematics and clearly some communications between engineers from different countries should have occurred.

I would like to propose a hypothesis that the major period of interaction of the ancient cultures with regards to mathematics happened during the period from 300 BC to 200 AD between Greeks, Egypt, Persia, India and China. Alexander's conquests brought the Greeks all the way east to Central Asia and the Indian subcontinent in 326 BC. It is likely that what the Chinese called the Da Yuezhi in Central Asia was the center of such cultural exchange. The Yuezhi people moved several times. They later conquered what is today Afghanistan and Pakistan in the northwest of India and formed the Kushan Empire from the period of 40 AD to 250 AD. For their likely importance in linking the different cultures, we shall spend some time discussing the Yuezhi people below.

### **Communications between ancient China and Central Asia**

Let us start by first commenting on the historical documents I can find on the communications between ancient China and Central Asia.

It should be noted that the written documents of ancient China are much less reliable for events that happened before 600 BC. It has been speculated that segments of some of the ancient books were written during the period of the Han Dynasty (around 200 BC - 200 AD). A lot of the books were dug up in tombs or walls during the ancient days. But often

the original disintegrated and some later authors made up the rest either from memory or by imagination. One has to be very careful how to use such books. There are of course recent discoveries in tombs dating back to two or three hundred BC that are very reliable. On the other hand, by and large, Chinese government-appointed historians are widely regarded to be very serious in their works. The government supported a large group of historians to collect data. In most cases, the government appointed scholars to study records of previous governments. These are high-level scholars who considered producing honest works to be noble acts in their life. Hence most records given by government historians are rather reliable. There are historical records which were read by Confucius and his disciples. For example, Confucius thought highly of the *I-Ching* and this book certainly should have existed before 500 BC. In fact, recently some ancient fragments of the book *I-Ching* were dug up on bamboos which can be traced back to 200 BC.

In contrast, Indian literature are more difficult to trace prior to 500 AD. For example, it is still not clear what is the exact date of the important Bakhshali manuscript. One may also say that while there is continuity in Chinese history, several parts of Indian subcontinent is less united. Several invasions occurred, especially in northern India. The ancient Harappan script remains today undeciphered. Hence very ancient Indian mathematics proficiency must be based on excavated artifacts.

About 1500 BC, the Aryans, who were pastoral people, took over the Harappan culture and developed a language called Sanskrit. This language was systematically organized by Panini around 500 BC. On a basis of 4000 sutras, he built the Sanskrit language. To many, Panini's work has had a lasting influence in leading Indian mathematics towards algebraic reasoning and the techniques of algorithm and recursion. It is interesting to know whether Panini's aim of building a complete structure for the Sanskrit language might have inspired in some way the writing of Euclid's *Elements* two hundred years later.

Let us now discuss some relevant facts written by Chinese historians.<sup>1</sup>

*I. King Mu, Emperor of Zhou and his western travel*

A classical history book written by Sima Qian around 180 BC is called *Shiji* or in English *Records of the Grand Scribe*. Sima Qian was born into a family of professional historians appointed by the emperor. Sima Qian traveled around ancient China to collect and verify historical events. Because of his family's connections, he was able to read many historical documents from centuries past in the possession of the government. In his writing

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<sup>1</sup>With few exceptions, I follow the Hanyu Pinyin system for romanization of Chinese names below.



about the Kingdom of Zhao, he described the story around 950 BC of the emperor of the Zhou Dynasty, King Mu, who rode fast running horses to western China, all the way to the Qilian Mountains (or in Chinese called Tian Shan, the Heavenly Mountains) to meet Xi Wang Mu, the “Queen Mother of the West,” and leader of a local tribe. There are many fairy tales or legends based on this trip. The area that the Zhou emperor went is speculated to be part of Da Yuezhi, a country in Central Asia. The story involves jade which is consistent with the traders of Da Yuezhi.

### *II. Westerners in ancient China, students of Confucius*

In the book of *Shiji*, there was an extensive discussion on the students of Confucius. Of the few commentaries on *Shiji*, an important one was written by Pei Yin called *Collected Explanations*. This was written around 420 AD. In his commentary, Pei Yin mentioned another commentary which was written by a group of scholars under the leadership of Cao Pei, the King of Wei, around 210 AD. (After a few hundred years, only a small part of this book survived.) In this commentary, it mentioned that when Confucius died, his students traveled from all over the country to stay for three years near his tomb. Each of them brought plants from their home countries to be planted in the tomb garden of Confucius. Some of these plants came from the “western area.” This means that students from the West came to study in China. The Han emperors were fond of the people from Central Asia. When the great Han emperor Wudi defeated the Huns (Xiongnu) in a major battle, a king of the Huns was killed by another fellow Huns king. The widow and her son ended up surrendering to the Han Dynasty. The son Jin Richan was eventually promoted to the second most powerful position in the imperial court of the Han around 140 BC.

Regarding King Mu and Westerners in ancient China, the historical evidences for them are not solid and we should be critical about their reliability. However, the historical events that I now describe are backed up with solid evidences.

### *III. Han Dynasty, Da Yuezhi, and the Kushan Empire*

The first major communication between China and the Central Asian countries is around the time of Wudi. Wudi was interested in defeating the Huns on the northwest boundary of China. In 138 BC, he sent an ambassador named Zhang Qian to a country called Da Yuezhi (moon people) to seek help to defeat the Huns. (The reason for this is that the Yuezhi were defeated by the Huns in a very cruel manner.) These people are believed to be of Indo-European descent and were called Tocharians by ancient Greeks. They eventually migrated to Transoxiana, Bactria and then northwestern India where they formed the Kushan Empire in India (100-300 AD). See Figure 2. It was said in *Guanzi*, a Chinese book written around 645 BC

that they supplied jade to the Chinese from the nearby mountains of Yuezhi at Gansu. (The queens of the Shang Dynasty were fond of jade and the Yuezhi were called the Di people, meaning western barbarians.)

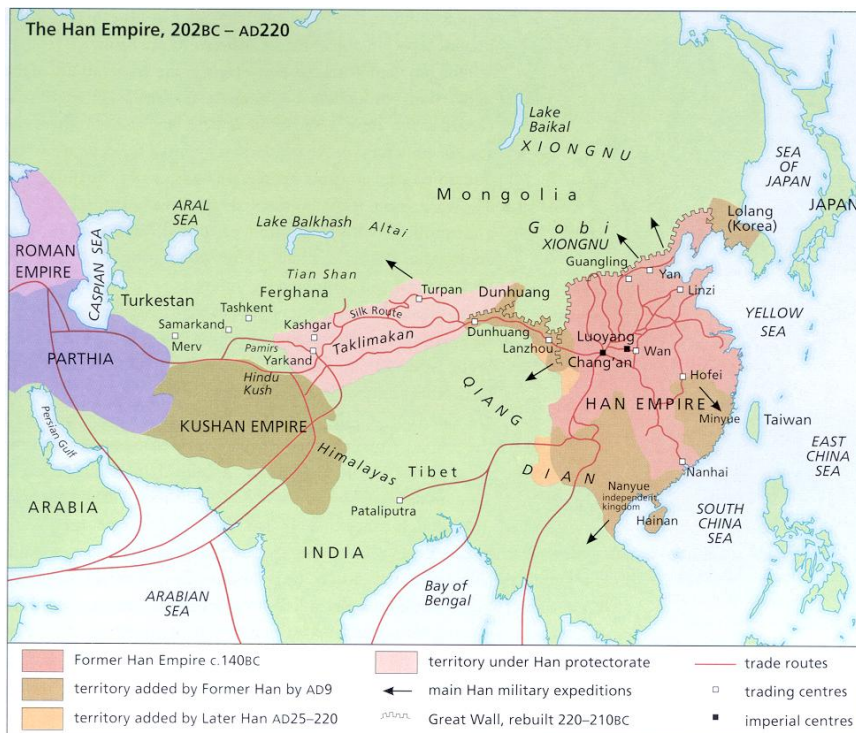


FIGURE 2. Han and Kushan Empires (from Kenyon College web resources).

Zhang Qian had a group of a hundred people that went with him. But he was captured by the Huns and after 10 years of imprisonment, he managed to escape (with one assistant) to see the King of Yuezhi who had no interest to help the Han to defeat the Huns. On the other hand, Zhang Qian spent a year in Yuezhi and wrote down detail descriptions of the area around this country, which is southwest of another country called Dayuan (Ferghana), south of the Gui (Oxus) River. He witnessed the last period of the Greco-Bactrian Kingdom when it was being subjugated by the Yuezhi. He reported about Shendu (or Sindhu in Sanskrit). This was a region of India bordering Persia and the Arabian Sea that was very advanced. He said that the inhabitants rode elephants when they went to battle. The kingdom is situated on the great Indus River. He also wrote about Anxi (Parthia), and to its west Tiaozhi (Mesopotamia). He visited Sogdiana, Kangju and

Yancai (the vast Steppe). He learnt from merchants in Yuezhi that there were traders from India who brought bamboo sticks and clothes made in the western part of China known as Sichuan. Zhang Qian concluded that there were direct routes that went from mainland China to India. (Wudi sent four different groups to look for this route for trading. The route was not found. But materials recently found in the tombs of southern China verified their existence.) Zhang Qian made another trip to Wu-Sun in 119 BC. He brought with him hundreds of people, ten thousand cattle, and gold and silk as gifts to give to the heads of countries in Central Asia. Altogether, he and his envoys visited thirty-six countries.

The rise of the Kushans (who the Chinese called Yuezhi) was well-documented in the Chinese historical chronicle *Hou Hanshu*. It recorded that Yuezhi envoys came to give oral teaching on the Buddhist sutras to a student in the Chinese capital around the first century BC. In fact, Yuezhi integrated Buddhism into a pantheon of many deities and became great promoters of Mahayana Buddhism. Their interactions with the Greek civilization helped the Gandharan culture and Greco-Buddhism flourish. Yuezhi is located about 3000 km north of India. The kings called themselves “Sons of Heaven.” Yet their city layouts and palaces are quite similar to those of the Daqin (Chinese name for the Roman Empire). There are records in the Western world of the invasion in 124 AD by Yuezhi against the Parthians, when King Artabanus I of Parthia was critically wounded and died. (Justin, *Epitomes*, XLII,2,2) As they settled in Bactria from around 125 BC, the Yuezhi became hellenized to some degree.

During the first century AD, the Kushan Empire traded with the Roman Empire and also with China. They joint forces with Chinese against nomadic incursions. The Chinese general, Ban Chao, worked with them to fight against the Sogdians in 84 AD and against the Turanians, east of the Tarim Basin in 85 AD. At that time, the Kushans requested a Han princess but was denied. They then marched against Ban Chao in 86 AD and were defeated. They paid tributes to the Chinese emperor Han He (89-106 AD). It is recorded in *Sanguozhi* (a Chinese history book of the Three Kingdoms) that in 229 AD, King of Da Yuezhi, Bodiao, sent his envoy to present tribute, and the Emperor Cao Rui gave him the title, “King of the Da Yuezhi Intimate with the Wei.”

During the stay of General Ban Chao in Yuezhi, he heard about Daqin (the Roman emperor) and he sent (in 89 AD) his deputy General Gan Ying to go to Rome. Gan Ying went through Iran and Iraq and either arrived at Antiochia, the tip of Persian Gulf, or the northeastern part of the Black Sea (Novorossiysk). Where exactly he ended up is still open to debate. Regardless, he was told by the Persians that the ocean is very

dangerous, and he turned back. But the fact that the Chinese traveled that far is significant. It should be noted that Ban Chao's elder brother wrote the famous historical chronicle *Hanzhu*. His sister wrote a chapter of this chronicle on astronomical events. She knew mathematics well and taught the emperors and noble ladies at the court. She should had been well-read in the ancient Chinese books on mathematics. By 166 AD, the Roman merchants acting as emissaries of the Roman emperor, were able to present gifts to the Chinese emperor and teach China about the West.

Therefore, since 140 BC, the Silk Road was already very active. According to *Shiji*, many Chinese missions were sent. "The largest of these emissaries to foreign nations numbered several hundred persons, while even the smaller parties included over 100 members .... In the course of one year anywhere from five to six to over ten parties would be sent out." Naturally there were many engineers with knowledge of mathematics in these parties.

#### IV. Buddhism and the Silk Road

The communication between China and India would have been much more active if not for the barrier of high mountains. However, the great interest in Buddhism led many monks to travel between these two ancient countries, starting around the first century. The monks back then were among the most learned men in both countries. It would not be surprising that they knew enough mathematics to communicate to each other.

Up to the Tang Dynasty, the major traffic between China and the West occurred on land through the Silk Road. The northern Silk Road went through the pastoral area which went through the Gobi Desert, through the mountain areas and finally ended up at the Black Sea, where the ancient Greeks had colonies. The second Silk Road went from Xian to Lanzhou, and through Dunhuang and then through the north of India, to Iran and to the Mediterranean Sea. See Figure 3.

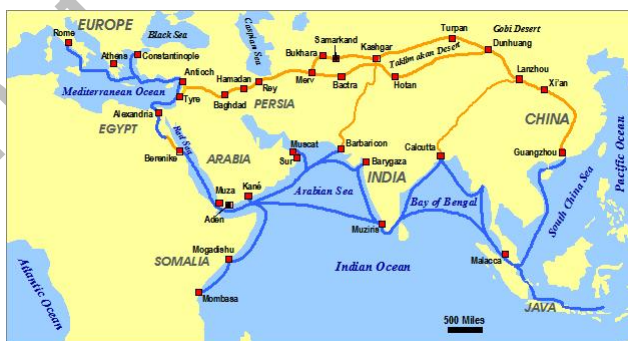


FIGURE 3. Silk Road(from Silk Road Foundation).

In Dunhuang, there are many caves where ancient manuscripts have been found. Some of them were on mathematics, though not very sophisticated. Since the manuscripts were scattered in different places and some important ones might have been lost, it is difficult to make conclusions from those that were found. On the other hand, it is evident that Buddhist disciples played rather important roles in the Dunhuang manuscripts.

Apparently, the earliest contact with Buddhism in China may have gone back to the time of the Qin Dynasty. It was recorded that Buddhist monks from Yuezhi also came to China by 2 BC. In the year 64 AD, the Han emperor heard about the Buddha and decided to send the government official Cai Yin to India to learn Buddhism. Two high-level Indian monks came back with him and built a big temple in the capital of Luoyang.

During the period of 400 AD and afterwards, a number of Buddhist scholars traveled between China and India: Fa Xian (399 AD) and Xuan Zang (650 AD) are the most notable ones. Among the places they traveled were monasteries such as Nalanda and Taxila which were Indian centers of scholarships, not only in religion, but also in astronomy and mathematics. Usually the monks went through the southern Silk Road to travel to India. Occasionally, they also used the Silk Road on the sea. The monks communicated with Chinese pilgrims extensively during the Sixteen Kingdoms Period (304-439 AD) and during the Tang Dynasty. The Sixteen Kingdoms were led by non-Chinese nomadic tribes, some of them are related to the Huns.

The travels of Fa Xian are well-documented. He walked from central China, across the Taklamakan Desert, over the Pamir Plateau, and through India to the mouth of the Hooghly River, in Bengal. He wrote about the Kingdom of Khotan (southern arm of the Silk Road) and the Kingdom of Kashgar where the northern and southern branches of the Silk Road reunite. He went through Afghanistan (Punjab). He spent six years in India during the Gupta Dynasty. It is remarkable that besides Buddhist writings, he also wrote about the differences between Chinese and Indian calendars.

The family of the Tang emperor came from the area in the west not far from the Dunhuang area, and cultural interactions between the West and East were extensive during the Tang Dynasty. By the end of Tang period and the period right afterwards - before Sui and Song Dynasty - the "barbarian state" was too strong for the Chinese merchants to go via land on the Silk Road to trade. They started to travel via the sea. The Song Dynasty had extensive trading with countries by the Indian Ocean and all the way to the Persian Gulf and even to Africa. The last major envoy Zheng He, admiral of the Chinese fleet, was sent by the Chinese emperor seven times from 1405 to 1435, to the Indian Ocean. After that, both Ming

and Qing Dynasties lost interest in interacting with the Western world in a strong manner. The direct consequence for mathematical development was very negative. Chinese mathematicians lost the broad vision and became very narrowly focused on studying the ancient developments of Chinese mathematics in its own right.

Now let us discuss the history of Indian and Chinese mathematics. We shall separate the discussion into three periods of time:

### Indian and Chinese Mathematics

#### I. The years before 500 AD

In India, we should start with the Vedic literature: the Samhitas (1000 BC) the Brahmanas (800 BC) the Aranyakas (700 BC) and the Upanishads (600-500 BC). These are collections of hymns, prayers, sacrificial and magical formulas. Rituals and sacrifices are important in ancient India. These texts were commonly learned by rote memory and transmitted orally from one generation to the next. The important ritual literature included *Srautasutras*, which described ways to construct sacrificial fires at different times of the year. It specifies the measurement and construction of sacrificial altars, see Figure 4. (The ritual literature are called *Sulbasutras*.) As was pointed out by A. Seidenberg, these sutras contain information of geometry where Pythagorean theorem can be found.

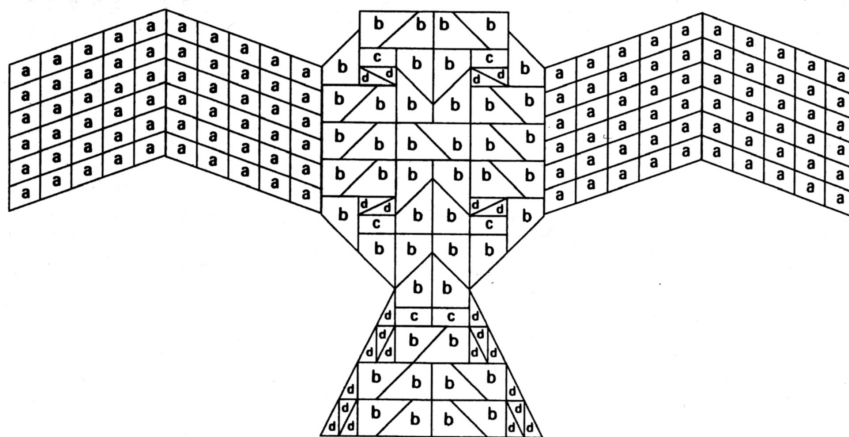


FIGURE 4. The first layer of a Vedic sacrificial altar in the shape of a falcon; the wings are each made from 60 bricks of type a, and the body from 46 type b, 6 type c, and 24 type d bricks (from G.G. Joseph, *The Crest of the Peacock*).

The three most important ones were recorded by Baudhayana, Apastamba and Katyayana. The one recorded by Baudhayana is the earliest, around

500 BC. This manuscript contains a general statement of the Pythagorean theorem, an approximate procedure for obtaining the square root of 2 accurate to five decimal places and squaring the circle (apparently he got the value of  $\pi$  to be 3.09), and also constructing rectilinear shapes whose area was equal to the sum or difference of areas of other shapes. Baudhayana was followed by Apastamba and Katyayana in the next few centuries. See Figure 5 for usage of Pythagorean triples in Apastamba's Sulbasutra. Babylonian mathematicians also discussed questions on the Pythagorean theorem and their triples, the square root of two, found the value of  $\pi$  to be 3.125 (as appeared in the Susa Tablet, 1600 BC). However, it is difficult to claim that the Sulbasutras derived their results from the Babylonians as there is no trace of the sexagesimal system in ancient India mathematics. As was mentioned earlier, there are also appendices to the main Vedas, where the ancient Indian paid a lot of attention to a form of writing that aimed at utmost brevity and used a poetic style to memorize literature. It may have influenced the later developments of axioms in Euclidean geometry.

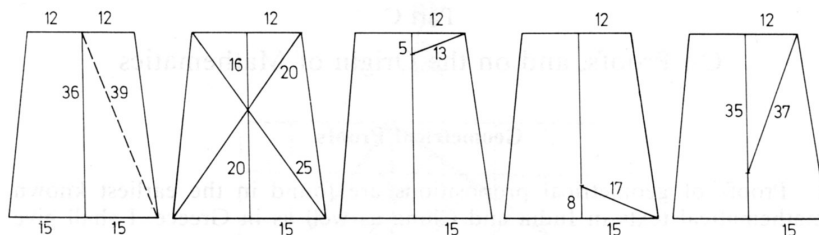


FIGURE 5. Trapezoid altars in the Apastamba Sulbasutra (from B. van der Waerden, *Geometry and Algebra in Ancient Civilizations*).

The first number system of India is the Kharothi around 400 BC to 200 AD, where the number “9” is not known to appear. The second system is called the Brahmi system which is more advanced. It appeared in the caves of Central India and is dated around 150 BC. The system eventually evolved into the Bakhshali (200-400 AD) and the Gwalior system (850 AD). They used the numbers 1 to 9 with zero. The place value principle for the number zero had to wait until it appeared in the works of the astronomer Varahamihira (587AD). This completed number system later appeared in the Arabian literature and is the current system we use today.

After the period of Vedic literature saw the rise of Buddhism and Jainism. The latter made contributions and studied mathematics for its own sake. The Jains realized different form of infinities and studied sequences.

(They were interested in cosmological structures that contained innumerable concentric rings. The diameter of each ring is twice that of the previous one.)

As we have mentioned, the records of ancient China before year 1000 BC is probably unreliable, with exceptions for those found on carvings on tortoise shells and on bronze vases. But there is no doubt that the most famous old book that is related to mathematics is the *I-Ching: Book of Changes*. The book is said to be written by Fu Xi in 2850 BC. But that is most likely wrong. It is not even clear that Fu Xi, the first emperor of China ever existed. It was quite probably written by Zhou Wen Wang around 1000 BC and was continuously improved. At any rate, Confucius did read this book and considered it a book of great virtue. In the book, some symbols were used and the concept of yin and yang was created. And there are also *sixiang* (four figures) and *bagua* (eight trigrams). From there, it was extended to sixty-four hexagrams. The mysticism of numbers is great in the book as it has been used to this day by many to explain almost everything in our daily life. Perhaps this is similar to the Pythagorean doctrine with respect to numbers.

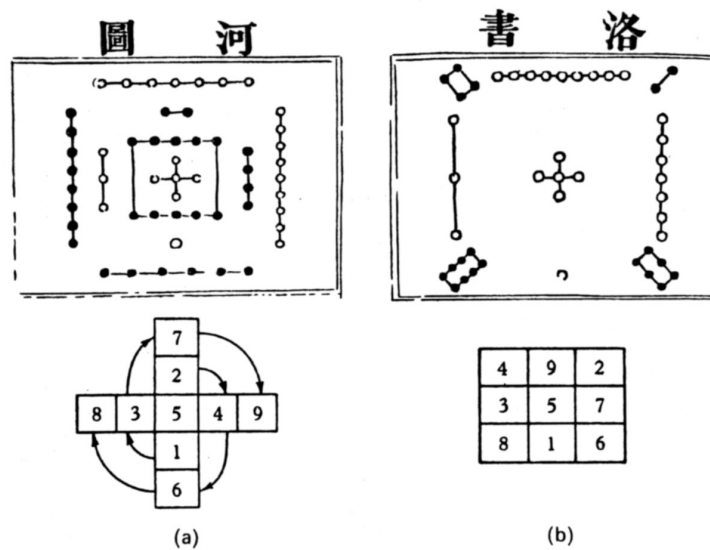


FIGURE 6. (a) Ho Thu and (b) Lo Shu (from G.G. Joseph, *The Crest of the Peacock*).

Magic squares also appeared in ancient China. A legend says that around 2000 BC, Emperor Yu acquired two diagrams: the first called Ho Thu (river chart) from a magical dragon horse which rose from the Yellow River and



the second diagram called Lo Shu (Lo River writing) copied from the design on the back of a sacred turtle found in Lo, a tributary of the Yellow River. See Figure 6. Lo Shu has the form of a three by three magic square. Ho Thu was mentioned by Confucius. But one has to be careful whether the two diagrams in Figure 6 were exactly those referred to by the *I-Ching* and what Confucius had in mind. Apparently, diagram (b) of Figure 6 was called the *Nine Halls* diagram during the period from the Qin Dynasty to the Han Dynasty. It was attributed to be the Lo Shu during the Sung Dynasty by Zhu Xi. In any case, Lo Shu is certainly the earliest recorded magic square.

The diagrams represented an important principle of Chinese philosophy: yin (female) and yang (male) in nature. The diagrams appeared in the Nine Halls of the Cosmic Temple, the Ming Tang. In 1275 AD, Yang Hui wrote in the book, *Continuation of Ancient Mathematical Methods for Elucidating the Strange Properties of Numbers*, that he has collected many magic squares constructed by the ancient Chinese. He also gave properties of the squares and classified the squares up to order 10. In 1880, Bao Qishou constructed magic cubes, spheres and tetrahedrons. Even now, some Chinese scholars are still fascinated in numerology and number mysticism related to *I-Ching*.

*I-Ching* is deeply influential in the intellectual circles through the teaching of Confucius. Its system of base 60 is used by many Chinese scholars even today. However, for more practical issues such as trading or bookkeeping, the decimal system was used in a large scale by 250 BC and perhaps even much earlier. (It had been found on the tortoise shells that dates back to 1200 BC.) In fact, some arithmetic instrument were used. (Archeologist had dug up bamboo or ivory sticks that were used for arithmetics as early as the Qin Dynasty, around 220 BC. See Figure 7). The ancient Chinese decimal system, see Figures 8 and 9, is interesting in that it operated without the “zero” number but yet it was still very effective and not ambiguous. This is in contrast to the number system without zero that created ambiguity in the Babylonian literature. (A simple form of zero did appear in the later Babylonian tablets of the Seleucid period.) The invention of “zero” by the Indian around 700 AD is an important contribution to mathematics. On the other hand, the Chinese bamboo sticks can be considered as the world’s oldest simple computing instrument.

After the *I-Ching*, there was *Zhou Bi Suan Jing*. This book’s written date has been much debated for a long time. It contains a conversation of Duke Zhou Gong and his minister Shang Kao around 1000 BC. (The conversation most likely never occurred. It may be an excuse for the author to give some importance for the mathematics in the book.) The Pythagorean triple (3,4,5) was mentioned in this book. This book was supposed to consist of government documents on how to use mathematics to understand

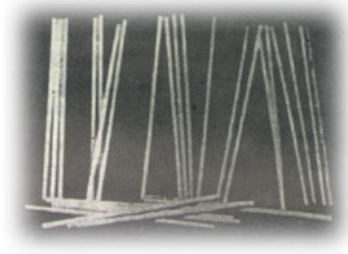


FIGURE 7. Ancient Chinese ivory calculating sticks (from Shanghai-Bavaria virtual campus telecenter).

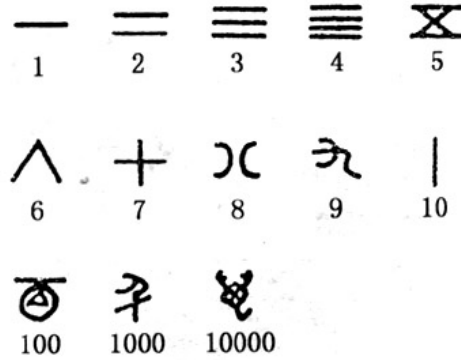


FIGURE 8. Thirteen number inscriptions found on Chinese oracle bones (from Shanghai-Bavaria virtual campus telecenter).

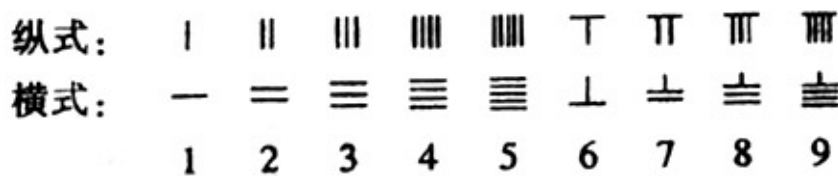


FIGURE 9. Ancient Chinese methods of denoting numbers: vertical (top) and horizontal (bottom) (from Shanghai-Bavaria virtual campus telecenter).

astronomical events. It was collected together to form a book around 235 BC to 156 BC. The editor was quite likely to be Zhang Chang around 165

BC. Between 190-220 AD, a famous mathematician called Zhao Shuang made detailed commentary on *Zhou Bi* and added in some original ideas, including the proof of the Pythagorean theorem as seen in Figure 10. (The general statement of the Pythagorean theorem was known much earlier as was described in *Zhou Bi*.)

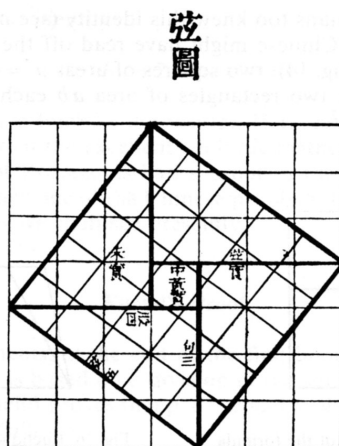


FIGURE 10. The proof of the Pythagoras Theorem in *Zhou Bi Suan Jing* (from B. van der Waerden, *Geometry and Algebra in Ancient Civilizations*).

The most important classic of Chinese mathematics is the *Nine Chapters on the Mathematical Art*. This book also consists of collections of government documents on mathematics related to events on earth. Nobody can be sure how old this book is. But it is fair to say that the book was also edited by Zhang Chang around 200 BC. Being a high-level government officer, it is likely that he made extensive use of ancient government documents, all the way back to Zhou Gong, about 1000 BC. In this book, there are questions asked on various topics in mathematics: calculations of volume, area, equations; approximations of square root and cubic root; and the Pythagorean theorem and its triples. The Euclidean algorithm was used extensively and the Gaussian elimination for matrices was introduced. (The Chinese calculations using bamboo sticks led naturally to matrices.) Answers to the questions raised in the book could had led to generalizations. But no general statements were stated.

Around 1984, a reasonably complete manuscript called the *Book of Mathematical Art* written on bamboo sticks (about 190 sticks altogether) was excavated from an old tomb of the Han Dynasty in Hubei Province. The

tomb was buried around 186 BC and therefore the book should have been completed before this date. There are many overlaps in the contents between this book and the *Nine Chapters*, though it is not as advanced. One may speculate that the editor of the *Nine Chapters* collected many ideas from these old manuscripts.

Several spectacular works can be found in the third century AD after the *Nine Chapters*. Liu Hui (208-263 AD) was the one who wrote a great detailed commentary on the *Nine Chapters*. His commentary exhibited a great deal of his own works. A rigorous proof of the Pythagorean theorem in the Euclidean manner was given. (Liu Hui did not actually draw pictures of his proof. Hence, modern understandings were needed to interpret this classical Chinese writing.) He computed  $\pi$  up to four decimals using a 192 sides regular polygon approximation of the circle. He discovered the principle that the volumes of two solids are the same if the cross sections with the same height has equal area. Extensive discussions of sub-dividing a solid into smaller and smaller prisms can be found. He also developed further the concept of a limit and gave geometric interpretations of finding the cubic root of a number. His computation of  $\pi$  was followed in 480 AD by a computation of Zu Chongzhi who computed  $\pi$  up to 7 digits using 24576 polygons.

The need of refining the calendar drove the Chinese to look into equations of undetermined type (of first degree). It was first written about by Sun Zi, in *Sun Zi Suan Jing*, where the concept of congruence was introduced and the Chinese remainder theorem was discovered. According to the investigations of Qian Bao Zong, this book was written around 400 AD, although there are suggestions that it was written around 300 AD.

The research was generalized to a system of linear equations of undetermined type in the Song Dynasty. The work of Sun Zi depended on the understanding of the Euclidean algorithm which was written in the *Nine Chapters*, and most likely went back to the ancient days of Chinese arithmetic.

Zu Chongzhi (429-500 AD) and his son are two remarkable mathematicians of this period. He devoted a great deal of his time to astronomical calculations. Besides the above-mentioned precise calculations of  $\pi$ , he was able to compute the volume of the sphere and solve any complicated system of undetermined linear equations of 12 variables. His son, Zu Gengzhi, found the formula for the volume of the sphere.

## II. The period between 500 AD and 1450 AD

There was a manuscript dug up in Bakhshali in 1881. It was written in the old form of Sanskrit. The date of this manuscript is still undetermined, presumably from around 500 AD, but it is the earliest evidence that

Indian mathematics started to be free of any religious and metaphysical associations. It has many resemblance with the Chinese *Nine Chapters*, including the style of writing. But the content is not as advanced as the *Nine Chapters*.

The first major mathematician-astronomer of India was Aryabhata I, who wrote the book *Aryabhatiya*. The work was around 500 AD. The work was done in Kusumapura, the imperial capital of the Gupta Empire. It was recorded by Umasvati (in 200 AD) that this is the place where a famous school of mathematics and astronomy was formed. The *Aryabhatiya* is concise and is essentially a systemization of the results contained in the older Siddhantas. The mathematics section of this book contains only 33 verses. Here we find the rules for arithmetic operations, simple and quadratic equations, and indeterminate equations of the first degree. Sine and cosine are introduced here. Aryabhata knew the finite difference form of the differential equation for sine and also computed  $\pi$  to be 3.1416. While trigonometry owed its foundation to Hipparchus (150 BC), Menelaus (100 AD) and Ptolemy (150 AD), Indian trigonometry began to take shape in the modern form during this period (partially due to the interest of astronomy). Important identities of the trigonometric functions were found by Aryabhata I, Varahamihira, Brahmagupta and Bhaskaracharya. Especially central to this development was Bhaskaracharya, who also studied indeterminate equations of second degree: Pell's equation. Based on the work of the astronomer Manjula (930 AD), Bhaskaracharya derived the differential of the sine function.

In the south, Kerala was the place where early form of calculus was developed. In 850 AD, Govindaswami produced a rule for second order interpolation to compute intermediate functional values. A particular case of the Newton's interpolation formula was known to them. Taylor series up to the second order term was also known to them. The important mathematician in this area was Madhava (1400 AD) who discovered power series for  $\pi$ , for sine, cosine and arc tangent functions. (Though the original sources have not been found, the discoveries were referred to by later writers.)

It appears that Indian mathematicians started the subject of mathematical analysis long before Newton and other Western workers. While the knowledge of differential calculus is impressive and perhaps had influenced Western works a few hundred years later, it is not as complete. Integral and differential calculus were not combined as well as by later workers. But nevertheless, the Indian advances during this period are truly impressive.

In the Sui Dynasty of China, there were records of Indian astronomical and mathematical texts, such as the *Brahmam Suan Fa* (the Brahman arithmetical rules) and *Brahman Suan Jing* (Brahman arithmetical classic).

The records of the Tang Dynasty contain the names of Indian astronomers. One of the Indians, whose Chinese name was Chu Tan Hsia Ta (Gautama Siddharta), was reputed to have constructed a calendar, based on the Indian Siddhantas, by order of the first emperor of the Tang Dynasty (in 718 AD). The text contains Indian numerals and operations and sine tables. Yabbuchi (1954) located a surviving block-print text which contains Indian numerals, including the use of a dot to indicate zero. There is also a sine table at intervals of 3 degree 45 arc for a radius of 3438 units, which are the values given in the Indian astronomical texts *Aryabhatiya* and *Surya Siddhanta*.

In both the Sui and the Tang Dynasties, the government appointed scholars to study mathematics, and for most part, mathematics was part of the civil service examinations given by the government. There were ten books on mathematics that were tested on. The rank of the Officer in Mathematics was the lowest in the government. Of these mathematicians, Wang Xiaotong (around 620 AD) was the most famous one. He transformed the problem of geometry into algebra. He was the first to study cubic equations which he claimed to be able to solve.

As was mentioned earlier, Admiral Zheng He brought more than twenty thousand people with him to travel to the Indian Ocean seven times during the period of 1405 to 1433. His envoy certainly went to Kerala, in southwest India, where Madhava and other Indian mathematicians were developing early forms of differential calculus. One naturally wonder whether there were any mathematical exchanges between the Chinese and Indians then?

It is a curiosity that despite the official examinations on mathematics, the Tang Dynasty did not produce as many good mathematicians as we would expect. I believe that the Chinese system of examinations discouraged innovative thinking and that the low rank of the officers in charge of the mathematics examinations could hardly measure up to the aspirations of young scholars of that period.

Building on the *Nine Chapters*, the Song Dynasty produced quite a few outstanding mathematicians. Qin Jiushao (1250) numerically solved equations of high order based on the use of what we call the Pascal's triangle method for the extraction of roots. The discussion of the Pascal triangle was treated by Yang Hui earlier. Yang Hui reported that this work was due to Jia Xian in 1050. Qin Jiushao also extended the Chinese remainder theorem to more variables. Another mathematician named Guo Shoujing (1275) derived the cubic interpolation formula which now go by the name of Newton-Sterling formula. And Zhu Shijie (1300) invented the equivalence of resultants, yet another high point of Chinese mathematics.

The above mentioned method of solving high degree equations is now called Horner's method. It dates back to the *Nine Chapters*. The method of extracting cubic root appeared in the Arabian mathematician al-Nasawi (1025). Later al-Kashi (1450) extracted roots of higher degree.

### *III. The period between 1450 and 1900*

It is still a mystery that both Indian and Chinese mathematics did not develop much during this period of time. Any progress made certainly was nowhere comparable to that made in the Western world during this period.

During this time, Europeans enjoyed tremendous economic growth and academic freedom compared with the Asians. Their pursuit of truth and beauty for its own sake, was not something that the Asians can compare to. A lot more study is needed to uncover the reason behind our decline during this period.

The mathematical developments after the Song Dynasty declined. By 1582, the Italian Jesuit Ricci came to China and translated the first six books of Euclid's *Elements*. A Chinese mathematician by the name Xu Guangqi helped Ricci with the translation. Xu expressed his amazement on the logical structure appearing in Euclid's book. It was unfortunate that Euclid's powerful deductive method was never developed in China until the last century. (In fact, the translation of the last chapters of Euclid's *Elements* was only completed in the 19th century by a Chinese mathematician called Li Shanlan.) The Jesuits may have brought back to Europe some of the mathematics of China. It may be interesting to note that the arrival of Ricci occurred roughly 100 years before the publishing of Newton's *Principia* in 1687. By then, the logical structure and the deductive power of Euclid's methods already had a deep influence in the Western world. Not only did they help clarify many puzzles, they also opened up new horizons in mathematics and physics. Such influence did not even begin to take root in Asia until almost two centuries later.

The Qing Dynasty mathematicians are not as original as those of the Han or Song Dynasty. And despite the efforts of the great emperor Kangxi of the Qing Dynasty, who was extremely interested in mathematics and hired several missionaries in his court, the glory of Chinese mathematics did not continue. Perhaps this exemplified the academic freedom in both the Han and the Song Dynasties. On the other hand, according to the 20th century historian, Chen Yinke, the Qing's court used a version of Euclid's *Elements* which omitted proofs, demonstrating a preference for the more practical side of Euclidean geometry. Besides that, the examination system during the Qing Dynasty did not pay enough attention to mathematics (while that of Song and Tang placed more emphasis). The overemphasis of practical use of mathematics finally showed its weakness. The overwhelming success

of mathematical development in Western world which combined the power of both abstract and practical application is so great that it took a long time for the Asians to catch up.

Among the three most famous ancient problems in mathematics were

1. the parallel postulate
2. squaring the circle
3. the trisection of an angle

The Chinese and Indian mathematics only touched the second one. The first one led to the introduction of hyperbolic geometry while the second one is related to Galois theory. Both of them have had a great impact on modern mathematics.

#### *IV. 1900 till today*

Both Indians and Chinese mathematics started to recover.

In modern India, the most original mathematician was no doubt Ramanujan, who changed our views on modular forms. His sheer mathematical power was tremendous and many of his ideas still need to be further explored even today. Ramanujan visited England and learned a lot from Hardy, who was able to explain some of Ramanujan's ideas. Ramanujan made decisive contributions to number theory at that time. The tradition of number theory was carried on by S.S. Pillai on the Waring's problem, by S. Chowla and by K. Chandrasekharan.

There are many Indians who left India to learn from the Western world. The great mathematician Harish-Chandra who went to England to study with Dirac, developed on his own the foundations for the theory of representation of noncompact groups. It is difficult to measure his deep influence in analysis and number theory. There are also many other Indian mathematicians who played very important roles in modern mathematics. The works of S. Minakshisundaram and Pleijel (Canadian) on eigenvalues of Laplacian through the heat kernel expansion is spectacular: a subject that Herman Weyl thought highly of. When Patodi further developed this subject, it finally led to a new analytic local form of the Atiyah-Singer index theorem which has had a tremendous influence in modern works. The contributions made by Indian mathematicians Raghunathan, Narasimhan and Seshadri on discrete groups and on vector bundles over algebraic curve were also fundamental.

Many outstanding Indian mathematicians have stayed in the West and became leaders in their respective fields. This list includes Abyankher (algebraic geometry), Chandrasekharan (number theory), Narasimhan (several complex variable), Kulkarni (geometry), Nori (algebraic geometry), Pandharipande (algebraic geometry), Parimala (algebra), Prasad (discrete group



theory), Varadarajin (representation theory) and Varadhan (probability and analysis). Chandrasekharan had great influence in the International Mathematics Union where he was the president for one term and secretary for two terms. Varadhan was awarded the prestigious Abel Prize. We are also seeing the growth of the second generation of mathematicians of Indian ancestry (born in the West). The works of young mathematicians such as Manjul Bhargava and Chandrashekar Khare, both on number theory, Ravi Vakil on Gromov-Witten invariants and that of (IIT Kanpur's) Manindra Agrawal, Neeraj Kayal and Nitin Saxena on an algorithm for testing prime numbers have received much international recognition recently.

The founders of Indian statistics are P. C. Mahalanobis and P. V. Sukhatme. The latter got his Ph. D. from the University of London in 1936. He settled in Delhi and formed a group of statisticians in the Indian Council for Agricultural Research. Mahalanobis founded the Statistical Institute in 1931 and also *Sankha, the India Journal of Statistics* in 1933. By the late 1950s, the Indian Statistical Institute became a center of national importance. Their associates such as R. C. Bose, C. R. Rao and S. N. Roy became prominent statisticians and are world famous. Nowadays, statistics plays an influential role in Indian society and government: there are the Ministry of Statistics and Programme Implementation, the Indian Statistical Service, the National Sample Survey Organization, the Indian Agricultural Statistics Research Institute, the Institute for Research in Medical Statistics, *etc.*

In the last hundred year of Chinese history, we saw the rise of several stars in mathematics: Shiing-Shen Chern, Loo-Keng Hua and Wei-Liang Chow. Chern obtained his bachelor's degree studying under Lifu Jiang at Nankai University and his master's under Guangyuan Sun at Tsinghua University. Hua was self-educated and was discovered by King-Lai Hiong and Wuzhi Yang, the leading professors at Tsinghua University. (Yang was the first Chinese mathematician to earn a PhD in number theory, from Chicago.) Hua was offered to work in the library as he had no high school degree. Both Chern and Hua were in Tsinghua at the same time around 1930. Besides the influences of Hiong, Sun and Yang at Tsinghua, there were important visits to China by some great professors from Europe and America in the 1930s. They included Norbert Wiener, Osgood, Blaschke and Hadamard. Chern studied with Blaschke in Hamburg and later went to Paris to study with Cartan. It may be interesting to note that Chern was offered a job at the Tata Institute in 1950 which he declined. Hua went to Cambridge to study with Hardy and later learned a lot from the Russian number theorist Vinogradov. Chow also went to Germany in 1932 and studied under van der Waerden. He returned to China and became a businessman before leaving for Princeton in 1947 to study with Lefschetz.

His great works were done in Princeton. These three great mathematicians were in China during the war with Japan.

A great number of brilliant students were trained by them. Chern helped his teacher run the Mathematics Institute of the Academia Sinica until 1949, when he left for America. During this period, he trained many first class students on topics related to geometry and modern topology. He had just finished his fundamental work on Chern classes and Gauss-Bonnet theorem in Princeton (1944). Hua was sent by the government to America to learn the secrets of the atomic bomb, which he never did. But when he returned to China in 1950, he did initiate several important branches of mathematics. Most notable were several complex variables and analytic number theory, where he trained a group of students on the theory of classical domains, the Waring's problem, and the Goldbach problem. In the latter problem, the Chinese gained the world's attention when Jingrun Chen proved that every sufficiently large even number can be written as either the sum of two primes or a prime and the product of two primes. Chow left China right after the war. His great insights into algebraic geometry have influenced the subject deeply. Unfortunately he did not train many students. There were other significant works in complex analysis and in Fourier analysis. But none of the achievements compared with the above mentioned figures.

On the other hand, the Chinese are very strong in the developments of applied mathematics. The leading figures are Chia-Chiao Lin, Yuan-Cheng Fung, Theodore Yao-Tsu Wu, Kang Feng and Pao-Lu Hsu. Lin did pioneering works on the density dynamics of galaxies. Fung pioneered bio-fluid dynamics and Wu contributed in a fundamental way to ocean fluids. Kang Feng is one of the founders of the finite element method which formed the foundation of numerical computations. And Pao-Lu Hsu is one of the founders of modern probability and statistics. He made fundamental contributions to the subject of probability and statistics such as partially balanced incomplete block designs and the asymptotic theory of empirical distributions. He is the role model for many outstanding Chinese statisticians that settled in America. Perhaps his return to China in 1947 has inspired some of these statisticians to contribute to Chinese society.

In the past century, Chinese mathematics had gone through three different stages of developments. The first is around the time of late 1920s to 1940s - when the first group of Ph.D. students came back home to teach, great mathematicians came to visit China, and new textbooks from Europe and England were used. When Chern and Hua came back in the mid-1930s, geometry, algebra and number theory started to take roots. There were also developments of complex analysis by King-Lai Hiong, who studied in

France, Fourier analysis by Jiangong Chen who was educated in Japan, differential geometry by Buqing Su who was trained in Japan and algebra by Chiungtze C. Tsen who was trained in Germany. During this period, many young people were trained by this group of leaders. Among them were Kuo-Tsai Chen, Kai Lai Chung, Ky Fan, Chuan-Chih Hsiung, Zhao Ke, Hsien-Chung Wang, Zhikun Wang, Wen-Tsun Wu, Zhida Yan and Chung-Tao Yang. Besides Ke, Wang, Wu and Yan, the rest left China. Hsiung is the founder of the *Journal of Differential Geometry*.

The second period was the time when Hua went back to China in 1950 to head its mathematical development. Many young mathematicians were trained by him, and also by Chen, Hiong and Su. While this group of mathematicians focused on pure mathematics, the prominent numerical analyst, Kang Feng, initiated the subject of computations in China. The great statistician, Pao-Lu Hsu, who had returned in the forties also led China to the forefront of statistics research. Some of the other most notable people of this period are Jingrun Chen, Xiayi Ding, Sheng Gong, Boling Guo, Chao-hao Gu, Jiaying Hong, Hesheng Hu, Shantao Liao, Qikeng Lu, Chendong Pan, Zhong-Ci Shi, Zhexian Wan, Guangyin Wang, Yuan Wang, Daoxing Xia, Lo Yang, Guanghou Zhang, Jiaqing Zhong and Yulin Zhou. This group of younger generation mathematicians did outstanding works in the early sixties and thereafter. Somewhat later, many outstanding Chinese mathematicians were produced in Hong Kong and Taiwan, such as Ching-Li Chai, Tony Chan, Gerald Chang, Sun-Yung Alice Chang, Mei-Chu Chang, Shiu-Yuen Cheng, I-Liang Chern, Fan Chung Graham, Wu-Chung Hsiang, Ming-chang Kang, Tze Leung Lai, Tsit-Yuen Lam, Peter Li, Wen-Ching Winnie Li, Chang-Shou Lin, Song-Sun Lin, Tai-Ping Liu, Chiang C. Mei, Ngaiming Mok, Tzuong-Tsieng Moh, Wei-Ming Ni, Yum-Tong Siu, Yung-sheng Tai, Luen-Fai Tam, Chuu-Lian Terng, Wing Hung Wong, Chien-Fu Jeff Wu, Hung-Hsi Wu, Andrew Chi-Chih Yao, Horng-Tzer Yau, Stephen Shing-Toung Yau, Lai-Sang Young and Jin Yu. Most of them were trained in graduate schools in America and many stayed in the US.

The third period was the time after the Cultural Revolution till today. Many young students were trained in American graduate schools and many of them have remained in the US. A good sample can be found in those who received prestigious awards: Jin-Yi Cai, Raymond Chan, Chiun-Chuan Chen, Chong-Qing Cheng, Ding-Zhu Du, Weinan E, Jianqing Fan, Pengfei Guan, Lei Guo, Thomas Yizhao Hou, Lizhen Ji, Shi Jin, Naichung Conan Leung, Jun Li, Bong Lian, Fanghua Lin, Xihong Lin, Ai-Ko Liu, Chiu-Chu Melissa Liu, Jun Liu, Kefeng Liu, Feng Luo, Xiaoli Meng, Sheng-Li Tan, Gang Tian, Daqing Wan, Chin-Lung Wang, Mu-Tao Wang, Xu-Jia Wang, Sijue Wu, Nanhua Xi, Jie Xiao, Zhou-Ping Xin, Zhiliang Ying, Guoliang

Yu, Jiu-Kang Yu, Shou-Wu Zhang, Xiangyu Zhou, Xin Zhou and Xiping Zhu.

The economic situation of China has improved tremendously in the past twenty years. It is expected that many brilliant Chinese mathematicians will go back in the future. This shall make a tremendous difference for the future of Chinese mathematics. As a notable example, about three years ago, Andrew Chi-Chih Yao, the first Chinese recipient of the Turing Award, resigned his job from Princeton University and settled down at Tsinghua University. This is significant as it demonstrates the desire of overseas Chinese to help and participate in the development of mathematics in China.

Overall, the overemphasis of Chinese mathematics on practical applications in its history has been an obstacle for the advancement for core mathematics, which in turn is the major obstacle for the advancement of applied mathematics. If we look at the past history of the development of mathematics in China, it is amazing that the Chinese placed so much emphases on the development of their forefathers as compared with creating innovative ideas for the new frontier of science. They are much less adventurous compared with our Western colleagues. A good example is the outstanding mathematician, Liu Hui, and later mathematicians of the Song Dynasty. They spent most of their time understanding the ancient book of the *Nine Chapters*. They made a great deal of advances in mathematics that way. But it is different from developing new ideas to understand the beauty of nature and creating new subjects. On the other hand, Sun Zi's development of the Chinese Remainder Theorem is original and spectacular.

### **Concluding Remarks**

Historical events have shown that when Indian and Chinese societies are open and interacting with foreign countries, outstanding mathematical developments took place. When the countries closed their borders or looked inward, the mathematical developments were quite different and often with no significant progress. A closed society not only loses the stimulation of fresh new ideas from the outside, but its scholars also tend to lose their objectivity in their judgements. Clearly cultural and economic exchanges are very important for mathematical advancement.

For the last couple of years, both India and China have been experiencing double-digit economic growth. Unless some unforeseen events arise, large-scale conflicts with other countries seem unlikely (and are of course, highly undesirable by all people in the region.) In a time of peace and with a reasonable accumulation of wealth, the governments of both countries shall continue to invest in mathematics. Highly talented mathematicians in India and in China will no doubt work toward making important contributions

for the good of mankind. Many prominent mathematicians will also come back and settle down in both countries. We have already seen many of our friends from other countries spending their precious time to visit us and deliver important lectures. In a short period of time, we will see our subject blossom again in our fertile land.

The great achievements of Ramanujan and Harish-Chandra have inspired generations of Indian mathematicians to work on both number theory and representation theory. The visits of Weil, Borel, Siegel, Samuel, Eilenberg, Mumford and Deligne have encouraged development of number theory, discrete group theory and algebraic geometry in India. Chern was an important role model and his works inspired much work in geometry and topology in China. Hua's work on analytic number theory and several complex variables have now established a "tradition" in Chinese mathematics. Chow's work is fundamental in algebraic geometry, but the impact of his work in China is not as clear as compared with that Chern and Hua, perhaps because Chow resided in America for most of his active years.

I am sure that a close collaboration between Indian and Chinese mathematicians will lead to great advancements in modern mathematics. Without regards to national boundaries, we can share our role models and communicate with each other, treating students from the other country like one's own.

In ancient days, the pursuit of truth and beauty was perhaps more focused on issues relating to those of agriculture, building constructions, ritual worship, music and astronomy. We expect that the globalization of economies will also promote the unification of different areas of mathematics. In fact, the unification of mathematics is a trend of modern mathematics. In the past century, great achievements in mathematics often spanned different disciplines of mathematics, physics and engineering. These include the works of Cartan, Hodge, Weyl, Lefschetz, Weil, Chern, Morse, Deligne, Hirzebruch, Atiyah, Singer, Langlands, Wiles, and more recently, the proof of the Poincaré conjecture which used geometric analysis, a subject that is not so familiar for classical topologists.

There are an abundance of mathematical developments that are influenced by the desire for a unifying theory in fundamental physics. String theory has inspired many different structures in mathematics to be merged together as one in a natural manner. This whole subject is rather complicated as it involves many different branches of mathematics and physics. I believe the unification of different disciplines of mathematics will occur in the future and that this will require the efforts of many, many mathematicians. Besides interactions with fundamental theories of physics, questions with practical applications have also inspired important developments in

mathematics. This includes fluid dynamics, numerical calculations, theory of complexity, graph theory, *etc.* I am sure our youth in India and China will contribute to this great endeavor and discover the beauty that we all love, that is mathematics.

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## HOMOGENIZATION

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Homogenization deals with the problem of approximating solutions of differential equations with rapidly oscillating coefficients by solutions of similar equations with constant coefficients. Behavior of many physical systems are described by differential equations. Substances, such as composite materials are often put together by arranging different materials. Sometimes the arrangements are periodic as in crystals but can often be random as in amorphous materials.

### 1. First order equations

Let us consider two very simple examples. If  $f(x)$  is a periodic function of period 1, and  $\epsilon > 0$  is small, then  $f(\frac{x}{\epsilon})$  is a rapidly oscillating function. We can consider the differential equation

$$\frac{dy}{dx} = f\left(\frac{x}{\epsilon}\right); y(0) = 0$$

whose solution is

$$y(x) = \int_0^x f\left(\frac{t}{\epsilon}\right) dt = \epsilon \int_0^{\frac{x}{\epsilon}} f(t) dt$$

As  $\epsilon \rightarrow 0$  the limit exists and equals  $y(x) = \bar{f}x$ , where

$$\bar{f} = \int_0^1 f(t) dt$$

On the other hand if the equation is

$$\frac{dy}{dx} = f\left(\frac{y}{\epsilon}\right); y(0) = 0$$

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rewriting the equation as

$$\frac{dx}{dy} = \frac{1}{f(\frac{y}{\epsilon})}$$

and assuming that  $f$  is positive and bounded away from 0, the limit satisfies  $x(y) = [\int_0^1 \frac{dx}{f(x)}] x$  or  $y(x) = \underline{f} x$ ,  $\underline{f}$  being the harmonic mean. It is far from clear what happens when we consider

$$\frac{dy}{dx} = F(\frac{x}{\epsilon}, \frac{y}{\epsilon})$$

with  $F$  that is periodic in  $x$  and  $y$ .

## 2. Linear second order partial differential equations

The solution  $u^\epsilon$  of the parabolic partial differential equation

$$u_t^\epsilon(t, x) = \frac{1}{2} a(\frac{x}{\epsilon}) u_{xx}^\epsilon(t, x) ; u^\epsilon(0, x) = f(x)$$

where  $a(x) > 0$  is periodic of period 1 converges as  $\epsilon \rightarrow 0$  to the solution  $u(t, x)$  of

$$u_t = \frac{1}{2} \bar{a} u_{xx} ; u(0, x) = f(x)$$

where the constant  $\bar{a}$  is the harmonic mean

$$\bar{a} = \left[ \int_0^1 \frac{dx}{a(x)} \right]^{-1}$$

We have diffusion process,  $x(t)$  that corresponds to the generator  $\frac{1}{2} a(x) D_{xx}$ . This process is martingale and

$$E[(x(t) - x(0))^2] = E\left[ \int_0^t a(x(s)) ds \right]$$

The scaling limit is essentially a central limit theorem for  $\frac{x(t) - x(0)}{\sqrt{t}}$ . This depends on a law of large numbers for the variance

$$\bar{a} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t a(x(s)) ds$$

Since  $a(x)$  is periodic we can consider the diffusion on the circle, i.e. the periodized line with 0 and 1 identified. The diffusion on the circle has the invariant measure

$$\frac{c}{a(x)} dx$$

where  $c$  is the normalizing constant  $\frac{1}{\int_0^1 \frac{1}{a(x)} dx}$ . The ergodic theorem tells that

$$\bar{a} = \int_0^1 \frac{c}{a(x)} a(x) dx = c.$$

See [7] for details of some early work that deals with similar one dimensional cases. Let us go on to consider some higher dimensional examples. These



problems were first considered in [1] and [2]. Solutions of equations  $u_t^\epsilon = \mathcal{L}^\epsilon u^\epsilon$  where  $\mathcal{L}^\epsilon$  is the operator in divergence form on  $R^d$ ,

$$\mathcal{L}^\epsilon = \frac{1}{2} \nabla \cdot a\left(\frac{x}{\epsilon}\right) \nabla$$

converge to solutions of the limiting equation  $u_t = \mathcal{L}u$  with

$$\mathcal{L} = \frac{1}{2} \sum_{i,j} \bar{a}_{i,j} D_{x_i} D_{x_j}$$

with  $\bar{a} = \bar{a}_{i,j}$  determined by

$$(2.1) \quad \langle \bar{a} \theta, \theta \rangle = \inf_u \int \langle a(x)(\nabla u - \theta), (\nabla u - \theta) \rangle dx$$

The variation is done on the torus  $\mathcal{T}^d$ , with  $u$  varying over periodic functions. What is really being approximated is the gradient of the linear functional  $\langle \theta, x \rangle$ , by the gradient  $\nabla u$  of a periodic function. On the other hand, solutions of

$$\mathcal{L}^\epsilon = \frac{1}{2} \sum_{i,j} a_{i,j}\left(\frac{x}{\epsilon}\right) D_{x_i} D_{x_j}$$

converge to a solution of  $u_t = \mathcal{L}u$  with  $\bar{a}$  now defined by

$$\bar{a} = \int_{\mathcal{T}^d} a(x) \Phi(x) dx$$

The invariant density  $\Phi(x)$  is the solution of the adjoint equation on the torus

$$\sum_{i,j} [a_{i,j}(x) \Phi(x)]_{x_i, x_j} = 0$$

normalized so that  $\int_{\mathcal{T}^d} \Phi(x) dx = 1$ .

### 3. Random Coefficients

Periodic coefficients vary regularly on  $R$  or  $R^d$ . Translating by a period does not change the function. If we replace the periodic function by a stationary random function, we then have an object that is irregular, but still regular in a statistical sense. Averages exist by the ergodic theorem and one can ask what the corresponding homogenization results are. The standard model for defining a stationary process is to start with a probability space  $(\Omega, \Sigma, P)$  and an action of  $R^d$  as a family  $\tau_x$  of measure preserving transformations on  $(\Omega, \Sigma, P)$ . One assumes the ergodicity of the action. If  $F : \Omega \rightarrow R^d$  is a measurable map, then  $F(t, \omega) = F(\tau_x \omega)$  defines a stationary process indexed by  $x \in R^d$ . On  $L_p(\Omega)$  one can define  $\nabla = \{D_i\}$  as the infinitesimal generators of the action by  $R^d$ .

$$D_i f = \lim_{h \rightarrow 0} \frac{1}{h} [f(\tau_{he_i} \omega) - f(\omega)]$$

where  $e_1, \dots, e_d$  are the unit vectors along the coordinate axes. They are densely defined operators, they commute with each other and functions of the form

$$g(\omega) = \int_{R^d} \phi(x) f(\tau_x \omega) dx$$

where  $\phi$  is a  $C^\infty$  function with compact support on  $R^d$  are all in the domain of the algebra generated by  $\{D_i\}$ . The operator

$$\mathcal{L} = \frac{1}{2} \sum_{i,j} a_{i,j}(\omega) D_i D_j + \sum_j b_j(\omega) D_j$$

is well defined on a dense set of functions and corresponds to a "diffusion" on  $\Omega$ . This diffusion is hard to define directly on  $\Omega$ , but with

$$b(x, \omega) = b(\tau_x \omega)$$

and

$$a(x, \omega) = a(\tau_x \omega)$$

$$\mathcal{L}_\omega = \frac{1}{2} \sum_{i,j} a_{i,j}(x, \omega) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x, \omega) \frac{\partial}{\partial x_j}$$

defines a generator of a diffusion on  $R^d$ . Given  $\omega$  we can consider the process  $x(t)$  that starts from 0 and has  $\mathcal{L}_\omega$  as the generator. We can lift the path  $x(t)$  to  $\Omega$  by

$$\omega(t) = \tau_{x(t)} \omega$$

which then defines a Markov process on  $\Omega$  whose generator is  $\mathcal{L}$ . We can consider examples with

$$cI \leq a(\omega) \leq CI$$

and  $\mathcal{L}$  given by either

$$\mathcal{L} = \frac{1}{2} \sum_{i,j} a_{i,j}(\omega) D_i D_j$$

or

$$(3.1) \quad \mathcal{L} = \frac{1}{2} \nabla a(\omega) \cdot \nabla$$

Studying the long time behavior of solutions with the random generator  $\mathcal{L}_\omega$  on  $R^d$  can be reduced to the study of the behavior of the evolution on  $\Omega$  with  $\mathcal{L}$ . There are results very similar to the ones in the periodic case. The solutions to the equations

$$u_t = \frac{1}{2} \sum_{i,j} a_{i,j}\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial^2}{\partial x_i \partial x_j}$$

converge to solutions of

$$u_t = \frac{1}{2} \sum_{i,j} \bar{a}_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$$

with  $\bar{a}$  given by an average

$$\bar{a} = \int a(\omega) \Phi(\omega) dP$$

with respect to the normalized invariant density

$$\sum_{i,j} D_i D_j [a_{i,j}(\omega) \Phi(\omega)] = 0$$

which can be shown to exist uniquely. See [6]. There is a similar homogenization result when  $\mathcal{L}$  is given by (3.1) and the variational formula

$$\langle \bar{a} \xi, \xi \rangle = \inf_u E^P [\langle a(\omega)(\xi - \nabla u), (\xi - \nabla u) \rangle]$$

holds for the limit just like (2.1).

#### 4. Nonlinear equations

Recently there has been some work [4], [5] and [6], on extending some of these results to the nonlinear context. We will describe a small part of it that can be found in [6].  $H(p, \omega)$  is a map of  $R^d \times \Omega \rightarrow R$ . For each  $\omega$ ,  $H(p, \omega)$  is a convex function of  $p$ . We assume reasonable growth conditions and regularity assumptions. The function  $L(p, \omega)$  is the convex conjugate of  $H$  defined as

$$L(\xi, \omega) = \sup_{p \in R^d} [\langle p, \xi \rangle - H(p, \omega)]$$

and by duality

$$H(p, \omega) = \sup_{\xi \in R^d} [\langle p, \xi \rangle - L(\xi, \omega)]$$

$$H(p, x, \omega) = H(p, \tau_x \omega)$$

defines a stationary random field of convex functions. The Hamilton-Jacobi-Bellman equation is defined as

$$u_t + \frac{1}{2} \Delta u + H(\nabla u, x, \omega) = 0; \quad 0 \leq t \leq T; \quad u(T, x) = f(x)$$

The solution has a variational representation

$$u(t, x) = \sup_{b(\cdot, \cdot)} v^{b(\cdot, \cdot)}(t, x, \omega)$$

where  $v = v^b$  is the solution of the linear equation

$$v_t + \frac{1}{2} \Delta v + \langle b(t, x), \nabla v \rangle - L(b(t, x), x, \omega) = 0$$

$$v(T, x) = f(x)$$

We now look at the rescaled problem with  $T = T\epsilon^{-1}$  and  $f(x) = \epsilon^{-1}f(\epsilon x)$ . We are interested in the behavior of  $\epsilon u(\epsilon^{-1}t, \epsilon^{-1}x, \omega)$  as  $\epsilon \rightarrow 0$ . The rescaled equation becomes

$$u_t + \frac{\epsilon}{2}\Delta u + H(\nabla u, x\epsilon^{-1}, \omega) = 0$$

$$u(T, x) = f(x)$$

Fix  $b(t, x, \omega) = b(\tau_x\omega)$ . Consider the solution of the SDE

$$dx(t) = b(x(t), \omega)dt + d\beta(t); x(0) = 0$$

that corresponds to the diffusion with generator

$$\mathcal{L}_\omega = \frac{1}{2}\Delta + b(\tau_x\omega) \cdot \nabla$$

on  $R^d$  or

$$\mathcal{L} = \frac{1}{2}\Delta + b(\omega) \cdot \nabla$$

on  $\Omega$ . Suppose

$$\lim_{\epsilon \rightarrow 0} \epsilon x(T\epsilon^{-1}) = Tm$$

exists and

$$\lim_{\epsilon \rightarrow 0} \epsilon \int_0^{T\epsilon^{-1}} L(b(t, x(t)), x(t), \omega) = T\ell$$

Then one can see that

$$\lim_{\epsilon \rightarrow 0} v_\epsilon^b(0, 0) = f(Tm) - T\ell$$

When will the limit exist? If we can find a positive weight  $\rho(\omega)$  on  $\Omega$  such that

$$(4.1) \quad \frac{1}{2}\Delta\rho = \nabla \cdot (b\rho)$$

Then by the ergodic theorem applied to the process on  $\Omega$ , the limits exist and

$$m = \int b(\omega)\rho(\omega)dP$$

and

$$\ell = \int L(b(\omega), \omega)\rho(\omega)dP$$

We cannot always find a  $\rho$  that solves (4.1) for a given  $b$ . But perhaps pairs  $\rho, b$  exist that solve the equation. We denote by  $\mathcal{A}$  the class of all  $(b, \rho)$  that satisfy (4.1). Then

$$\liminf_{\epsilon \rightarrow 0} u_\epsilon(0, 0) \geq \sup_{(b, \rho) \in \mathcal{A}} [f(Tm(b, \rho)) - T\ell(b, \rho)]$$

The problem is to prove that

$$\limsup_{\epsilon \rightarrow 0} u_\epsilon(0, 0) \leq \sup_{(b, \rho) \in \mathcal{A}} [f(Tm(b, \rho)) - T\ell(b, \rho)]$$

and then strengthen it with some uniformity. We need a characterization of

$$\sup_{(b,\rho) \in \mathcal{A}} [f(Tm(b, \rho)) - T\ell(b, \rho)]$$

The condition  $(b, \rho) \in \mathcal{A}$  is not very nice. How bad is it? We use Lagrange multipliers to get rid of the restraint. Add a term  $\int \mathcal{L}u \rho dP$  and minimize over  $u$ . This forces  $(b, \rho)$  to be in  $\mathcal{A}$ .

Take  $f(x) = \langle \theta, x \rangle$ . Then

$$\begin{aligned} \bar{H}(\theta) &= \sup_{(b,\rho) \in \mathcal{A}} [\int \langle \theta, b \rangle \rho dP - \int L(b(\omega), \omega) \rho dP] \\ &= \sup_{(b,\rho)} \inf_u [\int \langle \theta, b \rangle \rho dP - \int L(b(\omega), \omega) \rho dP + \int \mathcal{L}u \cdot \rho dP] \\ &= \sup_{\rho} \inf_u \sup_b [\int \langle \theta, b \rangle \rho dP + \int \mathcal{L}u \cdot \rho dP - \int L(b(\omega), \omega) \rho dP] \\ &= \sup_{\rho} \inf_u \int [\frac{1}{2} \Delta u + H(\theta + \nabla u, \omega)] \rho dP \\ &= \inf_u \sup_{\rho} \int [\frac{1}{2} \Delta u + H(\theta + \nabla u, \omega)] \rho dP \\ &= \inf_u \text{ess sup}_{\omega} [\frac{1}{2} \Delta u + H(\theta + \nabla u, \omega)] \end{aligned}$$

So for any  $\delta > 0$  there exists a  $u = u_{\delta}$  such that

$$\frac{1}{2} \Delta u + H(\theta + \nabla u, \omega) \leq \bar{H}(\theta) + \delta$$

$u$  will be a very weak sub solution. What regularity can one expect of  $u$ ?

$$E[H(\theta + \nabla u, \omega)] \leq \bar{H}(\theta) + \delta < \infty$$

This means that  $g = \nabla u \in L_{\alpha}(\Omega, \Sigma, P)$  provided we have a lower bound on the growth of  $H$  for large  $p$ .  $\Delta u = \nabla \cdot g$  is a distribution. We need to convolute in space by convoluting with  $\phi(x)$  to get  $u_t^{\phi} = f^{\phi}$  and  $\nabla u^{\phi} = g^{\phi}$ . One can expect  $\nabla \cdot g^{\phi}$  to be nice. Also  $E[g^{\phi}] = 0$ .  $v = \langle \theta, x \rangle + u^{\phi}(x)$ ,  $\lim_{\epsilon \rightarrow 0} \epsilon u^{\phi}(x\epsilon^{-1}) = 0$ .  $v + [\bar{H}(\theta) + \delta](T - t)$  is a super solution. With  $v(T, x) \simeq \langle \theta, x \rangle$ .  $v(0, 0) = 0$ . Compare  $v$  with  $\epsilon w(x\epsilon^{-1})$  where  $w$  is the solution of

$$\frac{1}{2} \Delta w + \langle b(t, x), \nabla w \rangle - L(b(t, x), x, \omega) = 0$$

Maximum principle yields a comparison.

$$w(0, 0) \leq v(0, 0) = T\bar{H}(\theta) + \delta$$

Since  $\delta > 0$  is arbitrary the upper bound is proved.

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## HARMONIC ANALYSIS ON THE UNIT CIRCLE: A PERSONAL PERSPECTIVE

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### Introduction

At the outset I must clarify that as indicated in the title of this lecture, my account is coloured by my prejudices, preferences and limitations. By no stretch of imagination must this be deemed a complete survey as many important discoveries and happenings in this realm of research have been left out. As regards the attribution of credit, I may have faltered on many occasions and if so I apologise in advance. I have included those references that were most relevant for the purposes of my talk and even for this limited purpose I may have missed out on some references.

An interesting coincidence prefaces our story: the centenary year of the founding of the Indian Mathematical Society coincides with the centenary of the award of the doctorate to a remarkable thesis authored by a student of Henri Lebesgue. I refer to Pierre Fatou, who obtained his doctorate exactly a hundred years ago in 1907. His thesis is remarkable since it contains some very insightful theorems that were to play an important role in creating the basis for a subject that continues to grow in myriad directions even in the twenty first century. In the course of this lecture we shall endeavour to justify this assertion. There are two remarkable theorems of Fatou that we cite here that were published in [35] but are part of his thesis:

**Fatou Theorem A:** *Let  $p$  be such that  $1 \leq p \leq \infty$ , and let  $u(re^{i\theta})$  be harmonic on the open unit disk  $D$  in the complex plane. If*

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$$\sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty, \quad (p < \infty) \quad (*)$$

or

$$\sup_{0 < r < 1, 0 \leq \theta \leq 2\pi} |u(re^{i\theta})| < \infty, \quad (p = \infty) \quad (**)$$

then the function  $u(e^{i\theta})$  can be defined and exists almost everywhere on the unit circle  $T$  by the the following

$$u(e^{i\theta}) = \lim_{re^{i\theta} \rightarrow e^{i\theta}} u(re^{i\theta})$$

Here the limit exists in the non-tangential sense and almost everywhere with respect to Lebesgue measure on the unit circle. Further, if  $1 < p \leq \infty$ , then  $u(e^{i\theta})$  is in the Lebesgue space  $L^p(T)$  and if  $p = 1$  then  $u(re^{i\theta})$  is the Poisson integral of some Borel measure  $\nu$  on the unit circle and the boundary function  $u(e^{i\theta})$  is the Radon-Nikodym derivative of the measure  $\nu$  i.e.  $u(e^{i\theta}) = \frac{d\nu}{d\theta}$ .

Note: The Poisson kernel is the function

$$P(re^{i\theta}) = \frac{1-r^2}{1-2r\cos\theta+r^2}, \quad 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$$

The above theorem of Fatou has a very interesting consequence when the function  $u(re^{i\theta})$  is actually holomorphic on the disk. In that event the limit function  $u(e^{i\theta})$  -in the case  $1 < p \leq \infty$  -will have a Fourier series of analytic type on the circle i.e.

$$u(e^{i\theta}) \sim \sum_{n=0}^{\infty} \alpha_n e^{in\theta}$$

and the disk function will have a Taylor series

$$u(re^{i\theta}) = \sum_{n=0}^{\infty} \alpha_n r^n e^{in\theta}.$$

What happens when  $p = 1$ ? Interestingly and fortuitously the situation is similar due to a famous theorem of the brothers Riesz. We shall state this result presently but for the moment we mention that it implies that if the disk function in the Fatou Theorem A is holomorphic then the measure  $\nu$  coincides with the boundary function  $u(e^{i\theta})$  that is to say  $\nu = u d\theta$  and then the Fourier series of  $u(e^{i\theta})$  is related to the Taylor series of  $u(re^{i\theta})$  exactly as in the case above when  $p > 1$ .

In addition, for all values of  $p$  the Lebesgue norm of the circle function coincides with the supremum associated with the disk function as expressed in the equation (\*) in the Fatou Theorem A. Further, for all values of  $p$  the



disk function can be recovered from its limit function on the circle simply by convolving the circle function with the Poisson kernel.

The situation is now rather elegant simply because a reverse process is also true. If we start with any function  $f$  in any of the  $L^p$  spaces ( $1 \leq p \leq \infty$ ) on the circle with Fourier series of analytic type then we can convolve it with the Poisson kernel to obtain a holomorphic function on the disk and the limits of such a holomorphic function as expressed in (\*) and (\*\*) above (in the Fatou Theorem A) shall coincide with the Lebesgue norms of the corresponding  $L^p$  spaces. This means that for any fixed value of  $p$  we can identify the class of all  $L^p$  functions with Fourier series of analytic type with the class of all holomorphic functions on the unit disk that satisfy (\*) or (\*\*) depending on whether  $p$  is finite or is equal to infinity respectively. Such a class of functions on the disk is not differentiated from the corresponding class on the circle and for each value of  $p$  this class is denoted by the symbol  $H^p$  in honour of G. H. Hardy. It was F. Riesz who named them thus and these spaces are Banach spaces for all values of  $p$  as above. In addition when  $p = 2$ ,  $H^2$  is a Hilbert space and when  $p = \infty$ ,  $H^\infty$  is a Banach algebra.

As mentioned, a critical role has been played in the above discussion by the F. and M. Riesz Theorem [69] and it is about time we stated the theorem. It is important for us to state here that the theorem is famous not only for its many applications but also for its numerous proofs and for its validity in very general contexts. More interestingly, as observed by Helson, the theorem is more general than any of its proofs. There is another reason for us to state this famous theorem here just before the statement of the other theorem from Fatou's thesis for they are intimately connected. However, we shall disclose this connection between the two theorems much later; almost at the end of our talk.

**The F. and M. Riesz Theorem:** *Let  $\mu$  be a finite Borel measure on the unit circle such that  $\mu$  has a Fourier series of analytic type i.e.*

$$\int_T e^{in\theta} d\mu = 0, \quad n = 1, 2, \dots$$

*Then  $\mu$  is absolutely continuous with respect to Lebesgue measure.*

The theorem says that such a  $\mu$  will satisfy  $d\mu = f d\theta$  where  $f$  shall be in the Hardy class  $H^1$ .

Here is the other theorem of Fatou that we want to mention:

**Fatou Theorem B:** *Let  $K$  be a closed set of Lebesgue measure zero on the unit circle. Then there is a continuous function on the closed unit disk*

which is holomorphic on the open unit disk and which vanishes precisely on  $K$ .

Just as in the case of the Fatou Theorem A and the F. and M. Riesz theorem, the Fatou Theorem B is also important for several reasons. An extension of this theorem has played a key role in characterising the closed ideals of the disk algebra  $A$  of functions continuous in the closed unit disk and holomorphic in the open unit disk. We shall have occasion to mention this disk algebra later. Suffice it to say here that it is a Banach algebra under the supremum norm and is a closed subalgebra of the Banach algebra  $H^\infty$ .

There are some other basic facts of the theory that we need to set aside and we shall do so in as brief a fashion as possible. We first observe that we shall make no distinction between the Hardy spaces on the circle and on the disk. The context shall make it clear as to whether we are referring to the disk or the circle though in general things are likely to remain valid in both contexts.

The operator of multiplication by the coordinate function  $z$  shall be denoted by  $S$ . This operator is well defined on all the  $L^p$  spaces and on each of the Hardy spaces. It is an isometry on each of the spaces and its adjoint on the Hilbert space  $H^2$  shall be denoted by  $S^*$ . The operator  $S^*$ , referred to as the backwards shift, is just as well defined on each of the Hardy spaces and its action is given by

$$(S^*f)(z) = \frac{f(z) - f(0)}{z}$$

It is also an important fact that each function  $f$  in every  $H^p$  space is the product in an essentially unique manner of two functions,  $f = IO$  where  $I$  and  $O$  are both in  $H^p$  and  $|I| = 1$  almost everywhere. The function  $O$  is a cyclic vector on  $H^p$  for the operator  $S$  i.e.  $H^p = \vee\{S^n O : n = 0, 1, \dots\}$ . (We are dealing with the closed linear span here). Any functions  $I$  and  $O$  with the above unimodularity and cyclicity properties respectively are called an inner function and an outer function. The only functions that are both inner and outer are the unimodular constants. An inner function  $I$  can be further factorised into two parts  $I = BJ$  where both  $B$  and  $J$  are inner and  $B$  is an infinite product of conformal maps of the unit disk onto itself with a suitable sequence of scalars attached to each conformal map. The zeros of  $B$  are precisely the zeros of the conformal maps and thus the conformal maps are repeated according to the multiplicity of the zeros. The inner function  $J$  has no zeros in the open unit disk. For details about all that has been said above and for other aspects of the Hardy spaces we refer to [6], [33], [34], [40], [42], [44], [46], [48], [49], [57], [71], [74], and [79].

**An overview:** The theory of Hardy spaces provides a fertile setting for that aspect of harmonic analysis on the unit circle where the problems and techniques of functional analysis and function theory intertwine with each other. As mentioned at the outset there are large parts of the theory that we shall mention not at all or only cursorily. Some of these topics that immediately come to mind are the rich and deep areas connected with Toeplitz and Hankel operators for which we refer to [2], [33], [52], [54], [57], [58], [59] and [60] where the theory is explained in detail and with depth. We are also not detailing the many similarities and contrasts that exist on several other related function spaces such as the Bergman spaces, the Dirichlet spaces and other function spaces that form a rich and active area of current research. We would refer the reader to [10], [27], [73], and [75]. While we are delineating our span for this lecture it will be worthwhile to mention that there is a very extensive and vibrant theory of the Hardy spaces that stems from the recognition that the setting of the unit circle is but one of the many examples of a compact Hausdorff space on which the disk algebra plays the role of a typical uniform algebra and the multiplicative functional of evaluation at zero is represented by a convenient measure. We shall refer to this aspect only in passing where the main theme of the talk blends with the area. In large measure we shall also not mention, except at a point relevant for the theme of our talk the well developed function theory on compact abelian groups with ordered duals details of which can be obtained from the wonderful works of Helson including but not limited to [43] and [44]. In similar fashion we shall not tread too much into the areas connected with function theory in the polydisk, [72], and the unit ball of  $\mathbb{C}^n$ , [73], where many problems remain and the field is exceptionally active. Finally, we have altogether avoided a discussion of function theory on planar domains that are different from the disk. A good idea of the nature of the problems can be had from [34] and [38] and [76]. Rest assured that there are other areas which have not even been mentioned by us that tie up nicely with the theme.

Here we are primarily visiting the following four broad sub-themes albeit in a limited manner.

- (1) The theory of invariant subspaces.
- (2) Factorisation of matrix valued holomorphic functions.
- (3) Some classical inequalities.
- (4) Interpolation .

### 1. The theory of invariant subspaces

For any bounded linear operator  $T$  from a Banach space  $X$  into itself, we shall call a subspace  $M$  of  $X$  as an invariant subspace of  $T$  if  $T$  maps  $M$

into itself and  $M$  is closed and is neither the trivial space nor all of  $X$ . The theory of invariant subspaces has proved to be important for many reasons and we shall not elaborate on that aspect over here.

For our purposes, the journey begins with the fundamental paper of Beurling [11] where he characterised the invariant subspaces of the operator  $S$  on the Hardy space  $H^2$ . Beurling proved the following:

**Theorem:** Let  $M$  be an invariant subspace of  $S$  on  $H^2$ . Then there is an inner function  $\phi$ , unique upto a factor of absolute value 1, such that  $M = \phi H^2$ .

Beurling's theorem is important for many reasons. It forms the cornerstone of the modern view point of the theory of Hardy spaces and generally of the functional analytic approach to function theory on the disk. Beurling proved his theorem using a great deal of the classical function theory on the disk developed by Riesz, Hardy, Littlewood and many distinguished mathematicians. The theorem is elegant and it has lent itself to many interpretations. One such perspective stems from the fact that  $H^2$  is a closed subspace of  $L^2$  and since  $S$  is well defined on  $L^2$  it can be interpreted as a statement concerning invariant subspaces in  $L^2$ . This was the approach of Helson and Lowdenslager. See [42], [46]. They generalised Beurling's theorem as follows:

**Theorem:** Let  $M$  be a simply invariant subspace of  $S$  on  $L^2$  so that  $S(M) \subsetneq M$ . Then there is a unimodular function  $\phi$ , unique upto a factor of absolute value 1, such that  $M = \phi H^2$ .

One of the most interesting aspects of the Helson-Lowdenslager theorem is its proof. Their ideas were independent of the function theory of the disk and were geometric in nature. There are many advantages to this. For one, once we have an independent proof of the theorem, it allows the recovery of a great deal of the function theory of the disk that was used by Beurling in his proof. So their approach was in a way the reverse of what Beurling had done. More importantly, their methods were geometrical in nature and thus they could be used in very general settings where there was no question of developing anything like a derivative at a point. Hence their proof allowed the characterisation of invariant subspaces on very general kinds of Hardy spaces such as in the context of uniform algebras associated with compact Hausdorff spaces and on compact Abelian groups with ordered duals. We refer to [39], [43] and [44]. We do remark here that the case  $S(M) = M$  had been tackled by Wiener much earlier, see [42]. His characterisation runs as follows:

**Theorem:** Let  $M$  be a doubly invariant subspace of  $S$  on  $L^2$  i.e.  $S(M) = M$ . Then there is a set  $E$  of positive Lebesgue measure on the circle such that  $M = I_E L^2$ .

Another interpretation of the Helson-Lowdenslager theorem looks at the fact that the space in question is  $L^2$  and since the operator  $S$  is well defined on each of the  $L^p$  spaces ( $p \geq 1$ ) the question of a characterisation of the invariant subspaces on  $L^p$  remains valid. Not surprisingly, the theorem remains valid in unchanged form except for the change in index from 2 to  $p$ . So a simply invariant subspace on  $L^p$  is of the form  $\phi H^p$  for a unimodular  $\phi$ . We must mention here that when  $p = \infty$  we deal with the weak star topology derived from the fact that  $L^\infty$  is the dual of  $L^1$ . The proof in this general  $L^p$  setting was discovered by T. P. Srinivasan and can be found in [42] and in [56]. There is an interesting comment that we would like to make here. In [42] Helson mentions that the proof of Srinivasan could not avoid the use of conjugate functions and that in a sense this is unavoidable. We shall be dealing with conjugate functions shortly. Over here we wish to point out that actually the proof can be made far more elementary by totally eliminating any use of conjugate functions. In fact Paulsen, Raghupathi and Singh in a manuscript under preparation [96] actually extend the same simplifications to the context of the proof of the invariant subspace theorem for  $L^p$  in the setting of uniform algebras.

There is a yet another and comparatively recent interpretation of the theorem of Beurling stemming from the work of L. de Branges [30] and [79]. De Branges observed that the invariant subspace  $M$  as in the theorem of Beurling could be replaced by an arbitrary Hilbert space  $M$  that was, instead of being a closed subspace of  $H^2$ , a vector subspace of  $H^2$  that was contractively contained in  $H^2$  i.e.  $\|f\|_{H^2} \leq \|f\|_M$  for all  $f$  in  $M$ . He further assumed that the linear transformation  $S$  was well defined from  $M$  to  $M$  and was an isometry on  $M$ .

**Theorem of de Branges (scalar version):** Let  $M$  be a Hilbert space that is contractively contained in  $H^2$  and on which  $S$  acts as an isometry. Then there is a unique  $b$  in the unit ball of  $H^\infty$  such that  $M = bH^2$  and  $\|bf\|_M = \|f\|_{H^2}$  for all  $f$  in  $H^2$ .

This theorem of de Branges is the starting point of a model theory for operators the ideas for which can be found in [30], [58], [59], and in [79]. Actually de Branges proved more than this. He generalised the Lax-Halmos extension of Beurling's theorem, see [41], [46] and [50]. The Lax-Halmos theorem was a generalisation of Beurling's theorem in the following sense: it replaces the Hardy space of scalar valued functions by a Hardy space of

functions taking values in a Hilbert space. Actually Lax had dealt with a finite dimensional Hilbert space.

Interestingly, in 1991, Dinesh Singh and U. N. Singh, [85], observed that the contractivity assumption in the theorem of de Branges (scalar version) could be dropped so that in some sense it was redundant. They showed that the conclusions of de Branges would remain valid even in the context of assuming the simple algebraic inclusion of  $M$  as a subspace of  $H^2$  devoid of any topological relationship between the two spaces as follows:

**Theorem (Singh and Singh):** Let  $M$  be a Hilbert space that is a vector subspace of  $H^2$  and on which  $S$  acts as an isometry. Then there is a unique  $b$  in  $H^\infty$  such that  $M = bH^2$  and  $\|bf\|_M = \|f\|_{H^2}$  for all  $f$  in  $H^2$ .

**Corollary:** The Theorems of de Branges and Beurling.

This line of reasoning has had many fruitful extensions as embodied in the work of Singh [82] where a polydisk version of the theorem of de Branges is derived. In addition, the following references indicate various extensions of the results of Beurling, Helson and Lowdenslager and of de Branges; Agrawal, Singh and Yadav [84], Agrawal and Singh [86], [87], Paulsen and Singh [89], Raghupathi and Singh [95] and Paulsen, Raghupathi and Singh [96]. In a series of recent papers Redett and Jupiter [47] and Redett, [47], [65], [66], [67], and [68] have extended the work of Helson and Lowdenslager [42], Singh [82], Agrawal and Singh [86] and Paulsen and Singh [89].

Over here we mention just two results from the above list in explicit form. The first one simultaneously generalises the theorems of Lax, de Branges and Singh and Singh. The second result is a generalisation of the Helson-Lowdenslager Theorem along the lines of de Branges' generalisation of Beurling's Theorem.

**Theorem (Singh and Thukral; [98]):** Let  $M$  be a Hilbert space that is a vector subspace of  $H^2$  and on which the operator of multiplication by a finite Blaschke factor  $B$ , with  $B(0) = 0$ , acts isometrically. Then there are functions  $\phi_1, \dots, \phi_n$ , in  $H^\infty$  such that

$$M = \phi_1 M(B) \oplus \dots \oplus \phi_n M(B)$$

and  $\|\phi_1 f_1 + \dots + \phi_n f_n\|_M^2 = \|f_1\|_{H^2}^2 + \dots + \|f_n\|_{H^2}^2$  for all  $f_i$ 's in  $H^2$ . Also  $n$  does not exceed the number of zeros of  $B$ . Note:  $M(B) = \vee\{B^n : n = 0, 1, \dots\}$ . The closure is in  $H^2$ .

Amongst other consequences, the above result actually leads to a very general inner-outer factorisation theory that includes the classical inner-outer factorisation as a special case.

**Theorem (Paulsen and Singh; [89]):** Let  $M$  be a Hilbert space that is contractively contained in the Lebesgue space  $L^2$  and is simply invariant under  $S$  which acts isometrically on it. Further assume that there is a constant  $\delta > 0$  such that  $\|f\|_M \leq \delta \|f\|_p$  for all  $f$  in  $M \cap L^p$  where  $p$  is some index larger than 2. (Note: we do not assume that  $M \cap L^p \neq \{0\}$ .) Then there is a unique  $b$  in the unit ball of  $L^\infty$  such that  $b$  does not vanish on any set of positive measure and  $M = bH^2$  with  $\|bf\|_M = \|f\|_{L^2}$  for all  $f$  in  $H^2$ .

**Corollary:** Helson-Lowdenslager.

We would like to add that, Redett [68], has recently extended the Paulsen-Singh theorem to the setting of other Lebesgue spaces.

#### Invariant subspaces in BMOA and VMOA:

There is a space of holomorphic functions sitting inside the space  $H^1$  and which acts as the dual of  $H^1$ . We refer to the space BMOA—short for analytic functions of bounded mean oscillation. Here is how it is defined. Let  $I$  be a subarc of the unit interval  $T$ . Let for  $f$  in  $H^1$

$$I(f) = \frac{1}{|I|} \int_I f d\theta$$

be the mean of  $f$  over  $I$  where  $|I|$  denotes the normalised Lebesgue measure of  $I$ . If the oscillation over the mean  $I(|f - I(f)|)$  satisfies

$$\|f\|_* = \sup_I I(|f - I(f)|) < \infty$$

then  $f$  is said to be in the class BMOA. Equipped with the norm  $\|f\| = \|f\|_* + |f(0)|$ , BMOA becomes a Banach space. Its claim to fame is the famous duality theorem of Fefferman:

**Fefferman's Duality Theorem:** BMOA is the dual of the Hardy space  $H^1$  and the action of a function  $f$  in BMOA on  $H^1$  is given by

$$\frac{1}{2\pi} \int_0^{2\pi} f \bar{p} d\theta$$

Here  $p$  runs through all the polynomials in  $H^1$ . For further details and insights we refer to [26], [36], [49], [57], [58], [59], [60], [74], [75], and [78].

Actually, Fefferman's program was to seek an appropriate mechanism to enable him to extend the theory of Hardy spaces to the setting of several variables. Since the theory of several complex variables did not seem to yield much ground for such purposes, a real variable approach was sought. The crucial breakthrough came via the deep and important theorem of

Burkholder-Gundy-Silverstein, [22], that enabled a real variable characterisation of the Hardy space  $H^1$ . Of course their setting was also in the context of functions of several variables.

Our exposition is very much in the context of the unit circle or the disk. Two comments are in order here. The theory that we expound here in the context of the disk runs almost parallel to the real and multivariable situation. More importantly, a program that started out to extend the theory to the multivariable situation came back to greatly enrich the classical situation of the unit disk bringing in its wake new and deep ideas that affected the classical theory in many significant ways that no one had anticipated.

Coming back to the Burkholder-Gundy-Silverstein theorem, we point out that it relies on the Hardy-Littlewood non-tangential maximal function. Here is how the function is defined: Fix an  $\alpha$  strictly between 0 and  $\pi/2$ . Let  $f$  be a harmonic function on the open unit disk. For a point  $e^{i\theta}$  on the unit circle, form the convex hull of the point  $e^{i\theta}$  and the circle with radius  $\sin \alpha$ . Delete the point  $e^{i\theta}$  from this hull and denote the region by  $\Gamma(\alpha; \theta)$ . The nontangential maximal function  $N_\alpha(f)$  is defined on the unit circle by  $N_\alpha(f)(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma(\alpha; \theta)\}$ .

The crucial point that needs to be noted is that if  $f$  is a real valued function in  $L^1$  then we can convolve it with the Poisson kernel and make it harmonic on the open disk. It can be shown that the harmonic conjugate  $\tilde{f}$  of  $f$  shall have non-tangential boundary values on the circle which we denote by  $\tilde{f}$  again. The function  $\tilde{f}$  lies in  $L^1$  if and only if  $f + i\tilde{f}$  is in  $H^1$ . One can in the reverse direction start with an  $H^1$  function on the circle and show that its imaginary part is the conjugate of the real part and lies in  $L^1$ . Thus, to characterise functions in  $H^1$  it is enough to know which real valued functions in  $L^1$  allow their conjugates to also lie in  $L^1$ . This will provide us with a real variable characterisation of  $H^1$ . Directly working with the conjugate function is difficult because of the nature of the conjugation operator. The Burkholder-Gundy-Silverstein theorem answers our question without recourse to the conjugation operator.

**Burkholder-Gundy-Silverstein Theorem:** Given  $f$  in  $L^1$ , its conjugate  $\tilde{f}$  lies in  $L^1$  if and only if  $N_\alpha(f) \in L^1$ .

Reverting to the space BMOA it has a very important subspace discovered by Sarason, [78], called VMOA and which is the closure of the continuous functions in BMOA in the norm of BMOA. Here is an interesting feature of both these spaces. The operator of multiplication by the coordinate function  $z$  is well defined and bounded on BMOA and on VMOA though it is far from being an isomtery given the difficult nature of the norm on BMOA. A natural question to ask is what are the closed invariant



subspaces of  $S$  on BMOA and on VMOA. Due to the non-separability of BMOA it is too much to answer. However, the situation takes a nice turn when we use the weak star topology on BMOA treating it as the dual of  $H^1$ .

**Theorem. (Singh and Singh; [85]):** Let  $M$  be a weak star closed subspace of BMOA invariant under  $S$ . Then there is a unique inner function  $I$  and an unique subspace  $N$  of BMOA such that  $M = IN$ . Further  $N$  is dense in BMOA and is equal to BMOA if and only if  $I$  is a finite Blaschke product.

A similar result holds for VMOA and the two results throw light on maximal ideals corresponding to fibres in the context of the Banach algebras  $H^\infty$  and  $H^\infty \cap VMOA$  respectively.

We must remark that Alexandrov, [8], has obtained equivalent versions of the BMOA result by different methods. Also Brown and Sadek [20] and [21] have obtained interesting proofs of some of our results.

We would like to mention briefly the maximality theorem of Wermer.

**Wermer's Maximality Theorem:** Let  $C(T)$  denote the algebra of all continuous functions on the circle  $T$  and let  $A$  denote the algebra of all functions in  $C(T)$  that have Fourier series of analytic type. Then  $A(T)$  is maximal as a closed subalgebra of  $C(T)$ .

Like some of its illustrious predecessors above this theorem is interesting for many reasons. One of them being the fact that it has many interesting proofs.

In [93] Paulsen and Singh have amongst other things obtained a far reaching generalisation of this result and we state a special case of our generalisation which is yet more general than the theorem of Wermer. For a closed set  $K$  of the circle  $T$ , we denote the ideal of functions in  $C(T)$  that vanish on  $K$  by  $Z(K)$ .

**Theorem. (Paulsen and Singh [93]):** Let  $K$  be a closed set of measure zero on the circle  $T$ . Then  $\mathbb{C} + A(T) \cap Z(K)$  is maximal among all proper closed subalgebras of  $\mathbb{C} + Z(K)$ .

## 2. Factorisation of matrix valued holomorphic functions

There is an extensive non-commutative vector version of the theory of Hardy spaces that stretches in many directions. Of particular interest is the recent work of Blecher and Labuschgne in this list below. We refer to [12], [13], [14], [15], [16], [44], [57], [58], [59], [64], [70], and [99]. Many of these references deal with the generalisation of the scalar valued inner-outer factorisation to a non-commutative situation. In particular we mention the important and pioneering work of Potapov who obtained a detailed analogue

of the scalar valued factorisation for matrix valued analytic functions that are in  $H^2$ .

In [98] Singh and Thukral present a non-commutative matricial analogue of the scalar valued factorization corresponding to zeros of holomorphic functions. We claim novelty on three counts. For one, our results are stated and proved in the context of non-square matrices as opposed to square matrices in the case of Potapov and others. Second, our results are in the context of the class  $H^1$  as opposed to  $H^2$ . Finally, our methods rely on the well known duality equation  $(VMOA)^* = H^1$  and on an invariant subspace theorem on  $H^1$ , first proved by us essentially in [58].

**Theorem(Singh and Thukral; [58]):** Let  $A(z)$  be a  $m \times n$  matrix of  $H^1$  functions such that there are distinct points  $\{\omega_i : 1 \leq i \leq l\}$  in the open disk for which  $A(\omega_i)$  has a nonzero kernel in  $\mathbb{C}^n$  for each  $i$ . Then there exist matrices of  $H^1$  functions  $\{G_i, F : 1 \leq i \leq l\}$ , such that for each  $i$ ,  $G_i^* G_i = I$  on  $T$ ,  $\ker G_i(\omega_i) \neq \{0\}$  and such that  $A = G_1 \cdots G_l F$ .

### 3. SOME CLASSICAL INEQUALITIES

There are two inequalities dealing with  $H^1$  and  $H^2$  respectively and which are intimately connected. We speak of the inequalities of Hady and of Hilbert respectively. Here is how they are stated:

**Hardy's Inequality:** Let  $f(z) = \sum_0^\infty a_n z^n$  be in  $H^1$ . Then  $\sum_0^\infty \left| \frac{a_n}{n+1} \right| \leq k \|f\|_1$  for some constant  $k$ .

**Hilbert's Inequality:** Let  $\{\alpha_n\}_0^\infty, \{\beta_n\}_0^\infty$  be two sequences in  $l^2$ . Then  $\sum \sum \frac{|\alpha_n \beta_n|}{n+m+1} \leq \delta \|\{\alpha_n\}\|_2 \|\{\beta_n\}\|_2$  for some constant  $\delta$ .

Many references in the literature obtain the inequality of Hardy from the inequality of Hilbert. We refer to [44], [58]. No one seems to have observed that one can derive the inequality of Hilbert from the inequality of Hardy. In fact in [97] Paulsen and Singh have obtained general versions of these two inequalities and established that they are equivalent. The method exploits the duality between  $H^1$  and BMOA and the fact that every  $H^1$  function can be factored as a product of two functions from  $H^2$  in a special way.

**Generalised Hardy's Inequality(Paulsen and Singh):** Let  $\{b_n\}$  be any sequence in  $l^2$  such that  $\sum_{k=0}^n k^2 |b_k|^2 = O(n)$ . Then for any  $f(z) = \sum a_n z^n$  in  $H^1$   $\sum |a_n b_n| \leq \delta \|f\|_1$ .

**Generalised Hilbert's Inequality (Paulsen and Singh):** Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  be in  $l^2$  and let  $\sum_{k=0}^n k^2 |\gamma_k|^2 = O(n)$ . Then

$$\sum |\alpha_n| |\beta_m| |\gamma_{n+m+1}| \leq \lambda \|\alpha_n\|_2 \|\beta_m\|_2$$

**Theorem (Paulsen and Singh):** Generalised Hardy and generalised Hilbert are equivalent.

**The Inequality of Bohr:** Let  $f(z) = \sum_0^\infty a_n z^n$  be holomorphic in the open disk and continuous on the closed disk. Then  $\sum |a_n| r^n \leq \|f\|_\infty$ ,  $0 \leq r \leq \frac{1}{3}$ .

This inequality of Bohr appears in [18] and he used it in connection with Dirichlet series for a problem in number theory. Subsequently the names of mathematicians who gave a proof reads like a who's who of analysis: Wiener, Schur, M. Riesz, Sidon, Tomic and many others. Interest in the inequality was revived recently when Dixon [32] used this inequality to settle in the negative the long standing conjecture for non-unital Banach algebras that says that if they satisfy von Neumann's inequality then they must be isomorphic to a norm closed subalgebra of  $B(H)$  for some Hilbert space  $H$ . We mention [61] and [100] for interesting facets of von Neumann's inequality.

Recently, Paulsen, Popescu and Singh [90] have provided various proofs and generalisations to many situations of this inequality including some multivariable and non-commutative settings. We mention here the fact that they have used this inequality to show that every non-unital Banach algebra has an equivalent norm under which it satisfies the von Neumann inequality thus providing an infinite number of counter examples to the conjecture on von Neumann's inequality.

In a subsequent paper, Paulsen and Singh [91], further generalise this inequality to the setting of operator valued analytic functions and to the context of real valued functions on the unit circle that are in  $L^1$  with an interesting consequence for Fourier series.

There are many interesting papers on this inequality and we mention a few such as [9], [17], [19], [31], [63], [81], and [101].

We would like to end this section by quoting from a letter that Professor Enrico Bombieri wrote us: "I want to thank you for sending me a preliminary copy of your new paper. I found it interesting and it suggests several new and nice problems, some of them quite intriguing. I did not suspect that the subject had become lively again and I had come to consider it only from the historical point of view, as a nice little paragraph of the early era of classical function theory."

Bombieri and Bourgain in [19] amongst other things, settle a question posed by Paulsen, Popescu and Singh in [91].

#### 4. Interpolation

We wish to clarify that we mention only in passing the interpolation problem concerning bounded analytic functions on the unit disk to which many luminaries have contributed, most notable of all Carleson, see [24] and [25]. Our concern is the interpolation problem of the Nevanlinna-Pick type which is also a very vibrant and important area of activity. It has proved to be a meeting ground for operator theory, invariant subspaces and function theory. We refer to [1], [3], [4], [5], [7], [28], [29], [37], [53], [55], [56], [57], [58], [59], [62], [77], and [94]. One of the main inspirations for the modern approach to this area has been Sarason [77].

The classic Nevanlinna-Pick interpolation result says that given  $n$  distinct points,  $z_1, \dots, z_n$ , in the open unit disk,  $\mathbb{D}$ , and  $n$  complex numbers,  $w_1, \dots, w_n$  and  $A > 0$ , then there exists an analytic function  $f$  on  $\mathbb{D}$  with  $f(z_i) = w_i$  for  $i = 1, \dots, n$  and  $\|f\|_\infty \leq A$  if and only if the  $n \times n$  matrix,

$$\left[ \frac{A^2 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right]$$

is positive semidefinite, where  $\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}$ .

In [91] Davidson, Paulsen, Raghupathi and Singh give two very different sets of necessary and sufficient conditions for the classical Nevanlinna-Pick problem with one additional constraint. Namely, that  $f'(0) = 0$ . We define the algebra

$$H_1^\infty = H_1^\infty(\mathbb{D}) := \{f \in H^\infty(\mathbb{D}) : f'(0) = 0\},$$

so that our constraint is simply the requirement that functions belong to this algebra.

Our main result is analogous to M.B. Abrahamse's, [1], interpolation results for finitely connected domains. If  $\mathcal{R}$  is a bounded domain in the complex plane whose boundary consisted of  $p + 1$  disjoint analytic Jordan curves, then Abrahamse identified a family of reproducing kernel Hilbert spaces  $H_\alpha^2(\mathcal{R})$  indexed by  $\alpha$  in the  $p$ -torus  $\mathbb{T}^p$ , with corresponding kernels  $K_\alpha(z, w)$ . He proved that if  $z_1, \dots, z_n$  are  $n$  distinct points in  $\mathcal{R}$  and  $w_1, \dots, w_n$  are complex numbers, then there exists an analytic function  $f$  on  $\mathcal{R}$  such that  $f(z_i) = w_i$  and  $\sup\{|f(z)| : z \in \mathcal{R}\} \leq A$  if and only if

$$\left[ (A^2 - w_i \overline{w_j}) K_\alpha(z_i, z_j) \right] \geq 0 \quad \text{for all } \alpha \in \mathbb{T}^p.$$

In a similar fashion, we identify a family of reproducing kernel Hilbert spaces of analytic functions on  $\mathbb{D}$ , denoted  $H_{\alpha, \beta}^2(\mathbb{D})$ , indexed by points on the sphere in complex 2-space,  $|\alpha|^2 + |\beta|^2 = 1$ . The corresponding kernels

are given by

$$K_{\alpha,\beta}(z, w) = (\alpha + \beta z)\overline{(\alpha + \beta w)} + \frac{z^2\overline{w^2}}{1 - z\overline{w}}.$$

We prove that these kernel functions play a similar role for our constrained interpolation problem to the role played by Abrahamse's kernel functions for interpolation on finitely connected domains.

**Theorem:** Let  $z_1, \dots, z_n$  be distinct points in  $\mathbb{D}$ , and let  $w_1, \dots, w_n$  be complex numbers. Then there exists an analytic function  $f$  on  $\mathbb{D}$  with  $\|f\|_\infty \leq A$  and  $f'(0) = 0$  such that  $f(z_i) = w_i$  for  $i = 1, \dots, n$  if and only if

$$\left[ (A^2 - w_i\overline{w_j})K^{\alpha,\beta}(z_i, z_j) \right]$$

is positive semidefinite for all  $|\alpha|^2 + |\beta|^2 = 1$ .

**Final comment:** In [93] Paulsen and Singh have established that the Fatou Theorem B and the Theorem of F. and M. Riesz mentioned at the beginning of this talk are in fact equivalent.

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## THE LOCATOR PROBLEM

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### 1. Introduction

This work has two settings:

- (1) a physical problem - approximating the location of an object, and
- (2) a best approximation problem from a specific non-linear set.

The locator problem models the following physical situation. Suppose one lands an unmanned space craft on an unobservable terrain (e.g., under the clouds of Venus or on the backside of the moon) and wishes to determine the location of the landing site. Suppose that the craft can sample the altitude at the landing site and at several other spots (say at 100 meters east and at 100 meters west). However, the craft does not have a map of the altitudes of the terrain (i.e.,  $a(x, y)$ ), but only a single function,  $p(x, y)$  that approximates the altitude. The locator problem is to find a function  $p$  from a family of functions  $\mathcal{P}$  that minimizes the error between the actual location of the craft and the computed location of the craft using the approximation  $p$ . The error is to be minimized over all possible locations. That is we seek the  $p \in \mathcal{P}$  to minimize

$$\|(x, y) - p^{-1}(a(x, y))\|$$

This physical problem is equivalent to classical approximation questions about existence and uniqueness of best approximations from this (non-linear) family of inverse functions. The question is most interesting when the elements in the setting are the most fundamental and basic: for example, when  $\mathcal{P}$  is the polynomials of degree  $n$  and the norm is the uniform norm or the  $L_1$ -norm.

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Although this is a rich theoretical setting with at least eight fundamental elements to define (various metrics, data collections and families of approximating functions) and potentially has useful applications, the results outlined here are the only proven setting that admits unique best locator functions. These are for  $\mathcal{P}$  the increasing polynomials of degree  $n$ ; the domain and range being the unit interval and the norm being either the uniform norm or the  $L_1$  norm. In this setting there exist best locator functions and they are unique.

This introduction to the Locator Problem and approximation from the inverses of monotone polynomials does not contain proofs or a thorough reference list.

This paper is a written form of an invited talk presented at the centenary meeting of the Indian Mathematical Society 27-30 December 2007 held in Pune. I am grateful to the organizing committee for inviting me and particularly to Professors Ajit Iqbal Singh and Ajay Kumar for their mathematical insights as well as for their kindness to me. I am also grateful to Professor J.R. Patadia and the editors of "The Mathematics Student" for their interest in this work and for inviting this submission.

## 2. General Best Approximation Theory

In this section we outline a few of the fundamental questions asked in the Theory of best approximations. The next sections will reduce the Locator Problem to a basic question about existence of unique best approximations from a non-linear set of inverse functions.

Let  $M$  be a subset of a Banach space  $B$ . We say that  $m \in M$  is a best approximation to  $b \in B$  if

$$\|b - m\| = \inf\{\|b - u\| : u \in M\} = \text{dist}(b, M).$$

Basic Questions include:

- Do best approximations exist?
- Are they unique?
- Can they be characterized?
- Is the best approximation operator continuous?

The answers to these questions are important if there are reasons to be interested in the space  $B$  and set  $M$ . But in general this has been a classically important setting in Mathematics. For example, these include the theorems on orthogonal projections with respect to an orthonormal basis in  $L_2$  as well as Chebyshev's prize winning characterization of best approximation by polynomials  $\mathcal{P}_n \subset C[0, 1]$  (of degree  $n$ )

In this talk we will look at

$$\{p^{-1} : p \in \mathcal{P}_n, p' \geq 0, p(0) = 0, p(1) = 1\}.$$

### 3. A Description of the Locator Problem

The Locator Problem is a basic question that is readily understood. However, a strict mathematical statement of the problem is almost incomprehensible. This is partly due to the variations possible for the problem and the resulting overly general statement.

Since this is an expository presentation intended to explain the problem, we will state the problem in one setting that has a physical description. Of course the mathematics doesn't begin until such settings are explicitly defined as a mathematics problem.

The description follows:

A craft lands in an unobservable terrain. We want to know the location  $(x_o, y_o)$ .

We can compute the altitude  $\alpha(x_o, y_o)$  (1) of the landing site, and maybe at nearby points (for example, say at one additional point  $\alpha(x_o, y_o + 1)$ ) (2).

But we don't have a map of  $\alpha(x, y)$ , only an approximation  $p(x, y)$  (taken from some family,  $\mathcal{P}$ , of easy to compute functions (3)) stored on an inboard computer. That is our computer "believes" the terrain is  $p(x, y)$ ,

The computer computes the location to be at the point  $(u, v)$  that minimizes  $|\alpha(x, y) - p(u, v)|$  (4).

We compute the error in location to be

$$E_\alpha(p)(x_o, y_o) = |(x_o, y_o) - (u, v)| \text{ (5)}.$$

Since  $E_\alpha(p)(x, y)$  is a function of  $(x, y)$ , and we can compute a norm for it (6).

The Locator Problem is to find the  $p \in \mathcal{P}$  to minimize  $E_\alpha(p)$ .

### 4. General Setting

The description in the last section evaded specifying norms, data points and families of functions involved. In fact, we put numbers (1) through (6) next to vague statements that need to be explicitly stated to make this a mathematical problem. The functions  $\alpha$  and  $p$  in the description of the last section had domain  $\mathcal{R}^2$  and range  $\mathcal{R}^1$ . Those two spaces can also be more general: thus giving us eight concepts to define for a precise problem. We list below these eight concepts:

Banach spaces  $(X, d)$  and  $(Y, \rho)$  to be the domain and range of the functions (7) and (8).

A class of functions  $A : X \rightarrow Y$  to play the role of the altitude **(1)**.

A class of approximating functions  $\mathcal{P}$  **(2)**.

For  $x \in X$ , a set of data points  $s(x)$  **(3)**.

A norm,  $\tilde{\rho}$ , for functions defined on  $s(x)$  **(4)**.

Put,  $e_{\alpha,p}(x) = \inf\{\tilde{\rho}[\alpha(s(x)), p(s(u))]; u \in X\}$   
 (how well  $p(u)$  approximates  $\alpha(x)$  on the data points  $s(u)$  and  $s(x)$ ).

Let  $T_{\alpha,p}(x) = \{u \in X : \tilde{\rho}[\alpha(s(x)), p(s(u))] = e_{\alpha,p}(x)\}$ .  
 (The points at which  $p(u)$  best approximates  $\alpha(x)$ )

A metric  $\tilde{d}$  on subsets of  $X$  **(5)**.

So  $\tilde{d}(\{x\}, T_{\alpha,p}(x))$  is a function on  $X$  that measures the distance between  $x$  and the points whose  $p$  values look the most like  $\alpha(x)$ .

Put  $E_{\alpha}(p) = \|\tilde{d}(\{x\}, T_{\alpha,p}(x))\|$  (the norm  $\|\cdot\|$  is to be designated **(6)**).

**Locator Problem:** Given an  $\alpha \in A$  find a  $p \in \mathcal{P}$  so that  $E_{\alpha}(p) = \inf\{E_{\alpha}(p) : p \in \mathcal{P}\}$ .

## 5. Background

The first papers on the Locator Problem appeared in the early 1990s, principally by V.I. Berdyshev (see 1992).

As I stated the Locator Problem—that of finding and determining the uniqueness of a best locator function—there were previously no solutions in any settings.

The only results of any kind are proven in the most basic settings such as  $X = Y = \mathcal{R}^1$ , and using only one data point. Berdyshev investigated questions that involved concepts such as the smoothness of the functions involved and the error of approximation. His approximates were  $\mathcal{P}_n - \mathcal{P}_0$  and often  $\mathcal{P}_1 - \mathcal{P}_0$  (the non-constant linear functions).

Without some restrictions on the setting even the simplest setting ( $\mathcal{P} = \mathcal{P}_1 - \mathcal{P}_0$ ) can yield no—or many— solutions.

**Examples on  $[-1, 1]$ :**

$\alpha(x) = 1 - \sqrt{1 - |x|}$  has no best locator functions.

$\alpha(x) = |x|$  has  $cx + 1$  as a best locator functions for every  $|c| \geq 1$ .

## 6. "A Correct" Setting

Let

$$\mathcal{A} = \{f \in C[0, 1]; f(0) = 0, f(1) = 1, f \text{ increasing}\},$$

and

$$\mathcal{P} = \mathcal{P}_n \cap \mathcal{A},$$

then for  $\alpha \in \mathcal{A}$  and  $p \in \mathcal{P}$ ,

$$\begin{aligned} e_{\alpha,p}(x) &= |x - p^{-1}(\alpha(x))|, \\ &= \|x - p^{-1}(\alpha(x))\|, \\ &= \|\alpha^{-1}(x) - p^{-1}(x)\|. \end{aligned}$$

Therefore in this setting the Locator Problem is equivalent to asking if every  $f \in \mathcal{A}$  has a unique best approximation from

$$\mathcal{P} = \{p^{-1} : p \in \mathcal{P} \cap \mathcal{A}\}$$

Note:  $\mathcal{P}$  is not convex. It is compact; so best approximations exist (and the results stated here show that they are unique).

## 7. The Approximation Problem

The last section brings us to the second question: do best approximations to  $f \in \mathcal{A}$  from

$$\{p^{-1} : p \in \mathcal{P}_n \cap \mathcal{A}\}$$

exist? If so, are they unique?

The answers are yes, but it is also true in a more general setting.

**Definition:** Let  $n > 0$ . Let  $1 = \ell_1 < \ell_2 < \dots < \ell_k \leq n$

Let  $s_1 = 1$  and  $s_i$  be either 1 or  $-1$  for  $i = 2, \dots, k$ .

Put

$$\begin{aligned} M &= \{p \in \mathcal{P}_n : s_i p^{(\ell_i)} \geq 0, p(0) = 0, p(1) = 1\}, \text{ and} \\ M^{-1} &= \{p^{-1} : p \in M\}. \end{aligned}$$

$M$  is called the monotone polynomials.

**Note:**  $M^{-1}$  is not convex.

**Theorem:** For  $f \in C[0, 1]$  such that  $f(0) = 0, f(1) = 1$ , best uniform approximations from  $M^{-1}$  exist and are unique.

**Theorem:** For  $f \in C[0, 1] \cap \mathcal{A}$ , best  $L_1$  approximations from  $M^{-1}$  exist and are unique.

**Note:** The approximations are not unique in any  $L_q[0, 1]$  for  $1 < q < \infty$ .

## 8. Questions

### (1) Approximation from Monotone Functions

- Is there a verifiable alternation characterization for the best approximation from  $M^{-1}$ ? What about from  $\{p^{-1} \in \mathcal{P}_n \cap \mathcal{A}\}$ ?
- Are there unique best approximations from other basic approximating families of functions such as:

$$\left\{ \frac{p}{q} \in \mathcal{A} : p \in \mathcal{P}_m, q \in \mathcal{P}_n, q > 0 \right\}?$$

- Are there interesting properties for approximation from the inverses of the above rational functions?

### (2) Locator

- Are there results for two data points ?
- Is there an interesting theory for the domain  $X = \mathcal{R}^2$  and  $\mathcal{P}$  is a finite dimensional space of polynomials in two variables?

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## RECENT DEVELOPMENTS IN FUNCTION FIELD ARITHMETIC

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### Notation

$\mathbb{Z}$	=	{integers}
$\mathbb{Q}$	=	{rational numbers}
$\mathbb{R}$	=	{real numbers}
$\mathbb{C}$	=	{complex numbers}
$\mathbb{Z}_+$	=	{ positive integers}
$q$	=	a power of a prime $p$
$\mathbb{F}_q$	=	A Finite field with $q$ elements
$A$	=	$\mathbb{F}_q[t]$
$A_+$	=	{monics in $A$ }
$K$	=	$\mathbb{F}_q(t)$
$K_\infty$	=	$\mathbb{F}_q((1/t))$ = completion of $K$ at $\infty$
$C_\infty$	=	completion of algebraic closure of $K_\infty$
$[n]$	=	$t^{q^n} - t$
$d_n$	=	$\prod_{i=0}^{n-1} (t^{q^n} - t^{q^i})$
$\ell_n$	=	$\prod_{i=1}^n (t - t^{q^i})$
deg	=	function assigning to $a \in A$ its degree in $t$

The purpose of this expository talk is to describe some fascinating recent developments in the Function Field Arithmetic and hopefully get some bright students working in this rapidly developing subject area. Though this name 'Function Field Arithmetic' of the subject might be unfamiliar to some, in fact, we all know the subject at some level.

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Let us look a little closely (and perhaps a little simplistically) at how the concept of numbers and arithmetic evolves in school. In school, we first learn the concept of and manipulations with the ‘counting numbers’. Then we see that many simple day-to-day problems can be formulated into linear equations involving counting numbers and to solve these it is helpful to introduce zero, negative numbers and fractions (i.e. rational numbers). But once we introduce the unknown or variable ‘ $x$ ’, just as starting from zero, one and addition, subtraction, multiplication leads us to ring of integers  $\mathbb{Z}$  and with division, the field of rational numbers  $\mathbb{Q}$ , adding  $x$  to zero and one, we are led to the polynomial ring and the field of rational functions (say with  $\mathbb{Q}$ -coefficients to start with). In addition to dealing with polynomial equations with unknown etc., we also learn to manipulate with polynomials and rational functions on equal footing with the integers and rational functions: Arithmetic operations, multiplication and division algorithms, factorizations into primes or irreducible polynomials respectively, greatest common divisors and so on. This is the introduction everybody has to the Function Field Arithmetic at its simplest level.

Soon, either through the geometric concept of length or the calculus concept of limits, we are introduced to real numbers and calculus also leads to power series and Laurent series through Taylor series developments.

Finally, we get the best analogies when we take the coefficient field of polynomials, rational functions or Laurent series to be a finite field, rather than say  $\mathbb{Q}$  or  $\mathbb{R}$  etc. for the simple reason that when we divide an integer by another non-zero integer  $n$  say, there are finitely many possibilities for the remainder, i.e. non-negative integers smaller than  $|n|$ , but when you divide by a non-zero polynomial  $n$ , a remainder can be any polynomial of smaller degree and hence there are infinitely many possibilities, unless the coefficient field is finite.

(While function fields with complex coefficients are also very useful sources of analogies with connections to Riemann surfaces and complex analysis techniques, for number theory, finite field coefficient are better for the reason explained).

In summary, we have the basic analogies:

$$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \iff A = \mathbb{F}_q[t], K = \mathbb{F}_q(t), K_\infty = \mathbb{F}_q((1/t)), C_\infty$$

where we have replaced  $x$  by more traditional choice  $t$  in this subject.

For usual numbers, the notion of the size is the usual absolute value while for rational functions it is the absolute value coming through ‘degree’, which is ‘non-archimedean’, in the sense that degree of the addition is at most the maximum of the degrees of the terms added, which is stronger than the usual ‘triangle inequality’. But the difference in the two cases is

not so huge once we realize that, in fact, in addition to these notions of ‘size at infinite prime’, there are notions of sizes for numbers and functions for each prime and irreducible polynomials respectively and all these so-called p-adic absolute values are non-archimedean. There is moreover a ‘product formula’  $\prod |k| = 1$  where the product is over all the absolute values suitably normalized.

A famous theorem of Artin-Whaples says roughly that any field with notions of sizes linked by the product formula is a finite extension of  $\mathbb{Q}$  (i.e. a number field) or of  $k(x)$  (i.e., a function field) for some field  $k$ . Thus (with  $k$  finite, as explained above) these fields, called global fields, are studied together in number theory. A nice parallel treatment of basic algebraic number theory and even the class field theory (i.e. theory of abelian extensions) was given for both global number and function fields in the first half of the last century.

Important success during that period is the proof, due to Hasse and Weil, of the Riemann hypothesis for function fields (for its higher dimensional generalization, Deligne got the 1978 Fields medal), for the zeta function defined by Artin by following an analogy with Riemann and Dedekind zeta functions. Namely, we associate a zeta function to a global field by the Euler product

$$\zeta(s) = \prod (1 - \text{Norm}(\wp)^{-s})^{-1},$$

where the product is over all ‘non-archimedean primes’  $\wp$  and the norm of a prime is number of residue (remainder) classes it has. This zeta function converges in certain half plane and can be continued to the whole complex plane. But it is a complex-valued (not  $C_\infty$ -valued) simple rational function in  $q^{-s}$  for function fields over  $\mathbb{F}_q$  and thus loses rich transcendental nature of Riemann zeta and special values involving  $\pi$  etc.

Work of Carlitz in 1930’s and work of Drinfeld in 1970’s brought in a new type of analogies introducing  $C_\infty$ -valued analogs of exponential and zeta. We will give quick introduction, but refer to the literature mentioned below for motivation and more properties and analogies.

The Carlitz exponential is

$$e(z) = \sum z^{q^i} / \prod_{j=0}^{i-1} (t^{q^i} - t^{q^j}) = \sum z^{q^i} / d_i$$

For  $a \in A$ , define the polynomial  $C_a(z)$  as follows: Put

$$C_1(z) = z, \quad C_t(z) := tz + z^q, \quad C_{t^n}(z) := C_{t^{n-1}}(C_t(z))$$

and extend by  $\mathbb{F}_q$ -linearity in  $a$ . Then the exponential satisfies for  $a \in A$ , a functional equation  $e(az) = C_a(e(z))$  analogous to classical  $e^{nz} = (e^z)^n$ , for

$n \in \mathbb{Z}$ . This leads to analogous situation

$$a \rightarrow z \mapsto C_a(z) : A \rightarrow \text{End } G_a \iff n \rightarrow (z \mapsto z^n) : \mathbb{Z} \rightarrow \text{End } G_m,$$

where  $G_a$  and  $G_m$  are the additive and multiplicative group respectively.

Just as the usual exponential has a period lattice  $2\pi i\mathbb{Z}$ , the Carlitz exponential has period lattice:  $\tilde{\pi}A$  for some  $\tilde{\pi} \in C_\infty$ . With this analogs of  $e$  and  $2\pi i$ , we have analogs of ‘ $a$ -th roots of unity’:

$$\{e(\tilde{\pi}b/a) : b \in A\}$$

indexed by  $a \in A$ . Adjoining these to  $K$  gives an abelian extension of  $K$  with Galois group  $(A/(a))^*$ . (Note that these are just the roots of the ‘cyclotomic polynomials’  $C_a(z)$ , by the functional equations above).

This cyclotomic theory was developed into an explicit class field theory for  $K$  and general function fields by work of Drinfeld and Hayes in 1970’s and 1980’s. In fact, generalizations (using similar objects with more general function fields and rank  $n$  period lattices) due to Drinfeld and many others eventually established Langlands correspondence between  $n$ -dimensional Galois representations and automorphic representations of  $GL_n$  in the function field case. This led to the Fields medal to Drinfeld in 1990 for  $n = 2$  case and to Lafforgue in 2002 for the general case.

In this talk, we focus instead on the arithmetic nature of special values of functions that come up in these new analogies. We focus on logarithms, Gamma (two of them) and Zeta. Let us introduce them. (Look at the references below for more details, motivation and properties).

The logarithm is the (multivalued) inverse function to the exponential and a simple branch is concretely given as

$$\log(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{\ell_n}$$

Comparison of the Taylor series of the Carlitz exponential with the usual one shows that  $d_i$  should be analog of  $q^i!$ . More generally, Carlitz-Goss factorial is defined by (where we divide by appropriate power of  $t$ ’s to make infinite product convergent to make an interpolation)

$$n! := \prod (d_i/t^{\deg(d_i)})^{n_i} \in K_\infty, \text{ where } n = \sum n_i q^i \in \mathbb{Z}_p, 0 \leq n_i < q$$

Another Gamma function, with poles at  $-A_+ \cup \{0\}$ , is defined by

$$\Gamma(z) = \frac{1}{z} \prod_{a \in A_+} (1 + \frac{z}{a})^{-1} \in C_\infty, \quad z \in C_\infty$$

Both have nice functional equations in (different) analogies with the classical case, analogs of  $(-1/2)! = \sqrt{\pi}$  have interpolations at finite primes with

special values of these connecting with algebraic Gauss sums analogs. Finally the Carlitz Zeta values are defined by

$$\zeta(s) = \sum_{a \in A_+} \frac{1}{a^s} \in K_\infty, \quad s \in \mathbb{Z}_+.$$

Carlitz' analog of Euler theorem is

$$\zeta(s)/\tilde{\pi}^s \in K \text{ for } s \text{ 'even', i.e multiple of } q - 1$$

Finally we can state the recent strong transcendence and algebraic independence results:

**Theorem 1.** *The logarithms of algebraic quantities, if linearly independent over  $K$ , are algebraically independent over  $\bar{K}$ .*

**Theorem 2.** *All algebraic monomials in factorial values at fractions are the known ones.*

**Theorem 3.** *Only algebraic relations for  $\Gamma$ -values at proper fractions are those explained by functional equations.*

**Theorem 4.** *Only algebraic relations among the zeta values together with  $\tilde{\pi}$  come from the Carlitz-Euler relation and  $\zeta(ps) = \zeta(s)^p$ .*

For complete statements and proofs we again refer to literature below.

Let us see comparison with what is known in the classical case as well the underlying structures involved and techniques used.

In the number fields case, in place of Theorem 1, Baker's famous theorem, for which he got Fields medal in 1970, proved 'linear' independence over  $\mathbb{Q}$  of logarithms of algebraic numbers, given the linear independence over  $\mathbb{Q}$ .

Theorem 1 proved by Matt Papanikolas uses the strong motivic machinery developed by Greg Anderson, which reduces algebraic dependency which is linear dependency of monomials to linear dependency questions by use of tensor powers of motives to make required monomials. The linear dependency techniques of Baker, Wustholz etc. were used and similar theorems were proved in the function field case by Jing Yu earlier.

Theorem 3 proved was proved earlier by Anderson, Brownawell and Papanikolas by similar techniques, using description of gamma values at fractions in terms of 'periods' of Anderson t-motives, to which Greg Anderson, his student Sinha and the author contributed. This algebraic incarnation of transcendental gamma values was achieved by method of 'solitons' and 'Fermat motives'.

For comparison with Theorem 2 and 3, it should be noted that classically even transcendence of individual gamma values at proper fractions

(except for denominators 2, 3, 4, 6) is not known, let alone the algebraic independence.

For the factorial function of Theorem 2, the techniques of periods led to results in close analogy with the classical case mentioned above, so the Theorem 2 was proved by the author by a different technique called ‘finite state automata’ using a theorem of Christol that a power/Laurent series  $\sum f_n t^{-n}$  in  $K_\infty$  is algebraic over  $K$  if and only if there a finite state  $q$ -automata which on input  $n$  gives output  $f_n$ .

For comparison with Theorem 4, note that classically we know  $\zeta(3)$  is irrational, but not whether it is transcendental, and we do not know what happens at other odd integers, neither do we know whether  $\zeta(3)/\pi^3$  is rational or irrational/transcendental.

The Theorem 4 was proved by Chieh-Yu Chang and Jing Yu using the techniques of Anderson-Brownawell-Papanikolas and Papanikolas and using the earlier result due to Anderson and the author which gave again algebraic incarnation of these transcendental zeta values in terms of Anderson’s  $t$ -motives (which are higher dimensional generalizations of Carlitz module we looked at, in the sense that we looked at certain embeddings  $A \rightarrow \text{End } G_a^d$ ).

Some of this may look too advanced, because it may be unfamiliar, so we end with mentioning a remarkable continued fraction formula, which can be proved by only high-school level mathematics, for  $e = e(1)$ .

Write  $[a_0, a_1, a_2, \dots]$  as a short-form for the continued fraction  $a_0 + 1/a_1 + (1/a_2 + \dots)$ .

Let us start by remarking that continued fraction expansion is unique and canonical, with no need for choice of a base. Its truncations give best possible approximations for their complexity. We understand them well for rational and quadratic irrationals, but we do not know a pattern for a single higher degree algebraic number or say for  $\pi$ .

On the other hand, Euler proved for Euler’s  $e$  the following exact formula:

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \dots].$$

In our case, we have completely different pattern (and proof). For simplicity, we specialize to  $A = \mathbb{F}_2[t]$  example. Write  $[n] = t^{2^n} - t$ . Then

$$e = [1, [1], [2], [1], [3], [1], [2], [1], [4], \dots],$$

where the pattern of block duplication

(whole block  $[1].[2].[1], [3], [1], [2], [1]$  is repeated after  $[4]$ , followed by  $[5]$  and so on) continues.

Here is another descriptions of the same pattern:

$$a_n = [\text{position of first 1 in } n \text{ base 2}] = [\text{ord}_2(2n)].$$

**References and Guide to the literature:** In our simplistic version here, we have, of course, omitted to mention many important contributions. Two books giving the background material are [6, 10], where you will find references and history. (This particular subject area started with [4]). Another nice book going at more relaxed pace is [9]. In particular, [10] gives quick motivated introduction to many objects we are considering such as exponential, zeta and gamma, which are then developed in detail later; and the chapter 10 there explains automata technique and the proof of Theorem 2.

For proofs of other main results, see the original articles listed below and expository accounts in the books as well [8] and [11].

Note added in the galley proofs(14<sup>th</sup> July 2008): Chang, Papanikolas, Thakur and Yu proved that all the algebraic relations between factorial values at proper functions together with zeta values are the known ones.

Chang, Papanikolas, Yu proved similar statement for  $\Gamma$ -values and zeta values. The proofs again use the breakthrough criterion of [2] and [7]. Pel-larin gave a simpler proof of Theorem 1 using ideas of Denis(together with those of Anderson and Papanikolas).

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## CONICS AS AXES AND JACOBIAN PROBLEM

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ABSTRACT. This is an expository article giving a modified version of my talk at the December 2007 Conference in Pune.

### 1. Introduction

Two given bivariate polynomials are said to form a jacobian pair if their jacobian equals a nonzero constant, and they are said to form an automorphic pair if the variables can be expressed as polynomials in the given polynomials. By the chain rule we see that every automorphic pair is a jacobian pair. The jacobian problem asks if conversely every jacobian pair is an automorphic pair. It turns out that a useful method for attacking this problem is to study the similarity of polynomials. Two bivariate polynomials are similar means their degree forms, i.e., highest degree terms, are powers of each other when they are multiplied by suitable nonzero constants. Geometrically this amounts to saying that the corresponding plane curves have the same points at infinity counting multiplicities. At any rate, the points at infinity correspond to the distinct irreducible factors of the degree form.

### 2. History

Before getting into technicalities, I shall first give a short history of the problem or rather the history of my acquaintance with the problem. For that we have to go back to 1965 when a German mathematician, Karl Stein who created Stein Manifolds, wrote me a letter asking a question. He said that there was an interesting 1955 paper in the *Mathematische Annalen* by Engel [14]. In this paper Engel claims to prove the jacobian theorem or what is now known as the jacobian problem or the jacobian conjecture

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or whatever. Karl Stein said to me that it is an interesting theorem but he cannot understand the proof. Can I help him? He also reduced it, or generalized it, to a conjecture about complex spaces. I wrote back to Stein giving a counterexample to his complex space conjecture. But I did not look at the Engel paper. Then in 1968, Max Rosenlicht of Berkeley asked me the same question and still I did not look at the Engel paper. Finally in 1970, my own guru (= venerable teacher) Oscar Zariski asked me the same question. Then, following the precept that one must obey one's guru, I looked up the Engel paper and found it full of mistakes and gaps.

The primary mistake in the Engel paper, which was repeated in a large number of published and unpublished wrong proofs of the jacobian problem in the last thirty-five years, is the presumed "obvious fact" that the order of the derivative of a univariate function is exactly one less than the order of the function. Being a prime characteristic person I never made this mistake. Indeed, the "fact" is correct only if the order of the function is nondivisible by the characteristic. Of course you could say that the jacobian problem is a characteristic zero problem, and zero does not divide anybody. But zero does divide zero. So the "fact" is incorrect if the order of the function is zero, i.e., if the value of the function is nonzero. Usually this mistake is well hidden inside a long argument, because you may start with a function which has a zero or pole at a given point and your calculation may lead to a function having a nonzero value at a resulting point.

A gap is a spot where you are not sure of the argument because of imprecise definitions or what have you. The gap in the Engel paper seems to be the uncritical use of the Zeuthen-Segre invariant. For this invariant of algebraic surfaces see the precious 1935 book of Zariski [23]. Over the years I have made several attempts to understand the somewhat mysterious theory of this invariant, and I still continue to do so.

In 1970-1977 I discussed the matter in my courses at Purdue and also in India and Japan. Mostly I was suggesting to the students to fix the proof and, to get them started, I proved a few small results. Notes of my lectures were taken down by Heinzer, van der Put, Sathaye, and Singh. These appeared in [2] and [5]. Then I put the matter aside for thirty years. Seeing that the problem has remained unsolved inspite of a continuous stream of wrong proofs announced practically every six months, I decided to write up my old results, together with some enhancements obtained recently, in the form of a series of three long papers [6], [9], [10], in the Journal of Algebra, dedicated to the fond memory of my good friend Walter Feit. The Pune Conference has given me a welcome opportunity of introducing these papers to the young students with an invitation to further investigate the problem.

Now one of my old results says that the jacobian conjecture is equivalent to the implication that each member of a jacobian pair can have only one point at infinity. Another says that each member of any jacobian pair has at most two points at infinity. Note that the first result is a funny statement; it only says that to prove the jacobian conjecture, it suffices to show that each member of any jacobian pair has only one point at infinity. The second result is of a more definitive nature, and it remains true even when we give weights to the variables which are different from the normal weights. Very recently I noticed that, and this is one of the enhancements, the weighted two point theorem yields a very short new proof of Jung's 1942 automorphism theorem [18]. This automorphism theorem says that every automorphism of a bivariate polynomial ring is composed of a finite number of linear automorphisms and elementary automorphisms. In a linear automorphism both variables are sent to linear expressions in them. In an elementary automorphism, one variable is unchanged and a polynomial in it is added to the second variable. In his 1972 lecture notes [19], Nagata declared the automorphism theorem to be very profound and so it did come as a pleasant surprise to me that the weighted two point theorem yields a five line proof of the automorphism theorem. For other recent enhancements let me refer to my Feit memorial papers cited above.

The present short note is only meant to whet the student's appetite. At any rate the material of this paper is based on my 2007 Pune Conference Talk and on a very short course I recently gave to Fergusson College and S. P. College students in that city.

### 3. Conics as Axes

In High-School Algebra we study factorization of polynomials. In College Analytic-Geometry we introduce the  $(X, Y)$ -axes to study geometric figures such as lines and conics. To put these subjects together we generalize the idea of axes .

**Definition:** Polynomials  $f(X, Y)$  and  $g(X, Y)$  are said to form an **axes-pair** (or automorphic pair) if  $X$  and  $Y$  can be expressed as polynomials in  $f$  and  $g$ , i.e., we can find polynomials  $u(X, Y)$  and  $v(X, Y)$  such that

$$(*) \quad X = u(f(X, Y), g(X, Y)) \quad \text{and} \quad Y = v(f(X, Y), g(X, Y)).$$

We call  $f(X, Y)$  an **axis** if  $f, g$  is an axes-pair for some  $g$ .

**Conics:** Let  $f(X, Y)$  be a nonconstant polynomial of degree  $N > 0$  and consider the plane curve  $C : f(X, Y) = 0$ . If  $N = 1$  then  $C$  is a line. If

$N = 2$  then  $C$  is a conic and

$$f(X, Y) = aY^2 + bXY + cX^2 + pX + qY + r$$

where  $a, b, c, p, q, r$  are constants with either  $a \neq 0$  or  $b \neq 0$  or  $c \neq 0$ . A conic is either a circle or an ellipse or a hyperbola or a parabola or a pair of lines (which may or not be distinct).

**First Exercise:** Show that a conic is an axis iff (= if and only if) it is a parabola.

**Hint:** The parabola  $Y^2 - X$  is an axis because  $Y^2 - X$  and  $Y$  form an axes-pair. If the conic  $f$  is not a parabola then it factors after adding a suitable constant; add 1 to the circle  $X^2 + Y^2 - 1$ , ellipse  $\frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1$ , hyperbola  $\frac{X^2}{a^2} - \frac{Y^2}{b^2} - 1$  or  $XY - 1$  (these standard forms are obtained after a suitable linear change of variables); if  $f$  is a pair of lines then take the constant to be zero. Therefore it only remains to show that (i) an axis must be irreducible (= not reducible = cannot be factored) and (ii) an axis remains an axis after adding a constant.

Assertion (ii) follows by noting that  $f, g$  axes-pair and  $a, b$  constants obviously implies  $f + a, g + b$  axes-pair.

To prove assertion (i), for any  $h(X, Y)$  let

$$h'(X, Y) = h(f(X, Y), g(X, Y))$$

where  $f, g$  is an axes-pair. Then  $(\bullet) h \mapsto h'$  is a  $k$ -automorphism of  $k[X, Y]$  where  $k$  is the field of coefficients, and hence  $h$  is irreducible iff  $h'$  is irreducible. Clearly  $f = X'$  [i.e., taking  $h(X, Y) = X$  we get  $h'(X, Y) = f(X, Y)$ ] and hence  $f$  is irreducible because  $X$  is irreducible; thus  $f, g$  axes-pair implies  $f$  irreducible.

The abstract assertion  $(\bullet)$  can be converted into an elementary argument thus. If (1)  $h(X, Y) = h_1(X, Y)h_2(X, Y)$  then substituting  $f, g$  for  $X, Y$  on both sides we get (2)  $h'(X, Y) = h'_1(X, Y)h'_2(X, Y)$  and conversely, in view of  $(*)$ , substituting  $u, v$  for  $X, Y$  in both sides of (2) we get back (1).

**Definition:** To generalize the First Exercise, for a polynomial  $f(X, Y)$ , by  $f^+$  we denote the **degree form** of  $f$ , i.e., the highest degree terms in  $f$ ; if  $f = 0$  then we put  $f^+ = 0$ .  $f^+$  gives the behaviour of  $f$  for large values of  $X, Y$ , and hence we call factors of  $f^+$  **points at infinity** of  $f$ . In particular, we say that  $f$  has **only one point at infinity** to mean that  $f$  is a nonconstant irreducible polynomial such that  $f^+ = a(bX + cY)^N$  where  $a, b, c$  are constants and  $N$  is the degree of  $f$ .

Now we observe that a circle and an ellipse have two complex points at infinity, a hyperbola has two real points at infinity, and a parabola has only

one point at infinity. This suggests the following generalization of the First Exercise.

**Second Exercise:** Show that  $f, g$  axes-pair implies  $f$  has only one point at infinity.

**Hint:** Every point of an irreducible projective variety is the center of a valuation of the function field of that variety, and conversely every valuation has a center. For details see [4] and [7].

**Future Tasks:** To make further progress in understanding the concept of an axis, I suggest that you read my Kyoto Paper [3]. You may disregard the first three pages of that paper, i.e., start with Chapter 2 on Decimal Expansion which is on pages 251-261. The Kyoto Paper is (except for the first 3 pages where it is summarized) completely self contained and can be understood by anybody with a high-school education. The other references at the end of the paper, in addition to some of my other books such as [1] and [8], contain a list of some classical algebra and analysis books which I highly recommend such as [11], [12], [13], [15], [16], [17], [20], [21], [22] and [23].

**4. Resultant or Alternative Hint to Second Exercise**

As an alternative solution of the Second Exercise, in the classical vein, we can use the Resultant introduced by Sylvester in 1840 thus; see Chapter 9 (pg. 374-404) of the Kyoto Paper [3], or Lecture 30 (pg. 267-273) of the Engineering Book [4] or Lecture 4§1 (pg. 100-104) of the Algebra Book I [7]. Assuming  $n, m$  to be nonnegative integers, the  $Y$ -Resultant of two polynomials

$$f(Y) = a_0Y^n + a_1Y^{n-1} + \dots + a_n$$

$$g(Y) = b_0Y^m + b_1Y^{m-1} + \dots + b_m$$

is the determinant  $\text{Res}_Y(f, g) = \det(\text{Resmat}_Y(f, g))$  of the  $n+m$  by  $n+m$  matrix

$$\text{Resmat}_Y(f, g) = \begin{pmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \\ b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & b_m & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & b_m & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & b_m \end{pmatrix}$$

where the first  $m$  rows consist of the coefficients of  $f$  and the last  $n$  rows consist of the coefficients of  $g$ . In detail, the first row starts with the coefficients of  $f$ , these are shifted one step to the right to get the second row, shifted two steps to the right to get the third row, and so on for the first  $m$  rows, then the  $(m + 1)$ -st row starts with the coefficients of  $g$ , these are shifted one step to the right to get the  $(m + 2)$ -nd row, and so on for the next  $n$  rows. The matrix is completed by stuffing zeroes elsewhere. The determinant  $\text{Res}_Y(f, g)$  is sometimes called the Sylvester resultant of  $f$  and  $g$  because it was introduced by Sylvester in his 1840 paper where he enunciated the following Basic Fact and Permuting, Isobaric, & Root Properties.

**Basic Fact** (formula (34) on page 391 of Kyoto Paper).

If the coefficients  $a_i, b_j$  belong to a domain  $R$  then we have:  
 $\text{Res}_Y(f, g) = 0 \Leftrightarrow n + m \neq 0$  and either  $a_0 = 0 = b_0$  or  $f$  and  $g$  have a common root in some overfield of  $R$ .

**Permuting Property** (formula (9) on page 377 of Kyoto Paper). We have

$$\text{Res}_Y(g, f) = (-1)^{mn} \text{Res}_Y(f, g).$$

**Isobaric Property** (formula (14) on p.378 of Kyoto Paper).

View the coefficients  $a_i, b_j$  as indeterminates over  $\mathbb{Z}$ . Give weight  $i$  to  $a_i$ , and  $j$  to  $b_j$ . Then  $0 \neq \text{Res}_Y(f, g) \in \mathbb{Z}[a_0, \dots, a_n, b_0, \dots, b_m]$  is isobaric of weight  $mn$ , i.e., for the weight of any monomial

$$a_0^{i_0} \dots a_n^{i_n} b_0^{j_0} \dots b_m^{j_m}$$

occurring in  $\text{Res}_Y(f, g)$  we have

$$\left( \sum_{0 \leq r \leq n} r i_r \right) + \left( \sum_{0 \leq s \leq m} s j_s \right) = mn.$$

In particular, the principal diagonal  $a_0^m b_m^n$  has weight  $mn$ , and it does not cancel out because there is no other term of  $b_m$ -degree  $n$  in the resultant; the principal diagonal of an  $N \times N$  matrix  $(A_{ij})$  is the term  $A_{11} A_{22} \dots A_{NN}$ . The resultant being isobaric of weight  $mn$  is the fundamental fact behind various cases of Bézout's Theorem.

**Root Property** (formula (28) on page 390 of Kyoto Paper).

If the coefficients  $a_i, b_j$  belong to a domain  $R$  and  $a_0 \neq 0$  then, upon writing

$$f(Y) = a_0 \prod_{1 \leq i \leq n} (Y - \alpha_i)$$

with  $\alpha_1, \dots, \alpha_n$  in an overfield of  $R$ , we have

$$\text{Res}_Y(f, g) = a_0^m \prod_{1 \leq i \leq n} g(\alpha_i).$$

By Root Prop we see that

$$(1) \quad \begin{cases} \text{if } a_0 \neq 0 \text{ and } g'(Y) = b'_0 Y^m + b'_1 Y^{m-1} + \dots + b'_m \text{ is such that} \\ g' = \nu g - hf \text{ for some constant } \nu \text{ and polynomial } h \\ \text{then } \text{Res}_Y(f, g') = \nu^n \text{Res}_Y(f, g). \end{cases}$$

Now if  $m = n$  with  $a_0 \neq 0 \neq a_n \neq 0 = a_1 = \dots = a_{n-1} = b_1 = \dots = b_{n-1}$  then

$$\left\{ \begin{array}{ll} \text{Res}_Y(f, g) \\ = a_n^{-n} \text{Res}_Y(f, a_n g - b_n f) & \text{taking } (\nu, h) = (a_n, b_n) \text{ in (1)} \\ = a_n^{-n} \text{Res}_Y(f, (a_n b_0 - b_n a_0) Y^n) & \text{by simplifying} \\ = (-1)^{n^2} (a_n b_0 - b_n a_0)^n & \text{by Permuting Prop and Root Prop} \\ & \text{if } a_n b_0 - b_n a_0 \neq 0 \\ & \text{and obviously otherwise} \end{array} \right.$$

and hence

$$\left\{ \begin{array}{l} \text{if } m = n \text{ with } a_0 \neq 0 = a_1 = \dots = a_{n-1} = b_1 = \dots = b_{n-1} \text{ then} \\ \text{(by above calculation in case } a_n \neq 0 \text{ and by Root Prop in case } a_n = 0) \\ \text{we get } \text{Res}_Y(f, g) = (-1)^{n^2} (a_n b_0 - b_n a_0)^n. \end{array} \right.$$

Thus

$$(2) \quad \begin{cases} \text{if } m = n \text{ with } a_0 \neq 0 = a_1 = \dots = a_{n-1} = b_1 = \dots = b_{n-1} \\ \text{then } \text{Res}_Y(f, g) = (-1)^{n^2} (a_n b_0 - b_n a_0)^n. \end{cases}$$

By (2) and the Isobaric Prop we get the following defo (= degree form) Property:

**Defo Property:** In the set-up of the Isobaric Property, assume  $m = n$ , and consider the bivariate (= two variable) polynomial

$$\Psi(a_n, b_n) = \text{Res}_Y(f, g) \in K[a_n, b_n]$$

over the field  $K = \mathbb{Q}(a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1})$ . Then for the (total)  $(a_n, b_n)$ -degree and degree form we have

$$\deg(\Psi) = n \quad \text{with} \quad \Psi^+ = (-1)^{n^2} (a_n b_0 - b_n a_0)^n.$$

**Proof:** By the Isobaric Property we see that  $\deg(\Psi) \leq n$  and any term of degree  $n$  in  $\Psi$  must be devoid of  $a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}$  and hence, by putting  $a_1 = \dots = a_{n-1} = b_1 = \dots = b_{n-1} = 0$  in  $f$  and  $g$ , our assertion

follows from (2).

**Corollary:** Assume that  $m + n > 0$ . Let  $A_0, \dots, A_n, B_0, \dots, B_m$  be elements in a field  $k$ , let  $X, Y, Z$  be indeterminates over  $k$ , and let

$$\begin{aligned} P(Z) &= A_0 Z^n + A_1 Z^{n-1} + \dots + A_n \\ Q(Z) &= B_0 Z^m + B_1 Z^{m-1} + \dots + B_m \\ \Phi(X, Y) &= \text{Res}_Z(P(Z) - X, Q(Z) - Y) \in k[X, Y]. \end{aligned}$$

Then we have

$$(3) \quad \Phi(P(Z), Q(Z)) = 0.$$

Moreover, if  $m = n$  with either  $A_0 \neq 0$  or  $B_0 \neq 0$  then

$$(4) \quad \deg(\Phi) = n \quad \text{and} \quad \Phi^+ = (-1)^{n^2} (A_0 Y - B_0 X)^n$$

and for any nonconstant irreducible  $F(X, Y) \in k[X, Y]$  with  $F(P(Z), Q(Z)) = 0$  we have

$$(5) \quad F^+ = -\theta (A_0 Y - B_0 X)^N$$

where  $\deg(F) = N$  and  $\theta = \mathbf{Abhyankar's\ nonzero}$  = a nonzero constant = an unspecified nonzero element of  $k$ .

**Proof:** Taking an indeterminate  $T$  over  $k(X, Y, Z)$  we obtain the equation  $\text{Res}_Z(P(Z) - P(T), Q(Z) - Q(T)) = \Phi(P(T), Q(T))$  and clearly  $Z = T$  is a common solution of  $P(Z) - P(T) = 0 = Q(Z) - Q(T)$ , and hence by the Basic Fact we get  $\Phi(P(T), Q(T)) = 0$  which yields (3). Taking  $a_i = A_i$  and  $b_i = B_i$  for  $0 \leq i \leq n-1$  with  $a_n = A_n - X$  and  $b_n = B_n - Y$ , by the Defo Property we get (4). By (3) and (4) we get (5).

**Polynomial Curve:** By a polynomial curve over a field  $k$  we mean a non-constant irreducible  $f(X, Y) \in k[X, Y]$  for which there exist  $P(Z), Q(Z)$  in  $k[Z]$ , at least one of which is not in  $k$ , such that  $f(P(Z), Q(Z)) = 0$ . Taking  $F(X, Y) = f(X, Y)$  and putting  $m = n = \max(\deg_Z P(Z), \deg_Z Q(Z))$  in the above Corollary we see that a polynomial curve has at most one point at infinity.

## 5. Equation Solving

The genesis of resultants is the other topic studied in high-school algebra which consists of the various solvings of polynomial equations. First linear equations are discussed culminating in Cramer's Rule for solving  $m$  equations in  $n$  variables. Then Bhaskaracharya's 1150 A.D. verse of Shreedharacharya's 500 A.D. completing the square method of solving one-variable



quadratic equations is studied. Turning to simultaneously solving two one-variable equations  $f(Y) = 0$  and  $g(Y) = 0$ , or rather to finding a condition for them to have a common solution, we get their resultant  $\text{Res}_Y(f, g)$  as depicted in the beginning of Section 2.

As reference for the above paragraph see pages 1-2, 100-104, 172-188 of my 2006 Algebra Book I [7], and for a classical treatment of the matter see the 1907 Higher Algebra Book of Bôcher [11].

In view of the description of Polynomial Curves given at the end of Section 2, to complete the explanation of the Second Exercise, it only remains to show that if  $f, g$  is an axes-pair then  $f$  is a polynomial curve. This can be done either abstractly or concretely thus.

**Strict Polynomial Curve:** Abstractly, for the polynomial ring  $R = k[X, Y]$ , as in (•) of the First Exercise,  $h \mapsto h'$  gives a  $k$ -homomorphism  $H : R \rightarrow R$ . Likewise, for any  $w(X, Y)$  let  $w^*(X, Y) = w(u(X, Y), v(X, Y))$ . Then we get the  $k$ -homomorphism  $W : R \rightarrow R$  given by  $w \mapsto w^*$ . The display (\*) at the beginning of Section 1 yields  $HW = WH =$  the identity map  $I : R \rightarrow R$ , and hence  $H$  and  $W$  are automorphisms. Consequently for the residue class epimorphism  $D : R \rightarrow R/(fR)$  we have  $D(R) = k[D(g)]$  and hence there is a unique  $k$ -epimorphism  $E : R \rightarrow k[Z]$  whose kernel is  $fR$  such that  $E(g) = Z$ . Let  $P(Z) = E(X)$  and  $Q(Z) = E(Y)$ . Then  $f(P(Z), Q(Z)) = 0$  and either  $P(Z) \notin k$  or  $Q(Z) \notin k$ . Thus  $f$  is a polynomial curve. Actually  $f$  is a **strict polynomial curve**, i.e.,  $(P(Z), Q(Z))$  is an **epimorphic pair** which means that

$$Z = G(P(Z), Q(Z))$$

for some  $G(X, Y) \in k[X, Y]$ ; indeed  $G(X, Y) = g(X, Y)$ .

Concretely speaking, putting  $P(Z) = u(0, Z)$  and  $Q(Z) = v(0, Z)$  gives us

$$f(P(Z), Q(Z)) = 0 \quad \text{and} \quad g(P(Z), Q(Z)) = Z.$$

**Epimorphic Pair:** In the Kyoto Paper it is proved that, in case of a characteristic zero field  $k$ , we have:  $(P(Z), Q(Z))$  epimorphic pair implies either the degree of  $P(Z)$  divides the degree of  $Q(Z)$  or the degree of  $Q(Z)$  divides the degree of  $P(Z)$ , and from this it is deduced that every strict polynomial curve is an axis. Examples show that both these are false for nonzero characteristic.

## 6. Newton's Theorem and Jacobian Problem

The above degree dividing property of an epimorphic pair gives us entry into the Jacobian Problem which is the high-school incarnation of the inverse function theorem of calculus. For a detailed treatment of the inverse

function theorem of calculus together with its mate the implicit function theorem, see Chapter II of my Local Analytic Book [1]. The said implicit function theorem is the often neglected foundation of the method of implicit differentiation as exemplified by the following example.

$$f(X, Y) = 0 \Rightarrow f_X dX + f_Y dY = 0 \Rightarrow \frac{dY}{dX} = \frac{-f_X}{f_Y}.$$

But what if  $f_X = f_Y = 0$  such as when  $f(X, Y) = Y^2 - X^3$  (cusp) or  $f(X, Y) = Y^2 - X^2 - X^3$  (node) and we are evaluating  $\frac{dY}{dX}$  at the origin  $(X, Y) = (0, 0)$ ? For the resulting theory of singularities see Lectures 1 and 5 of my Engineering Book [4].

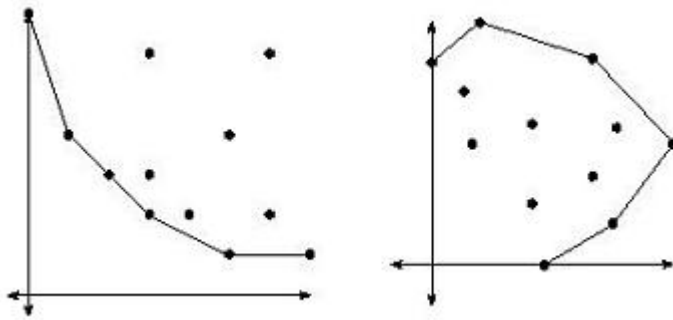
At any rate, the inverse function theorem of calculus says that if  $n$  functions of  $n$  variables (with sufficiently many continuous partial derivatives) are zero at the origin but their jacobian is not then, locally near the origin, the variables are functions of the functions. The Jacobian Problem asks if this remains true if the only permissible functions are polynomials. For instance, in case of  $n = 2$ , given polynomials  $f(X, Y), g(X, Y)$  with  $J(f, g) = \neq 0$  we are asking if  $X, Y$  are polynomials in  $f, g$ , i.e., if  $f, g$  is an axes-pair. Here  $J(f, g) = f_X g_Y - g_X f_Y$  with subscripts denoting partial derivatives. Surprisingly, the answer is not known even for  $n = 2$ . More about this later on. At present see Lectures 22-23 of the Engineering Book as well as my recent papers [6] and [9]. At any rate, according to these references, the 2 variable Jacobian Problem is equivalent to showing that if  $J(f, g) = \neq 0$  then out of the (total) degrees of  $f$  and  $g$ , one divides the other.

To prove the degree dividing property of an epimorphic pair, we apply Newton's Theorem on fractional expansion to the minimal equation  $\Phi(X, Y) = 0$  satisfied by the pair. For details see [3].

## 7. Degreewise Newton Polygon

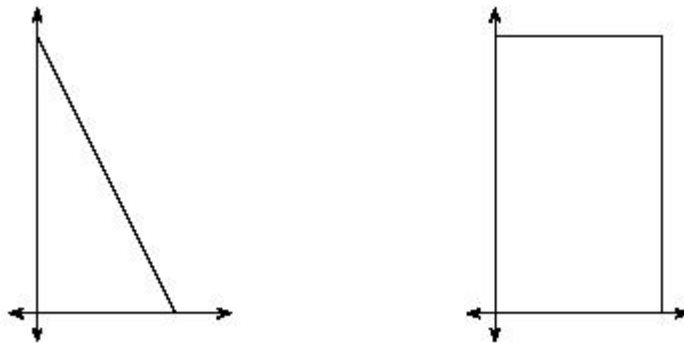
As an aid to proving his theorem, Newton considered the support  $\text{Supp}(\Phi) =$  the set of all integer pairs  $(i, j)$  such that the coefficient of  $X^i Y^j$  in  $\Phi(X, Y)$  is nonzero, and he drew the polygon which is the boundary of the convex hull of that support, as depicted in the left hand picture below. Good discussions of the Newton Polygon can be found on pages 373-396 of volume II of Chrystal's book [12] and on pages 98-106 of Walker's book [22]. One important property of the Newton Polygon is the fact that the slopes of the various sides are the orders of the various roots of  $\Phi$ . Moreover, the integral part of the slope of the first side equals the number of Quadratic Transforms required to decrease multiplicity of the plane analytic curve  $\Phi(X, Y) = 0$ . See the books [4] and [7].

Now assume that  $f$  is a nonzero polynomial in  $X$  and  $Y$ . Then the support is a nonempty finite set. So the convex hull will look something like the following diagram on the right, and we call it the Degree-wise Newton Polygon or DNP. Since this was introduced in my lecture notes [2] and [5], people also call it the Abhyankar Polygon.



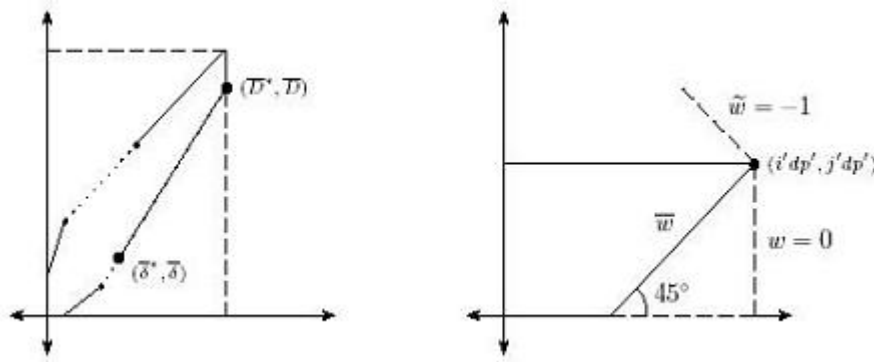
### 8. Triangle and Full Rectangle

The significance of the DNP for the Jacobian Problem is the fact that a positive answer to that problem is equivalent to showing that if  $J(f, g) = -\theta$  then the DNP of  $f$  is a triangle as depicted in the following picture on the left, while a negative answer is equivalent to showing that for some pair  $f, g$  with  $J(f, g) = -\theta$  the DNP of  $f$  is a full rectangle as depicted in the following picture on the right.



### 9. Squeezed Rectangle and Shortsqueezed Rectangle

Now if  $J(f, g) = -\theta$  but  $f$  has two points at infinity then, by a change of coordinates, its DNP can be arranged to be squeezed vertical rectangle as depicted in the following picture on the left, or a shortsqueezed horizontal rectangle as depicted in the following picture on the right where the exhibited angle could be smaller than  $45^\circ$ .



### Epilogue

MANGALACHARAN  
 ATA VISHVATMAKE DEVE | YENE VAGYADNYE TOSHAVE  
 TOSHONI MAJA DYAVE | PASAYDANA HE  
 GANITAVIDYECHEE JAGRUTEE | KARONIYA SARVA JAGATEE  
 PRADNYASURYE UJALATEE | SUKHAVAYA SAKALA JANA

Here is a free Paraphrase of the above MAGALACHARAN = INVOCATION in my mother tongue MARATHI whose founding father DNYANESHVAR composed the first two lines around 1250 A.D. to which I added the last two lines.

**Paraphrase:** May the Lord God of the Universe be pleased with my recounting of the story of algebra and geometry which are the essence of our beloved subject of mathematics. Being pleased may he shower his blessings upon us and make our endeavor pleasurable.

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## ANTI HOPFIAN AND ANTI CO-HOPFIAN MODULES

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### 1. Introduction

Throughout we deal with unital right modules. Hirano and Mogami refer to a module  $M$  as anti hopfian if  $M \neq 0$ ,  $M$  is not simple and every non-zero factor module  $\overline{M}$  of  $M$  is isomorphic to  $M$ . Referring to a ring as  $CH$  if all cyclic modules over the ring are hopfian, they completely characterise anti hopfian modules over  $CH$  rings by properties of the lattice  $L(M)$  of all the submodules of  $M$  and use this characterization to study the endomorphism ring  $End(M)$  of such a module [7]

A non-zero module  $M$  is said to be subdirectly irreducible if the intersection of all non-zero submodules of  $M$  is itself non-zero. A module  $M$  is called cocyclic if  $M$  possesses a simple submodule  $S$  which is essential in  $M$ . On page 115 of [12] it is shown that the class of subdirectly irreducible modules is the same as the class of cocyclic modules. The proofs in Hirano-Mogami [7] rely heavily on the fact that any non-zero module  $M$  admits a cocyclic quotient.

Dualizing the concept of an anti hopfian module we introduce the concept of an anti co-hopfian module. A non-zero module  $M$  is said to be anti co-hopfian if  $M$  is not simple and every non-zero submodule of  $M$  is isomorphic to  $M$ . The notion dual to that of a cocyclic module is that of a local module. A module  $M$  is called local if the sum of all proper submodules of  $M$  is a proper submodule of  $M$ . This largest proper submodule of  $M$  is then the Jacobson radical  $J(M)$  of  $M$  and  $M/J(M)$  is simple. While it is true that every non-zero module admits a cocyclic factor module, it is not true

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*Key words and phrases:* anti-hopfian, anti co-hopfian, cocyclic, local, lattice, well-ordered, anti well-ordered.

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that every non-zero module admits a local submodule. We give a complete characterization of rings possessing the property that every non-zero module admits a local submodule. We show that this class of rings is closed under the formation of finite direct products. By means of examples we show that this class is not closed under the formation of infinite direct products. Also we show that semi perfect rings belong to this class.

Define a ring to be coch if every cocyclic module is co-hopfian. All principal ideal domains have this property. Let  $R$  be a coch ring and  $M$  an  $R$ -module admitting a local submodule. Then we find necessary and sufficient conditions for  $M$  to be anti co-hopfian in terms of the lattice  $L(M)$  of submodules of  $M$  and use these to study the endomorphism ring  $End(M)$  of a such a module. These dualize the results of Hirano and Mogami.

Let  $R$  be a ring. As a consequence of the celebrated Hopkins and Levitzki theorem we will show that  $L(R_R)$  is well ordered if and only if  $L(R_R)$  is isomorphic to a finite ordinal. More genreally as a consequence of the generalised Hopkins-Levitzki theorem ([8] pg 59) we will show that if  $R$  is semi-primary, for a right module  $M$ , the lattice  $L(M)$  will be well-ordered if and only if  $L(M)$  is isomorphic to a finite ordinal. The case of a commutative discrete valuation ring  $R$  shows that  $L(R_R)$  could be anti well-ordered without  $L(R_R)$  being anti isomorphic to a finite ordinal.

The results presented by me in the centenary conference are appearing in Contemporary mathematics, AMS [11]. For more details the reader should refer to [11].

## 2. Results of Szélpál [10], Szele [9], Hirano and Mogami [7]

In [10], I. Szélpál determined all the abelian groups  $A$  possessing property  $E_1$  mentioned below.

$E_1$ : Any non-zero factor group  $\bar{A}$  of  $A$  is isomorphic to  $A$ .

In [9], T. Szele determined all the abelian groups  $A$  possessing property  $E_2$  mentioned below.

$E_2$ : Every non-zero endomorphism of  $A$  is an automorphism of  $A$ .

We will include easy proofs of these results.

**Theorem 2.1. (Szélpál)** The only non-zero abelian groups possessing property  $E_1$  are  $\mathbb{Z}_p$  (the cyclic group of order  $p$ ) and the Prüfer groups  $\mathbb{Z}_{p^\infty}$  with  $p$  a prime (upto an isomorphism).

**Proof:** When  $p$  is a prime,  $\mathbb{Z}_p$  is a simple abelian group and these are the only simple abelian groups. The proper subgroups of  $\mathbb{Z}_{p^\infty}$  are  $\mathbb{Z}_{p^k}$  with  $k$  an integer  $\geq 1$  and  $\mathbb{Z}_{p^\infty}/\mathbb{Z}_{p^k} \approx \mathbb{Z}_{p^\infty}$ .



Conversely, let  $A \neq 0$  be an abelian group possessing property  $E_1$ . Let  $D$  denote the largest divisible abelian subgroup of  $A$ . Then  $A = D \oplus B$  with  $B$  reduced. In case  $D \neq 0$ ,  $D$  admits a factor module isomorphic to  $\mathbb{Z}_{p^\infty}$  for some prime  $p$ . From property  $E_1$  we see that  $A \approx \mathbb{Z}_{p^\infty}$ . In case  $D = 0$ , we should have  $B \neq 0$ . Since  $B$  is reduced,  $pB \subsetneq B$  for some prime  $p$ . In this case  $B/pB$  is a direct sum of copies of  $\mathbb{Z}_p$ . Since  $B/pB \neq 0$ ,  $\mathbb{Z}_p$  is a non-zero factor module of  $B$ . From property  $E_1$  we get  $A \approx \mathbb{Z}_p$ . This completes the proof of Theorem 2.1.  $\square$

**Theorem 2.2. (Szele)** The only non-zero abelian groups  $A$  possessing property  $E_2$  are  $\mathbb{Z}_p$ ,  $p$  a prime and  $\mathbb{Q}$  the additive group of rational numbers.

**Proof:** Since  $\mathbb{Z}_p$  is simple for any prime  $p$ , any non-zero homomorphism of  $\mathbb{Z}_p$  to  $\mathbb{Z}_p$  is an automorphism. Also using the fact that  $\mathbb{Q}$  is torsion free it is easily checked that any homomorphism  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  in mod- $\mathbb{Z}$  is a homomorphism in mod- $\mathbb{Q}$ . Since  $\mathbb{Q}$  is a 1-dimensional vector space over  $\mathbb{Q}$ , we see that any non-zero homomorphism  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  in mod- $\mathbb{Q}$  is an isomorphism.

Conversely, let  $A \neq 0$  be an abelian group possessing property  $E_2$ . Let  $n$  be any integer  $\geq 2$ . Let  $\theta_n : A \rightarrow A$  be given by  $\theta_n(a) = na$  for any  $a \in A$ . From property  $E_2$ , either  $\theta_n = 0$  or  $\theta_n : A \rightarrow A$  is an isomorphism.

Suppose there exist an integer  $n \geq 2$  with  $\theta_n = 0$ . We claim that the smallest integer  $d \geq 2$  with  $\theta_d = 0$  is a prime. If  $d$  is not a prime,  $d = kl$  with  $1 < k < d$ ,  $1 < l < d$ . Also  $\theta_k \neq 0$  and  $\theta_l \neq 0$ . In particular  $\theta_k : A \rightarrow A$  and  $\theta_l : A \rightarrow A$  are isomorphisms by property  $E_2$ . It follows that  $\theta_d(A) = \theta_k \circ \theta_l(A) = A \neq 0$ , a contradiction. Thus  $\theta_p = 0$  for a certain prime  $p$ . This means  $A$  is a vector space over  $\mathbb{Z}_p$ . If  $\dim A$  over  $\mathbb{Z}_p > 1$ , there exist non-zero  $\mathbb{Z}_p$ -homomorphisms  $f : A \rightarrow A$  which are not isomorphisms, contradicting  $E_2$ . Hence  $A \approx \mathbb{Z}_p$ .

If on the other hand  $\theta_n : A \rightarrow A$  is an isomorphism for all  $n \geq 2$ ,  $A$  is a torsion free divisible abelian group, hence  $A$  is a vector space over  $\mathbb{Q}$ . If  $\dim A$  over  $\mathbb{Q} > 1$ , there exist non-zero  $\mathbb{Q}$ -homomorphisms  $f : A \rightarrow A$  which are not isomorphisms, contradicting  $E_2$ . Hence  $A \approx \mathbb{Q}$ . This completes the proof of Theorem 2.2.  $\square$

Recall that an  $R$ -module  $M$  is said to be hopfian if every surjective endomorphism  $f : M \rightarrow M$  of  $M$  is an automorphism. In [7] Hirano and Mogami refer to a submodule  $N$  of  $M$  to be a non-hopf kernel for  $M$  if  $M/N$  is isomorphic to  $M$ . Refer to a submodule  $N$  of  $M$  as a proper submodule of  $M$  if  $N \neq M$ . Hirano and Mogami define a module  $M$  to be anti hopfian if  $M \neq 0$ ,  $M$  is not simple and every proper submodule of  $M$  is a non hopf kernel for  $M$ . As an immediate consequence of Theorem 2.1 we

get the following result.

**Theorem 2.3.** In  $\text{mod-}\mathbb{Z}$  the only anti hopfian modules (upto an isomorphism) are  $\mathbb{Z}_{p^\infty}$  with  $p$  any prime.

Let  $\omega$  denote the first limit ordinal and  $\omega^+ = \{\xi \mid \xi \text{ an ordinal } \leq \omega\}$ . The  $\mathbb{Z}$ -submodules of  $\mathbb{Z}_{p^\infty}$  are  $0, \mathbb{Z}_{p^k}$  with  $k$  any integer  $\geq 1$  and  $\mathbb{Z}_{p^\infty}$ . It follows that  $L(\mathbb{Z}_{p^\infty})$  is lattice isomorphic to  $\omega^+$ . Also  $\text{Soc}(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_p$  and  $\mathbb{Z}_{p^\infty}/\mathbb{Z}_p \approx \mathbb{Z}_{p^\infty}$ . Thus  $\text{Soc}(\mathbb{Z}_{p^\infty})$  is a non-hopf kernel for  $\mathbb{Z}_{p^\infty}$ .

Inspired by the above observation, Hirano and Mogami proved the following significant result in [7].

**Theorem 2.4.** Let  $R$  be a right CH-ring (i.e every cyclic right  $R$ -module is hopfian) and  $M$  a right  $R$ -module. Then the following are equivalent.

- (1)  $M$  is anti hopfian.
- (2)  $L(M)$  is isomorphic to  $\omega^+$  and  $\text{Soc}(M)$  is a non-hopf kernel for  $M$ .

This is Theorem 2 in [7]. We end this section by stating the dual of Szélpal's result (Theorem 2.1).

**Theorem 2.5.** The only non-zero abelian groups  $A$  possessing the property that any non-zero subgroup  $B$  of  $A$  is isomorphic to  $A$ , are upto isomorphism,  $\{\mathbb{Z}_p \mid p \text{ a prime}\}$  and  $\mathbb{Z}$ .

This is Proposition 3.6 in [11]

### 3. Characterization of modules with $L(M)$ well-ordered (resp. anti well-ordered)

The proof of Theorem 2 of [7] uses an auxiliary proposition :

**Proposition 3.1.** Let  $M$  be an anti hopfian module over an arbitrary ring. Then  $L(M)$  is well-ordered.

This is Proposition 1 in [7]. This led me to the following question. Characterize modules  $M$  satisfying the condition that  $L(M)$  is well-ordered. The following result proved in [11] answers this question completely.

**Theorem 3.1.** Let  $M \neq 0$ . Then  $L(M)$  is well-ordered if and only if all the non-zero factor modules of  $M$  are cocyclic.

This is Theorem 2.2 in [11]. Naturally one is led to the following dual question. Characterize the modules  $M$  satisfying the condition that  $L(M)$  is anti well-ordered. The following result proved in [11] completely answers this question.

**Theorem 3.2.** Let  $M \neq 0$ . Then  $L(M)$  is anti well-ordered if and only if every non-zero submodule  $N$  of  $M$  is local.

This is Theorem 2.3 in [11].

**Remark 3.1.** If  $L(M)$  is well-ordered, then  $M$  is artinian, hence  $M$  is cohopfian. If  $L(M)$  is anti well-ordered, then  $M$  is noetherian, hence  $M$  is hopfian.

**Proposition 3.2.** (a) Suppose  $M$  is noetherian. Then  $L(M)$  is well-ordered if and only if  $L(M)$  is isomorphic to a finite ordinal.

(b) Suppose  $M$  is artinian. Then  $L(M)$  is anti well-ordered if and only if  $L(M)$  is anti isomorphic to a finite ordinal.(equivalently isomorphic to a finite ordinal)

(c) If  $L(M)$  is isomorphic to a finite ordinal, all the submodules and factor modules of  $M$  will share the same property. In particular the non-zero ones among these will simultaneously be cocyclic and local.

This is Proposition 2.5 in [11].

#### 4. Rings satisfying the condition that any non-zero $M \in \text{mod} - R$ possesses a local submodule

By a right ideal of  $R$  we mean a proper right ideal  $I$  of  $R$ .

**Theorem 4.1.** The following conditions are equivalent for a ring  $R$ .

- (a) Every non-zero  $M \in \text{mod} - R$  contains a local submodule.
- (b) For every right ideal  $I$  of  $R$ , there exists a  $\lambda \in R \setminus I$  (depending on  $I$ ) satisfying the condition that there is only one maximal right ideal in  $R$  containing  $(I : \lambda) = \{r \in R \mid \lambda r \in I\}$ .

**Corollary 4.1.** Let  $R$  be a commutative ring. Then every non-zero  $R$ -module  $M$  contains a local module if and only if for any ideal  $I$  of  $R$ , we can find a  $\lambda \in R \setminus I$  satisfying the condition that  $R/(I : \lambda)$  is a local ring.

See Theorem 4.5 and Corollary 4.8 in [11].

Let  $(P)$  denote the property mentioned below :

Every non-zero  $M \in \text{mod} - R$  contains a local module.

**Proposition 4.1.** The class of rings possessing property  $(P)$  is closed under the formation of finite direct products.

This is Proposition 4.9 in [11].

**Example 4.1.** Let  $J$  be an infinite set and  $R_\alpha = K$  for every  $\alpha \in J$ , where  $K$  is a given field. Let  $R = \prod_{\alpha \in J} R_\alpha$ . Then each  $R_\alpha$  possess property  $(P)$ . It is shown in [11] that  $R$  does not have property  $(P)$ . See example 4.11 [11].

**Theorem 4.2.** Any semi-perfect ring possesses property  $(P)$ .

This is Theorem 4.2 in [11].

### 5. Hirano-Mogami revisited

Theorem 2.4 stated in the present article can be strengthened to yield the following

**Theorem 5.1.** Let  $R$  be a ring satisfying the condition that local right  $R$ -modules are hopfian. Let  $M \in \text{mod-}R$ . Then the following are equivalent

- (1)  $M$  is anti hopfian.
- (2)  $L(M)$  is isomorphic to  $\omega^+$  and  $\text{Soc}(M)$  is a non-hopfian kernel for  $M$ .

### 6. Anti co-hopfian modules

A Ring  $R$  will be called (right) coch if every cocyclic right  $R$ -module is co-hopfian. All commutative principal ideal domains are coch. One of the major results in [11] is the following dual to Theorem 2.4 in the present article.

**Theorem 6.1.** Let  $R$  be a coch ring and  $M \in \text{mod-}R$  satisfy the condition that  $M$  admits a local submodule. Then the following conditions are equivalent.

- (1)  $M$  is anti co-hopfian.
- (2)  $L(M)$  is anti isomorphic to  $\omega^+$  and  $M \approx J(M)$ .

For a complete proof of this Theorem please refer to § 6 of [11].

**Theorem 6.2.** Let  $R$  be a coch ring and  $M \in \text{mod-}R$  an anti co-hopfian module admitting a local submodule. Then the following hold

- (a) For every  $0 \neq N \subseteq M$  the factor module  $M/N$  is a cocyclic module of finite length with a unique composition series.
- (b)  $S = \text{End}(M_R)$  is an integral domain with a unique maximal right ideal.

- (c) There exists an element  $\theta \in S$  satisfying the condition that  $\theta S$  is the unique maximal right ideal of  $S$  and  $\{\theta^k S/k \text{ an integer } \geq 1\}$  is the family of all proper right ideals of  $S$ .
- (d)  $S$  is right duo in the sense that all the right ideals of  $S$  are two sided.

This is Theorem 6.4 in [11].

### 7. Some Examples

In this section we describe some examples and counter examples constructed using the theory of formal triangular matrix rings and modules over them ([5] and [6]).

**Example 7.1.** Let  $p$  be a prime and let  $\mathbb{Z}_{p^\infty}$  be regarded as a  $(\mathbb{Z}, \mathbb{Z})$ -bimodule in the usual way. Let  $T$  be the formal triangular matrix ring

$$T = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_{p^\infty} & \mathbb{Z} \end{pmatrix}.$$

Then the right  $T$ -module  $(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z})_T$  corresponding to the triple  $(\mathbb{Z}_{p^\infty}, \mathbb{Z})_f$  where  $f : \mathbb{Z} \otimes \mathbb{Z}_{p^\infty} \rightarrow \mathbb{Z}_{p^\infty}$  is the map  $f(k \otimes x) = kx$  for  $k \in \mathbb{Z}$  and  $x \in \mathbb{Z}_{p^\infty}$  is a cocyclic module which is not co-hopfian.

**Example 7.2.** Let  $K$  be a field and  $K(x)$  the field of rational functions in one indeterminate  $x$  over  $K$ . Let  $K(x^2)$  be the subfield of rational functions in  $x^2$ . Regard  $K(x)$  as a  $(K(x^2), K(x))$ -bimodule in the usual way. Let

$$T = \begin{pmatrix} K(x) & 0 \\ K(x) & K(x^2) \end{pmatrix}.$$

Then  $T$  is simultaneously left and right artinian. Consider the right  $T$ -module  $(X \oplus Y)_T$  corresponding to the triple  $(X_{K(x)}, Y_{K(x^2)})_f$  where  $X = K(x)$  in mod- $K(x)$ ,  $Y = K(x)$  in mod- $K(x^2)$  and

$$f : Y \otimes_{K(x^2)} X = K(x) \otimes_{K(x^2)} K(x) \longrightarrow X = K(x)$$

corresponds to  $f(\lambda \otimes \mu) = \lambda\mu$ . Then  $(X \oplus Y)_T$  is artinian cocyclic admitting a non-zero non cocyclic factor module.

**Example 7.3.** Let  $T = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ . Consider the right  $T$ -module  $(\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q})_T$  corresponding to the triple  $(\mathbb{Q}/\mathbb{Z}, \mathbb{Q})_f$  where  $f : \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$  correspond to the canonical quotient map  $\eta : \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$  when we identify  $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$  with  $\mathbb{Q}$  via  $r \otimes s \mapsto rs$ . Then  $(\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q})_T$  is a non-hopfian local module.

**Example 7.4.** Let  $K$  be a field and  $x$  an indeterminate over  $K$ .

$$T = \begin{pmatrix} K(x^2) & 0 \\ K(x) & K(x^2) \end{pmatrix}.$$

The right  $T$ -module  $(X \oplus Y)_T$  corresponding to the triple  $(X, Y)_f$  where  $X = K(x)$  in  $\text{mod-}K(x^2)$ ,  $Y = K(x^2)$  in  $\text{mod-}K(x^2)$  and

$$f : K(x^2) \otimes_{K(x^2)} K(x) \longrightarrow K(x)$$

is the map  $f(\lambda \otimes \mu) = \lambda\mu$  is a noetherian local  $T$ -module admitting a non-zero non-local submodule

Examples 7.1 to 7.4 are respectively examples 3.1 to 3.4 of [11]. For more details the reader should consult [11].

## 8. Some Open Problems

F. Dischinger [2], [3] and independently E. P. Armendariz, J. W. Fisher and R. L. Snider [1] have given complete characterization of rings over which all  $k$ -generated modules are co-hopfian for any given integer  $k \geq 1$ . K. R. Goodearl [4] has completely characterized rings over which all finitely generated modules are hopfian.

In the same spirit it would be interesting to characterise all the rings over which

- (a) all cocyclic modules are co-hopfian
- (b) all finitely embedded modules are co-hopfian
- (c) all cocyclic modules are hopfian and
- (d) all finitely embedded modules are hopfian.

For any integer  $k \geq 1$  we could define a  $k$  co-generated module  $M$  to be a finitely embedded module of Goldie dimension  $\leq k$ ; equivalently there exist  $l$  simple submodules  $S_1, \dots, S_l$  of  $M$  with their sum  $\sum_{i=1}^l S_i$  in  $M$  direct and  $\sum_{i=1}^l S_i = \bigoplus_{i=1}^l S_i$  essential in  $M$  with  $l \leq k$ . We could add the following to the above list.

- (e) Characterise all the rings satisfying the condition that all  $k$  co-generated modules are cohopfian(resp. hopfian).

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## OPERATOR SPACE STRUCTURE OF BANACH SPACES

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The theory of operator spaces grew out of the analysis of completely positive and completely bounded mappings. These maps were first studied on  $C^*$ -algebras, and on suitable subspaces of  $C^*$ -algebras, see [1, 11, 23, 33, 35]. Just as the theory of  $C^*$ -algebras can be viewed as noncommutative topology and the theory of von Neumann algebras as noncommutative measure theory, one can think of the theory of operator spaces as noncommutative functional analysis. The current theory has taken shape in the last 20 years but there are still several unknown things. This talk has been divided into following parts:

- (1) Quantum analogue of function spaces.
- (2) Introduction to operator spaces.
- (3) Tensor products of operator spaces.
- (4) Harmonic Analysis and Operator spaces.

### 1. Quantum analogue of function spaces

In order to quantize mechanics, Heisenberg replaced the functions of classical physics by Hilbert space operators. In early 1930's von Neumann concluded that the same principle could be applied to various areas of mathematics. In collaboration with Murray he began by quantizing integration theory. Quantization is a micro phenomenon whereas classical theory is a macro phenomenon. All the quantum theory includes the classical theory:

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**Classical Spaces****Quantum Analogue**

$l_n^\infty$ –space of  $n$ –tuples

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_\infty = \sup_{1 \leq j \leq n} |\alpha_j|$$

$$M_n = B(l_n^2) = B(\mathbb{C}^n)$$

$l^\infty$ – space of bounded sequences

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_\infty = \sup |\alpha_j|$$

$$M_\infty = B(l^2)$$

$l^\infty(S)$ – space of complex bounded function on a set with supremum norm

$B(H)$ – space of bounded operator on a Hilbert space

$c_0 \subseteq l^\infty$  (Space of sequences converging to zero)

$K_\infty = K(l^2)$   
compact operators on  $l^2$

$C(X) \subseteq l^\infty(X)$

$X$  being compact Hausdorff topological space

$C^*$ –algebra  $\subset B(H)$

$L^1(X)$

$R_*$ – predual of a von Neumann algebra;  $R_*$  is the set of normal functionals on  $R$

$l_n^1$   
 $\|\alpha\|_1 = \sum |\alpha_i|$

$T_n$ –matrices with  $\|T\|_1 = \text{tr} |T|$

$(l_n^1)^* = l_n^\infty$

$T_n^* = M_n$

Function space on a set

$S$ – a closed linear subspace of  $l^\infty(S)$

Operator space- a closed linear subspace  $X$  of  $B(H)$  together with the natural norms on  $M_n(X)$  inherited from  $M_n(B(H)) \simeq B(H^n)$

**2. Introduction to operator spaces**

**Definition 2.1.** A matrix norm on a linear space  $E$  is a family  $\{\|\cdot\|_n\}_{n=1}^\infty$  such that  $\|\cdot\|_n$  is norm on  $M_n(E)$  for each  $n \in \mathbb{N}$ .

An operator space matrix norm on a linear space  $E$  is a matrix norm on  $E$  satisfying

$$(M.1) \quad \|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\} \text{ and}$$

(M. 2)  $\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_m \|\beta\|$ , for all  $x \in M_m(E), y \in M_m(E), \alpha \in M_{n,m}$  and  $\beta \in M_{m,n}$ .

**Definition 2.2.** An operator space  $X$  is a closed subspace of  $B(H)$ . The natural inclusion  $M_n(X) \subset M_n(B(H)) \simeq B(H^n)$  determines norms  $\|\cdot\|_n$  on  $M_n(X)$ . If  $A$  is a  $C^*$ -algebra, then  $A \subset B(H)$ , so  $A$  gets its operator space structure as a subspace of  $B(H)$  by  $M_n(A) \subset M_n(B(H)) \simeq B(H^n)$ .

Since we can think of  $C^*$ -algebras as closed self-adjoint subalgebras of  $B(H)$ , we can think of operator spaces as a closed subspaces of  $C^*$ -algebras. Observe that any Banach space can appear as a closed subspace of a  $C^*$ -algebra. Indeed, for any Banach space  $X$ , if we let  $T = (B_{X^*}, \sigma(X^*, X))$  then  $T$  is compact and we have an isometric embedding  $X \subset C(T)$ . Since  $C(T)$  is a  $C^*$ -algebra and  $C(T) \subset B(l_2(T))$ , so  $X$  appears among operator spaces. Thus operator spaces are just ordinary Banach spaces  $X$  but equipped with an extra structure in the form of an embedding  $X \subset B(H)$ . Operator space theory can be considered as a merger of  $C^*$ -algebra theory and Banach space theory.

In this new category, the objects remain Banach spaces but the morphisms become the completely bounded maps (instead of bounded linear maps).

**Definition 2.3.** [24] Let  $E$  and  $F$  be two operator spaces and  $\varphi : E \rightarrow F$  be a linear map. For each  $n \in \mathbb{N}$ ,  $\varphi$  induces a linear map  $\varphi_n : M_n(E) \rightarrow M_n(F)$  given by

$$\varphi_n(x) = (\varphi(x_{i,j})), \quad \forall x = (x_{i,j}) \in M_n(E).$$

If  $\varphi$  is bounded, then each  $\varphi_n$  is bounded.

Let  $\|\varphi\|_{cb} = \sup \{\|\varphi_n\| : n \in \mathbb{N}\}$ , then

$$\|\varphi\| = \|\varphi_1\| \leq \|\varphi_2\| \dots \leq \|\varphi\|_{cb}.$$

The linear map  $\varphi : E \rightarrow F$  is called

- (1) completely bounded if  $\|\varphi\|_{cb} < \infty$ .
- (2) completely contractive if for each  $n, \|\varphi_n\| \leq 1$ .
- (3) complete isometry if each  $\varphi_n$  is an isometry.
- (4) complete quotient mapping if each  $\varphi_n$  is a quotient mapping.

Results:

- (1) If  $\varphi : E \rightarrow M_n$ , then  $\|\varphi\|_{cb} = \|\varphi_n\|$  (see [32]).
- (2) If  $\varphi : E \rightarrow \mathbb{C}$ , then  $\|\varphi\|_{cb} = \|\varphi\|$ , i.e.,  $BL(E, \mathbb{C}) = CB(E, \mathbb{C})$ .
- (3) If  $\varphi : E \rightarrow A$ , where  $A$  is a commutative  $C^*$ -algebra, then  $\|\varphi\|_{cb} = \|\varphi\|$ .

**Subspace:** If  $F$  is linear subspace of an operator space  $E$ , then the inclusion  $M_n(F) \subseteq M_n(E)$  ( $n \in \mathbb{N}$ ) and the corresponding relative norms determine an operator space matrix norm for  $F$ .

**Quotient space:** Let  $F$  be a closed subspace of  $E$ , then  $M_n(F)$  is closed in  $M_n(E)$ ,  $\forall n \in \mathbb{N}$ . We use  $M_n(E/F) = M_n(E)/M_n(F)$  to define a norm on  $M_n(E/F)$ . For  $\tilde{v} \in M_n(E/F)$

$$\|\tilde{v}\| = \inf \{\|v\| : v \in M_n(E), \pi_n(v) = \tilde{v}\}$$

where  $\pi_n : M_n(E) \rightarrow M_n(E/F)$  is a quotient map.

**Product spaces:** For a collection of operator spaces  $\{E_s, s \in I\}$ , we have an identification  $M_n(\prod_s E_s) = \prod_s M_n(E_s)$ , giving an operator space structure to  $\prod E_s$ .

We also have the operator space  $CB(E, F)$ , the space of completely bounded maps from  $E$  into  $F$ , under the identification  $M_n(CB(E, F)) \simeq CB(E, M_n(F))$ . In particular,  $E^*$  is an operator space as  $M_n(E^*) = CB(E, M_n)$ . One can see [5, 6] for a detailed study of all these concepts.

### 3. Tensor Product of Operator Spaces

The theory of operator space tensor products follows the lines of the theory of tensor products of Banach spaces but at some points tensor products of operator spaces have new properties not found for tensor products of Banach spaces or even better properties as their counterparts. So in some cases the theory of operator space tensor products gives solutions to problems not solvable within the theory of Banach spaces, see [12].

An operator space tensor norm  $\otimes_\alpha$  is called

- *Symmetric* if  $X \otimes_\alpha Y \stackrel{cb}{=} Y \otimes_\alpha X$  is a complete isometry.
- *Associative* if  $(X \otimes_\alpha Y) \otimes_\alpha Z \stackrel{cb}{=} X \otimes_\alpha (Y \otimes_\alpha Z)$  is a complete isometry.
- *Injective* if for all subspaces  $X_1 \subseteq X, Y_1 \subseteq Y$  the map  $X_1 \otimes_\alpha Y_1 \rightarrow X \otimes_\alpha Y$  is a complete isometry.
- *Projective* if for all subspaces  $X_1 \subseteq X, Y_1 \subseteq Y$  the map  $X \otimes_\alpha Y \rightarrow X/X_1 \otimes_\alpha Y/Y_1$  is a complete quotient map.

We introduce here six tensor norms to discuss above properties of tensor norms.

(1) **Injective operator space tensor product:** The injective operator space norm is the least cross norm whose dual norm is a cross norm. Explicitly for an element  $u \in M_n(X \otimes Y)$  which is representation free:

$$\|u\|_{\vee} = \sup \|\langle u, \varphi \otimes \psi \rangle\|_{M_{nkl}},$$

where  $k, l \in \mathbb{N}, \varphi \in Ball(M_k(X^*))$  and  $\psi \in Ball(M_l(Y^*))$ . The injective operator space tensor product is symmetric, associative and injective but is

not projective, see [3].

(2) **Projective operator space tensor product:** Projective operator space tensor product is characterized by

$$X \hat{\otimes} Y \stackrel{cb}{=} CB(X, Y^*) \stackrel{cb}{=} CB(Y, X^*)$$

One also has an explicit expression for the projective operator space tensor norm of an element  $u \in M_n(X \otimes Y)$  as

$$\|u\|_{\wedge} = \inf \left\{ \|\alpha\| \|x\|_p \|y\|_q \|\beta\| : u = \alpha(x \otimes y)\beta \right\},$$

where  $p, q \in \mathbb{N}, x \in M_p(X), y \in M_q(Y), \alpha \in M_{n,pq}$  and  $\beta \in M_{pq,n}$ .

The projective operator space tensor norm is symmetric, associative and projective but not injective. The projective operator space is the greatest operator space tensor norm which is a cross norm. For  $C^*$ - algebras  $A$  and  $B, A \hat{\otimes} B$  is a  $*$ - semi simple Banach algebra, see [3, 5, 18].

(3) **The Haagerup tensor product:** This is characterized, for instance see [9, 10, 13], by the complete isometry

$$(X \otimes_h Y)^* \stackrel{cb}{=} CB(X \times Y, \mathbb{C})$$

If  $A$  and  $B$  are  $C^*$ - algebras, then for  $u \in A \otimes B$

$$\|u\|_h = \inf \left\{ \left\| \sum_{v=1}^n a_v a_v^* \right\|^{1/2} \left\| \sum_{v=1}^n b_v^* b_v \right\|^{1/2} : u = \sum_{v=1}^n a_v \otimes b_v \right\}$$

The Haagerup tensor product is not symmetric but is associative, injective and projective, see [3, 4, 7, 11, 17].

(4) **Banach space projective norm:** For Banach spaces  $A$  and  $B,$  the Banach space projective norm for an element  $u \in A \otimes B$  is defined as

$$\|u\|_{\gamma} = \inf \left\{ \sum_{i=1}^n \|a_i\| \|b_i\| : u = \sum_{i=1}^n a_i \otimes b_i \right\}.$$

$\|\cdot\|_{\gamma}$  is the largest cross norm. The relation between all these norms is given by

$$\|\cdot\|_{\vee} \leq \|\cdot\|_{\min} \leq \|\cdot\|_{\max} \leq \|\cdot\|_h \leq \|\cdot\|_{\wedge} \leq \|\cdot\|_{\gamma}$$

where

$$\|u\|_{\min} = \sup \{ \|\pi_1 \otimes \pi_2(u)\| : \pi_1 \text{ and } \pi_2 \text{ are the representations of } A \text{ and } B \} \text{ and}$$

$$\|u\|_{\max} = \sup \{ \|\pi(u)\| : \pi \text{ is a representation of } A \otimes B \},$$

see [34].

**Definition 3.1.** (*Joint Complete boundedness:*) Let  $X, Y$  and  $Z$  be operator spaces. A bilinear mapping  $\phi : X \otimes Y \rightarrow Z$  is called jointly completely bounded (see [3]) if  $\|\phi\|_{jcb} = \sup \|\phi^{(p \times q)}(x \otimes y)\| < \infty$ , where  $p, q \in \mathbb{N}$ ,  $x \in \text{Ball}(M_p(X)), y \in \text{Ball}(M_q(Y))$ .

For a jointly completely bounded map  $\phi$ , we can associate a completely bounded map  $\tilde{\phi} : X \hat{\otimes} Y \rightarrow Z$  such that  $\|\phi\|_{jcb} = \|\tilde{\phi}\|_{cb}$  and vice-versa. We end this discussion with recent most result of Haagerup and Musat [13]:

**Theorem:** Let  $A$  and  $B$  be  $C^*$ -algebras and  $u : A \times B \rightarrow \mathbb{C}$  be a jointly completely bounded bilinear form. Then there exist states  $f_1, f_2$  on  $A$  and states  $g_1, g_2$  on  $B$  such that for all  $a \in A$  and  $b \in B$ ,

$$|u(a, b)| \leq \|u\|_{jcb} (f_1(aa^*)^{1/2} g_1(bb^*)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2}).$$

It follows from this result that every completely bounded linear map  $T : A \rightarrow B^*$  from a  $C^*$ -algebra  $A$  to the dual of a  $C^*$ -algebra  $B$  has a factorization  $T = vw$  through  $H_r \oplus K_c$  (the direct sum of a row Hilbert space and a column Hilbert space), such that  $\|v\|_{cb} \|w\|_{cb} \leq 2 \|T\|_{cb}$ .

Pisier and Shlyakhtenko [28] proved the same result when at least one of  $A$  and  $B$  is exact with  $K = 2^{2/3}$ . Earlier versions were obtained by Pisier [26] and Haagerup [12].

#### 4. Harmonic analysis and Operator spaces

The Fourier algebra  $A(G)$  ( $G$  being a locally compact group) consists of all the coefficient functions of the left regular representation  $\lambda$  of  $G$ , i.e.,

$$A(G) = \{w = \langle \lambda\xi, \eta \rangle : \xi, \eta \in L^2(G)\}$$

Given  $w \in A(G)$ ,  $\|w\|_{A(G)} = \inf\{\|\xi\| \|\eta\|\}$

where the infimum is taken over all possible representation  $w = \langle \lambda\xi, \eta \rangle$ . This algebra was introduced and studied by Eymard [8]. If  $G$  is abelian then  $A(G)$  is identified via the Fourier transform with  $L_1(\hat{G})$ , where  $\hat{G}$  denotes the dual group of  $G$ .

If  $G$  and  $H$  are locally compact groups with Fourier algebra  $A(G)$  and  $A(H)$ , then  $A(G \times H)$  is not isometric to the Banach space projective tensor product.  $A(G)$  obtains its natural operator space structure by regarding it as the operator predual  $VN(G)_*$  of the von Neumann algebra  $VN(G)$  (von Neumann algebra generated by the left regular representation of  $G$  in  $B(L_2(G))$ ). For locally compact groups  $G$  and  $H$  the identification  $A(G) \hat{\otimes} A(H) \stackrel{cb}{=} A(G \times H)$  can be thought of as a non commutative analogue of the well known classical identification  $L_1(G) \otimes_\gamma L_1(H) = L_1(G \times H)$ . This provides convincing evidence that Fourier algebra should be regarded as an operator space rather than just a normed space.

A Banach algebra  $A$  is said to be amenable if for any Banach  $A$ -bimodule  $E$ , every bounded derivation from  $A$  into  $E^*$  is inner.

A classical result of Johnson [15] asserts that a locally compact group is amenable if and only if the group algebra is amenable as a Banach algebra. The amenability of  $A(G)$  has been an important field of research see Johnson [16], Lau, Loy, Willis [20], Runde [31], Forrest and Runde [10]. Amenability of  $A(G)$  does not capture the amenability of  $G$ . This is because it ignores an important additional structure of Fourier algebras. They are not only Banach Algebras, but completely contractive Banach Algebras, in a canonical way. Ruan [29] showed that a locally compact group is amenable if and only if its Fourier algebra is operator amenable, thereby establishing the natural analogue of Johnson's fundamental result on  $L'(G)$ .

Operator synthesis and connection with spectral synthesis for  $A(G)$  have been obtained by Ludwig and Turowska [21]. This gives us another proof that one-point subset of  $G$  is a set of spectral synthesis and that any closed subgroup is a set of local spectral synthesis.

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## GRAPH STRUCTURE VIA ITS LAPLACIAN MATRIX

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ABSTRACT. The Laplacian matrix of a graph has many applications in different branches of Science and Engineering. We first describe some basic properties of this matrix in relation to the structure of graphs, especially trees. We will discuss some recent developments related to this matrix.

### 1. Introduction and Preliminaries

Let  $G$  be a simple undirected graph with vertex set  $V = \{v_1, v_2, \dots, v_n\}$  and edge set  $E$ . The *adjacency matrix*  $A(G)$  of a graph  $G$  is defined as  $A(G) = [a_{ij}]$ , where  $a_{ij}$  is equal to 1 if the unordered pair  $(v_i, v_j)$  is an edge of  $G$  and 0 otherwise. Let  $D(G)$  be the diagonal matrix of vertex degrees of  $G$ . The *Laplacian matrix* of a graph  $G$  is defined as  $L(G) = D(G) - A(G)$ .

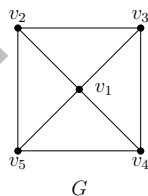


FIGURE 1. Laplacian matrix

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*Key words and phrases:* Laplacian matrix; Algebraic connectivity; Fiedler vector; Characteristic set; Perron component; Laplacian spectral radius.

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**Example 1.1.** Consider the graph  $G$  on five vertices in Figure 1. Then the Laplacian matrix of the graph  $G$  is

$$L(G) = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}.$$

We write  $L, D$  and  $A$  instead of  $L(G), D(G)$  and  $A(G)$ , respectively, whenever there is no chance of confusion about the graph  $G$ . In 1847, Kirchhoff proved the following result that related the Laplacian matrix of a connected graph with the number of spanning trees in it. The result is popularly known as Kirchhoff's *matrix tree theorem*.

**Theorem 1.1.** [16] *Let  $G$  be a connected graph on  $n$  vertices. Then all cofactors of  $L$  are equal and the common value is the number of spanning trees in  $G$ .*

Since then several authors from different disciplines have enriched the subject. Among the studies of different properties and uses of Laplacian matrices, the study of Laplacian spectrum and its relation with the structural properties of graphs has been one of the most attracting features of the subject. It is well known (see [8]) that  $L$  is a symmetric, positive semi-definite matrix. So all the eigenvalues are real and nonnegative. It can be easily checked that the sum of each row of  $L$  is zero. So the vector of all ones, which we denote by  $\mathbf{e}$  is an eigenvector of  $L$  corresponding to the eigenvalue zero. Thus 0 is the smallest eigenvalue of  $L$ . All vectors in this paper are column vectors. The readers are requested to see the book Graph Theory by Harary [16] for all graph theory notations that have not been explained in this writeup.

To know some interesting facts about Laplacian matrix and its eigenvalues, we refer the reader to the survey articles [25, 27]. Let  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  be the eigenvalues of  $L(G)$ , which are enumerated in non-decreasing order and repeated according to multiplicity. Let  $W = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T \mathbf{x} = 1, \mathbf{x}^T \mathbf{e} = 0\}$ . By *Rayleigh-Ritz* theorem (see [17]) the second smallest eigenvalue  $\lambda_2 = \min_{\mathbf{x} \in W} \mathbf{x}^T L(G) \mathbf{x}$  and the largest eigenvalue  $\lambda_n = \max_{\mathbf{x} \in W} \mathbf{x}^T L(G) \mathbf{x}$ .

So, it is comparatively easy to study the second smallest and the largest Laplacian eigenvalue than the other Laplacian eigenvalues.

Fiedler was the first to notice that  $\lambda_2 = 0$  if and only if  $G$  is disconnected. More generally, he observes that the multiplicity of the eigenvalue 0 is the same as the number of connected components  $\omega(G)$  of  $G$ . Thus the rank of  $L(G)$  is  $n - \omega(G)$ , as it is diagonalizable. Viewing  $\lambda_2$  as an algebraic measure of the connectivity of a graph, Fiedler termed this eigenvalue as the *algebraic connectivity* of  $G$ . We shall use  $\mu(G)$  or simply  $\mu$  to denote it. Fiedler also proved some remarkable results (see [10]) and showed that further information about the graph structures can be extracted from an eigenvector corresponding to the algebraic connectivity of a connected graph. Many researchers have studied the relationship of this eigenvector with the graph structure after Fiedler's observations and many interesting results have been proved. The eigenvectors corresponding to algebraic connectivity are now popularly known as *Fiedler vectors*.

It is interesting to note that if

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

are the Laplacian eigenvalues of a graph  $G$ , then

$$\lambda'_1 = 0 \leq \lambda'_2 = n - \lambda_n \leq \lambda'_3 = n - \lambda_{n-1} \leq \dots \leq \lambda'_n = n - \lambda_2$$

are the Laplacian eigenvalues of the complement of  $G$ . Thus the study of Laplacian spectral radius is very much related to the study of algebraic connectivity.

Graphs which have integer Laplacian eigenvalues have always remained very special to researchers, see for example [18, 19], for some related results. The complete graph  $K_n$  is trivially a graph with all Laplacian eigenvalue integers. With some effort, one may show that the graph  $K_n - e$  obtained by deleting an edge  $e$  from  $K_n$  has the same property. A nontrivial graph with all Laplacian eigenvalues integers is the cycle on 6 vertices. Characterizing graphs with all Laplacian eigenvalues integers is rather a very difficult problem and a complete solution is not yet known. A related problem is to study graphs with some fixed integers as eigenvalues. There has been some research in this direction, see [14]. We shall discuss a few results of this type, later.

The organization of the paper is as follow. In Section 2, we discuss some recent developments in the research relating the graph structure with the algebraic connectivity, specially trees. Some results related to the study of Laplacian spectral radius is presented in Section 3. In section 4, we mention some results about graphs  $G$  with 1 as a Laplacian eigenvalue.

2. Algebraic Connectivity, Characteristic Set and Perron Component

Let  $L = [l_{ij}]$  be the Laplacian matrix of a connected graph  $G$  and  $\mu$  be its algebraic connectivity. Let  $\mathbf{y}$  be a Fiedler vector. A subvector of  $\mathbf{y}$  is called a *Fiedler subvector*. By  $\mathbf{y}_v$ , we mean the co-ordinate of  $\mathbf{y}$  corresponding to vertex  $v$ . The condition  $L\mathbf{y} = \mu\mathbf{y}$  implies that for each  $v \in V$

$$\sum_{(u,v) \in E} l_{v,u}\mathbf{y}_u = (\mu - l_{v,v})\mathbf{y}_v.$$

This condition is called the eigencondition at the vertex  $v$ . Suppose  $\mathbf{y}_v = 0$  for some  $v \in V$ . As  $\mathbf{y} \neq \mathbf{0}$ , there exists a vertex  $v_0 \in V$  and edges  $(v_0, u)$  and  $(v_0, w)$  such that  $\mathbf{y}_{v_0} = 0, \mathbf{y}_u > 0$  and  $\mathbf{y}_w < 0$ . With this observation we have the following two definitions.

**Definition 2.1.** A vertex  $v$  of  $G$  is called a *characteristic vertex* of  $G$  if  $\mathbf{y}_v = 0$  and there exists a vertex  $w$  adjacent to  $v$  such that  $\mathbf{y}_w \neq 0$ . An edge  $e = (u, w)$  is called a *characteristic edge* of  $G$  if  $\mathbf{y}_u\mathbf{y}_w < 0$ . The *characteristic set* of  $G$  is the collection of all characteristic vertices and characteristic edges of  $G$  and is denoted by  $\mathcal{C}(G, \mathbf{y})$ .

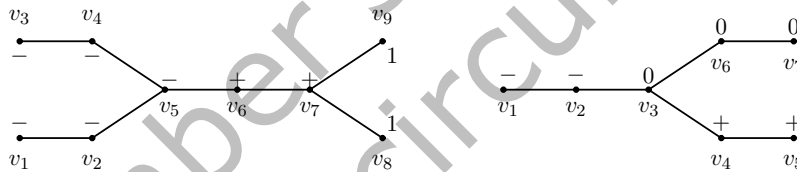


FIGURE 2. Characteristic set

**Example 2.1.** In Figure 2, Let  $G_1$  be the graph on 9 vertices and  $G_2$  be the graph on 7 vertices. Using ‘Matlab’ one can check that  $\mathbf{y}_1 \approx [-.75, -.61, -.75, -.61, -.34, .26, .80, 1, 1]^T$  is a Fiedler vector of  $G_1$  and  $\mathbf{y}_2 \approx [-4.71, -2.91, 0, 2.91, 4.71, 0, 0]^T$  is a Fiedler vector of  $G_2$ . Therefore,  $(v_5, v_6)$  is the characteristic edge of  $G_1$  and  $v_3$  is the characteristic vertex of  $G_2$ . So,  $\mathcal{C}(G_1, \mathbf{y}_1) = \{(v_5, v_6)\}$  and  $\mathcal{C}(G_2, \mathbf{y}_2) = \{v_3\}$ .

Let  $G$  be a connected graph and let  $\mathbf{y}$  be a Fiedler vector. Since  $\mathbf{e}^T\mathbf{y} = 0$ , we know that  $\mathbf{y}$  has at least one entry negative and at least one entry positive. The following result is an interesting fact.

**Lemma 2.1.** [2] *Let  $G$  be a connected graph. Let  $V_+$  be the set of vertices  $v$  for which  $\mathbf{y}_v \geq 0$  and  $V_-$  be the set of vertices  $u$  for which  $\mathbf{y}_u \leq 0$ . Then the subgraph induced by  $V_+$  (respectively  $V_-$ ) is connected.*

It is well known (see [8]) that removing an edge from a connected graph  $G$  can not increase the algebraic connectivity. Hence over all the connected graphs on  $n$  vertices, the maximum algebraic connectivity occurs for the complete graph  $K_n$  and the minimum algebraic connectivity occurs for a tree. It is also known (see [8]) that over all trees on  $n$  vertices the path  $P_n$  has the minimum algebraic connectivity which equals  $2(1 - \cos \frac{\pi}{n})$ . Thus over all connected graphs (or in particular over all trees) on  $n$  vertices the path has the minimum algebraic connectivity. Merris proved (see [26]) that for a tree  $T$  on  $n > 2$  vertices,  $\mu(T) \leq 1$  with equality if and only if  $T$  is a star graph on  $n$  vertices. Thus among all trees on  $n > 2$  vertices, the star has the maximum algebraic connectivity which equals 1. There are also other works on extremizing the algebraic connectivity subject to  $G$  having certain graph theoretic constraints (see [6]).

The following lemma gives a nice relation between the algebraic connectivity and the smallest eigenvalue of some principal submatrix of the Laplacian matrix  $L$ . This lemma has many applications as well.

**Lemma 2.2.** [2] *Let  $G$  be a connected graph and let  $\mu$  be its algebraic connectivity. Let  $W$  be a set of vertices of  $G$  such that  $G - W$  is disconnected. Let  $G_1, G_2$  be two components of  $G - W$  and let  $\hat{L}_1, \hat{L}_2$  be the principal sub-matrices of  $L$  corresponding to  $G_1, G_2$ , respectively. For  $i = 1, 2$ , let  $\tau(\hat{L}_i)$  denote the smallest eigenvalue of  $\hat{L}_i$  and let  $\tau(\hat{L}_1) \leq \tau(\hat{L}_2)$ . Then either  $\tau(\hat{L}_2) > \mu$  or  $\tau(\hat{L}_1) = \tau(\hat{L}_2) = \mu$ .*

By a result of Fiedler ([10]) either  $|\mathcal{C}(G, \mathbf{y})| = 1$  and  $\mathcal{C}(G, \mathbf{y})$  contains a vertex or  $\mathcal{C}(G, \mathbf{y})$  is contained in a block of  $G$ . Several authors have investigated the elements of  $\mathcal{C}(G, \mathbf{y})$  and their location (see, for example [11, 20, 22, 25]). In [2], Bapat and Pati gave a bound for the cardinality of the characteristic set. They have shown that if  $G$  is a connected graph with  $n$  vertices and  $m$  edges then the characteristic set has at most  $m - n + 2$  elements. As a corollary to the above result, they show that a tree possesses only one characteristic element which is either a vertex or an edge. Note here that the characteristic set of a graph in general changes with the selection of the Fiedler vector. Thus it is very interesting to study class of graphs which have the same characteristic set, independent of the Fiedler vectors. It turns out that all trees belong to this class. Thus if  $T$  has a characteristic vertex  $v$  with respect to a Fiedler vector, then that vertex  $v$  is a characteristic vertex with respect to all Fiedler vectors. Such trees are called *Type I*. Fiedler showed that if a tree has a characteristic edge with respect to a Fiedler vector, then the multiplicity of the algebraic connectivity is one. These type of trees are called *Type II*.

We now discuss some results on the inverses of the principal submatrices of the Laplacian matrix and its relation with algebraic connectivity of trees. It turns out that this relationship is very important in the study of tree structure through its Laplacian matrix.

A real square matrix  $A$  is called an  $M$ -matrix if all its off-diagonal entries are nonpositive and all its eigenvalues have nonnegative real part. An important result in the theory of  $M$ -matrix states that  $M$ -matrices are closed under the extraction of principal submatrices and the inverse of an irreducible nonsingular  $M$ -matrix has positive entries. So one fruitful technique for studying  $\mu$  and Fiedler vectors is to exploit the fact that  $L$  is an  $M$ -matrix.

A vertex  $v$  of  $G$  is called a *cut-vertex* if  $G - v$  has more connected components than  $G$ . Let  $v$  be a cut-vertex of a connected graph  $G$ . We denote the connected components of  $G - v$  (called the connected components of  $G$  at  $v$ ) by  $C_1, C_2, \dots, C_k$  ( $k \geq 2$  as  $v$  is a cut-vertex). For each such component, let  $\widehat{L}(C_i), i = 1, 2, \dots, k$ , denote the principal submatrix of the Laplacian matrix  $L$  corresponding to the vertices of  $C_i$ . Since  $L$  has nullity 1, the matrix  $\widehat{L}(C_i)$  is invertible. As  $L$  is an  $M$ -matrix and  $C_i$  is connected, it follows that  $\widehat{L}(C_i)^{-1}$  is a positive matrix, which is called the *bottleneck matrix* for  $C_i$ . The next proposition gives the description of the entries of a bottleneck matrix and is very useful.

**Proposition 2.1.** [22] *Let  $T$  be a tree and let  $v$  be any vertex. Let  $T_1$  be a connected component of  $T - v$ . Then  $\widehat{L}(T_1)^{-1} = [m_{ij}]$ , where  $m_{ij}$  is the number of edges in common between the path  $P_{iv}$  joining the vertices  $i$  and  $v$  and the path  $P_{jv}$  joining the vertices  $j$  and  $v$ .*

By Perron-Frobenius theorem,  $\widehat{L}(C_i)^{-1}$  has a simple dominant eigenvalue, called the *Perron value* of  $C_i$ . For a square nonnegative matrix  $A$ , we denote its *spectral radius* by  $\rho(A)$ . Then the Perron value is the spectral radius of  $\widehat{L}(C_i)^{-1}$ . So we denote the Perron value of  $C_i$  by  $\rho(\widehat{L}(C_i)^{-1})$ . The eigenvector corresponding to  $\rho(\widehat{L}(C_i)^{-1})$  has all entries positive and is called the *Perron vector*. A component  $C_j$  is called a *Perron component* at  $v$  if its Perron value is maximal among  $C_1, C_2, \dots, C_k$ , the connected components at  $v$ .

A connection between Perron components, bottleneck matrices and algebraic connectivity of a tree is described in the next two results (see [22]). In particular, they give a relation between Fiedler subvectors and Perron vectors of bottleneck matrices. We denote the vector having 1 in the  $k$ -th position and 0 elsewhere by  $\mathbf{e}_k$ . We also use  $J$  to denote the matrix  $\mathbf{e}\mathbf{e}^T$ . The order of these matrices will be clear from the context.

**Theorem 2.1.** [22] *Let  $T$  be a tree on  $n$  vertices. Then  $T$  is a Type II tree with the characteristic edge  $(i, j)$  if and only if the component  $C_i$  at vertex  $j$  containing the vertex  $i$  is the unique Perron component at  $j$ , while the component  $C_j$  at vertex  $i$  containing the vertex  $j$  is the unique Perron component at  $i$ . Moreover, in this case there exists a  $\gamma \in (0, 1)$  such that*

$$\frac{1}{\mu(T)} = \rho(\widehat{L}(C_i)^{-1} - \gamma J) = \rho(\widehat{L}(C_j)^{-1} - (1 - \gamma)J).$$

Furthermore, any eigenvector of  $L$  corresponding to  $\mu(T)$  can be permuted so that it has the block form  $\begin{bmatrix} \mathbf{y}_1 \\ -\mathbf{y}_2 \end{bmatrix}$ , where  $\mathbf{y}_1$  is a Perron vector for  $\rho(\widehat{L}(C_i)^{-1} - \gamma J)$  and  $\mathbf{y}_2$  is a Perron vector for  $\rho(\widehat{L}(C_j)^{-1} - (1 - \gamma)J)$ .

**Theorem 2.2.** [22] *Let  $T$  be a tree on  $n$  vertices. Then  $T$  is a Type I tree with the characteristic vertex  $v$  if and only if there are two or more Perron components of  $T$  at  $v$ . Moreover, in this case*

$$\mu(T) = \frac{1}{\rho(L_v^{-1})},$$

whenever  $L_v$  is a Perron component at  $v$ . Furthermore, given any two Perron components  $C_1, C_2$  of  $T$  at  $v$ , an eigenvector  $\mathbf{y}$  corresponding to  $\mu(T)$  can be chosen so that  $\mathbf{y}$  can be permuted and partitioned into the block form  $\mathbf{y}^T = [\mathbf{y}_1^T \mid -\mathbf{y}_2^T \mid \mathbf{0}^T]$ , where  $\mathbf{y}_1$  and  $\mathbf{y}_2$  are Perron vectors for the bottleneck matrices  $\widehat{L}(C_1)^{-1}$  and  $\widehat{L}(C_2)^{-1}$ , respectively and  $\mathbf{0}$  is the column vector of an appropriate order.

Identification of Perron components at a vertex helps to determine the location of characteristic set. By Theorem 2.1, for a tree  $T$ , the edge  $(u, v)$  is the characteristic edge if the component containing  $u$  is the only Perron component at the vertex  $v$  and the component containing  $v$  is the only Perron component at the vertex  $u$ . The next result gives the location of the characteristic set of a tree.

**Proposition 2.2.** [22] *Let  $T$  be a tree. Then for any vertex  $v$  that is neither a characteristic vertex nor an end vertex of the characteristic edge, the unique Perron component at  $v$  contains the characteristic set of  $T$ .*

Using the results we have discussed up to now, one can easily prove that for a tree with a characteristic edge, the multiplicity of  $\mu$  is one; i.e.,  $\mu$  is simple. Also it is not difficult to prove that for a tree, the characteristic vertex (if any) does not change with the change in Fiedler vector. So, for any Fiedler vector of a tree, the characteristic set is fixed. For a tree with a characteristic edge, at any vertex  $v$ , the number of Perron component at  $v$  is exactly one. Similarly for a tree with a characteristic vertex  $v$ , at any vertex

$u$  other than  $v$ , the number of Perron component is exactly one. At the characteristic vertex of a tree, the number of Perron components determines the multiplicity of  $\mu$ . It is known that at the characteristic vertex  $v$  of a tree, the number of Perron components is  $t + 1$  if and only if the multiplicity of  $\mu$  is  $t$  (see [3] for a more general result). We next discuss two graph operations and their effect on algebraic connectivity.

Let  $G$  be a connected graph on  $n$  vertices with  $n \geq 2$ . Let  $v$  be a vertex of  $G$ . For  $l, k \geq 1$ , let  $G_{k,l}$  be the graph obtained from  $G$  by attaching two new paths  $P : vv_1v_2 \dots v_k$  and  $Q : vu_1u_2 \dots u_l$  of lengths  $k$  and  $l$ , respectively at  $v$ , where  $u_1, u_2, \dots, u_l$  and  $v_1, v_2, \dots, v_k$  are distinct new vertices. Let  $\tilde{G}_{k,l}$  be the graph obtained by removing the edge  $(v_{k-1}, v_k)$  and adding the edge  $(u_l, v_k)$  (see Figure 3). Observe that the graph  $\tilde{G}_{k,l}$  is isomorphic to the graph  $G_{k-1,l+1}$ . We say that  $\tilde{G}_{k,l}$  is obtained from  $G_{k,l}$  by *grafting* an edge.

Let  $G$  be a tree. The next result compares the algebraic connectivity of the trees  $G_{k,l}$  and  $G_{k-1,l+1}$  (see Figure 3).

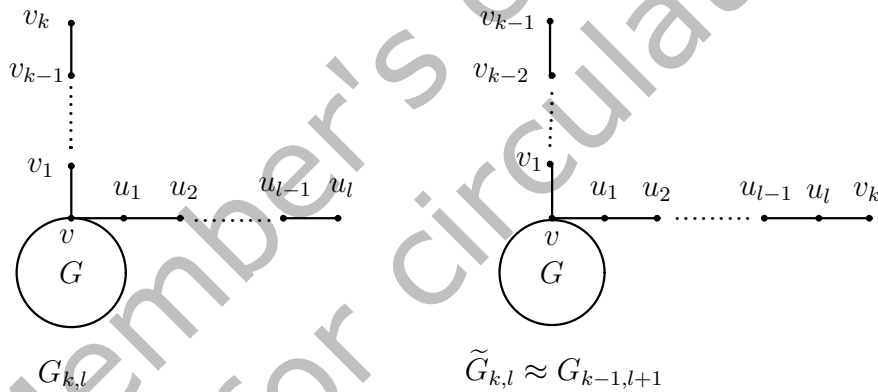


FIGURE 3. Grafting an edge

**Theorem 2.3.** [29] *Let  $G$  be a tree on  $n \geq 2$  vertices and let  $v$  be a vertex of  $G$ . Let  $G_{k,l}$  be the graph as above. If  $l \geq k \geq 1$ , then  $\mu(G_{k-1,l+1}) \leq \mu(G_{k,l})$ .*

Let  $G = (V, E)$  be a graph with an edge  $e = (v_1, v_2)$  not lying on a cycle in  $G$ . Let  $\tilde{G} = (\tilde{V}, \tilde{E})$  be the graph obtained from  $G$  by deleting the edge  $e$  and identifying  $v_1$  and  $v_2$ . We say  $\tilde{G}$  is obtained from  $G$  by *collapsing* an edge (see Figure 4).

**Theorem 2.4.** [29] *Let  $(u, v)$  be an edge of a tree  $T$ . Let  $\tilde{T}$  be the tree obtained from  $T$  by collapsing the edge  $(u, v)$ . Then  $\mu(\tilde{T}) \geq \mu(T)$ .*



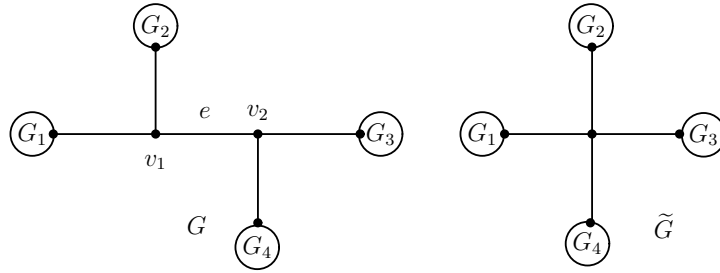


FIGURE 4. Collapsing an edge

### 3. Laplacian Spectral Radius

Let  $G = (V, E)$  be a graph on  $n$  vertices. Let  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  be the eigenvalues of  $L(G)$ . We call  $\lambda(G) = \lambda_n(G)$ , the largest eigenvalue of  $L(G)$ , the *Laplacian spectral radius* of the graph  $G$ , and denote it by  $\lambda(G)$ . By definition of the graph complement of a graph  $G$ , denoted  $\bar{G}$ , we have

$$L(G) + L(\bar{G}) = nI - J.$$

Therefore, for any  $\mathbf{x} \in W$ ,  $\mathbf{x}^T(L(G) + L(\bar{G}))\mathbf{x} = \mathbf{x}^T(nI - J)\mathbf{x} = n$  and hence

$$\lambda(G) = \max_{\mathbf{x} \in W} \mathbf{x}^T L(G) \mathbf{x} = n - \min_{\mathbf{x} \in W} \mathbf{x}^T L(\bar{G}) \mathbf{x} = n - \mu(\bar{G}).$$

This observation with the result of Fiedler [8] that  $\mu(G) = 0$  if and only if  $G$  is disconnected, gives the following theorem.

**Theorem 3.1.** *Let  $G$  be a graph on  $n$  vertices and  $\bar{G}$  be its complement. Then  $\lambda(G) \leq n$  and equality holds if and only if  $\bar{G}$  is disconnected.*

The above observation and *Rayleigh-Ritz* theorem motivate researcher to study graph structure related to Laplacian spectral radius. The next result is the first one in that direction and it follows directly from *Courant-Weyl* theorem (see [17]).

**Lemma 3.1.** *Let  $G$  be a non complete graph. Let  $\hat{G}$  be the graph obtained from  $G$  by joining two non adjacent vertices of  $G$ . Then  $\lambda(G) \leq \lambda(\hat{G})$ .*

The next two results give bounds on the Laplacian spectral radius of a graph. These two results are important for the study of graph structure related to the Laplacian spectral radius.

**Lemma 3.2.** [1] *Let  $G = (V, E)$  be a connected graph. Then*

$$\lambda(G) \leq \max\{d(u) + d(v) : (u, v) \in E\}$$

and equality holds if and only if  $G$  is bipartite and the degree is constant on each class of vertices.

**Lemma 3.3.** [13] *Let  $G = (V, E)$  be a connected graph. Then  $\lambda(G) \geq \Delta(G) + 1$ , where  $\Delta(G)$  is the maximum degree of the graph  $G$ , with equality if and only if  $\Delta(G) = n - 1$ .*

Let  $G$  be a connected graph and  $G_{k,l}$  be the graph (see Figure 3) obtained from  $G$  as defined earlier. Let  $G_{k-1,l+1}$  be the graph obtained from  $G_{k,l}$  by grafting an edge. The next theorem compares the Laplacian spectral radius of  $G_{k,l}$  and  $G_{k-1,l+1}$ .

**Theorem 3.2.** [15] *Let  $G$  be a connected graph on  $n \geq 2$  vertices and let  $v$  be a vertex of  $G$ . Let  $G_{k,l}$  be the graph defined as above. If  $l \geq k \geq 1$ , then*

$$\lambda(G_{k-1,l+1}) \leq \lambda(G_{k,l}),$$

with equality if and only if there exists an eigenvector of  $G_{k,l}$  corresponding to  $\lambda(G_{k,l})$  which takes the value 0 on the fixed vertex  $v$ .

The next proposition relates the Laplacian matrix of a bipartite graph with a nonnegative matrix. For the sake of completeness, we give the proof.

**Proposition 3.1.** [12] *Let  $G$  be a connected bipartite graph. Then  $B(G) = D(G) + A(G)$  and  $L(G) = D(G) - A(G)$  are unitarily similar. In particular, the largest eigenvalue of  $L(G)$  is simple.*

*Proof.* As  $G$  is a bipartite graph, the vertex set  $V(G)$  can be partitioned into two disjoint subsets  $V_1$  and  $V_2$  such that every edge of  $G$  is incident with exactly one element of  $V_1$  and exactly one element of  $V_2$ . Define a diagonal matrix  $U = [u_{ij}]$  by

$$u_{ii} = \begin{cases} 1, & \text{if } v_i \in V_1, \\ -1, & \text{if } v_i \in V_2. \end{cases}$$

Then it can be checked that  $UA(G)U^{-1} = -A(G)$  and hence the result follows. □

Using Proposition 3.1 and the Perron-Frobenius theory, the proof of the next corollary is immediate.

**Corollary 3.1.** *Let  $G$  be a bipartite graph on  $n$  vertices. Let  $\tilde{G}$  be the bipartite graph obtained from  $G$  by adjoining a new vertex to a vertex of  $G$ . Then  $\lambda(\tilde{G}) > \lambda(G)$ .*

Let  $G$  be a connected bipartite graph. Then by Proposition 3.1 and the Perron-Frobenius theorem, there exists a unique positive vector  $\mathbf{y} =$

$(y_1, y_2, \dots, y_n)^T$  such that  $B(G)\mathbf{y} = \lambda(G)\mathbf{y}$ . Since  $UB(G)U^{-1} = L(G)$ , where  $U$  is a diagonal matrix defined in the proof of Proposition 3.1, we have

$$L(G)U\mathbf{y} = U B(G) U^{-1} U\mathbf{y} = U B(G) \mathbf{y} = \lambda(G) U\mathbf{y}.$$

So, the co-ordinates of the eigenvector corresponding to the eigenvalue  $\lambda(G)$  of a bipartite graph  $G$  are nonzero. The above discussion with Theorem 3.2 leads to the next corollary.

**Corollary 3.2.** [15] *If  $G$  is a connected bipartite graph on  $n \geq 2$  vertices. Then for  $l \geq k \geq 1$ ,  $\lambda(G_{k,l}) > \lambda(G_{k-1,l+1})$ .*

The next theorem compares the Laplacian spectral radius of two trees  $T$  and  $\tilde{T}$ , where  $\tilde{T}$  is obtained from  $T$  by collapsing of edges.

**Theorem 3.3.** [14] *Let  $T$  be a tree. Suppose  $v$  and  $w$  are vertices each of degree at least 3. Suppose the path from  $v$  to  $w$  is homeomorphic to an edge (i.e., apart from the vertices  $v$  and  $w$ , each vertex of the path has degree two). Let  $\tilde{T}$  be the tree obtain from  $T$  by collapsing the entire  $v$ - $w$  path. Then  $\lambda(\tilde{T}) > \lambda(T)$ .*

By Lemma 3.1, it is clear that among all connected graphs on  $n$  vertices the maximum Laplacian spectral radius occurs for the complete graph  $K_n$  and the minimum Laplacian spectral radius occurs for a tree. From Theorem 3.2, it follows directly that the path  $P_n$  has the minimum Laplacian spectral radius among all trees on  $n$  vertices. As every tree is a bipartite graph, by using Theorem 3.3, Corollaries 3.1 and 3.2, one can easily prove that the star  $S_n$  has the maximum Laplacian spectral radius among all trees on  $n$  vertices.

For the path  $P_n$ , the Laplacian eigenvalues are  $2[1 - \cos \frac{(k-1)\pi}{n}]$  for  $k = 1, 2, \dots, n$ . So,  $\lambda(P_n) = 2[1 - \cos \frac{(n-1)\pi}{n}]$ . For the star  $S_n$ , the Laplacian eigenvalues are 0, 1 and  $n$  with multiplicity 1,  $n - 2$  and 1, respectively. Therefore, for any tree  $T$ ,

$$2[1 - \cos \frac{(n-1)\pi}{n}] \leq \lambda(T) \leq n.$$

The above inequality is also true for any connected graph  $T$ . The next result gives the description of the tree that maximizes the Laplacian spectral radius among all trees on  $n$  vertices with fixed diameter  $d$ . To state the theorem, recall that a path  $P_n$  on  $n$  vertices is a tree  $T$  with vertex set,  $V(T) = \{1, 2, \dots, n\}$  and edge set,  $E(T) = \{(i, i + 1) : 1 \leq i \leq n - 1\}$ .

**Theorem 3.4.** [24] *Let  $T$  be a tree on  $n$  vertices and let  $\text{diam}(T) = d$ . Among all such trees, the maximal Laplacian spectral radius is obtained uniquely at  $Q_{n,d}^c$ , where  $Q_{n,d}^c$  is a tree obtained by taking a path  $P_{d+1}$  on  $d + 1$  vertices and adding  $n - d - 1$  pendant vertices to the vertex  $\lfloor \frac{d+2}{2} \rfloor$ .*

Lots of researchers are presently working in problems related to the understanding of the graph structure via its Laplacian matrix. The results related with the study of graph structure via its adjacency matrix are well known. The interested readers are advised to see the book by Cvetkovic et al. [5].

#### 4. Trees with integral eigenvalues

The study of graphs whose Laplacian spectrum consists of integers has attracted many researchers. One of the first papers [14] in this area looks at the problem of “multiplicity of 1 or 2 as an eigenvalue of the Laplacian matrix”. In particular, they show that if  $m_T(\lambda)$  denotes the multiplicity of  $\lambda$  as an eigenvalue of  $L(T)$  then  $m_T(1)$  can be arbitrarily large while  $m_T(2)$  can be at most one.

For a tree  $T$ , let  $p(T)$  denote the number of pendant vertices and let  $q(T)$  denote the number of *quasipendant* vertices (a quasipendant vertex is a vertex which is adjacent to a pendant vertex). Then it was shown by Faria [7] that  $m_T(1) \geq p(T) - q(T)$ . This result can be easily understood as follows:

Let  $v_1, v_2, \dots, v_k$   $k \geq 2$  be the pendant vertices of  $T$  adjacent to vertex  $T$ . Define vectors  $\mathbf{y}_i$ ,  $i = 1, \dots, k - 1$  as  $\mathbf{y}_i(v_1), \mathbf{y}_i(v_{i+1}) = -1$ ,  $\mathbf{y}(w) = 0$ , for all other  $w \in T$ . It is easy to see that  $\mathbf{y}_i$ 's are linearly independent eigenvectors of  $L(T)$  corresponding to the eigenvalue 1. It follows that

$$p(T) - q(T) \leq m_T(1).$$

Note that in case of a star  $S_n$ , the above inequality becomes equality.

Now question arises whether it is possible for a tree with  $p(T) = q(T)$  to have an eigenvalue 1. One can easily show that  $P_{3n}$  has an eigenvalue 1 for any  $n > 1$ . Considering the problem of studying the trees with  $p(T) = q(T)$  possessing an eigenvalue 1, Barik, Lal and Pati in [4] proved several results in that direction. The interested reader can look at the paper [4].

#### 5. Conclusion

In Sections 2 and 3, we studied the graph structure related to algebraic connectivity and Laplacian spectral radius. One observes that there are some similarities between the graph structure related to  $\mu(G)$  and  $\lambda(G)$ . For example, among all trees on  $n$  vertices, the path  $P_n$  has the minimum algebraic connectivity and the minimum Laplacian spectral radius. Similarly, the star  $S_n$  has the maximum algebraic connectivity and the maximum Laplacian spectral radius, among all trees on  $n$  vertices. Further study is needed to say more about similarities and dissimilarities between the graph structures related to both  $\mu(G)$  and  $\lambda(G)$ .

The second chapter of the Ph.D. thesis of Patra [28] contains a survey of results related with Sections 2 and 3. For the results related to Section 4, the readers can see [4].

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## ON SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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(Received: )

ABSTRACT. In the present talk we define some new sequence spaces using Orlicz functions and examine some properties of these spaces. We also establish some inclusion relations.

### 1. Introduction

Let  $w$  be the family of all real or complex sequences. Any subspace of  $w$  is called a sequence space. A sequence space  $X$  with linear topology is called a  $K$ -space provided each of the maps  $\tau_i(x) = x_i$  is continuous,  $i \geq 1$ . A complete normed linear  $K$ -space is known as  $BK$ -space.

A function  $M : [0, \infty) \rightarrow [0, \infty)$  is said to be an Orlicz function if it is continuous, non-decreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , if convexity of Orlicz function is replaced by

$$M(x+y) \leq M(x) + M(y),$$

then this function is called modulus function defined and discussed by Ruckle [4] and Maddox [3].

### 2. New Sequence Spaces

Let  $p = (p_k)$  be any sequence of positive real numbers; let  $(\Delta x_i) = (x_i - x_{i+1})$ . We define the following new sequence spaces:

$$l(M, \Delta, p) = \left\{ x \in \omega : \sum_{k=1}^{\infty} \left( M \left( \frac{|\Delta x_k|}{\rho} \right) \right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}$$

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$$W(M, \Delta, p) = \left\{ \begin{array}{l} x \in \omega : \frac{1}{n} \sum_{k=1}^n \left( M \left( \frac{|\Delta x_{k-1}|}{\rho} \right) \right)^{p_k} \rightarrow 0, \\ \text{as } n \rightarrow \infty, \text{ for some } \rho > 0 \text{ and } l \in C \end{array} \right\}$$

$$W_0(M, \Delta, p) = \left\{ \begin{array}{l} x \in \omega : \frac{1}{n} \sum_{k=1}^n \left( M \left( \frac{|\Delta x_k|}{\rho} \right) \right)^{p_k} \rightarrow 0, \\ \text{as } n \rightarrow \infty, \text{ for some } \rho > 0 \end{array} \right\}$$

$$W_\infty(M, \Delta, p) = \left\{ \begin{array}{l} x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^n \left( M \left( \frac{|\Delta x_k|}{\rho} \right) \right)^{p_k} < \infty, \\ \text{for some } \rho > 0 \end{array} \right\}.$$

When  $p_k = 1$  for all  $k$ , we write  $l(M, \Delta, p)$ ,  $W(M, \Delta, p)$ ,  $W_0(M, \Delta, p)$  and  $W_\infty(M, \Delta, p)$  respectively as  $l(M, \Delta)$ ,  $W(M, \Delta)$ ,  $W_0(M, \Delta)$  and  $W_\infty(M, \Delta)$ .

When  $M(x) = x$ , then the sequence spaces defined above become  $bv(p)$ ,  $W(\Delta, p)$ ,  $W_0(\Delta, p)$  respectively. When  $p_k = 1$  for each  $k$ , then  $bv(p)$  becomes

$$bv = \left\{ x \in \omega : \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty \right\}$$

Note that  $bv$  is a BK-space normed by

$$\|x\| = |x_1| + \sum_{k=1}^{\infty} |x_k - x_{k+1}|.$$

An Orlicz function  $M$  can always be represented ([2]) in the following integral form:

$$M(x) = \int_0^x N(t) dt.$$

where  $N$ , known as the Kernel of  $M$ , is right differentiable for  $t \geq 0$ ,  $N(0) = 0$ ,  $N(t) > 0$  for  $t > 0$ ,  $N$  is non-decreasing and  $N(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

### 3. MAIN RESULTS

**Theorem 3.1:**  $l(M, \Delta, p)$  is a linear space over the set of complex numbers  $C$ .

**Proof:** Let  $x, y \in l(M, \Delta, p)$  and  $\alpha, \beta \in C$ . In order to prove the result we need to find some  $\rho_3 > 0$  such that

$$\sum_{k=1}^{\infty} \left( M \left( \frac{|\alpha \Delta x_k + \beta \Delta y_k|}{\rho_3} \right) \right) < \infty.$$

Since  $x, y \in l(M, \Delta, p)$ , there exist positive numbers  $\rho_1$  and  $\rho_2$  such that

$$\sum_{k=1}^{\infty} \left( M \left( \frac{|\Delta x_k|}{\rho_1} \right) \right)^{p_k} < \infty$$



and

$$\sum_{k=1}^{\infty} \left( M \left( \frac{|\Delta y_k|}{\rho_2} \right) \right)^{p_k} < \infty$$

Define  $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$ . Since  $M$  is non-decreasing and convex, we have the result.

**Theorem 3.2:**  $l(M, \Delta)$  is a K-Space nomred by

$$g(x) = |x_1| + \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|\Delta x_k|}{\rho} \right) \leq 1 \right\}$$

**Proof:** Let  $r > 0$  and  $x_0$  be fixed such that  $M(rx_0) \leq 1$  and  $rx_0 \geq 1$ . Then for a given  $\epsilon > 0$ , there exists  $n_0 > 0$  such that

$$g(x^i - x) < \frac{\epsilon}{rx_0}, \text{ for all } i \geq n_0.$$

This means that

$$|x_1^j - x_1| + \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|\Delta x_k^j|}{\rho} \right) \leq 1 \right\} < \frac{\epsilon}{rx_0}, \tag{1}$$

for all  $i \geq n_0$ . From (1), we have

$$|x_1^j - x_1| < \frac{\epsilon}{rx_0} \leq \epsilon, \text{ for all } i \geq n_0. \tag{2}$$

$$\lim_{i \rightarrow \infty} |x_1^i - x_1| = 0$$

Also from (1), we have

$$\sum_{k=1}^{\infty} M \left( \frac{|\Delta(x_k^j - x_k)|}{\rho} \right) \leq 1, \text{ for all } i \geq n_0.$$

This implies that

$$M \left( \frac{|\Delta(x_k^j - x_k)|}{\rho} \right) \leq 1 \leq M(rx_0), \forall i \geq n_0 \text{ and } \forall k \in \mathbb{N}.$$

It now follows from (1) that

$$|\Delta(x_k^j - x_k)| < rx_0 \left( \frac{\epsilon}{rx_0} \right) = \epsilon, \forall i \geq n_0 \text{ and } \forall k \in \mathbb{N}. \tag{3}$$

Now consider  $k = 1$ ; then we have

$$|(x_1^j - x_1) - (x_2^i - x_2)| < \epsilon, \forall i \geq n_0$$

that is

$$\lim_{i \rightarrow \infty} |(x_1^i - x_1) - (x_2^i - x_2)| = 0, \text{ for all } i \geq n_0. \tag{4}$$

It follows from (2) and (4) that

$$\lim_{i \rightarrow \infty} |x_2^i - x_2| = 0 \tag{5}$$

Proceeding in this way inductively, we have

$$\lim_{i \rightarrow \infty} |x_k^i - x_k| = 0, \text{ for each } k \in \mathbb{N}. \quad (6)$$

Hence,  $l(M, \Delta)$  is a K-space.

**Theorem 3.3:**  $l(M, \Delta)$  is a BK-space normed by

$$g(x) = |x_1| + \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \frac{|\Delta x_k|}{\rho} \leq 1 \right\}$$

**Proof:** We have proved in Theorem 3.2 that  $l(M, \Delta)$  is a K-space. We now complete the proof that it is a BK-space. Clearly

$$g(x) = g(-x), \text{ and } g(0) = 0.$$

To prove  $g(x+y) \leq g(x) + g(y)$ , we define

$$g(x+y) = |x_1 + y_1| + \inf \left\{ \rho : \sum_{k=1}^{\infty} M \left( \frac{|\Delta x_k + \Delta y_k|}{\rho} \right) \leq 1 \right\} = N_1 \quad (7)$$

$$g(x) = |x_1| + \inf \left\{ \rho_1 : \sum_{k=1}^{\infty} \left( M \left( \frac{|\Delta x_k|}{\rho_1} \right) \right) \leq 1 \right\} = N_2 \quad (8)$$

$$g(y) = |y_1| + \inf \left\{ \rho_2 : \sum_{k=1}^{\infty} \left( M \left( \frac{|\Delta y_k|}{\rho_2} \right) \leq 1 \right) \right\} = N_3 \quad (9)$$

Then it is sufficient to prove that

$$\sum_{k=1}^{\infty} M \left( \frac{|\Delta x_k + \Delta y_k|}{N_2 + N_3} \right) \leq 1$$

Now

$$\begin{aligned} \sum_{k=1}^{\infty} M \left( \frac{|\Delta x_k + \Delta y_k|}{N_2 + N_3} \right) &\leq \sum_{k=1}^{\infty} M \left( \frac{|\Delta x_k|}{N_2 + N_3} + \frac{|\Delta y_k|}{N_2 + N_3} \right) \\ &= \sum_{k=1}^{\infty} \alpha M \left( \frac{|\Delta x_k|}{N_2 + N_3} \right) + \sum_{k=1}^{\infty} (1 - \alpha) M \left( \frac{|\Delta x_k|}{N_2 + N_3} \right) \\ &\quad + \alpha \sum_{k=1}^{\infty} M \left( \frac{|\Delta y_k|}{N_2 + N_3} \right) + (1 - \alpha) \sum_{k=1}^{\infty} M \left( \frac{|\Delta y_k|}{N_2 + N_3} \right) \\ &\leq \alpha + (1 - \alpha) = 1. \end{aligned}$$

Now it is left to prove the completeness.

Let  $(x^i)$  be any Cauchy sequence in  $l(M, \Delta)$ . Let  $r$  and  $x_0$  be fixed. Then for each  $\frac{\epsilon}{rx_0} > 0$  there exists a positive integer  $m$  such that  $g(x^i - x^j) < \frac{\epsilon}{rx_0}$ , for all  $i, j \geq m$ .

Using definition of norm, we get

$$\sum_{k=1}^{\infty} \left( M \left( \frac{|(\Delta x_k)^i - (\Delta x_k)^j|}{g(x^i - x)} \right) \right) \leq 1, \text{ for all } i, j \geq m.$$

This implies

$$M \left( \frac{|(\Delta x_k)^i - (\Delta x_k)^j|}{g(x^i - x^j)} \right) \leq 1, k \geq 1 \text{ and } i, j \geq m.$$

Hence, one can find a number  $r > 0$  with

$$\left(\frac{x_0}{2}\right) r N \left(\frac{x_0}{2}\right) \geq 1,$$

where  $N$  is the Kernel associated with  $M$  such that

$$M \left( \frac{|(\Delta x_k)^i - (\Delta x_k)^j|}{g(x^i - x^j)} \right) \leq r \left(\frac{x_0}{2}\right) N \left(\frac{x_0}{2}\right)$$

This gives

$$|(\Delta x_k)^i - (\Delta x_k)^j| < \left(\frac{x_0 r}{2}\right) \frac{\epsilon}{x_0 r} = \frac{\epsilon}{2}$$

Hence,  $(\Delta x_k)^i$  is Cauchy sequence in  $\mathbf{R}$ . Therefore, for each  $\epsilon$  ( $0 < \epsilon < 1$ ), there exists a positive integer  $m$  such that

$$|(\Delta x)^i - \Delta x| < \epsilon, \text{ for all } i \geq m.$$

Using continuity of  $M$  we have

$$\sum_{k=1}^{\infty} M \left( \frac{|(\Delta x_k)^i - \lim_{j \rightarrow \infty} (\Delta x_k)^j|}{\rho} \right) \leq 1.$$

Thus,

$$\sum_{k=1}^{\infty} \left( M \left( \frac{|(\Delta x_k)^i - \Delta x_k|}{\rho} \right) \right) \leq 1.$$

Taking infimum of such  $\rho$ 's, we get

$$\inf \left\{ \rho : \sum_{k=1}^{\infty} M \left( \frac{|(\Delta x_k)^i - \Delta x|}{\rho} \right) \leq 1 \right\} < \epsilon$$

for all  $i \geq m$  and for  $j \rightarrow \infty$ .

This shows that  $x_k^i - x_k \in l(M, \Delta)$ . Since  $(x') \in l(M, \Delta)$  and  $M$  is continuous, it follows that  $x \in l(M, \Delta)$ . This completes the proof.

**Remark:** If  $M(x) = x$ , then the norm defined in  $l(M, \Delta)$  and the norm defined in  $l(M, \Delta)$  are the same.

**Theorem 3.4:** Let  $0 < p_k < q_k < \infty$  for each  $k$ , then

$$l(M, \Delta, p) \subseteq l(M, \Delta, q).$$

**Proof:** Let  $x \in l(M, \Delta, p)$ . Then there exists some  $\rho > 0$  such that

$$\sum_{k=1}^{\infty} M\left(\frac{|\Delta x_k|}{\rho}\right)^{p_k} < \infty.$$

Then  $M\left(\frac{|\Delta x_k|}{\rho}\right) \leq 1$  for sufficiently large value of  $k$ . Since  $M$  is non-decreasing

$$\sum_{k>K} \left(M\left(\frac{|\Delta x_k|}{\rho}\right)\right)^{q_k} \leq \sum_{k=1}^K \left(M\left(\frac{|\Delta x_k|}{\rho}\right)\right)^{p_k}$$

for sufficiently large values of  $K$ . Hence,  $x \in l(M, \Delta, q)$ .

**Theorem 3.5:** Let  $p_k$  be bounded. Then  $W(M, \Delta, p)$ ,  $W_0(M, \Delta, p)$  and  $W_{\infty}(M, \Delta, p)$  are linear spaces over the set of complex numbers.

**Proof:** The proof of this Theorem is similar to the proof of Theorem 3.1 and hence we omit the proof.

**Theorem 3.6:**  $W(M, \Delta)$ ,  $W_0(M, \Delta)$  and  $W_{\infty}(M, \Delta)$  are BK-spaces normed by

$$g(x) = \inf \left\{ \rho > 0 : \frac{1}{n} \sum_{k=1}^n \left(M \frac{|\Delta x_k|}{\rho}\right) \leq 1, n = 1, 2, \dots \right\} + |x_1|$$

**Proof:** The proof of this Theorem is similar to the proof of Theorem 3.3.

**Definition (Krasnoselskii and Rutisky [2]).** An Orlicz function  $M$  is said to satisfy  $\Delta_2$ -condition for all values of  $u$ , if there exists constant  $K > 0$  such that  $M(2u) \leq KM(u)$  ( $u \neq 0$ ).

The  $\Delta_2$ -condition is equivalent to the satisfaction of inequality

$$M(lu) \leq KlM(u)$$

for all values of  $u$  and for  $l > 1$ .

**Theorem 3.7:** Let  $M$  be an Orlicz function which satisfies the  $\Delta_2$ -condition. Then,

$$W(\Delta) \subseteq W(M, \Delta), W_0(\Delta) \subseteq W_0(M, \Delta), W_{\infty}(\Delta) \subseteq W_{\infty}(M, \Delta).$$

**Proof:** Let  $s \in W(\Delta)$ , so that

$$s_n = \frac{1}{n} \sum_{k=1}^n |\Delta x_k - l| \rightarrow 0 \text{ as } n \rightarrow \infty$$

Let  $\epsilon > 0$  and choose  $\delta$  with  $0 < \delta < 1$ , such that  $M(t) < \epsilon$  for  $0 < t < \delta$ . Write  $y_k = |\Delta x_k - l|$  and consider

$$\sum_{k=1}^n M(y_k) = \sum_1 + \sum_2$$

where first summation is over  $y_k < \delta$  and second is over  $y_k > \delta$ . Then

$$\sum_1 < \epsilon n$$

and for  $y_k > \delta$  we use

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$$

Since  $M$  is non-decreasing and convex, it follows that

$$\begin{aligned} M(y_k) &< M\left(1 + \frac{y_k}{\delta}\right) \\ &< \frac{1}{2}M(2) + \frac{1}{2}M\left(2\frac{y_k}{\delta}\right). \end{aligned}$$

Since,  $M$  satisfies  $\Delta_2$ -condition, we have

$$\begin{aligned} M(y_k) &< \frac{1}{2}K\frac{y_k}{\delta}M(2) + \frac{1}{2}K\frac{y_k}{\delta}M(2) \\ &= K y_k \delta^{-1}M(2) \quad \text{for some } k > 0. \end{aligned}$$

Hence,

$$\sum_2 M(y_k) \leq K\delta^{-1}M(2)n.s.$$

which together with

$$\sum_2 M(y_k) \leq (\epsilon n)$$

yields

$$W(\Delta) \leq W(M, \Delta).$$

Following similar arguments we can prove that  $W_0(\Delta) \subseteq W_0(M, \Delta)$  and  $W_\infty(\Delta) \subseteq W_\infty(M, \Delta)$ .

**Theorem 3.8:** Let  $0 < p_k < q_k$  and  $\frac{q_k}{p_k}$  be bounded. Then  $W(M, \Delta, q) \subseteq W(M, \Delta, p)$

**Proof:** Let  $x \in W(M, \Delta, q)$ . Write

$$t_k = \left( M \frac{(|\Delta x_k - l|)}{\rho} \right)^{q_k} \quad \text{and} \quad \lambda_k = \frac{p_k}{q_k}.$$

Since,  $p_k \leq q_k, 0 < \lambda_k \leq 1$ , for  $0 < \lambda < \lambda_k$ , define

$$u_k = \begin{cases} t_k & \text{when } (t_k \geq 1) \\ 0 & \text{when } (t_k < 1) \end{cases}$$

and

$$v_k = \begin{cases} t_k & \text{when } (t_k < 1) \\ 0 & \text{when } (t_k \geq 1) \end{cases}$$

Then

$$t_k = u_k + v_k \quad \text{and} \quad v_k^{\lambda_k} = u_k^{\lambda_k}.$$

Note that

$$u_k^{\lambda_k} \leq u_k \leq t_k \quad \text{and} \quad v_k^{\lambda_k} \leq u_k^\lambda.$$

Therefore,

$$\frac{1}{n} \sum_{k=1}^n t_k^{\lambda_k} \leq \frac{1}{n} \sum_{k=1}^n t_k + \left( \frac{1}{n} \sum_{k=1}^n v_k \right)^\lambda$$

Hence,  $x \in W(M, \Delta, p)$ .

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## BURGERS EQUATIONS: INTRODUCTION AND APPLICABILITY

KANTI PANDEY

ABSTRACT. In the present talk an attempt is made to give an introduction of Burgers equation, its origin and applicability in various fields and one example of weak-time dependent wave is illustrated. Numerical solutions are obtained for two problems by using restrictive Taylor approximation and it is concluded that Douglas method also performs well.

### NOMENCLATURE

$u$  – velocity  
 $\rho$  – density  
 $c$  – wave-velocity  
 $k$  – wave-number  
 $w$  – wave-frequency  
 $S$  – entropy  
 $a_0$  – speed of sound  
 $v$  – diffusivity coefficient  
 $h$  – enthalpy  
 $q_i$  –  $i^{th}$  species of internal variable  
 $a_e$  – equilibrium speed  
 $a_f$  – frozen speed  
 $\hat{q}_i$  – equilibrium value of  $q_i$   
 $\tilde{r}$  – equilibrium value of  $r$   
 $\tau_i$  – relaxation time for  $i$ -th species

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}$$

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Comma (,) followed by an index denotes partial derivative with respect to that index.

Subscript  $0(0)$  denotes initial condition.

## 1. Introduction

Steady and unsteady motion of continuous medium are usually studied through a discussion of waves. The simplest intuitive motion of a wave involves the concept of disturbances that varies with time throughout the same region in space. However, in general, waves can be regarded as the solutions of partial differential equations and can be distinguished by two main classes. The first is formulated mathematically in terms of hyperbolic and the second kind is referred to as dispersive. There are certain wave motions that exhibit both types of behaviour and there are certain exceptions that fit neither. There is another type of wave called diffusive wave and is epitomized by equations called Burgers equations.

### MOTIVATION

The motivation is due to its application in the following fields:

- Nonlinear waves,
- Blood flow,
- Environmental sciences.

### SIMULTANEOUS TOOLS

- Numerical Method
- Analytical Method
- Similarity Method.

Let us begin with linear and non-linear system of equations.

### LINEAR SYSTEM

- (a)  $u_{,t} - cu_{,x} = 0$ ,  $c$  being constant.
- (b)  $u_{,t} + cu_{,x} - vu_{,xx} = 0$   $c, v$  being constant,  $v > 0$ ,
- (c)  $u_{,t} + uu_{,x} + Ku_{,xxx} = 0$   $c, K$  being constant,  $K > 0$ .

### NON-LINEAR SYSTEM

- (a)  $u_{,t} + uu_{,x} = 0$ , (b)  $u_{,t} + uu_{,x} - vu_{,xx} = 0$  (Burgers equation),
- (c)  $u_{,t} + uu_{,x} + Ku_{,xxx} = 0$  (Korteweg-de Vries (K-dV) equation).

Let us start with the celebrated wave equation.

$$u_{,t} = c^2 u_{,xx} \quad (1)$$

where  $u$  is some property associated with the wave and  $c^2$  is a positive constant. The general solution of equation (1) is given by

$$u(x, t) = f(x - ct) + g(x + ct), \quad (2)$$



where  $f$  and  $g$  are arbitrary functions. Equation (1) can be rewritten as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0 \quad (3)$$

and retaining one factor of the above equation, we have

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0 \quad (4)$$

which is a linear-partial differential of the form

$$u_{,t} + cu_{,x} = 0$$

having solution of the form

$$u(x, t) = f(x - ct) \quad (5)$$

as shown in Fig. 1.

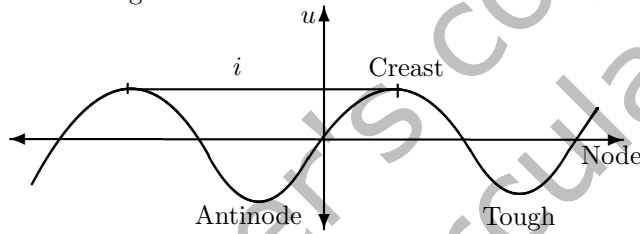


Fig. 1. Plot of  $u$  against  $x$  for a given value of  $t$

The characteristics corresponding to waves in this case are parallel straight lines as shown in Fig. 2 (the case for linear waves).

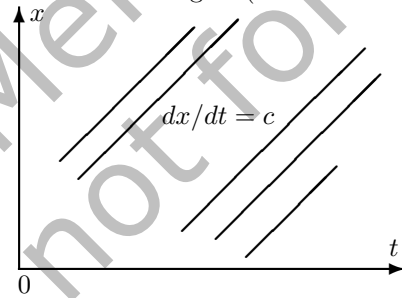


Fig. 2. Characteristics for linear waves

In a similar way for non-linear waves we consider a wave equation of the form

$$u_{,tt} = u^2 u_{,xx} \quad (6)$$

which can be written as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u = 0 \quad (7)$$

and retaining one factor of the above equation, we have

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right) u = 0 \quad (8)$$

which is a non-linear-partial differential having solution of the form

$$u(x, t) = f(x - ut) \quad (9)$$

and due to non-linearity, the wave profile shows a breaking as shown in Fig. 3.

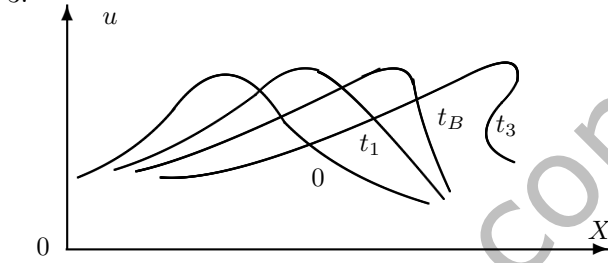


Fig. 3. Breaking wave: successive profiles corresponding to the times  $0, t_1, t_B, t_3$

The characteristics in this case are non-parallel straight lines as shown in Fig. 4.

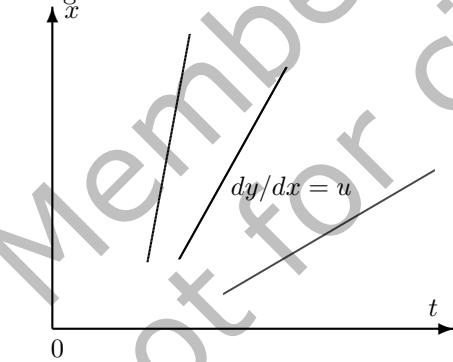


Fig. 4. Characteristics for non-linear waves

Now we want to see the effect produced by the term  $u_{,xx}$  occurring in equations

$$u_{,t} + cu_{,x} - vu_{,xx} = 0 \quad (10)$$

and

$$u_{,t} + uu_{,x} - vu_{,xx} = 0 \quad (\text{Burger's equation}) \quad (11)$$

Putting  $u(x, t) = ae^{i(kx - wt)}$  in equation (10), we have the dispersion relation as

$$w = ck - ivk^2$$

and substituting  $w$  in equation (11), we have

$$u(x, t) = ae^{-t/t_0} e^{ik(x-ct)} \quad (12)$$

representing harmonic-wave with wave-number  $k$  and velocity  $c$  whose amplitude decays exponentially with time and decay time is given by

$$t_0 = \frac{1}{vk^2}. \quad (13)$$

Thus, we conclude that :

- (1) for given  $v$ ,  $t_0$  becomes smaller and smaller as  $k$  increases which shows that waves of smaller length decay faster than waves of longer length;
- (2) when  $k$  is fixed,  $t_0$  decreases as  $v$  increases; thus waves of a given wave-length decay faster in a medium with larger  $v$  value.

In the present talk we are interested in equations of the form

$$u_t + uu_x - vu_{xx} = 0,$$

known as *Burgers equation* and is the simplest model of diffusive waves which under certain simplifying assumptions covers turbulence, sound-waves in viscous media, waves in fluid-filled visco-elastic tubes and magneto-hydrodynamic waves in media having finite electrical conductivity.

Major applications of non-linear diffusive equations are frequently applicable to areas such as plasma physics, biomathematics, blood-flow, traffic-flow and environmental science.

Let us begin our discussion with a system describing plane compressible flow in ideal (polytropic) gas ignoring dissipative effects given by

$$\rho_t + u\rho_x + \rho u_x = 0 \quad (14)$$

$$\rho(u_t + uu_x + \rho_x) = 0 \quad (15)$$

$$p = k\rho^\gamma, \quad S = \text{constant}. \quad (16)$$

Linearising the system of equations (14) to (16) and by suitable elimination, we have

$$\rho_{,tt} - a_0^2 \rho_{,xx} = 0 \quad (17)$$

where

$$a_0^2 = a^2(\rho_0, S_0),$$

being speed of sound, thus representing a linear wave.

Equations (14) and (15) can be written as

$$\rho_t + u\rho_x + \rho u_x = 0, \quad (18)$$

$$(u_t + uu_x) + \frac{a^2(\rho)\rho_x}{\rho} = 0, \quad (19)$$

where  $a^2(\rho) = \text{square of velocity of sound} = (\partial\rho/\partial p)_{S=S_0} = k\gamma\rho^{\gamma-1}$ .

If we assume that  $u = U(\rho)$ , equations (18) and (19) reduce to

$$\rho_{,t} + (U + \rho U')\rho_{,x} = 0, \quad (20)$$

$$\rho_{,t} + \left( U + \left( \frac{a^2}{\rho U'} \right) \right) \rho_{,x} = 0 \quad (21)$$

where prime denotes differentiations with respect to  $\rho$ . For non-trivial solution of system of equations (20) and (21), its coefficients matrix vanishes, which provides relations

$$U' = \pm \frac{a}{\rho} = \pm a_0 \left( \frac{\rho}{\rho_0} \right)^{\left( \frac{\gamma-1}{2} \right)} \left( \frac{1}{\rho} \right). \quad (22)$$

Thus the system of equations (18) and (19) reduces to one of the equations

$$\rho_{,t} + (U \pm a)\rho_{,x} = 0, \quad (23)$$

where

$$U(\rho) = \int_{\rho_0}^{\rho} \left( \frac{a(\rho)}{\rho} \right) d\rho = \frac{2}{\gamma-1} (a(\rho) - a_0) \quad (24)$$

If we restrict to the wave moving to the right and hence choosing positive sign in equation (23), then equation (20) or (21) can be written as

$$u_{,t} + \left[ a_0 + \frac{(\gamma+1)u}{2} \right] u_{,x} = 0, \quad (25)$$

which is typically non-linear and reduces the problem to solving an initial value problem for single first order non-linear partial differential equation (25) consistent with the intermediate integral (24) relating  $u$  and  $\rho$ .

Comparing equations (17) and (25), we see that solution corresponding to equation (25) ceases to be single-valued due to large velocity gradient. Thus, in a thin neighbourhood of this point, irreversible thermodynamic processes such as viscosity and heat conduction which were ignored in deriving equation (25) intervene.

Thus, a shock with a small thickness leads the smooth parts of the profile. The details of sock formation and its subsequent decay are discussed by Whitham [11]. Thus, the model equation (25) is inadequate to describe flows with shocks and to improve it, the effect of viscosity and heat conduction may be included which is improved by Cole [6] and the improved model is Burgers equation given as below:

$$u_{,t} + \left[ a_0 + \frac{(\gamma+1)u}{2} \right] u_{,x} = \left( \frac{v}{2} \right) u_{,xx}. \quad (26)$$

The equation

$$u_{,t} + uu_{,x} = \left( \frac{v}{2} \right) u_{,xx}$$

was first mooted by Bateman [1] who found its steady solutions for certain descriptive viscous flows. Later Burgers<sup>4</sup> proposed it as one of a class equations to describe the mathematical model of turbulence. In the context of gas dynamics, it was discussed by Hopf [8] and Cole [6], who by applying the non-linear transformation

$$u = -v(\log\Phi)_{,x} \quad (27)$$

reduce the equation,  $u_{,t} + uu_{,x} = (\frac{v}{2})u_{,xx}$  into the heat-equation

$$\Phi_{,t} = \frac{v}{2}(\Phi_{,xx}). \quad (28)$$

Burgers equations as well as Hopf-Cole transformation and equation (28) have the following properties which make it a popular model for the discussion of modern techniques of the transformation theory of non-linear partial differential equations.

1. The shift of origin

$$(x - x_0) \rightarrow x, (t - t_0) \rightarrow t, u \rightarrow u, \Phi \rightarrow \Phi,$$

where  $x_0$  and  $t_0$  are arbitrary independent constants.

2. The change of scale

$$\frac{x}{\alpha} \rightarrow x, \frac{t}{\alpha^2} \rightarrow t, (\alpha u) \rightarrow u, (\beta\Phi) \rightarrow \Phi.$$

where  $\alpha$  and  $\beta$  are arbitrary independent scale factors.

3. The Galilean transformation

$$(x - Ut) \rightarrow x, t \rightarrow t, (u - U) \rightarrow u.$$

### Applications

In the present talk we are applying Burgers equations for weak-time-dependent waves in fluids with internal state variables.

First let us define what we mean by weak-time-dependence.

"If  $t^*$  is the time scale of local changes and  $l^*$  is the length scale of spacewise changes, then the condition of weak-time-dependence is satisfied if  $t^* \gg l^* a_{e0}$ ".

Secondly, we define fluid with internal state variables: "If the thermodynamic state of a gas particle changes rapidly in the flow field, the state which it attains can no longer be considered as unconstrained equilibrium state. A complete description of the thermodynamic state then requires the introduction of additional independent variables  $q_i (i = 1, 2, 3, \dots)$ . These variables are usually so chosen that they may have a physical meaning or they may be progressive variables of chemical reactions going on in the flowing gas or they may describe the energy of internal degree of freedom." Many

authors [2,5,7] have discussed different properties of fluids with internal state variables.

Neglecting viscosity and heat-conduction, equations governing one-dimensional motion of a gas with internal state variables are given by

$$\frac{D\rho}{Dt} + \rho u_{,x} = 0, \quad (29)$$

$$\frac{Du}{Dt} + \frac{1}{\rho} p_{,x} = 0, \quad (30)$$

$$\frac{Dh}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} = 0, \quad (31)$$

$$\frac{Dq_i}{Dt} = L_i(p, S, q_i) \quad (i = 1, 2, 3 \dots), \quad (32)$$

Following linear transformation used by Meixner [9] equation of state and relaxation equation reduces to

$$h = h(p, S, r_i) \quad (i = 1, 2, 3 \dots) \quad (33)$$

and

$$\frac{Dr_i}{Dt} = \frac{1}{\tau_i} (\check{r}_i - r_i), \quad \text{respectively.}$$

Equation (31) with the help of equations (29) and (3) and applying certain thermodynamic properties and the relation

$$\frac{1}{a_e^2} = \frac{1}{a_f^2} - \rho^2 \sum_{i=1}^n h_{,pri} \check{r}_{i,p}$$

reduces to

$$\begin{aligned} \frac{1}{\rho} \left[ \frac{D\rho}{Dt} - \frac{1}{a_e^2} \frac{Dp}{Dt} \right] + \rho \left[ h_{,pS} + \sum_{i=1}^n h_{,pri} \check{r}_{i,S} \right] \frac{DS}{Dt} \\ + \rho \sum_{i=1}^n h_{,pri} \frac{D}{Dt} (r_i - \check{r}_i) = 0 \end{aligned} \quad (34)$$

and the entropy equation reduces to

$$T \frac{DS}{Dt} = \sum_{i=1}^n \frac{(r_i - \check{r}_i)^2}{\tau_i} \quad (35)$$

Applying weakly-time-dependence, equation (34) reduces to

$$2uu_{,t} + (u^2 - a_e^2)u_{,x} - v_d uu_{,xx} = 0 \quad (36)$$

where  $v_d = [\rho^2 a_e^4 \sum \tau_i (h_{,pri})^2]$ ,

$$(u - a_e) = \frac{u_0 - a_{e0}(u - a_e^*)}{(u_0 - a_e^*)}$$

and  $(u + a_e) = 2u$  approximately,  $a_e^*$  being the value of  $u$  for which  $u = a_e$ .

With the help of the above relations, eqn. (36) reduces to

$$u_{,t} + (u - a_e)u_{,x} - \frac{v_d}{2}u_{,xx} = 0. \quad (37)$$

If we put  $(u - a_e) = \eta$ , eqn. (37) reduces to

$$\eta_{,t} + \eta\eta_{,x} - \frac{v_d}{2}u_{,xx} = 0. \quad (38)$$

the *Burgers equation*.

Various authors [2,3,8,10] have solved Burgers equation for different cases. However, here we give two examples.

Case 1. For initial condition  $\eta(x, 0) = \sin(\Pi x)$ ,  $0 < x < 1$ .

Solution of equation (38) is given by

$$\eta(x, t) = \frac{\Pi v_d \sum_{n=1}^{\infty} A_n \exp(-n^2 \pi^2 \frac{v_d}{2} t) n \sin(\pi x)}{[A_0 + \sum_{n=1}^{\infty} A_n \exp(-n^2 \pi^2 \frac{v_d}{2} t) \cos(n \pi x)]} \quad (39)$$

where

$$A_0 = \int_0^1 \exp[(-\pi v_d)^{-1} \{1 - \cos(\pi x)\}] dx$$

$$A_n = \int_0^1 \exp[-(\pi v_d)^{-1} \{1 - \cos(\pi x)\}] \cos(n \pi x) dx \quad (n = 1, 2, 3, \dots)$$

Case 2. For initial condition  $\eta(x, 0) = 4x(1 - x)$ ,  $0 < x < 1$ .

Exact solution corresponding to eqn. (38) in this case is given by eqn. (39) with modified value of  $A_0$  and  $A_n$  given by

$$A_0 = \int_0^1 \exp\{-x^2 (\frac{3}{2} v_d)^{-1} (3 - 2x)\} dx$$

$$A_n = \int_0^1 \exp\{-x^2 (\frac{3}{2} v_d)^{-1} (3 - 2x)\} \cos(n \Pi x) dx, \quad (n = 1, 2, 3, \dots)$$

using restrictive Taylor variation of  $\eta$  for fixed value of  $v_d$  and different values of  $t$  are shown in Figs. 5 to 8. However, using Douglas variation of  $\eta$  is shown in Fig. 9.

### Generalized Burgers equation

The Burgers equation which we have discussed is an idealized equation which combines a simple non-linearity with a small linear viscous term. In actual physical situations there are other complicating physical factors which alter Burgers equations. These contributing terms may be lower order source or sink terms or geometrical expansion terms like spherical or cylindrical waves which arise due to point explosion such as blast-wave or disturbance produced along a line called cylindrical wave.

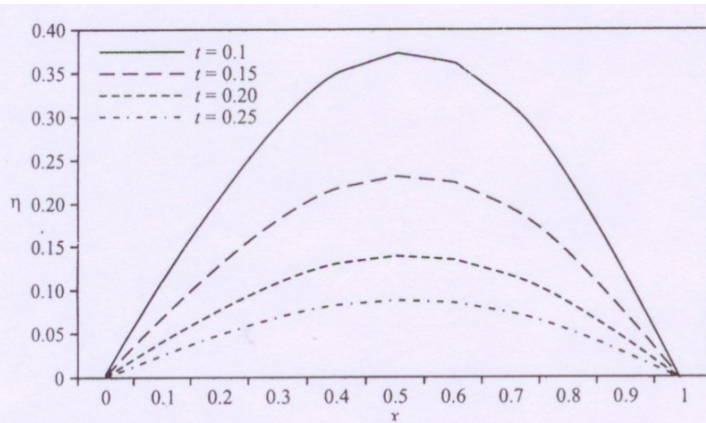


Fig. 5. Solutions of Problem 1 at different times for  $\nu = 2$ ,  $dt = 0.00001$ ,  $h = 0.0125$

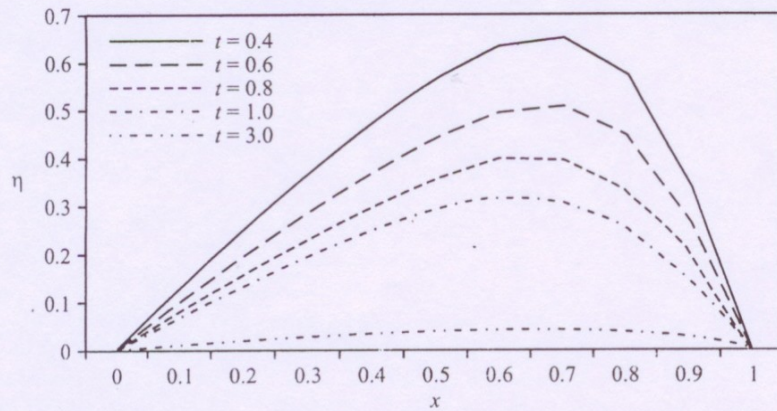


Fig. 6. Solutions of Problem 1 at different times for  $\nu = 0.2$ ,  $dt = 0.00001$ ,  $h = 0.0125$

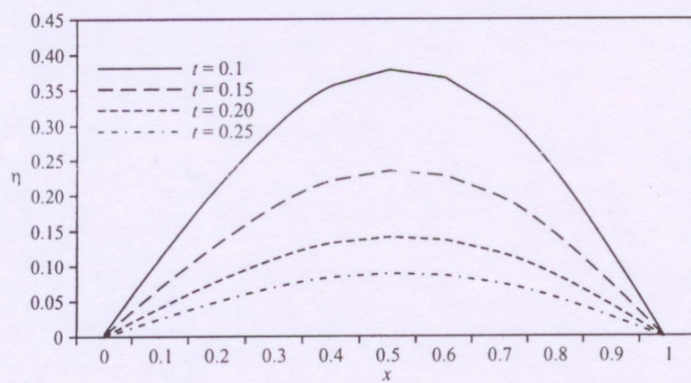


Fig. 7. Solutions of Problem 2 at different times for  $\nu = 2$ ,  $dt = 0.00001$ ,  $h = 0.0125$



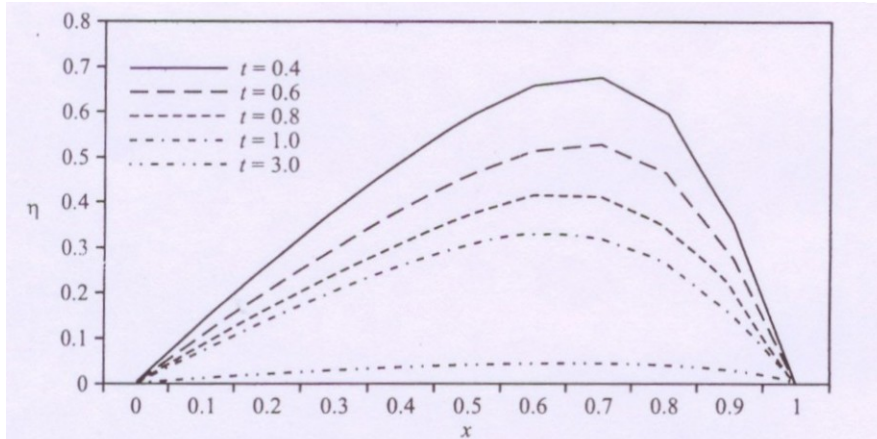


Fig. 8. Solutions of Problem 2 at different times for  $v = 0.2, dt = 0.0001, h = 0.0125$

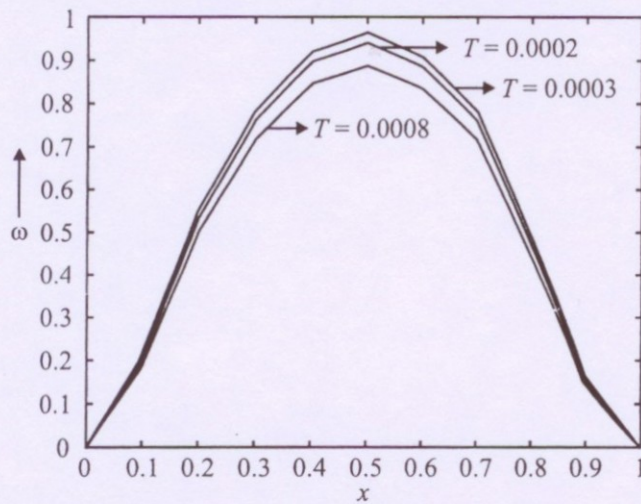


Fig. 9. Numerical solution of Problem 1 at different times for  $k = 0.0001, v_d = 20$

Generalized Burgers equations are given by

$$u_{,t} + uu_{,x} + \lambda u = \frac{v}{2} u_{,xx}, \tag{40}$$

and

$$u_{,t} + uu_{,x} + j \frac{u}{2t} = \frac{v}{2} u_{,xx}, (j = 1, 2) \tag{41}$$

Considering non-planar Navier-Stokes equation including the terms due to cylindrical or spherical symmetry in equation of continuity, we can obtain the generalized Burgers equation. For detail the reader is referred to [10].

**Major Applications:** In this talk the major applications of the non-linear diffusive equations have been drawn from gas dynamics which will frequently occur in other areas such as plasma physics, heat conduction, elasticity, biomathematics, blood flow and environmental sciences. Thus the talk is important for scientists, engineers and doctors working in their respective areas.

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## CORDIAL LABELING FOR TWO CYCLE RELATED GRAPHS

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ABSTRACT. In the present work we discuss cordiality of two cycle related graphs.

### 1. Introduction

We begin with simple, finite undirected graph  $G = (V, E)$ . In the present work  $C_n$ ,  $P_k$  and  $d(u)$  denote cycle with  $n$  vertices, path with  $k$  vertices and degree of vertex  $u$  respectively. For all other terminology and notations we follow Harary[1]. We will give brief summary of definitions which are useful for the present investigations.

**Definition 1.1:** A *chord* of a cycle  $C_n$  is an edge, which is not in  $C_n$ , but its end points lie on  $C_n$ .

**Definition 1.2:** Two chords of a cycle are said to be *twin chords* if they form a triangle with an edge of the cycle  $C_n$ .

For positive integers  $n$  and  $p$  with  $3 \leq p \leq n - 2$ ,  $C_{n,p}$  is the graph consisting of a cycle  $C_n$  with a pair of twin chords with which the edges of  $C_n$  form cycles  $C_p$ ,  $C_3$  and  $C_{n+1-p}$  without chords.

**Definition 1.3:** If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

One can refer Gallian[4] for detail survey on graph labeling.

Labeled graphs have many diversified applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal auto-correlation properties. A detailed study of variety of applications of graph labeling is given by Bloom and Golomb[2]. According to Beineke and Hegde[5] graph labeling serves as a frontier between number theory and structure of graphs.

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**Definition 1.4:** Let  $G = (V, E)$  be a graph. A mapping  $f : V(G) \rightarrow \{0,1\}$  is called *binary vertex labeling* of  $G$  and  $f(v)$  is called the *label* of the vertex  $v$  of  $G$  under  $f$ .

For an edge  $e = uv$ , the induced edge labeling  $f^* : E(G) \rightarrow \{0,1\}$  is given by  $f^*(e) = |f(u) - f(v)|$ . Let  $v_f(0), v_f(1)$  be the number of vertices of  $G$  having labels 0 and 1 respectively under  $f$  and let  $e_f(0), e_f(1)$  be the number of edges having labels 0 and 1 respectively under  $f^*$ .

**Definition 1.5:** A binary vertex labeling of a graph  $G$  is called a *cordial labeling* if  $|v_f(0) - v_f(1)| \leq 1$  and  $|e_f(0) - e_f(1)| \leq 1$ .

A graph  $G$  is cordial if it admits cordial labeling. This concept was introduced by Cahit[3]. Many researchers have studied cordiality of graphs. e.g. Ho, Lee and Shee[8] proved that unicyclic graph is cordial unless it is  $C_{4k+2}$ . Andar et al.[6] proved cordiality of multiple shells. Let graphs  $G_1, G_2, \dots, G_n, n \geq 2$  be all copies of a fixed graph  $G$ . Adding an edge between  $G_i$  to  $G_{i+1}$  for  $i = 1, 2, \dots, n - 1$  is called path union of  $G$ . Shee and Ho[7] proved that path union of cycles, Petersen graphs, trees, wheels, unicyclic graphs are cordial.

In present investigations we consider two copies of cycle  $C_n$  and join them by a path of arbitrary length, two copies of cycle with twin chords and join them by a path of arbitrary length. We prove that such graphs are cordial.

## 2. Main Results

**Theorem 2.1:** The graph  $G$  obtained by joining two copies of cycle  $C_n$  by a path of arbitrary length is cordial.

**Proof:** Let  $u_1, \dots, u_n$  be the vertices of first copy of cycle  $C_n$ ,  $v_1, \dots, v_k$  be the vertices of path  $P_k$  with  $u_1 = v_1$  and  $w_1, \dots, w_n$  be the vertices of second copy of cycle  $C_n$  with  $v_k = w_1$ . To define labeling function  $f : V(G) \rightarrow \{0, 1\}$  we consider all possible cases as mentioned in first eight columns of the following *Table 2.1*. These cases cover all possible arrangement of vertices. In each case the graph  $G$  under consideration satisfies the conditions for cordiality as shown in last two columns of the same *Table 2.1*. Thus  $G$  admits cordial labeling.

### 2.2 Illustrations

For better understanding of above defined labeling pattern, let us consider few examples.

**Example 1** Consider a graph obtained by joining two copies of cycles  $C_5$  by a path  $P_5$  (it is the case related with  $n \equiv 1(mod 4), k \equiv 1(mod 4)$ ). The labeling pattern is shown in *Figure-1*.

Let  $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$ , where  $n, k, i, j \in N$

$b$	$d$	$r$	$y$	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt seperately	vertex conditions	edge conditions	
0	0	0	0	0	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	1	0				
		2	2	1	1	0				
	1	0	0	0	1	1	1	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	0	0			
			2	2	0	0	0			
	2	0	0	0	1	1	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	0	0	0			
			2	2	0	0	0			
	3	0	0	0	1	1	0	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	0	1			
			2	2	0	0	1			
1	0	0	0	1	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	0	0	0				
		2	2	0	1	0				
	1	0	0	0	1	0	1	$f(v_k)=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	0	0	1			
			2	2	0	1	0			
	2	0	0	0	1	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	0	0	1			
			2	2	0	1	0			
	3	0	0	0	0	0	0	$f(v_k)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	0	0			
			2	2	1	1	1			
2	0	0	0	1	1	1	$f(u_n)=0,$ $f(u_{n-1})=1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	0	0	1				
		2	2	0	0	0				
	1	0	0	0	0	1	1	$f(w_n)=0,$ $f(w_{n-1})=1$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	0	0			
			2	2	1	0	0			
	2	0	0	0	1	0	1	$f(u_n)=0,$ $f(u_{n-1})=1$	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	0	0	1			
			2	2	0	1	0			
	3	0	0	0	1	0	1	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	0	1			
			2	2	0	1	0			
3	0	0	0	1	1	0	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	
		1	1	0	0	1				
		2	2	0	0	1				
	1	0	0	0	0	0	0	$f(v_k)=0$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	1	1	0			
			2	2	1	1	1			
	2	0	0	0	1	1	0	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	0	0	0			
			2	2	0	0	1			
	3	0	0	0	1	1	1	$f(v_k)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	0	0			
			2	2	0	0	1			

Table 2.1

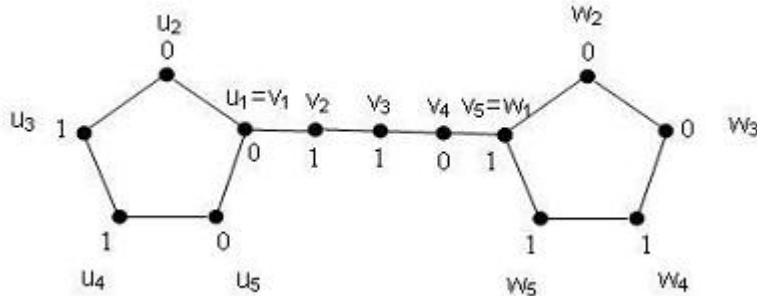


FIGURE 1

**Example 2** Consider a graph obtained by joining two copies of cycles  $C_7$  by a path  $P_7$  (it is the case related with  $n \equiv 3(mod 4)$ ,  $k \equiv 3(mod 4)$ ). The labeling pattern is shown in Figure-2.

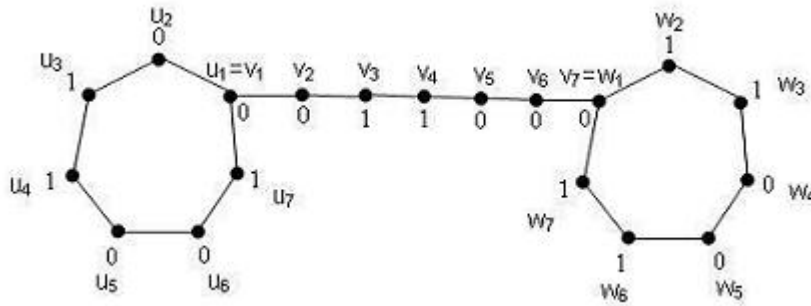


FIGURE 2

**Theorem 2.3** The graph  $G$  obtained by joining two cycles with twin chords by a path of arbitrary length is cordial, where chords form two triangles and one cycle  $C_{n-2}$ .

**Proof:** Let  $u_1, \dots, u_n$  be successive vertices of first copy of cycle  $C_n$  such that  $u_1, u_2, u_3$  form a triangle with one of the twin chords and  $d(u_1) = 4$ ,  $d(u_3) = d(u_4) = 3$  while  $d(u_2) = 2$  and  $d(u_i) = 2$ , for  $5 \leq i \leq n$ . Let  $w_1, \dots, w_n$  be the successive vertices of second copy of cycle  $C_n$  such that  $w_1, w_2, w_3$  form a triangle with one of the twin chords and  $d(w_1) = 4$ ,  $d(w_3) = d(w_4) = 3$  while  $d(w_2) = 2$  and  $d(w_i) = 2$ , for  $5 \leq i \leq n$ . Let  $v_1, \dots, v_k$  be the successive vertices of path  $P_k$  with  $v_1 = u_i$ , for  $i = 3$  or  $i = 1$  or  $i = 4$  and  $v_k = w_1$ . To define labeling function  $f : V(G) \rightarrow \{0, 1\}$  we consider all possible cases as shown in first eight columns of Table 2.2,

Table 2.3 and Table 2.4 which covers all possible arrangement of vertices. In each case the graph  $G$  under consideration satisfies the conditions for cordiality as recorded in last two columns of respective tables. Thus  $G$  admits cordial labeling.

**Case-A**  $v_1 = u_3$

Let  $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$ , where  $n \in N$  and  $n \geq 5$

$b$	$d$	$r$	$y$	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt seperately	vertex conditions	edge conditions	
0	0	0	0	1	1	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	0	1				
		2	2	0	0	0				
	1	1	0	0	1	1	0	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	1	0	0			
			2	2	0	0	1			
	2	2	0	0	1	1	1	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	1	0	1			
			2	2	0	1	0			
	3	3	0	0	1	1	0	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	1	0	1			
			2	2	0	0	0			
1	0	0	0	1	1	0	$f(w_n)=0$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	0	1				
		2	2	0	0	1				
	1	1	0	0	1	0	1	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	1	1			
			2	2	0	1	0			
	2	2	0	0	1	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	0	1	1			
			2	2	0	1	0			
	3	3	0	0	0	0	1	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	1	0			
			2	2	1	1	0			
2	0	0	0	0	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	0	1	1				
		2	2	1	1	1				
	1	1	0	0	1	0	0	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	1	1			
			2	2	0	1	1			
	2	2	0	0	1	0	0	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	0	1	1			
			2	2	0	1	1			
	3	3	0	0	1	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	0	1	1			
			2	2	0	1	1			
3	0	0	0	0	0	0	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	
		1	1	0	1	1				
		2	2	1	1	1				
	1	1	0	0	0	0	1	$f(w_1)=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	0	1	0			
			2	2	1	1	0			
	2	2	0	0	0	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	0	1	1			
			2	2	1	1	0			
	3	3	0	0	0	0	1	$f(v_k)=1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	0	1	1			
			2	2	1	1	0			

Table 2.2

**Case-B**  $v_1 = u_1$

Let  $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$ , where  $n \in N$  and  $n \geq 5$

$b$	$d$	$r$	$y$	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt separately	vertex conditions	edge conditions	
0	0	0	0	1	0	0	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	1	0				
		2	2	0	1	1				
		3	3	0	0	1				
	1	1	0	0	1	0	1	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	1	1	1			
			2	2	0	1	0			
			3	3	0	0	0			
	2	2	0	0	1	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	1	1	1			
			2	2	0	1	0			
			3	3	0	0	0			
3	3	0	0	1	0	0	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$	
		1	1	1	1	0				
		2	2	0	1	1				
		3	3	0	0	1				
1	0	0	0	1	1	1	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	
		1	1	1	1	1				
		2	2	0	0	0				
		3	3	0	0	0				
	1	1	0	0	0	0	1	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	1	1	1			
			2	2	1	1	0			
			3	3	0	0	0			
	2	2	0	0	0	1	0	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	1	1	0			
			2	2	1	0	1			
			3	3	0	0	1			
3	3	0	0	0	0	0	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$	
		1	1	1	1	0				
		2	2	1	1	1				
		3	3	0	0	1				
2	0	0	0	0	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	1	0				
		2	2	1	1	0				
		3	3	0	0	1				
	1	1	0	0	0	0	0	$f(v_k)=0$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	1	1	0			
			2	2	1	1	1			
			3	3	0	0	1			
	2	2	0	0	0	1	1	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$
			1	1	1	1	0			
			2	2	1	0	0			
			3	3	0	0	1			
3	3	0	0	0	0	1	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$	
		1	1	1	1	0				
		2	2	1	1	0				
		3	3	0	0	1				
3	0	0	0	1	0	0	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	
		1	1	1	1	0				
		2	2	0	1	1				
		3	3	0	0	1				
	1	1	0	0	1	0	0	$f(v_k)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	1	1	0			
			2	2	0	1	1			
			3	3	0	0	1			
	2	2	0	0	1	0	1	$f(w_1)=1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	1	1	0			
			2	2	0	1	0			
			3	3	0	0	1			
3	3	0	0	1	0	1	$f(v_k)=1$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$	
		1	1	1	1	1				
		2	2	0	1	0				
		3	3	0	0	0				

Table 2.3



Case-C  $v_1 = u_4$

Let  $n = 4a + b, k = 4c + d, i = 4s + r, j = 4x + y$ , where  $n \in N$  and  $n \geq 5$

$b$	$d$	$r$	$y$	$f(u_i)$	$f(v_j)$	$f(w_i)$	vertex labeling to be dealt separately	vertex conditions	edge conditions	
0	0	0	0	1	0	0	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	1	0				
		2	2	0	1	1				
		3	3	0	0	1				
	1	1	0	0	1	0	1	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	1	1	1			
			2	2	0	1	0			
			3	3	0	0	0			
	2	2	0	0	1	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	1	1	1			
			2	2	0	1	0			
			3	3	0	0	0			
3	3	0	0	1	0	0	-	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$	
		1	1	1	1	0				
		2	2	0	1	1				
		3	3	0	0	1				
1	0	0	0	1	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	1	0				
		2	2	0	1	0				
		3	3	0	0	1				
	1	1	0	0	1	1	0	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	1	1	1			
			2	2	0	0	1			
			3	3	0	0	0			
	2	2	0	0	0	0	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	0	0	1			
			2	2	1	1	0			
			3	3	1	1	0			
3	3	0	0	1	0	1	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$	
		1	1	1	1	0				
		2	2	0	1	0				
		3	3	0	0	1				
2	0	0	0	1	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	
		1	1	1	1	1				
		2	2	0	1	1				
		3	3	0	0	0				
	1	1	0	0	0	0	1	-	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			1	1	1	0	0			
			2	2	1	1	0			
			3	3	0	1	1			
	2	2	0	0	0	1	1	-	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	1	0	0			
			2	2	1	0	0			
			3	3	0	1	1			
3	3	0	0	0	1	0	$f(w_k)=0$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$	
		1	1	1	0	0				
		2	2	1	0	1				
		3	3	0	1	1				
3	0	0	0	1	0	0	-	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	
		1	1	1	1	0				
		2	2	0	1	1				
		3	3	0	0	1				
	1	1	0	0	1	0	1	$f(w_1)=1$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			1	1	1	1	0			
			2	2	0	1	0			
			3	3	0	0	1			
	2	2	0	0	1	0	1	$f(w_1)=1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$
			1	1	1	1	0			
			2	2	0	1	0			
			3	3	0	0	1			
3	3	0	0	1	0	0	$f(w_1)=0$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$	
		1	1	1	1	1				
		2	2	0	1	1				
		3	3	0	0	0				

Table 2.4

**2.4 Illustrations**

Let us demonstrate above labeling patterns by means of following examples.

**Example 1** Consider a graph obtained by joining two copies of cycles  $C_5$  by a path  $P_4$ (it is the case related with **case-A**,  $n \equiv 1(mod4)$ ,  $k \equiv 0(mod4)$ ). The labeling pattern is shown in *Figure-3*.

**Example 2** Consider a graph obtained by joining two copies of cycles  $C_6$  by a path  $P_6$ (it is the case related with **case-B**,  $n \equiv 2(mod4)$ ,  $k \equiv 2(mod4)$ ).The labeling pattern is shown in *Figure-4*.

**Example 3** Consider a graph obtained by joining two copies of cycles  $C_8$  by a path  $P_7$ (it is the case related with **case-C**,  $n \equiv 0(mod4)$ ,  $k \equiv 3(mod4)$ ).The labeling pattern is shown in *Figure-5*.

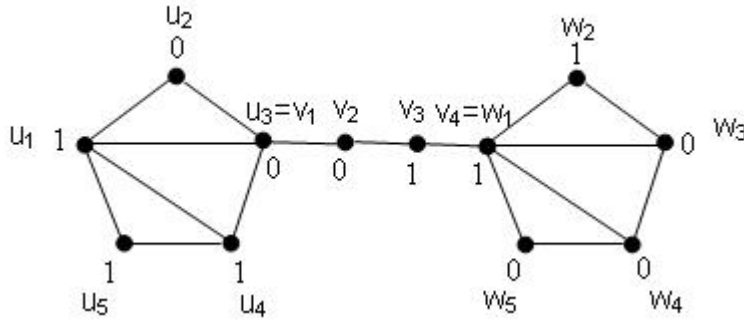


FIGURE 3

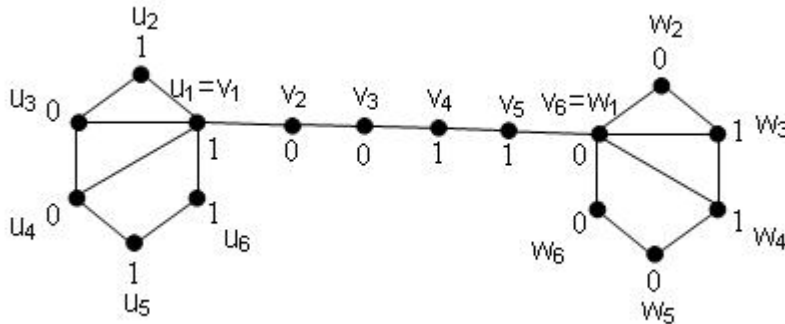


FIGURE 4

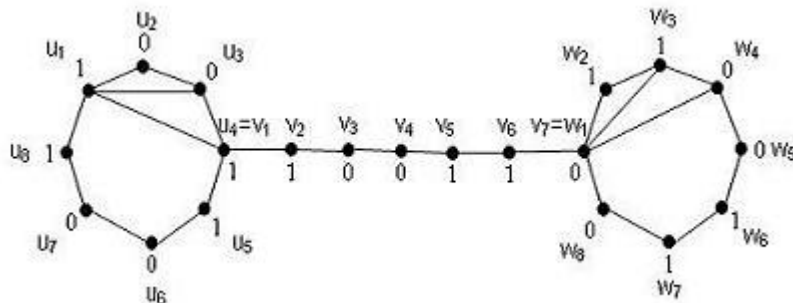


FIGURE 5

### 3. Concluding Remarks

We consider the path of arbitrary length between end vertices of chords only but one can consider path between any two vertices of the graphs. Arbitrariness of path is the key feature of present work. Moreover the investigations carried out here are new. The labeling pattern is given in detail and proofs are given in very elegant way. The defined labeling pattern is explained by various illustrations which gives better understanding of the results derived. This work contribute two new graphs to the theory of cordial graphs.

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## ON SQUARES OF SQUARES

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ABSTRACT. This paper is concerned with the problem of finding a  $3 \times 3$  magic square all of whose entries are perfect squares. The paper describes elementary methods of obtaining infinitely many parametric solutions of  $3 \times 3$  pseudo-magic squares of squares, that is, squares constructed from 9 integer squares in which the entries in all the rows and columns as well as one of the diagonals add up to the same magic sum. The problem of finding a  $3 \times 3$  magic square all of whose entries are perfect squares remains open.

### 1. Introduction

A magic square is a square array of  $n^2$  distinct positive integers such that the sums of the  $n$  integers in each row and column and each of the two diagonals are all equal. A square that fails to be magic only because one or both of the diagonal sums differs from the common sum is called a semimagic square. A semimagic square in which the entries in one of the diagonals also add up to the the common sum will be called a pseudo-magic square. The common sum of the rows and columns of a semimagic, pseudo-magic or magic square is called the magic sum.

Magic squares have attracted the attention of both amateurs and mathematicians for centuries. A detailed exposition of the theory on the subject is given by Andrews [1] and Kraitchik [8, Chapter 7, pp. 142–192]. Despite the long history of the subject, there are several open problems concerning magic squares. One such problem, posed by LaBar [9] and again mentioned in ([6, p. 269], [7]) is to prove or disprove that a  $3 \times 3$  magic square can be constructed from nine distinct integer squares. Sallows [12] and Schweitzer (as quoted in [6, p. 269]) found numerical examples of pseudo-magic squares of perfect squares. Bremner [3, 4] has studied the problem using techniques of algebraic geometry, and has obtained numerous parametric solutions of

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pseudo-magic squares of squares. The simplest solution found by Bremner has entries of degree 8 in terms of a single parameter. LaBar's problem has also been considered in [2, 10, 11].

As all the earlier known examples of pseudo-magic squares of squares had a perfect square as their magic sum, Guy [6, p. 269] asked the question "Does this have to happen?". A solitary numerical counterexample to this question, attributed to Schweitzer, is given in [2]. In this paper, we first obtain in Section 2 all semimagic squares whose entries are perfect squares. We then give in Section 3.1 an elementary method of obtaining infinitely many pseudo-magic squares of perfect squares, in parametric form, in which the magic sum is, in general, not a perfect square. We also give certain additional numerical examples of pseudo-magic squares of squares that are not obtained by giving numerical values to the parameters in the parametric solution. All these pseudo-magic squares provide new counterexamples to Guy's aforementioned question. Finally in Section 3.2 we give another elementary method of generating infinitely many pseudo-magic squares of perfect squares, in parametric form, in which the magic sum is always a perfect square.

It is well-known that the magic sum of a  $3 \times 3$  magic square is three times the central number. It follows that the solutions of Section 3.2 can never generate a magic square of perfect squares, but the possibility of a magic square of perfect squares being generated from solutions of the type obtained in Section 3.1 cannot be ruled out. However, no magic square of perfect squares could be found so far, and the problem of LaBar remains an open problem.

## 2. Semimagic squares of squares

Euler was apparently the first to discover the following semimagic square of perfect squares (cf. Dickson [5, p.530]):

$$(1) \quad \begin{bmatrix} (p^2 + q^2 - r^2 - s^2)^2 & (2qr + 2ps)^2 & (2qs - 2pr)^2 \\ (2qr - 2ps)^2 & (p^2 - q^2 + r^2 - s^2)^2 & (2pq + 2rs)^2 \\ (2qs + 2pr)^2 & (2rs - 2pq)^2 & (p^2 - q^2 - r^2 + s^2)^2 \end{bmatrix}.$$

This square does not represent all semimagic squares of squares as its magic sum is  $(p^2 + q^2 + r^2 + s^2)^2$  whereas there exist semimagic squares of squares whose magic sum is not a perfect square. We will obtain such counterexamples later in this paper.

We will now obtain all semimagic squares whose entries are squares of rational numbers. It suffices to find rational solutions as all the entries of such squares can be multiplied by a suitable perfect square to obtain semimagic squares of integer squares.

**Lemma:** A necessary and sufficient condition that the square of squares

$$(2) \quad \begin{bmatrix} x_3^2 & z_2^2 & y_1^2 \\ z_1^2 & y_3^2 & x_2^2 \\ y_2^2 & x_1^2 & z_3^2 \end{bmatrix}$$

is a semimagic square is that the following equations have a rational solution:

$$(3) \quad x_1^2 - x_2^2 = y_1^2 - y_2^2 = z_1^2 - z_2^2,$$

$$(4) \quad x_1^2 - x_3^2 = y_1^2 - y_3^2 = z_1^2 - z_3^2.$$

**Proof:** If the equations (3) and (4) are satisfied, by writing  $x_1^2 - x_2^2 = a$  and  $x_1^2 - x_3^2 = b$ , we easily find that the sum of each of the rows and columns of the square (2) is  $x_1^2 + y_1^2 + z_1^2 - a - b$ . Hence (2) is a semimagic square which shows that the condition stated in the lemma is sufficient. To show that the condition is necessary, we denote by  $R_i$  and  $C_i$ ,  $i = 1, 2, 3$ , the sums of the entries in the  $i^{\text{th}}$  row and column respectively. The conditions  $R_3 = C_3$  and  $R_1 = C_1$  yield the equation (3) while the conditions  $R_1 = C_2$  and  $R_2 = C_3$  yield the condition (4). This completes the proof.

**Theorem:** All semimagic squares with entries that are squares of rational numbers are given by (2) where

$$(5) \quad \begin{aligned} x_1 &= k(p^2 q r s^2 - p q^2 r^2 s - p^2 q r + p q^2 s + p r^2 s - q r s^2 - p s + q r), \\ x_2 &= k(-p^2 q r s^2 + p q^2 r^2 s + p^2 q r - p q^2 s - 2 p q r^2 \\ &\quad + 2 p q s^2 + p r^2 s - q r s^2 - p s + q r), \\ x_3 &= k(-p^2 q r s^2 + p q^2 r^2 s - p^2 q r + 2 p^2 r s + p q^2 s \\ &\quad - p r^2 s - 2 q^2 r s + q r s^2 - p s + q r), \\ y_1 &= k(-p^2 q r^2 + p^2 q s^2 + p^2 r^2 s - q^2 r^2 s - p^2 s + q^2 s + q r^2 - q s^2), \\ y_2 &= k(-p^2 q r^2 + p^2 q s^2 + p^2 r^2 s - 2 p q r s^2 + q^2 r^2 s \\ &\quad - p^2 s + 2 p q r - q^2 s - q r^2 + q s^2), \\ y_3 &= k(-p^2 q r^2 - p^2 q s^2 + p^2 r^2 s + 2 p q^2 r s \\ &\quad - q^2 r^2 s + p^2 s - 2 p r s - q^2 s + q r^2 + q s^2), \\ z_1 &= k(p^2 r s^2 - p q^2 r^2 + p q^2 s^2 - q^2 r s^2 - p^2 r + p r^2 - p s^2 + q^2 r), \\ z_2 &= k(-p^2 r s^2 - p q^2 r^2 + p q^2 s^2 + 2 p q r^2 s - q^2 r s^2 \\ &\quad + p^2 r - 2 p q s - p r^2 + p s^2 + q^2 r), \\ z_3 &= k(-2 p^2 q r s + p^2 r s^2 + p q^2 r^2 + p q^2 s^2 - q^2 r s^2 \\ &\quad + p^2 r - p r^2 - p s^2 - q^2 r + 2 q r s), \end{aligned}$$

where  $p, q, r, s$  and  $k$  are arbitrary rational parameters.

**Proof:** It is easily seen that equation (3) will have a rational solution if and only if there exist rational numbers  $p$  and  $q$  such that

$$(6) \quad x_1 - x_2 = p(y_1 - y_2), \quad p(x_1 + x_2) = y_1 + y_2,$$

$$(7) \quad x_1 - x_2 = q(z_1 - z_2), \quad q(x_1 + x_2) = z_1 + z_2.$$

Similarly, (4) is equivalent to the equations

$$(8) \quad x_1 - x_3 = r(y_1 - y_3), \quad r(x_1 + x_3) = y_1 + y_3,$$

$$(9) \quad x_1 - x_3 = s(z_1 - z_3), \quad s(x_1 + x_3) = z_1 + z_3,$$

where  $r$  and  $s$  are rational numbers. Equations (6), (7), (8), (9) are 8 linear equations in the 9 variables  $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ , and hence are readily solved to give the solution (5) and, in view of the lemma, all semimagic squares of squares are now given by (2).

### 3. Pseudo-magic squares of squares

We will now give elementary methods of generating infinitely many pseudo-magic squares in which the entries are perfect squares of rational numbers. As in the case of semimagic squares, it suffices to find rational solutions as these easily lead to pseudo-magic squares of integer squares by multiplying by a suitable integer square. Further, while the definition of pseudo-magic squares requires that the entries in either of the diagonals should add up to the magic sum, in all the pseudo-magic squares considered in this paper, it is the principal diagonal that adds up to the magic sum.

#### 3.1 Pseudo-magic squares of squares in which the magic sum is not a perfect square

**Lemma:** A necessary and sufficient condition that the square of squares (2) is a pseudo-magic square is that the following equations have a rational solution:

$$(10) \quad x_1^2 - x_2^2 = y_1^2 - y_2^2 = z_1^2 - z_2^2,$$

$$(11) \quad x_1^2 + x_2^2 - 2x_3^2 = y_1^2 + y_2^2 - 2y_3^2 = z_1^2 + z_2^2 - 2z_3^2 = 0.$$

**Proof:** If (10) and (11) are satisfied, we get  $x_1^2 + x_2^2 = 2x_3^2$ , and writing  $x_1^2 - x_2^2 = 2a$ , we get  $x_1^2 = x_3^2 + a$ ,  $x_2^2 = x_3^2 - a$ . Similarly, we get  $y_1^2 = y_3^2 + a$ ,  $y_2^2 = y_3^2 - a$ , and  $z_1^2 = z_3^2 + a$ ,  $z_2^2 = z_3^2 - a$ , and it is now readily seen that the sum of each row, column and the principal diagonal of the square (2) is  $x_3^2 + y_3^2 + z_3^2$ . Thus the condition stated in the lemma is sufficient. To show that it is also necessary, we note that since every pseudo-magic square is necessarily a semimagic square, (3) (which is same as equation (10)) and (4) must be satisfied. If the square (2) is a pseudo-magic square then, in addition to (3) and (4), the following condition obtained by equating the sum of the entries of the principal diagonal to the sum of the entries in the first row, is also satisfied:

$$(12) \quad y_1^2 - y_3^2 = z_3^2 - z_2^2.$$



Combining the last part of (4) with (12), we get

$$(13) \quad z_1^2 + z_2^2 - 2z_3^2 = 0.$$

Further, on subtracting two times (4) from (3), we get

$$(14) \quad x_1^2 + x_2^2 - 2x_3^2 = y_1^2 + y_2^2 - 2y_3^2 = z_1^2 + z_2^2 - 2z_3^2.$$

Now (11) follows from (13) and (14) and the proof is complete.

Equations (10) and (11) can be solved in various ways leading to pseudo-magic squares in which the magic sum is either a perfect square or not a perfect square. In this section we will consider solutions of these equations such that the magic sum of the resulting pseudo-magic square is not a perfect square.

We may consider equation (11) as three independent quadratic equations of the type  $x_1^2 + x_2^2 - 2x_3^2 = 0$ , and hence obtain its complete solution which may be written as follows:

$$(15) \quad \begin{aligned} x_1 &= k_1(X_1^2 - 2X_1X_2 - X_2^2), \\ x_2 &= k_1(X_1^2 + 2X_1X_2 - X_2^2), \\ x_3 &= k_1(X_1^2 + X_2^2), \\ y_1 &= k_2(Y_1^2 - 2Y_1Y_2 - Y_2^2), \\ y_2 &= k_2(Y_1^2 + 2Y_1Y_2 - Y_2^2), \\ y_3 &= k_2(Y_1^2 + Y_2^2), \\ z_1 &= k_3(Z_1^2 - 2Z_1Z_2 - Z_2^2), \\ z_2 &= k_3(Z_1^2 + 2Z_1Z_2 - Z_2^2), \\ z_3 &= k_3(Z_1^2 + Z_2^2), \end{aligned}$$

where  $X_i, Y_i, Z_i, i = 1, 2$  as well as  $k_1, k_2, k_3$  are arbitrary parameters. Substituting these values of  $X_i, Y_i, Z_i, i = 1, 2$  in (10), we get

$$(16) \quad k_1^2 X_1 X_2 (X_1^2 - X_2^2) = k_2^2 Y_1 Y_2 (Y_1^2 - Y_2^2) = k_3^2 Z_1 Z_2 (Z_1^2 - Z_2^2).$$

We write  $Y_1 = mX_1, Y_2 = -m(X_1 + X_2)$ , when the first part of equation (16) reduces to a linear equation in  $X_1$  and  $X_2$ , and we get the following solution for this part of equation (16):

$$(17) \quad \begin{aligned} X_1 &= (k_1^2 + k_2^2 m^4), & X_2 &= (k_1^2 - 2k_2^2 m^4), \\ Y_1 &= m(k_1^2 + k_2^2 m^4), & Y_2 &= -m(2k_1^2 - k_2^2 m^4), \end{aligned}$$

where  $m$  is an arbitrary parameter. The remaining part of equation (16) may now be written as

$$(18) \quad \begin{aligned} &3m^4(k_1^2 - 2m^4 k_2^2)(2k_1^2 - m^4 k_2^2)(k_1^2 + m^4 k_2^2)k_1^2 k_2^2 \\ &= k_3^2 Z_1 Z_2 (Z_1 - Z_2)(Z_1 + Z_2). \end{aligned}$$

To solve equation (18), we note that  $k_3$  will be rational if  $Z_1, Z_2$  are so chosen that

$$(19) \quad 3(k_1^2 - 2m^4 k_2^2)(2k_1^2 - m^4 k_2^2)(k_1^2 + m^4 k_2^2)Z_1 Z_2 (Z_1 - Z_2)(Z_1 + Z_2),$$

is a perfect square. Further, it follows from (16) that a trivial solution of (18) is given by  $Z_1 = X_1, Z_2 = X_2, k_3 = k_1$ . Since (19) is a quartic form in  $Z_1, Z_2$ , we can easily find the desired values of  $Z_1, Z_2$ , by using the known trivial solution and following the usual methods described in [5, p. 639]. Substituting the values of  $X_i, Y_i, Z_i, i = 1, 2$ , in (15), we get a solution of equations (10) and (11) and hence a pseudo-magic square in parametric form. We note that while we have apparently three parameters in the final solution, two of the parameters are not independent. Writing the polynomial  $c_0t^n + c_1t^{n-1} + \dots + c_n$  briefly as  $(c_0, c_1, \dots, c_n)$ , this pseudo-magic square may be written as (2) where  $x_i, y_i, z_i$  are polynomials in terms of a single variable  $t$  defined as follows:

$$\begin{aligned} x_1 &= (-32, 0, 224, 0, -128, 0, -1648, 0, 3760, 0, -5944, 0, 3136, 0, 1052, 0, 70, 0), \\ x_2 &= (32, 0, -32, 0, -544, 0, 496, 0, 800, 0, -3896, 0, 6728, 0, -3164, 0, -490, 0), \\ x_3 &= (32, 0, -128, 0, -64, 0, 640, 0, -2560, 0, 3904, 0, -4144, 0, 2320, 0, 350, 0), \\ y_1 &= (16, 0, 80, 0, -536, 0, -544, 0, 2140, 0, -5428, 0, 5326, 0, -424, 0, -140), \\ y_2 &= (-112, 0, 400, 0, 680, 0, -2816, 0, 5660, 0, -5444, 0, 158, 0, 1264, 0, 140), \\ y_3 &= (80, 0, -272, 0, -472, 0, 1744, 0, -4180, 0, 4420, 0, -1954, 0, 844, 0, 140), \\ z_1 &= (16, 0, -256, 0, 352, 0, -256, 0, 3448, 0, -5632, 0, 1240, 0, 3488, 0, -1151, 0), \\ z_2 &= (16, 0, 128, 0, -992, 0, 2624, 0, -392, 0, -4576, 0, 2824, 0, -2032, 0, 1249, 0), \\ z_3 &= (16, 0, -64, 0, 832, 0, -2272, 0, 664, 0, 2384, 0, 1168, 0, -2728, 0, 1201, 0). \end{aligned}$$

It is easily verified that the magic sum of this pseudo-magic square is not a perfect square. As a numerical example, if we take  $t = 2$  in the above solution, we get the following pseudo-magic square

$$\begin{bmatrix} 68962^2 & 1414561^2 & 54694^2 \\ 1411199^2 & 87986^2 & 97498^2 \\ 111766^2 & 2378^2 & 1412881^2 \end{bmatrix}$$

whose magic sum, expressed as a product of prime numbers, may be written as  $3.7.37.15077.171469$  and hence is not a perfect square.

We can find infinitely many values of  $Z_1, Z_2$  that make (19) a perfect square and hence obtain more such pseudo-magic squares in parametric form. The next pseudo-magic square thus obtained has polynomials of degree 74 in a single variable as its entries. Its magic sum is again not a perfect square.

We present below another pseudo-magic square, obtained by solving equations (10) and (11) in a different way:

$$\begin{bmatrix} \{(x^4 + 22x^2 + 9)z\}^2 & \{(x^4 + 4x^3 - 10x^2 - 12x + 9)y\}^2 & \{2(x^4 + 4x^2 + 12x - 9)z\}^2 \\ \{(x^4 - 4x^3 - 10x^2 + 12x + 9)y\}^2 & \{2(x^4 + 2x^3 + 2x^2 - 6x + 9)z\}^2 & \{(x^4 + 8x^3 - 10x^2 + 24x + 9)z\}^2 \\ \{2(x^4 + 4x^3 - 4x^2 - 9)z\}^2 & \{(x^4 - 8x^3 - 10x^2 - 24x + 9)z\}^2 & \{(x^4 - 2x^2 + 9)y\}^2 \end{bmatrix}$$

where  $z = x^2 - 3$  and  $x, y$  are related by the quartic equation

$$(20) \quad y^2 = 2x^4 - 18.$$

It is readily verified that the above indeed gives a pseudo-magic square. Now (20) is a quartic model of an elliptic curve of rank 1 and accordingly infinitely many rational solutions of this equation may be obtained, one solution being  $x = 99/47, y = 10212/2209$  which leads to the pseudo-magic square

$$\begin{bmatrix} 98434526^2 & 8968837^2 & 83721863^2 \\ 69684283^2 & 67982503^2 & 85449554^2 \\ 47264057^2 & 109895794^2 & 49680677^2 \end{bmatrix}$$

whose magic sum is  $2.3^2.386051.2414641573$  which is not a perfect square.

### 3.2 Pseudo-magic squares of squares in which the magic sum is a perfect square

While we can obtain pseudo-magic squares of squares in which the magic sum is a perfect square by solving equations (10) and (11), we will give a simpler method of obtaining such solutions. The method illustrates why such squares are more easily found as compared to the case in which the magic sum is not a perfect square.

In the semimagic square (1) we interchange the second and third columns to obtain the following semimagic square which naturally also has the magic sum  $(p^2 + q^2 + r^2 + s^2)^2$ :

$$(21) \quad \begin{bmatrix} (p^2 + q^2 - r^2 - s^2)^2 & (2qs - 2pr)^2 & (2qr + 2ps)^2 \\ (2qr - 2ps)^2 & (2pq + 2rs)^2 & (p^2 - q^2 + r^2 - s^2)^2 \\ (2qs + 2pr)^2 & (p^2 - q^2 - r^2 + s^2)^2 & (2rs - 2pq)^2 \end{bmatrix}.$$

The condition that the sum of entries of the principal diagonal of this square is also equal to the magic sum reduces to the following equation:

$$(22) \quad (2q^2 - r^2 - s^2)p^2 - q^2r^2 + 2r^2s^2 - q^2s^2 = 0.$$

Computer trials readily yield a number of numerical solutions of (22) leading to pseudo-magic squares of squares of relatively small integers. To obtain a parametric solution, we substitute  $p = 2r - q$  in (22) when this equation reduces to

$$(23) \quad q^2 - 2qr - 2r^2 - s^2 = 0.$$

The complete solution of (23) is readily found and we thus obtain the following pseudo-magic square with entries of degree 8 in the parameter  $t$ :

$$\begin{bmatrix} 4(t^4 - 2t^3 + 2t^2 + 6t + 9)^2 & 4(t^4 + 4t^3 - 4t^2 - 9)^2 & (t^4 - 8t^3 - 10t^2 - 24t + 9)^2 \\ 4(t^4 + 4t^2 + 12t - 9)^2 & (t^4 + 22t^2 + 9)^2 & 4(t^4 + 4t^2 - 12t - 9)^2 \\ (t^4 + 8t^3 - 10t^2 + 24t + 9)^2 & 4(t^4 - 4t^3 - 4t^2 - 9)^2 & 4(t^4 + 2t^3 + 2t^2 - 6t + 9)^2 \end{bmatrix}.$$

This square is readily seen to be equivalent to the square with entries of degree 8 given by Bremner in [3].

We now describe a method of obtaining other parametric solutions of (22) using a given solution. If  $(p_1, q_1, r_1, s_1)$  is a known solution of (22), we substitute in this equation,

$$(24) \quad p = gx + p_1, \quad q = gx + q_1, \quad r = gx + r_1, \quad s = hx + s_1,$$

where  $g, h,$  and  $x$  are arbitrary, when (22) reduces to the following quadratic equation in  $x$ :

$$(25) \quad \begin{aligned} & 2g(g^2 - h^2)(p_1 + q_1 - 2r_1)x^2 + \{(p_1^2 + 8p_1q_1 - 4p_1r_1 + q_1^2 \\ & - 4q_1r_1 - 2r_1^2)g^2 - 4(p_1 + q_1 - 2r_1)s_1gh - (p_1^2 + q_1^2 - 2r_1^2)h^2\}x \\ & - (2p_1^2 + 2q_1^2 - 4r_1^2)s_1h + (4p_1^2q_1 - 2p_1^2r_1 + 4p_1q_1^2 - 2p_1r_1^2 \\ & - 2p_1s_1^2 - 2q_1^2r_1 - 2q_1r_1^2 - 2q_1s_1^2 + 4r_1s_1^2)g = 0. \end{aligned}$$

We choose

$$(26) \quad \begin{aligned} g &= (2p_1^2 + 2q_1^2 - 4r_1^2)s_1, \\ h &= (4p_1^2q_1 - 2p_1^2r_1 + 4p_1q_1^2 - 2p_1r_1^2 - 2p_1s_1^2 \\ & \quad - 2q_1^2r_1 - 2q_1r_1^2 - 2q_1s_1^2 + 4r_1s_1^2), \end{aligned}$$

when the constant term in (25) vanishes and the equation is readily solved to obtain a nonzero value of  $x$  which, on using (24), gives a rational solution of (22). The trivial solution  $(p_1, q_1, r_1, s_1) = (1, t, -1, 1)$  of equation (22) yields the parametric solution obtained earlier by substituting  $p = 2r - q$  in (22). However, another trivial solution of (22) may be obtained by taking  $p = 0$  and solving the resulting simple equation for  $q, r$  and  $s$ . Using this trivial solution, we can obtain a non-trivial solution of (22) which leads to a pseudo-magic square of squares, in parametric form, with entries of degree 56 in one parameter. Further, using the non-trivial solution of (22) now obtained, we can apply the method described above to find another parametric solution of (22) and this process may be continued. We may thus obtain infinitely many pseudo-magic squares of squares in parametric form.

Equation (22) may also be solved by noting that a rational solution for  $p$  will exist if  $(2q^2 - r^2 - s^2)(q^2r^2 - 2r^2s^2 + q^2s^2)$  could be made a perfect square. This may be considered as a quartic in  $r$  and a trivial solution is given by  $r = q$ . Now using the elementary methods given in [5, p. 639], we can find infinitely many values of  $r$  that will make the above quartic a perfect square. This leads to new parametric solutions of equation (22) and hence we obtain more pseudo-magic squares of squares, in parametric form,

the first such pseudo-magic square having entries of degree 20 in a single parameter.

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## BOOK REVIEW

**Collected works of V. K. Patodi.**

**Edited by M. F. Atiyah and M. S. Narasimhan.**

**World Scientific, 1996; xii+294; ISBN No. 981-02-2659-4.**

**Reviewer: Ravi S. Kulkarni.**

This is a collection of 11 papers of V. K. Patodi, written in a short span from 1971 to 1976 in his meteoric career. Patodi was born on 12th March, 1945, in Guna, a town in Madhya Pradesh. He came from a traditional Jain family. He did his B.Sc. from Vikram University, Ujjain, and M.Sc. from Banaras Hindu University, in Varanasi. He spent a year at the Centre for Advanced Study in Mathematics in the university of Mumbai, when he moved to TIFR in 1967. After early promise, cf. [1], [2],[3], he went to work with Atiyah, Bott and Singer, in the Institute for Advanced Study in Princeton, NJ, USA. Atiyah and Singer had proved their celebrated Index Theorem for elliptic operators on compact differentiable manifolds, a few years earlier by topological methods using cobordism theory and K-theory. This theorem is a vast generalization of some beautiful classical theorems, such as Gauss-Bonnet-Chern theorem, Hirzebruch signature theorem, Hirzebruch-Riemann-Roch theorem,  $\hat{A}$ -genus theorem for Dirac operators, etc. There was wide interest in proving and extending even the special cases of this theorem in the Riemannian, complex-analytic, or algebro-geometric contexts by different means. Moreover the case of differentiable manifolds with non-empty boundary, or still more generally arbitrary non-compact manifolds, where the answer would not be purely topological, was open. That is where Patodi came in. There resulted a remarkable collaboration of Patodi with Atiyah, Bott, and Singer, cf. [4], [6], [7] (the last one is a collection of 3 papers). He also came in contact with some of the younger mathematicians Dodziuk and Donnelly which resulted in the papers [10], [11]. The papers [5], [8], [9] are essentially announcement of results.

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It so happened that the reviewer was also a member of the Institute for Advanced Study in Princeton during the same years as Patodi. We immediately became friends. We discussed a lot of mathematics, although we did not write a joint paper. R. Bott, my advisor who was also a member of the Institute for one year, and Patodi, stimulated me to write an expository account of the then developing work, on which R. Bott lectured at a summer meeting in Montreal, Canada. This account appeared in [Ku]. It served as a “low-brow” introduction to this area for some years.

Patodi was at the top of his creative powers, when he suddenly passed away on 23rd December, 1976. Patodi’s death was not entirely unpredictable. His health was poor. He died of renal failure. Still, to most of us, who knew him, his death came as a shock.

Patodi’s career somewhat resembles Ramanujan’s. Atiyah writes about his collaboration with Patodi as follows: “When Patodi came to Princeton, I was already struggling to formulate some version of the Hirzebruch signature theorem for manifolds with boundary, which would include various motivating special cases. Once the problem had reached the relevant stage, Patodi simply retired for a few days to carry out the kind of detailed calculation that was his forte. He returned in triumph to announce that he had verified the desired conjecture. Thus encouraged, Singer and I then had to grapple with understanding Patodi’s calculations on our own terms, so that we could formulate and prove the appropriate generalizations. It was a great collaboration.” Sounds similar to what Hardy wrote about Ramanujan! Except that Patodi’s way of thinking and arriving at definitive results, with hindsight, has not remained as obscure as Ramanujan’s!! The fundamental theorems of invariant theory for  $GL(n)$ ,  $O(n)$  and other classical groups do go a long way towards explaining Patodi’s tour de force calculations.

In this review we would like to give mainly the background of Patodi’s works, and what he achieved, in special cases, and some later development. With hindsight, for a modern reader, the paper [4] with Atiyah and Bott is a good entry point, which explains Patodi’s earlier works, and also gives a comparative overview of topological, analytic, differential geometric, and classical group-theoretic methods. Patodi’s work is devoted to understanding the interconnections of some of the most basic invariants of differentiable, Riemannian, complex-analytic, and combinatorial manifolds. A good example is the Gauss-Bonnet-Chern theorem. Recall that this theorem states that the integral of a certain expression constructed out of the Riemann curvature tensor, in the case of a compact Riemannian manifold



(without boundary) is a purely topological invariant, namely the Euler characteristic. Now the curvature is a local invariant of the Riemannian metric. When a compact  $n$ -dimensional manifold is triangulated, the Euler characteristic can be computed as  $\# \text{ vertices} - \# \text{ edges} + \# \text{ 2-faces} - \# \text{ 3-faces} + \dots + (-1)^n \# \text{ } n\text{-faces}$ . Indeed the latter expression makes sense even for a finite cell-complex, not necessarily a manifold. It is known that it is independent of the way one puts the finite cell-structure. More generally, for any Hausdorff, paracompact, topological space with finitely generated homology, Euler characteristic can be defined as  $\sum_i (-1)^i \dim H_i(X; k)$ , where  $k$  is any coefficient field. Still more generally, given a vector bundle  $\xi$  of rank  $n$  on a Hausdorff, paracompact topological space  $X$  we have a well-defined cohomology class  $e(\xi)$  in  $H^n(X; k)$ . Its vanishing is equivalent to the existence of no-where vanishing section of  $\xi$ . If the  $n$ -th homology group is 1-dimensional, and there is a specific way of choosing its generator  $[X]$ , we can define an element  $e(\xi)[X]$  in  $k$ , which we may call the Euler characteristic of the bundle  $\xi$ , with coefficients in  $k$ . Such is the case when  $X$  is a compact manifold which is (i.e. by definition, its tangent bundle is) orientable over  $k$ . It is indeed remarkable that the numbers obtained in such widely different ways turn out to be equal under appropriate conditions. It is a challenge to understand “why”. The answers to such open-ended question may not be unique. Different viewpoints provide different understandings.

The classical Gauss-Bonnet theorem, was a theorem for compact surfaces with boundary, that is a 2-dimensional compact Riemannian manifold with boundary. In this case the integrand is the Gaussian curvature, together with geodesic curvature terms for the boundary. Its higher dimensional extension (without boundary) is due to Chern, cf. [C] where the integrand is a certain expression constructed out of the Riemann curvature tensor. An important point is that this expression involves only the components of the curvature tensor, and not its derivatives. A good way to understand this expression is that it is the Pfaffian of the skew-symmetric matrix of curvature 2-forms. Whereas the matrix is only locally defined, the Pfaffian is globally defined. (This is an important point. Consider the very definition of a differentiable manifold. Although the space is  $n$ -dimensional, there may not exist  $n$  real-valued functions (“local coordinates”) which separate points. The new work involved in studying manifolds, compared with just open subsets of  $\mathbb{R}^n$ , is to sensitise oneself, as to which locally made constructions have a global meaning.)

Patodi's work is in the area of a major and larger theme in differential geometry, namely the "geometry of the Laplacian, (more generally) elliptic operators, and the associated heat operator". Curiously, this is an area to which another Indian mathematician, S. Minakshisundaram [Mi], [MiP], (the second jointly with A. Pleijel) made major contributions in the late 1940's, which were greatly appreciated by H. Weyl, a great mathematician, physicist, and philosopher. It seems that the seeds he sowed fructified in Patodi's works. There was however no connection – academic or otherwise – between the two. In fact in the reviewer's opinion, Minakshisundaram's work did not get the due recognition it deserved in his own life-time. Now, many years after Patodi's Collected Works have been published, Ramanujan Mathematical Society is making efforts in bringing out the Collected Works of Minakshisundaram.

Consider a bounded plane domain  $D$  with smooth boundary  $\partial D$ . Let  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  be the Laplacian, and consider the Dirichlet's problem: Given  $f : \partial D \rightarrow \mathbb{R}$ , find  $F : D \rightarrow \mathbb{R}$  such that  $F$  is harmonic (i.e.  $\Delta F = 0$ ), and  $F|_{\partial D} = f$ . One knows that a solution exists and is unique. Thus, for example, if  $f = a$  constant  $c$ , then we must have,  $F = c$ . For a round disk or a rectangle, one can exhibit explicit solutions. In general it is an existence result, and one can speak of only the qualitative features of the solution. On physical or geometric ground, one would expect that the qualitative features of the solutions should be related to geometric shape of the domain.

More generally, one asks for such information for a Riemannian manifold, which is a generalization of the Euclidean space. A Riemannian metric allows definitions of a "Laplacian" on functions, and more generally on all tensor-fields, on the manifold. It is a generalization of the Euclidean Laplacian = " $div \circ grad$ ". In fact, given a smooth real valued function  $f$  on a Riemannian manifold  $M$  ( $n$  denotes the dimension of  $M$ ), " $grad f$ " is a vector field orthogonal to the generic level sets of  $f$ , and points out the direction of steepest ascent of  $f$ . Moreover if  $X$  is any vector field on  $M$ , and  $\phi_t$  is the local flow generated by  $X$ , then " $div X$ " is the Lie (or Radon-Nikodym) derivative, i. e. the rate of change of the volume-form w.r.t  $\phi_t$ . Thus harmonic functions are precisely those smooth functions such that the local flows generated by their gradient vector fields are volume-preserving! This geometric motivation for the study of the Laplacian in the Euclidean space carries over to arbitrary Riemannian manifold. A simple direct generalization of " $f = c \Rightarrow F = c$ ", is: for a compact Riemannian (connected) manifold without boundary, the only real-valued harmonic functions on  $M$  are constants. In turn, in terms of de Rham cohomology, this statement

may be related to the fact that the 0-th dimensional cohomology group “ $H^0(M; \mathbb{R})$ ”, with real coefficients, is isomorphic to  $\mathbb{R}$ . Hodge [H] generalised this statement to all cohomology groups as follows. A remarkable theorem of de Rham in differential topology is: the topologically defined cohomology groups with real coefficients for any differentiable manifold coincide with the de Rham cohomology groups defined in terms of differential forms. Hodge’s equally remarkable extension of this result is that for a compact *Riemannian* manifold, each de Rham cohomology class is represented by a *unique harmonic form* defined in terms of the associated Laplacian on the spaces of  $p$ -forms,  $p = 0, 1, 2, \dots, n = \text{dimension of the manifold}$ . This is the story for the eigenvalue 0 of the Laplacian on Riemannian manifolds.

A starting point of Patodi’s work was a converse question posed by Kac [K], in a paper with an amusing title, “Can one hear the shape of the drum”? Let  $M^n$  be a compact Riemannian manifold. Let  $\Delta$  be the Laplacian of the Riemannian metric. Then  $-\Delta$  is known to be a non-negative operator considered as acting on the vector space of  $C^\infty$ -functions, and more generally on all tensor-fields. Its “almost” inverse is a compact operator. So on tensor-fields of a fixed type, the spectrum of  $-\Delta$ , i.e. eigenvalues of  $-\Delta$ , consists of countably many real numbers, multiplicity of each eigenvalue being finite. Let  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \lambda_k \rightarrow \infty$  be the eigenvalues of  $-\Delta$  each repeated as many times as its multiplicity, and  $\phi_k(x), k = 1, 2, \dots$  the corresponding eigenfunctions ( $-\Delta\phi_k(x) = \lambda_k\phi_k(x)$ ) which form a complete orthonormal set. Kac’s question for plane domain is: The  $\lambda_i$ s are “fundamental tones of vibration” of the “membrane”  $D$ . Do they determine  $D$ ? More generally, for a Riemannian manifold  $M^n$  for each tensor-type,  $\lambda_i$ s are analytic invariants of the Riemannian manifold  $M^n$ . *Do  $\lambda_i$ s determine  $M^n$  up to isometry?* Or more cautiously, let us pose an open-ended question: *Which topological/geometric invariants, such as Euler characteristic, volume, .... are determined by the analytic invariants  $\lambda_i$ s?* In the ensuing investigations over the years, one knows that iso-spectral manifolds are in general not isometric. A general construction is given by Sunada [Su]. Interestingly, the analytic invariant, the spectrum (that is, the eigenvalues) of the Laplacian can be related to the geometric invariant, the “length spectrum”, that is, the lengths of closed geodesics, cf. Selberg [Se]. There are many deep connections of these spectra, to the geometry and arithmetic, cf. for example, Gopal Prasad and Rapinchuk [PR], for a recent development. When Patodi started, Milnor [M] had given 16-dimensional examples of iso-spectral but non-isometric flat tori. (For flat tori, iso-spectrality on the space of  $C^\infty$ -functions, implies iso-spectrality on the space of all tensor-fields of fixed type.) People were also studying the associated heat operator

$\frac{\partial}{\partial t} - \Delta$ . In particular, one knew the fundamental solution of the heat operator,  $e(t, x, y) = \sum_k e^{-\lambda_k t} \phi_k(x) \phi_k(y)$  which converges uniformly on compact subsets of  $(0, \infty) \times M \times M$ . Moreover,

$$\int_M \text{trace } e(t, x, y) dV = \sum_k e^{-\lambda_k t}, \quad t > 0.$$

On the space of  $p$ -forms, we have Minakshisundaram expansions

$$\text{trace } e(t, x, y) = (4\pi t)^{n/2} \left[ \binom{n}{p} + tk_1(x) + \dots + t^N k_N(x) + O(t^{N+1}) \right].$$

Here  $k_i(x)$  are universal functions of the Riemann curvature tensor *and its covariant derivatives*. From the first term in the expansion, one sees that one can read the dimension and the volume of  $M^n$ . Patodi developed a detailed understanding of these expansions. In the initial paper published in the Journal of the Indian Math Society, he improved upon the results of McKean and Singer [MS], and Berger [Be]. For example, [MS] shows that if  $k_i$ 's, for the Laplacian on smooth functions, are identically zero, for  $i \geq 1$  and  $n \leq 3$  then the manifold is flat. Patodi [3] extended the result for  $n \leq 5$ , and showed that it is false for  $n \geq 6$ . However if one knows the  $k_i$ 's for the Laplacian on all  $p$ -forms, then in fact the result holds for manifolds of constant curvature, and even more generally for manifolds of constant Ricci curvature (Einstein manifolds). In fact only the information for  $k_i$ 's for  $(i \leq 2)$  is needed in these theorems.

His major works [1] and [2] followed. In these papers, he proved the Gauss-Bonnet-Chern theorem, and the Riemann-Roch-Hirzebruch theorem for Kähler manifolds, by the heat-operator technique. Specifically, for example in the first of these papers, he proved that the integrand of the Gauss-Bonnet-Chern theorem given by the heat operator techniques, is only the familiar one: the Pfaffian of the matrix of the curvature 2-forms. There is a remarkable cancellation so that the higher derivatives do not appear. M. S. Narasimhan in the foreword to Patodi's Collected Works writes how Patodi's calculations, in local geodesic coordinates, at first were not convincing to the experts. S. Ramanan, and M. S Narasimhan helped him to cast some of the calculations in a global form, and prove some "super"-algebraic lemmas, which made the work understandable!

In the joint paper of Patodi with Atiyah and Bott [4], the authors give a new proof of the Index Theorem by the heat operator technique. Around the same time of Patodi's papers [1], [2], Gilkey [G] had also shown that the higher derivatives of the curvature tensor in the heat-operator technique can be eliminated on a priori grounds. He also extended his argument to prove

the Hirzebruch's signature theorem for oriented  $4k$ -dimensional manifolds, in particular giving an intrinsic characterization of the Pontryagin forms. For a simplified account of Gilkey's theorem see the reviewer's lecture notes [Ku]. In [4], the authors offer a masterly overview of different techniques leading to the Index Theorem, without disguising their enthusiasm for all the techniques, and discussing their merits and drawbacks. They develop a functorial notion of "a regular invariant of Riemannian metrics", cf. [4], and point out how on a priori grounds of the main theorem of invariant theory of orthogonal group, the higher derivatives of the Riemann curvature tensor must cancel out. Such breath-takingly audacious a-priori arguments can occur only after the detailed evidence in many special cases! Atiyah remarks: "Interestingly enough, the interaction with theoretical physics of the past two decades, has produced yet more variants of the proof and these are closer to Patodi's original proof. "Super-symmetry" appears to lie behind the sophisticated algebraic identities of Patodi's approach". It should be remarked that the original proof of the index theorem used cobordism theory to establish a generalized signature theorem, and then used K-theory. In the new proof, the cobordism theory is replaced by local differential-geometric techniques. But the K-theory still remains to get the general index theorem, cf. [4], §7. M. S. Raghunathan informed me that he has given a variant of the original proof of Atiyah and Singer which sidesteps the theory of pseudo-differential operators and effects some simplifications in the way cobordism is applied. He deduces "bordism invariance" of the analytic index from qualitative information on the asymptotics of the heat kernel of the twisted index (or Dirac) operators.

The long paper [7], by Patodi with Atiyah and Singer, is an extension of the above work to manifolds with boundary, extending the Gauss-Bonnet theorem for surfaces with boundary. Significantly, they obtain a version of Hirzebruch's signature theorem for manifolds with boundary. The conclusion is not purely topological. There is also a significant difference with the Gauss-Bonnet theorem. Even when the metric near the boundary is a product metric, the boundary terms do not disappear (as they do in the Gauss-Bonnet case.) There is a wealth of ideas in these papers which should stimulate much further research. For recent developments of local index theorems and their super-symmetric basis, see for example, the survey article by Bismut [Bi], and the textbook [CS].

In [9] Patodi proves a combinatorial formula for Pontryagin forms on a *smooth* manifold. In a long paper [10], J. Dodziuk and Patodi prove that the eigenvalues of the combinatorial Laplacians converge to the eigenvalues

of the smooth Laplacian. The paper was motivated by a desire to relate the analytic torsion defined by Ray and Singer [24] to the Reidemeister-Franz torsion. There are many open problems in this area. In [11] Donnelly and Patodi relate the heat operator with the fixed point set of an isometry of a compact Riemannian manifold. In particular they provide a heat-operator-theoretic new proof of the G-signature theorem of Atiyah and Singer [AS].

The reviewer finds it significant that Patodi, after his spectacular work, did not continue in the US. Unlike Ramanujan, considering the global sociological changes that have taken place in the past 30-40 years, he might have gotten a good position at a good university in the US. Perhaps it might have been good for him, personally, for in the late 70's, his medical problem might have received better treatment. However, he returned to India, and TIFR, where he received his doctoral training. He was soon promoted to full professor. Narasimhan writes that he took active interest in the working of the mathematics faculty in TIFR. The reviewer also finds it significant, that he published a long paper with Dodziuk in the Journal of the Indian Mathematical Society, cf. [10]. This is unlike many of our research mathematicians even today. There is still a strong peer pressure in India to get their papers published in foreign journals. This situation may change in the 21st century. The Chinese, Japanese, and the Korean mathematicians seem more self-confident in this regard.

In this review we have tried to give just a flavor of Patodi's basic work following from Minakshisundaram's expansions of the trace of heat operator. This work will undoubtedly remain as a permanent part of mathematics. These techniques have been assimilated in an area now called "Geometric analysis" in the US and Europe. However in the land of birth of Minakshisundaram and Patodi, the subject has not attracted as much attention as it deserves. How to remedy this situation? The reviewer would like to make a suggestion to major funding agencies for higher mathematics such as NBHM or DST in India. The Clay Mathematics Institute in the US actually commissions some experts to write a report on a broad area of mathematics, such as the Poincaré Conjecture. A classical example is Hilbert's famous report on Number Theory in 1890s, commissioned by the German Mathematical Society. NBHM or DST support higher mathematics in various ways in India. Would they like to commission some experts in India or abroad to bring about specific works? There are many loose ends in Patodi's work. To bring about annotated edition of Patodi's work may well be such a project!

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## 73<sup>rd</sup> IMS CONFERENCE: A BRIEF REPORT

B. N. WAPHARE

The Indian Mathematical Society (IMS) is the oldest mathematical society in India founded by Late Shri Ramaswamy Aiyer with its headquarters at Pune in 1907. The Department of Mathematics, University of Pune, had a great privilege in organizing 73<sup>rd</sup> IMS Conference in its Centenary Year during December 27 - 30, 2007 under the Presidentship of Prof. R. B. Bapat (ISI, New Delhi). The academic programs were organized at the College of Engineering, Pune (a 153 years old premier institute for Technical Education in India) and at the Fergusson College, Pune (a 125 years old well known institute in India, where the IMS library was housed from its very inception to till 1950).

The Conference was inaugurated by Shri Prithviraj Chavan, Hon. Minister of State (Prime Ministers Office), Government of India. Dr. Narendra Jadhav, Vice-Chancellor, University of Pune and Chief Patron, Mr. Pratap Pawar, Member of Board of Governors COEP, Prof. A. D. Sahasrabudhe, Director COEP, Prof. V. M. Shah, General Secretary IMS, Prof. N. K. Thakare, Editor JIMS, Prof. S. B. Nimse, Administrative Secretary IMS, Prof. J. R. Patadia, Editor Mathematics Student, Dr. (Miss) S. P. Arya, Treasurer IMS and Prof. B. N. Waphare, Local Secretary, were present on the Dias.

The unique feature of the Conference was that the eminent mathematicians Prof. S. T. Yau (Fields Medalist), Prof. Richard Hamilton (Clay Awardee), Prof. S. R. S. Varadhan (Abel Prize winner) delivered Plenary Talks. The details are as under:

1. **Professor S. T. Yau**(Fields Medalist, Harvard University, USA) delivered a Plenary Lecture under M. K. Singal Memorial on "*PAST, PRESENT AND FUTURE OF INDIAN AND CHINESE MATHEMATICS*". Chairperson: Professor Satya Deo.
2. **Professor Richard Hamilton**(Clay Award Winner, Columbia University, USA) delivered a Plenary Lecture on "*SURGERED RICCI FLOW*". Chairperson: Professor R. S. Kulkarni.

3. **Professor S. R. S. Varadhan**(Abel Prize Winner, Courant Institute, USA) delivered a Plenary Lecture on “*HOMOGENIZATION*”.  
Chairperson: Professor N. K. Thakare.

The details of the Memorial Award Lectures delivered are as follows:

1. 21<sup>st</sup> **P. L. Bhatnagar Memorial Award Lecture** was delivered by Prof. T. M. Karade, Ex. Head, Department of Mathematics, University of Nagpur. The title of the talk was: “*Relativistic Considerations for Modified Newtonian Dynamics*”.
2. 18<sup>th</sup> **Ramaswamy Memorial Award Lecture** was delivered by Prof. Dinesh Singh (University of Delhi). The title of the talk was “*Harmonic Analysis on the Unit Circle - A Personal Perspective*”.
3. 18<sup>th</sup> **Srinivasa Ramanujan Memorial Award Lecture** was delivered by J. Radhakrishnan (TIFR, Mumbai). The title of the talk was “*The List Decoding Radius of Reed – Solomon Codes*”.
4. 8<sup>th</sup> **Ganesh Prasad Memorial Award Lecture** was delivered by Siddhartha Gadgil (IISc, Bangalore). The title of the talk was “*Orders on Manifolds and Surgery*”.

### Invited Talks

The details of the half an hour invited talks delivered are as under:

1. **S. S. Abhyankar**(Purdue University, USA)  
“*On the Jacobian Problem*”
2. **Ajay Kumar**(Delhi University, Delhi)  
“*Operator space structure of Banach spaces*”
3. **T. Parthasarathy**(Chennai)  
“*Semi definite Programming Problem - A Survey*”
4. **K. Varadarajan**(University of Calgary, Canada)  
“*Anti hopfian and anti co-hopfian modules*”
5. **V. Balaji**(CMI, Chennai)  
“*Principal bundles on projective varieties*”
6. **T. B. Singh**(Delhi University, Delhi)  
“*Free involutions on projective spaces*”
7. **Madhu Raka**(Panjab University, Chandigarh)  
“*Idempotent generators of irreducible cyclic codes*”
8. **Dinesh Thakur**(Arizona State, USA)  
“*Recent developments in Function Field Arithmetic*”
9. **Rafiqul Alam**(IIT, Guwahati)  
“*Numerics of structured eigenvalue problems*”
10. **K. Srivastava**(BHU, Varanasi)  
“*Some coseparators in L-topology and related areas*”

11. **C. M. Joshi**(Udaipur)  
*"On some generalized Baily type transforms"*

## Symposia

The details of the Symposia organized are as under:

### 1. Complex Analysis

**Organizer:** A. P. Singh(University of Jammu, Jammu)

**Speakers:**

A. P. Singh(University of Jammu)

*"Spiraling Baker Domain"*

R. K. Srivastava(Dr. B. R. Ambedkar University, Agra)

*"Certain Topological Aspects of the Bicomplex Space"*

O. P. Ahuja(Kent State University, USA)

*"Theory of Univalent Harmonic Mappings as an Active Domain of Research"*

Poonam Sharma(Lucknow University)

*"On Some Complex Harmonic Functions"*

M. G. P. Prasad(IIT Guwahati)

*"Iteration of Meromorphic Functions"*

### 2. Harmonic Analysis and Operator Spaces

**Organizer:** Ajay Kumar(University of Delhi)

**Speakers:**

S. G. Dani(TIFR, Mumbai) & K. B. Sinha(ISI, New Delhi)

*"Krein's shift function and index theorems"*

Daniel E. Wulbert(University of California, Sam Diego, USA)

*"On the location problem"*

S. K. Khanduja (Punjab University, Chandigarh)

*"On irreducible factor of polynomials in two variables"*

### 3. Recent Advances in Ring Theory

**Organizer:** S. A. Katre(Pune University, Pune)

**Speakers:**

J. B. Srivastava(IIT, Delhi)

*"Some research problems on Group Algebras of certain infinite groups"*

S. A. Katre(University of Pune, Pune)

*"Warings problem for Matrices over commutative rings"*

K. M. Rangaswamy(University of Colarado, USA)

*"On modules of finite torsion-free rank over valuation domains"*

### 4. Computer visualization and shape modeling

**Organizer:** A. Ojha(PDPM IIT-DM, Jabalpur)

**Speakers:**

Vijay Natarajan (IISc, Bangalore)

*“Topological Analysis for Data Exploration”*

Suman Mitra(DAIICT, Gandhi Nagar)

*“Detection and Tracking of Objects in Low Contrast Condition”*

Roshan Rammohan(New Mexico, University, USA)

*“Context Sensitive Probabilistic Modeling and Applications”*

Aparajita Ojha(PDPM IIT DM Jabalpur)

*“Rational Patches for Smooth Surface Construction”*

#### 5. Algebraic Coding Theory

**Organizer:** M. Raka(Punjab University, Chandigarh)

**Speakers:**

Sudhir Ghorpade(IIT, Mumbai)

*“Higher weights and Grassmann Codes”*

Gurmeet K. Bakshi(Punjab University, Chandigarh)

*“Polyadic Codes and their Generalizations”*

Anuradha Narasimhan

*“Exponential sums and linear codes over finite fields”*

#### 6. Topology and Geometry

**Organizer:** R. S. Kulkarni(IIT, Mumbai) & Satya Deo(HRI, Allahabad)

**Speakers:**

C. S. Aravinda(TIFR, Bangalore)

*“Khinchine Type Theorems for closed negatively curved manifolds”*

J. K. Verma(IIT, Bombay)

*“Mixed multiplicities of ideals and mixed volumes of polytopes”*

R. S. Kulkarni(IIT, Bombay)

*“Z-classes in Quaternionic Geometries”*

Kishore Marathe(CUNY)

*“Quantum Topology”*

#### 7. Biomathematics

**Organizer:** Preeyush Chandra(IIT, Kanpur)

**Speakers:**

Girija Jayaraman(IIT, Delhi)

*“Physiological Applications of Fluid Dynamics”*

S. Gakhar(IIT, Roorkee)

*“Complex Dynamical Behavior of Epidemiological Models”*

M. Ramchandra Kaimal(University of Kerala)

*“Graph Theoretical and computational approaches in Biological Networks”*

Joydeep Dhar(IITM, Gwalior)

*“Spatio-temporal models of spreading of disease: Role of Incubation Period and Movement of Carrier Population”*

V. Sundararajan(C-DAC, Pune)

*“Population genetics based algorithms and their applications”*

8. **Discrete Mathematics**

**Organizer:** B. N. Waphare(Pune University, Pune)

**Speakers:**

B. N. Waphare(University of Pune)

*“Removable ears in Graphs”*

Y. M. Borse(University of Pune)

*“Lovasz removal path Conjecture”*

M. M. Shikare (University of Pune)

*“Splitting in Matroids”*

Vinayak Joshi(University of Pune)

*“Frankel’s Conjecture”*

9. **Ramanujan Mathematics**

**Organizer:** A. K. Agarwal(Punjab University, Chandigarh)

**Speakers:**

A. K. Agarwal(Punjab University, Chandigarh)

*“Ramanujan’s mock theta functions and combinatorics”*

M. A. Pathan

*“Lie theory and Generalized Hermite Polynomials”*

S. N. Singh

*“Certain results involving cubic theta functions”*

S. Ahmad Ali

*“On partial sums of eight orders mock theta functions”*

N. K. Baruah

*“Ramanujan’s Eisenstein series and hypergeometric-like series for  $1/\pi^2$ ”*

M. S. Mahadeva Naika

*“some new explicit evaluations of class invariants”*

S. P. Singh

*“Evaluation of Ramanujan’s theta functions”*

R. K. Yadav

*“On applications of q-fractional calculus”*

10. **Generalized Measure and Integration**

**Organizer:** S. B. Nimse(Ahmednagar)

**Speakers:**

S. B. Nimse(Ahmednagar)

*“Choquet and Sugeno Integrals”*

Rajeshwar Andale(Mumbai)

*“Mathematical Theory of Evidence”*

M. S. Chaudhari (Shivaji University, Kolhapur)

K. Varadarajan(University of Calgary, Canada)  
“Harr measure”

### SPECIAL PAPER PRESENTATION SESSION FOR PRIZES

During the special session of paper presentation competition, research papers were presented for the award of Six IMS Prizes (in the areas of Algebra, Discrete Mathematics, Topology, Number Theory, Operations Research and Fluid Dynamics), V. M. Shah Prize in Analysis and AMU Prize in Differential geometry. The result was as follows:

1. **IMS Prize for Algebra:** No prize was awarded.
2. **IMS Prize for Discrete Mathematics** awarded to Mr. Santosh Dhotre.
3. **IMS Prize for Number Theory** awarded to Smt. Meenakshi Rana.
4. **IMS Prize for Operations Research:** No prize was awarded.
5. **IMS Prize for Fluid Dynamics** awarded to S. Saravanan.
6. **IMS Prize for Geometry** awarded to Anandteerth Mangasuli.
7. **A. M. U. Prize:** There was no entry.
8. **V. M. Shah Prize for Analysis** awarded to B. L. Ghadodra.

### P. L. BHATNAGAR MEMORIAL PRIZE

To encourage the young participants at the Mathematical Olympiads, the Indian Mathematical Society instituted in 1987 an annual prize in the memory of the Late Professor P. L. Bhatnagar who did pioneering work in organizing Olympiads in the country, out of an endowment made by the P. L. Bhatnagar Memorial Fund Committee. The prize is awarded every year during the inauguration session of the Indian Mathematical Society Annual Conference to the top scorer of the Indian team for IMO provided he/she wins a medal (from 1987 to 1990 the prize was awarded to the top scorer at IMO). The prize consists of a cash award of Rs. 1000/- and a certificate.

This year Abhishek H. Dang (Pune) was the top scorer. He was awarded the P. L. Bhatnagar Memorial Prize for 2007 in the inaugural session.

Apart from academic sessions, Cultural Evenings were also arranged on 28-29, December, 2007 at 7.30 p.m. in COEP Auditorium.

B. N. WAPHARE  
CHAIRMAN, LOCAL ORGANIZING COMMITTEE,  
73<sup>rd</sup> ANNUAL CONFERENCE OF IMS.

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V. M. SHAH  
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The Society publishes two periodicals. THE JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY and THE MATHEMATICS STUDENT, which appear quarterly. The annual subscription for the JOURNAL is US Dollars Eighty (Rs.1000/- for libraries in India) and that for the STUDENT is US Dollars Seventy five (Rs. 800/- for libraries in India). Subscriptions are payable in advance.

Back volumes of both the periodicals, except for a few numbers out of stock, are available. The following publications of the Society are also available: (1). Memoir on cubic transformation associated with a desmic system, by R. Vaidyanathswamy, pp. 92, Rs. 250/- (or, \$ 10/-), and (2). Tables of partitions, by Hansaraj Gupta, pp. 81, Rs. 350/- (or, \$ 15/-).

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