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Edited by M. M. SHIKARE

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HOMOMORPHISMS BETWEEN C(X) AND C(Y)

SOMNATH HAZRA AND BASILA P
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Abstract. In this short note, we give an elementary proof of the following well-known theorem. Let X and Y be compact Hausdorff spaces. If $\rho : C(X) \to C(Y)$ is a unital homomorphism, then there exists a continuous function $p : Y \to X$ such that $(\rho(f))(y) = f(p(y))$, $x \in X$, $y \in Y$, $f \in C(X)$.

1. Introduction

Given a compact Hausdorff topological space X, let $C(X)$ be the set of all continuous functions from X to $\mathbb{C}$. For $f, g \in C(X)$ and $c \in \mathbb{C}$, setting $f + g$ and cf to be the functions: $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$, $x \in X$, respectively, we see that $C(X)$ is a vector space. If f and g are in $C(X)$, then their point-wise product fg is a continuous function $(fg)(x) = f(x)g(x)$, $x \in X$. The vector space $C(X)$ equipped with this multiplication is an algebra. The constant function 1_X, which is the function $1_X(x) = 1$, $x \in X$, is the multiplicative identity of the algebra $C(X)$. Let Y be a compact Hausdorff topological spaces and $p : Y \to X$ be a continuous function. For $f \in C(X)$, the composition $p^*(f) := f \circ p$ is continuous and hence p^* maps $C(X)$ to $C(Y)$. Also, p^* is multiplicative linear map, that is,

$$p^*(fg) = p^*(f)p^*(g), \text{ and } p^*(af + bg) = ap^*(f) + bp^*(g),$$

for $a, b \in \mathbb{C}$, $f, g \in C(X)$. Moreover, it is unital, that is, $p^*(1_X) = 1_Y$.

The article is divided into three sections. In the first section, we describe all the homomorphisms from $C(X)$ to $C(Y)$ where X and Y are two finite sets. In the second section, we prove that if $\rho : C(X) \to C(Y)$ is a nonzero multiplicative linear map, where X is a finite set and Y is a compact,

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connected, Hausdorff space, then there is a continuous function $p : Y \to X$ such that $\rho = p^*$. In the final section, for any two compact Hausdorff spaces X and Y, we show that if $\rho : C(X) \to C(Y)$ is a unital multiplicative linear map, then there exists a continuous function $p : Y \to X$ such that $\rho = p^*$.

2. X and Y are finite

In what follows, if X is a finite set, then we assume that it is also a topological space and the topology is the discrete topology, namely, every subset of X is an open set.

For $n \in \mathbb{N}$, the vector space $\mathbb{C}^n$ is an algebra with coordinate-wise multiplication: For any two points $(z_1, \dots, z_n)$ and $(w_1, \dots, w_n) \in \mathbb{C}^n$, the multiplication in the vector space $\mathbb{C}^n$ is given by

$$(z_1, \dots, z_n)(w_1, \dots, w_n) = (z_1 w_1, \dots, z_n w_n).$$

Lemma 2.1. *If $X = \{x_1, x_2, \dots, x_n\}$ be a finite set, then there exists a multiplicative isomorphism between $C(X)$ and $\mathbb{C}^n$.*

Proof. Let $f_i : X \to \mathbb{C}$, $1 \le i \le n$, be the function:

$$f_i(x_j) = \begin{cases} 1 & \text{if } i = j,\ x_j \in X,\ 1 \le j \le n, \\ 0 & \text{if } i \ne j,\ x_j \in X,\ 1 \le j \le n. \end{cases}$$

The set of functions $\{f_1, f_2, \dots, f_n\}$ is a basis of $C(X)$. In particular if $f \in C(X)$, then $f = \sum_{i=1}^{n} f(x_i) f_i$.

To complete the proof, for $f \in C(X)$, define $U : C(X) \to \mathbb{C}^n$ by $U(f) = (f(x_1), f(x_2), \dots, f(x_n))$, and verify that U is a multiplicative isomorphism. □

Recall that $L : \mathbb{C}^n \to \mathbb{C}$ is said to be a linear functional if for every pair of vectors $\boldsymbol{z}$ and $\boldsymbol{w}$ in $\mathbb{C}^n$ and complex numbers a, b, we have

$$L(a\boldsymbol{z} + b\boldsymbol{w}) = aL(\boldsymbol{z}) + bL(\boldsymbol{w}).$$

Now, we describe all the linear functionals $L : \mathbb{C}^n \to \mathbb{C}$.

Lemma 2.2. *Suppose that $L : \mathbb{C}^n \to \mathbb{C}$ is a linear map. Then there exists $a_i \in \mathbb{C}$, $1 \le i \le n$, such that*

$$L(z_1, z_2, \dots, z_n) = \sum_{i=1}^{n} z_i a_i,\ (z_1, z_2, \dots, z_n) \in \mathbb{C}^n.$$

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$.Clearly we have $(z_1, z_2, \ldots, z_n) = \sum_{i=1}^n z_i e_i$ for any $(z_1, z_2, \ldots, z_n) \in \mathbb{C}$. Suppose that $L(e_i) = a_i$, $1 \leq i \leq n$. Since L is a linear map, it follows that

$$L\left(\sum_{i=1}^n z_i e_i\right) = \sum_{i=1}^n z_i L(e_i) = \sum_{i=1}^n z_i a_i$$

completing the proof. □

Recall that if $\mathcal{A}$ is an algebra, then a a multiplicative linear functional $L : \mathcal{A} \to \mathbb{C}$ is a linear functional L with the property that

$$L(ab) = L(a)L(b),\ a, b \in \mathcal{A}.$$

The following Lemma describes all the multiplicative linear functionals on $\mathbb{C}^n$.

Lemma 2.3. *Suppose that $L : \mathbb{C}^n \to \mathbb{C}$ is a non-zero multiplicative linear functional. Then there exists a k, $1 \leq k \leq n$, such that $L(\boldsymbol{z}) = z_k$, $\boldsymbol{z} := (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$.*

Proof. It follows from Lemma 2.2 that $L(\boldsymbol{z}) = \sum_{i=1}^n z_i a_i$, $\boldsymbol{z} \in \mathbb{C}^n$ for a suitable choice of $a_i \in \mathbb{C},\ 1 \leq i \leq n$. Now suppose there exists $1 \leq p, q \leq n$ such that $a_p \neq 0$ and $a_q \neq 0$. Since L is multiplicative, we have

$$\begin{aligned} &L((0, \ldots, z_p, \ldots, z_q, \ldots, 0)(0, \ldots, w_p, \ldots, w_q, \ldots, 0)) \\ &\qquad = L(0, \ldots, z_p, \ldots, z_q, \ldots, 0)L(0, \ldots, w_p, \ldots, w_q, \ldots, 0), \end{aligned} \tag{2.1}$$

where z_p, w_p and z_q, w_q are in p and qth slots of $\boldsymbol{z}$ and $\boldsymbol{w}$, respectively. Now expanding Equation (2.1), we obtain

$$z_p w_p a_p + z_q w_q a_q = z_p w_p a_p^2 + z_p w_q a_p a_q + z_q w_p a_p a_q + z_q w_q a_q^2. \tag{2.2}$$

Since the Equation (2.2) is actually an identity in the four variables z_p, z_q, w_p, $w_q \in \mathbb{C}$, it follows that $a_p a_q = 0$ which contradicts our assumption of $a_p \neq 0$ and $a_q \neq 0$. If L is not zero, then this proves the existence of a k such that $a_k \neq 0$ and $a_i = 0$ for $i \neq k$, $1 \leq i, k \leq n$.

Again, using the multiplicative property of L, it follows that $a_k^2 = a_k$. Since L is not zero, $a_k^2 = a_k$ implies that $a_k = 1$. □

The following corollary is a direct consequence of Lemma 2.1 and Lemma 2.3.

Corollary 2.4. *Let $X = \{x_1, x_2, \dots, x_n\}$. Suppose that $L : C(X) \to \mathbb{C}$ is a non-zero multiplicative linear functional. Then there exists k such that $L(f) = f(x_k)$ for all $f \in C(X)$.*

Now we are ready to prove the main theorem of this section, which describes all multiplicative linear functionals from $C(X)$ to $C(Y)$ assuming that X and Y are finite sets.

Theorem 2.5. *Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ be two finite sets. Suppose that $\rho : C(X) \to C(Y)$ is a non-zero multiplicative linear map. Then there exists a function $p : Y \to X$ such that $\rho = p^*$.*

Proof. For $1 \leq j \leq m$, let $ev_{y_j} : C(Y) \to \mathbb{C}$ be the evaluation map at the point y_j, that is, $ev_{y_j}(g) = g(y_j),\ g \in C(Y)$. The evaluation map ev_{y_j} is multiplicative.

Since ρ is a multiplicative linear map and ev_{y_j} is a multiplicative linear functionals, it follows that $ev_{y_j} \circ \rho$ is also a multiplicative linear map from $C(X)$ to $\mathbb{C}$. Consequently, using Corollary 2.4, we find x_i in X such that

$$(ev_{y_j} \circ \rho)(f) = f(x_i),\ f \in C(X).$$

Define $p : Y \to X$ by $p(y_j) = x_i$. For $f \in C(X)$, we have

$$(\rho(f))(y_j) = (ev_{y_j} \circ \rho)(f) = f(x_i) = f(p(y_j)).$$

This proves that $\rho = p^*$. □

3. X is finite and Y is connected

In this section, we describe all homomorphisms from $C(X)$ to $C(Y)$ where X is a finite set and Y is a compact connected Housdorff space.

Lemma 3.1. *Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set and Y be a compact, connected, Housdorff space with more than one point. Suppose $\rho : C(X) \to C(Y)$ is a non-zero multiplicative linear map. Then for any $f \in C(X)$, the function $\rho(f)$ in $C(Y)$ is a constant function.*

Proof. Suppose $\rho : C(X) \to C(Y)$ is a non-zero multiplicative linear map. By Lemma 2.1, $\dim C(X) = n$ and therefore $\dim(\rho(C(X))) \leq n$, where $\rho(C(X))$ denotes the image of ρ.

Assume that there exists f in $C(X)$ such that $\rho(f)$ is not constant. Let a, b be two complex numbers such that $a, b \in \rho(f)(Y)$. Since $a \neq b$, we have either $\mathrm{Re}\ a \neq \mathrm{Re}\ b$ or $\mathrm{Im}\ a \neq \mathrm{Im}\ b$, where Re and Im stand for real and imaginary part of a complex number, respectively. Without loss of generality, assume that $\mathrm{Re}\, a \neq \mathrm{Re}\, b$. Note that $\mathrm{Re}\,\rho(f) : Y \to \mathbb{R}$ is a continuous function. Since Y is a connected space and $\mathrm{Re}\,\rho(f)$ is continuous, the interval $[\mathrm{Re}\, a, \mathrm{Re}\, b]$ is contained in the image of $\mathrm{Re}\,\rho(f)$. Thus the image of the continuous function $\mathrm{Re}\,\rho(f)$, defined on Y, contains uncountably many points and therefore the image of $\rho(f)$ also contains uncountably many points. Let $g := \rho(f)$.

Now we prove that the set of functions $\{g, g^2, g^3, \dots\}$ is a linearly independent. Note that it is enough to prove that the set $\{g, g^2, \dots, g^n\}$ is linearly independent for every $n \geq 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be scalars such that

$$\sum_{i=1}^{n} \alpha_i g^i = 0. \tag{3.1}$$

Since $g(Y)$ contains uncountably many distinct points, there exists $y_1, \dots, y_n$ in Y such that $g(y_i) \neq g(y_j)$, $i \neq j$, and $g(y_i) \neq 0$ for all $i = 1, 2, \dots, n$. Let $g(y_j) = z_j$, $j = 1, 2, \dots, n$. Then Equation (3.1) implies that

$$\sum_{i=1}^{n} \alpha_i z_j^i = 0, \;\; 1 \leq j \leq n. \tag{3.2}$$

From the well known formula for the determinant of the Vandermonde matrix, we have

$$\det \;\left(\!\left(z_j^i \right)\!\right)_{i,j=1}^{n} = z_1 z_2 \cdots z_n \prod_{i \neq j, i<j} (z_i - z_j),$$

which is non-zero. Thus it follows from Equation (3.2) that $\alpha_i = 0$ for $i = 1, 2, \dots, n$. This proves that $\{g, g^2, \dots, g^n\}$ is a linearly independent set.

Since ρ is multiplicative, it follows that the set $\{g, g^2, g^3, \dots\}$ is contained in $\mathrm{Im}\,\rho$. This contradicts the assumption that $\dim(\mathrm{Im}\,\rho) \leq n$. Therefore $\rho(f)$ is constant for every $f \in C(X)$. $\square$

Theorem 3.2. *Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set and Y be a compact, connected, Housdorff space with more than one point. If $\rho : C(X) \to C(Y)$ is a non-zero multiplicative linear map, then there exists a constant function $p : Y \to X$ such that $\rho = p^*$.*

Proof. Let $f_i : X \to \mathbb{C}$ be the function $f_i(x_j) = \delta_{ij}$, $1 \leq i, j \leq n$. By Lemma 3.1, there exist scalars a_i, $1 \leq i \leq n$, such that $\rho(f_i)(y) = a_i$ for every $y \in Y$.

Now if f is in $C(X)$, then $f = \sum_{i=1}^{n} f(x_i) f_i$. Linearity of ρ implieas that

$$\rho(f) = \sum_{i=1}^{n} f(x_i)\rho(f_i). \tag{3.3}$$

Since $\rho(f_i)(y) = a_i$ for every $y \in Y$, we can rewrite Equation (3.3) as

$$\rho(f) = \sum_{i=1}^{n} f(x_i) a_i. \tag{3.4}$$

A computation similar to the one in the proof of Lemma 2.3 is enough to establish the existence of a k such that $a_k = 1$ and $a_i = 0$ if $i \neq k$.. Thus we have $\rho(f)(y) = f(x_k)$ for every y in Y. Setting $p : Y \to X$ to be the function given by the formula $p(y) = x_k$, $y \in Y$, we see that $\rho = p^*$. □

4. X and Y are arbitrary compact Hausdorff spaces

In this final section, we describe all homomorphisms from $C(X)$ to $C(Y)$ where X and Y are arbitrary compact Hausdorff spaces. To prove the main Theorem of this section, we need some properties of weak topology.

Given a topological space Y and a family of functions $\mathcal{F} = \{f : X \to Y\}$, the weak topology on X induced by $\mathcal{F}$ is defined to be the smallest topology generated by the sets

$$\left\{ f^{-1}(V) : f \in \mathcal{F} \text{ and } V \text{ is open in } Y \right\}.$$

The weak topology on X is the smallest topology on X such that every functions in $\mathcal{F}$ is continuous.

Given a compact, Hausdorff space X, taking $\mathcal{F}$ to be the set of continuous functions on X, we obtain the weak topology induced by $C(X)$ on X. In the following lemma, we describe the relationship between the original topology on X and the weak topology induced by $C(X)$ on it.

Lemma 4.1. *If (X, τ) is a compact, Hausdorff space, then the weak topology τ_w on X induced by $C(X)$ and the topology τ coincide.*

Proof. Let τ_w be the weak topology on X induced by $C(X)$. Clearly, the topology τ_w is contained in τ. Now we prove that τ is contained in τ_w.

Suppose $U \in \tau$. Since (X, τ) is a compact, Hausdorff space, by Urysohn metrization theorem there exists a metric d such that the topology τ is induced by the metric d.

Let $f : X \to \mathbb{C}$ be the function defined by

$$f(x) = d(x, U^c) := \inf \{d(x, y) : y \in U^c\}.$$

Suppose x_1 and x_2 are two elements of X. For any $y' \in U^c$, we have

$$\begin{aligned} f(x_1) &= \inf \{d(x_1, y) : y \in U^c\} \\ &\leq d(x_1, y') \\ &\leq d(x_1, x_2) + d(x_2, y'), \end{aligned}$$

where the last inequality follows by applying triangle inequality. Thus we have

$$f(x_1) \leq d(x_1, x_2) + d(x_2, y')$$

for any $y' \in U^c$. By taking infimum over y' aforementioned, we obtain

$$f(x_1) \leq d(x_1, x_2) + f(x_2). \tag{4.1}$$

Interchanging the role of x_1 and x_2 in the inequality (4.1) and then applying symmetry property of the metric, we obtain

$$f(x_2) \leq d(x_1, x_2) + f(x_1). \tag{4.2}$$

Now Equations (4.1) and (4.2) gives us

$$|f(x_1) - f(x_2)| \leq d(x_1, x_2).$$

This proves that the function f is continuous.

Since U^c is a closed set in τ, therefore $x \in U^c$ if and only if $d(x, U^c) = 0$. This implies that $U^c = f^{-1}\{0\}$. Therefore $U \in \tau_w$. This proves that the topology τ is contained in τ_w. □

One more ingredient in the proof of the main theorem of this section is the following lemma, which describes the maximal ideals of $C(X)$. Note that an ideal $\mathcal{M}$ of an algebra $\mathcal{A}$ is said to be a maximal ideal if $\mathcal{M}$ is not equal to $\mathcal{A}$ and if $\mathcal{I}$ is any ideal of $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{I}$, then either $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} = \mathcal{A}$.

Lemma 4.2. *Let X be a compact, Hausdorff space and M be a maximal ideal of $C(X)$. Then there exists unique point x_0 in X such that $M = \{f \in C(X) : f(x_0) = 0\}$.*

Proof. Suppose that M is a maximal ideal and that M is not of the form $\{f \in C(X) : f(x) = 0\}$ for any $x \in X$. Then for every x in X, there exists f_x

in M such $f_x(x) \neq 0$. Since f_x is continuous, there exists a neighbourhood U_x of x such that $f_x(y) \neq 0$ for every $y \in U_x$. The collection $\{U_x : x \in X\}$ is an open cover of X. Now compactness of X implies that there exists a finite subcover $\{U_{x_i} : i = 1, 2, \ldots, n\}$ of $\{U_x : x \in X\}$.

Let $g : X \to \mathbb{C}$ be the function defined by

$$g(x) = \sum_{i=1}^{n} f_{x_i}(x)\overline{f_{x_i}(x)}.$$

Since M is an ideal, it follows that $g \in M$. Also since f_{x_i} is non-zero on U_{x_i}, the function g does not vanish on any point of X. Therefore $\frac{1}{g}$ is a continuous function on X. This implies that $1 = g \cdot \frac{1}{g} \in M$, which is a contradiction.

Thus there exists at least one point x_0 such that $M = \{f \in C(X) : f(x_0) = 0\}$. Now if there exists two points x_0 and $\tilde{x}_0$ in X such that $M = \{f \in C(X) : f(x_0) = 0, f(\tilde{x}_0) = 0\}$, then M is no longer a maximal ideal. Therefore there exists $x_0 \in X$ such that $M = \{f \in C(X) : f(x_0) = 0\}$.

Since X is a compact Hausdorff space, therefore X is normal (See [2, Theorem 32.3]). Now applying Uryshon's Lemma ([2, Theorem 33.1]), uniqueness of the point x_0 follows. □

If X is a compact, Hausdorff space, then there is a natural norm on $C(X)$, namely,

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\}, \ f \in C(X).$$

It is well known that $(C(X), \|\cdot\|_\infty)$ is a Banach space (complete normed linear space). The topology on $C(X)$ is assumed to be the one induced by the sup norm $\|\cdot\|_\infty$.

For $x \in X$, recall that $ev_x : X \to \mathbb{C}$, the evaluation functional which maps f in $C(X)$ to $f(x)$, is a homomorphism. Also we have

$$|ev_x(f)| = |f(x)| \leq \|f\|_\infty.$$

Thus ev_x is a continuous homomorphism from $C(X)$ to $\mathbb{C}$. Therefore $\ker(ev_x)$ is a closed subspace of $C(X)$. Since $\ker(ev_x) = \{f \in C(x) : f(x) = 0\}$, it follows form the Lemma 4.2 that $\ker(ev_x)$ is a maximal ideal of $C(X)$.

Theorem 4.3. *Let X and Y be compact, Hausdorff spaces. Suppose that $\rho : C(X) \to C(Y)$ be a continuous unital homomorphism. Then there exists a continuous function $p : Y \to X$ such that $f = p^*$.*

Proof. For $y \in C(Y)$, let $ev_y : C(Y) \to \mathbb{C}$ be the evaluational functional at y. Since $\rho : C(X) \to C(Y)$ is a homomorphism, the map $ev_y \circ \rho : C(X) \to \mathbb{C}$ is also a homomorphism. Therefore $\ker(ev_y \circ \rho)$ is a maximal ideal in $C(X)$. Using Lemma 4.2, we obtain $x \in X$ such that $\ker ev_y \circ \rho = M_x$ where $M_x = \{f \in C(X) : f(x) = 0\}$.

Now define a map $p : Y \to X$, by setting $p(y) = x$, where $\ker(ev_y \circ \rho) = M_x$. The map p is well defined, thanks to Lemma 4.2.

Claim 1: The homomorphisms ρ and p^* are equal.

It is evident that $\ker ev_y \circ p^* = M_x$. Thus $\ker ev_y \circ \rho$ and $\ker ev_y \circ p^*$ are equal and therefore, by [1, Lemma, p.110], there exists a constant c such that $ev_y \circ \rho = c\ ev_y \circ p^*$. Since ρ is unital, it follows that $c = 1$. This implies that $ev_y \circ \rho = ev_y \circ p^*$. Thus we have $ev_y \circ \rho = ev_y \circ p^*$ for every $y \in Y$. This implies that $\rho = p^*$.

Claim 2: p is continuous.

Let U be an open subset of X and $f \in C(X)$. By Claim 1, we see that

$$p^{-1}\left(f^{-1}(U)\right) = \rho(f)^{-1}(U). \tag{4.3}$$

Since $\rho(f)$ is continuous for every $f \in C(X)$, therefore $\rho(f)^{-1}(U)$ is an open subset of Y. Now Equation (4.3) implies that $p^{-1}\left(f^{-1}(U)\right)$ is an open subset of Y. Thus $p^{-1}\left(f^{-1}(U)\right)$ is an open subset of Y for every $f \in C(X)$ and every open subset U of X. Now by Lemma 4.1, we conclude that p is continuous.

□

Remark 4.4. An abelian unital C^*-algebra $\mathcal{A}$ is $*$-isomprphic to $C(X)$ via the Gelfand map ([3, Theorem 4.29]), where X is the maximal ideal space of $\mathcal{A}$. Therefore, using Theorem 4.3, we can describe every continuous unital homomorphism between two abelian unital C^*-algebra.

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HYERS-ULAM-RASSIAS STABILITY OF n^{th} ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION

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Abstract. This paper deals with the Hyers-Ulam-Rassias stability of n^{th} order linear partial differential equation. Result is obtained by using Laplace Transform.

1. Introduction

The question raised by S.M.Ulam [17] during discussion on stability problems, in 1940 and D.H.Hyers [5] partial answer to it in terms of stability of linear functional equations, in 1941, opened up new avenues for research in the field of functional equations and differential equations. Paper by Alsina and Ger [3] on the Hyers–Ulam (HU) stability of the differential equation $y' = y$ and its generalization by Takahasi et al [16] in 2002 for the complex Banach space valued differential equation $y' = \lambda y$ together with [13] lead to series of papers on the topic. These include [6, 7, 8, 9] on HU and Hyers-Ulam-Rassias (HUR) stability. Gordji et al. [4] generalized the result of [9] to first and second order non linear partial differential equations. N. Lungu and C. Cracium [10] established HU stability and HUR stability of non linear hyperbolic partial differential equation in general form. M.N. Qarawani [12] studied the HUR stability for Heat equation. H. Rezaei [14] established the HU stability of linear differential equation of n^{th} order. HUR stability for special types of non linear equations have been studied in [1] , [2] and [11]. In this paper, we shall establish the HUR stability of n^{th} order linear partial differential equation

$$\frac{\partial u}{\partial t} = a^n \frac{\partial^n u}{\partial x^n}, \qquad t > 0, 0 < x < l, a > 0 \tag{1.1}$$

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with initial condition

$$u(x,0) = \mu(x), \qquad 0 \leq x \leq l \tag{1.2}$$

and boundary conditions

$$u(0,t) = v_0(t), u_x(0,t) = v_1(t), u_{xx}(0,t) = v_2(t), \cdots, u_{xx\cdots x}(0,t) = v_{n-1}(t), \tag{1.3}$$

where $l \in \mathbb{R}, \mu(x) \in C[0,l], v_0(t), v_1(t), v_2(t), \cdots, v_{n-1}(t) \in C(-\infty,\infty)$ and $u(x,t) \in C_1^n((0,l) \times (0,\infty))$.

We need following definitions.

Definition 1.1. : We shall say that the equation (1.1) is HUR stable with respect to $\phi(x,t) > 0$ if $\exists\ \psi(x,t) > 0$ such that for each $\epsilon > 0$ and for each solution $w(x,t) \in C_1^n((0,l) \times (0,\infty))$ of the inequality

$$|\frac{\partial u}{\partial t} - a^n \frac{\partial^n u}{\partial x^n}| \leq \epsilon \phi(x,t), \tag{1.4}$$

with conditions (1.2) and (1.3), $\exists$ a solution $u(x,t) \in C_1^n((0,l) \times (0,\infty))$ of the equation (1.1) such that

$$|w(x,t) - u(x,t)| \leq \epsilon \psi(x,t), \tag{1.5}$$

$\forall (x,t) \in ((0,l) \times (0,\infty)), \phi(x,t) \in C((0,l) \times (0,\infty))$ and $\psi(x,t) \in C((0,l) \times (0,\infty))$.

Definition 1.2. :[13] For each function $f : (0,\infty) \to \mathbb{F}$ ($\mathbb{R}$ *or* $\mathbb{C}$) of exponential order, the Laplace transform of f is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt.$$

There exists a unique number $-\infty \leq \sigma < \infty$ such that this integral converges if $\Re(s) > \sigma$ and diverges if $\Re(s) < \sigma$. The number σ is called abscissa of convergence and is denoted by σ_f.

Definition 1.3. :[13] Let $f(t)$ be a continuous function whose Laplace transform $F(s)$ has the abscissa of convergence σ_f. Then the inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{(\alpha+iy)t} F(\alpha + iy) dy, \text{ for any real } \alpha > \sigma_f.$$

2. Main result

In this section we prove the HUR stability of n^{th} order linear partial differential equation (1.1). We obtain the result by using Laplace transform.

Theorem 2.1. *: If $w(x,t) \in C_1^n((0,l)\times(0,\infty))$ is an approximate solution of the I-BVP* (1.1) - (1.3) *, then I-BVP* (1.1) - (1.3) *is HUR stable.*

Proof. : Given $\epsilon > 0$. Suppose $w(x,t)$ is an approximate solution of the I-BVP (1.1) - (1.3).
We have to show that $\exists$ an exact solution $u(x,t) \in C_1^n((0,l)\times(0,\infty))$ of equation (1.1) such that $|w(x,t) - u(x,t)| \leq \epsilon\psi(x,t)$, where $\psi(x,t) \in C((0,l)\times(0,\infty))$.
From the definition of HUR stability we have

$$|\tfrac{\partial w}{\partial t} - a^n \tfrac{\partial^n w}{\partial x^n}| \leq \epsilon\alpha(t - \tfrac{l^n}{a^n}).$$

This yields

$$-\epsilon\alpha(t-\frac{l^n}{a^n}) \leq \frac{\partial w}{\partial t} - a^n\frac{\partial^n w}{\partial x^n} \leq \epsilon\alpha(t-\frac{l^n}{a^n}), \tag{2.1}$$

where $\alpha(t-c) = 0$, for $t \leq c$ and $\alpha(t-c) = x(t-c)$, for $t \geq c, c \geq 0$.
Taking Laplace transform of equation (2.1), we get

$$-\epsilon\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\} \leq \mathcal{L}\{\frac{\partial w}{\partial t} - a^n\frac{\partial^n w}{\partial x^n}\} \leq \epsilon\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\},$$

and hence

$$|\mathcal{L}\{\frac{\partial w}{\partial t} - a^n\frac{\partial^n w}{\partial x^n}\}| \leq \epsilon\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\}.$$

This gives,

$$|\mathcal{L}\{\frac{\partial w}{\partial t}\} - a^n\mathcal{L}\{\frac{\partial^n w}{\partial x^n}\}| \leq \epsilon\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\}. \tag{2.2}$$

Also since $w(x,t)$ satisfies boundary conditions (1.3), we get

$\mathcal{L}\{w(0,t)\} = \mathcal{L}\{v_0(t)\} = W(0,p) = V_0(p)$,
$\mathcal{L}\{w_x(0,t)\} = \mathcal{L}\{v_1(t)\} = W_x(0,p) = V_1(p)$,
$\mathcal{L}\{w_{xx}(0,t)\} = \mathcal{L}\{v_2(t)\} = W_{xx}(0,p) = V_2(p), \cdots$,
$\cdots, \mathcal{L}\{w_{xx\cdots x}(0,t)\} = \mathcal{L}\{v_{n-1}(t)\} = W_{xx\cdots x}(0,p) = V_{n-1}(p)$.

As $\mathcal{L}\{\frac{\partial^n w}{\partial x^n}\} = \frac{d^n W}{dx^n}(x,p)$, $\mathcal{L}\{\frac{\partial w}{\partial t}\} = pW(x,p) - w(x,0)$ and $\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\} = \frac{x}{p^2}e^{-p\frac{l^n}{a^n}}$, with (2.2), we get

$$|-a^n\{\tfrac{d^n W}{dx^n}(x,p) - \tfrac{pW(x,p)}{a^n} + \tfrac{\mu(x)}{a^n}\}| \leq \epsilon\tfrac{x}{p^2}e^{-p\frac{l^n}{a^n}}.$$

$$\Rightarrow |a^n\{\tfrac{d^n W}{dx^n}(x,p) - \tfrac{pW(x,p)}{a^n} + \tfrac{\mu(x)}{a^n}\}| \leq \epsilon\tfrac{x}{p^2}e^{-p\frac{l^n}{a^n}}.$$

$$\Rightarrow |\tfrac{d^n W}{dx^n}(x,p) - \tfrac{pW(x,p)}{a^n} + \tfrac{\mu(x)}{a^n}| \leq \tfrac{\epsilon}{a^n}\tfrac{x}{p^2}e^{-p\frac{l^n}{a^n}}.$$

Hence we get

$$-\frac{\epsilon x}{a^n p^2}e^{\frac{-pl^n}{a^n}} \leq \frac{d^n W}{dx^n}(x,p) - \frac{pW(x,p)}{a^n} + \frac{\mu(x)}{a^n} \leq \frac{\epsilon x}{a^n p^2}e^{\frac{-pl^n}{a^n}}. \tag{2.3}$$

Integrating the inequality (2.3) n times, from 0 to x, we get

$$\begin{aligned}
&-\frac{\epsilon x^{n+1}}{(n+1)!\, p^2\, a^n} e^{\frac{-pl^n}{a^n}} \leq W(x,p) - W(0,p) - \frac{dW(0,p)}{dx}x \\
&-\frac{d^2W}{dx^2}(0,p)\frac{x^2}{2!} - \frac{d^3W}{dx^3}(0,p)\frac{x^3}{3!}\cdots - \frac{d^{n-1}W}{dx^{n-1}}(0,p)\frac{x^{n-1}}{(n-1)!} \\
&-\frac{p}{(n-1)!\, a^n}\int_0^x W(s,p)(x-s)^{n-1}ds \\
&+\frac{1}{(n-1)!\, a^n}\int_0^x \mu(s)(x-s)^{n-1}ds \leq \frac{\epsilon x^{n+1}}{(n+1)!\, p^2\, a^n} e^{-\frac{pl^n}{a^n}},
\end{aligned}$$

i.e.

$$\begin{aligned}
&-\frac{\epsilon x^{n+1}}{(n+1)!\, p^2\, a^n} e^{\frac{-pl^n}{a^n}} \leq W(x,p) - V_0(p) - V_1(p)x \\
&-V_2(p)\frac{x^2}{2!} - V_3(p)\frac{x^3}{3!}\cdots - V_{n-1}(p)\frac{x^{n-1}}{(n-1)!} \\
&-\frac{p}{(n-1)!\, a^n}\int_0^x W(s,p)(x-s)^{n-1}ds \\
&+\frac{1}{(n-1)!\, a^n}\int_0^x \mu(s)(x-s)^{n-1}ds \leq \frac{\epsilon x^{n+1}}{(n+1)!\, p^2\, a^n} e^{-\frac{pl^n}{a^n}}. \qquad (2.4)
\end{aligned}$$

It is easily verified that the function $U(x,p) = \mathcal{L}\{u(x,t)\}$ which is given by

$$\begin{aligned}
U(x,p) &= V_0(p) + V_1(p)x + V_2(p)\tfrac{x^2}{2!} + \cdots + V_{n-1}(p)\tfrac{x^{n-1}}{(n-1)!} \\
&\quad + \tfrac{p}{(n-1)!a^n}\int_0^x U(s,p)(x-s)^{n-1}ds - \tfrac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds
\end{aligned}$$

has to satisfy the equation $\frac{d^nW}{dx^n}(x,p) - \frac{pW}{a^n}(x,p) + \frac{\mu(x)}{a^n} = 0$, with the boundary conditions

$$W(0,p) = V_0(p), W_x(0,p) = V_1(p), \cdots, W_{xx\cdots x}(0,p) = V_{n-1}(p). \qquad (2.5)$$

Next consider, the difference

$$\begin{aligned}
\Delta &= |W(x,p) - U(x,p)| \\
&= |W(x,p) - V_0(p) - V_1(p)x - V_2(p)\tfrac{x^2}{2!} - \cdots\cdots\cdots - V_{n-1}(p)\tfrac{x^{n-1}}{(n-1)!} \\
&\quad -\tfrac{p}{(n-1)!a^n}\int_0^x U(s,p)(x-s)^{n-1}ds + \tfrac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds|. \\
&= |W(x,p) - V_0(p) - V_1(p)x - V_2(p)\tfrac{x^2}{2!} - \cdots - V_{n-1}(p)\tfrac{x^{n-1}}{(n-1)!} \\
&\quad -\tfrac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds + \tfrac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds \\
&\quad +\tfrac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds - \tfrac{p}{(n-1)!a^n}\int_0^x U(s,p)(x-s)^{n-1}ds|.
\end{aligned}$$

$$\leq |W(x,p) - V_0(p) - V_1(p)x - V_2(p)\frac{x^2}{2!} - \cdots\cdots\cdots - V_{n-1}(p)\frac{x^{n-1}}{(n-1)!}$$
$$-\frac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds + \frac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds|$$
$$+\frac{p}{(n-1)!a^n}\int_0^x |W(s,p) - U(s,p)|(x-s)^{n-1}ds.$$
$$\leq \frac{\epsilon x^{n+1}}{(n+1)!\,p^2\,a^n}e^{-\frac{pl^n}{a^n}} + \frac{p}{(n-1)!a^n}\int_0^x |W(s,p) - U(s,p)|(x-s)^{n-1}ds. \quad \text{(Using(2.4))}.$$
$$\leq \frac{\epsilon l^n}{(n+1)!a^n}\frac{x}{p^2}e^{-\frac{pl^n}{a^n}} + \frac{p}{(n-1)!a^n}\int_0^x |W(s,p) - U(s,p)|(x-s)^{n-1}ds.$$

By using Grownwall inequality we get

$$|W(x,p) - U(x,p)| \leq \frac{\epsilon l^n}{(n+1)!a^n}\frac{x}{p^2}e^{-\frac{pl^n}{a^n}} \quad e^{\int_0^x \frac{p}{(n-1)!a^n}(x-s)^{n-1}ds}.$$
$$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\epsilon l^n}{(n+1)!a^n}\frac{x}{p^2}e^{-\frac{pl^n}{a^n}} \quad e^{\frac{pl^n}{n!a^n}}.$$
$$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\epsilon l^n}{(n+1)!a^n}\frac{x}{p^2} \quad e^{-\frac{p(n!-1)l^n}{a^n n!}}.$$
$$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\epsilon l^n}{(n+1)!a^n} \quad \mathcal{L}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\}.$$
$$\Rightarrow -\frac{\epsilon l^n}{(n+1)!a^n} \quad \mathcal{L}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\} \leq W(x,p) - U(x,p)$$
$$\leq \frac{\epsilon l^n}{(n+1)!a^n} \quad \mathcal{L}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\}.$$
$$\Rightarrow -\frac{\epsilon l^n}{(n+1)!a^n}\mathcal{L}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\} \leq \mathcal{L}\{w(x,t) - u(x,t)\}$$
$$\leq \frac{\epsilon l^n}{(n+1)!a^n}\mathcal{L}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\}.$$

Taking inverse Laplace transform, we get,

$$-\frac{\epsilon l^n}{(n+1)!a^n}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\} \leq w(x,t) - u(x,t) \leq \frac{\epsilon l^n}{(n+1)!a^n}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\}.$$

$$\Rightarrow |w(x,t) - u(x,t)| \leq \frac{\epsilon l^n}{(n+1)!a^n}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\}.$$

Consequently, we have

$$\max_{o\leq x\leq l}|w(x,t) - u(x,t)| \leq \frac{\epsilon l^n}{(n+1)!a^n}\{\alpha(t - \frac{(n!-1)l^n}{n!a^n})\}.$$

Hence the I-BVP $(1.1) - (1.3)$ is HUR stable. □

Remark 2.2. The above result is an extension of the results for the HUR stability of the first and third order linear partilal differential equations, proved in [15] .

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SINGULAR MULTIPLICATIVE CALCULUS USING MULTIPLICATIVE MODULUS FUNCTION

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Abstract. There is a usual multiplicative calculus apart from classical Newton-Leibnitz additive calculus. This article provides a new multiplicative calculus, which includes singular multiplicative differentiation, singular multiplicative Riemann integration, and singular multiplicative Lebesgue integration based on newly introduced multiplicative measure. Properties of multiplicative modulus function have been used.

1. Introduction

Calculus, which is based on addition and summation, is known to high school students. There was no difficult to introduce multiplication based calculus, but it became an active research area only after appearance of the article [2]. The author also wrote an article [4] in usual multiplicative calculus about multiplicative differentiation, multiplicative Riemann integration, and multiplicative Lebesgue integration with respect usual measures, by using multiplicative modulus function. The concept of multiplicative modulus function was just mentioned in the article [2]. The first extensive usage of this concept was done in the recent article [3] to study infinite products. The second usage was done in the article [4]. The present article also uses the concept of multiplicative modulus function. Since the concepts introduced in this article are different from usual concepts in multiplicative calculus, the word "singular" is prefixed before the names of the concepts. There are some advantages in introduction of new concepts, and because of these advantages a deviation takes place. One deviation is introduction of multiplicative measures. It should be mentioned that if usual metrics

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are considered as additive metrics, there are other multiplicative metrics which have been introduced just for derivation of fixed point theorems; see [1]. There are topologies induced by these multiplicative metrics which can be obtained by exponentiation of usual metrics. Exponential function and logarithmic function play important role in transforming fixed point results from metric spaces to multiplicative metric spaces, as they play a role in transforming results from series to infinite products.

A real valued function f on an open interval (a, b) containing a point x_0 is said to be differentiable at x_0, if the limit $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} - k$ exists and it is equal to 0, for some real number k. That is, f is said to be differentiable at x_0, if for given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0) - k(x - x_0)| < \varepsilon|x - x_0|$ whenever $|x - x_0| < \delta$, and $x \in (a, b)$. This particular format is to be used in the third section to introduce a new concept of singular multiplicative differentiation. The fourth section introduces a concept of singular multiplicative Riemann integration by interchanging existing roles of bases and exponents. A similar modification is done in the fifth section to introduce a new concept of singular multiplicative Lebesgue integration by using newly introduced multiplicative measures. The next section provides some fundamental properties of multiplicative metrics and multiplicative modulus function. The author provides only results which are derivable by him.

2. Multiplicative modular function and EL-metrics

The following different name EL-metrics has been given, because of usage of Exponential function and Logarithmic function in transforming results and concepts. Definitions of many such metrics may be seen in [1]. The definition of EL-metrics may also be seen in the survey article [1] with a different name.

Definition 2.1. Let X be a non empty set. A mapping $D : X \times X \to [1, \infty)$ is called an EL-metric, if the following axioms are true.

(a) $D(x, y) = 1$ if and only if $x = y$ in X.
(b) $D(x, y) = D(y, x)$ for all x, y in X.
(c) $D(x, y) \leq D(x, z)D(y, z)$ for all x, y, z in X.

The pair (X, D) is called an EL-metric space.

Definition 2.2. For each x in $(0,\infty)$, let us associate a value $|x|_\times$ in $[1,\infty)$, which is defined by $|x|_\times = max\{x, \frac{1}{x}\}$. This function $|\ |_\times : (0,\infty) \to [1,\infty)$ is called multiplicative modulus function. Let us declare for convention that $|0|_\times = |\infty|_\times = \infty$.

Example 2.3. Define D in $(0,\infty)$ by $D(x,y) = |\frac{x}{y}|_\times$. Then D is an EL-metric on $(0,\infty) \times (0,\infty)$.

Remark 2.4.
(a) If d is a metric on X, and if D is defined on $X \times X$ by the relation $D(x,y) = \exp d(x,y)$, for all $x, y \in X$, then D is an EL-metric on X. Let us write it $D = \exp d$.
(b) If D is an EL-metric on X, and if d is defined on $X \times X$ by the relation $d(x,y) = \log D(x,y)$, for all $x, y \in X$, then d is a metric on X. Let us write it $d = \log D$.
(c) If D is an EL-metric, then $D = \exp(\log D)$. If d is a metric, then $d = \log(\exp d)$.
(d) Let D be an EL-metric on X. For each $r \geq 1$ and for each $x \in X$, let us define $B(x,r) = \{y \in X : D(x,y) < r\}$. Let $\tau = \{U \subseteq X :$ For each $x \in U$, there is a $r > 1$ such that $B(x,r) \subseteq U\}$. Then τ is a topology on X. Let us call it the topology induced by D. If $d = \log D$, then the topology induced by d coincides with the topology induced by D.
(e) If d is a metric on X, and if $D = \exp d$, then the topology induced by d coincides with the topology induced by D.
(f) Let (X,D) be an EL-metric space. A sequence $(x_n)_{n=1}^{\infty}$ is said to be Cauchy in (X,D), if for given $\varepsilon > 1$ there is a positive integer n_0 such that $D(x_n, x_m) < \varepsilon, \forall n, m \geq n_0$. A sequence $(x_n)_{n=1}^{\infty}$ is said to converge to x in (X,D), if for given $\varepsilon > 1$, there is a positive integer n_0 such that $D(x_n, x) < \varepsilon, \forall n \geq n_0$. Let $d = \log D$. A sequence $(x_n)_{n=1}^{\infty}$ is Cauchy ($(x_n)_{n=1}^{\infty}$ converges to x) in (X,D) if and only if it is Cauchy (it converges to x) in (X,d). If it is defined that (X,D) is a complete EL-metric space whenever every Cauchy sequence converges in (X,D), then (X,D) is complete if and only if (X,d) is complete.
(g) Let (X,d) be a metric space. If $D = \exp d$, then a sequence $(x_n)_{n=1}^{\infty}$ is Cauchy ($(x_n)_{n=1}^{\infty}$ converges to x) in (X,D) if and only if it is Cauchy (it converges to x) in (X,d). Thus, (X,D) is complete if and only if (X,d) is complete.

(h) A function f from an EL-metric space (X, D_X) to an EL-metric space (Y, D_Y) is said to be uniformly continuous, if for given $\varepsilon > 1$, there is a $\delta > 1$ such that $D_Y(f(x), f(y)) < \varepsilon$ whenever $D_X(x, y) < \delta$, and $x, y \in X$. A function f from an EL-metric space (X, D_X) to a metric space (Y, d_Y) is said to be uniformly continuous, if for given $\varepsilon > 0$, there is a $\delta > 1$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $D_X(x, y) < \delta$, and $x, y \in X$. A function f from a metric space (X, d_X) to an EL-metric space (Y, D_Y) is said to be uniformly continuous, if for given $\varepsilon > 1$, there is a $\delta > 0$ such that $D_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$, and $x, y \in X$. If these three statements given above are considered as definitions, then it can be concluded that f is uniformly continuous, whenever f is continuous on X, and whenever X is compact with respect to the topology induced by D_X or d_X.

All these properties of EL-metrics can be verified. The following properties of multiplicative modular function can also be verified directly.

Remark 2.5. (a) $|x|_\times = \exp(\log |x|_\times)$, for all $x \in (0, \infty)$.

(b) $|x| = \log(\exp |x|)$, for all $x \in (-\infty, +\infty)$.

(c) If $d(x, y) = |x - y|$ for all $x, y \in (-\infty, +\infty)$, and if $D(x, y) = |\frac{x}{y}|_\times$ for all $x, y \in (0, \infty)$, then $D = \exp(\log D)$, and $d = \log(\exp d)$.

(d) $|xy|_\times \leq |x|_\times |y|_\times$ and $\left|\frac{x}{y}\right|_\times \leq |x|_\times |y|_\times$, for all $x, y \in (0, \infty)$.

(e) $|x^y|_\times = |x|_\times^{|y|}$, for all $x \in (0, \infty)$ and for all $y \in (-\infty, +\infty)$.

(f) $|x^y|_\times \leq x^{|y|_\times}$, for all $x \in [1, \infty)$ and for all $y \in (0, \infty)$.

(g) For a given $x > 0$, if $p = (|x|_\times x)^{1/2}, q = \left(\frac{|x|_\times}{x}\right)^{1/2}$, then $\frac{p}{q} = x, pq = |x|_\times, p \geq 1$, and $q \geq 1$.

Remark 2.6. The conventions given in [6] for algebraic operations for extended real number system are to be followed in this article, in addition to the followings. $\infty^0 = 1$, and $1^x = 1$ and $0.x = 0, \forall x \in [-\infty, +\infty]$. If $x > 0, a \in (-\infty, +\infty)$, then $x^a > 0$.

3. Singular multiplicative differentiation

Let us begin with a new definition guessed from definition of classical differentiation as mentioned in the first section.

Definition 3.1. Let (a,b) be an interval in $(0,\infty)$. Let f be a positive real valued function defined on (a,b). Then f is said to be sm-differentiable at a point $x_0 \in (a,b)$, if there is a real number constant k such that for given $\varepsilon > 0$ there is a $\delta > 1$ such that

$$\left|\frac{f(x)}{f(x_0)}\left(\frac{x_0}{x}\right)^k\right|_\times = \left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^k\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^\varepsilon = \left|\frac{x}{x_0}\right|_\times^\varepsilon,$$

whenever $\left|\frac{x_0}{x}\right|_\times < \delta$ and $x \in (a,b)$.

Proposition 3.2. *The constant k mentioned in Definition 3.1 is unique.*

Proof. Suppose there are two different real constants k and s for Definition 3.1. Fix $\varepsilon = \frac{|k-s|}{4} > 0$. Then there is a $\delta > 1$ such that $\left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^k\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^\varepsilon$ and $\left|\frac{f(x)}{f(x_0)}\left(\frac{x_0}{x}\right)^s\right|_\times \leq \left|\frac{x}{x_0}\right|_\times^\varepsilon$, whenever $\left|\frac{x_0}{x}\right|_\times < \delta$ and $x \in (a,b)$. Then

$$\left|\frac{x}{x_0}\right|_\times^{|k-s|} = \left|\left(\frac{x}{x_0}\right)^{(k-s)}\right|_\times = \left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^k \frac{f(x)}{f(x_0)}\left(\frac{x_0}{x}\right)^s\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^{2\varepsilon} = \left|\frac{x_0}{x}\right|_\times^{(\frac{|k-s|}{2})},$$

whenever $\left|\frac{x_0}{x}\right|_\times < \delta$ and $x \in (a,b)$, by Remark 2.5. This is impossible. So, $k = s$. □

Definition 3.3. The constant k mentioned in Definition 3.1 is called sm-derivative of f at x_0 and it is denoted by $f^{(sm)}(x_0)$.

Example 3.4. If $f(x) = x^a, x > 0$, for some real constant a, then

$$\left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^a\right|_\times = \left|\left(\frac{x_0}{x}\right)^a\left(\frac{x}{x_0}\right)^a\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^\varepsilon,$$

for every $\varepsilon > 0$. Thus $f^{(sm)}(x) = a, \forall x \in (0,\infty)$.

Remark 3.5. Suppose f and g are positive real valued functions on $(a,b) \subseteq (0,\infty)$. Suppose f and g are sm-differentiable (continuous) at a point $x_0 \in (a,b)$. Let c be a real constant. Then the followings are true.

(a) If $c > 0$, then cf is sm- differentiable at x_0 , and $(cf)^{(sm)}(x_0) = f^{(sm)}(x_0)$.

(b) The function f^c is sm- differentiable at x_0, and $(f^c)^{(sm)}(x_0) = c(f^{(sm)}(x_0))$, where $f^c(x) = f(x)^c$.

Proof. Let $k = f^{(sm)}(x_0)$. Let $\varepsilon > 0$ be given. If $c \neq 0$, then let $\delta > 1$ be such that $\left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^k\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^{(\frac{\varepsilon}{|c|})}$, whenever $\left|\frac{x_0}{x}\right|_\times < \delta$

and $x \in (a,b)$. Then

$$\left|\frac{(f(x_0))^c}{(f(x))^c}\left(\frac{x}{x_0}\right)^{kc}\right|_\times = \left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^{k}\right|_\times^{|c|} \leq \left|\frac{x_0}{x}\right|_\times^{\varepsilon},$$

whenever $\left|\frac{x_0}{x}\right|_\times < \delta$ and $x \in (a,b)$. If $c = 0$, then again

$$1 = \left|\frac{(f(x_0))^c}{(f(x))^c}\left(\frac{x}{x_0}\right)^{kc}\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^{\varepsilon},$$

whenever $\left|\frac{x_0}{x}\right|_\times < \delta$ and $x \in (a,b)$. This proves that $(f^c)^{(sm)}(x_0) = c(f^{(sm)}(x_0))$. $\square$

(c) The function fg is sm-differentiable at x_0, and $(fg)^{(sm)}(x_0) = f^{(sm)}(x_0) + g^{(sm)}(x_0)$.

Proof. Let $k = f^{(sm)}(x_0)$ and $s = g^{(sm)}(x_0)$. Let $\varepsilon > 0$ be given. There is a $\delta > 1$ such that $\left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^k\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^{\frac{\varepsilon}{2}}$ and $\left|\frac{g(x_0)}{g(x)}\left(\frac{x}{x_0}\right)^s\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^{\frac{\varepsilon}{2}}$, whenever $\left|\frac{x_0}{x}\right|_\times < \delta$ and $x \in (a,b)$. Then $\left|\frac{f(x_0)g(x_0)}{f(x)g(x)}\left(\frac{x}{x_0}\right)^{k+s}\right|_\times \leq \left|\frac{f(x_0)}{f(x)}\left(\frac{x}{x_0}\right)^k\right|_\times \left|\frac{g(x_0)}{g(x)}\left(\frac{x}{x_0}\right)^s\right|_\times \leq \left|\frac{x_0}{x}\right|_\times^{\varepsilon}$, whenever $\left|\frac{x_0}{x}\right|_\times < \delta$ and $x \in (a,b)$ $\square$

(d) The inverse function $h = \frac{1}{f}$ is sm-differentiable at x_0, and $(h)^{(sm)}(x_0) = -f^{(sm)}(x_0)$.

Proof. Use the relation

$$\frac{h(x_o)}{h(x)}\left(\frac{x}{x_0}\right)^{-k} = \frac{f(x)}{f(x_0)}\left(\frac{x_0}{x}\right)^{k}.$$

$\square$

4. Singular multiplicative Riemann integration

Let $[a,b] \subseteq (0,\infty)$. Let $f : [a,b] \to [m,M] \subseteq (-\infty,+\infty)$ be a function. Let $\mathbb{P}$ be the collection of all partitions $P = \{x_0, x_1, ..., x_n\}, a = x_0 < x_1 < ... < x_n = b, n = 1,2,....$ Then $\mathbb{P}$ is a directed set under the inclusion relation. For a fixed partition $P = \{x_0, x_1, ..., x_n\}$, let $t_i \in [x_i, x_{i+1}]$ for $i = 0,1,2,...,n-1$, and consider the finite product $\prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{f(t_i)}$. Suppose that the net $\left(\prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{f(t_i)}\right)_{P\in\mathbb{P}}$ converges uniformly to a positive real number $s > 0$ for all possible $t_i \in [x_i, x_{i+1}]$. That is, for given $\varepsilon > 0$, there is a partition P_0 of $[a,b]$ such that $\left|\prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{f(t_i)} - s\right| < \varepsilon$, for all $P \supseteq P_0$ in $\mathbb{P}$, and for all possible points $t_i \in [x_i, x_{i+1}]$. Then f is said to be sm-Riemann integrable, and let us write the value s as $SM_a^b f(x)dx$. Note that this s is unique.

Let $[a,b] \subseteq (0,\infty)$. Fix a function $f : [a,b] \to [m,M] \subseteq (0,\infty)$. Fix a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$. Let $m_i = \inf_{t_i \in [x_i, x_{i+1}]} f(t_i)$ and $M_i = \sup_{t_i \in [x_i, x_{i+1}]} f(t_i)$, for $i = 0, 1, 2, ..., n-1$. Let $L(P,f) = \prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{m_i}$ and $U(P,f) = \prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{M_i}$. Here $m_i \geq 0, \forall i$. Then f is sm-integrable if and only if for given $\varepsilon > 1$, there is a partition P such that $\frac{U(P,f)}{L(P,f)} < \varepsilon$.

Remark 4.1. Let $[a,b] \subseteq (0,\infty)$. Suppose $f : [a,b] \to [m,M] \subseteq (0,\infty)$ and $g : [a,b] \to [m,M] \subseteq (0,\infty)$ be two sm-Riemann integrable functions over $[a,b]$. Let c be a positive constant. Then $f+g$ is sm-Riemann integrable over $[a,b]$, and the constant function c is integrable over $[a,b]$. Also, $SM_a^b(f+g)(x)dx = (SM_a^b f(x)dx)(SM_a^b g(x)dx)$ and $SM_a^b(cf)(x)dx = (SM_a^b f(x)dx)^c$.

The following Proposition 4.2 provides an immediate example for a class of sm-Riemann integrable functions. This proposition will be generalized in Theorem 5.15.

Proposition 4.2. *Let $[a,b] \subseteq (0,\infty)$. Let $f : [a,b] \to [m,M] \subseteq (0,\infty)$ be a continuous function. Then f is sm-Riemann integrable over $[a,b]$.*

Proof. The function f is uniformly continuous, when $[a,b]$ is endowed with the EL-metric $D(x,y) = \left|\frac{x}{y}\right|_\times$ for all $x, y \in [a,b]$, and $[m,M]$ is endowed with the metric $d(x,y) = |x-y|$ for all $x, y \in [m,M]$. Fix $\varepsilon > 1$. Find $\eta > 0$ such that $\left(\frac{b}{a}\right)^\eta < \varepsilon$. For this $\eta > 0$, there is a $\delta > 1$ such that $|f(x) - f(y)| < \eta$ whenever $\left|\frac{x}{y}\right|_\times < \delta$. Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[a,b]$ such that $\left(\frac{x_{i+1}}{x_i}\right) < \delta, \forall i$. Then $|M_i - m_i| \leq \eta$ when $m_i = \inf_{t \in [x_i, x_{i+}]} f(t)$ and $M_i = \sup_{t \in [x_i, x_{i+1}]} f(t)$, $\forall i$. For this P,

$$\frac{U(P,f)}{L(P,f)} = \prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{M_i - m_i} \leq \prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^\eta = \left(\frac{b}{a}\right)^\eta < \varepsilon.$$

This proves that f is sm-Riemann integrable over $[a,b]$. □

Lemma 4.3. *Let $[a,b] \subseteq (0,\infty)$. Let $f : [a,b] \to [m,M] \subseteq (0,\infty)$ be a function which is sm-Riemann integrable over $[a,b]$. For each $c \in [a,b]$, then the function $f : [a,c] \to [m,M]$ is sm-Riemann integrable over $[a,c]$.*

Proof. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$ such that $c \in \{x_0, x_1, ..., x_n\}$ and $c = x_r$ for some r. Then $1 \le \prod_{i=0}^{r-1} \left(\frac{x_{i+1}}{x_i}\right)^{M_i - m_i} \le \prod_{i=0}^{n-1} \left(\frac{x_{i+1}}{x_i}\right)^{M_i - m_i} = \frac{U(P,f)}{L(P,f)}$. This relation proves that $f : [a,c] \to [m,M]$ is sm-Riemann integrable over $[a,c]$. □

Theorem 4.4. *Let $[a,b] \subseteq (0,\infty)$. Let $f : [a,b] \to [m,M] \subseteq (0,\infty)$ be a function which is sm-Riemann integrable over $[a,b]$. For $a \le x \le b$, define $F(x) = SM_a^x f(t)dt$. Then F is continuous on $[a,b]$. If f is continuous at a point x_0, then F is sm-differentiable at x_0 and $F^{(sm)}(x_0) = f(x_0)$.*

Proof. For $a \le x < y \le b$ and for $c = \max\{m, M, m^{-1}, M^{-1}\}$,

$$\left|\frac{F(x)}{F(y)}\right|_\times = |SM_x^y f(t)dt|_\times \le SM_x^y c dt = \left(\frac{y}{x}\right)^c.$$

This proves uniform continuity of F over $[a,b]$.

Suppose further that f is continuous at $x_0 \in [a,b]$. Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that $|f(u) - f(x_0)| < \varepsilon$ whenever $|u - x_0| < \delta$. Then for $y \in [a,b]$ satisfying $x_0 \le y < x_0 + \delta$,

$$\begin{aligned}\left|\frac{F(y)}{F(x_0)}\left(\frac{x_0}{y}\right)^{f(x_0)}\right|_\times &= |(SM_{x_0}^y f(u)du)(SM_{x_0}^y(-f(x_0))du)|_\times \\ &= |SM_{x_0}^y (f(u) - f(x_0))du|_\times \le \left|\frac{y}{x_0}\right|_\times^\varepsilon.\end{aligned}$$

Similarly, for $y \in [a,b]$ satisfying $x_0 - \delta < y \le x_0$,

$$\begin{aligned}\left|\frac{F(x_0)}{F(y)}\left(\frac{y}{x_0}\right)^{f(x_0)}\right|_\times &= |(SM_y^{x_0} f(u)du)(SM_y^{x_0}(-f(x_0))du)|_\times \\ &= |SM_y^{x_0} (f(u) - f(x_0))du|_\times \le \left|\frac{y}{x_0}\right|_\times^\varepsilon.\end{aligned}$$

This proves that F is sm-differentiable at x_0 and $F^{(sm)}(x_0) = f(x_0)$. □

Remark 4.5. Since all positive constant functions c are sm-Riemann integrable over $[a,b] \subseteq (0,\infty)$, there are many continuous functions between c and $c+1$ which are sm-Riemann integrable, and hence there are many sm-differentiable functions in view of the previous Theorem 4.4.

5. Singular multiplicative integration by multiplicative measure

For arguments and similarities of arguments, it may be referred to the books [5, 6], mainly to the book [6]. In this section, $(X, \mathfrak{M})$ will denote a measurable space, in which X will denote a nonempty set, and $\mathfrak{M}$ will denote a σ-algebra. All subsets of X to be considered will be only measurable sets, and all extended real valued functions on X which are to be considered will be only measurable functions, when $\mathfrak{M}$ is also considered.

Definition 5.1. Let $\mu : \mathfrak{M} \to [1, \infty]$ be a function such that $\mu(A) < \infty$, for some $A \in \mathfrak{M}$, and such that $\mu(\bigcup_{i=1}^{\infty} A_i) = \prod_{i=1}^{\infty} \mu(A_i)$ whenever $A_i \in \mathfrak{M}$, $\forall i$, and $A_i \bigcap A_j = \emptyset$ for $i \neq j$. Let us call the function μ as a sm-measure. The triple $(X, \mathfrak{M}, \mu)$ is called sm-measure space.

Remark 5.2. Let $\mu : \mathfrak{M} \to [1, \infty]$ be a sm-measure. Define $\lambda : \mathfrak{M} \to [0, \infty]$ by $\lambda(E) = \log \mu(E)$, when $\mu(E) < \infty$, and $\lambda(E) = \infty$, otherwise, for $E \in \mathfrak{M}$. Then λ is a usual countably additive positive measure (or, simply measure). Let us write $\lambda = \log \mu$, in this case. On the other hand, if $\lambda : \mathfrak{M} \to [0, \infty]$ is a given countably additive positive measure, and if $\mu : \mathfrak{M} \to [1, \infty]$ is defined by $\mu(E) = \exp \lambda(E)$, when $\lambda(E) < \infty$, and $\mu(E) = \infty$, otherwise, for $E \in \mathfrak{M}$, then μ is a sm-measure. Let us write $\mu = \exp \lambda$, in this case. A common fact regarding "almost everywhere" is that $\mu(E) = 1$ if and only if $\lambda(E) = 0$.

Proposition 5.3. *Let μ be a sm-measure on a measurable space $(X, \mathfrak{M})$. Then the followings are true.*

(a) $\mu(\emptyset) = 1$.
(b) $\mu(\bigcup_{i=1}^{r} A_i) = \prod_{i=1}^{r} \mu(A_i)$, *for all pair wise disjoint $A_i \in \mathfrak{M}$, and for all $r = 1, 2, \ldots$.*
(c) *If $A \subseteq B, A \in \mathfrak{M}$, and $B \in \mathfrak{M}$, then $\mu(A) \leq \mu(B)$.*
(d) *If $A_1 \subseteq A_2 \subseteq \ldots$ in $\mathfrak{M}$, and if $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.*
(e) *If $A_1 \supseteq A_2 \supseteq \ldots$ in $\mathfrak{M}$, $\mu(A_1) < \infty$, and if $A = \bigcap_{i=1}^{\infty} A_i$, then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.*

Constructing Lebesgue measure can be done by means of Lebesgue outer measure, as it is explained in the book [5]. This method is used in the next example.

Example 5.4. Let $X = (0, \infty)$. For each interval $[a, b]$ in $(0, \infty)$ with $a \leq b$, let us define $\mu^*([a, b]) = \mu^*((a, b]) = \mu^*([a, b)) = \mu^*((a, b)) = \frac{b}{a}$. If $a = 0$ or $b = \infty$ with $a < b$, let us define $\mu^*((a, b)) = \infty$. If $a = 0$ and $b \in (0, \infty)$, let us define $\mu^*((a, b]) = \infty$. If $a \in (0, \infty)$ and $b = \infty$, let us define $\mu^*([a, b)) = \infty$. For a subset A of $(0, \infty)$, let us define $\mu^*(A) = \inf \prod_{i=1}^{\infty} \mu^*(I_i)$, where the infimum is taken over all countable collections $(I_i)_{i=1}^{\infty}$ of intervals of the types mentioned above satisfying $\bigcup_{i=1}^{\infty} I_i \supseteq A$. Let $\mathfrak{M}$ be the collection of all (classical) Lebesgue measurable subsets of $(0, \infty)$. Define $\mu : \mathfrak{M} \to [1, \infty]$ by $\mu(A) = \mu^*(A)$, for all $A \in \mathfrak{M}$. Then μ is a sm-measure on $\mathfrak{M}$. Let us call it Lebesgue sm-measure.

Definition 5.5. Let $(X, \mathfrak{M}, \mu)$ be a sm-measure space. Let $s : X \to (0, \infty)$ be a simple measurable function of the form $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where α_i are distinct positive numbers and $A_i = \{x \in X : s(x) = \alpha_i\}$. Let $A \in \mathfrak{M}$. Define sm-Lebesgue integral of s over A by $SM_A s d\mu = \prod_{i=1}^{n} (\mu(A_i \bigcap A))^{\alpha_i}$. Let $f : X \to [0, \infty]$ be a measurable function. Then, let us define the sm-Lebesgue integral of f over A by $SM_A f d\mu = \sup SM_A s d\mu$, where the supremum is taken over all simple measurable functions $s : X \to (0, \infty)$ satisfying $0 < s(x) \leq f(x)$, $\forall x \in X$.

Proposition 5.6. *Let $(X, \mathfrak{M}, \mu)$ be a sm-measure space. Let $f : X \to [0, \infty]$ and $g : X \to [0, \infty]$ be measurable functions. Then the followings are true.*

(a) *If $0 \leq f \leq g$, then $1 \leq SM_E f d\mu \leq SM_E \, g d\mu, \forall E \in \mathfrak{M}$.*
(b) *If $A \subseteq B$ in $\mathfrak{M}$, then $SM_A f d\mu \leq SM_B f d\mu$.*
(c) *If $c \in [0, \infty)$ is a constant, then $SM_E(cf) d\mu \leq (SM_E f d\mu)^c$, $\forall E \in \mathfrak{M}$.*
(d) *If $E \in \mathfrak{M}$, and $f(x) = 1$, $\forall x \in E$, then $SM_E f d\mu = \mu(E)$.*
(e) *If $\mu(E) = 1$, then $SM_E f d\mu = 1$ even if $f(x) = \infty$, $\forall x \in E$.*
(f) *If $E \in \mathfrak{M}$, then $SM_E f d\mu = SM_X \chi_E f d\mu$.*

Let $(X, \mathfrak{M}, \mu)$ be a sm-measure space. Let $\lambda = \log \mu$. Then $(X, \mathfrak{M}, \lambda)$ is a positive measure space. If $f : X \to [0, \infty]$ is a measurable function, then $SM_E f d\mu = \exp \int_E f d\lambda, \forall E \in \mathfrak{M}$. Similarly, for a given positive measure space $(X, \mathfrak{M}, \lambda)$, if $\mu = \exp \lambda$, then $(X, \mathfrak{M}, \mu)$ is a sm-measure space, and $\int_E f d\lambda = \log SM_E f d\mu, \forall E \in M$, for every measurable function $f : X \to [0, \infty]$. One may use these transformations to derive results. Next Proposition 5.7 is a counterpart of Proposition 1.25 in [6], but a direct proof is given without using exponential-logarithmic transformations.

Proposition 5.7. *Let $(X, \mathfrak{M}, \mu)$ be a sm-measure space. Let $s : X \to (0, \infty)$ and $t : X \to (0, \infty)$ be simple measurable functions. Define $\varphi : \mathfrak{M} \to [1, \infty]$ by $\varphi(E) = SM_E s d\mu, \forall E \in \mathfrak{M}$. Then φ is a sm-measure on $\mathfrak{M}$. Also, $SM_E(s + t)d\mu = (SM_E s d\mu)(SM_E t d\mu), \forall E \in \mathfrak{M}$.*

Proof. Let $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where α_i are distinct positive numbers and $A_i = \{x \in X : s(x) = \alpha_i\}$. Let $E_1, E_2,...$ be disjoint measurable sets. Let $E = \bigcup_{r=1}^{\infty} E_r$. Then

$$\begin{aligned} \varphi(E) = \prod_{i=1}^{n} (\mu(A_i \cap E))^{\alpha_i} &= \prod_{i=1}^{n} \Big(\prod_{r=1}^{\infty} (\mu(A_i \cap E_r))^{\alpha_i} \Big) \\ &= \prod_{r=1}^{\infty} \Big(\prod_{i=1}^{n} (\mu(A_i \cap E_r))^{\alpha_i} \Big) = \prod_{r=1}^{\infty} \varphi(E_r). \end{aligned}$$

This proves that φ is a sm-measure. Since $E \mapsto SM_E(s + t)d\mu, E \mapsto SM_E s d\mu$, and $E \mapsto SM_E t d\mu$ are sm- measures, it follows as in the proof of Proposition 1.25 in [6] that $SM_E(s+t)d\mu = (SM_E s d\mu)(SM_E t d\mu), \forall E \in \mathfrak{M}$. □

Proposition 5.8. *Let $(X, \mathfrak{M}, \mu)$ be a sm-measure space. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions on X such that $0 \le f_1(x) \le f_2(x) \le ... \le \infty$, and such that $f_n(x) \to f(x)$ as $n \to \infty$. Then $SM_X f_n d\mu \to SM_X f d\mu$ as $n \to \infty$.*

Proof. Let $\lambda = \log \mu$. Then, by the classical monotone convergence theorem, $\int_X f_n d\lambda \to \int_X f d\lambda$ as $n \to \infty$. It can be concluded with a convention for special cases that $SM_X f_n d\mu = \exp \int_X f_n d\lambda \to \exp \int_X f d\lambda = SM_X f d\mu$ as $n \to \infty$. □

Proposition 5.8 is a counterpart of the classical monotone convergence theorem. The next Theorem 5.9 is a counterpart of Theorem 1.27 in [6].

Theorem 5.9. *Let $(X, \mathfrak{M}, \mu)$ be a sm-measure space. Let $f_n : X \to [0, \infty]$ be measurable, for $n = 1, 2,....$ Let $f(x) = \sum_{n=1}^{\infty} f_n(x), \forall x \in X$. Then $SM_X f d\mu = \prod_{n=1}^{\infty} SM_X f_n d\mu$.*

The next one is a counterpart of the classical Fatou's lemma.

Theorem 5.10. *Let $(X, \mathfrak{M}, \mu)$ be a sm-measure space. Let $f_n : X \to [0, \infty]$ be measurable, for $n = 1, 2,...$*
Then $SM_X(\liminf_{n\to\infty} f_n)d\mu \le \liminf_{n\to\infty} SM_X f_n d\mu$.

Next Theorem 5.11 is a counterpart of Theorem 1.29 in [6].

Theorem 5.11. *Let* $(X, \mathfrak{M}, \mu)$ *be a sm-measure space. Let* $f : X \to [0, \infty]$ *be a measurable function. Define* $\varphi : \mathfrak{M} \to [1, \infty]$ *by* $\varphi(E) = SM_E f d\mu, \forall E \in \mathfrak{M}$. *Then* φ *is a sm-measure on* $\mathfrak{M}$. *Also, if* $g : X \to [0, \infty]$ *is measurable, then* $SM_X\ gd\varphi = SM_X\ gfd\mu$.

Proof. Let E_1, E_2,... be disjoint measurable sets. Let $E = \bigcup_{i=1}^{\infty} E_i$. Then $\chi_E f = \sum_{i=1}^{\infty} \chi_{E_i} f$. By Theorem 5.9, $\varphi(E) = SM_X \chi_E f d\mu = \prod_{i=1}^{\infty} SM_{E_i} f d\mu = \prod_{i=1}^{\infty} \varphi(E_i)$. Also, $\varphi(\emptyset) = 1$. Thus, φ is a sm-measure. Since $SM_X \chi_E d\varphi = \varphi(E) = SM_X \chi_E f d\mu$, then $SM_X s d\varphi = SM_X \chi_E s d\mu, \forall E \in \mathfrak{M}$, for every simple measurable function $s : X \to (0, \infty)$. Now, Proposition 5.8 implies that $SM_X g d\varphi = SM_X g f d\mu$. □

Next Theorem 5.12 is a counterpart of Theorem 1.39(a) in [6].

Theorem 5.12. *Let* $(X, \mathfrak{M}, \mu)$ *be a sm-measure space. Let* $f : X \to [0, \infty]$ *be a measurable function. Let* $E \in \mathfrak{M}$. *Suppose* $SM_E f d\mu = 1$. *Then* $f = 0$ *almost everywhere on* E *(see Remark 5.2).*

Proof. Let $\lambda = \log \mu$. Then $\int_E f d\lambda = 0$. Theorem 1.39 (a) in [6] implies that $f = 0$ almost everywhere on E. That is, $\mu(\{x \in E : f(x) \neq 0\}) = 1$. □

For a given sm-measure space $(X, \mathfrak{M}, \mu)$, and for all functions $f : X \to [0, \infty]$, if $\Lambda f = \Lambda(f) = SM_X f d\mu$, then $\Lambda(f + g) = (\Lambda f)(\Lambda g), \Lambda(cf) = (\Lambda f)^c$, and $\Lambda(f) \geq 1$, for all $f : X \to [0, \infty], g : X \to [0, \infty]$, and $c \geq 0$. Next Theorem 5.13 is a counter part of Theorem 2.14 in [6], the Riesz representation theorem.

Theorem 5.13. *Let* X *be a locally compact Hausdorff space. Let* $C_c^+(X) = \{f : X \to [0, \infty)$*:*
f *is continuous on* X *with compact support* $\} = \{f : X \to [0, \infty) : f \in C_c(X)\}$. *Let* $\Lambda : C_c^+(X) \to [1, \infty)$ *be a functional satisfying* $\Lambda(f + g) = (\Lambda f)(\Lambda g)$,*and* $\Lambda(cf) = (\Lambda f)^c, \forall f, g \in C_c^+(X)$, *and* $\forall c \geq 0$. *Then there is a* σ*-algebra* $\mathfrak{M}$ *in* X *which contains all Borel subsets of* X, *and there exists a unique sm-measure* μ *on* $\mathfrak{M}$ *having the following properties.*

(a) $\Lambda f = SM_X f d\mu, \forall f \in C_c^+(X)$.
(b) $\mu(K) < \infty$, *for every compact set* $K \subseteq X$.
(c) $\mu(E) = \inf\{\mu(V) : E \subseteq V, V\, open\}, \forall E \in \mathfrak{M}$
(d) *The relation* $\mu(E) = \sup\{\mu(K) : K \subseteq E, K$ *compact*$\}$ *holds for every open set* E, *and for every* $E \in \mathfrak{M}$ *with* $\mu(E) < \infty$.

(e) *If* $E \in \mathfrak{M}, A \subseteq E$, *and* $\mu(E) = 1$, *then* $A \in \mathfrak{M}$.

Proof. Define $\widetilde{\Lambda} : C_c^+(X) \to [0, \infty)$ by $\widetilde{\Lambda}(f) = \log \Lambda f, \forall f \in C_c^+(X)$. Then $\widetilde{\Lambda}(f + g) = \widetilde{\Lambda}(f) + \widetilde{\Lambda}(g)$, and $\widetilde{\Lambda}(cf) = c\widetilde{\Lambda}(f), \forall f, g \in C_c^+(X), \forall c \geq 0$. For each $f \in C_c^+(X)$, define $\widetilde{\Lambda}(f) = \widetilde{\Lambda}(f^+) - \widetilde{\Lambda}(f^-)$, where f^+ and f^- are positive part and negative part of f, respectively. Then $\widetilde{\Lambda} : C_c(X) \to (-\infty, +\infty)$ is a positive linear functional, as it was shown in the proof of Theorem 1.32 in [6]. By Theorem 2.14 in [6], there is a σ-algebra $\mathfrak{M}$ in X which contains all Borel subsets of X, and there exists a unique measure λ on $\mathfrak{M}$ satisfying the properties (b), (c), (d), (e) with replacement of μ by λ, and 1 by 0, and satisfying the relation $\widetilde{\Lambda} f = \int_X f d\lambda, \forall f \in C_c(X)$. Then $\Lambda f = \exp \widetilde{\Lambda} f = \exp \int_X f d\lambda = SM_X f d\mu, \forall f \in C_c^+(X)$ ☐

Remark 5.14. There is another method to construct Lebesgue sm-measure in $(0, \infty)$, by using the Riesz representation theorem, Theorem 5.13, apart from the method used in Example 5.4.

Let $X = (0, \infty)$ be endowed with the usual Euclidean topology so that it is a locally compact Hausdorff space. For each $f \in C_c^+(X)$, there is a bounded interval $[a, b] \subseteq (0, \infty)$ such that $f(x) = 0$ for all $x \notin [a, b]$. Define $\Lambda : C_c^+(X) \to [1, \infty)$ by using sm-Riemann integration: $\Lambda(f) = SM_a^b f(x)dx$, when $f(x) = 0$ for all $x \notin [a, b]$. Then $\Lambda(f+g) = (\Lambda f)(\Lambda g)$,and $\Lambda(cf) = (\Lambda f)^c, \forall f, g \in C_c^+(X)$,and $\forall c \geq 0$. Then $\mathfrak{M}$ constructed indirectly in Theorem 5.13 is the collection of all Lebesgue measurable subsets of X and the sm-measure μ constructed is the Lebesgue sm-measure on $\mathfrak{M}$.

There are obvious counterparts of Theorem 2.17, Theorem 2.18, and Theorem 1.41 in [6], and there are partial counterparts of Theorem 2.24, Theorem 6.10(a), and Theorem 6.11 in [6]. They are not to be stated. Next Theorem 5.15 is a counterpart of Theorem 11.33 in [5], and this is a generalization of Proposition 4.2.

Theorem 5.15. *Let* $[a, b] \subseteq (0, \infty)$. *Let* $f : [a, b] \to [m, M] \subseteq (0, \infty)$ *be a given function. Then* f *is sm-Riemann integrable over* $[a, b]$ *if and only if it is continuous almost everywhere on* $[a, b]$. *Moreover, in this case,* $SM_{[a,b]} f d\mu = SM_a^b f(x)dx$, *where* μ *is the Lebesgue sm-measure.*

Proof. If $P = \{x_0, x_1, ..., x_n\}, a = x_0 < x_1 < ... < x_n = b$ is a partition, define $U_P(a) = L_P(a) = f(a)$, and define $U_P(x) = M_i = \sup_{t \in [x_i, x_{i+1}]} f(t)$ and $L_P(x) = m_i = \inf_{t \in [x_i, x_{i+1}]} f(t)$ for $x_i < x \leq x_{i+1}$, when $i = 0, 1, 2, ..., n-1$.

Then $L(P,f) = SM_{[a,b]} L_P d\mu$ and $U(P,f) = SM_{[a,b]} U_P d\mu$. Suppose f is sm-Riemann integrable over $[a,b]$. Then there is a sequence of partitions $P_1 \subseteq P_2 \subseteq ...$ of $[a,b]$ such that $\frac{U(P,f)}{L(P,f)} < \frac{k+1}{k}$ whenever P is a partition such that $P \supseteq P_k$, for $k = 1, 2, \ldots$. Let $L(x) = \lim_{n\to\infty} L_{P_k}(x) = \sup_{k=1,2,\ldots} L_{P_k}(x)$ and $U(x) = \lim_{n\to\infty} U_{P_k}(x) = \inf_{k=1,2,\ldots} U_{P_k}(x)$, for $x \in [a,b]$. Then $L \le f \le U$ and, by the monotone convergence Theorem 5.8, $SM_{[a,b]} L d\mu = SM_a^b f(x)dx = SM_{[a,b]} U d\mu$. Since $SM_{[a,b]}(U-L)d\mu = 1$, then $U - L = 0$ almost everywhere in $[a,b]$. Since, for $x \notin \bigcup_{k=1}^{\infty} P_k$, $U(x) = L(x)$ if and only if f is continuous at x, then f continuous almost everywhere in $[a,b]$. Also, $SM_{[a,b]} f d\mu = SM_a^b f(x)dx$.

Suppose f is continuous almost everywhere in $[a,b]$. Then there is a sequence of partitions $P_1 \subseteq P_2 \subseteq ...$ of $[a,b]$ such that $\frac{SM_{[a,b]} U_{P_k} d\mu}{SM_{[a,b]} L_{P_k} d\mu} \to 1$ as $k \to \infty$, for functions U_{P_k} and L_{P_k} defined above. Thus, $\frac{U(P_k,f)}{L(P_k,f)} \to 1$ as $k \to \infty$ This proves that f is sm-Riemann integrable over $[a,b]$. $\square$

Corollary 5.16. *Let $[a,b] \subseteq (0,\infty)$. Let $f : [a,b] \to [m,M] \subseteq (0,\infty)$ be a monotone function. Then f is sm-Riemann integrable over $[a,b]$.*

Concluding comments

If a positive real valued function f is sm-differentiable at a point x_0, then one may try to prove that $lim_{x\to x_0} \frac{\log f(x) - \log f(x_0)}{\log x - \log x_0} = f^{(sm)}(x_0)$, and one may try to extend this as a definition of a g-singular derivative as a limit $lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{g(x) - g(x_0)}$. This article is successful in providing a definition for sm-differentiability such that the definition does not depend on any specific function g. This article has illustrated methods of using exponential-logarithmic transformations as well as methods of non-using exponential-logarithmic transformations, in arguments. However, it is suggested to avoid using exponential-logarithmic transformations in arguments, just to avoid missing something. Introduction of concepts should not involve specific functions. It is expected that one should know such singular analysis exists in nature of mathematics.

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IRRATIONAL NUMBERS AND STURM-LIOUVILLE PROBLEMS

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Abstract. In this paper, we establish a relationship between boundary value problems and irrational numbers. In particular, we exploit eigenvalues of Sturm-Liouville problems to give a new proof of the irrationality of tan$r$ for nonzero rational r.

1. Introduction

The idea of combining the theory of linear differential equations with the question of irrationality of certain numbers is very appealing. Ram Murty and Kumar Murty consider a solution $y(x)$ of a linear differential equation of order n,

$$p_0y^{(n)} + p_1y^{(n-1)} + ... + p_ny = 0$$

where p_i are rational numbers with $p_n \neq 0$ in [7]. y is defined on $[0, r]$. They show that if $y^{(i)}(0)$, $y^{(i)}(r)$ are rational for $i = 0, ..., n-1$ then r is irrational. We note the following corollaries of the theorem in [7].

Corollary 1.1. π^2 *is irrational.*

Corollary 1.2. e^r, $\sin r$, $\cos r$ *are irrational for nonzero rational* r.

Trigonometric functions $\sin r$ and $\cos r$ can be seen as solutions of linear differential equations. But, $\tan r$ cannot be a solution of linear differential equations with constant coefficients. Because of that, the irrationality of the tangent function for nonzero rational values of the arguments cannot be proved by a direct calculation using the result in [7].

In this study, we reformulate the special case $n = 2$ of the result of Ram Murty and Kumar Murty. We give new proofs of the irrationality

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of π^2 and $\tan r$ for nonzero rational r. In addition to we show that the relationship between spectral theory and irrationality by using boundary value problems.

Consider the Sturm-Liouville problem

$$\ell y := -y'' + cy = \lambda y,\ x \in [0, r], \tag{1.1}$$

$$y'(0) - hy(0) = 0, \tag{1.2}$$

$$y'(r) + Hy(r) = 0, \tag{1.3}$$

where $r, h \in \mathbb{Q}$, $c, H \in \mathbb{R}$ and λ is a spectral parameter. The values of the λ parameter for which (1.1)-(1.3) has nonzero solutions are called eigenvalues $\{\lambda_n\}_{n\geq 1}$ and the corresponding nontrivial solutions are called eigenfunctions $\{y_n\}_{n\geq 1}$. Some important results on the properties of eigenvalues and eigenfunctions of Sturm–Liouville problem have been published in various publications (see, [5, 6, 9]) and the references therein). It is known that the spectrum of such problems consists of countably many real eigenvalues, which have no finite limit point.

We can show that if r is rational, $y(0)$ is rational and $y'(0) - hy(0) = 0$ for some rational h, then $y(r)$ is irrational or the number H given by $y'(r) + Hy(r) = 0$ is irrational. In this way one obtains precisely the result of Ram Murty and Kumar Murty for $n = 2$. As a result, we can give following theorem by using same techniques in [7].

Theorem 1.3. *Assume that h and r are rational. If $(c - \lambda_n) \in \mathbb{Q}\setminus\{0\}$ and $y_n(0) \in \mathbb{Q}$ for some $n \in \mathbb{N}$ then $y_n(r)$ or H is irrational.*

We make the proof simpler and give it in Section 3. The following examples and corollaries illustrate how Theorem 1.3 can be used to advantage.

Corollary 1.4. $\tan r$ *is irrational for nonzero rational r.*

Proof. As an application, we can present the proof by two steps.

Step 1: Let us consider the following boundary value problem

$$-y'' = \lambda y,\ x \in [0, r]$$

$$y'(0) = 0$$

$$y'(r) + (\tan r)\, y(r) = 0.$$

It is clear that $\lambda_1 = 1$ and $y_1(x) = \cos x$ for this problem. This problem satisfies the assumptions of Theorem 1.3 by λ_1. Hence, we have that if r is rational then at least one of the $y_1(r) = \cos r$ and $H = (\tan r)$ is irrational.

Step 2: Now, we assume that $\tan r$ is rational for rational $r \neq 0$, so $\cos 2r = \frac{1-\tan^2 r}{1+\tan^2 r}$ must be rational. Similar to Niven's proof in [8, pp. 21], it is concluded that if r is rational then $\cos r$ is rational. This contradiction completes the proof. □

Remark 1.5. Lambert in 1761 proved that $\tan r$ is irrational for nonzero rational r [1, pp. 129-146]. Since then, this result has been given with various techniques (see, for example, [8, Corollary 2.7], [10]). It can be seen explanations about the necessity of irrationality of $\cos r$ for nonzero rational r in Niven's proof [8, pp. 21]. In the proof of Corollary 1.4, we do not need the knowledge of irrationality of $\cos r$ for nonzero rational r.

Corollary 1.6. *Since* $\tan \pi = 0 \in \mathbb{Q}$, π *is irrational.*

On the other hand, if condition (1.3) is replaced by

$$y'(r) + f(\lambda)y(r) = 0,$$

we obtain the Sturm-Liouville problem with eigenparameter-dependent boundary condition for a class of functions f. The properties of eigenvalues of this kind of the problem have been studied in various publications (see, for example, [2, 3, 4]). We note that, we can determine suitable $H = f(\lambda_n)$ to show the irrationality of certain eigenvalue λ_n by using Theorem 1.3.

Example 1.7. Consider the problem

$$\begin{aligned} &-y'' = \lambda y, \;\; x \in [0,1] \\ &y'(0) = 0 \\ &y'(1) + \left(\lambda - \frac{\lambda^2}{\pi^2}\right) y(1) = 0. \end{aligned}$$

It is easy to check that $\lambda_1 = \lambda_2 = 0$ is a double eigenvalue and all other simple eigenvalues are the solutions of the equation

$$\tan\sqrt{\lambda} = \sqrt{\lambda}\left(1 - \frac{\lambda}{\pi^2}\right).$$

$\lambda_3 = \pi^2$ is a simple eigenvalue, $h = 0$ and $y_3(x) = \cos \pi x$. $y_3(0)$, $y_3(1)$ and $H = f(\lambda_3) \in \mathbb{Q}$.

Corollary 1.8. *According to Example 1.7 and Theorem 1.3, π^2 is irrational.*

By using Theorem 1.3, we can determine suitable coefficients of the problem (1.1)-(1.3) from knowledge of the eigenvalues for $\lambda < c$. Thus, we show the irrationality of certain exponential values. In a similar way, we find that following example.

Example 1.9. Let us consider

$$\begin{aligned} &-y'' + 2y = \lambda y, \ x \in [0, r] \\ &y'(0) - y(0) = 0 \\ &y'(r) - y(r) = 0. \end{aligned}$$

This problem satisfies the assumptions of the Theorem 1.3 by $\lambda_1 = 1$ and $y_1(x) = e^x$.

Corollary 1.10. *According to Example 1.9 and Theorem 1.3, e^r is irrational for nonzero rational r.*

2. Eigenvalues and the Tangent Function

We have that the eigenvalues of the problem (1.1)-(1.3) satisfy the following equation [5, pp. 78]

$$\tan\left(r\sqrt{\lambda - c}\right) = \frac{h + H}{\sqrt{\lambda - c} - \frac{hH}{\sqrt{\lambda - c}}} \tag{2.1}$$

for $\lambda > c$. Thus, we show the relation between irrationality of $\tan r$ and the eigenvalues of the problem (1.1)-(1.3). In doing this, we obtain the following corollaries.

Corollary 2.1. *According to (2.1) and Corollary 1.4, if $\sqrt{\lambda_n - c}, r \neq 0$ and h are rational for $\exists\ n \in \mathbb{N}$ then H is irrational.*

Example 2.2. Consider the problem

$$-y'' = \lambda^2 y, \ x \in [0, r] \tag{2.2}$$

$$y'(0) = 0 \tag{2.3}$$

$$y'(r) + \lambda^2 y(r) = 0. \tag{2.4}$$

The positive eigenvalues are the roots of the following equation

$$\tan r\lambda = \lambda. \tag{2.5}$$

Corollary 2.3. *According to (2.5) and Corollary 1.4, if r is nonzero rational, then the positive eigenvaules $\{\lambda_n\}_{n\geq 1}$ of (2.2)-(2.4) are irrational.*

By considering parameter-dependent boundary conditions instead of (1.2), (1.3), we can obtain some stronger results on eigenvalues.

Example 2.4. Let $g(\lambda) = h$ and $H = f(\lambda)$ for f and g arbitary function. Consider the problem

$$-y'' = \lambda^2 y,\ x \in [0, r] \tag{2.6}$$

$$y'(0) - g(\lambda)y(0) = 0 \tag{2.7}$$

$$y'(r) + f(\lambda)y(r) = 0 \tag{2.8}$$

The eigenvalues are the roots of the following equation

$$\tan r\lambda = \frac{(f(\lambda) + g(\lambda))\,\lambda}{\lambda^2 - f(\lambda)g(\lambda)}. \tag{2.9}$$

Corollary 2.5. *Assume that $g(\lambda_n)$ and $f(\lambda_n)$ are rational for all λ_n. According to (2.9) and Corollary 1.4, if r is nonzero rational, then the problem (2.6)-(2.8) has no rational eigenvalue.*

3. Proof of the Theorem 1.3

Proof. Let $r = \frac{a}{b}$, $(a, b) = 1$, $a, b \in \mathbb{Z}^+$ and the eigenfunction $y_n(x)$ corresponding to eigenvalue λ_n. Under the assumptions of the Theorem , we define the functions

$$f_m(x) = \frac{(bx)^m(a - bx)^m}{m!},$$

$$F_m(x) = f_m(x) - \frac{f_m^{(2)}(x)}{\lambda_n - c} + \frac{f_m^{(4)}(x)}{(\lambda_n - c)^2} - \cdots + \frac{(-1)^m\, f_m^{(2m)}(x)}{(\lambda_n - c)^m}$$

where $m \in \mathbb{N}$. Put

$$Ly = -y'' + (c - \lambda_n)\, y.$$

Integrating by parts twice, we obtain

$$\int_0^r F_m(x)Ly_n(x)dx - \int_0^r y_n(x)LF_m(x)dx = W_r\,[y_n, F_m] - W_0\,[y_n, F_m]$$

where

$$W_x\left[y_n, F_m\right] = \begin{vmatrix} y_n(x) & F_m(x) \\ y_n'(x) & F_m'(x) \end{vmatrix}.$$

We have $Ly_n(x) = 0$ and $LF_m(x) = (c - \lambda_n) f_m(x)$, so that

$$-\int_0^r y_n(x) LF_m(x)dx = F_m'(r)y_n(r) - F_m(r)y_n'(r) + y_n'(0)F_m(0) - y_n(0)F_m'(0).$$

Since $y_n(x)$ satisfies the boundary conditions (1.2) and (1.3), one can obtain

$$\begin{aligned} -(c-\lambda_n)\int_0^r y_n(x)f_m(x)dx &= F_m'(r)y_n(r) + F_m(r)Hy_n(r) \\ &+ hy_n(0)F_m(0) - y_n(0)F_m'(0). \end{aligned}$$

It is easy to check that $f_m^{(k)}(0)$ and $f_m^{(k)}(r)$ are integers for $k \geq 0$. It follows that $\left((c-\lambda_n)^m F_m^{(k)}(x)\right)$ is an integer for $x = 0$ and $x = r$, $k \geq 0$.

We assume that the $y_n(r)$ and H are rational. A denotes the products of the denominators of r, $(c-\lambda_n)^{m+1}$, $y_n(0)$, h, $y_n(r)$ and H. Thus,

$$\left(A\left(c-\lambda_n\right)^m \left(F_m'(r)y_n(r) + F_m(r)Hy_n(r) + hy_n(0)F_m(0) - y_n(0)F_m'(0)\right)\right) \tag{3.1}$$

is an integer. For $|y_n(x)| \leq B$ and $|bx(a-bx)| \leq C$ on $[0, r]$, we have that

$$0 < \left| A\left(c-\lambda_n\right)^{m+1} \int_0^r y_n(x)f_m(x)dx \right| < \frac{(c-\lambda_n)^{m+1} rABC^m}{m!}.$$

We obtain

$$0 < \left| A\left(c-\lambda_n\right)^{m+1} \int_0^r y_n(x)f_m(x)dx \right| < 1$$

for sufficiently large m. This contradicts with (3.1). Thus, both $y_n(r)$ and H cannot be rational at the same time. The proof is complete. □

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A NOTE ON APPLICATIONS OF THE GREGORY-LEIBNIZ SERIES FOR π AND ITS GENERALIZATION

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Abstract. It is shown how different groupings of the terms in the Gregory-Leibniz series for π and some other series can be evaluated using a hypergeometric series approach. A natural generalization of the Gregory-Leibniz series is also considered. Several interesting special cases are given.

1. Introduction

The well-known Gregory-Leibniz series for π is given by

$$S_1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}. \tag{1}$$

A more subtle problem is the evaluation of the series

$$S_m = \sum_{n=0}^{\infty} \frac{(-1)^{\lfloor n/m \rfloor}}{2n+1} \tag{2}$$

for positive integer m, when the terms in (1) are taken in groups of m with alternating signs between the groups. For example, when $m = 2$ and $m = 3$ we have

$$S_2 = \left(1 + \frac{1}{3}\right) - \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{9} + \frac{1}{11}\right) - \cdots$$

and

$$S_3 = \left(1 + \frac{1}{3} + \frac{1}{5}\right) - \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right) + \cdots .$$

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In 1978, Lew [5] proposed the evaluation of S_2 and S_3 in the Problems Section of *The American Mathematical Monthly*. Berndt [2] employed a Fourier series representation combined with Cauchy's residue theorem applied to a contour integral to evaluate these series. Subsequently Cohen [4] determined the sum of these series by a Fourier series approach combined with use of the Chebyshev polynomials.

Both these approaches used quite sophisticated analysis to evaluate cases of S_m. Our aim in this note is to demonstrate that these sums can be established in an alternative way by using a hypergeometric series approach. To achieve this we exploit two classical summation formulas for the hypergeometric series of negative unit argument. We show the details of this evaluation in the cases $m = 2, 3$ and 4, and how the procedure can then be extended quite naturally to determine the sum S_m for arbitrary positive integer m.

We first recall the definition of the generalized hypergeometric function with p numerator and q denominator parameters

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{x^n}{n!}, \tag{3} $$

where $(a)_n$ denotes the Pochhammer symbol (or rising factorial since $(1)_n = n!$) defined by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\ldots(a+n-1) & (n = 1, 2, \ldots) \\ 1 & (n = 0). \end{cases} $$

For more details about the hypergeometric function and its convergence conditions, we refer to the standard texts of Andrews *et al.* [1] and Slater [8]. We shall make use of the following two classical summation formulas [1, p. 126, 148], [8, p. 243]:

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ 1+a-b \end{matrix}; -1\right) = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)} \tag{4} $$

and

$$ {}_4F_3\left(\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c \end{matrix}; -1\right) = \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)}. \tag{5} $$

The summation formula (4) is known in the literature as Kummer's summation theorem.

In Section 2, we discuss the evaluation of the series S_m in (2) by making use of the summation formulas (4) and (5). In Section 3 some additional examples are presented. In Section 4, a summation formula for the series ${}_3F_2(1)$, when the parameters depend on a non-negative integer n, is obtained as a consequence of examining the tail of the series S_1 truncated after n terms. We conclude with a natural extension of the Gregory-Leibniz series.

2. Summation of certain π-series

To illustrate the hypergeometric series approach, we first show that $S_1 = \pi/4$. This sum may be written in hypergeometric form as

$$\begin{aligned} S_1 &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{2}} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \\ &= \sum_{n=0}^{\infty}(-1)^n\,\frac{(\frac{1}{2})_n\,(1)_n}{(\frac{3}{2})_n\,n!} \\ &= {}_2F_1\left(\begin{matrix}1,\ 1/2\\ 3/2\end{matrix};-1\right). \end{aligned}$$

The ${}_2F_1$ series can now be evaluated with the help of (4) by taking $a = 1$ and $b = \frac{1}{2}$ and we immediately obtain $S_1 = \pi/4$.

Evaluation of S_2. The nth term of S_2 is $(-1)^n\left\{\frac{1}{(4n+1)}+\frac{1}{(4n+3)}\right\}$, so that

$$S_2 = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{4n+1}+\frac{1}{4n+3}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\,(n+\frac{1}{2})}{(n+\frac{1}{4})\,(n+\frac{3}{4})}.$$

Proceeding as above, we arrive at

$$\begin{aligned} S_2 &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\,\frac{\Gamma(n+\frac{3}{2})\,\Gamma(n+\frac{1}{4})\,\Gamma(n+\frac{3}{4})}{\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{5}{4})\,\Gamma(n+\frac{7}{4})} \\ &= \frac{4}{3}\,{}_4F_3\left(\begin{matrix}1,\ 3/2,\ 1/4,\ 3/4\\ 1/2,\ 5/4,\ 7/4\end{matrix};-1\right). \end{aligned}$$

The ${}_4F_3$ series can now be evaluated by (5) by taking $a = 1$, $b = 1/4$ and $c = 3/4$ to find after some simplification the result $S_2 = \pi\sqrt{2}/4$.

Evaluation of S_3. The nth term of S_3 is $(-1)^n\left\{\frac{1}{(6n+1)}+\frac{1}{(6n+3)}+\frac{1}{(6n+5)}\right\}$, so that

$$\begin{aligned} S_3 &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{6n+1}+\frac{1}{6n+3}+\frac{1}{6n+5}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{6n+3}+\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{6n+1}+\frac{1}{6n+5}\right). \end{aligned}$$

Now, as above, it is not difficult to see that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{6n+3}=\frac{1}{3}\,{}_2F_1\left(\begin{matrix}1,\ 1/2\\ 3/2\end{matrix};-1\right)=\frac{\pi}{12}$$

by (4), and

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{6n+1}+\frac{1}{6n+5}\right)=\frac{8}{5}\,{}_4F_3\left(\begin{matrix}1,\ 3/2,\ 1/6,\ 5/6\\ 1/2,\ 7/6,\ 11/6\end{matrix};-1\right)=\frac{\pi}{3}$$

by (5). Thus we obtain the result $S_3=5\pi/12$.

Evaluation of S_4. In the case $m=4$, we have the following series

$$\begin{aligned} S_4 &= \left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)-\left(\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}\right)+\left(\frac{1}{17}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}\right)-\cdots\\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{8n+1}+\frac{1}{8n+3}+\frac{1}{8n+5}+\frac{1}{8n+7}\right)\\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{8n+1}+\frac{1}{8n+7}\right)+\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{8n+3}+\frac{1}{8n+5}\right). \end{aligned}$$

As above, we find that

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{8n+1}+\frac{1}{8n+7}\right) &= \frac{8}{7}\,{}_4F_3\left(\begin{matrix}1,\ 3/2,\ 1/8,\ 7/8\\ 1/2,\ 9/8,\ 15/8\end{matrix};-1\right)\\ &= \frac{1}{8}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{7}{8}\right) \end{aligned}$$

by (5). Similarly, we have

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{8n+3}+\frac{1}{8n+5}\right) &= \frac{8}{15}\,{}_4F_3\left(\begin{matrix}1,\ 3/2,\ 3/8,\ 5/8\\ 1/2,\ 11/8,\ 13/8\end{matrix};-1\right)\\ &= \frac{1}{8}\,\Gamma\left(\frac{3}{8}\right)\Gamma\left(\frac{5}{8}\right) \end{aligned}$$

so that, upon use of the reflection formula for the gamma function

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin \pi x},$$

we obtain

$$S_4=\frac{\pi}{8}\left(\operatorname{cosec}\frac{\pi}{8}+\operatorname{cosec}\frac{3\pi}{8}\right).$$

Having shown how to evaluate S_m for $1\le m\le 4$, we can now deal with the general case. We have, for positive integer m,

$$\begin{aligned} S_m &= \left(1+\frac{1}{3}+\cdots+\frac{1}{2m-1}\right)-\left(\frac{1}{2m+1}+\frac{1}{2m+3}+\cdots+\frac{1}{4m-1}\right)+\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2mn+1}+\frac{1}{2mn+3}+\cdots+\frac{1}{2mn+2m-1}\right). \end{aligned}$$

If we define the following sum involving pairs of the above fractions

$$H(p,2m-p):=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2mn+p}+\frac{1}{2mn+2m-p}\right)$$

for integer p satisfying $1\le p\le 2m-1$, we observe that, with $d:=\frac{p}{2m}$,

$$\begin{aligned} H(p,2m-p) &= \frac{1}{m}\sum_{n=0}^{\infty}\frac{(-1)^n(n+\frac{1}{2})}{(n+d)(n+1-d)} \\ &= \frac{2m}{(2m-p)p}\,{}_4F_3\left(\begin{matrix}1,\ 3/2,\ d,\ 1-d\\ 1/2,\ 1+d,\ 2-d\end{matrix};-1\right) \\ &= \frac{2m}{p\,(2m-p)}\Gamma(2-d)\Gamma(1+d) \\ &= \frac{\pi}{2m}\operatorname{cosec}\frac{\pi p}{2m} \end{aligned}$$

by (5) and use of the reflection formula for the gamma function.

Then we may write the series S_m, making use of the symmetry property $H(p,2m-p)=H(2m-p,p)$, as

$$S_m=\frac{1}{2}\sum_{k=0}^{m-1}H(2k+1,2m-2k-1)=\frac{\pi}{4m}\sum_{k=0}^{m-1}\operatorname{cosec}\frac{(2k+1)\pi}{2m} \tag{6}$$

for positive integer $m\ge 1$. It can be readily verified that this result reduces to the above evaluations for S_m when $1\le m\le 4$. We remark that (6) agrees with the result stated in the editorial section associated with [2].

3. A summation formula for a ${}_3F_2(1)$ series

As an application of the Gregory-Leibniz series for π in (1), we shall show how it can be employed to establish the following summation formula for the particular ${}_3F_2$ series of positive unit argument given by

$$ {}_3F_2\left(\begin{matrix} 1, \frac{2n+1}{4}, \frac{2n+3}{4} \\ \frac{2n+5}{4}, \frac{2n+7}{4} \end{matrix}; 1\right) = \frac{1}{2}(-1)^n\ (2n+1)\ (2n+3) \left\{\frac{\pi}{4} - \sum_{r=0}^{n-1} \frac{(-1)^r}{2r+1}\right\}, \tag{7} $$

where n is a non-negative integer. We note that in series form this can be written as

$$ \frac{1}{(2n+1)(2n+3)} + \frac{1}{(2n+5)(2n+7)} + \frac{1}{(2n+9)(2n+11)} + \cdots $$
$$ = \frac{1}{2}(-1)^n \left\{\frac{\pi}{4} - \sum_{r=0}^{n-1} \frac{(-1)^r}{2r+1}\right\}. $$

From (1), we subtract off the first n terms of the series to obtain

$$ \begin{aligned} \frac{\pi}{4} - \sum_{r=0}^{n-1} \frac{(-1)^r}{2r+1} &= \sum_{r=n}^{\infty} \frac{(-1)^r}{2r+1} \\ &= (-1)^n \left\{\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) + \left(\frac{1}{2n+5} - \frac{1}{2n+7}\right) + \cdots\right\} \\ &= (-1)^n \sum_{s=0}^{\infty} \left\{\frac{1}{4s+2n+1} - \frac{1}{4s+2n+3}\right\}. \end{aligned} $$

Proceeding as in Section 2, we then find that

$$ \begin{aligned} \frac{\pi}{4} - \sum_{r=0}^{n-1} \frac{(-1)^r}{2r+1} &= \frac{(-1)^n}{8} \sum_{s=0}^{\infty} \frac{1}{(\frac{2n+1}{4}+s)(\frac{2n+3}{4}+s)} \\ &= \frac{2\ (-1)^n}{(2n+1)\ (2n+3)} \sum_{s=0}^{\infty} \frac{(1)_s\ (\frac{2n+1}{4})_s\ (\frac{2n+3}{4})_s}{(\frac{2n+5}{4})_s\ (\frac{2n+7}{4})_s\ s!}. \end{aligned} $$

Identification of the above sum as a ${}_3F_2(1)$ series then produces the result stated in (7).

As an example, we give the following evaluations for $n = 0, 1, 2, 3$ (where the cases corresponding to $n = 0, 1$ are recorded in [6]):

$$\begin{aligned}
{}_3F_2\left(\begin{matrix}1,\ 1/4,\ 3/4\\ 5/4,\ 7/4\end{matrix};1\right) &= \frac{3\pi}{8},\\
{}_3F_2\left(\begin{matrix}1,\ 3/4,\ 5/4\\ 7/4,\ 9/4\end{matrix};1\right) &= \frac{15}{8}(4-\pi),\\
{}_3F_2\left(\begin{matrix}1,\ 5/4,\ 7/4\\ 9/4,\ 11/4\end{matrix};1\right) &= \frac{35}{24}(3\pi-8),\\
{}_3F_2\left(\begin{matrix}1,\ 7/4,\ 9/4\\ 11/4,\ 13/4\end{matrix};1\right) &= \frac{21}{40}(52-15\pi).
\end{aligned}$$

Alternatively, the above results can be written in the form:

$$\begin{aligned}
\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\cdots &= \frac{\pi}{8},\\
\frac{1}{3\cdot 5}+\frac{1}{7\cdot 9}+\frac{1}{11\cdot 13}+\cdots &= \frac{1}{8}(4-\pi),\\
\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\frac{1}{13\cdot 15}+\cdots &= \frac{1}{24}(3\pi-8),\\
\frac{1}{7\cdot 9}+\frac{1}{11\cdot 13}+\frac{1}{15\cdot 17}+\cdots &= \frac{1}{120}(52-15\pi).
\end{aligned}$$

Similarly other results may be obtained.

The same procedure can be employed on other series. For example, the well-known evaluation $\sum_{r=1}^{\infty} r^{-2} = \frac{\pi^2}{6}$ yields

$$\frac{1}{n^2}\ {}_3F_2\left(\begin{matrix}1,\ n, n\\ n+1,\ n+1\end{matrix};1\right) = \frac{\pi^2}{6} - \sum_{r=1}^{n-1}\frac{1}{r^2},$$

where n is a positive integer. A different approach using continued fraction representations of the tails of hypergeometric series to obtain similar evaluations has been discussed in [3].

4. Some related series

The series

$$S_1 = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \cdots \tag{8}$$

was considered by Cohen [4] who determined the sum by means of a Fourier series approach combined with use of the Chebyshev polynomials. This series may be written as

$$\begin{aligned} S_1 &= \sum_{n=0}^{\infty} \left(\frac{1}{6n+1} - \frac{1}{6n+5} \right) \\ &= 4 \sum_{n=0}^{\infty} \frac{1}{(6n+1)\,(6n+5)} \\ &= \frac{4}{5}\, {}_3F_2 \left(\begin{matrix} 1,\ 1/6,\ 5/6 \\ 7/6,\ 11/6 \end{matrix} ; 1 \right) . \end{aligned}$$

By means of Dixon's summation theorem for a hypergeometric series of positive unit argument [8, p. 243]

$$\begin{aligned} &{}_3F_2 \left(\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix} ; 1 \right) \\ &\quad = \frac{\Gamma(1+\frac{1}{2}a)\ \Gamma(1+a-b)\ \Gamma(1+a-c)\ \Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\ \Gamma(1+\frac{1}{2}a-c)\ \Gamma(1+a-b-c)}, \end{aligned}$$

provided $\Re(\frac{1}{2}a - b - c) > -1$, we immediately deduce that $S = \frac{\pi}{2\sqrt{3}}$.

A variation of the series (8) is obtained by grouping the terms in pairs, namely

$$S_2 = 1 + \frac{1}{5} - \left(\frac{1}{7} + \frac{1}{11} \right) + \left(\frac{1}{13} + \frac{1}{17} \right) - \cdots .$$

This series can be expressed in the form

$$\begin{aligned} S_2 &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{6n+1} + \frac{1}{6n+5} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n\ \frac{12n+6}{(6n+1)\,(6n+5)} \\ &= \frac{6}{5}\, {}_4F_3 \left(\begin{matrix} 1,\ 3/2,\ 1/6,\ 5/6 \\ 1/2,\ 7/6,\ 11/6 \end{matrix} ; -1 \right) = \frac{\pi}{3} \end{aligned}$$

by (4).

As a final example, consider the series

$$\begin{aligned} T &= \left(1+\frac{1}{5}+\frac{1}{9}\right)-\left(\frac{1}{11}+\frac{1}{15}+\frac{1}{19}\right)+\left(\frac{1}{21}+\frac{1}{25}+\frac{1}{29}\right)-\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{10n+1}+\frac{1}{10n+5}+\frac{1}{10n+9}\right) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{10n+5}+\sum_{n=0}^{\infty}\frac{(-1)^n\ (20n+10)}{(10n+1)\ (10n+9)}. \end{aligned}$$

The first series has the value

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{10n+5}=\frac{1}{5}\,{}_2F_1\left(\begin{matrix}1,\ 1/2\\ 3/2\end{matrix};-1\right)=\frac{\pi}{20}$$

by (4), and the second series has the value

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(-1)^n\ (20n+10)}{(10n+1)\ (10n+9)} &= \frac{10}{9}\ {}_4F_3\left(\begin{matrix}1,\ 3/2,\ 1/10,\ 9/10\\ 1/2,\ 11/10,\ 19/10\end{matrix};-1\right) \\ &= \frac{\pi}{10\sin(\pi/10)} \end{aligned}$$

by (5). Thus we have

$$T=\frac{\pi}{20}+\frac{\pi}{10\sin(\pi/10)}=(3+2\sqrt{5})\frac{\pi}{20}.$$

Concluding comments

To conclude, we mention that a natural generalization of the Gregory-Leibniz series (1) is

$$\begin{aligned} U(n) &= 1-\frac{1}{2n{+}3}+\frac{1\cdot 3}{(2n{+}3)\ (2n{+}5)}-\frac{1\cdot 3\cdot 5}{(2n{+}3)\ (2n{+}5)\ (2n{+}7)}+\cdots \\ &= {}_2F_1\left(\begin{matrix}1,\ \frac{1}{2}\\ n+\frac{3}{2}\end{matrix};-1\right) \end{aligned}$$

for positive integer n (when $n=0$ the series reduces to S_1 given in (1)). If we substitute the values $a=1$, $b=\frac{1}{2}$ into the generalization of Kummer's theorem (4) in the form given in [6] (see also [7]):

$$
{}_2F_1\left(\begin{matrix} a,\ b \\ 1+a-b+n \end{matrix};-1\right) = \frac{2^{n-2b}\,\Gamma(1+a-b+n)\,\Gamma(b-n)}{\Gamma(b)\,\Gamma(a-2b+n+1)}
$$
$$
\times\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{\Gamma(\frac{1}{2}a-b+\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}r)}{\Gamma(\frac{1}{2}a+\frac{1}{2}+\frac{1}{2}r-\frac{1}{2}n)}
$$
$$
(n=1,2,3,\ldots),
$$

we immediately obtain after use of the reflection formula for the gamma function the result

$$
U(n) = \frac{2^{n-1}\sqrt{\pi}\,(-1)^n}{n!}\left(n+\frac{1}{2}\right)\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}r+\frac{1}{2}n)}{\Gamma(1+\frac{1}{2}r-\frac{1}{2}n)}. \qquad (9)
$$

Examples of these evaluations for $n = 1, 2, 3$ are:

$$
\begin{aligned}
1-\frac{1}{5}+\frac{1\cdot 3}{5\cdot 7}-\frac{1\cdot 3}{7\cdot 9}+\cdots &= \frac{3\pi}{4}-\frac{3}{2} \\
1-\frac{1}{7}+\frac{1\cdot 3}{7\cdot 9}-\frac{1\cdot 3\cdot 5}{7\cdot 9\cdot 11}+\cdots &= \frac{15\pi}{8}-5 \\
1-\frac{1}{9}+\frac{1\cdot 3}{9\cdot 11}-\frac{1\cdot 3\cdot 5}{9\cdot 11\cdot 13}+\cdots &= \frac{35\pi}{8}-\frac{77}{6},
\end{aligned}
$$

where the first evaluation is recorded in [6].

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PRIMES DIVIDING VALUES OF A GIVEN POLYNOMIAL

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Abstract. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial. We give an easy and new proof of the fact that the set of primes p such that $p \mid P(n)$, for some $n \in \mathbb{Z}$, is infinite. We also get analog of this result for some special domains.

1. Main results

Let $P(x) \in \mathbb{Z}[x]$ be a polynomial. Consider the set of primes p such that $p \mid f(n)$, for some $n \in \mathbb{Z}$. It is known that this set is infinite. This was proved for the first time by Schur and is called as Schur's theorem (see Schur [1]). In this article, we give a new proof of this fact. For a given prime p and given number d, we denote the highest power of p dividing d by $w_p(d)$. For instance, $w_2(12) = 2^2$. Now, we state our main result.

Theorem 1.1. *Let $P(x) \in \mathbb{Z}[x]$ be a polynomial. Then the set of primes p, such that there exists $n \in \mathbb{Z}$ such that $p \mid P(n)$ is infinite.*

Proof. We prove by contradiction. Assume there are only finitely many primes. We can label them as $p_1, p_2, \dots, p_r$. There exist integers k_i and e_i such that $w_{p_i}(f(k_i)) = p_i^{e_i}$ $\forall$ $1 \le i \le r$. If n_0 is a solution of congruence equations $x \equiv k_i \pmod{p_i^{e_i+1}}$, then $f(n_0)$ is divisible by $p_i^{e_i}$ but is not divisible by $p_i^{e_i+1}$ for any $0 \le i \le r$. This holds since $m \equiv n \pmod{\prod p_i^{ei+1}}$ implies $P(m) \equiv P(n) \pmod{\prod p_i^{ei+1}}$, so that if $w_{p_i}(P(m)) = p_i^{e_i}$, then $w_{p_i}(P(n))$ is also $p_i^{e_i}$. Since n_0 is a solution of the above congruence, $n = n_0 + k \prod p_i^{e_i+1}$ is also a solution for every integer k. The equality $P(n) = \prod_{i=1}^r p_i^{e_i}$ or $P(n) = -\prod_{i=1}^r p_i^{e_i}$ can hold only for finitely many solutions n of $x \equiv k_i \pmod{p_i^{e_i+1}}$ Hence, there exist a solution n_k of $x \equiv k_i \pmod{p_i^{e_i+1}}$

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such that $\prod_{i=0}^{r} P_i^{e_i}$ is a proper divisor of $P(n_k)$. Since $p_i^{e_i+1}$ cannot divide $P(n_k)$, so there exists a prime other than $p_1, p_2, \dots p_r$ dividing $P(n_k)$, which is a contradiction. Hence the result follows. $\square$

By our approach, it is evident that the result holds for polynomials with coefficients in a Dedekind domain or sometimes a domain.

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CHARACTERIZATIONS OF K-FRAMES IN 2-HILBERT SPACES

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Abstract. In this paper our interest is to discuss a few properties of K-frames in 2-Hilbert spaces and verify that sum of two K-frames is also a K-frame in 2-Hilbert space. Also we shall describe the concept of tight K-frames in 2-Hilbert spaces and some of their characterizations.

1. Introduction

In 1952, Duffin and Schaeffer introduced frames in Hilbert spaces in their fundamental paper [1], they used frames as a tool in the study of nonharmonic Fourier series. Later in 1986, frame theory was popularized by Daubechies, Grossman, Meyer [2]. A frame for a Hilbert space is a generalization of an orthonormal basis and this is such a tool that also allows each vector in this space can be written as a linear combination of elements from the frame but, linear independence among the frame elements is not required. Such frames play an important role in Gabor and wavelet analysis . Several generalizations of frames namely, G-frame [18], K-frames [4] etc. have been introduced in recent times. K-frames for a separable Hilbert space were introduced by Lara Gavruta to study the basic notions about atomic system for a bounded linear operator. K-frames are more generalization than the ordinary frames and many properties of ordinary frames may not holds for such generalization of frames.

The concept of 2-inner product space was first introduced by Diminnie, Gahler, White [9, 10], in 1970's and thereafter notion of 2-norm induced by the concept of 2-inner product space was first introduced by S. Gahler [13]. The notion of a frame in a 2-inner product space has been introduced

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by A. Arefijamaal and G. Sadeghi [15] and they also established some fundamental properties of 2-frames for 2-inner product space. The concept of 2-atomic systems which is a generalization of families of local 2-atoms in a 2-inner product spaces were introduced by B. Dastourian and M. Janfada [17] and they also defined 2-K-frames as the generalization of 2-frames.

In this paper, we shall discuss some properties of K-frame in 2-inner product space and it will be seen that the family of all K-frames in 2-inner product space is closed with respect addition. Further we also give the notion of a tight K-frames relative to 2-inner product spaces.

In the entire paper, H will denote a separable Hilbert space with the inner product $\langle \cdot , \cdot \rangle$ and $\mathcal{B}(H)$ denote the space of all bounded linear operator on H. We also denote $\mathcal{R}(T)$ for range set of T where $T \in \mathcal{B}(H)$ and $l^2(\mathbb{N})$ denote the space of square summable scalar-valued sequences with index set $\mathbb{N}$.

2. Preliminaries

Definition 2.1. [3] A sequence $\{f_i\}_{i=1}^{\infty}$ of elements in H is said to be a frame for H if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B \|f\|^2 \tag{2.1}$$

for all $f \in H$. The constants A and B are called frame bounds. If the collection $\{f_i\}_{i=1}^{\infty}$ satisfies only the right inequality of (2.1) then it is called a Bessel sequence.

Definition 2.2. [3] Let $\{f_i\}_{i=1}^{\infty}$ be a frame for H. Then the bounded linear operator $T : l^2(\mathbb{N}) \to H$, defined by $T\{c_i\} = \sum\limits_{i=1}^{\infty} c_i f_i$ is called pre-frame operator and its adjoint operator $T^* : H \to l^2(\mathbb{N})$, given by $T^* f = \{\langle f, f_i \rangle\}_{i=1}^{\infty}$ is called the analysis operator. The operator $S : H \to H$ defined by $S f = T T^* f = \sum\limits_{i=1}^{\infty} \langle f, f_i \rangle f_i$ is called the frame operator.

The frame operator S is bounded, positive, self-adjoint and invertible [3].

Definition 2.3. [4] Let $K : H \to H$ be a bounded linear operator. Then a sequence $\{f_i\}_{i=1}^{\infty}$ in H is said to be a K-frame for H if there exist constants $A, B > 0$ such that

$$A \, \| K^* f \|^2 \leq \sum_{i=1}^{\infty} | \langle f \, , \, f_i \rangle |^2 \leq B \, \| f \|^2$$

for all $f \in H$. If $A = B$ then it is said to be a tight K-frame and if $A = B = 1$ then it is called Parseval K-frame.

2.1. **Example.**

(i) Let H be an infinite dimensional separable Hilbert space and $\{ e_i \}_{i=1}^{\infty}$ be an orthonormal basis for H. Define $K : H \to H$ by $K f = \sum_{i=1}^{m} \langle f, e_i \rangle \, e_i$, $f \in H$, where m is a fixed positive integer.Now, for each $f \in H$, we have

$$\| K^* f \|^2 = \sum_{i=1}^{m} | \langle f, e_i \rangle |^2 \leq \sum_{i=1}^{\infty} | \langle f, e_i \rangle |^2$$
$$\leq 2 \, | \langle f, e_1 \rangle |^2 + 2 \, | \langle f, e_2 \rangle |^2 + | \langle f, e_3 \rangle |^2 + \cdots\cdots$$
$$\leq 2 \sum_{i=1}^{\infty} | \langle f, e_i \rangle |^2 = 2 \| f \|^2.$$

Thus, $\{ e_1, e_1, e_2, e_2, e_3, e_4, \cdots \}$ is a K-frame for H with bounds 1 and 2.

(ii) Let $H = \mathbb{R}^3$ and $\{ e_1, e_2, e_3 \}$ be its standard orthonormal basis. Define $K : H \to H$ by $K e_1 = e_1, K e_2 = e_2, K e_3 = e_3$. It is easy to verify that $K^* e_1 = e_1, K^* e_2 = e_2, K^* e_3 = e_3$. Then $\| K^* f \|^2 = \| f \|^2$ for all $f \in H$. Let $\{ f_i \}_{i=1}^{3} = \{ e_1, e_1, e_2, e_2, e_3, e_3 \}$. Then for each $f \in H$, we have

$$\sum_{i=1}^{3} | \langle f, f_i \rangle |^2 = 2 \sum_{i=1}^{3} | \langle f, e_i \rangle |^2 = 2 \, \| f \|^2 = 2 \, \| K^* f \|^2 .$$

So, $\{ f_i \}_{i=1}^{3}$ is a tight K-frame for H with bound 2.

Theorem 2.4. [5] *Let $K : H \to H$ be a bounded linear operator. Then a Bessel sequence $\{ f_i \}_{i=1}^{\infty}$ in H is a K-frame if and only if there exists $\lambda > 0$ such that $S \geq \lambda K K^*$, where S is the frame operator for $\{ f_i \}_{i=1}^{\infty}$.*

Theorem 2.5. [3] *Let U be a bounded linear operator from a Hilbert space H_1 to another Hilbert space H_2 with closed range $\mathcal{R}_U$. Then there exists a bounded linear operator $U^{\dagger} : H_2 \to H_1$ such that $U U^{\dagger} f = f$, for all $f \in \mathcal{R}_U$.*

The operator $U^{\dagger}$ is called the pseudo-inverse of U.

Theorem 2.6. *(Douglas' factorization theorem)* [6] *Let U, V be two bounded linear operators on H. Then the following conditions are equivalent:*

(I) $\mathcal{R}(U) \subseteq \mathcal{R}(V)$.

(II) $U U^{*} \leq \lambda^{2} V V^{*}$ *for some* $\lambda > 0$.

(III) $U = V W$ *for some bounded linear operator* W *on* H.

Theorem 2.7. [7] *Let $S, T, U \in \mathcal{B}(H)$. Then the following are equivalent:*

(I) $\mathcal{R}(S) \subseteq \mathcal{R}(T) + \mathcal{R}(U)$.

(II) $S S^{*} \leq \lambda^{2} (T T^{*} + U U^{*})$ *for some* $\lambda > 0$.

(III) $S = T A + U B$ *for some* $A, B \in \mathcal{B}(H)$.

Theorem 2.8. [12] *The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined as for $T, S \in \mathcal{S}(H)$*

$$T \leq S \Leftrightarrow \langle T f , f \rangle \leq \langle S f , f \rangle \;\; \forall f \in H.$$

Definition 2.9. [8, 9] Let X be a linear space of dimension greater than 1 over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. A function $\langle \cdot , \cdot | \cdot \rangle : X \times X \times X \to \mathbb{K}$ is said to be an 2-inner product on X if it satisfies the following conditions:

(*I*) $\langle x , x | z \rangle \geq 0$ and $\langle x , x | z \rangle = 0$ if and only if x and z are linearly dependent,
(*II*) $\langle x , x | z \rangle = \langle z , z | x \rangle$,
(*III*) $\langle x , y | z \rangle = \overline{\langle y , x | z \rangle}$,
(*IV*) $\langle \alpha x , y | z \rangle = \alpha \langle x , y | z \rangle$, for all $\alpha \in \mathbb{K}$,
(*V*) $\langle x_1 + x_2 , y | z \rangle = \langle x_1 , y | z \rangle + \langle x_2 , y | z \rangle$.

A linear space X equipped with an 2-inner product $\langle \cdot , \cdot | \cdot \rangle$ defined on X is called a 2-inner product space.

2.2. **Example.** Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then the standard 2-inner product $\langle \cdot , \cdot | \cdot \rangle$ on X is defined by

$$\langle x , y | z \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle ,$$

for all $x, y, z, \in X$.

Definition 2.10. [13] Let X be a linear space of dimension greater than 1 over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. A real valued function $\|\cdot, \cdot\|$ defined on X is said to be a 2-norm on X if it satisfies the following conditions:

(I) $\|x, y\| = 0$ if and only if x, y are linearly dependent,

(II) $\|x, y\| = \|y, x\|$,

(III) $\|\alpha x, y\| = |\alpha| \|x, y\|$, for all $\alpha \in \mathbb{K}$,

(IV) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

A linear space X together with a 2-norm $\|\cdot, \cdot\|$ is called a linear 2-normed space.

Theorem 2.11. [14] *Let* $(X, \langle \cdot, \cdot | \cdot \rangle)$ *be a* 2*-inner product space then*

$$|\langle x, y | z \rangle| \leq \|x, z\| \|y, z\|$$

holds for all $x, y, z \in X$*, where*

$$|\langle x, y | z \rangle| \leq \|x, z\| \|y, z\|$$

Definition 2.12. [11] Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over real or complex number field $\mathbb{K}$. Let $\{e_i\}_{i=1}^{n}$ be linearly independent vectors in X. Then for a given $h \in X$, if

$$\langle e_i, e_j | h \rangle = \delta_{ij} \quad i, j \in \{1, 2, \cdots, n\}$$

$$\text{where,} \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

the family $\{e_i\}_{i=1}^{n}$ is said to be h-orthonormal. If an h-orthonormal set is countable, we can arrange it in the form of a sequence $\{e_i\}$ and call it h-orthonormal sequence.

Definition 2.13. [16] Let $(X, \|\cdot, \cdot\|)$ be a linear 2-normed space. A sequence $\{x_n\}$ in X is said to converge to some $x \in X$ if

$$\lim_{n \to \infty} \|x_n - x, y\| = 0$$

for every $y \in X$ and it is called a Cauchy sequence if

$$\lim_{n, m \to \infty} \|x_n - x_m, z\| = 0$$

for every $z \in X$. The space X is said to be complete if every Cauchy sequence in this space is convergent in X. An 2-inner product space is called 2-Hilbert space if it is complete with respect to its induce norm.

Throughout this paper, X is considered to be a 2-Hilbert space associated with the 2-inner product $\langle \cdot , \cdot | \cdot \rangle$.

Definition 2.14. [15]. A sequence $\{ f_i \}_{i=1}^{\infty} \subseteq X$ is said to be a 2-frame associated to $h \in X$ if there exist constants $A, B > 0$ such that

$$A \| f , h \|^2 \leq \sum_{i=1}^{\infty} | \langle f , f_i | h \rangle |^2 \leq B \| f , h \|^2 \tag{2.2}$$

for all $f \in H$. If the sequence $\{ f_i \}_{i=1}^{\infty}$ satisfying the right inequality of (2.2) then it is called a 2-Bessel sequence associated to h.

Theorem 2.15. [15] *Let L_h denote the linear subspace of X generated by a fixed $h \in X$. Let M_h be the algebraic complement of L_h. Define $\langle x , y \rangle_h = \langle x , y | h \rangle$ on X. Then $\langle \cdot , \cdot \rangle_h$ is a semi-inner product on X and this semi-inner product induces an inner product on the quotient space X / L_h which is given by*

$$\langle x + L_h , y + L_h \rangle_h = \langle x , y \rangle_h = \langle x , y | h \rangle ,$$

for all $x, y \in X$. By identifying X / L_h with M_h in an obvious way, we obtain an inner product on M_h. Define $\| x \|_h = \sqrt{\langle x , x \rangle_h}$ $(x \in M_h)$. Then $(M_h , \| \cdot \|_h)$ is a norm space.

Let X_h be the completion of the inner product space M_h.

Theorem 2.16. [15] *A sequence $\{ f_i \}_{i=1}^{\infty}$ in X is a 2-frame associated to h with bounds A and B if and only if it is a frame for the Hilbert space X_h with bounds A and B.*

Definition 2.17. [15] Let $\{ f_i \}_{i=1}^{\infty}$ be a 2-Bessel sequence associated to h. Then the 2-pre frame operator

$$T_h : l^2 (\mathbb{N}) \to X_h , \; T_h (\{ c_i \}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i$$

is well-defined and bounded and its adjoint operator given by

$$T_h^* : X_h \to l^2 (\mathbb{N}) , \; T_h^* (f) = \{ \langle f , f_i | h \rangle \}_{i=1}^{\infty}$$

is also well-defined and bounded.

Definition 2.18. [15] Let $\{f_i\}_{i=1}^{\infty}$ be a 2-frame associated to h. Then the operator $S_h : X_h \to X_h$ defined by

$$S_h f = T_h T_h^* f = \sum_{i=1}^{\infty} \langle f, f_i \,|\, h \rangle f_i,$$

for all $f \in X_h$ is called the 2-frame operator for $\{f_i\}_{i=1}^{\infty}$.

Theorem 2.19. [15] *Let $\{f_i\}_{i=1}^{\infty}$ be a 2-frame associated to h. Then the corresponding 2-frame operator is bounded, invertible, self-adjoint, and positive.*

Definition 2.20. [17] Let $K_h : X_h \to X_h$ be a bounded linear operator. Then a sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is said to be a 2-K-frame for X if there exist constants $A, B > 0$ such that

$$A \, \| K_h^* f, h \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \,|\, h \rangle|^2 \leq B \, \| f, h \|^2$$

for all $f \in X_h$. In particular when K_h is the identity operator on X_h then $\{f_i\}_{i=1}^{\infty}$ becomes a 2-frame.

3. Some properties of K-frames in 2-Hilbert spaces

In this section, we first give an example of 2-K_h-frame and discuss various properties of K-frame relative 2-Hilbert space.

3.1. **Example.** Consider $X = \mathbb{C}^3$ with the standard 2-inner product

$$\langle x, y \,|\, z \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix}$$

for all $x, y, z, \in X$, where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathbb{C}^3$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of X and $h = e_3$. In this case $X_h = \mathbb{C}^2$. Now, we define $K_h : X_h \to X_h$ by $K_h e_1 = e_1$, $K_h e_2 = e_1$, $K_h e_3 = e_2$. It is easy to verify that $K_h^* e_1 = e_1$, $K_h^* e_2 = e_1$, $K_h^* e_3 = e_2$. Then $\{f_i\}_{i=1}^{3} = \{e_1, e_1, e_2\}$ is a 2-K_h-frame for X.

Theorem 3.1. *Let $K_h : X_h \to X_h$ be a bounded linear operator. If $\{f_i\}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X and $T_h \in \mathcal{B}(X_h)$ with $\mathcal{R}(T_h) \subset \mathcal{R}(K_h)$, then $\{f_i\}_{i=1}^{\infty}$ is also a 2-T-frame associated to h for X.*

Proof. Suppose that $\{ f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X. Then there exist constants $A, B > 0$ such that

$$A \, \| K_h^* f, h \|^2 \leq \sum_{i=1}^{\infty} | \langle f, f_i | h \rangle |^2 \leq B \, \| f, h \|^2, \tag{3.1}$$

for all $f \in X_h$. Since $R(T_h) \subset R(K_h)$, by Theorem 2.6, there exists $\lambda > 0$ such that $T_h T_h^* \leq \lambda^2 K_h K_h^*$. Therefore

$$\frac{A}{\lambda^2} \, \| T_h^* f, h \|^2 = \frac{A}{\lambda^2} \, \langle T_h T_h^* f, f | h \rangle = \left\langle \frac{A}{\lambda^2} T_h T_h^* f, f | h \right\rangle$$
$$\leq \langle A \, K_h K_h^* f, f | h \rangle = A \, \| K_h^* f, h \|^2.$$

Using this, for each $f \in X_h$, (3.1) can be rewrite as

$$\frac{A}{\lambda^2} \, \| T_h^* f, h \|^2 \leq A \, \| K_h^* f, h \|^2 \leq \sum_{i=1}^{\infty} | \langle f, f_i | h \rangle |^2 \leq B \, \| f, h \|^2.$$

This shows that $\{ f_i \}_{i=1}^{\infty}$ is a 2-T-frame for X. □

Theorem 3.2. *Let $K_h \in \mathcal{B}(X_h)$ and $\{ f_i \}_{i=1}^{\infty}$ be the corresponding 2-K-frame associated to h for X. If T_h is a surjective bounded linear operator on X_h such that it has closed range and $T_h K_h = K_h T_h$, then $\{ T_h f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X.*

Proof. Since T_h has a closed range so the pseudo-inverse of T_h exists say $T_h^{\dagger}$, by Theorem 2.5, we have $T T_h^{\dagger} = I_h$ where I_h is the identity operator on X_h. Then for each $f \in X_h$, $K_h^* f = \left(T_h^{\dagger} \right)^* T_h^* K_h^* f$. Therefore, for each $f \in X_h$, we have

$$\| K_h^* f, h \| = \left\| \left(T_h^{\dagger} \right)^* T_h^* K_h^* f, h \right\| \leq \left\| \left(T_h^{\dagger} \right)^* \right\| \| T_h^* K_h^* f, h \|.$$

Thus, for each $f \in X_h$, we have

$$\left\| \left(T_h^{\dagger} \right)^* \right\|^{-1} \| K_h^* f, h \| \leq \| T_h^* K_h^* f, h \|. \tag{3.2}$$

Now, for each $f \in X_h$,

$$\sum_{i=1}^{\infty} | \langle f, T_h f_i | h \rangle |^2 = \sum_{i=1}^{\infty} | \langle T_h^* f, f_i | h \rangle |^2$$
$$\geq A \, \| K_h^* T_h^* f, h \|^2 \quad [\text{ since } \{ f_i \}_{i=1}^{\infty} \text{ is 2-}K\text{-frame }]$$
$$= A \, \| T_h^* K_h^* f, h \|^2 \quad [\text{ using } T_h K_h = K_h T_h].$$

$$\geq A \left\| \left(T_h^{\dagger} \right)^* \right\|^{-2} \| K_h^* f, h \|^2 \;\; [\text{ using (3.2) }]. \tag{3.3}$$

On the other hand, for each $f \in X_h$, we have

$$\begin{aligned} \sum_{i=1}^{\infty} |\langle f, T_h f_i \,|\, h \rangle|^2 &= \sum_{i=1}^{\infty} |\langle T_h^* f, f_i \,|\, h \rangle|^2 \\ &\leq B \, \| T_h^* f, h \|^2 \;\; [\text{ since } \{ f_i \}_{i=1}^{\infty} \text{ is a 2-}K\text{-frame }] \\ &\leq B \, \| T_h^* \|^2 \, \| f, h \|^2 \\ &= B \, \| T_h \|^2 \, \| f, h \|^2. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), for each $f \in X_h$, we have

$$\begin{aligned} A \left\| \left(T_h^{\dagger} \right)^* \right\|^{-2} \| K_h^* f, h \|^2 &\leq \sum_{i=1}^{\infty} |\langle f, T_h f_i \,|\, h \rangle|^2 \\ &\leq B \, \| T_h \|^2 \, \| f, h \|^2. \end{aligned}$$

This shows that $\{ T_h f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X. $\square$

Theorem 3.3. *Let $K_h : X_h \to X_h$ be a bounded linear operator and $\{ f_i \}_{i=1}^{\infty}$ be the corresponding 2-K-frame associated to h for X. If $T_h \in \mathcal{B}(X_h)$ such that $T_h T_h^* = I_h$ and $T_h K_h = K_h T_h$, then $\{ T_h f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X.*

Proof. Since $T_h T_h^* = I_h$, for $f \in X_h$, $\| T_h^* f, h \|^2 = \| f, h \|^2$. Thus,

$$\| T_h^* K_h^* f, h \|^2 = \| K_h^* f, h \|^2, \;\; f \in X_h. \tag{3.5}$$

Now, for each $f \in X_h$, we have

$$\begin{aligned} \sum_{i=1}^{\infty} |\langle f, T_h f_i \,|\, h \rangle|^2 &= \sum_{i=1}^{\infty} |\langle T_h^* f, f_i \,|\, h \rangle|^2 \\ &\geq A \, \| K_h^* T_h^* f, h \|^2 \;\; [\text{ since } \{ f_i \}_{i=1}^{\infty} \text{ is a 2-}K\text{-frame }] \\ &= A \, \| T_h^* K_h^* f, h \|^2 \;\; [\text{ using } T_h K_h = K_h T_h] \\ &= A \, \| K_h^* f, h \|^2 \;\; [\text{ using (3.5) }]. \end{aligned}$$

Thus we see that $\{ T_h f_i \}_{i=1}^{\infty}$ satisfies lower 2-K-frame condition. Following the proof of the Theorem 3.2, it can be shown that it also satisfies upper 2-K-frame condition and therefore it is a 2-K-frame associated to h for X. $\square$

Theorem 3.4. *Let $K_h \in \mathcal{B}(X_h)$. Then $\{ f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h if and only if there exists a bounded linear operator T_h :*

$l^2(\mathbb{N}) \to X_h$ such that $f_i = T_h e_i$ and $\mathcal{R}(K_h) \subset \mathcal{R}(T_h)$, where $\{e_i\}_{i=1}^{\infty}$ is an h-orthonormal basis for $l^2(\mathbb{N})$.

Proof. Let $\{f_i\}_{i=1}^{\infty}$ be a 2-K-frame associated to h. Then for each $f \in X_h$, there exist $A, B > 0$ such that

$$A \, \| K_h^* f, h \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i | h \rangle|^2 \leq B \, \| f, h \|^2. \tag{3.6}$$

Now we consider the linear operator $L_h : X_h \to l^2(\mathbb{N})$ defined by

$$L_h(f) = \sum_{i=1}^{\infty} \langle f, f_i | h \rangle \, e_i.$$

Since $\{e_i\}_{i=1}^{\infty}$ is an h-orthonormal basis for $l^2(\mathbb{N})$, we can write

$$\| L_h(f) \|_{l^2}^2 = \sum_{i=1}^{\infty} |\langle f, f_i | h \rangle|^2 \leq B \, \| f, h \|^2 \ \text{[by (3.6)]}.$$

Thus, for each $f \in X_h$, we get $\| L_h(f) \|_{l^2}^2 \leq B \| f \|_h^2$. This shows that L_h is well-defined and bounded linear operator on X_h. So, adjoint of L_h, $L_h^* : l^2(\mathbb{N}) \to X_h$ exists and it is also a bounded linear operator. Now, for each $f \in X_h$, we have

$$\begin{aligned} \langle L_h^* e_i, f | h \rangle &= \langle e_i, L_h f | h \rangle = \left\langle e_i, \sum_{i=1}^{\infty} \langle f, f_i | h \rangle \, e_i | h \right\rangle \\ &= \overline{\langle f, f_i | h \rangle} = \langle f_i, f | h \rangle. \end{aligned}$$

This implies that, $L_h^*(e_i) = f_i$. Using the operator L_h, (3.6) can be written as

$$A \| K_h^* f \|_h^2 \leq \| L_h(f) \|_{l^2}^2 \Rightarrow \langle A K_h K_h^* f, f | h \rangle \leq \langle L_h^* L_h f, f | h \rangle$$

Thus, $A K_h K_h^* \leq T_h T_h^*$, where $T_h = L_h^*$ and hence by Theorem 2.6, $\mathcal{R}(K_h) \subset \mathcal{R}(T_h)$.

Conversely, suppose that $T_h : l^2(\mathbb{N}) \to X_h$ be a bounded linear operator such that $f_i = T_h e_i$ and $\mathcal{R}(K_h) \subset \mathcal{R}(T_h)$. We have to show that $\{f_i\}_{i=1}^{\infty}$ is a 2-K-frame. Now, for each $f \in X_h$, we have

$$\begin{aligned}
\langle T_h^* f, g \,|\, h \rangle &= \left\langle T_h^* f, \sum_{i=1}^{\infty} c_i e_i \,|\, h \right\rangle, \text{ where } g = \sum_{i=1}^{\infty} c_i e_i \\
&= \sum_{i=1}^{\infty} \overline{c_i} \, \langle f, T_h e_i \,|\, h \rangle = \sum_{i=1}^{\infty} \overline{c_i} \, \langle f, f_i \,|\, h \rangle \\
&= \sum_{i=1}^{\infty} \overline{\langle g, e_i \,|\, h \rangle} \, \langle f, f_i \,|\, h \rangle, \; [\text{ since } c_i = \langle g, e_i \,|\, h \rangle] \\
&= \sum_{i=1}^{\infty} \langle e_i, g \,|\, h \rangle \, \langle f, f_i \,|\, h \rangle \\
&= \left\langle \sum_{i=1}^{\infty} \langle f, f_i \,|\, h \rangle \, e_i, \; g \,|\, h \right\rangle, \quad \text{for all } g \in l^2(\mathbb{N}).
\end{aligned}$$

Thus, for $f \in X_h$, we get

$$T_h^*(f) = \sum_{i=1}^{\infty} \langle f, f_i \,|\, h \rangle \, e_i.$$

Now, for each $f \in X_h$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} |\langle f, f_i \,|\, h \rangle|^2 &= \sum_{i=1}^{\infty} |\langle f, T_h e_i \,|\, h \rangle|^2 \\
&= \sum_{i=1}^{\infty} |\langle T_h^* f, e_i \,|\, h \rangle|^2 \\
&= \| T_h^* f, h \|^2 \leq \| T_h^* \|^2 \, \| f, h \|^2.
\end{aligned}$$

This shows that $\{ f_i \}_{i=1}^{\infty}$ is a 2-Bessel sequence associated to h. Since $\mathcal{R}(K_h) \subset \mathcal{R}(T_h)$, by Theorem 2.6, there exists $A > 0$ such that $A K_h K_h^* \leq T_h T_h^*$. Hence following the proof of the Theorem 3.1, for each $f \in X_h$, we have

$$A \, \| K_h^* f, h \|^2 \leq \| T_h^* f, h \|^2 = \sum_{i=1}^{\infty} |\langle f, f_i \,|\, h \rangle|^2.$$

Therefore, $\{ f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X. □

Theorem 3.5. *Let $\{ f_i \}_{i=1}^{\infty}$ and $\{ g_i \}_{i=1}^{\infty}$ be two 2-K-frame associated to h for X with corresponding 2-pre frame operators T_h and L_h, respectively. If $T_h L_h^*$ and $L_h T_h^*$ are positive operators, then $\{ f_i + g_i \}_{i=1}^{\infty}$ is 2-K-frame associated to h for X.*

Proof. Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be 2-K-frames associated to h for X. Then by Theorem 3.4, there exist bounded linear operators T_h and L_h such that $T_h e_i = f_i$, $L_h e_i = g_i$ and $\mathcal{R}(K_h) \subset \mathcal{R}(T_h)$, $\mathcal{R}(K_h) \subset \mathcal{R}(L_h)$, where $\{e_i\}_{i=1}^{\infty}$ is an h-orthonormal basis for $l^2(\mathbb{N})$. Now we have $\mathcal{R}(K_h) \subset \mathcal{R}(T_h) + \mathcal{R}(L_h)$. Therefore by Theorem 2.7, $K_h K_h^* \leq \lambda^2 (T_h T_h^* + L_h L_h^*)$, for some $\lambda > 0$. Now, for each $f \in X_h$, we have

$$\sum_{i=1}^{\infty} |\langle f, f_i + g_i | h \rangle|^2 = \sum_{i=1}^{\infty} |\langle f, T_h e_i + L_h e_i | h \rangle|^2$$

$$= \sum_{i=1}^{\infty} |\langle f, (T_h + L_h) e_i | h \rangle|^2 = \sum_{i=1}^{\infty} |\langle (T_h + L_h)^* f, e_i | h \rangle|^2$$

$$= \|(T_h + L_h)^* f, h\|^2 \, [\text{ since } \{e_i\}_{i=1}^{\infty} \text{ is an } h\text{-orthonormal basis }]$$

$$= \langle (T_h + L_h)^* f, (T_h + L_h)^* f | h \rangle$$

$$= \langle T_h^* f + L_h^* f, T_h^* f + L_h^* f | h \rangle$$

$$= \langle T_h^* f, T_h^* f | h \rangle + \langle L_h^* f, T_h^* f | h \rangle + \langle T_h^* f, L_h^* f | h \rangle + \langle L_h^* f, L_h^* f | h \rangle$$

$$= \langle T_h T_h^* f, f | h \rangle + \langle T_h L_h^* f, f | h \rangle + \langle L_h T_h^* f, f | h \rangle + \langle L_h L_h^* f, f | h \rangle$$

$$\geq \langle (T_h T_h^* + L_h L_h^*) f, f | h \rangle \quad [\text{ since } T_h L_h^*, L_h T_h^* \text{ are positive }]$$

$$\geq \frac{1}{\lambda^2} \langle K_h K_h^* f, f | h \rangle \quad [\text{ since } K_h K_h^* \leq \lambda^2 (T_h T_h^* + L_h L_h^*)]$$

$$= \frac{1}{\lambda^2} \langle K_h^* f, K_h^* f | h \rangle = \frac{1}{\lambda^2} \|K_h^* f, h\|^2.$$

Also, for each $f \in X_h$, using the Minkowski's inequality, we have

$$\left(\sum_{i=1}^{\infty} |\langle f, f_i + g_i | h \rangle|^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^{\infty} |\langle f, f_i | h \rangle|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} |\langle f, g_i | h \rangle|^2 \right)^{\frac{1}{2}}$$

$$\leq \sqrt{A} \|f, h\| + \sqrt{B} \|f, h\| \quad [\text{ since } \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \text{ are 2-}K\text{-frame }]$$

$$= \left(\sqrt{A} + \sqrt{B} \right) \|f, h\|.$$

Thus, for each $f \in X_h$, we have

$$\sum_{i=1}^{\infty} |\langle f, f_i + g_i | h \rangle|^2 \leq \left(\sqrt{A} + \sqrt{B} \right)^2 \|f, h\|^2.$$

Hence, $\{f_i + g_i\}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X. $\square$

Theorem 3.6. *Let $\{f_i\}_{i=1}^{\infty}$ be 2-K-frame associated to $h \in X$ with corresponding 2-frame operator S_h and $A_h : X_h \to X_h$ be a positive operator. Then $\{f_i + A_h f_i\}_{i=1}^{\infty}$ is also a 2-K-frame associated to h for X.*

Proof. Let $\{f_i\}_{i=1}^{\infty}$ be 2-K-frame with corresponding 2-frame operator S_h. Then for each $f \in X_h$, we have $\langle S_h f, f \mid h \rangle = \sum_{i=1}^{\infty} |\langle f, f_i \mid h \rangle|^2$ and

$$A \, \| K_h^* f, h \|^2 \leq \langle S_h f, f \mid h \rangle \leq B \, \| f, h \|^2$$

with symbols this can be written as $A K_h K_h^* \leq S_h \leq B I_h$, where I_h is the identity operator on X_h. Now, for each $f \in X_h$,

$$\begin{aligned}
&\sum_{i=1}^{\infty} \langle f, f_i + A_h f_i \mid h \rangle (f_i + A_h f_i) \\
&= \sum_{i=1}^{\infty} \langle f, (I_h + A_h) f_i \mid h \rangle (I_h + A_h) f_i \\
&= (I_h + A_h) \sum_{i=1}^{\infty} \langle f, (I_h + A_h) f_i \mid h \rangle f_i \\
&= (I_h + A_h) \sum_{i=1}^{\infty} \langle (I_h + A_h)^* f, f_i \mid h \rangle f_i \\
&= (I_h + A_h) S_h (I_h + A_h)^* f.
\end{aligned}$$

This shows that the frame operator for $\{f_i + A_h f_i\}_{i=1}^{\infty}$ is $(I_h + A_h) S_h (I_h + A_h)^*$. Since S_h, A_h are positive, $(I_h + A_h) S_h (I_h + A_h)^* \geq S_h \geq A K_h K_h^*$. Hence, by Theorem 2.4, $\{f_i + A_h f_i\}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X. $\square$

Theorem 3.7. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be two 2-Bessel sequences associated to h in X with bounds C and D, respectively. Suppose that T_h and T'_h be their 2-pre frame operators such that $T_h (T'_h)^* = K_h^*$. Then $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are 2-K-frames associated to h in X.*

Proof. Since T_h and T'_h are the 2-pre frame operator for $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$, respectively. Then for each $f \in X_h$, we have

$$\| T_h^* f, h \|^2 = \sum_{i=1}^{\infty} |\langle f, f_i \mid h \rangle|^2, \; \left\| \left(T'_h\right)^* f, h \right\|^2 = \sum_{i=1}^{\infty} |\langle f, g_i \mid h \rangle|^2.$$

Now, for each $f \in X_h$, we have

$$\|K_h^* f, h\|^4 = (\langle K_h^* f, K_h^* f \mid h \rangle)^2 = \left(\left\langle T_h \left(T_h'\right)^* f, K_h^* f \mid h \right\rangle\right)^2$$
$$= \left(\left\langle \left(T_h'\right)^* f, T_h^* K_h^* f \mid h \right\rangle\right)^2 \leq \left\|\left(T_h'\right)^* f, h\right\|^2 \|T_h^* K_h^* f, h\|^2$$
$$= \sum_{i=1}^{\infty} |\langle f, g_i \mid h \rangle|^2 \sum_{i=1}^{\infty} |\langle K_h^* f, f_i \mid h \rangle|^2$$
$$\leq \sum_{i=1}^{\infty} |\langle f, g_i \mid h \rangle|^2 \, C \, \|K_h^* f, h\|^2,$$

[since $\{f_i\}_{i=1}^{\infty}$ is 2-Bessel sequence].

Thus, $\{g_i\}_{k=1}^{\infty}$ is a 2-K-frame associated to h in X with bounds $1/C$ and D. Similarly, it can be shown that $\{f_i\}_{i=1}^{\infty}$ is a 2-K-frame associated to h with the lower bound $1/D$. □

4. Tight K-frame in 2-Hilbert space

Definition 4.1. A sequence $\{f_i\}_{i=1}^{\infty}$ in X is said to be a tight 2-K-frame associated to h for X, if for each $f \in X_h$, there exists constant $A > 0$ such that

$$\sum_{i=1}^{\infty} |\langle f, f_i \mid h \rangle|^2 = A \, \|K_h^* f, h\|^2.$$

If $A = 1$, then $\{f_i\}_{i=1}^{\infty}$ is called Parseval 2-K-frame. From above we get that

$$\sum_{i=1}^{\infty} \left|\left\langle \frac{1}{\sqrt{A}} f, f_i \mid h \right\rangle\right|^2 = \|K_h^* f, h\|^2.$$

Therefore if $\{f_i\}_{i=1}^{\infty}$ is a tight 2-K-frame associated to h with bound A then family $\left\{\dfrac{1}{\sqrt{A}} f_i\right\}_{i=1}^{\infty}$ is a Parseval 2-K-frame associated to h.

Theorem 4.2. *Let $\{f_i\}_{i=1}^{\infty}$ is a tight 2-frame associated to h for X with bound A and $K_h \in \mathcal{B}(X_h)$, then $\{K_h f_i\}_{i=1}^{\infty}$ is a tight 2-K-frame associated to h for X with bound A.*

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a tight 2-frame associated to h for X with bound A, for each $f \in X_h$, we have

$$\sum_{i=1}^{\infty} |\langle f, K_h f_i \mid h \rangle|^2 = \sum_{i=1}^{\infty} |\langle K_h^* f, f_i \mid h \rangle|^2 = A \, \|K_h^* f, h\|^2.$$

Hence, $\{K_h f_i\}_{i=1}^{\infty}$ is a tight 2-K-frame associated to h for X with bound A. $\square$

Theorem 4.3. *Let $K_h, T \in \mathcal{B}(X_h)$ and $\{f_i\}_{i=1}^{\infty}$ is a tight 2-K-frame associated to h for X with bound A. Then $\{T f_i\}_{i=1}^{\infty}$ is a tight 2-TK-frame associated to h for X with bound A.*

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a tight 2-K-frame associated to h for X with bound A, for each $f \in X_h$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle f, T f_i | h\rangle|^2 &= \sum_{i=1}^{\infty} |\langle T^* f, f_i | h\rangle|^2 \\ &= A\| K_h^*(T^* f), h\|^2 = A\|(T K_h)^* f, h\|^2.\end{aligned}$$

Hence, $\{T f_i\}_{i=1}^{\infty}$ is a tight 2-TK-frame associated to h for X with bound A. $\square$

Theorem 4.4. *Let $\{f_i\}_{i=1}^{\infty}$ & $\{g_i\}_{i=1}^{\infty}$ be two Parseval 2-K-frames associated to h for X with corresponding 2-pre frame operators T_h and L_h, respectively. If $T_h L_h^* = \theta$, where θ is the null operator on X_h then $\{f_i + g_i\}_{i=1}^{\infty}$ is a tight 2-K-frame associated to h with frame bound 2.*

Proof. Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be two Parseval 2-K-frames associated to h for X. Then by Theorem 3.4, there exist 2-pre frame operators T_h and L_h such that $T_h e_i = f_i$, $L_h e_i = g_i$, (where $\{e_i\}_{i=1}^{\infty}$ is an h-orthonormal basis for $l^2(\mathbb{N})$) with $\mathcal{R}(K_h) \subset \mathcal{R}(T_h)$, $\mathcal{R}(K_h) \subset \mathcal{R}(L_h)$, respectively. Then the corresponding adjoint operators can be defined as for all $f \in X_h$,

$$T_h^* : X_h \to l^2(\mathbb{N}),\ T_h^*(f) = \sum_{i=1}^{\infty} \langle f, f_i | h\rangle e_i,$$

$$L_h^* : X_h \to l^2(\mathbb{N}),\ L_h^*(f) = \sum_{i=1}^{\infty} \langle f, g_i | h\rangle e_i.$$

Now from the definition of Parseval 2-K-frame, we can write

$$\| K_h^* f, h\|^2 = \sum_{i=1}^{\infty} |\langle f, f_i | h\rangle|^2 = \| T_h^* f, h\|^2, \tag{4.1}$$

$$\| K_h^* f, h\|^2 = \sum_{i=1}^{\infty} |\langle f, g_i | h\rangle|^2 = \| L_h^* f, h\|^2. \tag{4.2}$$

Following the proof of the Theorem 3.5, it can be shown that for each $f \in X_h$,

$$\sum_{i=1}^{\infty} |\langle f, f_i + g_i | h \rangle|^2 = \sum_{i=1}^{\infty} |\langle f, (T_h + L_h) e_i | h \rangle|^2$$
$$= \|(T_h + L_h)^* f, , h\|^2$$
$$= \langle T_h T_h^* f, f | h \rangle + \langle T_h L_h^* f, f | h \rangle + \langle L_h T_h^* f, f | h \rangle + \langle L_h L_h^* f, f | h \rangle$$
$$= \langle T_h T_h^* f, f | h \rangle + \langle L_h L_h^* f, f | h \rangle \text{ [since } T_h L_h^* = \theta = L_h T_h^* \text{]}$$
$$= \langle T_h^* f, T_h^* f | h \rangle + \langle L_h^* f, L_h^* f | h \rangle = \|T_h^* f, h\|^2 + \|L_h^* f, h\|^2$$
$$= \|K_h^* f, h\|^2 + \|K_h^* f, h\|^2 \text{ [using (4.1) and (4.2)]}$$
$$= 2 \|K_h^* f, h\|^2.$$

This shows that $\{f_i + g_i\}_{i=1}^{\infty}$ is a tight 2-K-frame associated to h with frame bound 2. □

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A COMBINATORIAL PROOF OF A GENERALIZATION OF A THEOREM OF FROBENIUS

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ABSTRACT. In this article, we shall generalize a theorem due to Frobenius in group theory, which asserts that if p is a prime and p^r divides the order of a finite group, then the number of subgroups of order p^r is $\equiv 1(\text{mod } p)$. Interestingly, our proof is purely combinatorial and does not use much group theory.

1. Introduction

Although Sylow's theorems are taught in almost all undergraduate courses in abstract algebra, a generalization due to Frobenius does not seem to be as well known as it ought to be. Frobenius' generalization states that if p is a prime and p^r divides the order N of a finite group G, the number of subgroups of G of order p^r is $\equiv 1 \pmod p$. The special case when p^r is the largest power of p dividing N is part of Sylow's third theorem. Many of the standard texts do not mention this theorem. One source is Ian Macdonald's 'Theory of Groups' [1]. In fact, a further generalization due to Snapper [2] asserts that for any subgroup K of order p^r and for any $s \geq r$ where p^s divides the order of G, the number of subgroups of order p^s containing K is also $\equiv 1 \pmod p$. In this article, we give a new proof of a further extension of Snapper's result that is purely combinatorial and does not use much group theory. Thus, we have a new combinatorial proof of Frobenius's theorem as well.

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2. Main results

We initially started by giving a combinatorial proof of Frobenius's result and, interestingly, our method of proof yields as a corollary an extension of Snapper's Theorem. Our proof builds on the famous combinatorial proof of Cauchy's theorem which asserts that if a prime divides the order of a group, there is an element of that prime order.

Theorem 1.1. *Let G be a finite group of order N, and let p be a prime. Let $b_0 < b_1 < \cdots < b_r$ be nonnegative integers such that p^{b_r} divides N and P_{b_0} be a subgroup of G of order p^{b_0}. Then the number of ordered tuples $(P_{b_1}, P_{b_2}, \cdots, P_{b_r})$ such that each P_{b_i} is subgroup of G of order p^{b_i} and*

$$P_{b_0} \subset P_{b_1} \subset \cdots \subset P_{b_r}$$

is $\equiv 1$ (mod p).

The case $r = 1$ is a Theorem due to Snapper [2] which is itself an extension of Frobenius's Theorem that corresponds to the case $r = 1, b_0 = 0$ in our Theorem.

Let us recall here the simple results in finite group theory that we will need.

(1) If H is a subgroup of a finite group G of order N, and the index $[G : H]$ is the smallest prime divisor of N, then H is normal in G.
(2) (Sylow's first theorem) If G is a finite group of order N, p a prime, $i \geq 0$ is an integer, $p^{i+1}|N$ and P is a subgroup of G of order p^i, then there is a subgroup Q of G containing P of order p^{i+1}.

We shall also use the following notations throughout.

(1) For a finite set S, $|S|$ denotes the number of elements (cardinality) of S.
(2) If G, H are finite groups, $H \leq G$ means H is a subgroup of G.
(3) If G, H are finite groups, $H \leq G$, $[G : H]$ denotes the index of H in G.
(4) If G is a finite group, the order of G is the number of elements of G.
(5) If H is a subgroup of a group G, $N_G(H)$ denotes the normalizer of H in G.
(6) For positive integers a, b, we write $a|b$ to mean a divides b.

Proof of Theorem.
For ease of understanding, we divide the proof into three steps.

Step 1: We tackle the case $r = 1, b_0 = 0, b_1 = 1$ first, which just says that if p divides the order of G, then the number of subgroups of G of order p is $\equiv 1 \pmod{p}$.
Let $T = \{(a_1, a_2, ..., a_p) \mid a_i \in G \ \forall i,\ a_1a_2...a_p = 1\}$.
Observe that $|T| = N^{p-1} \equiv 0 \pmod{p}$, as any choice of $a_1, ..., a_{p-1}$ uniquely determines a_p. Also, if not all a_i's are equal, then $(a_1, a_2, ..., a_p) \in T$ implies $(a_i, a_{i+1}, ..., a_{i+p-1})$ for $i = 1, 2, ..., p$ (indices are modulo p) are p distinct elements of T. The reason is as follows:
If $(a_i, a_{i+1}, ..., a_{i+p-1}) = (a_j, a_{j+1}, ..., a_{j+p-1})$ for some $i \neq j$, then $a_k = a_{k+j-i} \forall k$. By induction, $a_k = a_{k+\alpha(j-i)}$ for any integer α. But $i \neq j$ implies $gcd(j-i, p) = 1$, as $0 < |i-j| < p$ and p is a prime. So, $j - i$ is invertible modulo p. So any $1 \leq l \leq p$ satisfies $l \equiv 1 + \alpha(j-i) \pmod{p}$ for some integer α. So, $a_l = a_{1+\alpha(j-i)} = a_1$ for any $1 \leq l \leq p$. So, a_l's are all equal, which leads to a contradiction.

So, if d is the number of elements of G of order p, then $0 \equiv |T| \equiv (1 + d) \pmod{p}$. So, $d \equiv -1 \pmod{p}$ (as there are exactly $1+d$ elements of T with all a_i's equal.) In each subgroup of order p, there are $p-1$ elements of order p, different subgroups of order p intersect at the identity. So,

$-1 \equiv d = (p-1)$(number of subgroups of order p) $\equiv -$(number of subgroups of order p) $\pmod{p}$.

So, number of subgroups of order p is $\equiv 1 \pmod{p}$, which finishes the proof for the case $r = 1, b_0 = 0, b_1 = 1$.

Step 2: Now come to a general case. First, we fix a notation. Let H be any group of order M, $p^n | M, p^{n+1} \nmid M$, $0 \leq r \leq n$. Let P_r be a subgroup of order p^r in H. Define

$$S(P_r, H) = \{(P_{r+1}, P_{r+2}, \cdots, P_n) | P_i \leq H, |P_i| = p^i \,\forall\, i, P_r \leq P_{r+1} \leq \cdots \leq P_n \leq H\}.$$

So, $S(P_n, H)$ is a singleton set, by convention.
For $r \leq i < n$ and a subgroup P_i of H of order p^i, there is a subgroup P'_{i+1} of H of order p^{i+1} containing P_i, by Sylow's theorems. $[P'_{i+1} : P_i] = p$, which is the smallest prime divisor of $|P'_{i+1}|$, so P_i is normal in P'_{i+1}. Hence $P'_{i+1} \leq N_G(P_i)$. So, $\frac{P'_{i+1}}{P_i}$ is a subgroup of order p in $\frac{N_G(P_i)}{P_i}$. So,

$p|[N_G(P_i) : P_i]$. By the same reasoning, any subgroup P_{i+1} of H of order p^{i+1} containing P_i must be a subgroup of $N_G(P_i)$, and so $\frac{P_{i+1}}{P_i}$ is a subgroup of order p in $\frac{N_G(P_i)}{P_i}$. Conversely, any subgroup of order p in $\frac{N_G(P_i)}{P_i}$ gives rise via pullback to a subgroup P_{i+1} of $N_G(P_i)$(hence of H) of order p^{i+1} containing P_i. So, there is a one-to-one correspondence between such P_{i+1} (subgroups of G of order p^{i+1} containing P_i) and the subgroups of order p of the quotient group $\frac{N_G(P_i)}{P_i}$.

So, the number of such P_{i+1} is the number of subgroups of order p in $\frac{N_G(P_i)}{P_i}$, which is $\equiv 1 \pmod p$, in view of Step 1. So, in mod p, we can choose P_{r+1} in 1 way, after each such choice we can choose P_{r+2} in 1 way, and so on. So, $|S(P_r, H)| \equiv 1 \pmod p$.

Step 3: Now come to the setup of our theorem. We have $|S(P_{b_0}, G)| \equiv 1 (\text{mod } p)$, by Step 2. Let us count $|S(P_{b_0}, G)|$ in another way. Let x be the number of ordered tuples as in the statement of our theorem. After choosing any of such x ordered tuples, we can choose $(P_{b_i+1}, ..., P_{b_{i+1}-1})$ in $|S(P_{b_i}, P_{b_{i+1}})| \equiv 1(\text{mod } p)$ ways, for each $0 \leq i \leq r-1$, and we can choose $(P_{b_r+1}, ..., P_n)$ in $|S(P_{b_r}, G)| \equiv 1(\text{mod } p)$ ways.
Now, p^n is the largest power of p dividing N, each P_i is a subgroup of G of order p^i and $P_i \leq P_{i+1}$ for all $b_0 \leq i < n$. So, we obtain $|S(P_{b_0}, G)| \equiv x \pmod p$. Hence finally we get $x \equiv 1 \pmod p$ which completes the proof.

Remarks

The case $r = 1$ is Snapper's result and the further special case $r = 1, b_0 = 0$ corresponds to Frobenius' theorem.

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AN INEQUALITY REGARDING DIFFERENTIAL POLYNOMIAL

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ABSTRACT. In this paper, we prove an inequality regarding the differential polynomial. This improves some recent results.

1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, we assume that the reader is familiar with the value distribution theory [6]. Further, it will be convenient to let that E denote any set of positive real numbers of finite Lebesgue measure, not necessarily same at each occurrence. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \text{ as } r \to \infty, \ r \notin E.$$

Let f be a non-constant meromorphic function. A meromorphic function $a(z)(\not\equiv 0, \infty)$ is called a "small function" with respect to f if $T(r, a(z)) = S(r, f)$. For example, polynomial functions are small functions with respect to any transcendental entire function.

Definition 1.1. [8]. *Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer k, we denote*

i) by $N_{k)}(r, a; f)$ the counting function of a-points of f whose multiplicities are not greater than k,

ii) by $N_{(k}(r, a; f)$ the counting function of a-points of f whose multiplicities are not less than k.

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Similarly, the reduced counting functions $\overline{N}_{k)}(r,a;f)$ *and* $\overline{N}_{(k}(r,a;f)$ *are defined.*

Definition 1.2. [7]. *For a positive integer* k*, we denote* $N_k(r,0;f)$ *the counting function of zeros of* f*, where a zero of* f *with multiplicity* q *is counted* q *times if* $q \leq k$*, and is counted* k *times if* $q > k$*.*

Definition 1.3. [1]. *Let* $n_{0j}, n_{1j}, \cdots, n_{kj}$ *be non-negative integers. Then the expression* $M_j[f] = (f)^{n_{0j}}(f')^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$ *is called a differential monomial generated by* f*. The quantities* $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ *and* $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i+1)n_{ij}$ *are known as the degree and weight of the monomial* M_j *respectively.*

The sum $P[f] = \sum\limits_{j=1}^{t} b_j M_j[f]$ *is called a differential polynomial generated by* f*, where* $T(r,b_j) = S(r,f)$ $(j = 1,2,\cdots,t)$*. The quantities* $\overline{d}(P) = \max\limits_{1\leq j\leq t}\{d(M_j)\}$ *and* $\Gamma_P = \max\limits_{1\leq j\leq t}\{\Gamma_{M_j}\}$ *are called degree and weight of the polynomial* $P[f]$ *respectively.*

The numbers $\underline{d}(P) = \min\limits_{1\leq j\leq t}\{d(M_j)\}$ *and* k *(the highest order of the derivative of* f *in* $P[f]$*) are known as the lower degree and order of the polynomial* $P[f]$*.*

$P[f]$ *is called homogeneous if* $\underline{d}(P) = \overline{d}(P)$*. Otherwise* $P[f]$ *is called a non-linear differential polynomial.* $P[f]$ *is called a linear differential polynomial if* $\overline{d}(P) = 1$*.*

In 2003, I. Lahiri and S. Dewan proved the following theorem:

Theorem A. [7]. *Let* f *be a transcendental meromorphic function and* $\alpha(\not\equiv 0,\infty)$ *be a small function of* f*. If* $\psi = \alpha(f)^n(f^{(k)})^p$*, where* $n(\geq 0), p(\geq 1), k(\geq 1)$ *are integers, then for any small function* $a(\not\equiv 0,\infty)$ *of* ψ *we have*

$$(p+n)T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + pN_k(r,0;f) + \overline{N}(r,a;\psi) + S(r,f).$$

In this paper we extend and improve the Theorem A. Now in the following we are stating our result:

Theorem 1.1. *Let* $f(z)$ *be a transcendental meromorphic function and* $\alpha(z)(\not\equiv 0,\infty)$ *be a small function of* $f(z)$*. Let* $P[f] = \alpha \sum\limits_{j=1}^{t} M_j[f]$ *be a differential polynomial generated by* f*, where* $M_j[f] = c_j(f)^{n_{0j}}(f')^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$ $(j = 1,2,\cdots,t)$ *such that* $k(\geq 1)$ *is the order of* $P[f]$*,* $t(\geq 1)$*,* $n_{ij}(i = 0,1,\cdots,k; j = 1,2,\cdots,t)$ *are non-negative integers and* $c_j(j = 1,2,\cdots,t)$

are small functions of f such that they do not have poles at the zeros of f. Then, for a small function $a(\not\equiv 0,\infty)$,

$$\begin{aligned}\underline{d}(P)T(r,f) \;\; &\leq \;\; \overline{N}(r,0;f)+\overline{N}(r,a;P[f])+\overline{N}(r,\infty;f)+n_{1e}N_1(r,0;f)\\ &\quad +n_{2e}N_2(r,0;f)+\cdots+n_{ke}N_k(r,0;f)+S(r,f),\end{aligned}$$

where $(1\cdot n_{1e}+2\cdot n_{2e}+\cdots+k\cdot n_{ke})=\max\limits_{1\leq j\leq t}(1\cdot n_{1j}+2\cdot n_{2j}+\cdots+k\cdot n_{kj})$.

2. Necessary Lemmas

Lemma 2.1. [5] *Let $A>1$, then there exists a set $M(A)$ of upper logarithmic density at most $\delta(A)=\min\{(2e^{(A-1)}-1)^{-1},1+e(A-1)\exp(e(1-A))\}$ such that for $k=1,2,3,\cdots$*

$$\limsup_{r\to\infty,r\notin M(A)}\frac{T(r,f)}{T(r,f^{(k)})}\leq 3eA.$$

Lemma 2.2. *Let f be a transcendental meromorphic function and α $(\not\equiv 0,\infty)$ be a small function of f. Let, $\psi=\alpha(f)^{q_0}(f')^{q_1}\cdots(f^{(k)})^{q_k}$, where $q_0,q_1,\cdots,q_k(\geq 1),k(\geq 1)$ are non-negative integers. Then ψ is not identically constant.*

Proof. Since, α is a small function of f, then $T(r,\alpha)=S(r,f)$. Therefore the proof follows from Lemma (3.4) of ([3]). ☐

Lemma 2.3. *Let f be a transcendental meromorphic function and α $(\not\equiv 0,\infty)$ be a small function of f. Let, $\psi=\alpha(f)^{q_0}(f')^{q_1}\cdots(f^{(k)})^{q_k}$, where $q_0,q_1,\cdots,q_k(\geq 1),k(\geq 1)$ are non-negative integers. Then*

$$T(r,\psi)\leq\{q_0+2q_1+\cdots+(k+1)q_k\}\,T(r,f)+S(r,f).$$

Proof. This result is very well known and the proof is similar to the Lemma (2.3) of [7] . So, we omit the details. ☐

Lemma 2.4 ([2, 4])**.** *Let, f be a meromorphic function and $P[f]$ be a differential polynomial. Then*

$$m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)\leq(\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)+S(r,f).$$

3. The Proofs

Proof of Theorem 1.1. Since $P[f]$ is a differential polynomial generated by f, then using Lemma 2.4, we have

$$\begin{aligned}
T(r,(f)^{\overline{d}(P)}) &= N(r,0;(f)^{\overline{d}(P)}) + m\left(r,\frac{1}{(f)^{\overline{d}(P)}}\right) + O(1)\\
&= N(r,0;(f)^{\overline{d}(P)}) + m\left(r,\frac{P[f]}{(f)^{\overline{d}(P)}}\frac{1}{P[f]}\right) + O(1)\\
&\leq N(r,0;(f)^{\overline{d}(P)}) + (\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)\\
&\quad +m\left(r,\frac{1}{P[f]}\right) + S(r,f)\\
&= N(r,0;(f)^{\overline{d}(P)}) + (\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right) + \ T(r,P[f])\\
&\quad -N(r,0;P[f]) + S(r,f).
\end{aligned}$$

Thus,

$$\begin{aligned}
T(r,(f)^{\overline{d}(P)}) &\leq N(r,0;(f)^{\overline{d}(P)}) + (\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)\\
&\quad + T(r,P[f]) - N(r,0;P[f]) + S(r,f). \qquad (3.1)
\end{aligned}$$

Using Nevanlinna's second fundamental theorem, from (3.1) we get

$$\begin{aligned}
T(r,(f)^{\overline{d}(P)}) &\leq N(r,0;(f)^{\overline{d}(P)}) + \overline{N}(r,0;P[f]) + \overline{N}(r,\infty;P[f])\\
&\quad +\overline{N}(r,a;P[f]) - N(r,0;P[f]) + (\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)\\
&\quad +S(r,P[f]) + S(r,f). \qquad (3.2)
\end{aligned}$$

From definition of $P[f]$ and using Lemma (2.3) we have

$$T(r,P[f]) \leq K_1 T(r,f) + S(r,f),$$

for some constant K_1. This implies $S(r,P[f]) = S(r,f)$. We also note that $\overline{N}(r,\infty;P[f]) = \overline{N}(r,\infty;f) + S(r,f)$.
Thus from (3.2),

$$\begin{aligned}
T(r,(f)^{\overline{d}(P)}) &\leq N(r,0;(f)^{\overline{d}(P)}) + \overline{N}(r,0;P[f]) + \overline{N}(r,\infty;f)\\
&\quad +\overline{N}(r,a;P[f]) - N(r,0;P[f]) + (\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)\\
&\quad +S(r,f). \qquad (3.3)
\end{aligned}$$

Let, z_0 be a zero of $f(z)$ with multiplicity q (≥ 1). For any k, $M_j[f]$ $(j = 1, 2, \cdots, t)$ has a zero at z_0 of order atleast

$$\begin{aligned}
& qn_{0j} + (q-1)n_{1j} + (q-2)n_{2j} + \cdots + 2n_{q-2\ j} + n_{q-1\ j} + r_j \\
= \;& q(n_{0j} + n_{1j} + \cdots + n_{q-1\ j}) - (1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + (q-1) \cdot n_{q-1j}) \\
& + r_j \\
= \;& q(n_{0j} + n_{1j} + \cdots + n_{q-1\ j} + \cdots + n_{kj}) - q(n_{qj} + n_{q+1\ j} + \cdots + n_{kj}) \\
& -(1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + (q-1) \cdot n_{q-1\ j}) + r_j \\
= \;& q(d(M_j)) - (1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + (q-1) \cdot n_{q-1\ j} + qn_{qj} + \cdots \\
& + qn_{kj}) + r_j \\
\geq \;& q(\underline{d}(P)) - (1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + (q-1) \cdot n_{q-1\ j} + qn_{qj} + \cdots \\
& + qn_{kj}), \text{ if } q \leq k
\end{aligned}$$

and

$$\begin{aligned}
& qn_{0j} + (q-1)n_{1j} + (q-2)n_{2j} + \cdots + (q-k)n_{kj} + r_j \\
= \;& q(n_{0j} + n_{1j} + \cdots + n_{kj}) - (1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + k \cdot n_{kj}) + r_j \\
= \;& q(d(M_j)) - (1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + k \cdot n_{kj}) + r_j \\
\geq \;& q(\underline{d}(P)) - (1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + k \cdot n_{kj}), \text{ if } q > k,
\end{aligned}$$

where r_j $(j = 1, 2, \cdots, t)$ is the multiplicity of zero of c_j $(j = 1, 2, \cdots, t)$ at z_0, which must be a non-negative quantity.

Therefore, $P[f]$ has a zero at z_0 of order

$$\begin{aligned}
\geq \;& q(\underline{d}(P)) + r - (1 \cdot n_{1e} + \cdots + (q-1) \cdot n_{q-1\ e} + qn_{qe} + \cdots \\
& + q \cdot n_{ke}) \text{ if } q \leq k \text{ and} \\
\geq \;& q(\underline{d}(P)) + r - (1 \cdot n_{1e} + 2 \cdot n_{2e} + \cdots + k \cdot n_{ke}) \text{ if } q > k,
\end{aligned}$$

where $r = 0$ if $\alpha(z)$ does not have a zero or pole at z_0, $r = s$ if $\alpha(z)$ has a zero of order s at z_0, $r = -s$ if $\alpha(z)$ has a pole of order s at z_0, s being a natural number and $\{n_{1e}, n_{2e}, \cdots, n_{ke}\}$ is the set of values such that $(1 \cdot n_{1e} + 2 \cdot n_{2e} + \cdots + k \cdot n_{ke}) = \max\limits_{1 \leq j \leq t}(1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + k \cdot n_{kj})$. [We see that whenever $q (\geq 1)$ and $k (\geq 1)$ are fixed, then $\max\limits_{1 \leq j \leq t}(1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + q \cdot n_{kj})$ and $\max\limits_{1 \leq j \leq t}(1 \cdot n_{1j} + 2 \cdot n_{2j} + \cdots + k \cdot n_{kj})$ will appear for same set of values

$\{n_{1j}, n_{2j}, \cdots, n_{kj}\}$ for some $j \in \{1, 2, \cdots, t\}$.]

Therefore,

$$\begin{aligned} & 1 + q(\overline{d}(P)) - q(\underline{d}(P)) - r \\ & +(1 \cdot n_{1e} + \cdots + (q-1) \cdot n_{q-1\ e} + qn_{qe} + \cdots + q \cdot n_{ke}) \\ = \ & q(\overline{d}(P) - \underline{d}(P)) + 1 - r \\ & +(1 \cdot n_{1e} + \cdots + (q-1) \cdot n_{q-1\ e} + qn_{qe} + \cdots + q \cdot n_{ke}) \text{ if } q \leq k \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & 1 + q(\overline{d}(P)) - q(\underline{d}(P)) - r + (1 \cdot n_{1e} + 2 \cdot n_{2e} + \cdots + k \cdot n_{ke}) \\ = \ & q(\overline{d}(P) - \underline{d}(P)) + 1 - r + (1 \cdot n_{1e} + 2 \cdot n_{2e} + \cdots + k \cdot n_{ke}) \text{ if } q > k. \end{aligned} \tag{3.5}$$

Therefore, from (3.4) and (3.5) we have

$$\begin{aligned} & N(r, 0; (f)^{\overline{d}(P)}) + \overline{N}(r, 0; P[f]) - N(r, 0; P[f]) \\ \leq \ & (\overline{d}(P) - \underline{d}(P))N(r, 0; f) + \overline{N}(r, 0; f) + n_{1e}N_1(r, 0; f) \\ & +n_{2e}N_2(r, 0; f) + \cdots + n_{ke}N_k(r, 0; f) + S(r, f). \end{aligned}$$

Therefore (3.3) gives

$$\begin{aligned} & \overline{d}(P)T(r, f) \\ \leq \ & \overline{N}(r, 0; f) + \overline{N}(r, a; P[f]) + \overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))N(r, 0; f) \\ & +(\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + n_{1e}N_1(r, 0; f) + n_{2e}N_2(r, 0; f) + \cdots \\ & +n_{ke}N_k(r, 0; f) + S(r, f) \\ = \ & \overline{N}(r, 0; f) + \overline{N}(r, a; P[f]) + \overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))T(r, f) \\ & +n_{1e}N_1(r, 0; f) + n_{2e}N_2(r, 0; f) + \cdots + n_{ke}N_k(r, 0; f) + S(r, f). \end{aligned}$$

Thus,

$$\begin{aligned} \underline{d}(P)T(r, f) \leq \ & \overline{N}(r, 0; f) + \overline{N}(r, a; P[f]) + \overline{N}(r, \infty; f) + n_{1e}N_1(r, 0; f) \\ & +n_{2e}N_2(r, 0; f) + \cdots + n_{ke}N_k(r, 0; f) + S(r, f). \end{aligned}$$

□

Concluding comments

In this short note, we have described a result(Theorem 1.1) which extends and improves Theorem A.

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RESULTS ON FINITE COLLECTION OF POLYGONS AND A PROOF OF THE JORDAN CURVE THEOREM

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Abstract. We introduce the notion of polygons and Jordan curves. We first provide a proof of the Jordan Curve Theorem for polygons, and then we answer the following questions: given a finite collection of polygonal regions in the plane, can we write their union as an almost disjoint union of polygonal regions? What do the boundaries of the connected components in the complement of these polygons look like? Having answered these questions, we construct a "regular" polygonal cover for arcs in the plane and use such a covering to prove a separation result about arcs inside discs. In the last section we provide a proof of the Jordan Curve Theorem using the methods developed in the previous sections.

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1. Introduction

In this paper, we provide a proof of the classic Jordan Curve Theorem. This is one of those theorems that is very simple to state and understand but notoriously difficult to prove. The first proof was given by Jordan (although there were some doubts, [1] claims that the original proof was indeed correct), and since then there have been many other proofs.

In the following two sections we provide a few definitions and a proof of the Jordan Curve Theorem for polygons using a parity function. In section 4 we answer the following question: can we write a finite union of polygonal regions as an almost disjoint union of other polygonal regions. The answer is yes, provided the collection satisfies a regularity condition as defined later. Furthermore, we also prove that under said regularity, all connected components in the complement of a union of polygons (just the boundaries) have polygonal boundaries.

In section 5 we spend some time proving certain Jordan like construction and results for finite collection of polygons, their boundaries and arcs connecting points on these boundaries.

In section 6 we prove a rather interesting theorem about arcs, which doesn't seem to be a direct consequence of the Jordan Curve Theorem. Suppose we have two arcs intersecting only at the end points. A bulk of this section is spent in explicitly constructing a polygonal covering of one of these arcs meeting the aforementioned regularity conditions and more. We then show that under a rather mild condition on these two arcs (see Theorem 6), given a point, say x, not on these arcs, we can cover one of these arcs by polygons such that x is in the unbounded component of the complement of these polygons (we shall prove that there is only one unbounded component in section 3).

In section 7 we prove that an arc lying inside the unit disc with ends on the boundary circle separates the disc into two regions and makes up their common boundary. This theorem has been proved before, but we provide a proof using the methods developed in the previous sections. Finally, in section 8 we provide a proof of the Jordan Curve theorem, mainly relying on the methods developed in the previous sections.

The core of the paper is contained in sections 4 through 6 and these concern finite collections of polygons. The results and methods involved here, specifically Theorems 4, 5, 6 and the construction detailed in section 6.1, are novel to the best of the author's knowledge.

While the end goal of this article is to prove the Jordan Curve Theorem, we mention here that the bulk of it is spent in developing certain results for finite collections of polygons and polygonal covers of arcs/curves. As far as proofs of the theorem go, [1] contains the original proof and [2] is a much shorter proof and both involve approximations of curves by polygons. The proofs detailed in [3, 4] are slightly different, but still concentrated on curves. As mentioned before, the focus of this article is on finite collections of polygons and polygonal covers and came out of an attempt by the author to provide a proof of Jordan Curve Theorem.

Throughout we shall use the same label to refer to a map or its image. It is assumed that the reader is familiar with basic real analysis and topology.

2. Definitions

A Jordan curve is a homeomorphic image of S^1 in $\mathbb{R}^2$. An arc is a homeomorphic image of $[0,1]$ in any $\mathbb{R}^n$, while a Jordan arc is an arc in $\mathbb{R}^2$. A path is any continuous image of $[0,1]$ in any $\mathbb{R}^n$, although we will mostly be confined to the plane.

A polygon P is a Jordan curve that is piecewise linear i.e., a map $P\colon S^1 \to \mathbb{R}^2$ with finitely many points in S^1 between which P is a line. Similarly, we define a polygonal arc.

Given a polygon P, let $V(P) = \{v_1, \ldots, v_n\}$ be a minimal set such that P is a line between $v_i, v_{i+1}, i = 1, \ldots, n$ where $v_{n+1} = v_1$. By minimality, the two lines at v_i must be non parallel. The points in V are called the vertices of P, and the lines are called the edges of P.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a finite collection of polygons. We define the finite set of "new vertices" $V(\mathcal{P})$ to be the set that contains

- $V(P_i), i = 1, \ldots, n$.
- Points of intersections of non parallel edges from different polygons in $\mathcal{P}$.

An edge of $\mathcal{P}$ refers to any edge of any $P \in \mathcal{P}$. Given $v \in V = V(\mathcal{P})$, take an open ball U around v disjoint from other points of V, and edges that don't contain v. Any such a ball shall be called the *zone* of v, and exists because both V and the set of edges of $\mathcal{P}$ are finite. Edges that pass through v induce a radius (or diameter) in the zone of v because U is an open disc with centre v.

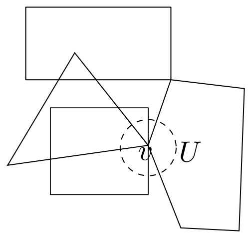

FIGURE 2.1. Zone U of vertex v with 6 radii

Two radii are *adjacent* if there is no other radii between them in at least one orientation (i.e., clockwise or anticlockwise). By the choice of $U, U \cap \mathcal{P}$ contains only radial lines. Similarly define a zone for $v \notin \cup_{P \in \mathcal{P}} P \setminus V(\mathcal{P})$ by avoiding all points in V and edges not passing through v. This time, there is only one diameter in U as only overlapping edges of P_i pass through v.

Suppose C_1, C_2 are compact subsets of $\mathbb{R}^2$. The distance d between C_1, C_2 refers to the minimum attained by the distance map (which is continuous) on $C_1 \times C_2$. It is zero if and only if $C_1 \cap C_2 \neq \emptyset$, and when $d \neq 0$, an open ball of radius d (or less) around any $x \in C_1$ does not intersect C_2.

3. Jordan curve theorem for polygons

In this section we prove that if P is a polygon, then $\mathbb{R}^2 \setminus P$ has two components, one bounded and the other unbounded, both having P as their boundaries.

3.1. **Parity function.** Let P be a polygon and f be a direction not parallel to any of the edges in P. Since there are finitely many edges, such an f always exists. Because rotation is a linear homeomorphism, sending polygons to polygons, we may suppose that f is along the positive x-axis. We will now define a parity function $n\colon \mathbb{R}^2 \setminus P \to \{0, 1\}$.

The zones at each point of P have two sectors given by non parallel radii at vertices, and a diameter at non vertices. For $x \notin P$, take R_x to be the ray (half-line) originating at x parallel to f. For each $p \in R_x \cap P$, we define a contribution $c(p)$ to $n(x)$ to be 1 if R_x intersects both sectors in the zone of p and 0 otherwise. Define

$$n(x) = \sum_{p \in R_x \cap P} c(p) \mod 2.$$

Here the empty sum is taken to be 0. The sum is finite because R_x is not parallel to any edge of P. Note that if p is not a vertex, then its contribution(see Figure 3.1) is 1, and if p is a vertex, it is 1 if and only if the edges at p lie on both sides of R_x.

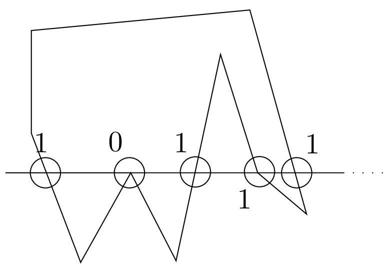

FIGURE 3.1. Zones and their contributions

Lemma 3.1.1. *The parity function constructed above is locally constant.*

Proof. Take $x = (a, b) \notin P$ and π to be the projection onto the y-axis. Suppose $R_x \cap P = \emptyset$. Consider the closed set $P_x = P \cap \{(c, d) \in \mathbb{R}^2 | c \geq a\}$, then $\pi(P_x)$ doesn't contain b, hence there is a neighbourhood $(b \pm \delta)$ disjoint from $\pi(P_x)$. It follows for p sufficiently close to x with $\pi(p) \in (b \pm \delta), R_p \cap P = \emptyset$, so n is identically zero in a neighbourhood of x.

Suppose $R_x \cap P = \{a_1, \ldots, a_m\}$. In the zone of a_i, the edges of P induce two radii, neither of which is parallel to the x-axis. Projecting both to the y-axis we get two positive lengths, of which r_i is the smaller one. Take $r = \min\{r_1, \ldots, r_m\}$.

Next, let η be the smallest y-length of the edges of P. Since no edge is parallel to the x-axis, $\eta > 0$. Lastly, let $\delta > 0$ be such that $B(x, \delta) \cap P = \emptyset$. Now, shift the line R_x vertically within $\epsilon = \min\{r, \eta, \delta\}$ of the original to get a ray R'_x, i.e., R'_x is the half-line parallel to positive x-axis originating from a point (a, b') with $|b' - b| < \epsilon$. When comparing $R'_x \cap P$ with $R_x \cap P$, by the choice of r, we see that (see Figure 3.1)

- If a_i is not a vertex, then it is replaced by another non vertex
- If a_i is a vertex that contributes 1 to $n(x)$, then it is replaced by a non vertex

- If a_i is a vertex that contributes 0 to $n(x)$, then either it is replaced by two non vertices or removed altogether

So this was about the edges that both R_x, R'_x intersect. Now, suppose R'_x was shifted upwards to $y = b'$ and intersects an edge e that R_x did not.

It follows that e lies above the line $y = b$. Let v be the lower vertex of e and e' the other edge of v. Let N be the union of R_x, R'_x and the vertical line segement l from (a, b) to (a, b'). Observe that N divides the plane into two parts.

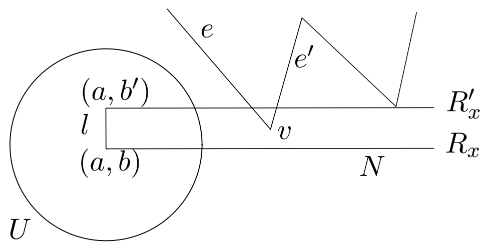

FIGURE 3.2. Parity is invariant under small vertical shifts

Since e touches R'_x but not R_x, v is either inside N or on its boundary. Note that e, e' cannot intersect l because l lies in $U = B(x, \epsilon)$. Since $\epsilon < \eta$, the other vertex of e' must be outside N. If e' doesn't intersect R_x, R'_x, then it intersects l as the other vertex is outside N. Thus, either R_x intersects e' or R'_x intersects e'.

If R_x intersects e', then by the choice of r, N cannot contain v. Thus, e' intersects R'_x but not R_x. So, e, e' together contribute 0 to n, by two non vertices or by v. Note that the lower vertex for e' is also v, so the edges that R'_x intersects but not R_x come in pairs.

So, n doesn't change under this vertical shift. Since every point of U is obtained by a vertical shift followed by a horizontal shift from x, and n remains invariant under these perturbations (by choice of ϵ), it is constant on U. The parity doesn't change under horizontal shifts because, by the choice of δ, the intersection points do not change. □

Lemma 3.1.2. *Parity function is surjective.*

Proof. Let p be a point on some edge e of P that is not a vertex. There are two sectors in the zone of p. The parity for points in those sectors differ by

1 because the horizontal ray passes through an extra edge, namely e, which contributes 1. □

As a consequence of the lemma, any small neighbourhood around non vertices has points of both parity. It follows that $\mathbb{R}^2 \setminus P$ is disconnected.

3.2. **Two components.** The following is taken from [2]. Cover P with a zone U_p at each $p \in P$ and obtain a finite subcover $U_1, \ldots, U_n$. Each U_i has two connected sectors, say U_i', U_i''. Label the sectors so that

$$U_i' \cap U_{i+1}' \neq \emptyset \text{ and } U_i'' \cap U_{i+1}'' \neq \emptyset.$$

Then the unions $U_1' \cup \cdots \cup U_n', U_1'' \cup \cdots \cup U_n''$ are connected sets and there is a line from any $p \notin P$ to one of these sets that doesn't intersect P (draw the line from p to any edge and look at the first time it meets P). Thus, $\mathbb{R}^2 \setminus P$ has at most two components, hence exactly two components. Since n is continuous, the components are given by $n^{-1}(0), n^{-1}(1)$.

Of these, $n^{-1}(0)$ is unbounded for we can enclose P in a rectangle whose outside remains connected after removing P and is part of $n^{-1}(0)$, whereas $n^{-1}(1)$ is inside, hence bounded.

The proof of the lemma above shows that all points of P that are not vertices lie in the boundary of both components. Since boundary is closed, both components have P as their boundary.

Thus, $\mathbb{R}^2 \setminus P$ has two components, the bounded "inside" $i(P)$ and the unbounded "outside" $o(P)$ with P as their common boundary. Observe that these are path components of the complement, hence independent of the choice of the ray used to compute parity, so we may choose any convenient ray and check whether the parity is 0 (outside) or 1 (inside).

Given a collection of polygons $\mathcal{P}$ outside of $\mathcal{P}$ refers to $\cap_{P \in \mathcal{P}} o(P)$ and inside refers to $\cup_{P \in \mathcal{P}} i(P)$.

3.3. **Approximations.**

Lemma 3.3.1. *Suppose U is an open set in $\mathbb{R}^n$ and $J\colon [0,1] \to U$ a continuous path, with $J(0) \neq J(1)$. Given $\epsilon > 0$, there is a polygonal arc $P\colon [0,1] \to U$ with $P(0) = J(0), P(1) = J(1)$ such that every point of P is within ϵ of some point of J.*

Proof. For $x \in J$ choose $\mu_x > 0$ such that $B(x, 2\mu_x) \subseteq U$. From the cover $\{B(x, \mu_x)\}_{x \in J}$ obtain a finite subcover $\{B(x_1, \mu_1), \ldots, B(x_m, \mu_m)\}$ and let $\mu = \min\{\mu_1, \ldots, \mu_m\}$. Then for $x \in B(x_i, \mu_i)$, $B(x, \mu) \subseteq B(x_i, 2\mu_i) \subseteq U$.

We may take $\epsilon < \mu$. By uniform continuity of J, choose N such that

$$|t_1 - t_2| < 2/N \Rightarrow |J(t_1) - J(t_2)| < \epsilon.$$

Take $J(0), J(1/N), \ldots, J(1)$ and draw line segments between consecutive points to get a piecewise linear path from $J(0)$ to $J(1)$. Observe that some of these lines may be degenerate because it is possible that $J(\frac{i}{N}) = J(\frac{i+1}{N})$, but $J(0) \neq J(1)$, so P is not altogether degenerate.

For $1 \leq i \leq N$,

$$\left| J\left(\frac{i-1}{N}\right) - J\left(\frac{i}{N}\right) \right| < \epsilon \Rightarrow J\left(\frac{i-1}{N}\right) \in B\left(J\left(\frac{i}{N}\right), \epsilon\right)$$

so the line between them is in $B(J(\frac{i}{N}), \epsilon) \subseteq U$ by the choice of $\epsilon < \mu$.

Thus, every point on the resulting union of lines is in U and within ϵ of some point of J. Next, replace any two overlapping lines by their union, this way the lines intersect at only finitely many points. Remove the loops as we go from $J(0)$ to $J(1)$ along the union of lines. Since there are finitely many loops (finitely many intersection points) this process terminates.

We will be left with a polygonal arc, i.e., a Jordan arc, $P \subset U$ such that every point of P is within ϵ of some point of J and ends of P are the ends of J. It is easy to obtain P as an injective image of $[0, 1]$ using piecewise definitions. □

Corollary 3.3.1. *If P is a polygon and $J \subset i(P)$ (or $J \subset o(P)$), then there is a polygonal approximation of J within given $\epsilon > 0$ that lies in $i(P)$ ($o(P)$).*

Corollary 3.3.2. *A connected open $U \subseteq \mathbb{R}^n$, is path connected, hence arc connected. In particular, for a polygon $P, i(P), o(P)$ are arc connected.*

Proof. If S_x denotes the path component of $x \in U$, then one can show that it is both closed and open, hence $S_x = U$. The rest follows from Lemma 3.3.1. □

Corollary 3.3.3. *For a polygon P, given $x, y \in \overline{i(P)}$, there is a polygonal arc between them which lies entirely in $i(P)$, except possibly at the ends.*

Proof. For $x \in P$ there is a straight line from x to a point $x' \in i(P)$ that, except for x, lies in $i(P)$. Similarly obtain a point $y' \in i(P)$ for y. Using Corollary 3.3.2, obtain a polygonal arc from x' to y'. We get the required polygonal arc by joining these paths. □

4. Some results about polygons

Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a collection of polygons, $V = V(\mathcal{P})$. Let U be a zone of some fixed $v \in V$. The edges of $\mathcal{P}$ determine radii in U which in turn determine open sectors between adjacent radii. These sectors remain connected in the complement of $\mathcal{P}$ and are either inside or outside $\mathcal{P}$.

We say that v is a *regular* vertex if for every component outside $\mathcal{P}$, there is at most one sector that intersects it, otherwise v is *singular*. This notion doesn't depend on U as long as it has it is disjoint from other points of V and edges that v doesn't lie on, i.e., as long as U is a zone of v.

When v is not a vertex, its zone has just a diameter, and one of the sectors lies in some $i(P_i)$, so non vertices are always regular. We say that $\mathcal{P}$ is regular if all points in $\mathcal{P}$ are regular.

Polygons P, Q are said to have *shallow intersection* if $i(P) \cap i(Q) = \emptyset$. The collection $\mathcal{P}$ is said to have shallow intersection if $i(P_j) \cap i(P_k) = \emptyset, \forall j \neq k$. In this case, no edge of P_i goes into $i(P_j)$ for any j. The union $\overline{i(P_1)} \cup \cdots \cup \overline{i(P_n)}$ is said to be a *shallow union* when $\mathcal{P}$ has a shallow intersection.

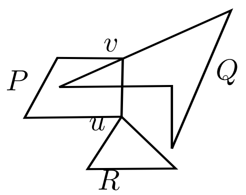

Figure 4.1. Here v is a regular vertex, u is singular, P, Q don't have shallow intersection, P, R and Q, R do have shallow intersection.

Theorem 4.0.1. *Let G be a graph. If every vertex of G has degree 2, then G is a disjoint union of cycles.*

For a proof see [5].

Theorem 4.0.2. *Let P, Q be two polygons, $V = V(\{P, Q\})$. Suppose $P \not\subset \overline{i(Q)}, Q \not\subset \overline{i(P)}$. Then $\overline{i(P)} \cup \overline{i(Q)}$ is a shallow union $\overline{i(R_1)} \cup \cdots \cup \overline{i(R_m)}$ for some polygons $R_1, \dots, R_m$ whose vertices come from V with edges subsegments of the edges of P, Q.*

Lemma 4.0.1. *With the setting as above, let $v \in P \cup Q$, then its zone has at most 4 sectors and no two of them are in the same component of $i(P) \setminus \overline{i(Q)}$.*

Proof. The zone U of v has at most 4 radii - two each from P, Q and therefore at most 4 sectors. Suppose a sector S bounded by r_1, r_2 is in $i(P) \setminus \overline{i(Q)} = i(P) \cap o(Q)$. Because r_1, r_2 come from P, Q the sectors adjacent to S cannot be in $i(P) \cap o(Q)$. For example, if r_1 is from P, then the other side of r_1 is in $o(P)$. Therefore, if U has 2 or 3 sectors then at most one is in $i(P) \setminus \overline{i(Q)}$.

We are left with the case when U has 4 sectors, two of which, S_1, S_2 are in $i(P) \cap o(Q)$ (by the discussion above, it is at most two). This can happen only when the radii from Q at v and the corresponding sector inside Q are all in $i(P)$. Now, suppose S_1, S_2 are in some component R of $i(P) \setminus o(Q)$, then there is a polygonal path from a point in S_1 to S_2 that lies inside R. Using radii from v, we construct a polygon $\tilde{P}$ that lies in $R \cup \{v\}$, hence in $i(P) \cup \{v\}$ with $\tilde{P} \cap P = \{v\}$.

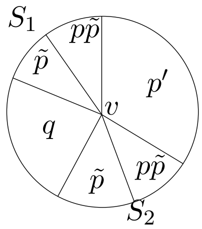

FIGURE 4.2. In the adjacent figure we use lower case letters to indicate sectors inside polygons, product to denote intersection and prime to denote outside.

We claim that $Q \subset \overline{i(\tilde{P})} \subseteq \overline{i(P)}$ which is a contradiction. For the first inclusion, observe that the radii from Q at v are in $i(\tilde{P})$, and Q doesn't intersect $\tilde{P}$ (except at v), therefore by connectedness, Q must be in $\overline{i(\tilde{P})}$. For the second inclusion, let $x \in i(\tilde{P})$, then there is a path from x to any $y \in \tilde{P}$ lying in $i(\tilde{P})$. This is a path from x to a point in $i(P)$, so it suffices to prove that it doesn't intersect P.

To this end, if it did intersect P, then because the path lies in $i(\tilde{P})$, we conclude that there is some $x_0 \in o(P) \cap i(\tilde{P})$. However, this is impossible for $o(P) \subset o(\tilde{P})$. $\square$

Proof of Theorem 4.0.2. The proof has the following steps: we show that components of $i(P) \setminus \overline{i(Q)}$ have boundaries that are unions of polygons, we then show that this boundary is actually a single polygon and the last step completes the proof.

Step 1: Boundary of components is a union of polygon

Let R be a component of $i(P) \setminus \overline{i(Q)}$. Given $x \in \partial R$, we know $x \in P \cup Q$ and let U be a zone of x. From the lemma, U has at most 4 sectors and exactly one is in R (because $x \in \partial R$).Let r_1, r_2 be the radii bounding this sector. If $x \notin V$, then $r_1 \parallel r_2$, hence an open line segment is part of $\partial R>$ When l is maximal, the ends must be in $\partial R \cap V$ as otherwise we can extend it. If $x \in V$, then $r_1 \not\parallel r_2$.

Thus, ∂R is a union of line segments with ends in V and at these points there are exactly two non parallel line segments that are part of ∂R (because of the previous lemma). Therefore, ∂R is a graph with vertices from V each of degree 2. By Theorem 4.0.1 it is a union of disjoint cycles, in this case polygons. Obeserve that if P' is one such boundary polygon, then one of the sectors at every point in P' is in R.

Step 2: Boundary has exactly one polygon

With R as above, suppose Q_1 is a boundary polygon of R. Then R must either lie inside or outside Q_1 because R is path connected. Suppose it lies outside Q_1. Any path from $i(Q_1)$ to $o(Q_1)$ must then pass through R, in particular through $i(P)$. However, observe that $o(P) \cap o(Q_1) \neq \emptyset$, so we conclude that $i(Q_1) \cap o(P) = \emptyset$.

The edges of Q_1 are subsegments of edges in $P \cup Q$ and for any such edge, we must have both sides in $i(P)$ (because the outside part is in R, hence in $i(P)$ and the inside part is disjoint from $o(P)$). Therefore, these edges must be subsegements of edges of Q which forces $Q = Q_1$ and in this case, Q is inside $\overline{i(P)}$ which is a contradiction. Therefore, R must be inside Q_1 and in this case, by connectedness of $R, \partial R = Q_1$ is a polygon.

Step 3:

So the components of $i(P) \setminus \overline{i(Q)}$ lie inside their polygonal boundaries whose vertices come from V. Since V is finite, there can be only finitely many polygons, say $R_1, \ldots, R_{m-1}$. The inside regions of R_i, R_j cannot intersect for $i \neq j$ because they would then describe the same component. By construction of R_i, the inside regions of R_i, Q cannot intersect. Set $R_m = Q$, then the collection $\{R_1, \ldots, R_m\}$ has shallow intersection. It is clear that $\overline{i(P)} \cup \overline{i(Q)} = \overline{i(R_1)} \cup \cdots \cup \overline{i(R_m)}$ and that the edges of R_i are subsegments of the original edges. $\square$

In the proof, we partitioned $i(P) \setminus \overline{i(Q)}$ into polygonal regions we call this the *reduction* of P by Q.

Theorem 4.0.3. *Let $P_1, \ldots, P_n$ be polygons, set $V = V(\{P_1, \ldots, P_n\})$. Suppose $P_i \not\subset \overline{i(P_j)}$ for $i \neq j$. Then $\overline{i(P_1)} \cup \cdots \cup \overline{i(P_n)}$ is a shallow union of $\overline{i(R_1)}, \ldots, \overline{i(R_m)}$ for some polygons $R_1, \ldots, R_m$ with vertices coming from V and edges subsegement of the original edges.*

Proof. By the previous theorem, the statment is true for $n = 2$. Assume $n \geq 3$ and that the statment is true for $n - 1$. First reduce P_1 by P_2 to obtain polygons $R_1^2, \ldots, R_k^2$. The set of new vertices is a subset of the original and the collection $R_1^2, \ldots, R_k^2, P_2$ has shallow intersection.

Ignoring any R_i^2 for which $R_i^2 \subset \overline{i(P_3)}$, which means $i(R_i^2) \subseteq i(P_3)$ we take $R_i^2 \not\subset \overline{i(P_3)}$. By construction, $\overline{i(R_i^2)} \subset \overline{i(P_1)}$, so $P_3 \not\subset \overline{i(R_i^2)}$. Reduce each R_i^2 by P_3 to obtain polygons $R_{i1}^{23}, \ldots, R_{ik_i}^{23}$ such that $i(R_{ij}^{23}) \subset i(R_i^2)$. Since $R_1^2, \ldots, R_k^2$ had shallow intersection, the collection $\{R_{ij}^{23} | 1 \leq i \leq k, 1 \leq j \leq k_i\}$ has shallow intersection. In fact, these R_{ij}^{23} also have shallow intersection with P_2, P_3.

Continuing this way we next reduce each R_{ij}^{23} by P_4 and so on to arrive at a collection $R_1, \ldots, R_p$ each contained in $\overline{i(P_1)}$ having shallow intersection among themselves and with $P_2, \ldots, P_n$. At each stage the union $\overline{i(P_1)} \cup \cdots \cup \overline{i(P_n)}$ is preserved and the set of vertices is a subset of the original.

Using induction hypothesis, obtain polygons $R_{p+1}, \ldots, R_m$ with vertices from $V(\{P_2, \ldots, P_n\}) \subset V$ and shallow intersection such that $\overline{R_{p+1}} \cup \ldots \overline{i(R_m)} = \overline{i(P_2)} \cup \cdots \cup \overline{i(P_n)}$.

At each stage above, the set of vertices is a subset of the original V and the edges are subsegments of the original edges. Together $R_1, \ldots, R_m$ satisfy the requirements of the statement, proving the theorem for n and by induction for every n. □

Remark. Since the set of new vertices is a subset of the original V and all edges are subsegments of the original edges, the zones around each new vertex doesn't change from the original. As a consequence, if $\{P_1, \ldots, P_n\}$ is regular, then so is the new collection of polygons (the common outside does not change from the original).

Given a collection of polygons $\mathcal{P}$, the *intersection graph* is a graph with a vertex v_P for every $P \in \mathcal{P}$ and an edge between v_P, v_Q if $\overline{i(P)} \cap \overline{i(Q)} \neq \emptyset$. If there is an edge between v_P, v_Q, then there is a path from any point in $\overline{i(P)}$ to any point in $\overline{i(Q)}$ that lies entirely in their union.

Theorem 4.0.4. *Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a regular collection of polygons with shallow intersection and connected intersection graph. Then all components in the complement of $\mathcal{P}$ have polygonal boundaries.*

Proof. Since $\mathcal{P}$ has shallow intersection, for every $j, i(P_j)$ is unaffected by the presence of other polygons, so it stays connected. If R is a component in the complement of $P = P_1 \cup \cdots \cup P_n$ that intersects some $i(P_j)$, then $R = i(P_j)$ for $i(P_j)$ is connected and there are no paths to points outside. Clearly $i(P_j)$ has a polygonal boundary.

Let R be a component on the outside and let $V = V(\mathcal{P})$. For $x \in \partial R \backslash V$, say from some edge e, the zone has two sectors. Since $x \in \partial R$, one of these sectors lies in the common outside. The other sector lies inside some P_i (that which e is a part of). Reasoning as before, there is an open segment around x in e that is part of ∂R.

For $x \in \partial R \cap V$ its zone has exactly one sector lying in R by regularity. As before, two radii at v are part of ∂R. We conclude that ∂R is a union of disjoint polygons.

Suppose Q_1, Q_2 are two boundary polygons, then they are disjoint, so either $Q_1 \subset i(Q_2)$ or $Q_1 \subset o(Q_2)$. In the first case R must lie between Q_1, Q_2 (although there may be other boundary polygons in between as well) and this contradicts the connectedness of the intersection graph because a path from inside of Q_1 (hence inside some P_i) to outside of Q_2 must pass through R (which lies in the common outside of $\mathcal{P}$). In the second case, R must lie outside both Q_1, Q_2 again contradicting the path connectedness of the intersection graph (this time there is no path from inside Q_1 to inside Q_2 which doesn't pass through R). Therefore R must have exactly one polygon in its boundary. □

Remark. Observe that the boundaries of components outside $\mathcal{P}$ are polygonal regardless of whether $\mathcal{P}$ has a shallow intersection.

5. Some more results about polygons

Let a, b be points in the plane, l_1, l_2, l_3 be polygonal arcs from a to b that intersect only at the ends. Using two of the three arcs, we form polygons P_1, P_2, P_3, where P_i doesn't use l_i. Let n_1, n_2, n_3 be the corresponding parity functions, calculated using a direction not parallel to any of the edges in l_1, l_2, l_3.

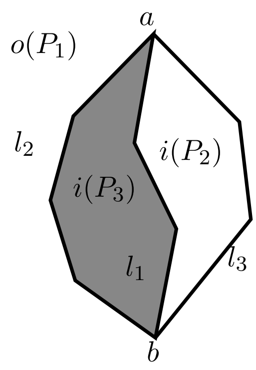

FIGURE 5.1.

Lemma 5.0.1. $n_1 + n_2 + n_3 = 0$.

Proof. Take $x \notin L = l_1 \cup l_2 \cup l_3$ and let R_x be the ray originating from x used to compute the parity.

Suppose R_x passes through a. At a, take a zone U small enough to avoid x. Denote by r_1, r_2, r_3 the radii in U induced by l_1, l_2, l_3 respectively. The diameter d induced by R_x cuts U in half. If two of the three radii, say r_1, r_2, lie on the same side then the contribution by a is 0 to n_3 and 1 to n_1, n_2. Otherwise all three radii are on the same half and the contribution to each n_i is 0. In both cases the contribution to the sum is 0. The same holds for b.

If p is a point other than a, b, then it appears in exactly one of l_1, l_2, l_3 and its zone has two sectors. Since each l_i is part of two polygons, the contribution from p is counted twice in the sum $n_1(x) + n_2(x) + n_3(x)$. It follows that $n_1 + n_2 + n_3$ is identically zero. □

As a consequence, every point in the complement of L is either inside exactly two polygons or outside all three. In the zone of a, there are three sectors and it is easy to see that one of the three sectors, say the one bounded by r_2, r_3, must lie outside all P_i. Then the radius r_1 lies inside P_1 and by connectedness, l_1 excluding a, b, lies inside P_1. So, some $l_i \setminus \{a, b\}$ is inside P_i.

Suppose l_1 lies inside P_1. Bound all the polygons in a large square and take a point p outside this square, so $p \in o(P_1) \cap o(P_2) \cap o(P_3)$. For $x \in o(P_1)$, there is a path from x to p lying outside P_1. Since l_1 is inside P_1, this path cannot intersect l_1, hence P_2, P_3. It follows that $x \in o(P_2) \cap o(P_3)$. Therefore, $o(P_1) \subseteq o(P_2) \cap o(P_3)$ and $n_1(x) = 0 \Rightarrow n_2(x) = n_3(x) = 0$.

Thus there are three components $o(P_1), i(P_2), i(P_3)$ in the complement of L with boundaries P_1, P_2, P_3 respectively. The zones at a, b have three sectors, one for each component. For points in l_1 other than a, b both sectors are in $i(P_1)$.

Corollary 5.0.1. *With the setting as above, if $\phi\colon [0,1] \to \mathbb{R}^2$ is a path with $\phi(0) \in o(P_1), \phi(1) \in i(P_2)$ that doesn't intersect $i(P_3)$, then it intersects l_3.*

Proof. Because $o(P_1) \subseteq o(P_2), \phi \cap P_2 \neq \emptyset$. Suppose $\phi \cap l_3 = \emptyset$, then it must intersect $l_1 \setminus \{a, b\}$. Let $t_1 = \inf\{\phi^{-1}(\phi \cap l_1)\}$. Since $\phi(0) \notin i(P_1)$, we have $t_1 \neq 0$. Let U be a zone of $\phi(t_1) \in l_1$, by continuity there is an interval (t_0, t_2) such that $\phi((t_0, t_1)) \subset U$.

One of the sectors of U is in $i(P_3)$ and the other in $i(P_2)$. For $t_0 < t < t_1$, by minimality of $t_1, \phi(t)$ cannot lie on the radii in U. By hypothesis, $\phi(t)$ cannot lie in the sector contained in $i(P_3)$. So, $\phi(t)$ is in the sector that is contained in $i(P_2)$ and we can find a $t' < t$ such that $\phi(t') \in P_2$. Since $\phi \cap l_3 = \emptyset$ by assumption, $\phi(t') \in l_1 \setminus \{a, b\}$ contradiction minimaity of t_1. So, $\phi \cap l_3 \neq \emptyset$. □

Corollary 5.0.2. *If $l_1, \ldots, l_n$ are n polygonal paths between a, b that intersect only at a, b, then the complement of $L = l_1 \cup \cdots \cup l_n$ has $n-1$ bounded components and one unbounded.*

Proof. Induction. □

Lemma 5.0.2. *Suppose P, Q are polygons with $P \not\subset \overline{i(Q)}$. Suppose $Q \cap i(P)$ is a non empty, connected (open) path whose ends are different. Then we can reduce P by Q in the sense that $i(P) \setminus \overline{i(Q)} = i(R)$ for some polygon R.*

Proof. $L = \overline{Q \cap i(P)}$ is a polygonal arc between two points $a, b \in P, a \neq b$. Let L_1, L_2 be the two paths between a, b along P. So L_1, L, L_2 are three polygonal arcs that intersect only at the ends and we know that L (except for the ends) is contained inside P. Let $P_1 = L_1 \cup L, P_2 = L_2 \cup L$.

We have $i(P_1) \cap Q = \emptyset$ because $i(P_1) \cap Q \subset i(P) \cap Q \subset L$. So, $i(P_1)$ is connected in the complement of Q, hence inside or outside Q. Similarly $i(P_2)$ is inside or outside Q. Take $z \in L \setminus \{a, b\}$, then the zone of z has two sectors, one from each $i(P_1), i(P_2)$ and these two sectors should also come from $i(Q), o(Q)$. So, one of $i(P_1), i(P_2)$ is inside Q and the other outside. Assume $i(P_2) \subset i(Q)$, then

$$i(P) \setminus \overline{i(Q)} = (i(P_1) \cup i(P_2) \cup L) \setminus \overline{i(Q)} = i(P_1).$$

So, the reduction of P by Q is the polygon P_1. □

5.1. **Removing singular vertices.** Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a collection of polygons, $V = V(\mathcal{P})$. If $v \in V$ is a singular vertex, with zone U, then there is more than one sector in U that comes from a component C outside $\mathcal{P}$.

Choose radii r_1, r_2 such that all sectors of U lying in C lie in a sector S determined by r_1, r_2. Let S to be minimal in the sense that no smaller sector in S contains all the sectors of U that intersect C. Then the sectors in S that have r_1 or r_2 in their boundary must lie in C. We note $r_1 \neq r_2$ and that they cannot lie inside any P_i.

By assumption $S \not\subseteq C$, so it contains some radii of U. In fact, there must be at least two radii in S as if there is only one radius s, then S has two parts - sectors sr_1, sr_2. By the assumptions on C, v at least two of the sectors in S must be in C outside $\mathcal{P}$, which means that both sides of s are outside $\mathcal{P}$ which is impossible.

As we go from r_1 to r_2 through S, let s_1, s_2 be the first and second radii we meet before reaching r_2. The sector $r_1 s_1$ in S must lie in C, therefore outside $\mathcal{P}$ and the sector S' determined by s_1, s_2 must lie in some $i(P_i)$ because s_1 cannot have both its sides outside $\mathcal{P}$.

Let a, b be the midpoints of s_1, s_2 respectively. Let l be a polygonal path from a to b that lies entirely in S'. Form the polygon $Q = av \cup l \cup bv$.

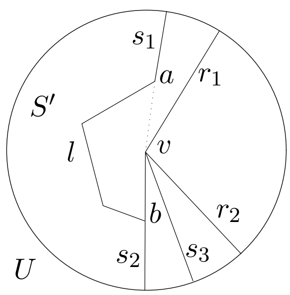

FIGURE 5.2. Line bv may also be removed, av is removed

For each i, by the choice of s_1, s_2 either $S' \subset \overline{i(P_i)}$ or $S' \subset o(P_i)$. Therefore, the intersection $Q \cap i(P_i)$ is either empty or the open path l or the union $l \cup bv$ in case s_2 is inside P_i. In the last two cases, the end points of the intersection are different, so we may apply Lemma 5.0.2 to reduce P_i by Q and obtain a polygon P_i' ($= P_i$ when $Q \cap i(P_i) = \emptyset$).

One side of av is outside $\mathcal{P}$ and the other in $i(Q)$, so av is outside all P_i', hence $i(Q)$ is now part of C. The set $V(P_1', \ldots, P_n')$ includes the vertices of Q in addition to the original V. At the zones of $a(b)$ we have three radii from $l, s_1(s_2)$. So, a, b are regular. The other points of Q other than v are regular as they have only two radii from l in their zones.

Lastly, note that the loss of interior from P_i is just $i(Q)$. So, anything not in $i(Q)$ covered by $\mathcal{P}$ is still covered by $\overline{i(P_1')} \cup \cdots \cup \overline{i(P_n')}$. Therefore, if w is a regular vertex outside U, it cannot become singular upon these reductions.

Now, shrink U to avoid l. The sectors contained in C still lie in S. Since av is removed, the number of radii between r_1, r_2 has reduced by at least 1. Repeating the steps above, we make sure that S has no other radius in between, giving us one sector in U that intersects C. Repeating this process for other sectors of U makes v a regular point.

The process terminates as there are finitely many radii and we are left with one less singular point. Modify $\mathcal{P}$ in a finite number of steps so that the resulting V has no singular points. Note that at each stage the cardinality of $\mathcal{P}$ does not change.

5.2. **Jordan-like results.** Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a regular collection of polygons with shallow intersection and a connected intersection graph. In this case, $V = V(\mathcal{P})$ is the collection of vertices in $\mathcal{P}$. Let C be a bounded component outside $\mathcal{P}$ with polygonal boundary R (Theorem 4.0.4).

Suppose $u_1, u_3 \in R$ are distinct points. Let L_1, L_2 be the two paths along R from u_1 to u_3. Fix polygons $R_1, R_3 \in \mathcal{P}$ that contain u_1, u_3 respectively. We may have $R_1 = R_3$. Traversing L_1 from u_1 to u_3 gives each $p \in L_1$ a backward edge and a forward edge. In the zone of p, these edges determine radii b, f respectively and two sectors one of which is inside R.

All other radii are in the other sector S. As we go from b to f through S, form the sequence of sectors that lie in $i(P)$ for some $P \in \mathcal{P}$. By shallow intersection, such P are unique. If this sequence has just one term, then points on b, f also have the same sequence, so an open segment of R around

p also has the same associated sequence. The ends of a maximal such segment must lie in V.

So the only $p \in L_1$ that can have more than one element in their associated sequence of polygons are those from $V(R) = V \cap R$, a finite set. Now start at the sector corresponding to R_1 at u_1 and go through the sequence of polygons associated to points in $V(R) \cap L_1$ along L_1 in the manner described, backward edge to forward edge outside R, till we reach the sector corresponding to R_3 at u_3. This sequence we call $S(L_1)$. Similarly define $S(L_2)$ starting at u_1 and going to u_3.

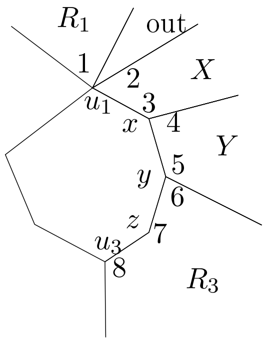

FIGURE 5.3. Obtaining $S(L_1)$

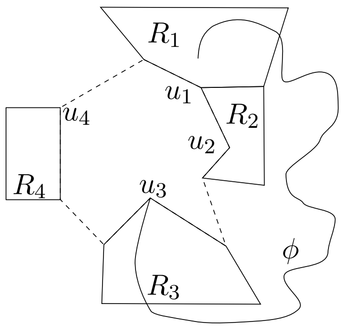

FIGURE 5.4. Set up for Theorem 5

For example, in the figure above, the sequence $S(L_1)$ will contain the sectors numbered in order, i.e., the sectors $1, 2$ at u_1 (skipping the sector lying outside the cover) then sectors $3, 4$ at x and so on till sector 8 at u_3.

Suppose we can choose polygons $R_2 \in S(L_1), R_4 \in S(L_2)$ different from R_1, R_3 (we can have $R_2 = R_4$) and points $u_2 \in L_1 \cap V(R), u_4 \in L_2 \cap V(R)$ that are part of R_2, R_4 respectively. Assume u_1, u_2, u_3, u_4 are distinct.

Let ϕ be a path (continuous image of $[0, 1]$) from a point in $\overline{i(R_1)}$ to one in $\overline{i(R_3)}$ that lies outside R, except possibly for the ends and if an end does lie on R, then it is u_1 or u_3. Let ψ be a path from a point in $\overline{i(R_2)}$ to one in $\overline{i(R_4)}$ with similar conditions. Furthermore, assume

- ϕ doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ with the exception of u_1, u_3
- ψ doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$ with the exception of u_2, u_4

Theorem 5.2.1. *The paths ϕ, ψ should intersect.*

Proof. Suppose they do not intersect. Assume $\phi(0) \in \overline{i(R_1)}, \phi(1) \in \overline{i(R_3)}$. If $\phi(0) \neq u_1$, let l_1 be a polygonal arc from u_1 to $\phi(0)$ that, except for the ends, lies in $i(R_1)$. Note that l_1 doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ except possibly at u_1. Extend ϕ by travelling along l_1 from u_1 till the first time we meet ϕ and from there along ϕ. If $\phi(1) \neq u_3$, extend ϕ using a similar polygonal arc l_3 from u_3 to $\phi(1)$.

Since the open segments of l_1, l_3 lie inside R_1, R_3, they cannot intersect ψ. So, ψ cannot intersect the extension of ϕ, which we continue to denote by ϕ. Even after the extension, ϕ lies outside R except for the ends and the ends are u_1, u_3 because l_1, l_3 can intersect R only at u_1, u_3 respectively. Furthermore, ϕ continues to not intersect $\overline{i(R_2)}, \overline{i(R_4)}$ except possibly at u_1, u_3.

Let $d > 0$ be the distance between (the extended) ϕ and ψ and $\epsilon = \min\{d, |u_1 - u_3|\}$. Around u_1, take a zone of radius $< \epsilon/2$ and a radial line r_1 from u_1 to a point $x \in \phi$ in this zone. Note that $x \neq u_3$ is outside R and does not lie in any sector corresponding to $\overline{i(R_2)}, \overline{i(R_4)}$. So, the line r_1 doesn't intersect $\overline{i(R)}, \overline{i(R_2)}, \overline{i(R_4)}$ except at u_1. Similarly, take a line r_3 from u_3 to a point $y \in \phi$ that does't intersect $\overline{i(R)}, \overline{i(R_2)}, \overline{i(R_4)}$ except at u_3.

Let ϕ' be the restriction of ϕ between x, y. Note that $x \neq y$ by the choice of ϵ. We know that ϕ' lies outside R and is at a distance of some $\mu > 0$ from the closed set $\overline{i(R_2)} \cup \overline{i(R_4)}$. Approximate ϕ' within $\min\{\epsilon, \mu\}$ by a polygonal arc P that lies outside R.

Let Q be the polygonal arc $r_1 + P + r_3$ from u_1 to u_3. We make the following observations

- Q lies outside R except for the ends u_1, u_3 which lie on R
- Q doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ except possibly at u_1, u_3. In particular $Q \cap i(R_2) = Q \cap i(R_4) = \emptyset$
- By the choice of ϵ, ψ doesn't intersect r_1, r_3 and P, hence it doesn't intersect Q

Extend ψ similar to ϕ using polygonal arcs lying in $i(R_2), i(R_4)$ to get a path from u_2 to u_4. Since Q doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ and u_i are distinct, Q doesn't intersect this extension of ψ.

The paths L_1, L_2, Q are all polygonal arcs between u_1, u_3 that intersect only at the ends and Q lies outside $L_1 \cup L_2$. We may assume that L_2 is inside $L_1 \cup Q$. When looking at L_1, L_2, Q, ψ, the zone of u_4 has two parts, one lying in $i(R)$ and the other in $i(L_2 \cup Q)$, and ψ enters the second sector. Similarly, u_2 has two sectors in its zone and ψ enters that sector lying outside $L_1 \cup Q$ (the other is inside R). By Corollary 5.0.1, ψ should intersect Q which is a contradiction. □

The main idea of the proof is to use Corollary 5.0.1. We can extend the result using a few assumptions. If ϕ doesn't have u_1 as an end for example, then the extension of ϕ must start at u_1 and enter $i(R_1)$. At the same time, if ψ doesn't have u_2 as an end, then the extension of ψ must start at u_2 and enter $i(R_2)$. It is easy to see that, in this case, we can deal with $u_1 = u_2$, because of the order in $S(L_1)$ which allows the use of Corollary 5.0.1. Similarly, we can have (observe that u_1, u_3 can be interchanged and intuitively this is just turning the diagram upside down)

- $u_1 = u_4$ when ϕ doesn't have u_1 as an end and ψ doesn't have u_4 as an end
- $u_3 = u_2$ when ϕ doesn't have u_3 as an end and ψ doesn't have u_2 as an end etc.

An extreme case is $u_1 = u_2 = u_4$ when ϕ doesn't have u_1 as an end and ψ doesn't have u_2, u_4 as its ends.

However, the proof doesn't apply directly to the case when all u_i are the same, because to construct L_1, L_2 we need $u_1 \neq u_3$. Although, such an extension can be similarly proved by looking at the sequence of polygons at u_1 and assuming that the R_i come in the appropriate order : R_1 between R_4, R_2 and R_2 between R_1, R_3. We will also need ϕ, ψ to be outside R. If they don't intersect, we obtain Q as before, and this time it is directly a polygon. At u_1 we have two sectors determined by Q and by the order forced on R_i, we see that ψ is a path from a point in one of these sectors to the other, i.e., a path from $i(Q)$ to $o(Q)$. So it must intersect Q, a contradiction.

Remark. Observe that we are close to having a version of the Jordan curve theorem because ϕ above corresponds to a curve and ψ is a path which we have forced to go from "inside" ϕ to outside and we have shown that ψ must intersect ϕ.

6. A theorem about arcs

Suppose we have points a, b in the plane and arcs ϕ_1, ϕ_2 from a to b that intersect only at a, b. Set I_1, I_2 to be $[0, 1]$ and suppose $\phi_i \colon I_i \to \mathbb{R}^2$ with $\phi_i(0) = a, \phi_i(1) = b, i = 1, 2$. Since ϕ_1 is a homeomorphism, ϕ_1 and its inverse are uniformly continuous. Fix $\epsilon > 0$, and choose $\delta > 0, \epsilon > \epsilon' > 0$ such that

$$|t_1 - t_2| < 3\delta \Rightarrow |\phi_1(t_1) - \phi_1(t_2)| < \epsilon$$
$$|\phi_1(t_1) - \phi_1(t_2)| \leq \epsilon' \Rightarrow |t_1 - t_2| < \delta.$$

6.1. **A special covering.** At a, b take open squares with disjoint closures and diameter $< \epsilon'$. For $t \in (0,1)$, pick an open square of diameter $< \epsilon'$ around $\phi_1(t)$ whose closure doesn't intersect ϕ_2. By compactness of ϕ_1 obtain a finite open subcover $\mathcal{P} = \{P_1, \ldots, P_n\}$.

We refine this cover in steps. At each step we ensure that there are unique polygons T_1, T_2 such that $a \in \overline{i(T_1)}, b \in \overline{i(T_2)}, \overline{i(T_1)} \cap \overline{i(T_2)} = \emptyset$ and if $\phi_2 \cap \overline{i(P)} \neq \emptyset$, then $P = T_1$ or $P = T_2$. This is true for $\mathcal{P}$.

(1) First we remove singular vertices (5.1) in the cover $\mathcal{P}$. A point $v \in \phi_1$ is inside some polygon $\tilde{P}$ which means that a zone of v has no sectors outside $\mathcal{P}$, hence v is regular. If $v \notin \phi_1$ is singular, then we choose a zone that avoids ϕ_1. The modifications to $\mathcal{P}$ while making v regular involves removing a part of this zone from each $P \in \mathcal{P}$, therefore even after these modifications ϕ_1 is covered by the resulting polygons (and their insides).

    If the original collection was $\{P_1, \ldots, P_n\}, T_1 = P_1, T_2 = P_n$, then each P_i is replaced (as detailed in 5.1) with a $P_i', \overline{i(P_i')} \subseteq \overline{i(P_i)}$. It is easy to see that $T_1 = P_1', T_2 = P_n'$ after making v regular. Similarly remove other singular vertices. At each stage we can find suitable T_1, T_2.

(2) Remove redundant polygons to arrive at a minimal cover $\{P_1, \ldots, P_n\}$ with $T_1 = P_1, T_2 = P_n$. Note that T_1, T_2 cannot be redundant. Now if $P_i \subset \overline{i(P_j)}$, then $i(P_i) \subset i(P_j)$ and P_i is redundant, so no $P_i \subset \overline{i(P_j)}$. Take the collection $P_2, \ldots, P_{n-1}$ and apply Theorem 4.0.3 to obtain a collection $R_1, \ldots, R_m$. Now $\overline{i(R_j)} \subseteq \overline{i(P_k)}$ for some $2 \leq k \leq n-1$, so $\overline{i(R_j)} \cap \phi_2 = \emptyset$.

(3) So, we have the collection $\{P_1, R_1, \ldots, R_m, P_n\}$. We cannot have $P_1 \subset \overline{i(R_j)}$ and remove any R_j for which $R_j \subset \overline{i(P_1)}$. So we may take $R_j \not\subset \overline{i(P_1)}, 1 \leq j \leq m$. Reduce (Theorem 4.0.2) each R_j by P_1 to get a collection $\{P_1, R_1', \ldots, R_s', P_n\}$.

(4) As in Step 3, reduce each R_j' by P_n and obtain $\{P_1, R_1'', \ldots, R_t'', P_n\}$. The sets $i(P_1), i(P_n)$ are unaltered and $\overline{i(P_1)} \cap \overline{i(P_n)} = \emptyset$. The union $\cup \overline{i(P_j)} \supset \phi_1$ is preserved and the collection has shallow intersection.

Furthermore, for any i, there are $j, k, 2 \leq l \leq n-1$ such that

$$\overline{i(R''_i)} \subseteq \overline{i(R'_j)} \subseteq \overline{i(R_k)} \subseteq \overline{i(P_l)} \Rightarrow \phi_2 \cap \overline{i(R''_i)} = \emptyset.$$

So, we can take $T_1 = P_1, T_2 = P_n$. For the sake of convenience let continue to call this collection $\mathcal{P} = \{P_1, \ldots, P_n\}$ and set $V = V(\mathcal{P})$.

Lastly, we make sure that if two polygons intersect, then the intersection contains points of ϕ_1. For polygons P, Q with shallow intersection, the intersection of edges $e \in P, f \in Q$ is either a point (short intersection), or a closed segment (long intersection). Henceforth short and long intersections refer to those that don't contain points of ϕ_1. This is a two step process.

First we make sure that long intersections have points of ϕ_1. Then we look at short intersections in $P \cap Q$ not part of any long intersection in $P \cap Q$. Henceforth, short intersections refer only to this specific type. The notion of short intersection now depends on the polygons P, Q for $e \cap f$ may be short in $P \cap Q$, but not in $P' \cap Q'$ for some P', Q' (see Figure 6.1). The zone of a short intersection $v \in P \cap Q$ has 2 radii each from P, Q.

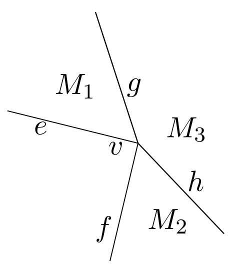

FIGURE 6.1. In the adjacent figure, the intersection $g \cap g$ is a long intersection. The vertex $v = e \cap f$ is a short intersection when considered as a point in $M_1 \cap M_2$, but the same vertex is not a short intersection in $M_1 \cap M_3$ and $M_2 \cap M_3$ because it appears in the long intersections $g \cap g, h \cap h$ respectively.

Removing long intersections:

Consider edges $e \in P, f \in P$, where $P, Q \in \mathcal{P}$ are arbitrary, and suppose $e \cap f$ is a long intersection. The ends of $e \cap f$ are in V and since one side of this segment lies in $i(P)$ and the other in $i(Q)$, by shallow intersection, there is no point of V in $e \cap f$ other than the ends.

Around $e \cap f$ take a rectangle R with one side parallel to e such that $\overline{i(R)}$ is disjoint from

- ϕ_1 and
- edges $e' \in \mathcal{P}$ with $e' \cap e \cap f = \emptyset$ and
- $V \setminus e \cap f$, so that $V \cap \overline{i(R)} = V \cap e \cap f \subset i(R)$.

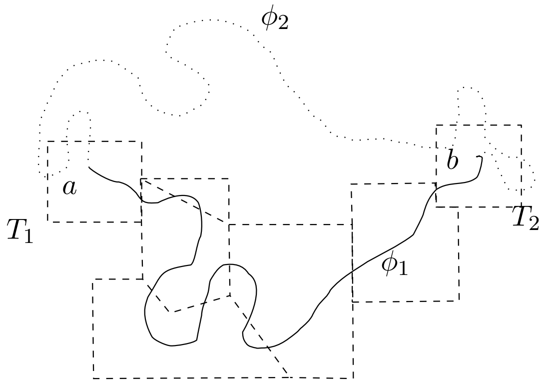

FIGURE 6.2. Example of a special covering

This is possible because $e \cap f$ is compact, so we take a finite cover by squares with one side parallel to e and then take a "minimum height" rectangle. The initial cover of $e \cap f$ is one that avoids $\phi_1 \cup (V \setminus (e \cap f))$ and edges of $\mathcal{P}$ that don't intersect $e \cap f$. Such a cover exists by the assumptions on $\mathcal{P}$ and $e \cap f$.

Now $V \cap R = V \cap e \cap f$ has 2 elements, so R does not contain any $\tilde{P} \in \mathcal{P}$. We have

$$e \cap f \subset i(R) \Rightarrow i(R) \cap i(P) \neq \emptyset, i(R) \cap i(Q) \neq \emptyset$$

so by shallow intersection $R \not\subseteq \overline{i(\tilde{P})}$ for any $\tilde{P} \in \mathcal{P}$. Reduce (Theorem 4.0.2) each $\tilde{P} \in \mathcal{P}$ by R giving us in place of $\tilde{P}$, a shallow union of some polygons. We show that each reduction gives one polygon.

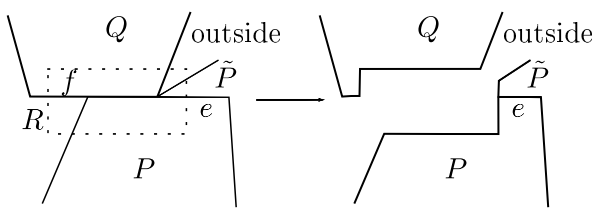

FIGURE 6.3. Removing long intersection $e \cap f$

Number of polygons doesn't change:

$P \cap i(R)$ is a connected path (involving 3 or fewer segments) and divides R into polygons R_1, R' with $i(R_1) \subset i(P)$. By shallow intersection $Q \cap i(R) \subset \overline{i(R')}$. By Lemma 5.0.2, $Q \cap i(R)$ divides R' into 3 or fewer polygons R_2, R_3, R_4 with $i(R_2) \subset i(Q)$ where R_3, R_4 may be degenerate, $\overline{i(R_3)} \cap \overline{i(R_4)} = \emptyset$. At each end of $e \cap f$, there is one sector each for R_1, R_2 and one for R_3 or R_4.

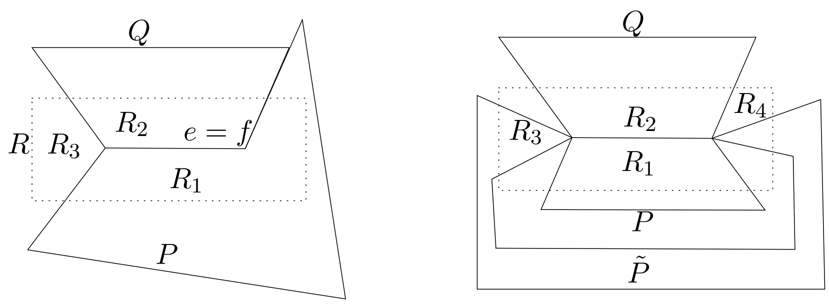

FIGURE 6.4. Examples of what R looks like. R_4 is degenerate in the first one

For $\tilde{P} \in \mathcal{P}$, if $\tilde{P} \cap \overline{i(R)} = \emptyset$ the reduction by R doesn't change $\tilde{P}$, so assume $\tilde{P} \cap \overline{i(R)} \neq \emptyset$, then we must have $\tilde{P} \cap e \cap f \neq \emptyset$. If $e \cap f \subset \tilde{P}$, by shallow intersection $\tilde{P} = P$ or $\tilde{P} = Q$ and it is easy to see that $R \cap i(\tilde{P})$ is a connected path.

For $\tilde{P} \neq P, Q$, $\tilde{P} \cap e \cap f$ contains only the ends of $e \cap f$. At each point of $\tilde{P} \cap e \cap f$, there are two segments induced by $\tilde{P}$ and both must be in $\overline{i(R_3)}$ or $\overline{i(R_4)}$ by shallow intersection. The reduction of $\tilde{P}$ by R happens in two steps - by R_3, R_4 separately. Observe that $R_3 \cap i(P), R_4 \cap i(\tilde{P})$ are connected in the step-wise reduction. After reducing by R_3 for example, $R_4 \cap i(\tilde{P})$ doesn't change.

In both cases, by Lemma 5.0.2, the number of polygons doesn't change. Each $\tilde{P} \in \mathcal{P}$ is replaced by a P' forming a collection $\mathcal{P}'$. Similar to step 1, we can find T_1, T_2.

Regularity:

$\mathcal{P}'$ covers ϕ_1 because $\overline{R} \cap \phi_1 = \emptyset$ and has a shallow intersection because the interiors have shrunk. So $V(\mathcal{P}') = V'$ is the collection of vertices in $\mathcal{P}'$ and consists of points of R that lie in $\overline{i(P_j)}, P_j \in \mathcal{P}$ and points of V outside R. As a consequence $i(R)$ is now in the common outside.

- Because R can intersect only those edges that intersect $e \cap f$, any edge intersecting R must go inside R. If $x \in R$ lies on an edge of P_i, then $x \notin V$ by construction of R. Its original zone had one diameter and now contains a radius on either side introduced by R. After reducing P_i by R, the sector that was in $i(R)$ is now in the common outside and the new zone has three sectors, with two radii from R and one from half of the original diameter.
- If $x \in R$ lies inside P_i, then its original "zone" (a ball around x that lies inside the polygon will do) had no radii and the new zone has two.

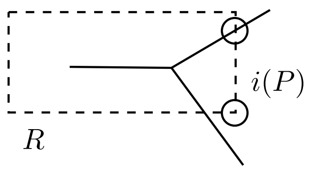

FIGURE 6.5. After reducing by R, the vertices introduced by R are regular.

The other points of V' are points of V outside R. If x is such a point, then the zone of x is unaltered because the line segments passing through x are unaltered (for they are outside R). Note that any loss of interiors comes from $i(R)$, so the sectors at x do not change. So, $\mathcal{P}'$ is regular.

<u>Number of long intersections:</u>

Every long intersection in $\mathcal{P}'$ must come from a long intersection in $\mathcal{P}$ because the new edges of $\mathcal{P}'$ lie inside polygons of $\mathcal{P}$ and cannot have long intersections by shallow intersection of $\mathcal{P}$.

Suppose e_1, f_1 are two edges in $\mathcal{P}$ such that $e_1 \cap f_1$ is long. If $e_1 \cap f_1 \subset i(R)$, then by the choice of R, we have $e_1 \cap f_1 = e \cap f$, which is removed. If $e_1 \cap f_1 \subset o(R)$, it is unaffected and is a long intersection in $\mathcal{P}'$. Lastly, if $e_1 \cap f_1$ enters R but not contained in it, then one end of $e_1 \cap f_1$ is an end of $e \cap f$ and the other is outside R. Because $i(R)$ is a convex shape, $\overline{e_1 \cap f_1 \setminus i(R)}$ is a connected closed segment, so $e_1 \cap f_1$ gives rise to exactly one long intersection in $\mathcal{P}'$.

Thus, the number of long intersections has decreased by 1. Repeating this process, we arrive at a collection $\mathcal{P}$ such that $\phi_1 \subset \cup_{P \in \mathcal{P}} \overline{i(P)}$ with shallow intersection and every long intersection of edges having points from ϕ_1. At each step, we can find T_1, T_2.

<u>Removing short intersections:</u>

Suppose $v = e \cap f$ is a short intersection for edges $e \in P, f \in Q$. Let U be a zone of $v \in V$ that avoids ϕ_1. In U, the sectors that lie inside P, Q are disjoint and since $v \in V$ one of them must be convex. Without loss of generality, assume the sector in $i(P)$ bounded by radii r_1, r_2 is convex. Join the midpoints of r_1, r_2 to get a triangle W contained inside $\overline{i(P)}$.

Reducing $\tilde{P} \in \mathcal{P} \setminus \{P\}$ doesn't change it as $\overline{i(W)} \cap i(\tilde{P}) = \emptyset$. Reducing P by W gives it two vertices in place of v. As before, these vertices and those outside U are regular. The component $i(W)$ is either a new component outside, or it merges with some other component outside all $\tilde{P}$ (depending on whether the other side of r_i is inside or outside $\mathcal{P}$). In either case, v stays regular. So, after reduction, points in $V(\mathcal{P}) \cup V(W)$ are regular.

Thus, upon reducing $\mathcal{P}$ by W, we have a regular polygonal cover (by the choice of U) of ϕ_1 with shallow intersection. Moreover, any long intersection is a subsegment of one in $\mathcal{P}$ and, by the choice of U, contains a point of ϕ_1.

The zones of a, b have two or three radii, so they cannot be short intersections. Moreover, v is no longer a vertex of the reduced P as there is a ball around v that doesn't intersect $i(P) \setminus \overline{i(W)}$. Thus, the number of short intersections has reduced by 1 as $v \in P \cap Q$ is no longer an intersection in the new collection. Again, since the number of polygons hasn't changed, we can find T_1, T_2.

This process terminates and we obtain a regular $\mathcal{P}$ containing ϕ_1 in its shallow union and if $P \cap Q \neq \emptyset, P, Q \in \mathcal{P}$, then $P \cap Q$ (in fact, every component of $P \cap Q$) contains a point of ϕ_1. Furthermore, there are polygons T_1, T_2 as described in the beginning.

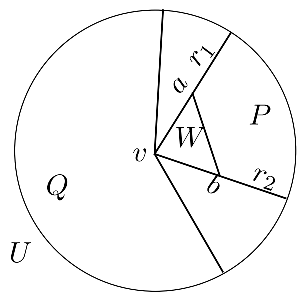

FIGURE 6.6. Removing short intersection $v \in P \cap Q$

6.2. **Consequence of the special covering.** Suppose we have the covering $\mathcal{P} = \{P_1, \ldots, P_n\}$ as above and C is a bounded component outside $\mathcal{P}$.

Let $Q_1, \ldots, Q_k \in \mathcal{P}$ be the polygons that intersect the polygon $R = \partial C$. To each Q_i, by minimality of the cover, we have the interval $[a_i, b_i] \subseteq I_1$ where a_i is the first time ϕ_1 intersects $\overline{i(Q_i)}$ and b_i last. By the refinement in the last subsection, if $Q_i \cap Q_j \neq \emptyset$, then $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$. We started with a cover where the diameter was less than ϵ' and refined this cover. At each step of the refinement the diameter never increased, so $b_i - a_i < \delta$ (assumptions made at the start of section 6).

Suppose in $\mathcal{P}$, the polygon P_1 contains a and P_n contains b, i.e., P_1 plays the role of T_1 and P_n that of T_2. We know that $\overline{i(P_1)} \cap \overline{i(P_n)} = \emptyset$ and $\phi_2 \cap \overline{i(P_j)} = \emptyset, 2 \leq j \leq n-1$. Now, $\overline{i(P_1)} \cap \phi_2, \overline{i(P_n)} \cap \phi_2$ are two non empty (because they contain a, b) disjoint closed sets in ϕ_2. Because ϕ_2 is connected, there is an $s \in (0,1) \subset I_2$ such that $\phi_2(s) \notin \overline{i(P_1)} \cup \overline{i(P_n)}$, hence it is outside $\mathcal{P}$, in particular outside $Q_1, \ldots, Q_k$.

Set

$$\alpha = \min_{1 \leq i \leq k} a_i, \beta = \max_{1 \leq i \leq k} b_i.$$

Look at the path $\phi_1(\alpha) \xrightarrow{\phi_1} a \xrightarrow{\phi_2} \phi_2(s)$. The first part, i.e. $\phi_1(\alpha) \xrightarrow{\phi_1} a$ intersects $\cup_i \overline{i(Q_i)}$ only at $\phi_1(\alpha)$ by definition, hence intersects R at most once. If the second part, i.e., $a \xrightarrow{\phi_2} \phi_2(s)$, intersects R, hence any of the $\overline{i(Q_j)}$, then, by construction of $\mathcal{P}$, that $Q_j = T_1$ or T_2 and we must have $\alpha = 0$ or $\beta = 1$.

Since $\phi_2(s) \notin R$, we can obtain a subpath, or more properly sub-arc, not intersecting R, except possibly at one end. This arc, ψ_1, goes from some Q_i to $\phi_2(s)$. By the arguments in the preceding paragraph, ψ_1 can intersect only those $\overline{i(Q_j)}$ which contain $\phi_1(\alpha)$ or $\phi(\beta)$.

Next, using the path $\phi_1(\beta) \xrightarrow{\phi_1} b \xrightarrow{\phi_2} \phi_2(s)$, obtain an arc ψ_2 not intersecting R, except possible at one end and intersecting only those $\overline{i(Q_j)}$ that contain $\phi(\alpha)$ or $\phi_1(\beta)$. Then the arc $\psi = \psi_1 \cup \psi_2$ goes from a point in some $\overline{i(Q_i)}$ to one in $\overline{i(Q_j)}$ and intersects only those $\overline{i(Q_l)}$ that contain $\phi_1(\alpha)$ or $\phi_1(\beta)$.

Now assume $\phi_2(s) \in o(R)$. Then the path ψ (except for the ends) must also lie outside R by definition of ψ_1, ψ_2 for if they meet $i(R)$ then ψ_1 or ψ_2 must meet R.

Lemma 6.2.1. $\beta - \alpha < 3\delta$.

Proof. Choose polygons R_1, R_3 from $Q_1, \ldots, Q_k$ with associated intervals $[\alpha, \alpha'], [\beta', \beta]$ respectively where α', β' are chosen so that $\alpha' - \alpha, \beta - \beta'$ are

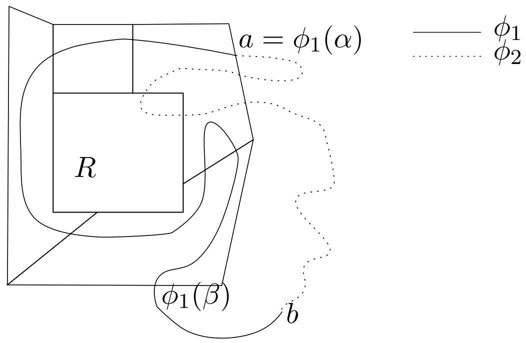

FIGURE 6.7. Set up for Lemma 7, only the necessary polygons are drawn

maximal. The ends of ψ are in these polygons, say $\psi(0) \in \overline{i(R_1)}, \psi(1) \in \overline{i(R_3)}$. If $\psi(0) \in R$, take $u = \psi(0)$, otherwise take u to be a vertex of R_1 in $V(R)$. Similarly, if $\psi(1) \in R$, take $v = \psi(1)$, otherwise take it to be a vertex of R_3 in $V(R)$.

If $u = v$, then these intervals should intersect (they may even be equal) and it follows that $\beta - \alpha < 2\delta < 3\delta$. So, assume that $u \neq v$ and that $[\alpha, \alpha'] \cap [\beta', \beta] = \emptyset$.

Let the two paths from u to v along R be L_1, L_2. Starting at u traverse the sectors in the sequence $S(L_1)$ (with ends R_1, R_3), till the first point in $V(R) \cap L_1$ that has a polygon R_2 with $[a_i, b_i], b_i > \alpha'$. Going along L_2, arrive at a point in $V(R) \cap L_2$ that has a polygon R_4 with interval $[a_j, b_j], b_j > \alpha'$.

By the maximality assumption, we must have $a_i, a_j > \alpha$. As we move sequentially from R_1, each polygon shares an edge or vertex with the previous one, so their intervals intersect. Therefore, we must have $a_i, a_j < \alpha'$. If $b_i \geq \beta'$ (or similarly $b_j \geq \beta'$), then

$$\beta - \alpha = \beta - b_i + b_i - a_i + a_i - \alpha < 3\delta.$$

So, assume $b_i, b_j < \beta'$ and without loss of generality, assume $b_i \leq b_j$.

For example, in Figure 6.8, we traverse the sectors from u to v according to the numbering, i.e., sectors $1, 2$ at u, then $3, 4$ at x. Sector numbered 4 is from the polygon Y whose associated interval is the first to go beyond $[\alpha, \alpha']$. Notice that in the sequence each polygon shares an edge or vertex with the previous one.

Let ϕ' be the restriction of ϕ_1 to $[b_i, b_j]$. By assumption (on b_i, b_j), ϕ' doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$. If $\phi' \cap L_1 \neq \emptyset$, then let x be the last time

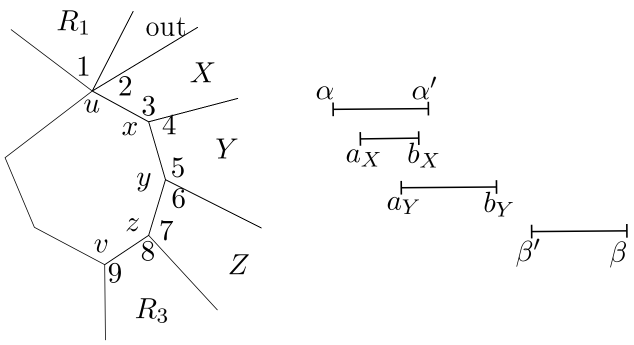

FIGURE 6.8. Example

(while going from b_i to b_j) that it hits L_1. Being closed, $\phi_1(x) \in L_1$ and by assumption $\phi_1(x) \neq u, v$. Otherwise, set $x = b_i$.

Next, if $\phi' \cap L_2 \neq \emptyset$, then set y to be the first time it hits L_2 while going from x to b_j. Again, $\phi_1(y) \in L_2$ and $\phi_1(y) \neq u, v$. Otherwise set $y = b_j$.

This way we obtain a path ϕ, given by the restriction of ϕ' to go from x to y, from a polygon R_2' with a vertex in L_1 to a polygon R_4' with a vertex in L_2, say $\phi(0) \in \overline{i(R_2')}, \phi(1) \in \overline{i(R_4')}$. Take $u_2 = \phi(0)$ if $\phi(0) \in R$, otherwise take it to be a vertex of R_2' in $V(R)$. Similarly, take $u_4 = \phi(1)$ if it is in R, otherwise take it to be a vertex of R_4' in $V(R)$. Note that R_2', R_4' are different from R_1, R_3 (because ϕ', hence ϕ, doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$). Observe

(1) Both ψ, ϕ lie outside R and the ends may lie on R or outside. We assumed ψ is outside, but ϕ is outside by definition of R. If the ends do lie on R, then it is u, v or u_2, u_4 respectively.
(2) By assumptions on b_i, b_j, ϕ doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$. In particular, ϕ cannot have u or v as its ends.
(3) ψ intersects only those $\overline{i(Q_i)}$ that contain $\phi_1(\alpha)$ or $\phi_1(\beta)$. Neither $\overline{i(R_2')}, \overline{i(R_4')}$ have these points by the maximality conditions on α', β' so ψ doesn't intersect $\overline{i(R_2')}, \overline{i(R_4')}$.
(4) By 3, if $\psi(0) = u$ ($\psi(1) = v$), then R_2', R_4' cannot have u (v) as a vertex.
(5) Lastly, note that ϕ, ψ do not intersect

We are in a possition to apply Theorem 5.2.1 and its extensions and we have a contradiction. Therefore, one of b_i, b_j is larger than β' and we conclude that $\beta - \alpha < 3\delta$. □

Theorem 6.2.1. *Suppose ϕ_1, ϕ_2 are two arcs meeting only at the ends a, b. Let B be a circle or a polygon such that $\phi_1 \cup \phi_2 \subset i(B)$. Suppose there is a point $c \in \phi_2 \setminus \{a, b\}$ with a path ϕ going from c to a point outside B such that $\phi \cap \phi_1 = \emptyset$. Then given any $x \notin \phi_1 \cup \phi_2$, there is a polygonal cover $\mathcal{P}$ (with each polygon inside B) of ϕ_1 such that x is in the unbounded component of $\mathbb{R}^2 \setminus \mathcal{P}$.*

Proof. Let $4\epsilon = \inf_{y \in \phi_1} |x - y| > 0$. With ϵ', δ defined as in the start of Section 6, around each point of ϕ_1 take open squares such that

- Each square has diameter $< \epsilon'$.
- The closure of inside of each square is inside B (possible because $\phi_1 \subset i(B)$).
- The closure of inside of each square is disjoint from ϕ (possible because $\phi \cap \phi_1 = \emptyset$), in particular c.
- a, b are in different squares whose interiors have disjoint closures.

From this cover obtain a finite subcover $\mathcal{S}$. Refine $\mathcal{S}$ and obtain a special covering $\mathcal{P}$. Now c is in the unbounded component of $\mathcal{P}$ because $o(B)$ is in the unbounded component of $\mathcal{P}$ and we have a path, namely ϕ, going from c to $o(B)$ avoiding $\mathcal{P}$. Notice that ϕ avoids the squares in $\mathcal{S}$ and $\mathcal{P}$ is obtained by shrinking these squares, so ϕ avoids $\mathcal{P}$ as well. It is clear that each polygon in $\mathcal{P}$ is inside B.

By the choice of ϵ, ϵ', the point x is outside $\mathcal{P}$. Suppose x is in a bounded component C (outside $\mathcal{P}$) with boundary polygon R. Continuing with the notation above, let $Q_1, \ldots, Q_k$ be the polygons surrounding R and $[a_i, b_i]$ be the interval associated to Q_i. By the lemma $\beta - \alpha < 3\delta$ where α, β are as defined above. Because c is in the unbounded component in the complement of $\mathcal{P}, c \in o(R)$ and the lemma is applicable.

Now, pass a line through x and let p_1, p_2 be the first time the two rays hit R (so x is between p_1, p_2). Since $x \in i(R)$, both rays must hit R. Suppose $p_1 \in Q_1$. Since Q_1 contains a point of ϕ_1 and has diameter less than ϵ', we know that p_1 is within ϵ' of some $q_1 \in \phi_1$. Similarly, p_2 is within ϵ' of some $q_2 \in \phi_1$. Since $\beta - \alpha < 3\delta$, we have $|q_1 - q_2| < \epsilon$ and

$$|p_1 - p_2| < \epsilon' + \epsilon + \epsilon' < 3\epsilon.$$

Since x is on the line segment between p_1, p_2, it is a convex linear combination of p_1, p_2. So,

$$|x-q_1| \leq |x-p_1|+|p_1-q_1| \leq |p_1-p_2|+|p_1-q_1| < 4\epsilon \Rightarrow 0 < \inf_{y\in\phi_1} |x-y| < 4\epsilon.$$

This contradicts the definition of 4ϵ, therefore x must be in the unbounded component in the complement of $\mathcal{P}$. □

7. Arcs in discs

Let $\overline{\mathbb{D}}$ be the open unit disc and $J\colon [0,1] \to \mathbb{R}^2$ be an arc such that $J(t) \in \mathbb{D} \forall t \in (0,1)$ and $J(0), J(1) \in S^1$. We will show that $\overline{\mathbb{D}} \setminus J$ has two components, determined by the arcs in $S^1 \setminus \{J(0), J(1)\}$, with common boundary J.

7.1. **Separation theorem.** Let A_1, A_2 be the two arcs of $S^1 \setminus \{J(0), J(1)\}$ and take $c \in A_1, d \in A_2$. Suppose there is a path ϕ from c to d in $\overline{\mathbb{D}}$ that avoids J. Let ϵ be the minimum distance between ϕ and J. Approximate ϕ by a polygonal path P within ϵ so that $P \subset \overline{\mathbb{D}}$ ($\overline{\mathbb{D}}$ is convex, so this is possible). Draw tangents at c, d to $S^1 = C$. If they do not intersect, then use a perpendicular line outside C to join them. This way, we get a polygonal path between c, d lying outside the circle. Together with P we have a polygon Q.

Using rays that go outside the circle, we conclude that $J(0), J(1)$ are on different sides of Q. More generally what we notice is that the two arcs determined by c, d are on different sides of Q, hence in particular $J(0), J(1)$ are on different sides. Since J is a path going from inside Q to the outside, it must intersect Q. Because J is inside the disc it must intersect P which is impossible by the choice of ϵ. Therefore, c, d, hence A_1, A_2, are in different components of $\overline{\mathbb{D}} \setminus J$.

7.2. **Exactly two components.** Let $x \in \mathbb{D} \setminus J$. Fix a $c \in A_1$ and take a normal to C going outwards at c. The arcs J, A_1 meet only at the ends (taking the closure of A_1). Applying Theorem 6.2.1 (taking B to be a circle of radius 2 centered at the origin for example), we obtain a polygonal cover $\mathcal{P}$ such that x is in the unbounded component in the complement of $\mathcal{P}$.

Then, there are paths from x to points far outside the circle that avoid $\mathcal{P}$ and hence J. Since $x \in \mathbb{D}$, any such path must pass through S^1 and therefore A_1 or A_2. Thus, x is in the same component of $\overline{\mathbb{D}} \setminus J$ as A_1 or A_2 and $\overline{\mathbb{D}} \setminus J$ has exactly two components.

7.3. **Boundary.** We have two components, C_1, C_2 corresponding to A_1, A_2 respectively. We will show that J is part of the boundary of C_1. First, $J(0), J(1) \in \partial C_1, \partial C_2$ because any neighbourhood of both these points intersects A_1, A_2 and $\partial C_1, \partial C_2$ are closed.

Lemma 7.3.1. *Let $R \subset \mathbb{R}^2$ with $Int(R), Ext(R) \neq \emptyset$, then any path from $Int(R)$ to $Ext(R)$ intersects ∂R.*

Proof. Suppose $\phi\colon [0,1] \to \mathbb{R}^2$ is a path from $x = \phi(0) \in Int(R)$ to $y = \phi(1) \in Ext(R)$. There is a neighbourhood around x that lies in $Int(R)$, hence a $t > 0$ such that $\phi([0,t)) \subseteq Int(R)$. Take

$$t_0 = \sup_{[0,1]}\{t : \phi([0,t)) \subseteq Int(R)\}.$$

If $\phi(t_0) \in Int(R)$, then $t_0 \neq 1$ and we can increase t_0. If $\phi(t_0) \in Ext(R)$, there is a $t < t_0$ such that $\phi((t,t_0)) \subseteq Ext(R)$. Both contradict the definition of t_0, hence $\phi(t_0) \in \partial R$. $\square$

By this lemma, we conclude that ∂R disconnects $Int(R)$ from $Ext(R)$.

Corollary 7.3.1. *Let $R \subset \mathbb{R}^2$ with $Int(R), Ext(R) \neq \emptyset$. Suppose there is a path from $x \in Int(R)$ to $y \in Ext(R)$ in $(\mathbb{R}^2 \setminus \partial R) \cup S$ for some subset S of the plane. Then $S \cap \partial R \neq \emptyset$.*

Suppose $x \in J$ is not in ∂C_1. Then there is a neighbourhood of x in J that is not in ∂C_1. We will show that adding this neighbourhood to $\overline{\mathbb{D}} \setminus J$ connects C_1, C_2. Essentially, we want to show that if $J' = J([0,1] \setminus (t_1, t_2))$, then $\overline{\mathbb{D}} \setminus J'$ is connected, where $0 < t_1 < t_2 < 1$.

Around each point of $J_1 = J([0,t_1])$ take open squares whose

- closures avoid $J_2 = J([t_2, 1])$ and
- for $0 < t \leq t_1$, the closure is contained in $\mathbb{D}$.

Obtain a finite subcover $\mathcal{S}$ of J_1. In $\mathcal{S}$, there is only one square that intersects S^1. We now remove singular vertices in such a way as to obtain a cover of J_1 by polygons $\mathcal{P}$ such that

- The boundary of polygons in $\mathcal{P}$ do not intersect J_2.
- Every point of J_1 lies inside at least one of the new polygons.
- Only one polygon intersects S^1 at exactly two points.

This is true for $\mathcal{S}$. Now the vertices that can be singular are inside C, as only one square in $\mathcal{S}$ goes outside, so points outside are regular. Since we are going to shrink the polygons, it is clear that the boundaries of the resulting polygons do not intersect J_2.

Given a singular vertex v, if $v \in J_1$, then take a zone that lies inside one of the polygons, else take a zone that avoids J_1. Ensure that the zone lies inside the circle C. Since the new edges and the consequent loss of the interiors happen inside this zone, the new edges do not intersect the circle and J_1 stays inside the polygons. The intersection with S^1 still has the original two points.

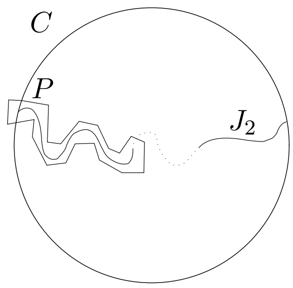

FIGURE 7.1. $J \subset \partial C_1 \cap \partial C_2$

Since J_1 is connected, the intersection graph of (the modified) $\mathcal{S}$ is connected. So the unbounded component of $\mathcal{S}$ has a polygonal boundary P which has edges going into the disc $\mathbb{D}$. Since every point of J_1 lies inside some $Q \in \mathcal{P}, P \cap J_1 = \emptyset$. By construction $P \cap J_2 = \emptyset$.

In $\mathcal{P}$ there is exactly one polygon that contains $J(0)$ and hence intersects A_1, A_2. We know that there are edges of P that go into $\mathbb{D}$, so these edges give a path from a point in A_1 to one in A_2 that lies entirely in $\mathbb{D}$ and avoids $J_1 \cup J_2$.

It lies entirely in $\mathbb{D}$ because $P \cap S^1$ has only two points. So, $\overline{\mathbb{D}} \setminus (J_1 \cup J_2)$ is connected, therefore $J((t_1, t_2)) \cap \partial C_1 \neq \emptyset$. This path along P must intersect $J((t_1, t_2))$ and the first point of intersection is then arcwise accessible from A_1.

We conclude that $J \subseteq \partial C_1, \partial C_2$ and that any open segment of J has a point accessible from A_1, i.e., an x with an arc from a point in A_1 to x that avoids $J \setminus \{x\}$.

8. Jordan curve theorem

Let J be a Jordan curve. Since J is bounded, it is inside a circle C. Take two different points on J and extend the line between them to a chord of the circle. We can talk of the "first" and "last" points on this chord, which corresponds to the points of J that lie farthest apart on this chord (such points exist because $J \cap \text{chord}$ is closed). Let these points be a, b and the ends of the chord c, d with a being between c, b and b between a, d. The line segments ca, bd are disjoint and $ac \cap J = \{a\}, bd \cap J = \{b\}$.

The curve J, being homeomorphic to S^1, gives us two arcs from a to b, call them ϕ_1, ϕ_2. Together with the segments ac, bd, we get two arcs

$$J_1 = ac \cup \phi_1 \cup bd; J_2 = ac \cup \phi_2 \cup bd$$

from c to d: they intersect only on the segements ac, bd. Let the arcs of $C \setminus \{c, d\}$ be A_1, A_2.

For $x \in ca, x \neq a, c$, there is a ball U around x that avoids J and the line segment bd. Because $U \cap (J_1 \cup J_2) = U \cap ac$ is a diameter in $U, U \setminus (J_1 \cup J_2)$ has two components. Around c, there is an open ball that avoids J, bd. This ball has three components of which one lies outside the circle C. We see that all three parts are connected in the complement of $J_1 \cup J_2$. Similar results hold true for points on bd different from b.

8.1. **At least two components.**

In $\overline{\mathbb{D}} \setminus J_1$ let the components of A_1, A_2 be C_1, C_2 respectively. Being connected in $\overline{\mathbb{D}} \setminus J_1$, ϕ_2 must lie in one of these components, say C_2. Now, a point of ϕ_1 is accessible from A_1 so this path, say ψ, therefore lies in C_1. So, $\psi \cap \phi_2 = \emptyset$ and $\psi \cap J_2 = \emptyset$ giving us $\psi \subset \overline{\mathbb{D}} \setminus J_2$.

In $\overline{\mathbb{D}} \setminus J_2$, let the components corresponding to A_1, A_2 be D_1, D_2 respectively. Since ψ is a path in $\overline{\mathbb{D}} \setminus J_2$, a point of ϕ_1 is in D_1 via ψ. We conclude that $\phi_1 \subset D_1$ as ϕ_1 is connected in $\overline{\mathbb{D}} \setminus J_2$.

Similarly, we have $C_1 \subseteq D_1$ and $D_2 \subseteq C_2$. For $x \in \phi_1$, choose an open U such that $x \in U \subset D_1$. Since ϕ_1 is contained in the boundaries of C_1, C_2, this neighbourhood has points from both C_1, C_2. In particular, U has a point from $C_2 \cap D_1$, so $C_2 \cap D_1 \neq \emptyset$. Note that $C_1 \cap D_2 = \emptyset$ as $C_1 \subseteq D_1$. Thus,

$$\overline{\mathbb{D}} \setminus (J_1 \cup J_2) = C_1 \sqcup (C_2 \cap D_1) \sqcup D_2.$$

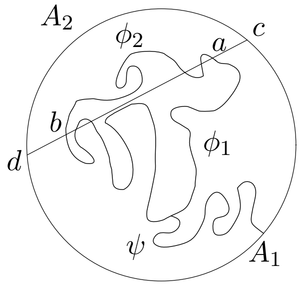

FIGURE 8.1. J inside C

Let $x \in C_2 \cap D_1$ and suppose ϕ is a path from x to a point $y \in C$ that avoids J. If ϕ goes outside the circle C, look at the first time it hits C (since x is inside, it must hit C at least once). With this as our new y, we may assume that ϕ, except for y, lies inside C.

Case 1: $y \in A_1$

Since $x \in C_2$ and ϕ is inside C, we have $\phi \cap J_1 \neq \emptyset$. Since $\phi \cap J = \emptyset$, it must intersect one of ca, db, say ac. Let $z \in ac$ be the first point of intersection. By the choice of $y, z \neq c$.

Parametrize the restricted path by $[0,1]$ with $\phi(0) = x, \phi(1) = z$. Take a ball U around z, disjoint from J and a $t_1 < 1$ such that $\phi(t_1) \in U$. By the choice of z, for any $t < 1, \phi(t) \notin J_1$ and since $\phi(t) \notin J$, we have $\phi(t) \notin J_1 \cup J_2$. Let ϕ' be the restriction of ϕ between $x, \phi(t_1)$.

$U \setminus (J_1 \cup J_2)$ has two components that lie inside the circle, say H_1, H_2. Because $z \in \partial C_1 \cap \partial C_2$, one of these lies in C_1 and the other in C_2, say $H_1 \subset C_1$. Because $C_1 \subset D_1$, we have $H_1 \subset D_1, H_2 \subset D_2$.

If $\phi(t_1) \in H_1 \subset C_1$ or $\phi(t_1) \in H_2$, then ϕ' must meet J_1, J_2 respectively, which is impossible.

Case 2: $y = c$

Take a ball U around y that avoids J, bd. As mentioned above, $U \setminus (J_1 \cup J_2)$ has three components, of which one lies outside C. Parametrize ϕ by $[0,1]$ with $\phi(0) = x, \phi(1) = y$ and pick a $t < 1$ such that $\phi(t) \in U$. We know that $\phi(t)$ must either be on the radius of U induced by line ac, or in one of the two components of $U \setminus (J_1 \cup J_2)$ inside C.

If it lies on the radius, then ϕ intersects ac and we continue as in case 1. If it lies in one of the other components, then as in case 1, $\phi(t)$ is in C_1 or in D_2 which also impossible.

All other cases, i.e., $y \in A_2, y = d$ can be treated similarly. We conclude that there is no path from x to a point in C avoiding J.

For $x \in C_1$ there is a path from x to points on A_1 avoiding J_1. Since $\phi_2 \in C_2$, such paths also avoid ϕ_2 and hence J. Similarly, for points in D_2, there are paths to A_2 that avoid J. Lastly, for points on ac, bd different from a, b the line itself is a path to the circle that avoids J.

So, we have the "inside" of J which is the set $C_2 \cap D_1$ and the "outside", which is everything else in the complement of J. Fix a point p outside the circle C. If x is on or outside the circle, there is a path from x to p avoiding the inside of C and hence J. If $x \in C_1 \cup D_2$ or $x \in ac \cup bd, x \neq a, b$ first go to C and then to p. So, the outside of J is path connected.

Let $i(J)$ denote the inside and $o(J)$ the outside. We have shown above that there is no path from a point in $i(J)$ to one in $o(J)$ that avoids J, so $\mathbb{R}^2 \setminus J$ has at least two components.

8.2. **Boundary.** Next, we show that all components in the complement of J have J as the boundary. Since the boundary is closed, it suffices to show that all components have $\phi_1 \cup \phi_2$ as the boundary. For points in ϕ_1, take neighbourhoods that avoid ϕ_2. We know that these neighbourhoods contain points from C_1, hence from $o(J)$. It follows that $J = \partial o(J)$.

Let $x \in i(J)$ and $y \in \phi_1$. We will show that in any neighbourhood of y, there is a point that is accessible from the component C_x of x in the complement of J. Then y is in the boundary of this component.

Parametrize ϕ_1 by $(0,1)$ with $\lim\limits_{t\to 0} \phi_1(t) = a, \lim\limits_{t\to 1} \phi_1(t) = b$. Suppose $y = \phi_1(t_1)$ where $t_1 \in (t_0, t_2) \subset (0,1)$ is from any neighbourhood of y in ϕ_1. We have arcs l_1 via ϕ_1 and l_2 through ϕ_2 between points $a_1 = \phi_1(t_0), b_1 = \phi_1(t_2)$.

Some point z in the open l_1 is accessible from A_1, through some arc ψ. This arc doesn't intersect J, hence $\overline{l_2}$. We can apply Theorem 6.2.1 (with $B = C$) to conclude that there is a polygonal covering $\mathcal{P}$ of l_2 such that x is in the unbounded component in the complement of $\mathcal{P}$. Thus, there are polygonal paths from x to any point outside the circle avoiding $\mathcal{P}$ and its inside, hence avoiding l_2.

Let ψ' be one such path. We chose $x \in i(J)$, so ψ' must intersect J. Since it cannot intersect $\overline{l_2}$, it must intersect the open l_1. Suppose ψ' is parametrized by $[0,1]$ with $\psi'(0) = x$. Let $t_3 > 0$ be the first time ψ'

intersects (closed) l_1, then observe that

$$\psi'([0,t_3)) \cap J = \emptyset.$$

Therefore, $\psi'|_{[0,t_3)}$ is a connected path in the complement of J, hence lies in C_x. Since $\psi'(t_3) \in l_1$, it follows that the neighbourhood $\phi_1((t_0,t_2))$ of y has points accessible from C_x and from $o(J)$ which lies exterior to C_x.

We conclude that $y \in \partial C_x$ and that any neighbourhood has a point (arcwise) accessible from x. It follows that ϕ_1, ϕ_2 and hence J are in the boundary of C_x. Since $\partial C_x \subseteq J$, we have $J = \partial C_x$.

8.3. **Two components.** This subsection is based on [4]. So far we have shown that if J is a Jordan curve, then $\mathbb{R}^2 \backslash J$ has one unbounded component, $o(J)$ and bounded components in $i(J)$. We know that all components have J as the boundary. All that is left is to show that there is exactly one bounded component.

Lemma 8.3.1. *Let γ_1, γ_2 be two Jordan curves. If*

$$\gamma_2 \cap i(\gamma_1) \neq \emptyset \textit{ and } \gamma_2 \cap o(\gamma_1) \neq \emptyset,$$

then

$$\gamma_1 \cap i(\gamma_2) \neq \emptyset \textit{ and } \gamma_1 \cap o(\gamma_2) \neq \emptyset.$$

Proof. First take $x \in \gamma_2 \cap i(\gamma_1)$ and $y \in \gamma_2 \cap o(\gamma_1)$. Choose neighbourhoods U_x, U_y of x, y respectively such that $U_x \subset i(\gamma_1), U_y \subset o(\gamma_1)$. Observe that $U_x \cap U_y = \emptyset$.

Since x, y are in the boundary of every component of the complement of γ_2, pick $z_1 \in U_x, z_2 \in U_y$ that lie in the same component. There is a path from z_1 to z_2 that avoids γ_2. Since $z_1 \in i(\gamma_1), z_2 \in o(\gamma_1)$, such a path intersects γ_1. However, this path is in a component of $\mathbb{R}^2 \setminus \gamma_2$, therefore γ_1 intersects every component in the complement of γ_2. $\square$

Take $p \in i(J)$ and draw two rays through it. Both rays intersect J because $p \in i(J)$ and J is bounded. Let $p', p'', p' \neq p''$ be the first points of intersection. Let $q' \in \phi_1, q'' \in \phi_2$ be accessible from A_1, A_2 respectively, say there is an arc from $z_1 \in A_1$ to q' and $z_2 \in A_2$ to q''. Fix a q outside the circle and take disjoint polygonal paths to z_1, z_2 outside the circle.

Together we get the Jordan curve $J' : q \to z_1 \to q' \to p' \to p \to p'' \to q'' \to z_2 \to q$. Since $p \in i(J), q \in o(J)$, by the lemma above, J' must intersect every component in the complement of J. However, by the choice

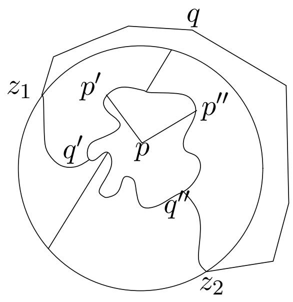

FIGURE 8.2. Curve J'

of p', p'', the $p' \to p \to p''$ arcs lie in the component of p and the other arcs lie on or outside J. So, there can be only two components, that of p and the outside of J. In particular the inside of $J, i(J)$ is a connected open set.

In conclusion, $\mathbb{R}^2 \setminus J$ has two connected components, one bounded and the other unbounded, both having J as the boundary. This completes the proof of the Jordan Curve Theorem.

9. Epilogue

In this article we have given proofs of some intuitive results on finite collection of polygons, using which we proved a few theorems on arcs in the plane and then the Jordan Curve Theorem. However, unlike most other proofs, we did not need the complete Jordan Arc Theorem. What we used was a special case when the arcs are parts of Jordan Curves.

Using stronger approximation theorems, we can envelope this polygonal arc to prove the connectedness of the complement. Proofs of the arc theorem can be found in the references listed.

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THE FAITHFUL REPRESENTATIONS OF RIGID MOTIONS OF A REGULAR POLYGON

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Abstract. Let n be a natural number. In this paper we characterize all degree n faithful representations of a dihedral group G of order $2m$, $m \geq 3$, over the field of complex numbers $\mathbb{C}$. The results are important due to their applications in the study of physical sciences.

1. Introduction

The study of faithful representations of a finite group acquires an important place in the Representation theory. So for a finite group G and a natural number n, the question of characterization of degree n faithful representation of G along with a combinatorial count of the number of all such representations becomes very much necessary to investigate.

Definition 1.1. For G a finite group, an injective representation $\rho : G \to GL(\mathbb{V})$ is called a faithful representation of the group G.

By Maschke's theorem (see [1], Corollary 4.9, p. 316) every degree n representation of G can be written as a direct sum of copies of it's irreducible representations.

The objective of this paper pertains to the following problems:
1. Characterization of all degree n faithful representations of G.
2. Deriving a formula to count the number of all degree n faithful representations of G.

The problems in concern have been dealt with in the literature in different points of view. Gongopadhyay, Kulkarni, and Tanti, in [5], [6], investigated about invariant bilinear spaces and the existence of non-degenerate

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invariant bilinear forms. Behravesh, Ghaffarzadeh and Delfani, in [2], [3], [4], calculated the minimal degree of faithful representations of a group by permutation matrix and the minimal degree of faithful representation of a group by quasi-permutation matrices over the rational field $\mathbb{Q}$ and the complex field $\mathbb{C}$ respectively.

In this paper we discuss all degree n faithful representations of the dihedral group D_m, $m \geq 3$.

$D_m = \{\mathbf{1}, a, a^2, \cdots, a^{m-1}, b, ab, a^2b, \cdots, a^{m-1}b \mid a^m = b^2 = \mathbf{1}, ba = a^{m-1}b\}$.

2. Preliminaries

The number of irreducible representation of D_m over $\mathbb{C}$ is $2|Z(D_m)| + \lfloor \frac{m-1}{2} \rfloor$. Again we know that $|G| = \sum_{i=1}^{r} d_i^2$, where d_i's are the degrees of the irreducible representation and we also have $d_i||G|$. So we conclude that there are $2|Z(D_m)|$ number of degree 1 representations and $\lfloor \frac{m-1}{2} \rfloor$ number of degree 2 irreducible representations of D_m. Let $\rho_{2|Z(D_m)|+1}$, $\rho_{2|Z(D_m)|+2}$, $\cdots$, $\rho_{2|Z(D_m)|+\lfloor \frac{m-1}{2} \rfloor}$ be all degree 2 irreducible representations of D_m, where the notations $|Z(D_m)|$ and $\lfloor . \rfloor$ denote order of the center $Z(D_m)$ and greatest integer function respectively.

2.1. **Counter-clockwise rotation and their composition with reflection can be seen as below for $1 \leq s \leq m$ and $1 \leq t \leq \lfloor \frac{m-1}{2} \rfloor$.**

$$\rho_{2|Z(G)|+t}(a^s) = \begin{bmatrix} Cos(\frac{2\pi}{m}ts) & -Sin(\frac{2\pi}{m}ts) \\ Sin(\frac{2\pi}{m}ts) & Cos(\frac{2\pi}{m}ts) \end{bmatrix} \qquad and$$

$$\rho_{2|Z(G)|+t}(a^sb) = \begin{bmatrix} Cos(\frac{2\pi}{m}ts) & Sin(\frac{2\pi}{m}ts) \\ Sin(\frac{2\pi}{m}ts) & -Cos(\frac{2\pi}{m}ts) \end{bmatrix}.$$

2.2. **For m odd and $1 \leq t \leq \frac{m-1}{2}$, all irreducible representations of G are recorded in the following table.**

	ρ_1	ρ_2	ρ_{2+t}
a	1	1	$\begin{bmatrix} Cos(\frac{2\pi}{m}t) & -Sin(\frac{2\pi}{m}t) \\ Sin(\frac{2\pi}{m}t) & Cos(\frac{2\pi}{m}t) \end{bmatrix}$
b	1	-1	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

.

2.3. **For m even and $1 \leq t \leq \frac{m}{2} - 1$, all irreducible representations of G are presented by the following table.**

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_{4+t}
a	1	1	-1	-1	$\begin{bmatrix} Cos(\frac{2\pi}{m}t) & -Sin(\frac{2\pi}{m}t) \\ Sin(\frac{2\pi}{m}t) & Cos(\frac{2\pi}{m}t) \end{bmatrix}$
b	1	-1	-1	1	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

.

Note 2.1 For $1 \leq t \leq \lfloor \frac{m-1}{2} \rfloor$, $\rho_{2|Z(D_m)|+t}$ is a faithful irreducible representation of D_m, if $gcd(m,t) = 1$. Fact that when $gcd(m,t) \neq 1$, then order of $\rho_{2|Z(D_m)|+t}(a)$ is a proper divisor of m (see subsection 2.1).

Lemma 2.1. *If $T = \{t \in \mathbb{N} \,|\, gcd(m,t) = 1 \,\&\, 1 \leq t \leq \frac{m}{2}\}$. Then $|T| = \frac{\phi(m)}{2}$, where ϕ is Euler's totient function.*

Proof. Let p be a prime divisor of m and $gcd(m,t) = 1$, then $p \nmid t$ implies that $p \nmid (m-t)$ and $gcd(m, m-t) = 1$. So with the concepts of Euler's totient function, we have

$$\phi(m) = |\{t \in \mathbb{N} \,|\, gcd(m,t) = 1 \,\&\, 1 \leq t < m\}|.$$

$$\Longrightarrow \phi(m) = \left|\left\{t \in \mathbb{N} \,|\, gcd(m,t) = 1 \,\&\, 1 \leq t \leq \tfrac{m}{2}\right\}\right| + \left|\left\{t \in \mathbb{N} \,|\, gcd(m,t) = 1 \,\&\, \tfrac{m}{2} < t < m\right\}\right|.$$

$$\Longrightarrow \phi(m) = \left|\left\{t \in \mathbb{N} \,|\, gcd(m,t) = 1 \,\&\, 1 \leq t \leq \tfrac{m}{2}\right\}\right| + \left|\left\{t \in \mathbb{N} \,|\, gcd(m,t) = 1 \,\&\, -m < -t < -\tfrac{m}{2}\right\}\right|.$$

$$\Longrightarrow \phi(m) = \left|\left\{t \in \mathbb{N} \,|\, gcd(m,t) = 1 \,\&\, 1 \leq t \leq \tfrac{m}{2}\right\}\right| + \left|\left\{t \in \mathbb{N} \,|\, gcd(m,m-t) = 1 \,\&\, 0 < m-t < \tfrac{m}{2}\right\}\right|.$$

$$\Longrightarrow \phi(m) = |T| + |T|.$$

$$\Longrightarrow |T| = \tfrac{\phi(m)}{2}.$$

□

Now by Maschke's theorem we have

$$\rho = k_1\rho_1 \oplus \cdots \oplus k_{2|Z(D_m)|}\rho_{2|Z(D_m)|} \oplus \cdots \oplus k_{2|Z(D_m)|+\lfloor \frac{m-1}{2} \rfloor}\rho_{2|Z(D_m)|+\lfloor \frac{m-1}{2} \rfloor}. \tag{2.1}$$

where for every $1 \leq i \leq 2|Z(D_m)| + \lfloor \frac{m-1}{2} \rfloor$, $k_i\rho_i$ stands for the direct sum of k_i copies of the irreducible representation ρ_i. Let χ be the corresponding character of the representation ρ, then

$$\chi = k_1\chi_1 + k_2\chi_2 + \cdots + k_{2|Z(D_m)|+\lfloor \frac{m-1}{2} \rfloor}\chi_{2|Z(D_m)|+\lfloor \frac{m-1}{2} \rfloor},$$

where χ_i is the character of ρ_i, $\forall\ 1 \leq i \leq 2|Z(D_m)| + \lfloor \frac{m-1}{2} \rfloor$. The degree of the character is being calculated at identity element of a group and is equal to degree of corresponding representation. i.e,

$$k_1 + \cdots + k_{2|Z(D_m)|} + 2k_{2|Z(D_m)|+1} + \cdots + 2k_{2|Z(D_m)|+\lfloor \frac{m-1}{2} \rfloor} = n. \tag{2.2}$$

Following Lemma is useful and its an standard result can be found in most of the text books of Representation theory [1], [7].

Lemma 2.2. *For G a finite group if $\rho = \oplus_{j=1}^{s} k_{i_j}\rho_{i_j}$ is a representation of G with $k_{i_j} \in \mathbb{N}$ and ρ_{i_j}, $1 \leq i_j \leq s$ irreducible representations, then $ord(\rho(g)) = lcm(ord(\rho_{i_1}(g)), \cdots, \rho_{i_s}(g))$ for every $g \in G$.*

Corollary 2.3. *For ρ a representation of D_m, ρ is a faithful representation if and only if the lcm of orders of its irreducible components evaluated at a is m.*

Proof. Immediate from the lemma. □

Note 2.2 Let N denotes the number of degree n representations, each of which consists of $t_1^{th}, t_2^{th}, \cdots, t_l^{th}$ irreducible representations of degree 2 such that

$$gcd(m, t_r) \neq 1 \text{ and } lcm\left\{\frac{m}{gcd(m,t_1)}, \frac{m}{gcd(m,t_2)}, \cdots, \frac{m}{gcd(m,t_l)}\right\} = m.$$

3. Main results

In this section we will prove main results. Our results are stated in the following two main theorems.

Theorem 3.1. *Let $Z(D_m)$ and ρ_i, $1 \leq i \leq 2|Z(D_m)| + \lfloor \frac{m-1}{2} \rfloor$ be the center and irreducible representation respectively of D_m, then degree n representation $\rho = \oplus_{i=1}^{2|Z(D_m)|+\lfloor \frac{m-1}{2} \rfloor} k_i\rho_i$ of D_m is faithful if ρ consists of at least one faithful irreducible representation.*

Converse of the above theorem is not true. For this we will produce one counter example just after the proof of Theorem 3.1.

Theorem 3.2. *Let ϕ be the Euler's totient function and $Z(D_m)$ the center of D_m, then the number of degree n faithful representations of D_m is*

$$\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{s + \lfloor \frac{m-3}{2} \rfloor}{\lfloor \frac{m-3}{2} \rfloor} - \binom{s + \lfloor \frac{m-1}{2} \rfloor - \frac{\phi(m)}{2} - 1}{\lfloor \frac{m-1}{2} \rfloor - \frac{\phi(m)}{2} - 1} \right] \binom{n - 2s + 2|Z(D_m)| - 1}{2|Z(D_m)| - 1} + N,$$

where N is described in the above Note **??**.

Proof of Theorem 3.1. Let ρ_j be a faithful irreducible representation appearing in ρ, then we have $\rho_j(D_m) \cong D_m$. Therefore m is the least positive integer such that $\rho_j(a)^m$ is the identity operator and so $\rho(a)^m$ is the identity operator, from Corollary 2.3. This completes the proof.

Example 1. *The converse of Theorem 3.1 is not true, in general.*

Let $m = 15$ and n even, then $\rho = \frac{n-2}{2}\rho_{2|Z(D_m)|+3} \oplus \rho_{2|Z(D_m)|+5}$ is a faithful representation, but $\rho_{2|Z(D_m)|+3}$ and $\rho_{2|Z(D_m)|+5}$ are not faithful, similarly we can see for odd n.

Proof of theorem 3.2. We begin the proof by calculating all degree n representations with non-trivial kernel. By using Maschke's theorem, we have from equation 2.2

$$\sum_{i=1}^{2|Z(D_m)|} k_i + 2 \sum_{t=1}^{\lfloor \frac{m-1}{2} \rfloor} k_{2|Z(D_m)|+t} = n. \tag{3.1}$$

$$\sum_{i=1}^{2|Z(D_m)|} k_i + 2 \sum_{\substack{1 \leq t \leq \lfloor \frac{m-1}{2} \rfloor \\ gcd(m,t) \neq 1}} k_{2|Z(D_m)|+t} + 2 \sum_{\substack{1 \leq t \leq \lfloor \frac{m-1}{2} \rfloor \\ gcd(m,t) = 1}} k_{2|Z(D_m)|+t} = n. \tag{3.2}$$

By Theorem 3.1, ρ has trivial kernel if it consists of a faithful irreducible representation. In present case we are looking for a representation with non-trivial kernel i.e, in the first step whenever $gcd(m,t)=1$, we must have $k_{2|Z(D_m)|+t}=0$ in 3.2, and get the following reduced equation

$$\sum_{i=1}^{2|Z(D_m)|} k_i + 2\sum_{\substack{1\le t\le \lfloor\frac{m-1}{2}\rfloor \\ gcd(m,t)\neq 1}} k_{2|Z(D_m)|+t} = n.$$

Now by Lemma 2.1, for $1\le t\le \lfloor\frac{m-1}{2}\rfloor$, the number of t each of which shares a common factor with m is $\lfloor\frac{m-1}{2}\rfloor - \frac{\phi(m)}{2}$. Thus the number of representations satisfying the above equation is

$$\sum_{s=0}^{\lfloor\frac{n}{2}\rfloor}\binom{s+\lfloor\frac{m-1}{2}\rfloor-\frac{\phi(m)}{2}-1}{\lfloor\frac{m-1}{2}\rfloor-\frac{\phi(m)}{2}-1}\binom{n-2s+2|Z(D_m)|-1}{2|Z(D_m)|-1}.$$

Now in view of Corollary 2.3, in the second step we have to remove those cases wherein the representation is with the copies of $t_1^{th}, t_2^{th}, \cdots, t_l^{th}$ irreducible representations of degree 2 such that

$$gcd(m,t_r)\neq 1 \text{ and } lcm\left\{\frac{m}{gcd(m,t_1)}, \frac{m}{gcd(m,t_2)}, \cdots, \frac{m}{gcd(m,t_l)}\right\} = m,$$

which are precisely N in counting from Note **??**.

Thus the number of degree n representations with non-trivial kernel is

$$\sum_{s=0}^{\lfloor\frac{n}{2}\rfloor}\binom{s+\lfloor\frac{m-1}{2}\rfloor-\frac{\phi(m)}{2}-1}{\lfloor\frac{m-1}{2}\rfloor-\frac{\phi(m)}{2}-1}\binom{n-2s+2|Z(D_m)|-1}{2|Z(D_m)|-1} - N. \tag{3.3}$$

To solve the equation 3.1

As $\underbrace{k_{2|Z(D_m)|+1}+\cdots+k_{2|Z(D_m)|+\lfloor\frac{m-1}{2}\rfloor}}_{\lfloor\frac{m-1}{2}\rfloor} = s$, $0\le s\le\lfloor\frac{n}{2}\rfloor$ we have $\lfloor\frac{n}{2}\rfloor+1$ equations, the s^{th} equation

$$\underbrace{k_{2|Z(D_m)|+1}+\cdots+k_{2|Z(D_m)|+\lfloor\frac{m-1}{2}\rfloor}}_{\lfloor\frac{m-1}{2}\rfloor} = s \tag{3.4}$$

The number of distinct solution to above equations 3.4 is $\binom{s+\lfloor\frac{m-1}{2}\rfloor-1}{\lfloor\frac{m-1}{2}\rfloor-1}$, $0\le s\le\lfloor\frac{n}{2}\rfloor$.

Thus the number of all distinct $2|Z(D_m)|+\lfloor\frac{m-1}{2}\rfloor$ tuples $(k_1,\cdots,k_{2|Z(D_m)|},\cdots, k_{2|Z(D_m)|+\lfloor\frac{m-1}{2}\rfloor})$ is $\sum_{s=0}^{\lfloor\frac{n}{2}\rfloor}\binom{s+\lfloor\frac{m-3}{2}\rfloor}{\lfloor\frac{m-3}{2}\rfloor}\binom{n-2s+2|Z(D_m)|-1}{2|Z(D_m)|-1}$.

Now subtracting equation 3.3 from this result, we have the result.

Example 2. *Here we want to find out all degree 10 faithful representations of the Dihedral group D_{15}. Out of* 9 *irreducible representations, 2 are of degree one and 7 are of degree two, namely ρ_1, ρ_2 and ρ_{2+t}, for* $1 \leq t \leq 7$. *Further a representation ρ of degree* 10 *is expressed as,*

$$\rho = \oplus_{i=1}^{9} k_i \rho_i.$$

Thus all the representations of degree 10 are $\sum_{s=0}^{5} \binom{s+6}{6}\binom{11-2s}{1} = 1782$ *and all the faithful representation of degree* 10 *are* $\sum_{s=0}^{5} \left[\binom{s+6}{6} - \binom{s+2}{2}\right]\binom{11-2s}{1} + N = 1586 + N$ *in counting, here N is the number of all distinct solutions of the equation*

$$k_1 + k_2 + 2k_5 + 2k_7 + 2k_8 = 10,\ k_7 \geq 1\, and\, (k_5, k_8) \neq (0, 0).$$

$$k_1 + k_2 = 10 - 2(k_5 + k_7 + k_8).$$

As $2 \leq k_5 + k_7 + k_8 \leq 5$, *we have 4 equations stated below*

$$k_5 + k_7 + k_8 = 2,\ k_5 + k_7 + k_8 = 3,\ k_5 + k_7 + k_8 = 4,\ k_5 + k_7 + k_8 = 5.$$

The number of distinct solutions to each of these equations is $\binom{s-1+3-1}{3-1} - 1$, $2 \leq s \leq 5$. *Thus the number of all distinct such 5-tuples* $(k_1, k_2, k_5, k_7, k_8)$ *is* $N = \sum_{s=2}^{5} \left[\binom{s+2}{2} - 1\right]\binom{n-2s+2-1}{2-1} = 80$. *Hence the number of distinct faithful representations for D_{15} of degree* 10 *is* $1586 + 80 = 1666$.

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FINITE GROUPS WITH SMALL AUTOMIZERS FOR EVERY ABELIAN SUBGROUP OF NON PRIME POWER ORDER

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Abstract. In this paper, we discuss about finite groups in which, $C_G H = N_G H$, for every abelian subgroup H of non prime power order. Also, we classify all such nilpotent and minimal non nilpotent groups

1. Introduction

Throughout this paper, G denotes a finite group and p and q denote a prime. By a p-group G, we mean G is a group of prime power order, for some prime p. By a non p-group G, we mean G is a group of non prime power order. Also, $C_G H$ and $N_G H$ denote the centralizer and the normalizer of H in G, respectively. Also, we use $F(G)$ and $Z(G)$ to denote the Fitting subgroup and the center of G, respectively. Further, $[H]K$ denotes a split extension of K by a normal subgroup H.

Recall that, the automizer of a subgroup H in G is defined as $Aut_G H = N_G H/C_G H$. Therefore $Aut_G H$ can be considered as a subgroup of $AutH$ and since $Aut_G H$ contains an isomorphic copy of $InnH$, we have $InnH \leq Aut_G H \leq AutH$. Now $Aut_G H$ is said to be small if $Aut_G H \cong InnH$ and $Aut_G H$ is said to be large if $Aut_G H \cong AutH$.

All finite groups with small automizers for their nonabelian subgroups are classified in [3]. Zassenhaus proved in [10] that $N_G H = C_G H$ for every abelian subgroup iff G is abelian (also see [6]). Therefore $Aut_G H$ is small for all abelian subgroups of G iff G is abelian. Further, Li [8] classified all finite groups in which any nonmaximal abelian subgroups do not have any nontrivial inner automorphisms. Such groups are called NC-groups. All

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finite groups, in which every non normal abelian subgroup has trivial automizer, are classified in [1]. Such groups are called quasi-NC group. Finite groups, in which for any non normal abelian subgroup A, either $Aut_G A = 1$ or $C_G A = A$, are called NNC-groups and are discussed in [2]. More generally, all finite groups in which $Aut_G H$ is either small or large for every abelian subgroup H, are discussed in [9].

The following result is taken from [8, Lemma 2.4.]:

Lemma 1.1. *Let G be a group. Then the following are equivalent:*
(1) G is an NC-group;
(2) $C_G A = A$ or $C_G A = N_G A$ holds for all abelian subgroups A of G of prime power order.

The above lemma asserts also that if every abelian subgroup of G of prime power order has small automizer, then infact every abelian subgroup of G of non prime power order has also small automizer and G become abelian. However, if every abelian subgroup of G of non prime power order has small automizer, an abelian p-subgroup of G need not have small automizer. For example, in the group $S_3 \times Z_3$, the only abelian subgroups of non prime power order are $\{I, \sigma\} \times Z_3$, where σ is a transposition in S_3. Clearly all such subgroups have small automizer in $S_3 \times Z_3$. However, $A_3 \times Z_3$ is an abelian subgroup of prime power order but it's automizer is not small. Therefore it is of natural interest to classify all finite groups in which, for every abelian subgroup H, either H is a p-subgroup (p depends on H) or $C_G H = N_G H$. We shall do this partially in the present paper. We call such groups as PNC-group. Clearly every finite abelian group is PNC. So we shall focus on finite non abelian groups, which are PNC. The aim of this paper is to study all finite solvable PNC-groups. Also, we give a characterization of nilpotent PNC-groups and minimal non nilpotent PNC-groups.

2. Nilpotent PNC-groups

Lemma 2.1. *The class of PNC-groups is subgroup closed.*

Proof. It is obvious. □

Proposition 2.2. *Let G be a decomposable group with $G \cong G_1 \times G_2$, for some finite non trivial groups G_1 and G_2. If, either $(|G_1|, |G_2|) = 1$ or both*

G_1 and G_2 are non p-groups, then the following are equivalent:
(i) G is abelian;
(ii) G is PNC.

Proof. Clearly, if G is abelian, then it is PNC. Conversely, assume that G is PNC. First we observe that, under the given conditions, $N_{G_i}H_i = C_{G_i}H_i$,$\forall\ i = 1, 2$ and for every abelian subgroup H_i of G_i . Now the result follows from [10, Theorem 7]. □

Corollary 2.3. *Let G be a non abelian group. Then G is a nilpotent PNC-group iff G is a p-group.*

Proof. If possible, suppose that G is a non p-group. Then we have $G \cong G_1 \times G_2$, where G_1 is a Sylow-p subgroup of G and G_2 is a Hall-p' subgroup of G. Also, we have $(|G_1|, |G_2|) = 1$. Now by above Proposition, G is an abelian group, a contradiction. Hence G is a p-group.
Converse is obvious. □

3. Solvable non nilpotent PNC-groups

Recall that the Fitting subgroup of G, denoted as $F(G)$, is defined as the largest normal nilpotent subgroup of G. Also, we denote the commutator subgroup of G as G' .

Lemma 3.1. *Let G be a solvable non abelian group. If G is a PNC-group, then $F(G)$ is a p-group.*

Proof. Note that $F(G)$ is a nilpotent PNC-group. Now if $F(G)$ is non abelian, then by Corollary 2.3, it is a p-group. Also, if $F(G)$ is an abelian but non p-group, then $C_GF(G) = N_GF(G)$. Since G is solvable, we have $F(G) = C_GF(G) = N_GF(G)$. This gives that G is abelian, a contradiction. This completes the proof. □

Lemma 3.2. *Let G be a solvable non abelian group. If G is a PNC-group, then $Z(G)$ is a p-group.*

Proof. First we note that $Z(G) \subseteq F(G)$. Now the result follows from Lemma 3.1. □

Lemma 3.3. *Let G be a non abelian group with G' nilpotent. If G is a PNC-group, then G' is a p-group.*

Proof. First we note that if G' is nilpotent then $G' \subseteq F(G)$. Now the result follows from Lemma 3.1. $\square$

Recall that a group is said to be supersolvable, if it has a normal series with cyclic factors.
If G is PNC, $F(G)$ need not be a Sylow subgroup. For example, S_4 is PNC but it's Fitting subgroup is not a Sylow subgroup. However we have the following result.

Lemma 3.4. *Let G be a non abelian supersolvable group. If G is PNC, then $F(G)$ is a Sylow-p subgroup.*

Proof. First we note that the commutator subgroup of a supersolvable group is nilpotent. Now by above Lemma, G' is a p-group and hence $F(G)$ is a Sylow-p subgroup. $\square$

Lemma 3.5. *Let G be a PNC-group and $H \leq Z(G)$. If $H \cap G' = 1$, then G/H is also PNC.*

Proof. Let K/H be an abelian non p-subgroup of G/H. Then K is also a non p-subgroup of G. Also, $K' \subseteq H \cap G' = 1$. This gives that K is abelian. Therefore $N_G K = C_G K$. Now, $N_{G/H}(K/H) = N_G K/H = C_G K/H \leq C_{G/H}(K/H)$. This implies that $N_{G/H}(K/H) = C_{G/H}(K/H)$. $\square$

Recall the following results from [1]:

Theorem 3.6. *(see[1], Theorem 4.1.) Let G be a non-nilpotent quasi-NC group. Then all Sylow subgroups of G are abelian.*

Theorem 3.7. *(see[1], Theorem 4.4.) Suppose that G is non nilpotent quasi-NC group. Then $G/Z(G)$ is also a quasi-NC group.*

Note that if G is a non nilpotent PNC-group, then Sylow subgroups of G need not be abelian. For example, S_4 is a non nilpotent PNC-group, but it's Sylow-2 subgroups are non abelian. However the following result is an analogous of Theorem 3.7. Recall that, an A-group is a finite group with the property that all of its Sylow subgroups are abelian.

Lemma 3.8. *Let G be a non nilpotent A-group. If G is PNC, then $G/Z(G)$ is also PNC.*

Proof. From [1, Lemma 2.7], $G' \cap Z(G) = 1$. Now the result follows from Lemma 3.5. $\square$

Recall that a finite group G is said to be SBP, if every proper subgroup H of G has prime power order (the prime depending on H). The following classification of SBP groups is given in [4; chapter 3, page 74/75].

Theorem 3.9. *Let $G \neq 1$ be an SBP group. Then precisely one of the following holds:*
(I) G is a p-group, for some prime p.
(II) $|G| = pq$, where p and q are distinct primes.
(III) $|G| = p^a q$, p and q are distinct primes, $a = exp(p, q)$ and $a \geq 2$. More ever, $G \cong Z_p^a \times_\theta Z_q$ for some monomorphism $\theta : Z_q \to AutZ_p^a$.

Recall that a group H is said to a central extension of G, if $H/A \cong G$, where $A \subseteq Z(H)$. For our convenience, we call a central extension H of G as a p-central extension, if $Z(H)$ is a p-group. Clearly, an SBP-group is a solvable PNC-group and if it is non abelian, then it's Fitting subgroup is a p-group. Now we prove the following result.

Lemma 3.10. *Let G be an SBP-group with $F(G)$ a p-group. Then a p-central extension of G is PNC.*

Proof. If G is an abelian group or a p-group, the result holds trivially. So let G be a non abelian group of order $p^n q$, with $F(G)$ a p-group. Also let H be a p-central extension of G with $H/A \cong G$, where $A \subseteq Z(H)$. Since G is SBP, $A = Z(H)$. Also, since we have $F(H/Z(H)) = F(H)/Z(H)$, $F(H)$ is a p-group. Now we claim that every proper non p-subgroup of H is of the form TQ, where $T \subseteq Z(H)$ and Q is a Sylow-q subgroup of H. Let K be a non p-subgroup of H. Then KA/A is a subgroup of H/A and since H/A is SBP, $|K/K \cap A| = |KA/A| = q$. This gives that $K = TQ$, where $T = K \cap A$ and Q is a Sylow-q subgroup of K. Therefore, for given any abelian non p-subgroup TQ of H, $N_H(TQ) = C_H(TQ) = AQ$. This completes the proof. □

Recall that a group is called a CP-group if every element of the group has prime power order (see [5]). Finite CP-groups were first studied by Higman [7] in 1957. The finite soluble CP-groups were classified by him as follows (see [7], Theorem 1):

Theorem 3.11. *Let G be a soluble group all of whose elements have prime power order. Let p be the prime such that G has a normal p-subgroup greater than 1, and let P be the greatest normal p-subgroup of G. Then G/P is either* *(i) a cyclic group whose order is a power of a prime other than p; or (ii) a generalized quaternion group, p being odd; or (iii) a group of order $p^a q^b$ with cyclic Sylow subgroups, q being a prime of the form $kp^a + 1$. Thus G has order divisible by at most two primes, and G/P is metabelian.*

Now we prove some results about CP-groups.

Lemma 3.12. *Every finite CP-group is PNC.*

Proof. Note that a finite group is CP iff every abelian subgroup of G is of prime power order. Now the result follows from definition. □

Remark. Unlike the case of SBP-groups, if G is solvable CP-group with $F(G)$ a p-group, a p-central extension of G need not be PNC. For example, S_4 is a solvable CP-group but $S_4 \times Z_2$ is not a PNC-group as $A_3 \times Z_2$ has not small automizer. However we have the following result.

Theorem 3.13. *Let G be a solvable CP-group of order $p^n q$ with $p > q$. Then a p-central extension of G is a PNC-group.*

Proof. Let H be a p-central extension of G, with $H/A \cong G$, where $A \subseteq Z(H)$. Since G is CP, A can not be a proper subgroup of $Z(H)$ and so $A = Z(H)$. Let K be an abelian non p-subgroup of H. Then KA/A is an abelian subgroup of H/A which is CP, being isomorphic to G. Therefore $|K/K \cap A| = |KA/A| = q$. This gives that $K = TQ$, where $T = K \cap A$ and Q is a Sylow-q subgroup of K. Clearly, $AQ \subseteq C_H(TQ)$. Now we shall show that $N_H(TQ) \subseteq AQ$. First note that $N_G S = S$, where S is a Sylow-q subgroup of G. For if $|N_G S| = p^r q$, for some positive integer $r \geq 1$. Then, since $p > q$, G will contain an element of order pq, which is not possible. Hence $AQ/A = N_{H/A}(AQ/A) = N_H(AQ)/A$. This implies that $N_H(AQ) = AQ$. Now we have $N_H(TQ) \subseteq N_H(AQ) = AQ$. Hence $N_H(TQ) \subseteq AQ \subseteq C_H(TQ)$. This completes the proof. □

Proposition 3.14. *Let G be a non abelian solvable group with an abelian maximal normal subgroup. If G is a PNC-group, then $|G| = p^n q$, where p, q are primes (need not be distinct) and n is some positive integer.*

Proof. First note that a maximal normal subgroup of a solvable group is of prime index. Now if H is an abelian maximal normal subgroup but not a p-group, then $N_G H = C_G H$. It gives that G is abelian, a contradiction. Hence H is a p-group. Now the result follows. □

Corollary 3.15. *D_{2n} is a PNC-group iff $n = p^k$, for some prime p and positive integer k.*

Proof. Since $< r >$ (where $r \in D_{2n}$ is an element of order n) is an abelian maximal normal subgroup of D_{2n}, from the above proposition $< r >$ is a p-group. Now the result follows. Conversely, it is easy to observe that, if $n = p^r$, then D_{2n} is PNC. In fact in this case D_{2n} is CP. □

We have seen in the section 2 that nilpotent PNC groups are p-group. However if we impose a weaker condition on G, namely the condition of minimal non nilpotent, then we have the following result.

Theorem 3.16. *G is a minimal non nilpotent PNC-group iff $G \cong [F(G)]Q$, where $F(G)$ is a Sylow-p subgroup of G and Q is a cyclic subgroup of order q acting non trivially on $[F(G)]$ and every proper non p-subgroup of G is abelian.*

Proof. Let G be a minimal non nilpotent PNC group. Then G is solvable and so $F(G)$ is a p-group. Since every proper subgroup is nilpotent, $F(G)$ is a maximal normal subgroup of G and so $[G : F(G)] = q$, for some prime $q \neq p$. Also from Corollary 2.3, it follows that every proper non p-subgroup of G is abelian. Since G is non nilpotent, Sylow-q subgroup is non normal. So let Q be a Sylow-q subgroup of G. Then clearly, $G \cong [F(G)]Q$ and Q, being non normal, acts non trivially on F(G).
Conversely, assume that $G \cong [F(G)]Q$, where $F(G)$ is a Sylow-p subgroup of G and Q is a cyclic subgroup of order q acting non trivially on $[F(G)]$ and every proper non p-subgroup of G is abelian. Since the action of Q on $F(G)$ is non trivial, G is non nilpotent. Also every non p-subgroup of G is abelian, so G is minimal non nilpotent. Now since $F(G)$ is a p-group, no proper non p-subgroup of G is normal. Therefore $N_G(H) = C_G(H)$, for every abelian non p-subgroup of G. This completes the proof. □

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ON MOTZKINS CHARACTERISATION FOR EUCLIDEAN DOMAIN

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Abstract. This is essentially an expository paper which sheds new light on existing knowledge of Euclidean domain due to Th. Motzkin. In [4], Motzkin gave a constructive criterion for the existence of a Euclidean algorithm within a given integral domain and from among the different possible Euclidean algorithms in an integral domain one is singled out. It is useful criterion for checking whether integral domains in general and rings of integers in particular are Euclidean or not.

1. Introduction

Generally, we find in the literature two definitions of a Euclidean domain R in which one definition says that a domain R with a Euclidean function d on $R-\{0\}$ satisfies the property

$$d(a) \leq d(ab)\ \forall a, b \neq 0 \in R$$

and in other definition the above inequality is missing.

Here we write N for the set of elements $0, 1, 2, \ldots$.

Definition 1.1. [3] An integral domain R is called *Euclidean* if there is a function
$d : R-\{0\} \to N$ with the following two properties:
(1) $d(a) \leq d(ab)$ for all non-zero a and b in R (the d-inequality)
(2) for all a and b in R with $b \neq 0$ we can find q and r in R such that
$a = bq + r,\ r = 0$ *or* $d(r) < d(b)$ (Euclidean function).

Examples of Euclidean domains (with both (1) and (2) being satisfied):

- Any field F. Define $d(a) = 1$ for all non-zero $a \in F$.

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- $\mathbb{Z}$(the ring of integers) is also a Euclidean domain. Define $d(n) = |n|$, the absolute value of n, where $n \in \mathbb{Z}$.
- $\mathbb{Z}[i]$, the ring of Gaussian integers. Define $d(a + ib) = a^2 + b^2$, the squared norm of the Gaussian integer $a + bi$.
- $\mathbb{Z}[w]$ (where w is a primitive cube root of unity), the ring of Eisenstein integers. Define $d(a + bw) = a^2 - ab + b^2$, the norm of the Eisenstein integer $a + bw$.

Definition 1.2. [2]
An integral domain R is called *Euclidean* if there is a function $d : R - \{0\} \to N$ such that for all a and b in R with $b \neq 0$, we can find q and r in R such that

$$a = bq + r, \ \ r = 0 \text{ or } d(r) < d(b). \tag{1.1}$$

Any function $d : R - \{0\} \to N$ that satisfies (1.1) will be called a Euclidean function on R. Thus a Euclidean domain in definition (1.2) is an integral domain that admits a Euclidean function, while a Euclidean domain in Definition (1.1) is an integral domain that admits a Euclidean function satisfying the d-inequality.

Definition 1.3. Let R be a Integral domain. Then R is said to be a *Euclidean domain* if there is a function $E : R \to \{0, 1, 2, 3, \cdots\}$ such that $E(0) = 0$ and $\forall a, b \in R, a \neq 0, \exists\, q, c$ with $b = qa + c, c = 0$ or $E(c) < E(a)$.

Remark 1.4. This definition is equivalent to Definition 1.2 (without d-inequality). The only difference is that the E-function is defined at 0 also.

Remark 1.5. [1] Suppose (R, d) is a Euclidean domain in the sense of definition 1.2. We will introduce a new Euclidean function $\bar{d} : R - \{0\} \to N$, built out of d, which satisfies $\bar{d}(a) \leq \bar{d}(ab)$. Then $(R, \bar{d})$ is Euclidean in the sense of 1.1.This new Euclidean function can be defined as follows:
For non-zero a in R, set $\bar{d}(a) = \min\ \{d(ab) : b \neq 0 \in R\}$.

Example 1.6. Let $R = F[x], F$ a field. Then R is a Euclidean domain with the Euclidean function $E(0) = 0$ and $E(f(x)) = deg(f(x))$, for non-zero $f(x) \in F[x]$.

Example 1.7. $R = \mathbb{Z}$ is a Euclidean domain with Euclidean function $E(a) = |a|$, for all $a \in \mathbb{Z}$.

With the above Euclidean function, let

$$X_0 = \{x \in R : E(x) = 0\}.$$

Then $0 \in X_0$. Let $a \in X_0$, then $E(a) = 0$. If $a \neq 0$, then for any $b \in R$, $b = qa + c$ for $q, c \in R$, with $c = 0$ or $E(c) < E(a) = 0$. It follows that $c = 0$, so a must be a unit. Thus $X_0 \subset \{0\} \bigcup \{\text{units of } R\}$. Let $X_i = \{x \in R : E(x) \leq i\}$. Then $X_0 \subset X_1 \subset \cdots$ and $\bigcup_{i=0}^{\infty} X_i = R$.

Example 1.8. If $R = \mathbb{Z}$, and $d(a) = |a|$, $X_0 = \{0\}$, so X_0 does not contain units of $\mathbb{Z}$ viz. ± 1. If $R = F[x]$, F field, then $X_0 = F = \{0\} \bigcup R^*$.

2. Motzkins Criterion for Euclidean Algorithm

Definition 2.1. (Motzkin's set) For a integral domain R, define $A_0 = \{0\}$. For $i \geq 1$, let

$$A_i = \{0\} \cup \{a \in R : \forall x \in R, \exists\, y \in A_{i-1} \text{ such that } x - y \in (a)\}.$$

Suppose $a \neq 0$. Then $a \in A_1$ if and only if for all $x \in R, x - 0 \in (a)$, i.e. if and only if $R = (a)$, i.e. if and only if a is a unit.
Thus $A_1 = \{0\} \cup$ {the set of all units}. We prove by induction that

$$A_0 \subset A_1 \subset A_2 \subset \cdots.$$

By definition, $A_0 \subset A_1$. Assume for $i \geq 1$, that $A_{i-1} \subset A_i$. Let $a \neq 0$ and $a \in A_i$. Then $\forall x \in R$, there is $y \in A_{i-1}$ such that $x - y \in (a)$. As $y \in A_{i-1}$ and $A_{i-1} \subset A_i$, we get that " $\forall x \in R$, there is $y \in A_i$ such that $x - y \in (a)$ ". Thus $a \in A_{i+1}$. Hence $A_i \subset A_{i+1}$. Thus by induction, $A_i \subset A_{i+1}$ for all $i \geq 0$.

Lemma 2.2. *Suppose R be a Euclidean domain. Then $A_1 - A_0 = \{a \in R :$ each $\bar{r} \in R/(a)$ contains $0\} = R^*$ (the set of all units of a R).*

Proof. Let each $\bar{r}$ in $R/(a)$ contains 0. That is for each $r \in R$

$$r + ka = 0, \text{ for some } k \in R.$$

Taking $r = 1$, we get

$$1 + ka = 0 \text{ for some } k \in R.$$

Thus $ab = 1$ in R with $b = -k$. Therefore $a \in R^*$. Then $(a) = R$, so $R/(a) = \{\bar{0}\}$. Conversely suppose that a is a unit in R. We want to show that any arbitrary class in $R/(a)$ contains 0. But $(a) = R$, so there is only one class in $R/(a)$, viz. $\bar{r} = \bar{0}$, i.e. $0 \in \bar{r}$.

$\square$

Let $A = \cup A_i$.

Lemma 2.3. *(Motzkin's lemma) An integral domain R has a Euclidean algorithm if and only if every element of R is in A.*

Proof. (i) Suppose R is a Euclidean domain. Then with the E-function as above, $X_0 \subset \{0\} \cup \{\text{units of R}\}$. By Lemma 2.2, we have $A_1 = \{0\} \cup \{\text{units of R}\}$. Thus $X_0 \subset A_1$.
Assume that $X_i \subset A_{i+1}$ for $0 \leq i \leq k-1$. We now show by induction that $X_k \subset A_{k+1}$. Let $a \in X_k$. For $x \in R$, we have

$$x = qa + r, \text{ where } E(r) < E(a) \leq k.$$

Therefore $E(r) \leq k-1$, so $r \in X_{k-1} \subset A_k$, hence $a \in A_{k+1}$. Thus $X_k \subset A_{k+1}$. But $\cup X_i = R$, so $\cup A_i = R$, i.e. $A = R$.
(ii) Conversely suppose that $A = R$. For $x \in R$, define
$E : R \to \{0, 1, 2, \cdots\}$ by

$$E(x) = \min\{i : x \in A_i\}.$$

Hence $E(0) = 0$. For $a, b \in R$, with $a \neq 0$, suppose $E(a) = i$, then $a \in A_i$. Hence given $x \in R$, $x = aq + r$, with $r \in A_{i-1}$. Hence $E(r) \leq i-1$, i.e. $E(r) < E(a)$. Hence R is a Euclidean domain. $\square$

Example 2.4. $R = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$.

By Definition of Motzkin's set, we have $A_0 = \{0\}$. Any element $\alpha = a + b(\frac{1+\sqrt{-19}}{2}) \in R$ is unit if and only if $N(\alpha) = \pm 1$. Now

$$N(\alpha) = a^2 + ab + 5b^2 = \big(a + \frac{b}{2}\big)^2 + \frac{19b^2}{4}.$$

If $b \neq 0$, then $N(\alpha) > 5$, hence $b = 0$. Thus $N(\alpha) = \pm 1$ if and only if $\alpha = \pm 1$. Therefore ± 1 are the only units of a ring R. Thus

$$A_1 = A_0 \cup \{\text{set of units}\} = \{0, 1, -1\}.$$

Now $A_2 = A_1 \cup \{\alpha \in R : \text{every residue class of } R/(a) \text{ is represented}$
$\text{by an element of } A_1\}$. Assume that α is a non-zero non-unit element. Since R is the ring of integers of an imaginary quadratic field, we have $|R/(\alpha)| = N(\alpha)$. For α to be in A_2, we must have $N(\alpha) \leq 3$. But it

is not possible for non-zero non-unit $\alpha \in R$, since for non-zero non-unit $\alpha \in R$, $N(\alpha) \geq 4$. Thus $A_2 = A_1$. Similarly we can show that $A_2 = A_3 = A_4 = \cdots = A_n = \cdots$. Therefore, we have

$$R \neq \bigcup_{i=0}^{\infty} A_i.$$

Hence by Motzkin's Lemma 2.3, $R = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is not a Euclidean domain.

Remark 2.5. It is known that for a given imaginary quadratic number field, its ring of quadratic integers is Euclidean if and only if the norm is a Euclidean function for it. We can also use the Motzkin's criterion to the rings of integers other than quadratic rings of integers. This idea of Motzkin's criterion is used by Weinberger in 1973 to prove the following theorem and hence we get the large number of Euclidean domain.

Theorem 2.6. *Let K be an algebraic number field whose ring of integers is both a PID and also has infinitely many units. Then a Generalized Riemann Hypothesis (GRH) implies that the ring of integers of K is Euclidean.*

In 1801, Gauss conjectured that there are infinitely many real quadratic fields with class number one, i.e. real quadratic fields which are PID. Such real quadratic fields satisfies the condition of the above theorem. If Gauss conjecture comes true, then we will get infinitely many PID's and under the assumption of GRH, we will get infinitely many examples of Euclidean domain.

But unfortunately, both Gauss conjecture and GRH (Generalized Riemann Hypothesis) are still open.

Example 2.7. $R = \mathbb{Z}[\sqrt{m}]$, where m is a positive square-free integer.

Joseph Louis Lagrange was the first European Mathematician who proved that $a^2 - mb^2 = 1$ has infinitely many solutions for $a, b \in \mathbb{Z}$. These solutions correspond to units $a + b\sqrt{m}$ of norm 1 in $\mathbb{Z}[\sqrt{m}]$. Bhaskaracharya, in 12^{th} century, gave a method to find one non-trivial solution to such an equation. This method is called 'cyclic' method (chakrawal). Once one solution is obtained, infinitely many solutions can be obtained. This was earlier shown by Brahmagupta in 7^{th} century using an identity. In this case, Motzkin's set A_1 is infinite, since there are infinitely many units in R. Thus in this case all $A_i, i \geq 1$ are infinite sets. If $m \equiv 1 \pmod 4$, $\mathbb{Z}[\sqrt{m}]$ is not a UFD, so not a Euclidean domain, so $\bigcup_{i=0}^{\infty} A_i \neq R$.

Example 2.8. Let $R = \mathbb{Z}$. Here for illustration sake, we use Motzkin's lemma to show that $\mathbb{Z}$ is a Euclidean domain. From (2.1), $A_0 = \{0\}$ and $A_1 = \{-1, 0, 1\}$, since the -1 and 1 are the only units in $\mathbb{Z}$. For $a = 0, \pm1, \pm2, \pm3$, every residue class $mod(a)$ is represented by element of A_1, but this does not hold if $|a| \geq 4$. Therefore,

$$A_2 = \{0\} \cup \{\pm1, \pm2, \pm3\} = \{-3, -2, -1, 0, 1, 2, 3\}.$$

In general, we have

$$A_i = \{-2^i + 1, -2^i, \cdots, -2, -1, 0, 1, 2, \cdots, 2^i, 2^i - 1\}.$$

Here $\cup A_i = \mathbb{Z}$. Hence by Motzkin's lemma (2.3), $\mathbb{Z}$ is a Euclidean domain. Note also that for the Euclidean function $E(a) = |a|$, we have

$$X_i = \{a \in \mathbb{Z} : E(a) \leq i\} = \{a \in \mathbb{Z} : |a| \leq i\} \subset A_i \; for \; all \; i.$$

Example 2.9. Let $R = F[x]$, where F a field. From (2.1),
$A_0 = \{0\}$ and A_1 is the set of all constant polynomials, since all non-zero elements of F are the only units of R. If $g(x) \in F[x]$ is of degree ≤ 1, every residue class *mod* $(g(x))$ is represented by elements of A_1. This is not true if $deg(g(x)) \geq 2$. Therefore,

$$A_2 = \{g(x) \in F[x] : deg(g(x)) \leq 1\}.$$

In general, by induction we can see that

$$A_i = \{0\} \;\cup\; \{g(x) \in F[x] : deg(g(x)) \leq i - 1\}.$$

Hence $\cup A_i = F[x]$. Hence $F[x]$ is Euclidean domain by Motzkin's lemma (2.3). Note also that $E(g(x)) = deg(g(x))$ works as a Euclidean function and

$$X_i = \{g(x) \in F[x] : E(g(x)) \leq i\} = \{g(x) : deg(g(x)) \leq i\}.$$

Here $X_i = A_{i+1}$, for all $i \geq 0$.

Note: The criterion is applied to some special rings, in particular rings of quadratic integers. Motzkin used his criterion to prove that of the nine imaginary quadratic fields of class number one, only five of them are Euclidean and for these fields, the norm map serves as the Euclidean function.

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SOME REMARKS ON THE DIFFERENTIAL 1-FORM

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Abstract. In this paper we will give some alternate proofs of some propositions about differential 1-form and path integration. We have noticed if for the given differential 1-form for a smooth function, in some subsets of the plane, then there does not exist a smooth function such that its differential is equal to the 1-form. In the end of this paper we construct subdivision and proof that for every closed 1-form, the path integral is equal to the sum of the endpoints in subintervals.

1. Introduction and auxiliary facts

We denote with U an open set in the plane $\mathbb{R}^2$.

Definition 1.1. A smooth function or C^∞ function on U is a function $f : U \to R$ such that all partial derivatives of all order exists and are continuous.

A differential 1-form, or just a 1-form, on U is given by a pair of smooth functions p and q on U. We will denote a 1-form by w and we will write $w = pdx + qdy$.
By a smooth path or just path in U, we mean a mapping $\gamma : [a, b] \to U$ from a bounded interval into U that is continuous on $[a, b]$ and differentiable in the open interval (a, b). So $\gamma(t) = (x(t), y(t))$ and $\gamma(a), \gamma(b)$ are called the endpoints.

With $w = pdx+qdy$ as above and γ a path given by the pair of functions $\gamma(t)$, the integral $\int_w = \int_a^b \left(p(x(t), y(t))\frac{dx}{dt} + q(x(t), y(t))\frac{dx}{dt}\right) dt$.

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We write $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ for this 1-form and say that w is the differential of f if $w = df$.

Proposition 1.2. *$df = dg$ on U if and only if $f - g$ is locally constant on U*

Proposition 1.3. *If γ is a segmented path in U from P to Q and $w = df$ in U then*

$$\int_\gamma w = f(Q) - f(P).$$

Proposition 1.4. *Let U be a product of two open finite or infinite intervals, i.e., $U = \{(x, y) : a < x < b \text{ and } c < y < d\}$, with $-\infty \le a < b \le \infty$ and $-\infty \le c < d \le \infty$. If w is any 1-form on U such that $dw = 0$, then there is function f on U with $w = df$.*

A 1-form w is called ***closed*** *if $dw = 0$ and is called* ***exact*** *if $w = df$.*

2. Main results

Proposition 2.1. *There is no smooth function $g \in C^\infty$ such that $dg = w_\theta$ on U where $w_\theta = \frac{-ydx+xdy}{x^2+y^2}$ and U is*

a) the upper half plane

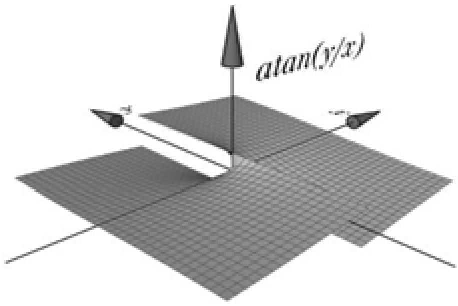
b) the union of the upper half plane and the right half plane

c) the complement of the negative x-axis.

Proof. If we take differential to the function $f(x, y) = \arctan(\frac{y}{x})$ we have $df = \frac{-ydx+xdy}{x^2+y^2}$. Let we denote this differential with w_θ. The function $f(x, y) = \arctan(\frac{y}{x})$ is not differential in $x = 0$ (because is composition of $z = \frac{y}{x}$ defined everywhere expect in the point $x = 0$, the function arctan is continuous). If there exist $g \in C^\infty$ such that $dg = w_\theta$ on U then the integral $\int_\gamma w_\theta$ depends on endpoints for every path γ on U (from Proposition 1.3), so we have

$$\int_\gamma w_\theta = f\left(\gamma(b) - \gamma(a)\right),$$

where $\gamma : [a, b] \to U, \gamma(t) = (x(t), y(t))$. The graph of the function $f(x, y) = \arctan \frac{y}{x}$ is:



arctan(y/x) graph.

From the graph we can decide that U can be the right and the left half plane, then there exist $g \in C^\infty$ such that $dg = w_\theta$ on U, while $f(x,y) = \arctan \frac{y}{x}$ is not defined in $x = 0$, while the surface is divided by $x = 0$ which means according to the y-axis. If U is the set in the cases a), b) or c) we can not find such a function. If we suppose that there exists a function $g \in C^\infty$, such that $dg = w\theta$ on U we obtain the case

a) Let $\gamma(t) = (\cos(t), \sin(t))\,, 0 \leq t \leq \pi$ is path on U, then
$$\int_\gamma w_\theta = \int_0^\pi df = \int_0^\pi \frac{-ydx + xdy}{x^2+y^2} = \int_0^\pi 1dt = \pi$$
On the other hand we have $\gamma(0) = (1,0), \gamma(\pi) = (-1,0)$. So we obtain
$$f(\gamma(\pi)) - f(\gamma(0)) = f(-1,0) - f(1,0) = \arctan 0^\circ - \arctan 0^\circ = 0$$
While $\int_\gamma w_\theta = \pi \neq 0 = f(\gamma(\pi)) - f(\gamma(0))$. So we get that there is no function $g \in C^\infty$, such that $dg = w_\theta$ on U.

b) Let $\gamma(t) = (\cos(t), \sin(t))\,, -\frac{\pi}{2} \leq t \leq \pi$ is path on U, then
$$\int_\gamma w_\theta = \int_{-\frac{\pi}{2}}^\pi df = \int_{-\frac{\pi}{2}}^\pi \frac{-ydx + xdy}{x^2+y^2} = \int_{-\frac{\pi}{2}}^\pi 1dt = \frac{3\pi}{2}$$
On the other hand we have $\gamma(-\frac{\pi}{2}) = (0,-1), \gamma(\pi) = (-1,0)$. So we obtain
$$f(\gamma(\pi)) - f\left(\gamma(-\frac{\pi}{2})\right) = f(-1,0) - f(0,-1) =$$
$$= \arctan \frac{0}{-1} - \arctan \frac{-1}{0} = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$
So there is no function $g \in C^\infty$, such that $dg = w_\theta$ on U, where U is the union of the upper and the right half plane.

c) Similarly there is no function $g \in C^{\infty}$. Here easy we can take $\gamma(t) = (\cos(t), \sin(t)), -\frac{3\pi}{4} \leq t \leq \frac{3\pi}{4}$ on the other hand from the graph we have that the surface is divided from the y-axis).

$\square$

Proposition 2.2. *There exists a function on $\mathbb{R}^2 / \{(0,0)\}$, such that $w = \frac{xdx+ydy}{(x^2+y^2)^2}$ represents its differential.*

Proof. If the differential form $w = \frac{xdx+ydy}{(x^2+y^2)^2}$ can be written as the differential of function f on $\mathbb{R}^2 / \{(0,0)\}$, than we have $df = w$.
If we integrate $df = w$, we obtain

$$f = \int \frac{xdx + ydy}{(x^2 + y^2)^2} = \frac{1}{2} \int \frac{d(x^2 + y^2)}{(x^2 + y^2)^2} = -\frac{1}{2} \frac{1}{x^2 + y^2}$$

So $f(x,y) = -\frac{1}{2(x^2+y^2)}$ is smooth function on $\mathbb{R}^2 / \{(0,0)\}$ that fulfills the required condition. $\square$

Proposition 2.3. *Let w be a closed 1-form on U and $\gamma : [a,b] \to U$, is a smooth path, then there exists a subdivision $a = t_0 < t_1 < ... < t_n = b$ and a collection of $U_i, i = 1, 2, ..., n$ where U_i is an open subset of U so that γ maps $[t_{i-1}, t_i]$ into U_i, and the restriction of w to U_i is the differential of a function f_i. For $P_i = \gamma(t_i)$ and for any such choices, the following relation is valid*

$$\int_{\gamma} w = \sum_{i=1}^{n} [f_i(P_i) - f_i(P_{i-1})]$$

Proof. For any point $P \in \gamma([a,b])$, we choose its neighborhood U_P on which the restriction of w is exact. It is clear that $\gamma^{-1}(U_P)$ form an open covering of the compact interval $[a,b]$, so the finite number of them cover the interval. From this it is not hard to construct the subdivision.
From $[a,b]$ compact set and $\gamma^{-1}(U_P)$ open covering then exist $\varepsilon > 0$ such that for every compact subset of $K = [a,b]$ with diameter less then ε , and is subset of any open set of covering.
Let we denote with A_n the segments of subdivision of $[a,b]$.
If it is not true, let us suppose that for A_n subset of $[a,b]$ with diameter less then $\frac{1}{n}$, $\left(d(A_n) < \frac{1}{n}\right)$ and is not a subset of any open set of covering $\gamma^{-1}(U_P)$.
From $[a,b]$ compact set, every infinitely subset have at least one boundary value, for example A. Let Ω is open set from open covering, let $A \in \Omega, r > 0$.

Then each point of $B\left(A, \frac{r}{2}\right)$ is element of Ω. We obtain $B\left(A, \frac{r}{2}\right) \cup A_n \neq \emptyset$ for infinitely terms, which is a contradiction.
So we have that every subset of $[a, b]$ with diameter $\varepsilon > 0 (\varepsilon = \frac{1}{n})$, is subset of any open set of covering.
If we fix such a subdivision, choose one of these open sets U_P which contain $\gamma(A_i)$ and denote them with U_i and f_i in U_i such that $df_i = w$ in U_i.
Let $\gamma_i : [t_{i-1}, t_i] \to U_i$ be the restriction of γ in A_i. From $df_i = w$ in U_i (from Proposition 1.3 we have that integral depends from the end points). So we have that

$$\begin{aligned}\int_\gamma w = \int_{\gamma_1} w + \int_{\gamma_2} w + ... + \int_{\gamma_n} w = \\ = \sum_{i=1}^{n} [f_i(\gamma_i(t_i)) - f_i(\gamma_i(t_{i-1}))] = \\ = \sum_{i=1}^{n} [f_i(P_i) - f_i(P_{i-1})].\end{aligned}$$

□

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TORSIONAL WAVES IN NONLOCAL ELASTIC SOLID HALF-SPACE WITH VOIDS

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Abstract. Torsional surface wave propagation in an elastic half-space with void pores is investigated within the context of Eringen's nonlocal theory of elasticity. Three types of torsional surface waves are found to travel with distinct speeds, which are frequency dependent and hence dispersive in nature. All the three kind of torsional surface waves are affected by the presence of non-locality present in the medium, where as torsional surface wave of second and third kind also depend on the presence of voids in the medium. To attain more clarity about the behavior of all the three kind of waves, results are simulated numerically. The dispersion curves are plotted and the effect of the presence of voids and non-locality is noticed and analyzed.

1. Introduction

The subject of surface waves has been of immense interest since long due to their applications in diverse fields. These waves confine themselves near the boundary surface of an elastic half-space and can penetrate very little into the half-space. The most popular surface waves in the literature are Rayleigh, Love and Stoneley waves and their literature are available in abundance. Besides these waves, there are another type of surface waves in the literature called 'torsional waves' or 'twisting waves', in which the particles of the host medium rotate around the direction of their propagation. These waves are found to be horizontally polarized but give a twist to the host medium. Due to the lack of sufficient information in the past, these disturbances were termed as "noise" and received no serious attention in the study of seismic waves. Shekhar and Parvez [15] observed these "noise" in the form of torsional surface waves that can propagate in the

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non-homogeneous earth.

Elastic material with voids is a continuum matter containing uniform distribution of small vacuous pores having nothing of mechanical or energetic significance. The linear and nonlinear theories of elastic material with voids were presented by Cowin and his coworker [3, 4]. This theory has a departure from the classical theory of elasticity in the sense that the deformation of the continuum is characterized by a displacement vector, in addition to a new kinematic variable corresponding to the change in void volume fraction from the reference state. The introduction of the new kinematic variable came into picture while expressing the bulk density of the material as a product of its matrix density and void volume fraction. The interaction between the two neighboring elements of the continuum body is governed by a force stress tensor and an equilibrated stress vector. A brief summary of equations and relations in linear elastic material with voids has been nicely reviewed by Dey et al. [6], while the literature review of some dynamical problems attempted by the early researchers has been given in Tomar [11]. Singh et al. [17] extended the theory of elastic material with voids within the context of Eringen's nonlocal theory of elasticity, presented the governing equations and explored the propagation of time harmonic plane waves.

Dey et al. [6] have shown the existence of two types of torsional modes in an elastic half-space with void pores. Few relevant papers on torsional waves are by Dey et al. [7, 8, 10], Chattaraj et al. [12], and Gupta et al. [13, 14, 16] among others. Recently, Tomar and Kaur [18] have studied the importance of sliding contact interface on torsional waves.

In the classical theory of elasticity, the stress at a point is a function of strains at that point only and this relation is expressed by well known generalized Hooke's law in the literature. The underlying idea of nonlocal continuum field theories is that the stress at an interior point of the continuum depends not only on the state of strains at that point but also at all other points of the continuum body. Thus the stress at a point of the continuum is the integral of the strain fields taken over the entire volume of the body. This is how constitutive relations are expressed within the context of nonlocal theory of elasticity. Non-locality in the theory of elasticity has been proposed by Edelen and Laws [1], Edelen and his co-workers [2] and Eringen and Edelen [5]. The book by Eringen [9] is a nice monograph on nonlocal continuum field theories for elastic solids and fluids.

In the present paper, the possibility of surface torsional waves in nonlocal elastic solid half-space with voids has been analyzed. It is concluded that such a half-space can allow three types of torsional surface waves propagating with distinct speeds. Two of the torsional modes are found to depend only on the parameter associated with voids, while the remaining one travels independently. All the three types of torsional modes are found to be dispersive and affected by the nonlocality of the half-space.

1.1. **Basic relations and equations.** Following Singh et al. [17], the equations of motion for a uniform nonlocal elastic material with void pores are given by

$$\beta\nabla\phi + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla^2\mathbf{u} + \rho(1 - \epsilon^2\nabla^2)\mathbf{f} = \rho(1 - \epsilon^2\nabla^2)\ddot{\mathbf{u}}, \quad (1.1)$$

$$\alpha\nabla^2\phi - \xi\phi - \beta\nabla \cdot \mathbf{u} - \omega\dot{\phi} + \rho(1 - \epsilon^2\nabla^2)l = \rho(1 - \epsilon^2\nabla^2)\chi\ddot{\phi}, \quad (1.2)$$

where $\epsilon = e_0 a$ is nonlocality parameter with 'e_0' as material parameter and 'a' as internal characteristic length; λ and μ are the Lame's constants; ρ is the mass density; χ is the equilibrated inertia; $\mathbf{f}$ is the body force; l is the equilibrated extrinsic body force; $\mathbf{u}(\mathbf{x}, t)$ is the displacement vector; $\phi(\mathbf{x}, t)$ is the volume fraction field. The quantities α, β, ξ are the void parameters. Symbol with a superposed dot denotes the partial derivative with respect to time variable t.

The constitutive relations are given by

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \beta\phi\delta_{ij}, \quad (1.3)$$

$$h_i = \alpha\phi,_i, \quad (1.4)$$

where σ_{ij} is the force stress tensor; h_i is the equilibrated stress vector; e_{ij} is the strain tensor; δ_{ij} is Kronecker delta and a comma in the subscript represents the spatial derivative, e.g.,

$$\phi,_i = \frac{\partial\phi}{\partial x_i}.$$

The other symbols have their usual meanings.

2. Torsional wave propagation

Consider an elastic half-space having uniform distribution of small void pores throughout. Introducing the cylindrical coordinate system (r, θ, z)

such that the $z-$axis is pointing vertically downward into the elastic half-space and $r-\theta$ plane is coincident with the boundary surface of the half-space. Let u, v and w denote the displacement components along radial, circumferential and axial directions, respectively. We shall consider a two-dimensional problem in $r-z$ plane and assume that the properties of the half-space are independent of θ coordinate. Thus, we have

$$u \ = w \ = 0, \quad v = \ v(r,z,t), \quad \phi = \phi(r,z,t), \quad \text{and} \quad \frac{\partial(\cdot)}{\partial\theta} \equiv 0.$$

With these considerations, the dynamical equations of motion (1.1) and (1.2) in the absence of body force $\mathbf{f}$ and extrinsic equilibrated body force l take the following form as

$$\mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right) + \beta\left(\frac{\partial\phi}{\partial r} + \frac{\partial\phi}{\partial z}\right) = \rho(1-\epsilon^2\nabla^2)\frac{\partial^2 v}{\partial t^2}, \tag{2.1}$$

$$\alpha\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2}\right) - \xi\phi = \rho\chi(1-\epsilon^2\nabla^2)\frac{\partial^2\phi}{\partial t^2}. \tag{2.2}$$

Introducing the non-dimensional quantities

$$s = \frac{r}{L}, \quad p = \xi_0 + \frac{z}{L}, \quad \omega = mT, \quad \text{and} \quad \tau = \frac{t}{T}, \tag{2.3}$$

where ξ_0 is a constant and m is the circular frequency having dimension T^{-1}. Here s is a dimensionless radial co-ordinate; p is a dimensionless depth co-ordinate; ω is dimensionless frequency; while T is the standard time; L is standard length such that L^{-1} shall represent the dimension of wavenumber. With these non-dimensional parameters, the equations (2.1) and (2.2) transform into

$$\begin{aligned}&\mu\left(\frac{\partial^2 v}{\partial s^2} + \frac{1}{s}\frac{\partial v}{\partial s} + \frac{\partial^2 v}{\partial p^2}\right) + \beta L\left(\frac{\partial\phi}{\partial s} + \frac{\partial\phi}{\partial p}\right)\\ &= \rho\frac{L^2}{T^2}\left[1 - \frac{\epsilon^2}{L^2}\left(\frac{\partial^2 v}{\partial s^2} + \frac{1}{s}\frac{\partial v}{\partial s} + \frac{\partial^2 v}{\partial p^2}\right)\right]\frac{\partial^2 v}{\partial\tau^2},\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\alpha\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{s}\frac{\partial\phi}{\partial s} + \frac{\partial^2\phi}{\partial p^2}\right) - L^2\xi\phi\\ &= \rho\chi\frac{L^2}{T^2}\left[1 - \frac{\epsilon^2}{L^2}\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial p^2}\right)\right]\frac{\partial^2\phi}{\partial\tau^2}.\end{aligned} \tag{2.5}$$

For the time harmonic propagation of torsional waves, the displacement and change in volume fraction fields can be taken as

$$\{v, \phi\}(p, s, \tau) \quad = \quad \{V, \Phi\}(p) J_0(s) e^{\iota\omega\tau}, \tag{2.6}$$

with J_0 as Bessel function of first kind and order zero, ω is the non-dimensional circular frequency of the wave and $\iota = \sqrt{-1}$.

Inserting (2.6) into equations (2.4) and (2.5), one can obtain

$$V''(p) - A_1 V(p) + B_1 \left[\Phi(p) \frac{J_0'(s)}{J_0(s)} + \Phi'(p)\right] \quad = \quad 0. \tag{2.7}$$

and

$$\Phi''(p) - A\Phi(p) = 0, \tag{2.8}$$

where

$$A_1 = \left[1 - (\epsilon^2 + L^2) \frac{\rho}{\mu} \frac{\omega^2}{T^2}\right] \left(1 - \epsilon^2 \frac{\rho}{\mu} \frac{\omega^2}{T^2}\right)^{-1}, \quad B_1 = \frac{\beta L}{\mu} \left(1 - \epsilon^2 \frac{\rho}{\mu} \frac{\omega^2}{T^2}\right)^{-1},$$

$$A = \left[1 - L^2 \frac{\xi}{\alpha} - (L^2 + \epsilon^2) \chi \frac{\rho}{\alpha} \frac{\omega^2}{T^2}\right] \left(1 - \epsilon^2 \chi \frac{\rho}{\alpha} \frac{\omega^2}{T^2}\right)^{-1}.$$

The general solution for equation (2.8) satisfying $\Phi \to 0$ as $p \to \infty$ is given by

$$\Phi = Ce^{-\sqrt{A}p}, \tag{2.9}$$

where C is an arbitrary constant.

Using (2.9) into (2.7), we obtain

$$V''(p) - A_1 V(p) = B_1 C \left(\sqrt{A} - \frac{J_0'(s)}{J_0(s)}\right) e^{-\sqrt{A}p}, \tag{2.10}$$

whose general solution that satisfies the condition $V(p) \to 0$ as $p \to \infty$ is given by

$$V = De^{-\sqrt{A_1}p} + \frac{B_1 C}{(A - A_1)} \left(\sqrt{A} - \frac{J_0'(s)}{J_0(s)}\right) e^{-\sqrt{A}p}, \tag{2.11}$$

where D is an arbitrary constant.

Utilizing (2.9) and (2.11) into (2.6), one can write the expressions of v and ϕ satisfying the radiation condition as

$$v = \left[De^{-\sqrt{A_1}p} + \frac{B_1 C}{(A - A_1)} \left(\sqrt{A} - \frac{J_0'(s)}{J_0(s)}\right) e^{-\sqrt{A}p}\right] J_0(s) e^{\iota\omega\tau}, \tag{2.12}$$

$$\phi = CJ_0(s)e^{-\sqrt{A}p}e^{\iota\omega\tau}. \tag{2.13}$$

3. Boundary conditions

The boundary surface of the half-space is assumed to be free from mechanical stresses. Accounting (1.3), (1.4) and (2.3), the appropriate boundary conditions are given by

$$(i)\ \sigma_{p\theta} = \frac{\partial v}{\partial p} = 0 \qquad \text{and} \qquad (ii)\ h_p = \frac{\partial \phi}{\partial p} = 0, \quad \text{at } p = \xi_0. \tag{3.1}$$

Note that the other components of stresses are automatically vanishing. These two boundary conditions give

$$\sqrt{A}\frac{\beta L}{\mu}\left(\sqrt{A} - \frac{J_0'(s)}{J_0(s)}\right)Ce^{\sqrt{A_1}\,\xi_0} + \sqrt{A_1}\left(1 - \epsilon^2\frac{\rho}{\mu}\frac{\omega^2}{T^2}\right)(A - A_1)\,De^{\sqrt{A}\,\xi_0} = 0,$$

and

$$\sqrt{A}J_0(s)Ce^{-\sqrt{A}\,\xi_0} = 0. \tag{3.2}$$

The non-zero value of constants C and D make the displacement v and change in void volume fraction ϕ to happen for the propagation of torsional waves. For the non-zero value of the constants C and D, the determinant of the coefficient matrix of homogeneous equations in (3.2) must vanish, which yields

$$\sqrt{A_1}\sqrt{A}(A - A_1) = 0. \tag{3.3}$$

Equation (3.3) gives

$$C_{T_1}^2 = \frac{c^2}{c_2^2} = \frac{1}{\omega^2\Gamma_1}, \qquad C_{T_2}^2 = \frac{c^2}{c_3^2} = \frac{1 + \frac{L^2}{R^2}}{\omega^2\Gamma_1}, \qquad C_{T_3}^2 = \frac{c^2}{c_3^2} = \frac{L^2}{\omega^2R^2\Gamma_2}, \tag{3.4}$$

where

$$\Gamma_1 = 1 + \frac{\epsilon^2}{L^2}, \qquad \Gamma_2 = 1 - \frac{c_3^2}{c_2^2}\left(1 - \frac{\epsilon^2}{R^2}\right), \qquad R = \sqrt{\frac{\alpha}{\xi}},$$

$$c = \frac{L}{T}, \qquad c_2^2 = \frac{\mu}{\rho} \qquad c_3^2 = \frac{\alpha}{\rho\chi}.$$

Here, C_{T_1}, C_{T_2} and C_{T_3} represent the non-dimensional speeds of three torsional modes in a nonlocal elastic half-space with voids. From the formulae obtained in (3.4), we note that:

(a) The speeds of all the three torsional modes are influenced by the nonlocality parameter (ϵ) of the half-space.

(b) From the first two formulae of (3.4), it is clear that

$$C_{T_2} = C_{T_1} \times \sqrt{1 + \frac{L^2}{R^2}}.$$

(c) The speed of one of the torsional modes, namely, C_{T_1} is independent of the presence of voids, while the remaining torsional modes are influenced by the presence of voids in the medium.
(d) The speeds of all the three torsional modes do depend on ω indicating that they are dispersive in nature.

3.1. **Special Case.** In the absence of nonlocality from the half-space, the problem reduces to the corresponding problem in the half-space with voids earlier discussed by Dey et al. [6]. For the purpose, we set the nonlocality parameter to zero, that is, $\epsilon = 0$. Using $\epsilon = 0$, we see that the formulae of C_{T_1}, C_{T_2} and C_{T_3} given in (3.4), reduce to

$$c^2 = \frac{c_2^2}{\omega^2}, \qquad c^2 = \frac{c_3^2}{\omega^2}\left(1 + \frac{L^2}{R^2}\right), \qquad c^2 = \frac{c_3^2}{\omega^2}\frac{L^2}{R^2}\left(\frac{c_2^2}{c_2^2 - c_3^2}\right),$$

respectively. Now owing to the relations $\omega = m\,T$ and $c = \frac{L}{T}$ defined earlier, it can be seen that $\omega\, c = m\, L$. Then the above formulae become

$$c_T = \sqrt{\frac{\mu}{\rho}}, \qquad c_T = c_3\sqrt{1 + \frac{1}{K_1^2\, R^2}}, \qquad c_T = \frac{1}{K_1\, R}\frac{c_2 c_3}{\sqrt{(c_2^2 - c_3^2)}}, \qquad (3.5)$$

respectively. Here the notations defined by $c_T = m\, L$ and $K_1 = L^{-1}$ have been used. These formulae are same as obtained by Dey et al. [6] for the corresponding problem in the relevant half-space apart from notations. Note that one of these formulae corresponds to shear wave speed in the elastic half-space, which is not influenced by the presence of voids.

4. Numerical results

In order to understand the dependence of torsional modes on the frequency parameter and to investigate the effect of various material parameters on the existing modes, numerical computations have been performed for a specific model with the following values of relevant parameters:

$$\mu = 7.5 \times 10^{10}\ Pa, \quad \rho = 2000\ kg/m^3, \quad \alpha = 8 \times 10^9 Pa\ m^2,$$

$$\xi = 12 \times 10^9\ Pa, \quad \chi = 0.16\ m^2, \quad e_0 = 0.39, \quad a = 0.5 \times 10^{-9}\ m.$$

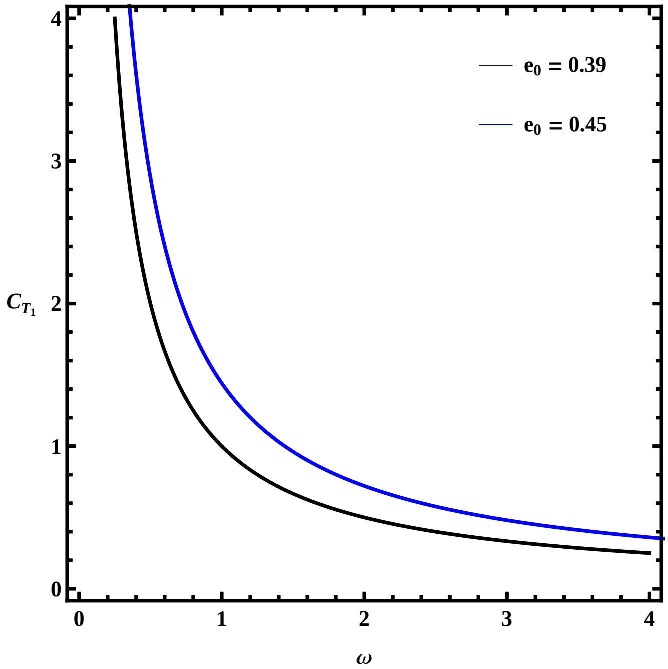

FIGURE 1. Variation of torsional mode speed C_{T_1} against frequency.

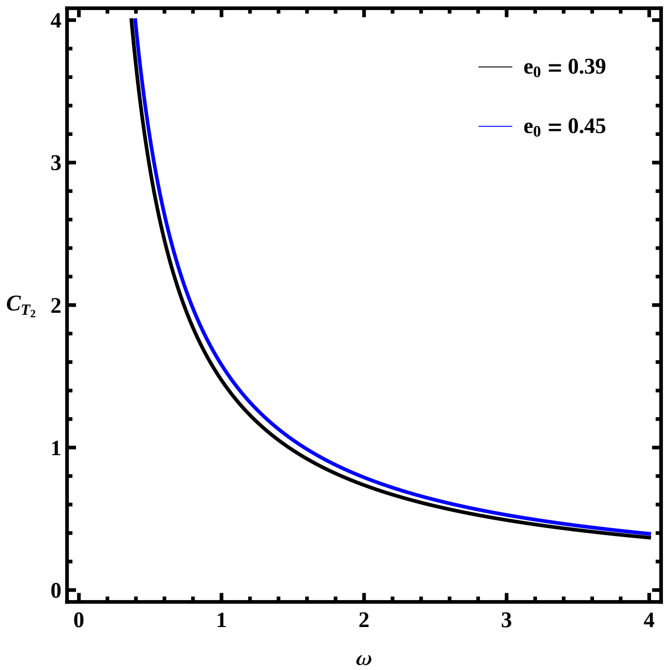

FIGURE 2. Variation of torsional mode speed C_{T_2} against frequency.

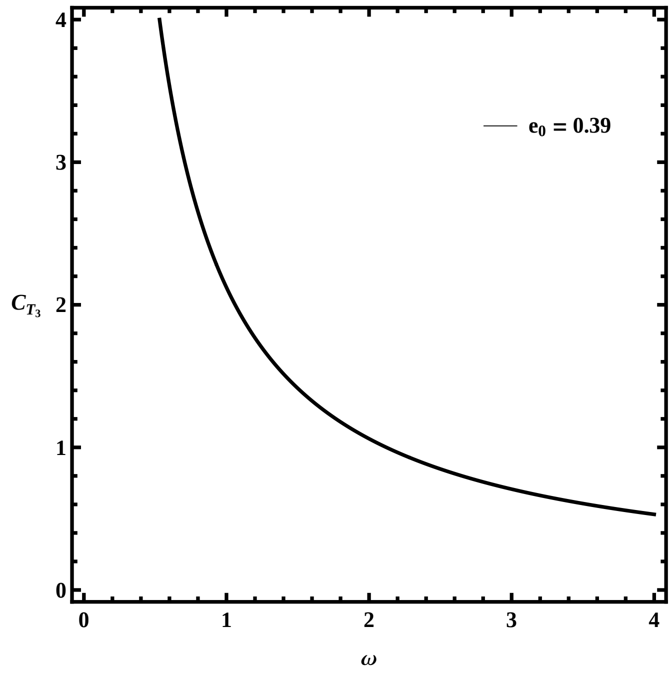

FIGURE 3. Variation of torsional mode speed C_{T_3} against frequency.

The non-dimensional speeds of torsional modes C_{T_i} $(i = 1, 2, 3)$ are computed from the formulae given in (3.4) for the considered model and plotted against the non-dimensional frequency parameter ω. The dispersion curves corresponding to the first, second and third torsional modes are shown through Figures 1, 2 & 3, respectively. In these figures, the black curve corresponds to $e_0 = 0.39$, while the blue curve corresponds to $e_0 = 0.45$. It is clear from these figures that the speeds of only two torsional modes, namely, C_{T_1} and C_{T_2} are affected significantly by the nonlocality of the medium, while the speed of the third torsional mode (C_{T_3}) is hardly affected. These plots show that the torsional wave speed (C_{T_1}) increases significantly with increase of e_0, while the torsional mode speed (C_{T_2}) increases slightly with increase of e_0. We note that the speeds C_{T_1} and C_{T_2} increase almost 45.5% and 8.5% respectively due to 15% increase of e_0. Whereas C_{T_3} decreases to very little extent upon changing e_0 from 0.39 to 0.45. It can be observed that all the three torsional modes are dispersive in nature.

In Figures 4, 5 & 6, the speeds of third torsional mode (C_{T_3}) is plotted

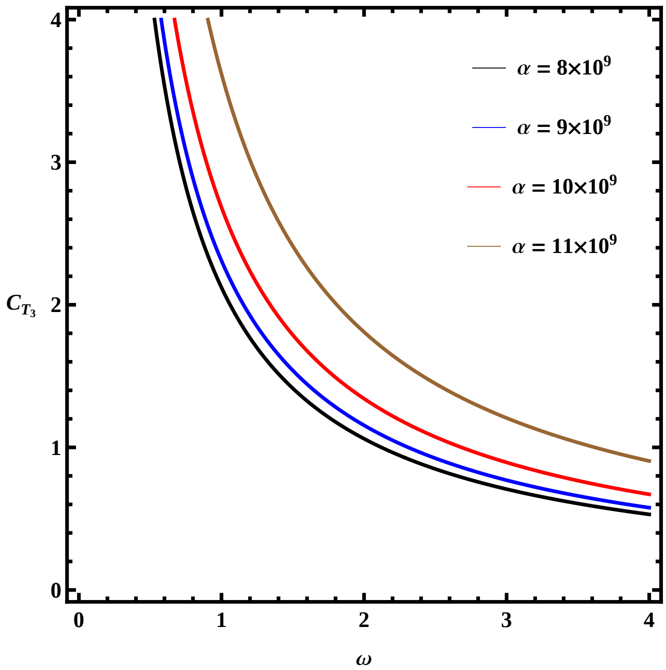

FIGURE 4. Variation of torsional mode speed C_{T_3} against frequency for different values of α.

against the non-dimensional frequency for different values of void parameters, respectively, α, χ and ξ. It is clear that this torsional wave speed is influenced by the void parameters. We note that the speed C_{T_3} increases with increase of parameter α, whereas it decreases with increase of parameter χ and ξ. The speed C_{T_3} increases around 8.7% when α enhances from 8×10^9 to 9×10^9; 16.1% when α enhances from 9×10^9 to 10×10^9 and 35.7% when α enhances from 10×10^9 to 11×10^9. Thus, we see that equal step size enhancement of α, almost doubles the speed of C_{T_3}. Further, the speed C_{T_3} decreases around 9.01% when χ enhances from 0.16 to 0.26 and 8.14% when α enhances from 0.26 to 0.36. It is also observed that the speed C_{T_3} shows decrease of around 13.8% when ξ decreases from 12×10^9 to 9×10^9 and 11.1% when ξ decreases from 9×10^9 to 7×10^9.

Figures 7 and 8, show the variation of the speed of torsional mode C_{T_3} against the dimensionless parameter $U = \frac{\alpha}{\xi L^2} (= K_1^2 R^2)$ for varying values of μ and α and at fixed value of frequency $\omega = 20\pi$. It can be observed from these figures that a decrease in the value of μ enhances the speed of torsional mode, while a decrease in the value of α slightly slowdown the

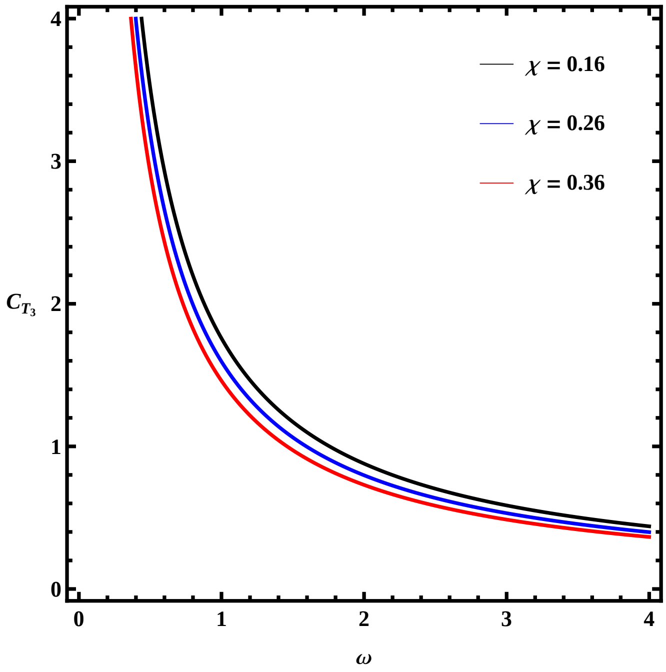

FIGURE 5. Variation of torsional mode speed C_{T_3} against frequency for different values of equilibrated inertia χ.

speed of torsional mode in the question. However, the respective deviation is very poor in the vicinity of zero frequency. We note that around 36% enhancement of μ results in around 71.7% decrease in the speed of C_{T_3}, while 60% enhancement of α results in increase of around 35.1% in the speed of C_{T_3}.

In Figure 9, all the three torsional wave speeds have been compared. This figure clearly shows that the speed C_{T_3} is the fastest as compared to C_{T_2} and C_{T_1} for the considered model. The magnitudes of the speeds of existing torsional modes can be compared as follows $C_{T_1} < C_{T_2} < C_{T_3}$. The speed C_{T_2} is almost midway between C_{T_1} and C_{T_3}.

5. Conclusions

A mathematical study for the propagation of torsional surface waves in a nonlocal elastic half-space with voids is performed. The following observations can be drawn from the analysis :

- It is found that there exist three torsional surface waves propagating with different speeds.

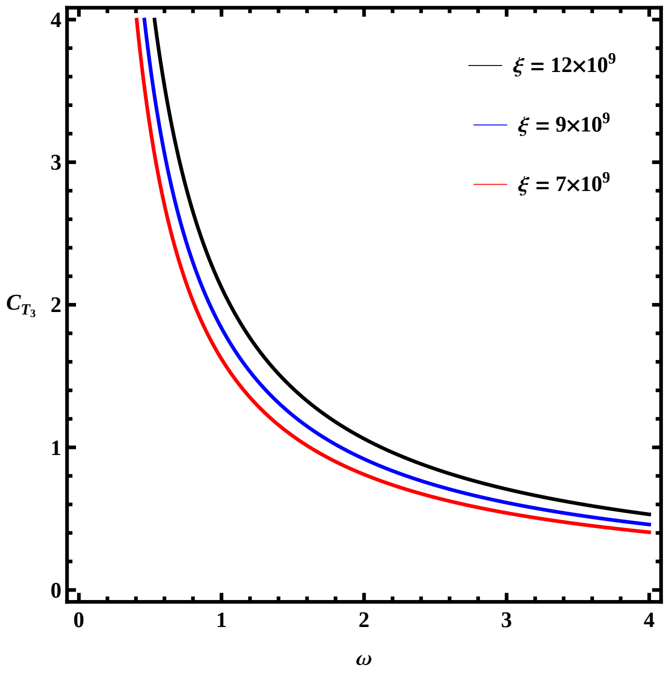

FIGURE 6. Variation of torsional mode speed C_{T_3} against frequency for different values of ξ.

- All the three torsional modes are found to be dispersive and depend on the non-locality parameter ϵ of the half-space.
- Two of the three torsional modes are found to be affected by the presence of voids in the medium, while the remaining mode travels independently.
- The speed of one of the torsional modes, namely, C_{T_2} is found to be $\left(1 + \frac{L^2}{R^2}\right)$ times multiple of C_{T_1}.
- Enhancing the value e_0 from 0.39 to 0.45, the speed C_{T_1} becomes almost double, the speed C_{T_2} becomes almost half, while the speed C_{T_3} undergoes slightest decrease.
- The speed C_{T_3} decreases around 9.01% when equilibrated inertia χ enhances from 0.16 to 0.26 and 8.14% when χ enhances from 0.26 to 0.36.
- The speed C_{T_3} decreases around 13.8% when the void parameter ξ decreases from 12×10^9 to 9×10^9 and 11.1% when ξ decreases from 9×10^9 to 7×10^9.

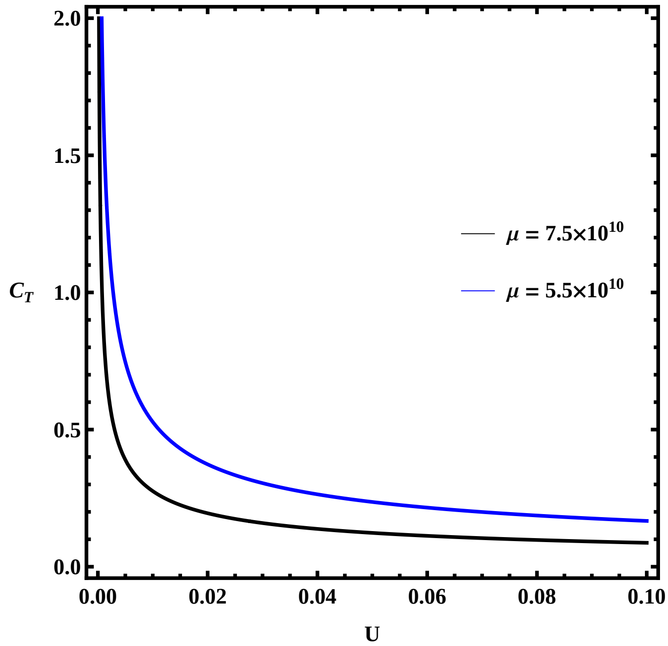

FIGURE 7. Variation of torsional mode speed C_{T_3} against non-dimensional quantity U for different values of μ.

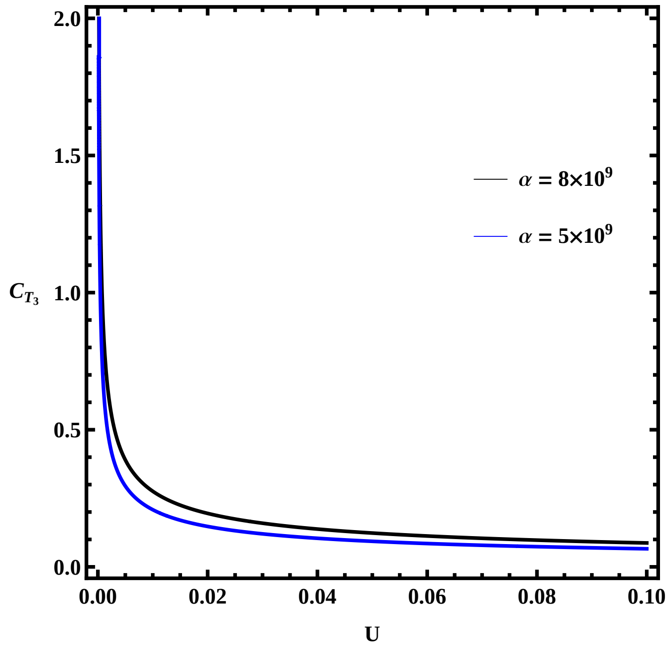

FIGURE 8. Variation of torsional mode speed C_{T_3} against non-dimensional quantity U for different values of α.

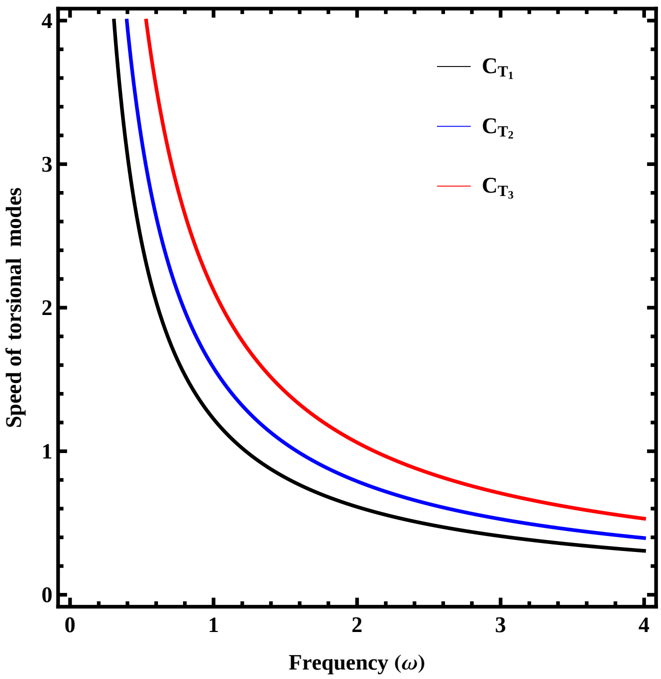

FIGURE 9. Comparison of the speeds of torsional modes

- In the absence of nonlocality, the torsional modes reduce to those obtained by Dey et al. [6] for the corresponding problem.

COMPETING INTERESTS STATEMENT

All authors have no competing interests.

DECLARATION OF INTEREST

'Declarations of interest: none'

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FIXED POINTS OF CONTRACTION MAPPINGS WITH VARIATIONS IN S-METRIC SPACE DOMAINS

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Abstract. This article has presented several generalized fixed point theorems with variations in S-metric space domains. In addition, several more fixed point results are obtained for different forms of contraction mappings that have closed graphs in S-metric spaces.

1. Introduction

Metric spaces are most important in pure and applied mathematics. So many authors were tried to find generalized metric spaces. B. C. Dhage introduced the concept of 2-metric space in [1]. The concept of D-metric space was introduced by S. G¨ahler in [3]. These two attempts have some drawbacks, see for example [9, 10]. So, Z. Mustafa, and B. Sims introduced the G-metric space in [8]. S. Sedghi et al modified concept of D-metric spaces to D^*-metric spaces in [17].

The concept of S-metric space was introduced by S. Sedghi et al in [16]. A S-metric is a real valued mapping on N^3, for some set $N \neq \emptyset$, where the map represents the perimeter of the triangle. Also given examples to every G-metric is a D^* metric and every D^*-metric is a S-metric in [16]. Because of introduction of this concept of S-metric spaces, many articles appeared for fixed point theory, see for example [2, 5, 4, 6, 11, 12, 13, 14, 15, 18].

A cycle of domains to derive some fixed point theorems for metric spaces was considered by C. G. Moorthy, and P. X. Raj in [7]. This article motivates us to consider an increasing sequence of subsets $N_1 \subseteq N_2 \subseteq ...$ of a S-metric space (N, S), and a map $T : N \to N$ satisfying a contraction condition such that $T(N_i) \subseteq N_{i+1}$, for all i, and $N = \bigcup\limits_{j=1}^{\infty} N_j$.

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In this paper, we prove some more fixed point theorems for various forms of contraction mappings having closed graph on S-metric spaces.

2. S-metric spaces

Definition 2.1. [16] Let $N \neq \emptyset$ be a set. The mapping $S : N^3 \to [0, \infty)$ is said to be a S-metric if

(i) $S(a, b, c) \geq 0$, for all $a, b, c \in N$ and
(ii) $S(a, b, c) = 0$ if and only if $a = b = c$, for all $a, b, c \in N$; and
(iii) $S(a, b, c) \leq S(a, a, e) + S(b, b, e) + S(c, c, e)$, for all $a, b, c, e \in N$.

Then (N, S) is called S-metric space(or, SMS).

Example 2.2. Let d be an ordinary metric on $N \neq \emptyset$, then $S(a, b, c) = d(a, b) + d(b, c) + d(c, a)$ is a S-metric on N.

Lemma 2.3. [16] *Let S be a S-metric on N, then $S(a, a, b) = S(b, b, a)$.*

Definition 2.4. [16] Suppose (N, S) is a SMS.
(I) A subset B of N is called S-bounded if there exists $r > 0$ such that $S(a, a, b) < r$, for all $a, b \in B$.
(II) A sequence $\{a_n\}$ in N is said to be convergent if for every $\epsilon > 0$, there is a positive integer n_0 such that for all $n > n_0$, $S(a_n, a_n, a) < \epsilon$, for some $a \in N$.
(III) A sequence $\{a_n\}$ in N is said to be Cauchy sequence if for any $\epsilon > 0$, there is a positive integer n_0 such that for all $n, m > n_0$, $S(a_n, a_n, a_m) < \epsilon$.

Remark 2.5. Let $\{a_n\}$ be a sequence in N. Then $\{a_n\}$ is said to be convergent to a if and only if $S(a_n, a_n, a) \to 0$ as $n \to \infty$, and $\{a_n\}$ is said to be Cauchy if and only if $S(a_n, a_n, a_m) \to 0$ as $n, m \to \infty$,

Lemma 2.6. [16] *Let (N, S) be a SMS. Let $\{a_n\}$ be a sequence in N. If $\{a_n\}$ converges to a and $\{a_n\}$ converges to b, then $a = b$.*

Definition 2.7. [16] A SMS (N, S) is called complete, if every Cauchy sequence is convergent in S.

Lemma 2.8. [16] *Let (N, S) be a SMS. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in N such that $a_n \to a$, $b_n \to b$ as $n \to \infty$. Then $S(a_n, a_n, b_n) \to S(a, a, b)$ as $n \to \infty$.*

Definition 2.9. Let T be a mapping from a SMS (N, S) into itself. If whenever $a_n \to a_0$ and $Ta_n \to b_0$ for some sequence $\{a_n\}$ in N and some a_0, b_0 in N, we have $b_0 = Ta_0$, then T is said to have a closed graph.

3. MAIN RESULTS

Theorem 3.1. *Let (N, S) be a complete SMS. Let $T : N \to N$ have a closed graph. Let $N_1 \subseteq N_2 \subseteq \dots$ be subsets of N such that $N = \bigcup_{j=1}^{\infty} N_j$, $T(N_i) \subseteq N_{i+1}$, for all i, and $S(Ta, Ta, Tb) \leq l_i S(a, a, b)$, for all $a, b \in N_i$, for all i, where $l_i \in (0, \infty)$ are constants such that $\sum_{n=1}^{\infty} l_1 l_2 \dots l_n < \infty$. Then, for any fixed $a_1 \in N$, $\{T^n a_1\}$ converges to a fixed point. Also, if $l_i \in (0, 1)$, for all i, then T has a unique fixed point(or, UFP) in N.*

Proof. Fix $a_1 \in N_1$, and define $a_{n+1} = Ta_n = T^n a_1$, for every $n = 1, 2, 3, \dots$. Then we have,

$$\begin{aligned} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) &\leq l_{n+1} S(T^n a_1, T^n a_1, T^{n-1} a_1) \\ &\leq l_{n+1} l_n l_{n-1} \dots l_2 S(Ta_1, Ta_1, a_1). \end{aligned}$$

Also, for $m > n \geq 1$, we have,

$$\begin{aligned} & S(T^m a_1, T^m a_1, T^n a_1) \\ \leq\ & 2S(T^m a_1, T^m a_1, T^{m-1} a_1) + 2S(T^{m-1} a_1, T^{m-1} a_1, T^{m-2} a_1) \\ & + \dots + S(T^{n+1} a_1, T^{n+1} a_1, T^n a_1) \\ \leq\ & 2 \sum_{i=n+1}^{m-1} S(T^{i+1} a_1, T^{i+1} a_1, T^i a_1) + S(T^{n+1} a_1, T^{n+1} a_1, T^n a_1) \\ \leq\ & 2\Big(\sum_{i=n+1}^{m-1} l_{i+1} l_i \dots l_2 \Big) S(Ta_1, Ta_1, a_1) + l_{n+1} l_n \dots l_2 S(Ta_1, Ta_1, a_1) \\ \leq\ & 2\Big(\sum_{i=n}^{m-1} l_{i+1} l_i \dots l_2 \Big) S(Ta_1, Ta_1, a_1). \end{aligned}$$

Therefore, $S(T^m a_1, T^m a_1, T^n a_1) \to 0$ $(n, m \to \infty)$. Thus, $\{T^m a_1\}_{m=1}^{\infty}$ is a Cauchy sequence in N. Since N is complete, $\{T^m a_1\}_{m=1}^{\infty}$ converges to a^* in N. It should be noted that $\{T^{m+1} a_1\}_{m=1}^{\infty}$ is also a Cauchy sequence and it converges to a^* in N. Since T has a closed graph, we should have $Ta^* = a^*$. Then a^* is a fixed point of T.
Such claims are relevant for the common case: $a_1 \in N_n$, for some n.
Suppose, further that $l_i \in (0, 1)$, for all i.
If a^*, b^* are fixed points of T, then let $a^*, b^* \in N_n$, for some n, so that

$$0 \leq S(a^*, a^*, b^*) = S(Ta^*, Ta^*, Tb^*) \leq l_n S(a^*, a^*, b^*).$$

Then, $S(a^*, a^*, b^*) \leq (l_n)^m S(a^*, a^*, b^*)$, for every $m = 1, 2, 3, ...$ Since $(l_n)^m \to 0$ as $m \to \infty$, $S(a^*, a^*, b^*) = 0$ and $a^* = b^*$. Therefore, T has a UFP. $\square$

Example 3.2. Let $N = [0, \infty)$ and $S : N^3 \to [0, \infty)$ be defined by $S(a, b, c) = |a - c| + |b - c|$. Then (N, S) is a complete SMS.
Let $N_n = [0, n]$, and $l_n = \frac{n^2}{(n+1)^2} \in (0, 1)$, for $n = 2, 3, ...$ Then $\sum_{n=2}^{\infty} l_2 ... l_n < \infty$.
Define $T : N \to N$ by $T(a) = 1 + \frac{1}{2^2} + \frac{2^2}{3^2}(a - 1)$, if $a \in N_n$, for $n = 2, 3, ...$
For $a, c \in N_n$, we have

$$\begin{aligned}
S(Ta, Ta, Tc) &= |Ta - Tc| + |Ta - Tc| \\
&= \left|\frac{2^2}{3^2}(a-1) - \frac{2^2}{3^2}(c-1)\right| + \left|\frac{2^2}{3^2}(a-1) - \frac{2^2}{3^2}(c-1)\right| \\
&= \frac{2^2}{3^2}\Big(|a - c| + |a - c|\Big) \\
&\leq l_n S(a, a, c), \text{ for all } n = 2, 3, ...
\end{aligned}$$

The assumptions in Theorem 3.1 are then fulfilled. Also, the UFP is $\frac{29}{20} = 1.45$.

Theorem 3.3. *Let (N, S) be a complete SMS and S-bounded. Let $T : N \to N$ have a closed graph. Let $l_i \in (0, 1)$, for all i, be such that $l_1 l_2 ... l_n \to 0$ as $n \to \infty$. Suppose $N_1 \subseteq N_2 \subseteq ...$ be subsets of N such that $T(N_i) \subseteq N_{i+1}$, for all i, and $S(Ta, Ta, Tb) \leq l_i d(a, a, b)$, for all $a \in N_i$, for all $b \in \bigcup_{j=1}^{\infty} N_j$, for all i. Let $a_1 \in \bigcup_{j=1}^{\infty} N_j$. Then the sequence $\{T^n a_1\}$ converges to a unique point in N, which is a fixed point of T. If $N = \bigcup_{j=1}^{\infty} N_j$, then T has a UFP in N.*

Proof. Fix $a_1, b_1 \in N_1$. Define $a_{n+1} = Ta_n = T^n a_1$, and $b_{n+1} = Tb_n = T^n b_1$, for every $n = 1, 2, 3, ...$ For $n > m$, we have

$$\begin{aligned}
S(a_n, a_n, b_m) &= S(Ta_{n-1}, Ta_{n-1}, Tb_{m-1}) \\
&\leq l_{m-1} S(a_{n-1}, a_{n-1}, b_{m-1}) \\
&\leq l_{m-1} l_{m-2} ... l_2 l_1 S(a_{n-m+1}, a_{n-m+1}, b_1) \\
&\leq l_{m-1} l_{m-2} ... l_2 l_1 r,
\end{aligned}$$

Therefore, $S(a_n, a_n, b_m) \to 0$ as $n, m \to \infty$. Also $S(a_n, a_n, a_m) \to 0$, and $S(b_n, b_n, b_m) \to 0$ as $n, m \to \infty$. So, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in N. Since, (S, d) is a complete, $\{a_n\}$ and $\{b_n\}$ converges to a unique point a^* in S, by Lemma 2.7. Since $\{a_n\} \to a^*$, we have $\{Ta_n\} \to a^*$. Since T has a closed graph, we have $Ta^* = a^*$.
Such claims are relevant for the common case: $a_1, b_1 \in N_n$, for some n.
Suppose now $N = \bigcup_{j=1}^{\infty} N_j$.
If a^*, b^* are fixed points of T, then let $a^* \in N_n$, and $b^* \in N_n$, for some n, so that

$$\begin{aligned} 0 \le S(a^*, a^*, b^*) &= d(Ta^*, Ta^*, Tb^*) \\ &\le l_n d(a^*, a^*, b^*) \\ &\le l_n l_{n+1} l_{n+2} \ldots l_m S(a^*, a^*, b^*), \text{ for all } m > n. \end{aligned}$$

So, $S(a^*, a^*, b^*) = 0$, because $l_1 l_2 \ldots l_m \to 0$ as $m \to \infty$. Therefore, T has a UFP, when $N = \bigcup_{j=1}^{\infty} N_j$. □

Example 3.4. Let $N = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, and $S : N^3 \to [0, \infty)$ be defined by $S(a, b, c) = |a - c| + |b - c|$. Then (N, S) is a complete SMS.
Let $N_n = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$, and $l_n = \frac{n}{(n+1)} \in (0, 1)$, for $n = 1, 2, 3, \ldots$. Then $l_1 l_2 \ldots l_n = \frac{1}{n+1} \to 0$ as $n \to \infty$.
Define $T : N \to N$ by $T(a) = 0$ if $a = 0$, and $T(a) = \frac{1}{n+1}$ if $a = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$.
For $n > m$, we have

$$\begin{aligned} S\Big(T\frac{1}{m}, T\frac{1}{m}, T\frac{1}{n}\Big) &= \Big|T\frac{1}{m} - T\frac{1}{n}\Big| + \Big|T\frac{1}{m} - T\frac{1}{n}\Big| \\ &= \frac{1}{m+1} - \frac{1}{n+1} + \frac{1}{m+1} - \frac{1}{n+1} \\ &= \frac{n-m}{nm}\frac{nm}{(m+1)(n+1)} + \frac{n-m}{nm}\frac{nm}{(m+1)(n+1)} \\ &\le l_m \frac{n-m}{nm} + l_m \frac{n-m}{nm} \\ &= l_m \Big|\frac{1}{m} - \frac{1}{n}\Big| + l_m \Big|\frac{1}{m} - \frac{1}{n}\Big| \\ &= l_m S\Big(\frac{1}{m}, \frac{1}{m}, \frac{1}{n}\Big). \end{aligned}$$

The assumptions in Theorem 3.3 are then fulfilled. Also, the UFP is 0.

Theorem 3.5. *Let (N, S) be a complete SMS. Let $T : N \to N$ have a closed graph. Let $N_1 \subseteq N_2 \subseteq \ldots$ be subsets of N such that $N = \bigcup\limits_{j=1}^{\infty} N_j$, $T(N_i) \subseteq N_{i+1}$, for all i, and $S(Ta, Ta, Tb) \leq l_i[S(Ta, Ta, a) + S(Tb, Tb, b)]$, for all $a, b \in N_i$, for all i, where $l_i \in (0, 1)$ are constants such that $\sum\limits_{n=1}^{\infty} h_1 h_2 \ldots h_n < \infty$, where $h_i = \frac{l_i}{1-l_i}$, for all i. Then T has a UFP in N. Also, for any fixed $a_1 \in N$, $\{T^n a_1\}$ converges to a UFP.*

Proof. Fix $a_1 \in N_1$, and define $a_{n+1} = Ta_n = T^n a_1$, for every $n = 1, 2, 3, \ldots$. Then we have,

$$\begin{aligned} & S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\ \leq\ & l_{n+1}[S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) + S(T^n a_1, T^n a_1, T^{n-1}a_1)] \\ \leq\ & l_{n+1}S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) + l_{n+1}S(T^n a_1, T^n a_1, T^{n-1}a_1). \end{aligned}$$

Now, we get

$$\begin{aligned} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) &\leq \frac{l_{n+1}}{1 - l_{n+1}} S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ &\leq h_{n+1} S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ &\leq h_{n+1} h_n h_{n-1} \ldots h_2 S(Ta_1, Ta_1, a_1). \end{aligned}$$

Also, for $m > n \geq 1$, we have,

$$\begin{aligned} & S(T^m a_1, T^m a_1, T^n a_1) \\ \leq\ & 2S(T^m a_1, T^m a_1, T^{m-1}a_1) + 2S(T^{m-1}a_1, T^{m-1}a_1, T^{m-2}a_1) \\ & + \ldots + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\ \leq\ & 2 \sum_{i=n+1}^{m-1} S(T^{i+1}a_1, T^{i+1}a_1, T^i a_1) + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\ \leq\ & 2\Big(\sum_{i=n+1}^{m-1} h_{i+1} h_i \ldots h_2 \Big) S(Ta_1, Ta_1, a_1) + l_{n+1} l_n \ldots l_2 S(Ta_1, Ta_1, a_1) \\ \leq\ & 2\Big(\sum_{i=n}^{m-1} h_{i+1} h_i \ldots h_2 \Big) S(Ta_1, Ta_1, a_1). \end{aligned}$$

Therefore, $S(T^m a_1, T^m a_1, T^n a_1) \to 0$ $(n, m \to \infty)$. Thus, $\{T^m a_1\}_{m=1}^{\infty}$ is a Cauchy sequence in N. Since N is complete, $\{T^m a_1\}_{m=1}^{\infty}$ converges to a^* in N. It should be noted that $\{T^{m+1} a_1\}_{m=1}^{\infty}$ is also a Cauchy sequence and it converges to a^* in N. Since T has a closed graph, we should have

$Ta^* = a^*$. Then a^* is a fixed point of T.
Such claims are relevant for the common case: $a_1 \in N_n$, for some n.
If a^*, b^* are fixed points of T, then let $a^*, b^* \in N_n$, for some n, so that

$$S(a^*, a^*, b^*) = S(Ta^*, Ta^*, Tb^*) \leq l_n[S(Ta^*, Ta^*, a^*) + S(Tb^*, Tb^*, b^*)] = 0.$$

Then, $S(a^*, a^*, b^*) = 0$ and $a^* = b^*$. Therefore, T has a UFP. $\square$

Example 3.6. Let $N = \{0, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, ...\}$, and $S : N^3 \to [0, \infty)$ be defined by $S(a, b, c) = |a - c| + |b - c|$. Then (N, S) is a complete SMS.
Let $N_n = \{0, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, ..., \frac{1}{3^{2n+1}}\}$, and $l_n = \frac{1}{3}$, for $n = 1, 2, 3, ...$ Then $\sum\limits_{n=1}^{\infty} h_1 h_2 ... h_n < \infty$, where $h_i = \frac{l_i}{1-l_i}$.
Define $T : N \to N$ by $T(a) = 0$ if $a = 0$, and $T(a) = \frac{1}{3^{n+2}}$ if $a = \frac{1}{3^n}$ for $n = 1, 2, 3, ...$
For $n > m$, we have

$$\begin{aligned}
& S\Big(T\frac{1}{3^m}, T\frac{1}{3^m}, T\frac{1}{3^n}\Big) \\
= \ & \Big|\frac{1}{3^{m+2}} - \frac{1}{3^{n+2}}\Big| + \Big|\frac{1}{3^{m+2}} - \frac{1}{3^{n+2}}\Big| \\
= \ & \Big|\frac{1}{3^{m+2}} - \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}} - \frac{1}{3^{n+2}}\Big| + \Big|\frac{1}{3^{m+2}} - \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}} - \frac{1}{3^{n+2}}\Big| \\
\leq \ & \frac{1}{3}\Big\{\Big|\frac{1}{3^{m+1}} - \frac{1}{3^n}\Big| + \Big|\frac{1}{3^n} - \frac{1}{3^{n+1}}\Big|\Big\} + \frac{1}{3}\Big\{\Big|\frac{1}{3^{m+1}} - \frac{1}{3^n}\Big| + \Big|\frac{1}{3^n} - \frac{1}{3^{n+1}}\Big|\Big\} \\
< \ & \frac{1}{3}\Big\{\frac{2}{3^{m+1}} + \Big|\frac{1}{3^n} - \frac{1}{3^{n+1}}\Big|\Big\} + \frac{1}{3}\Big\{\frac{2}{3^{m+1}} + \Big|\frac{1}{3^n} - \frac{1}{3^{n+1}}\Big|\Big\} \\
= \ & \frac{1}{3}\Big\{\Big|\frac{1}{3^m} - \frac{1}{3^{m+1}}\Big| + \Big|\frac{1}{3^n} - \frac{1}{3^{n+1}}\Big|\Big\} + \frac{1}{3}\Big\{\Big|\frac{1}{3^m} - \frac{1}{3^{m+1}}\Big| + \Big|\frac{1}{3^n} - \frac{1}{3^{n+1}}\Big|\Big\} \\
< \ & \frac{1}{3}\Big\{\Big|\frac{1}{3^m} - \frac{1}{3^{m+2}}\Big| + \Big|\frac{1}{3^n} - \frac{1}{3^{n+2}}\Big|\Big\} + \frac{1}{3}\Big\{\Big|\frac{1}{3^m} - \frac{1}{3^{m+2}}\Big| + \Big|\frac{1}{3^n} - \frac{1}{3^{n+2}}\Big|\Big\} \\
= \ & \frac{1}{3}\Big[S\Big(T\frac{1}{3^m}, T\frac{1}{3^m}, \frac{1}{3^m}\Big) + S\Big(T\frac{1}{3^n}, T\frac{1}{3^n}, \frac{1}{3^n}\Big)\Big].
\end{aligned}$$

The assumptions in Theorem 3.5 are then fulfilled. Also, the UFP is 0.

Theorem 3.7. *Let (N, S) be a complete SMS. Let $T : N \to N$ have a closed graph. Let $N_1 \subseteq N_2 \subseteq ...$ be subsets of N such that $N = \bigcup\limits_{j=1}^{\infty} N_j$, $T(N_i) \subseteq N_{i+1}$, for all i, and $S(Ta, Ta, Tb) \leq l_i[S(Ta, Ta, b) + S(Tb, Tb, a)]$, for all $a, b \in N_i$, for all i, where $l_i \in (0, \frac{1}{2})$ are constants such that $\sum\limits_{n=1}^{\infty} h_1 h_2 ... h_n < \infty$, where $h_i = \frac{l_i}{1-2l_i}$, for all i. Then T has a UFP in N. Also, for any fixed $a_1 \in N$, $\{T^n a_1\}$ converges to a UFP.*

Proof. Fix $a_1 \in N_1$, and define $a_{n+1} = Ta_n = T^n a_1$, for every $n = 1, 2, 3, ...$ Then we have,

$$\begin{aligned}
& S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\
\leq\ & l_{n+1}[S(T^{n+1}a_1, T^{n+1}a_1, T^{n-1}a_1) + S(T^n a_1, T^n a_1, T^n a_1)] \\
\leq\ & l_{n+1}[2S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) + S(T^{n-1}a_1, T^{n-1}a_1, T^n a_1)].
\end{aligned}$$

Now, we get

$$\begin{aligned}
S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) &\leq \frac{l_{n+1}}{1-2l_{n+1}} S(T^n a_1, T^n a_1, T^{n-1}a_1) \\
&\leq h_{n+1} S(T^n a_1, T^n a_1, T^{n-1}a_1) \\
&\leq h_{n+1} h_n h_{n-1} ... h_2 S(Ta_1, Ta_1, a_1).
\end{aligned}$$

Also, for $m > n \geq 1$, we have,

$$\begin{aligned}
& S(T^m a_1, T^m a_1, T^n a_1) \\
\leq\ & 2S(T^m a_1, T^m a_1, T^{m-1}a_1) + 2S(T^{m-1}a_1, T^{m-1}a_1, T^{m-2}a_1) \\
& +... + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\
\leq\ & 2\sum_{i=n+1}^{m-1} S(T^{i+1}a_1, T^{i+1}a_1, T^i a_1) + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\
\leq\ & 2\Big(\sum_{i=n+1}^{m-1} h_{i+1}h_i...h_2\Big) S(Ta_1, Ta_1, a_1) + l_{n+1}l_n...l_2 S(Ta_1, Ta_1, a_1) \\
\leq\ & 2\Big(\sum_{i=n}^{m-1} h_{i+1}h_i...h_2\Big) S(Ta_1, Ta_1, a_1).
\end{aligned}$$

Therefore, $S(T^m a_1, T^m a_1, T^n a_1) \to 0$ $(n, m \to \infty)$. Thus, $\{T^m a_1\}_{m=1}^{\infty}$ is a Cauchy sequence in N. Since N is complete, $\{T^m a_1\}_{m=1}^{\infty}$ converges to a^* in N. It should be noted that $\{T^{m+1}a_1\}_{m=1}^{\infty}$ is also a Cauchy sequence and it converges to a^* in N. Since T has a closed graph, we should have $Ta^* = a^*$.

Then a^* is a fixed point of T. Such claims are relevant for the common case: $a_1 \in N_n$, for some n. If a^*, b^* are fixed points of T, then let $a^*, b^* \in N_n$, for some n, so that

$$\begin{aligned} S(a^*,a^*,b^*) = S(Ta^*,Ta^*,Tb^*) &\leq l_n[S(Ta^*,Ta^*,b^*) + S(Tb^*,Tb^*,a^*)] \\ &= l_n[S(a^*,a^*,b^*) + S(b^*,b^*,a^*)] \\ &= 2l_n S(a^*,a^*,b^*). \end{aligned}$$

Then, $S(a^*,a^*,b^*) \leq (2l_n)^m S(a^*,a^*,b^*)$, for every $m = 1,2,3,....$ Since $(2l_n)^m \to 0$ as $m \to \infty$, $S(a^*,a^*,b^*) = 0$ and $a^* = b^*$. Therefore, T has a UFP.

□

Theorem 3.8. *Let (N,S) be a complete SMS. Let $T : N \to N$ have a closed graph. Let $N_1 \subseteq N_2 \subseteq ...$ be subsets of N such that $N = \bigcup_{j=1}^{\infty} N_j$, $T(N_i) \subseteq N_{i+1}$, for all i, and $S(Ta,Ta,Tb) \leq l_i S(a,a,b) + t_i S(b,b,Ta)$, for all $a,b \in N_i$, for all i, where $l_i, t_i \in (0,1)$ are constants such that $l_i + t_i < 1$, for all i, and $\sum_{n=1}^{\infty} h_1 h_2 ... h_n < \infty$, where $h_i = \frac{l_i+2t_i}{1-t_i}$, for all i. Then T has a UFP in N. Also, for any fixed $a_1 \in N$, $\{T^n a_1\}$ converges to a UFP.*

Proof. Fix $a_1 \in N_1$, and define $a_{n+1} = Ta_n = T^n a_1$, for every $n = 1,2,3,....$ Then we have,

$$\begin{aligned} & S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\ \leq\ & l_{n+1} S(T^n a_1, T^n a_1, T^{n-1}a_1) + t_{n+1} S(T^{n-1}a_1, T^{n-1}a_1, T^{n+1}a_1)] \\ \leq\ & l_{n+1} S(T^n a_1, T^n a_1, T^{n-1}a_1) + t_{n+1} 2S(T^{n-1}a_1, T^{n-1}a_1, T^n a_1) \\ & + t_{n+1} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\ =\ & (l_{n+1} + 2t_{n+1}) S(T^n a_1, T^n a_1, T^{n-1}a_1) + t_{n+1} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \end{aligned}$$

Now, we get

$$\begin{aligned} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) &\leq \frac{l_{n+1} + 2t_{n+1}}{1 - t_{n+1}} S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ &\leq h_{n+1} S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ &\leq h_{n+1} h_n h_{n-1} ... h_2 S(Ta_1, Ta_1, a_1). \end{aligned}$$

Also, for $m > n \geq 1$, we have,

$$\begin{aligned}
& S(T^m a_1, T^m a_1, T^n a_1) \\
\leq \;& 2S(T^m a_1, T^m a_1, T^{m-1}a_1) + 2S(T^{m-1}a_1, T^{m-1}a_1, T^{m-2}a_1) \\
& +... + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\
\leq \;& 2\sum_{i=n+1}^{m-1} S(T^{i+1}a_1, T^{i+1}a_1, T^i a_1) + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\
\leq \;& 2\Big(\sum_{i=n+1}^{m-1} h_{i+1}h_i...h_2\Big) S(Ta_1, Ta_1, a_1) + l_{n+1}l_n...l_2 S(Ta_1, Ta_1, a_1) \\
\leq \;& 2\Big(\sum_{i=n}^{m-1} h_{i+1}h_i...h_2\Big) S(Ta_1, Ta_1, a_1).
\end{aligned}$$

Therefore, $S(T^m a_1, T^m a_1, T^n a_1) \to 0$ $(n, m \to \infty)$. Thus, $\{T^m a_1\}_{m=1}^{\infty}$ is a Cauchy sequence in N. Since N is complete, $\{T^m a_1\}_{m=1}^{\infty}$ converges to a^* in N. It should be noted that $\{T^{m+1}a_1\}_{m=1}^{\infty}$ is also a Cauchy sequence and it converges to a^* in N. Since T has a closed graph, we should have $Ta^* = a^*$. Then a^* is a fixed point of T.
Such claims are relevant for the common case: $a_1 \in N_n$, for some n.
If a^*, b^* are fixed points of T, then let $a^*, b^* \in N_n$, for some n, so that

$$\begin{aligned}
S(a^*, a^*, b^*) = S(Ta^*, Ta^*, Tb^*) &\leq l_n S(a^*, a^*, b^*) + t_n S(b^*, b^*, Ta^*)] \\
&= (l_n + t_n) S(a^*, a^*, b^*).
\end{aligned}$$

Then, $S(a^*, a^*, b^*) \leq (l_n + t_n)^m S(a^*, a^*, b^*)$, for every $m = 1, 2, 3, ...$ Since $(l_n + t_n)^m \to 0$ as $m \to \infty$, $S(a^*, a^*, b^*) = 0$ and $a^* = b^*$. Therefore, T has a UFP. $\square$

Theorem 3.9. *Let (N, S) be a complete SMS. Let $T : N \to N$ have a closed graph. Let $N_1 \subseteq N_2 \subseteq ...$ be subsets of N such that $N = \bigcup_{j=1}^{\infty} N_j$, $T(N_i) \subseteq N_{i+1}$, for all i, and $S(Ta, Ta, Tb) \leq q_i S(a, a, b) + l_i[S(Ta, Ta, a) + S(Tb, Tb, b)] + r_i[S(Ta, Ta, b) + S(Tb, Tb, a)]$, for all $a, b \in N_i$, for all i, where $q_i, l_i, r_i \in (0, \frac{1}{2})$ are constants such that $q_i + 2r_i < 1$, for all i, and*

$\sum_{n=1}^{\infty} h_1h_2...h_n < \infty$, *where* $h_i = \frac{q_i+l_i+r_i}{1-l_i-r_i}$, *for all* i. *Then* T *has a UFP in* N. *Also, for any fixed* $a_1 \in N$, $\{T^n a_1\}$ *converges to a UFP.*

Proof. Fix $a_1 \in N_1$, and define $a_{n+1} = Ta_n = T^n a_1$, for every $n = 1, 2, 3, ...$ Then we have,

$$\begin{aligned} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) &\leq q_{n+1}S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ +l_{n+1}[S(T^{n+1}a_1, &T^{n+1}a_1, T^n a_1) + S(T^n a_1, T^n a_1, T^{n-1}a_1)] \\ +r_{n+1}[S(T^{n+1}a_1, &T^{n+1}a_1, T^{n-1}a_1) + S(T^n a_1, T^n a_1, T^n a_1)] \end{aligned}$$

$$\begin{aligned} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) &\leq q_{n+1}S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ +l_{n+1}S(T^{n+1}a_1, T^{n+1}a_1, &T^n a_1) + l_{n+1}S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ +r_{n+1}[2S(T^{n+1}a_1, T^{n+1}a_1, &T^n a_1) + S(T^{n-1}a_1, T^{n-1}a_1, T^n a_1)]. \end{aligned}$$

Now, we get

$$\begin{aligned} S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) &\leq (\frac{q_{n+1}+l_{n+1}+r_{n+1}}{1-l_{n+1}-r_{n+1}})S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ &\leq h_{n+1}S(T^n a_1, T^n a_1, T^{n-1}a_1) \\ &\leq h_{n+1}h_n h_{n-1}...h_2 S(Ta_1, Ta_1, a_1). \end{aligned}$$

Also, for $m > n \geq 1$, we have,

$$\begin{aligned} & S(T^m a_1, T^m a_1, T^n a_1) \\ \leq\ & 2S(T^m a_1, T^m a_1, T^{m-1}a_1) + 2S(T^{m-1}a_1, T^{m-1}a_1, T^{m-2}a_1) \\ & +... + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\ \leq\ & 2\sum_{i=n+1}^{m-1} S(T^{i+1}a_1, T^{i+1}a_1, T^i a_1) + S(T^{n+1}a_1, T^{n+1}a_1, T^n a_1) \\ \leq\ & 2\Big(\sum_{i=n+1}^{m-1} h_{i+1}h_i...h_2\Big)S(Ta_1, Ta_1, a_1) + l_{n+1}l_n...l_2 S(Ta_1, Ta_1, a_1) \\ \leq\ & 2\Big(\sum_{i=n}^{m-1} h_{i+1}h_i...h_2\Big)S(Ta_1, Ta_1, a_1). \end{aligned}$$

Therefore, $S(T^m a_1, T^m a_1, T^n a_1) \to 0$ $(n, m \to \infty)$. Thus, $\{T^m a_1\}_{m=1}^{\infty}$ is a Cauchy sequence in N. Since N is complete, $\{T^m a_1\}_{m=1}^{\infty}$ converges to a^* in N. It should be noted that $\{T^{m+1}a_1\}_{m=1}^{\infty}$ is also a Cauchy sequence and it converges to a^* in N. Since T has a closed graph, we should have

$Ta^* = a^*$. Then a^* is a fixed point of T.
Such claims are relevant for the common case: $a_1 \in N_n$, for some n.
If a^*, b^* are fixed points of T, then let $a^*, b^* \in N_n$, for some n, so that

$$\begin{aligned} S(a^*, a^*, b^*) &= S(Ta^*, Ta^*, Tb^*) \\ &\leq q_n S(a^*, a^*, b^*) + l_n[S(Ta^*, Ta^*, a^*) + S(Tb^*, Tb^*, b^*)] \\ &\quad + r_n[S(Ta^*, Ta^*, b^*) + S(Tb^*, Tb^*, a^*)] \\ &= q_n S(a^*, a^*, b^*) + r_n S(a^*, a^*, b^*) + r_n S(b^*, b^*, a^*) \\ &= (\frac{q_n}{1-2r_n}) S(a^*, a^*, b^*) \end{aligned}$$

Then, $S(a^*, a^*, b^*) \leq (\frac{q_n}{1-2r_n})^m S(a^*, a^*, b^*)$, for every $m = 1, 2, 3, \ldots$. Since $(\frac{q_n}{1-2r_n})^m \to 0$ as $m \to \infty$, Thus, $S(a^*, a^*, b^*) = 0$ and $a^* = b^*$. Therefore, T has a UFP. □

Corollary 3.10. *Let (N, S) be a complete SMS. Let $T : N \to N$ have a closed graph. Let $N_1 \subseteq N_2 \subseteq \ldots$ be subsets of N such that $N = \bigcup\limits_{j=1}^{\infty} N_j$, $T(N_i) \subseteq N_{i+1}$, for all i, and $S(Ta, Ta, Tb) \leq q_i S(a, a, b) + l_i[S(Ta, Ta, a) + S(Tb, Tb, b)]$, for all $a, b \in N_i$, for all i, where $q_i, l_i \in (0, 1)$ are constants such that $\sum\limits_{n=1}^{\infty} h_1 h_2 \ldots h_n < \infty$, where $h_i = \frac{q_i + l_i}{1 - l_i}$, for all i. Then T has a UFP in N. Also, for any fixed $a_1 \in N$, $\{T^n a_1\}$ converges to a UFP.*

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A NOTE ON ORTHOGONAL AND ALTERNATE DUAL G-FRAME PAIRS

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ABSTRACT. W. Sun introduced the concept of g- frames which are generalized frames in Hilbert spaces. In this paper, we consider linear combination of g-frames with coefficients as bounded linear operators and construct some new g-frames from existing g-frames, considering the cases of orthogonal and alternate dual g-frame pair in a Hilbert space and give alternate proofs of some previously proved results. We use our result to construct Gabor frames.

1. INTRODUCTION

Frames in Hilbert spaces have been introduced in 1952 by J. Duffin and A.C. Schaeffer [7] while studying non harmonic Fourier series. The work of Daubechies, Grossmann and Meyer [6] in 1986 reintroduced the Frames.

In [14], W. Sun introduced the concept of generalized frames (or g-frames) in Hilbert spaces, which are generalizations of frames and cover many other recent generalizations of frames such as bounded quasi-projections, fusion frames, and pseudo frames. Constructions of frames and g-frames is an interesting problem and is useful in applications. Therefore, algebraic operations are considered among frames to construct new frames from existing frames. For more details see [1, 3, 11] and the references therein.

In this paper, we construct new g-frames by considering sum and difference of orthogonal and alternate dual g-frames. Our work generalizes the work done in [2]. We give here alternate proofs of ([9], *Theorem 31, Theorem 32*) using the concept of R-duality of g-frames.

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2. Preliminaries

Throughout this paper, $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces and $\{\mathscr{H}_i\}_{i\in I}$ is a sequence of closed subspaces of $\mathscr{K}$, where I is a subset of Z and $L(\mathscr{H}, \mathscr{H}_i)$ is the collection of all bounded linear operators from $\mathscr{H}$ into $\mathscr{H}_i$. And we denote by $I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$. Hilbert space adjoint operator of T is denoted by T^*. By $T^\dagger$ we denote the pseudo-inverse of the operator T.

Operators on $L^2(\mathbb{R})$:
Translation by $a \in \mathbb{R}$, $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (T_a f)(x) = f(x-a)$;
Modulation by $b \in \mathbb{R}$, $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (E_b f)(x) = e^{2\pi ibx} f(x)$;
Dilation $a \neq 0$, $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (D_a f)(x) = \dfrac{1}{\sqrt{|a|}} f(\dfrac{x}{a})$.

Definition 2.1. A sequence $\{f_i : i \in I\}$ of elements in $\mathscr{H}$ is called a frame for $\mathscr{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \;\; \forall\, f \in \mathscr{H}.$$

The constants A and B are called lower and upper frame bounds.

Definition 2.2. $\Big(\sum_{i\in I} \oplus \mathscr{H}_i\Big)_{l^2}$ is a Hilbert space and is defined by

$$\left(\sum_{i\in I} \oplus \mathscr{H}_i\right)_{l^2} = \left\{ \{f_i\}_{i\in I} : f_i \in \mathscr{H}_i, i \in I, \; \|\{f_i\}_{i\in I}\|^2 = \sum_{i\in I} \|f_i\|^2 < \infty \right\}.$$

with the inner product defined by: $\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$.

Definition 2.3. [14] A sequence $\{\Lambda_i \in L(\mathscr{H}, \mathscr{H}_i) : i \in I\}$ of bounded operators is said to be a generalized frame or simply a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in I}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \;\; \forall\, f \in \mathscr{H}.$$

we call A and B the lower and upper g-frame bounds, respectively. We call $\{\Lambda_i\}_{i\in I}$ a tight g-frame if $A = B$ and a Parseval g-frame or a normalized tight g-frame if $A = B = 1$.

We call $\{\Lambda_i : i \in I\}$ an exact g-frame if it ceases to be a g-frame whenever any one of its element is removed.

We call $\{\Lambda_i : i \in I\}$ a g-frame for $\mathscr{H}$ whenever $\mathscr{H}_i = \mathscr{H}$, $\forall\, i \in I$.

The synthesis(g-pre frame) operator of $\{\Lambda_i\}_{i\in I}; T_\Lambda : \left(\sum_{i\in I} \oplus \mathscr{H}_i\right)_{l^2} \to \mathscr{H}$ is defined by

$$T_\Lambda\Big(\{f_i\}_{i\in I}\Big) = \sum_{i\in I} \Lambda_i^* f_i.$$

We call the adjoint T_Λ^*, where $T_\Lambda^* : \mathscr{H} \to \Big(\sum_{i\in I} \oplus \mathscr{H}_i\Big)_{l^2}$ of the synthesis operator, the analysis operator which is given by

$$T_\Lambda^* f = \{\Lambda_i f\}_{i\in I}, \ \ \forall\, f \in \mathscr{H}.$$

By composing T_Λ and T_Λ^*, we obtain the g-frame operator $S_\Lambda : \mathscr{H} \to \mathscr{H}$ given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f$$

which is bounded, positive, self adjoint, invertible operator and satisfies $AI_{\mathscr{H}} \leq S_\Lambda \leq BI_{\mathscr{H}}$. Then the following reconstruction formula takes place for all $f \in \mathscr{H}$

$$f = S_\Lambda^{-1} S_\Lambda f = S_\Lambda S_\Lambda^{-1} f.$$

$\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ is also a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in I}$ with bounds B^{-1} and A^{-1} and it is said to be the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Definition 2.4. A g-frame $\{\Theta_i\}_{i\in I}$ of $\mathscr{H}$ is called an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$ if it satisfies

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f, \qquad \forall\, f \in \mathscr{H}.$$

In terms of synthesis operators

$$T_\Lambda T_\Theta^* = I_{\mathscr{H}} \quad \text{or} \quad T_\Theta T_\Lambda^* = I_{\mathscr{H}}.$$

where T_Λ and T_Θ are the synthesis operators for $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ respectively.

It is easy to show that if $\{\Theta_i\}_{i\in I}$ is an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i\}_{i\in I}$ is an alternate dual g-frame of $\{\Theta_i\}_{i\in I}$.

Definition 2.5. We call two g-Bessel sequences $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ to be orthogonal if

$$\sum_{i\in I}\Lambda_i^*\Theta_i f=0 \text{ or } \sum_{i\in I}\Theta_i^*\Lambda_i f=0, \qquad \forall\, f\in\mathscr{H}.$$

In terms of synthesis operators

$$T_\Lambda T_\Theta^*=0 \quad\text{or}\quad T_\Theta T_\Lambda^*=0.$$

Definition 2.6. [14] Let $\Lambda_i\in L(\mathscr{H},\mathscr{H}_i), i\in I$.

(1) If the right hand inequality of (2) holds, then we say that $\{\Lambda_i : i\in I\}$ is a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{H}_i : i\in I\}$.
(2) If $\{f : \Lambda_i f=0,\ i\in I\}=\{0\}$, then we say that $\{\Lambda_i : i\in I\}$ is g-complete.
(3) If $\{\Lambda_i : i\in I\}$ is g-complete and there are positive constants A and B such that for any finite subset $I_1\subset I$ and $g_i\in\mathscr{H}_i,\ i\in I_1$,

$$A\sum_{i\in I_1}\|g_i\|^2\le\left\|\sum_{i\in I_1}\Lambda_i^*g_i\right\|^2\le B\sum_{i\in I_1}\|g_i\|^2$$

then we say that $\{\Lambda_i : i\in I\}$ is g-Riesz basis for $\mathscr{H}$ with respect to $\{\mathscr{H}_i : i\in I\}$.
(4) We say that $\{\Lambda_i : i\in I\}$ is a g-orthonormal basis for $\mathscr{H}$ with respect to $\{\mathscr{H}_i : i\in I\}$ if it satisfies the following:

$$\langle\Lambda_{i_1}^*g_{i_1},\Lambda_{i_2}^*g_{i_2}\rangle=\delta_{i_1,i_2}\langle g_{i_1},g_{i_2}\rangle,\ \ \forall\, i_1,i_2\in I,\ g_{i_1}\in\mathscr{H}_{i_1},g_{i_2}\in\mathscr{H}_{i_2},$$

$$\sum_{i\in I}\|\Lambda_i f^2\|=\|f\|^2,\ \ \forall\, f\in\mathscr{H}.$$

Definition 2.7. [8] Let $\{\Xi_i\}_{i\in I}$ and $\{\Psi_i\}_{i\in I}$ be g-orthonormal bases for $\mathscr{H}$ with respect to $\{\mathscr{W}_i\}_{i\in I}$ and $\{\mathscr{H}_i\}_{i\in I}$, respectively. Let $\{\Lambda_i\in L(\mathscr{H},\mathscr{H}_i) : i\in I\}$ be such that the series $\sum_{i\in I}\Lambda_i^*g_i^{'}$ is convergent for all $\{g_i^{'}\}_{i\in I}\in\left(\sum_{i\in I}\oplus\mathscr{H}_i\right)_{l^2}$.

The g-R-dual sequence for the sequence $\{\Lambda_i\}_{i\in I}$ is $\Gamma_j^\Lambda : \mathscr{H}\to\mathscr{W}_j$ which is defined as

$$\Gamma_j^\Lambda=\sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i, \qquad \forall\, j\in I.$$

Definition 2.8. Let $g\in L^2(\mathbb{R})$ and $a,b>0$. The sequence $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is called a Gabor system. A Gabor system is called a Gabor frame (resp.

Gabor Riesz basis) if it is a frame (resp. Riesz basis) for $L^2(\mathbb{R})$. A Gabor system is called a Gabor sequence (resp. Gabor Riesz sequence) if it is a frame sequence (resp. Riesz sequence). Gabor frames are concrete realization of frames.

Definition 2.9. Let D and T be the standard dilation and translation operators, respectively, on $L^2(\mathbb{R})$, defined by $(Df)(x) = \sqrt{2}f(2x)$ and $(Tf)(x) = f(x-1)$ for any $f \in L^2(\mathbb{R})$.
A function $\phi \in L^2(\mathbb{R})$ is called a frame wavelet of $L^2(\mathbb{R})$ if

$$\{\phi_{n,l}(x)\} = \{2^{n/2}\phi(2^n x - l) : n, l \in Z\} = \{D^n T^l \phi : n, l \in Z\}$$

is a frame of $L^2(\mathbb{R})$, i.e., if there exist two positive constants $0 < A \leq B$ such that

$$A\,\|f\|^2 \leq \sum_{n,l\in Z} |\langle f, D^n T^l \phi\rangle|^2 \leq B\,\|f\|^2$$

for all $f \in L^2(\mathbb{R})$. ϕ is called a tight frame wavelet if this frame is tight. Similarly, ϕ is called a normalized tight frame wavelet if this frame is a normalized tight frame.

The following results which are referred to in this paper are listed in the form of lemmas.

Lemma 2.9. [9]. *Let $\{\Theta_i \in L(\mathscr{H}, \mathscr{H}_i) : i \in I\}$ be a g-orthonormal basis for $\mathscr{H}, T \in L(\mathscr{H})$. Define the transformation $\Phi_T : \{\Lambda_i \in L(\mathscr{H}, \mathscr{H}_i) : i \in I\} \to \{\Lambda_i T^* : i \in I\}$, then*

1. (1) *It transforms g-frames to g-frames if and only if T is onto.*
2. (2) *It transforms normalized tight g-frame to normalized tight g-frame if and only if T is a coisometry.*
3. (3) *It transforms g-Riesz bases to g-Riesz bases if and only if T is invertible.*
4. (4) *It transforms g-orthonormal bases to g-orthonormal bases if and only if T is unitary.*

Lemma 2.10. [5]. *Let $T : \mathscr{K} \to \mathscr{H}$ be a bounded surjective operator. Then there exists a bounded operator (called the pseudoinverse of T) $T^\dagger : \mathscr{H} \to \mathscr{K}$ for which $TT^\dagger f = f$, $\forall\, f \in \mathscr{H}$.*

Lemma 2.11. [8]. Let $\{\Lambda_i\}_{i\in I}$ *and* $\{\Omega_i\}_{i\in I}$ *be g-frames for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i\}_{i\in I}$. *Then* $\{\Omega_i\}_{i\in I}$ *is a dual g-frame of* $\{\Lambda_i\}_{i\in I}$ *if and only if g-R-dual sequences* $\left\{\Gamma_j^{\Lambda}\right\}_{j\in I}$ *and* $\left\{\Gamma_j^{\Omega}\right\}_{j\in I}$ *are g-biorthogonal; that is,*

$$\Gamma_i^{\Lambda}\left(\Gamma_j^{\Omega}\right)^* g_j = \Gamma_i^{\Omega}\left(\Gamma_j^{\Lambda}\right)^* g_j = \delta_{ij} g_j, \quad \forall\, i, j \in I,\ g_j \in \mathscr{W}_j$$

Lemma 2.12. [13]. *Let* $U : \mathscr{K} \to \mathscr{H}$ *be a bounded operator. Then the following hold:*

(1) $\|U\| = \|U^*\|$ *and* $\|UU^*\| = \|U\|^2$.
(2) *Range of* U *is closed in* $\mathscr{H}$ *if and only if range of* U^* *is closed in* $\mathscr{K}$.
(3) U *is surjective if and only if there exists a constant* $C > 0$ *such that* $\|U^* y\| \geq C\|y\|$, $\forall\, y \in \mathscr{H}$.

3. Main Result

This section deals with the following problem:
When is $\{\Lambda_i U^* + \Gamma_i V^* : i \in I\}$ a g-frame, given $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are orthogonal g-frames for $\mathscr{H}$ and $U, V \in L(\mathscr{H})$?

Theorem 3.1. *Let* $\{\Lambda_i : i \in I\}$ *and* $\{\Gamma_i : i \in I\}$ *be two g-frames for Hilbert space* $\mathscr{H}$, *and let* T_Λ *and* T_Γ *be synthesis operators of* $\{\Lambda_i : i \in I\}$ *and* $\{\Gamma_i : i \in I\}$ *respectively. Let* $U, V \in L(\mathscr{H})$. *If* $T_\Lambda T_\Gamma^* = 0$ *and* U *or* V *is surjective, then* $\left\{\Lambda_i U^* + \Gamma_i V^* : i \in I\right\}$ *is a g-frame for* $\mathscr{H}$.

Proof. Let $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ be two g-frames for $\mathscr{H}$. Then there exist $0 < A_1 \leq B_1 < \infty$ and $0 < A_2 \leq B_2 < \infty$ such that

$$A_1\|f\|^2 \leq \sum_{i\in I}\|\Lambda_i f\|^2 \leq B_1\|f\|^2, \quad A_2\|f\|^2 \leq \sum_{i\in I}\|\Gamma_i f\|^2 \leq B_2\|f\|^2.$$

Since $T_\Lambda T_\Gamma^* = 0$, for any $f \in \mathscr{H}$, we have

$$\sum_{i\in I}\Lambda_i^*\Gamma_i f = \sum_{i\in I}\Gamma_i^*\Lambda_i f = 0.$$

Hence for any $f \in \mathscr{H}$, we have

$$\sum_{i\in I}\|(\Lambda_i U^* + \Gamma_i V^*)f\|^2 = \sum_{i\in I}\langle(\Lambda_i U^* + \Gamma_i V^*)f, (\Lambda_i U^* + \Gamma_i V^*)f\rangle$$

$$\begin{aligned}
&= \sum_{i\in I}\|\Lambda_i U^* f\|^2 + \sum_{i\in I}\|\Gamma_i V^* f\|^2 + 2Re\Big(\sum_{i\in I}\langle \Lambda_i U^* f, \Gamma_i V^* f\rangle\Big)\\
&= \sum_{i\in I}\|\Lambda_i U^* f\|^2 + \sum_{i\in I}\|\Gamma_i V^* f\|^2 + 2Re\Big\langle \sum_{i\in I}\Gamma_i^*\Lambda_i U^* f, V^* f\Big\rangle\\
&= \sum_{i\in I}\|\Lambda_i U^* f\|^2 + \sum_{i\in I}\|\Gamma_i V^* f\|^2\\
&\leq B_1\|U^* f\|^2 + B_2\|V^* f\|^2\\
&\leq \Big(B_1\|U^*\|^2 + B_2\|V^*\|^2\Big)\|f\|^2
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{i\in I}\|(\Lambda_i U^* + \Gamma_i V^*)f\|^2 &= \sum_{i\in I}\|\Lambda_i U^* f\|^2 + \sum_{i\in I}\|\Gamma_i V^* f\|^2\\
&\geq \sum_{i\in I}\|\Lambda_i U^* f\|^2\\
&\geq A_1\|U^* f\|^2
\end{aligned}$$

Assume that U is surjective. Then by lemma [2.12], we have

$$\sum_{i\in I}\|(\Lambda_i U^* + \Gamma_i V^*)f\|^2 \geq A_1 C^2\|f\|^2$$

Thus $\{\Lambda_i U^* + \Gamma_i V^* : i \in I\}$ is a g-frame for $\mathscr{H}$. □

When $U = I_{\mathscr{H}}$, we have the following result:

Corollary 3.2. *Let $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ be two g-frames for a Hilbert space $\mathscr{H}$, and let T_Λ and T_Γ be synthesis operators of $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ respectively. Let $V \in L(\mathscr{H})$. If $T_\Lambda T_\Gamma^* = 0$, then $\{\Lambda_i + \Gamma_i V^* : i \in I\}$ is a g-frame for $\mathscr{H}$. Moreover, $\{\Lambda_i + \Gamma_i (V^*)^a : i \in I\}$ is also a g-frame for $\mathscr{H}$ for any natural number a.*

The following corollary can be immediately derived from Theorem [3.1], when $U = V = I_{\mathscr{H}}$.

Corollary 3.3. *Let $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ be two g-frames for Hilbert space $\mathscr{H}$, and let T_Λ and T_Γ be synthesis operators of $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$, respectively. If $T_\Lambda T_\Gamma^* = 0$, then $\{\Lambda_i + \Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.*

The next theorem provides a necessary and sufficient condition for the new g-frame to be a tight g-frame.

Theorem 3.4. *Let* $\{\Lambda_i : i \in I\}$ *and* $\{\Gamma_i : i \in I\}$ *be two Parseval g-frames for* $\mathscr{H}$*, and let* T_Λ *and* T_Γ *be synthesis operators of* $\{\Lambda_i : i \in I\}$ *and* $\{\Gamma_i : i \in I\}$*, respectively, such that* $T_\Lambda T_\Gamma^* = 0$*. Let* $U, V \in L(\mathscr{H})$*. Then* $\{\Lambda_i U^* + \Gamma_i V^* : i \in I\}$ *is a* λ*-tight g-frame for* $\mathscr{H}$ *if and only if* $(UU^* + VV^*) = \lambda I_{\mathscr{H}}$*.*

Proof. Since $T_\Lambda T_\Gamma^* = 0$, for any $f \in \mathscr{H}$, we have

$$\sum_{i\in I} \|(\Lambda_i U^* + \Gamma_i V^*)f\|^2 = \sum_{i\in I} \|\Lambda_i U^* f\|^2 + \sum_{i\in I} \|\Gamma_i V^* f\|^2$$
$$= \|U^* f\|^2 + \|V^* f\|^2,$$

(Since $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are Parseval g-frames.)

$$= \langle U^* f, U^* f\rangle + \langle V^* f, V^* f\rangle$$
$$= \langle UU^* f, f\rangle + \langle VV^* f, f\rangle$$
$$= \langle (UU^* + VV^*)f, f\rangle$$

It follows that $\{\Lambda_i U^* + \Gamma_i V^* : i \in I\}$ is a λ-tight g-frame for $\mathscr{H}$ if and only if $(UU^* + VV^*) = \lambda I_{\mathscr{H}}$. □

Next we consider the case when $\{(\Lambda_i + \Gamma_i)U^* : i \in I\}$ is a g-frame, where $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are alternate dual g-frame pair for $\mathscr{H}$ and $U \in L(\mathscr{H})$.

Theorem 3.5. *Let* $\{\Lambda_i : i \in I\}$ *and* $\{\Gamma_i : i \in I\}$ *be alternate dual g-frame pair for Hilbert space* $\mathscr{H}$*. Let* $U \in L(\mathscr{H})$*. If* U *is surjective, then* $\{(\Lambda_i + \Gamma_i)U^* : i \in I\}$ *is a g-frame for* $\mathscr{H}$*.*

Proof. Since $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are g-frame for $\mathscr{H}$, there exist $0 < A_1 \le B_1 < \infty$ and $0 < A_2 \le B_2 < \infty$ such that

$$A_1 \|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B_1 \|f\|^2,\ A_2 \|f\|^2 \le \sum_{i\in I} \|\Gamma_i f\|^2 \le B_2 \|f\|^2.$$

Now $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ is an alternate dual g-frame pair, therefore $T_\Lambda T_\Gamma^* = T_\Gamma T_\Lambda^* = I_{\mathscr{H}}$ i.e

$$\sum_{i\in I} \Lambda_i^* \Gamma_i f = \sum_{i\in I} \Gamma_i^* \Lambda_i f = f,\ \ \forall\ f \in \mathscr{H}.$$

Hence for all $f \in \mathscr{H}$, we have

$$\sum_{i\in I} \|(\Lambda_i + \Gamma_i)U^* f\|^2 = \sum_{i\in I} \langle (\Lambda_i + \Gamma_i)U^* f, (\Lambda_i + \Gamma_i)U^* f\rangle$$

$$
\begin{aligned}
&= \sum_{i\in I} \|\Lambda_i U^* f\|^2 + \sum_{i\in I} \|\Gamma_i U^* f\|^2 + 2Re \sum_{i\in I} \langle \Lambda_i U^* f, \Gamma_i U^* f\rangle \\
&= \sum_{i\in I} \|\Lambda_i U^* f\|^2 + \sum_{i\in I} \|\Gamma_i U^* f\|^2 + 2Re \Big\langle \sum_{i\in I} \Gamma_i^* \Lambda_i U^* f, U^* f \Big\rangle \\
&= \sum_{i\in I} \|\Lambda_i U^* f\|^2 + \sum_{i\in I} \|\Gamma_i U^* f\|^2 + 2\langle U^* f, U^* f\rangle \\
&\le B_1 \|U^* f\|^2 + B_2 \|U^* f\|^2 + 2 \|U^* f\|^2 \\
&\le \Big(B_1 \|U^*\|^2 + B_2 \|U^*\|^2 + 2 \|U^*\|^2 \Big) \|f\|^2
\end{aligned}
$$

Since U is surjective, we have by lemma[2.12] that there exists some constant $C > 0$ such that $\|U^* f\|^2 \ge C^2 \|f\|^2$ for any $f \in \mathscr{H}$.
Now we have,

$$
\begin{aligned}
\sum_{i\in I} \|(\Lambda_i + \Gamma_i) U^* f\|^2 &= \sum_{i\in I} \|\Lambda_i U^* f\|^2 + \sum_{i\in I} \|\Gamma_i V^* f\|^2 + 2 \|U^* f\|^2 \\
&\ge \sum_{i\in I} \|\Lambda_i U^* f\|^2 \\
&\ge A_1 \|U^* f\|^2 \\
&\ge A_1 C^2 \|f\|^2
\end{aligned}
$$

Thus $\{(\Lambda_i + \Gamma_i) U^* : i \in I\}$ is a g-frame for $\mathscr{H}$. □

The following corollary can be immediately derived from Theorem[3.5], when $U = I_{\mathscr{H}}$.

Corollary 3.6. *Let $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ be two alternate dual g-frame pair for Hilbert space $\mathscr{H}$. Then $\{\Lambda_i + \Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.*

In the following results, we always assume that there exists a g-orthonormal basis $\{\Theta_i : i \in I\}$ for $\mathscr{H}$ with respect to $\{\mathscr{H}_i : i \in I\}$.

Theorem 3.7. *Let $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ be two g-frames for Hilbert space $\mathscr{H}$, and let T_Λ and T_Γ be synthesis operators of $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$, respectively, such that $T_\Lambda T_\Gamma^* = 0$. Then $\{\Lambda_i - \Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$. Moreover, if $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are normalized* *tight g-frames and* $T_\Lambda T_\Gamma^* = 0$*, then* $\{\Lambda_i - \Gamma_i : i \in I\}$ *is a tight g-frame for* $\mathscr{H}$ *with bound 2.*

Proof. Since T_Λ and T_Γ are synthesis operators associated with g-frames $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ respectively, $\Theta_i T_\Lambda^* = \Lambda_i$ and $\Theta_i T_\Gamma^* = \Gamma_i$ for any $i \in I$, where $\{\Theta_i\}_{i\in I}$ is a g-orthonormal basis for $\mathscr{H}$. Hence $(\Lambda_i - \Gamma_i) = \Theta_i(T_\Lambda - T_\Gamma)^*$, for any $i \in I$. Also

$$\sum_{i\in I} \Lambda_i^* \Gamma_i f = 0 = \sum_{i\in I} \Gamma_i^* \Lambda_i f = 0, \qquad \forall\, f \in \mathscr{H}.$$

To show that $\{(\Lambda_i - \Gamma_i)f : i \in I\}$ is a g-frame, it is sufficient to show that $(T_\Lambda - T_\Gamma)$ is onto by lemma [2.9].

Since $T_\Gamma T_\Lambda^* = 0$, we have $(T_\Lambda - T_\Gamma)T_\Lambda^* = T_\Lambda T_\Lambda^* - T_\Gamma T_\Lambda^* = T_\Lambda T_\Lambda^*$.

Since $T_\Lambda T_\Lambda^*$ is invertible, for any $z \in \mathscr{H}$, there exists $x = T_\Lambda^*(T_\Lambda T_\Lambda^*)^{-1} z \in \mathscr{H}$ such that

$$\begin{aligned}
(T_\Lambda - T_\Gamma)x &= (T_\Lambda - T_\Gamma)T_\Lambda^*(T_\Lambda T_\Lambda^*)^{-1} z \\
&= (T_\Lambda T_\Lambda^*)(T_\Lambda T_\Lambda^*)^{-1} z \\
&= z
\end{aligned}$$

Therefore, $(T_\Lambda - T_\Gamma)$ is an onto operator. If $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are normalized tight g-frames and $T_\Lambda T_\Gamma^* = 0$, then for any $x \in \mathscr{H}$, we have

$$\begin{aligned}
\sum_{i\in I} \|(\Lambda_i - \Gamma_i)x\|^2 &= \sum_{i\in I} \|\Lambda_i x\|^2 + \sum_{i\in I} \|\Gamma_i x\|^2 - \sum_{i\in I} \langle \Lambda_i x, \Gamma_i x\rangle - \sum_{i\in I} \langle \Gamma_i x, \Lambda_i x\rangle \\
&= 2\,\|x\|^2 - \sum_{i\in I} \langle \Theta_i T_\Lambda^* x, \Theta_i T_\Gamma^* x\rangle - \sum_{i\in I} \langle \Theta_i T_\Gamma^* x, \Theta_i T_\Lambda^* x\rangle \\
&= 2\,\|x\|^2 - \sum_{i\in I} \langle T_\Gamma \Theta_i^* \Theta_i T_\Lambda^* x, x\rangle - \sum_{i\in I} \langle T_\Lambda \Theta_i^* \Theta_i T_\Gamma^* x, x\rangle \\
&= 2\,\|x\|^2 - \left\langle T_\Gamma \sum_{i\in I} \Theta_i^* \Theta_i T_\Lambda^* x, x \right\rangle - \left\langle T_\Lambda \sum_{i\in I} \Theta_i^* \Theta_i T_\Gamma^* x, x \right\rangle
\end{aligned}$$

$$= 2\|x\|^2 - \langle T_\Gamma T_\Lambda^* x, x\rangle - \langle T_\Lambda T_\Gamma^* x, x\rangle$$
$$= 2\|x\|^2$$

Thus $\{\Lambda_i - \Gamma_i : i \in I\}$ is a tight g-frame for $\mathscr{H}$ with bound 2. $\square$

Example : A difference of Gabor frames in $L_2(\mathbb{R})$. In [2] author has shown that the sum of two orthogonal frames is a frame. Here we are using the same example to generate frame by taking the difference of the two frames. For $x, y \in \mathbb{R}$, let E_x, and T_y be operators defined on $L_2(\mathbb{R})$ by

$E_x(f(t)) = e^{2nixt} f(t)$ and $T_y(f(t)) = f(t-y)$.

Since the polynomial $1 + pz$ does not have root on the unit circle for $p \leq 1$, the set $[0,1) \cup [1,2) = [0,2)$ forms a Gabor frame wavelet set [4, 10] Likewise, the set $[2,1) \cup [1,0) = [2,0)$ forms a Gabor frame wavelet set. Let $g_1(t) = \chi_{[0,1)} + p\chi_{[1,2)}$, and $g_2(t) = \chi_{[1,0)} + p\chi_{[2,1)}$.

The families

$$\mathbb{X} = \{E_{mT_n} g_1(t)\}_{m,n\in Z}, \text{ and } \mathbb{Y} = \{E_{mT_n g_2(t)}\}_{m,n\in Z}$$

form frames for the space $L_2(\mathbb{R})$. Since the support$(\mathbb{X}) \cap$ support$(\mathbb{Y}) = \phi\ \ \forall m, n \in Z$, it follows that $\forall f \in L_2(\mathbb{R})$, we have

$$\sum_{m,n\in Z} \langle f(t), E_m T_n g_1(t)\rangle E_m T_n g_2(t) = 0,$$

So $\mathbb{X}$ and $\mathbb{Y}$ form a pair of orthogonal frames for the space $L_2(\mathbb{R})$. Therefore the difference

$h(t) = g_1(t) - g_2(t) = \chi_{[0,1)} + p\chi_{[1,2)} - (\chi_{[1,0)} + p\chi_{[2,1)})$

forms a frame for $L_2(\mathbb{R})$.

In [12], it is shown that the difference of two alternate dual g-frame pair is also a g-frame. In fact, sum of two alternate dual g-frames is also a g-frame. We give a different proof of corollary[3.6].

Proof. Since $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are alternate dual g-frames, we have $\sum_{i\in I}\Lambda_i^*\Gamma_i f=\sum_{i\in I}\Gamma_i^*\Lambda_i f=f, \quad \forall\, f\in\mathscr{H}$. For any $f\in\mathscr{H}$, we have

$$\begin{aligned}\sum_{i\in I}\|(\Lambda_i+\Gamma_i)f\|^2 &= \sum_{i\in I}\|\Lambda_i f\|^2+\sum_{i\in I}\|\Gamma_i f\|^2+2Re\sum_{i\in I}\langle\Lambda_i f,\Gamma_i f\rangle\\ &= \sum_{i\in I}\|\Lambda_i f\|^2+\sum_{i\in I}\|\Gamma_i f\|^2+2Re\Big\langle\sum_{i\in I}\Gamma_i^*\Lambda_i f,f\Big\rangle\\ &= \sum_{i\in I}\|\Lambda_i f\|^2+\sum_{i\in I}\|\Gamma_i f\|^2+2\langle f,f\rangle\\ &\le (B_1+B_2+2)\,\|f\|^2\end{aligned}$$

Let $\{\Theta_i\}_{i\in I}$ be g-orthonormal basis for $\mathscr{H}$.

$$\begin{aligned}\sum_{i\in I}\|(\Lambda_i+\Gamma_i)f\|^2 &= \sum_{i\in I}\|\Lambda_i f\|^2+\sum_{i\in I}\|\Gamma_i f\|^2+2\,\|f\|^2\\ &= \sum_{i\in I}\|\Theta_i T_\Lambda^* f\|^2+\sum_{i\in I}\|\Theta_i T_\Gamma^* f\|^2+2\,\|f\|^2\\ &= \sum_{i\in I}\|T_\Lambda^* f\|^2+\sum_{i\in I}\|T_\Gamma^* f\|^2+2\,\|f\|^2\end{aligned}$$

Now T_Λ is onto, so there exists an operator $(T_\Lambda)^\dagger$ such that $T_\Lambda(T_\Lambda)^\dagger = I_{\mathscr{H}} \Rightarrow \left(T_\Lambda^\dagger\right)^* T_\Lambda^* = I_{\mathscr{H}}$.

So for any $f\in\mathscr{H}$, we have

$$\begin{aligned}\|f\|^2 &= \left\|\left(T_\Lambda^\dagger\right)^* T_\Lambda^* f\right\|^2\\ &\le \left\|\left(T_\Lambda^\dagger\right)^*\right\|^2\|T_\Lambda^* f\|^2\end{aligned}$$

which implies that $\|T_\Lambda^* f\|^2 \ge \frac{\|f\|^2}{\left\|\left(T_\Lambda^\dagger\right)^*\right\|^2}$.

Also T_Γ is onto, so there exists an operator $T_\Gamma^\dagger$ such that $T_\Gamma(T_\Gamma)^\dagger = I_{\mathscr{H}} \Rightarrow \left(T_\Gamma^\dagger\right)^* T_\Gamma^* = I_{\mathscr{H}}$.

So for any $f \in \mathscr{H}$, we have

$$\begin{aligned}\|f\|^2 &= \left\| \left(T_\Gamma^\dagger\right)^* T_\Gamma^* f \right\|^2 \\ &\leq \left\| \left(T_\Gamma^\dagger\right)^* \right\|^2 \|T_\Gamma^* f\|^2\end{aligned}$$

which implies that $\|T_\Gamma^* f\|^2 \geq \frac{\|f\|^2}{\left\|\left(T_\Gamma^\dagger\right)^*\right\|^2}$.

Therefore

$\sum_{i\in I} \|T_\Lambda^* f\|^2 + \sum_{i\in I} \|T_\Gamma^* f\|^2 + 2\|f\|^2 \geq \left(\frac{1}{\left\|\left(T_\Lambda^\dagger\right)^*\right\|^2} + \frac{1}{\left\|\left(T_\Gamma^\dagger\right)^*\right\|^2} + 2 \right) \|f\|^2$

or $\sum_{i\in I} \|(\Lambda_i + \Gamma_i) f\|^2 \geq \left(\frac{1}{\left\|\left(T_\Lambda^\dagger\right)^*\right\|^2} + \frac{1}{\left\|\left(T_\Gamma^\dagger\right)^*\right\|^2} + 2 \right) \|f\|^2$

Thus $\{\Lambda_i + \Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.

In [9] author proved that, when $\{\Lambda_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ is an alternate dual g-frame pair for Hilbert space $\mathscr{H}$ and T is a coisometry, $\{\Lambda_i T^* : i \in I\}$ and $\{\Theta_i T^* : i \in I\}$ form an alternate dual g-frame pair. We give here an alternate proof by using the concept of g-R-duality of g-frames. We shall prove the duality of $\{\Lambda_i T^* : i \in I\}$ and $\{\Theta_i T^* : i \in I\}$ by showing that the their g-R-duals sequences are biorthogonal by lemma[2.11].

Theorem 3.8. *Let $\{\Lambda_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ be alternate dual g-frames for Hilbert space $\mathscr{H}$, and T be a coisometry in $L(\mathscr{H})$. Then $\{\Lambda_i T^* : i \in I\}$ and $\{\Theta_i T^* : i \in I\}$ are alternate dual g-frames for $\mathscr{H}$.*

Proof. Let $\{\Lambda_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ be alternate dual g-frame pair for $\mathscr{H}$ and T be a coisometry. Therefore by lemma[2.9], $\{\Lambda_i T^*\}_{i\in I}$ and $\{\Theta_i T^*\}_{i\in I}$ are also g-frames for $\mathscr{H}$.

Let $\{\Xi_i\}_{i\in I}$ and $\{\Psi_i\}_{i\in I}$ be g-orthonormal basis for $\mathscr{H}$ with respect to $\{\mathscr{W}_i\}_{i\in I}$ and $\{\mathscr{H}_i\}_{i\in I}$, respectively and $\left\{\Gamma_j^{\Lambda}\right\}_{j\in I}$ and $\left\{\Gamma_j^{\Theta}\right\}_{j\in I}$ denote the g-R-duals sequences of $\{\Lambda_i T^*\}_{i\in I}$ and $\{\Theta_i T^*\}_{i\in I}$ respectively. By definition of $\left\{\Gamma_j^{\Lambda}\right\}_{j\in I}$ and $\left\{\Gamma_j^{\Theta}\right\}_{j\in I}$, for every $i,j\in I$ and $\{g_j\}_{j\in I}\in\mathscr{W}_j$, we have

$$\Gamma_j^{\Lambda} = \sum_{i\in I} \Xi_j\{\Lambda_i T^*\}^*\Psi_i, \qquad \forall\, j\in I.$$

and

$$\Gamma_j^{\Theta} = \sum_{i\in I} \Xi_j\{\Theta_i T^*\}^*\Psi_i, \qquad \forall\, j\in I.$$

Now

$$\begin{aligned}
\Gamma_i^{\Lambda}\left(\Gamma_j^{\Theta}\right)^* g_j &= \sum_{k\in I}\Xi_i\left(\Lambda_k T^*\right)^*\Psi_k\left\{\sum_{m\in I}\Xi_j\left(\Theta_m T^*\right)^*\Psi_m\right\}^* g_j\\
&= \sum_{k\in I}\sum_{m\in I}\Xi_i T\Lambda_k^*\Psi_k\Psi_m^*\Theta_m T^*\Xi_j^* g_j\\
&= \sum_{k\in I}\Xi_i T\Lambda_k^*\Theta_k T^*\Xi_j^* g_j\\
&= \Xi_i T\left(\sum_{k\in I}\Lambda_k^*\Theta_k T^*\Xi_j^* g_j\right)\\
&= \Xi_i T T^*\Xi_j^* g_j
\end{aligned}$$

Because T is a coisometry, $TT^* = I_{\mathscr{H}}$. Hence, $\Gamma_i^{\Lambda}\left(\Gamma_j^{\Theta}\right)^* g_j = \Xi_i\Xi_j^* g_j = \delta_{ij}g_j$. □

Theorem 3.9. *Let $\{\Lambda_i : i\in I\}$ and $\{\Theta_i : i\in I\}$ be alternate dual g-frames for Hilbert space $\mathscr{H}$, and T be a surjective operator in $L(\mathscr{H})$. Then $\{\Lambda_i T^* : i\in I\}$ and $\{\Theta_i T^{\dagger} : i\in I\}$ are alternate dual g-frames for $\mathscr{H}$. Similarly, $\{\Lambda_i T^{\dagger} : i\in I\}$ and $\{\Theta_i T^* : i\in I\}$ are alternate dual g-frames for $\mathscr{H}$.*

Proof. Since T is onto, $TT^{\dagger} = I_{\mathscr{H}}$ by lemma[2.10]. It follows that $(T^{\dagger})^{*}T^{*} = I_{\mathscr{H}}$. It follows that $T^{\dagger *}$ is onto. Since $\{\Lambda_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are g-frames for $\mathscr{H}$, $\{\Lambda_i T^* : i \in I\}$, $\{\Theta_i T^* : i \in I\}$, $\{\Lambda_i T^{\dagger} : i \in I\}$ and $\{\Theta_i T^{\dagger} : i \in I\}$ are g-frames for $\mathscr{H}$ by lemma[2.9]. Let g-R-dual sequences of $\{\Lambda_i T^* : i \in I\}$, $\{\Theta_i T^* : i \in I\}$, $\{\Lambda_i T^{\dagger} : i \in I\}$ and $\{\Theta_i T^{\dagger} : i \in I\}$ be denoted by $\{\Gamma_i^1\}_{i\in I}, \{\Gamma_i^2\}_{i\in I}, \{\Gamma_i^3\}_{i\in I}$ and $\{\Gamma_i^4\}_{i\in I}$ respectively. For every $i, j \in I$ and $\{g_j\}_{i\in I} \in \mathscr{W}_j$, we have

$$\{\Gamma_i^1\} = \sum_{i\in I} \Xi_j \big\{\Lambda_i T^*\big\}^* \Psi_i, \qquad \forall\, j \in I.$$

$$\{\Gamma_i^2\} = \sum_{i\in I} \Xi_j \big\{\Theta_i T^*\big\}^* \Psi_i, \qquad \forall\, j \in I.$$

$$\{\Gamma_i^3\} = \sum_{i\in I} \Xi_j \big\{\Lambda_i T^{\dagger}\big\}^* \Psi_i, \qquad \forall\, j \in I.$$

$$\{\Gamma_i^4\} = \sum_{i\in I} \Xi_j \big\{\Theta_i T^{\dagger}\big\}^* \Psi_i, \qquad \forall\, j \in I.$$

Now

$$\begin{aligned}
\{\Gamma_i^1\}\big(\{\Gamma_i^4\}\big)^* g_j &= \sum_{k\in I} \Xi_i \big(T\Lambda_k^*\big)\Psi_k \left\{ \sum_{m\in I} \Xi_j \big(\Theta_m T^{\dagger}\big)^* \Psi_m \right\}^* g_j \\
&= \sum_{k\in I}\sum_{m\in I} \Xi_i T \Lambda_k^* \Psi_k \Psi_m^* \Theta_m T^{\dagger} \Xi_j^* g_j \\
&= \sum_{k\in I} \Xi_i T \Lambda_k^* \Theta_k T^{\dagger} \Xi_j^* g_j \\
&= \Xi_i T \left(\sum_{k\in I} \Lambda_k^* \Theta_k T^{\dagger} \Xi_j^* g_j \right) \\
&= \Xi_i T T^{\dagger} \Xi_j^* g_j = \Xi_i \Xi_j^* g_j = \delta_{ij} g_j
\end{aligned}$$

So g-R-dual sequences of $\{\Lambda_i T^* : i \in I\}$, $\{\Theta_i T^{\dagger} : i \in I\}$ are biorthogonal. It follows that $\{\Lambda_i T^* : i \in I\}$, $\{\Theta_i T^{\dagger} : i \in I\}$ are alternate dual g-frames for $\mathscr{H}$. Similarly, we can show that g-R-dual sequences of $\{\Theta_i T^* : i \in I\}$ and $\{\Lambda_i T^\dagger : i \in I\}$ are biorthogonal. $\square$

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CONTINUOUS FUNCTIONS AND THE GAUSS LEMMA

B SURY

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Abstract. In Article 42 of his celebrated book 'Disquisitiones Arithmeticae', Gauss proved the following result:
If the coefficients $A, B, C, \cdots, N; a, b, c, \cdots n$ of two functions of the form

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \cdots + N \qquad (P)$$

$$x^\mu + ax^{\mu-1} + bx^{\mu-2} + cx^{\mu-3} + \cdots + n \qquad (Q)$$

are all rational and not all integers, and if the product of (P) and (Q)

$$= x^{m+\mu} + \mathfrak{A}x^{m+\mu-1} + \mathfrak{B}x^{m+\mu-2} + etc. + \mathfrak{Z}$$

then not all the coefficients $\mathfrak{A}, \mathfrak{B}, \cdots, \mathfrak{Z}$ can be integers.
This is the famous Gauss lemma which has been rephrased and generalized in several ways over 150 years. Some of the statements have only existential proofs while some have surprisingly explicit proofs. We discuss these aspects of the Gauss lemma and its generalizations.

1. Introduction

If f, g are polynomials in one variable over any commutative ring with unity, a lemma due (independently) to Dedekind and Mertens from 1892 generalizes the classical Gauss lemma and asserts that

$$c(f)^{deg(g)}c(fg) = c(f)^{deg(g)}c(f)c(g).$$

Here, for a polynomial f, one defines the content of f to be the ideal $c(f)$ generated by its coefficients. However, one thing that is true over ANY commutative ring with unity is that, for any f and g, the equality $c(fg) = c(f)c(g)$ holds *if $c(f), c(g)$ are unit ideals.* We start first by recalling that the statement "$c(fg) = c(f)c(g)$ if $c(f), c(g)$ are unit ideals" has a

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purely existential proof, and that indeed, no proof is known that is either constructive or, is accomplished by some algebraic manipulations. Following that, we provide a twist in the tale for a certain ring of functions where a Gauss-lemma-like proof does work. Finally, in the next few sections, we give a brief tour of some generalizations that the subject of Gauss's lemma has led to over the years.

2. Thus spake Gauss

In Article 42 of his celebrated book 'Disquisitiones Arithmeticae', Gauss proved the following result (here is an English translation of his statement):

If the coefficients $A, B, C, \cdots, N; a, b, c, \cdots n$ of two functions of the form

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \cdots + N \qquad (P)$$

$$x^\mu + ax^{\mu-1} + bx^{\mu-2} + cx^{\mu-3} + \cdots + n \qquad (Q)$$

are all rational and not all integers, and if the product of (P) and (Q)

$$= x^{m+\mu} + \mathfrak{A}x^{m+\mu-1} + \mathfrak{B}x^{m+\mu-2} + etc. + \mathfrak{Z}$$

then not all the coefficients $\mathfrak{A}, \mathfrak{B}, \cdots, \mathfrak{Z}$ can be integers.

This is the famous Gauss lemma which is often re-phrased in several ways, one of which is the following statement:
Over a unique factorization domain (abbreviated as UFD), the product of primitive polynomials is a primitive polynomial.
Here, the adjective 'primitive' refers to a polynomial whose coefficients have no common divisor in the UFD other than units. The Gauss lemma has been generalized over time. For instance, Kaplansky showed that the above statement holds over any integral domain in which any two elements admit a GCD (greatest common divisor) - these are now known as GCD domains and we discuss them in a later section here.
Note that over a UFD, any two non-zero non-units have a GCD which is unique up to multiplication by units. The Gauss lemma can also be thought of as the assertion that over a UFD, the product of the GCDs of polynomials f and g is the GCD of the polynomial fg (up to multiplication by units).
The main implication of Gauss's lemma is that for any UFD A, the polynomial ring $A[X]$ is also a UFD.

For a polynomial f, one may define the content of f to be the ideal $c(f)$ generated by its coefficients - this definition makes sense over any commutative ring with unity. It is evident that we have an inclusion $c(fg) \subseteq c(f)c(g)$ for polynomials f, g.
The content ideal is the same as the ideal generated by the GCD when the ring is a PID (principal ideal domain); hence the above is an equality in this case.
However, it is interesting to observe the subtlety that the inclusion $c(fg) \subseteq c(f)c(g)$ could be proper for polynomials f, g over UFDs A.
For instance, if $A = K[X, Y]$ for a field A, the polynomials $f(t) = X + Yt$ and $g(t) = X - Yt$ have the property that $fg = X^2 - Y^2t^2$ and hence

$$c(f)c(g) = (X, Y)^2 = (X^2, XY, Y^2) \supset (X^2, Y^2) = c(fg)$$

where the inclusion is proper.
Another example is $A = \mathbb{Z}[X]$ where $f(t) = 2 + Xt, g(t) = 2 - Xt$ give

$$c(f)c(g) = (2, X)^2 = (4, 2X, X^2) \supset (4, X^2) = c(fg)$$

which is a strict inclusion.

3. Existential proofs - a twist in the tale

If f, g are polynomials in one variable over any commutative ring with unity, a lemma due (independently) to Dedekind and Mertens from 1892 which will be discussed in detail in the next section asserts that

$$c(f)^{deg(g)} c(fg) = c(f)^{deg(g)} c(f)c(g).$$

However, one thing that is true over ANY commutative ring with unity is that, for any f and g, the equality $c(fg) = c(f)c(g)$ holds *if* $c(f), c(g)$ *are unit ideals.*

Our purpose is to start first by recalling that the statement "$c(fg) = c(f)c(g)$ if $c(f), c(g)$ are unit ideals" has a purely existential proof, and that indeed, no proof is known that is either constructive or, is accomplished by some algebraic manipulations. This may be instructive to bring to the notice of the students. Following that, we provide a twist in the tale for a certain ring of functions where a Gauss-lemma-like proof does work. Finally, in the next two sections, we give a brief tour of some generalizations that the subject of Gauss's lemma has led to over the years.

First, we recall the existential argument alluded to:

Let R be a commutative ring with unity. Let $f = \sum_{i=0}^{n} a_i X^i, g = \sum_{j=0}^{m} b_j X^j \in R[X]$ be such that $c(f), c(g)$ are unit ideals; that is,

$$1 = \sum_{i=0}^{n} a_i A_i = \sum_{j=0}^{m} b_j B_j$$

for some $A_i, B_j \in R$. If $fg = \sum_{k=0}^{m+n} c_k X^k$, then $c(fg)$ is the unit ideal; that is, there exist $C_k \in R$ so that $\sum_{k=0}^{m+n} c_k C_k = 1$.

To prove this, suppose the ideal generated by $c_0, \cdots, c_{m+n}$ is a proper ideal, and let M be a maximal ideal containing it. Then, under the natural ring homomorphism from $R[X]$ to $(R/M)[X]$, the polynomial fg maps to zero. However, neither the image of f nor that of g maps to zero which contradicts the fact that $(R/M)[X]$ is an integral domain.

As mentioned above, the proof is purely existential. Having said this, we observe now that for a ring like $C[0,1]$, the ring of real-valued continuous functions on $[0,1]$, which is far from being even an integral domain, we provide a twist in the tale by showing that a proof akin to Gauss's lemma works.

Here is the result and a constructive proof.

Lemma. Let $R = C[0,1]$ with addition and multiplication of functions given in terms of their values. Let $F = \sum_{i=0}^{n} f_i X_i, G = \sum_{i=0}^{m} g_i X^i \in R[X]$. If $c(F) = c(G) = R$, then $c(FG) = R$; further, one can prove this constructively.

Note that if $FG = \sum_{i=0}^{m+n} h_i X^i$, then $c(FG) = R$ if, and only if, $h_0, \cdots, h_{m+n}$ have no common zero in $[0,1]$. This is because if h_i's have no common zero, the elements $H_i = \frac{h_i}{\sum_i h_i^2} \in R$ satisfy $\sum_i h_i H_i = 1$, the constant function 1, which is the unity of R. Therefore, the assumptions $c(F) = c(G) = R$ imply that the f_i's have no common zero and the g_j's have no common zero as well. Consider an arbitrary $a \in [0,1]$. Then we would have a smallest r with $0 \le r \le n$ for which $f_r(a) \neq 0$; similarly, we would have a smallest s with $0 \le s \le m$ so that $g_s(a) \neq 0$. Evidently $h_{r+s}(a) = f_r(a) g_s(a) \neq 0$, which means all the h_i's cannot have a common zero. Hence $c(FG) = R$. This proof is just like the Gauss-lemma proof for $\mathbb{Z}$.

4. Dedekind-Mertens

As mentioned in the previous section, if f, g are polynomials in one variable over any commutative ring with unity, Dedekind and Mertens independently, proved the so-called (by Krull) Dedekind-Mertens Lemma. It has been generalized by Prüfer and many others in diverse directions. The readers can refer to [6] for a recent description of some beautiful generalizations. The paper [4] which defines and studies something called the Dedekind-Mertens number mentions the interesting history of Dedekind and Mertens's works. One form of the original lemma asserts that

$$c(f)^{deg(g)}c(fg) = c(f)^{deg(g)}c(f)c(g).$$

Here is a lovely, simple Gauss-lemma-like proof due to Coquand - who champions the cause of constructive mathematics.

Coquand's Proof of Dedekind-Mertens

Let A be a commutative ring with unity. Suppose $f = \sum_{i=0}^{n} f_i X^i, g = \sum_{j=0}^{m} g_j X^j$ and $h = fg = \sum_{r=0}^{m+n} h_r X^r$ in $A[X]$. Write the content ideals $c(f) = (f_0, \cdots, f_n), c(g) = (g_), \cdots, g_m)$ and $c(h) = (h_0, \cdots, h_{m+n})$. We may take the ring A to be $\mathbb{Z}[f_0, \cdots, f_n, g_0, \cdots, g_m]$ where the f_i's and g_j's can be regarded as indeterminates. We wish to prove $c(f)^{m+1}c(g) \subseteq c(f)^m c(h)$ because the reverse inclusion is evident. Let F, G, H denote, respectively, the abelian subgroup of A generated by the coefficients of f, g, h. We wish to prove:

$$F^{m+1}G \subseteq F^m H.$$

This will be proved by induction on m where it is obvious when $m = 0$. Assume $m > 0$ and let G_m denote the additive subgroup of G generated by $g_0, g_1, \cdots, g_{m-1}$. As usual, a symbol f_k for $k < 0$ or $k > n$ stands for 0. Note

$$h_r = f_{r-m}g_m + \sum_{s<m} f_{r-s}g_s.$$

Therefore,

$$\sum_{s<m} f_{r-s}g_s = h_r - f_{r-m}g_m \in H + Fg_m$$

which gives, by the definition of G_m that

$$FG_m \subseteq H + Fg_m.$$

So, inductively, $F^2G_m \subseteq FH + F^2g_m$,

$$F^3G_m \subseteq F^2H + F^3g_m$$

etc. Inductively, we obtain

$$F^mG_m \subseteq F^{m-1}H + F^mg_m.$$

Therefore, for $0 \leq i \leq n$, we have

$$f_iF^mG_m \subseteq f_iF^{m-1}H + f_iF^mg_m \subseteq F^mH + f_iF^mg_m.$$

On the other hand, since

$$f_ig_m = h_{i+m} - \sum_{s<m} f_{i+m-s}g_s \in H + f_{i+1}G_m + \cdots + f_nG_m.$$

This implies that for all $0 \leq i \leq n$,

$$f_iF^mG_m \subseteq F^mH + f_{i+1}F^mG_m + \cdots + f_nF^mG_m.$$

Taking respectively $i = n, n-1, \cdots$ etc., we have for all $0 \leq i \leq n$ that

$$f_iF^mG_m \subseteq F^mH.$$

Hence $F^{m+1}G_m \subseteq F^mH$ which proves the assertion.

5. GCD Domains

As we saw, some versions of Gauss's lemma involve the GCD of elements. The notions of GCD and LCM can be generalized to any integral domain D in an obvious manner but they do not always exist for two given elements and there are also some surprises. Before starting a discussion, recall that the GCD and LCM of a set of integers is defined only up to sign; so, in reality, one should call it “A" GCD (but understand that it is unique up to multiplication by a unit).

In an integral domain D, define “a" GCD of two non-zero elements $a \neq b$ in D to be an element d such that $d|a, d|b$ and any c dividing both a and b divides d also. It is clear that if c, d are two GCDs of a and b, then they are associates as we are in a domain. A similar definition of “a" LCM is easily given. The first fact which may not be all that surprising is that two elements may not have a GCD at all (because there is no reason to expect they should). But, a fact that is surprising is that two elements may have a GCD but may not an LCM. Moreover, the opposite implication is not true.

For instance, it is a little exercise to check that in the domain $K[X^2, X^3]$ for a field K, the elements X^2, X^3 have GCD 1 (and its associates) but do not have any LCM. We discuss these aspects in some detail now. The readers are invited to read the beautiful exposition by D D Anderson in [1]. For other interesting exercises on GCD domains, readers may refer to Kaplansky's book [5].

We shall use the symbol (a, b) for the ideal generated by a and b and write the qualifiers GCD, LCM etc. explicitly. Anderson uses the symbols $[a, b]$ and $]a, b[$ for GCD and LCM respectively but these are not so common. We first state the following obvious lemma:

Lemma. Let D be an integral domain, and let $0 \neq a, b \in D$. Then,
$GCD(a, b)$ exists if, and only if, the ideal $\cap\{(c) : (c) \supset (a, b)\}$ is principal;
$LCM(a, b)$ exists if, and only if, the ideal $\sum\{(c) : (c) \subset (a) \cap (b)\}$ is principal.
In the respective cases, a generator of the corresponding principal ideal is, respectively, a GCD and an LCM of a and b.
The statements generalize to a finite number of elements.

Proposition. Let D be an integral domain and let $0 \neq a, b \in D$.
(i) If $LCM(a, b)$ exists, then $GCD(a, b)$ also exists and they satisfy

$$GCD(a, b)LCM(a, b) = ab$$

up to units.
(ii) If $c \in D$, and if $GCD(ca, cb)$ exists, then $GCD(a, b)$ exists and

$$c.GCD(a, b) = GCD(ca, cb).$$

Consequently, if $GCD(a, b)$ exists, say d, then the GCD of a/d and b/d exists, and equals 1.
(iii) $LCM(a, b)$ exists if, and only if, $GCD(ca, cb)$ exists for all $c \in D$.
(iv) $GCD(a, b)$ exists for all $0 \neq a, b \in D$ if, and only if, $LCM(c, d)$ exists for all $0 \neq c, d \in D$.
Proof. We prove (i) first.
Suppose $LCM(a, b)$ exists; say ℓ. We want to show that $d := ab/\ell$ equals $GCD(a, b)$. As $a = d\ell/b$ and $b = d\ell/a$, it follows that d divides both a and b. Now suppose that h is a common divisor of a and b. Now as a, b both divide ab/h, ℓ divides ab/h which implies that h divides $ab/\ell = d$. Thus,

we have proved (i).
The proof of (ii) is obvious, and we skip it.
Now, we prove (iii). We first show that if $LCM(a, b)$ exists, then so does $LCM(ca, cb)$ for all $c \in D$. Note that both ca, cb divide $cLCM(a, b)$. Now suppose m is a common multiple of ca, cb. Then c divides m and both a, b divide m/c. Thus $LCM(a, b)$ divides m/c and so $cLCM(a, b)$ divides m. Thus $LCM(ca, cb)$ exists, and equals $cLCM(a, b)$. In particular, by (i), $GCD(ca, cb)$ exists for every c.
Now, we claim that if $GCD(ca, cb)$ exists for every c, then $LCM(a, b)$ exists and equals $ab/GCD(a, b)$. Clearly both a, b divide $ab/GCD(a, b)$. Now, suppose both a, b divide m. Then ab is a common divisor of ma and mb and so ab divides $GCD(ma, mb) = mGCD(a, b)$ by (ii) above. This implies that $ab/GCD(a, b)$ divides m. Thus (iii) follows.
Finally, (iv) is an immediate consequence of (i),(ii),(iii).

Definition. A GCD-domain is an integral domain D such that the equivalent properties in (iv) of the proposition holds; that is, each pair of non-zero elements has a GCD as well as an LCM. The nomenclature is due to I. Kaplansky.

Remarks.
(a) In a commutative ring that is not an integral domain, there is no relation between the existence of an LCM of two elements and the existence of a GCD. For example, in the ring $K[X^2, X^3]/(X^9, X^{10}]$, an LCM of X^5 and X^6 is X^8 whereas these elements do not have a GCD.
(b) In contrast with the polynomial ring over a UFD, it is known that there exist UFDs D such that $D[[X]]$ is not a UFD. It is a fact that these power series rings cannot be GCD-domains also. The proof of this needs other characterizations of GCD domains that we do not go into here, and refer to Anderson's article.

Since UFDs are GCD domains, one can show certain domains such as $\mathbb{Z}[\sqrt{-d}]$ ($d \geq 3$), are not GCD domains and hence not UFDs by exhibiting two elements which do not have an LCM. In a proposition below, we will observe that GCD domains are integrally closed; 'one-third' of the domains in the corollary below (namely, when $-d \equiv 1 \bmod 4$) are not even integrally closed.

Corollary. In each of the domains $\mathbb{Z}[\sqrt{-d}]$ ($d \geq 3$) with d square-free, there exist two elements a, b such that $GCD(a, b)$ exists but $LCM(a, b)$ does not exist. In particular, $\mathbb{Z}[\sqrt{-d}]$ ($d \geq 3$), is not a GCD domain and hence, is not a UFD.

Proof. Here $\mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} : a, b \in \mathbb{Z}\}$.

Firstly, suppose that $d + 1$ is not a prime number. Let $d + 1 = pk$, where p is a prime and $k \geq 2$. Clearly $a^2 + db^2 \neq p$ for any $a, b \in \mathbb{Z}$ because the left hand side is bigger than p if $b \neq 0$. If $p = (a + b\sqrt{-d})(u + v\sqrt{-d})$ in $\mathbb{Z}[\sqrt{-d}]$, then taking complex conjugates we see that $u = a, v = -b$. Thus, $p = a^2 + db^2$, which is impossible as observed above. Therefore, p is an irreducible element in $\mathbb{Z}[\sqrt{-d}]$. Also p does not divide $1 + \sqrt{-d}$ because $p(a + b\sqrt{-d}) = 1 + \sqrt{-d}$ gives $pa = 1$ which is impossible. Thus, $GCD(p, 1 + \sqrt{-d})$ exists, and equals 1.

We shall show that $GCD(pk, (1 + \sqrt{-d})k)$ does not exist. If it did, then by the proposition, $GCD(pk, (1+\sqrt{-d})k) = k$. As $1+\sqrt{-d}$ divides $pk = 1+d$, both $(1 + \sqrt{-d})k$, $1 + \sqrt{-d}$ divide k. Let $k = (1 + \sqrt{-d})(a + b\sqrt{-d}) = (a - bd) + (a + b)\sqrt{-d}$. This gives $a = -b$ and $a - bd = a + ad = k$. Thus $apk = a(1 + d) = k$ which is a contradiction. In view of the proposition, it follows that $LCM(p, 1 + \sqrt{-d})$ does not exist.

Suppose now that $d \geq 3$ and $d + 1$ is a prime. Then d is. Let $d + 4 = 2k$, for some $k > 1$. As above, one easily checks that 2 is irreducible and 2 does not divide $2 + \sqrt{-d}$. Thus $GCD(2, 2 + \sqrt{-d})$ exists and equals 1. We show that $GCD(2k, (2+\sqrt{-d})k)$ does not exist. If it did, then as above, $2+\sqrt{-d}$ divides k and which in turn implies that $4 + d$ divides $k = (4 + d)/2$ in $\mathbb{Z}$, a contradiction which shows that $LCM(2, 2 + \sqrt{-d})$ does not exist.

Remark. In the above proof, note that when $d+1 = pk$, p divides $d+1 = (1 + \sqrt{-d})(1 - \sqrt{-d})$ but p clearly does not divide either of $1 + \sqrt{-d}$ and $1 - \sqrt{-d}$, showing that p, which is irreducible, is not prime. Similarly in the second part of the proof, 2 divides $d + 4 = (2 + \sqrt{-d})(2 - \sqrt{-d})$ but does not divide either of them, which shows that 2 is not prime. This also proves that $\mathbb{Z}[\sqrt{-d}]$ ($d \geq 3$), is not a UFD.

Proposition. *GCD domains are integrally closed.*

Proof. Let D be a GCD domain, with quotient field K. Let $a/b \in K$

satisfy

$$(a/b)^n + a_{n-1}(a/b)^{n-1} + \cdots + a_0 = 0$$

where $a_i \in D$, $a_0 \neq 0$ and $GCD(a,b) = 1$. The last condition can be assumed without loss of generality because we have by a proposition above $GCD(a/d, b/d) = 1$ if $GCD(a,b) = d$ in a GCD domain. So, we get

$$a^n + a_{n-1}a^{n-1}b + a_{n-2}a^{n-2}b^2 + \cdots + a_0 b^n = 0.$$

Then $b|a^n$. But $GCD(a,b) = 1$ implies $GCD(a^m, b) = 1$ for all $m \geq 1$ by induction on m; indeed, if this is true for m, then any common divisor c of a^{m+1} and b divides a^{m+1} and ab but $GCD(a^{m+1}, ab) = aGCD(a^m, b) = a$. This shows that $b|1$; that is, it is a unit. Hence $a/b \in D$.

5.1. **Gauss Lemma in GCD domains.** In any GCD domain D, Gauss's lemma is valid. Indeed, if we define $f \in D[X]$ to be primitive if GCD of its coefficients is 1, then over a GCD domain D, the polynomial $fg \in D[X]$ is primitive if f, g are. This is an easy exercise - the usual proof for UFDs can be adapted here. But, now we mention another version of Gauss's lemma that is valid over integrally closed domains. This version is the closest in spirit to what Gauss actually stated in his article 42 - albeit, in the case of $\mathbb{Z}$ and $\mathbb{Q}$. The proof is an easy exercise (indeed, it is Ex.8, P.42 of [5]).

Gauss Lemma for Integrally closed domains. *If D is an integrally closed domain with quotient field K, and if $f \in D[X]$ is a monic polynomial such that $f = gh$ with $g, h \in K[X]$ monic, then $g, h \in A[X]$.*

6. Kaplansky's conjecture

Over any commutative ring A with unity, one defines a polynomial $f \in A[X]$ to be Gaussian if $c(f)c(g) = c(fg)$ holds for all polynomials $g \in A[X]$. One calls A a *Gaussian ring* if every polynomial $f \in A[X]$ is Gaussian. Several papers in the last six decades have been written on possible characterizations of Gaussian rings or Gaussian polynomials. It is known that being Gaussian is a local property. In particular, it was known for a long time that if $c(f)$ is locally principal, then f is Gaussian. Similarly, over a domain, it was known that if $c(f)$ is an invertible ideal, then f is Gaussian. Kaplansky conjectured that the converse holds:

Kaplansky's Conjecture If A is a commutative ring with unity and $f \in A[X]$ is Gaussian, then the ideal $c(f)$ is either invertible or locally principal.

The authors of [3] mention that this was a question one of them heard in the 1960's from Kaplansky. In fact, this conjecture also appeared in the PhD thesis of Kaplansky's student H. Tsang in 1965 but has not appeared in print. Many cases of the conjecture have been proved by Sarah Glaz and others but it is not completely proved yet, along with other questions raised by Glaz and others.

Acknowledgments. I am grateful to the referee for informing me about Thierry Coquand's work - especially with reference to the Dedekind-Mertens lemma; I was not aware of his writings. It is also a pleasure to thank him for suggesting a short discussion of GCD domains and, more generally, of the version of Gauss's lemma for integrally closed domains that is closest to Gauss's original statement. I also want to record my thanks to Dinesh Khurana who brought to my attention more than a decade back, the intricacies concerning GCDs and LCMs in domains.

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PROBLEM SECTION

In Volume 91 (1-2) 2022 of The Mathematics Student, we had invited solutions from the readers to the Problems 2, 3, 4, 5, 6, 8, 9 and 10 mentioned in MS 90 (3-4) 2021, as well as solutions to the twelve new problems, till April 20, 2022.

As regards to solutions to the eight Problems mentioned in MS 90 (3-4) 2021, we did not receive any solution to any of the eight problems. We feel that several readers can provide solutions to Problems 2, 3, 4 and 5 and therefore we give one more opportunity to the readers to provide their solutions to these problems until January 10, 2023. Solutions provided by the proposers to Problems 6, 8, 9 and 10 are printed in this section.

As far as solutions to the twelve new problems mentioned in MS 91 (1-2) 2022 are concerned, we received solutions from the readers to problems 2, 3, 6, 10 and 11. These solutions are being presented in this section.

We pose eight new problems in this section. We invite Solutions from the readers to the Problems 2, 3, 4 and 5 of MS 90 (3-4) 2021, solutions to the remaining seven problems viz. 1, 4, 5, 7, 8, 9, and 12 of MS 91 (1-2) 2022 and solutions to the eight new problems till January 10, 2023. Correct solutions received from the readers by this date will be published in Volume 92 (1-2) 2023 of The Mathematics Student. This volume is scheduled to be published in March 2023.

New Problems.

Dr. Anup Dixit, Institute of Mathematical Sciences, Chennai proposed the following two problems.

MS 91 (3-4) 2022: Problem 1. Suppose a_n is a sequence of positive integers such that

$$\sum_{n=1}^{\infty} \frac{\sin(1/a_n)}{\log a_n}$$

diverges. Show that for infinitely many n, $lcm\{a_1, \cdots, a_n\} = lcm\{a_1, \cdots, a_{n+1}\}$.

MS 91 (3-4) 2022: Problem 2. Let $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ be two permutations of $\{1, 2, \cdots, n\}$. Show that the set $\{a_1b_1, a_2b_2, \cdots, a_nb_n\}$ does not form a complete residue system modulo n.

Dr. Mohsen Soltanifar, University of Toronto, Canada proposed the following two problems.

MS 91 (3-4) 2022: Problem 3. Let X be a real valued random variable on the real line with finite mean. Assume for some $-\infty < \alpha < \infty$ we have:

$$E(\min(X, \alpha)) = E(\max(X, \alpha)).$$

Calculate the distribution of X.

MS 91 (3-4) 2022: Problem 4. Let $X_1, \cdots, X_n$ be i.i.d random variables with common uniform distribution on $(0, 1)$. Let $p, q > 0$ and define a random variable:

$$S_n(p, q) = (\prod_{i=1}^{n} X_i^p)^{\frac{1}{n^q}}, \quad (n \geq 1).$$

Compute

$$\lim_{n\to\infty} S_n(p, q)$$

if it exists and find values of $p,\ q$ for which it does.

Yathiraj Sharma, M. V. Sarada Vilas College, Mysuru, Karnataka suggested the next problem.

MS 91 (3-4) 2022: Problem 5. Consider the sequence $d_n = 3n + 1$. Prove that the sum of the Legendre symbols $(k/7)$ as k runs through divisors of 12 (7d-4) is 0 whenever $d \neq d_n$. Show further that for infinitely many (but not all) n, the sum is not 0 as k runs over divisors of $12(7d_n - 4)$.

Prof. Shpetim Rexhepi and Ilir Demiri, Mother Teresa University, Skopje, North Macedonia proposed the following two problems.

MS 91 (3-4) 2022: Problem 6. Prove that

$$\int_0^\infty \frac{u^3 du}{e^{\sqrt[4]{\frac{15}{4}}u} - 1} = \frac{4\pi^4}{225}.$$

MS 91 (3-4) 2022: Problem 7. For $a>$ $b>e$, e-Euler number, prove that

$$\frac{ln\Gamma(b^a)}{ln\Gamma(a^b)} > \frac{lnb}{lna}.$$

Mr. Toyesh Prakash Sharma of Agra College, Agra suggested the folowing problem.

MS 91 (3-4) 2022: Problem 8. If $n > 0$ and α is the positive root of quadratic equation $x^2 - x - 1 = 0$ then show that the following inequality

$$F_n\alpha^{F_n} + L_n\alpha^{L_n} \geq 2F_{n+1}\alpha^{F_{n+1}}$$

holds.

Further, obtain the above inequality using the convexity of a suitable function where the Fibonacci numbers F_n and the Lucas numbers L_n satisfy the conditiond.

$$F_{n+2} = F_{n+1} + F_n \ , \quad F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n \ , \quad L_0 = 2, \ L_1 = 1.$$

There was a mistake in Problem 1 of MS 91 (1-2) 2022. The correct problem is produced here.

MS 91 (3-4) 2022: Problem 1. (Proposed by Demiri and Rexhepi)

Prove that

$$\int\limits_0^1 \left(t^{\frac{-1}{n}} - t^{1-\frac{1}{n}}\right)^{n-1} dt = \frac{n^n}{(n+1)(n+\frac{1}{2})(n+\frac{1}{3})...(n+\frac{1}{n-1})}$$

where $n \in \mathbb{N}$ *and* $n > 1$.

Solutions to the Old Problems

MS 90 (1-2) 2021: Problem 2. (Proposed by Prof. B. Sury, ISI, Bangalore)

Let $f\colon [0,1] \to \mathbb{R}$ be differentiable, and let $f(0) = 0, f(1) = 1$. Prove that there exist $t_1, \ldots, t_{2021} \in [0,1]$ such that $2021 = \sum_{i=1}^{2021} \frac{1}{f'(t_i)}$.

Dr. Henry Ricardo, Westchester Area Math Circle, New York, USA provided a solution to the problem as given below.

Solution. The intermediate value theorem and continuity guarantee the existence of t_i, the smallest number in $[0,1]$ such that $f(t_i) = i/2021$. If we define $t_0 = 0$ and $t_{2021} = 1$, we have $0 = t_0 < t_1 < t_2 < \cdots < t_{2020} < t_{2021} = 1$. For each interval (t_{i-1}, t_i), $i = 1, 2, \ldots, 2021$, we may choose x_i such that

$$f'(x_i) = \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}$$

by the mean value theorem. Then

$$f'(x_i) = \frac{\frac{i}{2021} - \frac{i-1}{2021}}{t_i - t_{i-1}} = \frac{1}{2021(t_i - t_{i-1})}$$

and

$$\sum_{i=1}^{2021} \frac{1}{f'(t_i)} = \sum_{i=1}^{2021} 2021(t_i - t_{i-1}) = 2021 \sum_{i=1}^{2021} (t_i - t_{i-1}) = 2021.$$

Comment: Clearly we can generalize this result, replacing 2021 by any positive integer n.

MS 90 (3-4) 2021: Problem 6. (Proposed by Dr. Anup Dixit)
Let $x_1, x_2, \cdots, x_n$ be distinct real numbers. Show that

$$\sum_{1\leq i\leq n} \prod_{j\neq i} \left(\frac{1 - x_i x_j}{x_i - x_j} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Solution. (By Dr. Anup Dixit)
We first show that the function

$$(i)\ f(x_1, \cdots, x_n) := \sum_{1\leq i\leq n} \prod_{j\neq i} \left(\frac{1 - x_i x_j}{x_i - x_j} \right)$$

is a polynomial. Clearly, f is a symmetric function, i.e., swapping x_i and x_j does not change f. Let

$$g(x_1, \cdots, x_n) := \prod_{1\leq i<j\leq n} (x_i - x_j).$$

Then the function $h := fg$ is a polynomial. Furthermore, g is an alternating function, i.e., swapping x_i and x_j changes the sign of g. Since f is symmetric, we deduce that h is an alternating polynomial. Alternating polynomials in n-variables vanish on taking $x_i = x_j$ for any distinct i, j.

Thus, $(x_i - x_j)$ divides h for any distinct i, j. Hence, g divides h implying that $h/g = f$ is a polynomial.

If $y_i := 1/x_i$, then note that

$$\frac{1 - y_i y_j}{y_i - y_j} = \frac{1 - x_i x_j}{x_i - x_j}.$$

Therefore,

$$f\left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right) = f(x_1, \cdots, x_n) \implies f \text{ is a constant},$$

as we showed above that f is a polynomial.

To determine this constant, we evaluate f at $x_j = \zeta_n^j$, where ζ_n is the primitive n-th root of unity. The contribution to the summation in (i) when $j \neq n$, $n/2$ is zero as the corresponding term has $1 - \zeta_n^j \zeta_n^{n-j} = 0$ in the numerator. When n is odd, the term $j = n$ gives

$$f(\zeta_n, \zeta_n^2, \cdots, 1) = \prod_{j \neq n} \left(\frac{1 - \zeta_n^j}{1 - \zeta_n^j}\right) = 1.$$

When n is even, the terms for $j = n/2, n$ need to be considered and we obtain that

$$\begin{aligned} f(\zeta_n, \zeta_n^2, \cdots, 1) &= \prod_{j \neq n/2} \left(\frac{1 - \zeta_n^{n/2} \zeta_n^j}{\zeta_n^{n/2} - \zeta_n^j}\right) + \prod_{j \neq n} \left(\frac{1 - \zeta_n^j}{1 - \zeta_n^j}\right) \\ &= 1 + \prod_{j \neq n/2} \left(\frac{1 + \zeta_n^j}{-1 - \zeta_n^j}\right) \\ &= 1 + (-1)^{n-1} \\ &= 0. \end{aligned}$$

MS 90 (3-4) 2021: Problem 8. (Proposed by Dr. Anup Dixit)

Show that among any 4 distinct positive real numbers a_1, a_2, a_3, a_4, we can find a_i, a_j such that $a_i > a_j$ and

$$a_i(\sqrt{3} - a_j) < (\sqrt{3} a_j - 1).$$

Solution. (By Dr. Anup Dixit)

The inequality above is equivalent to showing that

$$0 < \frac{a_i - a_j}{1 + a_i a_j} < \frac{1}{\sqrt{3}}.$$

Define $\theta_i \in (0, \pi/2)$, such that

$$a_i = \tan \theta_i.$$

Then, we have

$$\begin{aligned} 0 < \frac{a_i - a_j}{1 + a_i a_j} < \frac{1}{\sqrt{3}} &\iff 0 < \tan(\theta_i - \theta_j) < \frac{1}{\sqrt{3}} \\ &\iff (\theta_i - \theta_j) < \frac{\pi}{6}. \end{aligned}$$

Since $\theta_i \in (0, \pi/2)$ and there are 4 of them, by pigeon hole principle, there exists θ_i, θ_j with the required property.

MS 90 (3-4) 2021: Problem 9. (Posed by Dr. Siddhi Pathak, Chennai Math. Inst., Chennai)

Let $a_1, a_2, \cdots, a_{2021}$ be real numbers such that

$$\sum_{i=1}^{2021} a_i = 0, \qquad \sum_{i=1}^{2021} a_i^2 = 1.$$

Let $c := \max\limits_{1 \le i \le 2021} a_i$ and $d := \min\limits_{1 \le i \le 2021} a_i$. Show that

$$-\frac{1}{2} \le c\, d \le -\frac{1}{2021}.$$

Solution. (By Dr. Pathak)

Since $\sum_{i=1}^{2021} a_i = 0$, $c > 0$ and $d < 0$. Also, as $\sum_{i=1}^{2021} a_i^2 = 1$, $c^2 + d^2 \le 1$. Thus, by the AM-GM inequality,

$$-c\, d = |c|\, |d| \le \frac{c^2 + d^2}{2} \le \frac{1}{2}.$$

Hence, $cd \ge -1/2$.

Let $P := \{i : a_i \ge 0\}$ and $N := \{j : a_j < 0\}$. Then we have that

$$\sum_{i \in P} a_i = -\sum_{j \in N} a_j.$$

Furthermore, $a_i \le c$ for all $i \in P$ and $-a_j \le -d$ for all $j \in N$. Therefore, we deduce that

$$\sum_{i\in P} a_i^2 \le c \sum_{i\in P} a_i = c \sum_{j\in N} (-a_j) \le -c\,d\,|N|.$$

Similarly, we have

$$\sum_{j\in N} a_j^2 \le -d \sum_{j\in N} (-a_j) = -d \sum_{i\in P} a_i \le -c\,d\,|P|.$$

Adding these two inequalities gives

$$1 = \sum_{i=1}^{n} a_i^2 \le -c\,d(|P| + |N|) = -2021\,c\,d.$$

Hence,

$$c\,d \le -\frac{1}{2021}.$$

MS 90 (3-4) 2021: Problem 10. (Posed by Dr. Siddhi Pathak)

Fix any positive integer $m > 1$. Show that if $f : (0,\infty) \to (0,\infty)$ satisfies

$$f(x)\,f(y) = m\,f\,(x + yf(x)) \text{ for all } x, y > 0,$$

then $f(x) = m$ for all $x > 0$.

Solution. (By Dr. Pathak)

(i) We first prove that any f satisfying the given condition must be *non-decreasing*. If not, then there exist positive real numbers x and z with $x < z$ but $f(x) > f(z)$. Now let $y := (z - x)/(f(x) - f(z))$. Note that $y > 0$ and $x + yf(x) = z + yf(z)$. Thus,

$$f(z)\,f(y) = m\,f\,(z + yf(z)) = m\,f\,(x + yf(x)) = f(x)\,f(y),$$

implying that $f(x) = f(z)$, which is a contradiction.

(ii) We now claim that f is *not strictly increasing*. Indeed, if f were to be strictly increasing, then f would be injective. Taking one of the arguments to be 1 in the given condition, we get

$$f(x)\,f(1) = m\,f\,(x + f(x)) = m\,f\,(1 + xf(1))\,.$$

Since f is injective, we obtain that

$$x + f(x) = 1 + xf(1) \implies f(x) = 1 + c\,x, \text{ with } c = f(1) - 1 \neq 0.$$

Substituting this in the given condition leads to

$$(1+cx)(1+cy) = m\Big(1 + c\big(x + y(1+cx)\big)\Big)$$
$$\implies 1 + cx + cy + c^2xy = m\Big(1 + cx + cy + c^2xy\Big)$$

for all x, $y > 0$. Since $f(x)$, $f(y) \neq 0$ and $m > 1$, this is a contradiction.

(iii) Next, we demonstrate the existence of *a* $y_0 \in (0, \infty)$ *such that* $f(y_0) = m$. Since f is not strictly increasing, there exist x, z with $x < z$ such that $f(x) = f(z)$. As f is non-decreasing, $f(w) = f(x)$ for all $x \leq w \leq z$. Choose y_0 to be any real number satisfying $0 < y_0 < (z - x)/f(x)$ so that $x < x + (y_0\, f(x)) < z$. Thus,

$$f(x) = f(x + y_0 f(x)) = \frac{1}{m}\, f(x)\, f(y_0) \implies f(y_0) = m.$$

(iv) Finally, we construct *an infinite sequence*, $\{y_n\}$ with $y_n \to \infty$ and $f(y_n) = m$. Since f is non-decreasing, this establishes that f is identically equal to m. From above, we have,

$$m^2 = f(y_0)^2 = m\, f(y_0 + (y_0 f(y_0))) = m\, f((m+1)y_0)$$
$$\implies f((m+1)y_0) = m.$$

Iterating the above process gives

$$f((m+1)^n\, y_0) = m \text{ for all positive integers } n \geq 1,$$

establishing the claim.

□

MS 91 (1-2) 2022: Problem 2. (Proposed by Ilir Demiri and Prof. Shpetim Rexhepi, Skopje, North Macedonia)

For the beta function, prove with usual meaning that

$$\sum_{n=0}^{\infty} (B(2,n) - B(3,n)) = \frac{1}{2} \quad \text{where} \quad n \in \mathbb{N}.$$

Dr. Henry Ricardo gave the following solution to this problem.

Solution. Starting the summation with 1 instead of 0, we have

$$\begin{aligned}\sum_{n=1}^{\infty}(B(2,n)-B(3,n)) &= \sum_{n=1}^{\infty}\left\{\int_0^1 t(1-t)^{n-1}dt - \int_0^1 t^2(1-t)^{n-1}dt\right\}\\ &= \sum_{n=1}^{\infty}\int_0^1 t(1-t)^n dt\\ &= \int_0^1 t\cdot\sum_{n=1}^{\infty}(1-t)^n dt\\ &= \int_0^1 t\cdot\frac{1-t}{t}dt = \int_0^1 (1-t)dt = \frac{1}{2},\end{aligned}$$

where the interchange of summation and integration is allowed for power series within the interval of convergence. If the summation starts with $n=0$, the sum becomes 1.

Mr. Anantha Krishna, Indian Inst. of Technology Bhubaneswar also provided a correct solution to the above problem

MS 91 (1-2) : Problem 3. (Proposed by Ilir Demiri and Prof. Shpetim Rexhepi, Skopje, North Macedonia)

For the beta function, prove with usual meaning that

$$B(k,n) = \frac{(k-1)B(k-1,n)}{n+k-1}$$

for $k,n \in \mathbb{N}$.

Mr. Anantha Krishna gave a correct solution to this problem. The solution is presented below.

Solution.

Claim : For $k,n \in \mathbb{N}$,

$$B(k,n) = \frac{(n-1)!(k-1)!}{(n+k-1)!}.$$

The above claim directly implies that the equation

$$B(k,n) = \frac{(k-1)B(k-1,n)}{n+k-1}$$

is true. For the case $k = 1$ we see that

$$B(1,n) = \int_0^1 x^{n-1}dx = \frac{1}{n} = \frac{(n-1)!(1-1)!}{(n+1-1)!}.$$

We use induction on k. Assume that the claim is true for all natural numbers less than k. Using integration by parts

$$B(k,n) = \int_0^1 x^{k-1}(1-x)^{n-1}dx = x^{k-1}\frac{-(1-x)^n}{n}\bigg|_0^1 + \frac{k-1}{n}\int_0^1 x^{k-2}(1-x)^n dx,$$

it follows that

$$B(k,n) = \frac{k-1}{n}B(k-1,n+1).$$

By inductive hypothesis,

$$B(k,n) = \frac{k-1}{n}\frac{(k-2)!(n)!}{(n+k-1)!} = \frac{(n-1)!(k-1)!}{(n+k-1)!}.$$

So our claim is verified.

Dr. Henry Ricardo also provided a correct solution to this problem.

MS 91 (1-2) 2022: Problem 6. (Proposed by Prof. B. Sury)
Let $f\colon [0,1] \to \mathbb{R}$ be differentiable and satisfy $f(0) = f'(0) = f'(1) = 0$. Then, show that there exists $t \in (0,1)$ such that $f(t) = tf'(t)$.

Dr. Henry Ricardo provided two solutions to this problem. One of the two solutions is given below.

Solution.

Consider the function $g\colon [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{f(x)}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases} \tag{1}$$

It is obvious that g is continuous on $[0,1]$ and differentiable on $(0,1]$. Furthermore, from (1), we see that

$$g'(x) = -\frac{f(x)}{x^2} + \frac{f'(x)}{x} = -\frac{g(x)}{x} + \frac{f'(x)}{x} \tag{2}$$

for all $x \in (0,1]$.

Now from (1), we see that $g(0) = 0$. If $g(1) = 0$, then by Rolle's theorem there exists an $\eta \in (0,1)$ such that $g'(\eta) = 0$, and the desired result is established. If $g(1) \neq 0$, then either $g(1) > 0$ or $g(1) < 0$. Suppose $g(1) > 0$. Then from (2) we have $g'(1) = -g(1) < 0$. Since g is continuous and $g'(1) < 0$, there exists a point x_1 in $(0,1)$ such that $g(x_1) > g(1)$.

Therefore we have $g(0) < g(1) < g(x_1)$, and by the Intermediate Value theorem there exists $x_0 \in (0, x_1)$ such that $g(x_0) = g(1)$. Applying Rolle's theorem to the function g on the interval $[x_0, 1]$, we have $g'(\eta) = 0$ for some $\eta \in (0,1)$. A similar argument applies if $g(1) < 0$, and now the proof of the problem statement is complete.

The second solution is given by using Flett's mean value theorem (*The Mathematical Gazette*, Vol. 42, No. 339, pp. 38-39). This theorem states that if $f\colon [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and $f'(a) = f'(b)$, then there exists a point $\eta \in (a,b)$ such that $f(\eta) - f(a) = (\eta - a)f'(\eta)$. Applying this to the given function on $(0,1)$, we get the desired result.

MS 91 (1-2) 2022: Problem 10.(Posed by Dr. Anup Dixit)
For a complex number $s = \sigma + it$ with $\sigma > 1$, let $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ denote the Riemann zeta function. Show that for a fixed $\sigma > 1$,

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |\zeta(\sigma + it)| \leq \zeta(\sigma).$$

Dr. Henry Ricardo provided provided two solutions to this problem. The solutions are presented below.

Solution 1.
The result $|1/\zeta(s)| \leq \zeta(\sigma)/\zeta(2\sigma)$ is known for $\sigma > 1$. (See, for example, p. 66 of *The Prime Number Theorem* by G. J. O. Jameson (London Mathematical Society, 2003).) Using this, we have

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \;\leq\; |\zeta(\sigma+it)| \;=\; \left|\sum_{n=1}^{\infty} \frac{1}{n^s}\right| \;\leq\; \sum_{n=1}^{\infty} \frac{1}{|n^s|} \;=\; \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \;=\; \zeta(\sigma).$$

Solution 2.
First of all, we see that

$$|\zeta(\sigma + it| \leq \sum_{n=1}^{\infty} \frac{1}{|n^{\sigma+it}|} = \sum_{n+1}^{\infty} \frac{1}{|n^\sigma||n^{it}|} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma)$$

since $n^\sigma > 0$ and $|n^{it}| = |e^{it\ln n}| = 1$

Next, we use Euler's identity, a well-known product representation of the Riemann zeta function:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where Re $s > 1$ and the product extends over all prime numbers p. The product is absolutely convergent for Re $s > 1$.

By Euler's identity we have

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} = \frac{\prod_p \frac{1}{1-p^{-2\sigma}}}{\prod_p \frac{1}{1-p^{-\sigma}}} = \prod_p \frac{p^{2\sigma}}{(p^\sigma - 1)(p^\sigma + 1)} \cdot \frac{p^\sigma - 1}{p^\sigma} = \prod_p \frac{p^\sigma}{p^\sigma + 1} = \prod_p \frac{1}{1 + p^{-\sigma}}.$$

Estimating the denominator of Euler's product for $\zeta(s)$, we see that

$$|1-p^{-s}| = |1-p^{-\sigma-it}| \leq 1+|p^{-\sigma}||p^{-it}| = 1+p^{-\sigma}, \text{ or } \frac{1}{|1-p^{-s}|} \geq \frac{1}{1+p^{-\sigma}}.$$

Now, to complete the proof, we use the last inequality to establish that

$$|\zeta(\sigma + it| = |\zeta(s)| = \prod_p \frac{1}{|1-p^{-s}|} \geq \prod_p \frac{1}{1+p^{-\sigma}} = \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

MS 91 (1-2) 2022 : **Problem 11**.(Posed by Dr. Siddhi Pathak, Chennai Mathematical Institute, Chennai.)

Fix an integer $k \geq 2$. A positive integer n is said to be k-free if it is not divisible by p^k for any prime p. Evaluate the sum

$$\sum_{\substack{n\geq 1,\\ n \text{ is } 4 \text{ free}}} \frac{1}{n^2}.$$

Mr. Ritesh Dwivedi, Prayagraj, U. P. gave a correct solution to the problem. The solution is presented below.

Solution: Recall that the Riemann Zeta function ζ is defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

Also by Euler's product formula we have

$$\frac{1}{\zeta(s)} = \prod_{p\in\mathbb{P}} (1 - \frac{1}{p^s}),$$

where $\mathbb{P}$ denotes the set of all primes.

Let k, $r \geq 2$ be positive integers. We prove in general that

$$\sum_{\substack{n\geq 1,\\ n \text{ is } k \text{ free}}} \frac{1}{n^r} = \frac{\zeta(r)}{\zeta(kr)}.$$

We have

$$\sum_{\substack{n\geq 1,\\ n \text{ is } k \text{ free}}} \frac{1}{n^r} = \prod_{p\in\mathbb{P}} (1 + \frac{1}{p^r} + \frac{1}{p^{2r}} + \cdots + \frac{1}{p^{(k-1)r}}).$$

$$= \prod_{p\in\mathbb{P}} \frac{(1 - \frac{1}{p^{kr}})}{(1 - \frac{1}{p^r})}.$$

$$= \frac{\zeta(r)}{\zeta(kr)}.$$

Now putting $r = 2$ and $k = 4$ in above equation we get the required sum as

$$\frac{\pi^2/6}{\pi^8/9450} = \frac{1575}{\pi^6}.$$

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