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# THE MATHEMATICS STUDENT

**Edited by M. M. SHIKARE**

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## CONTENTS

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The Mathematics Student ISSN: 0025-5742
Vol. 90, Nos. 3-4, July-December (2021), 1–17

# A COMMON FIXED POINT THEOREM IN BICOMPLEX VALUED b-METRIC SPACES

SANJIB KUMAR DATTA, DIPANKAR PAL, RAKESH SARKAR AND ARGHYATANU MANNA

(Received : 26 - 06 - 2020 ; Revised : 07 - 01 - 2021)

Abstract. The main purpose of this paper is to investigate a common fixed point theorem in bicomplex valued $b$-metric spaces for four maps. Some concepts of Choi et al.[12] and Jebril et al.[17] are used here.

## 1. Introduction

Segre's[28] paper, published in 1892 made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Unfortunately this significant work of Segre failed to earn the attention of the mathematicians for almost a century. However, recently a renewed interest in this subject contributes a lot in the different fields of mathematical sciences and other branches of science and technology.

Price[24] developed the bicomplex algebra and function theory. In this field an impressive body of work has been developed by different researchers during the last few years. One can see some of the attempts in {cf.[3]- [5],[7]-[9],[13],[15],[16],[18]-[25],[29] & [30]}.

Azam et al.[1] introduced a concept of complex valued metric space and established a common fixed point theorem for a pair of self contracting mappings. Rouzkard & Imdad[26] generalized the result obtained by Azam et al.[1] and they proved another common fixed point theorem satisfying some rational inequality in complex valued metric space. Banach contraction principle is the main tool to prove the fixed point theorems. It states

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that "If $(X, d)$ be a complete metric space and $T : X \to X$ is a self-map then $d(Tx, Ty) \leq ad(x, y)$ where $0 \leq a < 1$, then $T$ has a unique fixed point in $X$. Banach proved his theory in 1922. Choudhury et al.{[10]&[11]} proved some fixed point results in partially ordered complex valued metric spaces for rational type expressions. Also one can see the attempts in {cf.[2],[6],[31]&[32]}.

Rao et al.[27] introduced the concept of complex-valued $b$-metric spaces and proved a common fixed point theorem in complex valued $b$-metric spaces.

We denote the set of real, complex and bicomplex numbers respectively as $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$.

Let $z_1, z_2 \in \mathbb{C}_1$ be any two complex numbers, then the partial order relation $\precsim$ on $\mathbb{C}_1$ is defined as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$,

i.e., $z_1 \precsim z_2$ if one of the following conditions is satisfied:

$$\begin{array}{llll} (1) & \mathrm{Re}(z_1) & = & \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) = \mathrm{Im}(z_2), \\ (2) & \mathrm{Re}(z_1) & < & \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) = \mathrm{Im}(z_2), \\ (3) & \mathrm{Re}(z_1) & = & \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) < \mathrm{Im}(z_2) \text{ and} \\ (4) & \mathrm{Re}(z_1) & < & \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) < \mathrm{Im}(z_2). \end{array}$$

In particular, we can say $z_1 \lnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (2), (3) and (4) is satisfied and $z_1 \prec z_2$ if only (4) is satisfied. We can easily check the following fundamental properties of partial order relation $\precsim$ on $\mathbb{C}_1$:

1. If $0 \precsim z_1 \lnsim z_2$, then $|z_1| < |z_2|$,
2. If $z_1 \precsim z_2, z_2 \prec z_3$then $z_1 \prec z_3$ and
3. If$z_1 \precsim z_2$ and $\lambda > 0$ is a real number then $\lambda z_1 \precsim \lambda z_2$.

Segre[28] defined the bicomplex number as:

$$\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units $i_1, i_2$ are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We denote $i_1 i_2 = j$ , which is known as the hyperbolic unit and such that $j^2 = 1$, $i_1 j = j i_1 = -i_2$, $i_2 j = j i_2 = -i_1$. Also the set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

i.e.,

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\}.$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

If $\xi = z_1 + i_2 z_2$ and $\eta = w_1 + i_2 w_2$ be any two bicomplex numbers then the sum is

$$\xi \pm \eta = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2 (z_2 \pm w_2)$$

and the product is

$$\xi.\eta = (z_1 + i_2 z_2) . (w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2 (z_1 w_2 + z_2 w_1).$$

There are four idempotent elements in $\mathbb{C}_2$, they are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ out of which $e_1$ and $e_2$ are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be expressed as the combination of $e_1$ and $e_2$, namely

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This representation of $\xi$ is known as the idempotent representation of bicomplex number and the complex coefficients $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$ are known as idempotent components of the bicomplex number $\xi$.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is said to be invertible if there exists another element $\eta$ in $\mathbb{C}_2$ such that $\xi\eta = 1$ and $\eta$ is said to be the inverse (multiplicative) of $\xi$. Consequently $\xi$ is said to be the inverse (multiplicative) of $\eta$. An element which has an inverse in $\mathbb{C}_2$ is said to be the nonsingular element of $\mathbb{C}_2$ and an element which does not have an inverse in $\mathbb{C}_2$ is said to be the singular element of $\mathbb{C}_2$.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular if and only if $\left|z_1^2 + z_2^2\right| \neq 0$ and singular if and only if $\left|z_1^2 + z_2^2\right| = 0$. The inverse of $\xi$ is defined as

$$\xi^{-1} = \eta = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the only element in $\mathbb{C}_0$ which does not have any multiplicative inverse and in $\mathbb{C}_1, 0 = 0 + i0$ is the only element which does not have any multiplicative inverse. We denote the set of singular elements of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $O_0$ and $O_1$ respectively. But there are more than one element in $\mathbb{C}_2$ which do not have any multiplicative inverse; we denote this set by $O_2$ and clearly $O_0 = O_1 \subset O_2$.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\xi\| &= \|z_1 + i_2 z_2\| = \left\{ |z_1|^2 + |z_2|^2 \right\}^{\frac{1}{2}} \\ &= \left[ \frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2} \right]^{\frac{1}{2}} = \left( a_1^2 + a_2^2 + a_3^2 + a_4^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space. If $\xi, \eta \in \mathbb{C}_2$ then $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ holds instead of $\|\xi\eta\| \leq \|\xi\|\,\|\eta\|$, therefore $\mathbb{C}_2$ is not the Banach algebra.

Now we define the partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows:

Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2$ then $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$ and $z_2 \precsim w_2$,

i.e., $\xi \precsim_{i_2} \eta$ if one of the following conditions is satisfied:

$$\begin{aligned} &(1) \quad z_1 = w_1,\ z_2 = w_2, \\ &(2) \quad z_1 \prec w_1,\ z_2 = w_2, \\ &(3) \quad z_1 = w_1,\ z_2 \prec w_2 \text{ and} \\ &(4) \quad z_1 \prec w_1,\ z_2 \prec w_2. \end{aligned}$$

In particular we can write $\xi \lnsim_{i_2} \eta$ if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$ i.e. one of (2), (3) and (4) is satisfied and we will write $\xi \prec_{i_2} \eta$ if only (4) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the followings:

(i) $\xi \precsim_{i_2} \eta \Rightarrow \|\xi\| \leq \|\eta\|$,

(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,

(iii) $\|a\xi\| = a\,\|\xi\|$, where $a$ is a non negative real number,

(iv) $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ and the equality holds only when at least one of $\xi$ and $\eta$ is equal to zero,

(v) $\left\|\xi^{-1}\right\| = \|\xi\|^{-1}$ if $\xi$ is a non-singular bicomplex number with $0 \prec \xi$,

(vi) $\left\|\frac{\xi}{\eta}\right\| = \frac{\|\xi\|}{\|\eta\|}$, if $\eta$ is a non-singular bicomplex number.

Choi et al.[12] defined the bicomplex valued metric space as follows:

**Definition 1.1.** [12] Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:

1. $0 \precsim_{i_2} d(x,y)$ for all $x, y \in X$,
2. $d(x,y) = 0$ if and only if $x = y$,
3. $d(x,y) = d(y,x)$ for all $x, y \in X$ and
4. $d(x,y) \precsim_{i_2} d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then $d$ is called a bicomplex valued metric on $X$ and $(X,d)$ is called the bicomplex valued metric space.

**Definition 1.2.** [14]Let $X$ be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:

1. $0 \precsim_{i_2} d(x,y)$ for all $x, y \in X$,
2. $d(x,y) = 0$ if and only if $x = y$,
3. $d(x,y) = d(y,x)$ for all $x, y \in X$ and
4. $d(x,y) \precsim_{i_2} s\,[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

Then $d$ is called a bicomplex valued $b$-metric on $X$ and $(X,d)$ is called the bicomplex valued $b$-metric space.

**Definition 1.3.** (i). Let $A \subseteq X$ and $a \in A$ is said to be an interior point of $A$ if there exists $0 \prec_{i_2} r \in \mathbb{C}_2$ such that

$$B(a,r) = \{x \in X : d(a,x) \prec_{i_2} r\} \subseteq A$$

and the subset $A \subseteq X$ is said to be an open set if each point of $A$ is an interior point of $A$.

(ii). A point $a \in X$ is said to be a limit point of $A$ if for all $0 \prec_{i_2} r \in \mathbb{C}_2$ such that

$$B(a,r) \cap \{A - \{a\}\} \neq \phi$$

and the subset $A \subseteq X$ is said to be a closed set if all the limit points of $A$ belong to $A$.

(iii). The family

$$F = \{B(a,r) : a \in X, 0 \prec_{i_2} r \in \mathbb{C}_2\}$$

is a sub-basis for a Hausdorff topology $\tau$ on X.

**Definition 1.4.** For a bicomplex valued metric space $(X,d)$

(i). A sequence $\{x_n\}$ in $X$ is said to be a convergent sequence and converges to a point $x$ if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec_{i_2} r$, for all $n > n_0$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii). A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence in $(X,d)$ if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec_{i_2} r$, for all $m, n \in \mathbb{N}$ and $n > n_0$.

(iii). If every Cauchy sequence in $X$ is convergent in $X$ then $(X,d)$ is said to be a complete bicomplex valued metric space.

**Definition 1.5.** Let $S, T : X \to X$ be two self-mappings then, $S$ and $T$ are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$.

**Definition 1.6.** Let $\mathbb{C}_2$ be the set of bicomplex numbers, then the max function on $\mathbb{C}_2$ for the patial order relation $\precsim_{i_2}$is defined by:

(1). $\max\{\xi, \eta\} = \eta$ if and only if $\xi \precsim_{i_2} \eta$,

(2). $\xi \precsim_{i_2} \max\{\eta, \zeta\}$ implies $\xi \precsim_{i_2} \eta$ or $\xi \precsim_{i_2} \zeta$.

From above definition we can say that if $\precsim_{i_2}$be the partial order relation on $\mathbb{C}_2$ and $\xi_1, \xi_1, \xi_1, .... \in \mathbb{C}_2$ then

(i). If $\xi_1 \precsim_{i_2} \max\{\xi_2, \xi_3\}$ then $\xi_1 \precsim_{i_2} \xi_2$ if $\xi_3 \precsim_{i_2} \xi_2$;

(ii). If $\xi_1 \precsim_{i_2} \max\{\xi_2, \xi_3, \xi_4\}$ then $\xi_1 \precsim_{i_2} \xi_2$ if $\max\{\xi_3, \xi_4\} \precsim_{i_2} \xi_2$;

(iii). If $\xi_1 \precsim_{i_2} \max\{\xi_2, \xi_3, \xi_4, \xi_5\}$ then $\xi_1 \precsim_{i_2} \xi_2$ if $\max\{\xi_3, \xi_4, \xi_5\} \precsim_{i_2} \xi_2$ and so on.

In this paper we use the concept of bicomplex valued $b$-metric space and prove a common fixed point theorem in bicomplex valued $b$-metric spaces.

## 2. The Proofs

In this section we present some lemmas which are needed in the sequel.

**Lemma 2.1.** *Let $(X,d)$ be a bicomplex valued metric space and $\{x_n\}$ a sequence in $X$. Then $\{x_n\}$ converges to a point $x$ if and only if $\lim_{n\to\infty} d(x_n, x) =$ 0.*

*Proof.* Let $\{x_n\}$ be a convergent sequence and converges to a point $x$ and let $\epsilon > 0$ be any real number. Suppose

$$r = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}.$$

Then clearly $0 \prec_{i_2} r \in \mathbb{C}_2$ and for this $r$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec_{i_2} r$ for all $n > n_0$.

Therefore,

$$\|d(x_n, x)\| < \|r\| = \epsilon \ \ \forall\ n > n_0,$$

which implies that

$$\lim_{n\to\infty} \|d(x_n, x)\| = 0$$

Conversely let $\lim\limits_{n\to\infty} \|d(x_n, x)\| = 0$.Then for $0 \prec_{i_2} r \in \mathbb{C}_2$, there exists a real $\epsilon > 0$ such that for all $\xi \in \mathbb{C}_2$,

$$\|\xi\| < \epsilon \Rightarrow \xi \prec_{i_2} r.$$

Then for this $\epsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\|d(x_n, x)\| < \epsilon \;\; \forall\, n > n_0.$$

Therefore,

$$d(x_n, x) \prec_{i_2} r \;\; \forall\, n > n_0.$$

Hence $\{x_n\}$ converges to a point $x$. □

**Lemma 2.2.** *Let $(X, d)$ be a bicomplex valued metric space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence in $X$ if and only if $\lim\limits_{n\to\infty} d(x_n, x_{n+m}) = 0$.*

*Proof.* Let $\{x_n\}$ be a Cauchy sequence in $X$ and let $\epsilon > 0$ be any real number. Suppose

$$r = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}.$$

Then clearly $0 \prec_{i_2} r \in \mathbb{C}_2$ and for this $r$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec_{i_2} r$ for all $n > n_0$.

Therefore,

$$\|d(x_n, x_{n+m})\| < \|r\| = \epsilon \;\; \forall\, n > n_0,$$

which implies that

$$\lim_{n\to\infty} \|d(x_n, x_{n+m})\| = 0$$

Conversely let $\lim\limits_{n\to\infty} \|d(x_n, x_{n+m})\| = 0$.Then for $0 \prec_{i_2} r \in \mathbb{C}_2$, there exists a real $\epsilon > 0$, such that for all $\xi \in \mathbb{C}_2$,

$$\|\xi\| < \epsilon \Rightarrow \xi \prec_{i_2} r.$$

Then for this $\epsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\|d(x_n, x_{n+m})\| < \epsilon \;\; \forall\, n > n_0.$$

Therefore,

$$d(x_n, x_{n+m}) \prec_{i_2} r \;\; \forall\, n > n_0.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

In this section we prove a fixed point theorem on bicomplex valued $b$-metric space for a pair of self contracting mappings.

**Theorem 2.3.** *Let $(X,d)$ be a complete bicomplex valued $b$-metric space with the coefficient $s \geq 1$. Let $S,T,f,g : X \to X$ be mappings such that*

$$S(X) \subseteq g(X) \text{ and } T(X) \subseteq f(X) \tag{2.1}$$

*also satisfying the condition*

$$d(Sx,Ty) \precsim_{i_2} a\max\{d(fx,gy), d(fx,Sx), d(gy,Ty), d(fx,Ty), d(gy,Sx)\} \tag{2.2}$$

*for all $x,y \in X$, where $0 \leq a < \frac{1}{s^2+s}$. Suppose the pairs $\{S,f\}$ and $\{T,g\}$ be weakly compatible and $T(X)$ be a closed subspace of $X$. Then $S,T,f$ and $g$ have a unique common fixed point.*

*Proof.* Since $a < \frac{1}{s^2+s}$, then $0 \leq a < 1$.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that

$$y_{2n} = Sx_{2n} = gx_{2n+1}, \quad y_{2n+1} = Tx_{2n+1} = fx_{2n+2}, \quad n = 0,1,2,...,$$

where $x_0$ is an arbitrary fixed point in $X$.

Therefore by using (2.2) we obtain that

$$\begin{aligned}
d(y_{2n},y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1}) \\
&\precsim_{i_2} a\max\left\{\begin{array}{c} d(fx_{2n},gx_{2n+1}), d(fx_{2n},Sx_{2n}), d(gx_{2n+1},Tx_{2n+1}), \\ d(fx_{2n},Tx_{2n+1}), d(gx_{2n+1},Sx_{2n}) \end{array}\right\} \\
&\precsim_{i_2} a\max\left\{\begin{array}{c} d(y_{2n-1},y_{2n}), d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}), \\ d(y_{2n-1},y_{2n+1}), d(y_{2n},y_{2n}) \end{array}\right\} \\
&\precsim_{i_2} a\max\left\{\begin{array}{c} d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}), \\ s[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})] \end{array}\right\}.
\end{aligned} \tag{2.3}$$

Now if $y_{2n-1} = y_{2n}$ for some $n$ then from (2.3) we get that

$$d(y_{2n},y_{2n+1}) \precsim_{i_2} sa\ d(y_{2n},y_{2n+1})$$

$$i.e., \ d(y_{2n},y_{2n+1}) = 0 \Rightarrow y_{2n} = y_{2n+1}.$$

Continuing this process we can show that $y_{2n-1} = y_{2n} = y_{2n+1} = ...$

Therefore $\{y_n\}$ is a Cauchy sequence. Again if $y_n \neq y_{n+1}$ for all $n$ then (2.3) can be written as

$$d_{2n} \precsim_{i_2} a \max\{d_{2n-1}, d_{2n}, s[d_{2n-1}+d_{2n}]\}, \tag{2.4}$$

where $d_{2n} = d(y_{2n}, y_{2n+1})$. Now if $a \max\{d_{2n-1}, d_{2n}, s[d_{2n-1}+d_{2n}]\} = d_{2n}$ then $d_{2n} \precsim_{i_2} ad_{2n}$, which is not true, as $0 \leq a < 1$. Therefore from (2.4) we get that $d_{2n} \precsim_{i_2} a \max\{d_{2n-1}, s[d_{2n-1}+d_{2n}]\} \Rightarrow d_{2n} \precsim_{i_2} \gamma d_{2n-1}$ for all $n \in \mathbb{N}$, where $\gamma = \max\left\{a, \frac{sa}{1-sa}\right\}$. Similarly, we can show that $d_{2n-1} \precsim_{i_2} \gamma d_{2n-2}$ for all $n \in \mathbb{N}$ and therefore

$$d_n \precsim_{i_2} \gamma d_{n-1} \text{ for all } n \in \mathbb{N}$$

$$\begin{aligned} i.e., \ \|d_n\| &\leq \gamma\|d_{n-1}\| \leq \gamma^2\|d_{n-2}\| \leq \ldots\ldots.. \leq \gamma^n\|d_0\| \\ i.e., \ \|d(y_n, y_{n+1})\| &\leq \gamma^n\|d(y_0, y_1)\| \text{ for all } n = 1, 2, 3, \ldots \end{aligned} \tag{2.5}$$

Since $\gamma = \max\left\{a, \frac{sa}{1-sa}\right\}$,

then $s\gamma = \begin{cases} sa < s\left[\frac{1}{s^2+s}\right] = \frac{1}{s+1} \leq \frac{1}{2} < 1 \text{ if } \gamma = a \\ s\frac{sa}{1-sa} < s\left[\frac{\frac{1}{s+1}}{1-\frac{1}{s+1}}\right] = 1 \text{ if } \gamma = \frac{sa}{1-sa}. \end{cases}$.

Therefore in both the cases $s\gamma < 1$.

Then for any two positive integers $m, n$ with $m > n$ and $s\gamma < 1$ we have

$$d(y_n, y_m) \precsim_{i_2} s[d(y_n, y_{n+1}) + d(y_{n+1}, y_m)].$$

Hence

$$\begin{aligned} \|d(y_n, y_m)\| &\leq s\|d(y_n, y_{n+1})\| + s\|d(y_{n+1}, y_m)\| \\ &\leq s\|d(y_n, y_{n+1})\| + s^2\|d(y_{n+1}, y_{n+2})\| + s^2\|d(y_{n+2}, y_m)\| \\ &\leq \left\{ \begin{array}{c} s\|d(y_n, y_{n+1})\| + s^2\|d(y_{n+1}, y_{n+2})\| + \\ s^3\|d(y_{n+2}, y_{n+3})\| + s^3\|d(y_{n+3}, y_m)\| \end{array} \right\} \end{aligned}$$

$$i.e.,\ \ \|d(y_n,y_m)\|$$
$$\leq \left\{ \begin{array}{c} s\|d(y_n,y_{n+1})\| + s^2\|d(y_{n+1},y_{n+2})\| + \\ s^3\|d(y_{n+2},y_{n+3})\| + ..... + s^{m-n-1}\|d(y_{m-1},y_m)\| \end{array} \right\}$$
$$i.e.,\ \ \|d(y_n,y_m)\|$$
$$\leq \left\{ \begin{array}{c} s\|d(y_n,y_{n+1})\| + s^2\|d(y_{n+1},y_{n+2})\| + \\ s^3\|d(y_{n+2},y_{n+3})\| + ..... + s^{m-n}\|d(y_{m-1},y_m)\| \end{array} \right\},\ \text{as } s \geq 1.$$

Therefore by using (2.5) we obtain that

$$\|d(y_n,y_m)\| \leq$$
$$\left\{ \begin{array}{c} s\gamma^n\|d(y_0,y_1)\| + s^2\gamma^{n+1}\|d(y_0,y_1)\| + s^3\gamma^{n+2}\|d(y_0,y_1)\| + \\ ..... + s^{m-n}\gamma^{m-1}\|d(y_0,y_1)\| \end{array} \right\}$$

$$i.e.,\ \ \|d(y_n,y_m)\| \leq \sum_{i=1}^{m-n} s^i\gamma^{i+n-1}\|d(y_0,y_1)\|$$

$$i.e.,\ \ \|d(y_n,y_m)\| \leq \sum_{i=1}^{m-n} s^{i+n-1}\gamma^{i+n-1}\|d(y_0,y_1)\|,\ \text{as } s \geq 1$$

$$i.e.,\ \ \|d(y_n,y_m)\| \leq \sum_{j=n}^{m-1} s^j\gamma^j\|d(y_0,y_1)\|$$

$$i.e.,\ \ \|d(y_n,y_m)\| \leq \sum_{j=n}^{\infty} (s\gamma)^j\|d(y_0,y_1)\|$$

$$i.e.,\ \ \|d(y_n,y_m)\| \leq \frac{(s\gamma)^n}{1-s\gamma}\|d(y_0,y_1)\|.$$

Since $\frac{(s\gamma)^n}{1-s\gamma} \longrightarrow 0$ as $n \longrightarrow \infty$, then for any $\varepsilon > 0$ there exists a positive integer $n_0$ such that $\|d(y_n,y_m)\| < \varepsilon$ for all $m,n > n_0$.Therefore $\{y_n\}$ is a Cauchy sequence in $X$. Also $X$ is a complete bicomplex valued $b$-metric space. Hence there exists $z \in X$ such that $\lim\limits_{n\longrightarrow\infty} Sx_{2n} = \lim\limits_{n\longrightarrow\infty} gx_{2n+1} = \lim\limits_{n\longrightarrow\infty} Tx_{2n+1} = \lim\limits_{n\longrightarrow\infty} fx_{2n+2} = z$. Again $T(X)$ is a closed subspace of $X$, therefore $z \in T(X)$.

Since, $T(X) \subseteq f(X)$, then there exists some $u \in X$ such that $z = fu$.

Now we show that $Su = z$.

We have

$d(Su,z) \precsim_{i_2} s\left[d(Su,Tx_{2n+1})+d(Tx_{2n+1},z)\right]$

$i.e.,\ \frac{1}{s}d(Su,z) \precsim_{i_2}$

$$a\max\left\{\begin{array}{c} d(fu,gx_{2n+1}),d(fu,Su),d(gx_{2n+1},Tx_{2n+1}), \\ d(fu,Tx_{2n+1}),d(gx_{2n+1},Su) \end{array}\right\}+d(Tx_{2n+1},z)$$

$i.e.,\ \frac{1}{s}d(Su,z) \precsim_{i_2}$

$$a\max\left\{\begin{array}{c} d(z,y_{2n}),d(z,Su),d(y_{2n},y_{2n+1}), \\ d(z,y_{2n+1}),d(y_{2n},Su) \end{array}\right\}+d(y_{2n+1},z)$$

$i.e.,\ \frac{1}{s}\|d(Su,z)\| \leq$

$$a\max\left\{\begin{array}{c} \|d(z,y_{2n})\|,\|d(z,Su)\|,\|d(y_{2n},y_{2n+1})\|, \\ \|d(z,y_{2n+1})\|,\|d(y_{2n},Su)\| \end{array}\right\}+\|d(y_{2n+1},z)\|$$

Taking limit as $n\to\infty$ we get that

$$\begin{aligned} \frac{1}{s}\|d(Su,z)\| &\leq a\max\{\|d(z,Su)\|,s\|d(z,Su)\|\} \\ i.e.,\ \frac{1}{s}\|d(Su,z)\| &\leq as\|d(z,Su)\| \\ i.e.,\ \|d(Su,z)\| &\leq s^2a\|d(Su,z)\|. \end{aligned}$$

Since $a<\frac{1}{s^2+s}\Rightarrow a\left(s^2+s\right)<1\Rightarrow s^2a<1$, then $\|d(Su,z)\|=0\Rightarrow Su=z=fu$.

Again since $z=Su\in(X)\subseteq g(X)$, then there exists some $v\in X$ such that $z=gv$.

Now we show that $Tv=z$.

We have

$$d(Tv,z)=d(Tv,Su)=d(Su,Tv)$$

$$\precsim_{i_2} a\max\{d(fu,gv),d(fu,Su),d(gv,Tv),d(fu,Tv),d(gv,Su)\}$$

$$\precsim_{i_2} a\max\{d(z,z),d(z,z),d(z,Tv),d(z,Tv),d(z,z)\}$$

$$\begin{aligned} i.e.,\ d(Tv,z) &\precsim_{i_2} ad(Tv,z) \\ i.e.,\ \|d(Tv,z)\| &\leq a\|d(Tv,z)\|. \end{aligned}$$

Since $a < 1$, then $\|d(Tv, z)\| = 0 \Rightarrow Tv = z$ and so $Su = fu = Tv = gv = z$.

Also $f$ and $S$ are weakly compatible mappings, therefore $Sfu = fSu \Rightarrow Sz = fz$. Now we show that $Sz = z$.

We have

$$\begin{aligned} d(Sz, z) &= d(Sz, Tv) \\ \precsim_{i_2} & \; a \max\{d(fz, gv), d(fz, Sz), d(gv, Tv), d(fz, Tv), d(gv, Sz)\} \\ \precsim_{i_2} & \; a \max\{d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(z, Sz)\} \\ \precsim_{i_2} & \; ad(Sz, z) \end{aligned}$$

$$i.e., \;\; \|d(Sz, z)\| \leq a \|d(Sz, z)\|.$$

Since $a < 1$, then $\|d(Sz, z)\| = 0 \Rightarrow Sz = z = fz$, which is a contradiction. Therefore $Sz = fz = z$.

Again since $g$ and $T$ are weakly compatible mappings, then $Tgv = gTv \Rightarrow Tz = gz$. Now we show that $Tz = z$. If not then $0 \prec_{i_2} d(Tz, z) \in \mathbb{C}_2$.

Therefore,

$$\begin{aligned} d(Tz, z) &= d(Tz, Sz) = d(Sz, Tz) \\ \precsim_{i_2} & \; a \max\{d(fz, gz), d(fz, Sz), d(gz, Tz), d(fz, Tz), d(gz, Sz)\} \\ \precsim_{i_2} & \; a \max\{d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(Tz, z)\} \\ \precsim_{i_2} & \; ad(Tz, z) \end{aligned}$$

i.e., $\|d(Tz, z)\| \leq a \|d(z, Tz)\|$

Since $a < 1$, then $\|d(Tz, z)\| = 0 \Rightarrow Tz = z = gz$ and so $Sz = Tz = fz = gz = z$. This shows that $z$ is a common fixed point of $S, T, f$ and $g$.

Now we show that $f, g, S$ and $T$ have a unique common fixed point. If possible suppose that $z^* \in X$ be another common fixed point of $S, T, f$ and $g$, i.e. $Sz^* = Tz^* = fz^* = gz^* = z^*$.

Then

$$
\begin{aligned}
d(z,z^*) &= d(Sz,Tz^*) \\
&\precsim_{i_2} a\max\left\{ \begin{array}{c} d(fz,gz^*), d(fz,Sz), d(gz^*,Tz^*), \\ d(fz,Tz^*), d(gz^*,Sz) \end{array} \right\} \\
&\precsim_{i_2} a\max\{d(z,z^*), d(z,z), d(z^*,z^*), d(z,z^*), d(z^*,z)\} \\
&\precsim_{i_2} ad(z,z^*).
\end{aligned}
$$

Therefore,

$$\|d(z,z^*)\| \leq a\|d(z,z^*)\|.$$

Since $a < 1$, then $\|d(z,z^*)\| = 0 \Rightarrow z = z^*$. Therefore $z$ is the unique common fixed point of $S, T, f$ and $g$. This completes the proof of the theorem. □

**Example 2.4.** Let $X = [0,1]$, and consider the mapping $d : X \times X \to \mathbb{C}_2$ as defined by $d(x,y) = (1 + i_1 + i_2 + i_1 i_2)|x - y|^2$

Then for all $x, y, z \in X$,

$$
\begin{aligned}
d(x,y) &= (1 + i_1 + i_2 + i_1 i_2)|x - y|^2 \\
&= (1 + i_1 + i_2 + i_1 i_2)|x - z + z - y|^2 \\
&= (1 + i_1 + i_2 + i_1 i_2)\left(|x - z|^2 + |z - y|^2 + 2|x - z||z - y|\right) \\
&\precsim_{i_2} (1 + i_1 + i_2 + i_1 i_2)\left(|x - z|^2 + |z - y|^2 + |x - z|^2 + |z - y|^2\right) \\
&\precsim_{i_2} 2[d(x,z) + d(z,y)]
\end{aligned}
$$

therefore $(X,d)$ is a bicomplex valued $b$-metric space as $s = 2$

Now, we consider the mappings $S, T, f, g : X \to X$ by $Sx = \frac{x}{4}, Tx = \frac{x^3}{8}, fx = x$ and $gx = \frac{x^3}{2}$,

Then, $d(Sx,Ty) = \frac{1}{16}(1 + i_1 + i_2 + i_1 i_2)\left|x - \frac{y^3}{2}\right|^2 = \frac{1}{16}d(fx,gy)$, here $a = \frac{1}{16} < \frac{1}{s^2+s} = \frac{1}{6}$.

Also $Sfx = fSx$ and $Tgx = gTx$ at the point of coincidence, therefore $\{S,f\}$ and $\{T,g\}$ are weakly compatible and $T(X)$ is a closed subspace of $X$.

Therefore, $S, T, f$ and $g$ satisfy all conditions of the Theorem(2.3).

Clearly, 0 is the unique common fixed point of $S, T, f$ and $g$.

**Corollary 2.5.** *Let $(X, d)$ be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let $S, T, f : X \to X$ be mappings such that*

$$S(X) \subseteq f(X) \text{ and } T(X) \subseteq f(X) \tag{2.6}$$

*also satisfying the condition*

$$d(Sx, Ty) \precsim_{i_2} a \max\{d(fx, fy), d(fx, Sx), d(fy, Ty), d(fx, Ty), d(fy, Sx)\} \tag{2.7}$$

*for all $x, y \in X$, where $0 \leq a < \frac{1}{s^2+s}$. Suppose that the pairs $\{S, f\}$ and $\{T, f\}$ be weakly compatible and $T(X)$ is a closed subspace of $X$. Then $S, T$ and $f$ have a unique common fixed point.*

## Concluding comments

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric, probabilistic metric, $p$-adic metric (where $p$ is a prime number), cone metric, quasi semi metric, convex metric, $D$-metric and other different types of metrics under the flavour of bicomplex analysis. This may be regarded as an active area of research to the future workers in this branch.

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# COMMON FIXED POINT THEOREM FOR TWO MAPPINGS IN B–METRIC SPACE

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Abstract. The aim of this paper is to obtain the common fixed point theorems for two mappings in b-–metric space. In this paper we prove that a unique common fixed point exists for two mappings in b-–metric space. Moreover an example is discussed to verify the main result.

## 1. Introduction

In many branches of Science, Economics, Computer Science, Engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool. In 1989, Bakhtin [2] introduced the concept of generalized b–metric spaces. Boriceanu [4], Mehmat Kir [9] extended the fixed point theorem in b–metric space. Borkar [5] obtained the common fixed point theorem for non expansive type mapping.

In 1993 Czerwik [8] extended the results of b-metric spaces. Using this idea many researchers presented generalization of renowned Banach fixed point theorem in b-metric space. Czerwik [8] 1998 presented the generalization of Banach fixed point theorem in b-metric spaces. Agrawal [1] presented the existence and uniqueness theorem in b-–Metric Space.

Chopade [6] dicussed the common fixed point theorems for contractive type mapping in metric space. Roshan [10] obtained common fixed point of four maps in b–Metric space. Suzuki [11] obtained some basic inequalities and applications in b–Metric space.

In this work we extend the well known fixed point theorems which are also valid in b–metric space.

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## 2. Some Basic Definitions and Preliminaries

**Definition 2.1**: Let $X$ be a non-empty set and $s \geq 1$ be a given real number. A function $\delta : X \times X \longrightarrow R$ is called a b–metric provided that for all $u, v, w \in X$

i. $\delta(u, v) \geq 0$
ii. $\delta(u, v) = 0$ if and only if $u = v$
iii. $\delta(u, v) = \delta(v, u)$
iv. $\delta(u, v) \leq s\{\delta(u, w) + \delta(w, v)\}$

A pair $(X, \delta)$ is called a b–metric space. It is clear that the definition of b–metric space is an extension of usual metric space.

**Remark:** If $s = 1$, then the b–metric space is a usual metric space.

**Example 2.1:** Let $(X, d)$ be a metric space and $\delta(u, v) = (d(u, v))^p$, where $p > 1$ is a real number. Clearly, $\delta(u, v)$ is a b–metric with $s = 2^{p-1}$.

**Example 2.2:** If $X = R$ , be the set of real numbers and $d(u, v) = |u - v|$ usual metric, then $\delta(u, v) = (u - v)^2$ is a b–metric on $R$ with $s = 2$, but not a metric on $R$.

**Example 2.3:** By Boriceanu [4], Let $M = \{0, 1, 2\}$ and $\delta : M \times M \to R$ is defined by,
$\delta(0, 2) = \delta(2, 0) = m \geq 2$
$\delta(0, 1) = \delta(1, 0) = \delta(1, 2) = \delta(2, 1) = 1$
$\delta(0, 0) = \delta(1, 1) = \delta(2, 2) = 0.$
Here $\delta(u, v)$ is a b–metric on $M$ with $s = \frac{m}{2}$

**Definition 2.2:** Let $(X, \delta)$ be a b–metric space then a sequence $\{u_n\}$ in $X$ is called a convergent sequence if there exists $u \in X$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in N$ such that $n \geq n(\epsilon)$ we have, $\delta(u_n, u) < \epsilon$.
In this case we write $\lim_{n\to\infty} u_n = u$.

**Definition 2.3:** Let $(X, \delta)$ be a b–metric space then a sequence $\{u_n\}$ in $X$ is called Cauchy sequence if for all $\epsilon > 0$ there exists $n(\epsilon) \in N$ such that

$m, n \geq n(\epsilon)$ we have, $\delta(u_n, u_m) < \epsilon$.

**Definition 2.4:** Let $(X, \delta)$ be a b–metric space then $X$ is said to be complete if every Cauchy sequence in $X$ is convergent sequence in $X$.

## 3. Main Result

We use the following Lemma to prove the main result.

**Lemma 3.1** [11] Let $(X, \delta)$ be a complete b–metric space and let $\{x_n\}$ be a sequence in $X$. Assume that there exist $r \in [0, 1)$ satisfying

$$\delta(x_{n+1}, x_{n+2}) \leq r\delta(x_n, x_{n+1}) \text{ for any } n \in N.$$

Then $\{x_n\}$ is a Cauchy sequence in $X$.

**Theorem 3.1** Let $(X, \delta)$ be a complete b–metric space. $A : X \to X$ and $B : X \to X$ be any two selfmaps on $X$ satisfying

$$\begin{aligned} \delta(Au, Bv) \leq \alpha\delta(u, v) \ + \ \beta max\{\delta(u, Bv), \delta(Au, v)\} \\ + \ \gamma[\delta(u, Au) \ + \ \delta(v, Bv)], \forall\, u, v \in X \end{aligned} \tag{3.1}$$

where $\alpha, \beta, \gamma \geq 0$ and $s \geq 1$ such that,

i. $\alpha + 2\beta s + 2\gamma < 1$
ii. $\gamma s + \beta s^2 < 1$

Then $A$ and $B$ have a unique common fixed point in X.

**Proof:** Let $u_0 \in X$, and $\{u_n\}$ be a sequence in X defined by the recursion

$$A(u_{2n}) = u_{2n+1} \qquad \textit{and} \qquad B(u_{2n-1}) = u_{2n}, n = 1, 2, ... \tag{3.2}$$

Using equations (3.1) and (3.2) we obtain,

$$\begin{aligned} \delta(u_1, u_2) &= \delta(Au_0, Bu_1) \\ &\leq \alpha\delta(u_0, u_1) + \beta max\{\delta(u_0, Bu_1), \delta(u_1, Au_0)\} \\ &+ \gamma[\delta(u_0, Au_0) + \delta(u_1, Bu_1)] \\ &\leq \alpha\delta(u_0, u_1) + \beta max\{\delta(u_0, u_2), \delta(u_1, u_1)\} \\ &+ \gamma[\delta(u_0, u_1) + \delta(u_1, u_2)] \\ &\leq \alpha\delta(u_0, u_1) + \beta max\{\delta(u_0, u_2), 0\} \\ &+ \gamma[\delta(u_0, u_1) + \delta(u_1, u_2)] \\ &\leq \alpha\delta(u_0, u_1) + \beta\delta(u_0, u_2) + \gamma\delta(u_0, u_1) + \gamma\delta(u_1, u_2) \\ (1 - \gamma)\delta(u_1, u_2) &\leq (\alpha + \gamma)\delta(u_0, u_1) + \beta s\delta(u_0, u_1) + \beta s\delta(u_1, u_2) \end{aligned}$$

$$(1-\beta s-\gamma)\delta(u_1,u_2) \leq (\alpha+\gamma+\beta s)\delta(u_0,u_1)$$
$$\delta(u_1,u_2) \leq \frac{(\alpha+\gamma+\beta s)}{(1-\beta s-\gamma)}\delta(u_0,u_1)$$
$$\delta(u_1,u_2) \leq r\delta(u_0,u_1)$$

where $r = \frac{(\alpha+\gamma+\beta s)}{(1-\beta s-\gamma)} < 1$.
Now consider,

$$\begin{aligned}
\delta(u_2,u_3) &= \delta(Bu_1,Au_2) = \delta(Au_2,Bu_1)\\
&\leq \alpha\delta(u_2,u_1) + \beta max\{\delta(u_2,Bu_1),\delta(u_1,Au_2)\}\\
&+ \gamma[\delta(u_2,Au_2) + \delta(u_1,Bu_1)]\\
&\leq \alpha\delta(u_2,u_1) + \beta max\{\delta(u_2,u_2),\delta(u_1,u_3)\}\\
&+ \gamma[\delta(u_2,u_3) + \delta(u_1,u_2)]\\
&\leq \alpha\delta(u_2,u_1) + \beta max\{0,\delta(u_1,u_3)\}\\
&+ \gamma[\delta(u_2,u_3) + \delta(u_1,u_2)]\\
&\leq \alpha\delta(u_2,u_1) + \beta\delta(u_1,u_3) + \gamma\delta(u_2,u_3) + \gamma\delta(u_1,u_2)\\
(1-\gamma)\delta(u_2,u_3) &\leq (\alpha+\gamma)\delta(u_1,u_2) + \beta s\delta(u_1,u_2) + \beta s\delta(u_2,u_3)\\
(1-\beta s-\gamma)\delta(u_2,u_3) &\leq (\alpha+\gamma+\beta s)\delta(u_1,u_2)\\
\delta(u_2,u_3) &\leq \frac{(\alpha+\gamma+\beta s)}{(1-\beta s-\gamma)}\delta(u_1,u_2)\\
\delta(u_2,u_3) &\leq r\delta(u_1,u_2)
\end{aligned}$$

where $r = \frac{(\alpha+\gamma+\beta s)}{(1-\beta s-\gamma)} < 1$.
In general, for any $n \in N$,

$$\delta(u_{n+1},u_{n+2}) \leq r\delta(u_n,u_{n+1}) \tag{3.3}$$

where $r = \frac{(\alpha+\gamma+\beta s)}{(1-\beta s-\gamma)} < 1$.

Therefore by Lemma 3.1, the sequence $\{u_n\}$ is a Cauchy Sequence in X. Since X is complete, sequence $\{u_n\}$ converges to $u^*$ in X.
Hence as,

$$n \to \infty,\ u_n \to u^* \Rightarrow u_{2n-1} \to u^*,\ u_{2n} \to u^*\ and\ u_{2n+1} \to u^*$$

Now we show that $u^*$ is a fixed point of both the mappings $A$ and $B$.

Consider,

$$\delta(Au^*,u^*) \leq s[\delta(Au^*,u_{2n}) + \delta(u_{2n},u^*)]$$

$$
\begin{aligned}
&\leq s\delta(Au^*, Bu_{2n-1}) + s\delta(u_{2n}, u^*)\\
&\leq s\{\alpha\delta(u^*, u_{2n-1}) + \beta max\{\delta(u^*, Bu_{2n-1}), \delta(u_{2n-1}, Au^*)\}\\
&+ \gamma[\delta(u^*, Au^*) + \delta(u_{2n-1}, Bu_{2n-1})]\} + s\delta(u_{2n}, u^*)
\end{aligned}
$$

$$
\begin{aligned}
\delta(Au^*, u^*) &\leq \alpha s\delta(u^*, u_{2n-1}) + \beta sM_1 + \gamma s\delta(u^*, Au^*)\\
&+ \gamma s\delta(u_{2n-1}, Bu_{2n-1}) + s\delta(u_{2n}, u^*)
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
(1-\gamma s)\delta(Au^*, u^*) &\leq \alpha s\delta(u^*, u_{2n-1}) + \beta sM_1\\
&+ \gamma s^2[\delta(u_{2n-1}, u^*) + \delta(u^*, Bu_{2n-1})] + s\delta(u_{2n}, u^*)\\
(1-\gamma s)\delta(Au^*, u^*) &\leq \alpha s\delta(u^*, u_{2n-1}) + \beta sM_1 + \gamma s^2\delta(u_{2n-1}, u^*)\\
&+ \gamma s^2\delta(u^*, u_{2n}) + s\delta(u_{2n}, u^*)\\
(1-\gamma s)\delta(Au^*, u^*) &\leq (\alpha s + \gamma s^2)\delta(u^*, u_{2n-1}) + \beta sM_1\\
&+ (s + \gamma s^2)\delta(u^*, u_{2n})
\end{aligned}
$$

where, $M_1 = max\{\delta(u^*, Bu_{2n-1}), \delta(u_{2n-1}, Au^*)\}$

**Case I:** If

$$
\begin{aligned}
M_1 &= max\{\delta(u^*, Bu_{2n-1}), \delta(u_{2n-1}, Au^*)\}\\
&= \delta(u^*, Bu_{2n-1})
\end{aligned}
$$

then by equation (3.4) we get,

$$
\begin{aligned}
(1-\gamma s)\delta(Au^*, u^*) &\leq (\alpha s + \gamma s^2)\delta(u^*, u_{2n-1}) + \beta s\delta(u^*, Bu_{2n-1})\\
&+ (s + \gamma s^2)\delta(u^*, u_{2n})
\end{aligned}
$$

$$
\begin{aligned}
(1-\gamma s)\delta(Au^*, u^*) &\leq (\alpha s + \gamma s^2)\delta(u^*, u_{2n-1}) + \beta s\delta(u^*, u_{2n})\\
&+ (s + \gamma s^2)\delta(u^*, u_{2n})\\
(1-\gamma s)\delta(Au^*, u^*) &\leq (\alpha s + \gamma s^2)\delta(u^*, u_{2n-1}) + (s + \beta s + \gamma s^2)\delta(u^*, u_{2n})\\
\delta(Au^*, u^*) &\leq \frac{(\alpha s + \gamma s^2)}{(1-\gamma s)}\delta(u^*, u_{2n-1}) + \frac{(s + \beta s + \gamma s^2)}{(1-\gamma s)}\delta(u^*, u_{2n})
\end{aligned}
$$

Hence,

$$
\begin{gathered}
\lim_{n\to\infty} \delta(Au^*, u^*) = 0\\
\delta(Au^*, u^*) = 0\\
Au^* = u^*.
\end{gathered}
$$

Therefore, $u^*$ is a fixed point of $A$.

**Case II:** If $M_1 = max\{\delta(u^*, Bu_{2n-1}), \delta(u_{2n-1}, Au^*)\} = \delta(u_{2n-1}, Au^*)$ then by equation (3.4) we get,

$$\begin{aligned}
(1-\gamma s)\delta(Au^*,u^*) &\leq (\alpha s+\gamma s^2)\delta(u^*,u_{2n-1})+\beta s\delta(u_{2n-1},Au^*)\\
&\quad+(s+\gamma s^2)\delta(u^*,u_{2n})\\
(1-\gamma s)\delta(Au^*,u^*) &\leq (\alpha s+\gamma s^2)\delta(u^*,u_{2n-1})+\beta s^2\delta(u_{2n-1},u^*)\\
&\quad+\beta s^2\delta(Au^*,u^*)+(s+\gamma s^2)\delta(u^*,u_{2n})\\
(1-\gamma s-\beta s^2)\delta(Au^*,u^*) &\leq (\alpha s+\gamma s^2+\beta s^2)\delta(u^*,u_{2n-1})\\
&\quad+(s+\gamma s^2)\delta(u^*,u_{2n})\\
\delta(Au^*,u^*) &\leq \frac{(\alpha s+\gamma s^2+\beta s^2)}{(1-\gamma s-\beta s^2)}\delta(u^*,u_{2n-1})+\frac{(s+\gamma s^2)}{(1-\gamma s-\beta s^2)}\delta(u^*,u_{2n})
\end{aligned}$$

Hence,

$$\begin{gathered}
\lim_{n\to\infty}\delta(Au^*,u^*)=0\\
\delta(Au^*,u^*)=0\\
Au^*=u^*
\end{gathered}$$

Therefore, $u^*$ is a fixed point of $A$.
Now we prove that, $u^*$ is a fixed point of $B$.
Consider,

$$\begin{aligned}
\delta(u^*,Bu^*) &= \delta(Au^*,Bu^*)\\
\delta(u^*,Bu^*) &\leq \alpha\delta(u^*,u^*)+\beta\max\{\delta(u^*,Au^*),\delta(u^*,Bu^*)\}\\
&\quad+\gamma\delta(u^*,Bu^*)\\
(1-\gamma)\delta(u^*,Bu^*) &\leq \beta\ \max\{0,\delta(u^*,Bu^*)\}\\
(1-\gamma)\delta(u^*,Bu^*) &\leq \beta\delta(u^*,Bu^*)\\
(1-\gamma-\beta)\delta(u^*,Bu^*) &\leq 0\\
\delta(u^*,Bu^*) &\leq 0\\
\delta(u^*,Bu^*) &= 0\\
Bu^* &= u^*
\end{aligned}$$

Therefore, $u^*$ is a fixed point of $B$.
Hence, $u^*$ is a common fixed point of the mappings $A$ and $B$.

**Uniqueness:**

Suppose, $u^*$ and $v^*$ be two common fixed points of the mappings $A$ and $B$.
Therefore, $Au^*=Bu^*=u^*$ and $Av^*=Bv^*=v^*$

Consider

$\delta(u^*, v^*) = \delta(Au^*, Bv^*)$

$\delta(u^*, v^*) \leq \alpha\delta(u^*, v^*) + \beta \max\{\delta(u^*, Bv^*), \delta(Au^*, v^*)\} + \gamma\delta(u^*, Au^*)$
$\qquad + \gamma\delta(v^*, Bv^*)$

$\delta(u^*, v^*) \leq \alpha\delta(u^*, v^*) + \beta\max\{\delta(u^*, v^*), \delta(u^*, v^*)\} + \gamma\delta(u^*, u^*) + \gamma\delta(v^*, v^*)$

$\delta(u^*, v^*) \leq (\alpha + \beta)\delta(u^*, v^*)$

But, $(\alpha + \beta) < 1$.
Therefore we have,

$$\delta(u^*, v^*) < \delta(u^*, v^*)$$

This is a contradiction. Hence, $A$ and $B$ have a unique common fixed point.

**Corollary 3.1:** Let $(X, \delta)$ be a complete b–metric space. $A : X \to X$ and $B : X \to X$ are self mappings on $X$ which satisfies condition 3.1 and if any one of $A$ or $B$ is continuous then $A$ and $B$ have a unique common fixed point in $X$.

## 4. Application

**Example 4.1:** Let $X = [0, 1]$ be a complete b–metric space with respect to the b-metric $\delta$ defined by $\delta(u, v) = (u-v)^2$ for all $u, v \in X$. Let $A : X \to X$ and $B : X \to X$ be defined $A(u) = (\frac{u}{8})^2$ for all $u \in X$, and $B(v) = (\frac{v}{4})^2$ for all $v \in X$

Choose $\alpha = \frac{1}{4}, \beta = \frac{1}{16}$ and $\gamma = \frac{1}{8}$ . Then

i. $\alpha + 2\beta s + 2\gamma = \frac{1}{4} + \frac{4}{16} + \frac{2}{8} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} < 1$
ii. $\gamma s + \beta s^2 = \frac{2}{8} + \frac{4}{16} = \frac{1}{4} + \frac{1}{4} < 1$

Moreover for all $u \leq v$ with $u, v \in X$, $A$ and $B$ satisfies the condition

$$\begin{aligned}
\delta(Au, Bv) &= \left[\left(\frac{u}{8}\right)^2 - \left(\frac{v}{4}\right)^2\right]^2 \\
&= \left[\frac{u}{8} - \frac{v}{4}\right]^2 \left[\frac{u}{8} + \frac{v}{4}\right]^2 \\
&\leq \left(\frac{3}{8}\right)^2 \left[\frac{u}{8} - \frac{v}{4}\right]^2 \\
&\leq \left(\frac{9}{64}\right) \left[\frac{u - 2v}{8}\right]^2
\end{aligned}$$

$$\begin{aligned}
\delta(Au, Bv) &\leq \left(\frac{9}{64}\right)\left[\frac{2u-2v}{8}\right]^2 \\
&\leq \left(\frac{9}{64}\right)\frac{(u-v)^2}{16} \\
&\leq \frac{4(u-v)^2}{16} \\
&\leq \frac{\delta(u,v)}{4} \\
\delta(Au, Bv) &\leq \frac{1}{4}\delta(u,v) + \frac{1}{16}\max\{\delta(u,Bv), \delta(Au,v)\} + \frac{1}{8}\delta(Au,u) \\
&+ \frac{1}{8}\delta(v,Bv)
\end{aligned}$$

$$\delta(Au, Bv) \leq \alpha\delta(u,v) + \beta\max\{\delta(Au,v), \delta(u,Bv)\} + \gamma\left[\delta(Au,u) + \delta(v,Bv)\right]$$

Hence by Theorem 3.1, $A$ and $B$ have a unique common fixed point in $[0, 1]$. The unique common fixed point for $A$ and $B$ is $x = 0$.

## 5. Discussion and the concluding Remarks

In this paper, we have proved the existence and uniqueness of common fixed points for two contractive type mappings. Also we have obtained an example to verify the main result.

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# SPACE-TIME ADMITTING GENERALIZED PROJECTIVE CURVATURE TENSOR

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Abstract. The object of the present paper is to study space-time admitting generalized projective curvature tensor.

## 1. Introduction

The aim of the present work is to study certain investigations in general theory of relativity and cosmology by the coordinate free method of differential geometry. The basic differences between Riemannian and semi-Riemannian geometry are $(i)$ the existence of null vector (i.e. $g(v,v) = 0$, for $v \neq 0$, where $g$ is metric tensor) in semi-Riemannian manifold but not in Riemannian manifold, $(ii)$ the signature of metric tensor $g$ in semi-Riemannian manifold is $(-,-,...-,+,+,...,+)$ but in a Riemannian manifold the signature of $g$ is $(+,+,...,+)$. Lorentzian manifold is a spacial case of semi-Riemannian manifold. The signature of metric tensor $g$ in Lorentzian manifold is $(-,+,+,...,+)$. A Lorentzian manifold consists of three types of vectors such as timelike (i.e. $g(v,v) < 0$), spacelike (i.e. $g(v,v) > 0$) and null vector (i.e. $g(v,v) = 0$, for $v \neq 0$). In general, a Lorentzian manifold $(M^n, g)$ may not have a globally timelike vector field. If $(M^n, g)$ admits a globally timelike vector field, it is called time orientable Lorentzian manifold, physically known as space-time. The foundation of general relativity is based on a 4-dimensional space-time which is the stage of present modeling of the physical world a torsionless, time-oriented Lorentzian manifold $(M, g)$.

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An $n$-dimensional generalized Robertson-Walker (GRW) space-time with $n \geq 3$ is a Lorentzian manifold which is a warped product of an open interval $I$ of $\mathbb{R}$ and an $(n-1)$-dimensional Riemannian manifold ([12], [13], [14]). These Lorentzian manifold broadly extends the classical Robertson-Walker (RW) space-time. RW space-time is regarded as cosmological model since it is spatially homogenous and spatially isotropic whereas GRW space-time serve as inhomogeneous extension of RW space-times that admit an isotropic radiation [22]. A Lorentzian manifold is perfect fluid space-time if its Ricci tensor $S$ has the form

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.1}$$

where $\alpha$ and $\beta$ are scalars, $A$ is a non-zero one-form such that $g(X,v) = A(X)$ for all $v$ and $v$ is the velocity vector field such that $g(v,v) = -1$. Perfect-fluid space-times in a language of differential geometry are called quasi-Einstein spaces where $A$ is metrically equivalent to a unit space-like vector field [5].

The Einstein's field equation with cosmological constant is given by [18]

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{1.2}$$

where $S$ and $r$ denotes the Ricci tensor and scalar curvature respectively, $\lambda$ is the cosmological constant, $T(X,Y)$ is the energy momentum tensor and $k \neq 0$. Einstein's field equation without cosmological constant is given by [18]

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y). \tag{1.3}$$

The Einstein's field equations (1.2) and (1.3) imply that the energy-momentum tensor is conservative. This requirement is satisfied if the energy-momentum tensor is covariant constant [2]. Chaki and Ray [2] showed that a general relativistic space-time with covariant constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$.

A symmetric $(0,2)$ type tensor field $E$ on a semi-Riemannian manifold $(M^n, g)$ is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_U E)(V,X) = (\nabla_V E)(U,X), \tag{1.4}$$

for arbitrary vector fields $U$, $V$ and $X$. The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen [6].

## 2. Preliminaries

The Weyl (Projective) curvature tensor is given as [16]

$$\begin{aligned} P(U,V,X,Y) =& R(U,V,X,Y) \\ & - \frac{1}{n-1}[S(V,X)g(U,Y) - S(U,X)g(V,Y)], \end{aligned} \tag{2.1}$$

where $R(U,V,X,Y)$ is Riemann curvature tensor and $r$ denotes the scalar curvature. Covariant derivative of projective curvature tensor is given as

$$\begin{aligned} (\nabla_W P)(U,V,X,Y) &= (\nabla_W R)(U,V,X,Y) \\ & - \frac{1}{n-1}[g(U,Y)(\nabla_W S)(V,X) - g(V,Y)(\nabla_W S)(U,X)]. \end{aligned} \tag{2.2}$$

Divergence of projective curvature tensor is given as

$$(divP)(U,V,X,Y) = \frac{n-2}{n-1}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X)]. \tag{2.3}$$

In 2012, Mantica and Suh [11] introduced a new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ and studied various geometric properties of it on Riemannian manifold. A new tensor $\mathcal{Z}$ is defined as

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y) \tag{2.4}$$

is called generalized $\mathcal{Z}-$tensor and $\psi$ is an arbitrary scalar function.

**Definition 2.1.** A Riemannian manifold $(M^n, g)$ is said to be of quasi-constant curvature [3] if the Riemannian curvature tensor $R(U,V,X,Y)$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} R(U,V,X,Y) =& p[g(V,X)g(U,Y) - g(U,X)g(V,Y)] \\ & + q[g(U,Y)A(V)A(X) + g(V,X)A(U)A(Y) \\ & - g(U,X)A(V)A(Y) - g(V,Y)A(U)A(X)], \end{aligned} \tag{2.5}$$

where $p, q$ are scalar functions and $A$ is non-zero 1-form. If $q = 0$, then the manifold reduces to manifold of constant curvature.

**Definition 2.2.** A $(0,4)$ tensor is called a generalized curvature tensor if it obeys the symmetries like Riemannian curvature tensor $R$ ([7], [20], [21]). Since the projective curvature tensor $P(U,V,X,Y)$ is skew-symmetric in first two slots, not skew-symmetric in last two slots and not symmetric in pair of slots [16], therefore projective curvature tensor is not a generalized curvature tensor.

## 3. Generalized Projective Curvature Tensor

In view of equation (2.4), equation (2.1) takes the form

$$P(U,V,X,Y) = R(U,V,X,Y) - \frac{1}{n-1}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y)] + \frac{\psi}{(n-1)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]. \tag{3.1}$$

Define

$$P^*(U,V,X,Y) = R(U,V,X,Y) - \frac{1}{n-1}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y)]. \tag{3.2}$$

Thus using equation (3.2) in equation (3.1), we have

$$P(U,V,X,Y) = P^*(U,V,X,Y) + \frac{\psi}{(n-1)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)],$$

which gives

$$P^*(U,V,X,Y) = P(U,V,X,Y) - \frac{\psi}{(n-1)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)], \tag{3.3}$$

where $P^*(U,V,X,Y)$ is called generalized projective curvature tensor.
If $\psi = 0$, then from equation (3.3), we obtain

$$P^*(U,V,X,Y) = P(U,V,X,Y). \tag{3.4}$$

Thus we have the following:
**Note:** A generalized projective curvature tensor reduces to projective curvature tensor provided that the scalar function $\psi$ vanishes.
Now, if $P^*(U,V,X,Y) = 0$, then from equation (3.3), we obtain

$$P(U,V,X,Y) = \frac{\psi}{(n-1)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)], \tag{3.5}$$

which is of constant projective curvature tensor. Thus we can state as follows:

**Proposition 3.1.** *A generalized projectively flat manifold is of constant projective curvature tensor.*

Again, if projective curvature tensor vanishes, then from equation (3.3), we obtain

$$P^*(U,V,X,Y) = -\frac{\psi}{(n-1)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)], \qquad (3.6)$$

which is of constant generalized projective curvature tensor. Thus we can state as follows:

**Proposition 3.2.** *A projectively flat manifold is of constant generalized projective curvature tensor.*

From equation (3.3), we obtain

$$P^*(V,U,X,Y) = P(V,U,X,Y) - \frac{\psi}{(n-1)}G(U,V,X,Y), \qquad (3.7)$$

where

$$G(U,V,X,Y) = g(V,X)g(U,Y) - g(U,X)g(V,Y) \qquad (3.8)$$

called Gaussian tensor [19] and [21]. Gaussion tensor $G(U,V,X,Y)$ is skew-symmetric first two slots, skew-symmetric last two slots and symmetric pair of slots [19] and [21]. By using the properties of Gaussian tensor and projective curvature tensor we can state as follows:

**Theorem 3.3.** *A generalized projective curvature tensor on* $(M^n, g)$ *is*
(1) *skew-symmetric in first two slots,*
(2) *not skew-symmetric in lost two slots,*
(3) *not symmetric in pair of slots.*

Now, writing two more equations by the cyclic permutations of $U$, $V$ and $X$ of equation (3.3), we obtain

$$P^*(V,X,U,Y) = P(V,X,U,Y) - \frac{\psi}{(n-1)}[g(X,U)g(V,Y) - g(V,U)g(X,Y)], \qquad (3.9)$$

and

$$P^*(X,U,V,Y) = P(X,U,V,Y) - \frac{\psi}{(n-1)}[g(U,V)g(X,Y) - g(X,V)g(U,Y)], \qquad (3.10)$$

Adding equations (3.3), (3.9) and (3.10), we obtain

$$\begin{aligned} P^*(U,V,X,Y) &+ P^*(V,X,U,Y) + P^*(X,U,V,Y) \\ &= P(U,V,X,Y) + P(V,X,U,Y) + P(X,U,V,Y). \end{aligned} \qquad (3.11)$$

In [21], Shaikh and Kundu have been proved that the projective curvature tensor $P$ satisfies Bianchi's first identity i.e. $P(U,V,X,Y)+P(V,X,U,Y)+P(X,U,V,Y)=0$. By using this fact, equation (3.11) reduces to

$$P^*(U,V,X,Y)+P^*(V,X,U,Y)+P^*(X,U,V,Y)=0, \tag{3.12}$$

which shows that generalized projective curvature tensor satisfied Bianchi's first identity. Thus we can state as follows:

**Theorem 3.4.** *A generalized projective curvature tensor on $(M^n,g)$ satisfies Bianchi's first identity.*

Now, taking the covariant derivative of equation (3.7), with respect to $U$, we obtain

$$(\nabla_U P^*)(V,X,Y,W)=(\nabla_U P)(V,X,Y,W). \tag{3.13}$$

Writing two more equations by the cyclic permutations of $U$, $V$ and $X$ from equation (3.13), we obtain

$$(\nabla_V P^*)(X,U,Y,W)=(\nabla_V P)(X,U,Y,W), \tag{3.14}$$

and

$$(\nabla_X P^*)(U,V,Y,W)=(\nabla_X P)(U,V,Y,W). \tag{3.15}$$

Adding equations (3.13), (3.14) and (3.15), we obtain

$$\begin{aligned}(\nabla_U P^*)(V,X,Y,W)&+(\nabla_V P^*)(X,U,Y,W)+(\nabla_X P^*)(U,V,Y,W)\\ &=(\nabla_U P)(V,X,Y,W)+(\nabla_V P)(X,U,Y,W)+(\nabla_X P)(U,V,Y,W).\end{aligned} \tag{3.16}$$

In [21], Shaikh and Kundu have been proved that the projective curvature tensor $P$ satisfies Bianchi's second identity i.e. $(\nabla_U P)(V,X,Y,W)+(\nabla_V P)(X,U,Y,W)+(\nabla_X P)(U,V,Y,W)=0$ if and only if the Ricci tensor of manifold is Codazzi tensor. By using this fact, equation (3.16) reduces to

$$(\nabla_U P^*)(V,X,Y,W)+(\nabla_V P^*)(X,U,Y,W)+(\nabla_X P^*)(U,V,Y,W)=0. \tag{3.17}$$

Thus we can state as follows:

**Theorem 3.5.** *A generalized projective curvature tensor on $(M^n,g)$ satisfies Bianchi's second identity if the Ricci tensor of $M$ is of codazzi type.*

## 4. Generalized Projectively Flat Space-time

Let $M$ be a 4-dimensional space-time of general relativity, then in view of equation (3.2), we have

$$P^*(U,V,X,Y) = R(U,V,X,Y) - \frac{1}{3}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y)]. \tag{4.1}$$

If $P^*(U,V,X,Y) = 0$, then from equation (4.1), we have

$$R(U,V,X,Y) = \frac{1}{3}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y)]. \tag{4.2}$$

Contracting $V$ and $X$ in above equation by using equation (2.4), we obtain

$$S(U,Y) = \frac{r+3\psi}{4}g(U,Y), \tag{4.3}$$

which on contraction gives $\psi = 0$. Thus equation (4.3) reduces to

$$S(U,Y) = \frac{r}{4}g(U,Y). \tag{4.4}$$

This shows that a generalized projectively flat space-time is an Einstein space-time. Thus we have the following corollary:

**Corollary 4.1.** *A* 4*-dimensional relativistic generalized projectively flat space-time is an Einstein space-time.*

Now, $\psi = 0$ so in view of equation (3.3), we have $P^* = P$. In view of equations (2.1) and (4.2), we get

$$R(U,V,X,Y) = \frac{1}{3}[S(V,X)g(U,Y) - S(U,X)g(V,Y)]. \tag{4.5}$$

Now, using equation (4.4) in equation (4.5), we obtain

$$R(U,V,X,Y) = \frac{r}{12}[g(U,Y)g(V,X) - g(U,X)g(V,Y)], \tag{4.6}$$

which shows that $M$ is of constant curvature. Thus we can state as follows:

**Theorem 4.2.** *The following conditions are equivalent in* 4*-dimensional relativistic space-time*
(1) $M$ *is generalized projectively flat space-time,*
(2) $M$ *is projectively flat space-time,*
(3) $M$ *is of constant curvature.*

It is known that a Lorentzian manifold of constant curvature is a manifold of conformally flat. Thus we have the following corollary:

**Corollary 4.3.** *A 4-dimensional relativistic generalized projectively flat space-time is a conformally flat space-time.*

In 1980, Kramer et al. [10] have been proved that a space is of O-type if the conformal curvature tensor vanishes on it. Thus we have a theorem as follows:

**Theorem 4.4.** *A 4-dimensional relativistic generalized projectively flat space-time is of O-type.*

Now, we consider a perfect fluid space-time with flat generalized projective curvature tensor having Einstein's field equation in the presence of cosmological constant. Let $\pounds_\xi$ be the Lie derivative operator along the vector field $\xi$ generating the symmetry. The matter collineation defined by $(\pounds_\xi T)(U, V) = 0$ represents the symmetry of energy momentum tensor $T$. In view of equation (4.4), equation (1.2) takes the form

$$(\lambda - \frac{r}{4})g(X, Y) = kT(X, Y). \tag{4.7}$$

If $\xi$ be a Killing vector field on the space-time with generalized projectively flat curvature tensor, then

$$(\pounds_\xi g)(X, Y) = 0. \tag{4.8}$$

Taking the Lie derivative of equation (4.7) along $\xi$, we obtain

$$(\lambda - \frac{r}{4})(\pounds_\xi g)(X, Y) = k(\pounds_\xi T)(X, Y). \tag{4.9}$$

In virtue of equation (4.8), equation (4.9) shows that $(\pounds_\xi T)(X, Y) = 0$, which shows that the space-time admits matter collineation. Conversely, If $(\pounds_\xi T)(X, Y) = 0$, it follows that from equation (4.9), that $\xi$ is Killing vector field. Hence we can state as follows:

**Theorem 4.5.** *If a 4-dimensional relativistic space-time having Einstein's field equation in the presence of cosmological constant generalized projectively flat curvature tensor, then the space-time satisfies matter collineation along a vector field $\xi$ if and only if $\xi$ is a Killing vector field.*

Next, Let us assume that the vector field $\xi$ is a conformal Killing vector field, then we obatin

$$(\pounds_\xi g)(X, Y) = 2\phi g(X, Y), \tag{4.10}$$

where $\phi$ is scalar, which is view of equation (4.9), gives

$$(\lambda - \frac{r}{4})2\phi g(X,Y) = k(\pounds_\xi T)(X,Y). \tag{4.11}$$

Using equation (4.7) in equation (4.11), we obtain

$$(\pounds_\xi T)(X,Y) = 2\phi T(X,Y). \tag{4.12}$$

From above equation, we see that the energy-momentum tensor has Lie inheritance property along $\xi$. Conversely, if equation (4.12) holds, then it follows that equation (4.10) holds, *i.e.* the vector field $\xi$ is a conformal Killing vector field. Thus we have a theorem as follows:

**Theorem 4.6.** *In a 4-dimensional relativistic space-time having Einstein's field equation in the presence of cosmological constant generalized projectively flat curvature tensor, a vector field $\xi$ is conformal Killing vector field if and only if the energy-momentum tensor $T$ has a symmetry inheritance property along $\xi$.*

Again, we consider a perfect fluid space-time with generalized projectively flat curvature tensor having Einstein's field equation in the absence of cosmological constant. The energy momentum tensor $T$ of a perfect fluid is given by [18]

$$T(X,Y) = (\sigma + \rho)A(X)A(Y) + \rho g(X,Y), \tag{4.13}$$

where $\rho$ is the isotropic pressure, $\sigma$ is the energy density and $A$ is the non-zero one-form such that $g(X,v) = A(X)$, for all $X$, $v$ being the velocity vector field of the flow, *i.e.* $g(v,v) = -1$. Using equations (4.4) and (4.13) in equation (1.3), we get

$$(\frac{r}{4} - k\rho)g(X,Y) = k(\rho + \sigma)A(X)A(Y), \tag{4.14}$$

which on contraction gives

$$r = k(\sigma - 3\rho). \tag{4.15}$$

Now, taking $X = Y = v$ in equation (4.14) and using $g(v,v) = -1$, we obtain

$$r = 4k\sigma. \tag{4.16}$$

From equations (4.15) and (4.16), we have

$$\sigma + \rho = 0, \tag{4.17}$$

which gives that the perfect fluid behave as a cosmological constant. Thus, in view of equation (4.17), equation (4.13) reduces to

$$T(X,Y) = \rho g(X,Y). \tag{4.18}$$

For a generalized projectively flat space-time, the scalar curvature is constant. Thus $\sigma$ is constant. Consequently, $\rho$ is constant. Therefore, on taking covariant derivative of equation (4.18), we obtain

$$(\nabla_U T)(X,Y) = 0, \tag{4.19}$$

which shows that the energy momentum tensor is covariantly constant. Thus we have theorem as follows:

**Theorem 4.7.** *In a 4-dimensional relativistic perfect fluid generalized projectively flat space-time following Einstein's field equation in the absence of cosmological constant, $\sigma + \rho = 0$ and the isotropic pressure and energy density are constants. Moreover, energy momentum tensor is covariantly constant.*

Taking the frame-field after contraction over $X$ and $Y$ of equation (1.3), we obtain

$$r = -kt, \tag{4.20}$$

where $t$ is $tr(T)$. Therefore, equation (1.3) can be written as:

$$S(X,Y) = k[T(X,Y) - \frac{t}{2}g(X,Y)]. \tag{4.21}$$

Einstein's field equation in the absence of cosmological constant for a purely electromagnetic distribution takes the form [18]

$$S(X,Y) = kT(X,Y). \tag{4.22}$$

From equations (4.21) and (4.22), we obtained $t = 0$. Thus from equation (4.20), we get $r = 0$. Therefore, from equation (4.6), we obtain $R(U,V,X,Y) = 0$, which shows that the space is flat. Thus we arrive at the following:

**Theorem 4.8.** *A 4-dimensional relativistic generalized projectively flat space-time having Einstein's field equation in the absence of cosmological constant for a purely electromagnetic distribution is an Euclidean space.*

## 5. Conservative Generalized Projective Space-time

From equation (3.3), generalized projective curvature tensor is given by

$$P^*(U,V)X = P(U,V)X - \frac{\psi}{(n-1)}[g(V,X)U - g(U,X)V]. \tag{5.1}$$

The divergence of $P^*(U,V)X$ is defined as

$$(divP^*)(U,V)X = g((\nabla_{e_i}P^*)(U,V)X, e_i)$$

i.e.

$$\begin{aligned}(divP^*)(U,V)X =& g((\nabla_{e_i}P)(U,V)X, e_i)\\ &- \frac{1}{n-1}[g((\nabla_{e_i}\psi)\{g(V,X)U - g(U,X)V\}, e_i)],\end{aligned} \tag{5.2}$$

which gives

$$(divP^*)(U,V)X = (divP)(U,V)X - \frac{1}{(n-1)}[(U\psi)g(V,X) - (V\psi)g(U,X)]. \tag{5.3}$$

From equations (2.3) and (5.3), we obtain

$$\begin{aligned}(divP^*)(U,V)X =& \frac{n-2}{n-1}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X)]\\ &- \frac{1}{(n-1)}[(U\psi)g(V,X) - (V\psi)g(U,X)].\end{aligned} \tag{5.4}$$

If scalar function $\psi$ is constant then from equation (5.3), we obtain

$$(divP^*)(U,V)X = (divP)(U,V)X. \tag{5.5}$$

If $(divP^*)(U,V)X = 0$ then from equation (5.5), we obtain

$$(divP)(U,V)X = 0,$$

which gives

$$\frac{n-2}{n-1}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X)] = 0. \tag{5.6}$$

From above equation, it follows that Ricci tensor is Codazzi tensor.

**Theorem 5.1.** *A 4-dimensional relativistic conservative generalized projective space-time $M$ with constant scalar function $\psi$ is conservative projective curvature tensor, provided that Ricci tensor is codazzi tensor.*

Guifoyle and Nolan [9], gave "Yang Pure Space", a 4-dimensional Lorentzian manifold $(M^n, g)$ whose metric tensor solves Yang's equation

$$(\nabla_U S)(V,X) - (\nabla_V S)(U,X) = 0.$$

Thus we can state as follows:

**Theorem 5.2.** *A 4-dimensional relativistic conservative generalized projective space-time $M$ with constant scalar function $\psi$ is a Yang Pure space.*

Since we known that [9], a 4-dimensional relativistic perfect fluid space-time with $\sigma + \rho \neq 0$ is a Yang Pure space-time if and only if space-time is RW space-time.

**Theorem 5.3.** *A 4-dimensional relativistic conservative generalized projective space-time $M$ with constant scalar function $\psi$ is a RW space-time.*

It is known that divergence of conformal (Weyl) curvature tensor can be written as

$$\begin{aligned}(divC)(U,V)X = \frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X) \\ - \frac{1}{2(n-1)}g(V,X)dr(U) - g(U,X)dr(V)].\end{aligned} \tag{5.7}$$

In virtue of equations (5.6) and (5.7), we observe that $(divC)(U,V)X = 0$ if and only if $(divP^*)(U,V)X = 0$. Thus we can state as follows:

**Theorem 5.4.** *A 4-dimensional relativistic space-time $M$ satisfying $divC = 0$ if and only if $(divP^*)(U,V)X = 0$ with constant scalar function $\psi$.*

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# THE CONJUGACY CLASSES OF BIJECTIVE CUBIC POLYNOMIALS

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Abstract. In this paper, we prove that there are only seven bijective cubic polynomials on the real line upto topological conjugacy.

## 1. Introduction and preliminaries

A dynamical system is a pair $(X, f)$ where $X$ is a metric space and $f$ is a continuous self map on $X$. Simply we call the map $f : X \to X$ a dynamical system. Two dynamical systems $f : X \to X$ and $g : Y \to Y$ are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$ (called topological conjugacy) such that $h \circ f = g \circ h$. A point $x \in X$ is said to be a fixed point of a dynamical system $f : X \to X$ if $f(x) = x$. A point $x \in X$ is said to be a periodic point of period $n$ of a dynamical system $f : X \to X$ if $f^n(x) = x$ and $f^p(x) \neq x$ for $1 \leq p < n$. If $f$ is an increasing cubic polynomial on the real, then we can associate a word over $\{0, 1\}$ to $f$ by looking at the graph of $f$ from left to right assigning 1 or 0 according as the graph is above or below the diagonal in each interval determined by the fixed points, we denote the word by $w(f)$. If $f$ is decreasing, then $f^2 := f \circ f$ is increasing and hence a word can be associated to $f$ by looking at the graph of $f^2$, this word is enough to determine the conjugacy class (separately for increasing and decreasing classes). If $f$ is a bijective linear polynomial on the real line, then there are only four such $f$ upto topological conjugacy. It may be noted that the conjugacy classes of the real linear polynomials $x+1$, $x$, $\frac{x}{2}$ and $\frac{3x}{2}$ are distinct. If $f(x) = ax^2 + bx + c$ is a real quadratic polynomial then $f$ has an extremum at $x = -\frac{b}{2a}$, thus there is no bijective quadratic polynomial on the real line. Refer [2] for more details.

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Let $\Sigma = \{0,1\}$, and $\tilde{0} = 1$ and $\tilde{1} = 0$. If $w = w_1w_2...w_n$ is a word over $\Sigma$ then the dual of $w$ is defined as $\tilde{w} = \tilde{w}_n\tilde{w}_{n-1}...\tilde{w}_1$. Two bijective real polynomials (bijections on $\mathbb{R}$ with unique fixed point) are conjugate to each other if and only if the associated words are either the same or one is the dual of the other [see [1], [2]]. When $f$ is a real cubic polynomial, the equation $f(x) = x$ has three solutions in the complex plane. Hence the possible words for the increasing bijective real cubic polynomials are 01, 001 and 0101. Also there are at the most three orbits of period two for $f$ (which are among the remaining six solutions of $f^2(x) = x$). This implies that the possible words for a decreasing cubic are 01, 0101, 0011, 010101, 011001, 001011, 01010101. Therefore, the number of conjugacy classes among bijective real cubic polynomials cannot exceed 10 (3+7).

In this paper, we provide seven examples of bijective cubic polynomials on the real line such that no two of them are topologically conjugate. This is established by giving some distinguishing conjugate invariant properties. We prove that these are the only bijective real cubic polynomials upto topological conjugacy with the help of properties of polynomials.

## 2. Main Results

For a polynomial $f$, we denote $f'$ for the derivative map. A fixed point $x$ of a polynomial $f$ is said to be attracting if $|f'(x)| < 1$, repelling if $|f'(x)| > 1$, and neutral if $|f'(x)| = 1$. A periodic point $x$ of period $n$ of a polynomial $f$ is said to be attracting if $|(f^n)'(x)| < 1$, repelling if $|(f^n)'(x)| > 1$, and neutral if $|(f^n)'(x)| = 1$.

### 2.1. **Increasing Cubics.** 

Here we provide examples for three increasing bijective real cubic polynomials which are not conjugate to each other.

**Example 2.1.** Let $f_1(x) = x^3 + x$. The increasing bijective cubic polynomial $f_1$ has unique fixed point, namely 0. Hence the associated word $w(f_1)$ is 01.

**Example 2.2.** Let $f_2(x) = x^3 + x^2 + x$. The increasing bijective cubic polynomial $f_2$ has exactly two fixed points, namely 0 and $-1$. Hence the associated word $w(f_2)$ is 001.

**Example 2.3.** Let $f_3(x) = x^3$. The increasing bijective cubic polynomial $f_3$ has exactly three fixed points, namely $-1, 0$, and $1$. Hence the associated word $w(f_3)$ is 0101.

2.2. **Decreasing Cubics.** Here we provide examples for four decreasing bijective real cubic polynomials which are not conjugate to each other.

**Example 2.4.** Let $f_4(x) = -x^3 - x$. The decreasing bijective polynomial $f_4$ has unique fixed point 0. Since $|f_4(x)| > x$ for all $x \neq 0$, there is no point of period 2. Hence the associated word $w(f_4)$ is 01.

**Example 2.5.** Let $f_5(x) = -x^3$. The decreasing bijective cubic polynomial $f_5$ has unique fixed point 0, which is attracting since $f_5'(0) = 0$. The only orbit of $f_5$ of period 2 is $\{-1, 1\}$, which is repelling (i.e., each periodic point in the orbit is repelling) since $f_5'(-1).f_5'(1) = 9$. Hence the associated word $w(f_5)$ is 0101.

**Example 2.6.** Let $f_6(x) = -x^3 + 1$. Then $f_6(\frac{2}{3}) = \frac{19}{27} > \frac{2}{3}$, $f_6(\frac{3}{4}) = \frac{37}{64} < \frac{3}{4}$ and $|f_6'(x)| = 3x^2 > 1$ for all $x \in [\frac{2}{3}, \frac{3}{4}]$. So the unique fixed point of $f_6$ is in $[\frac{2}{3}, \frac{3}{4}]$ and it is repelling. Also $\{0, 1\}$ is an orbit of period two, which is attracting since $f_6'(0).f_6'(1) = 0$. If $x_0$ is the largest fixed point of $f_6^2$, then $(f_6^2)'(x_0) \geq 1$ since $f_6^2(x) > x$ for large $x$. But $(f_6^2)'(1) = 0 < 1$. Therefore, $f_6$ must have one more orbit of period two. Another argument for the same is $f_6^2(1.1) = f_6(-0.331) < f_6(\frac{1}{3}) = \frac{1}{27} + 1 < 1.1$; and $f_6^2(x) > x$ for large $x$. Let $p > 1$ be a point of period two and let $q = f_6(p)$. Then $q < -\frac{1}{3}$. Therefore, $|f_6'(p)|.|f_6'(q)| > |f_6'(1)|.|f_6'(-\frac{1}{3})| = 3.\dfrac{1}{3} = 1$. Thus the orbit $\{p, q\}$ is repelling. Also $f_6$ does not have a third orbit of period two because the attracting and the repelling types should come alternately and the outermost one cannot be attracting. Hence the associated word $w(f_6)$ is 010101.

**Example 2.7.** Let $f_7(x) = -x^3 - \frac{x}{2} + \frac{1}{\sqrt{2}}$. Then $f_7'(x) = -3x^2 - \frac{1}{2} < 0$ for all $x$ and hence $f_7$ is bijective. Also $\{0, \frac{1}{\sqrt{2}}\}$ is an orbit of period two; $f_7'(0).f_7'(\frac{1}{\sqrt{2}}) = (-\dfrac{1}{2}).(-2) = 1$ implies it is neutral (hence it accounts for four solutions of $f_7^2(x) = x$). Note that $|f_7'(x)| > 1$ if and only if $|x| > \dfrac{1}{\sqrt{6}}$ and $f_7(\frac{1}{\sqrt{6}}) = \dfrac{-4 + 6\sqrt{3}}{6\sqrt{6}} > \dfrac{-4 + 6.(\dfrac{5}{3})}{6\sqrt{6}} = \dfrac{1}{\sqrt{6}}$. Therefore the fixed point is in $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}})$ and it is repelling. Now, $(f_7^2)''(\frac{1}{\sqrt{2}}) =$

$f_7''(0).\left[f_7'(\frac{1}{\sqrt{2}})\right]^2 + f_7'(0).f_7''(\frac{1}{\sqrt{2}}) = \dfrac{3}{\sqrt{2}} > 0$. This shows that the graph of $f_7^2$ lies above the diagonal $y = x$ in a deleted neighbourhood of $\frac{1}{\sqrt{2}}$. Also, $(f_7^2)'(x) > (f_7^2)'(\frac{1}{\sqrt{2}}) = 1$ for all $x > \frac{1}{\sqrt{2}}$. These facts ensure that there is no more orbit of period two. Hence the associated word for the decreasing bijective polynomial $f_7$ is 0011.

2.3. **The class of cubic polynomials.**
Now we are ready to state our Main Theorem:

**Theorem 2.8.** *The polynomials $f_i$, $i = 1, ..., 7$ are the only bijective real cubic polynomials upto topological conjugacy.*

For a proof, first we will consider the following four lemmas. Later, we will prove the theorem.

**Lemma 2.9.** *If $ax^3 + bx^2 + cx + d$ is a decreasing bijective real cubic polynomial then $a < 0$ and $c \leq 0$.*

*Proof.* Let $f(x) = ax^3 + bx^2 + cx + d$ be a decreasing bijective real cubic polynomial. Then $f'(x) = 3ax^2 + 2bx + c < 0$ for all $x \in \mathbb{R}$. Hence $a$ should be negative. Now $2bx \to 0$ and $-3ax^2 \to 0$ as $x \to 0$. But $2bx + c < -3ax^2$ for all $x \in \mathbb{R}$. Hence $c \leq 0$. $\square$

**Lemma 2.10.** *If $ax^3 + bx^2 + cx + d$ is a decreasing bijective real cubic polynomial then it is conjugate to $a'x^3 + c'x + d'$ for some $a', c', d' \in \mathbb{R}$.*

*Proof.* Let $f(x) = ax^3 + bx^2 + cx + d$ be a decreasing bijective real cubic polynomial. Put $x = y - \frac{b}{3a}$. Then the polynomial becomes $g(y) = a'y^3 + c'y + d'$ for some $a', c', d' \in \mathbb{R}$. Since two bijective real polynomials are conjugate to each other if and only if the associated words are either the same or one is the dual of the other, the proof follows. $\square$

**Lemma 2.11.** *If $-ax^3 + cx + d$ with $a > 0$ is a decreasing bijective real polynomial then it is conjugate to $-x^3 + c'x + d'$ for some $c', d' \in \mathbb{R}$.*

*Proof.* Let $f(x) = -ax^3 + cx + d$ with $a > 0$ be a decreasing bijective real polynomial. Put $x = a^{-\frac{1}{3}}y$. Then the polynomial becomes $g(y) = -y^3 + c.a^{-\frac{1}{3}}y + d$. Since two bijective real polynomials are conjugate to each other if and only if the associated words are either the same or one is the dual of the other, the proof follows. $\square$

Now we have:

**Lemma 2.12.** *Let $f : \mathbb{R} \to \mathbb{R}$ be any decreasing bijection with unique fixed point and let $c \in \mathbb{R}$. Let $a, b$ be the fixed points of $f$ and $f - \mathbf{c}$ respectively, where $\mathbf{c}$ denotes the constant function on $\mathbb{R}$ with value $c$. Then $f$ is topologically conjugate to $f - \mathbf{c}$ if and only if $f^2|_{[a,\infty)}$ is topologically conjugate to $(f - \mathbf{c})^2|_{[b,\infty)}$.*

*Proof.* First observe that, if $f : \mathbb{R} \to \mathbb{R}$ is a decreasing bijection with unique fixed point then $f - \mathbf{c}$ is also a decreasing bijection with unique fixed point for any real number $c$. Fix $c > 0$ and let $a, b$ be the fixed points $f$ and $f - \mathbf{c}$ respectively with $b > a$. Since two decreasing bijections with unique fixed point are conjugate to each other if and only if the associated words are either the same or one is the dual of the other, we have that $f$ is topologically conjugate to $f - \mathbf{c}$ if and only if $f^2|_{[a,\infty)}$ is topologically conjugate to $(f - \mathbf{c})^2|_{[b,\infty)}$. Hence the proof follows. □

**Proof of Theorem 2.8**: By Lemmas 2.9, 2.11 and 2.12, it is enough to consider the decreasing bijective real cubic polynomials of the form $f(x) = -x^3 - cx + d$ with $c \geq 0$. Now let $f(x) = -x^3 - cx + d$ be a decreasing bijective real cubic polynomial with $c > 0$ and $d \neq 0$. By similar arguments involved as in Example 2.7, the map $f$ has unique fixed point and exactly one orbit of period 2. If $c > 1$ then the unique fixed point of $f$ should not be neutral. If $0 < c \leq 1$ then the unique fixed point of the polynomial $g(x) = -x^3 - cx + \sqrt{c}$ is also not neutral. Hence because of Lemma 2.12, $w(f)$ is 0011 whenever $f(x) = -x^3 - cx + d$ with $c > 0$ and $d \neq 0$. If $f(x) = -x^3 - cx$ with $c \geq 1$ then the associated word $w(f)$ is 01. This is because, if $f(x) = -x^3 - cx$ with $c \geq 1$ then $|f(x)| > |x|$ for all $x \neq 0$. If $0 < c < 1$ then by Lemma 2.12, $w(f)$ is either 0011 or 01. If $f(x) = -x^3 + d$ with $d > 0$ then by similar arguments involved as in Example 2.6 and by Lemma 2.12, the associated word $w(f)$ is 010101. Hence there are only seven bijective real cubic polynomials upto topological conjugacy.

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# APPLICATIONS OF HYPERGEOMETRIC FUNCTIONS TO EXACTLY SOLVABLE QUANTUM MECHANICAL POTENTIALS

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Abstract. Most textbooks on Quantum Mechanics only discuss the harmonic oscillator, potential well, hydrogen atom and the Morse potential as examples of exactly solvable systems under the Schrödinger equation. However, there is significant ongoing research in which exact solutions of more complex quantum mechanical potentials have been obtained using theory of Heun functions. In this article, we describe one such potential whose solutions are given by the hypergeometric functions. In addition, an accessible introduction to hypergeometric and Heun's function is given that familiarises the reader with advanced mathematical terminology and motivates to conduct research in mathematical and physical applications

## 1. Introduction

The behaviour of any system being treated under the non-relativistic theory of quantum mechanics is described by a linear, second order differential equation called the Schrödinger equation. For a particle of mass $m$, energy $E$ in the presence of a potential $V(x)$, the one-dimensional, time-independent Schrödinger's equation is:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - V(x))\psi = 0, \tag{1.1}$$

Here, $\hbar = h/2\pi = 1.054571817 \times 10^{-34} Js$ is the reduced Planck's constant and $\psi$ is the wavefunction that completely describes all the features of the system. A complete treatment of the physical and mathematical features

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of this equation is beyond the scope of this paper, so we refer the reader to [12].

Now, the treatments in most books like [12] focus on potentials of the potential well ($V_{\text{pw}}$), harmonic oscillator ($V_{\text{ho}}$), hydrogen atom ($V_{\text{hyd}}$) and the Morse potential $V_{\text{M}}$ describing the potential energy of a diatomic molecule, that have a rather simple mathematical form:

$$V_{\text{pw}}(x) = \text{constant}, \tag{1.2}$$

$$V_{\text{ho}}(x) = \frac{kx^2}{2}, \tag{1.3}$$

$$V_{\text{hyd}}(x) = -\frac{e^2}{4\pi\epsilon x}, \tag{1.4}$$

$$V_{\text{M}}(x) = D_e(1 - \exp^{-\beta(x-x_e)})^2 \tag{1.5}$$

where, $k$ is the spring constant, $e$ is the charge of the electron $(9.1\times10^{-31}C)$, $\epsilon$ is the permittivity of the medium, and in $V_M(x)$, $D_e$ is the well-depth for the potential, $x$ is the internuclear distance and $x_e$ is the bond length and $\beta$ is a function that depends on the vibrational constant, reduced mass and $D_e$. Even for these simple cases, the solutions to the Schrödinger equation are described by special mathematical functions such as Hermite polynomials for the eigenfunction of the harmonic oscillator and Laguerre polynomials for the radial part of the hydrogen wavefunction. The important mathematical properties of these special functions are given in [2].

The importance of having exactly solvable forms of potential for the Schrodinger equation cannot be overemphasized, since it is pivotal for extracting physical parameters to be tested in experiments. From a mathematical point of view, it is desired that an exact solution can be obtained for potentials where all the involved parameters can be varied independently. This provides a rich analytical structure to the solutions and ascribes to the unsaid rule of modern mathematics: the more general the better.

As far as we know, only a handful of such potentials are known which are solvable in terms of the most well studied class of special functions: the hypergeometric functions. There are five known solutions based on the Kummer hypergeometric functions and three on the basis of the more general Gauss hypergeometric functions. In this paper, we describe a solution that has rather recently been obtained for the following potential:

$$V_{\text{gh}}(x) = V_0 + \frac{V_1}{\sqrt{1 + e^{2(x-x_0)/\sigma}}} \tag{1.6}$$

The solution for this potential is written in terms of Gauss hypergeometric functions. The motives for discussing this particular potential are several: to introduce the reader to a mathematically rich class of functions, namely the hypergeometric functions by studying their application to modern quantum physics, to demonstrate the analytical complexity that remains to be explored of seemingly simple equations such as the Schrödinger equation and to encourage further research in the mathematical features as well as the physical applications of the Heun functions.

The solution for this potential, along with most of the current research in this area is conducted by Ishkhanyan(see [8, 9] and references therein) along with notable exceptions such as [24]. Furthermore, a full classification of all potentials solvable by Heun functions is given in [10]. However before we describe the solutions, we shall give a compact and largely self contained description of the special functions, in particular the Heun functions.

## 2. Hypergeometric and Heun Equations

2.1. **Fuchsian Equations.** The primary object of study here are ordinary, linear, homogenous second order differential equations with polynomial coefficients $P(z), Q(z)$ and $R(z)$ of the form:

$$P(z)\frac{d^2y(z)}{dz^2} + Q(z)\frac{dy(z)}{dz} + R(z)y(z) = 0, z \in \mathbb{CP}^1 \tag{2.1}$$

where $\mathbb{CP}^1$ is the Riemann sphere, which contains the entire complex $z$-plane along with the point $z = \infty$. The singular points of the equation, the points at which the equation is not analytic, are classified into regular (or *Fuchsian*) and irregular points. In the above equation, if the function $K_{QP} = Q(z)/P(z)$ has a pole of at most first order and $K_{RP} = R(z)/P(z)$ has a pole of at most second order at some singularity $z = z_0$, then $z_0$ is called a *Fuchsian* singularity, otherwise it is an *irregular* singularity. The above equation is a *Fuchsian equation* if all its singularities are Fuchsian singularities. A rather complete albeit slightly advanced treatment of Fuchsian equations is given in [16].

Any Fuchsian equation which has exactly three singular points can be transformed into the hypergeometric equation by transformations in dependent or independent variables. Similarly, any Fuchsian equation with exactly four singular points can be transformed into a Heun equation. The transformation in dependent and independent variables are called s-homotopic and Möbius transformations respectively.

For each of the regular singularities, we can construct a polynomial equation called the *indicial equation*, the roots of which are called *characteristic exponents*. The indicial equation is given by:

$$\rho(\rho-1)+p_i\rho+q_i=0, \tag{2.2}$$

where,

$$p_i=\mathrm{Residue}_{z=z_i}\frac{Q(z)}{P(z)}, q_i=\mathrm{Residue}_{z=z_i}(z-z_i)\frac{R(x)}{P(x)} \tag{2.3}$$

The introduction of these mathematical terminologies would assist in systematically studying the relevant mathematical features of the hypergeometric and Heun equations.

2.2. **Hypergeometric equations.** The hypergeometric equation, also called Gauss hypergeometric equation, is linear, second order differential equations that can be obtained from performing s-homotopic and Möbius transformations to the Riemann equation-a Fuchsian equation with three singularities (see Chapter 15 in [1]). For the Gauss hypergeometric equations, these are: $z=0,1,\infty$. The explicit form of the equation is as follows:

$$z(1-z)\frac{d^2y(z)}{dz^2}+[c-(a+b+1)z]\frac{dy(z)}{dz}-aby(z)=0 \tag{2.4}$$

Here, $a,b,c,d$ are free parameters. The standard solution to this equation is given in terms of a power series ${}_2F_1(a,b;c;z)$ called the *hypergeometric function*. It can be expanded as a Frobenius series (see [16]) about the three singular points of the equation. For example, under the condition that parameter $c\neq 0,-1,-2.....$the Frobenius series about $z=0$ is of the form:

$${}_2F_1(a,b;c;z=0,z)=\sum_{k=0}^{\infty}g_kz^k, \tag{2.5}$$

where the coefficients $g_k$ are given by a ratio of gamma functions:

$$g_k=\frac{\Gamma(k+a)\Gamma(k+b)\Gamma(c)}{\Gamma(k+1)\Gamma(k+c)\Gamma(a)\Gamma(b)} \tag{2.6}$$

This series converges within the unit circle $|z| \leq 1$. Similar series solutions exist for singularities $z = 1, z = \infty$. For a comprehensive treatment of the properties of functions like the gamma function, we refer the reader to [16, 23].

A compact method of describing Fuchsian-type equations is by using the Riemann-P symbol. For the hypergeometric equation, the Riemann-P symbol is:

$$\begin{pmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & ;z \\ 1-c & b & c-a-b & \end{pmatrix} \tag{2.7}$$

The first row indicates the singularities of the equation and the two subsequent rows (excluding the $z$ term in the second row, which indicates the complex variable for the equation) denote the characteristic exponents corresponding to the singularities. The advantage of this notation is that the underlying equation can be completely recovered from the P-symbol. These symbols are extremely useful in obtaining identities related to hypergeometric and Heun equations (see page 139-159 in [14]).

2.3. **Heun equations.** The Heun class of second order ODEs can be obtained from the general Heun[3]-a Fuchsian equation with four singularities. They are a straightforward extension of the hypergeometric equation. The canonical form of the general Heun equation is:

$$\frac{d^2y(z)}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right]\frac{dy(z)}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y(z) = 0 \tag{2.8}$$

where $q \in \mathbb{C}$ is called the *accessory parameter*. The corresponding Riemann-P symbol is as follows:

$$\begin{pmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & ;z \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{pmatrix} \tag{2.9}$$

where the parameters satisfy the Fuch's condition:

$$1 + \alpha + \beta = \gamma + \delta + \epsilon \tag{2.10}$$

Several features are worth mentioning. Firstly, the equation is singular at the following four points: $z = 0, 1, a, \infty$. Secondly, in total the Heun equation has 192 solutions as compared to the mere 24 solutions of the Gauss Hypergeometric equation found by Kummer. In other words, according to

Theorem 4.3 in [13], the solutions to the Heun equation have the Coxeter Group $D_4$ as the automorphism group, which means that all 192 solutions can be generated by using the symmetries of $D_4$. All solutions are explicitly listed in [13].

If we set $\epsilon = 0, \gamma = \delta = 1/2$ in the Heun equation, we get another prominent equation, the Lamé equation which arises in the separation of variables for the Laplace equation in elliptic coordinates. For an interesting study on the relationship between Lamé and Heun polynomials, we refer the reader to [15]

The study of Heun equations, both in mathematics and physics is an active area of research and we would discuss applications in both subjects in the last section. For now, we have introduced enough formalism to understand their applicability to quantum mechanical potentials.

## 3. The Quantum Mechanical Potential

In this section we learn in detail the applicability of the afore mentioned hypergeometric and Heun equations to the quantum mechanical potential given in equation 1.6, that can be exactly solved using the Schrödinger's equation. We note that although the transformations might seem *ad hoc*, the techniques are based on the results in [11].

We first start with the transformation of the wavefunction $\psi = \varphi(z)u(z)$ and $z = z(x)$ with,

$$\varphi(z) = \rho(z)^{-1/2} exp\left(\frac{1}{2}\int f(z)dz\right), \tag{3.1}$$

$$\rho(z) = \frac{dz}{dx} \tag{3.2}$$

This reduces the Schrödinger equation to:

$$\begin{aligned} I(z) = g - \frac{f_z}{2} - \frac{f^2}{4} = \\ -\frac{1}{2}\left(\frac{\rho_z}{\rho}\right)_z - \frac{1}{4}\left(\frac{\rho_z}{\rho}\right)^2 + \frac{2m}{\hbar^2}\frac{E - V(z)}{\rho^2} \end{aligned} \tag{3.3}$$

where, $f(z)$ and $g(z)$ are coefficients of the Heun equation and $I(z)$ is the invariant of the equation [7] when rewritten in *Liouvillian* form. A consequence of the Theorem 27 in [11] is that the logarithmic derivative of $(\rho_z/\rho)$ should contain only the poles corresponding to the finite singularities of the Heun equation. Therefore, if we label the finite singularities by

$z = a_1, a_2, a_3$, the function $\rho$ is given as:

$$\rho = (z - a_1)^m_1 (z - a_2)^m_2 (z - a_3)^m_3 / \sigma \tag{3.4}$$

where $m_1, m_2, m_3$ are integers or half integers and $\sigma$ is an arbitrary scaling constant. Now, the invariant form of the Heun equation is a fourth-order polynomial in $z$ divided by the factors corresponding to the singularities $a_1, a_2, a_3$, i.e $(z - a_1)^2(z - a_2)^2(z - a_3)^2$.

The next step is to match the $(\rho_z/\rho)$ terms in equation . This leads to the set of equations given in [10]. For the potential in equation 1.6, if $a_1, a_2, a_3 = -1, 1, 0$ respectively (as is the case for the finite singularities of the Heun equation), the solutions would be a combination of two hypergeometric functions [10]. This would in turn correspond to the following conditions for the Heun parameters:

$$q(q + \gamma - \delta) = \alpha\beta \tag{3.5}$$

The values of $m_1, m_2, m_3$ corresponding to these conditions are $1, 1$ and $-1$ respectively. Now, equation (16) suggests the substitution:

$$\varphi(z) = (z - a_1)^{\alpha_1}(z - a_2)^{\alpha_2}(z - a_3)^{\alpha_3} \tag{3.6}$$

By requiring,

$$\begin{aligned}(z - a_1)^2(z - a_2)^2(z - a_3)^2 V(z)/\rho^2 &= \\ \nu_0 + \nu_1 z + \nu_2 z^2 + \nu_3 z^3 + \nu_4 z^4,&\end{aligned} \tag{3.7}$$

and that $\nu_0, \nu_1, \nu_2, \nu_3$ be independent, we obtain the following solutions to the Schrödinger equation:

$$\psi(x) = (z + 1)^{\alpha_1}(z - 1)^{\alpha^2} u(z), \tag{3.8}$$

$$z = \sqrt{1 + e^{2(x - x_0)/\sigma}}, \tag{3.9}$$

$$\begin{aligned}u(z) &= {}_2F_1\left(\alpha - 1, \beta; \gamma - 1; \frac{z+1}{2}\right) \\ &+ \frac{(\alpha_2 - \alpha_1 + \beta z)}{2(\gamma - 1)} {}_2F_1(\alpha, \beta + 1; \gamma; \frac{z+1}{2}),\end{aligned} \tag{3.10}$$

where the parameters are given by:

$$(\alpha, \beta, \gamma) = (\alpha + 1 + \alpha_2 - \alpha_0, \alpha_1 + \alpha_2 + \alpha_0, 1 + 2\alpha_1), \tag{3.11}$$

$$\alpha_0 = \pm\sqrt{\frac{-2m\sigma^2}{\hbar^2}(E - V_0)}, \tag{3.12}$$

$$\alpha_1 = \pm\sqrt{\frac{-m\sigma^2}{2\hbar^2}(E - V_0 + V_1)}, \tag{3.13}$$

$$\alpha_2 = \pm\sqrt{\frac{-m\sigma^2}{2\hbar^2}(E - V_0 - V_1)}, \tag{3.14}$$

Thus, by reducing the Schrödinger's equation to the Heun equation and analysing it, we have obtained an exact solution in terms of hypergeometric functions, given by equation 3.8, for the potential in equation 1.6.

## 4. Further Avenues for Research

The preceding discussion demonstrates the analytical richness as well the power of solutions based on hypergeometric functions. Here, we very briefly discuss their applicability to two other avenues, one in physics and one in mathematics, both of which are areas of active current research.

4.1. **In Mathematics.** Until now, all the mathematical functions that we dealt with are called *transcendental* functions, i.e. a class of functions that do not satisfy a polynomial equation. This makes them analytically rich, but at the same time rather difficult to study. Therefore, mathematicians are working on whether solutions to the Heun equation can be obtained by functions that are not transcendental. It is also interesting to see whether the theory of Heun equations can shed light on other differential equations that are also known to have transcendental solutions.

One of the methods of doing so has been pioneered by Takemura(see [20, 21] and references therein). In [20], the Hermite-Krichever Ansatz, which essentially deals with solutions of a differential equation as a finite series in the derivatives of an elliptic function, was applied to the Heun equation which provided useful formulas for reduction of hyperelliptic to elliptic integrals. This was done by studying the monodromy behaviour (behaviour near the singularity of the equation) of two singularities of the Heun equation. In [21], the aforementioned result was generalised to a broader class of Fuchsian equations that contained additional (removable) singularities as compared to the Heun equation. By performing an elliptic transformation to this class, which preserves the monodromy behaviour of the equations, and by utilising the Hermite-Krichever Ansatz, the equation was converted to a Painleve-VI equation, that has applications in advanced mathematical physics.

This relationship between Heun equations and Painlevé equations has been extensively studied in [17, 18, 19].

4.2. **In Physics.** In addition to obtaining exactly solvable quantum mechanical potentials, another active area of research in the applicability of hypergeometric functions to physics is in black hole perturbation theory. While the complete treatment of this subject is beyond the scope of this paper, we briefly mention the important points. The perturbation dynamics of a Kerr Black Hole (a rotating black hole which is obtained as a solution of Einstein's General Relativity equations) by a particle with any spin value ($s = 0,1/2,1,3/2$) are governed by the Teukolsky Equation [22]. It has been shown that the Teukolsky equation for Kerr Black Holes can be analytically solved by using confluent Heun functions [4]. This has proved to be extremely crucial for studying quantities of astrophysical relevance, in particular quasinormal modes [5, 6]. A detailed treatment of the matter is given in [4].

## 5. Conclusion

In this paper, we have described the application of hypergeometric functions, in particular as solutions to Heun equations, in obtaining exactly solvable potentials for the Schrödinger's equation. This demonstrates the analytical richness of these functions. Furthermore, their applications in modern mathematics and black hole astrophysics are discussed to motivate further research in their applicability.

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# HURWITZ COMPLEX CONTINUED FRACTION EXPANSION OF SOME COMPLEX QUADRATIC IRRATIONALS

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Abstract. In this paper, we study Hurwitz complex continued fraction (HCCF) expansion of some complex quadratic irrationals. We observed that some complex quadratic irrationals can be expanded into two HCCFs. On comparing the Hurwitz complex continued fraction with simple continued fraction of any real quadratic irrational, it was found that there are differences between the two.

## 1. Introduction

**Definition 1.1.** A simple continued fraction (infinite) or regular continued fraction [2, 3, 8, 9] is defined as an expression of the form,

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}} \tag{1.1}$$

where $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N} \; \forall \; k = 1, 2, 3, ..$

In short, it is written as $x = [a_0; a_1, a_2, ...]$. A finite simple continued fraction (or a terminating regular continued fraction) has only a finite number of terms and it is written as

$$[a_0; a_1, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{... + \cfrac{1}{a_n}}} \tag{1.2}$$

Simple continued fraction is best described from Euclidean algorithm (in real numbers) [9]: The Euclidean algorithm, also called Euclid's algorithm,

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is an algorithm for finding the greatest common divisor (gcd) [4, 8, 9], of two numbers $a \in \mathbb{N}$ and $b \in \mathbb{N}$: there exist $q, r \in \mathbb{N}$ such that $a = bq + r$; $0 \leq r < b$.

A truncated simple continued fraction $[a_0; a_1, a_2, ..., a_k], 0 \leq k < \infty$ (or, $0 \leq k \leq n$, in case of finite simple continued fraction) is called the $k^{th}$ convergent of the continued fraction which we denote it as $C_k$ or $\dfrac{p_k}{q_k}$ where $p_k, q_k$ can be determined recursively as follows [8, 9]:

$$p_0 = a_0,\ p_1 = a_0 a_1 + 1,\ p_{m+1} = a_{m+1} p_m + p_{m-1}$$

$$q_0 = 1,\ q_1 = a_1,\ q_{m+1} = a_{m+1} a_m + a_{m-1} \text{ for all } m \geq 1.$$

The terms $a_0, a_1, ..., a_n, ...$ are called partial quotients or partial denominators of the simple continued fraction.

Convergents of a simple continued fraction (finite or infinite) satisfy the following condition:

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1} \text{ for all } 0 \leq k < \infty.$$

The following are the important properties from simple continued fraction [8, 9]:

**Theorem 1.2.** *The sequence $\{q_n\}_{n \geq 2}$ of denominators of convergents $\dfrac{p_n}{q_n}$ is a strictly increasing sequence.*

**Theorem 1.3.** *Any finite simple continued fraction represents a rational number. Conversely, any rational number $p/q$ can be expressed as finite simple continued fraction in exactly two ways one with the last term greater than* 1 *and the other with the last term equals to* 1.

**Theorem 1.4.** *Every infinite simple continued fraction (non-terminating regular continued fraction) represents (converges to) a positive irrational number and to every positive irrational number there is a unique infinte simple continued fraction converging to that number.*

An infinite simple continued fraction $[a_0; a_1, ..., a_k, ...]$ is periodic if $\exists n \in \mathbb{N}$ such that $a_r = a_{n+r}$ for all sufficiently large $r$ [8, 9]. We denote it as,

$$z = [a_0; a_1, ..., a_{r-1}, \overline{a_r, a_{r+1}, ..., a_{r+n-1}}]$$

The minimum number of the repeating terms of a periodic continued fraction is called the period (period length) of the continued fraction, which we denoted by $l$.

A real number $x$ is called a quadratic irrational, if $x$ is the irrational solution of a quadratic equation $ax^2+bx+c$ with rational coefficients $a, b, c$ [8].

**Theorem 1.5.** *A real number $x$ is quadratic irrational if and only if, its simple continued continued fraction expansion is periodic.*

Euclidean algorithm for Gaussian integers, is the algorithm to compute gcd of two or more Gaussian integers. It is briefly discussed as [1, 4]:

Let $p, q \in \mathbb{G}$, not both zero, suppose one of them is zero, without loss of generality let $q = 0$ then $gcd(p, q) = p$. Suppose both of them are not zero, arrange $p, q$ such that $|p| \geq |q|$, we define $g$, a Gaussian integer as $g = \left[\frac{p}{q}\right]$ where $[u] := [Re(u)] + i[Im(u)]$ is the Gaussian integer nearest to $u$. We then compute pairs $(p_{j+1}, q_{j+1})$ successively as $(p_{j+1}, q_{j+1}) := (q_j, p_j - gq_j)$ with $(p_0, q_0) = (p, q)$ the process is carried on, if at some $k \in \mathbb{N}$ we have $q_k = 0$ then we stop. We clearly see that this is a fast algorithm since, $|p_{j+1}q_{j+1}| \leq \frac{1}{\sqrt{2}}|p_jq_j|$ [1, 4].

For example, consider two Gaussian integers $219+47i$ and $67-59i$. Now, $g = \left[\frac{p}{q}\right] = \left[\frac{219+47i}{67-59i}\right] = 1+2i$, then, $(p_1, q_1) = (67-59i, 34-28i)$. Again, $g = \left[\frac{p_1}{q_1}\right] = \left[\frac{67-59i}{34-28i}\right] = 2$. We then have, $(p_2, q_2) = (34-28i, -1-3i)$ now, $g = \left[\frac{p_2}{q_2}\right] = \left[\frac{34-28i}{-1-3i}\right] = 5+13i$. $(p_3, q_3) = (-1-3i, 0)$. Therefore, $gcd(219+47i, 67-59i) = -1-3i$. Gcd of a pair of Gaussian integers is not unique, if $d$ is one of the gcd of a pair of Gaussian integers then $-d, id, -id$ are also gcds of that given pair.

From the idea of computing gcd of Gaussian integers, we understand the theory of Hurwitz complex continued fraction. The Hurwitz complex continued fraction (HCCF) was developed by Hurwitz from an algorithm which we now known as Hurwitz algorithm [4, 5, 10]:
Hurwitz start from a random complex number $z \neq 0$, to obtained the equations $a_0 = [z]$ $([z] := [Re(z)] + i[Im(z)])$, with $z_0 = z - a_0$ and $a_n = [z_{n-1}^{-1}]$; $z_n = z_{n-1}^{-1} - a_n$ $\forall n \geq 1$, the process is continued indefinitely if $z$ is not

a complex rational and the complex continued fraction (which we abbreviated as HCCF) is infinite, we denote it as $[a_0; a_1, ..., a_k, ...]$. If $z$ is a complex rational i.e. $z \in \mathbb{Q}(i)$, the algorithm terminates and the HCCF is a finite complex continued fraction also called finite HCCF. The Hurwitz complex continued fraction (infinite) is given by the expression

$$z = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}} \tag{1.3}$$

and we say that HCCF of $z$ is $[a_0; a_1, ..., a_n, ...]$.

When the input $z$ is a real number, the Hurwitz algorithm is commonly known as the nearest integer algorithm and the continued fraction developed is a regular continued fraction in terms of ordinary integers [4].

**Definition 1.6.** A complex number $z$ is called a complex quadratic irrational, if $z$ is the irrational solution of a quadratic equation $az^2 + bz + c$ with complex rational coefficients $a, b, c$ [8].

**Definition 1.7.** Sign of a real number is defined as

$$sign : \mathbb{R} \longrightarrow \{-1, 0, 1\}$$

$$sign(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

**Remark 1.8.** $\sqrt{a + ib} = \pm(\alpha + i\beta)\ \ a, b \in \mathbb{R}$, where $\alpha = \sqrt{\dfrac{a + \sqrt{a^2 + b^2}}{2}}$ and $\beta = sign(b)\sqrt{\dfrac{-a + \sqrt{a^2 + b^2}}{2}}$.

In Hurwitz complex continued fraction, partial quotients, convergents and other terms are also defined in the same manner as in simple continued fraction.

An infinite HCCF $[a_0; a_1, ..., a_k, ...]$ is periodic if $\exists n \in \mathbb{N}$ such that $a_r = a_{n+r}$ for all sufficiently large $r$ [5, 11]. We denote it as,

$$z = [a_0; a_1, ..., a_{r-1}, \overline{a_r, a_{r+1}, ..., a_{r+n-1}}]$$

The minimum number of the repeating terms of a periodic HCCF is called the period (period length) of the continued fraction, usually denoted by $l$. A complex quadratic irrational has periodic HCCF and vice versa [6, 7].

## 2. HCCF of some complex quadratic irrationals

In this section, we study Hurwitz complex continued fraction of some specific complex quadratic irrational $\sqrt{D}$ and found interesting properties. Following are some results and properties for some specific values of $D$:

**Proposition 2.1.** *Let $D = k^2 + i$; $k \geq 1$ is integer.*

*(1) HCCFs of $\sqrt{k^2+i}$ are $\big[k; \overline{-2ki, 2k}\big]$ and, $\big[-k; \overline{2ki, -2k}\big]$ and period length is $l = 2$.*

*(2) The convergents of the HCCFs are:*
$$\frac{p_0}{q_0} = \pm\left(\frac{k}{1}\right), \frac{p_1}{q_1} = \pm\left(\frac{-2k^2i+1}{-2ki}\right), \frac{p_2}{q_2} = \pm\left(\frac{-4k^3i+3k}{-4k^2i+1}\right), \ldots$$

*Proof.* Let $\alpha = \sqrt{k^2+i} = a+ib$, where $a, b \in \mathbb{R}$; solving for $a$ and $b$ we get, $a^2 = \dfrac{k^2+\sqrt{k^4+1}}{2}$ and $b^2 = \dfrac{-k^2+\sqrt{k^4+1}}{2}$ and so, $a = \pm\sqrt{\dfrac{k^2+\sqrt{k^4+1}}{2}}$ and $b = \pm\sqrt{\dfrac{-k^2+\sqrt{k^4+1}}{2}}$. □

**Example 2.2.** Consider $D = 1 + i$

(1) Here $z = \sqrt{1+i} = \pm\left(\sqrt{\dfrac{1+\sqrt{2}}{2}} + \sqrt{\dfrac{-1+\sqrt{2}}{2}}\right)$. Without loss of generality, let us consider only $'+'$ sign. Therefore, $a_0 := [z] = 1$, $z_0 := z - a_0 = \sqrt{1+i} - 1$ which gives $z_0^{-1} = (-i)(\sqrt{1+i}+1)$ so that, $a_1 := [z_0^{-1}] = -2i$, $z_1 := z_0^{-1} - a_1 = i(1-\sqrt{1+i})$ which gives $z_1^{-1} = \sqrt{1+i}+1$ so that, $a_2 := [z_1^{-1}] = 2$, $z_2 := z_1^{-1} - a_2 = \sqrt{1+i} - 1 = z_0$ which gives $a_3 = a_1$. Hence, HCCFs of $\sqrt{1+i}$ are $\big[1; \overline{-2i, 2}\big]$ and, $\big[-1; \overline{2i, -2}\big]$ and period length is $l = 2$.

(2) The convergents of the HCCFs are:
$$\frac{p_0}{q_0} = \pm 1, \frac{p_1}{q_1} = \pm\left(\frac{-2i+1}{-2i}\right), \frac{p_2}{q_2} = \pm\left(\frac{-4i+3k}{-4i+1}\right), \ldots$$

**Proposition 2.3.** *Let $D = k^2 - i$; $k \geq 1$ is integer.*

*(1) HCCFs of $\sqrt{k^2-i}$ are $\big[k; \overline{2ki, 2k}\big]$ and, $\big[-k; \overline{-2ki, -2k}\big]$ period length is $l = 2$.*

*(2) The convergents of the HCCFs are:*
$$\frac{p_0}{q_0} = \pm\left(\frac{k}{1}\right), \frac{p_1}{q_1} = \pm\left(\frac{2k^2i+1}{2ki}\right), \frac{p_2}{q_2} = \pm\left(\frac{4k^3i+3k}{4k^2i+1}\right), \ldots$$

*Proof.* The proof can be referred from proposition 2.1. □

**Example 2.4.** Consider $D = 4 - i$

(1) Here $z = \sqrt{4-i} = \pm\left(\sqrt{\frac{4+\sqrt{17}}{2}} + \sqrt{\frac{-4+\sqrt{17}}{2}}\right)$. Without loss of generality, let us consider only $'+'$ sign. Therefore, $a_0 := [z] = 2$, $z_0 := z - a_0 = \sqrt{4-i} - 2$ which gives $z_0^{-1} = (i)(\sqrt{4-i}+2)$ so that, $a_1 := [z_0^{-1}] = 4i$, $z_1 := z_0^{-1} - a_1 = i(\sqrt{4-i}-2)$ which gives $z_1^{-1} = \sqrt{4-i}+2$ so that, $a_2 := [z_1^{-1}] = 4$, $z_2 := z_1^{-1} - a_2 = \sqrt{4-i} - 2 = z_0$ which gives $a_3 = a_1$. Hence, HCCFs of $\sqrt{4-i}$ are $\big[2; \overline{4i, 4}\big]$ and, $\big[-2; \overline{-4i, -4}\big]$ and period length is $l = 2$.

(2) The convergents of the HCCFs are:
$\frac{p_0}{q_0} = \pm\left(\frac{2}{1}\right), \frac{p_1}{q_1} = \pm\left(\frac{8i+1}{4i}\right), \frac{p_2}{q_2} = \pm\left(\frac{32i+6}{16i+1}\right), \ldots$

**Corollary 2.5.** *Let $D = -k^2 + i$; $k \geq 1$ is integer.*

*(1) HCCF of $\sqrt{-k^2+i}$ is $\big[ki; \overline{2k, 2ki}\big]$, period length is $l = 2$.*

*(2) The convergents of the HCCF are:*
$\frac{p_0}{q_0} = \frac{ki}{1}, \frac{p_1}{q_1} = \frac{2k^2i+1}{2k}, \frac{p_2}{q_2} = \frac{(-4k^3+3ki)}{4k^2i+1}, \ldots$

*Proof.* Let $\alpha = \sqrt{-k^2+i} = a + ib$, where $a, b \in \mathbb{R}$; solving for $a$ and $b$ we get, $a^2 = \frac{-k^2+\sqrt{k^4+1}}{2}$ and $b^2 = \frac{k^2+\sqrt{k^4+1}}{2}$ and so, $a = \pm\sqrt{\frac{-k^2+\sqrt{k^4+1}}{2}}$ and $b = \pm\sqrt{\frac{k^2+\sqrt{k^4+1}}{2}}$.
Further steps can be proceeded in the same way as in proposition 2.1. □

**Corollary 2.6.** *Let $D = -k^2 - i$; $k \geq 1$ is integer.*

*(1) HCCF of $\sqrt{-k^2-i}$ is $\big[ki; \overline{-2k, 2ki}\big]$, period length is $l = 2$.*

*(2) The convergents of the HCCF are:*
$\frac{p_0}{q_0} = \frac{ki}{1}, \frac{p_1}{q_1} = \frac{-2k^2i+1}{-2k}, \frac{p_2}{q_2} = \frac{(4k^3+3ki)}{-4k^2i+1}, \ldots$

*Proof.* Let $\alpha = \sqrt{-k^2-i} = a + ib$, where $a, b \in \mathbb{R}$; solving for $a$ and $b$ we get, $a^2 = \frac{-k^2+\sqrt{k^4+1}}{2}$ and $b^2 = \frac{k^2+\sqrt{k^4+1}}{2}$ and so, $a = \pm\sqrt{\frac{-k^2+\sqrt{k^4+1}}{2}}$ and $b = \pm\sqrt{\frac{k^2+\sqrt{k^4+1}}{2}}$. Note that real numbers $a$ and $b$ can be chosen in such a way that, $ab < 0$.
Further steps can be proceeded as referred to proposition 2.1. □

**Theorem 2.7.** *Let $D \in \mathbb{G}$, then HCCF of a complex quadratic irrational $\sqrt{D}$ is $[a_0; \overline{2a_0}]$, if and only if $D = a_0^2 + 1$ and $D \neq 0$.*

*Proof.* Given, HCCF of $\sqrt{D}$ is $[a_0; \overline{2a_0}]$, let $\theta = [\overline{2a_0}]$, then we have a quadratic equation with complex integral coefficients,
$\theta^2 - 2a_0\theta - 1 = 0$ so that, $\theta = \dfrac{-1}{a_0 - \sqrt{a_0^2+1}}$ which implies that, $\sqrt{D} = a_0 + \dfrac{1}{\theta} = \sqrt{a_0^2+1}$, that is, $D = a_0^2 + 1$.
Conversely, let $D = a_0^2 + 1$, by Hurwitz algorithm, $a_0 = [\sqrt{D}], z_0 = \sqrt{a_0^2+1} - a_0$ and $\dfrac{-1}{2} \leq Re(z_0), Im(z_0) < \dfrac{1}{2}$, so, $z_0^{-1} = \sqrt{a_0^2+1} + a_0$ which implies, $a_1 = 2a_0$ and $z_1 = \sqrt{a_0^2+1} - a_0 = z_0$.
Hence, HCCF of $\sqrt{D}$ is $[a_0; \overline{2a_0}]$.

□

**Corollary 2.8.** *(1) If $a_0 = k \in \mathbb{N}$, then HCCF of $\sqrt{k^2+1}$ is $\left[k; \overline{2k}\right]$, period length is $l = 1$.*

*(2) The convergents of the HCCF are:*
$\dfrac{p_0}{q_0} = \dfrac{k}{1}, \dfrac{p_1}{q_1} = \dfrac{2k^2+1}{2k}, \dfrac{p_2}{q_2} = \dfrac{(4k^3+3k)}{4k^2+1}, \dots$

**Corollary 2.9.** *(1) If $a_0 = ki, 1mmk \in \mathbb{N}$, then HCCF of $\sqrt{a_0^2+1}$ is $\left[ki; \overline{2ki}\right]$, with period length $l = 1$.*

*(2) The convergents of the HCCF are:*
$\dfrac{p_0}{q_0} = \dfrac{ki}{1}, \dfrac{p_1}{q_1} = \dfrac{-2k^2+1}{2ki}, \dfrac{p_2}{q_2} = \dfrac{(-4k^3+3k)i}{-4k^2+1}, \dots$

**Theorem 2.10.** *Let $D \in \mathbb{G}$, then HCCF of a complex quadratic irrational $\sqrt{D}$ is $[a_0; \overline{-2a_0, 2a_0}]$, if and only if $D = a_0^2 - 1$ and $D \neq 0$.*

**Corollary 2.11.** *Let $D = (k^2 - 1)$, $k \geq 2$.*

*(1) HCCF of $\sqrt{k^2-1}$ is $\left[k; \overline{-2k, 2k}\right]$ or, $[-k; \overline{2k, -2k}]$, period length is $l = 2$.*

*(2) The convergents of the HCCF are:*
$\dfrac{p_0}{q_0} = \dfrac{k}{1}, \dfrac{p_1}{q_1} = \dfrac{-2k^2+1}{-2k}, \dfrac{p_2}{q_2} = \dfrac{(-4k^3+3k)}{-4k^2+1}, \dots$

*Proof.* The proof can be referred from proposition 2.9 and proposition 2.8.

□

**Corollary 2.12.** *Let $D = -(k^2 + 1)$, $k \geq 1$.*

*(1) HCCF of $i\sqrt{k^2+1}$ is $\left[ki; \overline{-2ki, 2ki}\right]$, period length is $l = 2$.*

(2) *The convergents of the HCCF are:*
$$\frac{p_0}{q_0}=\frac{ki}{1},\frac{p_1}{q_1}=\frac{2k^2+1}{-2ki},\frac{p_2}{q_2}=\frac{(4k^3+3k)i}{4k^2+1},\ldots$$

**Theorem 2.13.** *Let $D \in \mathbb{G}$, then HCCF of a complex quadratic irrational $\sqrt{D}$ is $[a_0;\overline{a_0,2a_0}]$, if and only if $D = a_0^2+2$ and $D \neq -2,1$.*

**Corollary 2.14.** *Let $D = (k^2+2)$, $k \in \mathbb{N}$, $k \geq 2$.*

(1) *HCCF of $\sqrt{k^2+2}$ is $\left[k;\overline{k,2k}\right]$, period length is $l = 2$.*

(2) *The convergents of the HCCF are:*
$$\frac{p_0}{q_0}=\frac{k}{1},\frac{p_1}{q_1}=\frac{k^2+1}{k},\frac{p_2}{q_2}=\frac{(2k^3+3k)}{2k^2+1},\ldots$$

**Corollary 2.15.** *Let $D = -(k^2-2)$, $k \in \mathbb{N}$, $k > 2$. Then*

(1) *HCCF of $i\sqrt{k^2-2}$ is $\left[ki;\overline{ki,2ki}\right]$, period length is $l = 2$.*

(2) *The convergents of the HCCF are:*
$$\frac{p_0}{q_0}=\frac{ki}{1},\frac{p_1}{q_1}=\frac{-k^2+1}{ki},\frac{p_2}{q_2}=\frac{(-2k^3+3k)i}{-2k^2+1},\ldots$$

**Theorem 2.16.** *Let $D \in \mathbb{G}$, then HCCF of a complex quadratic irrational $\sqrt{D}$ is $[a_0;\overline{-a_0,2a_0}]$, if and only if $D = a_0^2-2$ and $D \neq -3,-2,1$.*

**Corollary 2.17.** *If $a_0 = ki$, $D = -(k^2+2)$, $k \geq 2$. Then*

(1) *HCCF of $i\sqrt{k^2+2}$ is $\left[ki;\overline{-ki,2ki}\right]$, period length is $l = 2$.*

(2) *The convergents of the HCCF are:*
$$\frac{p_0}{q_0}=\frac{ki}{1},\frac{p_1}{q_1}=\frac{k^2+1}{-ki},\frac{p_2}{q_2}=\frac{(2k^3+3k)i}{2k^2+1},\ldots$$

**Theorem 2.18.** *Let $D \in \mathbb{G}$, the HCCF of $\sqrt{D}$ is $[a_0;\overline{2,2a_0}]$, if and only if $D = a_0^2+a_0$ and $D \neq 0$.*

**Corollary 2.19.** *If $a_0 = k \in \mathbb{N}$, then HCCF of $\sqrt{D} = \sqrt{k^2+k}$ is $[k;\overline{2,2k}]$.*

**Corollary 2.20.** *If $a_0 = \pm ki$, $k \in \mathbb{N}$, then HCCF of $\sqrt{D} = \sqrt{-k^2+ki}$ is $[ki;\overline{2,2ki}]$ or, $[-ki;\overline{-2,-2ki}]$.*

**Corollary 2.21.** *The HCCF of $\sqrt{D} = \sqrt{-k^2-ki}$ is $[ki;\overline{-2,2ki}]$ or, $[-ki;\overline{2,-2ki}]$.*

**Proposition 2.22.** *Let $D = -(k^2+k)$, $k \geq 1$.*

(1) *HCCF of $i\sqrt{k^2+k}$ is $\left[ki;\overline{-2i,2ki}\right]$, period length is $l = 2$.*

(2) *The convergents of the HCCF are:*
$$\frac{p_0}{q_0}=\frac{ki}{1},\frac{p_1}{q_1}=\frac{2k+1}{-2i},\frac{p_2}{q_2}=\frac{(4k^2+3k)i}{4k+1},\ldots$$

**Example 2.23.** Let us consider $k = 2$, then $D = -6$.
Now $z = i\sqrt{6}$, $a_0 := [z] = 2i$, $z_0 = z - a_0 = i(\sqrt{6}-2)$, $z_0^{-1} = \dfrac{-i(\sqrt{6}+2)}{2}$ so that, $a_1 := [z_0^{-1}] = -2i$, $z_1 = z_0^{-1} - a_1 = \dfrac{-i(\sqrt{6}-2)}{2}$, $z_1^{-1} = i(\sqrt{6}+2)$ so that, $a_2 := [z_1^{-1}] = 4i$, $z_2 = z_1^{-1} - a_2 = i(\sqrt{6}-2) = z_0$, this gives, $a_3 = a_1$.
Hence, HCCF of $i\sqrt{6}$ is $[2i; \overline{-2i, 4i}]$.

The convergents of the HCCF are:
$\dfrac{p_0}{q_0} = \dfrac{2i}{1}, \dfrac{p_1}{q_1} = \dfrac{5}{-2i}, \dfrac{p_2}{q_2} = \dfrac{22i}{9}, \ldots$

**Theorem 2.24.** *Let $k \in \mathbb{N}$, then HCCF of $\sqrt{D}$ is $[k(1+i); \overline{2k, 2k(1+i)}]$, if and only if $D = 1 + (2k^2+1)i$.*

**Corollary 2.25.** *The HCCF of $\sqrt{-1+i(2k^2-1)}$ is $[k(1+i); \overline{-2k, 2k(1+i)}]$.*

*Proof.* HCCF of $z = \sqrt{D}$ be $z = [k(1+i); \overline{2k, 2k(1+i)}]$, let $\theta = [\overline{-2k, 2k(1+i)}]$, then we have a quadratic equation with complex integeral coefficients
$(1+i)\theta^2 + 2k(1+i)\theta + 1 = 0$ so that, $\theta = \dfrac{-k(1+i) \pm \sqrt{-1+i(2k^2-1)}}{(1+i)}$
$\theta = \dfrac{-1}{k(1+i) \pm \sqrt{-1+i(2k^2-1)}}$
then, $z = \pm\sqrt{-1+i(2k^2-1)}$ which implies that, $D = z^2 = -1+i(2k^2-1)$. □

**Theorem 2.26.** *Let $k \in \mathbb{N}, k \neq 1$, then HCCF of $\sqrt{D}$ is $[k(1+i); \overline{2ki, 2k(1+i)}]$, if and only if $D = 1 + (2k^2-1)i$.*

**Corollary 2.27.** *Let $k \in \mathbb{N}$, the HCCF of $\sqrt{-1+i(2k^2+1)}$ is $[k(1+i); \overline{-2ki, 2k(1+i)}]$ and conversely.*

*Proof.* Let HCCF of $z = \sqrt{D}$ be $z = [k(1+i); \overline{-2ki, 2k(1+i)}]$, let $\theta = [\overline{-2ki, 2k(1+i)}]$, then we have a quadratic equation with complex integeral coefficients
$(1+i)\theta^2 + 2ki(1+i)\theta + i = 0$ so that, $\theta = \dfrac{-ki(1+i) \pm \sqrt{1-i(2k^2+1)}}{(1+i)}$
$\theta = \dfrac{1}{\pm i\sqrt{1-i(2k^2+1)} - k(1+i)}$
then, $z = \pm i\sqrt{1-i(2k^2+1)}$ which implies that, $D = z^2 = -1+i(2k^2+1)$. □

**Theorem 2.28.** *Let $k \in \mathbb{N}$, if the HCCF of $\sqrt{D}$ is $[k(1+i); \overline{k, 2k(1+i)}]$, then $D = 2 + 2i(k^2+1)$.*

*Proof.* Given, HCCF of $z$ is $[k(1+i);\overline{k, 2k(1+i)}]$, $k \in \mathbb{N}$. Let $\theta = [\overline{k, 2k(1+i)}]$, then we have a binary quadratic equation with complex integral coefficients (in $\theta$), $2(1+i)\theta^2 - 2k(1+i)\theta - 1 = 0$ so that, $\theta = \dfrac{k(1+i) \pm \sqrt{2+2i(k^2+1)}}{2(1+i)} = \dfrac{-1}{k(1+i) \pm \sqrt{2+2i(k^2+1)}}$, we have $z = k(1+i) + \dfrac{1}{\theta} = \pm\sqrt{2+2i(k^2+1)}$. Hence, $D = 2+2i(k^2+1)$. ∎

**Theorem 2.29.** *Let $k \in \mathbb{N}, k \neq 1$, if the HCCF of $\sqrt{D}$ is $[k(1+i);\overline{-k, 2k(1+i)}]$, then $D = -2+2(k^2-1)i$.*

*Proof.* Given, HCCF of $z$ is $[k(1+i);\overline{-k, 2k(1+i)}]$, $k \in \mathbb{N}$. Let $\theta = [\overline{-k, 2k(1+i)}]$, then we have a binary quadratic equation with complex integral coefficients (in $\theta$), $2(1+i)\theta^2 + 2k(1+i)\theta + 1 = 0$ so that, $\theta = \dfrac{k(1+i) \pm \sqrt{-2+2i(k^2-1)}}{2(1+i)} = \dfrac{-1}{k(1+i) \pm \sqrt{-2+2i(k^2-1)}}$, we have $z = k(1+i) + \dfrac{1}{\theta} = \pm\sqrt{-2+2i(k^2-1)}$. Hence, $D = -2+2i(k^2-1)$. ∎

The above expansion of complex quadratic irrationals is necessary for us to study Pell's equation in Gaussian integers. For example, does a Pell's equation $x^2+6y^2 = 4$ has solution in Gaussian integers? The answer is yes.

**Example 2.30.** Here, Pell's equation $x^2+6y^2 = 4$. We find HCCF of $\sqrt{D} = i\sqrt{6}$ is $[2i;\overline{-2i, 4i}]$, here, convergents are $\dfrac{p_0}{q_0} = \dfrac{2i}{1}, \dfrac{p_1}{q_1} = \dfrac{5}{-2i}, \dfrac{p_2}{q_2} = \dfrac{22i}{9}, \ldots$

We have, $p_0^2+6q_0^2 = 2$, $p_1^2+6q_1^2 = 1$, $p_2^2+6q_2^2 = 2$,..., hence, one of the solutions of Pell's equation $x^2+6y^2 = 4$ is $(p_1, 2q_1)$.

## Concluding comments

Throughout this paper we have seen Hurwitz complex continued fraction expansion of different types of complex quadratic irrationals and their periodic behaviors, these HCCF expansions will help us study the complex theory of Pell's equation, that is, the study of Pell's equation in Gaussian integers, as well as binary quadratic forms with complex integral coefficients.

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# ON SOME SERIES APPROXIMATIONS ARISING FROM THE PROBABILISTIC PROOF OF WEIERSTRASS APPROXIMATION

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Abstract. In this paper, we first present Bernstein's proof of the Weierstrass approximation theorem using probabilistic arguments. We then apply the method to develop approximations for a Dirichlet series on the critical strip. We prove two limit theorems, the first one by considering sums of independent Bernoulli random variables, and the second one by looking at sums of independent geometric random variables.

## 1. Introduction

The **Riemann Hypothesis** (see [1]) is the conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part equal to $\frac{1}{2}$. It is considered to be one of the most important unsolved problems in Mathematics. It is of great interest in number theory as it has important connections to the distribution of prime numbers. It was proposed by Bernhard Riemann in 1859, after whom it is named. The Riemann zeta function $\zeta(s)$ is a function whose argument $s$ may be any complex number other than 1, and whose values are also complex. It has zeros at the negative even integers -2, -4, -6,. . . , which are called the trivial zeros. However, these are not the only values for which $\zeta(s) = 0$, and the other ones are called the non-trivial zeros. The Riemann hypothesis is concerned with the locations of these non-trivial zeros, and it states that the real part of every non-trivial zero of the Riemann zeta function is $\frac{1}{2}$.

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Dirichlet series may be regarded as natural generalizations of the Riemann zeta function, and one may ask for the locations of the zeros of these closely related Dirichlet series. In this context, the Dirichlet eta function, also called the alternating zeta function, plays a prominent role. This paper makes approximations to a certain Dirichlet series using probabilistic arguments, and one can now ask for the locations of the zeros of these approximating polynomials and functions.

## 2. Preliminaries

In this section, we state and prove the Weierstrass approximation theorem. The proof is due to Bernstein and may be found in [2]. Another exposition of Bernstein's proof may also be found in [3]. We present some preliminaries from probability theory that we will need, such as Chebyshev's inequality and some results concerning Bernoulli random variables.

**Chebyshev's Inequality:** For $c \geq 0$ and $X \in \mathcal{L}^2$, we have

$$c^2 P\left(|X - \mu| > c\right) \leq Var\left(X\right),$$

where $\mu = E\left[X\right]$.

**Bernoulli random variables:** We consider a sequence $\{X_n\}$ of independent and identically distributed Bernoulli random variables (IID RVs) with values in $\{0, 1\}$, with

$$p = P\left(X_n = 1\right) = 1 - P\left(X_n = 0\right).$$

Then, evidently, $E\left[X_n\right] = p$ and $Var\left[X_n\right] = p(1-p) \leq \frac{1}{4}$. Now consider sums of IID Bernoulli RVs of the form

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Then, $S_n$ is known to have a binomial distribution with expectation $np$ and variance $np(1-p) \leq \frac{n}{4}$. Furthermore, we have $E\left[\frac{S_n}{n}\right] = p$ and $Var\left[\frac{S_n}{n}\right] = \frac{1}{n^2} Var\left[S_n\right] \leq \frac{1}{4n}$. Chebyshev's inequality then readily yields

$$P\left(\left|\frac{S_n}{n} - p\right| > \delta\right) \leq \frac{1}{\delta^2} Var\left[\frac{S_n}{n}\right] \leq \frac{1}{4n\delta^2}.$$

With these in hand, we now state the following theorem, due to Weierstrass.

**Weierstrass Approximation Theorem:** If $f$ is a continuous function on $[0,1]$ and $\epsilon > 0$, then there exists a polynomial $B$ such that

$$\sup_{x \in [0,1]} |B(x) - f(x)| \leq \epsilon.$$

**Proof:** Consider a sequence $\{X_n\}$ of IID Bernoulli RVs, and let

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Then, the probability of $k$ successes in $n$ Bernoulli trials is given by the binomial probability

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots n. \tag{2.1}$$

From (2.1), we define the following polynomial of degree $n$:

$$B_n(p) = E\left[f\left(\frac{S_n}{n}\right)\right] = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} p^k (1-p)^{n-k},$$

with '$B$' being in deference to Bernstein, whose proof we are presenting.

Now, $f$ is bounded on $[0,1]$, for there is a $K$ such that $|f(y)| \leq K$ for all $y \in [0,1]$. Also, $f$ is uniformly continuous on $[0,1]$; for given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - y| \leq \delta \text{ implies that } |f(x) - f(y)| < \frac{1}{2}\epsilon. \tag{2.2}$$

Now, for $p \in [0,1]$,

$$|B_n(p) - f(p)| = \left| E\left\{ f\left(\frac{S_n}{n}\right) - f(p) \right\} \right|.$$

Let us write $Y_n = \left|f\left(\frac{S_n}{n}\right) - f(p)\right|$ and $Z_n = \left|\frac{S_n}{n} - p\right|$. Then, $Z_n \leq \delta$ implies that $Y_n < \frac{1}{2}\epsilon$. Moreover, we note that $B_n(p)$ is a bounded function on $[0,1]$ since

$$|B_n(p)| \leq \sum_{i=0}^{n} K \cdot \binom{n}{i} p^i (1-p)^{n-i} = K \cdot (p + (1-p))^n = K.$$

Thus, we have, by conditioning on $Z_n$:

$$\begin{aligned} |B_n(p) - f(p)| &\leq E[Y_n] \\ &= E[Y_n|\ Z_n \leq \delta] P(Z_n \leq \delta) + E[Y_n|\ Z_n > \delta] P(Z_n > \delta) \\ &\leq \frac{1}{2}\epsilon\ P(Z_n \leq \delta) + 2K\ P(Z_n > \delta) \\ &\leq \frac{1}{2}\epsilon + 2K \cdot \frac{1}{4n\delta^2}, \end{aligned}$$

where the last inequality follows from Chebyshev's inequality, as outlined above.

Earlier, we chose a fixed $\delta$ in (2.2). We can now choose $n$ so that

$$\frac{2K}{4n\delta^2} < \frac{1}{2}\epsilon.$$

Then, $|B_n(p) - f(p)| < \epsilon$ for all $p \in [0,1]$, which completes the proof.

## 3. Application to approximation of Dirichlet series

In this section, we again consider a sequence of independent and identically distributed Bernoulli random variables, and use the Strong Law of Large Numbers to make estimates of a Dirichlet series related to the Riemann zeta function. We first state the Strong Law of Large Numbers, which can be found in many textbooks like [2].

**Strong Law of Large Numbers:** Let $X_1, X_2, \ldots, X_n$ be IID random variables with a finite expected value $E[X_i] = \mu < \infty$, and let $S_n = X_1 + X_2 + \cdots + X_n$. Then,

$$P\left(\frac{S_n}{n} \to \mu\right) = 1 \text{ or } \frac{S_n}{n} \to \mu \text{ almost surely.}$$

Before continuing with our probabilistic arguments, we first describe some preliminaries from number theory that we need, for subsequent developments.

**Preliminaries from Number Theory:** Let

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} \tag{3.1}$$

be the Riemann zeta function, where $s$ is a complex variable. It is known (see [4]) that (3.1) converges only for $Re(s) > 1$, but we are interested in approximating the zeta function on the critical strip, $0 < Re(s) < 1$. We wish to work with a Dirichlet series that converges on $0 < Re(s) < 1$, and for this reason we consider the Dirichlet eta function, defined as (see [5])

$$\eta(s) = \sum_{m \geq 1} \frac{(-1)^{m+1}}{m^s}. \tag{3.2}$$

The Dirichlet eta function is also known as the alternating zeta function, and (3.2) is convergent (albeit not absolutely) for any complex number $s$ with $Re(s) > 0$. The following relation holds between the Dirichlet eta function and the Riemann zeta function, as conjectured by Euler in 1749 (see [4]):

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s). \tag{3.3}$$

From (3.3), we see that the zeros of the eta function include all the zeros of the zeta function. The factor of $1 - 2^{1-s}$ adds an infinite number of complex simple zeros $s_n$, with

$$s_n = 1 + \frac{2n\pi i}{\log 2},$$

where $n$ is any non-zero integer. We are interested in the zeros of $\eta(s)$ in the critical strip, $0 < Re(s) < 1$. We do not work with the eta function directly, but we consider instead the following closely related Dirichlet series:

$$f(s, a) = \sum_{m \geq 1} \frac{a^{m+1}}{m^s}, \qquad |a| \leq 1. \tag{3.4}$$

The parameter $a$ interpolates between $\eta(s)$ and $\zeta(s)$, and in particular $f(s, -1) = \eta(s)$ and $f(s, 1) = \zeta(s)$. We are now interested in the domain of convergence of $f(s, a)$, and so we state the following convergence theorem for Dirichlet series (see [6]).

**Theorem:** Consider an arbitrary sequence $\{a_m\}$ of complex numbers, and consider also the following Dirichlet series formed from this sequence:

$$f(s) = \sum_{m \geq 1} \frac{a_m}{m^s}.$$

Regard $f$ as a function of the complex variable $s$. If the set of sums

$$a_m + a_{m+1} + \cdots + a_{m+k}$$

is bounded for all $m$ and $k \geq 0$, then the Dirichlet series converges on the open half plane of $s$ such that $Re(s) > 0$.

In (3.4), this convergence condition is satisfied for $a = -1$, $|a| < 1$, but not for $a = 1$, the case of the Riemann zeta function. Having $|a| < 1$ is very useful for the convergence properties of the series that are discussed below.

We let

$$f_M(s, a) = \sum_{m=1}^{M} \frac{a^{m+1}}{m^s} \tag{3.5}$$

be the sum of the first $M$ terms of the series in (3.4), and we use (3.5) in the arguments that follow. We will specifically use the Strong Law of Large Numbers to make approximations to (3.5) on the critical strip, $0 < Re(s) < 1$.

Returning to our probabilistic arguments, we now consider, in analogy with the Bernstein polynomials of Section 2, the following expected value of $f_M$ :

$$E\left[f_M\left(\frac{S_n}{n}, a\right)\right], \tag{3.6}$$

where $S_n$ is the sum of $n$ IID Bernoulli RVs, as in the preceding section. The first step is to evaluate (3.6) using the Strong Law of Large Numbers. Thus,

$$\begin{aligned}\lim_{n\to\infty} E\left[f_M\left(\frac{S_n}{n},a\right)\right] &= E\left[\lim_{n\to\infty} f_M\left(\frac{S_n}{n},a\right)\right] = E\left[f_M(p,a)\right] \\ &= f_M\left(p,a\right) = \sum_{m=1}^{M}\frac{a^{m+1}}{m^p} \qquad (3.7)\end{aligned}$$

On the other hand, we can evaluate (3.6) directly, using the binomial distribution of $S_n$, as follows:

$$\begin{aligned}E\left[f_M\left(\frac{S_n}{n},a\right)\right] &= \sum_{k=0}^{n} f_M\left(\frac{k}{n},a\right)P(S_n=k) \\ &= \sum_{k=0}^{n}\left(\sum_{m=1}^{M}\frac{a^{m+1}}{m^{\frac{k}{n}}}\right)\binom{n}{k}p^k(1-p)^{n-k}. \qquad (3.8)\end{aligned}$$

Interchanging the sums, we obtain:

$$\begin{aligned}&\sum_{m=1}^{M} a^{m+1}\sum_{k=0}^{n}\frac{1}{\left(m^{\frac{1}{n}}\right)^k}\binom{n}{k}p^k(1-p)^{n-k} \\ &=\sum_{m=1}^{M} a^{m+1}\sum_{k=0}^{n}\binom{n}{k}\left(p\cdot m^{\frac{-1}{n}}\right)^k(1-p)^{n-k} \\ &=\sum_{m=1}^{M} a^{m+1}\left(p\cdot m^{\frac{-1}{n}}+(1-p)\right)^n, \qquad (3.9)\end{aligned}$$

from the binomial theorem. Observe that the last line of (3.9) is a polynomial in $p$ of degree $n$. We may realize (3.7) by taking the limit as $n\to\infty$ of (3.9). Thus, we have established the following result.

**Theorem 1:**

$$\sum_{m=1}^{M}\frac{a^{m+1}}{m^p} = \lim_{n\to\infty}\sum_{m=1}^{M} a^{m+1}\left(p\cdot m^{\frac{-1}{n}}+(1-p)\right)^n. \qquad (3.10)$$

Now, (3.10) is valid for all $p\in[0,1]$, but we may extend this domain of convergence by taking the limit term by term in (3.10). Firstly, we note that

$$m^{\frac{-1}{n}} = e^{(\log m)\frac{-1}{n}} = 1 - \frac{\log m}{n} + O\left(\frac{1}{n^2}\right),$$

and so

$$\lim_{n\to\infty}\left(p \cdot m^{\frac{-1}{n}} + (1-p)\right)^n = \lim_{n\to\infty}\left(p\left(1 - \frac{\log m}{n} + O\left(\frac{1}{n^2}\right)\right) + (1-p)\right)^n$$
$$= \lim_{n\to\infty}\left(1 - \frac{p \cdot \log m}{n} + O\left(\frac{1}{n^2}\right)\right)^n$$
$$= \lim_{n\to\infty}\left(1 - \frac{p \cdot \log m}{n}\right)^{\frac{n}{p\cdot\log m}p\cdot\log m} = e^{-p\cdot\log m} = m^{-p}. \tag{3.11}$$

This argument does not depend on $p \in [0,1]$ and is valid for all $p \in \mathbb{C}$, and so (3.10) is true for any complex number $p$. Now, let

$$g_M(x,a) = \sum_{m=1}^{M} a^{m+1}\left(x \cdot m^{\frac{-1}{n}} + (1-x)\right)^n. \tag{3.12}$$

As $g_M(x,a)$ is a polynomial of degree $n$, it would be prudent to investigate its zeros in the critical strip for large $n$, maybe numerically. We can simplify $g_M(x,a)$ somewhat from the binomial theorem. Note that

$$(ax + (1-x))^n = (1 + (a-1)\,x)^n = \sum_{k=0}^{n}\binom{n}{k}(a-1)^k x^k. \tag{3.13}$$

From (3.13), it is clear that we can write (3.12) as

$$g_M(x,a) = \sum_{m=1}^{M} a^{m+1}\sum_{k=0}^{n}\binom{n}{k}\left(m^{\frac{-1}{n}} - 1\right)^k x^k.$$

Returning to (3.11), we can neglect $O\left(\frac{1}{n^2}\right)$ terms for large $n$, to obtain the approximation

$$\left(x \cdot m^{\frac{-1}{n}} + (1-x)\right)^n \approx \left(1 - \frac{\log m \cdot x}{n}\right)^n$$

for large $n$. Summing over $m$, we can approximate $g_M(x,a)$ by $h_M(x,a)$ for large $n$, where $h_M(x,a)$ is the following $n$-th degree polynomial:

$$h_M(x,a) = \sum_{m=1}^{M} a^{m+1}\left(1 - \frac{\log m \cdot x}{n}\right)^n.$$

It is also possible to simplify (3.9) without interchanging the sums. Returning to (3.8) and factoring out $p^n$, we obtain the following:

$$E\left[f_M\left(\frac{S_n}{n}, a\right)\right] = p^n \cdot \sum_{k=0}^{n}\left(\sum_{m=1}^{M}\frac{a^{m+1}}{m^{\frac{k}{n}}}\right)\binom{n}{k}\left(\frac{1-p}{p}\right)^{n-k}.$$

Equating the right hand side to zero, dividing out the factor of $p^n$, and letting $z = \frac{1-p}{p}$, we obtain the following $n$-th degree polynomial in $z$:

$$\sum_{k=0}^{n}\left(\sum_{m=1}^{M}\frac{a^{m+1}}{m^{\frac{k}{n}}}\right)\binom{n}{k} z^{n-k} = 0.$$

It will be of interest to study the zeros of this approximating polynomial for large values of $n$.

We complete this discussion by showing that $g_M(x,a)$ is a good approximation to $f(x,a)$ for $|a| < 1$, or $a = -1$. Let $f(x,a)$, $f_M(x,a)$, and $g_M(x,a)$ be as given above. Given $\epsilon > 0$, choose $M$ large enough that

$$|f\,(x,a) - f_M(x,a)| < \frac{\epsilon}{2},$$

and choose $n$ large enough that

$$|f_M\,(x,a) - g_M(x,a)| < \frac{\epsilon}{2}.$$

Then, it is easy to see that $|f\,(x,a) - g_M(x,a)| < \epsilon$ directly from the triangle inequality.

## 4. Application to sums of geometric random variables

In this section, we extend the approach in Section 3 by considering sums of independent geometric random variables instead of Bernoulli variables. First, we review some properties of geometric and negative binomial random variables; see Scheaffer et al. [7] and Balakrishnan et al. [8] for details.

**Geometric Random Variables:** Consider a sequence of independent Bernoulli trials with success probability $p$, and define $X$ to be the number of failures prior to the first success. Then, it is clear that $X$ has the following distribution, where $q = 1 - p$:

$$P(X=k) = q^k p, \quad k = 0, 1, 2, \ldots$$

It is also evident that

$$E[X] = \sum_{k \geq 0} kP(X=k) = \sum_{k \geq 0} kq^k p = \frac{q}{p}.$$

We are now interested in sums of independent geometric random variables, and so let us consider

$$S_n = X_1 + X_2 + \cdots + X_n,$$

where $\{X_i\}$ are IID geometric RVs. Clearly, $S_n$ represents the number of failures prior to the $n$-th success in a sequence of Bernoulli trials, and it is known that $S_n$ has the following **negative binomial distribution:**

$$P(S_n = k) = \binom{k+n-1}{n-1} p^n q^k, \quad k = 0, 1, 2, \ldots$$

We now recall the negative binomial theorem, which states that

$$\frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k, \quad |x| < 1.$$

As in Section 3, we next evaluate

$$E\left[f_M\left(\frac{S_n}{n}, a\right)\right], \tag{4.1}$$

using the Strong Law of Large Numbers. Thus,

$$\begin{aligned} \lim_{n\to\infty} E\left[f_M\left(\frac{S_n}{n}, a\right)\right] &= E\left[\lim_{n\to\infty} f_M\left(\frac{S_n}{n}, a\right)\right] = E\left[f_M(\frac{q}{p}, a)\right] \\ &= f_M\left(\frac{q}{p}, a\right) = \sum_{m=1}^{M} \frac{a^{m+1}}{m^{\frac{q}{p}}} \end{aligned} \tag{4.2}$$

We now evaluate (4.1) using the distribution of $S_n$ as follows:

$$
\begin{aligned}
E\left[f_M\left(\frac{S_n}{n}, a\right)\right] &= \sum_{k=0}^{\infty} f_M\left(\frac{k}{n}, a\right) P(S_n = k) \\
&= \sum_{k=0}^{\infty}\left(\sum_{m=1}^{M} \frac{a^{m+1}}{m^{\frac{k}{n}}}\right)\binom{k+n-1}{n-1} p^n (1-p)^k.
\end{aligned}
$$

Interchanging the sums, we obtain

$$
\begin{aligned}
&\sum_{m=1}^{M} a^{m+1} \sum_{k=0}^{\infty} \frac{1}{\left(m^{\frac{1}{n}}\right)^k}\binom{k+n-1}{n-1} p^n (1-p)^k \\
&= \sum_{m=1}^{M} a^{m+1} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} p^n \left(\frac{1-p}{m^{\frac{1}{n}}}\right)^k \\
&= \sum_{m=1}^{M} a^{m+1} \frac{p^n}{\left(1-\frac{1-p}{m^{\frac{1}{n}}}\right)^n}, \qquad (4.3)
\end{aligned}
$$

from the negative binomial theorem.

Taking the limit as $n \to \infty$ in (4.3) and comparing to (4.2), we arrive at the following result.

**Theorem 2:**

$$
\sum_{m=1}^{M} \frac{a^{m+1}}{m^{\frac{q}{p}}} = \lim_{n\to\infty} \sum_{m=1}^{M} a^{m+1} \frac{p^n}{\left(1-\frac{1-p}{m^{\frac{1}{n}}}\right)^n} \qquad (4.4)
$$

It will be of interest to study the zeros of the right hand sides of (3.10) and (4.4) when $a = -1$ and for large $n$.

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# SOME NECESSARY AND SUFFICIENT CONDITIONS FOR INVARIANT SUBMANIFOLDS OF LP-SASAKIAN MANIFOLDS WITH COEFFICIENT $\alpha$ TO BE TOTALLY GEODESIC

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Abstract. The object of the present paper is to deduce some necessary and sufficient conditions for submanifolds of LP-Sasakian manifolds with coefficient $\alpha$ to be totally geodesic. Obtained resultes are supported by illustrative examples.

## 1. Introduction

In differential geometry, submanifold theory has become a growing topic of research. B. Y. Chen ([3],[4]) has done many works in this line. Invariant and semi-invariant submanifolds has been studied by several authors ([1], [2], [8],[15]). In general an invariant submanifold is not necessarily totally geodesic. In the paper ([15]), the authors attempted to prove an invariant submanifold of an LP-Sasakian manifold is totally geodesic. In the paper ([16]) it has been shown that a recurrent submanifold of a Kenmotsu manifold is totally geodesic. Keeping this works in mind, in the present paper we would like to study invariant and totally geodesic submanifolds of LP-Sasakian manifolds with coefficient $\alpha$. For details about LP-Sasakian manifolds with coefficient $\alpha$, we refer ([7],[9],[12],[13]). LP-Sasakian manifolds with coefficient $\alpha$ are generalizations of LP-Sasakian manifolds. So our results are more general than the results of ([15]). Submanifolds with second fundamental form satisfying some parallelity and symmetry conditions have been studied in ([6]). Invariant submanifolds of LP-Sasakian manifolds have also been studied in ([14]) and in this paper it is proved that an

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invariant submanifold of an LP-Sasakian manifold is also LP-Sasakian.

The present paper is organized as follows : After the introduction in Section 1, we give the required preliminaries in Section 2. In Section 3 we show that if a 3-dimensional submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is invariant then it is totally geodesic. The converse is also true. Interestingly, in Section 4, we prove that the same is not true for 4-dimensional submanifold. We prove it by an example. It is also shown in Section 5 that in 4-dimensional case if we assume the submanifold is invariant and recurrent then it is totally geodesic. Totally umbilical submanifolds of LP-Sasakian manifolds with coefficient $\alpha$ have been studied in Section 6. The last section contains the study of submanifolds whose second fundamental forms satisfy some parallelity and pseudo symmetry conditions.

## 2. PRELIMINARIES

Let $\widetilde{M}$ be an $n$-dimensional real differentiable manifold of differentiability class $C^\infty$ endowed with a $C^\infty$-vector valued linear function $\phi$, a $C^\infty$-vector field $\xi$, an one form $\eta$ and a Lorentzian metric g of type $(0,2)$ such that for each p$\in\widetilde{M}$, the tensor $g_p$ : $T_p\widetilde{M}\times T_p\widetilde{M}\rightarrow$R is a non-degenerate inner product of signature $(-,+,+,+,+,......,+)$, where $T_p\widetilde{M}$ denotes the tangent vector space of $\widetilde{M}$ at p and $\mathbb{R}$ is the field of real numbers, which satisfies

$$\phi^2X = X + \eta(X)\xi, \qquad \eta(\xi) = -1 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(X,\xi) = \eta(X) \tag{2.2}$$

for all vector fields $X, Y$ tangent to $\widetilde{M}$. Such a structure ($\phi$,$\xi$,$\eta$,g) is termed as Lorentzian para contact structure. In Lorentzian para contact structure the following results hold :

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \tag{2.3}$$

$$\text{rank}\phi = (n-1), \tag{2.4}$$

A Lorentzian para contact manifold $\widetilde{M}$ is called Lorentzian para-Sasakian manifold or LP-Sasakian manifold if

$$(\widetilde{\nabla}_X\phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$\widetilde{\nabla}_X\xi = \phi X \tag{2.6}$$

for all $X, Y$ tangent to $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Levi-civita connection with respect to $g$.

In the Lorentzian para contact manifold $\widetilde{M}$, if the relations

$$\begin{aligned}(\widetilde{\nabla}_Z\Omega)(X,Y) &= \alpha[\{g(X,Y)+\eta(X)\eta(Z)\}\eta(Y) \\ &+ \{g(Y,Z)+\eta(Y)\eta(Z)\}\eta(X)],\end{aligned} \tag{2.7}$$

$$\Omega(X,Y) = (1/\alpha)(\widetilde{\nabla}_X\eta)(Y), \tag{2.8}$$

hold, where, $\Omega(X,Y) = \Omega(Y,X)$ and $\Omega(X,Y) = g(X,\phi Y)$ and $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to Lorentzian metric $g$ and $\alpha$ is a non-zero scalar function , then $\widetilde{M}$ is called an LP-Sasakian manifold with a coefficient $\alpha$. An LP-Sasakian manifold with a coefficient 1 is an LP-Sasakian manifold.

Let $M$ be a submanifold immersed in an $n$-dimensional Riemannian manifold $\widetilde{M}$, we denote by the same symbol $g$ induced metric on $M$. Let $TM$ be the tangent space of $M$ and $T^{\perp}M$ is the set of all vector fields normal to $M$. Then Gauss and Weingarten formulae are given by ([3])

$$\widetilde{\nabla}_XY = \nabla_XY + \sigma(X,Y), \tag{2.9}$$

$$\widetilde{\nabla}_XN = -A_NX + \nabla^{\perp}_XN \tag{2.10}$$

for any tangent vector fields $X, Y$ of $M$ and normal vector fields $N$ of $M$, where $\nabla^{\perp}$ is the connection in the normal bundle. The second fundamental form $\sigma$ and $A_N$ are related by

$$g(A_NX,Y) = g(\sigma(X,Y),N). \tag{2.11}$$

It is also noted that $\sigma(X,Y)$ is bilinear, and since $\nabla_{fX}\mathrm{Y}=f\nabla_X\mathrm{Y}$, for a $\mathrm{C}^{\infty}$ function $f$ on a manifold we have

$$\sigma(fX,Y) = f\sigma(X,Y). \tag{2.12}$$

Let us now recall the following:

**Definition 2.1.** *Let, $\widetilde{M}$ be an $n$-dimensional LP-Sasakian manifold with coefficient $\alpha$ and $M$ be a submanifold of $\widetilde{M}$. The submanifold $M$ of $\widetilde{M}$ is said to be invariant if the structure vector field $\xi$ is tangent to $M$, at every point of $M$ and $\phi X$ is tangent to $M$ for any vector field $X$ tangent to $M$,* *at every point on $M$, that is, $\phi TM \subset TM$ at every point on $M$.*

**Definition 2.2.** *A submanifold of an LP-Sasakian manifold with a coefficient $\alpha$ is called totally geodesic if $\sigma(X,Y) = 0$, for any $X, Y \in TM$.*
**Definition 2.3.** *The second fundamental form $\sigma$ is said to be recurrent, respectively, 2-recurrent if the following conditions hold :*

$$(\nabla_W \sigma)(Y,Z) = A(W)\sigma(Y,Z), \tag{2.13}$$

$$(\nabla_U \nabla_W \sigma)(Y,Z) = B(U,W)\sigma(Y,Z), \tag{2.14}$$

*where $A$ is a 1-form on $M$ and $B$ is 2-form on $M$.*

## 3. Invariant submanifolds of LP-Sasakian manifolds with coefficient $\alpha$

**Proposition 3.1.** *An invariant submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is also an LP-Sasakian manifold with coefficient $\alpha$.*
**Proof:** Let $\widetilde{M}$ be an LP-Sasakian manifold with coefficient $\alpha$. And also let, $M$ be an invariant submanifold of $\widetilde{M}$.
We shall prove that, $M$ is also an LP-Sasakian manifold with coefficient $\alpha$. Since, $\widetilde{M}$ is an LP-Sasakian manifold with coefficient $\alpha$, we get

$$\begin{aligned}(\widetilde{\nabla}_Z \Omega)(X,Y) &= \alpha[\{g(X,Z) + \eta(X)\eta(Z)\}\eta(Y) \\ &+ \{g(Y,Z) + \eta(Y)\eta(Z)\}\eta(X)].\end{aligned} \tag{3.1}$$

By covariant differentiation, we get

$$(\widetilde{\nabla}_Z \Omega)(X,Y) = \widetilde{\nabla}_Z \Omega(X,Y) - \Omega(\widetilde{\nabla}_Z X, Y) - \Omega(X, \widetilde{\nabla}_Z Y). \tag{3.2}$$

Again, by (2.9)

$$\widetilde{\nabla}_Z \Omega(X,Y) = \nabla_Z \Omega(X,Y) - \sigma(Z, \Omega(X,Y)), \tag{3.3}$$

$$\widetilde{\nabla}_Z X = \nabla_Z X - \sigma(Z,X). \tag{3.4}$$

Combining (3.2),(3.3) and (3.4), we get

$$(\widetilde{\nabla}_Z \Omega)(X,Y) = (\nabla_Z \Omega)(X,Y). \tag{3.5}$$

Using (3.1) and (3.4), we get

$$\begin{aligned}(\nabla_Z \Omega)(X,Y) &= \alpha[\{g(X,Z) + \eta(X)\eta(Z)\}\eta(Y) \\ &+ \{g(Y,Z) + \eta(Y)\eta(Z)\}\eta(X)].\end{aligned} \tag{3.6}$$

This shows that, the invariant submanifold $M$ is also LP-Sasakian manifold with coefficient $\alpha$. Hence, the proposition follows.

**Proposition 3.2.** *Let, $M$ be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$ with coefficient $\alpha$. Then there exists two differentiable orthogonal distributions $D$ and $D^{\perp}$ on $M$ such that*
$TM = D \oplus D^{\perp} \oplus < \xi >,$ *and* $\phi(D) \subset D^{\perp}$, $\phi(D^{\perp}) \subset D$.

*Proof.* For an invariant submanifold $M$, $\xi$ is tangent to $M$. Hence, we can write $TM = D^1 \oplus < \xi >$. Let $X_1 \in D^1$. Now $g(X_1, \phi X_1) = 0$ and $g(\xi, \phi X_1) = 0$, so $\phi X_1$ is orthogonal to $X_1$ and $\xi$. Consequently, it is possible to write $D^1 = D \oplus D^{\perp}$, where $X_1 \in D \subset D^1$ and $\phi X_1 \in D^{\perp} \subset D^1$. For $\phi X_1 \in D^{\perp}$, we note that
$\phi(\phi X_1) = \phi^2 X_1 = X_1 + \eta(X_1)\xi = X_1 \in D$.
Let, $\phi X_1 = X_2 \in D^{\perp}$. Hence for $X_1 \in D$, $\phi X_1 \in D^{\perp}$ and for $X_2 \in D^{\perp}$, $\phi X_2 \in D$. Hence, the proposition follows. □

**Proposition 3.3.** *For an invariant submanifold $M$ of an LP-Sasakian manifold $\widetilde{M}$ with a coefficient $\alpha$ , we have for the two differentiable tangent vector fields $X, Y$ of $M$*
$\sigma(X, \xi) = 0$,
$\sigma(X, \phi Y) = \phi\sigma(X, Y) = \sigma(\phi X, Y)$.

*Proof.* $\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$, Putting, $Y = \xi$, we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi). \tag{3.7}$$

Again, for LP-Sasakian manifold with coefficient $\alpha$, we get

$$\widetilde{\nabla}_X \xi = \alpha\phi X. \tag{3.8}$$

Combining (3.7) and (3.8) we get,

$$\alpha\phi X = \nabla_X \xi + \sigma(X, \xi). \tag{3.9}$$

Comparing tangential component and normal component, we get

$$\nabla_X \xi = \alpha\phi X \qquad \text{and} \qquad \sigma(X, \xi) = 0. \tag{3.10}$$

Again since, $\nabla_{fX} Y = f\nabla_X Y$, for a $C^{\infty}$ function $f$ on a manifold we have, $\sigma(X, \phi Y) = \phi\sigma(X, Y) = \sigma(\phi X, Y)$. This completes the proof. □

**Theorem 3.1.** *Every odd dimensional invariant submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is totally geodesic.*

*Proof.* Let us consider an odd dimensional invariant submanifold $M$ of an LP-Sasakian manifold $\widetilde{M}$ with coefficient $\alpha$. Then $M$ is also LP-Sasakian manifold with coefficient $\alpha$ by Proposition 3.1.
Now it is obvious that $\sigma(X,Y)$ satisfies

$$\phi^2\sigma(X,Y) = \sigma(X,Y) + \eta(\sigma(X,Y))\xi. \tag{3.11}$$

Let $X_1$, $Y_1 \in D$, then we have $\sigma(X_1,\phi Y_1)=\phi\sigma(X_1,Y_1)$.
Then $\phi\sigma(X_1,\phi Y_1)=\phi^2\sigma(X_1,Y_1)=\sigma(X_1,Y_1)+\eta(\sigma(X_1,Y_1))\xi$.
Since $\sigma(X_1,Y_1)\in(TM)^{\perp}$, $\sigma(X_1,Y_1)$ is orthogonal to $\xi\in TM$. Hence we obtain, $\eta(\sigma(X_1,Y_1))=0$. Thus we have $\sigma(\phi X_1,\phi Y_1)=\sigma(X_1,Y_1)$.
Let $\phi X_1=X_2$ , $\phi Y_1=Y_2$ . We note that $X_2=\phi X_1\in D^{\perp}$ and $Y_2=\phi Y_1\in D^{\perp}$. Then

$$\sigma(X_2,Y_2) = \sigma(X_1,Y_1) \tag{3.12}$$

for $X_1$ ,$Y_1\in D$ and $X_2$, $Y_2\in D^{\perp}$. Since $\sigma$ is bilinear, for $X_1$, $Y_1\in D$ and $X_2$ ,$Y_2\in D^{\perp}$, it follows that

$$\sigma(X_1 + X_2 + \xi, Y_1) = \sigma(X_1,Y_1) + \sigma(X_2,Y_1) + \sigma(\xi,Y_1), \tag{3.13}$$

$$\sigma(X_1 + X_2 + \xi, -Y_2) = -\sigma(X_1,Y_2) - \sigma(X_2,Y_2) - \sigma(\xi,Y_2), \tag{3.14}$$

$$\sigma(X_1 + X_2 + \xi, \xi) = \sigma(X_1,\xi) + \sigma(X_2,\xi) + \sigma(\xi,\xi). \tag{3.15}$$

Keeping in mind that $\sigma$(X,$\xi$)=0, for $X\in TM$. Using (3.13), (3.14) and (3.15) we get, by virtue of (3.12),

$$\sigma(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = \sigma(X_2,Y_1) - \sigma(X_1,Y_2). \tag{3.16}$$

Now, $TM = D\oplus D^{\perp}\oplus < \xi >$.
So, $U$=$X_1$+$X_2$+$\xi \in TM$ and $V$=$Y_1$-$Y_2$+$\xi \in TM$.
Thus, $\sigma(U,V)$=$\sigma(X_2,Y_1)$-$\sigma(X_1,Y_2)$.
From the above equation it follows that
$\phi\sigma(U,V)$=$\sigma(X_2,\phi Y_1)$-$\sigma(\phi X_1,Y_2)$=$\sigma(X_2,Y_2)$-$\sigma(X_2,Y_2)$=0.
The above equation gives $\phi^2\sigma(U,V)$=0. Consequently, $\sigma(U,V)$=0. □

Now we shall show that the converse is true irrespective of dimension.
**Theorem 3.2.** *Every totally geodesic submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is invariant.*

*Proof.* Let, the submanifold be totally geodesic. So, $\sigma(X,Y)$=0 for any tangent vector fields $X$, $Y$ of $M$. Now we know that

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y). \tag{3.17}$$

Putting $Y = \xi$, we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi). \tag{3.18}$$

Again, for LP-Sasakian manifold with coefficient $\alpha$, we get

$$\widetilde{\nabla}_X \xi = \alpha\phi X. \tag{3.19}$$

Since $\sigma$(X,$\xi$)=0, then from (3.18) and (3.19) we get

$$\alpha\phi X = \nabla_X \xi. \tag{3.20}$$

From (3.20) it is clear that $\phi$X$\in TM$. So, the submanifold is invariant. □

## 4. Examples

In this section we would like to construct an example of a four-dimensional LP-Sasakian manifold with coefficient $\alpha$ and there on an example of two-dimensional invariant submanifold of the manifold. We shall show that every even dimensional submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is not necessarily totally geodesic.

Let us consider the 4-dimensional manifold $\widetilde{M}$=$\{(x,y,z,w)\in\mathbb{R}^4 | w \neq 0\}$, where $(x,y,z,w)$ are the standard coordinates in $\mathbb{R}^4$. The vector fields

$e_1 = w(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, $e_2 = w\frac{\partial}{\partial y}$, $e_3$= $w(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$, $e_4 = w^3\frac{\partial}{\partial w}$

are linearly independent at each point of $\widetilde{M}$. Let, $g$ be the Lorentzian metric defined by

$g(e_1,e_1)$=$g(e_2,e_2)$=$g(e_3,e_3)$=1, $g(e_4,e_4)$ = -1.

$g(e_i,e_j)$=0 for $i\neq j$ and $i,j = 1,2,3,4$.

Let, $\eta$ be the 1-form defined by

$\eta(Z)$=$g(Z,e_4)$ for any $Z\in \chi(\widetilde{M})$. Let, $\phi$ be the (1,1) tensor field defined by

$\phi(e_1) = e_1$, $\phi(e_2) = e_2$, $\phi(e_3) = e_3$, $\phi(e_4) = 0$.

Then using the linearity of $\phi$ and $g$ we have

$$\eta(e_4) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_4,$$

$$g(\phi Z, \phi W) = g(Z,W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(\widetilde{M})$.

Then for $e_4 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines a Lorentzian para contact structure on $\widetilde{M}$. Let, $\widetilde{\nabla}$ be the Levi-Civita connection with respect to the Lorentzian metric $g$. Then we have

$[e_1, e_2] = \text{-}we_2$, $[e_1, e_3] = \text{-}we_2$, $[e_1, e_4] = \text{-}w^2 e_1$,
$[e_2, e_3] = 0$, $[e_2, e_4] = \text{-}w^2 e_2$, $[e_3, e_4] = \text{-}w^2 e_3$.

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate the following

$\widetilde{\nabla}_{e_1} e_1 = \text{-}w^2 e_4$, $\widetilde{\nabla}_{e_1} e_2 = \text{-}\frac{w}{2} e_3$, $\widetilde{\nabla}_{e_1} e_3 = \text{-}\frac{w}{2} e_2$, $\widetilde{\nabla}_{e_1} e_4 = \text{-}w^2 e_1$,
$\widetilde{\nabla}_{e_2} e_1 = we_2 + \frac{w}{2} e_3$, $\widetilde{\nabla}_{e_2} e_2 = w^2 e_4\text{-}we_1$, $\widetilde{\nabla}_{e_2} e_3 = \text{-}\frac{w}{2} e_1$, $\widetilde{\nabla}_{e_2} e_4 = \text{-}w^2 e_2$,
$\widetilde{\nabla}_{e_3} e_1 = \frac{w}{2} e_2$, $\widetilde{\nabla}_{e_3} e_2 = \text{-}we_1$, $\widetilde{\nabla}_{e_3} e_3 = w^2 e_4$, $\widetilde{\nabla}_{e_3} e_4 = \text{-}w^2 e_3$,
$\widetilde{\nabla}_{e_4} e_1 = 0$, $\widetilde{\nabla}_{e_4} e_2 = 0$, $\widetilde{\nabla}_{e_4} e_3 = 0$, $\widetilde{\nabla}_{e_4} e_4 = 0$.

From the above it can be easily seen that $\widetilde{M}^4(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold with a coefficient $\alpha = \text{-}w^2 \neq 0$.

Let, $f$ be an isometric immersion from $M$ to $\widetilde{M}$ defined by $f(y, w) = (0, y, 0, w)$. Let, $M = \{(y, w) \in \mathbb{R}^2 \mid w \neq 0\}$, where $(y, w)$ are the standard coordinates in $\mathbb{R}^2$.

The vector fields $e_2 = w\frac{\partial}{\partial y}$ and $e_4 = w^3 \frac{\partial}{\partial w}$ are linearly independent at each point of $M$.

Let, $g$ be the Lorentzian metric defined by

$g(e_2{,}e_2) = 1$, $g(e_2{,}e_4) = 0$, $g(e_4{,}e_4) = \text{-}1$.

Let, $\eta$ be the 1-form defined by $\eta(Z){=}g(Z{,}e_4)$ for any vector field $Z$ tangent to $M$.

Let, $\phi$ be the (1,1) tensor field defined by $\phi(e_2) = e_2$ and $\phi(e_4) = 0$.

Then, using the linearity of $\phi$ and $g$ we have

$$\eta(e_4) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_4,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W)$$

for any vector fields $Z, W$ tangent to $M$.

Then for $e_4 = \xi$, the structure $M(\phi, \xi, \eta, g)$ defines a Lorentzian para contact structure on $M$. Let, $\nabla$ be the Levi-Civita connection on $M$ with respect to the Lorentzian metric $g$. Then we have $[e_2, e_4] = \text{-}w^2 e_2$.

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate the following :

$\nabla_{e_2} e_2 = \text{-}w^2 e_4$, $\nabla_{e_2} e_4 = \text{-}w^2 e_2$,

$\nabla_{e_4} e_2 = 0, \qquad \nabla_{e_4} e_4 = 0.$

We see that the $M(\phi, \xi, \eta, g)$ structure satisfies the formula

$\nabla_X \xi = (-w^2)\phi X = \alpha\phi X$. Hence, $M(\phi, \xi, \eta, g)$ is a two-dimensional LP-Sasakian manifold with a coefficient $\alpha = -w^2 \neq 0$. It is obvious that the manifold $M$ under consideration is a submanifold of the manifold $\widetilde{M}$. Let us take $X \in TM$ , then we can write $X = \lambda e_2 + \mu e_4$, where, $\lambda$, $\mu$ are two scalars. Now, $\phi X = \phi(\lambda e_2 + \mu e_4) = \lambda\phi(e_2) + \mu\phi(e_4) = \lambda e_2 + 0 = \lambda e_2 \in TM$. Hence, the two-dimensional submanifold is invariant submanifold. Let, $U = \lambda_1 e_2 + \lambda_2 e_4 \in TM$ and $V = \mu_1 e_2 + \mu_2 e_4 \in TM$ where, $\lambda_i$ and $\mu_i$ are scalars, $i$=1,2 . Then

$$\begin{aligned} \sigma(U, V) &= \sigma(\lambda_1 e_2 + \lambda_2 e_4, \mu_1 e_2 + \mu_2 e_4) \\ &= \lambda_1\mu_1\sigma(e_2, e_2) + \lambda_1\mu_2\sigma(e_2, e_4) + \lambda_2\mu_1\sigma(e_4, e_2) + \lambda_2\mu_2\sigma(e_4, e_4). \end{aligned}$$

From the values of $\widetilde{\nabla}_{e_i} e_j$ and $\nabla_{e_i} e_j$ calculated before and from the relation $\sigma(e_i, e_j) = \widetilde{\nabla}_{e_i} e_j - \nabla_{e_i} e_j$ , we see that, $\sigma(e_2, e_2) = 2\mathrm{w}^2 e_4 - \mathrm{w} e_1 \neq 0$. So, $\sigma(U, V) \neq 0$, for all $U, V \in TM$.

So, the submanifold is not totally geodesic.

**Remark 4.1.** *From the above example it is seen that the Theorem 3.1 is not true for 4-dimensional manifold, the Theorem 3.1 is true only for odd dimensional manifold. So, every even dimensional invariant submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is not necessarily totally geodesic. In the next section we will prove that in even-dimensional case if we assume the submanifold is invariant as well as recurrent then it will be totally geodesic.*

## 5. Recurrent submanifolds of LP-Sasakian manifolds with coefficient $\alpha$

A submanifold is called recurrent, if its second fundamental form is recurrent. To prove the main theorem of this section, first we prove the following lemma.

**Lemma 5.1.** *Let, $M$ be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$ with coefficient $\alpha$. Then*

$$\sigma(X, \xi) = 0 \quad \text{and} \quad \nabla_X \xi = \alpha\phi X.$$

*Proof.* From equation (2.9), we get

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{5.1}$$

Putting, $Y{=}\xi$, we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi). \tag{5.2}$$

Again, for LP-Sasakian manifold with coefficient $\alpha$, we get

$$\widetilde{\nabla}_X \xi = \alpha\phi X. \tag{5.3}$$

Combining (5.2) and (5.3), we get

$$\alpha\phi X = \nabla_X \xi + \sigma(X, \xi). \tag{5.4}$$

Since, the submanifold is invariant then $\phi X \in TM$. Now comparing the tangential and normal component, we have

$$\sigma(X, \xi) = 0 \quad \text{and} \quad \nabla_X \xi = \alpha\phi X.$$ □

**Theorem 5.1.** Let, $M$ be a submanifold of an LP-Sasakian manifold $\widetilde{M}$ with a coefficient $\alpha$ tangent to $\xi$. If $M$ is invariant and recurrent then $M$ is totally geodesic.

*Proof.* For recurrent submanifolds we get

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = A(X)\sigma(Y, Z), \tag{5.5}$$

where $A$ is a 1-form on $M$. Again, by covariant differentiation we have

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}{}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{5.6}$$

Combining (5.5) and (5.6) we get

$$\nabla^{\perp}{}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = A(X)\sigma(Y, Z). \tag{5.7}$$

Taking $Z = \xi$ in (5.7) we have

$$\nabla^{\perp}{}_X \sigma(Y, \xi) - \sigma(\nabla_X Y, \xi) - \sigma(Y, \nabla_X \xi) = A(X)\sigma(Y, \xi). \tag{5.8}$$

Using Lemma 5.1, in equation (5.8) we have

$$\sigma(Y, \nabla_X \xi) = 0. \tag{5.9}$$

Again, by Lemma 5.1, $\sigma(Y, \alpha\phi X) = 0$. So for $\alpha \neq 0$, $\sigma(X, Y) = 0$. Hence $M$ is totally geodesic.

The converse part of the theorem is also true. The proof of the converse part is trivial. □

**Remark 5.1.** *In* [16], *the authors studied some submanifolds of Kenmotsu manifolds with second fundamental form satisfying some conditions. All results of that paper has analogue in case of submanifolds of LP-Sasakian manifolds with coefficient $\alpha$. All the proofs are similar to the proof of the above theorem.*

## 6. Totally umbilical submanifolds of LP-sasakian manifolds with coefficient $\alpha$

**Definition 6.1.** *A submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is called totally umbilical if it satisfies*

$$\sigma(X,Y) = g(X,Y)H. \tag{6.1}$$

*Here $\sigma$ is second fundamental form of the submanifold, $g$ is the induced metric, $H$ is mean curvature vector. $X, Y$ are tangent to$M$* [5].

In this section, we shall prove the following:

**Theorem 6.1.** *A totally umbilical invariant submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is totally geodesic.*

*Proof.* From Codazzi equation [5] we get

$$\widetilde{R}^{\perp}(X,Y)Z = g(Y,Z)\nabla^{\perp}_{X}H - g(X,Z)\nabla^{\perp}_{Y}H. \tag{6.2}$$

Here $\widetilde{R}$ is the curvature tensor of the ambient manifold and $\widetilde{R}^{\perp}$ is its normal part. Putting $Z = \xi$ in (6.2), we obtain

$$\widetilde{R}^{\perp}(X,Y)\xi = \eta(Y)\nabla^{\perp}_{X}H - \eta(X)\nabla^{\perp}_{Y}H. \tag{6.3}$$

Now from ([9]), we know

$$\widetilde{R}(X,Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y). \tag{6.4}$$

Here $\rho$ is a scalar. Since the right hand side of (6.4) is tangential to the submanifold, we get

$$\widetilde{R}^{\perp}(X,Y)\xi = 0. \tag{6.5}$$

By virtue of (6.3) and (6.5)

$$\eta(Y)\nabla^{\perp}_{X}H = \eta(X)\nabla^{\perp}_{Y}H. \tag{6.6}$$

Replacing $X$ by $\phi X$ in the above equation, we have

$$\nabla^{\perp}_{\phi X} H = 0. \tag{6.7}$$

Again by covariant differentiation of (6.1), we see that

$$\nabla^{\perp}_{W}\sigma(X,Y) - \sigma(\nabla_W X, Y) - \sigma(X, \nabla_W Y) = g(X,Y)\nabla^{\perp}_{W} H. \tag{6.8}$$

If the submanifold is invariant, then putting $Y = \xi$ in the above equation and using Proposition 3.3, equations (6.7) and (6.8) we immediately get $\sigma(X, \alpha\phi W) = 0$. Hence, for $\alpha \neq 0$, $\sigma(X, W) = 0$. So, the submanifold is totally geodesic. This completes the proof. □

## 7. Submanifolds of LP-Sasakian manifolds with coefficient $\alpha$ with parallel semi parallel and pseudo parallel second fundamental forms

**Definition 7.1.** *The second fundamental form $\sigma$ of a submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is called parallel if $\nabla\sigma$=0.*
**Definition 7.2.** *The second fundamental form $\sigma$ of a submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is called semi parallel if $\widetilde{R}(X,Y).\sigma = 0$.*
**Definition 7.3.** *The second fundamental form $\sigma$ of a submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is called pseudo parallel if*

$$(\widetilde{R}(X,Y).\sigma)(U,V) = fQ(g,\sigma)(X,Y,U,V),$$

*where $f$ denotes a real function on $\widetilde{M}$. Here $Q$ is given by*

$$\begin{aligned} Q(E,T)(X,Y,Z,W) &= -(X\wedge_E Y)T(Z,W) - T((X\wedge_E Y)Z,W) \\ &- T(Z,(X\wedge_E Y)W), \end{aligned}$$

$$(X\wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y$$

*for a (0,2) tensor $E$ and an arbitrary tensor $T$.* For details about parallel and pscudo symmetric tensor, we refer ([10], [11]).

Submanifolds of trans-Sasakian manifolds with such properties have been studied in the paper ([6]). Following the similar method of the paper ([6]) we obtain the following
**Theorem 7.1.** *An invariant submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is totally geodesic if and only if the second fundamental form of the submanifold is parallel.*

*Proof.* Since $\sigma$ is parallel, we have
$(\nabla_W \sigma)(X, Y) = 0$, which implies

$$\nabla^{\perp}_{W}\sigma(X, Y) - \sigma(\nabla_W X, Y) - \sigma(X, \nabla_W Y) = 0. \tag{7.1}$$

Putting $Z = \xi$ in the above equation and applying (3.11) we obtain

$$\sigma(X, \nabla_W \xi) = 0. \tag{7.2}$$

So from (3.9) and the above equation (7.2) we obtain

$$\sigma(X, \alpha\phi W) = 0. \tag{7.3}$$

Hence, for $\alpha \neq 0$,
$\sigma(X, W) = 0$. So the submanifold is totally geodesic. The converse part is trivial. Hence the result. □

**Theorem 7.2.** *An invariant submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is totally geodesic if and only if the second fundamental form of the submanifold is semi-parallel.*

*Proof.* Since $\sigma$ is semi-parallel, we have $(\widetilde{R}(X, Y).\sigma)(U, V)$=0, which implies

$$\widetilde{R}^{\perp}(X, Y)\sigma(U, V) - \sigma(\widetilde{R}(X, Y)U, V) - \sigma(U, \widetilde{R}(X, Y)V) = 0. \tag{7.4}$$

Putting $V = \xi = Y$ and using Proposition 3.3 we get from equation (7.4)

$$\sigma(U, \widetilde{R}(X, \xi)\xi) = 0. \tag{7.5}$$

Now from [9], we know

$$\widetilde{R}(X, Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y). \tag{7.6}$$

Here $\rho$ is a scalar. The above equation implies

$$\widetilde{R}(X, \xi)\xi = -(\alpha^2 - \rho)\phi^2 X. \tag{7.7}$$

From (7.5) and (7.7) we get

$$\sigma(U, \phi^2 X) = 0, \tag{7.8}$$

provided $(\alpha^2 - \rho) \neq 0$. Hence from (7.8) we have $\sigma(X, U) = 0$. Thus the submanifold is totally geodesic. The converse part is trivial. Hence the result. □

**Theorem 7.3.** *An invariant submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is totally geodesic if and only if the second fundamental form of the submanifold is pseudo parallel, provided $(\alpha^2 - \rho - f) \neq 0$.*

*Proof.* Since $\sigma$ is pseudo parallel, we have

$(\widetilde{R}(X, Y).\sigma)(U, V) = fQ(g, \sigma)(X, Y, U, V)$, which implies

$$\begin{aligned}\widetilde{R}^{\perp}(X, Y)\sigma(U, V) \;&-\; \sigma(\widetilde{R}(X, Y)U, V) - \sigma(U, \widetilde{R}(X, Y)V) \\ &= \; -f\{g(Y, \sigma(U, V))X - g(X, \sigma(U, V))Y \\ &+ \; \sigma(g(Y, U)X - g(X, U)Y, V) \\ &+ \; \sigma(U, g(Y, V)X - g(X, V)Y)\}. \end{aligned} \tag{7.9}$$

Putting $V = \xi = Y$ in equation (7.9) and applying Proposition 3.3 we obtain

$$\sigma(U, \widetilde{R}(X, \xi)\xi) = f\sigma(U, g(\xi, \xi)X - g(X, \xi)\xi). \tag{7.10}$$

Now from [9], we know

$$\widetilde{R}(X, Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y). \tag{7.11}$$

Here $\rho$ is a scalar. The above equation implies

$$\widetilde{R}(X, \xi)\xi = -(\alpha^2 - \rho)\phi^2 X. \tag{7.12}$$

From (7.10) and (7.12) we get

$$\sigma(U, \phi^2 X) = 0, \tag{7.13}$$

provided $(\alpha^2 - \rho - f) \neq 0$. Hence from (7.13) we have $\sigma(X, U) = 0$. So, the submanifold is totally geodesic. The converse part is trivial. □

**Conclusion.** *Any totally geodesic submanifold of an LP-Sasakian manifold with coefficient $\alpha$ is invariant. The converse is true only for odd dimensional submanifolds. Some other necessary and sufficient conditions for invariant submanifolds of LP-Sasakian manifolds with coefficient $\alpha$ to be totally geodesic are that the second fundamental form of the submanifold should be any one of the following types*

(1) *parallel*
(2) *semi-parallel*
(3) *pseudo parallel*
(4) *recurrent.*

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# CONVOLUTION PRODUCT OF FUNCTIONS OF GENERALIZED WIENER CLASS

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Abstract. Let $L^1(\overline{\mathbb{T}}^N)$ be the class of all $2\pi$-periodic, Lebesgue integrable complex functions on $\overline{\mathbb{T}}^N$, where $\mathbb{T} = [0, 2\pi)$ is the one−dimensional torus. Here, for any $f \in L^1(\overline{\mathbb{T}}^N)$ and $g \in (\Lambda^1, ..., \Lambda^N)^* BV^{(p_1,...,p_N)}(\overline{\mathbb{T}}^N)$, where $1 \leq p_1 \leq ... \leq p_N < \infty$ and $N > 1$, it is observed that their convolution product $f * g \in (\Lambda^1, ..., \Lambda^N)^* BV^{(p_1,...,p_N)}(\overline{\mathbb{T}}^N)$. Thus, the convolution product inherits the generalized Wiener class property and the class $\bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$ can be regarded as a module over the ring $L^1(\overline{\mathbb{T}}^N)$.

## 1. Introduction

In 1961, Benedek and Panzone [1] finer the classical structure of Banach space $L^p(\mathbb{R}^N)$ $(p \geqslant 1)$ by the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^N)$, replacing the constant exponent $p$ of the $L^p$-norm by an exponent vector $\vec{p} := (p_1, p_2, ..., p_N) \in [1, \infty]^N$.

**Definition 1.1.** The mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^N)$ $(\vec{p} := (p_1, p_2, ..., p_N) \in [1, \infty]^N)$ is defined as the set of all measurable functions $f$ such that
$\|f\|_{L^{\vec{p}}(\mathbb{R}^N)}$

$$= \left( \int_{\mathbb{R}} \cdots \left( \int_{\mathbb{R}} |f(x_1, \ldots, x_N)|^{p_1} \, dx_1 \right)^{\frac{p_2}{p_1}} \ldots \, dx_N \right)^{\frac{1}{p_N}} < \infty.$$

Benedek and Panzone [1, Theorem 1] proved that the mixed Lebesgue space $(L^{\vec{p}}(\mathbb{R}^N), \| \, \|_{L^{\vec{p}}(\mathbb{R}^N)})$ is a Banach space and $(L^{\vec{p}}(\mathbb{R}^N))^* = (L^{\vec{q}}(\mathbb{R}^N)$, where $\vec{p} \in (1, \infty)^N$, $\vec{q} := (q_1, q_2, ..., q_N)$ denote the conjugate exponent of $\vec{p}$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $\forall i \in \{1, \ldots, N\}$. Moreover, Holder's inequality holds for

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the mixed norm [3, Lemma 1].

If $\vec{p} = (p, p, ..., p)$ with some $p \in [1, \infty]$ then the space coincides with the classical Banach space $L^p(\mathbb{R}^N)$.

Property of various function spaces with mixed norm are studied, like Besov spaces, Orlicz spaces, Sobolev spaces, Hardy spaces and Bessel potential spaces. Recently, J. Zhao, Wei-Shih Du and Y. Chen [5, Theorem 4] obtained mixed norm convolution inequality

$$||f{*}g||_{L^{\vec{p}}(\mathbb{R}^N)} \leq ||f||_{L^{\vec{p}}(\mathbb{R}^N)}||g||_{L^1(\mathbb{R}^N)},\ for\ any\ f \in L^{\vec{p}}(\mathbb{R}^N)\ and\ g \in L^1(\mathbb{R}^N).$$

**Definition 1.2.** For any $f, g \in L^1(\overline{\mathbb{T}}^N)$ and for any $(x_1, \ldots, x_N) \in \overline{\mathbb{T}}^N$, the convolution product of $f$ and $g$ (that is $f * g$) is defined as
$(f * g)(x_1, \ldots, x_N) =$

$$\frac{1}{(2\pi)^N} \int \cdots \int_{\overline{\mathbb{T}}^N} f(s_1, \ldots, s_N) g(x_1 - s_1, \ldots, x_N - s_N)\ ds_1 \ldots\ ds_n.$$

It is observed that the Banach space (C[a,b], $\| \ \|_\infty$) as well as spaces of functions of generalized bounded variations with compact support and a suitable variation norm, are closed under the convolution product. Moreover [4], $L^1(\overline{\mathbb{T}}^N) * E = \{f * g : \ f \in L^1(\overline{\mathbb{T}}^N)\ and\ g \in E\ over\ \overline{\mathbb{T}}^N\} \subset E$ for $E = \Lambda BV^{(p)}(\overline{\mathbb{T}}^N)$.

Here, we have extended this result for more generalized Wiener class $(\Lambda^1, ..., \Lambda^N)BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$. For the simplicity, first we discuss the result for functions of two variables.

In the sequel, $\mathbb{L}$ is the class of non-decreasing sequences $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n^{-1}$ diverges. Throughout the paper, $C$ represents a constant which would vary from time to time.

## 2. New result for functions of two variables

Consider a function $f$ on a compact subset of $\mathbb{R}^k$. For $k = 1$ and $I = [a, b]$, define $\Delta f_a^b = f(I) = f(b) - f(a)$. For $k = 2$, $I = [a, b]$ and $J = [c, d]$, define

$$\Delta f_{(a,c)}^{(b,d)} = f(I \times J) = f(I, d) - f(I, c) = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

**Definition 2.1.** Given a pair of sequences $\bigwedge = (\Lambda^1, \Lambda^2)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^{\infty} \in \mathbb{L}$, for $k = 1,\ 2$; and given an index vector $\vec{p} = (p, q) \in [1, \infty]^2$. A complex−valued function $f$ defined on $\overline{\mathbb{T}}^2$ is said to be of $\vec{p}-\bigwedge-$ bounded variation (that is, $f \in \bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$) if

$$V_{\bigwedge_{\vec{p}}}(f, \overline{\mathbb{T}}^2) = \sup_{P = P_1 \times P_2} \left\{ V_{\bigwedge_{\vec{p}}}(f, \overline{\mathbb{T}}^2, P) \right\} < \infty,$$

where $P_1 : 0 = x_0 < x_1 < \cdots < x_m = 2\pi$, $P_2 : 0 = y_0 < y_1 < \cdots < y_n = 2\pi$ and

$$V_{\bigwedge_{\vec{p}}}(f, \overline{\mathbb{T}}^2, P) = \left( \sum_j \frac{\left( \sum_i \frac{|\Delta f(x_i, y_j)|^p}{\lambda_i^1} \right)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}}, \ \textit{in which}$$

$$\Delta f(x_i, y_j) = f([x_i, x_{i+1}] \times [y_j, y_{j+1}]).$$

For any two arbitrary functions $g$ and $h$ need not be bounded (or need not be measurable) from $\overline{\mathbb{T}}$ into $\mathbb{R}$. Define the function $f : \overline{\mathbb{T}}^2 \to \mathbb{R}$ as $f(x, y) = g(x) + h(y)$, then $V_{\bigwedge_{\vec{p}}}(f, \overline{\mathbb{T}}^2) = 0$ implies $f \in \bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$. Thus, a function $f \in \bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$ need not be bounded (or need not be measurable).

If a measurable function $f \in \bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$ is such that the marginal functions $f(0, .) \in \Lambda^2 BV^{(q)}(\overline{\mathbb{T}})$ and $f(., 0) \in \Lambda^1 BV^{(p)}(\overline{\mathbb{T}})$ then $f$ is said to be of $\vec{p}-\bigwedge^*-$ bounded variation (that is, $f \in \bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$). Recently [2, Theorem 3.1], it is observed that the space $\bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$), the class of functions of $\vec{p} - \bigwedge^* -$ bounded variation, is a Banach space with a suitable variation norm.

**Theorem 2.2.** *If $f \in L^1(\overline{\mathbb{T}}^2)$ and $g \in \bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$, where $\vec{p} = (p, q), 1 \leq p \leq q < \infty$; then $f * g \in \bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$.*

*Proof.* Since, $g \in \bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$, from the result [2, Lemma 3.1] $g$ is bounded on $\overline{\mathbb{T}}^2$. Hence, $g \in L^{\vec{p}}(\overline{\mathbb{T}}^2)$.

Consider a partition $P = P_1 \times P_2$ of the given rectangle $\overline{\mathbb{T}}^2$, where

$$P_1 : 0 = x_0 < x_1 < \cdots < x_m = 2\pi \textit{ and } P_2 : 0 = y_0 < y_1 < \cdots < y_n = 2\pi.$$

For any $\mathbf{x}_{ij} = (x_i, y_j) \in P$, from Hölder's inequality, we have

$|\Delta f * g(\mathbf{x}_{ij})|^p = |\frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x,y)\Delta g(x_i - x, y_j - y)dx\ dy|^p$

$$\leq \left(\frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} |f(x,y)|^{1-1/p}|f(x,y)|^{1/p}|\Delta g(x_i - x, y_j - y)|dx\ dy\right)^p$$

$$\leq C\ \left(||f||_{1,\overline{\mathbb{T}}^2}\right)^{p-1}\left(\int\int_{\overline{\mathbb{T}}^2} |f(x,y)||\Delta g(x_i - x, y_j - y)|^p\ dx\ dy\right).$$

Dividing both the sides of above inequality by $\lambda_i^1$ and then summing over $i = 1$ to $m$, we get

$\sum_{i=1}^m \frac{|\Delta f * g(\mathbf{x}_{ij})|^p}{\lambda_i^1}$

$$\leq C\ \left(\int\int_{\overline{\mathbb{T}}^2} |f(x,y)|\sum_{i=1}^m \frac{|\Delta g(x_i - x, y_j - y)|^p}{\lambda_i^1}\ dx\ dy\right)$$

For $p = q$ theorem can be easily follows from the above inequality.
If $p < q$, then for $r = \frac{q}{p} > 1$. In view of Hölder's inequality, , we have

$\left(\sum_{i=1}^m \frac{|\Delta f * g(\mathbf{x}_{ij})|^p}{\lambda_i^1}\right)^r$

$$\leq\ C\ \left(\int\int_{\overline{\mathbb{T}}^2} |f(x,y)|^{1-1/r}|f(x,y)|^{1/r}\sum_{i=1}^m \frac{|\Delta g(x_i - x, y_j - y)|^p}{\lambda_i^1}\ dx\ dy\right)^r$$

$$\leq\ C\left(\int\int_{\overline{\mathbb{T}}^2} |f(x,y)|\left(\sum_{i=1}^m \frac{|\Delta g(x_i - x, y_j - y)|^p}{\lambda_i^1}\right)^r\ dx\ dy\right).$$

Dividing both the sides of above inequality by $\lambda_j^2$ and then summing over $j = 1$ to $n$, we have

$\sum_{j=1}^n \left(\frac{\left(\sum_{i=1}^m \frac{|\Delta f * g(\mathbf{x}_{ij})|^p}{\lambda_i^1}\right)^r}{\lambda_j^2}\right)$

$$\leq\ C\left(\int\int_{\overline{\mathbb{T}}^2} |f(x,y)|\sum_{j=1}^n\left(\frac{\left(\sum_{i=1}^m \frac{|\Delta g(x_i - x, y_j - y)|^p}{\lambda_i^1}\right)^r}{\lambda_j^2}\right)\ dx\ dy\right)$$

$$\leq\ C\ \left(V_{\Lambda_{(p,q)}}(g, \overline{\mathbb{T}}^2)\right)^q.$$

Thus, $V_{\Lambda_{(p,q)}}(f * g, \overline{\mathbb{T}}^2) < \infty$.

It is easy to show that, $g(0,.) \in \Lambda^2 BV^{(q)}(\overline{\mathbb{T}})$ and $g(.,0) \in \Lambda^1 BV^{(p)}(\overline{\mathbb{T}})$ imply $(f*g)(0,.) \in \Lambda^2 BV^{(q)}(\overline{\mathbb{T}})$ and $(f*g)(.,0) \in \Lambda^1 BV^{(p)}(\overline{\mathbb{T}})$ respectively. Hence, the result follows. □

Theorem 2.2 with $p = q$ reduces to [4, Theorem 2.4] as a particular case.

**Remark 2.3.** Since $L^1(\overline{\mathbb{T}}^2)$ is a ring, for the convolution product as a ring product. From the above theorem, the class $\bigwedge^* BV^{(p,q)}(\overline{\mathbb{T}}^2)$ can be regarded as a module over the ring $L^1(\overline{\mathbb{T}}^2)$.

## 3. Extension of the result for functions of several variables

Let $I^k = [a_k, b_k] \subset \mathbb{R}$, for $k = 1, 2, \cdots, N$. In the above Section 2, we defined $f(I^1)$ and $f(I^1 \times I^2)$ for a function of one variable and a function of two variables respectively. Similarly, for a function $f$ on $\mathbb{R}^N$, by induction, defining the expression $f(I^1 \times \cdots \times I^{N-1})$ for a function of $N-1$ variables, one gets

$$f(I^1 \times \cdots \times I^N) = f(I^1 \times \cdots \times I^{N-1}, b_N) - f(I^1 \times \cdots \times I^{N-1}, a_N).$$

Observe that, $f(I^1 \times \cdots \times I^N)$ can also be expressed as

$$f(I^1 \times \cdots \times I^N) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \textstyle\sum_{\mathbf{c}} k(\mathbf{c}) f(\mathbf{c}),$$

where $\mathbf{a} = (a_1, a_2, \cdots, a_N)$, $\mathbf{b} = (b_1, b_2, \cdots, b_N) \in \mathbb{R}^N$, the summation is over all $\mathbf{c} = (c_1, c_2, \cdots, c_N) \in \mathbb{R}^N$ such that $c_i \in \{a_i, b_i\}$, for $i = 1, \cdots, N$, and for any such $\mathbf{c}$, $k(\mathbf{c}) = k_1 \cdots k_N,\ in\ which,\ for\ 1 \le i \le N,$

$$k_i = \begin{cases} 1, & \text{if } c_i = b_i, \\ -1, & \text{if } c_i = a_i. \end{cases}$$

For $N = 1$, we get

$$f(I^1) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{a_1}^{b_1} = \textstyle\sum_{c_1} k(\mathbf{c}) f(\mathbf{c}) = f(b_1) - f(a_1).$$

For $N = 2$,we get

$$f(I^1 \times I^2) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{(a_1,a_2)}^{(b_1,b_2)} = \textstyle\sum_{(c_1,c_2)} k(\mathbf{c}) f(\mathbf{c})$$

$$= f(b_1, b_2) + f(a_1, a_2) - f(b_1, a_2) - f(a_1, b_2).$$

Similarly, for $N = 3$, we get

$$f(I^1 \times I^2 \times I^3) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{(a_1,a_2,a_3)}^{(b_1,b_2,b_3)} = \sum_{(c_1,c_2,c_3)} k(\mathbf{c}) f(\mathbf{c})$$

$$= f(b_1, b_2, b_3) + f(b_1, a_2, a_3) + f(a_1, b_2, a_3) + f(a_1, a_2, b_3)$$
$$-f(b_1, b_2, a_3) - f(a_1, b_2, b_3) - f(b_1, a_2, b_3) - f(a_1, a_2, a_3).$$

Given $\bigwedge = (\Lambda^1, \cdots, \Lambda^N)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^{\infty} \in \mathbb{L}$, for $k = 1, \cdots, N$, and $\vec{p} := (p_1, p_2 \cdots, p_N) \in [1, \infty]$, a complex valued function $f$ defined on $\overline{\mathbb{T}}^N$ is said to be of $\vec{p} - \bigwedge -$ bounded variation (that is, $f \in \bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$) if

$$V_{\bigwedge_{\vec{p}}}(f, \overline{\mathbb{T}}^N) = \sup_{P = P_1 \times \cdots \times P_N} \left\{ V_{\bigwedge_{\vec{p}}}(f, \overline{\mathbb{T}}^N, P) \right\} < \infty,$$

where $P_i : 0 = x_i^0 < x_i^1 < \cdots < x_i^{k_i} = 2\pi;$ *for all* $i = 1, 2, \cdots, N$, and

$$V_{\bigwedge_{\vec{p}}}(f, \overline{\mathbb{T}}^N, P)$$
$$= \left( \sum_{s_N} \frac{1}{\lambda_{s_N}^N} \left( \cdots \left( \sum_{s_2} \frac{1}{\lambda_{s_2}^2} \left( \sum_{s_1} \frac{|\Delta f(x_1^{s_1}, \cdots, x_N^{s_N})|^{p_1}}{\lambda_{s_1}^1} \right)^{\frac{p_2}{p_1}} \right)^{\frac{p_3}{p_2}} \cdots \right)^{\frac{p_{N-1}}{p_{N-2}}} \right)^{\frac{1}{p_N}},$$

in which $\Delta f(x_1^{s_1}, \cdots, x_N^{s_N}) = f([x_1^{s_1}, x_1^{s_1+1}] \times \cdots \times [x_N^{s_N}, x_N^{s_N+1}])$.

Note that, $f \in \bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$) need not be bounded (or need not be measurable).

A measurable function $f \in \bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$ is said to be of $\vec{p} - \bigwedge^*$ bounded variation (that is, $f \in \bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$) if for each of its marginal functions

$$f(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_N) \in (\Lambda^1, \cdots, \Lambda^{i-1}, \Lambda^{i+1}, \cdots, \Lambda^N)^* BV^{(p_1, \cdots, p_N)}(\overline{\mathbb{T}}^N(0_i)),$$

$\forall\ i = 1, 2, \cdots, N$, where

$$\overline{\mathbb{T}}^N(0_i)$$
$$= \{(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \overline{\mathbb{T}}^{N-1} : x_k \in \overline{\mathbb{T}} \text{ for } k = 1, \cdots, i-1, i+1, \cdots, N\}.$$

Note that, for $p_1 = \cdots = p_N = p$, the classes $\bigwedge BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$ and $\bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$ reduce to the classes $\bigwedge BV^{(p)}(\overline{\mathbb{T}}^N)$ and $\bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^N)$ respectively.

We say $f \in L^{(\vec{p})}(\overline{\mathbb{T}}^N)$ if
$\|f\|_{(\vec{p})} =$
$$\left( \int_0^{2\pi} \left( \cdots \left( \int_0^{2\pi} \left( \int_0^{2\pi} |f(x_1, \cdots, x_N)|^{p_1}\ dx_1 \right)^{\frac{p_2}{p_1}}\ dx_2 \right)^{\frac{p_3}{p_2}} \cdots \right)^{p_{N-1}}\ dx_N \right)^{\frac{1}{p_N}}$$
is finite.

Note that, for $p_1 = \cdots = p_N = p$, the class $L^{(\vec{p})}(\overline{\mathbb{T}}^N)$ reduces to the class $L^p(\overline{\mathbb{T}}^N)$. It is observed that the space $\left(L^{(\vec{p})}(\overline{\mathbb{T}}^N), \|.\|_{(\vec{p})}\right)$ is a Banach space [1].

**Theorem 3.1.** *If $f \in L^1(\overline{\mathbb{T}}^N)$ and $g \in \bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$, where $1 \leq p_1 \leq p_2, \leq, ..., \leq p_N < \infty$, then $f * g \in \bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^2)$.*

The above extended theorem of this section can be easily proved by the induction of the arguments to N from Theorem 2.2.

Remark. Since $L^1(\overline{\mathbb{T}}^N)$ is a ring for the convolution product as a ring product. From the above theorem the class $\bigwedge^* BV^{(\vec{p})}(\overline{\mathbb{T}}^N)$ can be regarded as a module over the ring $L^1(\overline{\mathbb{T}}^N)$.

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# SOME RESULTS ON GENERALIZED $(\kappa, \mu)$-SPACE FORMS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. In the present paper, we have studied curvature tensor of generalized $(\kappa, \mu)$-space forms with respect to the semi-symmetric metric connection. We have established a relation between conformal curvature tensors of a generalized $(\kappa, \mu)$-space form with respect to the Levi-Civita connection and semi-symmetric metric connection and construct the condition for which a generalized $(\kappa, \mu)$-space form with respect to semi-symmetric metric connection to be $\xi$-conformally flat. We have also derived the expression of Weyl projective curvature tensor of a generalized $(\kappa, \mu)$-space form on the same metric connection. Also we give an example.

## 1. Introduction

In 2004, Alegre, Blair and Carriazo introduced the notion of generalized Sasakian-space forms in the paper [1]. The same topic has also been studied in the papers [2], [3], [9], [13], [15] etc. by several authors. Generalized $(\kappa, \mu)$-space forms are the generalizations of generalized Sasakian-space forms. The notion of generalized $(\kappa, \mu)$-space forms was introduced in the paper [8]. A generalized $(\kappa, \mu)$-space form is an almost contact metric manifold whose curvature tensor satisfies the following:

$$\begin{aligned} R(X,Y)Z = \quad & f_1R_1(X,Y)Z + f_2R_2(X,Y)Z + f_3R_3(X,Y)Z \\ & +f_4R_4(X,Y)Z + f_5R_5(X,Y)Z + f_6R_6(X,Y)Z, \quad (1.1) \end{aligned}$$

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where $f_i (i = 1, 2, 3, 4, 5, 6)$ are some differentiable functions on the manifold and $R_i (i = 1, 2, 3, 4, 5, 6)$ are defined below

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,$$

$$\begin{aligned} R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned}$$

$$\begin{aligned} R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X \\ &- g(hX,Z)Y, \end{aligned}$$

$$\begin{aligned} R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY \\ &+ g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX, \end{aligned}$$

$$\begin{aligned} R_6(X,Y)Z = &\quad \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\ &+ g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi. \end{aligned}$$

In 1970, Yano introduced semi-symmetric metric connection in the paper [16]. Let $M$ be a $(2n+1)$-dimensional generalized $(\kappa,\mu)$-space form and $\nabla$ denotes the Levi-Civita connection on $M$. A linear connection $\bar{\nabla}$ on $M$ is called a semi-symmetric connection if the torsion tensor $\bar{T}$ of type $(1,2)$ defined as

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.2}$$

satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.3}$$

for all vector fields $X$, $Y$ on $M$.

A semi-symmetric connection $\bar{\nabla}$ is said to be semi-symmetric metric connection if

$$\bar{\nabla}g = 0. \tag{1.4}$$

A relation between semi-symmetric metric connection $\bar{\nabla}$ and Levi-Civita connection $\nabla$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{1.5}$$

Further, a relation between the curvature tensors $R$ and $\bar{R}$ of type $(1,3)$ of $\nabla$ and $\bar{\nabla}$, respectively, is given by

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z+\alpha(X,Z)Y-\alpha(Y,Z)X\\ &+ g(X,Z)AY-g(Y,Z)AX, \qquad (1.6)\end{aligned}$$

where

$$\alpha(X,Y)=g(AX,Y)=(\nabla_X\eta)(Y)-\eta(X)\eta(Y)+\frac{1}{2}g(X,Y). \qquad (1.7)$$

Several authors such as De and Sengupta [11], Sharfuddin and Hussain [14] and many more authors have also been studied semi-symmetric metric connection in different types of manifolds.

In this paper we would like to study some properties of generalized $(\kappa,\mu)$-space forms with semi-symmetric metric connection.

The present paper is organized as follows: In Section 2, we have discussed some preliminary results. In Section 3, we have derived curvature tensor of a generalized $(\kappa,\mu)$-space forms with respect to the semi-symmetric metric connection. In Section 4 and 5, we study Weyl conformal curvature tensor and Weyl projective curvature tensor with respect to the same connection, respectively. Last section contains an example.

## 2. PRELIMINARIES

A differentiable manifold of dimension $(2n+1)$ is said to be an almost contact metric manifold if it satisfies the following conditions [9]:

$$\phi^2(X)=-X+\eta(X)\xi, \quad \eta(\xi)=1. \qquad (2.1)$$

As a consequence, we get the following:

$$\phi\xi=0, \quad g(X,\xi)=\eta(X), \quad \eta(\phi X)=0, \qquad (2.2)$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \qquad (2.3)$$

$$g(\phi X,Y)=-g(X,\phi Y), \quad g(\phi X,X)=0, \qquad (2.4)$$

$$(\nabla_X\eta)(Y)=g(\nabla_X\xi,Y), \qquad (2.5)$$

where $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric on the manifold.

An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y)=\Phi(X,Y)=g(X,\phi Y), \qquad (2.6)$$

where $\Phi$ is the fundamental 2-form on the manifold.

On a contact metric manifold $(M, \phi, \xi, \eta, g)$, the tensor field $h$, defined by $h = \frac{1}{2}L_\xi \phi$, is symmetric and satisfies the following relations [8]:

$$h\xi = 0, \quad \nabla_X \xi = -\phi X - \phi h X, \quad h\phi = -\phi h, \tag{2.7}$$

$$tr(h) = 0, \quad tr(\phi h) = 0, \quad \eta \circ h = 0.$$

On a generalized $(\kappa, \mu)$-space form, we have the following relations [12]:

$$\begin{aligned} R(X,Y)\xi &= (f_1 - f_3)[\eta(Y)X - \eta(X)Y] \\ &+ (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \end{aligned} \tag{2.8}$$

$$\begin{aligned} R(\xi,Y)Z &= (f_1 - f_3)[g(Y,Z) - \eta(Z)Y] \\ &+ (f_4 - f_6)[g(hY,Z)\xi - \eta(Z)hY], \end{aligned} \tag{2.9}$$

$$R(\xi,X)\xi = (f_1 - f_3)[\eta(X)\xi - X] - (f_4 - f_6)hX, \tag{2.10}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+ (f_4 - f_6)[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)], \end{aligned} \tag{2.11}$$

$$\begin{aligned} S(X,Y) &= (2nf_1 + 3f_2 - f_3)g(X,Y) \\ &- (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \\ &+ ((2n-1)f_4 - f_6)g(hX,Y), \end{aligned} \tag{2.12}$$

$$\begin{aligned} QX &= (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \\ &+ ((2n-1)f_4 - f_6)hX, \end{aligned} \tag{2.13}$$

$$\begin{aligned} S(\phi X, \phi Y) &= S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y) \\ &- 2((2n-1)f_4 - f_6)g(hX,Y), \end{aligned} \tag{2.14}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.15}$$

$$\begin{aligned} (\nabla_X \phi)(Y) &= (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X] \\ &+ (f_4 - f_6)[g(hX,Y)\xi - \eta(Y)hX], \end{aligned} \tag{2.16}$$

$$(\nabla_X \eta)(Y) = -(f_1 - f_3)g(\phi X, Y) - (f_4 - f_6)g(\phi hX, Y), \tag{2.17}$$

$$\nabla_X \xi = -(f_1 - f_3)\phi X - (f_4 - f_6)\phi hX, \tag{2.18}$$

where $S$, $Q$ and $r$ are the Ricci tensor of type$(0,2)$, Ricci operator and scalar curvature of M, respectively.

The Weyl conformal curvature tensor $C$ of type $(1,3)$ of an $(2n+1)(n>1)$-dimensional Riemannian manifold $M$ is defined by [10]

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.19}$$

where $R$, $S$, $Q$, $r$ denotes the Riemannian curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator and the scalar curvature of the manifold, respectively.

The Weyl projective curvature tensor $P$ of type $(1,3)$ on a Riemannian manifold $M$ of dimension $(2n+1)$ is defined by [10]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{2.20}$$

## 3. CURVATURE TENSOR OF A GENERALIZED $(\kappa,\mu)$-SPACE FORM WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

We have, from (1.6)

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + \alpha(X,Z)g(Y,W) \\ &- \alpha(Y,Z)g(X,W) + g(X,Z)\alpha(Y,W) \\ &- g(Y,Z)\alpha(X,W). \end{aligned} \tag{3.1}$$

Using (1.7) and (2.17), we get

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) - [(f_1-f_3)g(\phi X,Z) \\ &+ (f_4-f_6)g(\phi hX,Z) + \eta(X)\eta(Z) \\ &- \frac{1}{2}g(X,Z)]g(Y,W) + [(f_1-f_3)g(\phi Y,Z) \\ &+ (f_4-f_6)g(\phi hY,Z) + \eta(Y)\eta(Z) \\ &- \frac{1}{2}g(Y,Z)]g(X,W) - [(f_1-f_3)g(\phi Y,W) \\ &+ (f_4-f_6)g(\phi hY,W) + \eta(Y)\eta(W) \\ &- \frac{1}{2}g(Y,W)]g(X,Z) + [(f_1-f_3)g(\phi X,W) \\ &+ (f_4-f_6)g(\phi hX,W) + \eta(X)\eta(W) \\ &- \frac{1}{2}g(X,W)]g(Y,Z) \end{aligned} \tag{3.2}$$

Therefore, from above, we get

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z-(f_1-f_3)[g(\phi X,Z)Y-g(\phi Y,Z)X\\ &+ g(X,Z)\phi Y-g(Y,Z)\phi X]-(f_4-f_6)[g(\phi hX,Z)Y\\ &- g(\phi hY,Z)X+g(X,Z)\phi hY-g(Y,Z)\phi hX]\\ &+ [g(X,Z)Y-g(Y,Z)X]+[\eta(Y)X-\eta(X)Y]\eta(Z)\\ &+ [g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi. \qquad (3.3)\end{aligned}$$

Now, putting $Z=\xi$ in (3.3) and using (2.8), we get

$$\begin{aligned}\bar{R}(X,Y)\xi &= (f_1-f_3)[\eta(Y)X-\eta(X)Y+\eta(Y)\phi X\\ &- \eta(X)\phi Y]+(f_4-f_6)[\eta(Y)hX-\eta(X)hY\\ &+ \eta(Y)\phi hX-\eta(X)\phi hY]. \qquad (3.4)\end{aligned}$$

Putting $X=\xi$ in (3.3) and using (2.9), we get

$$\begin{aligned}\bar{R}(\xi,Y)Z &= (f_1-f_3)[g(Y,Z)\xi-\eta(Z)Y+g(\phi Y,Z)\xi\\ &- \eta(Z)\phi Y]+(f_4-f_6)[g(hY,Z)\xi-\eta(Z)hY\\ &+ g(\phi hY,Z)\xi-\eta(Z)\phi hY]. \qquad (3.5)\end{aligned}$$

Putting $Z=\xi$ in (3.5), we get

$$\bar{R}(\xi,Y)\xi=(f_1-f_3)[\eta(Y)\xi-Y-\phi Y]-(f_4-f_6)[hY+\phi hY]. \qquad (3.6)$$

Taking inner product (3.3) with the vector field $\xi$, we get

$$\begin{aligned}\eta(\bar{R}(X,Y)Z) &= (f_1-f_3)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\\ &+ (f_4-f_6)[g(hY,Z)\eta(X)-g(hX,Z)\eta(Y)]\\ &+ [g(\phi Y,Z)\eta(X)-g(\phi X,Z)\eta(Y)]\\ &+ [g(\phi hY,Z)\eta(X)-g(\phi hX,Z)\eta(Y)]. \qquad (3.7)\end{aligned}$$

From (3.3), we get

$$\begin{aligned}g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W)-(f_1-f_3)[g(\phi X,Z)g(Y,W)\\ &- g(\phi Y,Z)g(X,W)+g(X,Z)g(\phi Y,W)\\ &- g(Y,Z)g(\phi X,W)]-(f_4-f_6)[g(\phi hX,Z)g(Y,W)\\ &- g(\phi hY,Z)g(X,W)+g(X,Z)g(\phi hY,W)\\ &- g(Y,Z)g(\phi hX,W)]+[g(X,Z)g(Y,W)\\ &- g(Y,Z)g(X,W)]+[g(X,W)\eta(Y)\end{aligned}$$

$$
\begin{aligned}
&- \quad g(Y,W)\eta(X)]\eta(Z) + [g(Y,Z)\eta(X) \\
&- \quad g(X,Z)\eta(Y)]\eta(W). \qquad (3.8)
\end{aligned}
$$

Let $\{e_1, e_2, ...., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in the manifold. Then, by putting $X = W = e_i$ in the above equation and taking summation over $i$, $1 \leq i \leq 2n$, we get

$$
\begin{aligned}
\bar{S}(Y,Z) &= S(Y,Z) + (2n-1)[(f_1 - f_3)g(\phi Y, Z) \\
&+ (f_4 - f_6)g(\phi h Y, Z) - g(Y,Z) + \eta(Y)\eta(Z)], \qquad (3.9)
\end{aligned}
$$

where $\bar{S}$ is the Ricci tensor of type $(0,2)$ with respect to semi-symmetric metric connection.

Contracting $Y$ and $Z$ in (3.9) and using (2.15), we get

$$
\bar{r} = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 - 2n(2n-1), \qquad (3.10)
$$

where $\bar{r}$ is the scalar curvature with respect to the semi-symmetric metric connection.

Now, from (3.9) and (2.13), we get

$$
\begin{aligned}
\bar{Q}Y &= QY + (2n-1)[(f_1 - f_3)\phi Y \\
&+ (f_4 - f_6)\phi h Y - Y + \eta(Y)\xi], \qquad (3.11)
\end{aligned}
$$

where $\bar{Q}$ is the Ricci operator with respect to the semi-symmetric metric connection.

Using (1.5), (2.2) and (2.16) we get

$$
\begin{aligned}
(\bar{\nabla}_X \phi)(Y) &= (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X] \\
&+ (f_4 - f_6)[g(hX,Y)\xi - \eta(Y)hX] \\
&+ g(\phi X, Y)\xi - \eta(Y)\phi X. \qquad (3.12)
\end{aligned}
$$

From the equations (1.5) and (2.18), we get

$$
\bar{\nabla}_X \xi = -(f_1 - f_3)\phi X - (f_4 - f_6)\phi h X + X - \eta(X)\xi. \qquad (3.13)
$$

Using (1.5), (2.2) and (2.18), we get

$$
\begin{aligned}
(\bar{\nabla}_X \eta)(Y) &= -(f_1 - f_3)g(\phi X, Y) - (f_4 - f_6)g(\phi h X, Y) \\
&+ g(X,Y) - \eta(X)\eta(Y). \qquad (3.14)
\end{aligned}
$$

Thus we can state the following:

**Theorem 3.1.** *Let $M$ be a generalized $(\kappa,\mu)$-space form of dimension $(2n+1)$. With respect to the semi-symmetric metric connection, the following hold in $M$:*

$$\begin{aligned}\bar{R}(X,Y)\xi &= (f_1-f_3)[\eta(Y)X-\eta(X)Y+\eta(Y)\phi X\\ &- \eta(X)\phi Y]+(f_4-f_6)[\eta(Y)hX-\eta(X)hY\\ &+ \eta(Y)\phi hX-\eta(X)\phi hY],\end{aligned}$$

$$\begin{aligned}\bar{R}(\xi,Y)Z &= (f_1-f_3)[g(Y,Z)\xi-\eta(Z)Y+g(\phi Y,Z)\xi\\ &- \eta(Z)\phi Y]+(f_4-f_6)[g(hY,Z)\xi-\eta(Z)hY\\ &+ g(\phi hY,Z)\xi-\eta(Z)\phi hY],\end{aligned}$$

$$\bar{R}(\xi,Y)\xi=(f_1-f_3)[\eta(Y)\xi-Y-\phi Y]-(f_4-f_6)[hY+\phi hY],$$

$$\begin{aligned}\eta(\bar{R}(X,Y)Z) &= (f_1-f_3)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\\ &+ (f_4-f_6)[g(hY,Z)\eta(X)-g(hX,Z)\eta(Y)]\\ &+ [g(\phi Y,Z)\eta(X)-g(\phi X,Z)\eta(Y)]\\ &+ [g(\phi hY,Z)\eta(X)-g(\phi hX,Z)\eta(Y)],\end{aligned}$$

$$\begin{aligned}\bar{S}(X,Y) &= S(X,Y)+(2n-1)[(f_1-f_3)g(\phi X,Y)\\ &+ (f_4-f_6)g(\phi hX,Y)-g(X,Y)+\eta(X)\eta(Y)],\end{aligned}$$

$$\bar{r}=2n(2n+1)f_1+6nf_2-4nf_3-2n(2n-1),$$

$$\begin{aligned}\bar{Q}X &= QX+(2n-1)[(f_1-f_3)\phi Y\\ &+ (f_4-f_6)\phi hY-Y+\eta(Y)\xi],\end{aligned}$$

$$\begin{aligned}(\bar{\nabla}_X\phi)(Y) &= (f_1-f_3)[g(X,Y)\xi-\eta(Y)X]\\ &+ (f_4-f_6)[g(hX,Y)\xi-\eta(Y)hX]\\ &+ g(\phi X,Y)\xi-\eta(Y)\phi X,\end{aligned}$$

$$\bar{\nabla}_X\xi=-(f_1-f_3)\phi X-(f_4-f_6)\phi hX+X-\eta(X)\xi,$$

$$\begin{aligned}(\bar{\nabla}_X\eta)(Y) &= -(f_1-f_3)g(\phi X,Y)-(f_4-f_6)g(\phi hX,Y)\\ &+ g(X,Y)-\eta(X)\eta(Y).\end{aligned}$$

## 4. Weyl conformal curvature tensor with respect to semi-symmetric metric connection

The Weyl conformal curvature tensor with respect to the semi-symmetric metric connection is given by [11]

$$\begin{aligned}\bar{C}(X,Y)Z &= \bar{R}(X,Y)Z-\frac{1}{2n-1}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y\\ &+ g(Y,Z)\bar{Q}X-g(X,Z)\bar{Q}Y]\\ &+ \frac{\bar{r}}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y]. \qquad (4.1)\end{aligned}$$

Using (3.3), (3.9), (3.10) and (3.11) in (4.1), we get

$$\begin{aligned}\bar{C}(X,Y)Z &= R(X,Y)Z+(f_1-f_3)[g(\phi Y,Z)X-g(\phi X,Z)Y\\ &+ g(Y,Z)\phi X-g(X,Z)\phi Y]+(f_4-f_6)[g(\phi hY,Z)X\\ &- g(\phi hX,Z)Y+g(Y,Z)\phi hX-g(X,Z)\phi hY]\\ &+ [g(X,Z)Y-g(Y,Z)X]+[\eta(Y)X-\eta(X)Y]\eta(Z)\\ &+ [g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi-\frac{1}{2n-1}[S(Y,Z)X\\ &+ (2n-1)\{(f_1-f_3)g(\phi Y,Z)X+(f_4-f_6)g(\phi hY,Z)X\\ &- g(Y,Z)X+\eta(Y)\eta(Z)X\}-S(X,Z)Y\\ &- (2n-1)\{(f_1-f_3)g(\phi X,Z)Y+(f_4-f_6)g(\phi hX,Z)Y\\ &- g(X,Z)Y+\eta(X)\eta(Z)Y\}+g(Y,Z)QX\\ &+ (2n-1)\{(f_1-f_3)g(Y,Z)\phi X+(f_4-f_6)g(Y,Z)\phi hX\\ &- g(Y,Z)X+g(Y,Z)\eta(X)\xi\}-g(X,Z)QY\\ &- (2n-1)\{(f_1-f_3)g(X,Z)\phi Y+(f_4-f_6)g(X,Z)\phi hY\\ &- g(X,Z)Y+g(X,Z)\eta(Y)\xi\}]\\ &+ \frac{r-2n(2n-1)}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y]\\ &= R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y\\ &+ g(Y,Z)QX-g(X,Z)QY]\\ &+ \frac{r}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y]\\ &= C(X,Y)Z. \qquad (4.2)\end{aligned}$$

Thus we can state the following:

**Theorem 4.1.** *The Weyl conformal curvature tensors of a generalized $(\kappa, \mu)$-space form with respect to the Levi-Civita connection and semi-symmetric metric connection are equal.*

**Definition 4.2.** A generalized $(\kappa, \mu)$-space form of dimension $(2n+1)$ is said to be $\xi$-conformally flat with respect to the semi-symmetric metric connection if

$$\bar{C}(X,Y)\xi = 0.$$

Putting $Z = \xi$ in (4.2), we get

$$\begin{aligned}\bar{C}(X,Y)\xi &= R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y \\ &+ \eta(Y)QX - \eta(X)QY] \\ &+ \frac{r}{2n(2n-1)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{4.3}$$

Using (2.8), (2.12), (2.13) and (2.15) in (4.3), we get

$$\bar{C}(X,Y)\xi = -\frac{2(n-1)f_6}{2n-1}(\eta(Y)hX - \eta(X)hY). \tag{4.4}$$

Thus we can state the following:

**Theorem 4.3.** *A generalized $(\kappa, \mu)$-space form of dimension $(2n+1)(n>1)$ is $\xi$-conformally flat with respect to the semi-symmetric metric connection if $f_6 = 0$.*

## 5. WEYL PROJECTIVE CURVATURE TENSOR WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

The Weyl projective curvature tensor of a generalized $(\kappa, \mu)$-space form of dimension $(2n+1)$ with respect to semi-symmetric metric connection is given by [11]

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y], \tag{5.1}$$

where $\bar{P}(X,Y)Z$ is the projective curvature tensor of type $(1,3)$ with respect to semi-symmetric metric connection.

A generalized $(\kappa, \mu)$-space form of dimension $(2n+1)$ is projectively flat with respect to the semi-symmetric metric connection if

$$\bar{P}(X,Y)Z = 0,$$

for all $X$, $Y$, $Z$ on $M$.
Using (3.3) and (3.9) in (5.1), we get

$$\begin{aligned}\bar{P}(X,Y)Z &= P(X,Y)Z+\frac{1}{2n}[(f_1-f_3)(g(\phi Y,Z)X-g(\phi X,Z)Y)\\ &+ (f_4-f_6)(g(\phi hY,Z)X-g(\phi hX,Z)Y-(g(Y,Z)X\\ &- g(X,Z)Y)+(\eta(Y)X-\eta(X)Y)\eta(Z)]\\ &+ (f_1-f_3)(g(Y,Z)\phi X-g(X,Z)\phi Y)\\ &+ (f_4-f_6)(g(Y,Z)\phi hX-g(X,Z)\phi hY)\\ &+ (g(Y,Z)\eta(X)-g(X,Z)\eta(Y))\xi. \qquad (5.2)\end{aligned}$$

By straight forward calculation, we can state the following proposition:
**Proposition 5.1.** *In a generalized $(\kappa,\mu)$-space form, the projective curvature tensor with respect to semi-symmetric metric connection satisfies the following:*

$$\bar{P}(X,Y)Z=-\bar{P}(Y,X)Z, \qquad (5.3)$$

$$\bar{P}(X,Y)Z+\bar{P}(Y,Z)X+\bar{P}(Z,X)Y=0, \qquad (5.4)$$

*if $\bar{P}(X,Y,Z,W)=g(\bar{P}(X,Y)Z,W)$,then*

$$\sum_{i=1}^{2n+1}\bar{P}(e_i,Y,Z,e_i)=\sum_{i=1}^{2n+1}\bar{P}(e_i,e_i,Z,W)=\sum_{i=1}^{2n+1}\bar{P}(X,Y,e_i,e_i)=0. \qquad (5.5)$$

Let a generalized $(\kappa,\mu)$-space form is projectively flat with respect to the semi-symmetric metric connection. Then from (5.1), we get

$$\bar{R}(X,Y)Z=\frac{1}{2n}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y]. \qquad (5.6)$$

This implies

$$\bar{R}(X,Y,Z,W)=\frac{1}{2n}[\bar{S}(Y,Z)g(X,W)-\bar{S}(X,Z)g(Y,W)], \qquad (5.7)$$

where $\bar{R}(X,Y,Z,W)=g(\bar{R}(X,Y)Z,W)$. Contracting $Y$, $Z$ in (5.7), we get

$$\bar{S}(X,W)=\frac{\bar{r}}{2n+1}g(X,W). \qquad (5.8)$$

Thus we can state the following:

**Theorem 5.1.** *Let $M$ be a generalized $(\kappa,\mu)$-space form of dimension $(2n+1)$. If $M$ be projectively flat with respect to the semi-symmetric metric connection, then $M$ is an Einstein manifold.*

## 6. EXAMPLE

Let us consider the manifold $M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_5 \neq 0\}$ of dimension 5, where $\{x_1, x_2, x_3, x_4, x_5\}$ are standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$e_1 = e^{-x_5}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-x_5}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-x_5}\frac{\partial}{\partial x_3},$$
$$e_4 = e^{-x_5}\frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial x_5},$$

which are linearly independent at each point of $M$, we get

$$[e_1, e_5] = e_1, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = e_3, \quad [e_4, e_5] = e_4$$

and the remaining $[e_i, e_j] = 0$, for all $1 \leq i, j \leq 5$.

Let $g$ be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij},\ i, j = 1, 2, 3, 4, 5$. Let $\nabla$ be the Riemannian connection and $R$ the curvature tensor of $g$. The 1-form $\eta$ is defined by $\eta(X) = g(X, e_5)$, for any $X$ on $M$, which is a contact form because $\eta \wedge d\eta \neq 0$. Let $\phi$ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_3, \quad \phi(e_2) = e_4, \quad \phi(e_3) = -e_1, \quad \phi(e_4) = -e_2, \quad \phi(e_5) = 0.$$

Then we find that

$$\eta(e_5) = 1, \quad \phi^2 X = -X + \eta(X)e_5, \quad d\eta(X, Y) = g(X, \phi Y),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X$, $Y$ on $M$. Hence $(\phi, e_5, \eta, g)$ defines an almost contact metric structure on $M$.

Using Koszul's formula, we obtain

$$\nabla_{e_1} e_1 = -e_5, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_4} e_4 = -e_5$$

and the remaining $\nabla_{e_i} e_j = 0$, for all $1 \leq i, j \leq 5$. The tensor field $h$ satisfies

$$he_1 = e_1, \quad he_2 = e_2, \quad he_3 = e_3, \quad he_4 = e_4, \quad he_5 = 0.$$

Now, from the definition of curvature tensor, we obtain

$$R(e_1, e_5)e_1 = e_5, \quad R(e_2, e_5)e_2 = e_5,$$
$$R(e_3, e_5)e_3 = e_5, \quad R(e_4, e_5)e_4 = e_5$$

and the remaining $R(e_i, e_j)e_k = 0$, for all $1 \leq i, j, k \leq 5$.

Thus $M$ is a generalized $(\kappa,\mu)$-space form with $f_1 = -\frac{1+f}{2}$, $f_2 = 0$, $f_3 = \frac{1-f}{2}$, $f_4 = 0$, $f_5 = 0$, $f_6 = 0$, where $f$ is an arbitrary differentiable function on the manifold.

Let $\bar{\nabla}$ be the semi-symmetric metric connection, we have

$$\bar{\nabla}_{e_1} e_1 = -2e_5, \quad \bar{\nabla}_{e_2} e_2 = -2e_5, \quad \bar{\nabla}_{e_3} e_3 = -2e_5,$$
$$\bar{\nabla}_{e_4} e_4 = -2e_5, \quad \bar{\nabla}_{e_1} e_5 = e_1, \quad \bar{\nabla}_{e_2} e_5 = e_2,$$
$$\bar{\nabla}_{e_3} e_5 = e_3, \quad \bar{\nabla}_{e_4} e_5 = e_4$$

and the remaining $\bar{\nabla}_{e_i} e_j = 0$, for all $1 \le i, j \le 5$.

The components of torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ are given by

$$\bar{T}(e_1, e_5) = -\bar{T}(e_5, e_1) = e_1, \quad \bar{T}(e_2, e_5) = -\bar{T}(e_5, e_2) = e_2,$$
$$\bar{T}(e_3, e_5) = -\bar{T}(e_5, e_3) = e_3, \quad \bar{T}(e_4, e_5) = -\bar{T}(e_5, e_4) = e_4$$

and the remaining $\bar{T}(e_i, e_j) = 0$, $1 \le i, j \le 5$.

The components of the curvature tensor $\bar{R}$ of the manifold $M$ with respect to the semi-symmetric metric connection $\bar{\nabla}$ are given by

$$\bar{R}(e_1, e_2)e_1 = 2e_2, \quad \bar{R}(e_1, e_2)e_2 = -2e_1, \quad \bar{R}(e_1, e_3)e_1 = 2e_3,$$
$$\bar{R}(e_1, e_3)e_3 = -2e_1, \quad \bar{R}(e_1, e_4)e_1 = 2e_4, \quad \bar{R}(e_1, e_4)e_4 = -2e_1,$$
$$\bar{R}(e_1, e_5)e_1 = 2e_5, \quad \bar{R}(e_1, e_5)e_5 = -e_1, \quad \bar{R}(e_2, e_3)e_2 = 2e_3,$$
$$\bar{R}(e_2, e_3)e_3 = -2e_2, \quad \bar{R}(e_2, e_4)e_2 = 2e_4, \quad \bar{R}(e_2, e_4)e_4 = -2e_2,$$
$$\bar{R}(e_2 e_5)e_2 = 2e_5, \quad \bar{R}(e_2, e_5)e_5 = -e_2, \quad \bar{R}(e_3, e_4)e_3 = 2e_4,$$
$$\bar{R}(e_3, e_4)e_4 = -2e_3, \quad \bar{R}(e_3, e_5)e_3 = 2e_5, \quad \bar{R}(e_3, e_5)e_5 = -e_3,$$
$$\bar{R}(e_4, e_5)e_4 = 2e_5, \quad \bar{R}(e_4, e_5)e_5 = -e_4$$

and the remaining $\bar{R}(e_i, e_j)e_k = 0$, $1 \le i, j, k \le 5$.

From above, we see that $\bar{C}(e_i, e_j)e_5 = 0$, for all $1 \le i, j \le 5$.
Hence the manifold $M$ is $\xi$-conformally flat.

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# $S\beta_\lambda$-CLOSED SETS AND SOME LOW SEPARATION AXIOMS IN GT-SPACES

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Abstract. Here we have studied the ideas of $s\lambda$- , $sg_\lambda$- and $s\beta_\lambda$-closed sets and investigated some of their topological properties in generalized topological spaces. We have also studied some low separation axioms namely $s\lambda T_{\frac{1}{4}}$, $s\lambda T_{\frac{3}{8}}$, $s\lambda T_{\frac{1}{2}}$ axioms and their mutual relation with $s\lambda T_0$ and $s\lambda T_1$ axioms.

## 1. Introduction

After generalization of a topological space by A.D. Alexandroff [1] in 1940 to $\sigma$-space (Alexandroff space), many topologists turned their attention to carry out their works in such direction viz. in $\sigma$-spaces and bispaces etc. [2, 4, 5, 6, 13] where several works in respect of topological properties had been studied. On the other hand, N.Levine [14] in 1970, introduced the concept of generalized closed sets (g-closed sets) in a topological space which opened the door of many aspects in topological spaces to generate different types of generalized sets. In 1987, Bhattacharyya and Lahiri [7] introduced the class of semi-generalizsed closed sets in a topological space. Since then many authors have contributed to the subsequent development of various topological properties on semi generalized closed sets ($sg$-closed sets) [12, 15, 16] where many more references are found. By taking an equivalent form of $g$-closed sets, M. S. Sarsak [17] introduced $g_\mu$-closed sets in a generalized topological space $(X, \mu)$ [9, 10] and studied the idea of new separation axioms namely $\mu$-$T_{\frac{1}{4}}$, $\mu$-$T_{\frac{3}{8}}$ and $\mu$-$T_{\frac{1}{2}}$ axioms by defining $\lambda_\mu$-closed sets and investigated their properties and relations among the new axioms with $\mu$-$T_0$ and $\mu$-$T_1$ axioms.

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Key words and phrases: $s\lambda$-closed set, $sg_\lambda$-closed set, $s\beta_\lambda$-closed set, $sA^\wedge_\lambda$-set, $s\lambda T_{\frac{1}{4}}$, $s\lambda T_{\frac{3}{8}}$, $s\lambda T_{\frac{1}{2}}$, $s\lambda T_0$ and $s\lambda T_1$ axioms

We will see in this paper that a collection of $s\lambda$-open sets forms generalized topology in GTspaces (Corollary-3.16). This idea leads to another generalization of closed sets viz., $sg_\lambda$-closed set with the help of $s\lambda$-closed sets by extending the notion of $g_\mu$-closed sets and $\lambda_\mu$-closed sets in a more general structure of generalized topological space. It may be investigated how far topological properties are affected with the defined sets in the setting of generalized topology. Then we have investigated some of the topological properties of the defined sets along with the low separation axioms in GTspaces by introducing $s\beta_\lambda$-closed set. We have also explored mutual relation among $s\lambda T_{\frac{1}{4}}, s\lambda T_{\frac{3}{8}}, s\lambda T_{\frac{1}{2}}$ axioms with $s\lambda T_0$ and $s\lambda T_1$ axioms. We have also studied $s\lambda$-homeomorphism in a generalized topological space.

## 2. **Preliminaries**

Let $X$ be a nonempty set. A generalized topology $\mu$ [17] is a collection of subsets of $X$ such that $\emptyset \in \mu$ and $\mu$ is closed under arbitrary unions. In a generalized topological space (GTspace in short) $(X, \mu)$, members of $\mu$ are called $\mu$-open sets and complements are $\mu$-closed sets, $\mu$-closure of a set $A$ denoted by $\overline{A_\mu}$ is similar to a topological space and $P(X)$ denotes all subsets of $X$. If otherwise not stated, GTspace $(X, \mu)$ will be denoted by $X$ and $R, Q$ respectively stand for the set of real and rational numbers.

**Definition 2.1.** (c.f.[13]). A set $A$ in $X$ is said to be semi $\mu$-open ($s\mu$-open in short) if there exists a $\mu$-open set $E$ in $X$ such that $E \subset A \subset \overline{E_\mu}$. A set $A$ is semi $\mu$-closed ($s\mu$-closed in short) if and only if $X - A$ is semi $\mu$-open.

**Definition 2.2.** (c.f.[17]). Let $A$ be a subset of $(X, \mu)$. We define the $s\mu$-kernel of $A$ is the set $\bigcap\{U : A \subset U, U \text{ is } s\mu\text{-open}\}$ and denote it by $sA^\wedge_\mu$. Assume $sA^\vee_\mu = \bigcup\{F : F \subset A, F \text{ is } s\mu\text{-closed}\}$. The set $A$ is called a $s\wedge_\mu$-set if $A = sA^\wedge_\mu$ and $A$ is called a $s\vee_\mu$-set if $A = sA^\vee_\mu$. Note that $sA^\wedge_\mu = X$ if there is no $s\mu$-open set containing $A$ and $sA^\vee_\mu = \emptyset$ if there is no $s\mu$-closed set contained in $A$.

**Definition 2.3.** (c.f.[17]). A subset $A$ of $(X, \mu)$ is said to be $s\lambda$-closed if $A = K \cap P$ where $K$ is a $s\wedge_\mu$-set and $P$ is a $s\mu$-closed set. $A$ is called $s\lambda$-open if $X - A$ is $s\lambda$-closed.

**Definition 2.4.** Suppose $A$ is a subset of $(X, \mu)$. We define $sA^\wedge_\lambda$ is the set $\bigcap\{U : A \subset U, U \text{ is } s\lambda\text{-open}\}$. Assume $sA^\vee_\lambda = \bigcup\{F : F \subset A, F \text{ is } s\lambda\text{-closed}\}$. The set $A$ is called a $s\wedge_\lambda$-set if $A = sA^\wedge_\lambda$ and $A$ is called a $s\vee_\lambda$-set if $A = sA^{\vee}_{\lambda}$. Note that $sA^{\wedge}_{\lambda} = X$ if there is no $s\lambda$-open set containing $A$ and $sA^{\vee}_{\lambda} = \emptyset$ if there is no $s\lambda$-closed set contained in $A$.

Obviously in a GTspace $(X, \mu)$, a $\mu$-open set is $s\mu$-open and a $\mu$-closed set is $s\mu$-closed but converses may not be always true. Again, arbitrary union of $s\mu$-open sets is $s\mu$-open. So collection of $s\mu$-open sets forms generalized topology in $X$. On the other hand, collection of all $s\vee_\mu$-sets in a GTspace $(X, \mu)$ also forms generalized topology. A set $A$ of $(X, \mu)$ is a $s\wedge_\mu$-set if and only if $X - A$ is a $s\vee_\mu$-set . Hence, arbitrary intersection of $s\lambda$-closed sets is $s\lambda$-closed [8].

We use the symbol $\tau$ in general sense to stand for $\mu$- or $\lambda$- in the rest of the section for the sake of repetition of similar types of results.

**Definition 2.5.** (c.f.[3]). Let $(X, \mu)$ be a GTspace then

(1) a point $x \in X$ is said to be a $s\tau$-adherence point of a subset $A$ of $X$, if for every $s\tau$-open set $U$ containing $x$ such that $A \cap U \neq \emptyset$. The set of all $s\tau$-adherence points of $A$ is called $s\tau$-closure of $A$ and is denoted by $\overline{sA_\tau}$;

(2) $s\tau$-interior of a set $A \subset X$ is defined as the union of all $s\tau$-open sets contained in $A$ and is denoted by $sInt_\tau(A)$.

**Theorem 2.6.** *(c.f.[8]). Let $A, B \subset X$. For $s\tau$-closure following hold:*

*(1) $\overline{sA_\tau} = \bigcap\{F : A \subset F; F$ is $s\tau$-closed$\}$;*

*(2) $\overline{sA_\tau}$ is $s\tau$-closed;*

*(3) $A$ is $s\tau$-closed if and only if $A = \overline{sA_\tau}$;*

*(4) $A \subset \overline{sA_\tau}$ and $\overline{s(\overline{sA_\tau})_\tau} = \overline{sA_\tau}$;*

*(5) if $A \subset B$, then $\overline{sA_\tau} \subset \overline{sB_\tau}$;*

*(6) $x \in \overline{sA_\tau}$ if and only if each $s\tau$-open set containing $x$ intersects $A$.*

**Remark 2.7.** (c.f.[8]). Let $A, B \subset X$. For $s\tau$-interior following hold:

(1) $sInt_\tau(A) \subset A$; (2) if $A \subset B$, $sInt_\tau(A) \subset sInt_\tau(B)$;

(3) $sInt_\tau(A)$ is $s\tau$-open; (4) $A$ is $s\tau$-open if and only if $A = sInt_\tau(A)$.

**Lemma 2.8.** *(c.f.[17]). Suppose $A, B \subset X$, then the following hold:*

*(1) $s\emptyset^{\wedge}_{\tau} = \emptyset$, $s\emptyset^{\vee}_{\tau} = \emptyset$, $sX^{\wedge}_{\tau} = X$, $sX^{\vee}_{\tau} = X$;*

*(2) $A \subset sA^{\wedge}_{\tau}$, $sA^{\vee}_{\tau} \subset A$;*

*(3) $A \subset B \Rightarrow sA^{\wedge}_{\tau} \subset sB^{\wedge}_{\tau}$, $A \subset B \Rightarrow sA^{\vee}_{\tau} \subset sB^{\vee}_{\tau}$;*

*(4) $s(sA^{\wedge}_{\tau})^{\wedge}_{\tau} = sA^{\wedge}_{\tau}$, $s(sA^{\vee}_{\tau})^{\vee}_{\tau} = sA^{\vee}_{\tau}$;*

*(5) $s(X \backslash A)^{\wedge}_{\tau} = X \backslash sA^{\vee}_{\tau}$, $s(X \backslash A)^{\vee}_{\tau} = X \backslash sA^{\wedge}_{\tau}$;*

*(6) $sA^{\wedge}_{\tau}$ is a $s\wedge_\tau$-set and $sA^{\vee}_{\tau}$ is a $s\vee_\tau$-set.*

## 3. $s\lambda$-closed sets and $sg_\lambda$-closed sets in GTspaces and $s\lambda T_{\frac{1}{2}}$ GTspace

In this section we will discuss some properties of $s\lambda$-closed sets, $s\lambda$-open sets, $sg_\lambda$-closed sets, $sg_\lambda$-open sets and $s\lambda T_{\frac{1}{2}}$ axiom in a GTspace which will be useful in the sequel.

**Definition 3.1.** ([14]). A subset $A$ of a topological space is said to be generalized closed ($g$-closed for short) if and only if $\overline{A} \subset U$ whenever $A \subset U$ and $U$ is open.

**Definition 3.2.** (c.f.[14]). A subset $A$ of $X$ is said to be a $sg_\lambda$-closed set if $\overline{sA_\lambda} \subset U$ whenever $A \subset U$ and $U$ is $s\lambda$-open. $A$ is called $sg_\lambda$-open if $X - A$ is $sg_\lambda$-closed.

**Note 3.3.** Clearly, a set $A$ of $X$ is $sg_\lambda$-closed if and only if $\overline{sA_\lambda} \subset sA^\wedge_\lambda$.

**Theorem 3.4.** *If $A \subset X$, then $X - \overline{s(X-A)_\lambda} = sInt_\lambda(A)$.*

**Theorem 3.5.** *A subset $A$ of $X$ is $sg_\lambda$-open if and only if $F \subset sInt_\lambda(A)$ whenever $F \subset A$ and $F$ is $s\lambda$-closed (or equivalently, $sA^\vee_\lambda \subset sInt_\lambda(A)$).*

*Proof.* For necessary part: suppose $A$ is a $sg_\lambda$-open set and $F$ is a $s\lambda$-closed set, $F \subset A$. Then $X - A \subset X - F$, a $s\lambda$-open set and $X - A$ is $sg_\lambda$-closed. So we have $\overline{s(X-A)_\lambda} \subset X - F$, by definition 3.2. By theorem 3.4, $X - sInt_\lambda(A) = \overline{s(X-A)_\lambda} \subset X - F$ and hence $F \subset sInt_\lambda(A)$.

For sufficient part: suppose $X - A \subset U, U$ is $s\lambda$-open. Then $X - U \subset A$ and $X - U$ is $s\lambda$-closed. By assumption, $X - U \subset sInt_\lambda(A)$ and so by theorem 3.4, $\overline{s(X-A)_\lambda} = X - sInt_\lambda(A) \subset U$ and hence $X - A$ is $sg_\lambda$-closed by definition 3.2. This implies that $A$ is $sg_\lambda$-open. □

**Lemma 3.6.** *For $A \subset X$, the following hold:*

*(1) $A$ is $s\lambda$-closed if and only if $A = sA^\wedge_\mu \cap \overline{sA_\mu}$.*
*(2) If $A$ is $s\mu$-closed then $A$ is $s\lambda$-closed.*
*(3) If $A$ is $s\lambda$-closed then $A = sA^\wedge_\mu \cap \overline{sA_\lambda}$.*

*Proof.* (1). The condition is necessary. For let $A$ be a $s\lambda$-closed set then by definition 2.3, $A = K \cap P$ where $K$ is a $s\wedge_\mu$-set i.e. $K = sK^\wedge_\mu$ by definition 2.2 and $P$ is a $s\mu$-closed set. Now $A \subset K \Rightarrow sA^\wedge_\mu \subset sK^\wedge_\mu$ and $A \subset P \Rightarrow \overline{sA_\mu} \subset \overline{sP_\mu}$. Hence $A \subset sA^\wedge_\mu \cap \overline{sA_\mu} \subset sK^\wedge_\mu \cap \overline{sP_\mu} = K \cap P = A \Rightarrow A = sA^\wedge_\mu \cap \overline{sA_\mu}$.

Sufficient part is obvious since $sA_\mu^\wedge$ is a $s\wedge_\mu$-set and $\overline{sA_\mu}$ is $s\mu$-closed.

(2) Let $A$ be a $s\mu$-closed set, then $sA_\mu^\wedge \cap \overline{sA_\mu} = sA_\mu^\wedge \cap A = A \Rightarrow A$ is $s\lambda$-closed by (1).

(3). Let $A$ be a $s\lambda$-closed set, then by definition 2.3, $A = K \cap P$ where $K$ is a $s\wedge_\mu$-set i.e. $K = sK_\mu^\wedge$ by definition 2.2 and $P$ is a $s\mu$-closed set, then by (2), $P$ is a $s\lambda$-closed set. Now $A \subset K \Rightarrow sA_\mu^\wedge \subset sK_\mu^\wedge$ and $A \subset P \Rightarrow \overline{sA_\lambda} \subset \overline{sP_\lambda}$. Hence $A \subset sA_\mu^\wedge \cap \overline{sA_\lambda} \subset sK_\mu^\wedge \cap \overline{sP_\lambda} = K \cap P = A \Rightarrow A = sA_\mu^\wedge \cap \overline{sA_\lambda}$. □

**Remark 3.7.** Obviously in a GTspace $(X, \mu)$, every $s\wedge_\mu$-set as well as every $s\mu$-closed set are $s\lambda$-closed and $s\lambda$-closed set is $sg_\lambda$-closed. Converse parts are not in general true as revealed from examples 3.8 (i), (ii) and (iii).

**Example 3.8.** (i): In a GTspace, example of a $s\lambda$-closed set which is not a $s\wedge_\mu$-set:

Suppose $X = \{a, b, c\}, \mu = \{\emptyset, \{a\}, \{a, b\}\}$. Then $(X, \mu)$ is a GTspace but not a topological space. Among $P(X)$, the family of $\mu$-closed sets is $\{X, \{b, c\}, \{c\}\}$; the family of $s\mu$-open sets is $\{\emptyset, \{a\}, \{c\}, \{a, b\}\}$, by definition 2.1; the family of $s\mu$-closed sets is $\{X, \{b, c\}, \{a, b\}, \{c\}\}$; the family of $s\lambda$-closed sets is $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$, by lemma 3.6 (1) and the family of $s\lambda$-open sets is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$. Hence $\{b\}$ is a $s\lambda$-closed set but not a $s\wedge_\mu$-set, since $s\{b\}_\mu^\wedge = \bigcap\{U : \{b\} \subset U, U$ is $s\mu$-open$\} = \{a, b\} \neq \{b\}$, by definition 2.2.

(ii): In a GTspace, example of a $s\lambda$-closed set which is not a $s\mu$-closed set:

Suppose $X = \{a, b, c\}, \mu = \{\emptyset, \{a, b\}, \{b, c\}, X\}$. Then $(X, \mu)$ is a GTspace but not a topological space. Among $P(X)$, the family of $\mu$-closed sets is $\{X, \{c\}, \{a\}, \emptyset\}$; the family of $s\mu$-open sets is $\{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, by definition 2.1 and the family of $s\mu$-closed sets is $\{X, \{c\}, \{a\}, \emptyset\}$. Then $\{a, b\}$ is $s\lambda$-closed as $s\{a, b\}_\mu^\wedge \cap \overline{s\{a, b\}_\mu} = \{a, b\} \cap \{a, b, c\} = \{a, b\}$, by lemma 3.6 (1). But $\{a, b\}$ is not $s\mu$-closed.

(iii): In a GTspace, example of a $sg_\lambda$-closed set which is not a $s\lambda$-closed set:

Let $X = \{a, b, c, d\}, \mu = \{\emptyset, \{a, b, c\}\}$. Then $(X, \mu)$ is a GTspace but not a topological space. Among $P(X)$, the family of $\mu$-closed sets is $\{X, \{d\}\}$; the family of $s\mu$-open sets is $\{\emptyset, \{d\}, \{a, b, c\}\}$, by definition 2.1; the family of $s\mu$-closed sets is $\{X, \{a, b, c\}, \{d\}\}$; the family of $s\lambda$-closed sets is $\{X, \{d\}, \{a, b, c\}\}$, by lemma 3.6 (1) and the family of $s\lambda$-open sets is

$\{\emptyset, \{d\}, \{a, b, c\}\}$. Consider the subset $\{a, d\} = B$ (say). Now by definition 2.4, $sB^{\wedge}_{\lambda} = \bigcap\{U : B \subset U, U$ is $s\lambda$-open$\}= X$ and $\overline{sB_{\lambda}} = X \subset X = sB^{\wedge}_{\lambda}$ and so by note 3.3, $B$ is $sg_{\lambda}$-closed but not $s\lambda$-closed.

The idea of $s\wedge_{\lambda}$-set has been used in section 4 to define $s\beta_{\lambda}$-closed set. Now we are giving the idea of $sg\wedge_{\lambda}$-set, a generalization of $s\wedge_{\lambda}$-set and it will be used to make a GTspace to be $s\lambda T_{\frac{1}{2}}$ in a different way.

**Definition 3.9.** (c.f.[17]). A set $A$ of a GTspace $(X, \mu)$ is called a semi generalised $\wedge_{\lambda}$-set ($sg\wedge_{\lambda}$-set in short ) if $sA^{\wedge}_{\lambda} \subset F$ whenever $F \supset A$ and $F$ is $s\lambda$-closed. $A$ is called a semi generalized $\vee_{\lambda}$-set ($sg\vee_{\lambda}$-set in short ) if $X - A$ is $sg\wedge_{\lambda}$-set.

**Note 3.10.** If a set $A$ of $X$ is a $s\wedge_{\lambda}$-set (resp. $s\vee_{\lambda}$-set) then $A$ is $sg\wedge_{\lambda}$-set (resp. $sg\vee_{\lambda}$-set). But converse may not be true as seen from example 3.20.

**Theorem 3.11.** *Let $(X, \mu)$ be a GTspace then*

*(1): For each $x \in X$, $\{x\}$ is either $s\lambda$-open or $sg\vee_{\lambda}$-set.*

*(2): For each $x \in X$, $\{x\}$ is either $s\lambda$-closed or $\{x\}$ is $sg_{\lambda}$-open.*

*Proof.* (1): Suppose $x \in X$ and $\{x\}$ is not $s\lambda$-open, then $X - \{x\}$ is not $s\lambda$-closed. But $X$ is $\mu$-closed $\Rightarrow X$ is $s\mu$-closed $\Rightarrow X$ is $s\lambda$-closed by remark 3.7, hence $\overline{s(X - \{x\})_{\lambda}} = X$. Now $s(X - \{x\})^{\wedge}_{\lambda} \subset \overline{s(X - \{x\})_{\lambda}} \Rightarrow X - \{x\}$ is $sg\wedge_{\lambda}$-set, by definition 3.9 and thus {x} is $sg\vee_{\lambda}$-set.

(2): Suppose $x \in X$ and $\{x\}$ is not $s\lambda$-closed, then $X - \{x\}$ is not $s\lambda$-open. By definition 2.4, $s(X-\{x\})^{\wedge}_{\lambda} = X \Rightarrow \overline{s(X - \{x\})_{\lambda}} \subset s(X-\{x\})^{\wedge}_{\lambda} \Rightarrow X - \{x\}$ is a $sg_{\lambda}$-closed set by note 3.3 and thus {x} is a $sg_{\lambda}$-open set. □

**Theorem 3.12.** *For a subset $A$ of a GTspace $(X, \mu)$ the following hold:*

*(1) if $A$ is $s\lambda$-closed then $A$ is $sg_{\lambda}$-closed;*

*(2) if $A$ is $sg_{\lambda}$-closed and $s\lambda$-open then $A$ is $s\lambda$-closed;*

*(3) if $A$ is $sg_{\lambda}$-closed and $A \subset B \subset \overline{sA_{\lambda}}$ then $B$ is $sg_{\lambda}$-closed.*

*Proof.* (1): This is obvious, by definition 3.2.

(2): Let $A$ be $sg_{\lambda}$-closed and $s\lambda$-open. Then $\overline{sA_{\lambda}} \subset A$, by definition 3.2 and so $A$ is $s\lambda$-closed.

(3): Let $B \subset U, U$ is $s\lambda$-open then $A \subset U$. By assumption $A$ is $sg_{\lambda}$-closed then $\overline{sA_{\lambda}} \subset U$. Again $A \subset B \subset \overline{sA_{\lambda}} \Rightarrow \overline{sA_{\lambda}} \subset \overline{sB_{\lambda}} \subset \overline{s(\overline{sA_{\lambda}})_{\lambda}} = \overline{sA_{\lambda}}$ then $\overline{sA_{\lambda}} = \overline{sB_{\lambda}}$ and hence $\overline{sB_{\lambda}} \subset U$. So $B$ is $sg_{\lambda}$-closed. □

**Theorem 3.13.** *Let $(X, \mu)$ be a GTspace and $A \subset X$ then following hold:*

*(1) $A$ is $sg_\lambda$-closed if and only if $\overline{sA_\lambda} - A$ does not contain any non-empty $s\lambda$-closed set;*

*(2) let $A$ be $s\wedge_\lambda$-set (resp. $s\vee_\lambda$-set) then $A$ is $sg_\lambda$-closed (resp. $sg_\lambda$-open ) if and only if $A$ is $s\lambda$-closed (resp. $s\lambda$-open);*

*(3) if $sA^\wedge_\lambda$ is $sg_\lambda$-closed (resp. $sA^\vee_\lambda$ is $sg_\lambda$-open) then $A$ is $sg_\lambda$-closed (resp. $sg_\lambda$-open).*

*Proof.* (1): The condition is necessary: suppose $A$ is a $sg_\lambda$-closed set and $P$ is a non-empty $s\lambda$-closed set, $P \subset \overline{sA_\lambda} - A$. Then $A \subset X - P$, a $s\lambda$-open set, so by definition 3.2, $\overline{sA_\lambda} \subset X - P \Rightarrow P \subset X - \overline{sA_\lambda}$. Thus $P \subset \overline{sA_\lambda} \cap (X - \overline{sA_\lambda}) = \emptyset$.

The condition is sufficient: assume the conditions hold and we shall prove that $A$ is $sg_\lambda$-closed. Let $A \subset V$ where $V$ is $s\lambda$-open. If $\overline{sA_\lambda} \not\subset V$ then $\overline{sA_\lambda} \cap (X - V)$ is a non-empty $s\lambda$-closed set contained in $\overline{sA_\lambda} - A$, a contradiction to the assumption. Hence $A$ is $sg_\lambda$-closed.

(2): Let $A$ be a $sg_\lambda$-closed set and a $s\wedge_\lambda$-set i.e. $A = sA^\wedge_\lambda$. Then by note 3.3, $\overline{sA_\lambda} \subset sA^\wedge_\lambda = A \Rightarrow \overline{sA_\lambda} = A$, hence $A$ is $s\lambda$-closed. On the other hand, a $s\lambda$-closed set is obviously a $sg_\lambda$-closed set. Hence the result follows.

Again, let $A$ be a $sg_\lambda$-open set and a $s\vee_\lambda$-set i.e. $A = sA^\vee_\lambda$. Then by theorem 3.5, $sInt_\lambda(A) \supset sA^\vee_\lambda = A \Rightarrow sInt_\lambda(A) = A$ and hence $A$ is $s\lambda$-open. On the other hand, a $s\lambda$-open set is obviously a $sg_\lambda$-open.

(3): Suppose $sA^\wedge_\lambda$ is $sg_\lambda$-closed. As $sA^\wedge_\lambda$ is a $s\wedge_\lambda$-set, by lemma 2.8 (6), then $sA^\wedge_\lambda$ is $s\lambda$-closed by (2). Now $A \subset sA^\wedge_\lambda \Rightarrow \overline{sA_\lambda} \subset \overline{s(sA^\wedge_\lambda)_\lambda} = sA^\wedge_\lambda$. Hence by note 3.3, $A$ is $sg_\lambda$-closed.

Again, suppose $sA^\vee_\lambda$ is $sg_\lambda$-open. As the set $sA^\vee_\lambda$ is a $s\vee_\lambda$-set by lemma 2.8 (6), then $sA^\vee_\lambda$ is $s\lambda$-open by (2). Now $A \supset sA^\vee_\lambda \Rightarrow sInt_\lambda(A) \supset sInt_\lambda(sA^\vee_\lambda) = sA^\vee_\lambda$. Hence $A$ is $sg_\lambda$-open by theorem 3.5. □

**Lemma 3.14.** *In a GTspace $(X, \mu)$ arbitrary intersection of $s\wedge_\mu$-sets is a $s\wedge_\mu$-set.*

*Proof.* Let $A_\alpha, \alpha \in \Delta, \Delta$ being an index set, be an arbitrary collection of $s\wedge_\mu$-sets and let $A = \bigcap\{A_\alpha; \alpha \in \Delta\}$. Then for each $\alpha \in \Delta, A_\alpha = s(A_\alpha)^\wedge_\mu$, by definition 2.2 and $A_\alpha \supset A \Rightarrow s(A_\alpha)^\wedge_\mu \supset sA^\wedge_\mu$ for all $\alpha \in \Delta$. This implies that $sA^\wedge_\mu \subset \bigcap s(A_\alpha)^\wedge_\mu = \bigcap A_\alpha = A \subset sA^\wedge_\mu$. Hence $A = sA^\wedge_\mu$. □

**Theorem 3.15.** *Suppose $(X, \mu)$ is a GTspace then the following hold:*

*(1) if $A_i, i \in I$, are $s\lambda$-closed sets of $X$, $I$ being an index set, then $\bigcap_i A_i$ is $s\lambda$-closed;*

*(2) if $A_i, i \in I$, are $s\lambda$-open sets of $X$, $I$ being an index set, then $\bigcup_i A_i$ is $s\lambda$-open.*

*Proof.* (1) Suppose $A_i, i \in I$, are $s\lambda$-closed sets, $I$ being an index set, then for each $i$ there exist a $s\wedge_\mu$-set $K_i$ and a $s\mu$-closed set $P_i$ such that $A_i = K_i \cap P_i$. Hence $\bigcap A_i = \bigcap(K_i \cap P_i) = (\bigcap K_i) \bigcap (\bigcap P_i)$. By lemma 3.14, $\bigcap K_i$ is a $s\wedge_\mu$-set and $\bigcap P_i$ is a $s\mu$-closed set. This shows that $\bigcap A_i$ is $s\lambda$-closed.

(2) Suppose $A_i, i \in I$, are $s\lambda$-open sets, $I$ being an index set. Then $X - A_i$ is $s\lambda$-closed set for each i and $X - \bigcup A_i = \bigcap(X - A_i)$. Therefore by (1), $X - \bigcup A_i$ is $s\lambda$-closed and hence $\bigcup A_i$ is $s\lambda$-open. □

**Corollary 3.16.** Collection of $s\lambda$-open sets in a GTspace $(X, \mu)$ forms generalized topology.

**Definition 3.17.** (c.f.[11]). A GTspace $(X, \mu)$ is said to be $s\lambda T_{\frac{1}{2}}$ if every $sg_\lambda$-closed set is $s\lambda$-closed.

**Theorem 3.18.** *(c.f.[17]). In a GTspace $(X, \mu)$, following are equivalent:*

*(1) $(X, \mu)$ is $s\lambda T_{\frac{1}{2}}$ GTspace;*

*(2) Every singleton of $X$ is either $s\lambda$-open or, $s\lambda$-closed;*

*(3) Every $sg\wedge_\lambda$-set is $s\wedge_\lambda$-set.*

*Proof.* (1) $\Rightarrow$ (2): Let $(X, \mu)$ be a $s\lambda T_{\frac{1}{2}}$ GTspace and $\{x\} \in X$. By theorem 3.11 (2), $\{x\}$ is either $s\lambda$-closed or $X - \{x\}$ is $sg_\lambda$-closed. If $X - \{x\}$ is $sg_\lambda$-closed then by assumption, $X - \{x\}$ is $s\lambda$-closed and so $\{x\}$ is $s\lambda$-open.

(2) $\Rightarrow$ (3): Let $A \subset X$ be a $sg\wedge_\lambda$-set which is not $s\wedge_\lambda$-set i.e. $A \neq sA^\wedge_\lambda$. Then $sA^\wedge_\lambda \not\subset A$, therefore there exists $x \in sA^\wedge_\lambda, x \notin A$. By supposition, $\{x\}$ is either $s\lambda$-open or $s\lambda$-closed.

Case (i): If $\{x\}$ is $s\lambda$-open, then $X - \{x\}$ is $s\lambda$-closed containing $A$. As $A$ is a $sg\wedge_\lambda$-set then by definition 3.9, $sA^\wedge_\lambda \subset X - \{x\} \Rightarrow x \notin sA^\wedge_\lambda$, a contradiction.

Case (ii): If $\{x\}$ is $s\lambda$-closed, then $X - \{x\}$ is $s\lambda$-open containing $A$. But $x \in sA^\wedge_\lambda \subset X - \{x\}$, a contradiction. Hence the result follows.

(3) $\Rightarrow$ (1) : Suppose every $sg\wedge_\lambda$-set is $s\wedge_\lambda$-set and $(X, \mu)$ is not $s\lambda T_{\frac{1}{2}}$ GTspace. Then there exists a $sg_\lambda$-closed set $A$ which is not $s\lambda$-closed i.e. $\overline{sA_\lambda} \neq A$, therefore there exists a point $x \in \overline{sA_\lambda}$ but $x \notin A$.

By theorem 3.11 (1), $\{x\}$ is either $s\lambda$-open or, $X - \{x\}$ is a $sg\wedge_\lambda$-set.

Case (i): suppose $\{x\}$ is $s\lambda$-open. Since $x \in \overline{sA_\lambda}, \{x\} \cap A \neq \emptyset \Rightarrow x \in A$, a contradiction, hence $A$ is $s\lambda$-closed.

Case (ii): if $X - \{x\}$ is a $sg\wedge_\lambda$ set, then by assumption, it is $s\wedge_\lambda$-set i.e. $(X - \{x\})^\wedge_\lambda = X - \{x\}$ by definition 2.4, hence $X - \{x\}$ is a $s\lambda$-open set containing $A$. Since $A$ is $sg_\lambda$-closed, $\overline{sA_\lambda} \subset X - \{x\}$ which contradicts that $x \in \overline{sA_\lambda}$. Hence $A$ is $s\lambda$-closed and $(X, \mu)$ is $s\lambda T_{\frac{1}{2}}$ GTspace. □

**Theorem 3.19.** *Following theorems for a $s\lambda$-open set $A$ of a GTspace $(X, \mu)$ can be deduced easily from the definitions.*

*(1) $A$ is $s\lambda$-open if and only if $A = N \cup H$, $N$ is a $s\vee_\mu$-set and $H$ is a $s\mu$-open set.*

*(2) $A$ is $s\lambda$-open if and only if $A = sA^\vee_\mu \cup sInt_\mu(A)$.*

*(3) $s\vee_\mu$-sets and $s\mu$-open sets are $s\lambda$-open sets.*

**Example 3.20.** In a GTspace, a $sg\wedge_\lambda$-set which is not a $s\wedge_\lambda$-set.

Let $X = R - Q$ and $\mu = \{\emptyset, (\{\sqrt{2}, \sqrt{7}\} \cup A; A \subset X, A \neq \emptyset)\}$. Then $(X, \mu)$ is a GTspace but not a topological space. Consider the set $B$ of all irrational numbers in $(1, 2)$. So $\sqrt{2} \in B$ but $\sqrt{7} \notin B$. As $B$ and $X - B$ do not contain a proper $\mu$-open set, $B$ is neither a $s\mu$-open set nor a $s\mu$-closed set, by definition 2.1. Hence $sB^\wedge_\mu = \bigcap\{X - \{x\} : B \subset X - \{x\}$, a $s\mu$-open set, $x \in X - B, x \neq \sqrt{7}\} = B \cup \{\sqrt{7}\}$, by definition 2.2 and $\overline{sB_\mu} = X$ as $\sqrt{2} \in B$. Therefore $\overline{sB_\mu} \cap sB^\wedge_\mu = B \cup \{\sqrt{7}\} \neq B$, so by lemma 3.6 (1), $B$ is not a $s\lambda$-closed set. Again, since each $\{y\} \subset B, y \neq \sqrt{2}$, is a $\mu$-closed set, it is a $s\mu$-closed set and since $B$ does not contain any $s\mu$-open set (except $\emptyset$), $sB^\vee_\mu \cup sInt_\mu(B) = (\bigcup\{\{y\} : \{y\} \subset B, y \neq \sqrt{2}, \{y\}$ is $s\mu$-closed$\})$ $\bigcup sInt_\mu(B) = (B - \{\sqrt{2}\}) \bigcup \emptyset = B - \{\sqrt{2}\} \neq B$ and hence $B$ is not a $s\lambda$-open set, by theorem 3.19 (2). For each $x \in X - B$ $(x \neq \sqrt{7}), X - \{x\}$ is a $s\mu$-open set so it is a $s\wedge_\mu$-set containing $B$, hence by remark 3.7, it is a $s\lambda$-closed set and these $s\mu$-open sets are also $s\lambda$-open, by theorem 3.19 (3). Therefore $sB^\wedge_\lambda = B \cup \{\sqrt{7}\} = \overline{sB_\lambda} \neq B \Rightarrow B$ is not a $s\wedge_\lambda$-set but a $sg\wedge_\lambda$-set since $sB^\wedge_\lambda \subset \overline{sB_\lambda}$, by definition 3.9.

**Remark 3.21.** In a GTspace $(X, \mu)$, intersection of two $\mu$-open sets may not be $\mu$-open can be seen from example 3.8 (ii). This leads by duality that union of two $\mu$-closed sets may not be $\mu$-closed which affects the topological closure property $\overline{A_\mu} \cup \overline{B_\mu} = \overline{(A \cup B)_\mu}; A, B \in X$.

## 4. $s\lambda T_{\frac{1}{4}}$ GTspace, $s\lambda T_{\frac{3}{8}}$ GTspace and $s\lambda$-homeomorphism

In this section we introduce $s\beta_\lambda$-closed set and $s\lambda T_{\frac{1}{4}}$, $s\lambda T_{\frac{3}{8}}$ axioms and investigate some of their properties including relation with $s\lambda T_0, s\lambda T_1$ and $s\lambda T_{\frac{1}{2}}$ axioms and discuss $s\lambda$-homeomorphism in a GTspace.

**Definition 4.1.** Suppose $(X, \mu)$ is a GTspace, then

(1) it is called $s\lambda T_0$ if for any two distinct points $x, y$ of $X$, there exists a $s\lambda$-open set $U$ which contains only one of the points;

(2) it is called $s\lambda T_1$ if for any two distinct points $x, y \in X$, there are $s\lambda$-open sets $U, V$ such that $x \in U, y \notin U, y \in V, x \notin V$.

**Theorem 4.2.** *$(X, \mu)$ is a $s\lambda T_0$ GTspace if and only if for any pair of distinct points $x, y \in X$, there is a set $A$ containing only one of the points such that $A$ is either $s\lambda$-open or $s\lambda$-closed.*

*Proof.* Condition necessary. Let a GTspace $(X, \mu)$ be $s\lambda T_0$ and $x, y \in X, x \neq y$. Then there is a $s\lambda$-open set $A$ such that $x \in A, y \notin A$ or there is a $s\lambda$-open set $B$ such that $y \in B, x \notin B$. Therefore $x \in X - B, y \notin X - B$ when $X - B$ is a $s\lambda$-closed set. Hence the result follows.

Condition sufficient. Suppose $x, y \in X, x \neq y$ and $A$ is a $s\lambda$-open set such that $x \in A$, and $y \notin A$. Then by definition, $(X, \mu)$ is a $s\lambda T_0$ GTspace. Now let $x \in A, y \notin A$ but $A$ is $s\lambda$-closed. Then $X - A$ is $s\lambda$-open and $y \in X - A, x \notin X - A$. In this case also the GTspace $(X, \mu)$ is a $s\lambda T_0$. □

**Theorem 4.3.** *A GTspace $(X, \mu)$ is $s\lambda T_1$ if and only if every singleton of $X$ is $s\lambda$-closed.*

*Proof.* First let $(X, \mu)$ be $s\lambda T_1$ and $x \in X$. Assume any point $y \in X$ such that $x \neq y$. Then there exists $s\lambda$-open set $V$ containing $y$ such that $x \notin V$. So $y$ can not be a $s\lambda$-adherence point of $\{x\}$. Therefore $\overline{s\{x\}_\lambda} = \{x\}$ and hence $\{x\}$ is $s\lambda$-closed.

Conversely, let every singleton of $X$ be $s\lambda$-closed and $x, y \in X, x \neq y$. Therefore $X - \{x\}$ and $X - \{y\}$ are $s\lambda$-open sets such that $y \in X - \{x\}, x \notin X - \{x\}$ and $x \in X - \{y\}, y \notin X - \{y\}$, so the GTspace $(X, \mu)$ is $s\lambda T_1$. □

Next we are going to define $s\beta_\lambda$-closed set which plays an important role for establishing a relation among $s\lambda T_0, s\lambda T_1, s\lambda T_{\frac{1}{2}}, s\lambda T_{\frac{1}{4}}, s\lambda T_{\frac{3}{8}}$ axioms.

**Definition 4.4.** A set $A$ of a GTspace $(X, \mu)$ is said to be $s\beta_\lambda$-closed if $A = H \cap Q$ where $H$ is a $s\wedge_\lambda$-set and $Q$ is a $s\lambda$-closed set. $A$ is $s\beta_\lambda$-open if $X - A$ is $s\beta_\lambda$-closed.

**Lemma 4.5.** *A set $A$ of a GTspace $(X, \mu)$ is $s\beta_\lambda$-closed if and only if* $A = sA^\wedge_\lambda \cap \overline{sA_\lambda}$.

Proof is similar to that of the lemma 3.6 (1) and so is omitted.

**Remark 4.6.** Clearly a $s\lambda$-closed set in a GTspace $(X, \mu)$ is both $sg_\lambda$-closed and $s\beta_\lambda$-closed. But converses may not be true as seen from examples 3.8 (iii) and (i). In example (i) it is seen that $\{a, c\}$ is $s\beta_\lambda$-closed, by lemma 4.5, but it is not $s\lambda$-closed. Necessary and sufficient condition is given here for a set to be $s\lambda$-closed.

**Theorem 4.7.** *A set $A$ of $X$ is $s\lambda$-closed if and only if $A$ is $sg_\lambda$-closed and $s\beta_\lambda$-closed.*

*Proof.* Necessary part is obvious.

Conversely, let $A$ be $sg_\lambda$-closed and $s\beta_\lambda$-closed. Since $A$ be $sg_\lambda$-closed, $\overline{sA_\lambda} \subset sA^\wedge_\lambda$, by note 3.3 and since $A$ is $s\beta_\lambda$-closed, by lemma 4.5, $A = sA^\wedge_\lambda \cap \overline{sA_\lambda} = \overline{sA_\lambda}$. Hence $A$ is $s\lambda$-closed. $\square$

We summarize the relationship among various types of generalized closed as well as open sets for a set $A \subset X$ in the following diagram; it is noted that, in general, none of the implications in the diagram is reversible.

(1)

$$\begin{tikzcd}[column sep=small]
 & \begin{matrix} s\vee_\mu\text{-set} \\ (A = sA^\vee_\mu) \end{matrix} & \begin{matrix} s\beta_\lambda\text{-closed} \\ (A = sA^\wedge_\lambda \cap \overline{sA_\lambda}) \end{matrix} & \begin{matrix} s\vee_\lambda\text{-set} \\ (A = sA^\vee_\lambda) \end{matrix} \arrow[r] & sg\vee_\lambda\text{-set} \\
\mu\text{-closed} \arrow[r] & s\mu\text{-closed} \arrow[r] \arrow[u] & \begin{matrix} s\lambda\text{-closed} \\ (A = sA^\wedge_\mu \cap \overline{sA_\mu}) \end{matrix} \arrow[r] \arrow[u] \arrow[ur] & \begin{matrix} sg_\lambda\text{-closed} \\ (\overline{sA_\lambda} \subset sA^\wedge_\lambda) \end{matrix} &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\mu\text{-open} \arrow[r] & s\mu\text{-open} \arrow[r] \arrow[d] & \begin{matrix} s\lambda\text{-open} \\ (A = sA^\vee_\mu \cup sInt_\mu(A)) \end{matrix} \arrow[r] \arrow[d] \arrow[dr] & \begin{matrix} sg_\lambda\text{-open} \\ (sA^\vee_\lambda \subset sInt_\lambda(A)) \end{matrix} & \\
 & \begin{matrix} s\wedge_\mu\text{-set} \\ (A = sA^\wedge_\mu) \end{matrix} & s\beta_\lambda\text{-open} & \begin{matrix} s\wedge_\lambda\text{-set} \\ (A = sA^\wedge_\lambda) \end{matrix} \arrow[r] & \begin{matrix} sg\wedge_\lambda\text{-set} \\ (sA^\wedge_\lambda \subset \overline{sA_\lambda}) \end{matrix}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
s\mu\text{-open} \arrow[r] & s\lambda\text{-closed}; \\
s\wedge_\mu\text{-set} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
s\mu\text{-closed} \arrow[r] & s\lambda\text{-open} \\
s\vee_\mu\text{-set} \arrow[ur] &
\end{tikzcd}$$

**Theorem 4.8.** *A GTspace $(X,\mu)$ is $s\lambda T_{\frac{1}{2}}$ if and only if every subset is $s\beta_\lambda$-closed.*

*Proof.* First assume that $(X,\mu)$ is a $s\lambda T_{\frac{1}{2}}$ GTspace and $A \subset X$. Then by theorem 3.18 (2), every singleton is either $s\lambda$-open or $s\lambda$-closed. Put $G = \bigcap\{X-\{x\} : x \in X-A, \{x\} \text{ is } s\lambda\text{-closed}\}$ and $H = \bigcap\{X-\{x\} : x \in X-A, \{x\} \text{ is } s\lambda\text{-open}\}$. Therefore, $G$ is a $s\wedge_\lambda$-set, by definition 2.4 and $H$ is a $s\lambda$-closed set and $A = G \cap H$. Hence $A$ is a $s\beta_\lambda$-closed set.

Conversely, suppose that every subset of $(X,\mu)$ is $s\beta_\lambda$-closed and $x \in X$, $\{x\}$ is not $s\lambda$-open. Then $X-\{x\}$ is not $s\lambda$-closed. Now $X$ is $\mu$-closed $\Rightarrow X$ is $s\mu$-closed $\Rightarrow X$ is $s\lambda$-closed, by remark 3.7 $\Rightarrow \overline{s(X-\{x\})_\lambda} = X$. By assumption, $X-\{x\}$ is a $s\beta_\lambda$-closed set and then by lemma 4.5, $X-\{x\} = s(X-\{x\})^\wedge_\lambda \cap \overline{s(X-\{x\})_\lambda} = s(X-\{x\})^\wedge_\lambda = \bigcap\{U : (X-\{x\}) \subset U, U \text{ is } s\lambda\text{-open}\}$. Therefore $X-\{x\}$ is $s\lambda$-open which implies that $\{x\}$ is $s\lambda$-closed and hence by theorem 3.18, $(X,\mu)$ is $s\lambda T_{\frac{1}{2}}$ GTspace. □

Levine [14] placed $T_{\frac{1}{2}}$ axiom in between $T_0$ and $T_1$ axioms in a topological space. Following are the definitions of new axioms which will be placed between $s\lambda T_0$ and $s\lambda T_{\frac{1}{2}}$ axioms.

**Definition 4.9.** (c.f.[17]). Suppose $(X,\mu)$ is a GTspace then it is called a

(1) $s\lambda T_{\frac{1}{4}}$ GTspace if for every finite subset $E$ of $X$ and for every $y \in X-E$, there exists a set $G_y$ containing $E$ and $G_y \cap \{y\} = \emptyset$ such that $G_y$ is either $s\lambda$-open or $s\lambda$-closed;

(2) $s\lambda T_{\frac{3}{8}}$ GTspace if for every countable subset $E$ of $X$ and for every $y \in X-E$, there exists a set $G_y$ containing $E$ and $G_y \cap \{y\} = \emptyset$ such that $G_y$ is either $s\lambda$-open or $s\lambda$-closed.

**Note 4.10.** Clearly if we take $E = \{x\}$ in the definition 4.9 (1), then in view of theorem 4.2 we see that every $s\lambda T_{\frac{1}{4}}$ GTspace is a $s\lambda T_0$, but converse may not be true as seen from example 4.12.

**Theorem 4.11.** *Following results hold in a GTspace $(X,\mu)$:*

*(1) $(X,\mu)$ is $s\lambda T_0$ if and only if every singleton of $X$ is $s\beta_\lambda$-closed;*

*(2) $(X,\mu)$ is $s\lambda T_{\frac{1}{4}}$ if and only if every finite subset of $X$ is $s\beta_\lambda$-closed;*

*(3) $(X,\mu)$ is $s\lambda T_{\frac{3}{8}}$ if and only if every countable subset of $X$ is $s\beta_\lambda$-closed.*

*Proof.* We prove the result (2) only; proofs of other are similar to that of (2) and so are omitted.

(2) Suppose $(X, \mu)$ is $s\lambda T_{\frac{1}{4}}$ GTspace and $E$ is a finite subset of $X$. Then by definition 4.9 (1), for every $y \in X - E$ there is a set $G_y$ containing $E$ and disjoint from $\{y\}$ such that $G_y$ is either $s\lambda$-open or $s\lambda$-closed. Let $K$ be the intersection of all such $s\lambda$-open sets $G_y$ and $P$ be the intersection of all such $s\lambda$-closed sets $G_y$ as $y$ runs over $X - E$. Then $K$ is a $s\wedge_\lambda$-set by definition 2.4 and $P$ is a $s\lambda$-closed set and $E = K \cap P$. Thus $E$ is $s\beta_\lambda$-closed.

Conversely, consider a finite subset $E \subset X$ and it is $s\beta_\lambda$-closed and $y \in X - E$. Then by definition 4.4, $E = K \cap P$ where $K$ is a $s\wedge_\lambda$-set and $P$ ia a $s\lambda$-closed set. If $y \notin P$ then the case is obvious since $P = \overline{sP_\lambda}$. If $y \in P$, then $y \notin K = sK_\lambda^\wedge = \cap\{U : K \subset U, U \text{ is } s\lambda\text{-open}\}$, so there exists some $s\lambda$-open set $U$ containing $E$ such that $y \notin U$. Hence $(X, \mu)$ is $s\lambda T_{\frac{1}{4}}$ GTspace. □

**Example 4.12.** A $s\lambda T_0$ GTspace which is not $s\lambda T_{\frac{1}{4}}$.

Let $X = R - Q$ and $\mu = \{\emptyset, (\{\sqrt{2}, \sqrt{3}\} \cup (X - (\{\sqrt{7}\} \cup A)); A \subset X, A$ is finite$)\}$. Then $(X, \mu)$ is a GTspace but not a topological space. We claim that every singleton is $s\beta_\lambda$-closed. Now $\{\sqrt{7}\}$ is $\mu$-closed $\Rightarrow$ $s\mu$-closed $\Rightarrow$ $s\lambda$-closed $\Rightarrow$ $s\beta_\lambda$-closed, by lemma 4.5. Again $s\{\sqrt{2}\}_\mu^\wedge = \bigcap\{U = X - \{\sqrt{7}, r\}, r \neq \sqrt{2}, \sqrt{3}; U$ is $s\mu$-open, $U \supset \{\sqrt{2}\}\} = \{\sqrt{2}, \sqrt{3}\}$ and $\overline{s\{\sqrt{2}\}_\mu} = \{\sqrt{2}, \sqrt{7}\}$, then $s\{\sqrt{2}\}_\mu^\wedge \cap \overline{s\{\sqrt{2}\}_\mu} = \{\sqrt{2}\} \Rightarrow \{\sqrt{2}\}$ is $s\lambda$-closed, by lemma 3.6 (1) $\Rightarrow$ it is $s\beta_\lambda$-closed. Similarly, $\{\sqrt{3}\}$ is $s\beta_\lambda$-closed. Again for each $x \in X, x \neq \sqrt{2}, \sqrt{3}, \sqrt{7}; s\{x\}_\mu^\wedge = \{x, \sqrt{2}, \sqrt{3}\}$ and $\overline{s\{x\}_\mu} = \{x, \sqrt{7}\}$ then $s\{x\}_\mu^\wedge \cap \overline{s\{x\}_\mu} = \{x\} \Rightarrow \{x\}$ is $s\lambda$-closed $\Rightarrow$ it is $s\beta_\lambda$-closed. Hence by theorem 4.11 (1), $(X, \mu)$ is a $s\lambda T_0$ GTspace.

Consider a finite set $D = \{\sqrt{2}, \sqrt{5}, \sqrt{11}\}$. Since $\sqrt{3}, \sqrt{7} \notin D, D$ is neither $s\mu$-open nor $s\mu$-closed. By theorem 3.19 (2), $D$ is not a $s\lambda$-open set. Now $sD_\mu^\wedge = \bigcap\{V = X - \{\sqrt{7}, r\}, r \neq \sqrt{3}, r \in X - D, V$ is $s\mu$-open, $V \supset D\} = D \cup \{\sqrt{3}\}$...........(Z) and $\overline{sD_\mu} = X$ as $\sqrt{2} \in D$, hence $sD_\mu^\wedge \cap \overline{sD_\mu} = D \cup \{\sqrt{3}\} \neq D \Rightarrow D$ is not $s\lambda$-closed. From (Z), we see that each $V$ is $s\mu$-open so it is $s\wedge_\mu$-set by definition 2.2, hence by remark 3.7, it is a $s\lambda$-closed set. Again each $V$ is also a $s\lambda$-open set by theorem 3.19 (3). Therefore, $sD_\lambda^\wedge = \bigcap\{V, V \supset D, V$ is $s\lambda$-open$\} = D \cup \{\sqrt{3}\} = \overline{sD_\lambda}$, so $sD_\lambda^\wedge \cap \overline{sD_\lambda} = D \cup \{\sqrt{3}\} \neq D \Rightarrow D$ is not $s\beta_\lambda$-closed. Hence by theorem 4.11 (2), $(X, \mu)$ is not a $s\lambda T_{\frac{1}{4}}$ GTspace.

**Remark 4.13.** Theorems 4.8 and 4.11 (1) show that a $s\lambda T_{\frac{1}{2}}$ GTspace is $s\lambda T_0$. It follows from theorems 4.8, 4.11 (3), 4.11 (2) that $s\lambda T_{\frac{1}{2}}$ axiom implies $s\lambda T_{\frac{3}{8}}$ axiom and $s\lambda T_{\frac{3}{8}}$ axiom implies $s\lambda T_{\frac{1}{4}}$ axiom. But those may not be reversible as seen from examples 4.14 and 4.15.

**Example 4.14.** A $s\lambda T_{\frac{1}{4}}$ GTspace which is not a $s\lambda T_{\frac{3}{8}}$ GTspace.

Let $X = R - Q$ and $\mu = \{\emptyset, (\{\sqrt{3}\} \cup (X - (\{\sqrt{2}\} \cup A)); A \subset X, A$ is finite$)\}$. Then $(X, \mu)$ is a GTspace but not a topological space. We assert that $(X, \mu)$ is a $s\lambda T_{\frac{1}{4}}$ GTspace. Assume $B$ is a finite set then consider following cases:

(i) if $\sqrt{3} \in B, \sqrt{2} \notin B$; then $B$ is not $\mu$-open as well as not $s\mu$-open and by definition 2.2, $sB^{\wedge}_{\mu} = \bigcap\{(X - \{\sqrt{2}, r\})$, a $s\mu$-open set containing $B, r \in X - B\} = B \Rightarrow B$ is $s\wedge_{\mu}$-set $\Rightarrow B$ is $s\lambda$-closed, by remark 3.7.

(ii) if $\sqrt{2} \in B, \sqrt{3} \notin B \Rightarrow B$ is $\mu$-closed $\Rightarrow B$ is $s\mu$-closed $\Rightarrow B$ is $s\lambda$-closed, by remark 3.7.

(iii) if $\sqrt{2}, \sqrt{3} \notin B$; then $sB^{\wedge}_{\mu} = \bigcap\{X - \{\sqrt{2}, r\}$, a $s\mu$-open set containing $B$, $r \in X - B, r \neq \sqrt{3}\} = B \cup \{\sqrt{3}\}$ and $\overline{sB_{\mu}} = B \cup \{\sqrt{2}\} \Rightarrow \overline{sB_{\mu}} \cap sB^{\wedge}_{\mu} = B \Rightarrow B$ is $s\lambda$-closed, by lemma 3.6 (1).

(iv) if $\sqrt{2}, \sqrt{3} \in B$; then $B$ is neither a $s\mu$-open set nor a $s\mu$-closed set. Now $sB^{\vee}_{\mu} \cup sInt_{\mu}(B) = \bigcup\{\{\sqrt{2}, r\}, r \in B, r \neq \sqrt{3}, \{\sqrt{2}, r\}$ is $s\mu$-closed$\}$ $\bigcup sInt_{\mu}(B) = (B - \{\sqrt{3}\}) \bigcup \emptyset = B - \{\sqrt{3}\} \neq B$; hence $B$ is not a $s\lambda$-open set by theorem 3.19 (2). Assume for each $p \in X - B$, $G_p = X - \{p\}$; then for each $r \neq \sqrt{3}, r \in B, \{\sqrt{2}, r, p\}$ is a $s\mu$-closed set and $G_p - \{\sqrt{2}, r\}$ is a $s\mu$-open set, hence $s(G_p)^{\vee}_{\mu} \cup sInt_{\mu}(G_p) = \{\sqrt{2}, r\} \cup (G_p - \{\sqrt{2}, r\}) = G_p$; so each $G_p$ is a $s\lambda$-open set containing $B$, this implies that $sB^{\wedge}_{\lambda} = B$.

Thus in all four cases we can easily assess by lemma 4.5 that $B$ (including each singleton) is a $s\beta_{\lambda}$-closed set and hence $(X, \mu)$ is $s\lambda T_{\frac{1}{4}}$ GTspace, by theorem 4.11 (2).

Now we are going to prove that $(X, \mu)$ is not a $s\lambda T_{\frac{3}{8}}$ GTspace. Consider a countable set $D$ of $X$ such that $\sqrt{2}, \sqrt{3} \notin D$, then $D$ is neither a $s\mu$-open set nor a $s\mu$-closed set. Therefore $sD^{\wedge}_{\mu} = \bigcap\{X - \{\sqrt{2}, r\}$, a $s\mu$-open set containing $D, r \in X - D, r \neq \sqrt{3}\} = D \cup \{\sqrt{3}\}$. Since $D$ is countable, $\overline{sD_{\mu}} = X$. Hence $\overline{sD_{\mu}} \cap sD^{\wedge}_{\mu} = D \cup \{\sqrt{3}\} \neq D \Rightarrow D$ is not a $s\lambda$-closed set, by lemma 3.6. Again $D$ contains none of $s\mu$-closed set and $s\mu$-open set (except $\emptyset$) , so by theorem 3.19 (2), $D$ is not a $s\lambda$-open set. Now for each $r \in X - D, r \neq \sqrt{3}, X - \{\sqrt{2}, r\}$ is a $s\mu$-open set so it is a $s\wedge_{\mu}$-set, hence a $s\lambda$-closed set containing $D$, by remark 3.7 and these $s\mu$-open sets are also $s\lambda$-open sets, by theorem 3.19 (3), hence $sD^{\wedge}_{\lambda} = D \cup \{\sqrt{3}\} = \overline{sD_{\lambda}}$. Thus

$sD^\wedge_\lambda \bigcap \overline{sD_\lambda} = \{\sqrt{3}\} \cup D \neq D$, then by lemma 4.5, $D$ is not $s\beta_\lambda$-closed and $(X, \mu)$ is not $s\lambda T_{\frac{3}{8}}$ GTspace, by theorem 4.11 (3).

**Example 4.15.** A $s\lambda T_{\frac{3}{8}}$ GTspace which is not a $s\lambda T_{\frac{1}{2}}$ GTspace.

Consider the GTspace $(X, \mu)$ as in example 4.14 and assume $A$ is countable in lieu of finite, $B$ is any countable set, $D$ is the set of all irrational numbers in $(0, 1)$, so $\{\sqrt{2}, \sqrt{3}\} \notin D$. Then it can be shown in the similar way followed in the said example that the GTspace is $s\lambda T_{\frac{3}{8}}$. To prove $(X, \mu)$ is not a $s\lambda T_{\frac{1}{2}}$ GTspace we follow the modality applied in the last part of the example 4.14 which affords us $sD^\wedge_\lambda = D \cup \{\sqrt{3}\} = \overline{sD_\lambda} \neq D \Rightarrow \overline{sD_\lambda} \subset sD^\wedge_\lambda$ and hence $D$ is a $sg_\lambda$-closed set, by note 3.3 but not a $s\lambda$-closed set, by lemma 3.6 (1). Hence the result follows.

**Theorem 4.16.** *If the GTspace $(X, \mu)$ is $s\lambda T_0$ then for every pair of distinct points $p, q \in X$, either $p \notin \overline{s\{q\}_\lambda}$ or $q \notin \overline{s\{p\}_\lambda}$.*

*Proof.* Let the GTspace $(X, \mu)$ be $s\lambda T_0$ and $p, q \in X, p \neq q$. Then there exists a $s\lambda$-open set $U$ which contains only one of $p, q$. Suppose $p \in U$ and $q \notin U$. Then the $s\lambda$-open set $U$ has an empty intersection with $\{q\}$ and hence $p \notin \overline{s\{q\}_\lambda}$. Similarly if $U$ contains the point $q$ but not $p$ then $q \notin \overline{s\{p\}_\lambda}$. □

Following definition will help us to convert a $s\lambda T_0$ GTspace to a $s\lambda T_1$.

**Definition 4.17.** A GTspace $(X, \mu)$ is said to be $s\lambda$-symmetric if $x, y \in X, x \in \overline{s\{y\}_\lambda} \Rightarrow y \in \overline{s\{x\}_\lambda}$.

**Theorem 4.18.** *A GTspace $(X, \mu)$ is $s\lambda$-symmetric if and only if $\{x\}$ is $sg_\lambda$-closed for each $x \in X$.*

*Proof.* Condition necessary. Assume $x \in X, \{x\} \subset U$, a $s\lambda$-open set, but $\overline{s\{x\}_\lambda} \not\subset U$. This implies that $\overline{s\{x\}_\lambda} \cap (X-U) \neq \emptyset$. Let $y \in \overline{s\{x\}_\lambda} \cap (X-U)$. Since the GTspace is $s\lambda$-symmetric, $x \in \overline{s\{y\}_\lambda}$. As $y \in X - U$, $\overline{s\{y\}_\lambda} \subset X - U$ and $x \notin U$, we arrive at a contradiction. Hence $\overline{s\{x\}_\lambda} \subset U$ and so by definition 3.2, $\{x\}$ is $sg_\lambda$-closed.

Condition sufficient. Suppose each singleton of $X$ is $sg_\lambda$-closed and for $x, y \in X, x \in \overline{s\{y\}_\lambda}$ but $y \notin \overline{s\{x\}_\lambda}$. Then $\{y\} \subset X - \overline{s\{x\}_\lambda}$, a $s\lambda$-open set. Since $\{y\}$ is $sg_\lambda$-closed, $\overline{s\{y\}_\lambda} \subset X - \overline{s\{x\}_\lambda}$, by definition 3.2 and hence $x \in X - \overline{s\{x\}_\lambda}$, a contradiction. Thus $y \in \overline{s\{x\}_\lambda}$. □

**Theorem 4.19.** *A GTspace $(X, \mu)$ is $s\lambda T_1$ if and only if it is $s\lambda$-symmetric and $s\lambda T_0$.*

*Proof.* Condition necessary. Let $(X, \mu)$ be $s\lambda T_1$ GTspace. Then obviously it is $s\lambda T_0$. Since $(X, \mu)$ is $s\lambda T_1$, every singleton is $s\lambda$-closed, by theorem 4.3 and so $sg_\lambda$-closed. Hence $(X, \mu)$ is $s\lambda$-symmetric by theorem 4.18.

Condition sufficient. Let $(X, \mu)$ be $s\lambda$-symmetric and $s\lambda T_0$ GTspace and let $x, y \in X, x \neq y$. Since $(X, \mu)$ is $s\lambda T_0$, either $x \notin \overline{s\{y\}_\lambda}$ or $y \notin \overline{s\{x\}_\lambda}$, by theorem 4.16. Let $x \notin \overline{s\{y\}_\lambda}$. Then $y \notin \overline{s\{x\}_\lambda}$. For if $y \in \overline{s\{x\}_\lambda}$ then it would imply $x \in \overline{s\{y\}_\lambda}$ since the GTspace is $s\lambda$-symmetric. Again since $x \notin \overline{s\{y\}_\lambda}$, there is a $s\lambda$-closed set $F$ such that $y \in F$ and $x \notin F$. So $x \in X - F$, a $s\lambda$-open set and $y \notin X - F$. Also since $y \notin \overline{s\{x\}_\lambda}$, there is a $s\lambda$-closed set $P$ such that $x \in P$ and $y \notin P$. So $y \in X - P$, a $s\lambda$-open set and $x \notin X - P$. Hence $(X, \mu)$ is $s\lambda T_1$. □

**Remark 4.20.** The main purpose of this paper to explore mutual relation among the axioms discussed herein above is now summarized as $s\lambda T_1 \subseteq s\lambda T_{\frac{1}{2}} \subseteq s\lambda T_{\frac{3}{8}} \subseteq s\lambda T_{\frac{1}{4}} \subseteq s\lambda T_0$ and it is not reversible and next theorem shows that all axioms are equivalent under certain condition.

**Theorem 4.21.** *If the GTspace $(X, \mu)$ is $s\lambda$-symmetric then the axioms $s\lambda T_0$, $s\lambda T_1$, $s\lambda T_{\frac{1}{2}}$, $s\lambda T_{\frac{3}{8}}$, $s\lambda T_{\frac{1}{4}}$ are all equivalent.*

*Proof.* (1): By theorem 4.19, $s\lambda$-symmetric $s\lambda T_0$ axiom implies $s\lambda T_1$ axiom.

(2): By theorem 4.3 and 3.18, $s\lambda T_1$ axiom implies $s\lambda T_{\frac{1}{2}}$ axiom.

(3): By theorem 4.8 and 4.11 (3), $s\lambda T_{\frac{1}{2}}$ axiom implies $s\lambda T_{\frac{3}{8}}$ axiom.

(4): By theorem 4.11 (3) and 4.11 (2), $s\lambda T_{\frac{3}{8}}$ axiom implies $s\lambda T_{\frac{1}{4}}$ axiom.

(5): By theorem 4.11 (2) and 4.11 (1), $s\lambda T_{\frac{1}{4}}$ axiom implies $s\lambda T_0$ axiom. □

**Definition 4.22.** (c.f.[2]). Let $(X, \mu_1)$ and $(Y, \mu_2)$ be two GTspaces. Then a function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is said to be $s\lambda$-continuous (respectively $s\beta_\lambda$-continuous and $sg_\lambda$-continuous) if the inverse image of each $\mu_2$-open set is $s\lambda$-open (respectively $s\beta_\lambda$-open and $sg_\lambda$-open) in $(X, \mu_1)$.

Clearly, a function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is $s\lambda$-continuous (respectively $s\beta_\lambda$-continuous and $sg_\lambda$-continuous) if the inverse image of each $\mu_2$-closed set is $s\lambda$-closed (respectively $s\beta_\lambda$-closed and $sg_\lambda$-closed) in $(X, \mu_1)$.

**Theorem 4.23.** *Suppose $(X, \mu_1)$ and $(Y, \mu_2)$ are two GTspaces. A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is $s\lambda$-continuous if and only if $f$ is both $s\beta_\lambda$-continuous and $sg_\lambda$-continuous.*

*Proof.* Condition necessary. Suppose a function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is $s\lambda$-continuous and $V$ is a $\mu_2$-closed set. Then $f^{-1}(V)$ is $s\lambda$-closed in $(X, \mu_1)$. Hence by remark 4.6, $f^{-1}(V)$ is both $s\beta_\lambda$-closed and $sg_\lambda$-closed in $(X, \mu_1)$. Hence $f$ is both $s\beta_\lambda$-continuous and $sg_\lambda$-continuous.

Condition sufficient. Suppose $f$ is both $s\beta_\lambda$-continuous and $sg_\lambda$-continuous and $V$ is a $\mu_2$-closed set. Then $f^{-1}(V)$ is both $s\beta_\lambda$-closed and $sg_\lambda$-closed in $(X, \mu_1)$. So by theorem 4.7, $f^{-1}(V)$ is $s\lambda$-closed in $(X, \mu_1)$. Hence the result follows. □

**Definition 4.24.** Let $(X, \mu_1)$ and $(Y, \mu_2)$ be two GTspaces. A bijective mapping $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called $s\lambda$-homeomorphism if the following conditions hold:

(i) inverse image of every $s\lambda$-open set in $(X, \mu_2)$ under $f$ is $s\lambda$-open in $(X, \mu_1)$ and

(ii) inverse image of every $s\lambda$-open set in $(X, \mu_1)$ under $f^{-1}$ is $s\lambda$-open in $(X, \mu_2)$.

Note that if $\mu_1 = \mu_2 = \mu$ then the conditions (i) and (ii) may be expressed as a single condition (B): "Inverse image of every $s\lambda$-open set in $(X, \mu)$ is $s\lambda$-open in $(X, \mu)$ under $f$ and $f^{-1}$."

For a set $E$ of a GTspace $(X, \mu)$ we denote the set of all maps $f : (X, \mu) \longrightarrow (X, \mu)$ such that $f$ is a $s\lambda$-homeomorphism and $f(E) = E$ by $s\lambda h(X, E, \mu)$. When $E = \emptyset$ we denote $s\lambda h(X, \emptyset, \mu)$ simply by $s\lambda h(X, \mu)$.

The composition $g \circ f$ for every $f, g \in s\lambda h(X, \mu)$ is clearly a binary operation on $s\lambda h(X, \mu)$.

**Theorem 4.25.** *Suppose $(X, \mu)$ is a GTspace. Then $s\lambda h(X, \mu)$ is a group and $s\lambda h(X, E, \mu)$ is a subgroup of $s\lambda h(X, \mu)$ for each $E \subset X$ with respect to the composition of maps of $s\lambda h(X, \mu)$ defined above.*

*Proof.* Clearly the operation is associative and closed. Since the identity map satisfies the condition (B), it belongs to $s\lambda h(X, \mu)$ and it is the identity element of the set. Again if any $f \in s\lambda h(X, \mu)$, then $f^{-1}$ satisfies the condition (B) and hence $f^{-1} \in s\lambda h(X, \mu)$. So $s\lambda h(X, \mu)$ is a group with this binary operation.

Since $s\lambda h(X, E, \mu) \subset s\lambda h(X, \mu)$ then we have to prove only that for any $f, g \in s\lambda h(X, E, \mu), g \circ f^{-1} \in s\lambda h(X, E, \mu)$. Since $f, g \in s\lambda h(X, E, \mu)$, then $f, f^{-1}, g, g^{-1}$ satisfy the condition (B) which map $E$ into $E$. Hence

$g \circ f^{-1}$ and $f \circ g^{-1}$ satisfy the condition (B). Again, $g \circ f^{-1}(E) = g(E) = E$. Therefore $s\lambda h(X, E, \mu)$ is a subgroup of $s\lambda h(X, \mu)$ for each $E \subset X$. $\square$

**Conclusion:** By introducing $s\lambda$-, $sg_\lambda$-, $s\beta_\lambda$-closed sets and $sg\wedge_\lambda$-set we have investigated some of their properties and also furnished mutual relation among low separation axioms viz. $s\lambda T_{\frac{1}{4}}, s\lambda T_{\frac{3}{8}}, s\lambda T_{\frac{1}{2}}$ axioms along with $s\lambda T_0$ and $s\lambda T_1$ axioms. Moreover, we have studied $s\lambda$-homeomorphism in a GTspace. There is a scope of further investigation in the field of separation axioms with ideal, compactness, connectedness, countability.

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# REPRODUCING KERNEL FOR ROBIN BOUNDARY CONDITIONS

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Abstract. In this paper, we introduced a reproducing kernel space which is a particular class of Hilbert space. We discuss various properties of the reproducing kernel. In particular, our aim to construct kernel in reproducing kernel Hilbert space of the specific function space (Sobolev space) with the inner product and norm. Also, we derive the reproducing kernel for Robin boundary conditions.

## 1. Introduction

Reproducing kernels were discovered during the initial stage of the twentieth century by Zeremba[17] in that effort the center of interest on harmonic function with boundary value. This was the earliest reproducing kernel with the reproducibility proved correlated with function family. Actually, in the early establishment, development of the reproducing kernel hypothesis, almost all the works were executed by Bergman[8, 9, 10, 11, 12] , and most of the kernels discussed in the 1930's and 1940's are Bergman kernels. Bergman raises the conversation of the kernels with one or several variables to the harmonic functions and utilized to solve the Laplace equation. It can be stated that this is the establishment of a particular trend of reproducing kernel. The next development of the reproducing kernel theory was pushed by Mercer [16]. He invented the positive definite property of reproducing kernel and known it as positive definite Hermition matrix:

$$\sum_{i,j=1}^{n} k(x_i, x_j)\gamma_i\gamma_j \geq 0 \tag{1.1}$$

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In 1950,N. Aronszajn[1] outlined the past works and gave a systematic reproducing kernel theory and laid a good foundation for the research of each special case and greatly simplified the proof. In this theory unifying the Bergman and Marces concept of reproducing kernel development.

Subsequently, reproducing kernel theory was used by the mathematician, scientist [2, 3, 4, 5, 6, 7, 14, 15] like to solve the theoretical problems of many special fields. In 1986, Cui [13] construct the reproducing kernel space and corresponding kernel in the Sobolev space.

Here, we review some aspects of reproducing kernel space and then construct the reproducing kernel for the inner product and norm of Sobolev space for $m = 2$ with Robin boundary conditions.

## 2. Preliminaries

In this section, we provide definition of reproducing kernel Hilbert space and necessary lemmas which are used in the proof of next section theorems.

**Definition 2.1.** *Consider $\mathcal{H} = \{f(\varrho) : f(\varrho) \in \mathbb{R}$ or $f(\varrho) \in \mathbb{C}$ , $\varrho$ is in abstract set$\}$ is endowed with $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{H}}$ , with respect to which $\mathcal{H}$ is a Hilbert space.*
*For an abstract set $X$, a function $\mathcal{R}(\varrho, \varphi) : X \times X \to \mathbb{F}$ ( $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$ )is called the reproducing kernel of Hilbert space $\mathcal{H}$ if its satisfies,*

$$\langle f(\varrho), \mathcal{R}(\varrho, \varphi)\rangle_{\mathcal{H}} = f(\varphi), \tag{2.1}$$

*for each fixed $\varphi \in X$.*

**Lemma 2.2.** In *reproducing kernel space $\mathcal{H}$, $\mathcal{R}(\varrho, \varphi) = \overline{\mathcal{R}(\varphi, \varrho)}$.*

*Proof.* We have

$$\mathcal{R}(\varrho, \varphi) = \langle \mathcal{R}(\cdot, \varphi), \mathcal{R}(\cdot, \varrho)\rangle_{\mathcal{H}} = \overline{\langle \mathcal{R}(\cdot, \varrho), \mathcal{R}(\cdot, \varphi)\rangle_{\mathcal{H}}} = \overline{\mathcal{R}(\varphi, \varrho)}.$$

Hence, $\mathcal{R}(\varrho, \varphi)$ is conjugate symmetric. □

**Lemma 2.3.** *The reproducing kernel $\mathcal{R}(\varrho, \varphi)$ is unique in reproducing kernel space $\mathcal{H}$.*

*Proof.* Let $Q(\varrho, \varphi)$ be also reproducing kernel , then

$$Q(\varrho, \varphi) = \langle Q(\cdot, \varphi), \mathcal{R}(\cdot, \varrho)\rangle_{\mathcal{H}} = \overline{\langle \mathcal{R}(\cdot, \varrho), Q(\cdot, \varphi)\rangle_{\mathcal{H}}} = \overline{\mathcal{R}(\varphi, \varrho)} = \mathcal{R}(\varrho, \varphi).$$

Hence, reproducing kernel is unique. □

**Lemma 2.4.** *If $\mathcal{R}(\varrho,\varphi)$ is the reproducing kernel in $\mathcal{H}$ , then for each $\varrho \in X$ ,$\mathcal{R}(\varrho,\varrho) \geq 0$ and $\mathcal{R}(\varrho,\varrho) = 0$ if and only if $\mathcal{H} = \{0\}$ .*

*Proof.* We have

$$\mathcal{R}(\varrho,\varrho) = \langle \mathcal{R}(\cdot,\varrho), \mathcal{R}(\cdot,\varrho)\rangle_{\mathcal{H}} = \|\mathcal{R}(\cdot,\varrho)\|^2_{\mathcal{H}}.$$

Which gives $\mathcal{R}(\varrho,\varrho) \geq 0$ and $\mathcal{R}(\varrho,\varrho) = 0$ if and only if $\mathcal{H} = \{0\}$. □

**Lemma 2.5.** *Reproducing kernel $\mathcal{R}(\varrho,\varphi)$ is a positive definite.*

*Proof.* For any complex number $\gamma_i$,

$$\begin{aligned}\sum_{\imath,\jmath=1}^{n} \overline{\gamma_i}\gamma_j \mathcal{R}(\varrho_i,\varrho_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\gamma_i}\gamma_j \left\langle \mathcal{R}(\cdot,\varrho_i), \mathcal{R}(\cdot,\varrho_j)\right\rangle_{\mathcal{H}} \\ &= \left\langle \sum_{j=1}^{n} \gamma_j \mathcal{R}(\cdot,\varrho_j), \sum_{i=1}^{n} \gamma_i \mathcal{R}(\cdot,\varrho_i) \right\rangle_{\mathcal{H}} \\ &= \left\langle \sum_{i=1}^{n} \gamma_i \mathcal{R}(\cdot,\varrho_i), \sum_{i=1}^{n} \gamma_i \mathcal{R}(\cdot,\varrho_i) \right\rangle_{\mathcal{H}} \\ &= \left\| \sum_{i=1}^{n} \gamma_i \mathcal{R}(\cdot,\varrho_i) \right\|^2_{\mathcal{H}} \geq 0.\end{aligned}$$

Hence, reproducing kernel is positive definite. □

**Lemma 2.6.** *For any fixed $\varrho \in X$, the linear functional $\mathcal{I}(f(\varrho)) = f(\varrho)$ is bounded if and only if Hilbert space $\mathcal{H}$ is a reproducing kernel space.*

*Proof.* Since $\mathcal{H}$ is a reproducing kernel space, there exists a reproducing kernel $\mathcal{R}(\varrho,\varphi)$.

$$\begin{aligned}|\mathcal{I}(f(\varrho))| = |f(\varrho)| &= |\langle f(\cdot), \mathcal{R}(\cdot,\varrho)\rangle_{\mathcal{H}}| \\ &\leq \|f(\cdot)\|_{\mathcal{H}} \|\mathcal{R}(\cdot,\varrho)\|_{\mathcal{H}} \\ &= \|f(\cdot)\|_{\mathcal{H}} \sqrt{\langle \mathcal{R}(\cdot,\varrho), \mathcal{R}(\cdot,\varrho)\rangle_{\mathcal{H}}} \\ &= \|f(\cdot)\|_{\mathcal{H}} \sqrt{\mathcal{R}(\varrho,\varrho)}.\end{aligned}$$

Therefore, $\mathcal{I}(f(\varrho)) = f(\varrho)$ is bounded.
Now, for every $f(\varrho) \in \mathcal{H}$, because of linear functional, by F. Riesz theorem there exists a unique $\mathcal{R}(\cdot,\varrho) \in \mathcal{H}$, whence $f(\varrho) = \mathcal{I}(f(\varrho)) = \langle f(\cdot), \mathcal{R}(\cdot,\varrho)\rangle_{\mathcal{H}}$.
Hence, the lemma is proved. □

## 3. Reproducing Kernel Space $\mathcal{W}_2^m[\alpha,\beta]$

In this section, the function space $\mathcal{W}_2^m[\alpha,\beta]=\{f(\varrho): f^{(m-1)}(\varrho)$ is absolutely continuous, $f^{(m)}(\varrho)\in L^2[\alpha,\beta], \varrho\in[\alpha,\beta]\}$.
For any functions $f(\varrho), g(\varrho)\in \mathcal{W}_2^m[\alpha,\beta]$,

$$\begin{aligned}\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m} = &\sum_{i=0}^{m-1}\left[\frac{d^i f(\alpha)}{d\varrho^i}\frac{d^i g(\alpha)}{d\varrho^i}+\frac{d^i f(\beta)}{d\varrho^i}\frac{d^i g(\beta)}{d\varrho^i}\right]\\ &+\int_\alpha^\beta \frac{d^m f(\varrho)}{d\varrho^m}\frac{d^m g(\varrho)}{d\varrho^m}d\varrho,\end{aligned} \tag{3.1}$$

$$\|f(\varrho)\|_{\mathcal{W}_2^m}=\sqrt{\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}}. \tag{3.2}$$

**Theorem 3.1.** *The space $\mathcal{W}_2^m[\alpha,\beta]$ is an inner product space.*

*Proof.* Let $f(\varrho), g(\varrho), h(\varrho)\in\mathcal{W}_2^m[\alpha,\beta]$.
Here,

$$\begin{aligned}\langle f(\varrho), f(\varrho)\rangle_{\mathcal{W}_2^m} = &\sum_{i=0}^{m-1}\left[\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2+\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2\right]\\ &+\int_\alpha^\beta\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho,\end{aligned}$$

since, $\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2>0$ and $\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2>0, 0\le i\le m-1$, also, $\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2=0$ and $\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2=0$ if and only if $\frac{d^i f(\alpha)}{d\varrho^i}=0$ and $\frac{d^i f(\beta)}{d\varrho^i}=0$, $0\le i\le m-1$ and $\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2>0$ and $\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2=0$ if and only if $\frac{d^m f(\varrho)}{d\varrho^m}=0, \forall\varrho\in[\alpha,\beta]$.
Therefore, $\int_\alpha^\beta\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho>0$ and $\int_\alpha^\beta\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho=0$ if and only if $\frac{d^m f(\varrho)}{d\varrho^m}=0, \forall\varrho\in[\alpha,\beta]$.
Thus, $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$ is positive definite.
Clearly, $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}=\langle g(\varrho), f(\varrho)\rangle_{\mathcal{W}_2^m}$ which gives $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$ is symmetric.
Now for linearity consider scalars $a$ and $b$,

$$\begin{aligned}\langle af(\varrho)+bg(\varrho), h(\varrho)\rangle_{\mathcal{W}_2^m} = &\sum_{i=0}^{m-1}\left[\left(a\frac{d^i f(\alpha)}{d\varrho^i}+b\frac{d^i g(\alpha)}{d\varrho^i}\right)\frac{d^i h(\alpha)}{d\varrho^i}\right.\\ &\left.+\left(a\frac{d^i f(\beta)}{d\varrho^i}+b\frac{d^i g(\beta)}{d\varrho^i}\right)\frac{d^i h(\beta)}{d\varrho^i}\right]\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&+\int_{\alpha}^{\beta}\left[a\frac{d^m f(\varrho)}{d\varrho^m}+b\frac{d^m g(\varrho)}{d\varrho^m}\right]\frac{d^m h(\varrho)}{d\varrho^m}d\varrho\\
=&\sum_{i=0}^{m-1}\left[a\frac{d^i f(\alpha)}{d\varrho^i}\frac{d^i h(\alpha)}{d\varrho^i}+a\frac{d^i f(\beta)}{d\varrho^i}\frac{d^i h(\beta)}{d\varrho^i}\right]\\
&+\int_{\alpha}^{\beta}a\frac{d^m f(\varrho)}{d\varrho^m}\frac{d^m h(\varrho)}{d\varrho^m}d\varrho\\
&+\sum_{i=0}^{m-1}\left[b\frac{d^i f(\alpha)}{d\varrho^i}\frac{d^i h(\alpha)}{d\varrho^i}+b\frac{d^i f(\beta)}{d\varrho^i}\frac{d^i h(\beta)}{d\varrho^i}\right]\\
&+\int_{\alpha}^{\beta}b\frac{d^m f(\varrho)}{d\varrho^m}\frac{d^m h(\varrho)}{d\varrho^m}d\varrho\\
=&a\langle f(\varrho),h(\varrho)\rangle_{\mathcal{W}_2^m}+b\langle g(\varrho),h(\varrho)\rangle_{\mathcal{W}_2^m}.
\end{aligned}
$$

Thus, $\langle f(\varrho),g(\varrho)\rangle_{\mathcal{W}_2^m}$ is linear.
This completes the proof. □

**Theorem 3.2.** *The space $\mathcal{W}_2^m[\alpha,\beta]$ is a Hilbert space.*

*Proof.* Consider $f_n(\varrho)$, $n=1,2,\dots$ is a Cauchy sequence in $\mathcal{W}_2^m[\alpha,\beta]$. Therefore,

$$
\begin{aligned}
\|f_{n+p}-f_n\|^2_{\mathcal{W}_2^m}=&\sum_{i=0}^{m-1}\left[\left(\frac{d^i f_{n+p}(\alpha)}{d\varrho^i}-\frac{d^i f_n(\alpha)}{d\varrho^i}\right)^2\right.\\
&\left.+\left(\frac{d^i f_{n+p}(\beta)}{d\varrho^i}-\frac{d^i f_n(\beta)}{d\varrho^i}\right)^2\right]\\
&+\int_{\alpha}^{\beta}\left(\frac{d^m f_{n+p}(\varrho)}{d\varrho^m}-\frac{d^m f_n(\varrho)}{d\varrho^m}\right)^2 d\varrho\to 0
\end{aligned}
$$

as $n\to\infty$.
Which gives, $\frac{d^i f_{n+p}(\alpha)}{d\varrho^i}-\frac{d^i f_n(\alpha)}{d\varrho^i}\to 0$ as $n\to\infty$, $0\le i\le m-1$,$n=1,2,\dots$.
Similarly, $\frac{d^i f_{n+p}(\beta)}{d\varrho^i}-\frac{d^i f_n(\beta)}{d\varrho^i}\to 0$ as $n\to\infty$, $0\le i\le m-1$,$n=1,2,\dots$ and

$$
\int_{\alpha}^{\beta}\left(\frac{d^m f_{n+p}(\varrho)}{d\varrho^m}-\frac{d^m f_n(\varrho)}{d\varrho^m}\right)^2 d\varrho\to 0\quad\text{as}\quad n\to\infty.
$$

Which indicates that for any $i$ $(0\le i\le m-1)$, the sequence $\frac{d^i f_n(\alpha)}{d\varrho^i}$ and $\frac{d^i f_n(\beta)}{d\varrho^i}$,$n=1,2,\dots$ are Cauchy sequences in R and $\frac{d^m f_n(\varrho)}{d\varrho^m}$,$n=1,2,\dots$ is a Cauchy sequence in space $L^2[\alpha,\beta]$.
So, there exists unique real numbers $c_i$ and $d_i$, $0\le i\le m-1$ and unique

function $h(\varrho) \in L^2[\alpha,\beta]$ such that, $\frac{d^i f_n(\alpha)}{d\varrho^i} \to c_i$ and $\frac{d^i f_n(\beta)}{d\varrho^i} \to d_i$ ,$0 \leq i \leq m-1$ and $\int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - h(\varrho)\right)^2 d\varrho \to 0$ as $n \to \infty$.
We must have $g(\varrho) \in \mathcal{W}_2^m[\alpha,\beta]$ with $\frac{d^i g(\alpha)}{d\varrho^i} = c_i$ , $\frac{d^i g(\beta)}{d\varrho^i} = d_i$ , $0 \leq i \leq m-1$ and $\frac{d^m g(\varrho)}{d\varrho^m} = h(\varrho)$.
Moreover,

$$\begin{aligned}
\|f_n(\varrho) - g(\varrho)\|^2_{\mathcal{W}_2^m} &= \sum_{i=0}^{m-1}\left[\left(\frac{d^i f_n(\alpha)}{d\varrho^i} - \frac{d^i g(\alpha)}{d\varrho^i}\right)^2 + \left(\frac{d^i f_n(\beta)}{d\varrho^i} - \frac{d^i g(\beta)}{d\varrho^i}\right)^2\right] \\
&\quad + \int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - \frac{d^m g(\varrho)}{d\varrho^m}\right)^2 d\varrho \\
&= \sum_{i=0}^{m-1}\left[\left(\frac{d^i f_n(\alpha)}{d\varrho^i} - c_i\right)^2 + \left(\frac{d^i f_n(\beta)}{d\varrho^i} - d_i\right)^2\right] \\
&\quad + \int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - h(\varrho)\right)^2 d\varrho \\
&\quad \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence, the function space $\mathcal{W}_2^m$ is a Hilbert space. □

**Theorem 3.3.** *The space $\mathcal{W}_2^m[\alpha,\beta]$ is a reproducing kernel Hilbert space.*

*Proof.* As per lemma 2.6, suppose that $\mathcal{I}(f) = f(\varrho)$, $\varrho \in [\alpha,\beta]$ is linear functional of $\mathcal{W}_2^m[\alpha,\beta]$ and $f(\varrho) \in \mathcal{W}_2^m$.
We have,

$$\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} = \frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}} + \int_\alpha^\varrho \frac{d^m f(\varrho)}{d\varrho^m} d\varrho,$$

and

$$\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} = \int_\varrho^\beta \frac{d^m f(\varrho)}{d\varrho^m} d\varrho - \frac{d^{m-1} f(\beta)}{d\varrho^{m-1}}.$$

Therefore,

$$\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} = \frac{1}{2}\left[\frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}} - \frac{d^{m-1} f(\beta)}{d\varrho^{m-1}}\right] + \frac{1}{2}\int_\alpha^\beta \frac{d^m f(\varrho)}{d\varrho^m} d\varrho.$$

Obviously,

$$\left|\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}}\right| \leq \left|\frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}}\right| + \left|\frac{d^{m-1} f(\beta)}{d\varrho^{m-1}}\right| + \int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right| d\varrho. \tag{3.4}$$

Since,

$$\begin{aligned}
\int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right| d\varrho &\le \left[(\beta-\alpha)\int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2 d\varrho\right]^{\frac{1}{2}} \\
&= K_0 \left[\int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2 d\varrho\right]^{\frac{1}{2}} \\
&\le K_0 \left[\sum_{i=0}^{m-1}\left(\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 + \left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2\right)\right. \qquad (3.5) \\
&\quad \left. + \int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2\right]^{\frac{1}{2}} \\
&= K_0 \|f\|_{\mathcal{W}_2^m}.
\end{aligned}$$

Now, for any $i$, $0 \le i \le m-1$,

$$\begin{aligned}
\left|\frac{d^i f(\alpha)}{d\varrho^i}\right| &\le \left[\sum_{i=0}^{m-1}\left(\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 + \left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2\right) + \int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2\right]^{\frac{1}{2}} \qquad (3.6) \\
&= \|f\|_{\mathcal{W}_2^m}.
\end{aligned}$$

Similarly,

$$\left|\frac{d^i f(\beta)}{d\varrho^i}\right| \le \|f\|_{\mathcal{W}_2^m}. \tag{3.7}$$

From (3.4) to (3.7),

$$\left|\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}}\right| \le K_1 \|f\|_{\mathcal{W}_2^m}. \tag{3.8}$$

Analogously,

$$\left|\frac{d^{m-2} f(\varrho)}{d\varrho^{m-2}}\right| \le K_2 \|f\|_{\mathcal{W}_2^m}. \tag{3.9}$$

Thus,$|\mathcal{I}(f)| = |f(\varrho)| \le K_m \|f\|_{\mathcal{W}_2^m}$.
Hence, $\mathcal{I}$ is bounded functional which provide that $\mathcal{W}_2^m[a,b]$ is reproducing kernel Hilbert space. □

## 4. Method to Construct Reproducing Kernel

Suppose $\mathcal{R}(\varrho,\varphi)$ is the reproducing kernel function of $\mathcal{W}_2^m[\alpha,\beta]$, then for any fixed $\varphi\in[\alpha,\beta]$ and any $f(\varrho)\in\mathcal{W}_2^m[\alpha,\beta]$, $\mathcal{R}(\varrho,\varphi)$ must satisfy

$$\langle f(\varrho),\mathcal{R}(\varrho,\varphi)\rangle_{\mathcal{W}_2^m}=f(\varphi). \tag{4.1}$$

Therefore,

$$\begin{aligned}\langle f(\varrho),\mathcal{R}(\varrho,\varphi)\rangle_{\mathcal{W}_2^m}=&\sum_{i=0}^{m-1}\left[\frac{d^i f(\alpha)}{d\varrho^i}\frac{\partial^i\mathcal{R}(\alpha,\varphi)}{\partial\varrho^i}+\frac{d^i f(\beta)}{d\varrho^i}\frac{\partial^i\mathcal{R}(\beta,\varphi)}{\partial\varrho^i}\right]\\&+\int_\alpha^\beta\frac{d^m f(\varrho)}{d\varrho^m}\frac{\partial^m\mathcal{R}(\varrho,\varphi)}{\partial\varrho^m}d\varrho.\end{aligned}\tag{4.2}$$

Since,

$$\begin{aligned}\int_\alpha^\beta&\frac{d^m f(\varrho)}{d\varrho^m}\frac{\partial^m\mathcal{R}(\varrho,\varphi)}{\partial\varrho^m}d\varrho\\&=\sum_{i=0}^{m-1}\left((-1)^i\frac{d^{m-i-1}f(\varrho)}{d\varrho^{m-i-1}}\frac{\partial^{m+i}\mathcal{R}(\varrho,\varphi)}{\partial\varrho^{m+i}}\right)_{\varrho=\alpha}^{\beta}\\&\quad+(-1)^m\int_\alpha^\beta f(\varrho)\frac{\partial^{2m}\mathcal{R}(\varrho,\varphi)}{\partial\varrho^{2m}}d\varrho.\end{aligned}\tag{4.3}$$

Also,

$$\begin{aligned}\sum_{i=0}^{m-1}&((-1)^i\frac{d^{m-i-1}f(\varrho)}{d\varrho^{m-i-1}}\frac{\partial^{m+i}\mathcal{R}(\varrho,\varphi)}{\partial\varrho^{m+i}}\\&=\sum_{i=0}^{m-1}(-1)^{m-i-1}\frac{d^i f(\varrho)}{d\varrho^i}\frac{\partial^{2m-i-1}\mathcal{R}(\varrho,\varphi)}{\partial\varrho^{2m-i-1}}.\end{aligned}\tag{4.4}$$

From equations (4.2) to (4.4), we get

$$\begin{aligned}\langle f(\varrho),&\mathcal{R}(\varrho,\varphi)\rangle_{\mathcal{W}_2^m}\\&=\sum_{i=0}^{m-1}\left[\frac{d^i f(\alpha)}{d\varrho^i}\left(\frac{\partial^i\mathcal{R}(\alpha,\varphi)}{\partial\varrho^i}-(-1)^{m-i-1}\frac{\partial^{2m-i-1}\mathcal{R}(\alpha,\varphi)}{\partial\varrho^{2m-i-1}}\right)\right.\\&\quad\left.+\frac{d^i f(\beta)}{d\varrho^i}\left((-1)^{m-i-1}\frac{\partial^{2m-i-1}\mathcal{R}(\beta,\varphi)}{\partial\varrho^{2m-i-1}}+\frac{\partial^i\mathcal{R}(\beta,\varphi)}{\partial\varrho^i}\right)\right]\\&\quad+(-1)^m\int_\alpha^\beta f(\varrho)\frac{\partial^{2m}\mathcal{R}(\varrho,\varphi)}{\partial\varrho^{2m}}d\varrho.\end{aligned}\tag{4.5}$$

Now, from equations (4.1), (4.5) and the Dirac delta function

$$(-1)^m \frac{\partial^{2m}\mathcal{R}(\varrho,\varphi)}{\partial\varrho^{2m}} = \delta(\varrho-\varphi), \tag{4.6}$$

$$\frac{\partial^i\mathcal{R}(\alpha,\varphi)}{\partial\varrho^i} - (-1)^{m-i-1}\frac{\partial^{2m-i-1}\mathcal{R}(\alpha,\varphi)}{\partial\varrho^{2m-i-1}} = 0, 0 \leq i \leq m-1, \tag{4.7}$$

$$(-1)^{m-i-1}\frac{\partial^{2m-i-1}\mathcal{R}(\beta,\varphi)}{\partial\varrho^{2m-i-1}} + \frac{\partial^i\mathcal{R}(\beta,\varphi)}{\partial\varrho^i} = 0, 0 \leq i \leq m-1. \tag{4.8}$$

Here, $\mathcal{R}(\varrho,\varphi)$ is the solution of the following constant coefficient $2m$ order differential equation with boundary conditions (4.7) and (4.8)

$$(-1)^m \frac{\partial^{2m}\mathcal{R}(\varrho,\varphi)}{\partial\varrho^{2m}} = 0. \tag{4.9}$$

The equation (4.9) has characteristic equation $\lambda^{2m} = 0$ whose the characteristic root $\lambda = 0$ with multiplicity $2m$.
Therefore,

$$\mathcal{R}(\varrho,\varphi) = \begin{cases} \mathcal{R}_1(\varrho,\varphi) = \sum_{i=1}^{2m} c_i(\varphi)\varrho^{i-1}, & \varrho < \varphi \\ \mathcal{R}_2(\varrho,\varphi) = \sum_{i=1}^{2m} d_i(\varphi)\varrho^{i-1}, & \varrho > \varphi. \end{cases} \tag{4.10}$$

Since, the solution (4.10) of (4.9) also satisfied the following conditions

$$\frac{\partial^i\mathcal{R}_1(\varphi,\varphi)}{\partial\varrho^i} = \frac{\partial^i\mathcal{R}_2(\varphi,\varphi)}{\partial\varrho^i}, 0 \leq i \leq 2m-2, \tag{4.11}$$

$$\frac{\partial^{2m-1}\mathcal{R}_1(\varphi^+,\varphi)}{\partial\varrho^{2m-1}} - \frac{\partial^{2m-1}\mathcal{R}_2(\varphi^-,\varphi)}{\partial\varrho^{2m-1}} = \frac{1}{(-1)^m}. \tag{4.12}$$

Using boundary conditions (4.7), (4.8), (4.11) and (4.12), we can derive the reproducing kernel $\mathcal{R}(\varrho,\varphi)$ for any $m$.

## 5. REPRODUCING KERNEL FOR ROBIN BOUNDARY CONDITIONS

In this section, we will derive reproducing kernel for $m = 2$ with Robin boundary conditions.
Therefore, the function space $\mathcal{W}_2^2[\alpha,\beta]$ is defined as follows,

### 5.1. Reproducing Kernel $\mathcal{R}(\varrho,\varphi)$ for $f(\alpha) = 0, f'(\beta) = 0$.

$\mathcal{W}_2^2[\alpha,\beta] = \{f(\varrho)|f(\varrho), f'(\varrho)$ are absolutely continuous, $f''(\varrho) \in L^2[\alpha,\beta], \varrho \in [\alpha,\beta], f(\alpha) = 0, f'(\beta) = 0\}$.
Described method in section 4 with the Robin boundary conditions, we get

fourth order differential equation with the boundary conditions in the form

$$\frac{\partial^4 \mathcal{R}(\varrho,\varphi)}{\partial \varrho^4} = 0, \tag{5.1}$$

$$\begin{aligned}
&\mathcal{R}(\alpha,\beta) = 0,\\
&\frac{\partial \mathcal{R}(\alpha,\varphi)}{\partial \varrho} - \frac{\partial^2 \mathcal{R}(\alpha,\varphi)}{\partial \varrho^2} = 0,\\
&\mathcal{R}(\beta,\varphi) - \frac{\partial^3 \mathcal{R}(\beta,\varphi)}{\partial \varrho^3} = 0,\\
&\frac{\partial \mathcal{R}(\beta,\varphi)}{\partial \varrho} = 0,\\
&\mathcal{R}_1(\varphi,\varphi) = \mathcal{R}_2(\varphi,\varphi),\\
&\frac{\partial \mathcal{R}_1(\varphi,\varphi)}{\partial \varrho} = \frac{\partial \mathcal{R}_2(\varphi,\varphi)}{\partial \varrho},\\
&\frac{\partial^2 \mathcal{R}_1(\varphi,\varphi)}{\partial \varrho^2} = \frac{\partial^2 \mathcal{R}_2(\varphi,\varphi)}{\partial \varrho^2},\\
&\frac{\partial^3 \mathcal{R}_1(\varphi^+,\varphi)}{\partial \varrho^3} - \frac{\partial^3 \mathcal{R}_2(\varphi^-,\varphi)}{\partial \varrho^3} = 1.
\end{aligned} \tag{5.2}$$

Hence, the solution of (5.1) and (5.2) is

$$\mathcal{R}(\varrho,\varphi) = \begin{cases} \mathcal{R}_1(\varrho,\varphi) = c_1(\varphi) + c_2(\varphi)\varrho + c_3(\varphi)\varrho^2 + c_4(\varphi)\varrho^3, & \varrho \le \varphi \\ \mathcal{R}_2(\varrho,\varphi) = d_1(\varphi) + d_2(\varphi)\varrho + d_3(\varphi)\varrho^2 + d_4(\varphi)\varrho^3, & \varrho > \varphi. \end{cases} \tag{5.3}$$

Where,

$$c_1(\varphi) = \frac{\alpha \begin{pmatrix} 2\,\alpha^3\,\beta^3 - 3\,\alpha^3\,\beta^2\,\varphi + \alpha^3\,\varphi^3 + 6\,\alpha^3 - 2\,\alpha^2\,\beta^4 - 8\,\alpha^2\,\beta^3 \\ +6\,\alpha^2\,\beta^2\,\varphi^2 + 12\,\alpha^2\,\beta^2\,\varphi - 4\,\alpha^2\,\beta\,\varphi^3 - 24\,\alpha^2\,\beta - 4\,\alpha^2\,\varphi^3 \\ -24\,\alpha^2 + 3\,\alpha\,\beta^4\,\varphi + 6\,\alpha\,\beta^4 - 6\,\alpha\,\beta^3\,\varphi^2 + 3\,\alpha\,\beta^2\,\varphi^3 \\ -18\,\alpha\,\beta^2\,\varphi^2 + 12\,\alpha\,\beta\,\varphi^3 + 36\,\alpha\,\beta\,\varphi + 72\,\alpha\,\beta - 18\,\alpha\,\varphi^2 \\ -6\,\beta^4\,\varphi + 12\,\beta^3\,\varphi^2 - 6\,\beta^2\,\varphi^3 - 72\,\beta\,\varphi + 36\,\varphi^2 \end{pmatrix}}{6\begin{pmatrix} \alpha^4 - 4\,\alpha^3\,\beta - 4\,\alpha^3 + 6\,\alpha^2\,\beta^2 + 12\,\alpha^2\,\beta \\ -4\,\alpha\,\beta^3 - 12\,\alpha\,\beta^2 - 12\,\alpha + \beta^4 + 4\,\beta^3 + 12\,\beta + 12 \end{pmatrix}},$$

$$c_2(\varphi) = -\frac{\begin{matrix} \alpha^4\beta^2 - 2\alpha^4\beta\varphi + \alpha^4\varphi^2 - 4\alpha^3\beta^2 + 8\alpha^3\beta\varphi - 4\alpha^3\varphi^2 \\ -\alpha^2\beta^4 + 3\alpha^2\beta^2\varphi^2 - 2\alpha^2\beta\varphi^3 - 12\alpha^2\beta + 2\alpha\beta^4\varphi + 2\alpha\beta^4 \\ -4\alpha\beta^3\varphi^2 + 2\alpha\beta^2\varphi^3 - 6\alpha\beta^2\varphi^2 + 4\alpha\beta\varphi^3 + 24\alpha\beta\varphi \\ +24\alpha\beta - 12\alpha\varphi^2 - 2\beta^4\varphi + 4\beta^3\varphi^2 - 2\beta^2\varphi^3 - 24\beta\varphi + 12\varphi^2 \end{matrix}}{2\begin{pmatrix} \alpha^4 - 4\alpha^3\beta - 4\alpha^3 + 6\alpha^2\beta^2 + 12\alpha^2\beta \\ -4\alpha\beta^3 - 12\alpha\beta^2 - 12\alpha + \beta^4 + 4\beta^3 + 12\beta + 12 \end{pmatrix}},$$

$$c_3(\varphi) = \frac{\varphi}{2} + \frac{(\alpha-\varphi)\begin{pmatrix} \alpha^3\varphi - 2\alpha^2\beta^2 - \alpha^2\varphi^2 - 4\alpha^2\varphi + 2\alpha\beta^3 \\ +\alpha\beta^2\varphi + 6\alpha\beta^2 + 2\alpha\varphi^2 + 6\alpha - 2\beta^3\varphi \\ -4\beta^3 + \beta^2\varphi^2 - 6\varphi - 12 \end{pmatrix}}{2\begin{pmatrix} 12\alpha + \beta^3(4\alpha - 4) + \beta^2(12\alpha - 6\alpha^2) \\ -\beta(-4\alpha^3 + 12\alpha^2 + 12) + 4\alpha^3 - \alpha^4 - \beta^4 - 12 \end{pmatrix}},$$

$$c_4(\varphi) = -\frac{(\alpha-\varphi)\begin{pmatrix} \alpha^3 - 4\alpha^2\beta + \alpha^2\varphi - 4\alpha^2 + 3\alpha\beta^2 \\ +2\alpha\beta\varphi + 12\alpha\beta - 2\alpha\varphi^2 - 4\alpha\varphi \\ -3\beta^2\varphi - 6\beta^2 + 2\beta\varphi^2 + 2\varphi^2 \end{pmatrix}}{6\begin{pmatrix} 12\alpha + \beta^3(4\alpha - 4) + \beta^2(12\alpha - 6\alpha^2) \\ -\beta(-4\alpha^3 + 12\alpha^2 + 12) + 4\alpha^3 - \alpha^4 - \beta^4 - 12 \end{pmatrix}} - \frac{1}{6},$$

$$d_1(\varphi) = \frac{(\alpha-\varphi)\begin{pmatrix} 2\alpha^3\beta^3 - 3\alpha^3\beta^2\varphi + 6\alpha^3 - 2\alpha^2\beta^4 + 2\alpha^2\beta^3\varphi \\ -8\alpha^2\beta^3 + 3\alpha^2\beta^2\varphi^2 + 12\alpha^2\beta^2\varphi - 24\alpha^2\beta \\ +6\alpha^2\varphi - 24\alpha^2 + \alpha\beta^4\varphi + 6\alpha\beta^4 - 4\alpha\beta^3\varphi^2 \\ -8\alpha\beta^3\varphi - 6\alpha\beta^2\varphi^2 + 12\alpha\beta\varphi + 72\alpha\beta - 12\alpha\varphi^2 \\ -24\alpha\varphi + \beta^4\varphi^2 + 4\beta^3\varphi^2 + 12\beta\varphi^2 + 12\varphi^2 \end{pmatrix}}{6\begin{pmatrix} \alpha^4 - 4\alpha^3\beta - 4\alpha^3 + 6\alpha^2\beta^2 + 12\alpha^2\beta - 4\alpha\beta^3 \\ -12\alpha\beta^2 - 12\alpha + \beta^4 + 4\beta^3 + 12\beta + 12 \end{pmatrix}},$$

$$d_2(\varphi) = -\frac{\beta(\alpha-\varphi)\begin{pmatrix} \alpha^3\beta - 2\alpha^3\varphi + \alpha^2\beta\varphi - 4\alpha^2\beta + 2\alpha^2\varphi^2 \\ +8\alpha^2\varphi - \alpha\beta^3 - 2\alpha\beta\varphi^2 - 4\alpha\beta\varphi - 4\alpha\varphi^2 \\ -12\alpha + \beta^3\varphi + 2\beta^3 + 2\beta\varphi^2 + 12\varphi + 24 \end{pmatrix}}{2\begin{pmatrix} \alpha^4 - 4\alpha^3\beta - 4\alpha^3 + 6\alpha^2\beta^2 + 12\alpha^2\beta - 4\alpha\beta^3 \\ -12\alpha\beta^2 - 12\alpha + \beta^4 + 4\beta^3 + 12\beta + 12 \end{pmatrix}},$$

$$d_3(\varphi) = -\frac{\frac{(\alpha-\varphi)\,(2\,\varphi-2\,\alpha+4)\,\beta^3}{2} - \frac{(\alpha-\varphi)\,\left(-2\,\alpha^2+\alpha\,\varphi+6\,\alpha+\varphi^2\right)\beta^2}{2}}{\begin{pmatrix} 12\,\alpha+\beta^3\,(4\,\alpha-4)+\beta^2\,\left(12\,\alpha-6\,\alpha^2\right)+4\,\alpha^3 \\ -\beta\,\left(-4\,\alpha^3+12\,\alpha^2+12\right)-\alpha^4-\beta^4-12 \end{pmatrix}}$$

$$-\frac{\frac{(\alpha-\varphi)\,\left(-\alpha^3\,\varphi+\alpha^2\,\varphi^2+4\,\alpha^2\,\varphi-2\,\alpha\,\varphi^2-6\,\alpha+6\,\varphi+12\right)}{2}}{\begin{pmatrix} 12\,\alpha+\beta^3\,(4\,\alpha-4)+\beta^2\,\left(12\,\alpha-6\,\alpha^2\right)+4\,\alpha^3 \\ -\beta\,\left(-4\,\alpha^3+12\,\alpha^2+12\right)-\alpha^4-\beta^4-12 \end{pmatrix}},$$

$$d_4(\varphi) = \frac{\frac{(\alpha-\varphi)\,(3\,\varphi-3\,\alpha+6)\,\beta^2}{6} - \frac{(\alpha-\varphi)\,\left(-4\,\alpha^2+2\,\alpha\,\varphi+12\,\alpha+2\,\varphi^2\right)\beta}{6}}{\begin{pmatrix} 12\,\alpha+\beta^3\,(4\,\alpha-4)+\beta^2\,\left(12\,\alpha-6\,\alpha^2\right)+4\,\alpha^3 \\ -\beta\,\left(-4\,\alpha^3+12\,\alpha^2+12\right)-\alpha^4-\beta^4-12 \end{pmatrix}}$$

$$+\frac{\frac{(\alpha-\varphi)\,\left(-\alpha^3-\alpha^2\,\varphi+4\,\alpha^2+2\,\alpha\,\varphi^2+4\,\alpha\,\varphi-2\,\varphi^2\right)}{6}}{\begin{pmatrix} 12\,\alpha+\beta^3\,(4\,\alpha-4)+\beta^2\,\left(12\,\alpha-6\,\alpha^2\right)+4\,\alpha^3 \\ -\beta\,\left(-4\,\alpha^3+12\,\alpha^2+12\right)-\alpha^4-\beta^4-12 \end{pmatrix}}.$$

5.2. **Reproducing Kernel $\mathcal{R}(\varrho,\varphi)$ for $f'(\alpha)=0, f(\beta)=0$.**
$\mathcal{W}_2^2[\alpha,\beta] = \{f(\varrho)|f(\varrho), f'(\varrho)$ are absolutely continuous, $f''(\varrho) \in L^2[\alpha,\beta], \varrho \in [\alpha,\beta], f'(\alpha)=0, f(\beta)=0\}$.
Described method in section 4 with the Robin boundary conditions, we get fourth order differential equation with the boundary conditions in the form

$$\frac{\partial^4 \mathcal{R}(\varrho,\varphi)}{\partial \varrho^4} = 0, \tag{5.4}$$

$$\begin{aligned}
&\mathcal{R}(\alpha,\varphi) + \frac{\partial^3 \mathcal{R}(\alpha,\varphi)}{\partial \varrho^3} = 0, \\
&\frac{\partial \mathcal{R}(\alpha,\varphi)}{\partial \varrho} = 0, \\
&\mathcal{R}(\beta,\varphi) = 0, \\
&\frac{\partial^2 \mathcal{R}(\beta,\varphi)}{\partial \varrho^2} + \frac{\partial \mathcal{R}(\beta,\varphi)}{\partial \varrho} = 0, \\
&\mathcal{R}_1(\varphi,\varphi) = \mathcal{R}_2(\varphi,\varphi),
\end{aligned} \tag{5.5}$$

$$\frac{\partial \mathcal{R}_1(\varphi,\varphi)}{\partial \varrho} = \frac{\partial \mathcal{R}_2(\varphi,\varphi)}{\partial \varrho},$$
$$\frac{\partial^2 \mathcal{R}_1(\varphi,\varphi)}{\partial \varrho^2} = \frac{\partial^2 \mathcal{R}_2(\varphi,\varphi)}{\partial \varrho^2},$$
$$\frac{\partial^3 \mathcal{R}_1(\varphi^+,\varphi)}{\partial \varrho^3} - \frac{\partial^3 \mathcal{R}_2(\varphi^-,\varphi)}{\partial \varrho^3} = 1.$$

Hence, the solution of (5.4) and (5.5) is

$$\mathcal{R}(\varrho,\varphi) = \begin{cases} \mathcal{R}_1(\varrho,\varphi) = c_1(\varphi) + c_2(\varphi)\varrho + c_3(\varphi)\varrho^2 + c_4(\varphi)\varrho^3, & \varrho \le \varphi \\ \mathcal{R}_2(\varrho,\varphi) = d_1(\varphi) + d_2(\varphi)\varrho + d_3(\varphi)\varrho^2 + d_4(\varphi)\varrho^3, & \varrho > \varphi. \end{cases} \tag{5.6}$$

Where,

$$c_1(\varphi) = -\frac{(\beta-\varphi)\left(\begin{array}{c} -2\,\alpha^4\,\beta^2 + \alpha^4\,\beta\,\varphi - 6\,\alpha^4\,\beta + \alpha^4\,\varphi^2 + 2\,\alpha^3\,\beta^3 \\ +2\,\alpha^3\,\beta^2\,\varphi + 8\,\alpha^3\,\beta^2 - 4\,\alpha^3\,\beta\,\varphi^2 + 8\,\alpha^3\,\beta\,\varphi \\ -4\,\alpha^3\,\varphi^2 - 3\,\alpha^2\,\beta^3\,\varphi + 3\,\alpha^2\,\beta^2\,\varphi^2 - 12\,\alpha^2\,\beta^2\,\varphi \\ +6\,\alpha^2\,\beta\,\varphi^2 + 24\,\alpha\,\beta^2 - 12\,\alpha\,\beta\,\varphi + 72\,\alpha\,\beta - 12\,\alpha\,\varphi^2 \\ -6\,\beta^3 - 6\,\beta^2\,\varphi - 24\,\beta^2 + 12\,\beta\,\varphi^2 - 24\,\beta\,\varphi + 12\,\varphi^2 \end{array}\right)}{6\left(\begin{array}{c} \alpha^4 - 4\,\alpha^3\,\beta - 4\,\alpha^3 + 6\,\alpha^2\,\beta^2 + 12\,\alpha^2\,\beta - 4\,\alpha\,\beta^3 \\ -12\,\alpha\,\beta^2 - 12\,\alpha + \beta^4 + 4\,\beta^3 + 12\,\beta + 12 \end{array}\right)},$$

$$c_2(\varphi) = \frac{\alpha\,(\beta-\varphi)\left(\begin{array}{c} -\alpha^3\,\beta + \alpha^3\,\varphi - 2\,\alpha^3 + \alpha\,\beta^3 + \alpha\,\beta^2\,\varphi \\ +4\,\alpha\,\beta^2 - 2\,\alpha\,\beta\,\varphi^2 + 4\,\alpha\,\beta\,\varphi - 2\,\alpha\,\varphi^2 - 2\,\beta^3\,\varphi \\ +2\,\beta^2\,\varphi^2 - 8\,\beta^2\,\varphi + 4\,\beta\,\varphi^2 + 12\,\beta - 12\,\varphi + 24 \end{array}\right)}{2\left(\begin{array}{c} \alpha^4 - 4\,\alpha^3\,\beta - 4\,\alpha^3 + 6\,\alpha^2\,\beta^2 + 12\,\alpha^2\,\beta - 4\,\alpha\,\beta^3 \\ -12\,\alpha\,\beta^2 - 12\,\alpha + \beta^4 + 4\,\beta^3 + 12\,\beta + 12 \end{array}\right)},$$

$$\begin{aligned} c_3(\varphi) = &-\frac{-\frac{(\beta-\varphi)\,(2\,\beta-2\,\varphi+4)\,\alpha^3}{2} + \frac{(\beta-\varphi)\,\left(2\,\beta^2-\beta\,\varphi+6\,\beta-\varphi^2\right)\,\alpha^2}{2}}{\left(\begin{array}{c} 12\,\beta - \alpha^3\,(4\,\beta+4) + \alpha^2\,\left(6\,\beta^2+12\,\beta\right) + 4\,\beta^3 \\ -\alpha\,(4\,\beta^3+12\,\beta^2+12) + \alpha^4 + \beta^4 + 12 \end{array}\right)} \\ &- \frac{\frac{(\beta-\varphi)\,\left(-\beta^3\,\varphi+\beta^2\,\varphi^2-4\,\beta^2\,\varphi+2\,\beta\,\varphi^2+6\,\beta-6\,\varphi+12\right)}{2}}{\left(\begin{array}{c} 12\,\beta - \alpha^3\,(4\,\beta+4) + \alpha^2\,\left(6\,\beta^2+12\,\beta\right) + 4\,\beta^3 \\ -\alpha\,(4\,\beta^3+12\,\beta^2+12) + \alpha^4 + \beta^4 + 12 \end{array}\right)}, \end{aligned}$$

$$c_4(\varphi) = -\frac{\frac{(\beta-\varphi)\,(3\,\beta-3\,\varphi+6)\,\alpha^2}{6} - \frac{(\beta-\varphi)\left(4\,\beta^2-2\,\beta\,\varphi+12\,\beta-2\,\varphi^2\right)\alpha}{6}}{\begin{pmatrix} 12\,\beta-\alpha^3\,(4\,\beta+4)+\alpha^2\,\left(6\,\beta^2+12\,\beta\right)+4\,\beta^3 \\ -\alpha\,(4\,\beta^3+12\,\beta^2+12)+\alpha^4+\beta^4+12 \end{pmatrix}} - \frac{\frac{(\beta-\varphi)\left(\beta^3+\beta^2\,\varphi+4\,\beta^2-2\,\beta\,\varphi^2+4\,\beta\,\varphi-2\,\varphi^2\right)}{6}}{\begin{pmatrix} 12\,\beta-\alpha^3\,(4\,\beta+4)+\alpha^2\,\left(6\,\beta^2+12\,\beta\right)+4\,\beta^3 \\ -\alpha\,(4\,\beta^3+12\,\beta^2+12)+\alpha^4+\beta^4+12 \end{pmatrix}},$$

$$d_1(\varphi) = -\frac{\beta\begin{pmatrix} -2\,\alpha^4\,\beta^2+3\,\alpha^4\,\beta\,\varphi-6\,\alpha^4\,\beta+6\,\alpha^4\,\varphi+2\,\alpha^3\,\beta^3+8\,\alpha^3\,\beta^2 \\ -6\,\alpha^3\,\beta\,\varphi^2-12\,\alpha^3\,\varphi^2-3\,\alpha^2\,\beta^3\,\varphi+6\,\alpha^2\,\beta^2\,\varphi^2 \\ +3\,\alpha^2\,\beta\,\varphi^3+18\,\alpha^2\,\beta\,\varphi^2+6\,\alpha^2\,\varphi^3-4\,\alpha\,\beta^2\,\varphi^3+24\,\alpha\,\beta^2 \\ -12\,\alpha\,\beta\,\varphi^3-36\,\alpha\,\beta\,\varphi+72\,\alpha\,\beta-72\,\alpha\,\varphi+\beta^3\,\varphi^3-6\,\beta^3 \\ +4\,\beta^2\,\varphi^3-24\,\beta^2+18\,\beta\,\varphi^2+36\,\varphi^2-12\,\alpha^2\,\beta^2\,\varphi \end{pmatrix}}{6\begin{pmatrix} \alpha^4-4\,\alpha^3\,\beta-4\,\alpha^3+6\,\alpha^2\,\beta^2+12\,\alpha^2\,\beta-4\,\alpha\,\beta^3 \\ -12\,\alpha\,\beta^2-12\,\alpha+\beta^4+4\,\beta^3+12\,\beta+12 \end{pmatrix}},$$

$$d_2(\varphi) = \frac{\begin{matrix} -\alpha^4\,\beta^2+2\,\alpha^4\,\beta\,\varphi-2\,\alpha^4\,\beta+2\,\alpha^4\,\varphi-4\,\alpha^3\,\beta\,\varphi^2-4\,\alpha^3\,\varphi^2 \\ +\alpha^2\,\beta^4+4\,\alpha^2\,\beta^3+3\,\alpha^2\,\beta^2\,\varphi^2+2\,\alpha^2\,\beta\,\varphi^3+6\,\alpha^2\,\beta\,\varphi^2 \\ +2\,\alpha^2\,\varphi^3-2\,\alpha\,\beta^4\,\varphi-8\,\alpha\,\beta^3\,\varphi-2\,\alpha\,\beta^2\,\varphi^3+12\,\alpha\,\beta^2-4\,\alpha\,\beta\,\varphi^3 \\ -24\,\alpha\,\beta\,\varphi+24\,\alpha\,\beta-24\,\alpha\,\varphi+\beta^4\,\varphi^2+4\,\beta^3\,\varphi^2+12\,\beta\,\varphi^2+12\,\varphi^2 \end{matrix}}{2\begin{pmatrix} \alpha^4-4\,\alpha^3\,\beta-4\,\alpha^3+6\,\alpha^2\,\beta^2+12\,\alpha^2\,\beta-4\,\alpha\,\beta^3 \\ -12\,\alpha\,\beta^2-12\,\alpha+\beta^4+4\,\beta^3+12\,\beta+12 \end{pmatrix}},$$

$$d_3(\varphi) = -\frac{(\beta-\varphi)\begin{pmatrix} -2\,\alpha^3\,\beta+2\,\alpha^3\,\varphi-4\,\alpha^3+2\,\alpha^2\,\beta^2 \\ -\alpha^2\,\beta\,\varphi+6\,\alpha^2\,\beta-\alpha^2\,\varphi^2-\beta^3\,\varphi \\ +\beta^2\,\varphi^2-4\,\beta^2\,\varphi+2\,\beta\,\varphi^2+6\,\beta-6\,\varphi+12 \end{pmatrix}}{2\begin{pmatrix} 12\,\beta-\alpha^3\,(4\,\beta+4)+\alpha^2\,\left(6\,\beta^2+12\,\beta\right) \\ -\alpha\,(4\,\beta^3+12\,\beta^2+12)+\alpha^4+4\,\beta^3+\beta^4+12 \end{pmatrix}} - \frac{\varphi}{2},$$

$$d_4(\varphi) = -\frac{(\beta-\varphi)\begin{pmatrix} 3\,\alpha^2\,\beta-3\,\alpha^2\,\varphi+6\,\alpha^2-4\,\alpha\,\beta^2 \\ +2\,\alpha\,\beta\,\varphi-12\,\alpha\,\beta+2\,\alpha\,\varphi^2+\beta^3+\beta^2\,\varphi \\ +4\,\beta^2-2\,\beta\,\varphi^2+4\,\beta\,\varphi-2\,\varphi^2 \end{pmatrix}}{6\begin{pmatrix} 12\,\beta-\alpha^3\,(4\,\beta+4)+\alpha^2\,\left(6\,\beta^2+12\,\beta\right) \\ -\alpha\,(4\,\beta^3+12\,\beta^2+12)+\alpha^4+4\,\beta^3+\beta^4+12 \end{pmatrix}} + \frac{1}{6}.$$

## 6. CONCLUSION

In this paper we derived a generalized reproducing kernel for Robin boundary conditions using an inner product. This reproducing kernel is used to solve the first order ordinary differential equations with Robin boundary condition in particular for $m = 2$. The derive reproducing kernel is generalized to $n^{th}$ order ordinary differential equations for substitute $m = n + 1$.
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# NONSTANDARD APPROACH TO HAUSDORFF OUTER MEASURE

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Abstract. We use nonstandard techniques, in the sense of Abraham Robinson, to define and give the exact Hausdorff outer measure.

## 1. Introduction

Developing a notion of dimension and measuring sets is foundational in mathematics (cf. [4, 7, 6, 8, 9, 10]). Originally studied by B. Riemann, they have been further developed by H. Schwarz, F. Klein, D. Hilbert, H. Lebesgue, F. Hausdorff, and others.

In this manuscript, we focus on Hausdorff measure, which is well-defined for *any* set. That is, a $d$-dimensional Hausdorff measure of a measurable subset of $\mathbb{R}^n$ is proportional to the dimension of the set. In particular, $d$-dimensional Hausdorff measure exists for any real number $d \geq 0$ (so $d$ is not necessarily an integer). This implies that the Hausdorff dimension of a set is greater than or equal to its topological dimension, and less than or equal to the dimension of the metric space imbedding the set (thus it is a refinement of an integral dimension).

We use a nonstandard approach to study Hausdorff measure since nonstandard techniques provide a richer insight to standard objects and sets. In particular, we show how discrete measure (cf. Definition 4.2) gives rise to Hausdorff outer measure, which is our main result:

**Theorem 1.1.** *Let $A$ be a subset in $[0,1]$. Let $\mathcal{H}^s(A)$ be the Hausdorff outer measure of dimension $s$. Then*

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*(a)* $\mathcal{H}^s(A) \leq \inf\{\operatorname{st} h^s_\delta(B) : \textit{internal}\, B \supseteq \operatorname{st}^{-1}(A)\}$ *for all infinitesimal* $\delta$*, and*

*(b)* $\mathcal{H}^s(A) = \lim_{\delta\to 0} \inf\{\operatorname{st} h^s_\delta(B) : \textit{internal}\, B \supseteq \operatorname{st}^{-1}(A)\}$*, where* $\delta$ *ranges over standard positive values in* $\mathbb{R}$ *in the limit.*

Theorem 1.1 introduces *counting* methods that calculate the correct Hausdorff measure. Other methods, such as Minkowski, *i.e.*, box-counting, methods, have been proven to overestimate or underestimate the correct dimension of a set, particularly if it is irregular (cf. [11, 14]). In fact, if a set or its complement is not self-similar, then the box-counting method fails since there is no dimension for which the limit converges.

Secondly, the nonstandard version is easier to compute than the standard Hausdorff version. This is because the discrete measure of $\mathcal{H}^s(A)$ has a fixed $\delta$, not a varying one. Also, taking the supremum over any $\delta$ is omitted to compute $\mathcal{H}^s(A)$ (see Definition 3.3) since the process of taking the supremum over all $\delta$'s has already been applied when choosing our $\delta$. So although this operation is omitted, we still obtain the accurate Hausdorff dimension, which is defined for any set.

Throughout this manuscript, we assume that the set $\mathbb{N}$ of natural numbers includes 0.

1.1. **Summary of the sections.** In §2, we introduce key ideas in nonstandard analysis needed to prove Theorem 1.1. In §3, we give a background on Hausdorff measure. In §4, we develop a notion of computing measure in the nonstandard universe. We give a relation between $\mathcal{H}^s(A)$ and Lebesgue outer measure $\overline{\lambda}(A)$ for nice sets $A \subseteq \mathbb{R}^n$ (cf. (4.1)), which provides an alternative (discretized) way to obtain Hausdorff outer measure.

In §5, we provide the proof of Theorem 1.1. Finally, we conclude with an example in §6, showing that Theorem 1.1(a) cannot be replaced by an equality.

## 2. Nonstandard analysis

We give a background on the theory of nonstandard analysis (also see, *e.g.*, [17, 16, 1]).

2.1. **Ultrafilters.** A filter over a set $I$ is a nonempty collection $\mathscr{U}$ of subsets of $I$ that satisfies the following:

(1) if $A, B \in \mathscr{U}$, then $A \cap B \in \mathscr{U}$, and
(2) if $A \in \mathscr{U}$ and $B \supseteq A$, then $B \in \mathscr{U}$.

Filters can be used to define a notion of *large* subsets of $I$. That is, a set $A \subseteq \mathbb{N}$ is large if and only if

(1) a finite subset of $\mathbb{N}$, including the empty set, is not large,
(2) the set of natural numbers is large,
(3) two subsets of $\mathbb{N}$ are large, then all supersets of their intersection are also large,
(4) its complement $A^c$ is not large.

This notion of largeness is a special kind of filter called an ultrafilter.

**Definition 2.1.** A nonprincipal ultrafilter $\mathscr{U}$ of subsets of a nonempty set $I$ contains all large subsets of $I$.

It is a maximal proper filter since it cannot contain any more subsets without including the empty set and becoming improper. Each infinite set $I$ contains at least one maximal nonprincipal proper filter, and if there are more than one nonprincipal ultrafilters, then they are isomorphic to each other. Let $a = (a_0, a_1, \ldots), b = (b_0, b_1, \ldots) \in \mathbb{R}^{\mathbb{N}}$, sequences of real numbers. We say $a = b$ if and only if $a_i = b_i$ for a large set of $i \in \mathbb{N}$, and we write $a \sim b$ if $\{i : a_i = b_i\} \in \mathscr{U}$.

Let $A \subseteq \mathbb{R}$ be a set. The starred version ${}^*A$ of $A$ is defined as $A^{\mathbb{N}}/{\sim}$, where $\sim$ is the equivalence relation given above; we say ${}^*A$ is the nonstandard version of $A$. With a slight abuse of notation, we write $a = [a_0, a_1, \ldots] \in {}^*\mathbb{R}$ to denote the equivalence class of $a = (a_0, a_1, \ldots)$.

2.2. **Nonstandard real and natural numbers.** Let ${}^*\mathbb{R}$ be the set of nonstandard reals. The algebraic operations on ${}^*\mathbb{R}$ are componentwise, and the partial order $<$ on ${}^*\mathbb{R}$ is defined to be $a < b$ if and only if $\{i : a_i < b_i\} \in \mathscr{U}$. Note that $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ via the constant-valued sequence embedding $r \mapsto {}^*r = [r, r, r, \ldots]$.

We will now describe hyperreal numbers. Suppose $a = [a_0, a_1, a_2, \ldots] \in {}^*\mathbb{R}$. Then for a large $i \in \mathbb{N}$, $a$ is a

(1) positive infinitesimal if $0 \leq a_i < t$ for every positive real $t$,
(2) finite if $s < a_i < t$ for some $s, t \in \mathbb{R}$,
(3) positive infinite if $s < a_i$ for every $s \in \mathbb{R}$, and
(4) standard if $a_i = a_j$ for any $i$ and $j$.

**Example 2.2.** Let $a = [1/n]_{n\in\mathbb{N}}$ and $b = [1/n^2]_{n\in\mathbb{N}}$ be hyperreals for $n \neq 0$. Then $a$ is an infinitesimal since $\{n : |1/n| < x\} \in \mathscr{U}$ for any positive real $x$. By the same reason, $b$ is also an infitesimal. The only difference between $a$ and $b$ is their rate of convergence: $a \geq b > 0$.

**Remark 2.3.** The number $^*0$ is the only number that is both an infinitesimal and standard.

We will also say a hyperreal number $x \in {}^*\mathbb{R}$ is an infinite real number if $x > {}^*n$ for every ${}^*n = [n, n, n, \ldots] \in {}^*\mathbb{N}$. We will write $x \in {}^*\mathbb{R} \setminus \mathbb{R}$ or $x > \mathbb{R}$ to say that $x$ is an infinite real number. Similarly, we will write $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ or $N > \mathbb{N}$ to say that $N$ is an infinite natural number.

2.3. **Nonstandard superstructure.** Let $A$ be a set. Let $\{V_i(A)\}_{i\in\mathbb{N}}$ be the sequence defined by $V_0(A) = A$ and $V_{i+1}(A) = V_i(A) \cup \mathscr{P}(V_i(A))$, where $\mathscr{P}(V_i(A))$ is the power set of $V_i(A)$. Then $V(A) := \bigcup\limits_{i\in\mathbb{N}} V_i(A)$, the superstructure over $A$. The rank of $x \in V(A)$ is the smallest $k$ such that $x \in V_k(A)$.

Let $V$ be the collection of all (pure) sets, *i.e.*, elements of pure sets are hereditary, and let $\in$ be the universal symbol *the element of*. Denote $\mathscr{V} = \langle V, \in\rangle$, the first-order structure in set theory. We define the ultrapower of $\mathscr{V}$ as ${}^*\mathscr{V} := \prod\limits_{\mathscr{U}} \mathscr{V} \succ \mathscr{V}$, where $\mathscr{U}$ is an ultrafilter.

The $*$-transformation is a technique to convert a language from the standard universe to the nonstandard world. That is, if $*$ proceeds an element, set, or function, then it represents that the element or set lives in the nonstandard world, or that the operation is computed in the nonstandard universe.

**Theorem 2.4** (Łoś' Theorem)**.** *Let $\mathscr{V} = \langle V, \in\rangle$, the superstructure of $V$. Let $\theta(a_0, a_1, \ldots, a_{n-1})$ be a first-order statement, and let $\theta({}^*a_0, {}^*a_1, \ldots, {}^*a_{n-1})$ be its $*$-transformation. Then $\theta(a_0,\, a_1, \ldots, a_{n-1})$ is true in $\mathscr{V}$ if and only if $\theta({}^*a_0, {}^*a_1, \ldots, {}^*a_{n-1})$ is true in ${}^*\mathscr{V}$.*

2.4. **Internal sets.** A subset $A$ of the nonstandard universe ${}^*V$ is called internal if there is a set $B \subseteq V$ such that

$$A = \{[a_0, a_1, \ldots] \in {}^*V : \{i : a_i \in B\} \in \mathscr{U}\}.$$

In particular, $A \subseteq {}^*\mathbb{R}$ is internal if and only if it is ${}^*B$ for some $B \subseteq \mathbb{R}$.

**Example 2.5.** A finite subset ${}^*A \subseteq {}^*\mathbb{R}$ such as $\{{}^*0, {}^*1, {}^*2\} = {}^*\{0, 1, 2\}$ is internal.

To build internal sets, produce a sequence $\{A_i\}_{i\in\mathbb{N}}$ of standard sets such that $A = [\{A_i\}_{i\in\mathbb{N}}] \subseteq {}^*\mathscr{V}$. To generate an internal function $f$ that maps an internal set $A$ to an internal $B$, form a sequence of functions $f = \{f_i\}_{i\in\mathbb{N}}$ such that each $f_i$ is well-defined between $A_i$ and $B_i$ for each $i$. Hence we have $f : A \to B$ if and only if $f_i : A_i \to B_i$ for almost all $i$, *i.e.*, for a large subset of the natural numbers.

Sets which are not internal are called external. Since basic principles of mathematics are broken down for external sets when moving between standard and nonstandard worlds, they are rarely of our interest.

To see how the reals are embedded in the hyperreals, we introduce monads.

**Definition 2.6.** The standard part $\mathrm{st}(x)$ of a finite $x \in {}^*\mathbb{R}$ is the unique $a \in \mathbb{R}$ that is closest to $x$, *i.e.*,

$$\mathrm{st}(x) := \inf\{a \in \mathbb{R} : {}^*a \geq x\} = \sup\{a \in \mathbb{R} : {}^*a \leq x\}.$$

The set consisting of $x \in {}^*\mathbb{R}$ such that $\mathrm{st}(x) = a$ is called the monad of $a$, and is written as

$$\mathrm{st}^{-1}(a) = \mu(a) = \{x \in {}^*\mathbb{R} : \mathrm{st}(x) = a\}.$$

Since the monad of 0 is not first-order definable, $\mu(0)$ is not internal. More generally, we have the following:

**Proposition 2.7.** *Let $a \in \mathbb{R}$ and let $\mu(a)$ be the monad of $a$. Then $\mu(a)$ is not internal.*

2.5. **Overspill principle and saturation.** In this section, we discuss two theorems frequently used in nonstandard analysis.

**Theorem 2.8** (Overspill principle). *Let $A \subseteq {}^*\mathbb{R}$ be a nonempty internal subset containing arbitrary large finite elements. Then $A$ contains an infinite element.*

*Proof.* Let $A$ be a nonempty internal subset of ${}^*\mathbb{R}$. If $A$ is unbounded, then $A$ must contain at least one infinite element, and we are done. So suppose $A$ is bounded. Let $a$ be the least upper bound of $A$. Since $A$ contains arbitrary large finite elements, $a$ must be infinite. If there is no $x \in A$ such that $a - \varepsilon \leq x \leq a$ for some positive $\varepsilon$, then $a - \varepsilon$ is the least upper bound of $A$, which is a contradiction. Thus, there is some $x \in A$ such that $a - \varepsilon \leq x \leq a$, and we see that $A$ contains an infinite element. □

An alternative way to think of the overspill principle is that if a statement is true for all infinitesimals, then it is true for some standard positive number.

**Proposition 2.9.** *Let $\mathbb{R}$ be a domain in $\mathscr{V} = \langle V, \in \rangle$. Suppose $A \subseteq {}^*\mathbb{R}$ is an internal subset and each $x \in A$ is a finite element. Then $A$ contains a least upper bound.*

The overspill principle is also used to distinguish sets that are not internal to ${}^*\mathscr{V}$ for if they were internal, then we would be assuming $\mathbb{R} = {}^*\mathbb{R}$, which is clearly false.

**Theorem 2.10** (Saturation). *Let $\{A^i\}_{i\in\mathbb{N}}$ be a sequence of internal sets satisfying $\displaystyle\bigcap_{i=0}^{n} A^i \neq \varnothing$ for each $n \in \mathbb{N}$, where each $A^i = [A^i_0, A^i_1, A^i_2, \ldots]$. Then $\displaystyle\bigcap_{i\in\mathbb{N}} A^i \neq \varnothing$.*

Theorem 2.10 is also known as $\aleph_1$-saturation (cf. [3]).

## 3. Hausdorff measure

We begin with some preliminary definitions. Also see [5, 12, 15, 18] for further background on Hausdorff measure. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the Euclidean magnitude $d(x, y) = \|x - y\| := \left(\displaystyle\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$ for $x, y \in \mathbb{R}^n$. Given $U \subseteq \mathbb{R}^n$, the diameter of $U$ is defined as $\operatorname{diam}(U) = \sup\{d(x, y) : x, y \in U\}$. Now let $A \subseteq \mathbb{R}^n$. For any $U_i \subseteq \mathbb{R}^n$, the collection $\{U_i : i \in \mathbb{N}\}$ is a $\delta$-cover of $A$ if

(1) $\bigcup_{i\in\mathbb{N}} U_i \supseteq A$, and

(2) $0 \leq \operatorname{diam}(U_i) \leq \delta$ for each $i \in \mathbb{N}$.

A $\delta$-covering of a set is chosen such that the differences $\bigcup_i U_i \backslash A$ and $A \backslash \bigcup_i U_i$ are negligible sets[1].

**Definition 3.1.** Let $A$ be a subset of $\mathbb{R}^n$, and let $s, \delta > 0$. The Hausdorff $\delta$-measure of $A$ is defined as

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=0}^{\infty} \operatorname{diam}(U_i)^s\right\},$$

where the infimum is taken over every $\delta$-cover $\{U_i\}_{i\in\mathbb{N}}$ of $A$.

We will simply write Hausdorff $\delta$-measure as Hausdorff measure.

**Example 3.2.** Let $C$ be the Cantor set on the unit interval. That is, it is generated by cutting the middle 1/3 from each of the previous connected line segments. So letting $C_0 = [0,1]$, $C_1 = [0,1/3] \cup [2/3,1]$, $C_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$, $\ldots$, $C_m = \bigcap_{i=0}^{m} C_i$. At $m$-th iteration, there are $2^m$ intervals of length $3^{-m}$ to cover $C_m$. So

$$\mathcal{H}^s_\delta(C) = 2^m 3^{-ms},$$

which implies the critical value $s$ must be $\log 2/\log 3$.

**Definition 3.3.** Let $A \subseteq \mathbb{R}^n$, and let $s > 0$. Then $s$-dimensional Hausdorff outer measure of $A$ is

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A).$$

The Hausdorff measure $\mathcal{H}^s_\delta(A)$ increases as $\delta$ decreases. In fact, $\mathcal{H}^s(A)$ is a nonincreasing function as $s \to \infty$.

**Definition 3.4.** Let $A$ be a subset of $\mathbb{R}^n$. The Hausdorff dimension of $A$ is

$$\dim_{\mathcal{H}}(A) = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

An interpretation of the infimum in Definition 3.4 is as follows: if $s$ is greater than the actual Hausdorff dimension of $A$, then $\mathcal{H}^s(A) = 0$, which

[1] Negligible sets form an ideal, and in this paper, we assume that negligible sets are $\sigma$-ideal, *i.e.*, a countable union of negligible sets is negligible.

is the reason for us to take the infimum over all such $s$. A similar argument holds for the second equality.

## 4. Nonstandard analysis techniques on Lebesgue and Hausdorff measure

In this section, we develop a notion of measure in the nonstandard world by using nonstandard analysis techniques to explore various ways to measure a set.

Consider the unit interval, and let $\Omega = \{0, 1/N, \ldots, i/N, \ldots, 1\}$, where $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ is a nonstandard natural. Note that the standard map for $\Omega$ sends all points to the interval $[0,1]$. Recall from Proposition 2.7 that, for some set $A \subseteq [0,1]$, $\mathrm{st}^{-1}(A) \subseteq \Omega$ is not necessary internal. In particular, studying the cardinality $\mathrm{card}(\mathrm{st}^{-1}(A))$ is meaningless. Hence we will approximate this set using internal sets.

### 4.1. **Lebesgue measure in the nonstandard universe.**

We refer to [13, 19, 2] for some background on Lebesgue measure. In this section, we define a nonstandard version of Lebesgue measure.

Let $A \subseteq [0,1]$ be a nonempty set. Let $B$ be an internal subset of $\Omega$ such that $B \subseteq \mathrm{st}^{-1}(A)$. Note that there may be many internal $B$ contained in $\mathrm{st}^{-1}(A)$. Each $B$ has a discrete measure in $\Omega$, which is the discrete probability measure, where each $x \in \Omega$ is equally likely. We define the discrete measure of $B$ as

$$d(B) = \frac{\mathrm{card}(B)}{N+1},$$

where $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Note that $d(B) \in {}^*\mathbb{Q} \cap {}^*[0,1]$.

**Definition 4.1.** Let $A$ be a subset of $[0,1]$, and let $\Omega$ be the set $\{0, 1/N, \ldots, i/N, \ldots, 1\}$, where $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $d(B)$ be defined as above for some internal set $B$.
Lower Lebesgue measure of $A$ is defined to be

$$\underline{\lambda}(A) = \sup\{\mathrm{st}\, d(B) : \text{internal } B \subseteq \mathrm{st}^{-1}(A)\},$$

and upper Lebesgue measure of $A$ is defined as

$$\overline{\lambda}(A) = \inf\{\mathrm{st}\, d(B) : \text{internal } B \supseteq \mathrm{st}^{-1}(A)\}.$$

We say $A$ is Lebesgue measurable if $\overline{\lambda}(A) = \underline{\lambda}(A)$, and we write $\lambda(A)$ when $A$ is Lebesgue measurable.

4.2. **Hausdorff measure in the nonstandard universe.** Hausdorff outer measure is related to Lebesgue outer measure by

$$\mathcal{H}^s(A) = c_s \overline{\lambda}(A) \tag{4.1}$$

for some constant $c_s$ that depends on $s$, and for nice sets $A$. That is, given a nice set embedded in $\mathbb{R}^n$ with positive integral dimension $s$, we use Lebesgue outer measure to find the measure of $A$.

We now give a discrete version of Hausdorff measure.

**Definition 4.2.** Given $\delta > 0$, $s$ in the unit interval, and $N > \mathbb{N}$, a $\delta$-interval of $\Omega$ is a set $\{i/N, (i+1)/N, \ldots, j/N\}$ with diameter $(j - i + 1)/N \leq \delta$. For an internal set $B \subseteq \Omega$, the discrete $s$-dimensional measure is defined as

$$h^s_\delta(B) = \min \sum_{i=1}^{L} (\operatorname{diam}(V_i))^s, \tag{4.2}$$

where we take the minimum over all partitions $\{V_1, \ldots, V_L\}$ of $B$ into $\delta$-intervals.

The finite $L \in {}^*\mathbb{N}$ varies, depending on the partition. In the discrete space $\Omega$, this definition is internal and since there are finitely-many nonstandard partitions to consider, the min in (4.2) is a true minimum. Moreover, since the $V_i$'s are subsets of $\Omega$ which have been normalized, the need for renormalization is unnecessary.

## 5. Proof of Theorem 1.1

*Proof.* We will prove (a) first. Assume that the infinitesimal $\delta$ is fixed, and that $h^s_\delta(B)$ is finite for some internal $B \supseteq \mathrm{st}^{-1}(A)$. Let $\eta > 0$ be standard. Our strategy is to find an $\eta$-cover $U_i$ for $i \in \mathbb{N}$ such that $\sum_i (\operatorname{diam}(U_i))^s \leq \mathrm{st}(h^s_\delta(B))$.

Take an optimal partition $\{V_j\}_{j=1}^K$ of $B$ in the sense of $h^s_\delta(B)$. Using an internal induction in the nonstandard universe, modify $\{V_j\}_{j=1}^K$ to some other partition $\{W_k\}_{k=1}^L$ as follows. When defining some $W_k$, given an internal interval $I \subseteq \Omega$, consider the leftmost $V_i$ that is to the right of $I$.

If $I \cup V_i$ is also an interval, replace $I$ with this interval and continue. This process stops when one of the following occurs:

(1) $I \cup V_i$ is not an interval, or there is no further $V_i$ to the right of $I$,
(2) $\mathrm{diam}(I) \geq \eta$.

If either (1) or (2) happens, we stop the construction and let $W_k = I$. We then construct $W_{k+1}$ starting with $I'$, the leftmost $V_i$ to the right of $I$ if any.

At the end of this construction, we will have nonstandard finitely-many intervals $W_k$ of length at most $\eta + \delta$ partitioning $B$. This gives a countable $\eta$-cover $\mathcal{U}$ of our original $A$ consisting of all sets $U = \mathrm{st}(W_k)$ for some $W_k$ such that the set $U$ has nonempty interior.

To see that this does indeed cover $A$, consider $a \in A$. Then the monad of $a$ is contained in $B$. By the overspill principle, this monad is contained in an interval $J \subseteq B$, where the length of $J$ is not infinitesimal. But then, by construction, the monad of $a$ is contained in either some set $W_k$ or else, in two neighboring sets $W_k$ and $W_{k+1}$ (the latter case is when the construction of the set $W_k$ was "finished" whilst inside the monad of $a$). Moreover, $W_k$ (or in the other case both of $W_k$ and $W_{k+1}$) have lengths that are non-infinitesimal, by construction and by choice of $\eta$. So $a$ is either in the interior of $\mathrm{st}(W_k)$ or else is an endpoint of both $\mathrm{st}(W_k)$ and $\mathrm{st}(W_{k+1})$, as required. The cover $\mathcal{U}$ is countable because any set of intervals in the real number line, all with nonempty interior, must necessarily be countable. Finally, since without loss of generality, $s \leq 1$ (since we are working on the unit interval) and hence $(a+b)^s \leq a^s + b^s$ for $a, b > 0$, we have

$$\sum_{U \in \mathcal{U}} (\mathrm{diam} U)^s = \mathrm{st} \sum_k (\mathrm{diam}(W_k))^s \leq \mathrm{st} \sum_i (\mathrm{diam}(V_i))^s,$$

as required.

We will now prove (b). We observe first that for $0 < \delta < \eta$, even if $\delta$ is not infinitesimal, then the argument just given shows that $\mathcal{H}^s_\eta(A) \leq \mathrm{st} h^s_\delta(B)$ for all internal $B \supseteq \mathrm{st}^{-1}(A)$. Thus we only have to prove the other direction. It suffices to show that for each standard $\delta > 0$ and each standard $\varepsilon > 0$, there is an internal $B \supseteq \mathrm{st}^{-1}(A)$ with $\mathcal{H}^s_\delta(A) \geq \mathrm{st}(h^s_\eta(B)) - \varepsilon$ for a certain $\eta$ depending only on $\delta$ and $\varepsilon$. Let $\lambda = \mathcal{H}^s_\delta(A)$.

Let $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ be a $\delta$-cover of $A$ such that

$$\sum_{i=1}^{N}(\mathrm{diam}(U_i))^s \leq \lambda + (\varepsilon/2),$$

and assume without loss of generality that each $U_i$ is an interval. We "enlarge" $U_i$ by increasing its length by $(\varepsilon/2)^{1/s}2^{-i/s}$ on each side, obtaining intervals $V_i$. By saturation, there is a sequence of intervals $W_i \subseteq \Omega$ such that $\mathrm{st}(W_i) = V_i$ and $W_i$ is defined for all $i < K$, where $K > \mathbb{N}$. Then for any $N > \mathbb{N}$, we have $\bigcup_{i=1}^{N} W_i \supseteq \mathrm{st}^{-1}(A)$ because of the "enlarging". Moreover, for each $N \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{i=1}^{N}(\mathrm{diam}(W_i))^s &\leq \sum_{i=1}^{N}(\mathrm{diam}(U_i) + (\varepsilon/2)^{1/s}2^{-i/s})^s \\ &\leq \sum_{i=1}^{N}(\mathrm{diam}(U_i))^s + (\varepsilon/2) + (\varepsilon/2)\sum_{i=1}^{N}2^{-i} \\ &\leq \lambda + \varepsilon.\end{aligned}$$

By the overspill principle, there is an infinite $N$ such that the above inequalities hold. Thus, some $B = \bigcup_{i=1}^{N} W_i$ has a partition showing $h^s_\eta(B) \leq \lambda + \varepsilon$. Finally, note that the maximum diameter of any $W_j$ is $\delta + (\varepsilon/2)^{1/s}2^{-1/s}$, which may be made as close to $\delta$ as we like by choosing $\varepsilon$ sufficiently small. This completes the proof. $\square$

## 6. Example to the main theorem

Unfortunately, it does not seem possible to replace the limit as $\delta \to 0$ over standard $\delta$ with an infinitesimal $\delta$ in Theorem 1.1(b). In other words, the inequality in Theorem 1.1(a) cannot be replaced by an equality, even for carefully chosen $\delta$, as shown by the following example.

**Example 6.1.** Consider the Cantor set $C \subseteq [0, 1]$ in Example 3.2, which has dimension $s = \log 2/\log 3$. Given an internal $B \supseteq \mathrm{st}^{-1}(C)$ and a positive infinitesimal $\delta$, we will estimate $h^s_\delta(B)$.

For each $a \in C$, the monad $\mathrm{st}^{-1}(a)$ is covered by an interval of non-infinitesimal length $I \subseteq B$. Taking all such intervals and mapping them back to the unit interval via the standard part map, and taking the interiors of these sets, we obtain an open cover of $C$. Since $C$ is closed and bounded (hence compact), there is a finite subcover. This shows that we may assume that $B$ is one of the sets of $2^m$ intervals of length $3^{-m}$ obtained at the $m$-th stage of the construction of $C$ (if not, such collection would be smaller than $B$). For an interval $I \subseteq \Omega$ of length $\ell$, we have that $h^s_\delta(I)$ is approximately $(\ell/\delta)\delta^s$, so

$$h^s_\delta(B) = 2^m \frac{3^{-m}}{\delta}\delta^s = (2/3)^m \delta^{s-1}.$$

Since $0 < s < 1$ and $\delta$ is an infinitesimal, this value is infinite for all standard $m \in \mathbb{N}$. Hence $\mathrm{st}(h^s_\delta(B)) = \infty$ for all internal $B \supseteq \mathrm{st}^{-1}(C)$.

One can speculate that the problem is that for an infinitesimal $\delta$, the function $h^s_\delta$ measures the size of $B$ in terms of $s < 1$ , whereas the local structure of $B$ shows that it has dimension 1. We leave it as future work to determine for which sets we have an equality in Theorem 1.1(a) for a suitably chosen infinitesimal $\delta$.

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# NOTE FOR THE THIRD HILBERT PROBLEM: A FRACTAL CONSTRUCTION

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(Received : 16 - 06 - 2020 ; Revised : 09 - 03 - 20201)

Abstract. Hilbert's Third problem questioned whether, given two polyhedrons with the same volume, it is possible to decompose the first one into a finite number of polyhedral parts that can be put together to yield the second one. This finite equidecomposition process had already been shown to be possible between polygons of the same area. Dehn solved the problem by showing that a regular tetrahedron and a cube with equal volume were not equidecomposable. In this paper, we present an infinite fractal process that allows the cube to be visually reconstructed from a tetrahedron with equal volume. We have proved that, given two tetrahedrons with the same volume, the first one can be decomposed into an infinite number of polyhedral parts that can be put together to yield the second one. This process makes it possible to obtain the volume of a tetrahedron from the volume of the parallelepiped, without the use of formulas or the Cavallieri Principle.

## 1. Introduction

In 1900, Hilbert proposed 23 problems that opened research lines in different branches of Mathematics [8]. Some of these yet unsolved problems remain as a challenge for current mathematicians. On the other hand, the Third problem was solved before Hilbert's lecture was delivered [3]. However, the interest on this matter remains.

In particular, the Third problem, in which prestigious mathematicians such as Gauss had already shown interest [6], stated that given two polyhedrons with the same volume, it is possible to decompose the first one into a finite number of polyhedral parts that can be put together to yield the second one. Dehn, [2], one of Hilbert's students, provided a negative answer

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to this problem and proved that a regular tetrahedron and a cube with the same volume are not equidecomposable. In other words, the former cannot be decomposed into finitely many polyhedral pieces that can be put together to obtain the latter. The main idea behind Dehn's proof was to define an invariant that remains unchanged in the process of decomposition into finite polyhedral pieces, and to show that the cube and the regular tetrahedron had different values of the invariant. In 1965, Sydler, [6], proved that two polyhedra are equidecomposable if and only if they have the same volume and the same Dehn invariant.

The above highlights the difference between the Euclidean plane and the Euclidean space, as the Wallace-Bolyai-Gerwein theorem [1] states: *Polygons are equidecomposable if and only if they have the same area.* Theile [7] has suggested that on Hilbert's view, the need of infinite processes could contradict the metaphysical principle that the Universe is governed in such a way that a maximum of simplicity and perfection is done. However, infinite processes can also result of great simplicity and perfection, as happens with fractals, which exhibit approximations to infinity of great beauty and mathematical regularity [5].

In this paper we describe an infinite fractal process to decompose a regular tetrahedron into infinite pieces that can be put together to form a cube of the same volume. The process is repeated successively in different scales showing a fractal nature.

This process is generalizable to any tetrahedron, obtaining an original visual demonstration of an "infinite" version of the Third problem for tetrahedrons: *given two tetrahedrons with the same volume, it is possible to decompose the first one into a finite number of polyhedral parts that can be put together to yield the second one.*

To ease the visual comprehension of this process without resorting to the use of formulas of volumes or the Cavalieri principle, the decomposition of a particular tetrahedron, the trirectangular tetrahedron, is initially shown. Subsequently the generalization to any tetrahedron is presented.

Before the main results we introducing some concepts and notations in order to clarify the paper.

## 2. Concepts and notations

- A trirectangular tetrahedron is a tetrahedron with all three faces angles at one vertex are right angles. The three edges that meet at the right angle of a trirectangular tetrahedron are called legs.
- $C_n$ denotes a cube of edge length $n$.
- $T_n$ denotes a trirectangular tetrahedron of equal legs of length $n$.
- $P_n$ denotes the regular tetrahedron obtained by joining four of the 8 vertices of the cube $C_n$. More specifically, picking every other vertex of a cube so that no two are joined by an edges but any pair is joined by a diagonal of the cube's face we can form a regular tetrahedron.

## 3. Main result

**Lemma 3.1.** *A Fractal Decomposition of a trirectangular tetrahedron in a cube is possible.*

*Proof.* The three diagonals of the faces of a cube that meet in a vertex comprise a trihedron angle which is the tip of a regular tetrahedron. By joining the other vertices of these diagonals with those from the other sides of the cube, we obtain a tetrahedron with equal angles and legs of the same length, the diagonal of the face of the cube $C_1$, therefore it is a regular tetrahedron $P_1$ (Figure 1). □

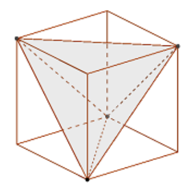
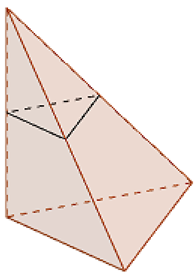


Figure 1. Regular tetrahedron $P_1$ within the cube $C_1$.

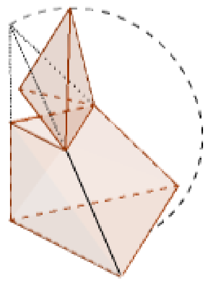
Outside the tetrahedron there remain four identical figures, namely triangular pyramids, with 3 faces comprising rectangular and isosceles triangles and one face comprising an equilateral triangle (the face of the tetrahedron). So, $C_1$ is decomposed in four trirectangular tetrahedrons $T_1$, and a tetrahedron $P_1$ (Figure 2).

FIGURE 2. Decomposition of $C_1$ in $P_1$ and four $T_1$.

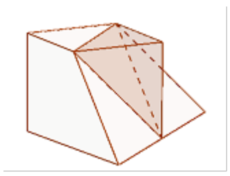
We start the fractal decomposition of the trirectangular tetrahedron $T_1$. In order to determine the portion of the cube occupied by $T_1$, we will truncate it by means of half-planes parallel to one of its bases. A new trirectangular tetrahedron then appears, $T_{\frac{1}{2}}$, in the upper part of the truncated one (Figure 3a). Letting such $T_{\frac{1}{2}}$ rotate (Figure 3b), with axis equal to the cutting line between the plane and the equilateral face we obtain three pieces: a cube $C_{\frac{1}{2}}$ and two $T_{\frac{1}{2}}$ adhered to it (Figures 3c, 3d).



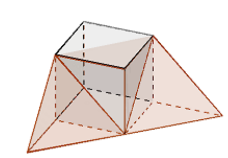
(A) Step 1



(B) Step 2



(C) Step 3



(D) Step 4

FIGURE 3. From left to right, top to bottom: Truncation of $T_1$ to obtain one cube $C_{\frac{1}{2}}$ and two $T_{\frac{1}{2}}$.

Repeating the truncation process for $T_{\frac{1}{2}}$, two cubes $C_{\frac{1}{4}}$ and four new $T_{\frac{1}{4}}$ are obtained (Figure 4).

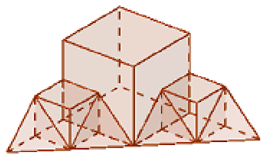


FIGURE 4. Truncation of the two $T_{\frac{1}{2}}$.

This process can be successively iterated as it is shown in Figure 5 obtaining a series of cubes and trirectangular tetrahedrons that can be summarized in Figure 6.

(A) Truncation of the 8 $T_{\frac{1}{4}}$.

(B) Recurrent process

FIGURE 5. Recurrent process of the truncation.

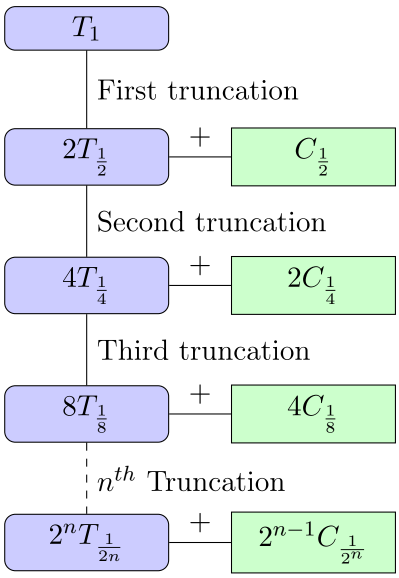
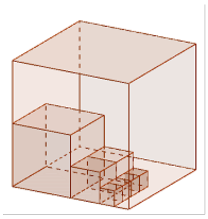


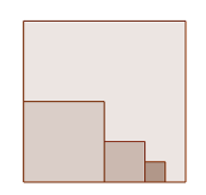

FIGURE 6. Number of trirectangular tetrahedrons and cubes obtained after each truncation.

By conveniently placing these cubes above half of the original cube, we obtain a sequence from which two views are shown (perspective and lateral) in Figures 7a-7b.

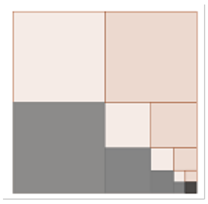
According to Nelsen's in [4], and by comparing quantities, the sum of the surface quantities of the square faces is equal to one-third of the surface quantity of the square (Figure 7c).



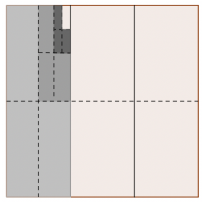
(A) Figure 7a



(B) Figure 7b



(C) Figure 7c



(D) Figure 7d

FIGURE 7. From left to right, top to bottom: location of the cubes in the truncation in order to obtain the volume.

We also note that this follows from the geometric series of ratio $\frac{1}{4}$ and first term $\frac{1}{4}$ and

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{3}.$$

This comparison allows us to say that the sum of the volume magnitudes of all the cubes in which we decomposed $T_1$ is one-third of half of the cube.

A new placement of these cubes (Figure 7d) divided into three pieces allows the construction of a prism of equal height of $T_1$ and rectangular base of dimensions of the base $\frac{1}{3}$ and $\frac{1}{2}$. To do so, each square of Figure

7c is divided into three equal rectangles. Two of them are left together and the third is placed on the first of those mentioned previously, as it can be perceived in the recursive process of Figure 7c.

Visualizing this construction in perspective as it is shown in Figure 7a, it is perceived that $T_1$ has decomposed into a right prism of $\frac{1}{6}$ of the volume of $C_1$. As any two given prisms of equal volume are equidecomposable [1], we have proved the following result.

**Proposition 3.2.** *$T_1$ has been decomposed into an infinite fractal process into a prism of which the volume is $\frac{1}{6}$ the volume of the cube with its same edge (Figure 8).*

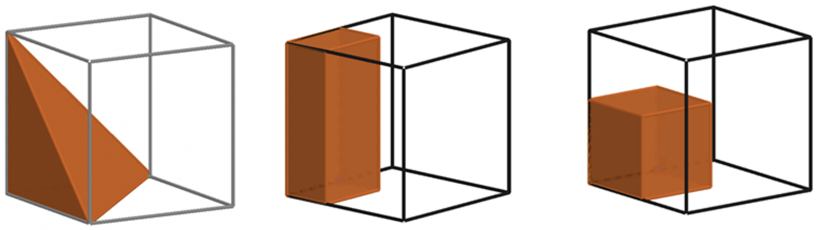
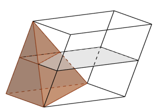


Figure 8. Equidecomposition from Ttri to a cube.

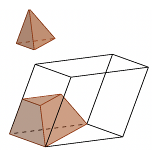
Now we can generalize the result for any tetrahedron.

**Theorem 3.3.** *A fractal decomposition of a tetrahedron in a cube is possible.*

*Proof.* The above fractal process is generalizable to any tetrahedron. We show with images the sequence of steps. The only difference is that in the truncation process it is necessary to perform a symmetry (Figures 9–10). □

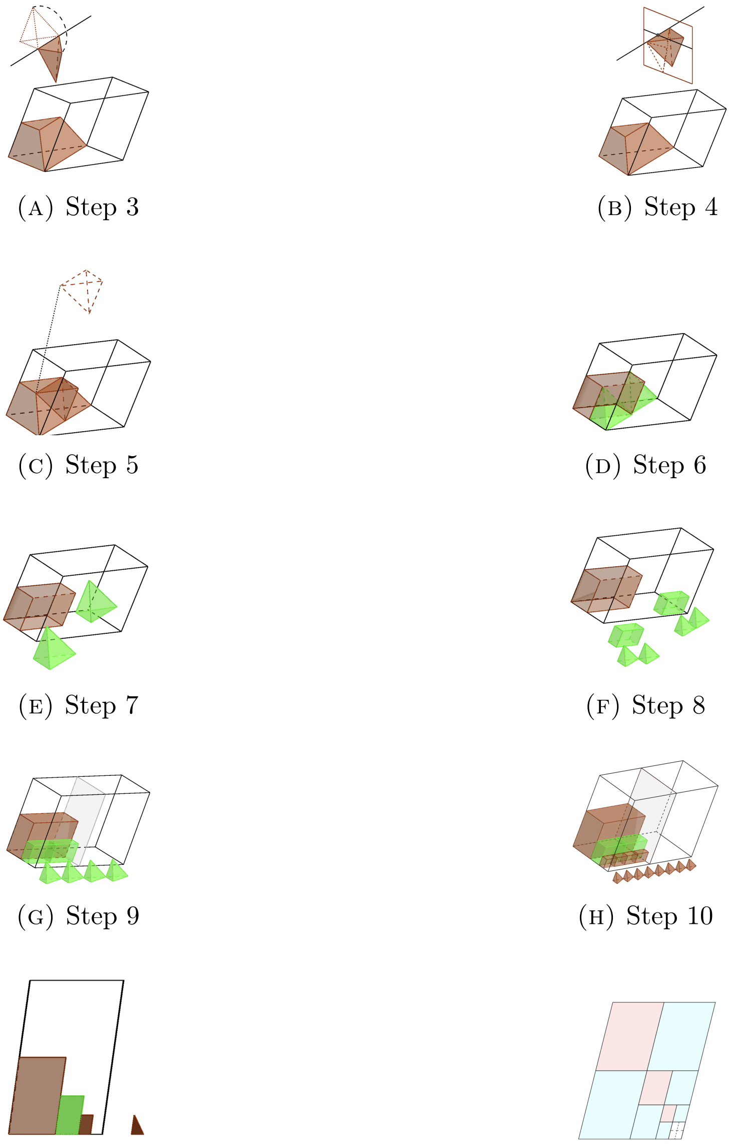


(a) Step 1



(b) Step 2

Figure 9. Fractal process: Steps 1 and 2.



(A) Step 3 (B) Step 4 (C) Step 5 (D) Step 6 (E) Step 7 (F) Step 8 (G) Step 9 (H) Step 10 (I) Step 11 (J) Step 12

FIGURE 10. Fractal process: Steps 3 to 12.

As in the $T_1$ case, these parallelepipeds divided into three pieces can be repositioned to obtain a new parallelepiped of volume 1/6 of the original parallelepiped.

Finally, by equidecomposition of prisms of equal volume, we obtain a cube of equal volume to the original tetrahedron, and therefore of volume equal to $1/6$ of the parallelepiped that contains the original tetrahedron. Therefore, we can formulate the following result:

**Theorem 3.4.** *Given two tetrahedra of equal volume, it is possible to decompose the first one into an infinite number of polyhedral pieces that can be put together so as to yield the second one.*

*Proof.* The proof is based on decomposing the first tetrahedron into a prism with volume $1/6$ of the parallelepiped and then repeating the reverse process to reconstruct the other tetrahedron. Furthermore, as a consequence of this process, what is obtained is that the volume of a tetrahedron is $1/6$ of the parallelepiped that contains it. An infinite process has been necessary, but neither the formulas nor the Cavalieri Principle have been used. This construction can also be used for the case of the regular square pyramids, because they can be decomposed in 4 $T_1$ (Figure 11). □

FIGURE 11. Regular square pyramid (left) and comprising four trirectangular tetrahedrons (right).

As pointed out by the Dehn Theorem answering to Hilbert's problem, the comparison of the volume of a pyramid requires an infinite process. The explained procedure makes an iterative comparison based on self-similarity, which allows to obtain the ratio between the volumes of the tetrahedron and the regular square pyramid, and between both and the cube.

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# PROBLEM SECTION

Through Volume 90 (1-2) 2021 of The Mathematics Student, we had invited solutions from readers to Problems 1, 2, 3, 5, 6, 8 and 9 mentioned in MS 89 (3-4) 2020, as well as solutions to the eight new problems till August 20, 2021.

As regards to solutions to the seven Problems mentioned in MS 89(1-2) 2020, we did not receive any solution to any of the seven problems. We believe that Problems 1, 2, 3 and 5 can be solved by several readers and we give some more time to the readers to solve these problems. Solutions provided by the proposers to Problems 6, 8 and 9 are being published in this section.

As far as solutions to the eight new Problems mentioned in MS 90 (1-2) 2021 are concerned, we received two correct solutions to Problem 3 and we publish one solution which is more precise and elegant. We also received two correct solutions to Problem 6 and we publish one solution which is short and better. Further, we received from readers one correct solution to Problem 7 and one correct solution to Problem 8. These solutions are being presented in this section.

We pose **ten new problems** in this section. We invite Solutions from the readers to these ten problems; solutions to Problems 1, 2, 3 and 5 of MS 89 (3-4) 2020 and also solutions to the rest of the Problems mentioned in MS 90 (1-2) 2021 till January 10, 2022. Correct solutions received from the readers by this date will be published in Volume 91 (1-2) 2022 of The Mathematics Student. This volume is scheduled to be published in February 2022.

**New Problems**

The following five problems are posed by **Prof. B. Sury**, Indian Statistical Institute, Bangalore.

**MS 90 (3-4) 2021: Problem 1.** Find all 4-tuples $(a_1, a_2, a_3, a_4)$ of positive integers such that $a_i$'s are distinct and each $a_i$ divides the sum of the other three.

**MS 90 (3-4) 2021: Problem 2.** Call a positive integer $N$ 'powerful' if it can be written as $a^3 + b^5 + c^7 + d^9 + e^{11} + f^{13}$ for positive integers $a, b, c, d, e, f$. Show that there exist arbitrarily large numbers that are NOT powerful.

**MS 90 (3-4) 2021: Problem 3.** Let $p$ be a prime and, for each $1 \le i \le p-1$, let $i^p \equiv a_i \bmod p^2$ where $0 < a_i < p^2$. Find the value of $\sum_{i=1}^{p-1} a_i$.

**MS 90 (3-4) 2021: Problem 4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $f(x_1, \cdots, x_n) = \frac{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}{x_1^{b_1} + \cdots + x_n^{b_n}}$ for $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ and $f(0, \cdots, 0) = 0$, where $a_i$'s are positive integers and $b_i$'s are even positive integers. Determine precisely all the $a_i$'s and $b_i$'s for which $f$ is discontinuous at 0.

**MS 90 (3-4) 2021: Problem 5.** Let $S$ be a finite set of points in the plane and suppose each of the points is given one of two colours red or blue. Prove that there exist $P, Q \in S$ such that all the points on the line joining $P$ and $Q$ have the same colour.

**Dr. Anup Dixit**, Institute of Mathematical Sciences, Chennai posed the following three problems.

**MS 90 (3-4) 2021: Problem 6.** Let $x_1, x_2, \cdots, x_n$ be distinct real numbers. Show that

$$\sum_{1 \le i \le n} \prod_{j \neq i} \left( \frac{1 - x_i x_j}{x_i - x_j} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

**MS 90 (3-4) 2021: Problem 7.** Show that

$$\zeta(3) < \frac{5}{4},$$

where $\zeta(k) := \sum_{n=1}^{\infty} 1/n^k$.

**MS 90 (3-4) 2021: Problem 8.** Show that among any 4 distinct positive real numbers $a_1, a_2, a_3, a_4$, we can find $a_i, a_j$ such that $a_i > a_j$ and

$$a_i(\sqrt{3} - a_j) < (\sqrt{3} a_j - 1).$$

The following two problems have been proposed by **Dr. Siddhi Pathak**, Chennai Mathematical Institute, Chennai.

**MS 90 (3-4) 2021: Problem 9.** Let $a_1, a_2, \cdots, a_{2021}$ be real numbers such that

$$\sum_{i=1}^{2021} a_i = 0, \qquad \sum_{i=1}^{2021} a_i^2 = 1.$$

Let $c := \max_{1 \le i \le 2021} a_i$ and $d := \min_{1 \le i \le 2021} a_i$. Show that

$$-\frac{1}{2} \le c\,d \le -\frac{1}{2021}.$$

**MS 90 (3-4) 2021: Problem 10.** Fix any positive integer $m > 1$. Show that if $f : (0, \infty) \to (0, \infty)$ satisfies

$$f(x)\, f(y) = m\, f\left(x + y f(x)\right) \text{ for all } x, y > 0,$$

then $f(x) = m$ for all $x > 0$.

**Solutions to the old Problems**

**MS 89 (3-4) 2020: Problem 6** (Posed by Dr. Siddhi Pathak).
Let $\mathbb{Q}^+$ denote the set of positive rational numbers, and $P : \mathbb{Q}^+ \to \mathbb{N}$ be defined as $P(m/n) = mn$ for $\gcd(m, n) = 1$. Show that

$$\sum_{q \in \mathbb{Q}^+} \frac{1}{P(q)^2} = \frac{5}{2}.$$

The solution provided by Dr. Pathak to this problem is given below.

**Solution:** Note that

$$\sum_{q \in \mathbb{Q}^+} \frac{1}{P(q)^2} = \sum_{\substack{m,n=1 \\ (m,n)=1}}^{\infty} \frac{1}{(mn)^2},$$

where $(m, n)$ denotes the $\gcd(m, n)$. Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ denote the Riemann zeta-function. Then we have

$$\zeta(2)^2 := \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^2 = \sum_{m,n=1}^{\infty} \frac{1}{(mn)^2} = \sum_{d=1}^{\infty} \sum_{\substack{m,n=1 \\ (m,n)=d}}^{\infty} \frac{1}{(mn)^2}$$

$$= \left( \sum_{d=1}^{\infty} \frac{1}{d^4} \right) \left( \sum_{\substack{m,n=1 \\ (m,n)=1}}^{\infty} \frac{1}{(mn)^2} \right) = \zeta(4) \left( \sum_{q \in \mathbb{Q}^+} \frac{1}{P(q)^2} \right).$$

Thus,
$$\sum_{q\in\mathbb{Q}^+}\frac{1}{P(q)^2}=\frac{\zeta(2)^2}{\zeta(4)}=\frac{5}{2},$$
since $\zeta(2)=\pi^2/6$ and $\zeta(4)=\pi^4/90$.

**Remark**. The above argument in fact shows that for $s\in\mathbb{C}$ with $\Re(s)>1$,
$$F(s):=\sum_{q\in\mathbb{Q}^+}\frac{1}{P(q)^s}=\frac{\zeta(s)^2}{\zeta(2s)}.$$
In particular, $F(2n)\in\mathbb{Q}$ as $\zeta(2n)\in\pi^{2n}\mathbb{Q}^\times$.

**MS 89 (3-4) 2020: Problem 8** (Posed by Dr. Anup Dixit).
Let $a_1,a_2,\cdots,a_n$ be positive real numbers all $\geq 1$. Let $G$ denote its geometric mean given by $G=(a_1a_2\cdots a_n)^{1/n}$. Show that
$$\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}\geq\frac{n}{1+G}.$$

The solution provided by Dr. Dixit is presented below.

**Solution**: We first show that the inequality holds for $n=2$, i.e., for $a_1,a_2\geq 1$,
$$\frac{1}{1+a_1}+\frac{1}{1+a_2}\geq\frac{2}{1+\sqrt{a_1a_2}}. \tag{1}$$
On simplification, this is equivalent to showing that
$$(2+a_1+a_2)\sqrt{a_1a_2}\geq a_1+a_2+2a_1a_2 \iff (\sqrt{a_1a_2}-1)(a_1+a_2-2\sqrt{a_1a_2})\geq 0.$$
This clearly holds because $a_1,a_2\geq 1$ and therefore $\sqrt{a_1a_2}-1\geq 0$. Furthermore, $(a_1+a_2-2\sqrt{a_1a_2})=(\sqrt{a_1}-\sqrt{a_2})^2\geq 0$.

For $n=4$, using (1) twice, we conclude that if $a_1,a_2,a_3,a_4\geq 1$, then
$$\frac{1}{1+a_1}+\frac{1}{1+a_2}+\frac{1}{1+a_3}+\frac{1}{1+a_4}\geq\frac{2}{1+\sqrt{a_1a_2}}+\frac{2}{1+\sqrt{a_3a_4}}\geq\frac{4}{1+(a_1a_2a_3a_4)^{1/4}}.$$
Similarly, using induction, it is easy to see that the required inequality holds for $n=2^k$ for any positive integer $k$, i.e., if $G$ is the geometric mean of $a_1,a_2,\cdots,a_{2^k}$ with all $a_i\geq 1$, then
$$\sum_{i=1}^{2^k}\frac{1}{1+a_i}\geq\frac{2^k}{1+G}. \tag{2}$$

Now, suppose $n$ is any positive integer satisfying $2^m \leq n < 2^{m+1}$. Let $G$ denote the geometric mean of $a_1, a_2, \cdots, a_n$. Set $a_{n+1} = a_{n+2} = \cdots = a_{2^{m+1}} = G$. Clearly, the geometric mean of $a_1, a_2, \cdots, a_{2^{m+1}}$ is also $G$. By (2), we have

$$\sum_{i=1}^{2^{m+1}} \frac{1}{1+a_i} = \sum_{i=1}^{n} \frac{1}{1+a_i} + \sum_{i=n+1}^{2^{m+1}} \frac{1}{1+G} \geq \frac{2^{m+1}}{1+G}$$

Hence,

$$\sum_{i=1}^{n} \frac{1}{1+a_i} \geq \frac{n}{1+G}$$

as required.

**MS 89 (3-4) 2020: Problem 9** (Posed by Prof. Ram Murty).
Prove that

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \alpha^n$$

is transcendental for every non-zero algebraic $\alpha$ satisfying $|\alpha| < 1/e$.

The solution given by Prof. Murty to the problem is as follows.

**Solution**. The above sum is connected to the tree function, which is the inverse of a certain well known function. Our first goal is to figure out the Taylor series expansion of inverse functions. If $f(z) = \sum_{i=0}^{\infty} c_i z^i$, then we use the notation $[z^n]f(z)$ to denote the co-efficient $c_n$.
**Theorem 1** (Lagrange's inversion formula). Suppose $f(z)$ is analytic in a neighborhood near zero, with $f(0) = 0$, $f'(0) \neq 0$. Then $f^{(-1)}(z)$ is analytic in a neighborhood near zero, and

$$[z^n]f^{(-1)}(z) = [z^{n-1}]\left(\frac{z^n}{nf(z)^n}\right).$$

Using the same ideas, for a positive integer $k$, we can also prove that
**Proposition 1**. Under the same assumptions on $f(z)$ and for $n \geq k$, we have

$$[z^n](f^{(-1)}(z))^k = [z^{n-k}]\left(\frac{kz^n}{nf(z)^n}\right)$$

.

The assumption of $n \geq k$ is required as $f^{-1}(z)$ has a zero of order one at $z = 0$, and therefore for the Taylor series of $f^{(-1)}(z)^k$ around $z = 0$ starts from $z^k$.

We also require a result on transcendence of the exponential function evaluated at a non-zero algebraic number.

**Theorem 2**. If $\alpha$ is a non-zero algebraic number, then $e^\alpha$ is transcendental.

This was essentially proved by Lindemann in 1882 and generalised by Weierstrass in 1885. We now proceed with the solution to the Problem.

*Proof.* We begin by describing the tree function $T(z)$ which is the inverse function of the map $z \to ze^{-z}$. We claim that the Taylor expansion of $T(z)$ around $z = 0$ is given by :

$$T(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n.$$

To prove this, we use the Lagrange inversion formula :

$$[z^n]T(z) = [z^{n-1}]\left(\frac{z^n}{n(ze^{-z})^n}\right) = [z^{n-1}]\left(\frac{e^{nz}}{n}\right) = \frac{n^{n-1}}{n(n-1)!} = \frac{n^{n-1}}{n!}. \quad (3)$$

This proves the claim. Furthermore, from Proposition **??** for any positive integer $k$, we have

$$T(z)^k = \sum_{n=0}^{\infty} \frac{k(n+k)^{n-1}}{n!} z^{n+k}.$$

In the above series, the term corresponding to $z^k$ is 1 and this result can formally be extended to non-negative integers $k$. Now we have :

$$\frac{1}{1-T(z)} = \sum_{k=0}^{\infty} T(z)^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{k(n+k)^{n-1}}{n!} z^{n+k} = \sum_{u=0}^{\infty}\sum_{n=0}^{u} \frac{(u-n)u^{n-1}}{n!} z^u,$$

where in the last step we are performing a change of variables by setting $n + k = u$, and since $k \geq 0$, the sum $n$ ranges from 0 to $u$. Rearranging the terms in the above sum, we obtain :

$$\frac{1}{1-T(z)} = \sum_{u=0}^{\infty}\left(\sum_{n=0}^{u} \frac{(u-n)u^{n-1}}{n!}\right)z^u = \sum_{u=0}^{\infty}\left(\sum_{n=0}^{u} \frac{u^n}{n!} - \sum_{n=0}^{u-1} \frac{u^n}{n!}\right)z^u = \sum_{u=0}^{\infty} \frac{u^u z^u}{u!}$$

where the term corresponding to $z^0$ is 1. If we set $S(\alpha) = \sum_{n=1}^{\infty} \frac{n^n}{n!}\alpha^n$, we have

$$\frac{1}{1-T(\alpha)} = 1 + S(\alpha) \tag{4}$$

We now assume that $S(\alpha)$ is algebraic for some non-zero algebraic number $\alpha$. From (4) we note that $T(\alpha)$ is algebraic. Since $T(z)$ is the inverse of the function $z \to ze^{-z}$, we have $T(\alpha)e^{-T(\alpha)} = \alpha$ and from this, we conclude that $e^{T(\alpha)}$ is algebraic. By Lindemann's theorem, this can happen only if $T(\alpha)$ is zero, and therefore $\alpha$ has to be zero. This is a contradiction to our assumption. $\square$

**Solutions Provided by the readers to the problems**

**MS 90 (1-2) 2021 : Problem 3** (posed by Prof. B. Sury, ISI, Bangalore). Let $A$ be an $n \times n$ matrix with rational entries. If the rank of $A$ is 1, show that $det(I_n + A) - trace(A) = 1$.

**Mr. Ritesh Dwivedi**, an M. Sc. Student of Allahabad University, Allahabad gave a solution to this problem. We also received a correct solution to this problem from **Ms Anupriya Shetty**, Department of PG Studies and Research in Mathematics, St Aloysius College, Mangaluru - 575 003, Karnataka. The solution provided by Dwivedi is shorter and it is presented below.

**Solution**. If $n = 1$, the result holds obviously. So, let $n > 1$. Since the rank of $A$ is 1, all the eigen values of $A$ are zero, except one which must be nonzero. So let, $\lambda$ be the non zero eigen value of $A$. Then, $trace(A) =$ sum of the eigen values of $A = \lambda$. Also, the eigen values of $I_n + A$ are 1 (with multiplicity $n-1$) and $\lambda + 1$. Therefore, $det(I_n + A) =$ product of the eigen values of $(I_n + A) = \lambda + 1$. Now the result follows.

**MS -90 (1-2) 2021: Problem 6**. (Proposed by Dr. Siddhi Pathak, Chennai Math. Inst., Chennai)
Let $\mathbb{Q}^+$ denote the set of positive rational numbers, and $P : \mathbb{Q}^+ \to \mathbb{N}$ be defined as $P(m/n) = mn$ for $\gcd(m, n) = 1$.
Show that

$$\sum_{q \in \mathbb{Q}^+} \frac{1}{P(q)^2} = \frac{5}{2}.$$

**Mr. Subhash Chand Bhoria**, Department of Mathematics, Pt. CLS Govt. PG College, Karnal, Haryana gave a solution to the problem. The solution given by him is presented below.

**Solution**. We prove a more general result involving two variables $a, b \in \mathbb{N}\backslash\{1\}$ namely,

$$\sum_{\substack{m,n=1\\ gcd(m,n)=1}}^{\infty} \frac{1}{m^a n^b} = \frac{\zeta(a)\zeta(b)}{\zeta(a+b)}.$$

In particular, $a = 2, b = 2$ in the above identity gives us the desired closed form for the sum $\sum_{q\in\mathbb{Q}^+} \frac{1}{P(q)^2}$. Let

$$\sum_{\substack{m,n=1\\ gcd(m,n)=1}}^{\infty} \frac{1}{m^a n^b} = S(a,b).$$

We know that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^a n^b} = \sum_{m=1}^{\infty} \frac{1}{m^a} \sum_{n=1}^{\infty} \frac{1}{n^b} = \zeta(a)\zeta(b). \tag{5}$$

Observe that sum in the LHS of the equation (5) can be written in the following manner,

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^a n^b} &= \sum_{\substack{m,n=1\\ gcd(m,n)=1}}^{\infty} \frac{1}{m^a n^b} + \sum_{\substack{m,n=2\\ gcd(m,n)=2}}^{\infty} \frac{1}{m^a n^b} + \sum_{\substack{m,n=3\\ gcd(m,n)=3}}^{\infty} \frac{1}{m^a n^b} + \cdots \\ &= \sum_{r=1}^{\infty} \sum_{\substack{m,n=r\\ gcd(m,n)=r}}^{\infty} \frac{1}{m^a n^b}. \end{aligned} \tag{6}$$

In order to re-index the sum in the RHS of equation (6), we make substititions $m = ru$ and $n = rv$ in the inner summation to receive,

$$\begin{aligned} \sum_{r=1}^{\infty} \sum_{\substack{m,n=r\\ gcd(m,n)=r}}^{\infty} \frac{1}{m^a n^b} &= \sum_{r=1}^{\infty} \sum_{\substack{u,v=1\\ gcd(u,v)=1}}^{\infty} \frac{1}{(ru)^a (rv)^b} \\ &= \sum_{r=1}^{\infty} \frac{1}{r^{a+b}} \sum_{\substack{u,v=1\\ gcd(u,v)=1}}^{\infty} \frac{1}{u^a v^b} \\ &= \zeta(a+b) \times S(a,b). \end{aligned} \tag{7}$$

Now, Combining equations (5), (6) and (7) we finally get,

$$\zeta(a)\zeta(b) = \zeta(a+b) \times S(a,b) \implies S(a,b) = \frac{\zeta(a)\zeta(b)}{\zeta(a+b)}.$$

Finally, taking the special values of the Riemann zeta-function that is, $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ yields $S(2,2) = \frac{\frac{\pi^2}{6} \times \frac{\pi^2}{6}}{\frac{\pi^4}{90}} = \frac{5}{2}$, which completes the proof.

**MS -90 (1-2) 2021: Problem 7** (Posed by Dr. Anup Dixit, IMSc, Chennai).
Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n^{\frac{5}{9}}}$$

is convergent. Here $[x]$ denotes the greatest integer $\leq x$.

**Mr. Rohit Yadav**, Mustafabad, Sangipur, Pratapgarh, Uttar Pradesh provided a solution to this problem. The solution due to him is given below.

**Solution**.
**Step 1**: Since $[\sqrt{n}]$ are same for such $n \in N$, which is satisfying -

$$k^2 \leq n < (k+1)^2, k \in N$$

$$\Rightarrow k^2 \leq n < k^2 + 2k + 1 ...........(1)$$

So, (2k+1) number of distinct $n \in N$ which satisfy (1).

**Step 2**: Now we grouping of terms of given series as $k^{th}$ group contains (2k+1) terms.

By step 1, $[\sqrt{n}]$ are same for each term of $k^{th}$ group and for $k^{th}$ group $[\sqrt{n}]$ = k

Hence by above discussion,

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n^{\frac{5}{9}}} = (-1)^1(\frac{1}{1^{\frac{5}{9}}} + \frac{1}{2^{\frac{5}{9}}} + \frac{1}{3^{\frac{5}{9}}}) + (-1)^2(\frac{1}{4^{\frac{5}{9}}} + \frac{1}{5^{\frac{5}{9}}} + \frac{1}{6^{\frac{5}{9}}} + \frac{1}{7^{\frac{5}{9}}} + \frac{1}{8^{\frac{5}{9}}})$$

$$+(-1)^3(\frac{1}{9^{\frac{5}{9}}} + \frac{1}{10^{\frac{5}{9}}} + \frac{1}{11^{\frac{5}{9}}} + \frac{1}{12^{\frac{5}{9}}} + \frac{1}{13^{\frac{5}{9}}} + \frac{1}{14^{\frac{5}{9}}} + \frac{1}{15^{\frac{5}{9}}}) + ......$$

$$= \sum_{m=1}^{\infty} (-1)^m (\frac{1}{(m^2)^{\frac{5}{9}}} + \frac{1}{(m^2+1)^{\frac{5}{9}}} + \frac{1}{(m^2+3)^{\frac{5}{9}}} + ...... + \frac{1}{(m^2+2m)^{\frac{5}{9}}}) ...........(2)$$

In the right hand side of equation (2), we have a alternating series of the form $\sum_{m=1}^{\infty}(-1)^m a_m$ where

$$a_m = (\frac{1}{(m^2)^{\frac{5}{9}}} + \frac{1}{(m^2+1)^{\frac{5}{9}}} + \frac{1}{(m^2+3)^{\frac{5}{9}}} + ...... + \frac{1}{(m^2+2m)^{\frac{5}{9}}}); m \in N$$

clearly,

$$\frac{2m+1}{(m^2)^{\frac{5}{9}}} \geq a_m \geq \frac{2m+1}{(m^2+2m)^{\frac{5}{9}}}; m \in N...........(3)$$

From above, by Squeeze Theorem, we have $\{a_m\}$ is converges to 0 and Also $\{a_m\}$ is strictly decreasing sequence. Hence, by Leibnizs test, given series

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n^{\frac{5}{9}}}$$

is convergent.

**MS -90 (1-2) 2021: Problem 8** (Proposed by Dr. Anup Dixit, IMSc, Chennai).
Show that

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 x^2 dx = 1 - \frac{1}{3}\big(\zeta(2) + \zeta(3)\big),$$

where $\{x\}$ is the fractional part of $x$ and $\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}$ denotes the Riemann zeta-function for $k > 1$.

**Mr. Subhash Chand Bhoria** whose affiliation is given above has also provided the solution to this problem. The solution provided by him is given below.

**Solution**. Let the given integral be

$$I = \int_0^1 \left\{\frac{1}{x}\right\}^2 x^2 dx.$$

We can write $\{\frac{1}{x}\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$, where $\left\lfloor \frac{1}{x} \right\rfloor$ denotes the greatest integer $\leq \frac{1}{x}$. Then the integral becomes

$$I = \int_0^1 \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor\right)^2 x^2 dx = \int_0^1 \left(\frac{1}{x^2} - \frac{2}{x}\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{1}{x} \right\rfloor^2\right) x^2 dx$$

$$= 1 - 2\int_0^1 x\left\lfloor \frac{1}{x} \right\rfloor dx + \int_0^1 x^2 \left\lfloor \frac{1}{x} \right\rfloor^2 dx = 1 - 2I_1 + I_2, \tag{8}$$

where $I_1 = \int_0^1 x\lfloor \frac{1}{x} dx$ and $I_2 = \int_0^1 x^2 \lfloor \frac{1}{x}^2 dx$. Now we would like to solve $I_1$ and $I_2$ independently. Note that, we can break the interval $(0, 1]$ into disjoint subintervals $(\frac{1}{n+1}, \frac{1}{n}]$ such that $(0, 1] = \cup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n}]$. This will give us that if

$$\frac{1}{n+1} < x \leq \frac{1}{n} \implies n \leq \frac{1}{x} < n+1 \implies \left\lfloor \frac{1}{x} \right\rfloor = n.$$

Consider

$$\begin{aligned} I_1 &= \int_0^1 x\left\lfloor \frac{1}{x} \right\rfloor dx = \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} x\left\lfloor \frac{1}{x} \right\rfloor dx = \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} xndx \\ &= \sum_{n=1}^{\infty} n\left(\frac{x^2}{2}\right)_{1/(n+1)}^{1/n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2n^2} - \frac{1}{2(n+1)^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)^2}. \end{aligned}$$

And

$$\begin{aligned} I_2 &= \int_0^1 x^2\left\lfloor \frac{1}{x} \right\rfloor^2 dx = \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} x^2\left\lfloor \frac{1}{x} \right\rfloor^2 dx = \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} x^2n^2dx \\ &= \sum_{n=1}^{\infty} n^2\left(\frac{x^3}{3}\right)_{1/(n+1)}^{1/n} = \sum_{n=1}^{\infty} n^2\left(\frac{1}{3n^3} - \frac{1}{3(n+1)^3}\right) = \frac{1}{3}\sum_{n=1}^{\infty} \frac{3n^3+3n+1}{n(n+1)^3}. \end{aligned}$$

Substituting the values of $I_1$ and $I_2$ in the equation (8) we arrive to

$$\begin{aligned} I &= 1 - \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)^2} + \frac{1}{3}\sum_{n=1}^{\infty} \frac{3n^3+3n+1}{n(n+1)^3} \\ &= 1 + \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{6n+3}{n(n+1)^2} - \frac{3n^3+3n+1}{n(n+1)^3}\right) \\ &= 1 - \frac{1}{3}\sum_{n=1}^{\infty} \frac{3n^2+6n+2}{n(n+1)^3} = 1 - \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{(n+1)(3n+2)}{n(n+1)^3} - \frac{n}{n(n+1)^3}\right) \\ &= 1 - \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{n+2(n+1)}{n(n+1)^2} + \frac{1}{(n+1)^3}\right) \\ &= 1 - \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2} + \frac{2}{n(n+1)} + \frac{1}{(n+1)^3}\right) \\ &= 1 - \frac{1}{3}\left(\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} + 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^3}\right). \end{aligned}$$

In the last step above observe that the sum

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = 1.$$

Utilizing the closed form of last sum above and the definition of Riemann zeta-function, $I$ finally becomes,

$$I = 1 - \frac{1}{3}\big(\zeta(2) - 1 + 2 + \zeta(3) - 1\big) = 1 - \frac{1}{3}\big(\zeta(2) + \zeta(3)\big).$$

This finishes the proof.

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