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Edited by

H. C. KHARE

(issued: December 2018*)

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THE MATHEMATICS STUDENT [1]

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H. C. KHARE

In keeping with the current periodical policy, THE MATHEMATICS STUDENT will seek to publish material of interest not just to mathematicians with specialized interest but to undergraduate and post-graduate students and teachers of mathematics in India. With this in view, it will publish material of the following type:

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The Mathematics Student
Vol. 69, Nos. 1-4 (2000)

CONTENTS

The Mathematics Student
Vol. 69, Numbers 1-4 (2000), 01–07

PRESIDENTIAL ADDRESS* (GENERAL)

SATYA DEO

Honourable Fellow Mathematicians and Friends,

On this occasion of 66th Annual Session of the Indian Mathematical Society, I feel greatly honoured in having been given the unique opportunity of addressing this august gathering of our fellow mathematicians and other invited guests. At the outset, I, on behalf of the Society, must thank the authorities of the Vivekanand Arts, Sardar Dalip Singh Commerce and Science College, Aurangabad and the Maratha Sikshan Sansthan who have volunteered to host the present Annual Session of the Indian Mathematical Society at their nice location in the famous city of Aurangabad. It is remarkable that even at the present time, when the mathematical societies of the advanced countries have become highly professional and have found ways so that the organizers have to devote very little time and money in organizing such conferences leaving the participants to take care of themselves, our Society is able to keep up the age old tradition of organizing such events mostly on cooperative and voluntary basis. In this country, the Local Secretary and the concerned host institution assume the entire burden and manage everything for the success of the conference. We hope that this spirit of selfless service to the cause of our subject will not only continue but will be further strengthened by users of mathematics and mathematicians in times to come. I say so simply because this is how some of our mathematician friends can always afford to be hosts and this is what has characterized our Indian society.

* The text of the Presidential Address (general) delivered at the 66^{th} Annual Conference of the Indian Mathematical Society held at Vivekanand Arts, Sardar Dalip Singh Commerce and Science College, Samarthnagar, Aurangabad - 431 001, Maharashtra, India during December 19-22, 2000.

I take this opportunity to thank all of you and other members of the Indian Mathematical Society who have chosen to elect me as President of the Society. It is indeed a great priviledge and honour to serve the Society specially in the capacity of its President. My worthy predecessors, who have been great scholars of mathematics, have performed this job with exemplary quality and dedication. They have expressed their opinion from this platform on almost every topic of our concern and there is little that can be added to their beliefs. Follow up actions should really be our biggest concern. It is in that spirit that I very much hope and aspire to add my own bit in the long tradition of service to the cause of teaching and reserach in mathematics. In fact, this annual meeting is one occasion when, keeping the changing scenerio in mind, we have the opportunity of putting our views for the development of mathematics before the fellow mathematicians for their consideration and adoption. This is also the time when all of us can be prompted to review our strong and weak points as mathematicians and to rededicate ourselves atleast for those decisions about which we feel convinced. I don't intend to deal with many issues affecting the development of teaching and research in mathematics. However, on this occasion I must say something about a few of my own observations on the state of art of our subject.

First I take up the teaching of Mathematics. In recent years we have witnessed a marked decrease in enrollment of graduates offering mathematics in their master's degree. The number of students going for Ph.D degrees or for a research career after completing their master's degree is still worse. Many research institutes equipped with modern facilities are not getting students. It is a matter of serious concern for all of us as members of the Indian Mathematical Society. We know that the most important single reason for this situation is the obvious shortage of employment opportunities for mathematics M.Sc's and Ph.D's. Most of the talented students prefer to go to those areas which have instantaneous job offers with very high salaries and there is hardly anything that we can do about this. Under the circumstances, what we should do to attract at least those who come to us for getting M.Sc. or Ph. D. in mathematics, is a good question to ask. This has been an important issue bothering the minds of all of us who are concerned about the growth and development of our subject. In my opinion, one of the most important reasons which seems to be responsible for making mathematics a dull or unattractive subject with very little employment potential is our inability to make students strong in fundamentals of the subject. One can easily discover that in a small crowd which has joined mathematics, there are merely an average of four to five percent of them who really understand what is going on in the class-room. Occasionally, a teacher himself is not clear about the concept that he is trying to explain. This is indeed a very diplorable situation. Our subject is known for its accuracy, rigour and logical beauty. There is hardly any room for confusion or difference of opinion on its facts and cocepts. Someone has rightly described the scene of having arguments on mathematical facts and concepts which is worth recalling. Two scientists (physical, biological, etc.) can always differ from each other on a scietific point and can have unending arguments. Following their own convictions they may never agree with one another. However, in the case of two mathematicians, the things are characteristically different. They can vehemently argue with one another on a mathematical point, but after a few minutes one of them will concede to the other saying, “yes, you are right, I had something else in mind!”

Clarity of basic ideas, concepts and methods make our subject interesting and attractive.This is one of the reasons why most of us believe that qualifying NET examination for lecturership in colleges and universities should be compulsory. If the basic concepts of mathematics of master’s degree level are not clear to some one, the subject will never be attractive either for him/her or for his/her students. As a matter of fact, this will necessarily be reflected in the quality of his/her research work also. It is here that the atmosphere of the institution where we get our own training of the subject becomes very crucial and significant. Therefore, it is very desirable that even if we were not lucky to get that sort of opportunity, we should try to provide that training to our young generation after catching up matters ourselves. The students who are thoroughly clear about the fundamentals of their subjects can easily secure a good job in the competitive world of today. There are enormous opportunities, if not in mathematics then in other walks of life, for our bright mathematics students.

The academic world around us has been changing very fast eversince the personal computer revolution started. Teaching and research, both are highly influenced by the computing skills which is becoming available to us amost on weekly basis. It is now apparent that if we lag behind in the proper use of these computational skills, we will definitely be left backward not only in learning our subject but also in its teaching and research. I don't think that any illustration is really needed to explain this point which is now well accepted fact all over the world. What really should be said instead in this context is that all of us should have more and more of these facilities at our disposal. In fact every mathematics department of a college or university should now be provided with a well developed computer laboratory along with an excellent library.The national funding agencies who have been supporting research proposals very generously on these matters will do well by devising a more comprehensive mechanism of further promoting and maintaining these facilities keeping the changing requirements in mind. Evidently mathematicians are more closely related to the development of computational facilities themselves than those belonging to any other subject. Consequently, we should take a lead in this wide open area of research and development. We have come to know that several mathematicians, including some Fields medalists, have already opted for such an adventure. I think this is the most appropriate time to do so.

The career of an average mathematician has always been a modest one. It has never been a sought after profession. Some of us are working in colleges and universities, some are working in research institutes, whereas few of us are doing mathematics as our personal hobby. Everywhere we seem to be facing problems of one kind or the other. Leaving aside a few exceptions, the libraries in our colleges as well as those of the university departments are not really up to the mark. The much needed opportunity of being able to talk to better mathematicians or scholars is just not there. We have only one season, such as the present one, of going out of our places of work to attend mathematical meetings, conferences, seminars and workshops and the same is just not enough. Looking to these I wish to draw your attention and also to emphasize that this has been the case all along, and yet the Indian contribution to research in mathematics has been quite satisfactory, if not good. A few of us have always won international recognition for our research work and also for our scholarship. It is admittedly not in proportion to population of working mathematicians in this country but it is definitely better than the performance of our conterparts in other areas of human competition such as Olympic Games, etc.

I must hasten to add here that the University Grants Commission, National Board for Higher Mathematics, Department of Science and Technology, the Concil of Scientific and Industrial Research and several other funding agencies are really doing their best to support teaching and research in mathematics. I must also admit that sometimes we are not able to make full use of their financial support or justify the same for one reason or the other.

Presently I am working in a small place called Rewa in Madhya Pradesh. That city, which has also gone into the 21st Century, has very little modern attraction, specially for very bright scholars. In spite of all the problems that people out there face, one can easily find hundreds of them from poor farmers to poor professors creating and writing Hindi Poetry. These poetries deal with all aspects of human concern and they hit upon the latest corruption world wide. Small Kavi Sammelans are very common where these people participate and derive a lot of pleasure and happiness by reciting their own poems and listening to those of others. I was surprized to watch a few of them and the way they enjoy it. They hardly get any reward or award for their works, but they definitely enjoy their creative output whatever be its worth. They are proud of the fact that renowned poets like Valmiki and Banbhatt were born in that region and the famous musician "Tansen" also came from that place. I feel this is precisely what we mathematicians have to do. Though our position as a mathematician is far better than the Rewa Hindi Poets, but the spirit of joy and happiness is worth cultivating.

While we should continue working in our areas of interest and expertise, we should also make sure as to what is worth pursuing and what is worth leaving. The hot topics of contemporary reasearch in mathematics have always been changing. New concepts, new theories and new directions have always attracted the attention of active mathematicians all over the world. There was a time when topics like shock waves in fluid dynamics, summability theory in real analysis, generalizations of Banach's fixed point theorem in functional analysis, metrization theorems in set topology, simple groups in finite group theory, homology-cohomology theories in algebraic topology, manifolds carrying special structure in geometry, etc. had engaged the attention of most of the mathematicians and some of them are still doing. Some very deep and interesting results have already been obtained in these areas. Important innovations in these topics can, however, be noticed everywhere. Presently, one finds a major shift in these topics. A glance at AMS Mathematical Reviews will bear this out. Dynamical systems and ergodic theory, problems in algebraic geometry surrounding the solution of Fermat's Last Theorem, questions relating to the Poincare Conjecture, Problems connected to Witten's work, and so on and so forth, are such topics where lot of challanging research activities can easily been seen. Indian mathematicians should not lag behind in understanding these areas and making contributions in solving contemporary problems of research. While it is true that we should continue to contribute in our areas of interest where we have kept ourselves in forefront, it is not good to stick to the same old stuff for too long specially when that topic has lost its current value and there are no challanges. It is always beneficial to oneself as well as to the growth of the subject that we learn and attempt to work on new mathematical challanges which are not entirely far removed from our own interests. We can combine our own expertise with the new concepts and methods and can refine/modify them so as to solve the problems of new field. It is also clear that now a days most of the good problems are becoming interdisciplinary in nature. Narrow specialixzation are proving to be fruitless. It is now a common belief that if one solves a new problem using new tools, it is a normal research whereas if one can solve an old problem or a new problem using old methods, it is really ingenuity. However, if one can solve an old problem which has defied solution, by using new methods, as in the case of Fermat's Last Theorem, it is a great human victory!

Finally, I wish to mention about an aspect of the methodology of contemporary teaching and research, which has proved its power beyond doubt. What I have experienced over the past several years of my career, is that conducting seminars on topics of research is indeed a very poweful method of learning and doing mathematics. We know that if we have to present a concept or a mathematical result before a well informed audience, we work hard whole heartedly and make sure that every bit of proposed lecture is perfectly correct and absolutely clear. There should be no ambiguity or confusion about any step or a point, and all the details should be well explained during presentation. Since our very reputation as a scholar of mathematics is at stake, we don't spare any effort in preparing ourselves to the best of our ability and knowledge. In a seminar everything is very informal and therefore one should be ready to answer all sorts of questions from the experts as well as from the learners. This is exactly what makes the seminar so useful. They are valuable for the Lecturer as well as for the audience, though the challange is generally for the lecturer. Most of the good places including the universities, colleges and research institutes, seminars have occupied an important place and are being encouraged everywhere. In my opinion, these are urgently required in all mathematics departments of various institutions of our country. It will go a long way in serving the cause of mathematics if every mathematics department makes it mandatory for their staff and research students to conduct and to participate in regular seminars. I myself have benefited the most from the seminars conducted at the university of Allahabad, university of Jammu, university of Delhi, university of Jabalpur, several universities abroad and elsewhere.

With above observations I now conclude this address by thanking the local secretary and the authorities of the Maratha Sikshan Sansthan for their excellent arrangements. I also thank you all once again for your patience.

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METHODS OF ALGEBRAIC TOPOLOGY IN GROUP ACTIONS AND RELATED AREAS*

SATYA DEO

Dear Fellow Mathematicians,

The area of Group Actions initiated by Dean Montgomery in the beginning fifties of the twentieth century continues to be one of the most dominant topics of mathematical research nowadays. In this address, I intend to take you to some of the questions of this field where Algebraic Topology has been successfully applied not only in solving some of the outstanding problems of Group Actions but also in creating some new and independent fields of research. Most of my own contributions from the year 1974 onwards, and collaborative works since then, can be classified to fall within this topic. In order to keep the contents of this address within reasonable limits, I will avoid giving details. However, one can always find them in full detail in the references provided at the end of the talk.

1. Alexander-Spanier Cohomology

We begin by briefly recalling a few properties of the Alexander-Spanier cohomology $H^p(X, A; G)$ for a topological pair (X, A) with coefficients in an R-Module G [136]. This is a cohomology theory satisfying all the seven axioms of Elienberg and Steenrod and is defined for all topological pairs. By an old result of C. H. Dowker, this cohomology theory is isomorphic to the classical Čech cohomology theory for all topological pairs, and in particular, for all topological spaces. As a consequence of this last result, the Alaxander-Spanier cohomology behaves very well with respect to the Lebesgue's covering dimension $dim(X)$ for a paracompact space X. In other

* The text of the Presidential Address (technical) delivered at the 66^{th} Annual Conference of the Indian Mathematical Society held at Vivekanand Arts, Sardar Dalip Singh Commerce and Science College, Samarthnagar, Aurangabad - 431 001, Maharashtra, India during December 19-22, 2000.

words, if $dim(X) < n$, then $H^p(X,G) = 0 \; \forall p \geq n$. It may be remarked here that the same result is not valid for the singular cohomology. In fact a result of Barrat and Milnor [116] says that there is a three-dimensional compact space X for which $H^p(X) \neq 0$ for infinitely many values of p.

One of the most important properties of the Alaxander-Spanier cohomology is its “tautness” property explained below: Let $A \subset X$ be a subspace. Then the set of all open nbds U of A in X form a directed set (directed downward by inclusion maps). Given a cohomology functor $H^*(;G)$, the cohomology groups $H^*(U;G)$ with homomorphisms $H^p(V;G) \to H^p(U;G)$ induced by inclusion maps $U \subset V$, therefore, will be a directed set of R-modules for each $p \geq 0$. We have

Definition 1.1. A subset A of a space X is said to be tautly embedded or taut in X with respect to the cohomology functor $H^*(;G)$ if the homomorphism

$$\eta : \lim_{\rightarrow} H^p(U;G) \to H^p(A;G)$$

induced by the restriction homomorphisms mentioned above is an isomorphism $\forall p \geq 0$ and for all coefficient groups G.

It is well known that the Alexander-Spanier cohomology, in contrast with other cohomologies, is very rich in tautness property [136]. The following results have already been proved.

Theorem 1.1. *(i). (E. H. Spanier) A closed subset of a paracompact Hausdorff space X is taut in X.*
(ii). (G. Sitnikov) Any subspace of a completely paracompact space is taut in that space.
(iii). (G. E. Bredon) A compact subspace of a Hausdorff space is taught in that space.

E. H. Spanier had observed in his book [136] that a point subspace of an arbitrary space X is taut in X with respect to the Alexander-Spanier cohomology. In contrast with this we have given example [34] of a space X so that none of its points is taut in that space with respect to the singular cohomology. Having examined some tautness results in singular cohomology, we proved the following result in [34].

Theorem 1.2. *A nbd retract of an arbitrary space is taut in that space with respect to the Alexander-Spanier cohomology.*

An alertnate proof of the above result was later on given by Spanier himself in [137].

W. Massey has developed [108] the Alexander-Spanier cohomology based on locally finitely valued cochains. Among other things, he has proved the two tautness results for this cohomology, viz, a closed subspace of a paracompact space, and an arbitrary subspace of completely paracompact space are tautly embedded with respect to this cohomology. In the joint paper [54] by myself and A. N. Roy, we have proved the remaining two results also. We have shown that a compact subspace of a Haudorff space, and a nbd retract of an arbitrary space is taut with respect to the Alexander-Spanier cohomology based on locally finitely valued cochains.

The tautness results stated in the preceeding theorem have been proved (see [15]) for the sheaf cohomology also. To be precise, the results are proved, more generally , for the sheaf theoretic cohomology functor $H^*_\phi(.;G)$, where ϕ is a paracompactifying family of supports. This raises a natural question, viz., is it true that all the tautness results are true for the sheaf cohomology functor $H^*_\phi(.;G)$, where ϕ is an arbitrary family of supports, not necessarily paracompactifying? This question has been studied by me in [45] in great detail where it has been shown, by constructing counter examples, that unless ϕ is paracompactifying family of supports, none of the results included in the last theorem remains valid. In fact, using one of my own results about the cohomological dimensions of n-manifolds, we have demonstrated the existence of a family of supports in the Euclidean space where all tautness results of the last Theorem are false.

Let us remark here that in the definition of tautness, we have required that the map η induced by the restriction homomorphisms should be an isomorphism. Ram Ji Lal asked a natural question in this context: is it possible that the group $\lim_{\rightarrow} H^p(U;G) \cong H^p(A;G)$, where U runs over all open nbds of A in X, but the homomorphism η is not an isomorphism? This question has been investigated by myself and K. Varadarajan in [56] where several examples, covering all cases, have been constructed to show that the subspace A is not tautly imbedded in X, and yet the groups $\lim_{\rightarrow} H^p(U;G)$ are isomorphic to $H^p(A;G)$ for all $p \geq 0$ and all coefficient groups G.

The topic of cohomology theories in the sense of Eilenberg-Steenrod ([84]) is now a classical one. The generalised cohomology theories have also occupied a similar space, and are expanding their appearances in other branches of mathematics besides algebraic topology. The cohomology of sheaves has become an independent discipline having its applications in complex analysis, algebraic geometry and so on. H. Cartan, R. Godement and A. Grothendeick developed the subject from various view points. While studying the cohomology of sheaves by canonical resolutions, one easily observes a close resemblance between Alexander-Spanier cochain complex and the cochain complex of sections of the canonical resolution of a given sheaf. An interesting case occurs when one considers the constant sheaves, viz., for a constant sheaf G of R-modules on a space X, the cochain complex G obtained by taking the sections of canonical resolution of G, is extremely similar to the Alexander-Spanier cochain complex of X with coefficients in G (see [38] for details). In fact, there is a canonical homomorphism from one chain complex to the other. It turns out that the induced homomorphism in cohomology becomes an isomorphism, if the space X is a paracompact Hausdorff space. In this context, the following is still an open problem.

Question. Find a space X for which the Alexander-Spanier cohomology of X with coefficients in a module G is not isomorphic to the sheaf-theoretic cohomology of the constant sheaf G of modules on X?

G. E. Bredon, long back, suggested to me that the long line may possibly be such a space. However, in a recent paper (see [72]) of myself and David Gauld, we have proved that the long line is indeed acyclic with respect to the Alexander-Spanier cohomology and the same result is true for the sheaf cohomology by the same argument. In fact, our result shows that the Alexander-Spanier cohomology and sheaf cohomology of the long line both are indeed isomorphic to the cohomology of an Euclidean interval.

2. Cohomology theory of Group Actions

Let G be a topological group and X be a topological space. A continuous map $\theta : G \times X \to X$ satisfying the following properties ($g.x$ stands for $\theta(g, x)$)

(i). $(g_1.g_2).x = g_1.(g_2.x)$ (ii). $e.x = x$ $(\forall\, g_1, g_2 \in G, x \in X)$

is called a (continuous) action of G on X. The pair (G, X) with the action θ is called a **transformation group**. Given a transformation group (G, X), the set $X^G = \{x \in X : g.x = x \;\forall g \in G\}$ is called the **fixed point set**. For a given point $x \in X$, the set $\{g \in G : gx = x\}$ is a subgroup of G and is called the **isotropy** subgroup of the point x. The subset $G(x) = \{gx : g \in G\}$ is called the **orbit** of x. Any two orbits of X are either identical or disjoint, and therefore these orbits decompose X in to mutually disjoint equivalence classes. The quotient space X/G is called the **orbit space** of the transformation group. For simplicity we will assue all topological groups and topological spaces to be Hausdorff.

The two objects, viz., the fixed point set and the orbit space, associated with a transformation group (X, G), have been the subject matter of intense study for more than the last sixty years now. Using the methods of set topology, the first book by Montgomery and Zippin [114] on transformation groups, which for the first time formalized the definition of a transformation group, published some basic results on group actions on Euclidean spaces. Before that, using the methods of algebraic topology, P. A. Smith had already published a few fundamental papers [126] to [133] on periodic maps on homological disks and homology spheres. These beautiful papers of P. A. Smith received immediate attention of the leading mathematicians of the time. The famous "Borel Seminars" by A. Borel and others was organized in the year 1960 and the proceedings of the seminar, published afterwords, were widely received for its contents as well as for its methods [11]. F. E. Floyd used sheaf cohomology to give an alternative proof of Smith theorems. A. Borel, while introducing the idea of "Borel Construction" via associated fibre bundles, used the powerful techniques of spectral sequences in studying group actions. Borel's method not only gave yet another proof of Smith theorems, but also laid the foundation of a far reaching generalizations of Smith theory. J. C. Su [138], [139], G. E. Bredon [17] and several other topologists exploited the Borel's method and proved some very basic results on the cohomology structure of the fixed point sets and orbit spaces. All these results are nicely compiled in Chapter III of the book on transformation groups [13] by G. E. Bredon.

Now let us recall the most general form of Smith theorems known as of today (see [13], Chapter VII). In what follows, we always use the Čech cohomology groups unless stated otherwise.

Theorem 2.1. *Let G be a p-group, p a prime and X be a finitistic G-space which is a mod p cohomology sphere. Then the fixed point set X^G is a mod p cohomology r-sphere for some $-1 \leq r \leq n$. If p is odd then $n - r$ is even.*

Theorem 2.2. *Let G be a p-group and X be a finitistic G-space. Let $A \subset X$ be a closed invariant subspace of X. If (X, A) is a mod p cohomology n-disk then the fixed set (X^G, A^G) is a mod p cohomolgy r-disk for some $0 \leq r \leq n$. If p is odd then $n - r$ is even.*

The other important theorem which follows from the Smith theorem is proved by E. E. Floyd and is on cohomology structure of orbit spaces.

Theorem 2.3. *Let G be a finite group and X be a finitistic G-space which is acyclic over integers. Then the orbit space X/G is also acyclic over integers.*

We recall that a paracompact space X is said to be **finitistic** (see [12] for details) if every open cover has a finite dimensional open refinement. These spaces were introduced by R. G. Swan (see [141] for details) who proved the Smith theorems for these general class of spaces. It must be remarked here that, by a result of C. H. Dowker, every paracompact space of finite covering dimension is finitistic. Every compact space is evidently finitistic. G. E. Bredon popularised these spaces, and rightly so, by proving all generalizations of Smith theorems for finitistic spaces. It must also be remarked here that Smith theorems are not true if the space is not finitistic (see [12] for example). These theorems are not valid if singular cohomology is used in place of Čech cohomology. Finally, the theorems are not true even when the coefficients are changed from mod p to something else (see [114] for examples).

The Smith theorems stated above for the finite p-groups were also proved for the Torus-groups T^k; here T stands for the circle group and T^k is simply k-fold product of T. The method of proof consists in developing the Smith-Gysin sequences and proving its exactness, and then the Smith technique of finite p-group actions yields the following Smith theorems for Torus group actions with rational coefficients. Here we recall that a group G is said to act on a space X **with finitely many orbit types** provided the number of conjugacy classes of subgroups of G occuring as isotropy subgroups at various points of the space X is finite. This turns out to be the necessary condition whenever one is dealing with infinite groups.

Theorem 2.4. *Let the circle group $G = T^k$ act on a space X with finitely many orbit types such that both X as well as the orbit space X/G are finitistic. Let A be a closed invariant subspace of X. With these conditions (a). If X is a rational cohomology n-sphere, then the fixed point set X^G is also a rational cohomology r-sphere, $-1 \leq r \leq n$ and $n - r$ even. (b). If (X, A) is a rational cohomology n-disk then the fixed set is also a rational cohomology r-disk, $-1 \leq r \leq n$ and $n - r$ even.*

The above theorem which was proved by Bredon in [13] uses very strong condition on the space X, viz., the space X as well as the orbit space X/G both are finitistic. It was indeed quite natural for Bredon to ask whether or not, assuming both X and X/G to be finitistic was necessary or X finitistic implies the orbit space X/G finitistic? Even for the circle group $G = T$, the question remained open for quite some time. I myself took up this question and answered it completely in our joint paper [40]. In fact our main result is very general.

Theorem 2.5. *Let G be a compact Lie group acting on a finitistic space X. Then the orbit space X/G is also finitistic.*

The above theorem not only answers the question raised by G. E. Bredon, but it improves a host of several other results proved by him generalizing Smith theorems using Borel's approach via spectral sequences (see [13]). The basic result needed to prove all these theorems by G. E. Bredon as well as by Wu-Yi Hsiang is the famous Borel-Segal-Quillen-Atiyah localization theorem. Now, using our previous theorem, that localization theorem has been proved (see [42]) for general finitistic spaces. First let us have some preliminaries.

Let G be a compact Lie group and $E_G \to B_G$ be the universal principle G-bundle. Suppose X is a G-space. Consider the product $X \times G$ with the diagonal action of G. The orbit space $(X \times E_G)/G$ is denoted by X_G and $H^*(X_G)$ is called the equivariant cohomology of the G-space X. An equivariant map $X \to Y$ will induce a homomorphism $f^* : H^*(Y_G) \to H^*(X_G)$, and it is easy to see that this equivariant cohomology is a generalized cohomology theory on the category of all G-spaces and G-maps. The two equivariant projections from $X \times G$ to X and B_G respectively, will induce two maps from the orbit space X_G to orbit spaces X/G and $B_G = E_G/G$ respectively. Let R denote the cohomology ring of the classifying space B/G. The continuous map induces a ring homomorphism $R \to H^*(X_G)$, and therefore $H^*(X_G)$ becomes an R-module. Now note that $\forall x' \in X/G, H^*(E_G/G_x) \sim H^*(B_{G_x})$, where x' is the image of x under the quotient map, is an R-module via the canonical quotient map $B_{G_x} \to B_G$. Now let $S \subset R$ be any multiplicative system in R and define $X^S = \{x \in X$: no element of S is mapped to zero in $R \to H^*(B_{G_x})\}$. Then, we have the following generalization of the Localisation Theorem for finitistic spaces (see [42]).

Theorem 2.6. *Let X be a finitistic space and G be a compact Lie group acting on X. Suppose S is a multiplicative system in $R = H^*(B_G)$ and $s \in S$. Then the localized restriction homomorphism $S^{-1}H^*_G(X) \to S^{-1}H^*_G(X^s)$*

is an isomorphism. Furthermore, if the member of the orbit types in X is finite, then the restriction homomorphism $S^{-1}H_G^*(X) \to S^{-1}H_G^*(X^s)$ *is also an isomorphism.*

The following results, which were known for compact spaces and paracompact spaces having finite cohomological dimensions, have been proved for the class of finitistic spaces (see [42] for details). In what follows the coefficient groups are in any of the groups $\Lambda = Z_p, Z, Q$.

Theorem 2.7. *Let* X *be a finitistic space and* G *be a compact Lie group acting on* X *with finitely many orbit types. Then if* X *is* Λ*-acyclic then so is* X/G*.*

Theorem 2.8. *Let* G *be a compact Lie group acting on finitistic space* X *with finitely many orbit types. Assume that* Y *is a closed invariant subspace of* X*. Then, if* $H^*(X, Y; \Lambda)$ *is finitely generated, then so is* $H^*(X/G, Y/G; \Lambda)$*.*

The above results were proved earlier by R. Oliver (see [119]) and T. Skjelbred (see [125]) respectively for compact spaces and spaces having finite cohomological dimensions.

Recall that a group G is said to **act freely** on a space X if for each point $x \in X$ and each $g(\neq e) \in G, g.x \neq x$. It is a consequence of the Lefschetz fixed point theorem that $G = Z_2$ is the only finite group which can act freely on a given even dimensional sphere. The question as to which finite groups can act freely on a given odd-dimensional sphere is still open. If we consider product of even-dimensional spheres, then evidently, the finite groups other than Z_2, can act freely on such products. On the other hand, given such a product, not every finite group can act freely on the product, and therefore, it is natual to ask as to what are those finite groups which can act freely on a given product of even spheres or other similar cases. L. Cusick (see [26] to [30]) and M. Hoffman (see [95] to [97]) have answered this question for the product of even spheres. It is well known that the product of symmetric powers of even spheres is more general class of spaces and includes the product of even spheres, the product of complex projective spaces, etc. In view of the known results about the product of even spheres mentioned above it is natural to ask as to which finite groups can act freely on products of symmetric powers of spheres. Interestingly, we have answered this question (see [70]) as follows:

Theorem 2.9. *A finite group* G *can act freely on a finite product of symmetric powers of even-dimensional spheres iff it can act freely on a suitable product of even-dimensional spheres themselves.*

Discontinuous actions have continued to be a very important topic of research and have obvious connections with geometry. Recall that a group G is said to act **properly discontinuously** on a space X if

1. each isptropy subgroup is finite.
2. each orbit is a closed subspace, and
3. for each $x \in X$ there is a open set U of X with $x \in U$ such that $\{g \in G : U \cap g(U) \neq \phi\}$ is finite.

We say that G acts on X **strongly properly discontinuously** if for any pair x, y of ponts of X, there exists open sets U, V in X with $x \in U, y \in V$, such that the set $\{g \in G\colon\ gU \cap V \neq \phi\ \}$ is finite.

It is well known (see [66]) that any discrete subgroup of the group $I(M)$ of isometries of a complete connected Riemannian manifold M acts strongly discontinuously on M. In this connection it is interesting to note that we have proved the same result for connected, locally compact separable metric space X which is far more general than a Riemannian manifold. As a matter of fact when we isolate all crucial aspects of the proof of that for the Riemannian manifold, and prove each one of them in most general setting, the final result is easily seen to be valid for a connected, locally compact separable metric space (see [66]). In the paper just cited we have also clarified all the definitions of several kind of discontinuous actions with lot of interesting illustrations and examples.

2.1. **Burnside Rings.** Let G be a finite group, and consider the class of all finite G-sets in X. Declare the two G-sets X, Y to be equivalent if there is a G-isomorphism between them. Now consider the set $\Omega(G)$ of equivalent classes $[X]$ of all such G-sets. Taking disjoint union and cartesian product of G-sets define two binary operations and make $\Omega(G)$ into half-ring.Applying Grothendiek construction on this half ring, we get a ring, denoted by $\Omega(G)$, called the Burnside ring of the finite group G.

It may be observed that the finite G-set X is simply a permutation representation of the group G, and therefore, the set of all finite G-sets is just the set of all permutation representations of the group G. Thus the resulting ring $\Omega(G)$ defined above arises from the class of all permutation reprentations of the group G. This was the view point proposed by Burnside in defining the Burnside ring of a finite group G for the first time.

Recall that if A is a commutative ring then the set of all prime ideals of A, with Zariski topology, is called the spectrum of the ring A and is denoted by Spec(A). Burnside rings of the finite groups and its generalizations became a hot topic of research after A. Dress proved (see [80]) the following beautiful result connecting group theory and topology.

Theorem 2.10. *A finite group G is solvable iff the space Spec $\Omega(G)$ is connected.*

In other words, a finite group G is solvable iff the only idempotent elements of the Burnside ring $\Omega(G)$ are zero and one. The above characterization of a solvable group by means of the connectedness of the spectrum of $\Omega(G)$ is really a rare example of a remarkable interplay between group theory and topology. People working in the representation theory of groups and in the field of transformation groups immediately get attracted towards the topic of the Burnside rings of more general groups. T. tom Dieck defined the Burnside ring of a compact Lie group in [73] using smooth G-manifolds and produced several interesting theorems on group actions, and about equivariant Euclidean nbd retracts. One complete chapter in his book [73] has been devoted to the study of Burnside rings.

An interesting question about Burnside rings of finite group G was left open by Dress. If two finite groups G and H are isomorphic then it follows obviously that their Burnside rings $\Omega(G)$ and $\Omega(H)$ are isomorphic as rings. What about the converse? In other words, suppose we are given that the Burnside rings $\Omega(G)$ and $\Omega(H)$ are isomorphic as rings. Can we then assert that the groups G and H are isomorphic as groups? The answer is "NO". This was seen for groups of very huge orders. However, later J. Thevenaz [142] gave an interesting example of two non-isomorphic groups of small orders having isomorphic Burnside rings. These groups were of course non-abelian. For abelian groups the question was still open. Myself and K. Varadarajan looked at this question and we proved the following result: Let A be the class of all finite abelian groups and for any $G \in A$ let $t_p(G)$ denote p-primary torsion subgroup of G. Suppose A_s=$\{G \in A : t_p(G)$ is a direct sum of at most two cyclic groups$\}$. Then we have the following:

Theorem 2.11. *Let G and H belong to A_p. Then $\Omega(G)$ and $\Omega(H)$ are isomorphic as rings iff G and H are isomorphic as groups.*

We also prove the following.

Theorem 2.12. *Let G and H be two finite cyclic groups. Then there exists a ring isomorphism $\alpha : \Omega(G) \to \Omega(H)$ satisfying $\Omega((G)^+) = \Omega((H)^+)$ iff G and H are isomorphic as groups.*

In a subsequent paper [31] Vallijo and Cardinas improved our result for all finite abelian groups as follows:

Theorem 2.13. *Let G and H be finite abelian groups. Then $\Omega(G)$ is isomorphic to $\Omega(H)$ as rings iff G and H are isomorphic as groups.*

3. Cohomological dimension Theory

The concept of cohomological dimension of a topological space was defined by P. S. Alexandroff using tools of Algebraic Topology. This was done way back in thirties of the twentieth century. In fact, he defined homological dimension rather than cohomological dimension and proved following important results: If X is a compact metric space having finite covering dimension, then cohomological dimension is less than or equal to the covering dimension; furthermore, if the latter is finite then the equality holds. It was not known at that time as to what happens when $dimX = \infty$. This question became famous by the name of "Alexandroff's problem". The question can be rephrased as follows: Does there exist a compact metric space of infinite covering dimension which has the finite cohomological dimension.

There are several definitions of cohomological dimension of a space X. We quote the one proposed by Okuyama [120]. The largest integer n such that there exists a closed subspace A of X such that the restriction homomorphism $H^n(X;G) \to H^n(A;G)$ is into, is called the cohomological dimension of X with coefficients G. A lot of work on this algebraic dimemsion, parallel to the cohomological dimension of a space X, depends on the coefficient group G. The case when $G = Z$ is of crucial significance. The definition of cohomological dimension using the sheaf cohomology is extremely useful in the study of group actions. Let X be a space with ϕ, a family of supports on X and R be a ring. The largest n such that there is a sheaf A of R-modules on X satisfying $H^n(X, A) \neq 0$ is called the ϕ-dimension of X over R and is denoted by $dim_{\phi,R}(X)$. If ϕ is a paracompactifying family of supports, with extent $\phi = X$, then $dim_{\phi,R}(X)$ is independent of ϕ and we denote it by $dim_R(X)$. In particular, if X is paracompact the $dim_R(X) = dim_{cld,R}(X)$, where cld is the family of all closed sets of X. If $R = Z$, then it is also known that $dim(X)$ defined earlier agrees with the sheaf theoretic cohomological dimension. It must be remarked here that locally paracompact spaces X admit a paracompactifying family of supports whose extent is X, and so $dim_R(X)$ is always defined for such spaces.

G. E. Bredon defined a new dimension called the **large cohomological dimension** $Dim_R(X)$ as follows: $Dim_R(X) = Sup_\phi\{dim_{\phi,R}(X)$, where supremum is taken over all families of supports. He also showed that

$$for\ an\ n\text{-}manifold\ M, \quad Dim_R(M) = n\ or\ n+1.$$

He was not able to prove which one it is even for real line. In one of my first papers [35], I proved that $Dim_Z(R) = 1$ and later on I proved [36] the general result that $Dim_Z(R^n) = n+1$, which immediately yields the result that for a n-manifold M, $dim_Z(M) = n+1$. The other properties of this large cohomological dimension are quite similar to the Alexandroff's cohomological dimension (see [50]).

One of the unique features of the cohomological dimension $dim_L(X)$ is that it is a local concept, i.e., if X is a locally paracompact space and each point of X has an open nbd N such that $dim_L(N) \leq n$, then $dim_L(X) \leq n$. This property is in contrast with the covering dimension $dim(X)$. In fact one can easily see that if X is a long line, then $dim_Z(X) = 1$. However, since for each n we can find a finite open cover of X which has no refinement of order less than n, it follows that the long line X is of infinite covering dimension. This provides a counter example to the Alexandroff's problem, but not quite, because Alexandroff's problem requires the space X to be compact metric space.

We made a systematic study [48] of the sheaf theoretc cohomological dimension $dim_L(X)$, here L is the ground ring, and proved various sum theorems for them. There is one question on these sum theorems which was asked by Kuzminov [105] and remains open even now. Recall that a subspace A of X is said to be locally closed if A is the intersection of a closed and open set. The subset theorem for cohomological dimension implies that $dim_L(A) \leq dimX$ for all locally closed subsets A of X.

Question (Kuzminov): Let $A_1, A_2, \cdots, A_n, ...$ be a countable covering by locally closed subsets of a space X. Can we assert that

$$dim_L(X) = sup\{dim_L(A_n) : n \in N\}?$$

We have answered the above question [49], but only partially.

3.1. **Topology of Finitistic Spaces.** We have already come across these spaces in the context of Smith theorems and their generalizations. Looking to their importance we wish to consider the topology of such spaces which is now well understood. Recall that a space X is called finitistic if every open cover of it has a finite dimensional open refinement. Note that we have, for the sake of generality, deliberately dropped paracompactness condition from X. Every compact space is evidently finitistic. By a result of C. H. Dowker, a paracmpact space of finite covering dimension is finitistic. By definition every finitistic space must be paracompact. It follows that the ordinal space $X = [0, \omega_1]$, which is not paracompact can not be finitistic, though $dim X = 0$. It follows that unless the space X is paracompact, a finite dimensional space need not be finitistic. This point must be carefully noted because, unless one has paid careful attention to this exceptional situation, one might make erroneous statements (see [73] for this oversight).

Theorem 3.1. *A paracompact Hausdorff space X is not finitistic iff there exists a closed subset G of X which can be expressed as a disjoint union of subsets $\{G_n\}$ such that each one of these sets is closed and open in G.*

The above characterization theorem immediately yields an answer to the question of G. E. Bredon regarding orbit spaces of finitistic spaces. From the fact that for a compact Lie group G, $dim X/G \leq dim X$, one easily derives the result, using the above characterization theorem, that the orbit space of a finitistic space under the action of a compact Lie group G is again finitistic.

After the proof of the above characterization theorem and the consequent solution of Bredon's problem, a series of papers on finitistic spaces were written by us as well as by others (see [41], [59], [44], etc.). Smith theorems were proved for two classes of spaces, viz. (i) for all compact spaces, and (ii) for all paracompact spaces having finite cohomological dimension. Tacitly assuming that perhaps all paracompact spaces having finite cohomological dimensions have also finite covering dimension, G. E. Bredon proved all his results for finitistic spaces which could possibly include both classes of spaces mentioned above. However, such an assertion was never made in writing because it was tied up with the Alexandroff problem [39]. In fact this indirectly left open the following question: Is it true that every paracompact space of finite cohomological dimension is finitistic? This question was answered negatively by me in [60] by exploiting the Dranishnikov's solution of Alexandroff's problem. We have shown in that paper that there are paracompact cohomology projective spaces of finite cohomological dimension which are not finitistic. Now, this result of ours makes the situation about general Smith type theorems quite clear, i.e., the two classes of spaces (i) the paracompact finitistic spaces and (ii) paracompact spaces having finite cohomological dimension, are indeed independent of each other.

4. Hopfian and co-Hopfian Groups and Spaces

Among several finiteness properties, the concept of Hopfian groups has been an old idea of research. K. Varadarajan [148] wrote an interesting pa-

paper dealing with general Hopfian and co-Hopfian objects in several categories. As an example, a group G is said to be Hopfian if every epimorphism $f : G \to G$ is an automorphism.Determining or identifying these objects in a given category is always an interesting problem. In certain categories, the characterization of these objects is almost trivial, as for example in the category of sets the Hopfian obects are precisely finite sets. However, in certain other categories, it may be a difficult problem. In the category of groups, it has already been a formidable problem [107]. Recently, G. Acuna, R. Litherland and W. Whitten jointly wrote a few interesting papers in which they attempted to determine which of the three-manifolds hace co-Hopfian fundamental groups (see [88]. [89]. [90]). Prompted by those papers, we went in slightly different direction and proved the following result [67]:

Theorem 4.1. *(i). Let N be a double cover of a closed nonorientable aspherical manifold M. If N has a co-Hopfian fundamental group then $\pi_1(M)$ is also co-Hopfian.*

(ii). Any finitely generated group can be expressed as a quotent of a finitely presented group which is simultaneously Hopfian and co-Hopfian.

(iii). There is no functorial embedding of groups, respectively of finitely generated groups, into Hopfian group.

Determining Hopfian and co-Hopfian topological spaces is also a difficult problem. Using the invariance of domain theorem, K. Varadrajan showed in [149] that any compact manifold without boundary is co-Hopfian. It was also shown by him in the same paper that the only Hopfian objects among compact totally disconnected manifolds are the discrete spaces. It was left an open question to decide what exactly are compact Hausdorff totally disconnected spaces, other than discrete ones, which are Hopfian or co-Hopfian. Varadarajan and I tackled this question and proved [62] the following result partially answering the above question:

Theorem 4.2. *(i). The only Hopfian and co-Hopfian objects in the category of compact totally disconnected metrizable spaces are the finite discrete spaces.*

(ii). Every infinite closed subspace of $\beta\mathbb{N}$ and hence any infinite closed subspce of $\beta\mathbb{N} - \mathbb{N}$ is nonco-Hopfian. However, $\mathbb{N}^*$ admits an abundance of non-closed subs-

paces which are simultaneously Hopfian and co-Hopfian.

(iii). There are atleast 2^c nonhomeomorphic compact totally disconnected co-Hopfian spaces which are not rigid.

In this context the following problem is still open:

Question. Does there exist a compact totally disconnected Hopfian space which is not discrete?

In algebraic topology, one frequently tries to solve a topological problem using algebraic methods. It is only rare that an algbraic problem is solved using topological methods. We recently came across a situation where our result mentioned above as (iii) in the preceeding theorem was put to an interesting application for answering the following question about polynomial rings: If ring R is Hopfian then can we assert that the ring $R[x]$ of polynomials is also Hopfian? Varadrajan attempted this question in [149] and proved the following:

Theorem 4.3. *If R is a Boolean Hopfian ring then the ring $R[T]$ of polynomials is also Hopfian.*

Recall that a ring R is said to be Boolean if $a^2 = a \; \forall a \in R$. Generalizing the above result of Varadarajan, S. P. Tripathi proved [147] the following interesting result where the crucial step is our earlier theorem where we have shown the existence of a compact totally disconnected co-Hopfian space.

Theorem 4.4. *Let R be a ring with no ($\neq 0$) nilpotent elements. Suppose there is an integer $n \geq 1$ such that $a^n = a \; \forall a \in R$. Then R is Hopfian implies that the polynomial ring $R[T]$ is also Hopfian.*

The author has constructed a class of examples to show that his generalization of Varadarajan's theorem covers more spaces than that of Varadarajan's theorem. The question regarding the existence of abundance of Hopfian groups was attempted by G. Baumslag when he claimed that for every cardinal α, there is an abelian Hopfian group G whose cardinality is α. However, it was pointed out by B. H. Neumann that Baumslag's proof was incorrect. In [71] we have proved the following

Theorem 4.5. *For every cardinal $\alpha, 1 \leq \alpha \leq c$, there exists an abelian group H such that $|H| = \alpha$ and H is simultaneously Hopfian and co-Hopfian.*

5. Projective Dimension of Spline Modules

The methods of algebraic topology, specially those of homological algebra, have found applications in answering some difficult questions regarding piecewise polynomials in several variables. This has been shown by the works of L. Billera [6], [7], L. Billera and L. Rose [8], [9] and our own works [64], [68].

A finite collection $\Box$ of convex polyhedra in the Euclidean space R^d is called a **polyhedral complex** if any face of a member of $\Box$ is again a member of $\Box$, and the intersection of any two members of $\Box$ is a face of both. A simplicial complex is obviously a special case of a polyhedral complex where the polyhedra are simplexes. Let a region Ω of the euclidean space $R^d, (d > 1)$, be decomposed as a polyhedral complex, and let $S^r(\Box)$ denote the set of all multivariate splines on Ω. Then with pointwise operations the set $S^r(\Box)$ turns out to be module over the ring $R[x_1, x_2, \cdots, x_d]$ of polynomials in d-variables. Billera [6], Billera and Rose [8] [9] initiated the study of this R-module. In the later two papers, they used the methods of commutative algebra to study the dimension problem of R-vector space $S^r_k(\Delta)$ of all splines on Δ of degree at most k when Δ is a simplicial complex. In [9] they studied the algebraic question as to under what conditions on r and d, the $R-$module $S^r(\Box)$ would be free. The case when $r = 0$, was solved for all d when the polyhedral complex is a simplicial complex; the case when $d = 1$ is trivially free, in the case $d = 2$, it is free iff Ω is a manifold with boundary.It follows from these results that the freeness of the $R-$module depends on topology of polyhedral complex rather than on its geometry. S. Yuzvinsky [154] considered the projective dimension $pr_R S^r(\Omega)$ of the module $S^r(\Box)$ and obtained interesting results generalizing the Billera-Rose results. He obtained a characterization of projective dimension in terms of the sheaf cohomology of certain posets associated with $\Box$ with coefficients in $R-$module $S^r(\Box)$. Note that $pr_R S^r(\Omega) = 0$ is equivalent to the statement $S^r(\Box)$ is free. In our paper [63], we extended the results of Billera-Rose and also of Yuzvinsky by studying the projective dimension of $S^r(\Box)$ on lines of other cohomological dimensions. For example, we have shown that "the subset theorem" and "the sum theorems" are valid for this projective dimension under suitable conditions. More importantly, we have proved that the projective dimension of $S^r(\Box)$ is a local concept when $\Box = \Delta$ is a simplicial complex, and we have constructed an example to show that the same is not a local concept for general polyhedral complex.

In yet another paper [68], we considered the computational problem of determining an $R-$basis for the $R-$module if it happens to be free. It is interesting to see that there is an algorithm for writing down a free basis for the above $R-$module in terms of suitable linear forms defining common faces of $\Box$. This is done for the case when $\Box$ consists of a finite number of parallelopipeds properly joined among themselves along the above linear form. In fact, we have given a proof of the existence of such an algorithm which may be of potential use in practical applications.

Finally, I take this rare opportunity of expressing gratitude to my teachers and colleagues at the University of Allahabad and to my doctoral supervisor late Prof. John Keesee of the University of Arkansas, USA for the encouragement and training which I received from them. I also express my thanks to my collaborators and my own Ph.D. students. It has been a great pleasure for me to collaborate with K. Varadarajan with whom my collaboration has been quite intensive and long. I could never meet G. E. Bredon, but I have followed him in his research interests from cohomology theory to cohomological dimensions and then to group actions. All along this route I acknowledge that I have always received encouragement from him. Thank you all for your patient hearing.

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DEFORMATION OF A STRATIFIED ELASTIC HALF-SPACE BY SURFACE LOADS AND BURIED SOURCES*

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ABSTRACT. In this study, we review the progress made in the recent past in the direction of finding the displacement and stress fields in stratified elastic half-space caused by either surface loads or buried sources, using matrix formulation. Both 2-D and 3-D cases are considered. Surface loads are represented by suitable boundary conditions at the boundary surface of the half-space; burried sources are represented as discontinuities in the displacements and stresses in the transform domain at the source level. The matrix formulation avoids the cumbersome nature of the problem and is well suited for numerical evaluation of the elastic field.

1. INTRODUCTION

The problem of finding fundamental elastostatic solutions has always occupied an important position in applied mathematics. Fundamental solutions represent the distribution of displacements and stresses in an elastic medium due to the action of concerned forces applied either at the boundary surface or in the interior of the elastic medium. Classical studies in electostatics is related to closed-form fundamental solutions for the action of concentrated point forces in a homogeneous medium of either infinite or semi-infinite extent. A logical extension of this study was to derive a closed-form electrostatic solution for a concentrated point force in a half space in welded contact with another half-space.

It is useful to develop similar fundamental solutions for stratified media. Stratified elastic media are often encountered in various branches of geo-

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physics, seismology, composite materials, highway and airport pavements and foundation engineering. However, closed form solutions are possible only for a homogeneous half-space or two half-spaces in contact. For more complex models closed-form solutions are not available. What one can obtain is the solution in the integral transform domain. The inverse tranform has to be evaluated by numerical methods.

There are three basic steps in the matrix method employed for finding the elastic field in stratified media. Firstly, the governing equations in a single layer are solved in the transform domain to get an algebraic solution. Secondly, the boundary conditions at the interface between two consecutive layers as well as at the outer boundary are applied in the transform domain instead of the physical domain. The solution in the transform domain is obtained by algebraic operations. Finally, the solution in the physical domain is obtained by finding inverse integral transform.

Thomson (1848) derived the solution for a concentrated point force in an unbounded elastic medium. This solution is now known as Kelvin's solution. Boussinesq (1885) provided the solution for a concentrated normal point force acting on the boundary of a uniform elastic half-space. The corresponding solution for a tangential point force was supplied by Cerruti (1882). Mindlin (1936) derived the solution for a concentrated point force acting at an interior point of a uniform elastic half-space. Mndlin and Cheng (1950) gave explicit expressions for the displacement and stress components of half-space nuclei of strain consisting of single force and double forces with or without moment. Mindlin's solution has been used very extensively for computing the deformation of a uniform half-space caused by seismic sources.

As a mathematical model of faulting in the earth, Steketee (1958) assumed a displacement dislocation surface, a surface across which the displacement vector is discontinuous. He developed a Green's function method to deal with the problem of dispacement location in a half-space and constructed the Green's function for a particular case. Maruyama (1964) calculated all the sets of Green's functions necessary for the displacement and stress fields around an arbitrary dislocation in a uniform half-space.

Rongved (1955) derived closed-form algebraic expressions for the Papkovich-Neuber displacement potentials for an arbitrary point force acting in an infinite medium consisting of two elastic half-spaces in welded contact. The expressions for these potentials are quite complex and Rongved did not present the expressions for the resulting displacement. Dundurs and Hetényi (1965) obtained these potentials when the half-spaces are in smooth contact and also obtained the corresponding displacements. Using the Rongved (1955) solution, Heaton and Heaton (1989) obtained the deformation field induced by point forces and point force couples embedded in two-half-spaces in welded contact, imposing a simplifying assumption that the Poisson's ratio is 1/4. Kumari *et al.* (1992) generalized the results of Heaton and Heaton (1989) by assuming arbitrary value for the Poisson's ratio and by obtaining analytical expressions for the stresses as well. The expressions for the displacements were given by Kumari *et el.* (1992) for nine basic nuclei of strain (three dipoles without moment and six single couples). Singh *et al.* (1993) and Rani *et al.* (1995) derived the expressions for the displacements due to a finite rectangular fault in the two welded half-spaces model by analytical integration over the fault area.

Tinti and Armigliato (1998) obtained analytical solution for a sigle force in the two welded half-spaces model, using the Galerkin vector approach. Since Tinti and Armigliato (1998) solved the problem *ab initio*, without exploiting the known Rongved (1995) solution, they had to perform lengthy algebraic calculations. This process will have to be repeated for any other source one wishes to consider. Indeed, Mindlin (1936) solution for a half-space and Rongved (1955) solution for two welded half-spaces are very general and powerful. While the former has been used extensively in seismological literature, the later has received comparatively less attention. Begining with the Rongved (1955) solution, Singh *et. al.* (1999) obtained closed form algebraic expressions for the displacements and stresses due to a single force acting at an arbitrary point of a two-phase medium consisting of two elastic half-spaces in welded contact. The force may be acting either parallel to or perpendicular to the interface.

The problem of the static deformation of a multilayered elastic half space by surface loads and buried sources has been discussed by several authors (see, e.g., Kuo (1969), Singh (1970, 1971); Bufler (1971), Singh and Garg (1985), Singh (1986), Kausel and Seale (1987), Shield and Bogy

(1989a, 1989b), Pan (1989a, 1989b), Garg *et. al.* (1991, 1992), Yue (1995), Pan (1997)) applied the method of layer matrices to solve the problem of the static deformation of a multilayered elastic half-space by buried sources. The problem of the static deformation of a multilayered half-space by two-dimensional souces has been formulated by Singh (1985) and Singh & Garg (1985). In this two-dimensional case, the plane strain problem and the antiplane strain problem are decoupled and, therefore, can be tackled separately. Singh (1986) studied the problem of axially symmetric deformation of a transversely isotropic multilayered half space by surface loads. The particular cases of a torque and a vertical force are considered in detail. Garg and Singh (1985) studied the two-dimensional problem of the static deformation of a multilayered isotropic half-space by surface loads in details. The particular cases of a normal line load and a shear line load are also considered. The corresponding problem of a transversely isotropic medium has been discussed by Garg and Singh (1987) and of an orthotropic medium by Garg *et al.* (1991).

2. Three-Dimensional Surface Loading

Kuo (1969) discussed the static deformation of a stratified elastic half-space by surface loads. He applied the Thomson-Haskel matrix method (Thomson (1950), Haskel (1953)) used earlier mainly in elastodynamics. Since Kuo did not separate the toroidal and spheroidal fields, he had to deal with 6×6 matrices. Following Singh (1970), Garg *et al.* (1992) formulated the problem of the deformation of a stratified elastic half-space by general surface loads in terms of a 4×4 and 2×2 matrices, after seperating the toroidal and spheroidal fields.

2.1. **Basic equations.** Consider a cylindrical co-ordinate system (r, θ, z) with $z-$axis vertically downwards. Let (u_r, u_θ, u_z) denote the components of the displacements vector $\mathbf{u}$ and τ_{rr}, $\tau_{r\theta}$, etc. the components of stress. The problem of the static deformation of a stratified half space with plane parallel boundaries of the form $z = constant$ splits in to two independent problems which we refer to as the toroidal and the spheroidal problems. In the toroidal problem

$$div\ \mathbf{u} = 0, u_z = 0,$$

whereas in the spheroidal problem

$$(curl\ \mathbf{u})_z = 0.$$

Spheroidal Problem

The diplacement $\mathbf{u}$ satisfies the Navier equation of static elasticity:

$$\nabla^2 \mathbf{u} + \beta \; grad \; div \; \mathbf{u} = 0,$$

where

$$\beta = \frac{\lambda + \mu}{\mu} = \frac{1}{1 - 2\sigma},$$

λ, μ being the Lamé parameters and σ the Poisson's ratio. In the case of the spheroidal problem the displacement $\mathbf{u}$ can be expressed in terms of Love's strain potential ϕ in the form (Fung (1965))

$$\mathbf{u} = \beta[2(1-\sigma)e_z \nabla^2 \phi - grad(\frac{\partial \phi}{\partial z}].$$

The displacement components are

$$\begin{aligned} u_r &= -\beta \frac{\partial^2 \phi}{\partial r \partial z}, \\ u_\theta &= -\frac{\beta}{r} \frac{\partial^2 \phi}{\partial \theta \partial z} \\ u_z &= (1+\beta)\nabla^2 \phi - \beta \frac{\partial^2 \phi}{\partial^2 z}. \end{aligned} \tag{2.1}$$

The corresponding stresses are

$$\begin{aligned} \tau_{rr} &= \mu \frac{\partial}{\partial z}[(\beta - 1)\nabla^2 \phi - 2\beta \frac{\partial^2 \phi}{\partial r^2}], \\ \tau_{\theta\theta} &= \frac{\partial}{\partial z}[(\beta - 1)\nabla^2 \phi - \frac{2\beta}{r}\frac{\partial \phi}{\partial r} - \frac{2\beta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}], \\ \tau_{zz} &= \mu \frac{\partial}{\partial z}[(3\beta + 1)\nabla^2 \phi - 2\beta \frac{\partial^2 \phi}{\partial z^2}, \\ \tau_{zr} &= \mu \frac{\partial}{\partial r}[(\beta + 1)\nabla^2 \phi - 2\beta \frac{\partial^2 \phi}{\partial z^2}], \\ \tau_{z\theta} &= \frac{\mu}{r}\frac{\partial}{\partial \theta}[(\beta + 1)\nabla^2 \phi - 2\beta \frac{\partial^2 \phi}{\partial z^2}, \\ \tau_{r\theta} &= -2\mu\beta \frac{\partial^3}{\partial r \partial \theta \partial z}\left(\frac{\phi}{r}\right). \end{aligned} \tag{2.2}$$

The Navier equation is satisfied provided ϕ is biharmonic, i.e., $\nabla^2 \nabla^2 \phi = 0$. A suitable solution of the biharmonic equation is of the form

$$\phi = \sum_n \int_0^\infty (Ae^{-kz} + Be^{kz} + Ckze^{-kz} + Dkze^{kz}) J_n(kr) \binom{sin \; n\theta}{cos \; n\theta} dk, \tag{2.3}$$

where n is a positive integer (or zero), A, B, C, D may be functions of n & k and $J_n(kr)$ is the Bessel function of the first kind and of order n. From equations (2.1) to (2.3), we obtain (Garg et al., (1992))

$$u_r = \sum_n \int_0^\infty V(z) J_n'(kr) \begin{pmatrix} sin\ n\theta \\ cos\ n\theta \end{pmatrix}\ k^2 dk,$$

$$u_\theta = \sum_n \int_0^\infty V(z) J_n(kr) \begin{pmatrix} cos\ n\theta \\ -sin\ n\theta \end{pmatrix} \frac{n}{r}\ kdk,$$

$$u_z = \sum_n \int_0^\infty W(z) J_n(kr) \begin{pmatrix} sin\ n\theta \\ cos\ n\theta \end{pmatrix}\ k^2 dk,$$

$$u_{zr} = \sum_n \int_0^\infty S(z) J_n'(kr) \begin{pmatrix} sin\ n\theta \\ cos\ n\theta \end{pmatrix}\ k^3 dk,$$

$$\tau_{z\theta} = \sum_n \int_0^\infty S(z) J_n(kr) \begin{pmatrix} cos\ n\theta \\ -sin\ n\theta \end{pmatrix} \frac{n}{r}\ k^2 dk,$$

$$\tau_{zz} = \sum_n \int_0^\infty N(z) J_n(kr) \begin{pmatrix} sin\ n\theta \\ cos\ n\theta \end{pmatrix}\ k^3 dk, \tag{2.4}$$

where a prime indicates differentiation with respect to the argument. The functions V, W, S, N are given by the matrix relation

$$[G(z)] = [Z(z)][K], \tag{2.5}$$

where

$$[G(z)] = [V, W, S, N]^T, \qquad [K] = [A, B, C, D]^T, \tag{2.6}$$

in which $[...]^T$ is the transpose of the matrix $[...]$. The elements of the 4×4 matrix $Z[z]$ are

$$(11) = -(21) = \beta e^{-kz},$$
$$(12) = (22) = -\beta e^{kz},$$
$$(13) = -\beta(1 - kz)e^{-kz},$$
$$(14) = -\beta(1 + kz)e^{kz},$$
$$(23) = -(2 + \beta kz)e^{-kz},$$
$$(24) = (2 - \beta kz)e^{kz},$$
$$(31) = -(41) = -2\mu\beta e^{-kz},$$
$$(32) = (42) = -2\mu\beta e^{kz},$$
$$(33) = 2\mu(\beta - 1 - \beta kz)e^{-kz},$$
$$(34) = -2\mu(\beta - 1 + \beta kz)e^{kz}$$
$$(43) = 2\mu(1 + \beta kz)e^{-kz},$$
$$(44) = 2\mu(1 - \beta kz)e^{kz}. \tag{2.7}$$

Toroidal Problem

For the toroidal problem the displacement $\mathbf{u}$ can be given by $\mathbf{u} = curl(e_z\psi)$

in which ψ is a scaler potential. The displacement components are

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r}, \qquad u_z = 0.$$

The corresponding stresses are

$$\begin{aligned}
\tau_{rr} &= 2\mu\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right),\\
\tau_{\theta\theta} &= -\tau_{rr},\\
\tau_{zz} &= 0,\\
\tau_{zr} &= \frac{\mu}{r}\frac{\partial^2\psi}{\partial\theta\partial z},\\
\tau_{z\theta} &= -\mu\frac{\partial^2\psi}{\partial r\partial z}\\
\tau_{r\theta} &= \mu\left(\frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right).
\end{aligned}$$

The Navier equation is satisfied if the scaler potential ψ is harmonic, i.e., $\nabla^2\psi = 0$. A suitable solution for ψ is of the form

$$\psi = \sum_n \int_0^\infty (Ee^{-kz} + Fe^{kz})J_n(kr)\begin{pmatrix} sin\ n\theta \\ cos\ n\theta \end{pmatrix} dk,$$

in which E, F may be functions of n and k. This yields

$$\begin{aligned}
u_r &= \sum_n \int_0^\infty U(z)J_n(kr)\begin{pmatrix} cos\ n\theta \\ -sin\ n\theta \end{pmatrix}\frac{n}{r}dk,\\
u_\theta &= -\sum_n \int_0^\infty U(z)J_n'(kr)\begin{pmatrix} sin\ n\theta \\ cos\ n\theta \end{pmatrix} kdk,\\
\tau_{zr} &= \sum_n \int_0^\infty T(z)J_n(kr)\begin{pmatrix} cos\ n\theta \\ -sin\ n\theta \end{pmatrix}\frac{n}{r}kdk,\\
\tau_{z\theta} &= -\sum_n \int_0^\infty T(z)J_n'(kr)\begin{pmatrix} sin\ n\theta \\ cos\ n\theta \end{pmatrix} k^2dk.
\end{aligned} \tag{2.8}$$

The functions U, T are given by the matrix relation

$$|G(z)| = |Z(z)||K|, \tag{2.9}$$

where

$$[G(z)]=[U(z), T(z)]^T, \quad [K] = [E, F]^T, \quad [Z(z)] = \begin{bmatrix} e^{-kz} & e^{kz} \\ -\mu e^{-kz} & \mu e^{kz} \end{bmatrix}. \tag{2.10}$$

2.2. **Startified half-space.** Consider a semi-infinite elastic medium consisting of $p-1$ parallel , homogeneous, isotropic layers lying over a homogeneous, sotropic half-space. The layers are numbered serially, the layer at the top being the layer 1 and the half-space layer p. We place the origin of the cylindrical coordinate system (r, θ, z) at the boundary of the semi-infinite medium and the z-axis is drawn into the medium. The m^{th} layer is of thickness d_m and is bounded by the interfaces $z = z_{m-1}, z_m$, so that $d_m = z_m - z_{m-1}$. Obviously, $z_0 = 0$ and $z_{p-1} = H$, where H is the depth of the last interface.

Using the subscript m for the quantities related to the m^{th} layer the equation (2.5) for the spheroidal problem or the equation (2.9) for the toroidal problem can be written in the form

$$[G_m(z)] = [Z_m(z)][K_m], \quad z_{m-1} \le z \le z_m. \tag{2.11}$$

Using the continuity of the displacement and stress vectors across the interface $z = z_{m-1}$, we obtain (Singh (1970))

$$[G_{m-1}(z_{m-1})] = [G_m(z_{m-1})] = [a_m][G_m(z_m)], \tag{2.12}$$

where the transfer matrix $[a_m]$ is given by

$$[a_m] = [Z_m(z_{m-1})][Z_m(z_m)]^{-1} = [Z_m(-d_m)][Z_m(0)]^{-1}. \tag{2.13}$$

The elements of the transfer matrix $[a_m]$, also known as *the layer matrix*, are given in the Appendix. The Layer matrix $[a_m]$ connects the elastic field at the bottom of the m^{th} layer with the elastic field at the top of this layer. A repeated use of the relation (2.12) yields

$$[G_1(0)] = [M][K_p], \tag{2.14}$$

where

$$[M] = [a_1][a_2] \cdots [a_{p-1}][Z_p(H)]. \tag{2.15}$$

Similarly, the field at any point of the m^{th} layer is given by

$$[G_m(z)] = [N_m][K_p], \; z_{m-1} \le z \le z_m, \tag{2.16}$$

where

$$[N_m] = [a_m(z_m - z)][a_{m+1}], \cdots [a_{p-1}][Z_p(H)], \tag{2.17}$$

and $[a_m(z_m - z)]$ is obtained from $[a_m]$ on replacing d_m by $z_m - z$.

Spheroidal problem

From equation (2.5) and (2.7) we note that the finiteness condition when applied to the field in the underlying homogeneous half-space (medium $p, z \ge H$), implies

$$[K_p] = [A_p, 0, C_p, 0]^T.$$

Equation (2.14) yields

$$S(0) = M_{31}A_p + M_{33}C_p, \quad N(0) = M_{41}A_p + M_{43}C_p. \tag{2.18}$$

When the surface tractions at the boundary $z = 0$ are prescribed, $S(0), N(0)$ can be determined from equation (3.3). On solving equation (2.18) we find

$$A_p = \Omega_1^{-1}[S(0)M_{43} - N(0)M_{33}],$$
$$C_p = \Omega_1^{-1}[N(0)'M_{31} - S(0)M_{41}], \tag{2.19}$$

where

$$\Omega_1 = (M_{31}M_{43} - M_{33}M_{41}).$$

This solves the spheroidal problem. Several examples of surface loading have been given by Garg *et al.* (1992).

For normal loading, the boundary conditions are of the form

$$\tau_{zr} = \tau_{z\theta} = 0, \quad \tau_{zz} = -f(r) \text{ at } z = 0. \tag{2.20}$$

Due to axial symmentry, $n = 0$. Let the Hankel transform of $f(r)$ be $\overline{f}(k)$ so that

$$\overline{f}(k) = \int_0^\infty f(r)J_0(kr)rdr, \tag{2.21}$$

$$f(r) = \int_0^\infty \overline{f}(k)J_0(kr)kdk. \tag{2.22}$$

From equations (3.3), (2.19), (2.20) and (2.22), we find

$$S(0) = 0, \qquad N(0) = -\overline{f}(k)/k^2,$$

$$A_p = \frac{\overline{f}(k)}{k^2\Omega_1}M_{33}, \qquad C_p = -\frac{\overline{f}(k)}{k^2\Omega_1}M_{31}.$$

For normal disk loading,

$$f(r) = \frac{P}{\pi a^2}H(a - r),$$

in which P denotes the total normal force acting over a circular area of radius a with center at the origin. Then

$$\overline{f}(k) = \frac{P}{\pi a_k}J_1(ak)$$

and the deformation field at aby point of the m^{th} layer is

$$u_r = \frac{P}{\pi a}\int_0^\infty (N_{11}M_{33} - N_{13}M_{31})J_0'(kr)J_1(ka)\frac{1}{\Omega_1 k}dk,$$
$$u_z = \frac{P}{\pi a}\int_0^\infty (N_{21}M_{33} - N_{23}M_{31})J_0(kr)J_1(ka))\frac{1}{\Omega_1 k}dk. \tag{2.23}$$

When $a \to 0$, the disk reduces to a concentrated force of magnitude P acting at the origin in the positive $z-$direction. Using the limit $\lim_{a\to 0}[\frac{J_1(ka)}{ka}] = \frac{1}{2}$, equation(2.23) gives the following expressions for the displacements due to a concentrated normal force:

$$u_r = \frac{P}{2\pi}\int_0^\infty (N_{11}M_{33} - N_{13}M_{31})\Omega_1^{-1} J_0'(kr)dk,$$

$$u_z = \frac{P}{2\pi}\int_0^\infty (N_{21}M_{33} - N_{23}M_{31})\Omega_1^{-1} J_0(kr)dk. \tag{2.24}$$

Toroidal Problem

In this case, the finiteness condition implies

$$[K_p] = [E_p, 0]^T.$$

Equation (2.14) shows that for surface loading $T(0)$,

$$E_p = T(0)/M_{21}. \tag{2.25}$$

This solves toroidal problem. For a torque of moment M applied at the origin about the $z-$axis (Garg *et al.* (1992))

$$n = 0, \qquad T(0) = -M/4\pi,$$

$$u_0 = -\frac{M}{4\pi}\int_0^\infty \frac{N_{11}}{M_{21}} J_1(kr)kdk, \tag{2.26}$$

selecting the lower solution in equation (2.8).

3. Two-Dimensional Surface Loading

Following Singh and Garg (1985), Garg and Singh (1985) formulated the problem of the static deformation of a stratified half-space by two dimensional surface loads in terms of 4×4 and 2×2 matrices, for plane strain and antiplane strain cases, respectively.

3.1. **Bacic equations.** Consider a two-dimensional approximation in which the displacement components (u_x, u_y, u_z) are independent of the cartesian coordinate x, so that $\frac{\partial}{\partial x} \equiv 0$.
Therefore the plane strain problem $(u_x = 0)$ and the antiplane strain problem $(u_y = y_z = 0)$ can be considered separately.

Plane Strain

The plane strain problem can be solved in terms of the Airy's stress function Φ such that

$$\tau_{yy} = \frac{\partial^2\Phi}{\partial z^2}, \quad \tau_{yz} = \frac{\partial^2\Phi}{\partial y\partial z}, \quad \tau_{zz} = \frac{\partial^2\Phi}{\partial y^2}, \quad \nabla^2\nabla^2\Phi = 0.$$

For plane strain, a suitable solution of the biharmonic equation is of the form (Little (1973))

$$\Phi = \int_0^\infty (Ae^{-kz} + Be^{kz} + Ckze^{-kz} + Dkze^{kz})\begin{pmatrix} sin\ ky \\ cos\ ky \end{pmatrix} dk,$$

in which A, B, C, D may be functions of k. The expressions for the stresses can be obtained by direct differentiation. The expressions for the displacements can then be obtained by integrating the stress-displacement relations:

$$\tau_{yy} = \frac{\partial^2\Phi}{\partial z^2} = (\lambda+2\mu)\frac{\partial u_y}{\partial y} + \lambda\frac{\partial u_z}{\partial z},$$
$$\tau_{zz} = \frac{\partial^2\Phi}{\partial y^2} = \lambda\frac{\partial u_y}{\partial y} + (\lambda+2\mu)\frac{\partial u_x}{\partial z},$$
$$\tau_{yz} = -\frac{\partial^2\Phi}{\partial y\partial z} = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right).$$

Following the indicated proceedure, we find that for a plane-strain problem, we may take (Singh and Garg (1985))

$$u_y = \int_0^\infty V\begin{pmatrix} cos\ ky \\ -sin\ ky \end{pmatrix} kdk,$$
$$u_z = \int_0^\infty W\begin{pmatrix} sin\ ky \\ cos\ ky \end{pmatrix} kdk,$$
$$\tau_{yz} = \int_0^\infty S\begin{pmatrix} cos\ ky \\ -sin\ ky \end{pmatrix} k^2dk,$$
$$\tau_{zz} = \int_0^\infty N\begin{pmatrix} sin\ ky \\ cos\ ky \end{pmatrix} k^2dk. \tag{3.1}$$

The functions V, W, S, N are given by the matrix relation

$$[G(z)] = [Z(z)][K], \tag{3.2}$$

where

$$[G(z)] = [V, W, S, N]^T, \qquad [K] = [A, B, C, D]^T$$

The elements of the 4×4 matrix $[Z(z)]$ are given by

$$(11) = -(21) = -(1/2\mu)e^{-kz},$$
$$(12) = (22) = -(1/2\mu)e^{kz},$$
$$(13) = (1/2\mu)(1/\alpha - kz)e^{-kz},$$
$$(14) = -(1/2\mu)(1/\alpha + kz)e^{kz},$$
$$(23) = (1/2\mu)(1/\alpha - 1 + kz)e^{-kz},$$
$$(24) = (1/2\mu)(1/\alpha - 1 - kz)e^{kz},$$
$$(31) = -(41) = e^{-kz},$$
$$(32) = (42) = -e^{kz},$$
$$(33) = -(1 - kz)e^{-kz},$$

$$
\begin{aligned}
(34) &= -(1+kz)e^{kz},\\
(43) &= -kze^{-kz},\\
(44) &= -kze^{kz}, \qquad (3.3)
\end{aligned}
$$

where

$$\alpha = \frac{\lambda+\mu}{\lambda+2\mu} = \frac{\beta}{1+\beta} = \frac{1}{2(1-\sigma)}. \qquad (3.4)$$

Antiplane Strain

For antiplane strain

$$u_x = \int_0^\infty U\binom{sin\ ky}{cos\ ky}dk, \qquad \tau_{xz} = \int_0^\infty T\binom{sin\ ky}{cos\ ky}kdk, \qquad (3.5)$$

The functions U and T are given by the matrix relation

$$[G(z)] = [Z(z)][K], \qquad (3.6)$$

where

$$[G(z)] = [U(z), T(z)]^T, \qquad [K] = [E, F]^T, \qquad (3.7)$$

in which E, F may be functions of k. The matrix $[Z(z)]$ is identified with the corresponding matrix for the toroidal problem given in (2.10).

3.2. **Stratified half space.** Equations (2.11) to (2.22) and (2.25) for the three dimensional problem are valid for the two dimensional problem as well. It may be noted that even though the matrix $[Z(z)]$ for the plane-strain problem differs from the matrix $[Z(z)]$ for the spheroidal problem, the layer matrix $[a_m]$ for the plane strain problem is identical with the layer matrix $[a_m]$ for the spheroidal problem given in the Appendix. This is as it should be because the layer matrix depends only on the thickness and the elastic properties of the particular layer and has nothing to do with the particular problem at hand. Several problems have been solved by Garg and Singh (1985). Let a normal line load P per unit length be applied at the origin to the boundary surface $z = 0$ in the positive z direction. Then the boundary conditions t $z = 0$ are

$$\tau_{yz} = 0, \ \ \tau_{zz} = -P\delta(y),$$

where $\delta(y)$ denotesthe Dirac delta function. Using the representation

$$\delta(y) = \frac{1}{\pi}\int_0^\infty coskydy,$$

equation (3.1) yields

$$S(0) = 0, \quad N(0) = -P/\pi k^2, \qquad (3.8)$$

and the lower solution in equation (3.1) is to be chosen. This yields

$$u_y(z) = -\frac{P}{\pi}\int_0^\infty (N_{11}M_{33} - N_{13}M_{31})\Omega_1^{-1} sin\ ky \frac{dk}{k},$$
$$u_z(z) = -\frac{P}{\pi}\int_0^\infty (N_{21}M_{33} - N_{23}M_{31})\Omega_1^{-1} cos\ ky \frac{dk}{k}, \qquad (3.9)$$

where Ω_1 is defined by equation (2.19). Similarly, for a shear line load Q per unit length in the positive y-direction

$$u_y(z) = -\frac{Q}{\pi}\int_0^\infty (N_{11}M_{43} - N_{13}M_{41})\Omega_1^{-1} cos\ ky \frac{dk}{k},$$
$$u_z(z) = -\frac{Q}{\pi}\int_0^\infty (N_{21}M_{43} - N_{23}M_{41})\Omega_1^{-1} sin\ ky \frac{dk}{k}. \qquad (3.10)$$

4. Burried Sources

The most convenient representation of the source in the matrix formulation of the problem of a point or line source in a stratified half-space is in terms of the discontinuities at the source level in the displacements and stresses in the transform domain due to the source in an infinite medium.

Let a point or line source be situated on the z-axis at a depth h below the free surface of a stratified semi-infinte medium. Let the source level be designated as layer s with boundaries $z = z_{s-1}, z_s$. We divide the source levyer in to two sub-layers, s_1 and s_2, of identical properties. The first sub-layer is bounded by the planes $z = z_{s-1}, h$ and the second layer by the planes $z = h, z_s$. Because of the presence of the source, the functions: V, W, S, N for the speroidal and plane strain problems and the functions U, T for the toroidal and antiplane strain problems may be discontinuous across $z = h$. We assume

$$[G_s^+(h)] - [G_s^-(h)] = [D], \qquad (4.1)$$

where the superscript + and - correspond, respectively, to $z > h$ and $z < h$. For a given source $[D]$ is known. Following Singh (1970), it follows from equations (2.11), (2.12) and (4.1) that

$$[G_1(0)] = [M][K_p] - [F], \qquad (4.2)$$

where M is defined by equation (2.15) and

$$[F] = [a_1][a_2]\cdots[a_{s-1}][a_s][D]. \qquad (4.3)$$

Applying the traction free boundary conditions at the free surface, we have

$$[G_1(0)] = [V(0), w(0), 0, 0]^T. \qquad (4.4)$$

for the plane strain (or spheroidal) problem. Similarly, the finiteness condition yields

$$[K_p] = [A_p, 0, C_p, 0]^T. \tag{4.5}$$

From equations (4.2), (4.4 (4.5), we find

$$[V(0), W(0), 0, 0]^T = [M][A_p, 0, C_p, 0]^T - [F]. \tag{4.6}$$

The four equations in (4.6) can be solved for the four unknown $V(0), W(0)$, A_p, C_p. This gives the formal solution of the plane strain (or spheroidal) problem. Similarly, for the antiplane strain (or the toroidal) problem, we find

$$U(0) = (M_{11}F_2 - M_{21}F_1)/M_{21}. \tag{4.7}$$

The source matrix $[D]$ can be determined by expressing the field due to the point (line) souce in an infinite medium as Hankel (Fourier sine / cosine) transform with the help of integral transform tables. For a horizontal line force of unit magnitude acting in the y-direction, we have (Singh and Garg (1986))

$$[D] = [0, 0, -1/\pi k^2, 0]^T \tag{4.8}$$

and the upper solution in equation (3.1) is to be used. Similarly, for a vertical line force of unit magnitude acting in the z-direction

$$[D] = [0, 0, 0, -1/\pi k^2]^T. \tag{4.9}$$

and the lower solution is to be chosen.

For the three-dimensional problem of vertical point force of unit magnitude acting in the positive z-direction (Garg *et al.* (1992))

$$[D] = [0, 0, 0, -1/2\pi k^2]^T, \quad n = 0 \tag{4.10}$$

and the lower solution in equation (2.3) is to be used.

5. Discussion

In sections 2-4, we have described a method of finding the solution in the transform domain. The solution in the physical domain can be obtained by numerical integration. One of the major difficulties with these integral expressions is evaluation of kernel functions. These kernal functions involve matrix products of highly disproportionate terms. To circumvent this problem, the layer matrix $[a_m]$ for the spheroidal (or plane strain) problem given in the Appendix is decomposed in to sum of four matrices (Jovanovich *et al.* (1974))

$$[a_m] = exp(-kd)[A_m^1] + kexp(-kd)[A_m^2] + exp(kd)[A_m^3] + kexp(kd)[A_m^4], \tag{5.1}$$

where the matrices $[A_m^n]$ contain only constant coefficients. The matrix $[Z_p(H)]$ is treated similarly. In this way, the $[M]$ matrix of equation (2.15) may be expressed as a sum of products of constant matrices in which all the exponentials and powers of k are carried outside the appropriate matrix product. The final form of $[M]$ matrix is

$$[M] = \sum_{q=1}^{Q} \sum_{s=0}^{p} k^s exp(-k\beta_q)[e_{sq}],$$

where $Q = 2^p$ and β_q are the exponential arguments from the product of the 'factored-out' exponentials. $[e_{sq}]$ are the coefficient matrices arising from the matrix product of the $[A_m^n]$ matrices associated with the s-th power of k and q-th exponential argument.

The occurrence of the polynomial-exponential series in the denominator of the kernel functions renders exact analytical integration impossible, except in the case of uniform half-space or two half-spaces in contact. In general, the denominator D of the kernel function may be expressed in the form

$$D = \sum_{q} \sum_{s} b_{sq} k^s exp(-kC_q). \tag{5.2}$$

Now D^{-1} is approximated by a truncated binomial series expansion. The method of least squares is employed to fit a sum of exponential-polynomial terms to the remainder series. By multiplying the exact numerator series by the approximate inverse denominator series, the kernel function becomes a finite sum of exponentials multiplied by polynomials. The elastic field integrals thus takes the form

$$\sum_{s,q} a_{sq} \int_0^\infty k^s exp(-k\beta_q) J_m(kr) dk. \tag{5.3}$$

The integral occurring in (5.3) is the standard Lipschitz-Hankel integral (Erdélyi (1954)).

Therefore, the proceedure described enables us to determine the elastic field by exact analytical integration of the approximated integrals.

Appendix

The elements of the layer matrix $[a_m]$ are (omitting the subscript m of α_m, μ_m and d_m)

$$(11) = (33) = ch\ kd + (\alpha kd)\ sh\ kd,$$

$$(12) = -(43) = \alpha\ kd\ ch\ kd + (1-\alpha)\ sh\ kd,$$

$$(13) = (-1/2\mu)\ [\alpha\ kd\ ch\ kd + (2-\alpha)\ sh\ kd],$$

$$(14) = -(23) = (-\alpha/2\mu)\ kd\ sh\ kd,$$

$$(21) = -(34) = -\alpha\ kd\ ch\ kd + (1-\alpha)\ sh\ kd,$$

$$(22) = (44) = ch\ kd - \alpha\ kd\ sh\ kd$$

$$(24) = (1/2\mu)[\alpha\ kd\ ch\ kd - (2-\alpha)\ sh\ kd],$$

$$(31) = -2\alpha\mu[kd\ ch\ kd + sh\ kd],$$

$$(32) = -(41) = -2\alpha\mu\ kd\ sh\ kd,$$

$$(42) = 2\alpha\mu\ [kd\ ch\ kd - sh\ kd].$$

Toroidal or antiplane strain problem

$$(11) = (22) = ch\ kd,$$

$$(12) = (-1/\mu)\ sh\ kd,$$

$$(21) = -\mu\ sh\ kd.$$

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LINEAR ALGEBRA TO QUANTUM COHOMOLOGY: THE STORY OF ALFRED HORN'S INEQUALITIES*+

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> We want first an overview of the aim and of the road; we want to understand the *idea* of the proof, the deeper context. A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.
>
> *Herman Weyl*

A long-standing problem in linear algebra — Alfred Horn's conjecture on eigenvalues of sums of Hermitian matrices — has been solved recently. The solution appeared in two papers, one by Alexander Klyachko [20] in 1998 and the other by Allen Knutson and Terene Tao [23] in 1999. This has been followed by a flurry of activity that has brought to the mathematical centerstage what for many years had been somewhat of a side-show. The aim of this article is to describe the problem, its origins, some of the early work on it, and some ideas that have gone in to its solution.

A substantial part of this article should be accessible to anyone who has had a second course on linear algebra. The reader who wants to know more

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will find it rewarding to read the comprehensive and advanced account [11] by William Fulton.

1. Linearity, Quasilinearity and convexity

The principal characters in our story are $n \times n$ Hermitian matrices A and B, their sum $C = A + B$, and the eigenvalues of A, B and C enumerated as $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, respectively. Sometimes we would like to emphasize the dependence of the eigenvalues on the matrix. We then use the notation $\lambda_j^{\downarrow}(A)$ for the jth eigenvalue of A when the eigenvalues are arranged in a (weakly) decreasing order. Thus $\alpha_j = \lambda_j^{\downarrow}(A)$. The n-tuple of eigenvalues of A as a whole is denoted by α or $\lambda^{\downarrow}(A)$.

The story begins with the simple question:

What are the relationships between α, β and γ ?

Now, the eigenvalues are *not* linear functions of A and no simple relation between α, β and γ is apparent, except one. The *trace* of A, denoted by tr A is the sum of the diagonal entries of A and also of the eigenvalues of A. So, tr C = tr A + tr B and hence

$$\sum_{j=1}^{n} \gamma_j = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_j. \tag{1}$$

We can think of A as a linear operator on the Complex Euclidean space $\mathbb{C}^n$ equipped with its usual inner product $\langle x, y \rangle$, written also as x^*y and the associated norm $\|x\| = (x^*x)^{1/2}$. The Spectral Theorem tells us that every Hermitian operator A can be diagonalized in some orthonormal basis; or equivalently, there exists a unitary matrix U such that UAU^*=diag $(\alpha_1, \cdots, \alpha_n)$, a diagonal matrix with diagonal entries $\alpha_1, \cdots, \alpha_n$. If u_j are the orthonormal eigenvectors corresponding to its eigenvalues α_j, we write $A = \sum \alpha_j u_j u_j^*$, and call this the *spectral resolution* of A. Using this, it is easy to see that the set $\{\langle x, Ax \rangle : \|x\| = 1\}$ (called the *numerical range* of A) is equal to the interval $[\alpha_n, \alpha_1]$. In particular, we have

$$\alpha_1 = \max_{\|x\|=1} \langle x, Ax \rangle, \tag{2}$$

$$\alpha_n = \min_{\|x\|=1} \langle x, Ax \rangle. \tag{3}$$

For each fixed vactor x, the quantity $\langle x, Ax \rangle$ depends linearly on A. Equations (2) and (3) express α_1, α_n as a maximum or minimum over these linear functions. Such expressions are called *quasilinear*. Very often, they lead to interesting inequalities. Thus, from (2) and (3) we have

$$\gamma_1 \leq \alpha_1 + \beta_1, \tag{4}$$

$$\gamma_n \geq \alpha_n + \beta_n. \tag{5}$$

In this way, we begin to get *linear inequalities* between α, β and γ. There is another way of looking at (4). The set of $n \times n$ Hermitian matrices is a real vector space. The inequality (4) says $\lambda_1^{\downarrow}(A)$ is a *convex* function on this space; the inequality (5) says that $\lambda_n^{\downarrow}(A)$ is *concave.*

The inequalities (4) and (5) are not independent. Note that the eigenvalues of $-A$ are the same as the negatives of the eigenvalues of A. But taking negative reverses the order; so for $1 \leq j \leq n$,

$$\lambda_j^{\downarrow}(-A) = -\lambda_{n-j+1}^{\downarrow}(A) = -\lambda_j^{\uparrow}(A), \tag{6}$$

where the notation $\lambda_j^{\uparrow}(A)$ indicates that we are now enumerating the eigenvalues of A in increasing order. Using this observation we can say that (2) and (3) are equivalent, as are (4) and (5). Many of the inequalities stated below lead to complementary inequalities by this argument.

2. The minimax principle and Weyl's inequalities

The relations (2) and (3) are subsumed in a variational principle called the *minimax principle.* It says that for all $1 \leq j \leq n$

$$\alpha_j = \max_{\substack{V \subset \mathbb{C}^n \\ dim V = j}} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle = \min_{\substack{V \subset \mathbb{C}^n \\ dim V = n-j+1}} \max_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle. \tag{7}$$

Here dim V stands for the dimension of a linear space V contained in $\mathbb{C}^n$. This principle was first mentioned in a 1905 paper of E. Fischer. Its proof is easy. Use the spectral resolution $A = \sum \alpha_j u_j u_j^*$. Let W be the space spanned by the vectors $u_j, \cdots, u_n$. Then dim $W = n - j + 1$. So, if V is any j-dimensional subspace of $\mathbb{C}^n$, then V and W have a nonzero intersection. If x is a unit vector in this intersection, then $\langle x, Ax \rangle$ lies in the interval $[\alpha_n, \alpha_j]$. This shows that

$$\min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle \leq \alpha_j.$$

If we choose V to be the subspace spanned by $u_1, \cdots, u_j$, we obtain equality here.This proves the first relation in (7). The second has a very similar proof.

This principle has several very interesting consequences. Hermitian matrices can be ordered in a natural way. we say that $A \leq B$ if $\langle x, Ax \rangle \leq \langle x, Bx \rangle$ for all x. One sees at once from (7) that if $A \leq B$, then $\lambda_j^{\downarrow}(A) \leq \lambda_j^{\downarrow}(B)$ for all j. This is called *Weyl's monotonicity principle.* (The Applied mathematics classic by Courant and Hilbert [8]) is full of applications of eigenvalue problems in physics. The Weyl monotonicity principle has the following physical interpretation: *If the system stiffens, the pitch of the fundamental tone and every overtone increases* [8, p. 286, Theorem IV]. This indeed is the experience of anyone tuning the wires of a musical instrument.)

Weyl's monotonocity principle, and several other relations between eigenvalues of A, B and $A+B$ were derived by H. Weyl in a famous paper in 1912 [33]. Particularly important for our story is the family of equations

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \quad \text{for} \quad i+j-1 \leq n. \tag{8}$$

These can be proved using the same idea as the one that gave us the minimax principle. Let A, B and $A+B$ have spectral resolutions $A = \sum \alpha_j u_j u_j^*$, $B = \sum \beta_j v_j v_j^*$, $A + B = \sum \gamma_j w_j w_j^*$. Consider the three subspaces spanned by $\{u_i, \cdots, u_n\}, \{v_j, \cdots, v_n\}$ and $\{w_1, \cdots, w_k\}$. These spaces have dimensions $n-i+1, n-j+1$ and k respectively. If $k = i+j-1$, these numbers add up to $2n+1$. This implies that these three subspaces of $\mathbb{C}^n$ have a nontrivial intersection. Let x be a unit vector in this intersection. Then $\langle x, Ax\rangle$ is in the interval $[\alpha_n, \alpha_i]$, $\langle x, Bx\rangle$ in $[\beta_n, \beta_j]$ and $\langle x, (A+B)x\rangle$ in $[\gamma_k, \gamma_1]$. Hence

$$\gamma_k \leq \langle x, (A+B)x\rangle = \langle x, Ax\rangle + \langle x, Bx\rangle \leq \alpha_i + \beta_j.$$

This proves (8).

Note that the inequality (4) is a very special case of (8). Another special consequence is the inequality

$$\alpha_i + \beta_n \leq \gamma_i \leq \alpha_i + \beta_1 \quad \text{for} \quad 1 \leq i \leq n. \tag{9}$$

The second inequality is derived from (8) simply by putting $j = 1$; the first by the sort of argument indicated at the end of Section 1.

As an aside, let us mention the interest such results have for numerical analysts. For any operator A on $\mathbb{C}^n$ define

$$\|A\| = \sup_{\|x\|=1} \|Ax\|. \tag{10}$$

If A is Hermitian then it is easy to see that

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle| \;=\; \max(|\alpha_1|, |\alpha_n|). \tag{11}$$

Using this, one can see from (9) that

$$\alpha_i - \|B\| \leq \gamma_i \leq \alpha_i + \|B\|. \tag{12}$$

By a change of labels (replace B by $B-A$) this leads to the *Weyl perturbation theorem*

$$\max_j |\alpha_j - \beta_j| \leq \|A - B\|. \tag{13}$$

In numerical analysis one often replaces a matrix A by a nearby matrix B whose eigenvalues might be easier to calculate. Inequalities like (13) then provide useful information on the error caused by such approximations.

Some of the inequalities in the following sections provide finer information of interest to the numerical analysts. We do not discuss this further in this article; see [5].

Convexity properties of eigenvalues and intersection properties of eigenspaces are closely related, as we have already seen. This is the *leitmotif* of our story.

3. The case $n = 2$

When $n = 2$, the statement (8) contains three inequalities

$$\gamma_1 \leq \alpha_1 + \beta_1, \qquad \gamma_2 \leq \alpha_1 + \beta_2, \qquad \gamma_2 \leq \alpha_2 + \beta_1. \tag{14}$$

It turns out that, together with the trace equality (1), these three inequalities are sufficient to characterise the possible eigenvalues of A, B and C; i.e., if three pair of real numbers $\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}, \{\gamma_1, \gamma_2\}$, each ordered decreasingly ($\alpha_1 \geq \alpha_2$, etc.), satisfy the relations (1) and (14), then there exists 2×2 Hermitian matrices A and B such that these pairs are the eigenvalues of A, B and $A + B$.

Let us indicate why this is so. Choose two pairs α, β, say

$$\alpha_1 = 4, \;\; \alpha_2 = 1, \;\; \beta_1 = 3, \;\; \beta_2 = -2.$$

What are the γ that satisfy (1) and (14)? The condition (1) says

$$\gamma_1 + \gamma_2 = 6.$$

This gives a line in the plane $\mathbb{R}^2$. The restriction $\gamma_1 \geq \gamma_2$ gives half of this line — its part in the half plane $\gamma_1 \geq \gamma_2$. One of the three inequalities in (14) is redundant; the other two are

$$\gamma_1 \leq 7, \quad \gamma_2 \leq 2.$$

So, the set of γ that satisfy (1) and (14) constitute the line segment with end points $(4, 2)$ and $(7, -1)$ (see FIGURE 1 on the next page). We want to show that each point on this segment corresponds to the two eigenvalues of a Hermitian matrix $C = A + B$, where A has eigenvalues $(4, 1)$ and B has eigenvalues $(3, -2)$.

Start with the diagonal matrices

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

Let U_θ be the 2×2 rotation matrix

$$U_\theta = \begin{bmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{bmatrix}$$

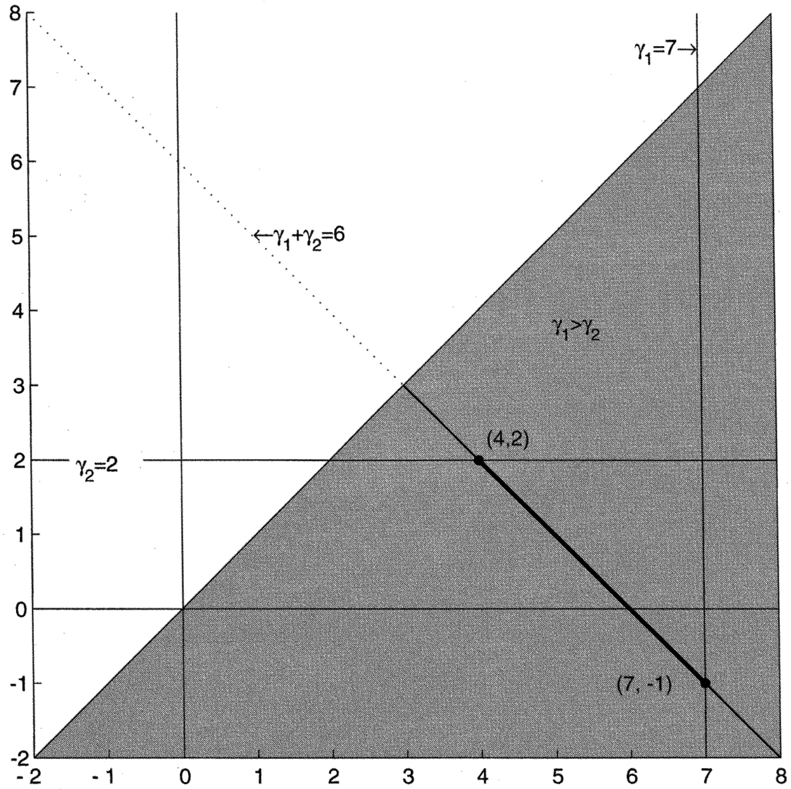

FIGURE 1. The line segment given by Weyl's inequalities

and let

$$B_\theta = U_\theta B_0 U_\theta^*, \quad C_\theta = A + B_\theta.$$

This gives a family of Hermitian matrices parametrized by the real number θ. Note that

$$C_0 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix},$$

$$C_{\pi/2} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus the two end points of our line segment correspond to $(\lambda_1^\downarrow(C_\theta), \lambda_2^\downarrow(C_\theta))$ for the values $\theta = 0$ and $\theta = \pi/2$. It is a fact that $\lambda_j^\downarrow(C_\theta)$ is a continuous function of θ; see [5, p. 154].

Condition (1) tells us that the eigenvalues of C_θ must lie on the line $\gamma_1 + \gamma_2 = 6$. So, by the intermediate value theorem each point of the line segment between $(7, -1)$and $(4, 2)$ must be pair of eigenvalues of C_θ for some $0 \le \theta \le \pi/2$.

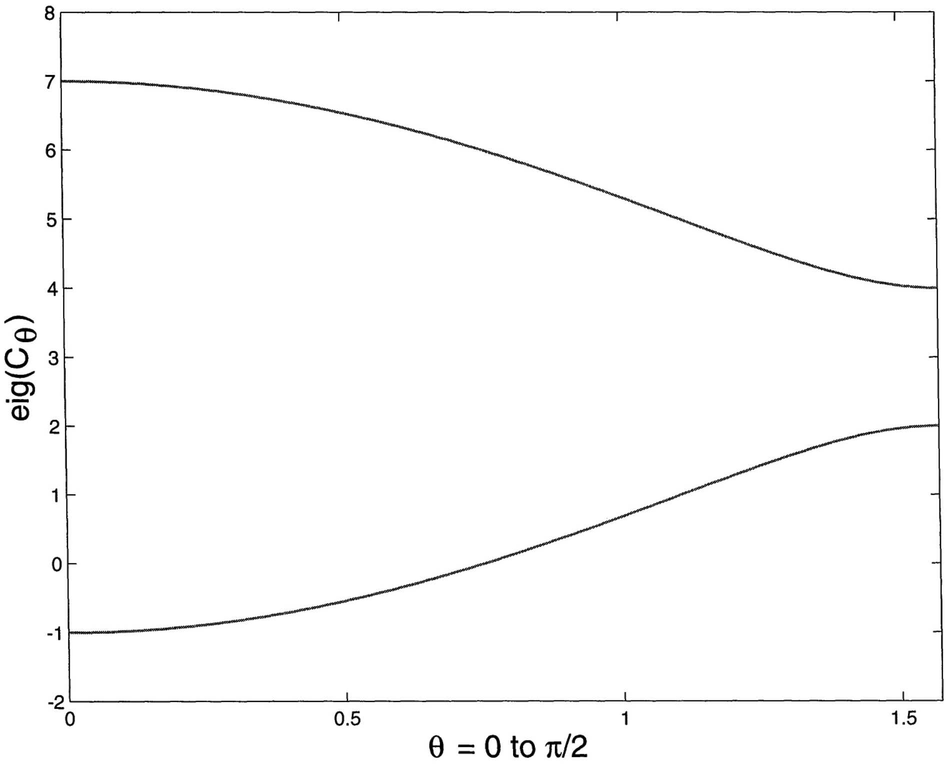

FIGURE 2. The two eigenvalues of family C_θ

FIGURE 2 shows a plot of the two eigenvalues $\lambda_1^\downarrow(C_\theta)$ and $\lambda_2^\downarrow(C_\theta), 0 \le \theta \le \pi/2$. The two curves are symmetric about the line $y = 3$ because of the trace condition (1).

Some comments are in order here. We chose numerical values for α, β for concrete illustrations. The same argument would work for any pairs. The matrices A and B we got are not just Hermitian; they are real symmetric. The condition (1) brought us down from the plane onto a line, the condition $\gamma_1 \ge \gamma_2$ to a part of this line and the inequalities (14) to a closed interval on it. We have proved the following theorem.

Theorem 1. *Let A, B be two real symmetric 2×2 matrices with eigenvalues $\alpha_1 \ge \alpha_2$ and $\beta_1 \ge \beta_2$, respectively. Then the set of (decreasing ordered) eigenvalues of the family $A + UBU^*$, where U varies over rotation matrices, is a convex set (actually a line segment). This convex set is described by Weyl's inequalities (14).*

This is also a good opportunity to comment on two features of Figure-2. Neither the smoothness of the two curves nor their avoidance of crossing each other is fortuitous. see the book [27, p. 113] (and the picture on its cover) for a discussion and explanation of these phenomena.

4. Majorisation

Before proceeding further, it would be helpful to introduce the concept of majorisation of vectors. The theorems of Ky Fan, Lidskii-Wielandt and Schur are best understood in the language of majorisation.

Let $x=(x_1, x_2, \cdots, x_n)$ be an element of $\mathbb{R}^n$. We write $x^{\downarrow}=(x_1^{\downarrow}, x_2^{\downarrow}, \cdots, x_n^{\downarrow})$ for the vector whose coordinates are obtained by rearranging the x_j in decreasing order $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$. If

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow} \quad \text{for} \quad 1 \leq k \leq n, \tag{15}$$

then we say x is *weakly majorised* by y, and write $x \prec_w y$. If, in addition to the inequalities (15), we have

$$\sum_{j=1}^{n} x_j^{\downarrow} = \sum_{j=1}^{n} y_j^{\downarrow} \tag{16}$$

then we say x is *majorised* by y and we write $x \prec y$.

As an example, let $p = (p_1, p_2, \cdots, p_n)$ be any probability vector; i.e., $p_j \geq 0$ and $\sum p_j = 1$. Then

$$(\frac{1}{n}, \cdots, \frac{1}{n}) \prec (p_1, p_2, \cdots, p_n) \prec (1, 0, \cdots, 0).$$

The notion of majorization is important. A good part of the classic [18] and all of the more recent book [29] are concerned with majorization. See also [5].

Among the several characterisations of majorisation the following two are especially interesting; see [5, p.33].

1. Let σ be a permutation on n symbols. Given $y \in \mathbb{R}^n$, let $y_\sigma = (y_{\sigma(1)}, \cdots, y_{\sigma(n)})$. Then $x \prec y$ if and only if x is in convex hull of the $n!$ points y_σ.

2. $x \prec y$ if and only if $x = Sy$ for a doubly stochastic matrix S.

Recall that a matrix $S = [s_{ij}]$ is *doubly stochastic* if $s_{ij} \geq 0$, $\sum_j s_{ij} = 1$ for all i, and $\sum_i s_{ij} = 1$ for all j.

Let us write $x^{\uparrow} = (x_1^{\uparrow}, \cdots, x_n^{\uparrow})$ for the vector whose corodinates are obtained by rearranging x_j in increasing order: $x_1^{\uparrow} \leq \cdots \leq x_n^{\uparrow}$. Note that $x_j^{\uparrow} = x_{n-j+1}^{\downarrow}$. Then x is majorised by y if and only if

$$\sum_{j=1}^{k} x_j^{\uparrow} \geq \sum_{j=1}^{k} y_j^{\uparrow}, \quad 1 \leq k \leq n \tag{17}$$

and the equality (16) holds.

One of the basic theorems about majorisation says that for any x, y in $\mathbb{R}^n$

$$x^{\downarrow} + y^{\uparrow} \prec x + y \prec x^{\downarrow} + y^{\downarrow}; \tag{18}$$

see [16, p. 49]. This relation describes the effect of rearrangement on addition of vectors. Some of the inequalities in the following sections have this form; the vectors involved are n-tuples of eigenvalues of Hermitian matrices.

5. The theorems of Schur and Fan

Return now to the Hermitian matrix A with eigenvalues α. Let $d = (a_{11}, \cdots, a_{nn})$ be the vector whose coordinates are the diagonal entries of A. Since $a_{jj} = \langle e_j, Ae_j \rangle$, the inequality

$$d_1^{\downarrow} \leq \alpha_1 \tag{19}$$

follows from (2). A famous theorem of Schur (1923), closely related to our main story, extends this inequality. This theorem says that we have the majorisation

$$d \prec \alpha. \tag{20}$$

Here is an easy proof. By the spectral theorem, there exists an unitary matrix U such that $A = UDU^*$, where $D = \text{diag}\ (\alpha_1, \cdots, \alpha_n)$. From this one sees that

$$a_{ii} = \sum_{j=1}^{n} |u_{ij}|^2 \alpha_j, \quad 1 \leq i \leq n.$$

This can be rewritten as $d = S_\alpha$, where S is the matrix with entries $s_{ij} = |u_{ij}|^2$. This matrix is doubly stochastic since U is unitary. Hence, by one of the characterisations of Section 4, we have the majorisation (20).

The eigenvalues of A do not change under a change of orthonormal basis. So, from the relation (20), we get the following extremal representation called *Fan's maximum principle*:

$$\sum_{j=1}^{k} a_j = \max_{orthonormal\ \{x_j\}} \sum_{j=1}^{k} \langle x_j, Ax_j \rangle, \quad 1 \leq k \leq n. \tag{21}$$

Here the maximum is taken over all orthonormal k-tuples $x_1, \cdots, x_k$. The summands on the right hand side of (21) are diagonal entries of a matrix representation of A. So, their sum is always less than or equal to $\sum_{j=1}^{k} \alpha_j$ by (20). For the special choice when x_j are eigenvectors of A with $Ax_j = \alpha_j x_j$, we have equality here.

When $k = 1$, (21) reduces to (2), and when $k = n$ both sides are equal to trA. This expression gives a quasilinear representation of the sum $\sum \alpha_j$. Among other things it tells us that for each k between 1 and n, $\sum_{j=1}^{k} \lambda_j^{\downarrow}(A)$ is a convex function of A. Thus each $\lambda_j^{\downarrow}(A)$ is a difference of two convex functions. Generalising (4) we now have inequalities

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \alpha_j + \sum_{j=1}^{k} \beta_j, \quad 1 \leq k \leq n, \tag{22}$$

proved by Ky Fan in 1949. Again, note that when $k = 1$, the inequality (22) reduces to (4) and when $k = n$, this is just the equality (1). In terms of majorisation we can express the family of inequalities (22) as

$$\lambda(A + B) \prec \lambda^{\downarrow}(A) + \lambda^{\downarrow}(B). \tag{23}$$

6. Inequalities of Lidskii and Wielandt

The next event of our story is quite dramatic. In 1950, V. B. Lidskii announced the following result: the vector γ lies in the convex hull of the $n!$ points $\alpha + \beta_\sigma$, where σ runs over all permutations σ of n indices. Lidskii, it would seem, was providing an elementary proof of this theorem that F. A. Berezin and I. M. Gel'fand had discovered in connection with their work on Lie groups. The paper of Berezin and Gel'fand appeared in 1956 and alludes to this. Lidskii's elementary proof may have been clear to the members of Gel'fand's famous Moscow seminar. However, the published version did not give all the details and it coul not be understood by many others. H. Wielandt saw the connection between Lidskii's theorem and Fan's inequalities (22) and provided another proof, very different in method from the one sketched by Lidskii.

Let $1 \leq k \leq n$ and $1 \leq i_1 < \cdots < i_k \leq n$. Then the assertion of Lidskii's theorem is equivalent to saying that for all such choices

$$\sum_{j=1}^{k} \gamma_{i_j} \leq \sum_{j=1}^{k} \alpha_{i_j} + \sum_{j=1}^{k} \beta_j. \tag{24}$$

The equivalence is readily seen using the characterisations of majorisation given in Section 4.

Note that Fan's inequalities (22) are included in (24). To derive these inequalities Wielandt proved a minimax principle that is far more general than (21). We return to this later.

Now seveal proofs of Lidskii's theorem are known. Some of them are fairly easy and are given in [5]. The *easiest* proof, however, is the following one due to C.-K. Lee and R. Mathias [28]:

Fix k and the indices $1 \le i_1 < \cdots < i_k \le n.$ we want to prove that

$$\sum_{j=1}^{k} [\lambda_{i_j}^{\downarrow}(A+B) - \lambda_{i_j}^{\downarrow}(A)] \le \sum_{j=1}^{k} \lambda_j^{\downarrow}(B). \tag{25}$$

We can replace B by $B - \lambda_k^{\downarrow}(B)I$, and thus may assume that $\lambda_k^{\downarrow}(B) = 0$. Let $B = B_+ - B_-$ be the decomposition of B in to its positive and negative parts (if B has the spectral resolution $\sum \beta_j u_j u_j^*$, then $B_+ = \sum \beta_j^+ u_j u_j^*$, where $\beta_j^+ = max(\beta_j, 0)$). Since $B \le B_+$, by Weyl's monotonocity principle $\lambda_{i_j}^{\downarrow}(A+B) \le \lambda_{i_j}^{\downarrow}(A+B_+)$. So, the left hand side of (25) is not bigger than

$$\sum_{j=1}^{k} [\lambda_{i_j}^{\downarrow}(A+B_+) - \lambda_{i_j}^{\downarrow}(A)].$$

By the same principle, this is not bigger than

$$\sum_{j=1}^{n} [\lambda_j^{\downarrow}(A+B_+) - \lambda_j^{\downarrow}(A)].$$

(All of the summands are nonnegative.) This sum is $tr\ B_+$, and since we assumed $\lambda_k^{\downarrow}(B) = 0$, it is equal to $\sum_{j=1}^{k} \lambda_j^{\downarrow}(B)$. This proves (25).

Using the observation (6), it is not difficult to obtain from the Lidskii-Wielandt inequalities (24) the relation

$$\lambda^{\downarrow}(A) + \lambda^{\uparrow}(B) \prec \lambda(A+B). \tag{26}$$

Together with (23), this gives *noncommutative analogue* of (18): If A, B were commuting Hermitian matrices the relations (23) and (26) would reduce to (18).

7. The case $n = 3$

Let us see what we have obtained so far when $n = 3$. We get six relations from Weyl's inequalities (8):

$$\begin{aligned} &\gamma_1 \le \alpha_1 + \beta_1, \quad \gamma_2 \le \alpha_1 + \beta_2, \quad \gamma_2 \le \alpha_2 + \beta_1, \\ &\gamma_3 \le \alpha_1 + \beta_3, \quad \gamma_3 \le \alpha_3 + \beta_1, \quad \gamma_3 \le \alpha_2 + \beta_2. \end{aligned} \tag{27}$$

One more follows from Fan's inequalities (22):

$$\gamma_1 + \gamma_2 \le \alpha_1 + \alpha_2 + \beta_1 + \beta_2. \tag{28}$$

Four more relations can be read off from the Lidskii-Wielandt inequalities (24):

$$\begin{aligned} &\gamma_1 + \gamma_3 \le \alpha_1 + \alpha_3 + \beta_1 + \beta_2, \\ &\gamma_2 + \gamma_3 \le \alpha_2 + \alpha_3 + \beta_1 + \beta_2, \\ &\gamma_1 + \gamma_3 \le \alpha_1 + \alpha_2 + \beta_1 + \beta_3, \quad \text{and} \\ &\gamma_2 + \gamma_3 \le \alpha_1 + \alpha_2 + \beta_2 + \beta_3. \end{aligned} \tag{29}$$

(Use the symmetry in A and B.)

It was shown by Horn [16] that one more inequality

$$\gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3 \tag{30}$$

is valid, and further, together with the trace equality (1), the twelve inequalities (27) to (30) are sufficient to characterize all triples α, β, γ that can be eigenvalues of A, B and $A+B$. The proof of this assertion is not as simple as the one we gave for the case $n = 2$ in Section 3.

Where does the inequality (30) come from? Horn derived all inequalities that sums like $\gamma_i + \gamma_j$ satisfy for any dimension n; the inequality (30) is one of them. For the special case $n = 3$, one can derive this inequality from the majorisation (26), which is a consequence of the Lidskii-Wielandt theorem. For $n = 3$, this says

$$(\alpha_1 + \beta_3, \alpha_2 + \beta_2, \alpha_3 + \beta_1) \prec (\gamma_1, \gamma_2, \gamma_3).$$

Now using (17) one sees that the last three inequalities in (27) are hidden in this assertion. (Only the first five inequalities in (27) can be derived from the Lidskii-Wielandt inequalities in their raw form (24).) The inequality (30) too follows from this majorisation: if $\alpha_2 + \beta_2$ is larger than $\alpha_1 + \beta_3$ and $\alpha_3 + \beta_1$, this is clear from (17); if it is smaller than one of them, this follows from (29).

Let us consider a simple example. Let

$$\alpha = (4, 3, -2), \qquad \beta = (2, -1, -6).$$

Then the condition (1) says

$$\gamma_1 + \gamma_2 + \gamma_3 = 0.$$

This is a plane in $\mathbb{R}^3$; See FIGURE 3. For convenience rotate it to the x-y

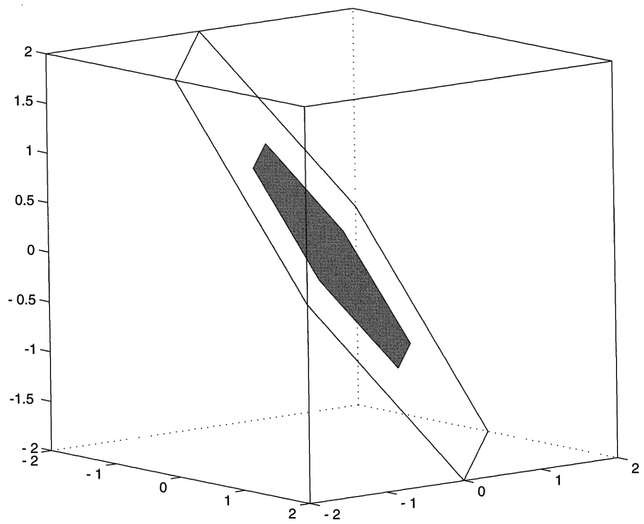

FIGURE 3. Part of the plane $\{\gamma_1 + \gamma_2 + \gamma_3 = 0\}$; small hexagon $= \{|\gamma_k| \leq 1\}$

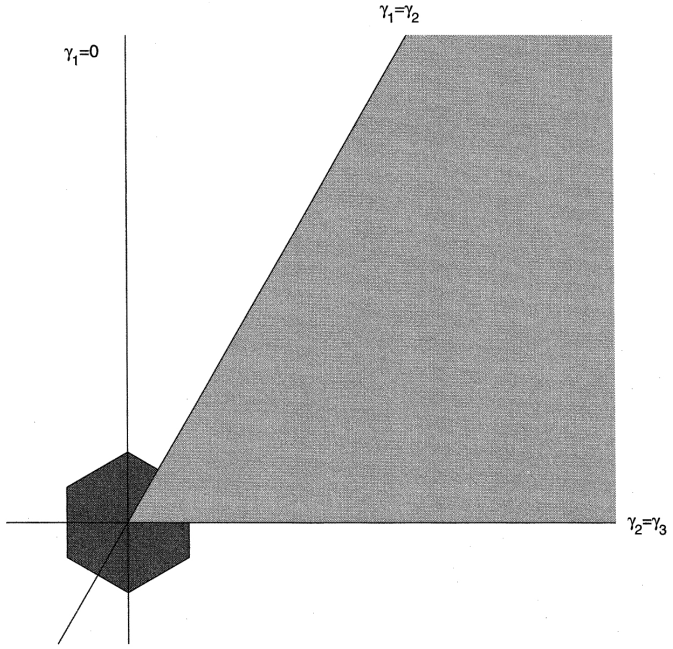

FIGURE 4. $\gamma_1 \geq \gamma_2 \geq \gamma_3 = 0$; small hexagon $= \{|\gamma_k| \leq 1\}$

plane. The condition $\gamma_1 \geq \gamma_2 \geq \gamma_3$ gives the part of the plane shown in FIGURE 4. The six inequalities of Weyl in (27) give three restrictions

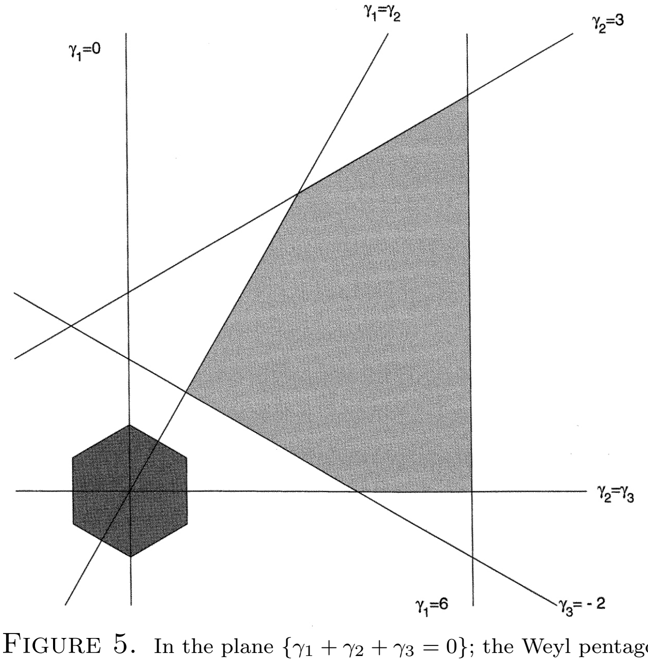

FIGURE 5. In the plane $\{\gamma_1 + \gamma_2 + \gamma_3 = 0\}$; the Weyl pentagon

$$\gamma_1 \leq 6, \quad \gamma_2 \leq 3, \quad \gamma_3 \leq -2.$$

This restrics γ further to the pentagon shown in FIGURE 5. A new restriction is imposed by Fan's inequality (28):

$$\gamma_1 + \gamma_2 \leq 8,$$

and this constrains γ to be in the hexagon in FIGURE 6. Of the four inequalities (29) of Lidskii-Wielandt, two are redundant. The remaining two are

$$\gamma_1 + \gamma_3 \leq 3, \qquad \gamma_2 + \gamma_3 \leq 0.$$

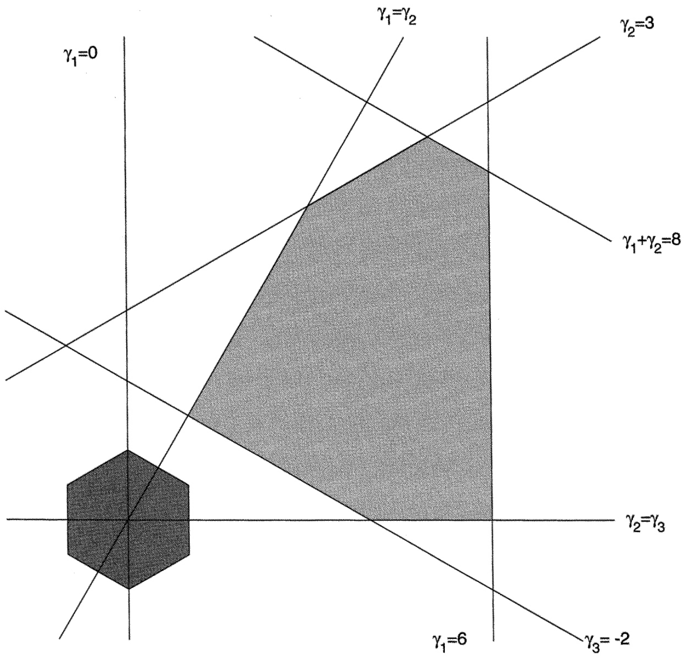

FIGURE 6. In the plane $\{\gamma_1 + \gamma_2 + \gamma_3 = 0\}$; the Ky Fan hexagon

However, they do not impose any new constraint; see FIGURE 7 on the next page. We have a new inequality from Horn's condition (30). This says

$$\gamma_2 + \gamma_3 \leq -2,$$

and cuts down the set of permissible γ to the septagon shown in FIGURE 8 on the next page.

Horn's theorem says that each point γ in this set is the eigenvalue tripple of a matrix $C = A + B$, where A, B are Hermitian matrices with eigenvalues α, β.

The majorisations (23) and (26) give in this example

$$(2, 0, -2) \prec \gamma \prec (6, 2, -8).$$

In the plane $\gamma_1 + \gamma_2 + \gamma_3 = 0$, the set of γ satisfying $\gamma \prec (6, 2, -8)$ is shown in FIGURE 9 on the page next to next; the set of γ satisfying $(2, 0, -2) \prec \gamma$ is shown in FIGURE 10 on the page next to next. The intersection of these two sets is a hexagon. The weyl inequality $\gamma_2 \leq 3$ imposes a constraint not included in these two majorisations. This additional constraint gives us the septagon of FIGURE 8.

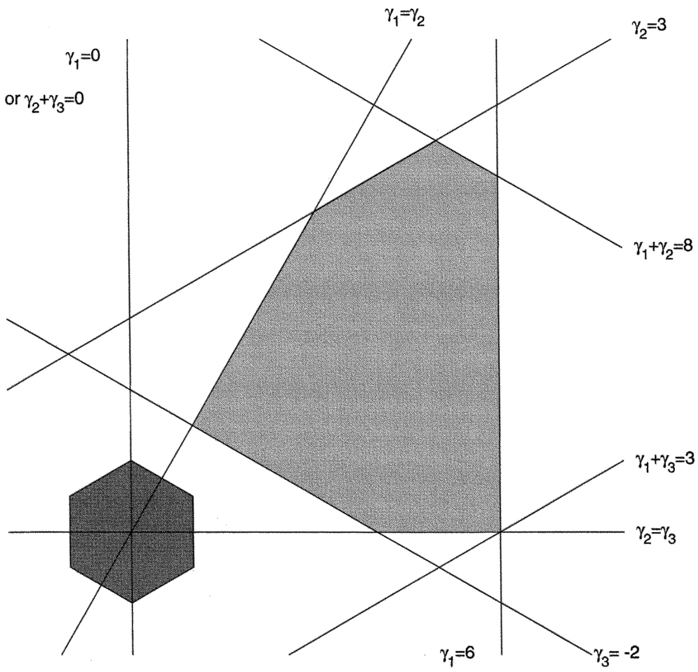

FIGURE 7. Lidskii-Wielandt inequalities have no effect in this example

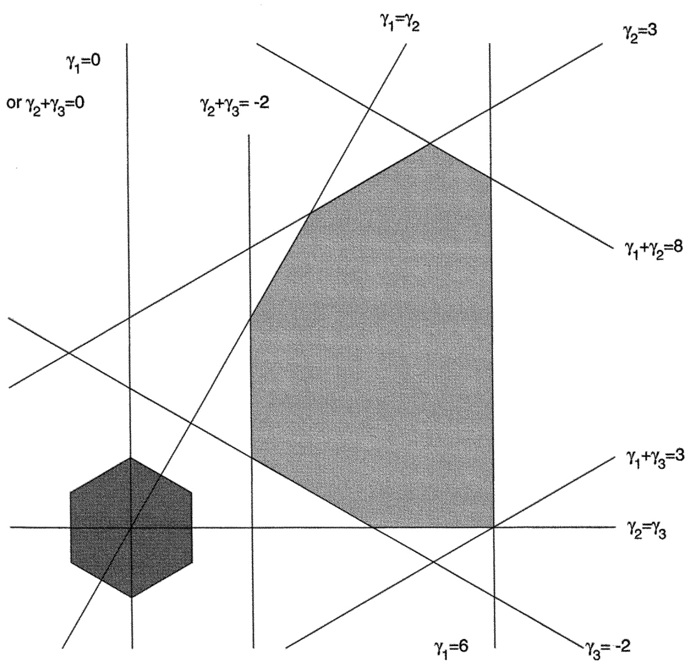

FIGURE 8. The Horn septagon

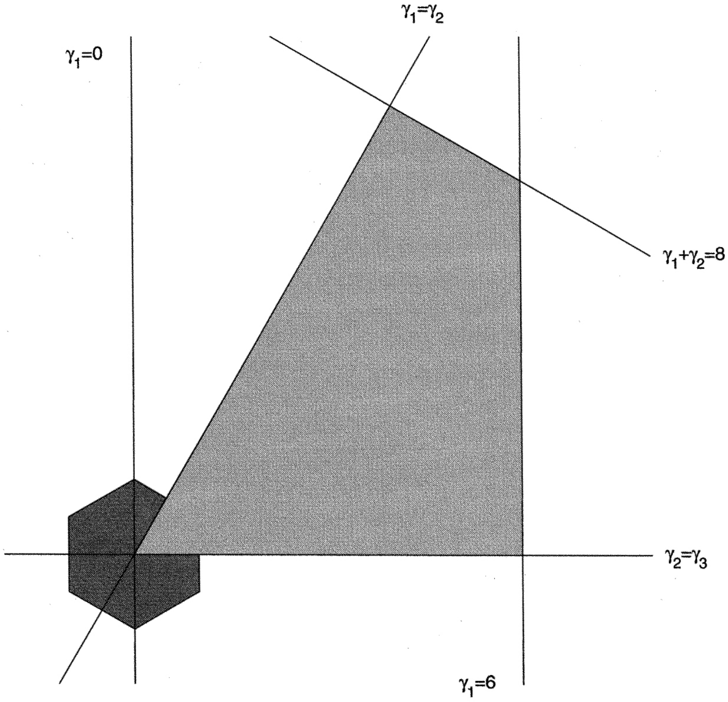

FIGURE 9. The quadrilateral containing all γ majorised by $(6, 2, -8)$

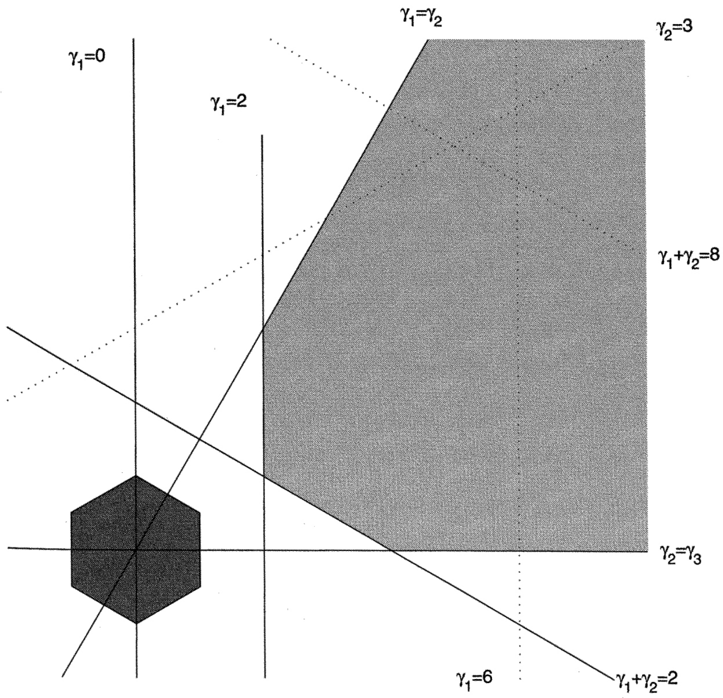

FIGURE 10. A part of the region containing γ that majorise (2, 0, -2)

8. The Horn Conjecture

The Lidskii-Wielandt Theorem aroused a lot of interest, and more inequalities connecting α, β, γ were discovered. Some of these looked very complicated. A particularly attractive one proved by R. C. Thompson and L. Freede in 1971 says

$$\sum_{j=1}^{k} \gamma_{i_j+p_j-j} \le \sum_{j=1}^{k} \alpha_{i_j} + \sum_{j=1}^{k} \beta_{p_j}, \tag{31}$$

for any choice of indices $1 \le i_1 < \cdots < i_k \le n$, $1 \le p_1 < \cdots < p_k \le n$ satisfying $i_k + p_k - k \le n$. This includes the Lidskii-Wielandt inequalities (24) (choose $p_j = j$) and treats α, β more symmetrically.

But where does the story end? Can one go on finding more and more inequalities like this? This question was considered, and an answer to it suggested, by A. Horn in a remarkable paper in 1962 [16]. This paper followed the ideas of Lidskii's original approach to the problem.

The inequalities (8), (22), (24) and (31) all have special form:

$$\sum_{k\in K} \gamma_k \le \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j, \tag{32}$$

where I, J, K are certain subsets of $\{1, 2, \cdots, n\}$ having the same cardinality. One may raise here two questions:

(i) What are triples (I, J, K) of subsets of $\{1, 2, \cdots, n\}$ for which inequalities (32) are true? Let us call such triples *admissible.*

(ii) Are these inequalities, together with (1), sufficient to characterise the α, β, γ that can be eigenvalues of Hermitian matrices A, B and $A + B$?

Horn conjectured that the answer to the second question is in the affirmative and that the set T_r^n of admissible triples (I, J, K) of cardinality r can be described by induction on r as follows.

Let us write $I = \{i_1 < i_2 < \cdots < i_r\}$ and likewise J and K. Then for $r = 1$, (I, J, K) is in T_1^n if $k_1 = i_1 + j_1 - 1$. For $r > 1$, (I, J, K) is in T_r^n if

$$\sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \binom{r+1}{2}, \tag{33}$$

and, for all $1 \le p \le r - 1$ and all $(U, V, W) \in T_p^r$,

$$\sum_{u\in U} i_u + \sum_{v\in V} j_v \le \sum_{w\in W} k_w + \binom{p+1}{2}. \tag{34}$$

Horn proved his conjecture for $n = 3$ and 4. When $n = 2$, these conditions just reduce to the three Weyl inequalities (14). When $n = 3$, they reduce to the twelve inequalities (27) – (30). When $n = 7$, there are 2062 inequalities given by these conditions, not all of which may be independent.

There is not much to explain about the conditions (33) and (34) themselves. The striking features of the conjecture - now a theorem - are the following. It says three things:

(1) Fix α, β and choose two Hermitian matrices A, B with eigenvalues α, β. Then the set of γ that are eigenvalues of $A + UBU^*$, as U varies over unitary matrices, is a convex polyhedron in $\mathbb{R}^n$.
(2) This convex polyhedron is described by Horn's inequalities.
(3) These inequalities can be obtained by an inductive proceedure.

We should emphasize that none of these is a statement of an obvious fact, and while each of them has now been proved the deeper reasons for their being true are still to be understood.

9. Schur-Horn Theorem and Convexity

A simple theorem like (20) is often an impetus for the development of several subjects. The theory of majorisation, a good part of matrix theory, and some important work in Lie groups and geometry, were inspired by this simple inequality.

In 1954 A. Horn [15] proved a converse to this theorem of Schur. Namely, if x and y are two real n-vectors such that $x \prec y$, then there exists a Hermitian matrix A such that x is the diagonal of A and y is its eigenvalues.

Using the properties of majorisation given in Section 4, we can state the theorem of Scur and its converse due to Horn as follows.

Theorem 2. *Let α be an n-tuple of real numbers and let $\mathcal{O}_\alpha$ be the set of Hermitian matrices with eigenvalues α. Let $\Phi : \mathcal{O}_\alpha \to \mathbb{R}^n$ be the map that takes a matrix to its diagonal. Then the image of Φ is a convex polyhedron, whose vertices are the $n!$ permutations of α.*

Now, the set of skew-Hermitian matrices $\mathcal{U}(n)$ is the Lie algebra associated with the compact Lie group $U(n)$ consisting of $n \times n$ unitary matrices. The set of Hermitian matrices is $i\mathcal{U}(n)$. The set $\mathcal{O}_\alpha$ is the *Orbit* of the diagonal matrix with diagonal α under the action of $U(n)$: it consists of all matrices $U\mathrm{diag}(\alpha)U^*$ as U varies over $U(n)$. This led B. Kostant in 1970 to interpret Theorem 2 as a special case of a general theorem for compact Lie groups. (The role of diagonal matrices is now played by a maximal compact abelian subgroup, that of the permutation group by the weyl group.) This in turn led to a much wider generalization in 1982 by M. Atiyah, and independently by

V. Guillemin and S. Sternberg. An explanation of these ideas is beyond our scope. However, let us state the theorem of Atiyah et la. to give a flavour of the subject.

Theorem 3. *Let M be a compact connected symplectic manifold, with an action of a torus T. Let $\Phi : M \to t^*$ be a moment map for this action. Then the image of Φ is a convex polytope, whose vertices are the imges of the T-fixed points on M.*

The curious reader should see the article [22] by A. Knutson (from where we have borrowed this formulation) for an explanation of the terms and the ideas. Another informative article is one by Atiyah [2].

For the present, we emphasize that the *moment map* and its convexity properties are now a major theme in geometry. Especially interesting for our story is the fact that the first part of Horn's conjecture stated at the end of Section 8 was proved in 1993 by A. H. Dooley, J. Repka and N. J. Wildberger [9], using convexity properties of the moment map.

10. Scubert Calculus and the heart of the matter

R. C. Thompson seems to have been the first one to realize that there are deep connections between the spectral inequalities we have been talking about and a topic in algebraic gepmetry called Schubert Calculus. Let us indicate these ideas briefly.

Start with the minimax principle (7). For convenience we rewrite it as

$$\alpha_j = \max_{dimV=j} \min_{x \in V;\ \|x\|=1} tr\ Axx^*. \tag{35}$$

Note that xx^*, the orthogonal projection operator onto the 1-dimensional space spanned by x, depends not on the *vector* x but on the *space* spanned by it.

The set of all 1-dimensional subspaces of $\mathbb{C}^{m+1}$ is known as the *complex projective space* $\mathbb{CP}_m$ *of dimension* m. These spaces are the basic objects studied by classical algebraic geometers and it is perhaps worth explaining briefly the geomeres' notation of homogeneous coordinates in projective spaces. Any non-zero vector of $\mathbb{C}^{m+1}$ determines a point in $\mathbb{CP}_m$; two points $(z_0, \cdots, z_m)$, $(z_0^{'}, \cdots, z_m^{'})$ determine the same 1-dimensional subspace (i.e., point of $\mathbb{CP}_m$) if and only if there is a non-zero $c \in \mathbb{C}$ such that $z_i^{'} = cz_i$ for each $i = 0, \cdots, m$. (The practice of using $\{0, \cdots, m\}$ to index the coordinates of $\mathbb{C}^{m+1}$ ensures that in m-dimensional projective space the last coordinate has index m rather than $m+1$.) In view of this the point ℓ of $\mathbb{CP}_m$ determined by $(z_0, \cdots, z_m)$ is denoted by $[z_0 : \cdots : z_m]$ and these are called the *homogeneous coordinates* of ℓ. Note that the homogeneous coordinates of a point in $\mathbb{CP}_m$ are not uniquely determined; they are defined only up to multiplication by non-zero complex numbers.

Now, if f is a nonconstant homogeneous polynomial in $z_0, \cdots, z_m$, then there is a well-defined zero locus of f:

$$Z_f = \{[z_0 : \cdots : z_m] \in \mathbb{CP}_m : \ f(z_0, \cdots, z_m) = 0\}.$$

This is known as the *projective hypersurface* defined by f. If f is a linear polynomial, Z_f is called a *hyperplane*, if f is quadratic, Z_f is called a *quadric hypersurface* and so on. *Projective varites* are intersections of a finite number of projective hypersurfaces.

These spaces enjoy interesting symmetry properties since it is easy to see that $\mathbb{CP}_m$ is homeomorphic to $U(m+1)/(U(1) \times U(m))$, where $U(1) \times U(m)$ is the subgroup of unitary matrices whose first row is $(1, 0, \cdots, 0)$.

A generalization of the notion of projective space is the *Grassmannian*, $G_k(\mathbb{C}^n)$, the set of k-dimensional subspaces of $\mathbb{C}^n$. From our perspective of matrices it is easy to get a model of these spaces. Associate with any k-dimensional subspace V of $\mathbb{C}^n$ the unitary operator $P_V - P_{V^\perp}$, where P_W is the orthogonal projection onto the subspace W. This sets up a bijective correspondence between $G_k(\mathbb{C}^n)$ and the set of $n \times n$ unitary matrices having trace equal to $2k - n$.

These Grassmanians can be embedded in projective spaces as subvarities in the following way. Given a subspace $V \subset \mathbb{C}^n$ of dimension k, choose a basis

$$u_1 = \begin{bmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_n^1 \end{bmatrix}, \cdots, u_k = \begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_n^k \end{bmatrix}$$

for V. Then the *Plücker coordinates* of V are the $\binom{n}{k}$ numbers

$$p_{i_1,\cdots,i_k}(V) = \det \begin{vmatrix} u_{i_1}^1 & u_{i_2}^1 & \cdots & u_{i_k}^1 \\ \vdots & \vdots & \ldots & \vdots \\ u_{i_1}^k & u_{i_2}^k & \cdots & u_{i_k}^k \end{vmatrix} \quad \text{for } \ 1 \leq i_1 < i_2 < \cdots < i_k \leq n.$$

If we choose a different basis $u_1^{'}, \cdots, u_k^{'}$ for V, then the Plücker coordinates are all multiplied by the same non-zero scaler factor (the determinant of the unitary transformation that takes each u_i to $u_i^{'}$ and is the identity map when restricted to $V^{\perp}$). So once an ordering has been chosen for the k-tuples $1 \leq i_1 < i_2 < \cdots < i_k \leq n$,

$$V \mapsto [\cdots : p_{i_1 \cdots i_k}(V) : \cdots]$$

yields an embedding of $G_k(\mathbb{C}^n)$ in $\mathbb{CP}_{\binom{n}{k}-1}$. In fact, the image of this embedding is a projective variety and its definig equations are well known.

Let us now return to matrix inequalities.

Given any Hermitian operator A on $\mathbb{C}^n$, and a subspace L of $\mathbb{C}^n$ (which we think of as a point in $G_k(\mathbb{C}^n)$), let $A_L = P_L A P_L$. Note that $tr\ A_L = tr\ P_L A P_L = tr\ AP_L$.

To prove the inequality (24), Wielandt invented a most remarkable minimax principle. This says that whenever $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, then

$$\sum_{j=1}^{k} \alpha_{i_j} = \max_{\substack{V_1 \subset \cdots \subset V_k \\ dim V_j = i_j}} \quad \min_{\substack{L \in G_k(\mathbb{C}^n) \\ dim(L \cap V_j) \geq j}} tr\ A_L. \tag{36}$$

When $k = 1$, this reduces to (12).

Another such principle was found by Hersch and Zwahlen. Let A have the spectral resolution $A = \sum \alpha_j v_j v_j^*$. For $1 \leq m \leq n$, let V_m be the linear span of $v_1, \cdots, v_m$. Then

$$\sum_{j=1}^{k} \alpha_{i_j} = \min_{L \in G_k(\mathbb{C}^n)} \{tr\ A_L : dim(L \cap V_{i_j}) \geq j, \quad j = 1, \cdots, k\}. \tag{37}$$

This can be proved using ideas familiar to us from Section 2. Let L be any k-dimensional subspace of $\mathbb{C}^n$ such that dim $(L \cap V_{i_j}) \geq j$. Since dim $(L \cap V_{i_1}) \geq 1$, we can find a unit vector x_1 in $L \cap V_{i_1}$. Since V_{i_1} is spanned by $\{v_1, \cdots, v_{i_1}\}$ we have the inequality $\alpha_{i_1} \leq \langle x_1, Ax_1 \rangle$. Since dim $(L \cap V_{i_2}) \geq 2$, we can find a unit vector x_2 in $L \cap V_{i_2}$ that is orthogonal to x_1. Then $\alpha_{i_2} \leq \langle x_2, Ax_2 \rangle$. Continuing in this way, we obtain an orthonormal basis $x_1, \cdots, x_k$ for L such that $\alpha_{i_j} \leq \langle x_j, Ax_j \rangle$ for $1 \leq j \leq k$. Thus

$$\sum_{j=1}^{k} \alpha_{i_j} \leq \sum_{j=1}^{k} \langle x_j, Ax_j \rangle = tr\ A_L.$$

For the special choice $L = \text{span}\{v_1, \cdots, v_{i_k}\}$, we have equality here. This proves the Hersh-Zwahlen principle (37).

The minimum in (37) is taken over a special kind of subset of $G_k(\mathbb{C}^n)$ studied by geometers and topologists for many years.

A sequence of nested subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n,$$

where $\dim V_j = j$, is called a *complete flag*. Given such a flag $\mathcal{F}$, for each multiindex $I = \{i_1 < \cdots < i_k\}$ the subset

$$S(I;\mathcal{F}) = \{W \in G_k(\mathbb{C}^n) : \dim(W \cap V_{i_j}) \geq j, \quad 1 \leq j \leq n\}$$

of the Grassmannian is called a *Schubert variety.*

The Hersh-Zwahlen principle says that the sum $\sum_{i\in I}\alpha_i$ is the minimal value of $tr\ A_L$ evaluated on the Schubert variety $S(I;\mathcal{F})$ corresponding to the flag constructed from the eigenvectors of A.

Hersch and Zhahlen developed a technique for obtaining inequalities like (32) using the principle (37). The essence of this technique can be described as follows. Consider the spectral resolutions

$$A = \sum \alpha_j u_j u_j^*, \quad B = \sum \beta_j v_j v_j^*, \quad C = A + B = \sum \gamma_j w_j w_j^*.$$

We find it convenient to write

$$-A - B + C = 0. \tag{38}$$

Recall that $\lambda_j^{\downarrow}(-A) = -\lambda_{n-j+1}^{\downarrow}(A)$. Given an index set $I = \{1 \leq i_1 < \cdots < i_k \leq n\}$ let $I^{'} = \{i : n-i+1 \in I\}$ and arrange the elements of $I^{'}$ in increasing order. For $1 \leq j \leq n$ consider the three families of subspaces

$$\begin{aligned} U_j &= span\{u_n, \cdots, u_{n-j+1}\}, \\ V_j &= span\{v_n, \cdots, v_{n-j+1}\}, \\ W_j &= span\{w_1, \cdots, w_j\}. \end{aligned}$$

Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be the complete flags formed by these three families. Now suppose our index sets I, J, K (of the same cardinality) are such that the Schubert varities $S(I^{'};\mathcal{F}), S(J^{'};\mathcal{G})$, and $S(K^{'};\mathcal{H})$ have a nonempty intersection. Choose a point L in this intersection. Then using (38) and (37) we get the inequality

$$\begin{aligned} 0 &= tr\ (-A_L - B_L + C_L) \\ &\geq \sum_{i\in I^{'}} \lambda_i^{\downarrow}(-A) + \sum_{j\in J^{'}} \lambda_j^{\downarrow}(-B) + \sum_{k\in K} \lambda_k^{\downarrow}(C). \end{aligned}$$

In other words

$$\sum_{k\in K} \lambda_k^{\downarrow}(C) \leq -\sum_{i\in I^{'}} \lambda_i^{\downarrow}(-A) - \sum_{j\in J^{'}} \lambda_j^{\downarrow}(-B) = \sum_{i\in I} \lambda_i^{\downarrow}(A) + \sum_{j\in J} \lambda_j^{\downarrow}(B).$$

This is the kind of equality (32) we are looking for, and we have now touched the heart of the matter. Whenever the Schubert varieties $S(I^{'};\mathcal{F})$, $S(J^{'};\mathcal{G})$, and $S(K;\mathcal{H})$ have a nontrivial intersection, the triple (I, J, K) is admissible. The simplest instance of this idea at work is the proof of Weyl's inequalities (8) that we gave in section 2. The triple $I = \{i\}, J = \{j\}, K = \{k\}$ is admissible if $k = i + j - 1 \leq n$.

The full significance of these ideas was grasped by R. C. Thompson; see especially the Ph.D. thesis of his student S. Johnson [19], and his unpublished lecture notes [32]; see also [14]. Among other things, Thompson asked whether the admissibility of three triples (I, J, K) in Horn's inequality was *equivalent* to the condition that Schubert varities $S(I'; \mathcal{F})$, $S(J'; \mathcal{G})$, and $S(K; \mathcal{H})$ corresponding to *any* three complete flags $\mathcal{F}, \mathcal{G}, \mathcal{H}$ (not necessarily constructed from eigenvectors of A, B and $A + B$) have a nontrivial intersection. This equivalence has now been proved.

Theorem 4. *The triple (I, J, K) is admissible if and only if for any three complete flags $\mathcal{F}, \mathcal{G}, \mathcal{H}$, the intersection of the Schubert varities $S(I'; \mathcal{F})$, $S(J'; \mathcal{G})$, and $S(K; \mathcal{H})$ is nonempty.*

The study of intersection property of Schubert varities is the subject of *Scubert calculus.* It reduces geometric questions about intersection of Scubert varities to algebraic questions about multiplication in a ring called the *integral cohomology ring $H^*(G_k(\mathbb{C}^n))$* associated with the Grassmanian. *Schubert cycles S_I* are eqivalence classes of Schbert varities (the dependence on $\mathcal{F}$ is removed). They form a basis for the ring $H^*(G_k(\mathbb{C}^n))$. Given triples I, J, K, consider the product $S_I \cdot S_J$ in this ring and expand it as

$$S_I \cdot S_J = \sum c_{I \cdot J}^L \; S_L, \tag{39}$$

where $c_{I \cdot J}^L$ are nonnegative integers. It turns out that the triple (I, J, K) is admissible if and only if the coefficiet $c_{I \cdot J}^K$ in (39) is nonzero (i.e., S_K occurs in the expanson of the product $S_I \cdot S_J$.)

It can now be said that the proof of Weyl's inequalities given in Section 2, and some others such as Wielandt's proof of (24) and the Thompson-Freede proof of (31), really amount to showing *using ideas from linear algebra alone* that certain Scubert varities always intersect. The full proof of Theorem 4 – and of Horn's conjecture that follows from it – needs advanced facts from Schubert calculus. However, to quote from [22], "In fact the details of the proofs are not actually very different from the hands-on techniques used e.g. by Horn himself."

Theorem 4 was proved by Klyachko [20] and Knutson and Tao [23]. Belkale [3] has shown that if $c_{I \cdot J}^K > 1$, then the inequalities (32) that correspond to triple (I, J, K) are redundant, that is, they can be derived from other inequalities in the list. On the other hand, Knutson, Tao and Woodward [25] have shown that the inequalities in the list (32) that correspond to those (I, J, K) for which $c_{I.J}^{K} = 1$ are independent.

Together, these results give the smallest set of inequalities needed to completely characterize the convex polyhedron whose points are eigenvalues of $A + UBU^*$, where A, B are given Hermitian matrices and U varies over unitaries.

11. Singular values of products of matrices

In this section A, B, etc. are arbitrary $n \times n$ matrices, not necessarily Hermitian any more.

The *singular values* of A are the non-negative numbers $s_1(A) \geq \cdots \geq s_n(A)$ that are the square roots of the eigenvalues of A^*A. It is easy to see that $s_1(A) = \|A\|$, and that

$$s_1(AB) \leq s_1(A)s_1(B). \tag{40}$$

Compare this with (4) and a natural problem stares at us: are there counterparts of inequalities for eigenvalues of sums of Hermitian matrices that are valid for products of singular values of arbitrary matrices? This question too has been of great interest and importance in linear algebra.

The k-fold antisymmetric tensor product $\Lambda^k A$ has singular values $s_{i_1}(A) \cdots s_{i_k}(A)$, where $1 \leq i_1 < \cdots < i_k \leq n$. Since $\Lambda^k(AB) = \Lambda^k(A)\Lambda^k(B)$, we get from (8) the inequality

$$\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A) \prod_{j=1}^{k} s_j(B). \tag{41}$$

This is the singular value analogue of (22). (Incidentally, there is a perfect analogy here. We have derived (41) by applying (40) to a tensor object. We can derive (22) from (4) by a quite similar argument [4, p. 23]). The analogue of (24) is the following inequality proved by Gel'fand and Naimark

$$\prod_{j=1}^{k} s_{i_j}(AB) \leq \prod_{j=1}^{k} s_{i_j}(A) \prod_{j=1}^{k} s_j(B). \tag{42}$$

Once again, the therorem was proved in connection with questions about Lie groups, a matrix-theoretic proof was given by V. B. Lidskii, the inequality was discussed and proved in [5], and the simplest proof was found by Li and Mathias [28] soon afterwards. More inequalities of this type had been discovered by others, notably by R. C. Thompson and his students. The conjecture parallel to that of Horn was discussed by Thompson. Now it has been proved:

Theorem 5. *Let* $a_1 \geq \cdots \geq a_n,\ b_1 \geq \cdots \geq b_n,\ c_1 \geq \cdots \geq c_n$, *be three tuples of nonnegative real numbers. Then there exists matrices* A, B *with singular values* $s_j(A) = a_j,\ s_j(B) = b_j,\ s_j(AB) = c_j$, *if and only if*

$$\prod_{k\in K} c_k \leq \prod_{i\in I} a_i \prod_{j\in J} b_j$$

for all admissible triples $\{I, J, K\}$.

This is stated as Theorem 16 in [11]. The reason why it is true and the connection with Horn's problem are provided by the following theorem [21].

Theorem 6. *Let* a, b, c *be three* n*-tuples of the decreasingly ordered real numbers. Then the following statements are equivalent:*

(i) *There exists non-singular matrices* A, B *with* $s_j(A) = a_j,\ s_j(B) = b_j,\ s_j(AB) = c_j$.
(ii) *There exists Hermitian matrices* X, Y *with* $\lambda_j^{\downarrow}(X) = log\ a_j,\ \lambda_j^{\downarrow}(Y) = log\ b_j,\quad \lambda_j^{\downarrow}(X+Y) = log\ c_j$.

12. Eigenvalues of products of unitary matrices

Eigenvalues of two unitary matrices and their products are the next objects we consider. Here the formulation of the problem is much more delicate and needs more advanced maschinary. We can indicate only somewhat vaguely what it involves.

To get rid of ambiguities arising from multiplication on the unit circle we restrict ourselves to the set $SU(n)$ of $n \times n$ unitary matrices with determinant one. For $A \in SU(n)$ let $Eig^{\downarrow}(A)$ be the set of its eigenvalues $\exp(2\pi i\lambda_j)$, labelled so that $\lambda_1 \geq \cdots \geq \lambda_n$. Since $\det A = 1$, we must have $\lambda_1 + \cdots + \lambda_n \equiv 0 (\text{mod} 1)$. Choose a normalisation that has $\lambda_1 + \cdots + \lambda_n = 0$ and $\lambda_1 - \lambda_n < 1$. With this normalization, call the numbers λ_j occuring here $\lambda_j^{\downarrow}(A)$.

Our problem is to find relations between $\lambda_j^{\downarrow}(A),\ \lambda_j^{\downarrow}(B)$ and $\lambda_j^{\downarrow}(AB)$ for two elements A, B of $SU(n)$.

The analogue of the Lidskii-Wielandt inequalities (24) in this context was discovered in 1958 by A. Nudel'man and P. Svarcman. This has exactly the form (24). However, the analogue of Horn's conjecture in this context involves some objects that arise in the study of vector bundles, and are related to *quantum Schubert calculus*, a subject of very recent origin.

In section 10 we alluded to the cohomology ring $H^*(G_k(\mathbb{C}^n))$ and how multiplication in this ring gives us information about intersection of Scubert cycles. Quantum cohomology associates with the Grassmannian the object

$$qH^*(G_k(\mathbb{C}^n)) = H^*(G_k(\mathbb{C}^n)) \otimes \mathbb{C}[[q]],$$

where $\mathbb{C}[[q]]$ is the ring of formal power series. Multiplication $S_I * S_J$ in this ring is more complicated. Instead of (39) we have an expansion that looks like

$$S_I * S_J = \sum_L \sum_{d \geq 0} \left(c_{I,J}^L\right)_d q^d S_L. \tag{43}$$

The new result on eigenvalues of unitary matrices is the following

Let (I, J, K) be triples such that the coefficient $(C_{I,J}^K)_d$ in the expansion (43) is nonzero. Then for all A, B in $SU(n)$

$$\sum_{i \in I} \lambda_i^{\downarrow}(A) + \sum_{j \in J} \lambda_j^{\downarrow}(B) \leq d + \sum_{k \in K} \lambda_k^{\downarrow}(AB). \tag{44}$$

Further, these inequalities give a *complete* set of restrictions (in the same sense as in Horn's problem).

This theorem has been proved by S. Agnihotri and C. Woodward [1] and by P. Belkale [3] with earlier contributions by I. Biswas [6]. A crucial componrent of the proof is a 1980 theorem of V. B. Mehta and C. S. Seshadri [30] on vector bundles on projective space $\mathbb{CP}_1$. Let us explain, in bare outline, this theorem, and the fascinating connection it has with our problem.

For brevity let $\mathbb{P}_1$ denote the projective space $\mathbb{CP}_1$ introduced in Section 10. This space can be identified with the two-dimensional sphere S^2. This, in turn, can be thought of as the Riemann sphere $\mathbb{C} \cup \{\infty\}$, the one point compactification of the complex plane. The point ∞ can be thought of as the north pole of the sphere and the point 0 as the south pole. To ponts in the open set $\mathbb{P}_1 \setminus \{\infty\}$ we assign the usual complex coordinate z while on the open set $\mathbb{P}_1 \setminus \{0\}$ we define the complex coordinate w by putting $w = 1/z$.

This space is simply connected: its fundamental group $\pi_1(\mathbb{P}_1)$ is trivial. $\mathbb{P}_1$ with one puncture (i.e., one of its point removed) can be identified by $\mathbb{C}$. This too is simply connected, and its fundamental group is trivial. $\mathbb{P}_1$ with two puncturs is isomorphic with the punctured plane $\mathbb{C} \setminus \{0\}$. The fundamental group of this space is $\mathbb{Z}$, a group generated by one element. Carry out this construction further. Let $S = \{p_1, \cdots, p_k\}$ be any finite subset of $\mathbb{P}_1$. Without loss of generality, think of p_k as the point at ∞. To identify the fundamental group of this space, chhose a base point p in $\mathbb{P}_1 \setminus S$. Loops, with fixed base point p, can be composed in the usual way. With this law of composition the product of the loops going counterclockwise around the points

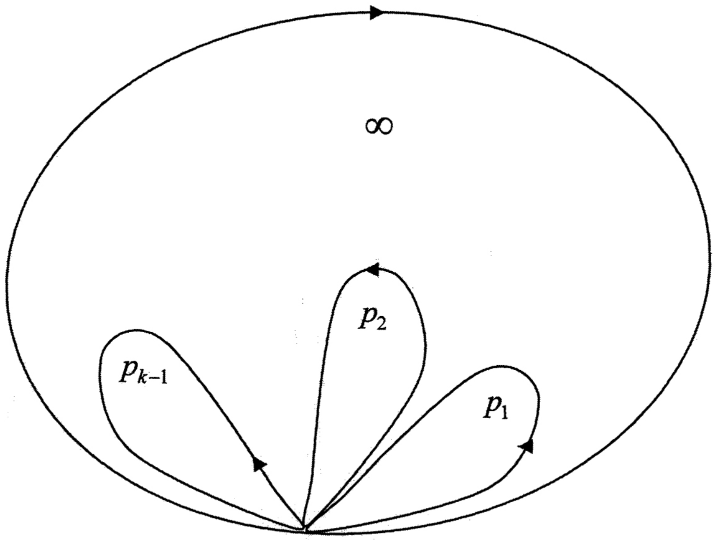

FIGURE 11. Computing the fundamental group of $\mathbb{P}_1 \setminus S$

p_j, $1 \leq j \leq k-1$, is the loop going clockwise around $p_k = \infty$; see FIGURE 11.

Thus the fundamental group of $\pi_1(\mathbb{P} \setminus S)$ is the free group with generators $g_1, \cdots, g_k$ with one relation $g_k = (g_1 \cdots g_{k-1})^{-1}$.

A homomorphism of a group G in to another group H is called a *representation* of G in H.

Let ρ be a representation of the fundamental group $\pi_1(\mathbb{P}_1 \setminus S)$ in the group $U(n)$ or $SU(n)$. If $A_j = \rho(g_j)$, this gives unitary matrices $A_1, \cdots, A_k$, with their product $A_1 A_2 \cdots A_k = I$.

In our original we are given three n-tuples of numbers and we want to know when they can be the eigenvalues of matrices A, B and AB in $SU(n)$. Prescribing eigenvalues means fixing the conjugacy class of A under unitary conjugations $A \mapsto UAU^*$. Thus our problem is to find conditions for the existence of three elements A, B, C of $SU(n)$ with prescribed conjugacy classes such that $ABC = I$. Instead of three matrices, we can equally well consider the same question for k matrices $A_1, \cdots, A_k$. In the preceding paragraph we saw how this problem is connected with representations of the fundamental group of $\mathbb{P}_1$ with k punctures.

Next we recall the notion of a vector bundle. For simplicity, we make some restrictions in our definitions; see [31] for splendid introduction. Let B be a compact connected Hausdorff topological space. A *vector bundle* over (the *base space*) B consists of the following:

(i) a topological spce $\mathcal{E}$ called the *total space*,

(ii) a continuous map $\Pi : \mathcal{E} \to B$ called the *projection map*,

(iii) on each set $E_b = \Pi^{-1}(b),\ b \in B$, the structre of an n-dimensional real or complex vector space. (the bundle is accordingly called real or complex.) The vector space E_b is called the *fibre* over b.

These objects are required to satisfy a restriction called *local triviality:* for each $b \in B$ there exists a neighbourhood U, and a homeomorphism $h : U \times \mathbb{K}^n \to \Pi^{-1}(U)$, such that for each $a \in U$ the map $x \mapsto h(a, x)$ from $\mathbb{K}^n$ to E_a is an isomorphism of vector spaces. Here, $\mathbb{K}^n$ is the space $\mathbb{R}^n$ or $\mathbb{C}^n$ depending on whether the bundle is real or complex. The pair (U, h) is called *local trivialisation* about b. If it is possible to choose U equal to the entire base space B, then the bundle $\mathcal{E}$ is called a *trivial bundle.* In this case $\mathcal{E} = B \times \mathbb{K}^n$.

The number n is called the *rank* of the bundle $\mathcal{E}$. If $n = 1$, the bundle is called a *line bundle.*

If B is a contractible space, every vector bundle on it is trivial. On the base space S^1 (the unit circle) the cylinder is a trivial line bundle while the Moebius strip is nontrivial line bundle.

Let U be any open set in B. A *section* over U is a continuous map $s : U \to \mathcal{E}$ such that $s(b) \in E_b$ for all $b \in U$.

Let (U, h) be a local trivialisation. Let $\{x_j\}$ be the standard basis for $\mathbb{K}^n$ and let $e_j^U(a) = h(a, x_j),\ a \in U$. Then $\{e_j^U(a)\}$ is a basis for the vector space E_a. The maps e_j^U are sections over U. The family $\{e_j^U\}$ is called a *local basis* for $\mathcal{E}$ over U. Let $\{e_j^U\}$ and $\{e_j^V\}$ be two local bases for $\mathcal{E}$ over open sets U and V. Then for each point $a \in U \cap V$, we can find an invertible matrix $g_{V,U}(a)$ that carries the basis $\{e_j^U(a)\}$ onto the basis $\{e_j^V(a)\}$ of E_a. This is called a *transition function.* Note that $a \to g_{V,U}(a)$ is a continuous map from $U \cap V$ into GL_n.

If the spaces involved have more stucture, we could define *smooth* bundles or *holomorphic* bundles by putting the appropriate conditions on the maps involved.

Let $\mathcal{E}$ and $\mathcal{F}$ be two bundles over the same base space B such that (the total space) $\mathcal{F}$ is contained in (the total space) $\mathcal{E}$ and each fibre F_b in the bundle $\mathcal{F}$ is a vector subspace of the corresponding fibre E_b. Then we say that $\mathcal{F}$ is a subbundle of $\mathcal{E}$. A trivial bundle may have nontrivial subbundles.

We are interested in complex vector bundles on the base space $\mathbb{P}_1$. One more notion that we need is the *degree* of a vector bundle on $\mathbb{P}_1$. We have identified $\mathbb{P}_1$ with the sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The complement of the north pole ∞ is an open set U that can be identified with the complex plane $\mathbb{C}$ with its coordinate z. This is a contractible space; so any vector bundle $\mathcal{E}$ on $\mathbb{P}_1$ admits a local trivialisation on U. Let $\{e_j^U\}$ be the corresponding local basis over U. Similarly, the set $V = \mathbb{P}_1 \setminus \{0\}$ – the complement of the south pole – is identified with the complex plane with coordinate $w = 1/z$. So $\mathcal{E}$ admits a local trivialization over V and a local basis $\{e_j^V\}$. The equator $\|z\| = 1$ lies in $U \cap V$. Let $g_{V,U}(z)$ be the trasnsition function between the two bases. Identifying the equator with S^1 with coordinate z, we get a map $g_{V,U}(z)$ from S^1 into $GL(n)$. Then $\psi(z)$=det $g_{V,U}(z)$ is a map from S^1 into nonzero complex numbers. The winding number of this map around 0 is called the *degree* of the vector bundle.

For example, consider the *tautological line bundle* on $\mathbb{P}_1$. This associates with each point of $\mathbb{P}_1$ the complex line through that point. In the open set $U = \mathbb{P}_1 \setminus \{\infty\}$ we associate with the point $[z:1]$ the line $\mathbb{C}(z,1)$ in $\mathbb{C}^2$. In the open set $V = \mathbb{P}_1 \setminus \{0\}$, we associate with the point $[1:1/z]$ the line $\mathbb{C}(1,1/z)$ in $\mathbb{C}^2$. The total space for this bundle is a subset of $\mathbb{P}_1 \times \mathbb{C}^2$. In the intersection $U \cap V$ the transition from the basis $(z,1)$ to $(1,1/z)$ is given by multiplication by $g(z) = 1/z$. This function on S^1 has winding number -1 around the origin. So this bundle has degree -1.

The *slope* of the vector bundle $\mathcal{E}$ is defined as

$$\text{slope}(\mathcal{E}) = \frac{\text{degree}(\mathcal{E})}{\text{rank}(\mathcal{E})}. \tag{45}$$

The bundle $\mathcal{E}$ is said to be *stable* if

$$\text{slope}(\mathcal{F}) < \text{slope}(\mathcal{E}) \tag{46}$$

for every subbundle $\mathcal{F}$ of $\mathcal{E}$, and *semistable* if

$$\text{slope}(\mathcal{F}) \leq \text{slope}(\mathcal{E}). \tag{47}$$

The bundle $\mathcal{E}$ is said to be *polystable* if it is isomorphic to a direct sum of stable bundles of the same slope. Polystable bundles are semistable. Each semistable bundle is equivalent to a canonoical polystable bundle (under an equivalence relation that we do not define here).

Let $\mathcal{E}$ be a vectoe bundle of rank n on the space $\mathbb{P}_1$. Let $S = \{p_1, \cdots, p_k\}$ be a given finite subset of $\mathbb{P}_1$. A *parabolic structure* on $\mathcal{E}$ consists of the following objects given at each point $p \in S$:

(i) in each fibre E_p, a complete flag

$$\{0\} = V_0^p \subset V_1^p \subset \cdots \subset V_n^p = \mathbb{C}^n, \tag{48}$$

(ii) an n-tuple of real numbers $\alpha_j^p,\ 1 \le j \le n$ satisfying

$$\alpha_1^p \ge \alpha_2^p \ge \cdots \ge \alpha_n^p > \alpha_1^p - 1. \tag{49}$$

The flag (48) is also called a *filtration*, and the sequence (49) is called a *weight sequence*. We should remark that in the original definition due to C. S. Seshadri, flags in (i) were not required to be complete, and the weights were restricted to be in the interval $[0, 1)$.

Let $\mathcal{F}$ be a subbundle of $\mathcal{E}$ with rank $(\mathcal{F}) = r$. At each point p, the fibre F_p is an r-dimensional subspace of the n-dimensional space E_p. For $p \in S$, consider the intersections $F_p \cap V_j^p,\ 0 \le j \le n$, where the V_j form the flag (48). If $r < n$, some of these spaces coincide. Retain only the distinct members of this sequence and label them as $W_i^p,\ 0 \le i \le r$. Assign to this W_i^p the highest possible weight allowed by this intersection, i.e., the weight $\beta_i^p = \alpha_j^p$, where j is the smallest number satisfying $W_i^p = F_p \cap V_j^p$. Then the subbubdle $\mathcal{F}$ with parabolic structure given by the filtration

$$\{0\} = W_0^p \subset W_1^p \subset \cdots \subset W_r^p = \mathbb{C}^r \tag{50}$$

and weights

$$\beta_1^p \ge \beta_2^p \ge \cdots \ge \beta_r^p \tag{51}$$

is called a *parabolic subbundle* of $\mathcal{E}$. For brevity, $\mathcal{E}$ is called a *sparabolic bundle*.

The *parabolic degree* of $\mathcal{E}$ is defined as

$$\text{par degree } (\mathcal{E}) = \text{degree } (\mathcal{E}) + \sum_{p \in S} \sum_{j=1}^{n} \alpha_j^p, \tag{52}$$

and its *parabolic slope* as

$$\text{par slope } (\mathcal{E}) = \frac{\text{par degree } (\mathcal{E})}{\text{rank } (\mathcal{E})}. \tag{53}$$

The notions of stability, semistability and polystability of a parabolic bundle are defined by replacing the quantity "slope" in the inequalities (46) or (47) by "parabolic slope".

Now we have all the pieces needed to describe the theorem of Mehta and Seshadri (as modified by Belkale and others to suit our needs).

We began by looking at $SU(n)$ representations of the fundamental group $\pi_1(\mathbb{P}_1 \setminus S)$, where $S = \{p_1, \cdots, p_k\}$. We saw that this amounts to finding matrices $A_1, \cdots, A_k$ in $SU(n)$ whose product is I. For $i = 1, 2, \cdots, k$, let $\alpha_j^i = \lambda_j^{\downarrow}(A)$, where $\lambda_j^{\downarrow}(A)$ is as defined at the begining of this section. For each $i = 1, 2, \cdots, k$, let V_m^i be the m-dimensional space spanned by the eigenvectors

$u_1^i, \cdots, u_m^i$ corresponding to the eigenvalues $\lambda_1^{\downarrow}(A_i), \cdots, \lambda_m^{\downarrow}(A_i)$. Use these data to give a parabolic structure to the trivial rank n bundle on $\mathbb{P}_1$ as follows:

(i) in each fibre E_{p_i} a filtration is given by $\{0\} = V_0^i \subset V_1^i \subset \cdots \subset V_n^i = \mathbb{C}^n$
(ii) the numbers $\alpha_1^i \geq \cdots \geq \alpha_n^i$ give a weight sequence.

The theorem of Mehta and Seshadri says that *the parabolic bundle obtained in this way is polystable*, and conversely *every polystable bundle arises in this way.*

Now to the denouement: families of inequalities such as (43) are used by Agnihotri-Woodwrd, Belkale and Biswas to prove that certain vector bundles on $\mathbb{P}_1 \setminus \{p_1, p_2, p_3\}$ are semistable. (Semistability is defined by a family of inequalities.) To each semistable parabolic bundle there corresponds a unique polystable parabolic bundle. The Mehta-Seshadri theorem then leads to the existence of unitary matrices whose eigenvalues are the given n-tuples.

The proof of Klyachko for the original Horn problem uses ideas similar to these, but it involves bundles on $\mathbb{P}_2$ and a theorem of Donaldson.

13. Rpresentations of GL_n

We began this story with Weyl's inequalities. It is befitting to end it with another subject in which Weyl was a pioneer - the theory of representations of groups. A fscinating connection between the two subjects has been discovered in recent years.

Let GL_n be the group consisting of $n \times n$ complex invertible matrices. By the *standard representation* of GL_n we mean the homomorphism from GL_n into the space $GL(V)$ of all linear operators on the space $V = \mathbb{C}^n$. If W is any m-dimensional complex vector space, a homomorphism $\rho : GL_n \to GL(W)$ is called a *representation* of GL_n in W. Such a representation is called an m-dimensional representation. For example, the map det gives a 1-dimensional representation. For brevity we denote a representation in W by W.

For simplicity, let us consider only *polynomial representations,* ones in which the entries of $\rho(A)$ are polynomials in the entries of A. The determinant representation is an example of such a representation. Another example is the tensor product, in which $W = \otimes^k V = V \otimes \cdots \otimes V$ (k times), and $\rho(A) = \otimes^k A$.

The space $\otimes^k V$ has several subspaces that are invariant under all the operators $\otimes^k A,\ A \in GL(V)$. Two examples are the spaces $\Lambda^k V$ and $\mathrm{Sym}^k V$ of antisymmetric and symmetric tensors, respectively. The restrictions of $\otimes^k A$ to these spaces are written as $\Lambda^k A$ and $\mathrm{Sym}^k A$. The spaces $\Lambda^k V$ and $\mathrm{Sym}^k V$ are eamples of *irreducible* representations of GL_n; they have no proper subspace invariant under all operators $\Lambda^k A$ or $\text{Sym}^k A$. These are subrepresentations of $\otimes^k V$. All polynomial representations are subrepresentations of $\otimes^k V$ for some k.

Let N_+ be the set of all upper triangular matrices with diagonal entries 1, N_- be the set of all lower triangular matrices with diagonal entries 1, and $\mathbf{D}$ the set of all nonsingular diagonal matrices. Each of these sets is a subgroup of GL_n. A matrix A is called *strongly non-singular* if all its leading principal minors are nonzero. (These are the minors of the top left $k \times k$ blocks of A, $1 \le k \le n$.) It is a basic fact that every such matrix can be factored as

$$A = LDR, \tag{54}$$

where L, D and R belong to $\mathbf{N}_-, \mathbf{D}$ and $\mathbf{N}_+$, respectively [17, pp. 158-165]. This is used in the Gaussian elimination method in solving linear equations, and (54) is called the the *Gauss decomposition* of A. For representation theory, its significance lies in the consequence that every irreducible representation of GL_n is induced by a one-dimensional unitary representation (character) of $\mathbf{D}$. The set $\mathbf{B}$ consisting of all nonsingular upper triangular matrices (or, equivalently, all products LR with $L \in \mathbf{D}, R \in \mathbf{N}_+$) is another subgroup of GL_n. This is a solvable group. It is known that every irreducible representation of such a group is 1-dimensional.

Let ρ be a representation of GL_n in W. A vector v in W is called a *weight vector* if it is a simultaneous eigenvector for $\rho(D)$ for all $D \in \mathbf{D}$. If v is such a vector let

$$\rho(D)v = \lambda(D)v, \quad D \in \mathbf{D}.$$

Then λ is a complex valued function on $\mathbf{D}$ such that

$$\lambda(DD^{'}) = \lambda(D)\ \lambda(D^{'}).$$

So, if D=diag$(d_1, \cdots, d_n)$ then

$$\lambda(D) = d_1^{m_1} \cdots d_n^{m_n}$$

for some nonnegative integers $m_1, \cdots, m_n$, called the *associated weights*. For example, if V is the standard representation, then the only weight vectors are the basis vectors e_i, and the associated weights are $(0, 0, \cdots, 1, 0, \cdots, 0),\ 1 \le i \le n$. If $W = \Lambda^k(\mathbb{C}^n)$, then $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is a weight vector with weight $(1, 1, \cdots, 1, 0, \cdots, 0)$ where 1 occurs k times. If $W = \text{Sym}^k\mathbb{C}^n,\ e_1 \vee e_1 \vee \cdots, \vee e_1$ is a weight vector with weight $(k, 0, \cdots, 0)$.

A weight vector is called a *maximal weight vector* if it is left fixed by all elements of $\rho(\mathbf{N}_+)$, or equivalently, if it is a simultaneous eigenvector for all elements of $\rho(\mathbf{B})$. Thus, for the standard representation the only such vector is e_1. The associated weights in this case are called *highest weights*.

A fundamental theorem of representation theory says that an irreducible representation ρ of GL_n is determined completely by a unique maximal weight vector and associated weights $m_1 \geq \cdots \geq m_n$.

This is a bare-bones summary of vast area; see [12] or [13] for details.

Decomposing representations into their irreducible components is a central problem in the theory of representations. In particular, the tensor product of two irreducible representations is not always irreducuble and one wants to find its irreducuble components. This is an intricate business. One important outcome of the recent work of Knutson-Tao, and others is the following theorem:

Theorem 7. *Let $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$, $\gamma_1 \geq \cdots \geq \gamma_n$ be three n-tuples of nonnegetive integers. Let $V_\alpha, V_\beta, V_\gamma$ be the irreducible representations of $GL(V)$ with highest weights α, β, γ. Then V_γ is a component of $V_\alpha \otimes V_\beta$ if and only if there exists Hermitian matrices A and B such that $\alpha = \lambda(A), \beta = \lambda(B), \gamma = \lambda(A+B)$.*

The motivation for Gel'fand and Berezin in their study that led to the Lidskii-Wielandt inequalities was to unravel properties of tensor products of representations. This, in turn, led to Horn's conjecture. So, the connection between these problems is not new.

let us show Theorem 7 in acction in a simple example.

Consider irreducible representations of GL_2 with highest weights $\alpha = (4,2)$ and $\beta = (3,1)$. By results in Section 3, the admissible γ (that can occur as eigenvalues of $C = A + B$, where A, B are 2×2 Hermitian matrices with eigenvalues α, β) are the ones that satisfy the condition

$$(5,5) \prec \gamma \prec (7,3).$$

If we restrict γ to have integral entries, there are three possibilities

$$\gamma = (5,5),\ (6,4),\ (7,3).$$

By the rules of calculations with highest weights, we write

$$\begin{aligned}\alpha &= (4,2) = (2,2) + (2,0) = 2(1,1) + (2,0),\\ \beta &= (3,1) = (1,1) + (2,0).\end{aligned}$$

The weights $(1,1)$ correspond to the representation $\wedge^2 V$; $2(1,1)$ to two copies of this; $(2,0)$ to $\mathrm{Sym}^2 V$. So,

$$V_\alpha = (\textstyle\bigwedge^2 V)^{\otimes 2} \otimes \mathrm{Sym}^2 V,$$
$$V_\beta = \textstyle\bigwedge^2 V \otimes \mathrm{Sym}^2 V,$$
$$V_\alpha \otimes V_\beta = (\textstyle\bigwedge^2 V)^{\otimes 3} \otimes (\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 V).$$

The last factor can be decomposed by using the *Clebsch-Gordon formula* [13, p. 306], which gives in our particular situation

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 V = \mathrm{Sym}^4 V \oplus [\textstyle\bigwedge^2 V \otimes \mathrm{Sym}^2 V] \oplus (\textstyle\bigwedge^2 V)^{\otimes 2}.$$

Thus, we have the direct sum decomposition

$$V_\alpha \otimes \mathcal{V}_\beta = [(\textstyle\bigwedge^2 V)^{\otimes 3} \otimes \mathrm{Sym}^4 V] \oplus [(\textstyle\bigwedge^2 V)^{\otimes 4} \otimes \mathrm{Sym}^2 V] \oplus (\textstyle\bigwedge^2 V)^{\otimes 5}.$$

The three distict summands are irreducible representations corresponding to highest weights

$$3(1,1) + (4,0) = (7,3)$$
$$4(1,1) + (2,0) = (6,4)$$
$$5(1,1) = (5,5)$$

respectively. This is what theorem 7 predicted.

It is not easy to write down irreducible components of representations; intricate calculations with Young tableaux enter the picture. Theorem 7 gives another way of making a list of such representations. Thus from results in Section 7 we know that representations with highest weights $(3,2,2), (3,3,1)$ and $(4,2,1)$ are the irreducible components of the two representations of GL_3 with weights $(2,1,0)$ and $(2,1,1)$. It is an interesting exercise to write this decomposition explicitely.

The general problem of finding irreducible components of tensor products of irreducible representations of Lie groups (including GL_n) has been studied under the name "PRV Conjecture" and solved [26]. Several proofs of this conjecture have been given, and one more has come out of the recent work on Horn's inequalities.

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ABSTRACTS OF THE PAPERS SUBMITTED FOR PRESENTATION AT THE 66TH ANNUAL CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY, HELD AT THE VIVEKANAND ARTS, SARDAR DALIP SINGH COMMERCE AND SCIENCE COLLEGE, SAMARTHNAGAR, AURANGABAD-431 001, MAHARASHTRA, INDIA, DURING DECEMBER 19-22, 2000.

A: Combinatorics, Graph Theory and Discrete Mathematics :

A-1: q_v monodiffric functions continued from basic continuation domain, *S. Anuradha Boopathy, Hindustan College of Arts & Science, Coimbatore - 641 028, Tamil Nadu.*

Since the introduction of Monodiffricity (1941), many notions got evolved to define and discuss discrete analyticities like preholomorphicity (1944), q-analyticity (1972) and q-monodiffricity (1982). In all these theories all the three approaches of classical complex analysis namely Cauchy-Riemann equations, Cauchy's Intregral Theorem and Weirstrass approach to infinite series were taken over as analogues and transplanted and theories were developed. Analogues for integral discrete powers and discrete product are usual practices in every such theory. But in recent developments and findings this type of development of discrete analysis is not desirable. Considering analytic continuation from basic continuation domain to finite domain, discrete theories are being constructed. For this approach some simple special functions are defined to create basis for this finite vector space structure. This is the basis and foundation of the whole theory. In this attempt, finite domain $D_{m,n}$ is considered for the finite vector space structure in q_v-monodiffric theory. One simple special function in this domain is considered as basis. Derivatives of these special functions are found out and analysed. This note is a part of an attempt to establish q_v-monodiffric theory in the present dimension avoiding the parallel analogues in q-monodiffric theory.

A-2: Characterization of semientire graphs with crossing number 2, *D. G. Akka and J. K. Bano, Gulbarga - 585 101, Karnataka.*

The purpose of this paper is to give characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2. In addition, we establish necessary and sufficient conditions for vertex-semientire graphs and edge-semientire graphs to have crossing number 2 in terms of forbidden subgraphs.

A-3: Graph-neural network approach in cellular manufacturing on the basis of non-binary system, *Iraj Madhavi, O. P. Kaushal and M. Chandra,* (*Department of Production Engineering, College of Technology, G. B. Pant University of Agriculture and Technology, Pantnagar - 263 145, Uttar Pradesh*).

Group technology is a trategy of production management which gives a significant improvement in profitability. For designing Group technology based cells, it is essential to analyse parts and machines. The object of this analysis is to identify a family of parts which could be manufactured completely on a group of machines. Most of the methods of cell formation are based on machine-parts incident matrix. In this paper graph-neural network approach is proposed for cell formation with sequence data. In this approach sequence based incidence matrix is the main source of basic information for such analysis,

in the oresent algorithm, there is no need of assulptions of any parameter as needed in the existing algorithm such as CAST and FUZZY ART neural network approach. Also, the proposed approach has advantages of fast computations and reliability in the presence of bottleneck machines or bottlenech parts.

Four problems from the literature have been solved to demonstrate the merits of the proposed algorithm with relevant computer program. In this method information is derived from the sequence based incidence matrix. One converts this matrix to matrix of multigraph whose vertices correspond to the machines and whose edges correspond to those parts which are processed from one machine to another machine sequentially. The matrix of multigraph is a square matrix with size equal to machines and entries of the matrix are number of parts processed by respective machines.

A-4: Determination of bases for splitting matroids, *M. M. Shikare and Ghodratollah Azadi* (*University of Pune, Pune - 411 007, Maharashtra.*).

The splitting operation in graphs is well known. In this paper we characterize this operation in terms of the spanning trees of graphs and then extend these results to binary matroids.

A-5: Characterization of third order difference equations with constant coefficients, *N. Parhi and A. K. Tripathy* (*Berhampur University, Berhampur - 760 007, Orissa*).

In this paper, the structure of the solution space of

$$y_{n+3} + ry_{n+2} + qy_{n+1} + py_n = 0,\ n \geq 0 \tag{0.1}$$

is studied in depth keeping oscillatory / non oscilatory behaviour of solutions in view, where p, q, r are constants. The characteristic equation of (0.1) is given by

$$\lambda^3 + r\lambda^2 + q\lambda + p = 0. \tag{0.2}$$

We state two of our results.

Theorem 0.1. *Let $p < 0, q < r^2/3, r > 0$ and*

$$0 < p - \frac{qr}{3} + \frac{2r^3}{27} < \frac{2}{3\sqrt{3}}(r^2/3 - q)^{3/2}. \tag{0.3}$$

Then equation (0.1) is strongly nonoscillatory if $\alpha - (r/3) > 0$ and $\beta - (r/3) > 0$. If $\alpha - (r/3) < 0$ and $\beta - (r/3) < 0$ then either nonoscillatory solutions of (0.1) along with its trivial solution form a one-dimensional subspace of the solution space of (0.1) or every non-oscillatory solution $\{y_n\}$ of (0.1) may be written as

$$y_n = c_2\left(\beta - \frac{r}{3}\right)^n + c_3\left(\gamma - \frac{r}{3}\right)^n,\ c_3 \neq 0,$$

where α, β, γ are distint roots of (0.2).

Theorem 0.2. *Let $p < 0, 0 \leq q < r^{2/3}, r > 0$ and (0.3) hold. Then equation (0.1) is either strongly oscillatory or non-oscillatory solutions of* *(ref(one)) along with its trivial solution form a one dimensional subspace of the solution space of (0.1).*

A-6: Some simple special functions in $q(1)$-analytic theory on discrete geometric space,

N. Murugesan (*Sri Ramkrishna Engineering College, Coimbatore - 641 022, Tamil Nadu*).

Disrete Analysis is the study of functions defined on discrete spaces. R. P. Isaac (1941) is the first mathematician who developed a discrete theory known as monodiffric theory on discrete arithmetic space of Gaussian intregers followed by Ferrand's (1944) theory of preholomorphicity. Later Herman (1972) and Velukutty (1982) considered geometric discrete space on the complex plane and established q-analytic and q-monodiffric theories respectively.

In all these theories the classical concepts, in fact the whole theory, are represented analogously. In other words, a discrete analogue approach to weirstrass thery of classical style is developed. Equivalently, Cauchy approach and Cauchy-Riemann approach are also treated analogously. Recent discrete analysts felt dissatisfied, it being analogous theory, and a new method of approach using finite vector space structure for discrete functions and a basis for the same in terms of very simple special functions is introduced. This may lead one to describe every entity of scientific models in terms of recurrence relations instead of differential or difference equations.

In this new scenerio, solutions of some simple special functions: $W' = 0$, $W' = K(constant)$, $W = Z^n$ are found in this paper. We get usual result for $W' = 0$ and recurrence relations for the remaining two equations.

B: Algebra, Number Theory and Lattice Theory:

B-1: Generalized near-fields and bi-ideals of near-rings,

T. Tamizh Chelvam and S. Jayalakshmi (*Manonmaniam Sundaranar University, Thirunelveli - 627 012. Tamil Nadu*).

A subgroup B of $(N, +)$ is said to a bi-ideal of N if $BNB \cap (BN)^*B \subset B$. A near-ring is said to be subcommutative if $xN = Nx$ for every $x \in N$. A near-ring N is called a generalized near-field if for each $a \in N$ there exista a unique $b \in N$ such that $a = aba$ and $b = bab$. In this paper, for a certain class of near-rings, we obtain fourteen equivalent conditions for a generalized near-field in terms of bi-ideals.

B-2: An element primary to another element, *Manjarekar C. S. and Chavan N. S.* (*Shivaji University, Kolhapur - 416 004, Maharashtra*).

In this paper we introduce a new concept 'An element primary to another element' and using this concept we have generalized some results proved by Anderson and others. Some results of join principally generated multiplicative lattices in which every semi-primary element is primary are proved. If S is a weak M-lattice then the condition for a prime element to be maximal is proved. We have proved the following main results:

Theorem 0.3. *Suppose L is a join principally generated and satisfying a certain condition (*). If a strong join principal element d is primary to b and p is a minimal prime over $d \vee b$, then p is maximal in L.*

Theorem 0.4. *Suppose L satisfies the condition (*). If a strong join principal element d is primary to b and p is a minimal prime over $d \vee b$, then there exists a p-primary element q such that $d \leq q$ and hence $d \vee b \leq q$.*

B-3: Stable range one for different classes of rings, *Manoranjan Manoj Kumar* (*B. N. M. V. College, Sahugarh, Madhepura - 852 113, Bihar*).

The main purpose of this paper is to generalize the stable range one condition for a number of classes of rings and algebras. Stable range condition is obtained on a ring R, if $x, y \in R$ there is a unit $u \in R$ such that $x-u$ and $y - u^{-1}$ are both units. We also verify the stable range one condition on a ring in the following different cases.

(1) Algebra over an uncountable field in which all non-zero divisors are units and have no uncountable direct sums of non-zero one sided ideals.
(2) Von Neumann regular algebras over an uncountable field.
(3) For a finite Rickart C^*-algebras, strongly regular Von Neumann regular rings and strongly regular rings in which every element is a sum of a unit plus a central limit, all have stable range one.
(4) A strongly-regular ring, in which all powers of every nilpotent Von neumann regular element are Von Neumann regular, has a stable range one.

B-4: On balancing numbers, *Vyawahare A. W. and Deshpande M. N.* (*M. Mohota Science College and Institute of Science, Nagpur - 440 015, Maharashtra*).

Balancing numbers are defined by A. Behera and G. K. Panda (1998), for the positive integers.

We extend this concept further for arithmetic progression (A. P.) and study the properties of a balancing number r, with respect to a positive integer n in reference to A. P. The Diophantine equations, thus obtained, are complex as expected from the nature of the problem. Hence computer is used to solve these equations. the solutions revealed many surprizing phenomenon.

The functions generating balancing numbers and conditions for their existence are obtained. The relations for the balancing numbers are obtained. It is noticed that the balancing pair (n, r) has relations with the triangular numbers . The numbers n and r are in A. P.

B-5: Determination of the Lattice of projections in $M_n(Z_m)$, *Patil P. P. and Waphare B. N.* (*Jai Hind College, Dhule - 424 002 and University of Pune, Pune - 411 007, Maharashtra*).

It is known that the posets of projections in a weakly Rickart *-ring forms a lattice. In this paper we prove that the set of projections in the *-ring $M_n(Z_m)$ forms a lattice for various positive integers m and n even though the corresponding ring need not be a weakly Rickart *-ring. We also obtain some properties of these lattices.

B-6: On modular pair and covering property in posets, *Pawar M. M. and Waphare B. N.* (*S. S. V. P.'s Science College, Dhule - 424 005 and University of Pune, Pune - 411 007, Maharashtra*).

In this paper we provide a new definition of modular pair in posets. A useful characterization of a modular pair in posets is obtained. We also introduce a concept of a covering property in posets and connections between these concepts are obtained in posets.

B-7: Fuzzy IFP ideals of Near-rings, *Satyanarayan Bhavanarai, Shyam Prasad Kunchan and Pradeep Kumar T. V. (Nagarjuna University and Engineering College, G. Valleru - 521 356, A. P.).*

The concept of Fuzzy IFP ideal in near-rings N is studeied and proved that

(1) A fuzzy ideal μ of N has IFP if and only if μ_t (the leval ideal) is an IFP ideal of N for all $0 \leq t \leq 1$;
(2) N has strong IFP if and only if every fuzzy ideal of N has IFP;
(3) μ is a fuzzy ideal of N if and only if the related ideal β_s is a fuzzy IFP ideal for all $s \in [0,1]$.

B-8: Some characterizations of completeness for trellises in terms of joins of cycles, *Parameshwara Bhatta S. and Shashirekha H. (Mangalore University, Mangalore - 574 199).*

In this paper we obtain some new characterizations of completeness for trellises by introducing the notion of a cycle-complete trellis. One of our results yields, in particular, a characterization of completeness for trellises of finite length due to K. Gladstien.

C: Real and Complex Analysis (Including Special Functions, Summability and Transforms):

C-1: On a translated Cesáro type summability method II, *D. Singh (Dehradun - 248 012, Uttarakhand).*

The paper deals with summability method $(D_\lambda, \alpha), \lambda > -1, \alpha > 0$ which is a generalization of $(D, \alpha), \alpha > 0$ summability method as defined by Ishiguro, K. [Acad. Roy. Belg. & Sci. Me'm coll in 8, **35** (1965)]. Main theorems of the paper are:

Theorem 0.5. *Let $\lambda > -1, \alpha > 0$ and $r \geq 0$. Suppose that $f(x)$ is summable (C, r) to the sum s and that the integral*

$$\int_1^\infty \frac{f(x)}{x^\lambda + 2} dx$$

converges. Then the function $f(x)$ is summable (D_λ, α) to the sum s.

Theorem 0.6. *Let $\lambda > \mu > -1, \alpha > \beta > 0$. Suppose that $f(x)$ is summable (D_λ, α) to the sum s. Then $f(x)$ is summable $(D\mu, \beta)$ to the sum s.*

C-2: On product summability methods $(D, \alpha, \beta)(C, \ell, m)$ **of functions,** *B. P. Mishra and Sanjal Kumar Singh* (*D. D. U. Gorakhpur University, Gorakhpur - 273 009, U. P.*).

The definition of product summability method $(D, k)(C, \ell)$ for functions was given by S. N. Pathak in his Ph.D. thesis, "Some investigation on summability of functions", gorakhpur university, Gorakhpur (1986), and investigated some of its properties. In this paper $(D, \alpha, \beta)(C, \ell, m)$ summability for functions are defined and some of their properties are investigated, $(\alpha > 0, \beta > -1, \ell > 0$ and $m > -1)$.

C-3: On sets with Baire property in topological spaces, *S. Basu* (*Maulana Azad College, Calcutta - 700 013, West Bangal*).

Steinhaus proved that if a set has a positive Lebesgue measure in the line then its distance set contains an interval. He obtained even more stronger forms of this result and they are concerned with the mutual distances between points in an infinite sequence of sets. Almost similar theorems, in the case where distance is replaced by mutual ratio, were established by Bose-Majumdar. In the present paper, we endeavour to set forth some results related to sets with the Baire property in locally compact topological spaces - particular cases of which yield the Baire-category analogues of the above results of Steinhaus and its corresponding form for ratios by Bose-Majumdar.

C-4: On a subclass of meromorphic univalent functions with positice coefficients, *S. R. Kulkarni & Sou. S. S. Joshi* (*Ferguson College, Pune - 411 004, Maharashtra*).

The aim of the present paper is to introduce a class $\sum_p(\alpha, \beta, \gamma)$ consisting of the functions of the form

$$f(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic and univalent in punctured unit disc $U^* = \{z : 0 < |z| < 1\}$. We have obtained representation formula, distortion theorem and coefficient estimates for the class: $\sum_p(\alpha, \beta, \gamma)$. Various results obtained in the present paper are shown to be sharp.

C-5: On a new sequence, *M. B. Dhakne* (*Dr. B. A. M. University, Aurangabad - 431 004, Maharashtra*).

Our aim is to define a new sequence and derive its mathematical properties.

C-6: A general Inversion formula - I, *B. I. Dave and Manisha Dalbhide* (*The M. S. University of Baroda, Vadodara - 390 002, Gujarat*).

In the year 1983 Gessel and Stanton (Tran. Amer. Math. Soc., Vol. 277 No. 1, 173–201) unified and proved some q-inverse series relations in more general form. Here we consider its ordinary version, provide a further extension and prove it. A variety of special cases including generalized orthogonal polynomials such as Racah polynomial, Wilson polynomial, etc. are illustrated.

C-7: On the inequalities for the derivative of a polynomial, *Prasanna Kumar N. and Shenoy B. G.* (*Mangalore University, Mangalore - 574 199, Karnataka*).

If $p(z)$ is a complex polynomial of degree n then concerning the estimate of $|P(z)|$ on the unit disc $|z| = 1$ we have a good number of results due to Tur*á*n, Schaeffer and Aziz.

Here we derive an inequality for the derivative of a complex polynomial, which improves all the earlier results.

C-8: Mellin transform of tempered Boehmians, *P. K. Banerji and Deshna Loonker* (*J. N. V. University, Jodhpur - 342 005, Rajasthan*).

The theories of generalized functions have been developed in the recent past by various concepts of distributions introduced by Soboleff and Schwartz in 1936 and 1935 respectively. Boehmians, which is discussed in the present article under the above title, is one of the generalizations of the generalized function, and is the motivation for studies of regular operators., introduced by Boehme (1973). Having given relevant definitions and terminologies we have established, in this article, two theorems on Mellin transform of tempered Boehmians.

C-9: I-function and its applications in two boundary value problems, *Ku. Rinu Shrivastava and S. S. Shrivastava* (*Govt. Model Science College, Rewa - 486 001, M. P.*).

In the present paper, we make application of I-function of one variable to solve two boundary value problems on heat conduction in a rod and deflection of vibrating formula involving the I-function.

C-10: A new class of three dimensional expansion for Fox's H-function involving Hermite polynomials, Bessel functions and Bessel polynomials, *Smt. Ritu Shrivastava* (*Govt. Model Science College, Rewa - 486 001, M. P.*).

In this paper we present a new class of three dimensional expansion of Fox's H-function involving Hermite polynomials, Bessel functions and Bessel polynomials.

C-11: Dual series equations and fractional calculus, *D. N. Vyas and P. K. Banerji* (*M. L. V. Textile Institute, Bhilwara - 311 001 and J. N. V. University, Jodhpur - 342 005, Rajasthan.*).

Noble (1963), Lowndes (1968) and Askey (1968) have considered dual series equations and obtained theor solutions by multiplying factor technique. Srivastava (1972) and Sneddon & Srivastava (1964) have also suggested solution of dual series equations by employing the same technique, as of Noble, Lowndes and Askey. Some dual series equations have also been solved by transforming them into Abel Integral equation. In the present paper, the dual series have been solved by employing fractional integral operators. The introduction of this paper includes all the relevant concepts.

C-12: On multiindex multivariable Hermite polynomials, *H. S. P. Shrivastava* (*Govt. Arts and Science P. G. College, Ratlam - 457 001, M. P.*).

In the present paper multiindex multivariable Hermite polynomial in term of series and generating function is defined. Their basic properties, differential and pure recurrence relations, differential equations, generating function relations and expansion have been established. few deductions are also obtained.

C-13: Index integral transform on L^p-spaces, *P. K. Banerji and Abha Purohit* (*J. N. V. University, Jodhpur - 342 005, Rajasthan*).

Certain integral transforms, involving the integration with respect to the index of the special functions occuring in the kernel, are called index integral transform. Until recently, not much literature is available except a monograph by Yakubovich (1996). In the present paper, we discuss one such index integral transform on L_p-spaces by suggesting its mapping properties, in the form of three theorems.

C-14: A generalized criterion for normal families, *Subhash S. Bhoosnurmath and Anand G. Puranik* (*Karnataka University, Dharwad - 580 003, Karnataka*).

In 1964 Hayman posed the following conjecture. Let a ($\neq 0$) and b be any two finite complex numbers and n (≥ 5) be an integer. If F is a family of meromorphic functions in a domain D and for each $f \in F$ and $z \in D$, if $f(z) - af^n(z) \neq b$, then F is normal in D. Drasin proved the above conjecture for analytic case. Later, Langley and others improved the results for meromorphic case and generalized the results.

In this paper, we obtain a result on normal families using the techniques of Langley and Yang Lo. The results of Langley on normal families are generalized. The techniques used here differ from the techniques of Drasin and Pang. We also use results of Ku and Yang C. C.

C-15: Strong summability of functions based on (D, k) (C, α, β) summability methods, *Babban Prasad Mishra and Narayan Mishra* (*D. D. U. Gorakhpur University, Gorakhpur - 273 009, U. P.*).

The definition of strong summability method (D, k) (C, ℓ) for functions was given in [Ph.D. thesis "some remarks on summability methods of functions", Gorakhpur University, Gorakhpur (1992)], and some of its properties were investigated by B. N. Singh. In this paper, (D, k) (C, α, β) strong summability method for functions are defined and some of their properties are investigated $(k > 0, \alpha \geq 1, \beta > -1)$.

C-16: Strong summability of infinite series on product summ-]linebreak ability methods A_λ (C, α, β), *Babban Prasad Mishra and Parthiweshwar Jee Pandey* (*D. D. U. Gorakhpur University, Gorakhpur - 273 009, U. P.*).

The definition of strong summability method A_λ (C, α) for series was given by Mishra and Khan ["Strong summability of infinte series: On product summability methods I (presented in the 88^{th} Indian Science Congress, Pune) and some of its properties are investigated. In this paper A_λ (C, α, β) strong summability for series is defined and some of its properties are investigated.

C-17: Strong summability of functions based on (D, α, β) (C, ℓ) summability methods, *Babban Prasad Mishra and Anupama Srivastava* (*D. D. U. Gorakhpur University, Gorakhpur - 273 009, U. P.*).

The definition of strong summability method $(D, k)(C, \ell)$ for functions was given by B. N. Singh in his Ph.D. Thesis [Some remarks on summability methods of functions, Gorakhpur University (1002)], and some of its properties were investigated. In this paper, (D, α, β) (C, ℓ) summability for functions are defined and some of their properties are investigated, $(\alpha > 0, \beta > -1, \ell \geq 1)$.

C-18: On bilinear generating integrals, *M. S. Chaudhary and P. G. Andhare* (*Shivaji University, Kolhapur - 416 004, Maharashtra*).

Like Hermite polynomials $H_n(x)$, S. K. Sinha [Maths. Edn. **34** June 2000] has defined three polynomials $H_{n,I}(X), H_{n,II}(X)$ and $H_{n,III}(X)$. In this paper we shall obtain usual bilinear generating integrals for these polynomials.

C-19: On Hermite polynomials, *T. B. Jagtap and P. G. Andhare* (*R. B. N. B. College, Srirampur - 413 709, Maharashtra*).

In the present paper we construct biorthogonal polynomials $S_n(x, k, 1)$ and $T_n(x, k, 1)$ suggested by general Hermite polynomials studied by Thakare and Karande. We also obtain generating functions, recurrence relations for these polynomials.

C-20: On Laplace transform of Air's functions, *V. D. Raghate* (*S. S. G. M. College of Engineering, Shegaon - 444 203, Maharashtra*).

In this paper we have established the Laplace transform of Air's functions and the L-transforms of integrals of Air's functions.

C-21: Some bilateral generating functions involving gegenbour polynomials, *M. I. Qureshi, N. U. Khan and M. A. Pathan* (*Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Milia Islamia, New Delhi - 110 025 and Aligarh Muslim University, Aligarh - 202 002, U. P.*).

In this paper we obtain three interesting bilateral generating functions gor gegenbour polynomials $C_m^b(x)$ associated with hypergeometric functions ${}_2F_1$, ${}_1F_2$ and ${}_3F_2$ which are not recorded earlier. Our results are obtained with the help of series manipulation technique. A few generalizations of these results are also considered.

C-22: Some transformations involving Exton's double hypergeometric function $\mathcal{H}$, *M. I. Qureshi and M. S. Khan* (*Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Milia Islamia, New Delhi - 110 025 and Aligarh Muslim University, Aligarh - 202 002, U. P.*).

This paper deals with two interesting transformation formulae involving Exton's double hypergeometric function

$$\mathcal{H}^{1:D:1:0}_{0:E:2:0}[-x, x] \quad \mathcal{H}^{1+D:0:0:0}_{E:0:1:0}[\frac{y^2}{4}, y].$$

These have been derived by utilizng Saalschotz and two of Gauss summation theorems. Well known quadratic transformations for Gauss ${}_2F_1$, Whipple transformation and Watson's summation theorem for ${}_3F_2$ are obtained as special cases of our main results.

C-23: Multicriteria and parametric optimization problems with bottleneck objectives, *V. H. Bajaj* (*Department of Statistics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad - 431 004, Maharashtra.*).

In this paper discrete multicriteria decision problems and parametric single criteria problems are analysed. In particular, such decision problems are considered where one criteria is sum function while the remaining criteria are bottleneck functions. In this case the multicriteria problem can be reduced to a linear parameter sum problem. By solving it parametrically we can generate the set of all efficient solutions to the original multicriteria problem.

C-24: A (T, S_1) policy inventory model for deteriorating items with time proportional demand, *Sunil Kawale and V. H. Bajaj* (*Department of Statistics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad - 431 004, Maharashtra.*).

The paper considers the Economic Order Quantity model in which the demand rate is changing linearly with time. The deterioration rates are assumed to be constant when commodity is in transportation and is on hand. Shortages are allowed for the situation of fixed-cycle time and increasing (or decreasing) levels of order quantities for the time proportional demand. The results are supported with the help of an example.

D: Functional Analysis:

D-1: A development of weaker forms of commuting maps, *Anita Tomar and S. L. Singh* (*Government Post Graduate College, Kodwara - 246 419 and Gurukul Kangri University, Hardwar - 249 404, Uttarakhand*).

Weak commutativity of a pair of maps was introduced by S. Seesa in 1982 in fixed point considerations. Thereafter, a number of generalzations of this notion has been obtained. The purpose of this note is to present a brief development of main weaker forms of commutativity.

D-2: Some generalizations of the spaces of almost convergent and strongly almost convergent functions, *Babban Prasad Mishra and Mamta Rani Jaiswal* (*D. D. U. Gorakhpur University, Gorakhpur - 273 009, U. P.*).

In this paper we generalize the definitions of almost convergent and strongly almost convergent functions defined by G. Das and S. Nanda [J. Indian Math. Soc. 59 (1993), 61–64] and investigate some of their properties.

D-3: Coincidence and fixed point theorems for a family of mappings on Menger spaces using asymptotic regularity, *P. K. Shivastava, N. P. S. Bava and Vivek Raich* (*Government P. G. College. Balaghat - 452 001 and Government P. G. College, Rewa - 486 001, M.P.*).

Using asymptotic regularity of mappings and combining ideas of Stojakovic, Singh and Pant, some coincidence and fixed point theorems for a family of mappings on an arbitrary set with values in Menger space have been established. A few general fixed point theorems for a family of mappings on Menger spaces can be derived from these results. The results of this paper generalize a number of known results as well.

D-4: Generalized vector valued difference sequence space defined using a modulus function, *P. D. Srivastava and C. Gnanseelan* (*I. I. T. Kharagpur, Kharagpur - 721 302, West Bengal*).

in this paper we introduce and study a generalized class of vector valued difference sequence space $F(X, \Delta, f)$ where f is a modulus function. Some topological and algebraic properties of the space $F(X, \Delta, f)$ are investigated. The properties such as separability, convergence criteria, etc. are studied in a particular class $F(x, \Delta)$ for suitable F.

D-5: Some fixed point theorems on product of uniform spaces, *Sunder Lal and Sanjay Kumar Gupta* (*Institute of Basic Science, Khandari, Agra - 282 202, U. P.*).

The Banach Contraction Principle guarentees a unique fixed point for each contraction mapping on a complete metric spaces. Nadler found a fixed point on the product of metric spaces $X \times Z$ for mappings on it which are uniformly continuous and also contraction in the first variable. Fora improved Nadler's result on larger class of spaces and for larger class of mappings. Tarafdar generalized the Banach Contraction Principle on a complete Haudorff uniform space. In this paper we generalize results of Nadler as well as Fora on uniform spaces.

D-6: Kothe-Toeplitz and topological duals of spaces of strongly summable vector sequences, *J. K. Srivastava* (*D. D. U. Gorakhpur University, Gorakhpur - 273 009, U.P.*).

In this paper we characterize generalized α-, β- and γ-duals of the spaces of strongly summable vector sequences $l(x,p), c_0(x,p), c(x,p)$ and $l_\infty(x,p)$, where sequences are taken from a locally convex space X. Besides the investigation of some topological properties of these spaces we determine topological duals of $l(x,p), c_0(x,p)$ and $c(x,p)$ when considered with the natural linear topologies on them generated by the family of paranorms.

D-7: Recent trend in metric fixed point theory, *P. P. Murthy* (*Dr. S. R. K. Goverment Arts College, Yanam - 533 464, Puducherry*).

The main intent of this paper is to display the present trend of obtaining common fixed points by using the concepts of *continuity, non-commutativity and completeness* of the space. The concept of commuting maps was introduced initially by Jungck during the generalization of most interesting and important theorem due to Banach who proved the existence theorem in a complete metric space. Our intention is to look in to certain types of non-commuting maps in metric and their related abstract spaces. We can find their applications in *Dynamic Programming, Approximation Theory, Variational Inequalities and Solution of Nonlinear Integral Equations, Fuzzy Set Theory* and many other branches of Science and Engineering. The most practical experience is in *Optimization Theory* where we use this theory to launch most sophisticated *Setellites* in to the right orbits. Also, I believe that this theory can be used at the time of operation of missiles during war for accurate penetration.

D-8: Common fixed points of weakly compatible maps on fuzzy metric spaces-II, *P. K. Shrivastava, N. P. S. Bava, V. V. Raich and Pankaj Singh* (*Government P. G. College, Balaghat - 452 001, Government P. G. College, Seoni - 480 001 and Government Science College, Rewa - 486 001, M. P.*).

Eliminating completely the continuity requirement and using the concept of weak compatibility, some common fixed point theorems for not necessarily continuous mappings are obtained in fuzzy metric spaces. The results of this paper are the variation / extension / generalization of the results of Cho. et. al. which require the space X to be complete, the pairs $\{P.S\}$ and $\{Q, T\}$ be compatibe of type (β) and S and T continuous.The above results, on the other hand, require that $ST(X)$ be complete, but weaken the compatibility of type (β) to weak compatibility and completely eliminate continuity of S and T.

D-9: Frechet algebras with Laurent series generator and characterization of function algebras on the annulus and the circle, *S. J. Bhatt, H. V. Dedania and S. R. Patel* (*Sardar Patel University, Vallabh Vidyanagar - 388 120, Gujarat*).

Banach and Frechet algebras with a Laurent series generator are introduced leading, via the discrete Beurling algebras, to the functional analytic characterizations of the holomorphic function algebra on the open annulus as well as the $C^\infty-$algebra on the unit circle.

D-10: Common fixed point of weakly compactible mappings, *Pankaj Singh, N. P. S. Bawa and Praveen Kumar Shrivastava* (*Government Model Science College, Rewa - 486 001, M. P.*).

In this paper, eliminating completely the continuity requirement and replacing the condition of semi compatibility by weak compatibility, some common fixed point theorems have been proved for following contracting conditions

$$d(Ax, Bv) \leq \phi(max\{d(Sx, Tv), d(Ax, Sx), \ d(Bv, Tv), \\ (1/2)[d(Ax, Tv) + d(Bv, Sx)]\}).$$

For all $x, y \in X$, $\phi : [0, \infty] \to [0, \infty]$ is a non-decreasing function, continuous from the right, with $\sum \phi^n(t) < \infty$ for all $t > 0$ (note that $\phi(t) < t$ for all $t > 0$).
These results generalize the results of Brijendra Singh et. al. and many others.

D-11: Fixed point for contraction involving single-valued and multi-valued mappings, *Pankaj Singh, Praveen Kumar Shrivastava and N. P. S. Bawa* (*Government Model Science College, Rewa - 486 001, M. P.*).

in this paper, using the concept of compability of type (N) and asymptotic regularity of mappings, some coincidence and fixed-point theorems are proved under following contractive conditions

$$H(Sx,Ty) \leq \phi(max\{D(fx,Sx),\ D(Fy,Ty),\ D(Fx,Ty), D(Fy,Sx), d(fx,fy)\}),$$

and $\phi(t) \leq qt$ for each $t > 0$ and some $q \in (0,1)$.

Our results generalize the results of Singh, H. A. and Cho, who have generalized the results of Rhoades, Singh and others.

E: Differential Equations, Integral Equations and Functional Equations:

E-1: Asymptotic behaviour of solutions of a second order quasilinear difference equation, *E. Thandapani and Lourdu Marian* (*Periyar University, salem - 636 011, Tamil Nadu*).

Consider the quasilinear difference equation

$$\Delta((|\Delta y_n|)^{\alpha-1}\Delta y_n) + q_{n+1}f(y_{n+1}) = 0,\ n = 0,1,2,\cdots, \tag{0.4}$$

under the condition that $\lim_{n\to\infty}\sum_{s=0}^{n} q_{s+1}$ exists and finite. Necessary and sufficient conditions are given for the equation (0.4) to have solutions which behave asymptotically like linear functions.

E-2: Oscillation theorems for general quasilinear second order difference equations, *E. Thandapani and K. Ravi* (*Periyar University, salem - 636 011, Tamil Nadu*).

The authors consider difference equation of the form

$$\Delta(a_n(\Delta y_n)^{\alpha}) + \phi(n, y_n, \Delta y_n) + q_n f(y_{n+1}) = 0, \tag{0.5}$$

where $a_n > 0, q_n > 0, f$ and ϕ are continuous real valued functions and $uf(u) > 0$ for $u \neq o$. They give oscilation results for the equation

(0.5). Examples are included to illustrate results.

E-3: Global existence result for an abstract nonlinear integrodifferential equation, *M. B. Dhakne and G. B. Lamb* (*Dr. B. A. M. University, Aurangabad - 431 004, Maharashtra*).

In this paper we study the global existence of solutions of nonlinear Volterra integrodifferential equations in Banach spaces. The main tool employed in our analysis is based on an application of the Leray-Schauder alternative and rely on a priori bounds of solutions.

E-4: On distance between zeros of solutions of third order differential equations, *N. Parhi and S. Panigrahi* (*Berhampur University, Berhampur - 760 007, Orissa*).

The lower bounds of spacings $(b-a)$ or $(a^{'}-a)$ of two consecutive zeros or three consecutive zeros of solution of third order differential equation of the form

$$y^{'''} + q(t)y^{'} + p(t) = 0 \tag{0.6}$$

are derived under very general assumptions on p and q. These results are then used to show that $(t_{n+1} - t_n) \to \infty$ or $(t_{n+2} - t_n) \to \infty$ as $n \to \infty$ under suitable assumptions on p and q, where $\{t_n\}$ is a sequence of zeros of an oscillatory solution of (0.6). The Opial-type inequalities are used to derive lower bounds of the spacings $(d-a)$ or $(b-d)$ for a solution $y(t)$ of (0.6) with $y(a) = 0 = y^{'}(a)$, $y^{'}(c) = 0$ and $y^{''}(d) = 0$, where $d \in (a, c)$ or $y^{'}(c) = 0, y(b) = 0 = y^{'}(b)$ and $y^{''}(d) = 0$, where $d \in (c, b)$.

E-5: Some oscillation and nonoscilation theorems for fourth order difference equations, *E. Thandapani and I. M. Arockiasamy* (*Periyar University, Salem - 636 011, Tamil Nadu*).

Sufficient conditions are established for the oscillation of all solutions of the fourth order difference equation

$$\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))) + q_n f(y_{n+1}) = h_n, \quad n = 0, 1, 2, 3, \cdots \tag{0.7}$$

where Δ is the forward difference operator given by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}, \{b_n\}, \{c_n\}, \{q_n\}$, & $\{h_n\}$ are real sequences and f is a real valued continuous function. Also, sufficient conditions are provided which ensure that all non-oscillatory solutions of (0.7) approach to zero as $n \to \infty$. Exampleas are inserted to illustrate the results.

E-6: Nonlinear oscillation of second order neutral delay differential equations, *E. Thandapani and R. Savithri* (*Periyar University, Salem - 636 011, Tamil Nadu*).

Some new oscillation criteria for the neutral differential equation

$$(a(t)(x(t) + p(t)x(t-\tau))^{'})' + q(t)f(x(t-\sigma)) = 0, \quad t \geq t_0,$$

where $\tau > 0, , \sigma > 0$ are established under the assumption $\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty$.

E-7: Oscillatory behaviour of nonlinear neutral delay difference equations, *R. Arul* (*Kandasamy Kandar College, Velur - 638 182, Tamil Nadu*).

Sufficient conditions are established for the oscillation of all solutions of a neutral difference equation of the form Sufficient conditions are established for the oscillation of all solutions of a neutral difference equation of the form

$$\Delta(y(n) - p(n)y(n-\tau)) + q(n)\prod_{j=1}^{m} |y(n-\sigma_i)|^{\alpha_i} \; sgn \; y(n-\sigma_i) = 0,$$

where $p(n) > 0, q(n) \geq 0,\ \tau,\ \sigma_i$ are non negative integers, $\alpha_i > 0$ and $\sum_{i=1}^{m} \alpha_i = 1$.

F: Geometry:

F-1: On the regular polygons inscribed by a circle, *Ashwani Kumar Sinha* (*M. M. Mahila College, Ara - 802 301, Bihar*).

The aim of the present note is to investigate the generalized formulae for the areas, perimeters and internal angles of the regilar polygons inscribed by a given circle and to deduce the expressions for the areas and circumference of the circle using limit concept.

F-2: On framed hyperbolic metric submanifolds, *R. N. Singh* (*A. P. S. University, Rewa - 486 003, M. P.*).

In this paper necessary and sufficient conditions have been obtained for hyperbolic framed metric submanifolds of almost hyperbolic Hermite manifold to have a certain structure. The conditions for hyperbolic framed metric submanifolds of hyperbolic Kahler manifold have been obtained. Also the framed hyperbolic pseudo-umbilical submanifolds have been defined and its various results have been studied.

G: Topology:

G-1: Semi generalized seperation axioms in topology, *G. B. Navalagi* (*Department of Mathematics, G. H. College, Haveri - 581 110, Karnataka*).

N. Levine, in 1963, defined the concept of semiopen set and in 1970 introduced the concept of generalized closed sets in a topology. Since then there is vast progress in thr field of generalized open sets. In 1975, S. N. Maheshwari and R. Prasad defined the concepts of semi separation axioms using the semi open sets of N. Levine, namely, semi-T_i, $i = 0, 1, 2$ axioms. In this paper we introduce and study the new concepts called semi generalized separation axioms namely, sg-$T_i, i = 0, 1, 2$ sg-$R_i, i = 0, 1$ axioms using the semi generalized open sets due to Bhattacharya and Lahiri. Also we define the $|Psi-$ open sets using the technique of $|Psi-$ closed sets in topology due to M. K. R. S. Veera Kumar and use them to define the Ψ-separation axioms, namely, Ψ-$T_i, i = 0, 1, 2$; Ψ-R_i, $i = 0, 1$. The class of semi separation axioms imply Ψ-separation axioms, and Ψ-separation axioms imply semi generalized seperation axioms . The basic properties, neighbourhoods and closures of semi generalized Ψ-open sets and open sets are studied in this paper.

G-2: Free actions of finite groups on products of symmetric powers of even spheres, *Satya Deo and Jitendra Kumar Maitra* (*R. D. University, Jabalpur - 482 001, M. P.*).

This paper answers a question on the existence of free actions on products of symmetric powers of even spheres. The main objective is to show that a finite group G acts freely on a finite product of symmetric powers of even-dimensional spheres iff it can act freely on a suitable product of even-dimensional spheres themselves.

G-3: Some fixed points theorems for commuting mappings in $2-$**uniform spaces,** *Sunder Lal and A. K. Goyal* (*Institute of Basic Science, Khandari, Agra - 282 202, U. P. and M. S. J. Government P. G. College, Bharatpur - 321 001, Rajasthan*).

Jungck (Amer. Math. Monthly, 1976) generalized the Banach Contraction Principle by introducing a contraction condition for a pair of commuting self mappings on a complete metric space. In recent years Jungck's theorem has been extended and generalized in various ways and in various structures such as uniform spaces, metric spaces, 2-metric spaces, etc.

In this paper some results on common fixed points for a pair of commuting mappings defined on sequentially complete Hausdorff 2-uniform space have been obtained. Our work extends fixed points theorems due to Ganguly and Bandopadhyay (Bull. Calutta Math. Soc., 1991) and Singh (Math. Sem. Notes, 1977).

G-4: Optimal solutions related to linear transformations of triangulations, *P. C. Tiwari, D. P. Sukla and N. P. S. Bawa* (*Government Model Science College, Rewa - 486 001, M. P.*).

In this paper we have tried to find the optimal solution of linear transformations of the triangulations $J_k^{'}(n)$ and $J_k^{''}(n)$ in the sense of minimizing their average directional density for a given mess size.

These optimal solutions are also extension of linear transformation of the triangulation J_1, K_1, and J' determined by M. J. Todd.

G-5: Common fixed point theorems for three mappings in 2−uniform spaces, *Sunder Lal and A. K. Goyal* (*Institute of Basic Science, Khandari, Agra - 282 202, U. P. and M. S. J. Government P. G. College, Bharatpur - 321 001, Rajasthan*).

In this note some results on fixed points for three maps are obtained in 2−uniform spaces.

Ganguly (Indian J. Pure Appl. Math., 1986) proved some fixed point theorems in a sequentially complete Hausdorff uniform space by considering contractive type condition for a triplet of mappings.

In this paper we establish some common fixed point theorems for three mappings in sequentially complete Hausdorff 2−uniform spaces. Our results generalize the results of Ganguly (Indian J. Pure Appl. Math., 1986), Rhoades (Math. Nachr., 1979) and Singh, Tiwari and Gupta (Math. Nachr., 1980).

H: Measure Theory, Probability Theory and Stochastic Processes, and Information Theory:

H-1: Use of fuzzy neural network to study material mix in cement plant, *W. B. Vasantha, N. R. Neelkantan and S. Ramathilakam* (*I. I. T. Madras, Chennai - 600 036, Tamilnadu*).

In this paper we adopt fuzzy neural networks and fuzzy relation equation to find the correct proportion of raw mix so that the desired quality of the clinker is achieved. This is done by taking experts opinion about the proportions by giving fuzzy weights. These weights are adjusted (or) changed a finite number of times till the error function reaches zero. This is equivalent to the study that the set point values and the measure value are identical. This paper consists of four sections. In the first section we give introduction and describe the statement of the problem. In the second section we give basic definitions of fuzzy theory and describe the equations used for singular value decomposition. In third section we adopt and use fuzzy neural network [FNN] and fuzzy relation equations to our problem. In section four conclusion is given about the problem.

H-2: Market promotion via electronic media, *J. Vaideeswaran and S. K. Srivasta* (*Department of Electrical Eingineering, Madras Institute of Technology, Chennai - 600 044 and School of ECE, Anna University, Chennai - 600 025, Tamil Nadu*).

Data and knowledge Engineering leads to information representation. Presenting the information in a number of ways to qualify a data is a challanging task in today's regime of e-commerce and marketing. The above task depends much on data mining activity which has to process a huge and dynamic statistical database for accurate measures. In the realm of computer, for the use of the subject topic discussed in this paper, time and space complexity have to be addressed. More often this involves a select end users on a chosen domain.

Thus, information theory is a quantified subject with the objective of how to quantify. An example case study is being brought out.

H-3: Fuzzy control to study crude fractionator, *W. B. Vasantha, N. R. Neelakanthan and S. R. Kannan* (*I. I. T. Madras, Chennai - 600 036, Tamil Nadu*).

Lars E. Ebbesen in 1992 has studied about the crude operating in kalundborg refinery. This refinery operates with 10 to 12 different crudes on a regular basis. In the study of the crude fractionator he mainly uses mass and enthalpy balance method. However, at the end of his study he has made it clear that in the case of Naphtha 95 % distillation stayed within 1 degree centigrade of its setpoints. After two hours the quality during crude switches was different indicating a lower quality. The main reason for this may be due to the improper prediction of the set point of temperature for naphtha. To overcome this situation we in this paper adopt, using the ranges of temperature for the naphtha distillation, the fuzzy control to obtain the probable set point.

We feel that using this method the prediction of setpoint for naphtha will be better as this gives a range within which the setpoit should fall thus consequenly giving the ranges of temperature for the maximum distillation of naphtha. Thus in this paper we have used only the temperature for naphtha by different distillation of crude refinery, and the method to obtain the setpoint is distictly different. The paper consists of three sections. In the first section we give fuzzy control technique used by us. In the second section we describe the crude refinery. And in the third section we adopt fuzzy control and obtain the setpoint for naphtha.

I: Numerical Analysis, Approximation Theory and Computer Science:

I-1: A simplification of authentication protocols in cryptology,

Sunder Lal, Anil Kumar and Sanjay Rawat (*Institute of Basic Science, Khandari, Agra - 282 002, U. P.*).

In traditional communication systems the parties can recognize each other in many ways, e.g., by writing, by voice, etc. but in electronic communication through computers, i.e., network of distributed systems, it is necessary to adopt ways by which communicating pairs can satisfy themselves about each other's identity. An authentication protocol guarentees this. A general proceedure to do so is through mathematical cryptography using its various tools. Some authentication protocols are shared encryption keys and others are based upon public key cryptosystems which distribute public key of principles and use them to establish shared secrets. A scheme for message authentication requires that communicating parties can efficiently produce an authentication tag and can verify whether a given sequence of characters is authentication string or not. As far as the signature schemes are concerned we do not require universal verification and so scheme for message authentication can be viewed as private key version of signature schemes. our work deals with the study of Fiat-Shamir and G-Q Protocols and use of geometrical structure of $e^{i\theta}$.

I-2: C-rational cubic spline involving tension parameters, *M. Shrivastava and J. Joseph* (*Rani Durgavati University, Jabalpur - 482 001, M. P.*).

In the present paper $C-$piecewise rational cubic spline function involving tension parameters is considered which produces a monotonic interpolant to a given monotonic data set. it is observed that under certain conditions the interpolant preserves the convexity property of the data set. The existence and uniqueness of a $C-$rational cubic spline interpolant are established. The error analysis of the spline interpolant is also given.

I-3: Searching for global stability for interference pattern, *M. Datta, S. K. Jain and Manu Pratap Singh* (*Department of Computer Science, Institute of Basic Science, Khandari, Agra - 282 002, U. P.*).

The global stability of any network for learning of any pattern is convergent to fixed points for all inputs and all choices of system parameters. That global stability involves some power and some limitation. The power comes from its dimension independence, its non-linear generality and its exponentially fast convergence to fixed points. But it is limited because in general it does not tell us where the equilibrium occur in the state space. Equilibration to fixed points provide pattern recall or decoding. If we provide any pattern to network for recalling it should search for earlier decoded patterns. It traverses in the basins of attractions that represent the stable states in the search of global

minima. If decoded correctly then it should converge to global stability state, i.e., the zero energy state or the zero Lyapunov function for the state, i.e., $1 \geq 0$.

In the present paper we will consider the recalling of two patterns simultaneosly and then search their global stability individually. We also study that on the search if they interfere with each other then what should be the behaviour of equilibrium for the interference pattern.

I-4: Some protocols for multisignatures, *Sunder Lal and Sanjay Rawat* (*Institute of Basic Science, Khandari, Agra - 282 002, U. P.*).

The main scheme discussed in this article is an extension of the double signature protocol to the situation where more than two signatories are required to sign a document with the condition that each one is able to read the message before signing it. The scheme is futher extended to the case where there are three groups of possible signatories and the document is to be signed by (any) one member of each group. Protocols are discussed in the presence as also in the absence of a trusted central authority.

I-5: A more representative and efficient electronic voting scheme,
Sunder Lal, Sanjay rawat and Kopal Rastogi (*Institute of Basic Science, Khandari, Agra - 282 005 and Dr. B. R. Ambedkar University, Agra - 282 002, U. P.*).

In any election Process there are several levels of communication and not in all types of elections but in certain specific ones it is very essential to keep these communications secret. Also every plan or scheme that is proposed for implementation by private or government sector requires some trusted authority or authorities. Our aim is to introduce cryptology to develop a mechanism of its refined usage in election process through electronic voting in our country. This is so as to provide leak-proof and fair voting services, smooth functioning of election process including non disruption, privacy, non duplication, authentication of voter, etc. which are our main points of analysis. Boyd's scheme (1990) was based on the assumption that each voter will follow all steps proposed, that is, voter will be co-operative. The salient features of our proposed scheme are that (i) it functions even under disruption, (ii) the authority can not falsify the ballots of eligible voters, (iii) necessarily zero-knowledge protocol is pursued when voter identifies itself to be authority, (iv) a better expression of public opinion by allowing voter to indicate most favoured and most nuisance candidate, that is, positive and negative voting respectively.

I-6: On a Banach space approximable by Jacobi polynomials, *Sarjoo Prasad Yadav* (*Govt. Model Science College, Rewa - 486 001, M. P.*).

We write X to represent either a space $C[-1,1]$ or $L_p^{(\alpha,\beta)}(w)$. Then X are the Banach spaces under the sup or the p norms. We prove that there exists a normalized Banach subspace $X_2^{(\alpha,\beta)}$ of X such that evry $f \in X_2^{(\alpha,\beta)}$ can be represented by a linear combination of Jacobi polynomials to any degree of accuracy. Our method to prove such approximation problem is Fourier-Jacobi analysis based on the convergence of Fourier-Jacobi expansions.

I-7: Encryption with permutation polynomial in F_{p^n} and a caution with zero element, *Sunder Lal and Sanjay Rawat* (*Institute of Basic Science, Khandari, Agra - 281 005, U. P.*).

In classical cryptology, transposition and substitution are two basic devices of cryptography. Each of the q elements of the finite field $F_q,\ q = p^n$, can be written in the form $\{\}$ and is expressible as n-tuple $\mathbf{a} = ()$. The process by which elements of the field are rearranged is termed as permutation and is accomplished by means of polynomial $P_n(x)$ over F_q. The letters of the alphabet could be associated with distinct n-tupples and encryption key (permutation polynomial) permutes components of vector giving rise to different letters of alphabet that is cryptogram. Zero element of F_q remains invarient and hence its vector can not be subjected to permutation process. Thus there exista a caution against this element that no letter of the plain text should be represented by zero vector.

I-8: Inverse result for modified Baskarov type operators, *Vijay Gupta* (*Netaji Subhash Institute of Technology, New Delhi - 110 078*).

Gupta and Srivastava (1997) defined a new modification of Baskarov operators with the weight function of Szász operators so as to approximate Lebesgue integrable functions on the interval $[0, \infty)$. They studied Voronovskaja type asymptotic formula and an estimate of error in simultaneous approximation for the linear combination of these operators. In the present paper we extend these results and study an inverse theorem in simultaneous approximation for the linear combination of these Baskarov-Szász type operators. We have used the technique of linear approximating method, viz, Petre's K-functionals to prove our main theorem.

J: Operations Research:

J-1: Sensitivity properties of variational and variational-like inequalities, *Vineeta Singh and R. N. Mukherjee* (*Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi - 221 005, U. P.*).

In the present paper we have introduced some problems connecting variational inequalities and variational-like inequalities. In some recent literature sensitivity properties of variational inequalities have been discussed elaborately and many applications have been proposed. We try to extend these ideas to incorporate the sensitivity studies to be done in the setting of variational-like inequalities. Moreover we also introduce a class of problems referred as quasi-variational inequalities.

Some further applications are given.

K: Solid Mechanics, Fluid Mechanics, Geophysics and Relativity:

K-1: Charged fluid distribution in higher dimensional speroidal space-time, *G. P. Singh and S. Kotambkar* (*V. Regional College of Engineering, Nagpur - 440 012 and G. H. R. College of Engineering, Nagpur - 440 019, Maharashtra*).

A general solution of Einstein field equations corresponding to a charged fluid distribution on the background of higher dimensional spheroidal space time is obtained. The solution describes interior field of a star in higher dimensions. The physical features of the solution are briefly discussed.

K-2: Collineations of the curvature tensor in general relativity, *R. K. Tiwari and J. P. Singh* (*Government Model Science College, Rewa - 486 001 and A. P. S. University, Rewa - 486 003, M. P.*).

Curvature collineation for curvature tensor, which is constructed from a fundamental Bianchi type-V metric, is studied.

K-3: Existence of a physically undetectable frame of reference at absolute rest and the absolute time in it, *Hirday Shanker Prasad* (*Hazaribagh - 825 301, Jharkhand*).

In this paper we prove that the theory of relativity has meaning if there exists an internal frame of reference at absolute rest with absolute time in it. Although the theory of relativity denies any such existence, it does not say that a physically undetectable inertial frame of reference at absolute rest with absolute time in it does not exist. We prove that such an absolute IRR with absolute time must exist. A state of pseudo-absolute time have also been defined to this effect.

K-4: Static deformation of an orthotropic multilayered elastic half space due to surface loads, *N. R. Garg* (*Maharshi Dayanand University, Rohtak - 124 001, Haryana*).

The static deformation of an orthotropic multilayered elastic half-space as a result of surface loads is studied. The layers are assumed to be horizontal and in welded contact. The plain strain problem of normal strip loading is discussed in detail. The method of layer matrices is used to obtain the displacements and stresses at any point of the medium. the proceedure developed is straight forward, conenient for numerical computation and avoids the complicated nature of the problem.

K-5: Bianchi type V perfect fluid cosmological models with constant active gravitational mass, *J. P. Singh, A. Prasad and R. K. Tiwari* (*A. P. S. University, Rewa - 486 003, M. P.*).

We investigate Biyanchi type V cosmological models representing perfect fluid and having active gravitational mass of the universe constant. Various physical and kinematical properties of the models have been discussed.

K-6: Analysis of the performance of a porous slider bearing with squeeze film formed by a magnetic fluid, *R. M. Patel and G. M. Deheri* (*Nirma Institute of Technology, Ahmedabad - 382 481 and Sardar Patel University, Vallabh Vidyanagar - 388 120, Gujarat*).

Efforts have been made to obtain the expressions for pressure, load carrying capacity, friction and the position of the centre of pressure for a general porous slider bearing with squeeze film formed by a magnetic fluid. Various slider bearing configurations are considered and the bearing characteristics are obtained. The results are presented numerically as well as graphically. The results show that magnetic fluid lubricant improves the performance of bearing considerably. Furthermore, it is observed that the effect of the magnetization parameter is quite significant in the case of the secant shaped bearing.

K-7: Separation of a binary mixture of viscous fluid due to a rotating heated disc, *A. C. Srivastava* (*Lucknow University, Lucknow - 226 001, U. P.*).

The mass transfer of a binary mixture of viscous fluids due to a temperature gradient created by the rotation of a heated disc has been discussed when there is a uniform suction at the disk. Due to the existance of the temperature gradient in the boundary layer region a thermal flux is set up such that the higher component moves to the hotter region and heaviour one to the colder region. The concentration of the higher component increases monotonically from its free stream value to a maximum at the disk. With the increase of the temperature difference between the disk and the ambient mixture more and more quantity of the lighter component gets collected at the disk. As there is a uniform suction at the disk, this process of temperature difussion provides a method of separating and collecting the lighter component of the mixture.

M: Bio-Mathematics:

M-1: A numerical study of the Casson fluid flow with homogeneous porous medium, *Sanjeev Kumar* (*Institute of Basic Science, Khandari, Agra - 282 002*).

The interest in the flow of time independent non-Newtonian fluids through tubes is increasing because of its applications in bio-dynamics and polymer processing industries. Here a model of Casson fluid flow in tube filled with a homogeneous porous medium is investigated by using suitable numerical technique. The initial motivation for studying this problem is to understand the blood flow in an artery under pathological situations when the fatty plaques of cholestrol and artery clogging blood clots are generated in the lumen of the coronary artery. We find that the frictional resistance will fall up to a certain level of permeability and then it will become steady.

M-2: A computational model for the blood flow in stenosed catheratized curved artery, *Narendra Kumar and Sanjeev Kumar* (*Institute of Basic Science, Khandari, Agra - 282 002, U.P.*).

There has been a considerable expansion in the use of catheter of various size due to the evolution of coronnary baloon angioplasty. In this paper we present a mathematical model of blood flow in a catheterized curved artery with stenosis, blood is considered as an incompressible Newtonian fluid and the flow is assumed to be steady and laminar. An analytical solution to the problem is obtained for the case of short curvature and mild stenosis. We study the effects of catheterization of the pressure drop, impedance and the wall shear stress for different values of the catheter size and Raynold's number of the flow. We came across that these flow characteristics vary marketedly across a stenotic lesion and their magnitude is proportional to the catheter size. The main object of this work is to understand the mechanics of blood flow in a catheter curved artery with stenosis and analyze the increase pressure drop and wall shear stress.

N: History and teaching of mathematics:

N-1: Mathematical creativity by teaching modern mathematics, *G. Rabbani and Miss Husn-Ara* (*Ranchi University, Ranchi - 834 008, Jharkhand*).

Earlier mathematicians and learning theorists were sympathetic to traditional teaching and drill approach while the modern mathematicians and newer learning theorists hold that conceptual and meaningful approach is preferable and helpful for competitive examinations, which have been proved in this investigation.

In this paper we have investigated the effect of conceptual teaching, compared with the results of traditional teaching and their background in the area of modern mathematics.

For this research, students and teachers from both the groups from plus two (+2) schools of Jharkhand Area were taken which plays an important role in engineering competitive examinations at national level.

ABSTRACT OF THE INVITED LECTURES:

1: Stability of functional differential equations, *G. R. Shendge* (*Ex. Reader, Department of Mathematics, Dr. B. A. M. University, Aurangabad - 431 004, Maharshtra*).

Different stability results for delay differential equations in terms of Lyapunov functions require different minimal sets over wcich the derivative of a Lyapunov function is to be estimated and the proofs vary accordingly. This is in contrast to the situation in ordinary differential equations where the different stability results can be combined in one theorem under a single set of conditions. With this kind of view a new comparision theorem for functional differential equations is obtained and it is shown that various stability properties of functional differential equations can be studied in a unified way.

2: The robust stability of distributed parameter control systems, *G. S. Ladde* (*Department of Mathematics, The University of Taxes at Arington, Arington, Taxes - 76019, USA*).

By employing feedback control laws generated by quadratic cost function, the robust stability of distributed parameter control system is investigated. The results are obtained by using vector Lupanov-like function/functional in the context of the comparision systems and the theory of differential inequalities. Examples are given to illustrate the usefulness of the results.

3: Stability of interactive proceedures in numerical praxes, *S. L. Singh* (*Gurukul Kangri University, Hardwar - 249 404, Uttarakhand*).

In solving fixed point equations by an iterative proceedure, we generally work with an approximative sequence instead of the actual sequence. The approximative sequence need not converge to a solution, even if it converges. The purpose of this talk is to address this problem. The survey presented here is not completely up to date.

4: Variation of parameters formulas for nonlinear hyperbolic partial differential equations, *Goskin Yakarn, S. G. Deo and D. Y. Kalsure* (*Florida Institute of technology, Melbourne, FL - 32901, USA; 12, Precy Building, mala, Panaji - 403 001, Goa; 3, Acharya Shree Colony, Konkanwadi, Aurangabad - 431 001, Maharashtra*).

The present paper investgates variation of parameter formula (VPF) for nonlinear hyperbolic partial differential equation of second order. Here the derivatives with respect to the initial data play the central role. It has been pointed out that the VPF for linear hyperbolic partial differential equation is a particular case.

5: Some recreational aspects of mathematics, *S. R. Joshi,* (*Block - 7, Vasudha Apartment, Jyotinagar, Aurangabad - 431 005, Maharashtra*).

In order to create interest in mathematics among the students, recreational aspects of mathematics play an important role. This is true not only for the high school students but also for the college and university students. Since mathematics is the queen of all the sciences, it is the duty of every mathematics teacher to create interest in mathematics among students so that they should never feel it is a difficult subject. For this purpose a teacher should be well versed in different recreational aspects of mathematics such as mathematics puzzles, paradoxes, etc.

In the present paper some recreational aspects of mathematics are discussed. The author does not claim any kind of originality of the results and examples discussed in the paper. The way of putting the results is different.

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66^{TH} IMS CONFERENCE : A BRIEF REPORT

The 66^{th} Annual Conference of the Indian Mathematical Society was held at the Vivekanand Arts, Sardar Dalip Singh Commerce and Science College, Samarthnagar, Aurangabad - 431 001, Maharashtra, India, during December 19-22, 2000 and was organized by the Principal G. Y. Pathrikar of the College. The conference was attended by more than 160 delegates from all over the country including foreign delegates. Two presidential addresses (General and Technical) and Four memorial award lectures were delivered during the Conference. Also, Eighteen invited lectures and about 86 reserach papers were presented in several parallel sessions during the four days that the Conference lasted. The academic programme of the Conference included one symposium, one panel discussion and one open session as well. The deliberations of the conference were held in the campus of Mahatma Gandhi Mission's Jawaharlal Nehru Engineering College, N-6, CIDCO, Aurangabad - 431 003. Prof. Satya Deo, Vice-Chancellor, Avadhesh Pratap Singh University, Rewa, Madhya Pradesh, was the President of the Indian Mathematical Society and Prof. S. K. Nimbhorkar, Head, Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, was the Local Secretary.

The inaugural function was held in the forenoon of December 19, 2000 at 10 a.m. and it was presided over by Shri K. P. Sonawane, Vice-Chancellor, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad. Principal G. Y. Pathrikar delivered the welcome address inviting the delegates and Prof. R. P. Agarwal, General Secretary, IMS presented his report before the audience. Prof. M. K. Singal, Administrative Secretary, IMS presented his report and on behalf of the Society expressed his sincere and profuse thanks to the host for organizing the Conference. The chief guest of the conference was Prof. S. Bhargava, Head, Department of Mathematics, University of Mysore, Mysore, Karnataka.

After the addresses of Prof. S. Bhargava and Shri K. P. Sonawane, the souvenir was released by Shri A. N. Kadam, Vice-President, Mahatma Gandhi Mission. The President of the Indian Mathematical Society, Prof. Satya Deo, then delivered the Presidential Address (General). Prof. S. K. Nimbhorkar presented the vote of thanks to the guests, delegates and all the colleagues involed. The function ended with the singing of the National Anthem.

After high-tea, Prof. Satya Deo delivered the Presidential Address (Technical) on *Methods of Algebraic Topology in Group Actions and related areas.* It was presided over by Prof. R. P. Agrawal, former Vice-Chancellor, Lucknow University and the senior most past president, IMS.

The details of the academic programme of the Conference are as follows.

MEMORIAL AWARD LECTURES

THE 14th P. L. BHATNAGAR MEMORIAL AWARD LECTURE

The 14th P. L. Bhatnagar Memorial Award Lecture was delivered by Prof. Sarva Jeet Singh, Kurukshetra University, Kurukshetra -136 119, Haryana, India. He spoke on "*Deformation of a stratified elastic half space by surface loads and buried sources*". The session was presided over by Prof. H. C. Khare, University of Allahabad, Allahabad, and the Editor, The Mathematics Student. (A memorial lecture to be delivered by a distinguished mathematician at the Annual Conference of the Society was instituted by the P. L. Bhatnagar Memorial Fund Committee in 1987. The lecture carries a token homourarium of Rs. 2000.)

Earlier lectures were delivered by the following.

N. MUKUNDA	1987	*KARMESHU*	1994
J. V. NARLIKAR	1988	*A. S. GUPTA*	1995
V. LAKSHMIKANTHAN	1989	*R. K. JAIN*	1996
J. N. KAPUR	1990	*MIHIR BANERJEE*	1997
H. C. KHARE	1991	*S. K. MALIK*	1998
D. K. SINHA	1992	*A. C. SRIVASTAVA*	1999
P. C. VAIDYA	1993		

THE 11th V. RAMASWAMI AIYAR MEMORIAL AWARD LECTURE

The 11th V. Ramaswami Aiyar Memorial Award Lecture was delivered by Prof. Ushadevi N. Bhosle, Tata Institute of Fundamental Research, Mumbai - 400 005, Maharashtra, India on "*Vector bundles and Principal bundles*". (In 1990 the Society instituted a lecture to be delivered annually by a distinguished mathematician as a part of the academic programme of its Annual Conference in the memory of its founder, Late Sri. V. Ramaswami Aiyar. The lecture carries a token homourarium of Rs. 2500.)

Earlier lectures in this series were delivered by the following.

M. K. SINGAL	1990	*A. R. SINGAL*	1995
R. S. MISHRA	1991	*S. P. ARYA*	1996
R. P. AGARWAL	1992	*GOVIND SWARUP*	1997
K. R. PARTHASARATHY	1993	*V. M. SHAH*	1998
V. KRISHNAMURTHY	1994	*SUNDER LAL*	1999

THE 11th HANSRAJ GUPTA MEMORIAL AWARD LECTURE

The 11th Hansraj Gupta Memorial Award Lecture was delivered by Prof. Rajendra Bhatia, Indian Statistical Institute, Delhii - 110 016, India on "*Linear algebra to quantum cohomology: A story of Horn's inequality*". The Session was presided over by Prof. Satya Deo, President, Indian Mathematical Society. (An annual lecture to be held in the memory of the distinguist number theorist, the Late Prof. Hansraj Gupta, was instituted in 1990 out of the contributions to the Hansraj Gupta Memorial Fund. The lecture is held as a part of the academic programme of the conference and carries a token honourarium of Rs. 2000.)

Earlier lectures in this series were delivered by the following.

A. M. VAIDYA	1990	*U. B. TIWARI*	1995
K. RAMCHANDRA	1991	*I. B. S. PASSI*	1996
H. P. DIKSHIT	1992	*R. PARIMALA*	1997
SATYA DEO	1993	*T. PARTHASARATHY*	1998
ARUN VERMA	1994	*R. S. PATHAK*	1999

THE 11th SRINIVASA RAMANUJAN MEMORIAL AWARD LECTURE

The 11th Srinivasa Ramanujan Memorial Award Lecture was delivered by Prof.V. Srinivas, Tata Institute of Fundamental Research, Mumbai - 400 005, India on "*On Diophantine equations*". (In 1990 the Society instituted a lecture to be delivered annually as a part of the academic programme of its Annual Conference in the memory of the mathematical genius Srinivasa Ramanujan. The lecture carries a token honourarium of Rs. 2500.)

Earlier lectures in this series were delivered by the following.

R. P. AGARWAL	1990	*B. V. LIMAYE*	1995
M. S. RAGHUNATHAN	1991	*N. K. THAKARE*	1996
S. BHARGAVA	1992	*V. S. SUNDER*	1997
R. P. BAMBAH	1993	*A. K. AGARWAL*	1998
V. KANNAN	1994	*DIPENDRA PRASAD*	1999

INVITED TALKS

Eighteen invited talks were delivered in parallel sessions during the Conference. The speakers and titles of the talks are given below.

	Speaker	*Title of the Talk*
1.	Hukum Singh	: *Geometry and its applications*
2.	H. K. Srivastava	: *Theory of functions of a bicomplex variable: Some recent developments*
3.	A. P. Singh	: *Dynamics of composition of entire functions*
4.	B. C. Gupta	: *Variables – Some recent developments*
5.	S. Sribala	: *Jordan Algebras*
6.	Manjul Gupta	: *On unconditional basis*
7.	G. R. Shendge	: *Stability in functional differential equations*
8.	R. Mukherjee	: *Optimization*
9.	M. A. Pathan	: *Lie theoretic approach to differential equations and special functions*
10.	D. Y. Kasture	: *Variation of parameters formulas for nonlinear hyperbolic partial differential equations*
11.	R. Y. Denis	: *On q-series and continued fractions*
12.	V. R. Kulli	: *Graph theory and some of its applications*

13.	G. S. Ladde	: *The robust stability of disturbed parameter control systems*
14.	S. R. Joshi	: *Some recreational aspects of mathematics)*
15.	J. M. C. Joshi	: *Mathematics Nature and theology*
16.	S. L. Singh	: *Stability of iterative procedures in numerical praxis*
17.	V. P. Saxena	: *New horizons in mathematics applications*
18.	B. L. Kirangi	

P. L. BHATNAGAR MEMORIAL PRIZE

To encourage the participants at the International Mathematical Olympiads, The Indian Mathematical Society instituted in 1987 an annual Prize in the memory of the Late Professor P. L. Bhatnagar, who did pioneering work in organizing Olympiads in the country, out of an endowment made by P. L. Bhatnagar Memorial Fund Committee. The prize is awarded every year during the Indian Mathematical Society annual conference to the top scorer of the Indian Team for IMO provided he/she wins a medal. (from 1987 to 1990 the prize was awarded to the top scorer at INMO.) The Prize consists of a cash award of Rs. 1000/- and a certificate.

At the 41st International Mathematical Olympiad held in Taejon, Republic of Korea, during July 13–25, 2000 all the six members of the Indian team bagged medals. Mr. Vaibhav Vaish (Score: 24), Swastik Koparty (Score: 24), Swarnendu Datta (Score: 24), Abhisk Saha (Score: 23) and Samik Basu(Score: 21) got Silver Medal each while Prasad Nagaraj Raghavendra (Score: 16) got the Bronz Medal; the score is out of total 42 points. **Mr. Vaibhav Vaish** (Lucknow), **Swastik Koparty** (Kolkata) and **Swarnendu Datta** (Kolkata) - the top scorers with 24 points each - were awarded the P. L. Bhatnagar prize for 2000.

The details of the earlier awardees so far are:

Year	*Name of the Awardee*	Place
1987	AMOL SHREERANG DIDHE	: Mumbai
1988	R. MURALIDHAR	: Mumbai

1989	ARCHISMAN RUDRA and	: Calcutta
	R. VENKATARAMAN	: Mumbai
1990	RUDRA SHYAM BANDHU	: Calcutta
1991	MOSES SAMSON CHARIKAR	: Mumbai
1992	KAUSTUBH NAMJOSHI	: Pune
1993	KAUSTUBH NAMJOSHI	: Pune
1994	ARAVIND SANKAR	: Mumbai
1995	SUBHASH AJIT KHOT	: Ichalkaranji
1996	AJAY C. RAMADOS	: Bangalore
1997	RISHI RAJ	: Ranchi
1998	ABHINAV KUMAR	Jamshedpur :
1999	VAIBHAV VAISH	: Lucknow

DETAILS OF OTHER ACADEMIC PROGRAMMES

1. SYMPOSIUM. On **"Algebraic and Differential Topology"**.
Coordinator: Satya Deo (President, IMS).

The morning deliberations on December 20, 2000 started with this symposia. The other speakers were
1. H. K. Mukherjee who spoke on *Aplications of surgery techniques in classification of manifolds*,
2. T. B. Singh who spoke on *Cohomology algebra of certain spaces*,
3. S. S. Khare who spoke on *on Grassmannian manifolds*,
4. A. K. Das who spoke on *A study of Dold and Milnor*, and
5. A. R. Rajan who spoke on *Simultaneous bilinear equations*.

2. PANEL DISCUSSION on **Mmathematics in the 21st Century**
This was organised in the forenoon of December 20, 2000. Following colleagues delivered talks expressing their views:
1. Prof. Satya Deo, President, IMS;
2. Prof. R. P. Agrawal, General Secretary, IMS;
3. Prof. M. K. Singal, Administrative Secretary, IMS;
4. Prof. V. M. Shah, Editor, J. Ind. Math. Soc.; and
5. Prof. H. C. Khare, Editor, The Math. Student.

3. OPEN SESSION on **the future of Mathematical Olympiads in India**

Coordinator: Prof. M. K. Singal (Administrative Secretary, IMS.) This was arranged in the evening of December 21, 2000.

The last day of the conference, i.e., December 22, 2000 started with **unvelinging of Ramanujan's portrait** and an invited talk by Prof. V. P. Saxena who spoke on *New horizons in mathematics applications.*

The conference concluded with a **valedictory function**.

S. K. Nimbhorkar
Local Organizing Secretary
66th IMS Conference, and
Head, Department of Mathematics
Dr. B. A. M. University, Aurangabad.

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(See Rule 8)

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I, N. K. Thakare, hereby declare that the particulars given above are true to the best of my knowledge and belief.

N. K. THAKARE

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Back volumes of our periodicals, except a few numbers out of stock, are available. The following publications of the Society are also available: (1) Memoir on cubic transformation associated with desmic system by R. Vaidyanathswamy, pp. 92, Rs. 250/- (or $ 10.00), and (2) Tables of partitions, by Hansraj Gupta, pp. 81, Rs. 350/- (or $ 15.00).

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