ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 89, Nos. 3-4, July - December (2020)
(Issued: October, 2020)

Editor-in-Chief
M. M. SHIKARE

EDITORS

Bruce C. Berndt	George E. Andrews	M. Ram Murty
N. K. Thakare	Satya Deo	Gadadhar Misra
B. Sury	Kaushal Verma	Krishnaswami Alladi
S. K. Tomar	Clare D′Cruz	L. Sunil Chandran
J. R. Patadia	C. S. Aravinda	A. S. Vasudeva Murthy
Indranil Biswas	Timothy Huber	T. S. S. R. K. Rao
Atul Dixit		

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
www.indianmathsociety.org.in

MEMBER'S COPY

THE MATHEMATICS STUDENT

Edited by M. M. SHIKARE

In keeping with the current periodical policy, THE MATHEMATICS STUDENT will seek to publish material of interest not just to mathematicians with specialized interest but to the postgraduate students and teachers of mathematics in India. With this in view, it will ordinarily publish material of the following type:

1. research papers,
2. the texts (written in a way accessible to students) of the Presidential Addresses, the Plenary talks and the Award Lectures delivered at the Annual Conferences.
3. general survey articles, popular articles, expository papers and Book-Reviews.
4. problems and solutions of the problems,
5. new, clever proofs of theorems that graduate / undergraduate students might see in their course work, and
6. articles that arouse curiosity and interest for learning mathematics among readers and motivate them for doing mathematics.

Articles of the above type are invited for publication in THE MATHEMATICS STUDENT. Manuscripts intended for publication should be submitted online in the LaTeX and .pdf file including figures and tables to the Editor M. M. Shikare on E-mail: msindianmathsociety@gmail.com along with a Declaration form downloadable from our website.

Manuscripts (including bibliographies, tables, *etc.*) should be typed double spaced on A4 size paper with 1 inch (2.5 cm.) margins on all sides with font size 11 pt. in LaTeX. Sections should appear in the following order: Title Page, Abstract, Text, Notes and References. Comments or replies to previously published articles should also follow this format. In LaTeX the following preamble be used as is required by the Press:

```
\ documentclass[11 pt,a4paper,twoside,reqno]{amsart}
\ usepackage {amsfonts, amssymb, amscd, amsmath, enumerate, verbatim, calc}
\ renewcommand{\ baselinestretch}{1.2}
\ textwidth=12.5 cm
\ textheight=20 cm
\ topmargin=0.5 cm
\ oddsidemargin=1 cm
\ evensidemargin=1 cm
\ pagestyle{plain}
```

The details are available on Society's website: www.indianmathsociety.org.in

Authors of articles / research papers printed in the the Mathematics Student as well as in the Journal shall be entitled to receive a *soft copy (PDF file)* of the paper published. There are no page charges.

All business correspondence should be addressed to S. K. Nimbhorkar, Treasurer, Indian Mathematical Society, C/O Dr. Mrs. Prachi Kulkarni, Ankur Hospital, Tilaknagar, Aurangabad 431 001 (MS), India. E-mail: treasurerindianmathsociety@gmail.com or sknimbhorkar@gmail.com

In case of any query, one may contact the Editor through the e-mail.

Copyright of the published articles lies with the Indian Mathematical Society.

MEMBER'S COPY

ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 89, Nos. 3-4, July - December (2020)
(Issued: October, 2020)

Editor-in-Chief
M. M. SHIKARE

EDITORS

Bruce C. Berndt	George E. Andrews	M. Ram Murty
N. K. Thakare	Satya Deo	Gadadhar Misra
B. Sury	Kaushal Verma	Krishnaswami Alladi
S. K. Tomar	Clare D′Cruz	L. Sunil Chandran
J. R. Patadia	C. S. Aravinda	A. S. Vasudeva Murthy
Indranil Biswas	Timothy Huber	T. S. S. R. K. Rao
Atul Dixit		

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
www.indianmathsociety.org.in

MEMBER'S COPY

ISSN: 0025-5742

© THE INDIAN MATHEMATICAL SOCIETY, 2020.

This volume or any part thereof may not be reproduced in any form without the written permission of the publisher.

This volume is not to be sold outside the Country to which it is consigned by the Indian Mathematical Society.

Member's copy is strictly for personal use. It is not intended for sale or circulation.

The volume is published by Prof. Satya Deo for the Indian Mathematical Society. Typeset of the articles is carried out by M. M. Shikare, "Krushnakali", Survey No. 73/6/1, Gulmohar Colony, Jagtap Patil Estate, Pimple Gurav, Pune 411061. The volume is printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune-411 041 (India).
Printed in India.

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July - December (2020)

CONTENTS

MEMBER'S COPY

Problems and Solutions

****

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 1–12

S. S. SHRIKHANDE-HIS LIFE AND SOME CONTRIBUTIONS TO MATHEMATICS

MOHAN S. SHRIKHANDE

1. Introduction

Sharadchandra Shankar Shrikhande (SSS) was born on October 19,1917 at Sagar, a town in the state of Madhya Pradesh. He received his B.Sc (Honors) degree in Mathematics from Nagpur University in 1939. The job situation at that time was very dire due to the world wide depression. In addition, all resources were being diverted by the British Government towards the war effort. Because of these circumstances SSS was unable to find a teaching position.

He decided to spend a year at the Indian Statistical Institute(ISI), Calcutta to learn the statistical techniques being advanced by Professor P. C. Mahalanobis in the Jute Sample Survey. Professor Raj Chandra Bose was a member of ISI, and had gained international fame in statistics, particularly in the area of design theory. But according to SSS, he did not have any contact with Bose during his stay at ISI. In 1942, SSS found a Lecturer's position in Mathematics at the Science College, Nagpur University. In 1943 he married Shakuntala Mohoni.

By the end of the War, the Government of India having amassed a large amount of US PL 480 funds, decided to use them to award scholarships for training students in statistics in USA. SSS was selected and began his studies at the University of North Carolina (UNC), Chapel Hill in 1947, leaving behind his wife and three young children.

He took some classes from Bose, who was on a visiting position at UNC. Bose returned to ISI, and in 1949 emigrated to USA, accepting a permanent

Key words and phrases: Life, Contributions

© *Indian Mathematical Society, 2020*.

MEMBER'S COPY

Figure 1. R. C. Bose (1901-1987)

professorship at UNC. SSS requested Bose to guide his Ph.D thesis. Bose became his thesis advisor and proposed some problems in combinatorial designs.

SSS completed his Ph.D work and became Bose's first Ph.D student in 1950. His thesis is regarded as a landmark by researchers in the field. It consisted of a result on the impossibility of certain symmetric balanced incomplete block designs by applying the techniques of Bruck and Ryser (1949), used for proving the non-existence of finite projective planes.

After completing his thesis, SSS returned to Science College, Nagpur and resumed his teaching duties. Due to the excellent training received under Bose, he was able to continue with his research and published several papers. He spent 1951-1953 on a visiting position at the University of Kansas, Lawrence.

After returning from Kansas to Science College, several of his students completed their Master's degrees. Among them was Damaraju Raghavarao, a gold medalist, who later received his Ph.D from Bombay University. He collaborated with SSS on several papers from the sixties to mid seventies. Many of their results are discussed in Raghavarao's book, Constructions and Combinatorial Problems in Design of Experiments (1971). During his tenure at Science College, he founded the Department of Statistics at Nagpur University and was Head of the Department.

SSS spent 1958-60 on a visiting position at UNC to collaborate with Bose. According to SSS, this period at Chapel Hill, proved to be among the most productive and satisfying time of his career. Refer to the next section for more details.

In 1960, SSS returned to India, accepting a Professorship in Statistics at Banaras Hindu University. While there he began his collaboration with Damaraju Raghavarao, who had then recently completed his Ph.D from Bombay University.

MEMBER'S COPY

FIGURE 2. Statistics Department UNC, 1958 front row: Nicholson, Roy, Hotelling, Bose, Hoeffding, back row: Connor, Ogawa, SSS among others

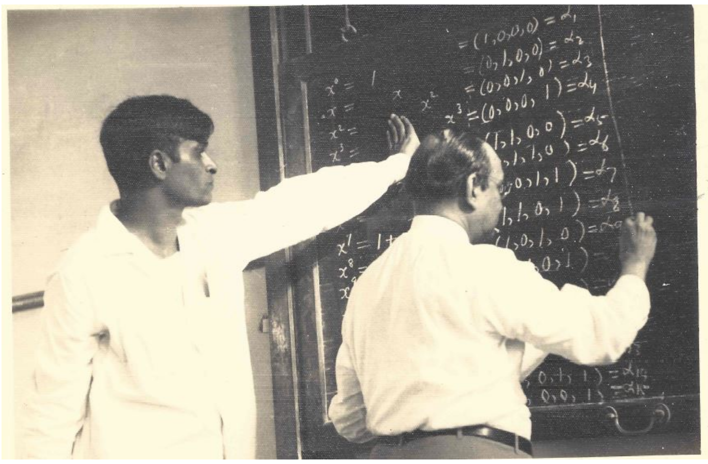


FIGURE 3. S. S. Shrikhande and R. C. Bose, 1958

In 1963, SSS joined Bombay University as Professor and Head of Department of Mathematics. There he established the Center of Advanced Study in Mathematics and served as its Director. There three students - Bhagwandas, Vasanti Bhat-Nayak, and Navin Singhi completed their Ph.Ds under him. Later Singhi was his collaborator on many significant problems in design theory. SSS officially retired from Bombay University in 1978.

MEMBER'S COPY

FIGURE 4. Mehta Institute, with Seidel, Hughes

In 1983, he came out of retirement to become Director of Mehta Research Institute (later renamed Harish Chandra Institute of Research) at Allahabad. This was a newly formed research centre in Mathematics and Mathematical Physics. Several internationally prominent mathematicians such as J. J. Seidel and D. R. Hughes visited the Institute. Seidel and SSS had many common research interests and had a profound influence on each other's work. In 1986, he returned to Nagpur.

In 1985 The Charles Babbage Research Centre in Canada published in two volumes, Selected Papers of S. S. Shrikhande [15]. This was modeled after their previously publication, Selected Papers of Sir R. A. Fisher. Its aim was to compile selected papers of contemporary prominent researchers, and to include in hindsight, a personal commentary of how they viewed their problems, and how they fit in the larger body of research. We highly recommend [15] to see how SSS thought about his own work.

After the death of his wife Shakuntala in 1988, he came to USA and stayed with his sons Mohan and Anil and grand daughter Nita.

He returned to India in 2009 and stayed with his son Anil in New Delhi. Many of his former colleagues and friends visited him there. Towards the end of his son's stay in Delhi, SSS suffered a fall and underwent major orthopedic surgery. After that SSS shifted to Chinmaya Ashram in Vijayawada. There he was looked after by his attendant Bhupinder. He spent his time listening to music and enjoyed solving Sudoku and other puzzles. His 100th birthday held on October 19, 2017, was attended by many of his former

MEMBER'S COPY

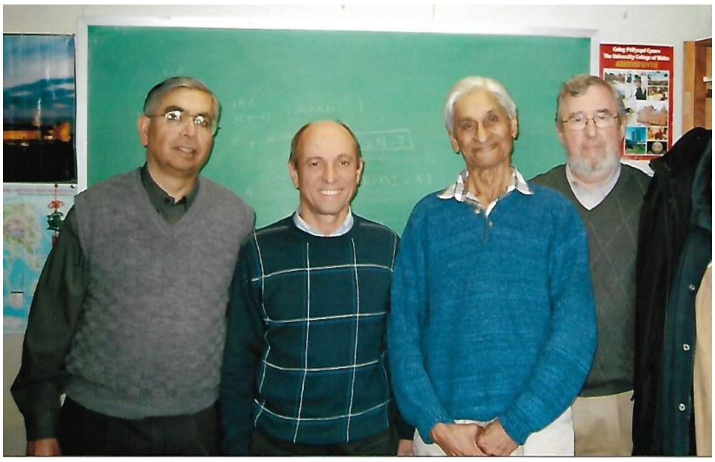


FIGURE 5. With MSS, Tonchev, Ionin in Mt. Pleasant, 2002

colleagues, relatives, and friends. SSS passed away peacefully in his sleep on April 21, 2020.

2. Some selected contributions of S. S. Shrikhande

We begin with Euler's Conjecture. The concept of Latin squares and orthogonal Latin squares goes back to Euler in late 1700s, when he served as court mathematician to the Czar of Russia. Euler was given the task of finding a suitable arrangement for a military parade of 36 officers from 6 ranks and 6 regiments in a 6 x 6 square, so that in each row and each column there was exactly one officer from each rank and each regiment. Formally, we make the following:

Definition 2.1. An n by n Latin square A is an array based on $\{1, 2, ..., n\}$ such that the entries in every row and column are distinct. Two Latin squares $A_1 = [a_{ij}^{(1)}]$ and $A_2 = [a_{ij}^{(2)}]$ of order $n \geq 3$ are orthogonal provided that each of the n^2 pairs $(a_{ij}^{(1)}, a_{ij}^{(2)}), (i, j = 1, 2, ..., n)$ are distinct. More generally, let $A_1, A_2, ..., A_t$ be a set of t Latin squares of order $n \geq 3$. They are called mutually orthogonal Latin squares (MOLS), if any pair of them are orthogonal.

In his quest for a solution to the 36 Officers problem, Euler was unable to find two orthogonal Latin squares of order 6 and thought that such a pair did not exist. More generally, he thought it was impossible find a solution which required two Latin squares of order $4t+2$ for any $t \geq 1$. This became

MEMBER'S COPY

known as Euler's Conjecture on Latin squares ([5]). In 1901, Tarry, a French mathematician verified that Euler was correct for the case $t = 1$ by covering all possible Latin squares of order 6.

The highlight of SSSs work with Bose was in proving the falsity of Euler's Conjecture on the non-existence of two mutually orthogonal Latin squares (MOLS)of order 4t+2. This was done by constructing two MOLS of order 22 ([2]). They introduced the concept of pairwise balanced designs, and used this idea in construction of MOLS ([3]). Finally in collaboration with E. T. Parker, they completely demolished Euler's Conjecture, by constructing two MOLS of order 4t+2, for all $t > 6$ ([5]). Bose, Shrikhande, and Parker became known as Euler's Spoilers, and their photo appeared on the front page of the New York Times on April 26, 1959.

The End of the Euler Conjecture:
Dr. E. T. Parker, Prof. S. S. Shrikhande and Prof. R. C. Bose.
(Reprinted from *The New York Times*, April 26, 1959, with permission from New York Times Pictures.)

S. S. Shrikhande
10 - 22 - 03

FIGURE 6. End of Euler Conjecture

An excellent commentary on this is provided by SSS himself ([18],[19]). We recommend the expository article by N. Rao [9]. In addition, refer to the very recent papers by Sane ([10],[11]) and Singhi and Vijayan ([14]).

MEMBER'S COPY

SSS's body of work in mathematics, aside from that on Euler's Conjecture deals with broad range of problems. These include non-existence problems(on designs), construction and structural problems, characterization of certain graphs and embedding problem of graphs and designs. Refer to ([15]) for a detailed exposition.

Next, after a reviewing a few very basic design theory concepts, we discuss some parts of SSS's Ph.D. thesis. His thesis is now regarded as a landmark by researchers in the field.

A design $\mathbf{D} = (\mathbf{V}, \mathcal{B})$ consists of a finite set V (called points or treatments), a set of subsets of V (called blocks), which satisfy some regularity conditions. A balanced incomplete block design (BIBD or 2-design) with parameters (v, b, r, k, λ) is a design on v points, b blocks each of size k, where each point occurs in r blocks and any two distinct points occur together in λ blocks. The parameters satisfy the relations $bk = vr, \lambda(v-1) = r(k-1)$ and Fisher's inequality $b \geq v$. The integer $n = r - \lambda$ is called the order of the design.

A 2-(v, k, λ) design with $v = b$ is called a symmetric 2-(v, k, λ). Using the usual zero-one incidence matrix N of a 2-(v, k, λ) design, it is easily shown that any two distinct blocks of a symmetric (v, k, λ)-design intersect in exactly λ points, i.e. it has exactly one block intersection number λ.

The well known Fano plane is the smallest symmetric design with parameters $v = 7, k = 3, \lambda = 1$.

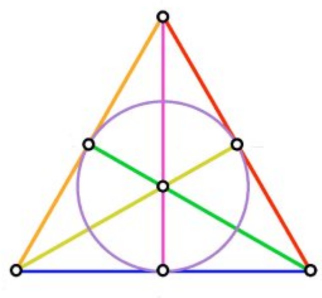


FIGURE 7. Fano Plane

Some facts about finite projective planes (FPP) are: A FPP of order n exists if and only iff a symmetric design with $v = n^2+n+1, k = n+1, \lambda = 1$ exists; A FPP exists for n a prime power; and the celebrated Bruck-Ryser

MEMBER'S COPY

Theorem(1949) which states that if there exists a FPP of order n, where $n \equiv 1, 2 \mod 4$, then n is a sum of two squares. As a consequence a FPP of order 6, or equivalently a symmetric - $(43, 7, 1)$ design does not exist.

Using the incidence matrix, SSS proved:

Theorem 2.2. *If a symmetric 2-(v, k, λ), exists for v even, then $k - \lambda$ is a square.*

As a corollary, the symmetric $(22, 7, 2)$ design does not exist.

This was shown earlier by Q. M. Hussain (1946), by a lengthy argument. Also, SSS ruled out a symmetric $(46, 10, 2)$-design, which was an open case in Fisher-Yates table (1949).

To resolve the situation for symmetric designs with odd v, SSS needed the use of Hasse-Minkowski invariants. We briefly give some background on this. For a more details, see [7]. Let A be a non-singular and symmetric matrix of order n. Let D_r be the leading principal minor determinant of order r and suppose $D_r \neq 0$, for $r = 1, 2, ..., n$. Define the Hasse-Minkowski invariant

$C_P(A) = (-1, -D_n)_p \prod_{j=1}^{n}(D_j, -D_{j+1})_p$, for every prime p, where $(m, m')_p$ is the Hilbert norm symbol.

The following non-existence result was proved by SSS:

Theorem 2.3. *Let a symmetric 2-(v, k, λ) design with odd v exist and let N be the incidence matrix of the design. Then,*

$$C_p(NN^t) = (k - \lambda, -1)_p^{v(v-1)/2}(k - \lambda, v)_p^{2v-2} = +1,$$

for all odd prime p.

As an application of his method he obtained,

Corollary 2.4. *A symmetric design with $v = 29, k = 8, \lambda = 2$ does not exist.*

Proof. $C_p(NN^t)$
$= (k - \lambda, -1)_p^{v(v-1)/2}(k - \lambda, v)_p^{2v-2}$
$= (6, -1)_p^{29\times 14}(6, 29)_p^{55}$
$= (3, 29)_p(2, 29)_p = (29/3)_p$, the Legendre symbol for $p = 3$
$=(2/3)_3 = -1$. Thus, a symmetric $(v = 29, k = 8, \lambda = 2)$ design does not exist. □

MEMBER'S COPY

This was proved earlier by Q. M. Hussain (1945). As further applications, SSS ruled out open cases $(137, 17, 2), (67, 12, 2), (43, 15, 5)$ in Fisher-Yates table (1949).

SSS's paper [15] was the first to systematically develop a method, using the incidence matrix of a symmetric design to investigate the non-existence of the design. It should be noted that some results of SSS were independently proved by (1949) and by Chowla and Ryser (1950). The Hasse-Minkowsi method was used for other types of designs, in S. S. Shrikhande, D. Raghavarao, and S. K. Tharthare (1963), and B. Bekker, Y. J. Ionin, and M. S. Shrikhande (1998).

We next discuss some work of SSS on strongly regular graphs (SRGs) and quasi-symmetric designs (q. s. designs). These were areas in which he provided some of the basic ideas and which have guided and influenced some of my own research. The concept of a SRG is implicit in early papers of Bose and Nair (1939) and Bose and Shimamoto (1952). Bose formally defined a SRG in a famous paper, titled Strongly regular graphs, partial geometries, and partially balanced designs (1963).

A SRG with parameters (n, a, c, d) is a graph on n vertices which is regular with valency a and has the following properties: any two adjacent vertices have exactly c common neighbors, and any two non-adjacent vertices have exactly d common neighbors.

The Lattice graphs $L_2(n)$ and Triangular graphs $T(n)$ are two important infinite families of SRGs. The vertices of $L_2(n)$ are the n^2 ordered pairs from $\{1, 2, \ldots, n\}$, with two vertices adjacent iff they agree in one coordinate. $L_2(n)$ is a SRG $(n^2, 2n-2, n-2, 2)$. The vertex set of $T(n), n \geq 4$ is the set of the 2-subsets from $\{1, 2, \ldots, n\}$, two vertices being adjacent iff they have an element in common. It is a SRG $(\binom{n}{2}, 2(n-2), n-2, 4)$.

An important research theme in the late fifties was to characterize certain graphs by their parameters. W. S. Connor, A. J. Hoffman, L. C. Chang, and SSS made independent contributions. The parameters of $T(n)$ uniquely characterize the graph for $n \neq 8$ and this result was proved by Connor (1958), SSS (1959, [16]), Hoffman (1960), and L. C. Chang (1960). In the case, $n = 8$, there are exactly three non-isomorphic graphs, besides $T(8)$ known as the Chang graphs.

For $L_2(n)$, SSS proved in [17] the following theorem,

MEMBER'S COPY

Theorem 2.5. *The $L_2(n)$ graph is uniquely characterized by its parameters for all $n \neq 4$. For $n = 4$, besides $L_2(4)$, there is a unique non-isomorphic SRG with parameters* $(16, 6, 2, 2)$.

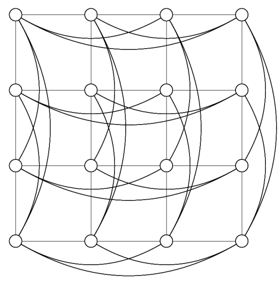


FIGURE 8. The Lattice graphs $L_2(4)$

Remark 2.6. The exceptional $L_2(4)$ graph is known as the Shrikhande graph. It appears in unexpected places, such as in representation of E_8 in Lie algebras. See the paper of Sane [12] on the Shrikhande graph.

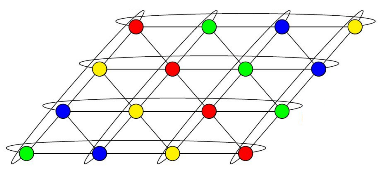


FIGURE 9. The Shrikhande graph on a torus

Another concept of SSS [20] is that of a quasi-symmetric (q. s.) design, though this was implicit in SSS's work on the dual of partially designs (1952). A q. s. design is a 2-(v, k, λ) with two block intersection numbers $x < y$. A 2-$(v, k, 1)$-design is q. s. with $x = 0, y = 1$. The parameters of a q. s. design with intersection numbers $x, y, x < y$ are $(v, b, r, k, \lambda; x, y)$.

Form the block graph Γ of a q. s. design by taking as vertices the blocks of D. Define two vertices adjacent iff the blocks intersect in y points. Further, assume Γ is connected. An important and useful result from [20] is:

Theorem 2.7. *The block graph Γ of a q. s. design is a SRG (n, a, c, d) graph whose parameters can be found from the eigenvalues of Γ.*

MEMBER'S COPY

Goethals and Seidel's paper [6] is another important paper on q. s. designs. The papers [20] and [6] were my first introduction to q. s. designs. There is by now a wide body of research on these designs. We refer to [13] which is the first monograph on q. s. designs.

We close with some remarks on the λ design conjecture, where SSS and Singhi had made significant contributions [21] and [22].

A λ-design $\mathbf{D} = (\mathbf{V}, \mathcal{B})$ on v points is a collection of v proper subsets (called blocks) of a point set with v points, such that every two blocks intersect in λ points, $\lambda < v$ and there are at least two block sizes. Recall that in a symmetric design all blocks have the same size and every two blocks intersect each other in λ points. The only known construction of a λ-design is via block complementation of a symmetric design. Such a λ-design is said to be of Type-1. The λ-design conjecture states that every λ-design is of Type-1. This conjecture has been open since the last fifty years. This has been proved true for many small values of λ. In 1976, Singhi and SSS [21] proved it true for $\lambda = p$, where p is a prime. More than a decade later, Seress proved it for $\lambda = 2p$. This problem is still unresolved and has currently been the focus of papers of Parulekar and Sane (2019, 2020); and Yadav, Pawale, and M. S. Shrikhande (2020). Refer to chapter 14, [7] for more details about the λ-designs.

References

[1] R. C. Bose, S. S. Shrikhande and E. T. Parker, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, Canad. J. Math., 12, 189-203 (1959).

[2] R. C. Bose and S. S. Shrikhande, On the falsity of Euler's Conjecture about the non-existence of two orthogonal Latin squares of order $4t + 2$, Pro. Nat. Acad. Sci. USA, 45, 734-737 (1959).

[3] R. C. Bose and S. S.Shrikhande, On the construction of mutually orthogonal Latin squares and the falsity of a conjecture of Euler, Trans. Amer. Math. Soc., 95, 191-209 (1960).

[4] R. C. Bose, S. S. Shrikhande and E. T. Parker, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's Conjecture. Canad. J. Math.,12,189-203 (1960).

[5] L. Euler, Recherches sur une nouvelle espace des quarres magiques, Verh. zeeuwsch Genoot. Weten. Vliss,9, 85-239 (1782).

[6] J. M. Goethals and J. J. Seidel, Strongly regular graphs derived from combinatorial designs, Canad. J. Math.,22,597-614 (1970).

MEMBER'S COPY

[7] Y. J. Ionin and M. S. Shrikhande, Combinatorics of Symmetric Designs, Cambridge University Press (2001).
[8] E. T. Parker, Orthogonal Latin Squares, Proc. Nat. Acad. Sci. USA, 45, 859-862 (1959).
[9] N. Rao, Shrikhande,"Euler's Spoiler" turns 100, Bh$\overline{a}$vn$\overline{a}$, vol.1, issue 4,Oct. 2017. (also at www.asiapacific-mathnews.com)
[10] S. S. Sane, Reminiscences of a heritage: Professor Sharadchandra Shankar Shrikhande, The Mathematical Consortium Bulletin (TMCB), Edited by Ravi S. Kulkarni, July, 2020, pages 25-29.
[11] S. S. Sane, Doing Combinatorics with passion: Life of Professor Sharadchandra Shankar Shrikhande, to appear in the Indian Journal of Discrete Mathematics (2020).
[12] S. S. Sane, The Shrikhande Graph, Resonance-Journal of Science Education, 20, issue 10, 903-918, 2015.
[13] M. S. Shrikhande and S. S. Sane, Quasi-Symmetric designs, Cambridge University Press, Cambridge (1998).
[14] N. M. Singhi and K. S. Vijayan, Sharad S. Shrikhande, (1917-2020), Current Science, 118,No.11, 10 June 2020
[15] S. S. Shrikhande,The impossibility of certain symmetrical balanced incomplete block designs, Annals of Math. Stat., 21, 106-111.
[16] S. S. Shrikhande, On a characterization of the triangular association scheme, Ann. Math. Stat., 30,39-47 (1959).
[17] S. S. Shrikhande, On the uniqueness of the L_2 association scheme, Ann. Math. Stat., 30, 781-798 (1959).
[18] S. S. Shrikhande, Selected Papers of S. S. Shrikhande, Vol. I and II, Edited by D. S. Meek and R. G. Stanton, The Charles Babbage Research Centre, Winnipeg, Canada.
[19] S. S. Shrikhande, R. C. Bose memorial conference, Fort Collins, Colorado, J. Statis. Plann. and Inference, 73, 273-276 (1998).
[20] S. S. Shrikhande and Bhagwandas, Duals of incomplete block designs, J. Indian Statist. Assoc. 3(1965).
[21] N. M. Singhi and S. S. Shrikhande, On the the λ-design conjecture, Util. Math., 9,301-318 (1976).
[22] S. S. Shrikhande and N. M. Singhi, Some combinatorial problems, in: Combinatorics and its Applications, K. S. Vijayan and N. M. Singhi (eds.), Calcutta: Indian Statistical Institute, 340-49 (1984).

Mohan S. Shrikhande
Mathematics Department
Central Michigan University
Mount Pleasant, MI, 48859, USA.
E-mail: Mohan.Shrikhande@cmich.edu

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 13–22

SHRIKHANDE AND DESIGNS

BHASKAR BAGCHI

Combinatorics is the branch of Mathematics dealing with its discrete aspects. Professor Sharadchandra Shrikhande (1917-2020) was a towering presence (both figuratively as well as literally!) among the leading combinatorialists of the twentieth century. An elegant and powerful mathematician and the gentlest and kindest of men, his chosen subfield of expertise was something known as **Design Theory**, aka 'Finite geometry'. What are these designs?

To appreciate Shrikhande's contributions, we need to delve into the background of design theory first. Although Euclid grappled (unsuccessfully) with the problem of defining points and lines, the modern point of view is to take a set theoretic approach. Thus we take it as an axiom :

(A1) There is a universe : a set of objects called points, and a special collection of subsets of this set whose members are called lines.

Thus, a line is to be thought of as a set whose elements are the points lying on it, whatever these 'points' may be. A second fundamental axiom, introduced by Euclid himself, is :

(A2) Any two distinct points are together in a unique line.

It follows that any two distinct lines have at most one point in common. Two lines are called **parallel** if they have no common point, i.e., if they are disjoint as sets. A third crucial axiom - Euclid's parallel postulate- involves this notion. Euclid's original formulation of this axiom is rather clumsy and not very intuitive (involving, as it does, angle measurement). The following equivalent formulation of this axiom is due to Playfair :

(A3)(Playfair's axiom) Given any line l and any point x not lying on l, there is a unique line passing through x which is parallel to l.

Notice that (A3) says : the set of lines is naturally broken up (partitioned) into **parallel classes** such that the lines in each parallel class

Key words and phrases: Designs, Latin squares, Linear spaces

© *Indian Mathematical Society, 2020.*

MEMBER'S COPY

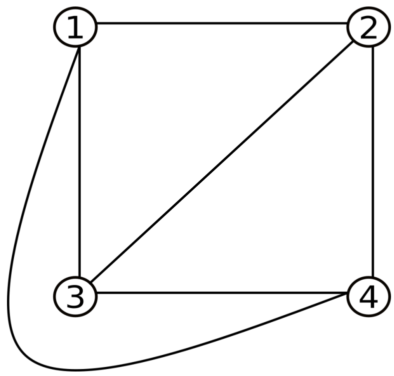


FIGURE 1. The Euclidean plane of order two.

partition the point set, and any two lines from different parallel classes intersect (at a unique point - by (A2)).

Of course, there are many more axioms in euclidean geometry. David Hilbert made a definitive study of these axioms, and he pointed out that Euclid had missed some of them (but used them implicitly all the same). But let us forget the other axioms and define an (abstract) **eucidean plane** to be any non-trivial 'model' satisfying (A1), (A2) and (A3). Of course, the classical euclidean plane of our intuition is one such. But there are many more, as we shall see.

For us, the key question is : what are the finite euclidean planes (i.e., those having finite sets of points as their universes)? Figure 1 tries to depict the smallest of them.

In this model there are four points, denoted 1,2,3,4, and six lines, namely $\{1,2\},\{3,4\}|\ \{1,3\},\{2,4\}|\ \{1,4\},\{2,3\}$. Each line in this model consists of two points and each point is in three lines. There are three parallel classes of lines (each containing two lines), separated by vertical bars in the list of lines. This model of the euclidean plane is known as the Fano plane. Note that, in our picture of this plane given in Figure 1, we have been forced to use a curved line to depict one of its lines.

Using elementary counting arguments, it is not hard to prove that we have :

Theorem 1. If π is a finite euclidean plane then there is a number $n \geq 2$ (called the **order** of π) such that (a) π has n^2 points and n^2+n lines, (b) each point of π is in $n+1$ lines, each line of π contains n points, and (c) π has $n+1$ parallel classes of lines, each containing n lines.

Thus the toy model shown above is an euclidean plane of order two. Figure 2 displays the lines of an euclidean plane of order three. Already,

MEMBER'S COPY

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

FIGURE 2. The Euclidean plane of order three.

this is too large for a picture as in Figure 1 to be very revealing. Instead, we represent the nine points of this plane by the nine positions in a 3×3 array. Each of the four squares in Figure 2 represents a parallel class in this plane. In each square the three positions marked i constitute the ith line in this parallel class ($i = 1, 2, 3$).

Descarte's introduction of co-ordinates in geometry is usually thought of as a great computational tool and nothing more. But it is also a great step in rigourising euclidean geometry by presenting a model of the classical euclidean plane built out of the real numbers alone. In the cartesean model, the universe of points is the set $R \times R$ of all ordered pairs of real numbers. The lines are the graphs of the linear functions. That is, the lines are the sets $\{(x, y) : y = mx + c\}$ and $\{(x, y) : x = c\}$, where m and c vary over all real numbers. This model "works" (satisfies the axioms A1, A2, A3) because R is a field : it has two binary operations (addition and multiplication) satisfying commutativity, associativity and distributivity, with an additive identity 0, a multiplicative identity 1, and each real number has an additive inverse (negative), each non-zero real number has a multiplicative inverse (reciprocal).

Using any field F (such as the field of rational numbers) in place of R in the cartesean model, we obtain a non-classical euclidean plane. If F is a finite field, say of order (size, cardinality) q, we get a finite euclidean plane of order q. It is well known since Evariste Galois that the order of any finite field is a prime power, and, conversely, for each prime power q, there is an essentially unique field of order q. So we have an infinite series of euclidean planes of prime power orders. When the prime power q is > 8 and not a prime, this construction can be perturbed to obtain other euclidean planes of order q. But, to date, no finite euclidean plane of non-prime-power order has been constructed. Indeed, one of the best known open problems in Combinatorics is to settle :

The prime-power conjecture. The orders of all finite euclidean planes are prime powers.

The only general result in this direction is the following non-existence theorem. (There is also a computer assisted proof that there is no euclidean

MEMBER'S COPY

plane of order ten.)

Theorem 2 (Bruck and Ryser, 1949). If $n \equiv 1$ or $2 \pmod 4$ and n is not a sum of (at most) two perfect squares, then there is no euclidean plane of order n.

For instance, $6 \equiv 2 \pmod 4$, and 6 can not be written as a sum of two squares. So there is no euclidean plane of order six. This theorem settles the problem for infinitely many orders, but infinitely many other orders remain undecided.

Many artists and mathematicians of Renaissance Europe were uncomfortable with the idea of there being exceptional pairs of lines in the plane which never meet. They pointed out that, in real life, when our vision is un-obstructed, so called parallel lines appear to meet at the horizon – think of a pair of railway tracks in an open field. So it was observed that an euclidean plane may be "completed" by a process of "projectivization". Namely, for each parallel class of lines, introduce a new point (a point at infinity or, an ideal point) through which all the lines in this class pass; also introduce a new line (the line at infinity or the horizon) on which all these points at infinity lie. What results is a new kind of geometry, called a **projective plane**. Formally, a projective plane is a non-trivial model satisfying the axioms (A1), (A2) and the following new axiom in place of Playfair's :
(A4) Any two distinct lines have a (unique) point in common.

Conversely, given any projective plane, one can use the inverse process of "affinization" to create an euclidean plane. Namely, select any line and remove it together with all the points lying on it. In the resulting euclidean plane, the removed line (respectively removed points) play the role of the line at infinity (respectively points at infinity). Coupling this construction with Theorem 1 (or by a counting argument similar to the one used in proving this theorem) one can prove :

Theorem 3. Let Π be a finite projective plane. Then there is a number $n \geq 2$ (called the **order** of Π) such that (a) Π has $n^2 + n + 1$ points and $n^2 + n + 1$ lines, and (b) each point of Π is in $n + 1$ lines; each line of Π contains $n + 1$ points.

Notice the simplicity and elegance of Theorem 3, which was missing in Theorem 1. In view of the projectivization-affinization process, for any number n, there is a projective plane of order n if and only if there is an euclidean plane of order n. Thus both the Bruck-Ryser theorem and the prime power conjecture may be (and usually is) stated in terms of projective rather than euclidean planes.

MEMBER'S COPY

The class of finite geometries we consider can be considerably expanded by introducing the notion of 2 -designs, as was done by Frank Yates, a famous statistician. An **incidence system** is a model satisfying (A1). In this generality one often uses the word "block" instead of "line". Let $v > k > \lambda > 0$ be integers. Then a **2-design** with parameters v, k, λ (in short, a $2-(v,k,\lambda)$ design) is an incidence system D satisfying (1) D has v points, (2) each line of D contains k points, and (3) any two distinct points of D are together in λ lines. Thus, Theorems 1 and 3 say that the finite euclidean (respectively projective) planes are just the $2-(n^2,n,1)$ designs (respectively $2-(n^2+n+1,n+1,1)$ designs), $n \geq 2$. Again, easy counting arguments show that we have :

Theorem 4. Let D be a $2-(v,k,\lambda)$ design. Then there are positive integers b and r such that D has b lines, and r is the number of lines through each point of D. These numbers are given by the formulae $r(k-1) = \lambda(v-1)$ and $bk = rv$.

Note that this theorem implies that, for the existence of 2-designs, the following conditions on their parameters are necessary : $k-1$ divides $\lambda(v-1)$ and $k(k-1)$ divides $\lambda v(v-1)$. But these divisibility conditions are far from sufficient for the existence of 2-designs with given parameters. The general existence question is, of course, wide open. The first progress in this direction was made by R.A. Fisher, the founding father of Statistics and Statistical Genetics. He used very ingenious counting arguments to prove :

Theorem 5 (Fisher, 1940). The parameters of any $2-(v,k,\lambda)$ design satisfy $b \geq v$. Equality holds here if and only if any two distinct lines of the design have λ points in common.

Later, R.C. Bose gave a much simpler proof of this theorem using linear algebra. A 2-design is said to be a **square 2-design** if it attains equality in Fisher's inequality. It is immediate from Theorem 4 that a 2-design is square if and only if its parameters satisfy $k(k-1) = \lambda(v-1)$. Thus the square 2-designs with $\lambda = 1$ are precisely the finite projective planes. We have seen that there are infinitely many of them. The following is an unpublished conjecture of Marshall Hall, jr.

Hall's Conjecture. For each fixed value of $\lambda > 1$, there are only finitely many square 2-designs.

While nobody could find a construction to disprove this conjecture, nobody has succeeded in proving it either. It remains open for each fixed value of λ, for instance, for $\lambda = 2$.

MEMBER'S COPY

Given this state of affairs, the following generalization of the Bruck-Ryser theorem due to Shrikhande is a major advance in the field. It was independently proved in the same year by Chowla and Ryser.

Theorem 6 (Shrikhande, Chowla-Ryser, 1950). For the existence of a square $2-(v,k,\lambda)$ design, the following conditions are necessary : (a) if v is even then $k-\lambda$ must be a perfect square, and (b) if v is odd then the equation $(k-\lambda)x^2+(-1)^{(v-1)/2}\lambda y^2 = z^2$ must have a solution in three integers x,y,z, not all three equal to zero.

Part (a) of this theorem, which is much simpler than part (b), was previously proved (in 1949) by M.P. Schutzenberger. The proof of part (b) required fairly sophisticated tools from Number Theory, such as p-adic Hilbert symbols and Hasse-Minkowski local-global theorem. This was the first time that such non-elementary number theory was used as a proof technique in Design Theory. This is all the more creditable on the part of Shrikhande because, unlike Sarvadaman Chowla, he was no number theorist. Shrikhande's work was a part of his Ph.D. thesis written under the supervision of Professor Raj Chandra Bose. R.C. Bose was one of the founders of the mathematical theory of designs, and was instrumental - among other achievements- in introducing finite fields for the construction of designs.

The Hasse Minskowski theory helps in proving parametric restrictions on square 2-designs partly because they are square incidence systems : have equally many points and lines. Bose and Connor used this technique to prove similar non-existence results for another class of square incidence systems, so called (square) regular divisible designs. Later, in another landmark paper of 1953, Shrikhande showed how to adopt this tool to prove parametric restrictions on a certain class of non-square 2-designs, known as affine resolvable designs. The full power of this approach remains to be exploited.

Design theory is not only about proving non-existence results. An important aspect of the theory is the construction of interesting designs. This is both a science and an art. Shrikhande was a consummate exponent of this art form. He had numerous elegant and important constructions to his credit. My personal favorite is his 1962 paper published in the journal Sankhya, in which he introduces a special class of 2-designs and shows how to "multiply" any two designs from this class to produce a third design from the same class. He also presents a construction of square 2-designs in this class using twin primes (i.e., pairs of prime numbers differing by two).

But Shrikhande's best known work is on **latin squares**. A latin square of order n is an $n\times n$ square array whose positions are filled in with n symbols in such a way that each symbol occurs (once) in each row and in each column of the array. For example, any correct solution of a Sudoku

MEMBER'S COPY

1/1	2/2	3/3
2/3	3/1	1/2
3/2	1/3	2/1

FIGURE 3. Orthogonal latin squares.

puzzle is a latin square of order 9. The third and fourth tables in Figure 2 (but not the first two squares there) are latin squares of order 3. If we superimpose these two squares, we get the square displayed in Figure 3.

Notice that, in the superimposed square of Figure 3, each of the nine ordered pairs $1/1, 1/2, \cdots, 3/2, 3/3$ occur once each. We say that the latin squares of Figure 2 are orthogonal. More generally, two latin squares of order n are said to be **orthogonal**, if, on superimposing them, one sees the n^2 ordered pairs of symbols once each. A **MOL** (mutually orthogonal set of latin squares) of order n is a set of $n \times n$ latin squares any two of which are orthogonal. It is not hard to see that any MOL of order n can contain at most $n-1$ squares. A MOL of order n and size $n-1$ is said to be a **complete MOL**.

The appearance of a pair of orthogonal latin squares in Figure 2, which was constructed to display a finite euclidean plane, is no accident. Figure 2 contains a clear hint as to how, given any complete MOL of order n, one can augment this set of squares by two trivial $n \times n$ squares (a "row square" and a "column square", like the first two squares in Figure 2) to construct an euclidean plane of order n. This process may easily be reversed to construct a complete MOL of order n starting with a given euclidean plane of order n. Thus complete MOLs of of order n and euclidean (or projective) planes of order n are co-extensive, almost like two names for the same combinatorial object.

The notion of latin squares and their orthogonality was introduced by Leonhard Euler in a paper published in 1782. In view of the close link between MOLs and finite euclidean planes, it is natural to consider the maximum size $N(n)$ of MOLs of order n. Thus, in general, $N(n) \leq n-1$, and $N(n) = n-1$ if and only if there is an euclidean plane of order n. Also, $N(q) = q-1$ for all prime powers q. Even though Euler knew nothing of design theory (finite planes were introduced by G.K.C. von Staudt only in 1856) his studies led him to conjecture that $N(n) = 1$ for all $n \equiv 2 \pmod 4$. In other words Euler's conjecture was that for every "singly even" number n (i.e., $n = 2, 6, 10, \cdots$), it is impossible to construct even two orthogonal latin squares of order n. This is easy to see for $n = 2$, and has turned out to be true for $n = 6$ as well. Since, besides being the greatest mathematician

MEMBER'S COPY

of all times, Euler was also a great human computer, it is likely that Euler verified by computation that $N(6) = 1$. Perhaps this is what led him to the general conjecture.

In a paper written in 1922, H.F. McNeish gave a nice product construction for latin squares. If A and B are two latin squares, of order m and n respectively, then his construction produces a latin square $A \otimes B$ of order mn. The construction is as follows. To find the (u, v)th entry of $A \otimes B$, where u, v are any two indices $(1 \leq u, v \leq mn)$, note that u and v can be written (uniquely) as $u = (i-1)n+k,\ \ v = (j-1)n+l$ where $1 \leq i, j \leq m$, and $1 \leq k, l \leq n$. If the (i, j)th entry of A is $a_{i,j}$ and the (k, l)th entry of B is $b_{k,l}$ then define the (u, v)th entry of $A \otimes B$ to be $(a_{i,j} - 1)n + b_{k,l}$. (In this construction, we have taken the symbols in A and B to be the numbers between 1 and m, and the numbers between 1 and n, respectively.) McNeish's construction has the useful property that, if A_1, A_2 are orthogonal latin squares of order m and B_1, B_2 are orthogonal latin squares of order n then $A_1 \otimes B_1$ and $A_2 \otimes B_2$ are orthogonal latin squares of order mn. It follows that, for any two numbers m, n, $N(mn) \geq \min(N(m), N(n))$. Iterating this inequality, we see that, for any factorisation $n = n_1 \cdots n_t$ of a number n, we have $N(n) \geq \min(N(n_1), \cdots, N(n_t))$. In particular, if $q_1 < q_2 \cdots < q_t$ are the powers of distinct primes occuring in the unique factorisation of n, we get $N(n) \geq q_1 - 1$. McNeish conjectured that actually equality holds here for all $n : N(n) = q - 1$, where q is the smallest prime power occurring in the unique factorisation of n. Note that $q = 2$ if and only if $n \equiv 2 \pmod 4$. Therefore McNeish's conjecture generalizes Euler's.

The first break through on this problem was achieved by E.T. Parker in 1957, when he disproved McNeish's conjecture by constructing three mutually orthogonal latin squares of order 21. Soon after this, when Shrikhande was visiting Bose in the university of North Carolina, they collaborated on this problem and disproved Euler's conjecture in a joint paper published in 1960. Here they constructed (for example) three mutually orthogonal latin squares of order 14. Finally, the three of them collaborated to disprove Euler's conjecture in a very strong form. They proved :

Theorem 7 (Bose- Parker- Shrikhande, 1960). For all $n \neq 1, 2.6$, there is a pair of orthogonal latin squares of order n.

Their proof rests on a construction of MOLs of order n using as inputs several MOLs of smaller order and a **linear space** with n points. Here a linear space is a model satisfying Axioms (A1) and (A2) (with possibly varying line sizes). The details of this construction are rather involved, and certainly beyond the scope of this expository article. Let me, instead, illustrate the kind of ingenious techniques used in these constructions by explaining Parker's construction of three mutually orthogonal latin squares of order 21.

MEMBER'S COPY

First observe that, given any latin square L, say of order m, we may obtain a second latin square L' by applying one or more of the following operations on L : (i) permute rows, (ii) permute columns, and (iii) substitute any m distinct symbols for the symbols of L. Moreover, if we apply these operations simultaneously to all the latin squares $L_1, \cdots, L_k$ in a MOL of order m and size k, then we obtain a second MOL $L'_1, \cdots, L'_k$ of the same order and size. Here, we must make sure that the same row/column permutation is applied to all the squares in the first MOL, but the symbol substitutions used may be entirely independent of each other. In particular, using a suitable row (or column) permutation, we can ensure that in the last square L'_k obtained, the entries in the main diagonal (i.e., the (i,i))th entry, $1 \leq i \leq m$) are all equal. Since the first $k-1$ squares in the resulting MOL are orthogonal to L'_k, it follows that each of these $k-1$ squares have m distinct entries in the main diagonal. Further, we can apply suitable symbol substitutions to ensure that, in each of the $k-1$ mutually orthogonal latin squares $L'_1, \cdots, L'_{k-1}$ (though not in L'_k), the (i,i)th entry is $= i$ for all i, $(1 \leq i \leq m)$. In this construction, we may, of course, take $k = N(m)$. Thus we have proved:

Lemma. For all m, there are $N(m) - 1$ mutually orthogonal latin squares of order m such that, in each of them, the ith entry in the main diagonal is i (for all i, $1 \leq i \leq m$).

Now take a projective plane Π of order 4. Since each line l of Π has size 5 and $N(5) = 4$, the lemma above gives us, for each line l of Π, three mutually orthogonal latin squares A_1^l, A_2^l, A_3^l such that the rows, columns and symbols of these squares are (without loss of generality) indexed by the five points of l (rather than the customary indices $1, 2, \cdots, 5$) and such that, for each point x in l, the (x,x)th entry in each of these three squares is $= x$. Now we can construct three 21×21 square arrays B_1, B_2, B_3, with rows, columns and symbols indexed by the 21 points of Π, as follows. For $i = 1, 2, 3$, and any two points x, y of Π, the (x,y)th entry of B_i is (a) the point x if $x = y$, and (b) if $x \neq y$, this entry is the (x,y)th entry of A_i^l, where l is the unique line of Π joining x and y. It is a routine matter to verify that B_1, B_2, B_3, thus constructed, are three mutually orthogonal latin squares of order 21.

Any reader who appreciates this beautiful construction will have no trouble convincing herself that this technique proves, more generally, that :

Theorem 8 (Bose-Parker-Shrikhande, 1960). If there is a linear space with v points, b lines and lines of size $k_1, k_2, \cdots, k_b$ then
$N(v) \geq -1 + \min\ (N(k_1), N(k_2), \cdots, N(k_b))$.

MEMBER'S COPY

It was found that, when the input linear space has certain desirable properties, the bound in this theorem can be improved upon slightly. Theorem 8, together with these improvements and the construction of suitable linear spaces, sufficed for Bose Parker and Shrikhande to disprove Euler's conjecture. In the same year, S. Chowla, P. Erdos and E. G. Strauss used these constructions together with feats of Number Theory to prove :

Theorem 9 (Chowla-Erdos-Strauss), 1960). There is a constant $c > 0$ such that $N(n) \geq n^c$ for all sufficiently large numbers n.

Note that, this theorem shows that as n goes to infinity, the number of mutually orthogonal latin squares of order n increases without bound – in dramatic contrast to the expectations of Euler and McNeish. Actually Theorem 9 was proved with the explicit constant $c = 1/91$. It may be conjectured that this theorem holds for all constants $c < 1/2$, or, perhaps, even with $c = 1/2$. But nobody has yet come anywhere close.

G. C. Rota, one of the leading exponents of Combinatorics, had this to say of Design Theory : "Progress in understanding and classification has been slow and proceeded by leaps and bounds, one ray of sunlight followed by years of darkness". To Professor Shrikhande we owe several of these leaps and bounds.

Bhaskar Bagchi
Professor of Mathematics (Retired)
Indian Statistical Institute, Bangalore
Email : bhaskarbagchi53@gmail.com

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 23–34

QUASI-ANALYTICITY AND THE FOURIER TRANSFORM*

SWAGATO K. RAY

ABSTRACT. It is a well known fact of Harmonic Analysis that very rapid decay of the Fourier transform of an integrable function, on Euclidean spaces, imposes real analyticity on the function. Consequently, such functions cannot vanish on nonempty open sets. In the context of real line, classical results of N. Levinson and A. E. Ingham characterized the decay of the Fourier transform which prohibits a nonzero function to vanish on nonempty open subsets. Results of this genre are related to the classical Denjoy-Carleman theorem regarding characterization of quasi-analytic functions on $\mathbb{R}$. In this article we will state some recent results regarding extension of a Denjoy-Carleman type theorem and the results of N. Levinson and A. E. Ingham to $\mathbb{R}^n$, $\mathbb{T}^n$ and Riemannian symmetric spaces of noncompact type which are generalizations of hyperbolic spaces.

1. INTRODUCTION

For $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform by the formula,

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i \langle x,y \rangle} dx.$$

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$ then we can reconstruct f from its Fourier transform by the Fourier inversion

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i \langle x,y \rangle} dy.$$

* This article is based on the text of the 30th Ramaswamy Aiyer Memorial Award Lecture delivered at the 85th Annual conference of the IMS - An International Meet held at IIT Kharagpur, W. B. during November 22-25, 2019.

© *Indian Mathematical Society, 2020.*

MEMBER'S COPY

For $n = 1$, it follows by differentiating inside the integral that

$$|\hat{f}(y)| \leq \frac{C}{(1+|y|)^3} \Rightarrow f' \text{ exists.}$$

More generally, for $m \geq 3$

$$|\hat{f}(y)| \leq \frac{C}{(1+|y|)^m} \Rightarrow f \text{ is } (m-2) \text{ times differentiable.}$$

So, faster decay of $\hat{f}$ increases differentiability of f. It is not hard to see that much faster decay of $\hat{f}$ can impose real analyticity on f. For instance, if for positive real numbers a, C and all large $|y|$

$$|\hat{f}(y)| \leq Ce^{-a|y|},$$

then f extends holomorphically on $\{z \in \mathbb{C} \mid |\text{Im } z| < a\}$. In particular, vanishing of f on any nonempty open subset of $\mathbb{R}$ implies that $f \equiv 0$. Analogous phenomena continue to occur in higher dimension.
Question: Is the above conclusion true if $\hat{f}$ has slower decay like

$$|\hat{f}(y)| \leq Ce^{-\frac{a|y|}{1+\log|y|}}?$$

Unlike the previous case, it is not immediately clear whether this slower decay of $\hat{f}$ imposes analyticity on f. Problems of this kind had been worked on by Paley, Wiener, Levinson and Ingham [20, 21, 18, 22] using the notion of quasi-analyticity in the 1930s.

We start with the notion of quasi-analyticity (see [24]). Given a sequence of positive real numbers $\{M_n\}$ we define a class $C\{M_n\}$ of smooth functions on $\mathbb{R}$ by

$$C\{M_n\} = \{f \in C^\infty(\mathbb{R}) \mid \|f^{(n)}\|_\infty \leq A_f \; B_f^n \; M_n, \; n = 0, 1, \ldots\}.$$

Definition 1.1. A class $C\{M_n\}$ is said to be quasi-analytic if the conditions

$$f \in C\{M_n\}, \quad f^{(n)}(0) = 0, \quad n = 0, 1, 2 \ldots,$$

imply that $f \equiv 0$.

It is known that $C\{n!\}$ is a quasi-analytic class, in fact, it consists of analytic functions [24]. A characterization of quasi-analytic classes were first proved by A. Denjoy in 1921 (under certain restriction) and then by T. Carleman in 1926 (see [24, 10]).

MEMBER'S COPY

Theorem 1.2 (Denjoy-Carleman). *$C\{M_n\}$ is a quasi-analytic class if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{M_n^{1/n}} = \infty. \tag{1.1}$$

Remark 1.3.
(1) All these started with a question of Hadamard: Under what conditions on $\{M_n\}$ can we conclude that vanishing of f with all its derivatives at a point $x_0 \in \mathbb{R}$ implies $f \equiv 0$?
(2) The Denjoy-Carleman theorem fails if we use only even powers of $\frac{d}{dx}$, the example being $f(x) = \sin x$, $x_0 = 0$.
(3) Denjoy proved the result under the conditions that $\{M_n\}$ is logarithmically convex. The general case is due to Carleman.

A result, somewhat closely related to the Denjoy-Carleman theorem is the Müntz-Szaz theorem.

Theorem 1.4. [24] *If $0 < \lambda_1 < \lambda_2 < \dots$ then span$\{x^{\lambda_n} \mid x \in [0,1],\ n = 0,1,2,\dots\}$, is dense in $C[0,1]$ if and only if*

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = \infty.$$

This connection will be important for us.

2. Fourier analytic implications

We now present some results proved in the 1930s which used the notion of quasi-analyticity for functions in $L^1(\mathbb{R})$. Until recently, these results were known only in one dimension.

Theorem 2.1. *Suppose $f \in L^1(\mathbb{R})$ and $\psi : [0,\infty) \to [0,\infty)$ is a locally integrable function and*

$$I = \int_1^{\infty} \frac{\psi(y)}{y^2} dy.$$

Suppose there exists a positive C such that for a.e. $y \geq 1$,

$$|\hat{f}(y)| \leq Ce^{-\psi(y)}. \tag{2.1}$$

(1) *(Ingham, [18]) Suppose $\psi(y) = \theta(y)y$ where θ decreases to zero, as $y \to \infty$. If $I = \infty$ and f vanishes on a nonempty open subset of $\mathbb{R}$ then $f \equiv 0$.*

MEMBER'S COPY

(2) *(Levinson,* [21, 20]*) Suppose* ψ *is an increasing function on* $[0, \infty)$ *and* $I = \infty$*. If* f *vanishes on a nonempty open subset of* $\mathbb{R}$ *then* $f \equiv 0$*.*
(3) *(Paley-wiener,* [22]*) If for some* $x_0 \in \mathbb{R}$*, supp* $f \subset (-\infty, x_0)$ *and* $I = \infty$ *then* $f \equiv 0$*.*

We come back to the question we started with: $f \in L^1(\mathbb{R})$ and for all lage $|y|$

$$|\hat{f}(y)| \leq Ce^{\frac{|y|}{1+\log|y|}}.$$

We can apply Ingham's theorem here by choosing $\theta(y) = 1/(1 + \log y)$ and observe that

$$\int_1^\infty \frac{1}{y(1+\log y)} dy = \infty.$$

Hence, vanishing of f on any nonempty open set implies that f is the zero function. We can also apply Levinson's result to draw the same conclusion by choosing

$$\psi(y) = \frac{y}{(1+\log y)}, \quad y \geq 1.$$

Remark 2.2. (1) These results are essentially sharp in the following sense. If $I < \infty$ then in all the cases there exists nonzero $f \in C_c(\mathbb{R})$ supported in $(-\alpha, \alpha)$ for any given $\alpha > 0$ satisfying (2.1).
(2) Ingham proved his result only for C_c functions. The current version is due to Mithun Bhowmik, who showed that Ingham's original proof is powerful enough to produce the current version.
(3) Though the theorems look similar but they are independent of each other.
(4) It is not immediately clear from the expression $\psi(y) = \theta(y)y$ that Ingham's condition implies decay of $\hat{f}$ at infinity. The same applies to the result of Paley-Wiener.
(5) Using divergence of I and the decay condition (2.1), Ingham showed that $f^{(n)}$ satisfy conditions of the Denjoy-Carleman theorem under certain restriction on the function θ. The general case was proved using the sharpness of the result.
(6) Paley-Wiener used ideas from complex function theory to prove their result and then used it to give an alternative proof of the Denjoy-Carleman theorem.
(7) Levinson reduced the problem to Paley-Wiener theorem. However, this method of reduction seems very special to one dimension. An alternative proof of this result can be found in [19]. Levinson also

MEMBER'S COPY

showed that his result is also valid for integrable functions on the unit circle.

(8) A. Beurling proved that Levinson's result remains true under the weaker hypothesis that f vanishes on a set of positive Lebesgue measure (see [19]). As of now, this result is known only in one dimension.

An obvious question now is to ask the following: *Are these results true for* $\mathbb{R}^n$, $\mathbb{T}^n$, $n \geq 2$ *and noncommutative Lie groups where the relevant notions make sense*? In the recent times we have obtained several results in commutative as well as noncommutative settings which seem to imply that these results are true in fairly general situation [3, 4, 6, 5, 1, 2, 7]. Due to lack of space, in this article we will focus only on the result of Ingham and show that it is true in most cases we are interested in, namely, $\mathbb{R}^n$, $\mathbb{T}^n$ and Riemannian symmetric spaces of noncompact type.

3. Euclidean spaces and the Torus

To extended the result of Ingham it is natural that we should start with the problem of generalization of the Denjoy-Carleman theorem to $\mathbb{R}^n$ and noncommutative spaces. Bochner and Taylor [9] proved several such generalizations of Denjoy-Carleman theorem for $\mathbb{R}^n$. We could use one of these results to prove the following extensions of Ingham's result.

Theorem 3.1. [3] *Let* $\theta : [0, \infty) \to [0, \infty)$ *be decreasing with* $\lim_{r\to\infty} \theta(r) = 0$ *and*

$$I = \int_{\{x\in\mathbb{R}^n \mid \|x\|\geq 1\}} \frac{\theta(\|x\|)}{\|x\|^n} dx.$$

Suppose $f \in L^1(\mathbb{R}^n)$ *is such that* $\hat{f}$ *satisfies the condition*

$$\int_{\mathbb{R}^n} |\hat{f}(x)| e^{\theta(\|x\|)\|x\|} dx < \infty. \tag{3.1}$$

If f *vanishes on a nonempty open subset of* $\mathbb{R}^n$ *and* $I = \infty$ *then* $f \equiv 0$. *If* $I < \infty$, *then given any positive real number* r *there exists a nontrivial continuous function* f *with supp* $f \subset B(0, r)$ *satisfying (3.1).*

Remark 3.2. Note that the assumed 'decay' of $\hat{f}$ is uniform in all direction, a weakness compared to the result on $\mathbb{R}$.

An interesting application of the Poisson summation formula (an idea already available in the work of Levinson) then produces the following extension of Ingham's result on the Torus $\mathbb{T}^n$.

MEMBER'S COPY

Theorem 3.3. [3] *Let θ be as in the previous theorem and*

$$S = \sum_{m \in \mathbb{N}} \frac{\theta(m)}{m}.$$

Suppose $f \in L^1(\mathbb{T}^n)$, $n \geq 1$ is such that $\hat{f}$ satisfies the condition

$$|\hat{f}(m)| \leq Ce^{-\theta(\|m\|)\|m\|}, \quad m \in \mathbb{Z}^n. \tag{3.2}$$

If f vanishes on a nonempty open subset of $\mathbb{T}^n$ and $S = \infty$ then $f \equiv 0$. If $S < \infty$ then given any nonempty open $U \subset \mathbb{T}^n$, there exists a nontrivial $f \in L^1(\mathbb{T}^n)$, supp $f \subset U$ satisfying (3.2).

The higher dimensional version of Theorem 1.2 which was used to prove Theorem 3.1 is not suitable for Riemannian symmetric spaces. However, Bochner proved extensions of Theorem 1.2 for $\mathbb{R}^n$ and Riemannian manifolds using iterates of the Laplace-beltrami operator Δ (see [8]). Unfortunately this does not quiet suffice for us. However, a variant of Bochner's result proved by P. R. Chernoff [11] turns out to be useful for us. The following are the results of Bochner and Chernoff.

Theorem 3.4. (1) *(Bochner, [8]) Suppose $f \in C^\infty(\mathbb{R}^n)$ and*

$$\sum_{m \in \mathbb{N}} \|\Delta^m f\|_\infty^{-\frac{1}{m}} = \infty.$$

If for all $m \in \mathbb{N} \cup \{0\}$,

$$\Delta^m f(x) = 0,$$

for all x in a set U of analytic determination (in particular, an open set) then $f \equiv 0$.

(2) *(Chernoff, [11]) Suppose $f \in C^\infty(\mathbb{R}^n)$ is such that for all $m \in \mathbb{N} \cup \{0\}$, $\Delta^m f \in L^2(\mathbb{R}^n)$ and*

$$\sum_{m \in \mathbb{N}} \|\Delta^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \cup \{0\}$,

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} f(x_0) = 0,$$

then $f \equiv 0$.

MEMBER'S COPY

Remark 3.5. (1) It is clear that the vanishing condition in the result of Chernoff is weaker than in the result of Bochner. But we find it more useful because of the use of L^2 norm which is much easier to obtain instead of the L^∞ norm used in Bochner's result.

(2) Chernoff has also provided us with an example which shows that mere vanishing of $\Delta^m f(x_0)$ is not enough to conclude that $f \equiv 0$. Since differential operators on $\mathbb{R}^n$ with translation as well as rotation invariance are polynomials in Δ it follows that any extension of Chernoff's result on symmetric spaces is likely to involve differential operators other than the Laplace-Beltrami operator. Given the method of proof in [11], this seems to be a rather nontrivial issue. In the next section we will resolve it by compromising the vanishing condition which will lead to a weaker version of Chernoff's result on Riemannian symmetric spaces of noncomapct type. This approach has been motivated by an alternative proof of Levinson's result on $\mathbb{R}$ using a Müntz-Szaz type theorem available in [19].

4. Riemannian symetric spaces of noncompact type

We start this section with the preliminaries on semisimple Lie groups and harmonic analysis on associated Riemannian symmetric spaces which will be necessary to state our results. This material is standard and can be found, for example, in [12, 13, 14, 15].

Let G be a connected, noncompact, real semisimple Lie group with finite centre and $\mathfrak{g}$ its Lie algebra. We fix a Cartan involution θ of $\mathfrak{g}$ and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces of θ respectively. Then $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a linear subspace of $\mathfrak{g}$. The Cartan involution θ induces an automorphism Θ of the group G and $K = \{g \in G \mid \Theta(g) = g\}$ is a maximal compact subgroup of G. Let $\mathfrak{a}$ be a maximal subalgebra in $\mathfrak{p}$; then $\mathfrak{a}$ is abelian. We assume that $\dim \mathfrak{a} = d$, called the real rank of G. Let B denote the Cartan Killing form of $\mathfrak{g}$. It is known that $B\mid_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite and hence induces an inner product and a norm $\|\cdot\|_B$ on $\mathfrak{p}$. The homogeneous space $X = G/K$ is a smooth manifold with $\mathrm{rank}(X) = d$. The tangent space of X at the point $o = eK$ can be naturally identified to $\mathfrak{p}$ and the restriction of B on $\mathfrak{p}$ then induces a G-invariant Riemannian metric d on X. We can identify $\mathfrak{a}$ with $\mathbb{R}^d$ endowed with the inner product induced from $\mathfrak{p}$ and let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$. The set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ. We choose a system of positive roots Σ_+ and with respect to Σ_+, the positive

MEMBER'S COPY

Weyl chamber $\mathfrak{a}_+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \quad \text{for all } \alpha \in \Sigma_+\}$. We denote by

$$\mathfrak{n} = \oplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha.$$

Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and we obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. If $N = \exp \mathfrak{n}$ and $A = \exp \mathfrak{a}$ then N is a Nilpotent Lie group and A normalizes N. For the group G, we now have the Iwasawa decomposition $G = KAN$, that is, every $g \in G$ can be uniquely written as

$$g = \kappa(g) \exp H(g) \eta(g), \quad \kappa(g) \in K, H(g) \in \mathfrak{a}, \eta(g) \in N,$$

and the map

$$(k, a, n) \mapsto kan$$

is a global diffeomorphism of $K \times A \times N$ onto G. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha$ be the half sum of positive roots counted with multiplicity. Let M' and M be the normalizer and centralizer of $\mathfrak{a}$ in K respectively. Then M is a normal subgroup of M' and normalizes N. The quotient group $W = M'/M$ is a finite group, called the Weyl group of the pair $(\mathfrak{g}, \mathfrak{k})$. W acts on $\mathfrak{a}$ by the adjoint action. It is known that W acts as a group of orthogonal transformation (preserving the Cartan-Killing form) on $\mathfrak{a}$. Each $w \in W$ permutes the Weyl chambers and the action of W on the Weyl chambers is simply transitive. Let $A_+ = \exp \mathfrak{a}_+$. Since $\exp : \mathfrak{a} \to A$ is an isomorphism we can identify A with $\mathbb{R}^d$. If $\overline{A_+}$ denotes the closure of A_+ in G, then one has the polar decomposition $G = KAK$, that is, each $g \in G$ can be written as

$$g = k_1(\exp Y)k_2, \ \ k_1, k_2 \in K, Y \in \mathfrak{a}.$$

In the above decomposition, the A component of $\mathfrak{g}$ is uniquely determined modulo W. In particular, it is well defined in $\overline{A_+}$. The map $(k_1, a, k_2) \mapsto k_1 a k_2$ of $K \times A \times K$ into G induces a diffeomorphism of $K/M \times A_+ \times K$ onto an open dense subset of G. It follows that if $gK = k_1(\exp Y)K \in X$ then

$$\mathsf{d}(o, gK) = \|Y\|_B. \tag{4.1}$$

We extend the inner product on $\mathfrak{a}$ induced by B to $\mathfrak{a}^*$ by duality, that is, we set

$$\langle \lambda, \mu \rangle = B(Y_\lambda, Y_\mu), \quad \lambda, \mu \in \mathfrak{a}^*, \ Y_\lambda, Y_\mu \in \mathfrak{a},$$

where Y_λ is the unique element in $\mathfrak{a}$ such that

$$\lambda(Y) = B(Y_\lambda, Y), \quad \text{for all } Y \in \mathfrak{a}.$$

This inner product induces a norm, again denoted by $\|\cdot\|_B$, on $\mathfrak{a}^*$,

$$\|\lambda\|_B = \langle \lambda, \lambda \rangle^{\frac{1}{2}}, \quad \lambda \in \mathfrak{a}^*.$$

MEMBER'S COPY

The elements of the Weyl group W acts on $\mathfrak{a}^*$ by the formula

$$sY_\lambda = Y_{s\lambda}, \quad s \in W, \lambda \in \mathfrak{a}^*.$$

Let $\mathfrak{a}^*_{\mathbb{C}}$ denote the complexification of $\mathfrak{a}^*$, that is, the set of all complex-valued real linear functionals on $\mathfrak{a}$. If $\lambda : \mathfrak{a} \to \mathbb{C}$ is a real linear functional then $\Re\lambda : \mathfrak{a} \to \mathbb{R}$ and $\Im\lambda : \mathfrak{a} \to \mathbb{R}$, given by

$$\begin{aligned} \Re\lambda(Y) &= \text{ Real part of } \lambda(Y), \quad \text{for all } Y \in \mathfrak{a}, \\ \Im\lambda(Y) &= \text{ Imaginary part of } \lambda(Y), \quad \text{for all } Y \in \mathfrak{a}, \end{aligned}$$

are real-valued linear functionals on $\mathfrak{a}$ and $\lambda = \Re\lambda + i\Im\lambda$. The usual extension of B to $\mathfrak{a}^*_{\mathbb{C}}$, using conjugate linearity is also denoted by B. Hence $\mathfrak{a}^*_{\mathbb{C}}$ can be naturally identified with $\mathbb{C}^d$ such that

$$\|\lambda\|_B = \left(\|\Re\lambda\|_B^2 + \|\Im\lambda\|_B^2\right)^{\frac{1}{2}}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

Through the identification of A with $\mathbb{R}^d$, we use the Lebesgue measure on $\mathbb{R}^d$ as the Haar measure da on A. As usual on the compact group K, we fix the normalized Haar measure dk and dn denotes a Haar measure on N. The following integral formulae describe the Haar measure of G corresponding to the Iwasawa and Polar decomposition respectively.

$$\begin{aligned} \int_G f(g)dg &= \int_K \int_{\mathfrak{a}} \int_N f(k \exp Yn)\, e^{2\rho(Y)}\, dn\, dY\, dk, \quad f \in C_c(G) \\ &= \int_K \int_{\overline{A_+}} \int_K f(k_1 a k_2)\, J(a)\, dk_1\, da\, dk_2, \end{aligned}$$

where dY is the Lebesgue measure on $\mathbb{R}^d$ and

$$J(\exp Y) = c \prod_{\alpha \in \Sigma_+} (\sinh \alpha(Y))^{m_\alpha}, \quad \text{for } Y \in \overline{\mathfrak{a}_+},$$

c being a normalizing constant. If f is a function on $X = G/K$ then f can be thought of as a function on G which is right invariant under the action of K. It follows that on X we have a G invariant measure dx such that

$$\int_X f(x)\, dx = \int_{K/M} \int_{\mathfrak{a}_+} f(k \exp Y)\, J(\exp Y)\, dY\, dk_M,$$

where dk_M is the K-invariant measure on K/M. For a sufficiently nice function f on X, its Fourier transform $\widetilde{f}$ is defined on $\mathfrak{a}^*_{\mathbb{C}} \times K$ by the formula

$$\widetilde{f}(\lambda, k) = \int_G f(g) e^{(i\lambda - \rho)H(g^{-1}k)} dg, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ k \in K, \tag{4.2}$$

MEMBER'S COPY

whenever the integral exists ([14, P. 199]). As M normalizes N, the function $k \mapsto \widetilde{f}(\lambda, k)$ is right M-invariant. If $\widetilde{f} \in L^1(\mathfrak{a}^* \times K, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dk)$ then the following Fourier inversion holds,

$$f(gK) = |W|^{-1} \int_{\mathfrak{a}^* \times K} \widetilde{f}(\lambda, k)\, e^{-(i\lambda+\rho)H(g^{-1}k)}\, |\mathbf{c}(\lambda)|^{-2} d\lambda\, dk, \tag{4.3}$$

for almost every $gK \in X$ ([14, Chapter III, Theorem 1.8, Theorem 1.9]). Here $\mathbf{c}(\lambda)$ denotes Harish Chandra's c-function. Moreover, $f \mapsto \widetilde{f}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}^*_+ \times K, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dk)$ ([14, Chapter III, Theorem 1.5]). The above results can also be thought of as results related to eigenfunction expansion. This is because if Δ denotes the Laplace-Beltrami operator on X and $e_{\lambda,k}(g) = e^{(i\lambda-\rho)H(g^{-1}k)}$, then the functions $e_{\lambda,k}$ are eigenfunctions of Δ. So, these functions basically play the role of exponentials in the context of Harmonic Analysis on X. The following is the analogue of the result of Chernoff for X (see Theorem 3.4)

Theorem 4.1. [7] *Let $f \in C^\infty(X)$ be such that $\Delta^m f \in L^2(X)$, for all $m \in \mathbb{N} \cup \{0\}$ and*

$$\sum_{m=1}^{\infty} \|\Delta^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If f vanishes on any non-empty open set in X then f is identically zero.

Though the assumption that f vanishes on a non-empty open set drastically changes the nature of the Euclidean version of the theorem but it is still sufficient to obtain the following version of Ingham's theorem (analogue of Theorem 3.1) on X.

Theorem 4.2. [7] *Let $\theta : [0, \infty) \to [0, \infty)$ be a decreasing function with $\lim_{r\to\infty} \theta(r) = 0$ and*

$$I = \int_{\lambda \in \mathfrak{a}^*_+ | \|\lambda\|_B \geq 1} \frac{\theta(\|\lambda\|_B)}{\|\lambda\|_B^d} d\lambda,$$

where $d = rank(X)$.

(1) *Suppose $f \in L^1(X)$ and its Fourier transform $\tilde{f}$ satisfies the estimate*

$$\int_{\mathfrak{a}^* \times K} |\tilde{f}(\lambda, k)| e^{\theta(\|\lambda\|_B)\|\lambda\|_B} |c(\lambda)|^{-2} d\lambda\, dk < \infty. \tag{4.4}$$

If f vanishes on a non-empty open set in X and I is infinite then f vanishes identically.

MEMBER'S COPY

(2) *If I is finite then given any positive number L, there exists a non-trivial $f \in C_c^\infty(X)$ with supp $f \subset B(o, L)$ (the ball of radius L with center at o) satisfying (4.4).*

We end this article with the following interesting questions.

(1) Is it possible to have a version of Ingham's result for functions without any K-invariance?
(2) What is the correct analogue of Chernoff's result for Riemannian symmetric spaces of noncompact type?
(3) A related question would be to ask the following: Suppose G is a connected Lie group with $f \in C^\infty(G)$ and there exists $g_0 \in G$ such that $Df(g_0) = 0$ for all $D \in \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra. Under what condition one can conclude that $f \equiv 0$?
(4) Is there an analogue of the result of Ingham for Riemannian symmetric spaces of compact type?
(5) Is it possible to prove Ingham's result (even for $\mathbb{R}$) under the assumption that f vanishes on a set of positive measure?

References

[1] Bhowmik, M., *A local Levinson theorem for compact symmetric spaces.* J. Lie Theory 29 (2019), no. 3, 787800.

[2] Bhowmik, M., *A Result of Paley and Wiener ON DamekRicci spaces*, J. Austral Math. Soc., 2019, doi:10.1017/S1446788719000077.

[3] Bhowmik, M.; Ray, Swagato K.; Sen, S., *Around theorems of Ingham-type regarding decay of Fourier transform on $\mathbb{R}^n$, $\mathbb{T}^n$ and two step nilpotent Lie Groups.*, to appear in Bull. Sci. Math.

[4] Bhowmik, M.; Ray, Swagato K., *A theorem of Levinson for Riemannian symmetric spaces of noncompact type*, arXiv:1808.09710.

[5] Bhowmik, Mithun; Sen, S., *An uncertainty principle of Paley and Wiener on Euclidean motion group*, J. Fourier Anal. Appl. 23 (2017), no. 6, 14451464.

[6] Bhowmik, M.; Sen, S., *Uncertainty principles of Ingham and Paley-Wiener on semisimple Lie groups*, Israel J. Math. 225 (2018), no. 1, 193221. MR3805646

[7] Bhowmik, M; Pusti, S; Ray, Swagato K., *Analogs of certain quasi-analyticity results on Riemannian symmetric spaces of noncompact type* , Preprint, 2018.

[8] Bochner, S., *Quasi-analytic functions, Laplace operator, positive kernels* Ann. of Math. (2) 51 (1950), 68-91. MR0032708 (11,334g)

[9] Bochner, S.; Taylor, A. E. *Some Theorems on Quasi-Analyticity for Functions of Several Variables* Amer. J. Math. 61 (1939), no. 2, 303-329. MR1507378 DML

[10] Carleman, T.; *Les functions quasi analytiques*, Paris, 1926, P. 20.

[11] Chernoff, Paul R., *Quasi-analytic vectors and quasi-analytic functions*, Bull. Amer. Math. Soc. 81 (1975), 637-646. MR0636276 (58#30432)

[12] Gangolli, R.; Varadarajan V. S., *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Springer-Verlag, Berlin, 1988. MR954385 (89m:22015)

MEMBER'S COPY

[13] Helgason, S., *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, 34, American Mathematical Society, Providence, RI, 2001. MR1834454 (2002b:53081)

[14] Helgason, S.,*Geometric Analysis on Symmetric Spaces*, Mathematical Surveys and Monographs 39. American Mathematical Society, Providence, RI, 1994. MR1280714 (96h:43009)

[15] Helgason, S., *Groups and geometric analysis, Integral geometry, invariant differential operators, and spherical functions*, Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2000. MR1790156 (2001h:22001)

[16] Helgason, S,. *A duality for symmetric spaces with applications to group representations*, Advances in Math. 5, 1-154 (1970). MR0263988 (41 #8587)

[17] Hirschman, I. I., *On the behaviour of Fourier transforms at infinity and on quasi-analytic classes of functions*, Amer. J. Math. 72 (1950), 200-213. MR0032816 (11,350f)

[18] Ingham, A. E., *A Note on Fourier Transforms*, J. London Math. Soc. S1-9 (1934), no. 1, 29-32. MR1574706

[19] Koosis, P., *The logarithmic integral I* , Cambridge Studies in Advanced Mathematics, 12. Cambridge University Press, Cambridge, 1998. xviii+606 pp. MR1670244 (99j:30001)

[20] Levinson, N., *On a Class of Non-Vanishing Functions*, Proc. London Math. Soc. (2) 41 (1936), no. 5, 393-407. MR1576177

[21] Levinson, N.; *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, v. 26. American Mathematical Society, New York, 1940. MR0003208 (2,180d)

[22] Paley, R. E. A. C.; Wiener, N., *Notes on the theory and application of Fourier transforms. I, II*, Trans. Amer. Math. Soc. 35 (1933), no. 2, 348-355. MR1501688

[23] Paris, R. B.; Kaminski, D., *Asymptotics and Mellin-Barnes integrals.* Encyclopedia of Mathematics and its Applications, 85. Cambridge University Press, Cambridge, 2001. MR1854469

[24] Rudin, W., *Real and complex analysis* Third edition. McGraw-Hill Book Co., New York, 1987. MR0924157 (88k:00002)

[25] Schapira, Bruno, *Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel*, Geom. Funct. Anal. 18 (2008), no. 1, 222250. MR2399102 (2009f:33019)

Swagato Ray
Stat-Math Unit
Indian Statistical Institute
203, B.T. Road, Kolkata-700108, India
E-mails: swagato@isical.ac.in, swagato.ray@gmail.com

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July - December (2020), 35–46

THE SPIRAL VORTEX : A STORY ABOUT TORNADOS AND BATHTUBS

KHRISTO N. BOYADZHIEV

(Received : 15 - 05 - 2019 ; Revised : 26 - 04 - 2020)

Abstract. In this paper, we show that equiangular spirals (also known as logarithmic spirals) appear naturally in fluids going through sinks. We provide a simple mathematical model that explains the spiral forms in some natural formations. For this model we need only small parts from the theories of complex variables and differential equations. The purpose of this educational paper is to stimulate the students for further studies in fluid theory and differential equations.

Introduction

When we drain the water from the bathtub we see a small whirlpool. Amazingly, some galaxies and hurricanes look like a whirlpool too (Figures 1 and 2).

Figure 1 Hurricane Bonnie, August 1998

2010 Mathematics Subject Classification: 26B20, 30B10, 34A30
Key words and phrases: Equiangular spiral, logarithmic spiral, fluid motion, Greens formula

© Indian Mathematical Society, 2020.

MEMBER'S COPY

Figure 2 Spiral Galaxy M51

This likeness is not coincidental - the spiral lines observed in such cases closely match equiangular spirals (Figure 3).

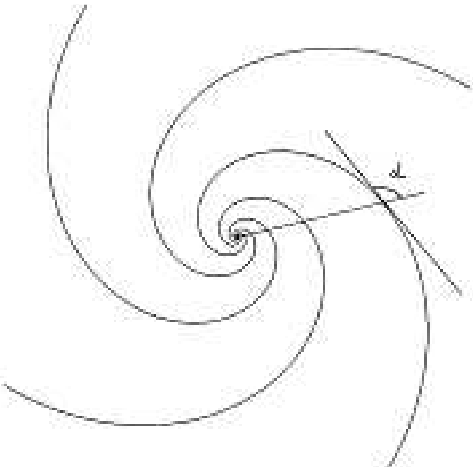


Figure 3 Equiangular spiral

The popular equiangular spiral is defined by the polar equation

$$r(\theta) = r_0 e^{k\theta}, \ -\infty < \theta < \infty \ .$$

It has the characteristic property that the angle α between the radius vector and the tangent vector is the same at any point on the curve. In the above equation, $k = cot\ \alpha$. This spiral appears in many formations in nature [2, 5, 7, 8, 13]. On Figure 3 we see several spirals with the same angle α and different values of r_0.

MEMBER'S COPY

Using complex variables, we shall give a simple explanation why vortex curves are equiangular spirals and not something else. First we consider two-dimensional velocity vector fields with one singularity at the origin O. Such fields can model a water surface, or a thin layer in the atmosphere. We also involve systems of differential equations which help to extend our simple model from two to three dimensions.
This article has an educational character and is designed for undergraduate student with background in complex variables and differential equation. The author believes it will be interesting also to all nature-curious mathematicians.

Vector fields on the plane

For convenience, we identify the -plane with the complex z-plane by setting $z = x + iy = (x, y)$. Vectors in standard position (starting from the origin), and complex numbers are also identified, as in equation (1) below.

Suppose now we have a fluid on the complex plane, described by the *velocity vector field*

$$V(z) = V(x, y) =< u, v >= u + iv, \tag{1}$$

defined everywhere except, possibly, at the origin. Every point (particle) $M(x, y)$ has velocity $V =< u(x, y), v(x, y) >$. Governed by this velocity vector field, the particles move on streamlines. A streamline (trajectory) is a curve, at each point of which, the velocity vector is tangent to that curve. If the smooth curve L is defined by the parametric equations

$$x = x(t),\ y = y(t),\ t_1 \leq t \leq t_2, \tag{2}$$

then L is a streamline for the velocity vector field (1) if and only if the tangent vector $< x', y' >$ and the velocity vector $< u, v >$ are parallel at every point (x, y) on L.
Some simple definitions from vector calculus are needed. Let G be a closed, positively oriented (counterclockwise) smooth curve surrounding a convex domain D. We can think that D is a just a disk. The integral

$$Cir(V : G) = \oint_G V dr = \oint_G u dx + v dy, \tag{3}$$

(where $dr =< dx, dy >$) defines the *circulation* of the vector field along the curve. Nonzero circulation indicates the presence of whirls inside D.

MEMBER'S COPY

We also define the *flux* through the curve

$$Flux(V:G) = \oint_G u\,dy - v\,dx. \qquad (4)$$

Nonzero flux indicates the appearance or disappearance of fluid inside , i.e. the presence of sources or sinks. We now formally define a *vortex*, a *source* and a *sink*.

Definition 1. A point M is called a *vortex* for the vector field V, if there is a neighborhood U of M such that the circulation $Cir(V:G)$ on any circle $G \subset U$ centered at M is nonzero.

Shortly, at a vortex the fluid spins, not necessarily appearing or disappearing.

Definition 2. With M and U as above, the point is called a *source*, if $Flux(V:G) > 0$ and a *sink*, if $Flux(V:G) < 0$ for every circle $G \subset U$ centered at M.

In these definitions we do not require M to be in the domain of the field. It may be a singular point.

Before proceeding further, recall that multiplying one complex number z by the exponential $e^{i\alpha}$ produces a counterclockwise rotation about O by the angle with radian measure α.

When considering a flat fluid with possibly one source/sink, it is reasonable to assume that the fluid is described by a two-dimensional velocity vector field $\boldsymbol{V}$ with possibly one singularity at the origin, which may be a sink, or a source, and/or a vortex (or neither), and there are no other sinks or sources except O. We also do not permit the presence of any whirls which are not centered at O. It is natural to assume that this field is rotationally invariant with respect to the origin O. Here are the exact conditions:

(A) The vector field $V = u + iv$ is smooth, i.e. u, v have continuous partial derivatives everywhere except, possibly, at the origin O .

(B) The vector field has no sinks or sources anywhere except, possibly, at the origin O, and also no whirls. This means both integrals (3), (4) are zero for every closed curve whose interior is separated from O.

(C) V is rotationally invariant, that is, $e^{i\alpha}V(z) = V(e^{i\alpha}z)$ for all $z \neq 0, \alpha \in (0,\pi)$. When we rotate the plane about the origin at angle α, the picture we see does not change.

MEMBER'S COPY

Now we will give a simple characterization of such vector fields and will show that their streamlines are equiangular spirals.

Spiral Vortex Theorem. A vector field V satisfies (A), (B) and (C), if and only if it has the form

$$V(z) = c/\bar{z} \tag{5}$$

where $c = a + ib$ is a complex constant. When $a \neq 0$ and $b \neq 0$, the streamlines of this field are equiangular spirals with polar equation

$$r = r_0 e^{kt}, \ -\infty < t < \infty \tag{6}$$

where $r = r(0)$ is an arbitrary real constant (initial condition) and $k = a/b$. When $b = 0$, the streamlines are rays going from O to infinity (Figure 5 below). When $a = 0$ the stream lines are concentric circles centered at the origin (Figure 4).

Proof of the Theorem. Suppose the vector field , (where $u = u(x,y)$, $v = v(x,y)$), satisfies the above conditions. Let D be an arbitrary disk separated from the origin with boundary G. Then condition (B) implies in view of Green's theorem that

$$Cir(V:G) = \oint_G u dx + v dy = \oint\oint_D (v_x - u_y) dx dy = 0$$

and also

$$Flux(V:G) = \oint_G u dy - v dx = \oint\oint_D (u_x + v_y) dx dy = 0$$

.

Since the disk D is arbitrary, we conclude that

$$u_x - v_y = 0, \ u_y + v_x = 0,$$

everywhere except possibly, at the origin. These two equations are the Cauchy-Riemann equations for the function $\overline{V(z)} = u - iv$. Therefore, this function is holomorphic on the entire complex plane indented at the origin. It has a Laurent series convergent for every $z \neq 0$,

$$\overline{V(z)} = \sum_{n=-\infty}^{\infty} c_n z^n.$$

MEMBER'S COPY

According to property (C), we have $e^{-i\alpha}\overline{V(z)} = \overline{V(e^{i\alpha}z)}$ for every $\alpha \in (0,\pi)$. This gives $e^{-i\alpha}c_n = e^{i\alpha n}c_n$, or $c_n = e^{i\alpha(n+a)}c_n$ for every integer n. This implies $c_n = 0$ for every $n \neq -1$ and, therefore, $\overline{V(z)} = c_{-1}/z$. We conclude that

$$V(z) = c/\bar{z}$$

with the complex constant $c = \bar{c}_{-1}$.

One surprising result from this theorem is that the vector field V is zero at infinity!

It is clear from the above considerations that any vector field of the form (5) satisfies (A), (B), and (C).

We continue now with the proof to show that the streamlines of (5) are equiangular spirals. Let $c = a + ib$, where a, b are real. With $z = x + iy$ equation (5) can be written in the form:

$$V(z) = V(x,y) = \Big\langle \frac{ax-by}{|z|^2}, \frac{bx+ay}{|z|^2} \Big\rangle.$$

.

Consider the system of differential equations:

$$x' = ax - by \ y' = bx + ay \tag{7}$$

where $x = x(t)$, $y = y(t)$. The vector field $< x', y' >$ defined by this system is parallel to $V(x,y)$ at each point, therefore the streamlines of V are determined by the solutions of (7). When $b \neq 0$ this system is equivalent to the second order differential equation

$$x'' - 2ax' + (a^2 + b^2)x = 0$$

together with $y = (ax - x')/b$. The general solution of (7) is easy to find:

$$x(t) = Ce^{at} cos\ (bt - \gamma)x,\ y(t) = Ce^{at} sin\ (bt - \gamma) \tag{8}$$

where $C \geq 0$, $0 \leq \gamma < 2\pi$ are constants. This is a family of equiangular spirals depending on the two parameters C and γ. When $a = 0$ we have concentric circles, which can be viewed as a particular case of (6) with $k = 0$ (Figure 4). When $b = 0$ we have rays going from the origin O to infinity with slope $y/x = -tan\ \gamma$ (Figure 5).

MEMBER'S COPY

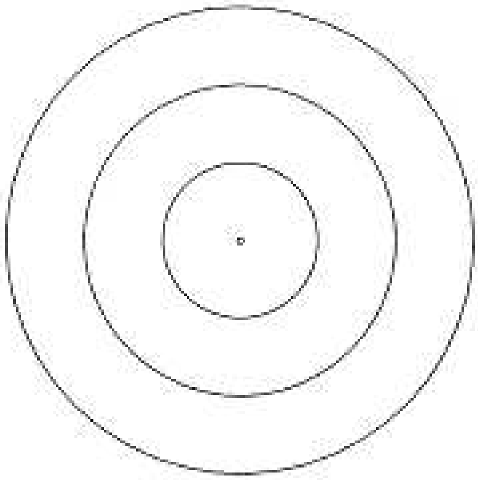


Figure 4

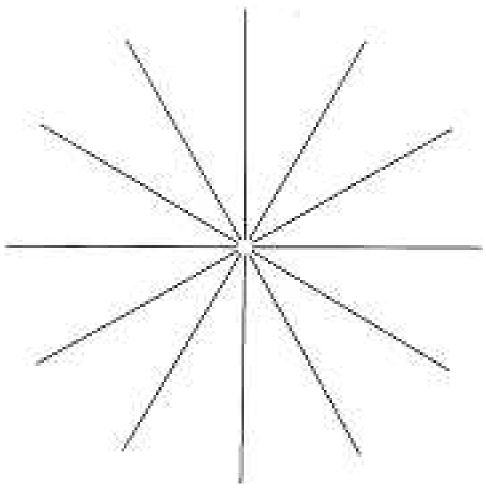


Figure 5

When $b \neq 0$, we can re-scale the paramete, $bt - \gamma \to t$ and setting $a/b = k$ we can write the solution in the form

$$x(t) = r_0 e^{kt} cos\ t,\ y(t) = r_0 e^{kt} sin\ t \tag{9}$$

which is equivalent to equation (6) in polar coordinates, if we consider t to be the polar angle. The proof is complete. □

MEMBER'S COPY

It is interesting to compute the circulation of the vector field (6) on some circle G with center at the origin O. Keeping in mind that the function

$$\overline{V(z)} = u - iv = \frac{\bar{c}}{z} = \frac{a - ib}{z}$$

is holomorphic we write:

$$Cir(V:G) = \oint_G udax + vdy = Re\oint_G (u - iv)d(x + iy) = Re\oint_G \frac{a - ib}{z}dz = 2\pi b$$

and the flux through that circle is

$$Flux(V:G) = \oint_G udy - vdx = Im\oint(u - iv)d(x + iy) = Im\oint_G \frac{a - ib}{z}dz = 2\pi a$$

Therefore, $Cir(V:G)$ and $Flux(V:G)$ are independent of the radius of G and also of each other. They are determined only by the complex constant $c = a + ib$ which can be arbitrary. When $a = 0$, the streamlines are concentric circles centered at O. We observe a vortex with no generation or disappearance of fluid. The case $a > 0$ corresponds to a source, and the case $a < 0$ to a sink. In our model, sources and sinks behave the same way, only the streamlines have different directions. It is clear from the equations (8) that when $a > 0$ the point $(x(t), y(t))$ moves away from the source at the origin O, and when $a < 0$ the point $(x(t), y(t))$ moves toward the sink at O. When $b = 0$ (no whirls!), the streamlines, as mentioned above, are radial rays; $b > 0$ corresponds to positive (counterclockwise) circulation (rotation about the vortex) and $b < 0$ indicates a clockwise circulation. The characteristic angle of the spirals is $\alpha = arccot\ (a/b)$, as represented in Figure 3.

A simple tornado

Tornadoes and hurricanes are three dimensional formations. In order to construct a simple model of tornado, we extend the spiral motion from the horizontal xy-plane to three dimensions, along the vertical z-axis (*the letter z plays now a different role - it will be used for the third Cartesian coordinate*). Our starting point is the system (7) which we replace by a 3×3 system of

MEMBER'S COPY

linear differential equations with real coefficients:

$$x' = a_{11}x + a_{12}y + a_{13}z,\ y' = a_{21}x + a_{22}y + a_{23}z,\ z' = a_{31}x + a_{32}y + a_{33}z \quad (10)$$

The characteristic equation of the real matrix $[a_{ij}]$ is a third order algebraic equation with real coefficients and it may have only real roots or two complex conjugate roots and one real. Since we want the xy-projection of the vector field $< x', y', z' >$ to satisfy two dimensional system (7), we exclude the case of all real roots, because this case brings to exponential growth or decay of the solutions in all dimensions. With two complex conjugate eigenvalues and one real eigenvalue, the system (10) splits into a direct product of one 2×2 real system of the form (7) for x, y and one single equation for the third variable z, which can be put in the simple form

$$z' = pz.$$

There are no other possibilities. The two complex eigenvalues bring equations (9) and the real eigenvalue, say, p, brings to the solution $z(t) = z_0e^{pt}$ of the third equation. The general solution is

$$x(t) = r_0e^{kt}cost$$

$$y(t) = r_0\, e^{k\,t} \sin t$$

$$z(t) = z_0e^{pt}$$

where $-\infty < t < \infty$ and r_0, z_0 are real constants. This is a family of spiral curves on the surface

$$x^2 + y^2 = M^2 z^{2k/p}$$

(M is a constant). When $k = p$ we have conchospirals, that is, spirals on the cone $x^2+y^2 = M^2z^2$ (see [2]). The tornado shape appears when we take $k < p < 0 < min\{r_0, z_0\}$.
A streamline is shown in Figure 6 together with its projection on the xy-plane.

MEMBER'S COPY

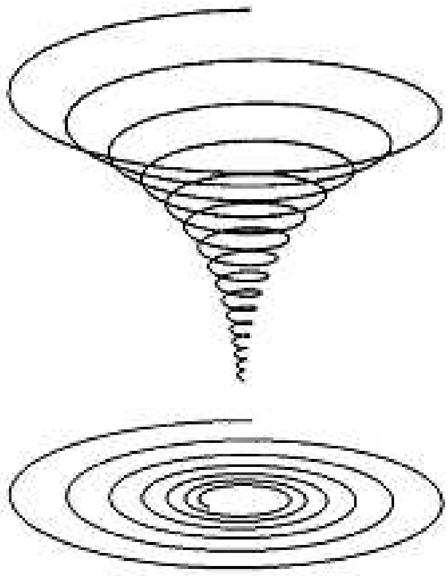


Figure 6

Final notes

Bathtub vortices have been discussed, for instance, in [6]. Many books on complex variables include two-dimensional fluid flows. This is done very well in the classical book [10]. The topic is included sometimes in books on fluid mechanics [11, 12]. Vortices and sinks/sources, however, are often treated separately and the spiral shape does not appear ([10] and [11] are exceptions). A combination of one vortex and one sink, also called a spiral vortex, is mentioned as a model of tornado in [12], p. 231: "A tornado may be approximated by a two-dimensional vortex and a sink, except in a region near the origin, and this is an example of good agreement between a real and ideal fluid".

In order to show that the vector field (5) has spiral streamlines we could integrate the function $\overline{V(z)} = \bar{c}/z$ and come to the complex potential $(a - ib)log(z)$, which provides the combination of one vortex and a source/sink [3, 10]. Using the system (7) instead, we avoid the multivalued complex logarithm. Systems of differential equations have been used very efficiently to describe vector fields with one singularity ([1, 4]).

Real world tornados and hurricanes are very complex formations while our model provides only a rough approximation. More about tornados

MEMBER'S COPY

can be found in [9]. There is a resemblance to a spiral vortex in the form of some spiral galaxies like M51, M81, M100 and M101. University of Rochester researchers have fitted recently the spiral arms of M51 with pieces of equiangular spirals and discovered a good match in [7] (see Figure 7, courtesy of the authors).

Figure 7. Galaxy M51

Some galaxies possibly behave like cosmic hurricanes in slow motion, where the density increases toward the center which plays the role of a sink/source. Interesting comments on this topic can be found on pp. 120-123 in [8].

Acknowledgment: Hurricane Bonnie and Galaxy M51 on Figures 1 and 2 are correspondingly from the NASA and the NASA/IPAC Extragalactic (NED) free databases.

MEMBER'S COPY

References

[1] V. I. Arnold, Ordinary Differential Equations, Springer-Verlag, New York, NY, 1997.

[2] Khristo N. Boyadzhiev, Spirals and conchospirals in the flight of insects, *College Math. J.*, 30(1), January (1999), 23-31.

[3] Khristo N. Boyadzhiev, Loxodromes on the plane (preprint).

[4] P. Blanchard, R. L. Devaney, G. R. Hall, Differential Equations, Brooks / Cole, Pacific Grove, CA, 2002.

[5] T. A. Cook, The Curves of Life, Dover, New York, NY, 1979.

[6] J. W. Hanneken, Coriolis myths and draining bathtubs, *Amer. J. Phys.*, 62(12), 1994, p. 1063.

[7] Alaina L. Henry, A. C. Quillen and Robert Gutermuth, Star formation and Asymmetry in the spiral arms of M51 - Variable star formation caused by more than one spiral density wave, *The Astronomical Journal*, 126, No 6, (2003), 2831-2839.

[8] Mario Livio, The Golden Ratio: The Story of Phi, the World's Most Astonishing Number by Broadway; 1st edition, 2002.

[9] H. J. Lugt, Vortex Flow in Nature and Technology, *J. Wiley & Sons*, New York, NY, 1983.

[10] A. L. Markushevich, Theory of Functions of a Complex Variable, Chelsea Publ. Co., New York, 1977.

[11] L. M. Milne-Thomson, Theoretical Hydrodynamics, 5-th ed, The Macmillan Company, 1969.

[12] R. M. Olson, S. J. Wright, Essentials of Engineering Fluid Mechanics, 5-th edition, Harper & Row, New York, NY, 1990.

[13] Vance A. Tucker, The deep fovea, sideways vision and spiral flight paths in raptors, *J. Exp. Biol.*, 203, (2000): 3745-3754.

Khristo N. Boyadzhiev
Department of Mathematics
Ohio Northern University
Ada, Ohio 45810, USA
k-boyadzhiev@onu.edu

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 47–66

GENERATION OF FAMILY OF PYTHAGOREAN N-TUPLES FROM A SEED PYTHAGOREAN N-TUPLE

NARINDER KUMAR WADHAWAN AND PRIYANKA WADHAWAN

(Received : 09 - 05 - 2019 ; Revised : 20 - 05 - 2020)

Abstract. In this paper, an innovative method has been devised to generate family of Pythagorean n-tuples from a given n-tuple called seed Pythagorean n-tuple. Since a given or seed Pythagorean n-tuple may always not be known, therefore, a formula for generating a seed Pythagorean n-tuple has also been derived and then used for generating n-tuples. While a seed Pythagorean n-tuple may have some terms repeating many times but family of Pythagorean n-tuples populated from a seed n-tuple will not have repeating terms unless our requirement needs so. Formula for a seed Pythagorean n-tuple is based on simple identity upon which certain mathematical operations have been performed. This method of generation of n-tuples is unattempted, unprecedented, easy to derive and hence is equally comprehensible to students and scholars alike.

1. INTRODUCTION

If $x_1,\ x_2$ and x_3 is a Pythagorean triple, then $x_1^2+\ x_2^2=x_3^2$. According to the Law of Trichotomy, either $x_1 > x_2$ or $x_1 < x_2$ or $x_1 = x_2$. If $x_1 = x_2$,, then $x_3^2 = 2\cdot x_1^2$. Since $x_1,\ x_2$ and x_3 all are integers, therefore, x_3 being equal to $\sqrt{2}\cdot x_1$ can never be rational. Hence two equal integers can never be a part of Pythagorean triple.

Let these numbers $x_1,\ x_2$ and x_3 be unequal. We can write, say x_1 as $(a_1\cdot x\ +a)$, x_2 as $(b_1\cdot x\ +b)$ and x_3 as $(c_1\cdot x\ +c)$ where $a_1,\ b_1,\ c_1$ and $a,\ b,\ c$ are rational numbers and $x\in\mathbb{R}$. For these numbers to be a Pythagorean triple, we must have

$$(a_1\cdot x+a)^2\ +\ (b_1\cdot x+b)^2=(c_1\cdot x+c)^2$$

Simplification yields a quadratic equation

$$x^2\cdot\left(a_1^2+b_1^2-c_1^2\right)+2\cdot x\ (a_1\cdot\ a+b_1\cdot b-c_1\cdot c)+a^2+b^2-c^2=0,$$

i.e.

$$P\cdot\ x^2+Q\cdot x\ +R=0,$$

2010 Mathematics Subject Classification: 11D09

Key words and phrases: Pythagorean Triples, Quadruples, Quintuples, n-tuples

© *Indian Mathematical Society, 2020.*

MEMBER'S COPY

where

$$\begin{aligned} P &= \left(a_1^2 + b_1^2 - c_1^2\right), \\ Q &= 2 \cdot (a_1 \cdot\ a + b_1 \cdot b - c_1 \cdot c), \\ R &= a^2 + b^2 - c^2. \end{aligned}$$

This quadratic equation has two roots

$$x = \frac{\left(-Q \pm\ \sqrt{Q^2 - 4P \cdot R}\right)}{2P}.$$

But our requirement is that the quadratic must have rational roots. This is possible if the quadratic is transformed into a linear equation. That requires coefficient of x^2 in the quadratic must be zero i.e. $P = 0$ or $a_1^2 + b_1^2 - c_1^2 = 0$. That further requires that the choice of $a_1,\ b_1$ and c_1 be such that

$$a_1^2 + b_1^2 - c_1^2 = 0$$

Therefore, we need a seed Pythagorean triple $a_1,\ b_1$ and c_1 which transforms the quadratic equation into a linear equation and then

$$x = -\frac{R}{Q}$$

where R and Q have values as already stated. Putting the value of x so obtained from above linear equation in $(a_1 \cdot x\ + a)$, $(b_1 \cdot x\ + b)$ and $(c_1 \cdot x\ + c)$ will generate Pythagorean triples. Since a , b and c can be assigned a number of rational values, therefore a number of Pythagorean triples can be generated. If the triples so generated are in fractions then multiplying these with the lowest common multiplier LCM, will give integer values. Multiplication with LCM used in the paper will be referred to as 'normalisation' henceforth. Above method can be generalised to generate Pythagorean n-tuples

$$(a_1 \cdot x + A_1),\ \ (a_2 \cdot x + A_2),\ \ (a_3 \cdot x + A_3),\ \ldots, (a_n \cdot\ x + A_n).$$

Being Pythagorean n-tuple,

$$(a_1 \cdot x + A_1)^2 + (a_2 \cdot x + A_2)^2 + (a_3 \cdot x + A_3)^2 + \ldots + (a_{n-1} \cdot x + A_{n-1})^2$$
$$= (a_n \cdot x + A_n)^2.$$

Putting the coefficient of x^2 equal to zero i.e.

$$a_1^2 +\ a_2^2 + a_3^2 + \ldots a_{n-1}^2 - a_n^2 = 0$$

means we need a seed Pythagorean n-tuple. On simplification of above said equation,

$$x = \frac{F}{G}$$

where

MEMBER'S COPY

$$
\begin{aligned}
F &= A_n^2 - (A_1^2 + A_2^2 + A_3^2 + \cdots + A_{n-1}^2) \\
G &= 2\left\{(a_1 \cdot A_1 + a_2 \cdot A_2 + a_3 \cdot A_3 + \cdots + a_{n-1} \cdot A_{n-1}) - a_n \cdot A_n\right\},
\end{aligned}
$$

$a_1,\ a_2,\ a_3, \ldots, a_n$ as already stated are a given seed Pythagorean n-tuple and $A_1,\ A_2,\ A_3,\ \ldots\ A_n$ have rational values as assigned by us. Assigning different rational values of $A_1,\ A_2,\ A_3,\ \ldots\ A_n$ will give different values of x and hence different sets of n-tuples. This proves Lemma 1.1.

Lemma 1.1. $(a_1 \cdot x + A_1), (a_2 \cdot x + A_2), (a_3 \cdot x + A_3), \ldots, (a_n \cdot x + A_n)$, *after normalisation, are Pythagorean n-tuples, when $x = \frac{F}{G}$ where F and G have values as already stated and $a_1,\ a_2,\ a_3, \ldots, a_n$ is a given seed Pythagorean n-tuple and $A_1,\ A_2,\ A_3,\ \ldots\ A_n$ have rational values as assigned by us. Putting different real rational values of $A_1, A_2, A_3, \ldots A_n$ will give different values of x and hence different sets of Pythagorean n-tuples after normalisation.*

It is, therefore, essential, according to Lemma 1.1 that one seed Pythagorean n-tuple must be known for populating n-tuples. But there are certain n-tuples where n has special values and n-tuples can be generated without a given seed Pythagorean n-tuple. One such special Pythagorean n-tuple is where $n = m^2 + 1$. Consider the identity

$$1^2 + 1^2 + 1^2 + \ldots \text{repeated } (n-1) \text{ times} = (n-1) = m^2 \tag{1.1}$$

Assuming first term as x_1, second term as $(x_1 + a_2)$, third term as $(x_1 + a_3)$ ...so on and $(n-1)\,th$ term as $(x_1 + a_{n-1})$ in left hand side and nth term in right hand side as $(m \cdot x_1 + a_n)$ where $a_2,\ a_3,\ a_4 \ldots a_n$ are real rational quantities assigned by us but all $\neq 0$, and putting these terms in equation (1.1) then

$$x_1^2 + (x_1 + a_2)^2 + (x_1 + a_3)^2 + \cdots + (x_1 + a_{n-1})^2 = (m \cdot x_1 + a_n)^2$$

and since $n = m^2 + 1$, on simplification,

$$x_1 = \frac{L}{M} \tag{1.2}$$

where

$$
\begin{aligned}
L &= a_n^2 - \left(a_2^2 + a_3^2 + a_4^2 + \cdots + a_{n-1}^2\right), \\
M &= 2\left(a_2 + a_3 + a_4 + \cdots + a_{n-1} - a_n \cdot m\right).
\end{aligned}
$$

Substituting the value of x_1 given by equation (1.2) in $x_1, (x_1 + a_2), (x_1 + a_3), \ldots (x_1 + a_{n-1})$ and $(m \cdot x_1 + a_n)$ gives unnormalised Pythagorean n-tuples as

$$\frac{L}{M}, \left[\frac{L}{M} + a_2\right], \left[\frac{L}{M} + a_3\right], \left[\frac{L}{M} + a_4\right] \ldots \left[\frac{L}{M} + a_{n-1}\right], \left[\frac{m \cdot L}{M} + a_n\right] \tag{1.3}$$

MEMBER'S COPY

After multiplying with M that is LCM, normalised Pythagorean n-tuples that satisfy the following relation (1.4) are generated

$$L^2 + (L + a_2 \cdot M)^2 + (L + a_3 \cdot M)^2 + \cdots + (L + a_{n-1} \cdot M)^2 \\ = (m \cdot L + a_n \cdot M)^2. \tag{1.4}$$

When $M = 0$ then all the terms in left hand side LHS of equation (1.4) will equal $(n-1) \cdot L^2$ and the term on the right hand side RHS will be $m^2 \cdot L^2$ i.e. $n - 1 = m^2$ which is the given condition. On the other hand when $L = 0$, that will make $\left\{a_n^2 - \left(a_2^2 + a_3^2 + a_4^2 + \ldots a_{n-1}^2\right)\right\} = 0$, then $a_2,\ a_3,\ a_4 \ldots a_n$ will be Pythagorean n-tuples. Otherwise for different values of $a_2,\ a_3,\ a_4 \ldots a_n$, then L and M have different values, therefore, by putting these values in equation (1.4), different sets of Pythagorean n-tuples will be generated. This proves Lemma 1.2.

Lemma 1.2. $x_1, (x_1 + a_2), (x_1 + a_3), \ldots\ (x_1 + a_{n-1})$ *and* $(m \cdot x_1 + a_n)$, *after normalisation by multiplying with least common multiplier (LCM), are Pythagorean n-tuples when $a_2,\ a_3,\ a_4 \ldots a_n$ are real rational quantities assigned by us but all $\neq 0$, m is a positive integer given by relation $n - 1 = m^2$ and $x_1 = L/M$ where L and M have values as already stated.*

Example 1.3. Generate Pythagorean n-tuples where $n - 1 = 4$.

Here $n - 1 = m^2 = 2^2 = 4$ and total terms are $2^2 + 1 = 5$, therefore, Lemma 1.2 is applicable and x_1, $(x_1 + a_2)$, $(x_1 + a_3)$, $(x_1 + a_4)$, and $(2.x_1 + a_5)$ after normalisation are also Pythagorean n-tuples where $x_1 = L/M$, $L = a_5^2 - (a_2^2 + a_3^2 + a_4^2)$ and $M = 2\{(a_2 + a_3 + a_4) - m \cdot a_5\}$. Assuming $a_5 = 5$, $a_2 = 1$, $a_3 = 2$,$a_4 = 3$ or any value one feels like but not $\{(\ a_2 + a_3 + a_4) - m.a_5\} = 0$, therefore, $L = 5^2 - \left(1^2 + 2^2 + 3^2\right) = 11$, $M = 2\{(1 + 2 + 3) - 10\} = -8$ and $x_1 = -11/8$. Pythagorean quintuples after normalisation are

$$11^2 + 3^2 + 5^2 + 13^2 = 18^2.$$

2. THEORY AND CONCEPT

With this introduction, we will proceed to populate Pythagorean n-tuples from a given seed Pythagorean triple or quadruple or quintuple so on. Unlike generation of Pythagorean n-tuples from roots of quadratic equations where n can have any positive integer value, here Pythagorean n-tuples will be generated from linear equations thus avoiding square root part that is inherently present in solution of quadratic equations. This has already been discussed in introduction. Obviously, this method is simple but is applicable to special type of n-tuples. Further, this method requires a known seed Pythagorean triple or quadruple likewise. Let the given seed Pythagorean triple be $a_1^2 + a_2^2 = a_3^2$, it will be proved that this triple will populate not only triples but special n-tuples also where n number of terms of n-tuples is given by

MEMBER'S COPY

$$\begin{aligned} n &= a_1^2 + 2 \\ \text{or } n &= a_2^2 + 2 \\ \text{or } n &= a_1^2 + a_2^2 + 1 \end{aligned}$$

If Pythagorean quadruples say $a_1^2 + a_2^2 + a_3^2 = a_4^2$ is given as seed, it will populate not only quadruples but special n-tuples where

$$\begin{aligned} n &= a_1^2 + 3 \\ \text{or } n &= a_2^2 + 3 \\ \text{or } n &= a_3^2 + 3 \\ \text{or } n &= a_1^2 + a_2^2 + 2 \\ n &= a_1^2 + a_3^2 + 2 \\ \text{or } n &= a_2^2 + a_3^2 + 2 \\ \text{or } n &= a_1^2 + a_2^2 + a_3^2 + 1 \end{aligned}$$

Similarly, if a seed Pythagorean quintuples is given, it will not only populate quintuples but also has more options of populating n-tuples.

2.1 Populating Pythagorean triples, quadruples so on up to n-tuples from a given seed Pythagorean triples or quadruples so on up to n-tuples.

As is evident from the above title, in this part, procedure is given for populating Pythagorean triples, quadruples so on up to n-tuples from a given respective Pythagorean triple or quadruple so on up to n-tuples.
Let a given seed Pythagorean triple a, b and c satisfy the equation $a^2 + b^2 = c^2$. Assume that $(a \cdot x + a_1)$, $(b \cdot x + b_1)$ *and* $(c \cdot x + c_1)$ as populated Pythagorean triples, therefore, $(a \cdot x + a_1)^2 + (b \cdot x + b_1)^2 = (c \cdot x + c_1)^2$. Since $a^2 + b^2 = c^2$, therefore, on simplification,

$$x = \frac{c_1^2 - b_1^2 - a_1^2}{2(a \cdot a_1 + b \cdot b_1 - c \cdot c_1)} \tag{2.1}$$

where a_1, b_1 *and* c_1 are real rational quantities assigned by us. By putting different values of a_1, b_1 *and* c_1, different sets of unnormalised Pythagorean triples $(a \cdot x + a_1)$, $(b \cdot x + b_1)$ and $(c \cdot x + c_1)$ are generated and on multiplying with least common multiplier LCM, normalised Pythagorean triples are generated.

Example 2.1. Let the given seed equation be $a^2 + b^2 = c^2$ is $3^2 + 4^2 = 5^2$. Assuming that $a_1 = 2$, $b_1 = 1$ *and* $c_1 = 4$. Therefore, from equation (2.1), $x = \frac{16-1-4}{2(3 \cdot 2 + 4 \cdot 1 - 5 \cdot 4)} = -\frac{11}{20}$ and $(a \cdot x + a_1) = \frac{7}{20}$, $(b \cdot x + b_1) = -\frac{24}{20}$ and $(c \cdot x + c_1) = \frac{25}{20}$.

MEMBER'S COPY

After normalisation, $7^2 + 24^2 = 25^2$. By assuming other values of a_1, b_1 *and* c_1 and after normalisation, different sets of Pythagorean triples are populated.

Coming to Pythagorean quadruples, let $a, b,\ c$ and d be a given seed Pythagorean quadruple, satisfying equation $a^2 + b^2 + c^2 = d^2$. Applying Lemma 1.1, $(a \cdot x + a_1)$, $(b \cdot x + b_1)$, $(c \cdot x + c_1)$ and $(d \cdot x + d_1)$ will be Pythagorean quadruple, when

$$(a \cdot x + a_1)^2 + (b \cdot x + b_1)^2 + (c \cdot x + c_1)^2 = (d \cdot x + d_1)^2.$$

Since $a^2 + b^2 + c^2 = d^2$, simplifying this equation

$$x = \frac{d_1^2 - c_1^2 - b_1^2 - a_1^2}{2\,(a \cdot a_1 + b \cdot b_1 + c \cdot c_1 - d \cdot d_1)} \tag{2.2}$$

where $a_1, b_1,\ c_1$ *and* d_1 are real rational quantities. By assuming different real rational values of $a_1, b_1,\ c_1$ *and* d_1, different sets of quadruples $(a \cdot x + a_1)$, $(b \cdot x + b_1)$, $(c \cdot x + c_1)$ $(d \cdot x + d_1)$ after normalisation are populated.

Example 2.2. Let the given equation $a^2+b^2+c^2 = d^2$ be $3^2+4^2+12^2 = 13^2$.

Assuming $a_1 = 2,\ b_1 = 1, c_1 = 4$ *and* $d_1 = 5$.

Therefore, from equation (2.2),

$$x = \frac{25 - 16 - 1 - 4}{2.\,(3 \cdot 2 + 4 \cdot 1 + 12 \cdot 4 - 13 \cdot 5)} = -\frac{2}{7} \text{ and } (a \cdot x + a_1) = \frac{8}{7},$$

$$(b \cdot x + b_1) = -\frac{1}{7},\ (c \cdot x + c_1) = \frac{4}{7}, (d \cdot x + d_1) = \frac{9}{7}\ .$$

After normalisation, $8^2 + 1^2 + 4^2 = 9^2$.

Proceeding in this way, Pythagorean quintuples, sextuples so on upto n-tuples can be populated from the given respective seed quintuple, sextuple so on upto n-tuples. For generalised formula for n-tuples, let the given seed Pythagorean n-tuple is $a_1,\ a_2,\ a_3, \ldots, a_n$ satisfying

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 = a_n^2$$

then $(a_1 \cdot x + A_1)$, $(a_2 \cdot x + A_2)$, $(a_3 \cdot x + A_3)$ upto $(a_n \cdot x + A_n)$ after normalisation are Pythagorean n-tuples satisfying the relation

$$(a_1 \cdot x + A_1)^2 + (a_2 \cdot x + A_2)^2 + (a_3 \cdot x + A_3)^2 + \cdots + (a_{n-1} \cdot x + A_{n-1})^2$$
$$= (a_n \cdot x + A_n)^2.$$

Since

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 = a_n^2,$$

simplification of the above equation gives

$$x = \frac{F}{G} \tag{2.3}$$

MEMBER'S COPY

where

$$\begin{aligned} F &= A_n^2 - \left(A_1^2 + A_2^2 + A_3^2 + \cdots + A_{n-1}^2\right), \\ G &= 2\left\{(a_1 \cdot A_1 + a_2 \cdot A_2 + a_3 \cdot A_3 + \cdots + a_{n-1} \cdot A_{n-1}) - a_n \cdot A_n\right\} \end{aligned}$$

and $A_1,\ A_2, A_3 \ldots A_{n-1},\ A_n$ are real rational quantities.
By assuming different values of $A_1,\ A_2,\ A_3 \ldots\ A_{n-1},\ A_n$ different sets of n-tuples $(a_1 \cdot x + A_1)$, $(a_2 \cdot x + A_2)$, $(a_3 \cdot x + A_3)$ upto $(a_n \cdot x + A_n)$ after normalisation are populated.

2.1a. Special Pythagorean n-tuples from a given Pythagorean triple, quadruple so on where $n = a_1^2 + 2$ *or* $n = a_2^2 + 2$ **or** $n = a_1^2 + a_2^2 + 1$ **or** $n = a_1^2 + 3$ **or** $n = a_2^2 + 3$ **or** $n = a_3^2 + 3$ **or** $n = a_1^2 + a_2^2 + 2$ **or** $n = a_1^2 + a_3^2 + 2$ **or** $n = a_2^2 + a_3^2 + 2$ **or** $n = p + 2$ as the case may be and $a_{1,}, a_2, a_3 \ldots$ are terms of given Pythagorean triple, quadruple so on.

For generation of n-tuples where the number of terms is $n = a_1^2 + 2$, let the given seed Pythagorean triple be $a_1^2 + a_2^2 = a_3^2$. This equation $a_1^2 + a_2^2 = a_3^2$ can also be written as

$$\left(1 + 1 + 1 \ldots \text{repeated up to } a_1^2 \text{ times }\right) + a_2^2 = a_3^2 \tag{2.4}$$

Thus equation (2.4) has $a_1^2 + 2$ terms if a_2^2 and a_3^2 each are not split up and each is kept as one term. If terms a_1^2 and a_3^2 are kept as each one and term a_2^2 is split up as $(1 + 1 + 1 \ldots$ repeated up to a_2^2 terms) then n will equal $a_2^2 + 2$.
Coming to Pythagorean n-tuple where $n = a_1^2 + 2$, terms a_1^2 in number are taken as $x, (x + A_2)\,(x + A_3) \ldots\ldots \left(x + A_{a_1^2}\right)$, single term corresponding to a_2 is taken as $(a_2.x + B)$ and single term corresponding to a_3 is $(a_3.x + C)$. For x, $(x + A_2)$, $(x + A_3) \ldots (x + A_{a_1^2})$, $(a_2 \cdot x + B)$ and $(a_3 \cdot x + C)$ to be Pythagorean n-tuples,

$$x^2 + (x + A_2)^2 + (x + A_3)^2 \ldots \left(x + A_{a_1^2}\right)^2 + (a_2 \cdot x + B)^2 = (a_3 \cdot x + C)^2$$

It is appropriate to define $A_{a_1^2}$. $A_{a_1^2}$ is $\left(a_1^2\right) th$ term of A and its value is $A_{a_1^2}$ assigned by us. If a_1 is 3 then $A_{a_1^2}$ in this case would be A_9 or 9^{th} term of A which has that value as assigned by us. On simplification of above equation,

$$\begin{aligned} &x^2\left(a_1^2 + a_2^2\right) + 2 \cdot x\left(A_2 + A_3 + A_4 \ldots\ A_{a_1^2} + B.a_2\right) + A_2^2 + A_3^2 + A_4^3 + ..A_{a_1^2}^2 \\ &\quad + B^2 \\ &= a_3^2 \cdot x^2 + 2 \cdot x \cdot C \cdot a_3 + C^2 \end{aligned}$$

Since $a_1^2 + a_2^2 = a_3^2$, therefore,

$$x = H/I \tag{2.5}$$

MEMBER'S COPY

where

$$H = C^2 - \left(A_2^2 + A_3^2 + A_4^2 + \cdots + A_{a_1^2}^2 + B^2\right),$$
$$I = 2\left(A_2 + A_3 + A_4 + \cdots + A_{a_1^2} + B \cdot a_2 - C \cdot a_3\right),$$

$A_2,\ A_3,\ \ A_4, \ldots, A_{a_1^2},\ B\ and\ C$ are rational quantities assigned by us. Putting different values of x obtained from equation (2.5) in x,$(x + A_2)$, $(x + A_3) \ldots (x + A_{a_1^2})\ (a_2 \cdot x + B)$ and $(a_3 \cdot x + C)$ and after normalisation will populate Pythagorean n-tuples.

For generating n-tuples where $n = a_1^2 + 3$, let the given Pythagorean quadruple be $a_1^2 + a_2^2 + a_3^2 = a_4^2$. This equation $a_1^2 + a_2^2 + a_3^2 = a_4^2$ can also be written as

$$\left(1 + 1 + 1 \ldots \text{ repeated up to } a_1^2 \text{ terms}\right) + a_2^2 + a_3^2 = a_4^2 \tag{2.6}$$

Thus equation (2.6) has $n = a_1^2 + 3$ terms if $a_2^2,\ a_3^2\ and\ a_4^2$ are taken as one term each. Similarly, on splitting $a_2^2\ \ or\ a_3^2,\ \ n$ can have value $a_2^2\ + 3$ or $a_3^2\ + 3$ terms.

In the case of terms $n = a_1^2 + 3$, a_1^2 split up as $\left(1 + 1 + 1 \ldots a_1^2 \text{ terms}\right)$ has terms a_1^2 in number and these are taken as $x, (x + A_2), (x + A_3) \ldots\ldots$ $\left(x + A_{a_1^2}\right)$, single term corresponding to a_2 is taken as $(a_2 \cdot x + B)$ and single terms corresponding to a_3 and a_4 are $(a_3 \cdot x + C)$ and $(a_4 \cdot x + D)$ respectively. For $x, (x + A_2), (x + A_3) \ldots (x + A_{a_1^2}), (a_2 \cdot x + B), (a_3 \cdot x + C)$ and $(a_4 \cdot x + D)$ to be Pythagorean n-tuples,

$$\begin{aligned} &\left\{x^2 + (x + A_2)^2 + \ (x + A_3)^2 + \cdots + \left(x + \ A_{a_1^2}\right)^2\right\} + (a_2 \cdot x + B)^{\ 2} \\ &\quad + (a_3 \cdot x + C)^2 \\ &= \ (a_4 \cdot x + D)^2 \end{aligned}$$

Since given seed quadruple is $a_1^2 + a_2^2 + a_3^2 = a_4^2$ where a_1^2 is split up, on simplification, it gives

$$x = S/T \tag{2.7}$$

where

$$S = D^2 - \left(A_2^2 + A_3^2 + A_4^2 + \cdots + A_{a_1^2}^2 + B^2 + C^2\right)$$
$$T = 2\left(A_2 + A_3 + A_4 + \cdots + A_{a_1^2} + B \cdot a_2 + C \cdot a_3 - D \cdot a_4\right),$$

$A_2, A_3, A_4 \ldots A_{a_1^2}, B, C$ and D are rational quantities assigned by us. Then x, $(x + A_2)$, $(x + A_3)$ $\ldots$ $\left(x + A_{a_1^2}\right)$, $(a_2 \cdot x + B)$,$(a_3 \cdot x + C)$ and $(a_4 \cdot x + D)$, on substitution of the value of x obtained from equation (2.7) and after normalisation, will be populated Pythagorean n-tuple for number $n = (a_1)^2 + 3$. For generating n-tuples where terms $n - 1 = p + 1 = \ m^2$.

MEMBER'S COPY

Let $a^2 + p = a_n^2$ where a^2 is assumed as single term and p is assumed as $1^2 + 1^2 + 1^2 + \dots$ repeated p times and a_n^2 is assumed as single term. For these terms to be Pythagorean n-tuples,

$$a^2 + (1^2 + 1^2 + 1^2 + \dots \text{repeated } p \text{ times}) = a_n^2 \tag{2.8}$$

Thus equation (2.8) has $2 + p$ terms. Single term a^2 is assumed as $(a \cdot x + A)^2$ and terms $1^2 + 1^2 + 1^2 + \dots$ repeated p times are assumed as $(x + B_1)^2 + (x + B_2)^2 + (x + B_3)^2 \dots (x + B_p)^2$ and nth term in right hand side as $(a_n \cdot x + C)$ then according to equation (2.8),

$$(a \cdot x + A)^2 + (x + B_1)^2 + (x + B_2)^2 + \dots + (x + B_p)^2 = (a_n \cdot x + C)^2$$

On simplification, since, $a^2 + p = a_n^2$, therefore,

$$x = U/V$$

where

$$\begin{aligned} U &= C^2 - \left(A^2 + B_1^2 + B_2^2 + B_3^2 + \dots + B_p^2\right), \\ V &= 2\left\{(a \cdot A + B_1 + B_2 + B_3 + \dots + B_p) - a_n \cdot C\right\}, \end{aligned}$$

$A, B_1, B_2, B_3 \dots B_p$ and C are real rational quantities assigned by us. After putting this value of x and normalisation $(a.x+A)$, $(x + B_1)$,$(x + B_2)$, $(x + B_3)$, . . . , $(x + B_p)$,$(a_n \cdot x + C)$ are populated Pythagorean n-tuples where $n - 1 = p + 1 = m^2$ and $a^2 + p = a_n^2$.

2.2. Generation of Pythagorean n-tuples where n can have any positive integer value

It is not always the case that n satisfies the conditions as detailed in the foregoing paragraphs and n can have any positive integer value. In such cases, the methods discussed and devised above may not be helpful. Further, it is also difficult to remember a seed Pythagorean triple or quadruple or n-tuple. To obviate such difficulties, a general solution is required for generating n-tuples. For that we need to generate a seed Pythagorean n-tuple ourselves. In paragraphs hereinafter, a method has been devised for generating a seed Pythagorean n-tuple.

Lemma 2.3. *If m is a positive integer such that $n - 1$* lies between $(m-1)^2$ *and m^2 then $(m + p)^2 - (n - 1)$ can always equal $x_2 \cdot (3) + x_3 \cdot (8) + x_4 \cdot (15) + \dots x_k \cdot (k^2 - 1)$ where $x_{2,}$ $x_3, \dots,$ x_k and p are positive integers (zero included) such that*

$$(x_2 + x_3 + x_4 + \dots + x_k) \le (n - 1).$$

Lemma 2.4. *If m is an integer such that $n - 1$ lies between $(m - 1)^2$ and m^2 and*

$$(m + p)^2 - (n - 1) = x_{2.} \cdot (3) + x_3 \cdot (8) + x_4 \cdot (15) + \dots x_k \cdot \left(k^2 - 1\right)$$

MEMBER'S COPY

then seed Pythagorean n-tuple is given by equation

$$(m+p)^2 = x_1 \cdot (1^2) + x_2 \cdot (2^2) + x_3 \cdot (3^2) + x_4 \cdot (4^2) + \cdots + x_k \cdot (k^2)$$

where $x_1,\ x_2,\ x_3, \ldots\ x_k$ and p are positive integers (zero included) such that

$$\begin{aligned} (x_2\ + x_3 + x_4 + \cdots + x_k) &\leq (n-1) \\ \textit{and}\ (x_1 + x_2\ + x_3 + \cdots + x_k) &= (n-1)\,. \end{aligned}$$

Lemma 2.5. *All Pythagorean n-tuples where $n > 3$ can have at least one term as one.*

Proof. It is obvious from Lemmas 2.3 and 2.4 that a seed n-tuple may have terms that repeat but populated n-tuples will have terms differing from one another unless our requirement demands so. Let Pythagorean n-tuples have each term as 1, then

$$1^2 + 1^2 + 1^2 \ldots \text{upto } n-1 \text{ terms} = n-1 \tag{2.9}$$

Left hand side (LHS) of identity (2.9) is 1^2 added $(n-1)$ times.

We will find a positive integer m so that $(n-1)$ lies between $(m-1)^2$ and m^2. That is $(n-1) > (m-1)^2$ and $(n-1) < m^2$ or $m < \left(\sqrt{n-1}+1\right)$ but $> \sqrt{n-1}$. Thus m can always be determined and with that determination, value of $m^2-(n-1)$ which is always a positive integer since m and n are both positive integers, can always be found. For generalised form, we take $(m+p)^2-(n-1)$ where p is any positive integer $0,\ 1,\ 2,\ 3,\ \ldots$ so on. Obviously, $(m+p)^2-(n-1)$ will again be a positive integer.

Putting $(m+p)^2-(n-1)$ in identity (2.9), we get

$$(m+p)^2 - \left(1^2 + 1^2 + 1^2 \ldots \text{upto } n-1 \text{ terms}\right) = (m+p)^2 - (n-1) \tag{2.10}$$

$$\textit{Let}\ \ (m+p)^2 - (n-1) = x_{2.} \cdot (3) + x_3 \cdot (8) + \cdots + x_k \cdot \left(k^2-1\right) \tag{2.11}$$

where $x_{2,}\ x_3,\ x_4, \ldots\ x_k$ are positive integers including 0 such that

$$(x_2\ + x_3 + x_4 + \cdots + x_k) \leq (n-1)\,.$$

Equation (2.11) is always achievable by increasing value of p from 0 to 1 to 2 to 3 so on till it is satisfied. That proves Lemma 2.3. On putting the value of $(m+p)^2-(n-1)$ from equation (2.11) into equation (2.10),

$$(m+p)^2 - \left(1^2 + 1^2 + 1^2 \text{...upto } n-1 \text{ terms}\right) = x_2 \cdot (3) + \cdots + x_k \cdot (k^2-1)$$

On transposing $\left(1^2 + 1^2 + 1^2 \ldots \text{upto } n-1 \text{ terms}\right)$ to right hand side and rearranging,

$$(m+p)^2 = (x_1) \cdot \left(1^2\right) + x_2 \cdot \left(2^2\right) + (x_3) \cdot \left(3^2\right) + \cdots + (x_k) \cdot (k^2) \tag{2.12}$$

where $x_1 = \{(n-1) - (x_2 + x_3 + x_4 + \cdots + x_k)\}$ and can have any integer value including 0. That makes

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_k = n-1 \tag{2.13}$$

MEMBER'S COPY

Derivation of equations (2.12) and (2.13) proves Lemma 2.4. On splitting up, equation (2.12) can be written as

$$\begin{aligned}(m+p)^2 = &\left(1^2+1^2+\ldots\ x_1 \text{ times}\right] + \left(2^2+2^2+\ldots\ x_2 \text{ times}\right)\\ &+\left(3^2+3^2+\ 3^2+\ldots\ x_3 \text{ times}\right)+\ldots\\ &+\ \left(k^2+k^2+k^2+\ldots\ x_k \text{ times}\right)\end{aligned} \tag{2.14}$$

Left hand side of equation (2.14) has a perfect square in the form of $(m+p)^2$ and right hand side has sum of squares of $(n-1)$ terms, therefore, total number of terms are n and it fulfills all essential ingredients to be a seed Pythagorean n-tuple. □

Given in Table 1 are some of seed Pythagorean n-tuple determined using above said method for n varying from 3 to 10^{11}. We have written only single seed equation for different n except for $n = 8,\ 13$ and 500 where two equations are provided but it is submitted, there can be infinite number of seed equations as the value of p is increased.

TABLE 1. Seed Pythagorean n-tuples with varying 'n'

N, Tuples (n)	Last Term	Seed Pythagorean's N-Tuples
n	$m+p$	$(m+p)^2$ $= x_1\cdot(1^2)+x_2\cdot(2^2)+x_3\cdot(3^2)+x_4\cdot(4^2)+\cdots+x_k\cdot(k^2)$
3	$2+3$	$5^2 = 1\cdot(3^2)+1\cdot(4^2)$
4	$2+1$	$3^2 = 1\cdot(1^2)+2\cdot(2^2)$
6	$3+1$	$4^2 = 3\cdot(1^2)+1\cdot(2^2)+1\cdot(3^2)$
7	$3+0$	$3^2 = 5\cdot(1^2)+1\cdot(2^2)$
8	$3+2$	$5^2 = 5\cdot(1^2)+\ 1\cdot(2^2)+1\cdot(4^2) = 1\cdot(1^2)+6\cdot(2^2)$
9	$3+1$	$4^2 = 7\cdot(1^2)+1\cdot(3^2)$
11	$4+0$	$4^2 = 8\cdot(1^2)+2\cdot(2^2)$
12	$4+1$	$5^2 = 8\cdot(1^2)+2\cdot(2^2)+1\cdot(3^2)$
13	$4+2$	$6^2 = 4\cdot(1^2)+8\cdot(2^2) =\ 8\cdot(1^2)+\ 3\cdot(2^2)+1\cdot(4^2)$
14	$4+0$	$4^2 = 12\cdot(1^2)+1\cdot(2^2)$
15	$4+2$	$6^2 = 10\cdot(1^2)+2\cdot(2^2)+2\cdot(3^2)$
20	$5+0$	$5^2 = 17\cdot(1^2)+2\cdot(2^2)$
30	$6+1$	$7^2 = 24\cdot(1^2)+4\cdot(2^2)+1\cdot(3^2)$
51	$8+0$	$8^2 = 47\cdot(1^2)+2\cdot(2^2)+1\cdot(3^2)$
100	$10+1$	$11^2 = 95\cdot(1^2)+2\cdot(2^2)+2\cdot(3^2)$
500	$23+0$	$23^2 = 489\cdot(1^2)+10\cdot(2^2) =\ 497\cdot(1^2)+\ 2\cdot(4^2)$ $=\ 494\cdot(1^2)+2\cdot(2^2)+3\cdot(3^2)$
2500	$50+1$	$51^2 = 2493\cdot(1^2)+2.\cdot(2^2)+4\cdot(5^2)$
10^5	$317+0$	$317^2 = 9996\cdot(1^2)+1\cdot(4^2)+1\cdot(6^2)+1\cdot(21^2)$
10^{11}	$316228+0$	$316228^2 = 99999999994\cdot(1^2)+\ 1\cdot(3^2)+1\cdot(4^2)$ $+1\cdot(5^2)+1\cdot(22^2)+1\cdot(384^2)$

MEMBER'S COPY

2.3. Populating Pythagorean n-tuples from a seed Pythagorean n-tuple

Lemma 2.6. *When integer $m \geq 2$ and $p, x_1,\ x_2 \ldots x_k$ are positive integers including* 0 *and*

$$(m+p)^2 = x_1 \cdot (1^2) + x_2 \cdot (2^2) + x_3 \cdot (3^2) + x_4 \cdot (4^2) + \cdots + x_k \cdot (k^2),$$

then the terms

$$\begin{aligned}
&\{(x+a_{11}),(x+a_{12}),(x+a_{13}),\ \ldots \text{upto}\,(x+a_{1x_1})\ \},\\
&\{(2x+a_{21}),\ (2x+a_{22}),\ (2x+a_{23}),\ldots \text{upto}\ (2x+a_{2x_2})\ \},\\
&\{(3x+a_{31}),(3x+a_{32}),(3x+a_{33}),\ \ldots \text{upto}\ \ (3x+a_{3x_3})\ \},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\{(k\cdot x+a_{k1}),\ (k\cdot x+a_{k2}),(k\cdot x+a_{k3}),\ldots \text{upto}\ (k\cdot x+a_{kx_k})\}
\end{aligned}$$

and $\{(m+p).x+a\}$ after normalisation are Pythagorean n-tuples when $x = Y/Z$ where

$$\begin{aligned}
Y &= a^2 - \Big[\Big(a_{11}^2 + a_{12}^2 + a_{13}^2 + \cdots + a_{1x_1}^2\Big) + \Big(a_{21}^2 + a_{22}^2 + a_{23}^2 + \cdots + a_{2x_2}^2\Big)\\
&\quad + \cdots + \Big(a_{k1}^2 + a_{k2}^2 + a_{k3}^2 + \cdots + a_{kx_k}^2\Big)\Big],\\
Z &= 2\Big[(a_{11}+a_{12}+a_{13}+\cdots+a_{1x_1}) + 2\,(a_{21}+a_{22}+a_{23}+\cdots+a_{2x_2})\\
&\qquad + \cdots + k\Big(a_{k1}+a_{k2}+a_{k3}+\cdots+a_{kx_k}\Big) - a\,(m+p)\Big]
\end{aligned}$$

and a, $\Big[(a_{11},\ a_{12},\ a_{13},\ \ldots a_{1x1}),\ (a_{21},\ a_{22},\ a_{23},\ \ldots a_{2x2}),$ $(a_{31},\ \ a_{32},\ \ a_{33},\ldots a_{3x3})\ldots\ (a_{k1},\ a_{k2},\ \ a_{k3},\ldots a_{kxk})\Big]$ are real rational quantities assumed by us.

Proof. To utilise seed Pythagorean n-tuples given by equation (2.14), we will take terms $x_1\left(1^2\right)$ i.e $(1^2+1^2+1^2+\ldots$ repeated x_1 times) as
$\Big\{(x+a_{11})^2+(x+a_{12})^2+(x+a_{13})^2+\cdots+(x+a_{1x_1})^2\Big\}$,

terms $x_2.\left(2^2\right)$ *i.e.*$(2^2+2^2+2^2+\ldots$ repeated; x_2 times) as
$\Big\{(2.x+a_{21})^2+(2.x+a_{22})^2+(2.x+a_{23})^2+\ldots+(2.x+a_{2x_2})^2\Big\}$,

terms $x_3.\left(3^2\right)$ i.e. $(3^2+3^2+3^2+\ldots$ repeated x_3 times) as
$\Big\{(3.x+a_{31})^2+(3.x+a_{32})^2+(3.x+a_{33})^2+\ldots+(3.x+a_{3x_3})^2\Big\}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

terms $x_k.\left(k^2\right)$ *i.e.*$(k^2+k^2+k^2+\ldots$ repeated x_k times) as
$\Big\{(k.x+a_{k1})^2+(k.x+a_{k2})^2+(k.x+a_{k3})^2+\ldots+(k.x+a_{kx_k})^2\Big\}$,

and term $(m+p)^2$ as single term $\{(m+p).x+a\}^2$.

MEMBER'S COPY

For these terms to be Pythagorean n- tuples, these must satisfy equation (2.14), therefore,

$$\begin{aligned}&\left\{(x+a_{11})^2+(x+a_{12})^2+(x+a_{13})^2+\ldots+(x+a_{1x_1})^{\,2}\right\}\\&\quad+\left\{(2x+a_{21})^2+(2x+a_{22})^2+(2x+a_{23})^2+\cdots+(2x+a_{2x_2})^{\,2}\right\}\\&\quad+\left\{(3x+a_{31})^2+(3x+a_{32})^2+(3x+a_{33})^2+\cdots+(3x+a_{3x_2})^{\,2}\right\}\\&\quad+\cdots+\left\{(k\cdot x+a_{k1})^2+(k\cdot x+a_{k2})^2+\cdots+(k\cdot x+a_{kx_k})^{\,2}\right\}\\&=\{(m+p)\cdot x+a\}^2\end{aligned} \tag{2.15}$$

On expanding and arranging coefficients of of x^2, x and constant term,

$$\begin{aligned}&x^2\left\{x_1+x_2\left(2^2\right)+x_3\left(3^2\right)+\cdots+x_k(k^2)\right\}\\&\quad+2x\Big[(a_{11}+a_{12}+a_{13}+\cdots+a_{1x1})\\&\quad+2\left(a_{21}+a_{22}+a_{23}+\cdots+a_{2x2}\right)+3\left(a_{31}+a_{32}+a_{33}+\cdots+a_{3x3}\right)\\&\quad+\cdots+k\left(a_{k1}+a_{k2}+a_{k3}+\cdots+a_{kxk}\right)\Big]\\&\quad+\Big[\left(a_{11}^2+a_{12}^2+a_{13}^2+\cdots+a_{1x1}^2\right)+\left(a_{21}^2+a_{22}^2+a_{23}^2+\cdots+a_{2x2}^2\right)\\&\quad+\left(a_{31}^2+a_{32}^2+a_{33}^2+\cdots+a_{3x3}^2\right)\\&\quad+\cdots+\left(a_{k1}^2+a_{k2}^2+a_{k3}^2\cdots+\cdots+a_{kxk}^2\right)\Big]\\&=x^2.(m+p)^2+2.x.a.\left(m+p\right)+a^2.\end{aligned} \tag{2.16}$$

□

As $(m+p)^2=x_1.(1^2)+x_2.(2^2)+x_3.(3^2)+x_4.(4^2)+\cdots+x_k.(k^2)$, on simplification,

$$x=Y/Z \tag{2.17}$$

where

$$\begin{aligned}Y&=a^2-\Big[\left(a_{11}^2+a_{12}^2+a_{13}^2+\cdots+a_{1x_1}^2\right)\\&\quad+\left(a_{21}^2+a_{22}^2+a_{23}^2+\cdots+a_{2x_2}^2\right)+\cdots+\left(a_{k1}^2+a_{k2}^2+a_{k3}^2+\cdots+a_{kx_k}^2\right)\Big],\end{aligned}$$

$$\begin{aligned}Z&=2\Big[\left(a_{11}+a_{12}+a_{13}+\cdots+a_{1x_1}\right)+2\left(a_{21}+a_{22}+a_{23}+\cdots+a_{2x_2}\right)\\&\quad+\cdots+k\left(a_{k1}+a_{k2}+a_{k3}+\cdots+a_{kx_k}\right)-a\left(m+p\right)\Big],\end{aligned}$$

MEMBER'S COPY

$p, a, \{(a_{11},\ a_{12},\ a_{13},\ \ldots a_{1x1})\,,\ (a_{21},\ a_{22},\ a_{23},\ \ldots a_{2x2})\,,$ $(a_{31},\ \ a_{32},\ \ a_{33}, \ldots a_{3x3}) \ldots (a_{k1},\ a_{k2},\ \ a_{k3}, \ldots a_{kxk})\}$ are real rational quantities assigned by us. Values of m and p are already known from seed equation. When x is determined according to equation (2.17), Pythagorean n-tuples after normalisation are given by equation (2.16).

To illustrate the formula (2.17) so derived, an example is given for populating n-tuples from seed Pythagorean n-tuple where $n = 20$.

Example 2.7. For $n = 20,\ \ n-1 = 20-1 = 19$, m must be an integer between $\sqrt{19}$ and $(\sqrt{19}+1)$ according to Lemma 4. Therefore, m must be an integer between 4.359 and 5.359 i.e. $m = 5$. Using equation (2.11),

$$(m+p)^2 - (n-1) = x_{2.} \cdot (3) + x_3 \cdot (8) + x_4 \cdot (15) + \ldots x_k \cdot \left(k^2-1\right).$$

First taking $p = 0$ and to find integer values of $x_2,\ \ x_3,\ x_4, \ldots,\ \ x_k$ satisfying this equation, values of m, p and n are put in equation (2.11),

$$\begin{aligned} (5+0)^2 - (20-1) &= x_{2.} \cdot (3) + x_3 \cdot (8) + x_4 \cdot (15) + \ldots x_k \cdot \left(k^2-1\right) \\ \text{or } 6 &= 2 \cdot (3) + 0 \cdot (8) + 0 \cdot (15) + \ldots 0 \cdot \left(k^2-1\right). \end{aligned}$$

In this way, $x_2 = 2,\ x_3 = 0,\ x_4 = 0,\ \ldots$ all other coefficients including x_k are zero. Using equation (2.13),

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + \cdots + x_k &= n-1 \\ \text{or } x_1 + 2 + 0 + 0 + \cdots + 0 &= 20-1. \end{aligned}$$

Therefore, $x_1 = 17$ and using equation (2.13), seed Pythagorean n-tuple when $n = 20$ is given by the identity,

$$\begin{aligned} (5+0)^2 &= 17 \cdot \left(1^2\right) + 2 \cdot \left(2^2\right) + 0 \cdot \left(3^2\right) + 0 \cdot \left(4^2\right) + \cdots + 0 \cdot (k^2) \\ \text{i.e. } 5^2 &= 17 \times \left(1^2\right) + 2 \times \left(2^2\right). \end{aligned}$$

Assuming

$a_{11} = 1,\ a_{12} = -1, a_{13} = 2,\ a_{14} = -2,\ \ a_{15} = 3,\ \ a_{16} = -3,\ a_{17} = 4,$

$a_{18} = -4,\ a_{19} = 5,\ \ a_{110} = -5,\ a_{111} = 6,\ \ a_{112} = -6, a_{113} = 7,\ \ a_{114} = -7,$

$a_{115} = 8,\ \ a_{116} = -8,\ \ a_{117} = 9,\ \ a_{21} = -9,\ a_{22} = 10,\ a = -10$

and using equation (2.17),

$$\begin{aligned} Y &= 100 - (1+1+4+4+9+9+16+16+25+25+36+36+49 \\ &\quad +49+64+64+81+81+100) \\ &= -570, \end{aligned}$$

$$\begin{aligned} Z &= 2\Big[(1-1+2-2+3-3+4-4+5-5+6-6+7-7+8-8+9) \\ &\quad + 2\,(-9+10) + 10(5)\Big] \\ &= 2(9+2+50) \\ &= 122, \end{aligned}$$

MEMBER'S COPY

$$x = Y/Z = -570/122 = -285/61.$$

Unnormalised Pythagorean n-tuples are

$$-\frac{224}{61}, -\frac{346}{61}, -\frac{163}{61}, -\frac{407}{61}, -\frac{102}{61}, -\frac{468}{61}, -\frac{41}{61}, -\frac{529}{61}, \frac{20}{61}, -\frac{590}{61}, \frac{81}{61}, -\frac{651}{61}, \frac{142}{61}, -\frac{712}{61}, \frac{203}{61}, -\frac{773}{61}, \frac{264}{61}, -\frac{1119}{61}, \frac{40}{61}, -\frac{2035}{61}.$$

Normalised Pythagorean n-tuples where $n = 20$, therefore, satisfy the equation,

$$\begin{aligned} &224^2 + 346^2 + 163^2 + 407^2 + 102^2 + 468^2 + 41^2 + 529^2 + 20^2 + 590^2 \\ &+ 81^2 + 651^2 + 142^2 + 712^2 + 203^2 + 773^2 + 264^2 + 1119^2 + 40^2 \\ &= 2035^2. \end{aligned} \tag{2.18}$$

By putting

$a_{11} = 1$, $a_{12} = -1$, $a_{13} = 2$, $a_{14} = -2$, $a_{15} = 3$, $a_{16} = -3$, $a_{17} = 4$, $a_{18} = -4$, $a_{19} = 5$, $a_{110} = -5$, $a_{111} = 6$, $a_{112} = -6$, $a_{113} = 7$, $a_{114} = -7$, $a_{115} = 8$, $a_{116} = -8$, $a_{117} = 9, a_{21} = -9$, $a_{22} = 10$ and $a = 11$

in equation (2.17), $x = \frac{549}{88}$ and Pythagorean n-tuples for $n = 20$ are

$$\begin{aligned} &637^2 + 461^2 + 725^2 + 373^2 + 813^2 + 285^2 + 901^2 + 197^2 + 989^2 + 109^2 \\ &+ 1077^2 + 21^2 + 1165^2 + 67^2 + 1253^2 + 155^2 + 1341^2 + 306^2 + 1978^2 \\ &= 3713^2. \end{aligned}$$

By putting different values of a_{11}, a_{12} , a_{13} , $\ldots\ldots.a$, different sets of n-tuples are populated.

In the Table 2, one set of n-tuples each for n varying from 3 to 30 are given using equation (2.17) and corresponding seed Pythagorean n-tuples are given in Table 1. For the sake of brevity, a set of n-tuples for each value of n that varies from 3 to 30 has not been calculated again in this paper. Also care has been exercised in choice of values of a_{1x_1}etc. that denominator of equation (2.17) does not turn out to be zero. It is explicit from the Table 2 that no term of any n-tuples repeats and also does not have a common factor. But at the same time, it is submitted that repeating terms of n-tuples can be had by choice of values of a_{1x_1}etc. if our requirement demands.

MEMBER'S COPY

Table 2: N-Tuples when n varies from 3 to 30

n	Values of a, $(a_{11},\ a_{12},\dots,a_{1x_1})$, $(a_{21},\ a_{22},\dots,a_{2x_2})$, $(a_{31},\ a_{32},\dots,a_{3x_3})\dots$ $(a_{k1},\ a_{k2},\dots,a_{kx_k})$	$m+p$	x	Normalised n-tuples
3	$a=3$, $a_{31}=1$, $a_{41}=-2$	$2+3$	$-\frac{1}{10}$	$25^2=7^2+24^2$
4	$a=4$, $a_{11}=-1$, $a_{21}=-2$, $a_{22}=3$.	$2+1$	$-\frac{1}{11}$	$41^2=12^2+24^2+31^2$
6	$a=-3$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{21}=-2$, $a_{31}=3$.	$3+1$	$-\frac{5}{19}$	$77^2=14^2+24^2+33^3+48^2+42^2$
7	$a=4$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{21}=-3$	$3+0$	$\frac{2}{5}$	$26^2=7^2+3^2+12^2+8^2+17^2+11^2$
8	$a=-4$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{21}=-3$, $a_{41}=4$	$3+2$	$-\frac{14}{33}$	$202^2=19^2+47^2+52^2+80^2+85^2+127^2+76^2$
9	$a=5$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{31}=-4$	$3+1$	$\frac{5}{8}$	$60^2=13^2+3^2+21^2+11^2+29^2+19^2+37^2+17^2$
11	$a=6$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{18}=-4$, $a_{21}=5$, $a_{22}=-5$	$4+0$	$\frac{37}{24}$	$292^2=61^2+13^2+85^2+11^2+109^2+35^2+133^2+59^2+194^2+46^2$
12	$a=-6$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{18}=-4$, $a_{21}=5$, $a_{22}=-5$, $a_{31}=6$	$4+1$	$-\frac{55}{48}$	$563^2=7^2+103^2+41^2+151^2+89^2+199^2+137^2+247^2+130^2+350^2+123^2$
13	$a=7$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{18}=-4$, $a_{21}=5$, $a_{22}=-5$, $a_{23}=6$, $a_{41}=-6$	$4+2$	$\frac{133}{108}$	$1554^2=241^2+25^2+349^2+83^2+457^2+191^2+565^2+299^2+806^2+274^2+914^2+116^2$

MEMBER'S COPY

14	$a=-7$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{18}=-4$, $a_{19}=5$, $a_{110}=-5$, $a_{111}=6$, $a_{112}=-6$, $a_{21}=7$	$4+0$	$-\frac{13}{6}$	$84^2=7^2+19^2$ $+1^2+25^2$ $+5^2+31^2+11^2$ $+37^2+17^2$ $+43^2+23^2$ $+49^2+16^2$
15	$a=9$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{18}=-4$, $a_{19}=5$, $a_{110}=-5$, $a_{21}=6$, $a_{22}=-6$, $a_{31}=7$, $a_{32}=-7$.	$4+2$	$\frac{199}{108}$	$2166^2=91^2+307^2$ $+17^2+415^2$ $+125^2+523^2$ $+233^2+631^2$ $+341^2+739^2$ $+250^2+1046^2$ $+139^2+1353^2$
20	$a=11, a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{18}=-4$, $a_{19}=5$, $a_{110}=-5$, $a_{111}=6$, $a_{112}=-6$, $a_{113}=7$, $a_{114}=-7$, $a_{115}=8$, $a_{116}=-8$, $a_{117}=9$, $a_{21}=-9$, $a_{22}=10$.	$5+0$	$-\frac{285}{61}$	$2035^2=224^2+346^2$ $+163^2+407^2$ $+102^2+468^2$ $+41^2+529^2$ $+20^2+590^2$ $+81^2+651^2$ $+142^2+712^2$ $+203^2+773^2$ $+264^2+1119^2$ $+40^2$
30	$a=-15$, $a_{11}=1$, $a_{12}=-1$, $a_{13}=2$, $a_{14}=-2$, $a_{15}=3$, $a_{16}=-3$, $a_{17}=4$, $a_{18}=-4$, $a_{19}=5$, $a_{110}=-5$, $a_{111}=6$, $a_{112}=-6$, $a_{113}=7$, $a_{114}=-7$, $a_{115}=8$, $a_{116}=-8$, $a_{117}=9$, $a_{118}=-9$, $a_{119}=10$, $a_{120}=-10$, $a_{121}=11$, $a_{122}=-11$, $a_{123}=12$, $a_{124}=-12$, $a_{21}=13$, $a_{22}=-13$, $a_{23}=14$, $a_{24}=-14$, $a_{31}=15$,	$6+1$	$-\frac{203}{30}$	$1871^2=173^2+233^2$ $+143^2+263^2$ $+113^2+293^2$ $+83^2+323^2$ $+53^2+353^2$ $+23^2+383^2$ $+7^2+413^2$ $+37^2+443^2$ $+67^2+473^2$ $+97^2+503^2$ $+127^2+533^2$ $+157^2+563^2$ $+16^2+796^2$ $+14^2+826^2$ $+159^2$

MEMBER'S COPY

3. RESULTS AND CONCLUSIONS

Pythagorean n-tuples $x_1^2 + x_2^2 + x_3^2 + \dots x_{n-1}^2 = x_n^2$ can always be expressed as a quadratic equation in x as

$$x^2 \left\{ \left(a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2\right) - a_n^2 \right\}$$
$$+ 2x \left\{ (a_1 \cdot A_1 + a_2 \cdot A_2 + a_3 \cdot A_3 + \dots a_{n-1} \cdot A_{n-1}) - a_n \cdot A_n \right\}$$
$$+ \left(A_1^2 + A_2^2 + A_3^2 + \dots A_{n-1}^2 - A_n^2\right)$$
$$= 0$$

where

$$x_1 = (a_1 \cdot x + A_1), \quad x_2 = (a_2 \cdot x + A_2), \quad x_3 = (a_3 \cdot x + A_3), \dots,$$
$$x_n = (a_n \cdot x + A_n).$$

This quadratic equation can be written as

$$K \cdot x^2 + M \cdot x + L = 0$$

and its roots as

$$\frac{-M \pm \sqrt{M^2 - 4K \cdot L}}{2K},$$

where

$$\begin{aligned} K &= \left\{ \left(a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2\right) - a_n^2 \right\}, \\ M &= 2x \left\{ (a_1 \cdot A_1 + a_2 \cdot A_2 + a_3 \cdot A_3 + \dots a_{n-1} \cdot A_{n-1}) - a_n \cdot A_n \right\} \\ L &= \left(A_1^2 + A_2^2 + A_3^2 + \dots A_{n-1}^2\right) - A_n^2 \end{aligned}$$

For roots to be rational requires choice of $a_1, a_2, a_3, \dots a_n$ and $A_1, A_2, A_3 \dots A_n$ which are rational quantities such that part under square root $M^2 - 4K \cdot L$ must be a perfect square. It may be cumbersome process to find such values of a's and A's. This difficulty is obviated if the quadratic is transformed into a linear equation, that can only be done if coefficient of x^2 which is K must be zero, or $\left\{ \left(a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2\right) - a_n^2 \right\}$ must equal to zero or $a_1, a_2, a_3, \dots a_n$ are so chosen that these are Pythagorean n-tuple. Quadratic equation then transforms into linear equation $x = -L/M$ where L and M have values as already stated. It is submitted different rational values of $A's$ and $a's$ will give different rational values of x. Hence different sets of n-tuples

$$(a_1 \cdot x + A_1), (a_2 \cdot x + A_2), (a_3 \cdot x + A_3), \dots, (a_n \cdot x + A_n)$$

are generated. It is essential that there must be at least one set of known seed Pythagorean n-tuple $\left\{ \left(a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2\right) = a_n^2 \right\}$ to populate further sets of Pythagorean n-tuples. It is also difficult to remember even a single set of n-tuples for all n varying 3 onwards. To tide over this difficulty, a formula has been devised to generate seed or known n-tuple.

$$(m+p)^2 = (x_1).(1^2) + x_2.(2^2) + (x_3).(3^2) + (x_4).(4^2) + \dots + (x_k).(k^2)$$

MEMBER'S COPY

where $x_1 = \{(n-1) - (x_2 + x_3 + x_4 + \cdots + x_k)\}$, it can have any integer value including 0, $(x_2 + x_3 + x_4 + \cdots + x_k) \leq n-1$, m is a positive integer such that $(n-1)$ lies between $(m-1)^2$ and m^2, p is a positive integer which can have any value $0,\ 1,\ 2, \ldots,$ so on, n is number of terms of n-tuples to be generated and $x_2,\ x_3,\ x_4, \ldots\ x_k$ are positive integers including 0. Once seed n-tuples are known, further sets of n-tuples can be populated after determining x by relation,

$$
\begin{aligned}
x &= \frac{Y}{Z} \\
\text{where } Y &= a^2 - \Big[\Big(a_{11}^2 + a_{12}^2 + a_{13}^2 + \cdots + a_{1x_1}^2 \Big) \\
&\quad + \Big(a_{21}^2 + a_{22}^2 + a_{23}^2 + \cdots + a_{2x_2}^2 \Big) \\
&\quad + \cdots + \Big(a_{k1}^2 + a_{k2}^2 + a_{k3}^2 + \cdots + a_{kx_k}^2 \Big) \Big] ,
\end{aligned}
$$

$$
\begin{aligned}
Z = 2\Big[& \left(a_{11} + a_{12} + a_{13} + \cdots + a_{1x_1}\right) \\
& + 2\left(a_{21} + a_{22} + a_{23} + \cdots + a_{2x_2}\right) \\
& + \cdots + k\Big(a_{k1} + a_{k2} + a_{k3} + \cdots + a_{kx_k}\Big) - a\,(m+p) \Big]
\end{aligned}
$$

and a, $\{(a_{11},\ a_{12},\ a_{13},\ \ldots a_{1x1}),\ (a_{21},\ a_{22},\ a_{23},\ \ldots a_{2x2}),$ $(a_{31},\ a_{32},\ a_{33}, \ldots a_{3x3}) \ldots (a_{k1},\ a_{k2},\ a_{k3}, \ldots a_{kxk})\}$ are real rational quantities assigned by us. Values of m and p are already known from seed equation. When x is determined, Pythagorean n-tuples given by the following relations are populated by putting the x and normalisation.

$$
\begin{aligned}
&\{(x+a_{11})^2 + (x+a_{12})^2 + (x+a_{13})^2 + \ldots + (x+a_{1x_1})^{\,2}\} \\
&+ \Big\{(2x+a_{21})^2 + (2x+a_{22})^2 + (2x+a_{23})^2 + \cdots + (2x+a_{2x_2})^{\,2}\Big\} \\
&+ \Big\{(3x+a_{31})^2 + (3x+a_{32})^2 + (3x+a_{33})^2 + \cdots + (3x+a_{3x_2})^{\,2}\Big\} \\
&+ \cdots + \Big\{(k\cdot x+a_{k1})^2 + (k\cdot x+a_{k2})^2 + \cdots + (k\cdot x+a_{kx_k})^{\,2}\Big\} \\
&= \{(m+p)\cdot x + a\}^2
\end{aligned}
$$

ACKNOWLEDGEMENTS

We acknowledge the help provided by the wonderful website https://www.desmos.com in calculating the values of tedious exponential terms.

References

[1] Edwin G. Landauer, A method of generating Pythagorean n-tuples, *International Journal of Mathematical Education in Science and Technology*, **10:2**(1979),293-294, DOI:10.1080/0020739790100221.

[2] Gareth A. Jones and J. Mary Jones. *Elementary Number Theory*, Springer, 2005.

[3] L. J. Mordell, *Diophantine Equations*, London: Academic Press, 1969.

MEMBER'S COPY

[4] http://en.wikipedia.org/wiki/Pythagorea-triple
[5] http://en.wikipedia.org/wiki/Pythagorean-quadruple
[6] http://en.wikipedia.org/wiki/Pythagorean-triplePythagorean-n-tuple

Narinder Kumar Wadhawan,
Civil Servant, Indian Administrative Services (Retired),
Chandigarh, India
Email: narinderkw@gmail.com

Priyanka Wadhawan,
Thapar Institute Of Engineering And Technology,
Patiala, India
Email: priyanka.wadhawan@gmail.com

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 67–89

THE SMALLEST ELLIPSE THAT COVERS TWO CIRCLES

JYOTIRMOY SARKAR
(Received : 26 - 02 - 2020 ; Revised : 15 - 04 - 2020)

Abstract. Given two externally-tangent, unequal circles in a plane, what are the lengths of the two axes of the smallest-area ellipse that covers them both? We solve the problem using single-variable calculus. We also pose several exercises and open problems.

1. Genesis of the Problem

When my wife invites guests for dinner, it befalls on me to set the table. On one occasion, she instructed me to set the round table for eight people, making sure that each person will have a dinner plate and a bread plate side-by-side. Very obediently, I set the 16 plates—alternately placing dinner plates and bread plates. When all the guests were seated at the table, each facing a dinner plate, a mild confusion arose as to which bread plate each person should use—the one to the left of their dinner plate, or the one to the right? Fortunately, the confusion evaporated when my wife served a gluten-free bread to one of the guests—who has dietary restriction—putting it on the bread plate to his left. Thereafter, by induction, everyone knew they should use the bread plate to the left of their dinner plate.

These guests were my wife's "Fiber Friends" (a neighborhood ladies sewing and knitting group) and their spouses. During the after-dinner conversation, I proposed that Fiber Friends take on the task of knitting elliptical place mats on which both the dinner plate (26 cm in diameter) and the bread plate (16 cm in diameter) would fit. This would bid goodbye to the confusion we had experienced earlier. Always ready for new challenges,

2010 Mathematics Subject Classification: $52C15$
Key words and phrases: Tangent, normal, in-circle, elliptical function, Heron's formula, intermediate value theorem

© *Indian Mathematical Society, 2020.*

MEMBER'S COPY

Fiber Friends were happy to entertain my proposal; but they demanded that I minimize their time on task by ensuring that the ellipse is as small as possible.

I retreated to my study; put on my mathematician's hat, and scribbled on many a note pad to determine the lengths of the two axes. Below is the result of my discovery. I suggest you drop this paper; and go to discovery mode, . . . after perhaps playing this one guessing game stated in Figure 1. All figures are drawn using the computing environment R.

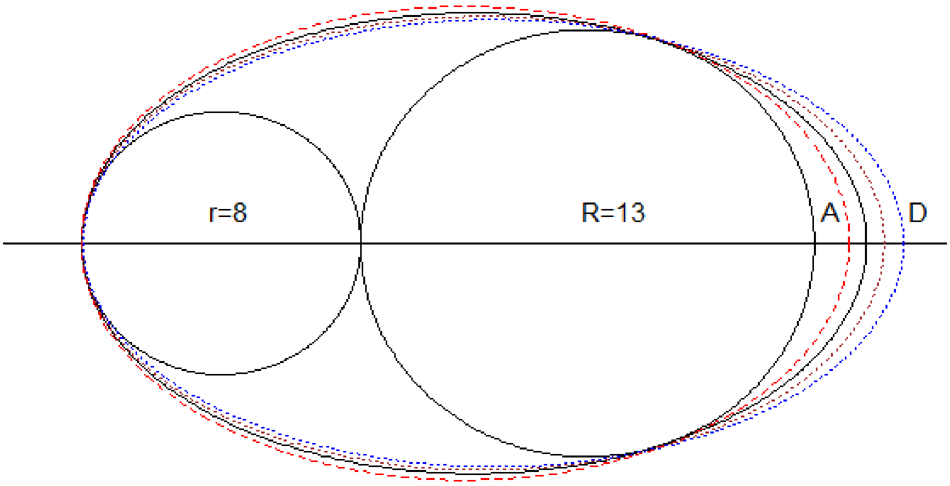


FIGURE 1. Which of the four suggested ellipses, each covering the two externally-tangent, coplanar circles of diameters 26 cm and 16 cm, has the smallest area? Name these ellipses A, B, C, D in increasing (decreasing) order of lengths of the major (minor) axis.

When you return to the paper, you will find in successive sections these seven topics:

(a) Some relevant properties of an ellipse;
(b) The largest circle that fits inside each part of an ellipse when partitioned by a line orthogonal to the major axis (and parallel to the minor axis);
(c) The solution to a simpler problem—one in which the two circles are equal in size;
(d) The solution to the general problem involving unequal circles;
(e) Some exercises solvable with ideas already in this paper;

MEMBER'S COPY

(f) Fitting additional circles inside the smallest-area ellipse; and
(g) Open problems involving several non-overlapping circles.

Problems of minimization and maximization, perhaps under one or more constraints, often require use of calculus, but not always. Personally I am partial to those optimization problems for which Euclidean plane geometry suffices. Here, we have used single-variable calculus. We invite readers to discover a more elementary solution. A good reference for other interesting geometric optimization problems is [3].

2. Properties of an ellipse

In a fixed plane, there are two fixed points (called the foci) at a distance of $2c \geq 0$ from each other. The set of all points in the plane whose distances from the two foci add up to a constant $2a$, with $a > c$, is called an ellipse. The midpoint between the two foci is called the center of the ellipse. Any line segment passing through the center and terminated by the ellipse is called a diameter. The largest diameter of an ellipse passes through the two foci; it is called the major axis; and it is of length $2a$. The shortest diameter of an ellipse is called the minor axis; and it is of length $2b$, where $b^2 = a^2 - c^2$. The two axes are orthogonal to each other. The eccentricity of this ellipse is defined by $e = c/a = \sqrt{1 - b^2/a^2}$. In the special case when eccentricity $e = 0$, we have $c = 0$ and $a = b$; that is, the two foci to coincide (at the center) and the ellipse reduces to a circle of radius $a = b$.

Next, let us look at the analytic expression for an ellipse. The total distance of any point (x, y) on an ellipse $\mathcal{E}$ from its two foci $(\mp c, 0)$ is a constant $2a$. That is,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a,$$

Or equivalently, (after some algebraic simplifications)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.1}$$

describes the equation of an ellipse with foci $(\mp c, 0)$ [hence, center $(0, 0)$], half major axis $a \geq c$ and half minor axis $b = \sqrt{a^2 - c^2} \geq 0$ in horizontal and vertical orientations respectively. Alternatively, given a and b, we can rewrite $c = \sqrt{a^2 - b^2}$. Hence, the two foci are at $(\pm c, 0) = (\pm ea, 0)$. See Figure 2 below.

MEMBER'S COPY

In particular, when $a = b$, (2.1) yields the equation of a circle of radius $r = a = b$. Equation (2.1) also tells us that an ellipse is a vertical compression (by a factor b/a) of a circle of radius a, or a horizontal expansion (by a factor a/b) of a circle of radius b. Consequently, the area of the ellipse $\mathcal{E}$ is $\pi\, ab$.

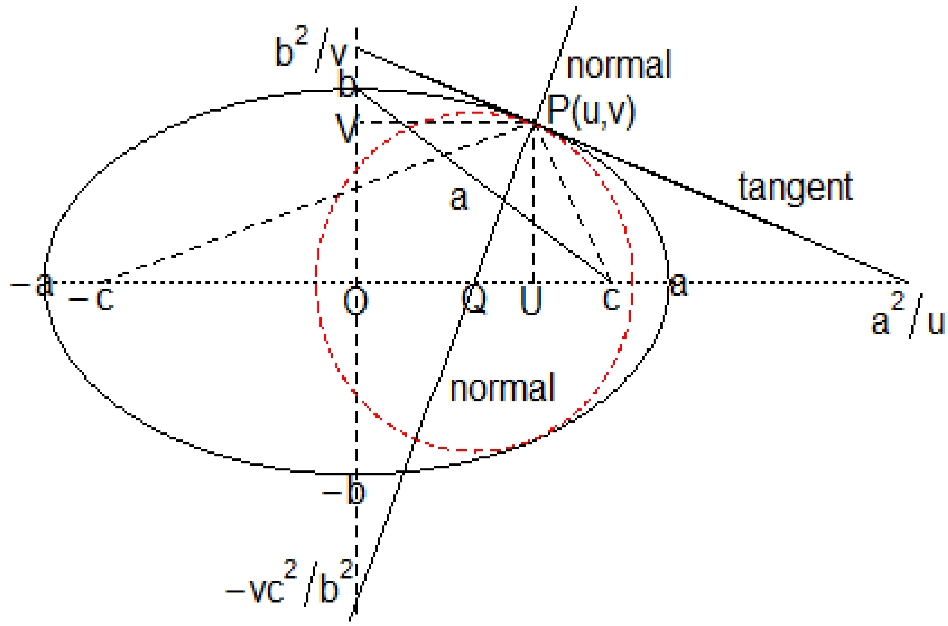


FIGURE 2. In a plane, the collection of all points that are equidistant from two fixed points (called foci) constitutes an ellipse. The tangent and the normal at a point on the ellipse can be identified by their intercepts.

2.1. **The tangent and the normal at a given point on the ellipse.** Differentiating the implicit function (2.1) and rearranging the terms, we note that the slope of the tangent at a point (x, y) on the ellipse $\mathcal{E}$ is

$$\frac{dy}{dx} = -\frac{x/a^2}{y/b^2} = -\frac{xb^2}{ya^2}. \tag{2.2}$$

Given a point $P(u, v)$ on $\mathcal{E}$ (assume, without loss of generality, that P is in the first quadrant), using (2.2), the tangent to the ellipse through P is given by

$$y = v - \frac{ub^2}{va^2}\,(x - u) = \frac{b^2}{v}\left[\frac{v^2}{b^2} - \frac{ux}{a^2} + \frac{u^2}{a^2}\right] = \frac{b^2}{v}\left[1 - \frac{ux}{a^2}\right];$$

MEMBER'S COPY

or equivalently,

$$\frac{ux}{a^2} + \frac{vy}{b^2} = 1. \tag{2.3}$$

The tangent intersects the x-axis at $(a^2/u, 0)$ and the y-axis at $(0, b^2/v)$.

The normal (the line orthogonal to the tangent) to the ellipse through $P(u, v)$ is given by

$$y = v + \frac{va^2}{ub^2}(x-u) = \frac{va^2}{ub^2}\,x - v\left[\frac{a^2}{b^2} - 1\right] = \frac{v}{b^2}\left[\frac{a^2x}{u} - c^2\right];$$

or equivalently,

$$\frac{a^2x}{uc^2} - \frac{b^2y}{vc^2} = 1. \tag{2.4}$$

The normal intersects the x-axis at $(uc^2/a^2, 0) = (ue^2, 0)$ and the y-axis at $(0, -vc^2/b^2)$. Thus, if the normal to the ellipse through P intersects the x-axis at $Q(q, 0)$, then

$$q = uc^2/a^2 = e^2u. \tag{2.5}$$

Thus, the points $U(u, 0)$ and $Q(q, 0)$ are linearly related: As u varies between 0 and a, the corresponding q varies between 0 and e^2a. Also, $QU = u - e^2u = ub^2/a^2$. Moreover, the normal at $P(u, v)$ bisects the angle formed by the lines joining $P(u, v)$ to the two foci.

2.2. **The largest in-circle passing through a point on the ellipse.** Given a point $P(u, v)$ on the ellipse $\mathcal{E}$, the largest circle $\mathcal{C}_u$ passing through P and residing inside $\mathcal{E}$ has a center at $Q(ue^2, 0)$ and a radius QP given by

$$QP = b\sqrt{1 - \frac{e^2u^2}{a^2}} \tag{2.6}$$

since in view of the Pythagorean theorem,

$$\begin{aligned} QP^2 = QU^2 + UP^2 \quad &= \quad \left[\frac{ub^2}{a^2}\right]^2 + v^2 = \frac{u^2b^4}{a^4} + b^2\left[1 - \frac{u^2}{a^2}\right] \\ &= \quad b^2\left[1 - \frac{u^2}{a^2}\left(1 - \frac{b^2}{a^2}\right)\right] = b^2\left[1 - \frac{e^2u^2}{a^2}\right]. \end{aligned}$$

Moreover, $\mathcal{C}_u$ is also internally tangent to $\mathcal{E}$ at $P'(u, -v)$, by vertical reflection symmetry.

MEMBER'S COPY

2.3. **The largest in-circle given its center on the major axis of the ellipse.** Let us invert the process discussed in the previous subsection. Here we want to construct the largest circle with center $Q(q,0)$ that fits inside the ellipse $\mathcal{E}$. By vertical reflection symmetry, the center must be on the major axis. Also, in view of horizontal symmetry, without loss of generality, we assume $q > 0$.

Given any $q \in [0, ec) = [0, e^2 a)$, the largest circle with center $(q,0)$ that fits inside the ellipse $\mathcal{E}$ is internally tangent to ellipse $\mathcal{E}$ at two points (u,v) and $(u,-v)$, where $u = q/e^2 = qa^2/c^2$ and $v = b\sqrt{1-u^2/a^2}$. In view of (2.6), this in-circle has radius r given by

$$r = b\sqrt{1-\frac{q^2}{e^2a^2}} = b\sqrt{1-\frac{q^2}{c^2}};$$

or equivalently, $q^2/c^2 + r^2/b^2 = 1$. Thus, r is an elliptic function of $q \in [0, ec) = [0, e^2 a)$. See the thick curve in Figure 3 to the left of the point Z given by

$$Z(ec, b\sqrt{1-e^2}) = a(e^2, 1-e^2).$$

On the other hand, given any $q \in [ec, a]$, the largest circle with center $(q,0)$ inside the ellipse $\mathcal{E}$ is internally tangent to ellipse $\mathcal{E}$ at only one point, $(a,0)$; and this circle has radius $a-q$, which is a linear function of $q \in [ec, a]$. See the thick line in Figure 3 to the right of the point Z.

Combining the two cases, Figure 3 depicts the radius $r(q)$ of the largest in-circle with center $(q,0)$, as a function of $q \in [0,a]$, given by

$$r(q) = \begin{cases} b\sqrt{1-q^2/c^2} & \text{if } q < ec \\ a-q & \text{if } ec \le q \le a. \end{cases} \tag{2.7}$$

Differentiating (2.7) with respect to q, we have

$$r'(q) = \begin{cases} -qb^2/(rc^2) & \text{if } q < ec \\ -1 & \text{if } ec \le q \le a. \end{cases} \tag{2.8}$$

In particular, $r'(ec-) = -1 = r'(ec+)$. Hence, Z is the unique point of contact between a line of slope -1 (shown by the dashed line) passing through $(a,0)$ and the inner ellipse $q^2/c^2 + r^2/b^2 = 1$ (shown by the dashed curve), which is a horizontal compression of the outer ellipse $\mathcal{E}$ by a factor e.

MEMBER'S COPY

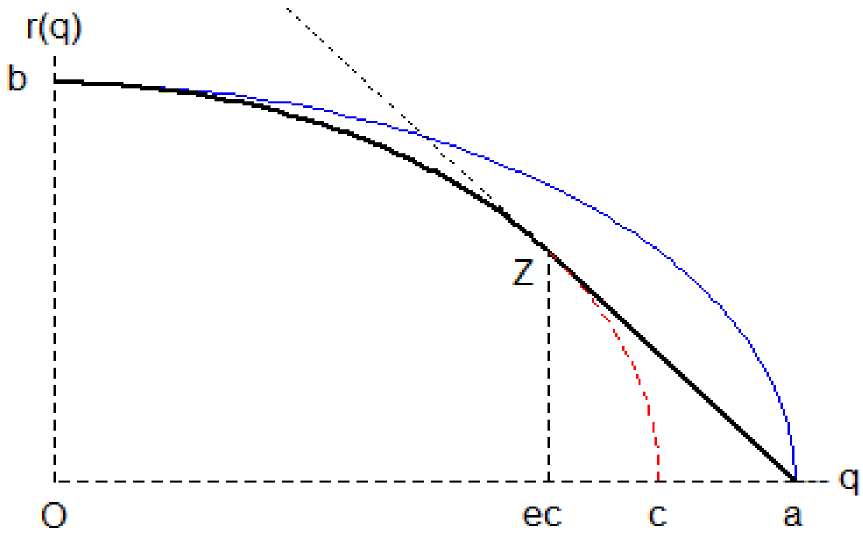


FIGURE 3. The radius $r(q)$ (thick curve) of the largest in-circle with center $(q, 0)$, is an elliptic function of $q \in [0, ec)$ and a linear function of $q \in [ec, 1]$.

3. THE LARGEST IN-CIRCLES ON THE TWO SIDES OF A VERTICAL PARTITIONING LINE

A vertical line (that is, a line orthogonal to the major axis and parallel to the minor axis) $x = l$, where $-a \leq l \leq a$, divides the ellipse $\mathcal{E}$ into two (unequal) parts. Let us fit the largest in-circles within the two parts of the ellipse on the two sides of the dividing line. See Figure 4.

Invoking vertical reflection symmetry, we note that the centers of the in-circles are on the major axis. Let the in-center in the part-ellipse on the right side of the dividing line be $Q(q, 0)$. We can find the in-radius in one of two ways: (a) we can find the distance between Q and $x = l$; that is, the in-radius is $q - l$; and (b) we can use (2.7). Equating these two expressions for the in-radius, we can solve for both the in-center and the in-radius of the largest circle that fits inside the right part-ellipse in the following three cases.

Case 1: $(a - 2b^2/a < l \leq a)$. The in-radius is $r = (a - l)/2 = q - l$; and the in-center is $q = (a + l)/2$. Thus, $ec < q \leq a$.

MEMBER'S COPY

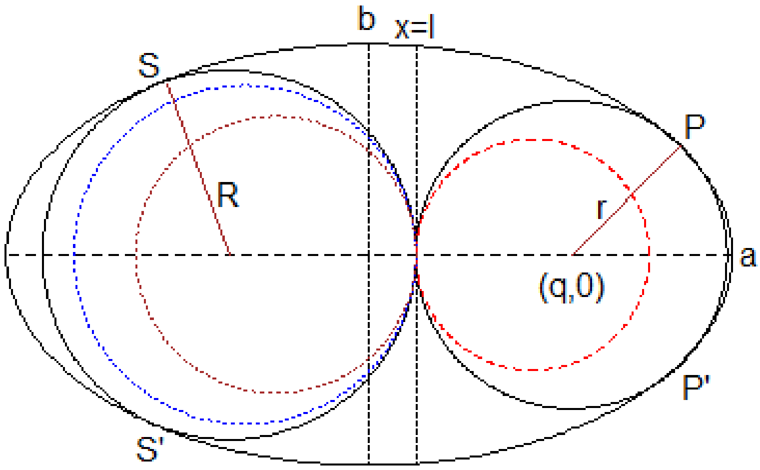


FIGURE 4. The largest in-circle within the right-part of an ellipse partitioned by a vertical line $x = l$ has center $(q, 0)$ and radius $r = q-l$. Likewise, for the left-part of the ellipse.

Case 2: $(-b < l \leq a - 2b^2/a)$. Starting from (2.7), we obtain the following chain of equations:

$$\begin{aligned} b\sqrt{1 - q^2/c^2} &= q - l, \\ \left(\frac{q}{c}\right)^2 + \left(\frac{q-l}{b}\right)^2 &= 1, \\ b^2q^2 + c^2(q-l)^2 &= b^2c^2, \\ a^2q^2 - 2c^2lq - c^2(b^2 - l^2) &= 0. \end{aligned} \tag{3.1}$$

Solving (3.1) for q, we obtain the center of the in-circle on the right part-ellipse. Since we must also have $q \geq \max(0, l)$, the only admissible solution is

$$q = e^2 l + eb\sqrt{1 - l^2/a^2}. \tag{3.2}$$

Thus, $0 < q \leq ec$. Moreover, the radius of the in-circle on the right part-ellipse is

$$r = q - l = -(1 - e^2)l + eb\sqrt{1 - l^2/a^2}. \tag{3.3}$$

Case 3: $(l \leq -b)$. The in-center is $(0, 0)$ and the in-radius is b. Thus, $q = 0$.

MEMBER'S COPY

Likewise, the center of the largest in-circle in the left-part of the ellipse can be obtained by reflecting the entire diagram on the y-axis, which causes the dividing line to become $x = -l$, and then fitting the largest in-circle to the right of $x = -l$. Again, let the in-center be $(q, 0)$. Note that irrespective of the value of l, positive or negative, we have $q \geq 0$. In fact, if $l \geq b$ (or $-l \leq -b$), then $q = 0$ and $R = b$. But if $-a + 2b^2/a < l < b$, then $q > 0$, and we can find the in-radius using (2.7); but now the in-radius equals $q+l$. Hence, simply replacing l by $-l$ in (3.2), we obtain q as

$$q = -e^2 l + eb\sqrt{1 - l^2/a^2}. \tag{3.4}$$

and replacing l by $-l$ in (3.3), we obtain the corresponding in-radius R as

$$R = q + l = (1 - e^2)l + eb\sqrt{1 - l^2/a^2}. \tag{3.5}$$

Note that when $l = 0$, by symmetry, we have $R = r$; and when $l > 0$, we have $R > r$. Finally, if $-a < l < -a + 2b^2/a$, the in-center is $((-a+l)/2, 0)$ and the in-radius is $(a + l)/2$.

4. The smallest ellipse covering two equal circles

In particular, if $l = 0$, then for the part-ellipse on the right of $x = 0$, we obtain from (3.1) the in-center $(q, 0)$, where $q = bc/a = be$, and from (3.3) the in-radius $r = be$. The same is true (by invoking horizontal reflection symmetry) for the part-ellipse on the left side of $x = 0$. Thus, the two in-circles are equal in radius; that is, $r = R = be$.

Conversely, any ellipse that covers the two externally tangent equal circles of radii $r = R$ must necessarily satisfy $r = be = (b/a)\sqrt{a^2 - b^2}$; or equivalently, $a^2 r^2 = b^2(a^2 - b^2)$. Hence,

$$a^2 = \frac{b^4}{b^2 - r^2}. \tag{4.1}$$

Our goal is to minimize the area πab of the covering ellipse. Equivalently, we must minimize $a^2 b^2$, subject to (4.1). Writing $B = b^2$, we must solve a single-variable minimization problem:

$$\underset{B > r^2}{\text{Minimize}} \ \frac{B^3}{B - r^2}. \tag{4.2}$$

Differentiating with respect to B the objective function $g(B) = B^3/(B - r^2)$ in (4.2), we get $g'(B) = 3B^2/(B - r^2) - B^3/(B - r^2)^2$. The first-order condition for minimization is $g'(B) = 0$; or equivalently, $B^2[3(B - r^2) - B] = 0$. Moreover, since $B > r^2$, the only admissible critical value is

MEMBER'S COPY

$B = 3r^2/2$. The second derivative of the objective function is $g''(B) = 6B/(B - r^2) - 6B^2/(B - r^2)^2 + 2B^3/(B - r^2)^3$, which when evaluated at the critical value yields $g''(3r^2/2) = 18 > 0$, satisfying the second-order condition for minimization. Hence, the optimal value of the half-minor axis is $b = \sqrt{B} = \sqrt{3/2}\, r$. Thereafter, using (4.1), the optimal value of the half-major axis is $a = (3/\sqrt{2})\, r = \sqrt{3}\, b$. Figure 5 shows the smallest-area ellipse that covers two externally-tangent, coplanar unit circles.

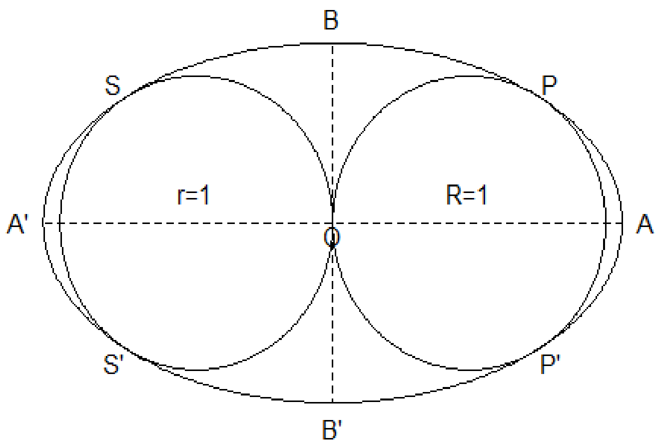


FIGURE 5. The smallest-area ellipse that covers two externally-tangent, coplanar unit circles has major axis $AA' = 3\sqrt{2}$, minor axis $BB' = \sqrt{6}$, and area $\pi\, 3\sqrt{3}/2$.

Remark 4.1. The smallest-area ellipse that covers two equal, coplanar circles satisfies the following properties: Triangles ABB' and $A'BB'$, constructed by joining each endpoint of the major axis to both endpoints of the minor axis BB' are equilateral, of side length $\sqrt{6}$. The two points of tangency between the ellipse and either circle, together with the origin O, form an equilateral triangle, $PP'O$ or $SS'O$, of side length $\sqrt{3}$. The four points of tangency between the ellipse and the two circles—P, P', S', S—form a rectangle of length 3 and width $\sqrt{3}$ respectively.

To learn about the smallest-area ellipse that covers three or more non-overlapping, coplanar equal circles, see [4].

MEMBER'S COPY

5. The smallest ellipse covering two unequal circles

Let us continue the discussion, which we started in Section 3, for the more general situation when $l \neq 0$. Indeed now, (3.1) is a genuine quadratic equation. We study three mutually exclusive and exhaustive cases:

Case 1 ($l \geq (2e^2-1)a = (c^2-b^2)/a$). The largest circle inside the part-ellipse on the right side of $x = l$ has center at $(q, 0)$, with $q \geq e^2 a$, and radius $r = (a-l)/2$.

Case 2 ($-b \leq l \leq (c^2-b^2)/a$). From (3.3), we have

$$\begin{aligned} r &= \frac{1}{2a}\left[-2b^2 l + \sqrt{4b^4 l^2 + 4a^2 b^2 (a^2 - b^2 - l^2)}\right] \\ &= -\frac{b^2}{a^2} l + \frac{b}{a^2}\sqrt{(a^2-b^2)(a^2-l^2)}. \end{aligned} \tag{5.1}$$

Case 3 ($l < -b$). The largest circle inside the part-ellipse on the right side of $x = l$ has center at $(0, 0)$ and radius $r = b$.

Likewise, if $l > b$, then the largest circle inside the part-ellipse on the left side of $x = l$ has center at $(0, 0)$ and radius $R = b$. If $(c^2-b^2)/a < l \leq b$, then replacing l by $-l$ in (5.1), we get

$$R = \frac{b^2}{a^2} l + \frac{b}{a^2}\sqrt{(a^2-b^2)(a^2-l^2)}. \tag{5.2}$$

Finally, if $l < (c^2-b^2)/a$, then $R = (a+l)/2$.

Adding (5.1) and (5.2), we get $R + r = (2b/a^2)\ \sqrt{(a^2-b^2)(a^2-l^2)}$; and subtracting (5.1) from (5.2), we get $R - r = (2b^2/a^2)\ l$. Then

$$\left(\frac{R+r}{2}\right)^2 \frac{a^4}{b^2} = (a^2-b^2)(a^2-l^2) = (a^2-b^2)\left[a^2 - \left(\frac{R-r}{2}\right)^2 \frac{a^4}{b^4}\right];$$

or equivalently,

$$\frac{\left(\frac{R+r}{2}\right)^2}{a^2-b^2} + \frac{\left(\frac{R-r}{2}\right)^2}{b^2} = \frac{b^2}{a^2}. \tag{5.3}$$

So far, we have fitted the largest-area in-circles within each part-ellipse of a given ellipse partitioned by a vertical line. Now let us reverse the process: Given two externally-tangent circles, we want to fit the smallest-area ellipse that covers them both. Henceforth, without loss of generality, let us assume $r = 1$ and $R > 1$. [If $r \neq 1$, we simply replace R by R/r; work out all quantities, and then at the end we replace them all by their r-multiples.]

MEMBER'S COPY

Suppose that there are two externally-tangent circles, with radii $r = 1$ and $R > 1$, whose diameters through the point of tangency is horizontal in orientation. The circle $\mathcal{C}_{R+1}$ with diameter given by the concatenation of the diameters of the two circles (and hence with radius $(R+1)$) covers them both. Our objective is to find the smallest-area ellipse that covers both circles. So, starting from $\mathcal{C}_{R+1}$, we reduce the vertical minor axis at the cost of possibly increasing the horizontal major axis, always ensuring that the modified ellipse barely covers the two given circles. In this continuum of choices, the ellipse at the other extreme, denoted by $\mathcal{E}_R$, has half-minor axis $b = R$. (The subscript of an ellipse denotes the length of the half-minor axis.) The corresponding half major-axis turns out to be $a = R + 2$ when $R \geq 2$. Moreover, if R is sufficiently large, say $R \geq \bar{R}$ (we will discover the threshold $\bar{R}$ in Subsection 5.4), then $\mathcal{E}_R$ is indeed the smallest-area ellipse that covers both the circles. On the other hand, if $R < \bar{R}$, then $\mathcal{E}_R$, though it covers both the circles, is not the smallest-area ellipse that covers both the circles. In this case, let us denote the smallest-area ellipse by $\mathcal{E}_*$, and its half-major and half-minor axes by a_* and b_* respectively. Note that $R < b_* < R + 1 < a_*$. But what are their exact values? We discover them using single-variable calculus.

Every ellipse, with half-major axis a and half-minor axis b, that covers both of the given circles must satisfy (5.3). To simplify the algebra, let us substitute

$$A = a^2;\ B = b^2;\ K = [(R-1)/2]^2;\ L = [(R+1)/2]^2.$$

Note that $L - K = R$. Then (5.3) becomes

$$\frac{L}{A-B} + \frac{K}{B} = \frac{B}{A};$$

or equivalently, $LBA + K(A-B)A = B^2(A-B)$; or $KA^2 - (B - L + K)BA + B^3 = 0$; or

$$\left(\frac{R-1}{2}\right)^2 A^2 - (B-R)BA + B^3 = 0. \tag{5.4}$$

Solving this quadratic equation in A as a function of B, and rejecting the larger solution (since our goal is to minimize the area πab of the covering ellipse, or equivalently, to minimize AB, subject to (5.4)), we only consider

MEMBER'S COPY

the smaller root

$$A = \frac{2B}{(R-1)^2}\ \left\{B - R - \sqrt{D}\right\} \tag{5.5}$$

where

$$D = (B-R)^2 - (R-1)^2 B = B^2 - (R^2+1)B + R^2 = (B-1)(B-R^2). \tag{5.6}$$

Thus, we face a single-variable minimization problem:

$$\underset{B > R^2}{\text{Minimize}}\ \frac{2B^2}{(R-1)^2}\ \left\{B - R - \sqrt{D}\right\}. \tag{5.7}$$

Differentiating the objective function in (5.7) with respect to B and equating the derivative to 0, we obtain the solution (leaving the details to interested readers)

$$B_{\text{opt}} = \frac{1}{48}\left\{\left(25R^2 + 2R + 25\right) + 5(R+1)\sqrt{25(R-1)^2 + 4R}\right\}. \tag{5.8}$$

Furthermore, we have the following bounds on B_{opt}

$$R^2 + \frac{R(R+1)}{24} < B_{\text{opt}} < R^2 + \frac{(R+1)^2}{24} + 1. \tag{5.9}$$

Indeed, B_{opt} is closer to the left bound than to the right bound in (5.9) as seen in Figure 6 where we graph the optimal half-minor axis $b_* = \sqrt{B_{\text{opt}}}$ as a function of $R \in [1, 4)$, together with its associated bounds obtained by taking square-roots in (5.9).

Substituting B_{opt}, given in (5.8), into (5.5), we obtain the optimal value A_{opt}. Thereafter, we compute the half major axis of the smallest-area ellipse $\mathcal{E}_*$ as $a_* = \sqrt{A_{\text{opt}}}$, and its area as $\pi\, a_* b_*$.

Remark 5.1. In (5.8), taking limit as $R \to 1+$, we see that $B_{\text{opt}} \to 3/2$. Thus, the result of Section 4 can be recovered from the results of this section.

Remark 5.2. For $1 < R < 2$, the smallest-area ellipse $\mathcal{E}_*$ is tangential to each of the given circles at two points vertically symmetric about the major axis. For $R \geq 2$, $\mathcal{E}_*$ is tangential to the larger circle at two symmetric points; and it is tangential to the smaller circle at only one point—the point farthest from the larger circle. The center of $\mathcal{E}_*$ coincides with the center of the larger circle only when R is sufficiently large, say $R \geq \bar{R}$, which we shall discover in Subsection 5.4.

MEMBER'S COPY

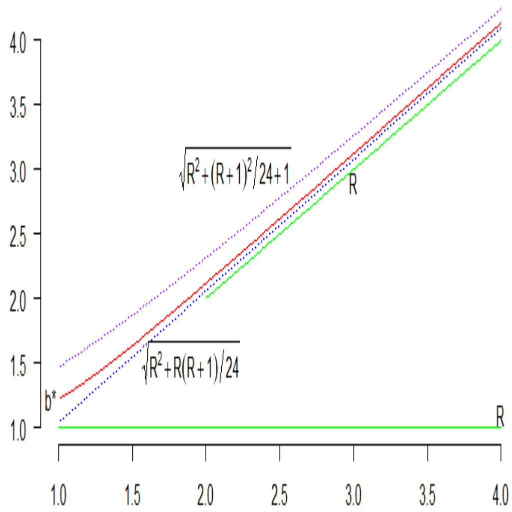


FIGURE 6. The length of the half-minor axis b_*, of the smallest-area ellipse covering two externally-tangent, coplanar circles with radii $1 : R$, as a function of $R \in [1, 4)$, and the associated bounds obtained from (5.9).

In the next few subsections, we illustrate the theory developed above by computing the dimensions of the smallest-area ellipse that covers two externally-tangent circles of given radii r and R, and depicting that ellipse. In Subsection 5.4, we discover the threshold $\bar{R}$.

5.1. **The smallest ellipse covering two externally-tangent circles of radii** $1 : 2$**.** Specializing to $R = 2$, from (5.8) we get $B_{\text{opt}} = (43 + 5\sqrt{33})/16 = 4.482676$. Thereafter, from (5.5), we obtain $A_{\text{opt}} = (61 + 19\sqrt{33})/16 = 10.63417$. Hence, the smallest-area ellipse that covers two circles of radii 1 and 2 has half-major axis $a = \sqrt{A} = 3.2610$ and half-minor axis $b = \sqrt{B} = 2.1172$, shown in Figure 7. Its area is $\pi ab = 6.9042\pi$.

5.2. **The smallest ellipse covering two externally-tangent circles of radii** $1 : 3$**.** Specializing to $R = 3$, from (5.8) we get $B_{\text{opt}} = (16+5\sqrt{7})/3 = 9.742919$. Thereafter, from (5.5), we obtain $A_{\text{opt}} = (86 + 37\sqrt{7})/9 = 20.43253$. Hence, the smallest-area ellipse that covers two circles of radii 1 and 2 respectively has half-major axis $a = \sqrt{A} = 4.5202$ and half-minor axis $b = \sqrt{B} = 3.1214$, shown in Figure 8. Its area is $\pi ab = 14.1094\pi$.

MEMBER'S COPY

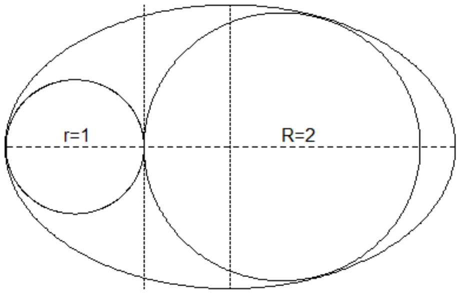


FIGURE 7. The smallest-area ellipse covering two externally-tangent, coplanar circles with radii 1 : 2 touches the smaller circle at one point.

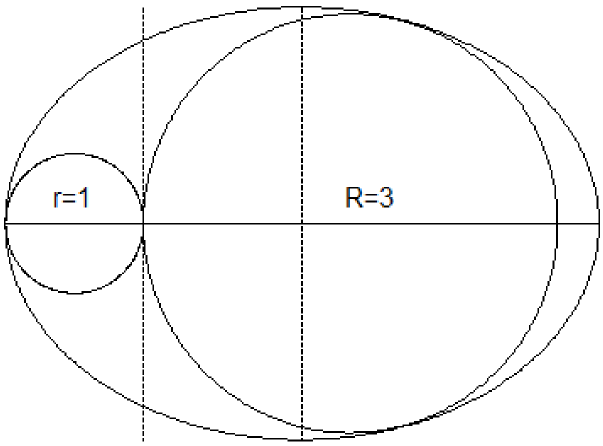


FIGURE 8. The smallest-area ellipse covering two externally-tangent, coplanar circles with radii 1 : 3 touches the smaller circle at one point.

5.3. **The smallest ellipse covering two externally-tangent circles of radii** 8 : 13**.** Since $r \neq 1$, we simply replace R by R/r=13/8=1.625; work out all quantities, and then replace them by their r-multiples. Specializing to $R = 13/8 = 1.625$, from (5.8) we get $B_{\text{opt}} = 3.0667$. Thereafter, from (5.5), we obtain $A_{\text{opt}} = 7.9029$. Hence, the smallest-area ellipse that covers two circles of radii 8 cm and 13 cm respectively has half-major axis $a = 8\sqrt{A_{\text{opt}}} = 22.48966$ cm and half-minor axis $b = 8\sqrt{B_{\text{opt}}} = 14.0095$ cm, shown in Figure 9. Its area is $\pi ab = 315.07\pi$.

MEMBER'S COPY

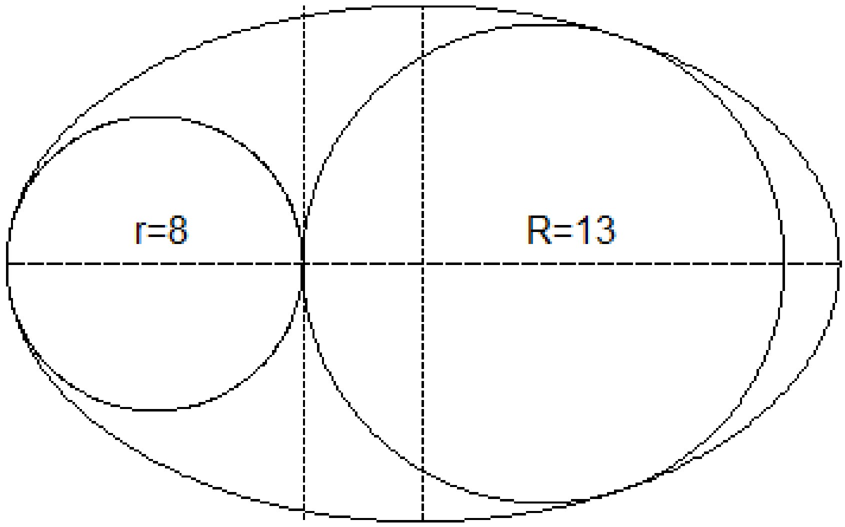


FIGURE 9. The smallest-area ellipse covering two externally-tangent, coplanar circles with diameters 16 cm and 26 cm has an approximate major axis 45 cm, an approximate minor axis 28 cm, and touches each of the given circles at two distinct points.

We are now ready to answer the quiz we had posed in Figure 1. Indeed, Ellipses A–D have half axes of lengths $(a, b) = (22.0, 14.4), (22.5, 14.01), (23.0, 13.8), (23.5, 13.6)$ respectively, with associated products $ab = 316.8, 315.2, 317.4, 319.6$ (correct to one decimal place). Hence, Ellipse B is the smallest-area ellipse (among A–D) that covers both plates. To keep matters simple, I recommended that Fiber Friends knit elliptical place mats with major axis 45 cm and minor axis 28 cm (instead of 44.979 cm and 28.019 cm respectively).

5.4. **The threshold $\bar{R}$ above which $\mathcal{E}_R$ is optimal.** We mentioned earlier that when R is sufficiently large, say $R \geq \bar{R}$, then the smallest-area ellipse that covers two externally-tangent, coplanar circles of radii 1 and R is $\mathcal{E}_R$, whose half-axes lengths are $a = (R+2)$ and $b = R$, and whose center is exactly at the center of the larger circle. What is this threshold $\bar{R}$?

We use the methodology of this section to compute the optimal half-axes lengths for some chosen values of $R \in [1.2, 4.5]$. These are depicted

MEMBER'S COPY

in Figure 10 and tabulated in Table 1. Extensive computations (not documented here) show that $\bar{R} = 3.970797\cdots$.

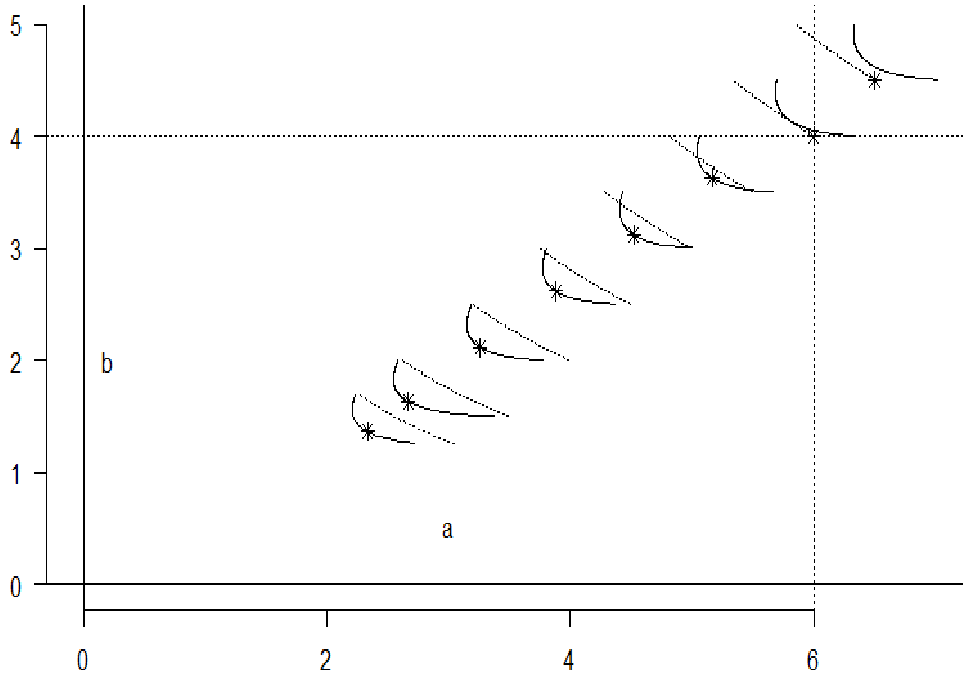


FIGURE 10. The symbol $*$ plots (a, b), the half-axes lengths of $\mathcal{E}_*$, for some choices of $R = 1.2, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5$ (going from bottom-left to top-right). The solid curve shows $a = \sqrt{A}$ as a function of $b = \sqrt{B}$ via (5.5) as b ranges over $(R, R + 0.5)$. The dotted curve shows the implicit function $ab = (R + 2)R$. When $R \geq \bar{R} \approx 3.9708$, the optimal ellipse is $\mathcal{E}_R$.

6. Exercises

We pose some exercises that readers can solve using no more advanced mathematics than those used in this paper.

Exercise 6.1. In Table 1, the ratio of optimal half-axes a_*/b_* decreases from 1.73 to 1.45 as R increases from 1 to 3. By the intermediate value theorem, there exists a unique number $R = R_1$ for which $a_*/b_* = R_1$. Discover R_1 [Answer: 1.60874, which is slightly smaller than the golden

MEMBER'S COPY

TABLE 1. The lengths of the half-axes of the smallest-area ellipse that covers two externally-tangent, coplanar circles of radii 1 and R

R	b_*	a_*	a_*/b_*	a_*b_*	$R(R+2)$
1.0	1.224745	2.121320	1.732051	2.598076	3.00
1.2	1.368481	2.334289	1.705752	3.194430	3.84
1.5	1.632993	2.666667	1.632993	4.354648	5.25
1.625	1.751188	2.811207	1.605314	4.922952	5.89
2.0	2.117233	3.261007	1.540220	6.904312	8.00
2.5	2.616811	3.883837	1.484187	10.16327	11.25
3.0	3.121365	4.520236	1.448160	14.10931	15.00
3.5	3.628101	5.163846	1.423291	18.73496	19.25
3.9	4.034341	5.681873	1.408377	22.92261	23.01
$R \geq 3.9708$	R	$R+2$	$1+2/R$	$R(R+2)$	$R(R+2)$
3.9708	3.9708	5.9708	1.503677	23.70885	23.70885
4.0	4.0	6.0	1.5	24.00	24.00
4.5	4.5	6.5	1.444444	29.25	29.25

ratio $(\sqrt{5}+1)/2 = 1.61803$. See [5] to learn about the definition and some fascinating properties of the golden ratio.]

Exercise 6.2. Assuming $r = 1$, what is the smallest value of $R > 1$ such that the smallest-area ellipse $\mathcal{E}_*$ covering the two externally-tangent circles is tangential to the smaller circle at only one point? [Answer: $R = 1.5$, in which case $a_* = 8/3 = b_*^2$, and $a_*/b_* = 1.633$.]

Exercise 6.3. A lamina is in the shape of an ellipse with its two axes measuring $2a$ and $2b$ respectively. Cut out from this lamina two discs whose total area is maximized. What are the radii of these two discs? Specialize your answer when the two axes of the lamina measure (a) 28 cm and 45 cm, (b) 28 cm and 38 cm.

Hint: Partition the ellipse with a vertical line $x = l$ (with $0 \leq l \leq b$) and draw in-circles in the right- and left-part ellipses with in-radii r and R given by (5.1) and (5.2) respectively. Choose l to maximize $\pi(R^2 + r^2)$; or equivalently, to maximize $(2b^2 - a^2)l^2$. [Answer: If $a \geq \sqrt{2}\,b$, then take $l = 0$; that is, cut out the largest circle from each half ellipse partitioned by the minor axis. But if $b \leq a < \sqrt{2}b$, then take $l = b$; that is, cut out the

MEMBER'S COPY

circle with the minor axis as diameter and then cut out another circle of diameter $(a-b)$ towards either extremity of the major axis. The circles in the special cases have diameters (a) 22.5 cm and 22.5 cm; (b) 28 cm and 5 cm.]

Exercise 6.4. The following optimization problems deal with spheres, hemispheres, cones and cylinders; but they can be solved using Euclidean plane geometry. Readers may formulate and solve other similar problems.

(1) What are the height and the radius of a right circular cone that covers a sphere of radius 1, and has the smallest volume? [Answer: The smallest-volume cone has height 4, radius $\sqrt{2}$ and volume $\pi\,8/3$.]

(2) What are the height and the radius of a right circular cone that covers a hemisphere of radius 1, and has the smallest volume? [Answer: The smallest-volume cone has height $\sqrt{3}$, radius $\sqrt{3/2}$ and volume $\pi\,3\sqrt{3}/2$.]

(3) What are the height and the radius of a right circular cylinder that is inscribed in a sphere of radius 1, and has the maximum volume? [Answer: The largest-volume cylinder has height $2/\sqrt{3}$, radius $\sqrt{2/3}$ and volume $\pi\,4/(3\sqrt{3})$.]

(4) What are the height and the radius of a right circular cone that is inscribed in a sphere of radius 1, and has the maximum volume? [Answer: The largest-volume cone has height $4/3$, radius $2\sqrt{2}/3$ and volume $\pi\,32/81$.]

7. Fitting additional circles within $\mathcal{E}_*$

Let us return to the motivating scenario which led us to the discovery of the smallest-area ellipse that covers two externally-tangent, coplanar (unequal) circles.

Among Fiber Friends, my wife was the first to knit an elliptical place mat with the recommended axes lengths of 28 cm and 45 cm. She tested if the two plates of diameters 26 cm and 16 cm would fit on it; and she was pleased with the result. Then all at once it dawned on her that she would like the guests to put also their water glass (frustum of a cone with base diameter 6 cm) and their cylindrical tea cup (with diameter 8 cm) on the

MEMBER'S COPY

same place mat when the two plates are already on it. Is it possible to do so without any part of the objects protruding outside the place mat?

My wife continued her experiment; and soon found that it is possible to place the glass, but not the cup. While she went to look for a smaller cup in the cupboard, my curiosity flared up: What is the largest diameter of a cylindrical cup that will fit on this place mat in the presence of these dinner plate and bread plate? Let us now solve this new problem.

Let us rotate by 180^o the place mat together with the dinner plate and the bread plate on it. See Figure 11, where the bread plate is to the right of the dinner plate—in violation of proper dinner etiquette! You will soon see that it is a small price to pay compared to the simplicity of notation that will result.

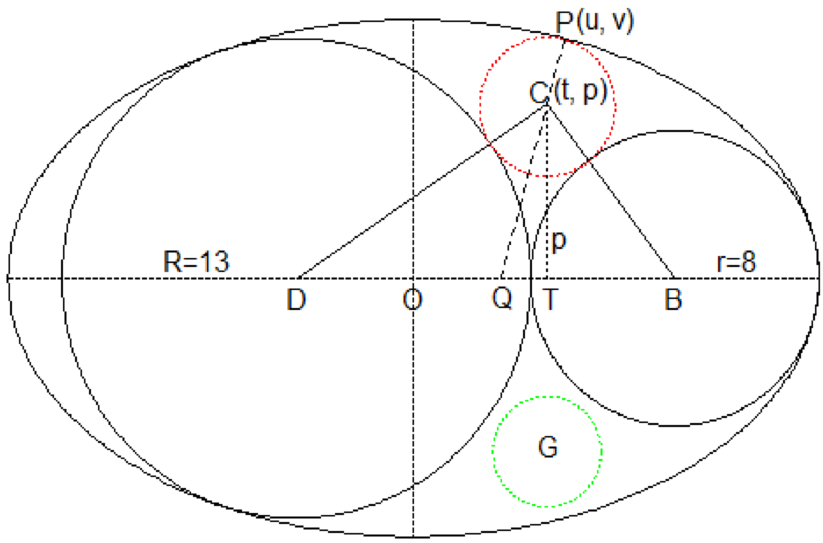


FIGURE 11. In the presence of the dinner plate and the bread plate, the water glass G with base-diameter 6 cm easily fits on the smallest-area elliptical place mat. But what is the largest-diameter cylindrical cup C that also fits on the place mat?

Let $O(0,0)$ denote the origin of the elliptical place mat with half axes $a = 22.5, b = 14$ in horizontal and vertical orientations respectively. Then the center of the dinner plate is at $D(-6.5,0)$ and that of the bread plate is at $B(14.5,0)$. We have plenty of flexibility to casually place the glass in

MEMBER'S COPY

the fourth quadrant. Let us place the largest possible cylindrical cup in the first quadrant, tangential to the two plates, and touching the periphery of the ellipse at $P(u, v)$. Let the radius of this largest cup be γ, and let its center be at $C(t, p)$, with $t, p > 0$.

Now we will show that p and t are functions of γ. From C drop a perpendicular which meets the major axis at $T(t, 0)$. Writing the area of triangle CDB in two different ways—first as half base ($DB = 21$) times height ($CT = p$), and then by using Heron's formula (see [6])—we note that

$$p = \frac{2}{21}\sqrt{8\,(13)\,\gamma\,(21+\gamma)}. \tag{7.1}$$

Also, using the Pythagorean theorem, we have $BT = \sqrt{(8+\gamma)^2 - p^2}$. Next, substituting (7.1), we get $BT = 8 - (5/21)\,\gamma$. Hence,

$$t = OT = OB - BT = \frac{13}{2} + \frac{5}{21}\gamma. \tag{7.2}$$

Next, we will show that u (and in view of (2.1), $v = b\sqrt{1 - u^2/a^2}$) is a function of p and t, and hence of γ. Let PC meet the major axis at $Q(q, 0)$. From (2.5), we know that $q = e^2 u$. Since P, C, Q are colinear, the slope of PQ equals the slope of CQ, or equivalently

$$v = \frac{p(u-q)}{t-q} = \frac{pu(1-e^2)}{t - e^2 u} = \frac{p(1-e^2)}{t/u - e^2}.$$

Equating the two expressions for v discussed above, we have

$$\frac{p(1-e^2)}{t/u - e^2} = b\sqrt{1 - \frac{u^2}{a^2}}. \tag{7.3}$$

The left-hand-side of (7.3) is an increasing function of $u \in [0, t/e^2]$ taking values in the range $[0, \infty)$. The right-hand-side, being the boundary in the first quadrant of an ellipse, is a decreasing, positive function on the same domain with a maximum value of b at $u = 0$. Hence, by the intermediate value theorem, there exists a unique $u \in (t, t/e^2)$ that satisfies (7.3); and the solution u can be expressed as a function of p and t, and hence of γ. Finally, v, being equal to the right hand side of (7.3), is also expressible as a function of γ.

We leave the analytic expression for γ to the interested reader. Here we evaluate it iteratively. We start with an initial proposed value for γ, say 3 cm. We calculate p using (7.1) and t using (7.2). Then we calculate u from the implicit equation (7.3). Next, we obtain v by evaluating either

MEMBER'S COPY

side of (7.3). Finally, we compute $PC = \sqrt{(u-t)^2 + (v-p)^2}$. If PC is more than (less than) the proposed value of γ, we increase (decrease) γ. We repeat the computations all over again, until PT is reasonably close to the proposed γ; and then we stop. We found the numerical solution to be $\gamma = 3.76$. Thus, the largest-diameter cup that fits on the smallest-area ellipse in the presence of the two given plates is 7.52 cm. If you know a tea cup salesperson, pass on the message that there is a customer in my household if only the salesperson will ensure the cups have just the right diameter.

8. Open Problems: Covering several non-overlapping circles

In this paper, we have found the smallest-area ellipse that covers two externally-tangent, coplanar circles, and then fitted one (or two) more biggest possible circles inside that ellipse. More generally, given three or more non-overlapping circular discs in a plane, what are the lengths of the two axes of the smallest-area ellipse that will cover all discs? This is a challenging global nonlinear optimization problem for which multistart strategies are studied in [1]. Website [4] depicts packing up to 24 unit discs within the ellipse with the smallest known area. Maplesoft (see [7]) has an application that optimizes packing circles in the smallest-area ellipse.

For packing ellipsoids in the n-dimensional space ($n \geq 2$), Birgin *et al.* in [2] introduces continuous and differentiable nonlinear programming models and algorithms, and gives illustrative numerical experiments using a simple multi-start strategy combined with a clever choice of starting guesses and a nonlinear programming local solver.

We are pleased to have opened the door to a new research for you. Won't you enter in?

Acknowledgments

I thank Fiber Friends for entertaining my proposal to knit an elliptical place mat on which to set the dinner plate and the bread plate side by side, and then demanding that I make the ellipse as small as possible. I thank Ms. Aparajeeta Guha for commenting on the manuscript. I am grateful to a referee for suggesting some improvements and for providing a reference.

MEMBER'S COPY

References

1. Ernesto G. Birgin, Luis Henrique Bustamante, Hector Flores Callisaya and Jose Mario Martnez. "Packing circles within ellipses." *International Transactions in Operational Research*, **20(3)**, 365–389, May 2013. First published online: 07 February 2013 `https://doi.org/10.1111/itor.12006`
2. Earnesto G. Birgin, Rafael D. Lobato and Jose Mario Martnez, "Packing ellipsoids by nonlinear optimization." *Journal of Global Optimization* **65(4)**, 709–743, August 2016.
3. Richard Courant and Herbert Robins (Revised by Ian Stewart), *What is Mathematics?* An Elementary Approach to Ideas and Methods. Oxford University Press, Inc. New York, pp. 373–378, 1996.
4. Erich Friedman. "Erich's Packing Center." Retrieved on January 16, 2020 from `https://www2.stetson.edu/ẽfriedma/cirinel/`
5. Wikipedia, the free encyclopedia. "Golden ratio." Retrieved on January 16, 2020 from `https://en.wikipedia.org/wiki/Golden_ratio`
6. Wikipedia, the free encyclopedia. "Heron's formula." Retrieved on January 16, 2020 from `https://en.wikipedia.org/wiki/Heron's_formula`
7. Maplesoft. "Circle Packing in an Ellipse." Retrieved on January 16, 2020 from `https://www.maplesoft.com/applications/view.aspx?sid=153598`
8. R Core Team (2019). "R: A language and environment for statistical computing." R Foundation for Statistical Computing, Vienna, Austria. `https://www.R-project.org/`.

Jyotirmoy Sarkar
Indiana University-Purdue University Indianapolis
Department of Mathematical Sciences
402 N Blackford Street
Indianapolis, IN 46202, USA
Email: jsarkar@iupui.edu

MEMBER'S COPY

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 91–95

SEVERAL NOVEL PROOFS OF THE INFINITUDE OF PRIMES

HAYDAR GÖRAL AND HIKMET BURAK ÖZCAN
(Received : 21 - 11 - 2019 ; Revised : 22 - 03 - 2020)

Abstract. In this note, our aim is to provide several short proofs of the infinitude of primes, and we believe that the proofs have novelty value.

1. Introduction

The first proof of the infinitude of primes is attributed to the ancient Greek mathematician Euclid. There are many proofs known for the infinitude of primes, and all proofs at least make use of the fact that any positive integer greater than 1 can be written as a product of prime numbers. For a nice historical survey and 183 different proofs of the infinitude of primes, we refer the reader to [5]. Our first proof will use the fundamental theorem of arithmetic which states that every integer greater than 1 is either a prime or it is the product of a unique combination of prime numbers. In our second and third proofs, we will make use of a significant property of the Jacobson radical.

In abstract algebra, rings are one of the most basic algebraic structures. Recall that a ring R is a set equipped with two binary operations called addition and multiplication such that the addition is associative and commutative while the multiplication is associative and distributive over the addition. Moreover, there exist an additive identity 0 and an additive inverse $-x$ for any $x \in R$. Furthermore, there exists a multiplicative identity 1. If the multiplication is also commutative, then R is said to be a commutative ring. For example, the set of integers $\mathbb{Z}$ is a commutative ring under the usual addition and multiplication. An element $u \in R$ is called a *unit*, if

2010 Mathematics Subject Classification: 11A41, 16N20
Key words and phrases: prime numbers, divisibility, ring

© *Indian Mathematical Society, 2020.*

MEMBER'S COPY

there exists an element $v \in R$ such that $uv = 1 = vu$. A subset I of R is said to be an *ideal* if $r(x-y)s \in I$ for all $x, y \in I$ and $r, s \in R$. An ideal M of R is called a *maximal ideal* if $M \neq R$ and there is no proper ideal of R containing M. One can show that every proper ideal is contained in a maximal ideal using Zorn's lemma.

The *Jacobson radical* of a commutative ring R, denoted by $J(R)$, is the intersection of all maximal ideals of R. For example, the Jacobson radical $J(\mathbb{Q})$ of rational numbers $\mathbb{Q}$ is $\{0\}$. The Jacobson radical of the finite ring $\mathbb{Z}/12\mathbb{Z}$ is $6\mathbb{Z}/12\mathbb{Z}$, because its all maximal ideals are $2\mathbb{Z}/12\mathbb{Z}$ and $3\mathbb{Z}/12\mathbb{Z}$. As all maximal ideals of integers are of the form $p\mathbb{Z}$ for a prime number p, the Jacobson radical $J(\mathbb{Z})$ of integers is

$$J(\mathbb{Z}) = \bigcap_{p \in \mathbb{P}} p\mathbb{Z},$$

where $\mathbb{P}$ is the set of prime numbers. This will be a key point for our second and third proofs.

2. The Proofs

The following lemma, which describes the elements of the Jacobson radical in terms of invertible elements, will be crucial in our second and third proofs. Details can be found in [4].

Lemma 2.1. *For any commutative ring R, we have*

(1) $1 - x$ *is a unit for each* $x \in J(R)$.
(2) The Jacobson radical is the largest ideal such that $1 - x$ *is a unit for each* $x \in J(R)$.

Proof. For the first part, suppose that $1 - x$ is a non-unit, where $x \in J(R)$. Then, the ideal I generated by $1 - x$ is a proper ideal of R. Hence, it is contained in a maximal ideal M. Now, we see that both x and $1 - x$ belong to M which in turn implies that $1 \in M$. It is a contradiction. Therefore, $1 - x$ is a unit.

For *(2)*, let A be an ideal such that $1 - a$ is a unit for each $a \in A$. We need to prove that $A \subseteq J(R)$. In order to see this, it is enough to show that $A \subseteq M$ for any maximal ideal M of R. Suppose that there exists a maximal ideal M which does not contain A. Then, we have $A + M = R$ and so $1 = a + m$ with $a \in A$ and $m \in M$. Since $m = 1 - a$ is a unit,

MEMBER'S COPY

this yields that $M = R$. It is a contradiction. Therefore, $A \subseteq M$ for any maximal ideal M of R. □

Now we are ready to give three different and new proofs of the infinitude of primes.

Theorem 2.2. *There are infinitely many prime numbers.*

1st Proof. Suppose that there are finitely many prime numbers and list them as $p_1, \ldots, p_n$. Choose an arbitrary integer $a \geq 0$ and let P be the product $p_1 \cdots p_n$ which is clearly not 0. Then, the positive integer $aP^2 + P$ is divisible by all prime numbers $p_1, \ldots, p_n$. However, for any prime p, we see that p^2 does not divide $aP^2 + P$ as p^2 does not divide P by the fundamental theorem of arithmetic. Hence, using the fundamental theorem of arithmetic again, we obtain that $aP^2 + P = P$, which in return yields that $a = 0$. In other words, all positive integers are equal to 0. Thus, 0 is the only integer. This is a contradiction, and the proof is completed. □

2nd Proof. Suppose that there are finitely many prime numbers and list them as $p_1, \ldots, p_n$, where $p_1 = 2$. Then, the ideal $p_i\mathbb{Z}$ is a maximal ideal for every prime number p_i. In fact, $p_1\mathbb{Z}, \ldots, p_n\mathbb{Z}$ are all the maximal ideals in the ring of integers $\mathbb{Z}$ as we mentioned in the introduction. Hence, the Jacobson radical $J(\mathbb{Z})$ is the intersection of all maximal ideals $p_1\mathbb{Z}, \ldots, p_n\mathbb{Z}$:

$$J(\mathbb{Z}) = \bigcap_{i=1}^{n} p_i\mathbb{Z} = (p_1 \cdots p_n)\mathbb{Z}.$$

Note that the only units of $\mathbb{Z}$ are -1 and 1. Since $p_1 \cdots p_n \in J(\mathbb{Z})$ and it is bigger than 1, we have that $1 - p_1 \cdots p_n$ is a negative unit by Lemma 2.1. So, $1 - p_1 \cdots p_n = -1$ which implies that $p_1 \cdots p_n = 2$. This yields that 2 is the only prime number. It is a contradiction and therefore there are infinitely many prime numbers. □

3rd Proof. Let x be an arbitrary integer in the Jacobson radical $J(\mathbb{Z})$. Then, the additive inverse $-x$ of x is also in $J(\mathbb{Z})$, because $J(\mathbb{Z})$ is an ideal. By Lemma 2.1, we have that $1-x$ and $1+x$ are units in $\mathbb{Z}$. In other words, for any $x \in J(\mathbb{Z})$ we have that $1 - x$ and $1 + x$ are in $\{-1, 1\}$. This is only possible when x is equal to 0. Thus, $J(\mathbb{Z}) = \{0\}$. However, we know that

$$J(\mathbb{Z}) = \bigcap_{p \in \mathbb{P}} p\mathbb{Z},$$

MEMBER'S COPY

where $\mathbb{P}$ denotes the set of all prime numbers. Therefore, this implies that there must be infinitely many primes. Because, if there were finitely many primes $p_1, \ldots, p_n$, then the non-zero product $p_1 \ldots p_n$ would be in $J(\mathbb{Z})$. $\square$

Concluding comments

In this short note, we gave several novel proofs of the infinitude of primes. There are various techniques to prove the infinitude of primes. To illusturate analysis, topology (see [2]), series and Ramsey theory can yield the infinitude of primes. Our proofs used the fundamental theorem of arithmetic and a fact from ring theory.

In the litarature, the infinitude of primes was also proved using infinite combinatorics, see [1] and [3]. In [1], the p-adic valuation is useful for the proof. In our first proof, a similar approach in an more elementary and short way was applied.

There are similarities between Furstenberg's topological proof and our second and third proofs. Furstenberg defined open sets of $\mathbb{Z}$ as arbitrary unions of infinite arithmetic progressions. Thus, any infinite arithmetic progression $a\mathbb{Z}+b$ is both open and closed. Then, if there are finitely many primes, it follows that

$$\bigcup_{p\in\mathbb{P}} p\mathbb{Z} = \mathbb{Z} \setminus \{\pm 1\}$$

is closed. Thus, $\{\pm 1\}$ must be open, but it does not contain any infinite arithmetic progression. In our proofs, we considered the intersection

$$J(\mathbb{Z}) = \bigcap_{p\in\mathbb{P}} p\mathbb{Z}.$$

If there are finitely many primes, then in the above topology $J(\mathbb{Z})$ should be open. Similarly, this yields a contradiction, as we indicated that $J(\mathbb{Z}) = \{0\}$, which is a finite set. In our second proof, one can obtain that the arithmetic progression $1 + p_1 \ldots p_n\mathbb{Z}$ (it is open in the mentioned topology) is finite and again one gets a contradiction.

Acknowledgement: We are grateful to the referee for the comments which improved the quality of the paper.

MEMBER'S COPY

References

[1] Alpoge, L., Van der Warden and the primes, *American Mathematical Monthly* **122** (2015), 784–785.

[2] Furstenberg, H., On the infinitude of primes, *American Mathematical Monthly* **62** (1955), 353.

[3] Granville, A., Squares in arithmetic progressions and infinitely many primes, *American Mathematical Monthly* **124** (2017), 951-954.

[4] Lam, T. Y., A First Course in Noncommutative Rings, Springer-Verlag New York, 2001.

[5] Mestrovic, R., Euclid's theorem on the infinitude of primes: A historical survey of its proofs, https://arxiv.org/pdf/1202.3670.pdf

Haydar Göral
Department of Mathematics
Faculty of Sciences
Dokuz Eylül University,
35390, Buca/İzmir, TURKEY.
E-mail: hgoral@gmail.com

Hikmet Burak Özcan
Department of Mathematics
Faculty of Sciences
İzmir Institute of Technology
35430, Urla/İzmir, TURKEY.
E-mail: hikmetburakozcan@gmail.com

MEMBER'S COPY

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 97– 102

A FEW π SERIES FROM KERALA AND A SIMPLE PROOF OF A SERIES BY MADHAVA

AMRIK SINGH NIMBRAN

(Received : 19 - 12 - 2019 ; Revised : 07 - 07 - 2020)

1. Historical Background

1.1. **Establishment of Royal Asiatic Society.** A positive development during the rule of the British East India Company in India was the creation of institutions for compiling knowledge about the county and its culture — its society, history, religion, laws, customs, languages and literature. The British civil servants played a pivotal role in this development. Sir William Jones (1746–94)[1], an Anglo-Welsh philologist and jurist who came to India in 1783 to serve as judge of the supreme court at Calcutta, established the Asiatic Society of Bengal on January 15, 1784 with the support and encouragement of Warren Hastings, the then Governor-General of Bengal. Jones learnt Sanskrit and in his 1786 presidential address to the Society, he postulated the common ancestory of Sanskrit, Latin, and Greek. Till 1828 only Europeans were allowed to be members of the Society; few Indians zamindars were admitted as members in 1829. Sir James Macikintosh, a British judge for Bombay, followed the example of Jones and founded the Literary Society of Bombay in 1804. It became an associate society of the Royal Asiatic Society[2] in 1838 and started publishing its journal in 1841. The Madras Literary Society came into existence in 1812 and became in 1830 an Auxiliary of the Royal Asiatic Society. It began to publish the *Madras Journal of Literature and Science* in mid thirties.

1.2. **Charles Whish on medieval Keralese mathematics.** Charles M. Whish (1794–1833), a member of the Madras Civil Service of the East India Company, in an article read on December 15, 1832 and published in 1834 *Transactions of the Royal Asiatic Society of Great Britain and Ireland*, brought to the notice of the West infinite series for the inverse tangent, cosine and sine functions and some series for π expressed in Sanskrit verses extracted from four books — Tantra Sangraha, Yuc(k)ti Bhasha, C(K)arana-Padhati, and Sadratnamala. Whish ascribed *Tantrasangraha* to

[1]He was the third child of William Jones (1675–1749), an able mathematician and a close friend of Sir Isaac Newton and Sir Edmund Halley, known for his first use (in 1706) of the symbol π for the ratio of the circumference of a circle to its diameter.

[2]The Royal Asiatic Society of Great Britain and Ireland founded on March 15, 1823 by the eminent Sanskrit scholar, Henry Thomas Colebrooke and received its charter from King George IV on the 11th August 1824 'for the investigation of subjects connected with and for the encouragement of science, literature and the arts in relation to Asia'.

MEMBER'S COPY

Talaculattúra Nambutiri of Kerala. This 1500 A.D. treatise on astronomy is now said to be composed by Nīlakantha Somayaji (1444–1545 AD). About the second book that he used, Whish promised at the end of his paper: "A farther account of the *Yucti-Bhasha,* the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper." However, he[3] could not do so as he died prematurely on 14 April, 1833. He attributed *Yukti-bhāsā*, a commentary on the *Tantrasangraha* in Malayalam (the native language of Kerala), to Cellalura Nambutiri who has now been identified as Jyesthadeva (1500–1610 AD), a student of Nīlkantha. *Karana-paddhati* (1733 AD), the third book from which Whish quoted, he attributed to Nutunagriha Sóma Súta: "The author of the *Carana Padhati,* whose grandson is now alive in his seventieth year, was Pathumana Sóma Yaji, a Nambutiri Brahmana of Tirusivapura (Trichur) in Malabar." He observed that "Sóma Yaji was not, however, inventor of the system by which he formed his infinite series."

1.3. **A sample of π series in medieval Keralese mathematics.** Whish quoted various extracts from the Tantra Sangraha. The first quotation is concerned with "proportion between the diameter and the circumference". Whish translated the quoted verses as: "Multiply the diameter by 4, and from it subtract and add alternately the quotients obtained by dividing four times the diameter by the odd numbers 3, 5, 7, 9, 11, &c, do thus to the extent required; and having fixed a limit, take half the even number next less than the last odd divisor for a multiplier, and its square plus one for a divisor. Multiply four times the diameter by the multiplier, and divide the product by the divisor, and add it or subtract it, according to the sign of the last quote in the series, from the sum of the series; thus the circumference of the given diameter will be is obtained very correctly."

The quotation converts into the following approximation formula:

$$c \approx 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots + (-1)^{k-1}\frac{4d}{2k-1} + (-1)^k \frac{4dk}{4k^2+1}. \tag{1.1}$$

[3]We find the following entry in the *The Madras Journal of Literature and Science,* published under the auspices of the Madras Literary Society and Auxiliary of the Royal Asiatic Society: "To J. C. Whish, Esq., the Library is indebted for a large collection of works, chiefly in Sanskrit, but in Malyalam character, written on palm-leaves, and principally comprising the Vedas, and other religious and philosophical works of the Hindus. This large collection was made by his late brother, C. M. Whish, Esq., of the Madras Civil Service during a course of many years that he was resident on the Western coast of the Southern Peninsula of India." Volume VIII, July–December 1838 (No 20 July 1838; No, 21 – October 1838, p.216) p.375

MEMBER'S COPY

Whish then quotes "a verse explaining more fully the correction by which this series is brought to greater perfection." We get the following formula:

$$\pi \approx 4\left[\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+(-1)^{k-1}\frac{1}{2k-1}\right\}+(-1)^k\frac{k^2+1}{k(4k^2+1)}\right]. \tag{1.2}$$

The quoted verses do not occur in Tantrasangraha but forms part of an anonymous commentary (*vyakhya*) on it.[4] An alternative reading is given in K. V. Sarma's edition (V.V.B. Institute of Sanskrit and Indological Studies, Panjab University, Hoshiarpur, 1977, p.10) of *Tantrasangraha* with *Yuktidipika* (a commentary in verse on the first four chapters) and *Laghuvivrti* (concise commentary in prose on the last four chapters) of Shankara, younger brother of Nīlakantha.

A commentary (c.1530) on Nīlakantha's *Tantrasangraha* (c.1500) in the regional Malayalam language of Kerala titled *Yukti-bhasa*[5] was written by Jyesthadeva (1500–1610 AD), a student of Nīlkantha. It presents detailed demonstrations of the famous results attributed to Madhava (c.1350–1425) of Sangamagrama –modern Irinjalakuda, located about 21 km. south of the district town of Thrissur (meaning the abode of Lord Siva), a part of the erstwhile Cochin state and now in the modern state of Kerala in South India — such as infinite series for the arc-tangent and the sine functions, π series and the estimation of correction terms and their use in the generation of faster convergent series. According to local tradition Mādhava was a teacher of Paramesvara (1360–1455 AD) who was the father of Damodara (1400–1500), the teacher of Nīlkantha. One of Nilakantha's pupils was Citrabhanu (1475–1550) whose student Narayana (c. 1500–1575) completed one of the major texts of the Kerala school, *Kriyakramakari.*

The next extract converts into the following famous formula:

$$\pi=\sqrt{12}\left[1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\frac{1}{9\cdot 3^4}-\frac{1}{11\cdot 3^5}+\cdots\right]. \tag{1.3}$$

2. A simple proof of a π series of Madhava

I now come to the series for which I have devised a very simple proof. Knopp [1, p.268, Ex. 107 (d) & (e)] gives these two related series as exercises for proof:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-1)^3}{(2n-1)^4+4}=0. \tag{2.1}$$

[4]The Sanskrit text is now available with English translation as *Tantrasaṅgraha of Nīlakantha Somayāji* (eds.) K. Ramasubramanian and M.S. Sriram, Springer and Hindustan Book Agency, 2011.

[5]An English translation as *Ganita-yukti-Bhāsā (Rationales in Mathematical Astronomy) of Jyesthadeva* by (eds.) K.V. Sarma, K. Ramasubramanian, M.D. Srinivas and M.S. Sriram, 2 volumes, Springer and Hindustan Book Agency, 2008.

MEMBER'S COPY

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)\{(2n-1)^4+4\}} = \frac{\pi}{16}. \tag{2.2}$$

Knopp gives no indication as to who gave these series. The series (2.2) occurs in [5, p.514] on the basis of which it appears in [2, p.70, eq(3)] [3, p.95] [4, p.2, B.3]. The series is attributed to Madhava. I have taken the following extract from a Sanskrit text edited by K.V. Sarma.

[परिधिसूक्ष्मतायै प्रकारान्तराणि]

"समपञ्चा[हतयो या][2] रूपाद्ययुजां चतुःघ्नमूलयुताः ।

ताभिः षोडशगुणिताद् व्यासात् पृथगाहृतेषु[3] विषमयुतेः ॥ २८७ ॥

समफलयोगे त्यक्ते स्यादिष्टव्याससम्भवः परिधिः" ॥ २८८ ॥

Whish translated these verses as: "Divide the diameter multiplied by 16 severally by the fifth power of the odd numbers $[1, 3, 5, 7, 9 \ldots]$, adding to each fifth power four times its root; of the quotes thus obtained subtract the sum of the second, fourth, sixth, &c. from that of the first, third, fifth, and seventh, the remainder will be the circumference of the circle whose diameter was taken." This gives an infinite series noted below:

$$c = \frac{16d}{1^5+4\cdot 1} - \frac{16d}{3^5+4\cdot 3} + \frac{16d}{5^5+4\cdot 5} - \frac{16d}{7^5+4\cdot 7} + \cdots$$

which results in the series (2.2).

Proof. The series (2.2) can be written as

$$\frac{\pi}{16} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)^3}{(2n-1)^4+4} = \frac{\pi}{16} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)^3}{(2n-1)^4+4}$$

which implies that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)^3}{(2n-1)^4+4} = 0.$$

To establish this, let

$$S_m = \sum_{n=1}^{m}\frac{(-1)^{n+1}(2n-1)^3}{(2n-1)^4+4}.$$

We notice that

$$S_1 = \frac{1}{5} = \frac{(-1)^{1+1}}{4\cdot 1^2+1},$$

$$S_2 = \frac{1}{5} - \frac{27}{17} = -\frac{2}{17} = \frac{(-1)^{2+1}2}{4\cdot 2^2+1},$$

MEMBER'S COPY

$$S_3 = \frac{2}{17} + \frac{125}{629} = \frac{3}{37} = \frac{(-1)^{3+1}3}{4 \cdot 3^2 + 1}$$

which suggests

$$S_k = \frac{(-1)^{k+1}k}{4 \cdot k^2 + 1}.$$

Assume the result true for $m = k$, we shall show it true for $m = k + 1$.

It is easy to verify that $(2n+1)^4 + 4 = (4n^2+1)(4n^2+8n+5)$. Now

$$S_{k+1} = S_k + \frac{(-1)^{k+2}(2k+1)^3}{(2k+1)^4+4} = \frac{(-1)^{k+1}k}{4k^2+1} + \frac{(-1)^{k+2}(2k+1)^3}{(4k^2+1)(4k^2+8k+5)}.$$

That is,

$$S_{k+1} = \frac{(-1)^{k+1}k}{4k^2+1} - \frac{(-1)^{k+1}(2k+1)^3}{(4k^2+1)(4k^2+8k+5)}.$$

Or,

$$S_{k+1} = \frac{(-1)^{k+1}}{4k^2+1}\left[\frac{(4k^3+8k^2+5k)-(8k^3+12k^2+6k+1)}{4k^2+8k+5}\right].$$

That is,

$$S_{k+1} = \frac{(-1)^{k+2}}{4k^2+1}\left[\frac{(4k^3+k)+(4k^2+1)}{4k^2+8k+5}\right].$$

Or,

$$S_{k+1} = \frac{(-1)^{k+2}(k+1)}{4(k+1)^2+1}.$$

So, the result assumed true for $m = k$ has been shown to be true for $m = k + 1$. Therefore,

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{(-1)^{k+1}k}{4k^2+1} = \lim_{k\to\infty} \frac{(-1)^{k+1}}{4k+\frac{1}{k}} = 0$$

which establishes Madhava's series. □

Acknowldegement: The author is thankful to Prof. Ranjan Roy of Beloit College, Wisconsin(US) for reading an earlier note containing the proofs given here. He remarked in his email of Jan 3, 2014 to the author that the proofs were 'fine' and 'nice'.

References

[1] K. Knopp, *Theory and Application of Infinite Series,* New York, Dover, 1990.

[2] K. Mukunda Marar & C. T. Rajagopal, *On the Hindu quadrature of the circle,* Journal of the Bombay Branch of the Royal Asiatic Society, N.S., 20 (1944). 65–82. `https://archive.org/details/in.ernet.dli.2015.280596/page/n74`

[3] C. T. Rajagopal & M. S. Rangachari, *On an Untapped Source of Medieval Keralese Mathematics,* Archive for History of Exact Sciences, Vol. 8, No. 2 (June 1978). Online `http://math.mit.edu/classes/18.01/F2011/school-of-kerala.pdf`

[4] R. Roy, *Sources in the Development of Mathematics: Infinite Series and Products from the Fifteenth to the Twenty-first Century,* Cambridge University Press, 2011.

MEMBER'S COPY

[5] C. M. Whish, *On the Hindu Quadrature of the Circle ...,* Transactions of the Royal Asiatic Society of Great Britain and Ireland, Vol. 3, No. 3 (1834), 509–523. https://ia601900.us.archive.org/14/items/jstor-25581775/25581775.pdf

Amrik Singh Nimbran
B3-304, Palm Grove Heights
Ardee City, Gurugram
Haryana, India
Email : amrikn622@gmail.com

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 103 – 109

GENOMIAL NUMBERS FOR SECOND ORDER GENERALIZED FIBONACCI NUMBERS

MANSI S. SHAH AND DEVBHADRA V. SHAH

(Received : 21 - 11 - 2019 ; Revised : 22 - 03 - 2020)

Abstract. In this paper, we have generalized the concept of binomial coefficient to the genomial numbers using the second order generalized Fibonacci numbers. We prove that these numbers are always integers and obtain some of its divisibility properties.

1. Introduction

In [1], Benjamin and Plott introduced the notion of Fibonomial numbers and studied various properties related to them. In this paper we generalize this concept and introduce genomial numbers for the second order generalized Fibonacci numbers. We also derive number of interesting results related with these numbers. The proofs involve the elementary concepts of binomial coefficients and the generalized Fibonacci numbers.Kalman and Mena [3] defined the generalized Fibonacci sequence as follows:
Definition: For any positive integers a and b, the generalized Fibonacci numbers are defined by the recurrence relation

$$F_n^{(a,b)} = aF_{n-1}^{(a,b)} + bF_{n-2}^{(a,b)}; n \geq 2 \tag{1.1}$$

Where $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$.

First few terms of this sequence are $0,\ 1,\ a,\text{‘}a^2 + b,\ a^3 + 2ab,\ a^4 + 3a^2b + b^2$. Clearly, $F_n^{(1,1)} = F_n$, the traditional n^{th} Fibonacci number. The extended Binet formula for $F_n^{(a,b)}$ is given by $F_n^{(a,b)} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$ and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$

2010 Mathematics Subject Classification: 11A41, 16N20
Key words and phrases: Fibonacci numbers, Lucas numbers

© Indian Mathematical Society, 2020.

MEMBER'S COPY

Shah and Shah [6] introduced the *generalized Lucas numbers* $L_n^{(a,b)}$ defined by the recurrence relation

$$L_n^{(a,b)} = aL_{n-1}^{(a,b)} + bL_{n-2}^{(a,b)}; n \geq 2 \tag{1.2}$$

where $L_0^{(a,b)} = 2$ and $L_1^{(a,b)} = a$.

They derived many interesting results related with these numbers. They also obtained the extended Binet formula for $L_n^{(a,b)}$ as $L_n^{(a,b)} = \alpha^n + \beta^n$, where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$ and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$. Clearly $L_n^{(1,1)} = L_n$, the traditional Lucas number.

We first derive some of the interesting relations between generalized Fibonacci number $F_n^{(a,b)}$ and generalized Lucas number $L_n^{(a,b)}$, which will be required for the study of genomial numbers.

2. Properties of generalized Fibonacci numbers

The following combination gives interesting result relating $F_n^{(a,b)}$ and $L_n^{(a,b)}$

Lemma 2.1. $2F_n^{(a,b)} = F_r^{(a,b)}L_{n-r}^{(a,b)} + L_r^{(a,b)}F_{n-r}^{(a,b)}$*; n and r are any positive integers such that* $r \leq n$

Proof. Using the extended Binet formula for both the sequences, the right side of the result can be expressed as:
$F_r^{(a,b)}L_{n-r}^{(a,b)}+L_r^{(a,b)}F_{n-r}^{(a,b)} = \left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)(\alpha^{n-r}+\beta^{n-r})+(\alpha^r+\beta^r)\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)$
$= \frac{2(\alpha^n-\beta^n)}{\alpha-\beta} = 2F_n^{(a,b)}$,which proves the required result. □

The following result expresses the product of a generalized Fibonacci number and generalized Lucas number as a generalised Fibonacci numbers.

Lemma 2.2. $L_n^{(a,b)}F_n^{(a,b)} = F_{2n}^{(a,b)}$

Proof. Since $F_n^{(a,b)} = \frac{\alpha^n-\beta^n}{\alpha-\beta}$, we write $\frac{F_{2n}^{(a,b)}}{F_n^{(a,b)}} = \frac{\alpha^{2n}-\beta^{2n}}{\alpha^n-\beta^n} = \frac{(\alpha^n+\beta^n)(\alpha^n-\beta^n)}{\alpha^n-\beta^n} = \alpha^n+\beta^n = L_n^{(a,b)}$, as required □

We use the above results for the study of a new class of genomial numbers for the second order generalized Fibonacci numbers introduced in the following section.

MEMBER'S COPY

3. Genomial numbers

In a one-page note [2] published in 1915, Fontene suggested a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence $\{A_n\}$ of real or complex numbers. Since 1964, there has been an accelerated interest in the Fibonomial coefficients $\begin{bmatrix} m \\ k \end{bmatrix}_F$, which correspond to the choice $A_n = F_n$, thus are defined, for $1 \leq k \leq m$,in the following way:

$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m^*}{F_k^* \times F_{m-k}^*}$ where $F_m^* = F_m \times F_{m-1} \times \cdots \times F_2 \times F_1$

In 2014, Koshy [5] introduced *Pellnomial numbers* $\begin{bmatrix} n \\ r \end{bmatrix}$ which corresponds to the choice $A_n = P_n$; where P_n is the n^{th} Pell number and discussed some of its interesting properties. Using the similar method, we introduce *genomial* numbers $\begin{bmatrix} n \\ r \end{bmatrix}$ which correspond to the choice .

We define

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{F_n^{(a,b)*}}{F_r^{(a,b)*} F_{n-r}^{(a,b)*}}; 0 \leq r \leq n \tag{3.1}$$

where $F_n^{(a,b)*} = F_n^{(a,b)} \times F_{n-1}^{(a,b)} \times \cdots \times F_2^{(a,b)} \times F_1^{(a,b)}, F_0^{(a,b)*} = 1$

The immediate question which now arises is whether every genomial number is an integer? Although it is not obvious from the definition, but we will prove later that genomial numbers are indeed (positive) integers.. It follows from (3.1) that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} n \\ 1 \end{bmatrix} = F_n^{(a,b)}$ and $\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}$

3.1. **A Recurrence Relation :** We use (3.1) to state the following recurrence relation for the genomial numbers.

Lemma 3.1. $\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} \times \frac{F_n^{(a,b)}}{F_r^{(a,b)}}$

From lemma 3.1 and 2.2, the following corollary follows immediately.

Corollary 3.2. $\begin{bmatrix} 2n \\ n \end{bmatrix} = \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} \times L_n^{(a,b)}$

MEMBER'S COPY

We next obtain the additional recurrence relation for $\begin{bmatrix} n \\ r \end{bmatrix}$ in terms of generalized Lucas numbers.

Lemma 3.3. $\begin{bmatrix} n \\ r \end{bmatrix} = \frac{1}{2}\left\{ \begin{bmatrix} n-1 \\ r \end{bmatrix} L_r^{(a,b)} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} L_{n-r}^{(a,b)} \right\}$

Proof. Using lemma 2.1, we write

$2\begin{bmatrix} n \\ r \end{bmatrix} F_n^{(a,b)} = \begin{bmatrix} n \\ r \end{bmatrix} F_r^{(a,b)} L_{n-r}^{(a,b)} + \begin{bmatrix} n \\ r \end{bmatrix} L_r^{(a,b)} F_{n-r}^{(a,b)} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} F_n^{(a,b)} L_{n-r}^{(a,b)} + \begin{bmatrix} n-1 \\ r \end{bmatrix} L_r^{(a,b)} F_n^{(a,b)}$.

Thus, $2\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} L_{n-r}^{(a,b)} + \begin{bmatrix} n-1 \\ r \end{bmatrix} L_r^{(a,b)}$, as desired.

□

We now use lemma 3.3 to confirm that every genomial number is a positive integer.

Theorem 3.4. *Every genomial number* $\begin{bmatrix} n \\ r \end{bmatrix}; n \geq 0$ *is a positive integer.*

Proof. We establish this using the strong version of induction. Result is clearly true for $n = 0, 1$. Assume that it is true for all nonnegative integers not exceeding $1 \leq r \leq n-1$.

Then by induction hypothesis, both $\begin{bmatrix} n-1 \\ r \end{bmatrix}$ *and* $\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}$ are positive integers.

Since $L_i^{(a,b)} \geq 1$ is an integer for every $i \geq 1$ and $\begin{bmatrix} n \\ 0 \end{bmatrix}$ is a positive integer, it now follows from lemma 3.3 that $\begin{bmatrix} n \\ r \end{bmatrix}$ is always a positive integer. □

As an illustration, if we consider $n = 6$ and $r = 2$. we have, $\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \frac{F_6^{(a,b)*}}{F_2^{(a,b)*} \times F_4^{(a,b)*}} = a^8 + 7a^6 b + 16a^4 b^2 + 13a^2 b^3 + 3b^4$ which is a positive integer for any choice of positive integers a and b.

What follows is an easy consequence of corollary 3.2 afte we prove theorem 3.4

MEMBER'S COPY

Corollary 3.5. $L_n^{(a,b)} \mid \begin{bmatrix} 2n \\ n \end{bmatrix}$

This result shows that every generalized Lucas number is a factor of some genomial number. Obviously, the result is dependent on the value of n only.

To illustrate this, consider $n = 3$. Then $L_3^{(a,b)} = a^3 + 3ab$ and $\begin{bmatrix} 6 \\ 3 \end{bmatrix} =$ $a^9 + 8a^7b + 22a^5b^2 + 23a^3b^3 + 6ab^4 = (a^6 + 5a^4b + 7a^2b^2 + 2b^3)(a^3 + 3ab)$. Thus $L_3^{(a,b)} \mid \begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

This divisibility property does not depend on the values of a and b.

3.2. **Divisibility:** In 2014, Koshy [5] gave two interesting divisibility properties $\frac{m}{gcd(m,n)} \mid \binom{m}{n}$ and $\frac{m-n+1}{gcd(m,n)} \mid \binom{m}{n}$
We show that they have their counterparts for genomial numbers too.

Lemma 3.6. $\frac{F_m^{(a,b)}}{gcd(F_m^{(a,b)}, F_n^{(a,b)})} \mid \begin{bmatrix} m \\ n \end{bmatrix}$

Proof. Let $d = gcd(F_m^{(a,b)}, F_n^{(a,b)})$. Then by means of Euclidean algorithm, we write $d = AF_m^{(a,b)} + BF_n^{(a,b)}$; for some integers A and B. So that
$$d \begin{bmatrix} m \\ n \end{bmatrix} = A \begin{bmatrix} m \\ n \end{bmatrix} F_m^{(a,b)} + B \begin{bmatrix} m \\ n \end{bmatrix} F_n^{(a,b)} = A \begin{bmatrix} m \\ n \end{bmatrix} F_m^{(a,b)} + B \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} F_m^{(a,b)}$$
$$= \left(A \begin{bmatrix} m \\ n \end{bmatrix} + B \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right) F_m^{(a,b)},$$ which yields the desired result.

□

Lemma 3.7. $\frac{F_{m-n+1}^{(a,b)}}{gcd(F_{m+1}^{(a,b)}, F_n^{(a,b)})} \mid \begin{bmatrix} m \\ n \end{bmatrix}$

Proof. Using definition (3.1) we observe that $\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m \\ n-1 \end{bmatrix} \frac{F_{m-n+1}^{(a,b)}}{F_n^{(a,b)}}$. Let $d = gcd\left(F_{m+1}^{(a,b)}, F_n^{(a,b)}\right)$. But by [3] it is known that $gcd\left(F_{m+1}^{(a,b)}, F_n^{(a,b)}\right) =$ $F_{gcd(m+1,n)}^{(a,b)}$. Then $d = F_{gcd(m+1,n)}^{(a,b)} = F_{gcd(m-n+1,n)}^{(a,b)} = gcd\left(F_{m-n+1}^{(a,b)}, F_n^{(a,b)}\right)$ $= AF_{m-n+1}^{(a,b)} + BF_n^{(a,b)}$; for some integers A and B. So that
$$d \begin{bmatrix} m \\ n \end{bmatrix} = A \begin{bmatrix} m \\ n \end{bmatrix} F_{m-n+1}^{(a,b)} + B \begin{bmatrix} m \\ n \end{bmatrix} F_n^{(a,b)}$$

MEMBER'S COPY

$$= A \begin{bmatrix} m \\ n \end{bmatrix} F^{(a,b)}_{m-n+1} + B \begin{bmatrix} m \\ n-1 \end{bmatrix} F^{(a,b)}_{m-n+1}$$

$= (A \begin{bmatrix} m \\ n \end{bmatrix} + B \begin{bmatrix} m \\ n-1 \end{bmatrix}) F^{(a,b)}_{m-n+1}$ which yields the desired result. □

3.3. **A Genomial Identity:** The following identity due to Gould [4] $\binom{n-a}{r-a}\binom{n}{r+a}\binom{n+a}{r} = \binom{n-a}{r}\binom{n+a}{r+a}\binom{n}{r-a}$ for binomial coefficients is well known. Interestingly, Gould's identity can be imitated for genomials too as shown in the following result:

Lemma 3.8. $\begin{bmatrix} n-a \\ r-a \end{bmatrix} \begin{bmatrix} n \\ r+a \end{bmatrix} \begin{bmatrix} n+a \\ r \end{bmatrix} = \begin{bmatrix} n-a \\ r \end{bmatrix} \begin{bmatrix} n+a \\ r+a \end{bmatrix} \begin{bmatrix} n \\ r-a \end{bmatrix}; r+a \leq n.$

Proof. Using the definition of genomial numbers, the left hand side of the result becomes,

$$\begin{bmatrix} n-a \\ r-a \end{bmatrix} \begin{bmatrix} n \\ r+a \end{bmatrix} \begin{bmatrix} n+a \\ r \end{bmatrix} = \frac{F^{(a,b)*}_{n-a}}{F^{(a,b)*}_{r-a} F^{(a,b)*}_{n-r}} \times \frac{F^{(a,b)*}_{n}}{F^{(a,b)*}_{r+a} F^{(a,b)*}_{n-r-a}} \times \frac{F^{(a,b)*}_{n+a}}{F^{(a,b)*}_{r} F^{(a,b)*}_{n-r+a}} =$$

$$\frac{F^{(a,b)*}_{n-a}}{F^{(a,b)*}_{r} F^{(a,b)*}_{n-r-a}} \times \frac{F^{(a,b)*}_{n+a}}{F^{(a,b)*}_{r+a} F^{(a,b)*}_{n-r}} \times \frac{F^{(a,b)*}_{n}}{F^{(a,b)*}_{r-a} F^{(a,b)*}_{n-r+a}} = \begin{bmatrix} n-a \\ r \end{bmatrix} \begin{bmatrix} n+a \\ r+a \end{bmatrix} \begin{bmatrix} n \\ r-a \end{bmatrix}$$,which proves the required result.

□

In particular, lemma 3.3 yields the following identity:

$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix} \begin{bmatrix} n \\ r+1 \end{bmatrix} \begin{bmatrix} n+1 \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r \end{bmatrix} \begin{bmatrix} n+1 \\ r+1 \end{bmatrix} \begin{bmatrix} n \\ r-1 \end{bmatrix}$. Thus the product of the six genomial numbers with all the possible adjacent permutations of n and r is always a perfect square.

References

[1] Benjamin Arthur, T., Plott Sean, S., A Combinatorial Approach to Fibonomial Coefficients, *Fib.Quar* Vol. **46** (2008), 7–9.

[2] Fontene, G.,Generalization d'uneformuleconnue Nouv. Ann. Math . **4(15)** (1915), 112.

[3] Kalman, D., and Mena, R.,The Fibonacci Numbers–Exposed *The Math. Magazine,* **2** (2002).

[4] Gould H, W., A New Greatest Common Divisor Property of the Binomial Coefficients,Fib.Quar., **9** (1971), 579-584,628.

[5] Koshy Thomas, Pellnomial Numbers, *Jr. Indian Acad. Math.,* **36** No., (2014), 189-201.

MEMBER'S COPY

[6] Shah Mansi, S., Shah Devbhadra, V., A New Class of Generalized Lucas Sequence, *Int. Jr. of Adv. Research in Eng., Sci. and Management* (2015), 1-7.

Mansi S. Shah
Department of Mathematics
Veer Narmad South Gujarat University,
Surat, India.
E-mail: mansi.shah_88@yahoo.co.in

Devbhadra V. Shah
Department of Mathematics
Veer Narmad South Gujarat University,
Surat, India.
E-mail: drdvshah@yahoo.com

MEMBER'S COPY

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 111–125

FIXED POINT RESULTS UNDER IMPLICIT RELATION IN S-METRIC SPACES

G. S. SALUJA

(Received : 21 - 11 - 2019 ; Revised : 22 - 03 - 2020)

Abstract. The aim of this paper is to establish some fixed point and common fixed points theorems in the setting of S-metric space under an implicit relation. Our results extend, unify and generalize several known results from the current existing literature.

1. Introduction and Preliminaries

Fixed point theory is one of the most important topic in the development of nonlinear analysis. It is well known that Banach's contraction mapping theorem is one of the pivotal results of functional analysis. A mapping $F\colon X \to X$ where (X, d) is a metric space, is said to be a contraction if there exists $a \in [0, 1)$ such that for all $x, y \in X$,

$$d(F(x), F(y)) \leq a\, d(x, y). \tag{1.1}$$

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of F. In recent time the study of fixed point theory in metric space is very interesting field and attract many researchers to investigated different results on it.

In 2006, Mustafa and Sims [17] introduced a new structure of generalized metric spaces which are called G-metric spaces as a generalization of metric spaces (X, d) to develop and introduce a new fixed point theory for various mappings in this new structure. In 2007, Sedghi et al. [20] introduced D^*-metric space which is a modification of D-metric spaces and proved some fixed point theorems in D^*-metric spaces. Later on many authors have studied the fixed point theorems in generalized metric spaces (see, for

2010 Mathematics Subject Classification: 54H25, 54E99.
Key words and phrases: Fixed point, common fixed point, implicit relation, S-metric space.

© Indian Mathematical Society, 2020.

MEMBER'S COPY

example [1, 2, 5, 6, 7, 9, 12, 13, 23, 24]). Also, we refer (see, [3], [11], [14], [15], [16], [27]) to the reader.

In 2012, Sedghi et al. [21] introduced the notion of S-metric space which is a generalization of a G-metric space and D^*-metric space. In [21] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in S-metric space for a self-map.

In 2013, Gupta [8] introduced the concept of cyclic contraction on S-metric space and proved some fixed point theorems on S-metric spaces which generalized the results of Sedghi et al. [21]. In 2014, Sedghi and Dung [22] have proved a general fixed point theorem in S-metric space using implicit relation and as application they obtained many analogous of fixed point theorems in metric spaces for S-metric spaces. In 2015, Prudhvi [18] proved some fixed point theorems on S-metric spaces which extend and improve the results of Sedghi and Dung [22].

The main purpose of this paper is to establish some fixed point and common fixed point theorems in S-metric space under an implicit relation. Our results extend, generalize and unify several known results from the existing literature.

We need the following definitions and lemmas in the sequel.

Definition 1.1. ([21]) Let X be a nonempty set and $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $t \in X$:

(SM_1) $S(x,y,z) = 0$ if and only if $x = y = z$;

(SM_2) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 1.2. ([21]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.

Example 1.3. ([21]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Example 1.4. ([22]) Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Lemma 1.5. *([21], Lemma 2.5) In an S-metric space, we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

MEMBER'S COPY

Lemma 1.6. *([21], Lemma 2.12) Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to +\infty$ then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to +\infty$.*

Definition 1.7. ([21]) Let (X,S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to +\infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n,x_n,x) < \varepsilon$. We denote this by $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to +\infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geq n_0$ we have $S(x_n,x_n,x_m) < \varepsilon$.

(3) The S-metric space (X,S) is called complete if every Cauchy sequence is a convergent sequence.

Definition 1.8. Let T be a self mapping on an S-metric space (X,S). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to +\infty$.

Definition 1.9. ([21]) Let (X,S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that

$$S(Tx,Tx,Ty) \leq L\, S(x,x,y)$$

for all $x,y \in X$. If the S-metric space (X,S) is complete then the mapping defined as above has a unique fixed point.

Now, we introduce an implicit relation to investigate some fixed point and common fixed point theorems in S-metric spaces.

Definition 1.10. (**Implicit Relation**) Let Ψ be the family of all real valued continuous functions $\psi\colon \mathbb{R}_+^4 \to \mathbb{R}_+$, for four variables. For some $k \in [0,1)$, we consider the following conditions.

(R_1) For $x,y \in \mathbb{R}_+$, if $x \leq \psi(y,y,x,\frac{x+2y}{2})$, then $x \leq ky$.

(R_2) For $x \in \mathbb{R}_+$, if $x \leq \psi(0,0,x,\frac{x}{2})$, then $x = 0$.

(R_3) For $x \in \mathbb{R}_+$, if $x \leq \psi(x,0,0,x)$, then $x = 0$, since $k \in [0,1)$.

2. Main Results

In this section we shall prove some fixed point and common fixed point theorems under an implicit relation in the setting of S-metric spaces.

MEMBER'S COPY

Theorem 2.1. *Let T be a self-map on a complete S-metric space (X,S) and*

$$S(Tx,Tx,Ty) \leq \psi\Big(S(x,x,y), S(x,x,Tx), S(y,y,Ty), \frac{S(x,x,Ty)+S(y,y,Tx)}{2}\Big) \tag{2.1}$$

for all $x,y \in X$ and some $\psi \in \Psi$. Then we have

(1) If ψ satisfies the condition (R_1) and (R_2), then T has a fixed point. Moreover, for any $x_0 \in X$ and the fixed point x, we have

$$S(Tx_n,Tx_n,x) \leq \Big(\frac{2k^n}{1-k}\Big)S(x_0,x_0,Tx_0).$$

(2) If ψ satisfies the condition (R_3) and T has a fixed point, then the fixed point is unique.

Proof. (1) For each $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. It follows from (2.1), (SM_2) and Lemma 1.5 that
$S(x_{n+1},x_{n+1},x_{n+2}) = S(Tx_n,Tx_n,Tx_{n+1})$

$$\leq \psi\Big(S(x_n,x_n,x_{n+1}), S(x_n,x_n,Tx_n), S(x_{n+1},x_{n+1},Tx_{n+1}), \mathrm{S}(\mathrm{x}_n,x_n,Tx_{n+1}) + S(x_{n+1},x_{n+1},Tx_n)/2\Big)$$

$$= \psi\Big(S(x_n,x_n,x_{n+1}), S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2}), \mathrm{S}(\mathrm{x}_n,x_n,x_{n+2}) + S(x_{n+1},x_{n+1},x_{n+1})/2\Big)$$

$$= \psi\Big(S(x_n,x_n,x_{n+1}), S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2}), \mathrm{S}(\mathrm{x}_n,x_n,x_{n+2})/2\Big)$$

$$\leq \psi\Big(S(x_n,x_n,x_{n+1}), S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2}), 2\mathrm{S}(\mathrm{x}_n,x_n,x_{n+1}) + S(x_{n+2},x_{n+2},x_{n+1})/2\Big)$$

$$= \psi\Big(S(x_n,x_n,x_{n+1}), S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2}), 2\mathrm{S}(\mathrm{x}_n,x_n,x_{n+1}) + S(x_{n+1},x_{n+1},x_{n+2})/2\Big).$$

Since ψ satisfies the condition (R_1), there exists $k \in [0,1)$ such that

$$S(x_{n+1},x_{n+1},x_{n+2}) \leq kS(x_n,x_n,x_{n+1}) \leq k^{n+1}S(x_0,x_0,x_1). \tag{2.2}$$

MEMBER'S COPY

Thus for all $n < m$, by using (SM_2), Lemma 1.5 and equation (2.2), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\cdots \\ &\leq 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1) \\ &\leq \Big(\frac{2k^n}{1-k}\Big)S(x_0, x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to +\infty$, we get $S(x_n, x_n, x_m) \to 0$, since $0 < k < 1$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, we have $\lim_{n \to +\infty} x_n = x \in X$. Moreover, taking the limit as $m \to +\infty$ we get

$$S(x_n, x_n, x) \leq \Big(\frac{2k^{n+1}}{1-k}\Big)S(x_0, x_0, x_1).$$

It implies that

$$S(Tx_n, Tx_n, x) \leq \Big(\frac{2k^n}{1-k}\Big)S(x_0, x_0, Tx_0).$$

Now we prove that x is a fixed point of T. By using inequality (2.1) again we get

$$\begin{aligned} S(x_{n+1}, x_{n+1}, Tx) &= S(Tx_n, Tx_n, Tx) \\ &\leq \psi\Big(S(x_n, x_n, x), S(x_n, x_n, Tx_n), S(x, x, Tx), \\ &\qquad \frac{S(x_n, x_n, Tx) + S(x, x, Tx_n)}{2}\Big) \\ &= \psi\Big(S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), \\ &\qquad \frac{S(x_n, x_n, Tx) + S(x, x, x_{n+1})}{2}\Big). \end{aligned}$$

Note that $\psi \in \Psi$, then using Lemma 1.6 and taking the limit as $n \to +\infty$, we get

$$S(x, x, Tx) \leq \psi\Big(0, 0, S(x, x, Tx), \frac{S(x, x, Tx)}{2}\Big).$$

Since ψ satisfies the condition (R_2), then $S(x, x, Tx) \leq k.0 = 0$. This shows that $x = Tx$. Thus x is a fixed point of T.

MEMBER'S COPY

(2) Let x_1, x_2 be fixed points of T with $x_1 \neq x_2$. We shall prove that $x_1 = x_2$. It follows from equation (2.1) and Lemma 1.5 that

$$\begin{aligned} S(x_1,x_1,x_2) &= S(Tx_1,Tx_1,Tx_2) \\ &\leq \psi\Big(S(x_1,x_1,x_2), S(x_1,x_1,Tx_1), S(x_2,x_2,Tx_2), \\ &\qquad \frac{S(x_1,x_1,Tx_2)+S(x_2,x_2,Tx_1)}{2}\Big) \\ &= \psi\Big(S(x_1,x_1,x_2), S(x_1,x_1,x_1), S(x_2,x_2,x_2), \\ &\qquad \frac{S(x_1,x_1,x_2)+S(x_2,x_2,x_1)}{2}\Big) \\ &= \psi\Big(S(x_1,x_1,x_2), 0, 0, \frac{S(x_1,x_1,x_2)+S(x_1,x_1,x_2)}{2}\Big) \\ &= \psi\Big(S(x_1,x_1,x_2), 0, 0, S(x_1,x_1,x_2)\Big). \end{aligned}$$

Since ψ satisfies the condition (R_3), then we get

$$S(x_1,x_1,x_2) \leq k\, S(x_1,x_1,x_2),$$

implies

$$S(x_1,x_1,x_2) = 0, \ \text{ since } 0 < k < 1.$$

This shows that $x_1 = x_2$. Thus the fixed point of T is unique. This completes the proof. $\square$

Theorem 2.2. *Let T_1 and T_2 be two self-maps on a complete S-metric space (X,S) and*

$$\begin{aligned} S(T_1x,T_1x,T_2y) &\leq \psi\Big(S(x,x,y), S(x,x,T_1x), S(y,y,T_2y), \\ &\qquad \frac{S(x,x,T_2y)+S(y,y,T_1x)}{2}\Big) \end{aligned} \tag{2.3}$$

for all $x, y \in X$ and some $\psi \in \Psi$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. For each $x_0 \in X$. Put $x_{2n+1} = T_1x_{2n}$ and $x_{2n+2} = T_2x_{2n+1}$ for $n = 0, 1, 2, \ldots$. It follows from (2.3), (SM_2) and Lemma 1.5 that

$$\begin{aligned} &S(x_{2n+1},x_{2n+1},x_{2n}) = S(T_1x_{2n},T_1x_{2n},T_2x_{2n-1}) \\ &\leq \psi\Big(S(x_{2n},x_{2n},x_{2n-1}), S(x_{2n},x_{2n},T_1x_{2n}), S(x_{2n-1},x_{2n-1},T_2x_{2n-1}), \\ &\qquad \mathrm{S}(\mathrm{x}_{2n},x_{2n},T_2x_{2n-1}) + S(x_{2n-1},x_{2n-1},T_1x_{2n})/2\Big) \\ &= \psi\Big(S(x_{2n},x_{2n},x_{2n-1}), S(x_{2n},x_{2n},x_{2n+1}), S(x_{2n-1},x_{2n-1},x_{2n}), \end{aligned}$$

MEMBER'S COPY

$$\begin{aligned}
&\qquad\qquad S(x_{2n}, x_{2n}, x_{2n}) + S(x_{2n-1}, x_{2n-1}, x_{2n+1})/2\Big)\\
&= \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n+1}), S(x_{2n-1}, x_{2n-1}, x_{2n}),\\
&\qquad\qquad S(x_{2n-1}, x_{2n-1}, x_{2n+1})/2\Big)\\
&\leq \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n+1}), S(x_{2n-1}, x_{2n-1}, x_{2n}),\\
&\qquad\qquad 2S(x_{2n-1}, x_{2n-1}, x_{2n}) + S(x_{2n+1}, x_{2n+1}, x_{2n})/2\Big)\\
&= \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n+1}, x_{2n+1}, x_{2n})/2\Big)
\end{aligned}$$

Since ψ satisfies the condition (R_1), there exists $k \in [0, 1)$ such that

$$S(x_{2n+1}, x_{2n+1}, x_{2n}) \leq kS(x_{2n}, x_{2n}, x_{2n-1}) \leq k^{2n} S(x_1, x_1, x_0). \qquad (2.4)$$

Thus for all $n < m$, by using (SM_2), Lemma 1.5 and equation (2.4), we have

$$\begin{aligned}
S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})\\
&= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m)\\
&\cdots\\
&\leq 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1)\\
&\leq \Big(\frac{2k^n}{1-k}\Big)S(x_0, x_0, x_1).
\end{aligned}$$

Taking the limit as $n, m \to +\infty$, we get $S(x_n, x_n, x_m) \to 0$, since $0 < k < 1$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, we have $\lim_{n\to+\infty} x_n = x \in X$. Now we have to prove that x is a common fixed point of T_1 and T_2. For this, consider

$$\begin{aligned}
S(x_{2n+1}, x_{2n+1}, T_1x) &= S(T_1x_{2n}, T_1x_{2n}, T_1x)\\
&\leq \psi\Big(S(x_{2n}, x_{2n}, x), S(x_{2n}, x_{2n}, T_1x_{2n}), S(x, x, T_1x),\\
&\qquad \frac{S(x_{2n}, x_{2n}, T_1x) + S(x, x, T_1x_{2n})}{2}\Big)\\
&= \psi\Big(S(x_{2n}, x_{2n}, x), S(x_{2n}, x_{2n}, x_{2n+1}), S(x, x, T_1x),\\
&\qquad \frac{S(x_{2n}, x_{2n}, T_1x) + S(x, x, x_{2n+1})}{2}\Big).
\end{aligned}$$

MEMBER'S COPY

Note that $\psi \in \Psi$, then using Lemma 1.6 and taking the limit as $n \to +\infty$, we get

$$S(x,x,T_1x) \leq \psi\Big(0,0,S(x,x,T_1x),\frac{S(x,x,T_1x)}{2}\Big).$$

Since ψ satisfies the condition (R_2), then $S(x,x,T_1x) \leq k.0 = 0$. This shows that $x = T_1x$ for all $x \in X$. Similarly, we can show that $x = T_2x$. Thus x is a common fixed point of T_1 and T_2.

Now to show that the common fixed point of T_1 and T_2 is unique. For this, let u be another common fixed point of T_1 and T_2, that is, $T_1u = T_2u = u$ with $x \neq u$. Then we have to show that $x = u$. It follows from equation (2.3) and Lemma 1.5 that

$$\begin{aligned}
S(x,x,u) &= S(T_1x,T_1x,T_2u) \\
&\leq \psi\Big(S(x,x,u),S(x,x,T_1x),S(u,u,T_2u), \\
&\qquad \frac{S(x,x,T_2u)+S(u,u,T_1x)}{2}\Big) \\
&= \psi\Big(S(x,x,u),S(x,x,x),S(u,u,u), \\
&\qquad \frac{S(x,x,u)+S(u,u,x)}{2}\Big) \\
&= \psi\Big(S(x,x,u),0,0,\frac{S(x,x,u)+S(x,x,u)}{2}\Big) \\
&= \psi\Big(S(x,x,u),0,0,S(x,x,u)\Big).
\end{aligned}$$

Since ψ satisfies the condition (R_3), then we get

$$S(x,x,u) \leq k\,S(x,x,u),$$

implies

$$S(x,x,u) = 0, \quad \text{since } 0 < k < 1.$$

Thus, we get $x = u$. This shows that x is the unique common fixed point of T_1 and T_2. This completes the proof. $\square$

Theorem 2.3. *Let T_1 and T_2 be two continuous self-maps on a complete S-metric space (X,S) and*

$$\begin{aligned}
S(T_1^px,T_1^px,T_2^qy) \;\leq\; & \psi\Big(S(x,x,y),S(x,x,T_1^px),S(y,y,T_2^qy), \\
& \frac{S(x,x,T_2^qy)+S(y,y,T_1^px)}{2}\Big) \qquad (2.5)
\end{aligned}$$

MEMBER'S COPY

for all $x, y \in X$, where p and q are some integers and some $\psi \in \Psi$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. Since T_1^p and T_2^q satisfy the conditions of Theorem 2.2. So T_1^p and T_2^q have a unique common fixed point. Let z be the common fixed point. Then, we have

$$\begin{aligned} T_1^p z = z &\Rightarrow T_1(T_1^p z) = T_1 z \\ &\Rightarrow T_1^p(T_1 z) = T_1 z. \end{aligned}$$

If $T_1 z = z_0$, then $T_1^p z_0 = z_0$. So, $T_1 z$ is a fixed point of T_1^p. Similarly, $T_2(T_2^q z) = T_2 z$. Now, using equation (2.5) and Lemma 1.5, we obtain

$$\begin{aligned} S(z, z, T_1 z) &= S(T_1^p z, T_1^p z, T_1^p(T_1 z)) \\ &\leq \psi\Big(S(z, z, T_1 z), S(z, z, T_1^p z), S(T_1 z, T_1 z, T_1^p(T_1 z)), \\ &\qquad \frac{S(z, z, T_1^p(T_1 z)) + S(T_1 z, T_1 z, T_1^p z)}{2}\Big) \\ &= \psi\Big(S(z, z, T_1 z), S(z, z, z), S(T_1 z, T_1 z, T_1 z), \\ &\qquad \frac{S(z, z, T_1 z) + S(T_1 z, T_1 z, z)}{2}\Big) \\ &= \psi\Big(S(z, z, T_1 z), 0, 0, \frac{S(z, z, T_1 z) + S(z, z, T_1 z)}{2}\Big) \\ &= \psi\Big(S(z, z, T_1 z), 0, 0, S(z, z, T_1 z)\Big). \end{aligned}$$

Since ψ satisfies the condition (R_3), then we get

$$S(z, z, T_1 z) \leq k\, S(z, z, T_1 z),$$

implies

$$S(z, z, T_1 z) = 0, \ \text{ since } 0 < k < 1.$$

Thus, we have $z = T_1 z$ for all $z \in X$. Similarly, we can show that $z = T_2 z$. This shows that z is a common fixed point of T_1 and T_2. For uniqueness of z, let $v \neq z$ be another common fixed point of T_1 and T_2. Then clearly v is also a common fixed point of T_1^p and T_2^q which implies $v = z$. Hence T_1 and T_2 have a unique common fixed point. This completes the proof. $\square$

MEMBER'S COPY

Theorem 2.4. *Let $\{T_\alpha\}$ be a family of continuous self mappings on a complete S-metric space (X, S) satisfying*

$$S(T_\alpha x, T_\alpha x, T_\beta y) \leq \psi\Big(S(x,x,y), S(x,x,T_\alpha x), S(y,y,T_\beta y), \frac{S(x,x,T_\beta y)+S(y,y,T_\alpha x)}{2}\Big) \tag{2.6}$$

for $\alpha, \beta \in \Psi$ with $\alpha \neq \beta$ and $x, y \in X$. Then there exists a unique $u \in X$ satisfying $T_\alpha u = u$ for all $\alpha \in \Psi$.

Proof. For $x_0 \in X$, we define a sequence as follows:

$$x_{2n+1} = T_\alpha x_{2n},\ x_{2n+2} = T_\beta x_{2n+1},\ n = 0, 1, 2, \ldots.$$

It follows from (2.6), (SM_2) and Lemma 1.5 that

$$S(x_{2n+1}, x_{2n+1}, x_{2n}) = S(T_\alpha x_{2n}, T_\alpha x_{2n}, T_\beta x_{2n-1})$$

$$\leq \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, T_\alpha x_{2n}), S(x_{2n-1}, x_{2n-1}, T_\beta x_{2n-1}), \mathrm{S}(\mathrm{x}_{2n}, x_{2n}, T_\beta x_{2n-1}) + S(x_{2n-1}, x_{2n-1}, T_\alpha x_{2n})/2\Big)$$

$$= \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n+1}), S(x_{2n-1}, x_{2n-1}, x_{2n}), \mathrm{S}(\mathrm{x}_{2n}, x_{2n}, x_{2n}) + S(x_{2n-1}, x_{2n-1}, x_{2n+1})/2\Big)$$

$$= \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n+1}), S(x_{2n-1}, x_{2n-1}, x_{2n}), \mathrm{S}(\mathrm{x}_{2n-1}, x_{2n-1}, x_{2n+1})/2\Big)$$

$$\leq \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n+1}), S(x_{2n-1}, x_{2n-1}, x_{2n}), 2\mathrm{S}(\mathrm{x}_{2n-1}, x_{2n-1}, x_{2n}) + S(x_{2n+1}, x_{2n+1}, x_{2n})/2\Big)$$

$$= \psi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n+1}, x_{2n+1}, x_{2n}), 2\mathrm{S}(\mathrm{x}_{2n}, x_{2n}, x_{2n-1}) + S(x_{2n+1}, x_{2n+1}, x_{2n})/2\Big).$$

Since ψ satisfies the condition (R_1), there exists $k \in (0, 1)$ such that

$$S(x_{2n+1}, x_{2n+1}, x_{2n}) \leq kS(x_{2n}, x_{2n}, x_{2n-1}) \leq k^{2n} S(x_1, x_1, x_0). \tag{2.7}$$

MEMBER'S COPY

Thus for all $n < m$, by using (SM_2), Lemma 1.5 and equation (2.7), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\cdots \\ &\leq 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1) \\ &\leq \Big(\frac{2k^n}{1-k}\Big) S(x_0, x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to +\infty$, we get $S(x_n, x_n, x_m) \to 0$, since $\left(\frac{2k^n}{1-k}\right) \to 0$ as $n \to +\infty$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, there exists $p \in X$ such that $x_n \to p \in X$ as $n \to +\infty$. By the continuity of T_α and T_β, it is clear that $T_\alpha p = T_\beta p = p$. Therefore p is a common fixed point of T_α for all $\alpha \in \Psi$.

In order to prove the uniqueness, let us take another common fixed point, w of T_α and T_β where $p \neq w$. Then using equation (2.6) and Lemma 1.5, we obtain

$$\begin{aligned} S(p, p, w) &= S(T_\alpha p, T_\alpha p, T_\beta w) \\ &\leq \psi\Big(S(p, p, w), S(p, p, T_\alpha p), S(w, w, T_\beta w), \\ &\qquad \frac{S(p, p, T_\beta w) + S(w, w, T_\alpha p)}{2}\Big) \\ &= \psi\Big(S(p, p, w), S(p, p, p), S(w, w, w), \\ &\qquad \frac{S(p, p, w) + S(w, w, p)}{2}\Big) \\ &= \psi\Big(S(p, p, w), 0, 0, \frac{S(p, p, w) + S(p, p, w)}{2}\Big) \\ &= \psi\Big(S(p, p, w), 0, 0, S(p, p, w)\Big). \end{aligned}$$

Since ψ satisfies the condition (R_3), then we get

$$S(p, p, w) \leq k\, S(p, p, w),$$

implies

$$S(p, p, w) = 0, \quad \text{since } 0 < k < 1.$$

MEMBER'S COPY

Thus, we get $p = w$ for all $w \in X$. This shows that p is a unique common fixed point of T_α for all $\alpha \in \Psi$. This completes the proof. $\square$

Next, we give an analogues of fixed point theorems in metric spaces for S-metric spaces by combining Theorem 2.1 with $\psi \in \Psi$ and ψ satisfies conditions (R_1), (R_2) and (R_3). The following corollary is an analogue of Banach's contraction principle.

Corollary 2.5. *Let (X, S) be a complete S-metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tx, Tx, Ty) \leq L\, S(x, x, y)$$

for all $x, y \in X$, where $L \in [0, 1)$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.1 with $\psi(u, v, w, z) = Lu$ for some $L \in [0, 1)$ and all $u, v, w, z \in \mathbb{R}_+$. $\square$

The following corollary is an analogue of R. Kannan's result in [10].

Corollary 2.6. *Let (X, S) be a complete S-metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tx, Tx, Ty) \leq L_1 \left[S(x, x, Tx) + S(y, y, Ty)\right]$$

for all $x, y \in X$, where $L_1 \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.1 with $\psi(u, v, w, z) = L_1(v + w)$ for some $L_1 \in [0, \frac{1}{2})$ and all $u, v, w, z \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(y, y, x, \frac{x+2y}{2}) = L_1(y + x)$. So, if $x \leq \psi(y, y, x, \frac{x+2y}{2})$, then $x \leq \left(\frac{L_1}{1-L_1}\right) y$ with $\left(\frac{L_1}{1-L_1}\right) < 1$. Thus, T satisfies the condition (R_1).

Next, if $x \leq \psi(0, 0, x, \frac{x}{2}) = L_1(0 + x) = L_1 x$, then $x = 0$, since $L_1 < \frac{1}{2} < 1$. Thus, T satisfies the condition (R_2).

Finally, if $x \leq \psi(x, 0, 0, x) = L_1.0 = 0$, then $x = 0$. Thus, T satisfies the condition (R_3). $\square$

The following corollary is an analogue of S. K. Chatterjae's result in [4].

Corollary 2.7. *Let (X, S) be a complete S-metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tx, Tx, Ty) \leq L_2 \left[S(x, x, Ty) + S(y, y, Tx)\right]$$

MEMBER'S COPY

for all $x, y \in X$, where $L_2 \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.1 with $\psi(u, v, w, z) = L_2 z$ for some $L_2 \in [0, 1)$ and all $u, v, w, z \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(y, y, x, \frac{x+2y}{2}) = L_2\left(\frac{x+2y}{2}\right)$. So, if $x \le \psi(y, y, x, \frac{x+2y}{2})$, then $x \le \left(\frac{2L_2}{2-L_2}\right)y$ with $\left(\frac{2L_2}{2-L_2}\right) < 1$. Thus, T satisfies the condition (R_1).

Next, if $x \le \psi(0, 0, x, \frac{x}{2})$, then $x = 0$ since $L_2 < 1$. Thus, T satisfies the condition (R_2).

Finally, if $x \le \psi(x, 0, 0, x) = L_2 x$, then $x = 0$ since $L_2 < 1$. Thus, T satisfies the condition (R_3). □

The following corollary is an analogue of S. Reich's result in [19].

Corollary 2.8. *Let (X, S) be a complete S-metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tx, Tx, Ty) \le a_1\, S(x, x, y) + a_2\, S(x, x, Tx) + a_3\, S(y, y, Ty)$$

for all $x, y \in X$, where $a_1, a_2, a_3 \ge 0$ are constants with $a_1 + a_2 + a_3 < 1$. Then T has a unique fixed point in X. Moreover, if $a_3 < \frac{1}{2}$, then T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.1 with $\psi(u, v, w, z) = a_1 u + a_2 v + a_3 w$ for some $a_1, a_2, a_3 \ge 0$ are constants with $a_1 + a_2 + a_3 < 1$ and all $u, v, w, z \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(y, y, x, \frac{x+2y}{2}) = a_1 y + a_2 y + a_3 x$. So, if $x \le \psi(y, y, x, \frac{x+2y}{2})$, then $x \le \left(\frac{a_1+a_2}{1-a_3}\right)y$ with $\left(\frac{a_1+a_2}{1-a_3}\right) < 1$. Thus, T satisfies the condition (R_1).

Next, if $x \le \psi(0, 0, x, \frac{x}{2}) = a_1.0 + a_2.0 + a_3.x = a_3 x$, then $x = 0$ since $a_3 < 1$. Thus, T satisfies the condition (R_2).

Finally, if $x \le \psi(x, 0, 0, x) = a_1.x + a_2.0 + a_3.0 = a_1 x$, then $x = 0$ since $a_1 < 1$. Thus, T satisfies the condition (R_3). □

Example 2.9. Let $X = \mathbb{R}$ be the usual S-metric space as in Example 1.4. Now, we consider the mapping $T\colon X \to X$ by $T(x) = \frac{x}{10}$ for all $x \in [0, 1]$.

MEMBER'S COPY

Then

$$\begin{aligned} S(Tx,Tx,Ty) &= |Tx-Ty| + |Tx-Ty| \\ &= 2|Tx-Ty| = 2\Big|\Big(\frac{x}{10}\Big) - \Big(\frac{y}{10}\Big)\Big| \\ &= \frac{1}{5}|x-y| \\ &\le \frac{2}{5}|x-y| \\ &= \frac{1}{5}\big(2|x-y|\big) \\ &= \mu\, S(x,x,y) \end{aligned}$$

where $\mu = \frac{1}{5} < 1$. Thus T satisfies all the conditions of Corollary 2.5 and clearly $0 \in X$ is the unique fixed point of T.

Example 2.10. Let $X = \mathbb{R}$ be the usual S-metric space as in Example 1.4. Now, we consider the mapping $T\colon X \to X$ by $T(x) = \frac{x}{5}$ for all $x \in [0,1]$. Then

$$\begin{aligned} S(Tx,Tx,Ty) &= |Tx-Ty| + |Tx-Ty| \\ &= 2|Tx-Ty| = 2\Big|\Big(\frac{x}{5}\Big) - \Big(\frac{y}{5}\Big)\Big| \\ &= \frac{2}{5}|x-y| \\ &\le \frac{4}{9}|x-y| \\ &\le \frac{1}{3}\Big[\frac{4}{3}|x| + \frac{4}{3}|y|\Big]. \end{aligned}$$

$$S(x,x,Tx) = 2|x-Tx| = \frac{4}{3}|x|.$$

$$S(y,y,Ty) = 2|y-Ty| = \frac{4}{3}|y|.$$

Now, we have

$$\begin{aligned} S(Tx,Tx,Ty) &\le \frac{1}{3}[S(x,x,Tx) + S(y,y,Ty)] \\ &= q[S(x,x,Tx) + S(y,y,Ty)] \end{aligned}$$

where $q = \frac{1}{3} < \frac{1}{2}$. Thus T satisfies all the conditions of Corollary 2.6 and clearly $0 \in X$ is the unique fixed point of T.

MEMBER'S COPY

Example 2.11. Let $X = \mathbb{R}$ be the usual S-metric space as in Example 1.4. Now, we consider the mapping $T\colon X \to X$ by $T(x) = \frac{x+2}{3}$ for all $x \in [0, 1]$. Then

$$\begin{aligned} S(Tx, Tx, Ty) &= |Tx - Ty| + |Tx - Ty| \\ &= 2|Tx - Ty| = 2\Big|\Big(\frac{x+2}{3}\Big) - \Big(\frac{y+2}{3}\Big)\Big| \\ &= \frac{2}{3}|x - y|. \end{aligned}$$

$$S(x, x, Ty) = 2|x - Ty| = \frac{2}{3}|3x - y - 2|.$$

$$S(y, y, Tx) = 2|y - Tx| = \frac{2}{3}|3y - x - 2|.$$

Now, we have

$$\begin{aligned} S(Tx, Tx, Ty) &\le \frac{2}{7}[S(x, x, Ty) + S(y, y, Tx)] \\ &= \delta[S(x, x, Ty) + S(y, y, Tx)] \end{aligned}$$

where $\delta = \frac{2}{7} < \frac{1}{2}$. Thus T satisfies all the conditions of Corollary 2.7 and clearly $1 \in X$ is the unique fixed point of T.

Conclusion

In this paper, we establish some fixed point and common fixed point theorems satisfying an implicit relation in the framework of S-metric spaces. Also, we give some examples in support of our results. Our results extend, unify and generalize several results from the existing literature.

Acknowledgements

The author is grateful to the anonymous referee for his/her careful reading and valuable suggestions to improve the manuscript.

References

[1] Agarwal, R. P., Meehan, M. and O'Regan, D., Fixed Point Theory and Applications, Cambridge University Press, 2004.

[2] Agarwal, R. P., Karapinar, E., O'Regan, D. and Roldan-Lopez-de-Hierro, A. F., Fixed Point Theory in Metric Type Spaces, Springer International Publishing, Switzerland, 2015.

MEMBER'S COPY

[3] Aleksić, Suzana, Mitrović, Z. D. and Radenović, S., Picard sequences in b-metric spaces, *Fixed Point Theory* **21** (2020), No. 1, 35-46.

[4] Chatterjae, S. K., Fixed point theorems compactes, *Rend. Acad. Bulgare Sci.* **25** (1972), 727-730.

[5] Ćirić, L., Some recent results in metrical fixed point theory, University of Belgrade, Beograd (Serbia), 2003.

[6] Dosenović, T., Radenović, S. and Sedghi, S., Generalized metric spaces: Survey, *TWMS. J. Pure Appl. Math.* Vol. **9** (2018), No. 1, 3-17.

[7] Esfahani, J., Mitrović, Z. D., Radenović, S. and Sedghi, S., Suzuki-type fixed point results in S-metric type spaces, *Commun. Appl. Nonlinear Anal.* **25** (2018), No. 3, 27-36.

[8] Gupta, A., Cyclic contraction on S-metric space, *Int. J. Anal. Appl.* **3(2)** (2013), 119-130.

[9] Jonanvić, M., Kadelburg, Z. and Radenović, S., Common fixed point results in metric-type spaces, *Fixed Point Theory Appl.* **2010** (2010), 1-15.

[10] Kannan, R., Some results on fixed point theorems, *Bull. Calcutta Math. Soc.* **60**(1969), 71-78.

[11] Kukić, K., Shatanawi, W. and Gardasević-Filipović, M., Khan and Ćirić contraction principles in almost b-metric spaces, *U.P.B. Sci. Bull. Series A*, Vol. **82** (2020), Iss. 1, 133-144.

[12] Khamsi, M. A., Remarks on cone metric spaces and fixed point theorems of contractive mappings, *Fixed Point Theory Appl.* **2010** (2010), 1-7.

[13] Malkowski, E. and Rakocević, V., Advanced Functional Analysis, CRS Press, Tayior and Francis Group, Boca Raton, FL, 2019.

[14] Miculescu, R. and Mihail, A., New fixed point theorems for set-valued contractions in b-metric spaces, *J. Fixed Point Theory* **19** (2017), 2153-2163.

[15] Mitrović, Z. D., On an open problem in rectangular b-metric spaces, *J. Anal.* **25**(2017), 135-137.

[16] Mitrović, Z. D., A note on the result of Suzuki, Miculescu and Mihail, *J. Fixed Point Theory Appl.* (2019), 21:24, http://doi.org/10.1007/s11784-019-0663-5.

[17] Mustafa, Z. and Sims, B. I., A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.* **7** (2006), 289-297.

[18] Prudhvi, K., Fixed point theorems in S-metric spaces, *Universal J. Comput. Math.* **3(2)** (2015), 19-21.

[19] Reich, S., Some remarks concerning contraction mappings, *Canad. Math. Bull.* **14** (1971), 121-124.

[20] Sedghi, S., Shobe, N. and Zhou, H., A common fixed point theorem in D^*-metric space, *Fixed Point Theory Appl.* (2007), 1-13.

[21] Sedghi, S., Shobe, N. and Aliouche, A., A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik* **64(3)** (2012), 258-266.

[22] Sedghi, S. and Dung, N. V., Fixed point theorems on S-metric spaces, *Mat. Vesnik* **66(1)** (2014), 113-124.

MEMBER'S COPY

[23] Sedghi, S., Gholidahneh, A., Dosenović, T., Esfahani, J. and Radenović, S., Common fixed point of four maps in S_b-metric spaces, *J. Linear and Topol. Algebra* Vol. **5** (2016), No. 2, 93-104.

[24] Sedghi, S., Rezaee, M. M., Dosenović, T. and Radenović, S., Common fixed point theorems for contractive mappings satisfying Φ-maps in S-metric spaces, *Acta Univ. Sapientiae Mathematica* **8** (2016), No. 2, 298-311.

[25] Shatanawi, W., Fixed point theory for contractive mappings satisfying ϕ-maps in G spaces, *Fixed Point Theory Appl.* Vol. 2010, Article ID 181650.

[26] Tas, N. and Yilmaz Ozgur, N., New generalized fixed point results on S_b-metric spaces, arxiv:1703.01868v2 [math.gn] 17 apr. 2017.

[27] Todorcević, Vesna, Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics, Springer Nature Switzerland AG, 2019.

G. S. Saluja
Department of Mathematics,
Govt. Kaktiya P. G. College Jagdalpur,
Jagdalpur-494001 (C.G.), India.
E-mail: saluja1963@gmail.com

MEMBER'S COPY

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 127–143

UNIQUENESS OF DIFFERENTIAL POLYNOMIALS OF A MEROMORPHIC FUNCTION CONCERNING WEIGHTED SHARING

DILIP CHANDRA PRAMANIK, JAYANTA ROY AND AFSAR HOSSAIN SARKAR
(Received : 19 - 05 - 2020 ; Revised : 07 - 09 - 2020)

Abstract. In this paper, we prove a uniqueness result of the differential polynomial $(f^n)^{(k)}$ and the differential polynomial $P[f^q]$ of meromorphic function f^q, when they share a small function. Our result generalize some earlier results.

1. Introduction and Main Results

Throughout this paper, we use the standard notations of the Nevanlinna theory of meromorphic functions as explained in ([12, 21, 22]). Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $b \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of b-points with the same multiplicities, then we say that f and g share the value b counting multiplicities (CM) and if we do not consider the multiplicities, then f and g are said to share the value b ignoring multiplicities (IM). If $b = \infty$, then the zeros of $f - b$ means the poles of f.
A meromorphic function $a = a(z)$ ($\not\equiv 0, \infty$) is called a small function with respect to f provided that $T(r, a) = S(r, f)$ as $r \to \infty$, $r \notin E$, where E is a set of positive real numbers with finite Lebesgue measure. We say that f and g share a IM(CM) according to $f - a$ and $g - a$ share 0 IM(CM). We denote by $S(f)$ the collection of all small functions with respect to f.

The subject on sharing values between entire function with its derivatives was first studies by Rubel and Yang ([20]). In 1977, they proved the following result.

2010 Mathematics Subject Classification: 30D35
Key words and phrases: Meromorphic function, Differential polynomial, Weighted sharing, Uniqueness

© Indian Mathematical Society, 2020.

MEMBER'S COPY

Theorem 1.1. *Let f be a non-constant entire function. If f and f' share two distinct finite complex numbers a, b CM, then $f = f'$.*

In 1979, Mues and Steinmetz ([19]) obtained the same result but in relax sharing condition as follows.

Theorem 1.2. *Let f be a non-constant entire function. If f and f' share two distinct finite complex numbers a, b IM, then $f = f'$.*

Subsequently, similar considerations have been made with respect to higher derivatives and more general differential expressions as well. Above theorem motivate researchers to study the relation between an entire function and its derivative counterpart for one CM shared value. In this direction, in 1996, the following famous conjecture was proposed by Brück ([8]).

Conjecture 1.3. Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite, where

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

If f and f' share a finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is a non-zero constant.

Very recently many results have been published concerning the above conjecture ([5, 6, 9, 11, 15, 16, 18]). Next we recall the following definitions.

Definition 1.4. Let p be a positive integer. Let f be a meromorphic function and $a \in S(f)$.

(i) $\overline{N}_{p)}(r, \frac{1}{f-a})$ denotes the counting function of those a-points of f whose multiplicities are not greater than p, where each a-point is counted only once.

(ii) $\overline{N}_{(p}(r, \frac{1}{f-a})$ denotes the counting function of those a-points of f whose multiplicities are not less than p, where each a-point is counted only once.

(iii) $N_p(r, \frac{1}{f-a})$ denotes the counting function of those a-points of f, where an a-point of f with multiplicity m is counted m times if $m \leq p$ and

MEMBER'S COPY

p times if $m > p$.

We denote $\delta_p(a, f)$ by the quantity

$$\delta_p(a, f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, \frac{1}{f-a})}{T(r, f)}.$$

Clearly $0 \leq \delta(a, f) \leq \delta_p(a, f) \leq \delta_{p-1}(a, f) \leq .... \leq \delta_2(a, f) \leq \delta_1(a, f) = \Theta(a, f)$.

Definition 1.5. Suppose f and g share a IM and let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q.

(i) By $\overline{N}_L(r, \frac{1}{f-a})$ we denote the reduced counting function of those a-points of f and g where $p > q \geq 1$; $\overline{N}_L(r, \frac{1}{g-a})$ is defined similarly.

(ii) By $N_E^{1)}(r, \frac{1}{f-a})$ the counting function of those a-points of f and g where $p = q = 1$,
and
(iii) by $\overline{N}_E^{(2}(r, \frac{1}{f-a})$ the counting function of those a-points of f and g where $p = q \geq 2$, where each such zero is counted only once.

Relaxation of the sharing is done on the basis of the following notion of weighted sharing as introduced in ([13, 14]).

Definition 1.6. Let l be a non-negative integer or infinity and $a \in S(f) \cap S(g)$. We denote by $E_l(a, f)$ the set of all zeros of $f - a$, where a zero of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that $f,\ g$ share the function a with weight l. We write f and g share (a, l) to mean that f and g share the function a with weight l. Since $E_l(a, f) = E_l(a, g)$ implies that $E_s(a, f) = E_s(a, g)$ for any integer $s\,(0 \leq s < l)$, if $f,\ g$ share (a, l), then $f,\ g$ share (a, s). Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

In 2008, Zhang and Lü ([25]) considered the uniqueness of the n-th power of a meromorphic function sharing a small function with its k-th derivative and proved the following theorem.

Theorem 1.7. *Let $k\ (\geq 1)$, $n\ (\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)\ (\not\equiv 0, \infty)$ be a small function with*

MEMBER'S COPY

respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0,l)$. If $l = \infty$ and

$$(3+k)\Theta(\infty,f) + 2\Theta(0,f) + \delta_{2+k}(0,f) > 6 + k - n$$

or, $l = 0$ and

$$(6+2k)\Theta(\infty,f) + 4\Theta(0,f) + 2\delta_{2+k}(0,f) > 12 + 2k - n,$$

then $f^n \equiv f^{(k)}$.

In the same paper, Zhang and Lü ([25]) posted the following question:

Question 1.8. What will happen if f^n and $\left(f^{(k)}\right)^s$ share a small function ?

In 2014, with the notion of weighted sharing of small function Banerjee and Majumder ([3]) prove two theorems, one of which improved Theorem 1.7 where together answer the Question 1.8.

Theorem 1.9. *Let k (≥ 1), n (≥ 1) be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \geq 2$ and*

$$(3+k)\Theta(\infty,f) + 2\Theta(0,f) + \delta_{2+k}(0,f) > 6 + k - n$$

or $l = 1$ and

$$(k+\frac{7}{2})\Theta(\infty,f) + \frac{5}{2}\Theta(0,f) + \delta_{2+k}(0,f) > 7 + k - n$$

or $l = 0$ and

$$(6+2k)\Theta(\infty,f) + 4\Theta(0,f) + \delta_{2+k}(0,f) + \delta_{1+k}(0,f) > 12 + 2k - n,$$

then $f^n \equiv f^{(k)}$.

Theorem 1.10. *Let k (≥ 1), n (≥ 1), m (≥ 2) be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $(f^{(k)})^m - a$ share $(0,l)$. If $l \geq 2$ and*

$$(3+2k)\Theta(\infty,f) + 2\Theta(0,f) + 2\delta_{1+k}(0,f) > 7 + 2k - n$$

or $l = 1$ and

$$(2k+\frac{7}{2})\Theta(\infty,f) + \frac{5}{2}\Theta(0,f) + 2\delta_{1+k}(0,f) > 8 + 2k - n$$

or $l = 0$ and

$$(6+3k)\Theta(\infty,f) + 4\Theta(0,f) + 3\delta_{1+k}(0,f) > 13 + 3k - n,$$

MEMBER'S COPY

then $f^n \equiv (f^{(k)})^m$.

In order to improve the results of Zhang ([23]), Li and Huang ([17]) obtain the following theorem.

Theorem 1.11. *Let* $k\ (\geq 1)$, $l\ (\geq 0)$ *be integers and* f *be a non-constant meromorphic function. Also let* $a(z)(\not\equiv 0,\infty)$ *be a small function with respect to* f. *Suppose* $f-a$ *and* $f^{(k)}-a$ *share (0,l). If* $l \geq 2$ *and*

$$(k+3)\Theta(\infty,f)+\delta_2(0,f)+\delta_{2+k}(0,f)>k+4$$

or $l=1$ *and*

$$(k+\frac{7}{2})\Theta(\infty,f)+\frac{1}{2}\Theta(0,f)+\delta_2(0,f)+\delta_{2+k}(0,f)>k+5$$

or $l=0$ *and*

$$(6+2k)\Theta(\infty,f)+2\Theta(0,f)+\delta_2(0,f)+\delta_{2+k}(0,f)+\delta_{1+k}(0,f)>2k+10,$$

then $f\equiv f^{(k)}$.

Next we give the following definition.

Definition 1.12. Let $n_{0j},\ n_{1j}\ ,n_{2j},...,\ n_{kj}$ be non-negative integers. The expression

$$M_j[f]=(f)^{n_{0j}}(f^{(1)})^{n_{1j}}(f^{(2)})^{n_{2j}}...(f^{(k)})^{n_{kj}}$$

is called a differential monomial generated by f of degree $d(M_j)=\sum\limits_{i=0}^{k}n_{ij}$ and weight $\Gamma_{M_j}=\sum\limits_{i=0}^{k}(i+1)n_{ij}$. Let $a_j\in S(f)$ and $a_j\not\equiv 0$ $(j=1,2,...,t)$. The sum $P[f]=\sum\limits_{j=1}^{t}a_jM_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P)=\max\{d(M_j):1\leq j\leq t\}$ and weight $\Gamma_P=\max\{\Gamma_{M_j}:1\leq j\leq t\}$. The numbers $\underline{d}(P)=\min\{d(M_j):1\leq j\leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and the order of $P[f]$. The differential polynomial $P[f]$ is said to be homogeneous if $\overline{d}(P)=\underline{d}(P)$, otherwise $P[f]$ is called a non-homogeneous differential polynomial. Also, we define $Q:=\max\{\Gamma_{M_j}-d(M_j):1\leq j\leq t\}$.

Using the notion of differential polynomial Bhoosnurmath and Kabbur ([7]) proved the following theorem.

MEMBER'S COPY

Theorem 1.13. *Let f be a non-constant meromorphic function and $a(z)$ be a small function such that $a(z) \not\equiv 0, \infty$. Let $P[f]$ be a non-constant differential polynomial in f. If f and $P[f]$ share the value a IM and*

$$(2Q+6)\Theta(\infty, f) + (2+3\underline{d}(P))\delta(0, f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7,$$

then $f \equiv P[f]$.

Motivated by such uniqueness investigations, Charak and Lal ([10]) considered the possible extension of Theorems 1.9-1.13 in the direction of the question of Zhang and Lü ([25]) up to differential polynomial considering weighted sharing.

Theorem 1.14. *Let f be a non-constant meromorphic function and n be a positive integer and $a(z)$ $(\not\equiv 0, \infty)$ be a small function with respect to f. Let $P[f]$ be a non-constant differential polynomial of f. Suppose $f^n - a$ and $P[f] - a$ share $(0, l)$. If $l \geq 2$ and*

$$(3+Q)\Theta(\infty, f) + 2\Theta(0, f) + \overline{d}(P)\delta(0, f) > Q + 5 + 2\overline{d}(P) - \underline{d}(P) - n,$$

or, $l = 1$ and

$$(\frac{7}{2}+Q)\Theta(\infty, f) + \frac{5}{2}\Theta(0, f) + \overline{d}(P)\delta(0, f) > Q + 6 + 2\overline{d}(P) - \underline{d}(P) - n,$$

or, $l = 0$ and

$$(6+2Q)\Theta(\infty, f) + 4\Theta(0, f) + 2\overline{d}(P)\delta(0, f) > 2Q + 4\overline{d}(P) - 2\underline{d}(P) + 10 - n,$$

then $f^n \equiv P[f]$.

Regarding above results, a natural question is what can be said when a non-constant differential polynomial $(f^n)^{(k)}$ share a small meromorphic function a with $P[f^q]$. In this paper we prove that under certain essential conditions $(f^n)^{(k)} \equiv P[f^q]$.

Theorem 1.15. *Let k (≥ 1), n $(\geq k+2)$, q (≥ 1), l (≥ 0) be integers and f be a non-constant meromorphic function. Also let a $(\not\equiv 0, \infty) \in S(f)$ and $P[f^q]$ be a non-constant differential polynomial of f^q such that $n \neq q\overline{d}(P)$. Suppose $(f^n)^{(k)} - a$ and $P[f^q] - a$ share $(0, l)$. If $l \geq 2$ and*

$$(Q+3)\Theta(\infty, f) + q\overline{d}(P)\delta_{k+2}(0, f) + n\delta_{k+2}(0, f^n) > Q + 3 + q\overline{d}(P), \quad (1.1)$$

MEMBER'S COPY

or, $l = 1$ *and*

$$(Q+\tfrac{k+7}{2})\Theta(\infty,f)+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n)$$
$$+n\delta_{k+1}(0,f^n) > Q+\tfrac{k+7}{2}+q\overline{d}(P)+n, \tag{1.2}$$

or $l = 0$ *and*

$$(2Q+6+2k)\Theta(\infty,f)+q\overline{d}(P)\delta_{k+1}(0,f)+2n\delta_{k+1}(0,f^n)$$
$$+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n) > 2Q+2k+2q\overline{d}(P)+6+2n, \tag{1.3}$$

then $(f^n)^{(k)} \equiv P[f^q]$.

2. Lemmas

In this section we present some lemmas needed in the sequel. Let F, G be two non-constant meromorphic functions. We shall denote by H the following function

$$H=\left(\frac{F''}{F'}-2\frac{F'}{F-1}\right)-\left(\frac{G''}{G'}-2\frac{G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. *Let* $q(\geq 1)$ *be an integer. Let* f *be a non-constant meromorphic function and* $P[f^q]$ *be differential polynomial of* f^q. *Then*

$$m\left(r,\frac{P[f^q]}{f^{q\overline{d}(P)}}\right)\leq(\overline{d}(P)-\underline{d}(P))m(r,\frac{1}{f^q})+S(r,f).$$

Proof. The proof can be obtain along the same line as the proof of Lemma 2.4 in [4]. □

Lemma 2.2. [1] *Let* f *and* g *be two non-constant meromorphic functions.*
(i) If f *and* g *share* $(1,0)$, *then*

$$\overline{N}_L\left(r,\frac{1}{f-1}\right)\leq\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+S(r,f).$$

(ii) If f *and* g *share* $(1,1)$, *then*

$$2\overline{N}_L\big(r,\tfrac{1}{f-1}\big)+2\overline{N}_L\big(r,\tfrac{1}{g-1}\big)+\overline{N}_E^{(2}\big(r,\tfrac{1}{f-1}\big)$$
$$-\overline{N}_{f>2}\big(r,\tfrac{1}{g-1}\big)\leq N(r,\tfrac{1}{g-1})-\overline{N}(r,\tfrac{1}{g-1}).$$

Lemma 2.3. [24] *Let* f *be a non-constant meromorphic function and* p, k *be two positive integers. Then*

$$N_p(r,\frac{1}{f^{(k)}})\leq T(r,f^{(k)})-T(r,f)+N_{p+k}(r,\frac{1}{f})+S(r,f),$$

MEMBER'S COPY

$$N_p(r,\frac{1}{f^{(k)}}) \le N_{p+k}(r,\frac{1}{f}) + k\overline{N}(r,\infty,f) + S(r,f),$$

where

$$\overline{N}(r,\frac{1}{f^{(k)}}) = N_1(r,\frac{1}{f^{(k)}}).$$

Lemma 2.4. [2] *Let F and G share $(1,l)$ and $H \not\equiv 0$, then*

$$N(r,H) \le \overline{N}(r,F) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}_{(2}(r,\tfrac{1}{G}) + \overline{N}_L(r,\tfrac{1}{F-1})$$
$$+\overline{N}_L(r,\tfrac{1}{G-1}) + N_0(r,\tfrac{1}{F'}) + N_0(r,\tfrac{1}{G'}) + S(r,f).$$

Lemma 2.5. [5] *For any two non-constant meromorphic functions f and g,*

$$N_p(r,fg) \le N_p(r,f) + N_p(r,g).$$

Lemma 2.6. *Let p, q be positive integers. Let f be a non-constant meromorphic function and $P[f^q]$ be a differential polynomial of f^q. Then*

$$N_p\left(r,\frac{1}{P[f^q]}\right) \le q\overline{d}(P)N_{p+k}(r,\frac{1}{f}) + Q\overline{N}(r,f) + S(r,f).$$

Proof. For any non-constant meromorphic function f, $N_p(r,f) \le N_s(r,f)$ if $p \le s$. Now by Lemma 2.3 and Lemma 2.5 we have

$$N_p\left(r,\tfrac{1}{P[f^q]}\right) \le \textstyle\sum_{j=1}^{t} N_p\left(r,\tfrac{1}{M_j[f^q]}\right) + S(r,f)$$
$$= N_p\left(r,\tfrac{1}{M_1[f^q]}\right) + N_p\left(r,\tfrac{1}{M_2[f^q]}\right) + ... + N_p\left(r,\tfrac{1}{M_t[f^q]}\right) + S(r,f)$$
$$= N_p\left(r,\tfrac{1}{\prod_{i=0}^{k}\left((f^q)^{(i)}\right)^{n_{i1}}}\right) + N_p\left(r,\tfrac{1}{\prod_{i=0}^{k}\left((f^q)^{(i)}\right)^{n_{i2}}}\right) + ... +$$
$$+N_p\left(r,\tfrac{1}{\prod_{i=0}^{k}\left((f^q)^{(i)}\right)^{n_{it}}}\right) + S(r,f)$$
$$\le \textstyle\sum_{i=0}^{k} n_{i1}N_p\left(r,\tfrac{1}{\left((f^q)^{(i)}\right)}\right) + \textstyle\sum_{i=0}^{k} n_{i2}N_p\left(r,\tfrac{1}{\left((f^q)^{(i)}\right)}\right) + ... +$$
$$+\textstyle\sum_{i=0}^{k} n_{it}N_p\left(r,\tfrac{1}{\left((f^q)^{(i)}\right)}\right) + S(r,f)$$
$$\le \textstyle\sum_{i=0}^{k}(n_{i1}+n_{i2}+...+n_{it})\left\{N_{p+i}\left(r,\tfrac{1}{f^q}\right) + i\overline{N}(r,f^q)\right\} + S(r,f)$$
$$\le \max_{1\le j\le t}\left\{\textstyle\sum_{i=0}^{k} n_{ij}qN_{p+k}\left(r,\tfrac{1}{f}\right)\right\} + \max_{1\le j\le t}\left(\textstyle\sum_{i=0}^{k} i.n_{ij}\right)\overline{N}(r,f)$$
$$+S(r,f) = q\overline{d}(P)N_{p+k}(r,\tfrac{1}{f}) + Q\overline{N}(r,f) + S(r,f).$$

This completes the proof of the lemma. ∎

MEMBER'S COPY

Lemma 2.7. *Let n, k, q be positive integers such that $n \geq (k+2)$. Let f be a non-constant meromorphic function and $P[f^q]$ be a differential polynomial of f^q. Let us define $F = \frac{(f^n)^{(k)}}{a}$ and $G = \frac{P[f^q]}{a}$, where $a \in S(f)$. Then $FG \not\equiv 1$.*

Proof. On contrary assume $FG \equiv 1$. So $(f^n)^{(k)}.P[f^q] = a^2$. Therefore $N(r, f) = S(r, f)$ and $N(r, \frac{1}{f}) = S(r, f)$. Now in view of 1st fundamental theorem and using Lemma 2.1 we have

$$\begin{aligned}
(n + q\overline{d}(P))T(r, f) &\leq T\left(r, \frac{a^2}{f^{n+q\overline{d}(P)}}\right) + S(r, f) \\
&\leq T\left(r, \frac{(f^n)^{(k)}}{f^n}.\frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + S(r, f) \\
&\leq T\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + T\left(r, \frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + S(r, f) \\
&\leq N\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + m\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + N\left(r, \frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + m\left(r, \frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + S(r, f) \\
&\leq k\left(\overline{N}(r, f) + \overline{N}(r, \tfrac{1}{f})\right) + N(r, P[f^q]) + q\overline{d}(P)N(r, \tfrac{1}{f}) \\
&\quad + q\left(\overline{d}(P) - \underline{d}(P)\right)T(r, f) + S(r, f).
\end{aligned}$$

That is $(n + q\underline{d}(P))T(r, f) \leq S(r, f)$, which is a contradiction. $\square$

Lemma 2.8. *Let q be a positive integer. Let f be a meromorphic function and $P[f^q]$ be a differential polynomial generated by f^q. Then*

$$m(r, P[f^q]) \leq \overline{d}(P)m(r, f^q) + S(r, f).$$

Proof. The proof can be obtain along the same line as the proof of Lemma 2.6 in [4]. $\square$

Lemma 2.9. *Let q be a positive integer. Let f be a non-constant meromorphic function and $P[f^q]$ be a differential polynomial of f^q. Then $S(r, P[f^q])$ can be replaced by $S(r, f)$.*

Proof. From Lemma 2.8 it is clear that $T(r, P[f^q]) = O(T(r, f))$ and so the lemma follows. $\square$

Lemma 2.10. *Let q be a positive integer. Let f be a meromorphic function with a pole of order $p \geq 1$ at z_0. If $P[f^q]$ be a differential polynomial of f^q whose coefficient are analytic at z_0, then $P[f^q]$ has a pole at z_0 of order at most $qp\overline{d}(P) + \Gamma_P - \overline{d}(P)$.*

Proof. Let f be a meromorphic function with a pole of order $p \geq 1$ at z_0 and $P[f]$ defined as in Definition 1.12. Then by simple calculation $P[f^q]$

MEMBER'S COPY

has a pole at z_0 and its order is at most

$$\begin{aligned}\max_{1\le j\le t}\left\{\sum_{s=0}^{k}(qp+s)n_{sj}\right\} &= \max_{1\le j\le t}\left\{(qp-1)\sum_{s=0}^{k}n_{sj}+\sum_{s=0}^{k}(s+1)n_{sj}\right\}\\ &\le (qp-1)\overline{d}(P)+\Gamma_P\\ &= qp\overline{d}(P)+\Gamma_P-\overline{d}(P).\end{aligned}$$

This completes the proof of the lemma. □

3. Proof of the Main Theorem

Proof of Theorem 1.15.

Proof. Let $F=\frac{(f^n)^{(k)}}{a}$ and $G=\frac{P[f^q]}{a}$. Then $F-1=\frac{(f^n)^{(k)}-a}{a}$ and $G-1=\frac{P[f^q]-a}{a}$. Since $(f^n)^{(k)}$ and $P[f^q]$ share (a,l), it follows that F and G share $(1,l)$ except at zeros and poles of a. By Lemma 2.9 we have $S(r,P[f^q])=S(r,f)$. Therefore $\overline{N}(r,G)=\overline{N}(r,f)+S(r,f)=\overline{N}(r,F)$.

Suppose $H\not\equiv 0$. Then from (2.1) and Lemma 2.4 we have $m(r,H)=S(r,f)$, and

$$\begin{aligned}&N_E{}^{1)}(r,\tfrac{1}{F-1})\le N(r,\tfrac{1}{H})+S(r,f)\\ &\le T(r,H)+S(r,f)=N(r,H)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G})+\overline{N}_L(r,\tfrac{1}{F-1})\\ &+\overline{N}_L(r,\tfrac{1}{G-1})+N_0(r,\tfrac{1}{F'})+N_0(r,\tfrac{1}{G'})+S(r,f).\end{aligned}\tag{3.1}$$

By the 2nd Fundamental Theorem of Nevanlinna, we have

$$\begin{aligned}T(r,F)+T(r,G)\le{}& 2\overline{N}(r,f)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,\tfrac{1}{G})\\ &+\overline{N}(r,\tfrac{1}{G-1})-N_0(r,\tfrac{1}{F'})-N_0(r,\tfrac{1}{G'})+S(r,f),\end{aligned}\tag{3.2}$$

where $N_0(r,\frac{1}{F'})$ denote the counting function of zeros of F', which are not the zeros of $F(F-1)$ and $N_0(r,\frac{1}{G'})$ is defined similarly.

MEMBER'S COPY

Case 1: $l \geq 1$. Then by (3.1), we obtain

$$\begin{aligned}
&\overline{N}\big(r, \tfrac{1}{F-1}\big) + \overline{N}\big(r, \tfrac{1}{G-1}\big) = N_E^{1)}\Big(r, \tfrac{1}{F-1}\Big) + \overline{N}_E^{(2}\Big(r, \tfrac{1}{F-1}\Big) \\
&\quad + \overline{N}_L\big(r, \tfrac{1}{F-1}\big) + \overline{N}_L\big(r, \tfrac{1}{G-1}\big) + \overline{N}\big(r, \tfrac{1}{G-1}\big) + S(r, f) \\
&\leq \overline{N}(r, F) + \overline{N}_{(2}(r, \tfrac{1}{F}) + \overline{N}_{(2}(r, \tfrac{1}{G}) + 2\overline{N}_L\big(r, \tfrac{1}{F-1}\big) + 2\overline{N}_L\big(r, \tfrac{1}{G-1}\big) \\
&\quad + N_0(r, \tfrac{1}{F'}) + N_0(r, \tfrac{1}{G'}) + \overline{N}_E^{(2}\big(r, \tfrac{1}{F-1}\big) + \overline{N}\big(r, \tfrac{1}{G-1}\big) + S(r, f). \qquad (3.3)
\end{aligned}$$

Subcase 1.1: $l = 1$. In this case we have,

$$\begin{aligned}
\overline{N}_L\big(r, \frac{1}{F-1}\big) &\leq \frac{1}{2} N(r, \frac{1}{F'} | F \neq 0) \\
&\leq \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}(r, \frac{1}{F}) + S(r, f), \qquad (3.4)
\end{aligned}$$

where $N(r, \frac{1}{F'} | F \neq 0)$ denotes the zeros of F', which are not the zeros of F.

Therefore by (3.4) and Lemmas 2.2, 2.3 we have

$$\begin{aligned}
&2\overline{N}_L(r, \tfrac{1}{F-1}) + 2\overline{N}_L(r, \tfrac{1}{G-1}) + \overline{N}_E^{(2}(r, \tfrac{1}{F-1}) + \overline{N}(r, \tfrac{1}{G-1}) \\
&\leq N(r, \tfrac{1}{G-1}) + \overline{N}_L(r, \tfrac{1}{F-1}) + S(r, f) \\
&\leq N(r, \tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r, F) + \tfrac{1}{2}\overline{N}(r, \tfrac{1}{F}) + S(r, f) \\
&\leq N(r, \tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r, f) + \tfrac{1}{2}\overline{N}(r, \tfrac{1}{(f^n)^{(k)}}) + S(r, f) \\
&\leq N(r, \tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r, f) + \tfrac{1}{2}N_{k+1}(r, \tfrac{1}{f^n}) + \tfrac{k}{2}\overline{N}(r, f) + S(r, f) \\
&\leq N(r, \tfrac{1}{G-1}) + \big(\tfrac{k+1}{2}\big)\overline{N}(r, f) + \tfrac{1}{2}N_{k+1}(r, \tfrac{1}{f^n}) + S(r, f). \qquad (3.5)
\end{aligned}$$

So from (3.3) and (3.5), we get

$$\begin{aligned}
&\overline{N}(r, \tfrac{1}{F-1}) + \overline{N}(r, \tfrac{1}{G-1}) \leq \overline{N}(r, f) + \overline{N}_{(2}(r, \tfrac{1}{F}) + \overline{N}_{(2}(r, \tfrac{1}{G}) \\
&\quad + N(r, \tfrac{1}{G-1}) + \tfrac{k+1}{2}\overline{N}(r, f) + N_{k+1}(r, \tfrac{1}{f^n}) \\
&\quad + N_0(r, \tfrac{1}{F'}) + N_0(r, \tfrac{1}{G'}) + S(r, f) \\
&\leq \tfrac{k+3}{2}\overline{N}(r, f) + \overline{N}_{(2}(r, \tfrac{1}{F}) + \overline{N}_{(2}(r, \tfrac{1}{G}) + T(r, G) \\
&\quad + N_{k+1}(r, \tfrac{1}{f^n}) + N_0(r, \tfrac{1}{F'}) + N_0(r, \tfrac{1}{G'}) + S(r, f). \qquad (3.6)
\end{aligned}$$

MEMBER'S COPY

Now from (3.2), (3.6), and Lemmas 2.3, 2.6 we get

$$\begin{aligned}
T(r,F) &\leq \left(\frac{k+3}{2}+2\right)\overline{N}(r,f)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G}) \\
&+ \overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+N_{k+1}(r,\frac{1}{f^n})+S(r,f) \\
&\leq \frac{k+7}{2}\overline{N}(r,f)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+N_{k+1}(r,\frac{1}{f^n})+S(r,f) \\
&\leq \frac{k+7}{2}\overline{N}(r,f)+T(r,F)-nT(r,f)+N_{k+2}(r,\frac{1}{f^n}) \\
&+ N_{k+1}(r,\frac{1}{f^n})+Q\overline{N}(r,f)+q\overline{d}(P)N_{k+2}(r,\frac{1}{f})+S(r,f),
\end{aligned}$$

which implies

$$\begin{gathered}
nT(r,f)\leq \left(Q+\tfrac{k+7}{2}\right)\overline{N}(r,f)+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f}) \\
+N_{k+2}(r,\tfrac{1}{f^n})+N_{k+1}(r,\tfrac{1}{f^n})+S(r,f).
\end{gathered}$$

So

$$\begin{gathered}
\left(Q+\tfrac{k+7}{2}\right)\Theta(\infty,f)+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n) \\
+n\delta_{k+1}(0,f^n)\leq Q+\tfrac{k+7}{2}+q\overline{d}(P)+n,
\end{gathered}$$

which contradicts (1.2).

Subcase 1.2: $l \geq 2$. In this case we have

$$\begin{gathered}
2\overline{N}_L(r,\tfrac{1}{F-1})+2\overline{N}_L(r,\tfrac{1}{G-1})+\overline{N_E}^{(2}(r,\tfrac{1}{F-1}) \\
+\overline{N}(r,\tfrac{1}{G-1})\leq N(r,\tfrac{1}{G-1})+S(r,f),
\end{gathered}$$

and from (3.3) we obtain

$$\begin{aligned}
\overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,\tfrac{1}{G-1}) &\leq \overline{N}(r,f)+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G}) \\
&+N(r,\tfrac{1}{G-1})+N_0(r,\tfrac{1}{F'})+N_0(r,\tfrac{1}{G'})+S(r,f) \\
&\leq \overline{N}(r,f)+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G})+T(r,G) \\
&+N_0(r,\tfrac{1}{F'})+N_0(r,\tfrac{1}{G'})+S(r,f). \qquad (3.7)
\end{aligned}$$

Now from (3.2), (3.7), and Lemmas 2.3, 2.6 we get

$$\begin{aligned}
T(r,F) &\leq 3\overline{N}(r,f)+\overline{N}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})+\overline{N}_{(2}(r,\tfrac{1}{G})+S(r,f) \\
&\leq 3\overline{N}(r,f)+N_2(r,\tfrac{1}{F})+N_2(r,\tfrac{1}{G})+S(r,f) \\
&\leq 3\overline{N}(r,f)+T(r,F)-nT(r,f)+N_{k+2}(r,\tfrac{1}{f^n})
\end{aligned}$$

MEMBER'S COPY

$$+Q\overline{N}(r,f)+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})+S(r,f),$$

which implies

$$\begin{aligned} nT(r,f) &\le (Q+3)\overline{N}(r,f)+N_{k+2}(r,\tfrac{1}{f^n})+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})+S(r,f)\\ &\le (Q+3)\{1-\Theta(\infty,f)\}T(r,f)+n(1-\delta_{k+2}(0,f^n))T(r,f)\\ &\quad +q\overline{d}(P)(1-\delta_{k+2}(0,f))T(r,f)+S(r,f) \end{aligned}$$

$$\Rightarrow (Q+3)\Theta(\infty,f)+q\overline{d}(P)\delta_{2+k}(0,f)+n\delta_{k+2}(0,f^n)\le Q+3+q\overline{d}(P),$$

which violates assumption (1.1).

Case 2 : $l=0$. Then we have

$$N_E^{1)}(r,\tfrac{1}{F-1})=N_E^{1)}(r,\tfrac{1}{G-1})+S(r,f),$$
$$\overline{N}_E^{(2}(r,\tfrac{1}{F-1})=\overline{N}_E^{(2}(r,\tfrac{1}{G-1})+S(r,f).$$

Now by (3.1), we have

$$\begin{aligned} \overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,\tfrac{1}{G-1}) &= N_E^{1)}(r,\tfrac{1}{F-1})+\overline{N}_E^{(2}(r,\tfrac{1}{F-1})\\ &\quad +\overline{N}_L(r,\tfrac{1}{F-1})+\overline{N}_L(r,\tfrac{1}{G-1})+\overline{N}(r,\tfrac{1}{G-1})+S(r,f)\\ &\le N_E^{1)}(r,\tfrac{1}{F-1})+\overline{N}_L(r,\tfrac{1}{F-1})+N(r,\tfrac{1}{G-1})+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G})+2\overline{N}_L(r,\tfrac{1}{F-1})+\overline{N}_L(r,\tfrac{1}{G-1})\\ &\quad +N(r,\tfrac{1}{G-1})+N_0(r,\tfrac{1}{F'})+N_0(r,\tfrac{1}{G'})+S(r,f). \end{aligned} \tag{3.8}$$

Therefore from (3.2), (3.8), and Lemmas 2.2, 2.3 and 2.6 we get

$$\begin{aligned} T(r,F) &\le 3\overline{N}(r,f)+\overline{N}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+\overline{N}_{(2}(r,\frac{1}{G})\\ &+ 2\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1})+S(r,f) \end{aligned}$$

$$\begin{aligned} &\le 3\overline{N}(r,f)+N_2(r,\tfrac{1}{F})+N_2(r,\tfrac{1}{G})+2\overline{N}_L(r,\tfrac{1}{F-1})+\overline{N}_L(r,\tfrac{1}{G-1})+S(r,f)\\ &\le 3\overline{N}(r,f)+T(r,F)-nT(r,f)+N_{k+2}(r,\tfrac{1}{f^n})+Q\overline{N}(r,f)\\ &\quad +q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})+2\overline{N}_L(r,\tfrac{1}{F-1})+\overline{N}_L(r,\tfrac{1}{G-1})+S(r,f), \end{aligned}$$

which implies

$$nT(r,f)\le (Q+3)\overline{N}(r,f)+N_{k+2}(r,\tfrac{1}{f^n})+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})+2\overline{N}(r,\tfrac{1}{F})$$

MEMBER'S COPY

$$
\begin{aligned}
&+2\overline{N}(r,F)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,G)+S(r,f).\\
&\leq (2Q+6+2k)\overline{N}(r,f)+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})+q\overline{d}(P)N_{k+1}(r,\tfrac{1}{f})\\
&+2N_{k+1}(r,\tfrac{1}{f^n})+N_{k+2}(r,\tfrac{1}{f^n})+S(r,f)\\
&\leq (2Q+6+2k)\{1-\Theta(\infty,f)\}T(r,f)+q\overline{d}(P)(1-\delta_{k+1}(0,f))T(r,f)\\
&+2n(1-\delta_{k+1}(0,f^n))T(r,f)+q\overline{d}(P)(1-\delta_{2+k}(0,f))T(r,f)\\
&+n(1-\delta_{2+k}(0,f^n))T(r,f)+S(r,f)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(2Q+6+2k)\Theta(\infty,f)+q\overline{d}(P)\delta_{k+1}(0,f)+2n\delta_{k+1}(0,f^n)\\
&+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n)\leq 2Q+2k+2q\overline{d}(P)+6+2n,
\end{aligned}
$$

which contradicts (1.3).

Next suppose $H\equiv 0$. Integrating twice we get

$$\frac{1}{F-1}=\frac{C}{G-1}+D, \tag{3.9}$$

where $C\neq 0$ and D are constants.
Here the following three cases can arise.
Case I: $D\neq 0,-1$. If z_0 be a pole of f with multiplicity p, by Lemma 2.10, F and G have a pole at z_0 with multiplicities $np+k$ and $qp\overline{d}(P)+\Gamma_P-\overline{d}(P)$ respectively. This contradicts (3.9). Thus it follows that $N(r,f)=S(r,f)$. Also it is clear that $\overline{N}(r,G)=\overline{N}(r,f)=S(r,f)$. From (3.9) we get

$$\frac{G-1}{C}=\frac{F-1}{D+1-DF}$$

Therefore $\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)=\overline{N}(r,G)=S(r,f)$.
Now by the Second Fundamental Theorem of Nevanlinna and Lemma 2.3, we have

$$
\begin{aligned}
T(r,F) &\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)+S(r,f)\\
&\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,G)+S(r,f)\\
&\leq 2\overline{N}(r,f)+T(r,F)-nT(r,f)+N_{k+1}(r,\frac{1}{f^n})+S(r,f).
\end{aligned}
$$

MEMBER'S COPY

$$\begin{aligned} nT(r,f) &\leq (k+1)\overline{N}(r,\frac{1}{f^n}) + S(r,f) \\ &\leq (k+1)T(r,f) + S(r,f), \end{aligned}$$

which contradicts $n \geq k+2$.

Case II: $D = 0$. Then from (3.9), we get $G = CF - (C-1)$. Therefore if $C \neq 1$ then
$\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{F - \frac{C-1}{C}})$.
Again by the Nevannlina 2nd Fundamental Theorem and Lemma 2.3, 2.6 we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) \\ &\leq \overline{N}(r,f) + T(r,F) - nT(r,f) + N_{k+1}(r,\frac{1}{f^n}) + \overline{N}(r,\frac{1}{G}) + S(r,f), \end{aligned}$$

which implies

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,f) + N_{k+1}(r,\frac{1}{f^n}) + Q\overline{N}(r,f) + q\overline{d}(P)N_{k+1}(r,\frac{1}{f}) + S(r,f) \\ &\leq (Q+1)\{1-\Theta(\infty,f)\}T(r,f) + n\{1-\delta_{k+1}(0,f^n)\}T(r,f) \\ &+ q\overline{d}(P)\{1-\delta_{k+1}(0,f)\}T(r,f) + S(r,f) \end{aligned}$$

$\Rightarrow (Q+1)\Theta(\infty,f) + n\delta_{k+1}(0,f^n) + q\overline{d}(P)\delta_{k+1}(0,f) \leq Q+1+q\overline{d}(P)$,
which is a contradiction to (1.3).

Therefore $C = 1$ and so $F \equiv G$ and hence $(f^n)^{(k)} \equiv P[f^q]$.
Case III: $D = -1$. Then from (3.9) we have

$$\frac{1}{F-1} = \frac{C}{G-1} - 1$$

Therefore if $C \neq -1$, then $\overline{N}(r, \frac{1}{G}) = \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right)$. Proceeding as in Case II we get a contradiction.

Therefore $C = -1$ and so $FG = 1$, which contradicts Lemma 2.7.

□

Acknowledgement: We are grateful to the referee for the comments which improved the quality of the paper.

MEMBER'S COPY

References

[1] Banerjee, A., Meromorphic functions sharing one value, *Int. J. Math. Sci.*, **22**(2005), 3587–3598.

[2] Banerjee, A. and Majumder, S., On the uniqueness of a power of a meromorphic function sharing a small function with the power of its derivative, *Comment. Math. Univ. Carolin.*, **51(4)**(2010), 565–576.

[3] Banerjee, A. and Majumder, S., Some uniqueness results related to meromorphic function that share a small function with its derivative, *Math Report.*, **66**(2014), 95–111.

[4] Banerjee, A. and Dhar, S., Meromorphic function with some power sharing a small function with the differential polynomial generated by the function, *Sci. Stud. Res. Ser. Math. Info.*, **25**(2015), 107–124.

[5] Banerjee, A. and Chakraborty, B., Further investigations on a questions of Zhang and Lu, *Ann. Univ. Paedagog. Crac. Stud. Math.*, **14**(2015), 105–119.

[6] Banerjee, A. and Chakraborty, B., On the generalizations of Brück conjecture, *Commun. Korean Math. Soc.*, **32**(2016), 311–327.

[7] Bhoosnurmath, S. and Kabbur, S. R., On entire and meromorphic functions that share one small function with their differential polynomial, Hindawi Publishing Corporation, *Int. J. Analysis*, (2013), Article ID 926340.

[8] Brück, R., On entire functions which share one value CM with their first derivative, *Results Math.*, **30(1)**(1996), 21–24.

[9] Chen, A. and Zhang, G., Unicity of meromorphic function and its derivative, *Kyungpook. Math. J.*, **50**(2010), 71–80.

[10] Charak, K. S. and Lal, B., Uniqueness of $p(f)$ and $P[f]$, *Turk J Math.*, **40**(2016), 569–581.

[11] Chakraborty, B., Some uniqueness results related to the brück conjecture, *Analysis*, **38(2)**(2018), 91–100.

[12] Hayman, W. K., *Meromorphic function*, Clarendon Press, Oxford, 1964.

[13] Lahiri, I., Weighted value sharing and uniqueness of meromorphic functions, *Complex Var. Theory Appl.*, **46**(2001), 241–253.

[14] Lahiri, I., Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.*, **161**(2001), 193–206.

[15] Lahiri, I. and Sarkar, A., Uniqueness of a meromorphic functions and its derivative, *J. Inequal. Pure. Appl. Math.*, **5(1)**(2004), Art. 20.

[16] Li, N. and Yang, L. Z., Meromorphic function that shares one small functions with its differential polynomial, *Kyunpook Math. J.*, **50**(2010), 447–454.

[17] Li, J. D. and Huang, G. X., On meromorphic functions that share one small function with their derivatives, *Palestine J. Math.*, **4(1)**(2015), 91–96.

[18] Li, N., Yang, L. and Liu, K., A further result related to a conjecture of R. Bruck, *Kyungpook. Math. J.*, **56**(2016), 451–464.

[19] Mues, E. and Steinmetz, N., Meromorphe Funktionen die unit ihrer Ableitung Werte teilen, *Manuscripta Math.*, **29**(1979), 195–206.

MEMBER'S COPY

[20] Rubel, L. A. and Yang, C. C., Values shared by an entire function and its derivative, *Lecture Notes in Math.*, **599**(1977), 101–103.

[21] Yang, L., *Value distributions theory*, Springer-Verlag, Berlin, 1993.

[22] Yi, H. X. and Yang, C. C., *Uniqueness theory of meromorphic functions(in Chinese)*, Science Press, Beijing, 1995.

[23] Zhang, Q. C., Meromorphic function that shares one small function with its derivative, *J. Inequal. Pure Appl. Math.*, **6(4)**(2005), Article Id. 116.

[24] Zhang, J. L. and Yang, L. Z., Some uniqueness results related to the br*ü*ck conjecture, *J. Inequal. Pure Appl. Math.*, **8(1)**(2007), Article Id. 18

[25] Zhang, T. D. and L*ü*, W. R., Notes on meromorphic function sharing one small function with its derivative, *Complex Var. Ellip. Eqn.*, **53(9)**(2008), 857–867.

Dilip Chandra Pramanik
Department of Mathematics
University of North Bengal
Raja Rammohunpur, Darjeeling
West Bengal, India-734013.
E-mail: dcpramanik.nbu2012@gmail.com

Jayanta Roy
Department of Mathematics, DDE
University of North Bengal
Raja Rammohunpur, Darjeeling
West Bengal, India-734013.
E-mail: jayantaroy983269@yahoo.com

Afsar Hossain Sarkar
Sahoorgachi BMNH High school,
Chanchal, Malda, West Bengal, India-732123
E-mail: afsarhsarkar1986@gmail.com

MEMBER'S COPY

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 145–164

SINGULARITIES OF GENERALISED MAXIMAL IMMERSIONS

PRADIP KUMAR, SAMIR KUMAR HAZRA AND VIDISHA KAIRAM
(Received : 14 - 02 - 2020 ; Revised : 15 - 05 - 2020)

Abstract. Maximal surfaces in the Lorentz-Minkowski space $\mathbb{E}_1^3$ are the natural counterparts of the minimal surfaces in $\mathbb{R}^3$ in the sense that both are critical points of the area functional giving rise to zero mean curvature surfaces. Though similar theories have been developed for these two kinds of objects, they are very different in terms of their singularities. In this article, we review various types of singularities arising in the theory of maximal surfaces till date. We also present special constructions related to these types of singularities.

1. Introduction

Minimal surfaces are the surfaces in $\mathbb{R}^3$ having zero mean curvature. For example the catenoid, helicoid and the Enneper surfaces in Figure 1 are few examples of minimal surfaces. Now in place of $\mathbb{R}^3$ if we consider the Lorentz-Minkowski space $\mathbb{E}_1^3$ (see Section

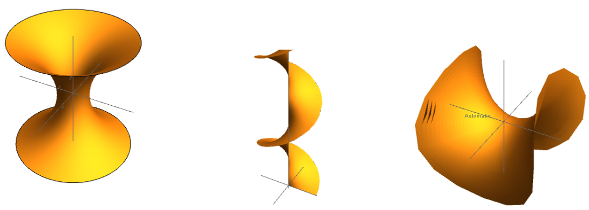


Figure 1. Minimal surfaces: Catenoid, Helicoid, Enneper surface

2) which is $\mathbb{R}^3$ as vector space but with a different metric called the Lorentz metric, we obtain the maximal surfaces as the spacelike zero mean curvature surfaces. These have the special property that they are locally area maximizing. But a striking fact here is, in

2010 Mathematics Subject Classification: 5302, 53A10, 53A40
Key words and phrases: Maximal surface, Lorentz Minkowski space, singularity, Generalised maximal immersions

© *Indian Mathematical Society, 2020.*

MEMBER'S COPY

contrast to the minimal surfaces, the maximal surfaces naturally allow singularities. The main purpose of this article is to explore various possible singularities arising in the existing theory of maximal surfaces.

A smooth map $X : \Omega \subset \mathbb{R}^2 \to \mathbb{E}_1^3$ defined on a domain Ω is said to be a generalised maximal immersion if almost everywhere on Ω, X is a spacelike immersion having zero mean curvature. These are precisely defined in Definition 2.2 and 3.1. It is in the same way the generalised minimal immersions were defined in [1]. Lorentzian catenoid, Lorentzian helicoid and Lorentzian Enneper surface in Figure 2

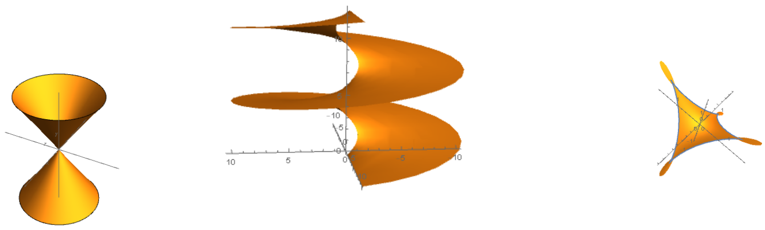


FIGURE 2. Lorentzian - Catenoid, Helicoid, Enneper surface

are few examples of generalised maximal immersions. We will discuss all these examples in detail along with many others in the later sections.

For a generalised minimal or maximal immersion X defined on a domain $\Omega \subset \mathbb{R}^2$, the points $p \in \Omega$ for which $Rank(dX|_p) < 2$ are called the singular points of X. Although maximal and minimal immersions have similar theories, their singularities behave very differently. In particular, for the generalised minimal immersions the singularities are all isolated whereas the singularities of generalised maximal immersions are mostly non-isolated. This makes the singularity theory important and rich in the case of generalised maximal immersions. In recent years it has drawn attention of various mathematicians and in [3], [4], [5], [6], [9], [12], [13], [19], [18], [20] etc. various aspects of singularities have been studied including detection of types and constructions of singularities. In this article we review many such results regarding singularities of generalised maximal immersions

The article is organised as follows. In Section 2 we introduce the *Lorentz-Minkowski space* $\mathbb{E}_1^3$ and define the maximal immersions. Section 3 starts with the definition of generalised maximal immersions followed by the introduction of various types of singularities which appear on a generalised maximal immersion. In Section 4 we discuss about constructions of generalised maximal immersions with prescribed set of singularities available from various above mentioned articles. In Section 5 we discuss about generic and nongeneric singularities.

MEMBER'S COPY

2. Maximal immersions

We start with an introduction to the *Lorentz-Minkowski space*, details can be found in [15], [7] for example.

The 3-dimensional *Lorentz-Minkowski space* $\mathbb{E}^3_1$ is the vector space $\mathbb{R}^3$ equipped with a bilinear form $\langle\ ,\ \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $\langle (u_1, u_2, u_3), (v_1, v_2, v_3)\rangle := u_1v_1 + u_2v_2 - u_3v_3$ for $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$. In the literature this is called the Lorentz metric. A vector $v \in \mathbb{E}^3_1$ is called

- a lightlike vector if $\langle v, v\rangle = 0$ and $v \neq 0$.
- a timelike vector if $\langle v, v\rangle < 0$
- a spacelike vector if $v = 0$ or $\langle v, v\rangle > 0$.

Let P be a 2-dimensional subspace of $\mathbb{E}^3_1$. Then the restriction of $\langle\ ,\ \rangle$ to P can be either positive definite or of index 1 or degenerate. P is called

- a lightlike subspace if restriction metric on P is degenerate
- a timelike subspace if restriction metric on P is of index 1
- a spacelike subspace if restriction metric on P is positive definite.

In particular the restriction metric on a timelike or spacelike subspace is nondegenerate. Such subspaces are called nondegenerate subspaces. Here we note that the metric on $\mathbb{E}^3_1$ is a nondegenerate metric. Now if $\{p_1, p_2\}$ is a basis of P, the nondegeneracy of the restriction metric is equivalent to the condition $det(g_{ij}) \neq 0$ where $g_{ij} := \langle p_i, p_j\rangle$ for $i, j \in \{1, 2\}$. It is also easy to see that for a spacelike subspace P, $\langle v, v\rangle > 0$ for $v \neq 0$, $v \in P$.

Example 2.1. *The subspaces* $V = span\{(0, 1, 0), (1, 1, 0)\}$*,* $W = span\{(1, 0, 1), (1, 1, 0)\}$ *and* $Z = span\{(0, 1, 1), (1, 0, 0)\}$ *of* $\mathbb{E}^3_1$ *are examples of spacelike, timelike and lightlike subspace respectively.*

A vector $w \in \mathbb{E}^3_1$ is said to be perpendicular to a subspace $V \subset \mathbb{E}^3_1$, written as $w \perp V$, if $\langle w, v\rangle = 0$ for all $v \in V$.

Proposition 2.1. [15] *Let P be a 2-dimensional spacelike subspace and $v \perp P, v \neq 0$. Then v is a timelike vector.*

Proof. Since the restriction metric is positive definite on P, we can find a basis $\{p_1, p_2\}$ of P such that $\langle p_1, p_2\rangle = 0$. Also since $v \neq 0$ and $v \perp P$, $v \notin P$. Therefore $\{p_1, p_2, v\}$ becomes a basis of $\mathbb{E}^3_1$. Now take a timelike vector $t \in \mathbb{E}^3_1$ and write it as $t = c_1p_1 + c_2p_2 + c_3v$. Then $\langle t, t\rangle = c_1^2\langle p_1, p_1\rangle + c_2^2\langle p_2, p_2\rangle + c_3^2\langle v, v\rangle < 0$. This implies $\langle v, v\rangle < 0$. □

The Lorentz metric on $\mathbb{E}^3_1$ gives many interesting and very different results (from those in the usual metric on $\mathbb{R}^3$) which change the theory completely. We mention one of such results here:

MEMBER'S COPY

Proposition 2.2 ([15] Proposition 2.2.). *Let $v, w \in \mathbb{E}_1^3$ be two lightlike vectors. Then $\langle v, w\rangle = 0$ if and only if v and w are dependent.*

Now let M and N be two smooth manifolds with $dim\, M = m$ and let $f : M \to N$ be a smooth map. Then $df|_p$: $T_p(M) \to T_{f(p)}(N)$ is a linear map for all $p \in M$. The maximum possible rank of this linear map is m and the set $Sing(f) := \{p \in M : rank(df|_p) < m\}$ is said to be the set of singular points of f.

For example if we take the map 'cross cap'

$$f_{cc}(u, v) = (u^2,\, v,\, uv)\, ;\, (u, v) \in \mathbb{R}^2 \tag{2.1}$$

then it has singularity at $(0, 0)$.

Definition 2.2 (**Maximal immersion** [15]). *Let $\Omega \subset \mathbb{R}^2$ be a domain with parameters (u, v). A smooth map $X : \Omega \subset \mathbb{R}^2 \to \mathbb{E}_1^3$ is said to be a maximal immersion if*

(i) $dX|_p = (X_u(p), X_v(p))$ has rank=2(full rank) for all $p \in \Omega$
(ii) $span\{X_u(p), X_v(p)\}$ is a spacelike subspace for all $p \in \Omega$
(iii) mean curvature of X is zero.

Remark 2.3. *A map $X : \Omega \subset \mathbb{R}^2 \to \mathbb{E}_1^3$ satisfying only first two conditions is said to be a spacelike immersion.*

Remark 2.4. *Our definition of maximal immersion is a local definition. More generally, maximal immersions are defined as spacelike immersions $X : M^2 \to \mathbb{E}_1^3$ with zero mean curvature. Here M^2 is a 2-dimensional manifold. In the case of a spacelike immersion, M^2 necessarily becomes orientable* [15]. *Moreover if one starts with a compact manifold then the boundary $\partial M^2 \neq \emptyset$* [15].

Our main goal in this article is to describe the singularities of a maximal immersion. Therefore we always assume our maximal immersions to be defined on a domain (open, connected) $\Omega \subset \mathbb{R}^2$.

Remark 2.5. *We recall here a fact proved in* [15]. *Let Ω be a bounded domain with boundary. Let $X : \Omega \to \mathbb{E}_1^3$ be a spacelike immersion. Then X has zero mean curvature if and only if it is a critical point of the area functional for all variations of X that fixes the boundary and these are locally area maximizing. Therefore the name "maximal" is justified. These maximal immersions in $\mathbb{E}_1^3$ are the counterparts of the minimal immersions in $\mathbb{R}^3$ (with metric given by the usual dot product), minimal immersions in $\mathbb{R}^3$ are locally area minimising.*

The parameters (u, v) are said to be isothermal for X if $\langle X_u, X_v\rangle = 0$ and $|X_u| = |X_v| > 0$ where $|-| = \sqrt{|\langle -, -\rangle|}$. The spacelike immersion in $\mathbb{E}_1^3$ is locally area maximising is

MEMBER'S COPY

equivalent to the fact that mean curvature of the immersion is zero. And this is equivalent to the fact that there are isothermal parameters (u, v) such that each coordinate map $X_i\,;\, i \in \{1, 2, 3\}$ is a harmonic map where $X = (X_1, X_2, X_3)$.

Let $X : \Omega \to \mathbb{E}_1^3$ be a maximal immersion in isothermal parameters (u, v). Let us define $\phi_i : \Omega \to \mathbb{C}^3$ by

$$\phi_i := \frac{\partial X_i}{\partial z}, \text{ where } \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right). \tag{2.2}$$

Since $\{X_i\}$ are harmonic $\{\phi_i\}$ become analytic. Now the parameters are isothermal implies ${\phi_1}^2 + {\phi_2}^2 - {\phi_3}^2 = 0$ and $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 > 0$. Conversely suppose $X : \Omega \to \mathbb{E}_1^3$ is a harmonic (i.e. each X_i is harmonic function) in some parameters (u, v) satisfying these two conditions. Then X turns out to be a maximal immersion. Therefore we have an equivalent definition of maximal immersions as follows:

Definition 2.6. [3] *Let $X \equiv (X_1, X_2, X_3) : \Omega \to \mathbb{E}_1^3$ be a smooth map. Then X is said to be a maximal immersion if X is harmonic and*

(i) ${\phi_1}^2 + {\phi_2}^2 - {\phi_3}^2 \equiv 0$
(ii) $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 > 0$

where $\{\phi_i\}$ are defined as above.

Example 2.7 (Lorentzian Catenoid)**.** *Let $\Omega := \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $X : \Omega \to \mathbb{E}_1^3$ be defined as*

$$X(u, v) = \left(\frac{u}{2}\left(1 - \frac{1}{u^2 + v^2}\right), \frac{v}{2}\left(1 - \frac{1}{u^2 + v^2}\right), -\frac{1}{2}\log(u^2 + v^2)\right). \tag{2.3}$$

Here we have $\phi_1 = \frac{1}{2}\left(\frac{1+z^2}{z^2}\right)$, $\phi_2 = \frac{i}{2}\left(\frac{1-z^2}{z^2}\right)$ and $\phi_3 = -\frac{1}{z}$. These ϕ_i's satisfy the conditions given in the Definition 2.6. Therefore the map X given by the equation (2.3) *is a maximal immersion. This is called the Lorentzian catenoid or sometimes maximal catenoid.*

Example 2.8 (Lorentizian Helicoid)**.** *Let us take Ω as in the previous example and define X by*

$$X(u, v) = \left(\frac{-v}{2}\left(1 + \frac{1}{u^2 + v^2}\right), \frac{u}{2}\left(1 + \frac{1}{u^2 + v^2}\right), tan^{-1}(v/u)\right) \tag{2.4}$$

In this case we have $\phi_1 = \frac{i}{2}\left(\frac{1+z^2}{z^2}\right)$, $\phi_2 = \frac{-1}{2}\left(\frac{1-z^2}{z^2}\right)$ and $\phi_3 = \frac{-i}{z}$. It can be checked that X defined in this way is a maximal immersion known as Lorentzian helicoid or maximal helicoid.

MEMBER'S COPY

Example 2.9 (Lorentzian Enneper Surface)**.** *Again we take the same Ω as above and define X by*

$$X(u,v) = \left(u + \frac{u^3}{3} - uv^2, -v - \frac{v^3}{3} + u^2 v, v^2 - u^2\right) \tag{2.5}$$

Here we have $\phi_1 = 1 + z^2$, $\phi_2 = i\left(1 - z^2\right)$ and $\phi_3 = -2$. Also, X becomes a maximal immersion as before, this is called the Lorentzian Enneper surface.

Let us now try to generalise the notion of maximal immersions by relaxing the first condition in Definition 2.2, that is we allow the map X to have singularities. So in this situation $rank(dX|_p) \leq 2$. The next proposition describes the singular points of X in terms of the functions $\{\phi_i\}$.

Proposition 2.3. *$rank(dX|_p) < 2$ if and only if $|\phi_1(p)|^2 + |\phi_2(p)|^2 - |\phi_3(p)|^2 = 0$.*

Proof. The first condition in Definition 2.6 implies that the parameters u, v(say) are isothermal. Since

$$\frac{\partial X_1}{\partial u}^2 + \frac{\partial X_2}{\partial u}^2 - \frac{\partial X_3}{\partial u}^2 + \frac{\partial X_1}{\partial v}^2 + \frac{\partial X_2}{\partial v}^2 - \frac{\partial X_3}{\partial v}^2 = \langle X_u, X_u\rangle + \langle X_v, X_v\rangle,$$

$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = 0$ if and only if $\langle X_u, X_u\rangle + \langle X_v, X_v\rangle = 0$. But since $\langle X_u, X_u\rangle = \langle X_v, X_v\rangle$, we obtain $\langle X_u, X_u\rangle = \langle X_v, X_v\rangle = 0$. Then either X_u is zero or X_v is zero or both the vectors X_u and X_v are lightlike. Now since $\langle X_u, X_v\rangle$ is also zero, this implies the vectors $\{X_u, X_v\}$ are dependent, i.e $rank(dX|_p) < 2$. Here we use the fact that two lightlike vectors $\{x, y\}$ are dependent if and only if $\langle x, y\rangle = 0$.

Conversely, if $\{X_u, X_v\}$ are dependent, without loss of generality assume $X_u = cX_v$ for some $c \in \mathbb{R}$. Then $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = \langle X_u, X_u\rangle + \langle X_v, X_v\rangle = 0$ as the parameters are isothermal.

□

3. Generalised maximal immersions and singularities

We are now ready to define the central objects of this article:

Definition 3.1 (Generalised maximal immersion [3])**.** *Let $X : \Omega \to \mathbb{E}^3_1$ be a non-constant harmonic map. Then X is said to be a generalised maximal immersion if*

(i) $\phi_1{}^2 + \phi_2{}^2 - \phi_3{}^2 \equiv 0$

(ii) $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 \not\equiv 0$

where $\{\phi_k\}$ are defined as in equation 2.2.

Now the Proposition 2.3 above allows us to define the singularity set of a generalised maximal immersion X as follows:

MEMBER'S COPY

Definition 3.2. *Let $X : \Omega \to \mathbb{E}_1^3$ be a generalised maximal immersion. The singularity set of X is the set*

$$Sing(X) := \{p \in \Omega : |\phi_1(p)|^2 + |\phi_2(p)|^2 - |\phi_3(p)|^2 = 0\} \tag{3.1}$$

A singular point $p \in Sing(X)$ is said to be an **isolated singular point** if there exist a neighbourhood $U(p)$ of p such that $U(p) \cap Sing(X) = \{p\}$. Otherwise p is called a **non-isolated singular point**.

Example 3.3. *Let $\Omega := \{z :\in \mathbb{C} : 0 < |z| < 2\}$, then the expressions of X as given in Example 2.7, 2.8 and 2.9 turn out to be generalised maximal immersions. In each case $Sing(X) := \{z \in \mathbb{C} : |z| = 1\}$.*

Remark 3.4. *In an analogous way as in Definition 3.1 one can consider the non-constant harmonic maps $X : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ satisfying the conditions*

(i) ${\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2 = 0$

(ii) $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \not\equiv 0$.

Such maps are called the generalised minimal immersions. Similarly the singularity set here is $Sing(X) := \{p \in \Omega : |\phi_1(p)|^2 + |\phi_2(p)|^2 + |\phi_3(p)|^2 = 0\}$. Clearly since $\{\phi_i\}$ are analytic, $Sing(X)$ in this case is discrete. Which is in clear contrast to the case of a generalised maximal immersion, the later are not always discrete but curves in most of the cases. This makes the singularities of generalised maximal immersions an important area of research.

3.1. **Types of singularities of generalised maximal immersions:** As our aim is to understand the nature of generalised maximal immersions $X : \Omega \to \mathbb{E}_1^3$ near the singular points, we can suitably take our domain Ω to be simply connected containing some singularities of X. In the starting of this section, we have seen that a generalised maximal immersions may have isolated singularities and non isolated singularities.

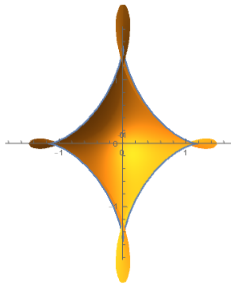
Here we introduce various types of non-isolated singularities which appear on X. These have been discussed in [6], [3], [12], [9], [4], [13], [19], [18], [20] etc. The following list is not exhaustive.

3.1.1. *Admissible singularity.* In this subsection we will define admissible singularities and various types of these which can appear on generalised maximal immersions.

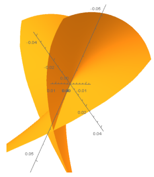
Definition 3.5. *Let us set $\mathcal{W} := \Omega \setminus Sing(X)$. Then $p \in Sing(X)$ is called an admissible singular point if the following conditions hold:*

(i) on a neighborhood U_p of p, there exists a C^1-differentiable function $\beta : U_p \cap \mathcal{W} \to \mathbb{R}_+$ such that the Riemannian metric $\beta \cdot ds^2$ extends to a C^1-differentiable Riemannian metric on U_p. Here ds^2 is the pullback metric on Ω via X.

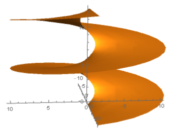
MEMBER'S COPY



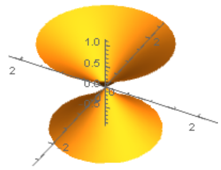
(A) Lorentz Enneper Surface: Weierstrass data: $\{g = z, f = 1\}$. This has all generic singularity on $|z|= 1$.



(B) Weierstrass data: $\{g = z, f = z\}$ and $z = 0$ is an isolated singularity. This is not a maxface around $z = 0$.



(C) Lorentzian helicoid: Weierstrass data $\{g = z, f = i/z^2\}$. This has folded singularity on each $|z|= 1$.



(D) Lorentzian catenoid: Weierstrass data $\{g = z, f = 1/z^2\}$. This has shrunken singularity on $|z|= 1$.

FIGURE 3. Maximal immersions

(ii) $dX|_p \neq 0$.

In a later part (discussion after the Definition 3.12) of this article we shall discuss a representation of generalized maximal immersions using which checking whether some singularities are admissible or not becomes easy. This was introduced by Umehara and Yamada in [12].

Following [12], we shall see in section 5.2 that any generalised maximal immersion $X : \Omega \to \mathbb{E}^3_1$ as a map $X : \Omega \to \mathbb{R}^3$ is a frontal (definition given in section 5.2). Few of them having admissible singularities become front. As a frontal (also as a front), there are various types of admissible singularities which can appear on generalised maximal immersions. Few (1 to 5 below) of these we mention below.

To define these singularities we need the notion of $\mathcal{A}-$equivalence discussed in [9], we define it here:

Definition 3.6 ($\mathcal{A}-$**equivalence**)**.** *Two smooth maps* $X : \Omega \subset \mathbb{R}^2 \to \mathbb{E}^3_1$ *and* $Y : \mathcal{V} \subset \mathbb{R}^2 \to \mathbb{E}^3_1$ *are said to be* $\mathcal{A}-$*equivalent at the points* $p \in \Omega$ *and* $q \in \mathcal{V}$ *if there exists a local diffeomorphism* η *of* $\mathbb{R}^2$ *with* $\eta(p) = q$ *and a local diffeomorphism* Φ *of* $\mathbb{E}^3_1$ *with* $\Phi(X(p)) = Y(q)$ *such that* $Y = \Phi \circ X \circ \eta^{-1}$.

(1) **Swallowtails**: X is said to have swallowtail singularity at $p \in \Omega$ if at $p \in \Omega$ and $(0, 0) \in \mathbb{R}^2$, X is $\mathcal{A}$-equivalent to $f_{sw}(u, v) = (3u^4 + u^2v,\, 4u^3 + 2uv,\, v)$; $(u, v) \in \mathbb{R}^2$
(2) **Cuspidal edge singularties** : X is said to have cuspidal edge singularities at $p \in \Omega$ if at $p \in \Omega$ and $(0, 0) \in \mathbb{R}^2$, X is $\mathcal{A}$-equivalent to $f_{ce}(u, v) = (u^2,\, u^3,\, v)$; $(u, v) \in \mathbb{R}^2$.

MEMBER'S COPY

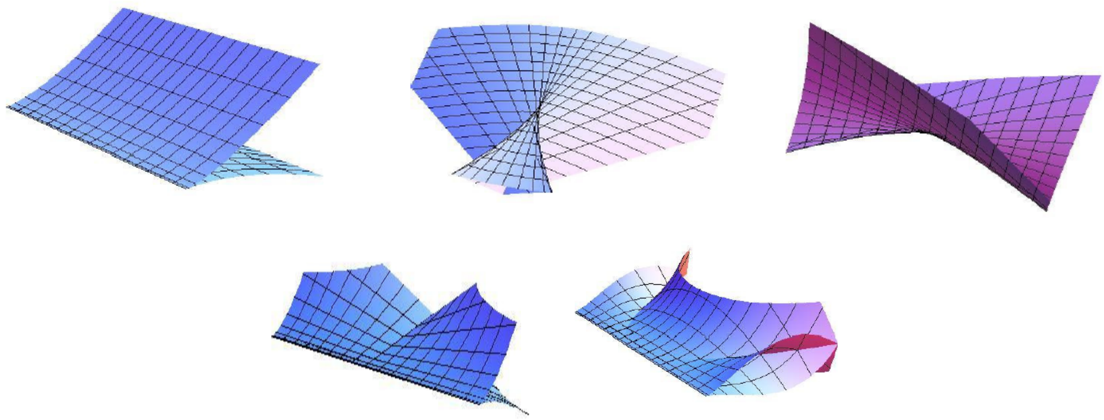


FIGURE 4. From top left to bottom right: cuspidal edge, swallowtail, cuspidal butterfly, cuspidal cross cap, and S_1^- singularity.

(3) **Cuspidal butterflies** X is said to have cuspidal butterflies singularity at $p \in \Omega$ if at $p \in \Omega$ and $(0,0) \in \mathbb{R}^2$, X is $\mathcal{A}$-equivalent to $f_{cb}(u,v) = (u, 4v^5 + uv^2, 5v^4 + 2uv)\,;\,(u,v) \in \mathbb{R}^2$.
(4) **Cuspidal crosscaps**: X is said to have cuspidal crosscaps at $p \in \Omega$ if at $p \in \Omega$ and $(0,0) \in \mathbb{R}^2$, X is $\mathcal{A}$-equivalent to $f_{ccc}(u,v) = (u,\, v^2,\, uv^3)\,;\,(u,v) \in \mathbb{R}^2$
(5) **Cuspidal S_1^- singularities** X is said to have cuspidal butterflies singularity at $p \in \Omega$ if at $p \in \Omega$ and $(0,0) \in \mathbb{R}^2$, X is $\mathcal{A}$-equivalent to $f_{cs}(u,v) = (u, v^2, v^3(u^2 - v^2))\,;\,(u,v) \in \mathbb{R}^2$.

The Lorentzian Enneper surface as in Example 2.9 has singularities at $\{z \in \mathbb{C} : |z| = 1\}$. The points $z = \pm 1, \pm i$ are swallowtails,
$e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ are cuspidal crosscaps and the remaining points on the unit circle are cuspidal edge singularities.

At this moment it is not clear how the types of singularities were determined here, this will be very clear from Table 1, [9] and [20] once we learn about the Weierstrass-Enneper representation (3.10) and Weierstrass data.

Now we define the following type of admissible singularities, we present the definition given in [18].

Definition 3.7 (Generalized cone-like singularity and cone-like singularity). *Let $\sum_0$ be the connected component of the set of all admissible singular points on a generalised maximal immersion $X : \Omega \to \mathbb{E}_1^3$. Then each point of $\sum_0$ is called a generalized cone-like singular point if $\sum_0$ is compact and the image $X(\sum_0)$ is a single point. Moreover, if there*

MEMBER'S COPY

is a neighborhood U of $\sum_0$ such that $X(U \setminus \sum_0)$ is embedded then each point of $\sum_0$ is called a cone-like singular point.

In [4], the authors use the name shrinking singularity for the cone-like singularity we defined here. For the Lorentzian catenoid as in example 2.7 see that every point z such that $|z|= 1$ is a cone-like singular point.

Cone-like singularity is also defined by Kobayashi in [6] but not as a point in the domain of $X : \Omega \to \mathbb{E}^3_1$ but as a point in the image set of X. We mention it here.

Conelike Singularity (as in [6]): Let $q = (x_0, y_0, z_0) \in \overline{X(\Omega)}$, the closure of the image set of X. Then q is called a cone-like singularity of X if the following conditions are satisfied:

(i) In a neighbourhood U_q of q, $X(\Omega)$ is the graph of a smooth function χ defined on $U \setminus (x_0, y_0)$, where U is a neighbourhood of (x_0, y_0) in the (x, y)-plane

(ii) On $U \setminus (x_0, y_0)$, $\chi < z_0$ or $\chi > z_0$. By setting $\chi(x_0, y_0) = z_0$, χ is continuous on U

(iii) $\lim_{(x,y)\to(x_0,y_0)}(\chi^2_x + \chi^2_y) = 1$.

Let $\sum_0 := X^{-1}(q) \subset \Omega$. It is clear from the condition (i) that there is a neighborhood U of $\sum_0$ such that $X(U\setminus\sum_0)$ is an embedding. Therefore by Lemma 2.1 [8], every point in $\sum_0$ is an admissible singularity. [Actually by lemma 2.1, the Gauss map has to be one-one and if $X = (\phi_1, \phi_2, \phi_3)$ then $\phi_3(z) \neq 0$. These two facts together imply that singularities are admissible.]

As per Kobayashi [6], for the Lorentzian catenoid in Example 2.7 the pre-image set (in the domain) of $(0, 0, 0)$ is $\{z : |z|= 1\}$ and all these points are cone-like singularities, these are admissible too.

Definition 3.8. ***Folded singularity:*** *We call an admissible singular point $p \in \Omega$ a folded singular point if it has a neighborhood $U(p)$ that can be reparametrised as $X : U(0) \subset \mathbb{R}^2 \to \mathbb{E}^3_1$ with $p = 0$ and parameters u and v such that the singular points are on the u-axis and $X_v(u, 0) \equiv 0$.*

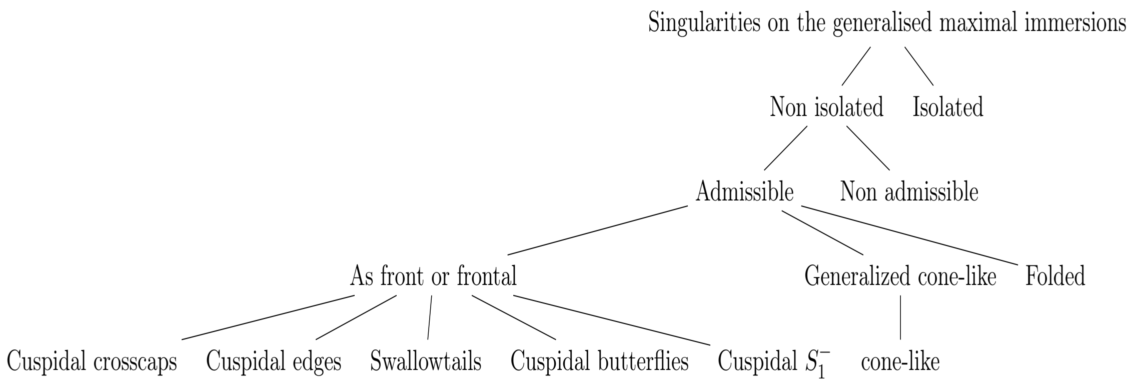
For the Lorentzian helicoid in example 2.8, every point z such that $|z|= 1$ is a folded singular point. Under the map X the set $\{z : |z|= 1\}$ is mapped to the helix.

3.1.2. ***Curvilinear singularity.*** This type of singularity may or may not be admissible, let us define it here.

Definition 3.9 (Curvilinear singularties). *We call $p \in \Omega$ a curvilinear singular point if there is a neighbourhood $U(p) \subset \Omega$ of p and a regular embedded curve $\gamma : I \to U(p)$; $\gamma(0) = p$ such that $\gamma(I) = U(p) \cap Sing(X)$ and $X \circ \gamma$ is one-to-one on I.*

MEMBER'S COPY

All the curvilinear singularities on the Lorentzian helicoid are folded singularities which are admissible too. However, as discussed in [4] (After Definition 2.2, [4]), there are generalised maximal immersions having curvilinear singularities which are not admissible. Construction of such generalised maximal immersions can be done using the singular Björling problem. We shall be discussing it in the next section.



3.2. **Weierstrass-Enneper representation and Maxfaces.** Any generalised maximal immersion locally can be written in a special form called the Weierstrass-Enneper representation, this can be found in [5], [12], [9] etc. Using this representation constructing examples of generalised maximal immersions with prescribed set of singularities becomes easy, we explain it here:

Theorem 3.10 (Weierstrass-Enneper representation [3, 12])**.** *A generalised maximal immersion $X : \Omega \subset \mathbb{R}^2 \to \mathbb{E}_1^3$ defined on simply connected domain Ω can be written as*

$$X(z) = Re \int_{z_\circ}^{z} \phi(z) \text{ where } \phi(z) = \left(\frac{1}{2}(1+g^2)\omega,\ \frac{1}{2}i(1-g^2)\omega,\ -gw\right) \ ; z \in \Omega \tag{3.2}$$

for some $z_0 \in \Omega$, some meromorphic function g with $|g| \not\equiv 1$ and a holomorphic 1-form ω on Ω such that the poles of g with order m are zeros of ω of order at least $2m$. Here $\{g, \omega = f(z)dz\}$ is called the Weierstrass data for X.

The Weierstrass data of the Lorentzian catenoid, Lorentzian helicoid and Lorentzian Enneper surface (with suitable simply connected domain Ω) described in the Examples 2.7, 2.8, 2.9 are given in the Figures 3a,3c, 3d.

For the generalised maximal immersion with Weierstrass data $\{g, fdz\}$, the singularity set (Definition 3.2) is given by the zeros of

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = \frac{1}{2}(1-|g|^2)^2 ||f|^2. \tag{3.3}$$

MEMBER'S COPY

Denote $\mathcal{A}_1 := \{p \in \Omega : |g(p)|= 1\}$ and $\mathcal{A}_2 := \{p \in \Omega : f(p) = g(p)^2 f(p) = 0\}$. Then by proposition 2.3, $p \in \Omega$ is a singular point if and only if $p \in \mathcal{A}_1 \cup \mathcal{A}_2$. In general, $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$. Singular points in $\mathcal{A}_2 \setminus \mathcal{A}_1$ are isolated but singular points in $\mathcal{A}_1$ can not be isolated (as proved by Romero in [3]).

Example 3.11.

- *In Example 2.7 the Lorentzian catenoid has Weierstrass data* $\{g, fdz\} \equiv \{z, \frac{dz}{z^2}\}$. *Clearly as* $f = \frac{1}{z^2}$ *has no zeros, all the singularities in this case lie in* $\mathcal{A}_1$.
- *Let us take the Weierstrass data to be* $\{g, fdz\} \equiv \{z, zdz\}$ *defined on the disk* $D = \{z : |z|< \frac{1}{2}\}$. *Then as* $|g|\neq 1$ *in the domain* D *the singularities come from the zeros of* f, *here the only singular point is* $z = 0$. *So in this case singularity lies in* $\mathcal{A}_2$ *only. This generalised maximal immersion is shown in Figure 3b.*
- *In the same way one can obtain many such examples with* $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$ *also, for example* $\{g, fdz\} = \{z, (z - 1)dz\}$.

As we have seen above, the singularities of a generalised maximal immersion can be easily obtained from the corresponding Weierstrass data. In fact we can say much more about the singularities by just analysing the Weierstrass data further, in particular in [12], [9] Fujimori *et al.* obtained very useful criteria to determine the types of admissible singularities. This is given in Table 1.

Definition 3.12. [12] *A generalised maximal immersion is called a maxface if all its singular points are admissible.*

The generalised maximal immersion X having the Weierstrass data $\{g, fdz\}$ is a maxface if and only if $(1 + |g|^2)^2|f|^2> 0$. This is because the map X is a maxface if and only if the map

$$F(z) := \int_{z_0}^{z} \left(\frac{1}{2}(1 + g^2)fdz,\ \frac{1}{2}i(1 - g^2)fdz,\ -gfdz\right) \tag{3.4}$$

is a Lorentzian null immersion [12]. Moreover F is an immersion is equivalent to the fact that

$$|dF_1|^2+|dF_2|^2+|dF_3|^2= (1 + |g|^2)^2|f|^2> 0.$$

So the above discussion implies that p is a singularity of the maxface X if and only if $|g(p)|= 1$ and $f(p) \neq 0$.

4. Constructing Generalised Maximal Immersions

So far we have seen examples of various types of generalised maximal immersions and their singularities but for the moment what we do not know is how to construct a

MEMBER'S COPY

Let $\alpha := \frac{g'}{g^2 f}$	$\beta := \frac{g}{g'}\alpha'$	$\gamma := \frac{g}{g'}\beta'$ and at p,		
$\mathrm{Re}(\alpha) \neq 0$	$\mathrm{Im}(\alpha) \neq 0$			$\Leftrightarrow p$ is a cuspidal edge
$\mathrm{Re}(\alpha) \neq 0$	$\mathrm{Im}(\alpha) = 0$	$\mathrm{Re}\ (\beta) \neq 0$		$\Leftrightarrow p$ is a swallotail
$\mathrm{Re}(\alpha) \neq 0$	$\mathrm{Im}(\alpha) = 0$	$\mathrm{Re}\ (\beta) = 0$	$\mathrm{Im}(\gamma) \neq 0$	$\Leftrightarrow p$ is a cuspidal butterlfly
$\mathrm{Re}(\alpha) \neq 0$	$\mathrm{Im}(\alpha) \neq 0$	$\mathrm{Im}(\beta) = 0$	$\mathrm{Re}(\gamma) \neq 0$	$\Leftrightarrow p$ is a cuspidal S_1^-
$\mathrm{Re}(\alpha) \neq 0$	$\mathrm{Im}(\alpha) \neq 0$	$\mathrm{Im}\ (\beta) \neq 0$		$\Leftrightarrow p$ is a cuspidal crosscap

TABLE 1

generalised maximal immersion with some prescribed type of singularity. In this section we will see how to do this in some of the cases.

4.1. **By selecting suitable Weierstrass data.** From the description given in Table 1 we know that by selecting suitable $\{g, fdz\}$ we can construct maxfaces which have swallowtail, cuspidal cross cap and cuspidal edge type singularities. In view of that Kim and Yang in [4] proposed the following construction to find the required $\{g, fdz\}$ for few singularities as in Table 2.

Let us consider the following functions defined on a simply connected domain Ω containing $(0, 0)$:

$$\alpha(z) := A_0 + A_1 z + A_2 z^2 + A_3 z^4 + \cdots$$

and

$$\beta(z) = B_0 + B_1 z + B_2 z^2 + B_3 z^4 + \cdots$$

where A_i, B_j are real numbers and at least one of A_0 and B_0 must be nonzero. That is we have $\alpha(0) - i\beta(0) \neq 0$. Now we set

$$g(z) = e^{iz}, \quad f(z) = (\alpha(z) - i\beta(z))\, e^{-iz}. \tag{4.1}$$

Then at $z = 0$, the functions $\{g, f\}$ satisfy the conditions to become the Weierstrass data of a maxface.

Now some easy calculation shows

$$\frac{g'}{g^2 f} = \frac{i}{\alpha(z) - i\beta(z)} \text{ and } \frac{g}{g'}\left(\frac{g'}{g^2 f}\right)' = \frac{\alpha'(z) - i\beta'(z)}{-(\alpha(z) - i\beta(z))^2}.$$

Moreover we have

$$\frac{i}{\alpha(0) - i\beta(0)} = \frac{-B_0 + iA_0}{A_0^2 + B_0^2} \quad \text{and}$$

$$\frac{\alpha'(0) - i\beta'(0)}{-(\alpha(0) - i\beta(0))^2} = \frac{[A_1(A_0^2 - B_0^2) + 2A_0 B_0 B_1] + i[2A_0 B_0 A_1 - B_1(A_0^2 - B_0^2)]}{-(A_0^2 + B_0^2)^2}.$$

MEMBER'S COPY

$B_0 \neq 0$	$A_0 \neq 0$		$\Leftrightarrow z=0$ is a cuspidal edge
$B_0 \neq 0$	$A_0 = 0$	$A_1 \neq 0$	$\Leftrightarrow z=0$ is a swallowtail
$B_0 = 0$	$A_0 \neq 0$	$B_1 \neq 0$	$\Leftrightarrow z=0$ is a cuspidal crosscap

TABLE 2

Now using the above calculation and the description in Table 1 one can easily obtain Table 2. By choosing suitable values for A_0, A_1, B_0 and B_1 in Table 2 we get the Weierstrass data from (4.1) and then putting this data in (3.2) we construct the generalised maximal immersion having some required type of singularity.

4.2. **From the singular Björling problem for the generalised maximal immersion.** Discussion of this section is based on [4] and [13].

Let I be an open interval and $\gamma : I \to \mathbb{E}_1^3$ be a null real analytic curve. Also let $L : I \to \mathbb{E}_1^3$ be a null real analytic vector field such that $\langle \gamma', L \rangle \equiv 0$ and at least one of them is not identically zero. Such $\{\gamma, L\}$ is known as a singular Björling data.

The singular Björling problem asks for existence of a generalised maximal immersion $X : \Omega \to \mathbb{E}_1^3$ where $I \subset \Omega$ and Ω a simply connected domain such that $X(u,0) = \gamma(u)$ and $\left.\dfrac{\partial X}{\partial v}\right|_{(u,0)} = L(u)$.

In [4],[13] we see that there exists a solution of the singular Björling problem if the analytic extension of the function g

$$g(u) := \begin{cases} \dfrac{L_1 + iL_2}{L_1 - iL_2}; & \text{if } \gamma' \text{ vanishes identically} \\ \dfrac{{\gamma_1}' + i{\gamma_2}'}{{\gamma_1}' - i{\gamma_2}'}; & \text{if } L \text{ vanishes identically} \end{cases}$$

satisfies $|g(z)| \not\equiv 1$.

Then the generalised maximal immersion $X : \Omega \to \mathbb{E}_1^3$ given by (for $u_0 \in I$ fixed),

$$X(z) = \gamma(u_0) + Re\left(\int_{u_0}^{z} \gamma'(w) - iL(w)dw\right) \tag{4.2}$$

is known to be the solution of the singular Björling problem.

It has singularity set at least on I. By choosing γ and L suitably we can construct many examples from (4.2), few of them can be seen in [4].

4.3. **By finding suitable Lorentzian null immersion.** A holomorphic map $F = (F_1, F_2, F_3) : \Omega \to \mathbb{C}^3$ is said to be a Lorentzian null immersion if

(1) $(dF_1)^2 + (dF_2)^2 - (dF_3)^2 = 0$ and
(2) $|dF_1|^2 + |dF_2|^2 + |dF_3|^2 > 0$.

MEMBER'S COPY

By [12], we know that if the Lorentzian null immersion satisfies

$$|dF_1|^2+|dF_2|^2-|dF_3|^2\not\equiv 0 \tag{4.3}$$

then $f = \mathrm{Re(F)}$ is a maxface. The singularities of f are the points where equation (4.3) vanishes.

4.4. **By finding a suitable harmonic map.** Discussion in this part is based on [13].

We can identify $\mathbb{E}_1^3$ by $\mathbb{C}\times\mathbb{R}$. Let $F = (h, w) : \Omega \to \mathbb{C}\times\mathbb{R}$ be a smooth map such that $h_{z\overline{z}} = 0$ and $w_{z\overline{z}} = 0$ with $h_z\overline{h}_{\overline{z}} - w_z^2 = 0$ and $\frac{|h_z|}{|h_{\overline{z}}|} \not\equiv 1$. Then in [13] it is proved that such F turns out to be a generalised maximal immersion.

For such h, as in [13], we have

$$w(z) = 2Re\int_{z_0}^{z}\sqrt{h_z\overline{h}_{\overline{z}}}dz + w(z_0) \tag{4.4}$$

where integration is on any path from z_0 to z.

The singularity set of such a generalised maximal immersion is $\{z \in \Omega : |h_z|= |h_{\overline{z}}|\}$. In this setting, to find a generalised maximal immersion, we find suitable harmonic map h and then w by the equation (4.4). The pair (h, w) will be a generalised maximal immersion. We illustrate this in the example below:

Example 4.1 (Lorentzian Catenoid)**.** $\Omega = \mathbb{C}-\{0\}$ and $h(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)$, $w(z) = \frac{1}{2}\log(z\overline{z})$. Then we define $F : \mathbb{C} - \{0\} \to \mathbb{C}\times\mathbb{R}$, $F(z) = (h(z), w(z))$. We have $h_z\overline{h}_{\overline{z}} - w_z^2 = 0$ and $h_{z\overline{z}} = w_{z\overline{z}} = 0$ for all $z \in \Omega$. Here $|h_z|$ is not identically equal to $|h_{\overline{z}}|$. Only on $|z|= 1$, $|h_z|= |h_{\overline{z}}|$.

4.5. **By finding conjugates of the generalised Maximal immersions.** To every generalised maximal immersion X defined on a simply connected domain Ω, there is a harmonic conjugate X^* of X on Ω. Since the sets $\{X_u, X_v\}$ and $\{X_u^*, X_v^*\}$ are simultaneously dependent or independent by the Cauchy-Riemann equations, the singularity sets of X and X^* are the same. The immersion X^* turns out to be a generalised maximal immersion with same set of singularities but the nature of singularities may get changed, some examples are given in Table 3 (see [4], [9]). General discussion about how does the the nature of singularities change after perturbation or conjugation is something not completely explored.

5. General discussion

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and $X : \Omega \to \mathbb{E}_1^3$ a generalised maximal immersion with Weierstrass data on Ω given by $\{g, fdz\}$. Then we have $X(z) = Re(F(z))$ where $F(z)$ is given by equation (3.4).

MEMBER'S COPY

X	X^*
Folded (as in Lorentzian helicoid)	Conelike (as in Lorentzian catenoid)
Swallowtail	Cuspidal crosscap
Cuspidal edge	Cuspidal edge
Cuspidal butterflies	Cuspidal S_1^- singularties

TABLE 3

5.1. **Generalised maximal immersion with isolated singularity.** Now let $p \in \Omega$ be an isolated singularity of X, then from the discussion following Theorem 3.10 we must have $f(p) = (g(p))^2 f(p) = 0$ and $|g(p)| \neq 1$. Also we have

$$X_z = \left(\frac{1}{2}(1+g^2)f,\, \frac{1}{2}i(1-g^2)f,\, -gf\right) \text{ and } X_{\overline{z}} = \left(\frac{1}{2}(1+\overline{g}^2)\overline{f},\, \frac{-1}{2}i(1-\overline{g}^2)\overline{f},\, -\overline{gf}\right).$$

As for an isolated singularity $X_z(p) = (0,0,0) = X_{\overline{z}}(p)$, dX_p must vanish. So at an isolated singular point of a generalised maximal immersion, every direction is a null direction as defined and discussed by Whitney in [14]. Here we quickly revise the main results from [14]:
We call a point $p \in \Omega$ a *'Whitney'* singularity for a map $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ if it is isolated and the following conditions are satisfied:

- $C1$: after a suitable change of coordinates around p, $f_x(p) = 0$ and $f_y(p) \neq 0$ in new coordinates (x, y) and
- $C2$: $\frac{\partial f}{\partial y}|_p$, $\frac{\partial^2 f}{\partial x^2}|_p$, $\frac{\partial^2 f}{\partial x \partial y}|_p$ are linearly independent vectors.

In [14], Whitney proved that any map which satisfies the above conditions $C1$ and $C2$ is $\mathcal{A}-$ equivalent to the cross cap

$$f_{cc}(u,v) = (u^2,\, v,\, uv)\,;\ (u,v) \in \mathbb{R}^2;\ \text{at the point } p \text{ and } (0,0). \tag{5.1}$$

Moreover by Theorem 2, [14], any smooth map f which does not satisfy C_1 or C_2, with a small perturbation f can be changed to one satisfying $C1$ and $C2$.

From the above discussion we see that any generalised maximal immersion as map from $\Omega \to \mathbb{R}^3$ with isolated singularity is very near (in sup norm) to the cross cap f_{cc}.

5.2. **Generalised maximal immersion with non-isolated singularity.** Discussion of this section is based on [9], [12], [16], [17] etc.

Let $X : \Omega \to \mathbb{E}_1^3$ be a generalised maximal surface with non-isolated singularity $p \in \Omega$. Then X has a representation given by (3.2) with F as in (3.4). We consider X as map

MEMBER'S COPY

from $\Omega \to \mathbb{R}^3$ given by (3.2). We have

$$X_u \times X_v = -2iF_z \times F_{\bar{z}} = (|g|^2 - 1)|f|^2(1 + |g|^2, 2\mathrm{Re}\,\mathrm{g}, 2\mathrm{Im}\,\mathrm{g})$$

where '$\times'$ is the formal Euclidean cross product. If we define $\mathbf{n} : \Omega \to \mathbb{R}^3$ given by

$$\mathbf{n} := \frac{1}{\sqrt{(1 + |g|^2)^2 + 4|g|^2}}(1 + |g|^2, 2\mathrm{Re}\,\mathrm{g}, 2\mathrm{Im}\,\mathrm{g})$$

then $\mathbf{n}$ is a unit vector field. Moreover X_u and X_v as vector in $\mathbb{R}^3$, we clearly have $\langle X_u, \mathbf{n}\rangle = \langle X_v, \mathbf{n}\rangle = 0$. Here $\langle, \rangle$ is formal Euclidean inner product.

Any smooth map $\chi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ having such $\mathbf{n}$ is said to be a frontal, it is defined in the following way:

Definition 5.1. *Let $\mathcal{U} \subset \mathbb{R}^2$ be a domain and $\chi : \mathcal{U} \to \mathbb{R}^3$ be any smooth map. Then χ is said to be a frontal if there exists a smooth unit vector field $\boldsymbol{n} : \mathcal{U} \to \mathbb{R}^3$ such that $\boldsymbol{n}_p \perp span\{\frac{\partial \chi}{\partial u}|_p, \frac{\partial \chi}{\partial v}|_p\}$ for all $p \in \mathcal{U}$.*

Corresponding to every frontal χ there is an obvious map called the Legendrian map $\mathcal{L} : \mathcal{U} \to \mathbb{R}^3 \times S^2$ defined by $\mathcal{L}(u, v) := (\chi(u, v), \mathbf{n}(u, v))$, clearly $\mathcal{L}$ is smooth. Using this map now we define fronts:

Definition 5.2. *A frontal $\chi : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a front if the Legendrian map $\mathcal{L}$ for χ is an immersion.*

When a smooth map $\chi : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$ is also a frontal then one often talks about singularities of χ in terms of a function $\lambda : \mathcal{U} \to \mathbb{R}$ defined by $\lambda(u, v) := \vec{\mathbf{n}} \cdot (\vec{\chi_u} \times \vec{\chi_v})$, precisely the zeros of the function λ are said to be the singular points of the frontal χ. In this case a singular point $p \in \mathcal{U}$ is said to be nondegenerate if $d\lambda|_p$ does not vanish, else it is said to be degenerate. Singularities of a front are defined as the singularities of the corresponding frontal.

In particular any generalised maximal immersion $X : \Omega \to \mathbb{R}^3$ as map is always a frontal. Then, as is expected, $p \in \Omega$ is a singularity of X as generalised maximal immersion if and only if p is a singularity of X as frontal. Moreover if p is a nondegenerate singularity of X as frontal then as a generalised maximal immersion p is not an isolated singularity and p has to satisfy the condition $|g(p)|= 1$ where g is the meromorphic function in the Weierstrass data of X.

Though the generalised maximal immersions can always be seen as frontals, they are not always fronts. In [12] Umehara and Yamada proved that for a maxface (particular class of generalised maximal immersions) X with Weierstrass data $\{g, fdz\}$, X is a front

MEMBER'S COPY

on a neighborhood of p and p is a nondegenerate singular point(as frontal) of X if and only if $Re\left(\frac{g'}{g^2 f}\right)\Big|_p \neq 0$.

In [16], up to $\mathcal{A}$-equivalence, the singularities of a front at a nondegenerate singular point have been studied and in [12] and [9] this equivalence has been described directly in terms of the corresponding Weierstrass data in the case of maxfaces. This description is given in Table 1.

For a maxface with Weierstrass data $\{g, w\}$, since $\omega(p) \neq 0$ for a singular point p, in some complex coordinate we can write $w = dz$, moreover $g(p) \neq 0$ as $|g(p)| = 1$. Therefore on a simply connected neighbourhood U of p there is an analytic function h such that $g = e^h$. In [9] a maxface generated by the Weierstrass data $\{e^h, dz\}$ is denoted by f_h. Let $\mathcal{O}(U)$ be the set of holomorphic functions defined on U. This set can be endowed with the compact open C^∞-topology. Now for an arbitrary compact subset $K \subset U$ define $S(K)$ to be the set of all holomorphic functions $h \in \mathcal{O}(U)$ such that the singularities on the set K of the corresponding maxface f_h are cuspidal edges, swallowtail or cuspidal cross caps. Then Fujimori *et al.* in [9] proved that $S(K)$ is an open and dense subset of $\mathcal{O}(U)$.

6. Acknowledgment

First Author thanks support by ICTS during a visit for participating in the program - Geometry and Topology for Lecturers (Code: ICTS/gtl2018/06). The second author would like to acknowledge NBHM Ph.D fellowship during the course of this study. Third Author thanks Shiv Nadar University's OUR program for the support. Authors are very grateful to Professors Fujimori, C.S. Aravinda, Rukmini Dey and Rahul Kumar Singh for giving suggestions and edits to improve the article.

References

[1] Osserman, R., A Survey of Minimal Surfaces, Dover Publications. ISBN–10: 0486649989

[2] Alías, L. J., Chaves, R. M. B. and Mira, P., Björing problem for maximal surfaces in Lorentz-Minkowski space, *Math. Proc. Cambridge Philos. Soc.* **134** (2003), 289-316.

[3] Estudillo, F. J. M. and Romero, A., Generalized maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3$, *Math. Proc. Cambridge Philos. Soc.* **111** (1992), 515-524.

[4] Kim, Y. W. and Yang, S.-D., Prescribing singularities of maximal surfaces via a singular Björling representation formula, *J. Geom. Phys.* **57** (2007), 2167-2177.

[5] Kobayashi, O., Maximal surfaces in the 3-dimensional Minkowski space $\mathbb{L}^3$, *Tokyo J. Math.* **6** (1983), 297-309.

[6] Kobayashi, O., Maximal surfaces with conelike singularities, *J. Math. Soc. Japan* **36** (1984), 609-617.

[7] ÓNeill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

MEMBER'S COPY

[8] Fernández, I., López, F. and Souam, R., The space of complete embedded maximal surfaces with isolated singularities in the 3-dimensional Lorentz - Minkowski space $\mathbb{L}^3$, *Math. Ann.* **332** (2005), 605-643.

[9] Fujimori, S., Saji, K., Umehara, M. and Yamada, K., Singularities of maximal surfaces, *Math. Z.*, **259** (2008), 827-848.

[10] Kim, Y. W., Koh, S.-E., Shin, H. and Yang, S.-D., Spacelike maximal surfaces, timelike minimal surfaces and Björling representation formulae, *Journal of Korean Math. Soc.* **48** (2011), 1083-1100.

[11] Kim, Y. W. and Yang, S.-D., A family of maximal surfaces in Lorentz-Minkowski three-space, *Proc. Amer. Math. Soc.* **134** (2006), 3379-3390.

[12] Umehara, M. and Yamada, K., Maximal surfaces with singularities in Minkowski space, *Hokkaido Math. J.* **35** (2006), 13-40.

[13] Dey, R., Kumar, P. and Singh, R. K., Existence of maximal surface containing given curve and special singularity, *J. Ramanujan Math. Soc.* **33** (2018), 455-471.

[14] Whitney, H., The general type of singularity of a set of $2n-1$ smooth functions of n variables. *Duke Math. J.* **10** (1943), 161-172.

[15] Rafael López, Differential geometry of curves and surfaces in Lorentz-Minkowski space, https://arxiv.org/abs/0810.3351

[16] Kokubu, M., Rossman, W., Saji,K., Umehara, M. and Yamada, K., Singularities of flat fronts in hyperbolic 3-space, *Pacific J. Math.* **221** (2005), 303-351.

[17] Arnol'd, V. I., Gusein-Zade, S. M. and Varchenko, A. N., Singularities of differentiable maps, Vol. 1, Monographs in Math. 82, Birkhäuser (1985)

[18] Fujimori,S., Rossman, W., Umehara, M., Yamada, K. and Yang, S.-D., New maximal surfaces in Minkowski 3-space with arbitrary genus and their cousins in de Sitter 3-space, *Results in Mathematics*, **56** (2009), 41-82.

[19] Fujimori, S., Kim, Y. W., Koh, S.-E., Rossman, W., Shin, H., Umehara, M., Yamada, K. and Yang, S., Zero mean curvature surfaces in Lorentz-Minkowski 3-space and 2-dimensional fluid mechanics, *Mathematical Journal of Okayama University*, **57** (2015), 173-200.

[20] Ogata, Y. and Teramoto, K., Duality between cuspidal butterflies and cuspidal S_1^- singularities on maximal surfaces, *Note Mat.* **38** (2018), 115-130

Pradip Kumar
Department of Mathematics
School of Natural Sciences
Shiv Nadar University,
Uttar Pradesh, INDIA-201314.
E-mail: pradip.kumar@snu.edu.in

Samir Kumar Hazra
Department of Mathematics
School of Natural Sciences

MEMBER'S COPY

Shiv Nadar University,
Uttar Pradesh, INDIA-201314.
E-mail: sk896@snu.edu.in

Vidisha Kairam
Department of Mathematics
School of Natural Sciences
Shiv Nadar University,
Uttar Pradesh, INDIA-201314.
E-mail: vk486@snu.edu.in

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 165–170

WEAK COMPACTNESS OF TRANSLATES OF THE WIENER MEASURE AND AN APPLICATION OF THE KAKUTANI DICHOTOMY

R. P. PAKSHIRAJAN

(Received : 31 - 12 - 2019 ; Revised : 29 - 05 - 2020)

Abstract Let μ denote the standard Wiener measure on $\mathbf{C}$ the space of all continuous functions defined on the interval $[0,\ 1]$ and all vanishing at 0. ν_n denotes the translate of μ by $f_n \in \mathbf{C}\ :\ \nu_n(A) = \mu(A - f_n)$. Let $P = \mu \times \mu \times ....$ and $Q = \nu_1 \times \nu_2 \times \nu_3 \times ......$ be the product measures on product space $\mathbf{C}^{(\infty)} = \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times .....$ Imposing certain natural conditions on the sequence (f_n) we prove $Q \ll P$ and, more importantly, exhibit explicitly the Radon-Nikodym derivative $\frac{\mathrm{d}Q}{\mathrm{d}P}(\mathbf{x}),\ \mathbf{x} \in \mathbf{C}^{(\infty)}$.

Introduction.

Let $\mathbf{C}$ denote the space of all continuous functions defined on $[0,\ 1]$, all vanishing at zero and equipped with the uniform metric ρ. The Borel σ-field of $\mathbf{C}$ is denoted by $\mathscr{C}$. Let $\mathbf{F} \subset \mathbf{C},\ \mathbf{F} = \{f\}$ be the family of all the absolutely continuous functions with square (Lebesgue) integrable Lebesgue deivatives $\{f'\}$. i.e.,

$$v_f = \int_0^1 \left(f'(t)\right)^2 \mathrm{d}t < \infty \tag{1}$$

Let μ denote the standard Wiener measure on $\mathscr{C}$. For $A \in \mathscr{C}$ and $f \in \mathbf{F}$, define $\nu(A) = \nu_f(A) = \mu(A - f)$. So defined, ν_f is a probability measure on $\mathscr{C}$.

Let $f_n \in \mathbf{F},\ n \geq 1$ be such that

$$v = \sum_n v_n = \sum_n \int_0^1 \left(f_n'(t)\right)^2 \mathrm{d}t < \infty \tag{2}$$

Write $$\nu_{f_n} = \nu_n \quad \text{and} \quad q_n = \sum_{j=1}^{n} v_j.\ \text{Note } q_n \to v \tag{3}$$

2010 Mathematics Subject Classification: 60F15

Key words and phrases: Function spaces, Measure theory

© *Indian Mathematical Society, 2020*.

MEMBER'S COPY

Facts

(a) Let $\mathbf{F}_\tau = \{f \ : \ f \in \mathbf{F}, \ v_f \leq \tau\}$.

(a1) If $f_n \in \mathbf{F}_\tau, \ n \geq 1$ and if $\rho(f_n, \ f) \to 0$, then $f \in \mathbf{F}_\tau$.

(a2) $\mathbf{F}_\tau$ is a compact subset of $\mathbf{C}$ (ref. pp 126 - 128, [5]). (4)

(b) **Kakutani dichotomy** (ref. [3])

Let $\mathcal{Q}_n, \ \mathcal{P}_n$ be probability measures on the measurable space $(\Omega_n, \ \mathscr{S}_n), \ n = 1, 2, 3, ...$ Let $\mathcal{Q} = \bigotimes \mathcal{Q}_n, \mathcal{P} = \bigotimes \mathcal{P}_n$ denote the product measures on the product measurable space $(\Omega, \ \mathscr{S})$. Let for each $n, \ \mathcal{Q}_n, \ \mathcal{P}_n$ be mutually absolutely continuous with respect to each other $\mathcal{Q}_n \sim \mathcal{P}_n$. Then $\mathcal{Q} \sim \mathcal{P}$ or the measures $\mathcal{Q}, \mathcal{P}$ will be mutually singular $(\mathcal{Q} \perp \mathcal{P})$. The first case occurs *iff* $\prod\limits_1^\infty \int\limits_{\Omega_n} \left(\frac{\mathrm{d}\mathcal{Q}_n}{\mathrm{d}\mathcal{P}_n}\right)^{\frac{1}{2}} \mathrm{d}\mathcal{P}_n > 0$. (5)

If $\mathcal{Q}_n = \beta$ and if $\mathcal{P}_n = \alpha$ for all n then $\mathcal{Q} \perp \mathcal{P}$ unless $\alpha \equiv \beta$.

In general, it is not easy to explicitly exhibit $\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}\mathcal{P}}$ when the condition (5) holds. In the paricular important case considered in the Theorem below, this has been possible.

(c) **Cameron-Martin** (ref. [1])

If $f \in \mathbf{F}$ then $\nu_f \ll \mu$ and the Radon-Nikodym derivative

$$\frac{\mathrm{d}\nu_f}{\mathrm{d}\mu}(x) = e^{-\frac{1}{2}v_f + u_f(x)}, \ x \in \mathbf{C} \tag{6}$$

where $v_f = \int\limits_0^1 \left(f'(t)\right)^2 \mathrm{d}t$ and $u_f(x) = \int\limits_0^1 f'(t)\, \mathrm{d}x(t)$ a stochastic integral. (7)

To avoid trivialities, assume with no loss of generality that none of the translaing functions is identically zero.

When μ is the underlying measure,

u_f has normal distribution with zero mean and variance v_f. (8)

We note v_f is necessarily positive since $v_f = 0$ would imply $f'(t) = 0$ for almost all t (Lebesgue measure). It would then follow from this that f is not absolutely continuous.

We introduce some notation. For identification convenience let $\mathbf{C}_n = \mathbf{C}, \ \mathscr{C}_n = \mathscr{C}, \ \mathbf{C}^{(n)}$ is the cartesian product of $\mathbf{C}_j, \ 1 \leq j \leq n$; $\mathbf{C}^{(\infty)}$ is the cartesian product of $\mathbf{C}_j, \ j \geq 1$; $\mathscr{C}^{(n)} = \bigotimes\limits_{j=1}^{n} \mathscr{C}_j$; $\mathscr{C}^{(\infty)} = \bigotimes\limits_{j=1}^{(\infty)}$; $\mu_n = \mu$; ν_n as in (3); $P_n = \mu_1 \times \mu_2 \times .... \times \mu_n, \ P = \mu_1 \times \mu_2 \times ...., \ Q_n = \nu_1 \times \nu_2 \times .... \times \nu_n, \ Q = \nu_1 \times \nu_2 \times ....$, the product measures.

Using (6) and properties of product measures we have

MEMBER'S COPY

$$\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}(x_1,\ x_2,\ ...,\ x_n) = e^{\sum\limits_{j=1}^{n}\{-\frac{1}{2}v_j+u_j(x_j)\}},\ x_j \in \mathbf{C}_j,\ 1 \leq j \leq n \qquad (9)$$

For $\mathbf{x} \in \mathbf{C}^{(\infty)}$, write $U_j(\mathbf{x}) = U_j(x_1,\ x_2,\ .....) = u_j(x_j)$. So defined on $\mathbf{C}^{(\infty)}$, the U_js are $\mathscr{C}^{(\infty)}$- measurable, they form a sequnce of independent variables, both under P as well as Q. Under P, U_j will have a zero mean normal distribution with v_j for its variance. Consider $\tilde{Q}_n$ defined on $\mathscr{C}^{(\infty)}$ by prescribing $\tilde{Q}_n(E) = Q_n(E \cap \mathbf{C}^{(n)})$ for $E \in \mathscr{C}^{(\infty)}$. Similar extension for P_n can be defined. We will then have:

$$\frac{\mathrm{d}\tilde{Q}_n}{\mathrm{d}\tilde{P}_n}(\mathbf{x}) = e^{-\frac{1}{2}q_n+\sum\limits_{j=1}^{n} U_j(\mathbf{x})},\ \mathbf{x} \in \mathbf{C}^{(\infty)} \qquad (10)$$

We note there exists a metric d for $\mathbf{C}^{(\infty)}$: ($d(\mathbf{x},\ \mathbf{y}) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}\frac{\rho(x_n,\ y_n)}{1+\rho(x_n,\ y_n)}$), ref. pp 197, 199, 243-244, [6]) under which it is a complete and separable metric space, the product topology being equivalent to the resulting metric topology; the product σ-field being the same as the Borel σ- field.

The object of this paper is to prove the following

Theorem. (i) $\mathcal{M}_\tau = \{\nu_f\ :\ f \in \mathbf{F}_\tau\}$ is a compact subset of $\mathcal{M}$ the space of all probability measures on $\mathscr{C}$ endowed with the Prohorov metric π.
(ii) $Q \perp P$ if $P = \mu \times\ \mu \times .....$ and $Q = \nu_f \times \nu_f \times ......,\ f \in \mathbf{F}$ arbitrary but fixed.
(iii) When the underlying measure is P,
Relation (2) $\Leftrightarrow \sum\limits_n U_n$ converges in distribution $\Leftrightarrow \sum\limits_n U_n(\mathbf{x})$ exists for P- almost all $\mathbf{x}$.
Further the limit variable U would be normally distributed with mean 0 and variance v.
(iv) If $\mu_n = \mu$ and $Q_n = \nu_n$ and if (2) holds, then $Q \sim P$
(v) If (2) holds, then with $P_n,\ Q_n$ as in (iv), $\frac{\mathrm{d}Q}{\mathrm{d}P}(\mathbf{x}) = e^{-\frac{1}{2}v+U(\mathbf{x})}$

Proof. (i) Let (ν_n) be an arbitrary sequence in $(\mathcal{M},\ \pi)$. Since $\mathbf{F}_\tau$ is a compact set (ref. Fact (a2)), sequence (f_n) (corresponding to the ν_ns) contains a ρ-convergent subsequence, say, (f_j) converging to, say, f_0. By **Fact** (a1), $f_0 \in \mathbf{F}_\tau$. Let $A \in \mathscr{C}$ be an arbitrary closed subset. Hence $A - f_0$ would be a closed set. Given $\varepsilon > 0$, there is N such that $\rho(f_n,\ f_0) < \varepsilon$ for all $n \geq N$.

MEMBER'S COPY

Let $S_\varepsilon = \{f \ : \ f \in \mathbf{C}, \ \|f\| < \varepsilon\}$. We note $A - f_0 + S_\varepsilon = \bigcup\limits_{f \in A} (f - f_0 + S_\varepsilon)$. Since S_ε is an open set, $f - f_0 + S_\varepsilon$ and hence $A - f_0 + S_\varepsilon$ would be open sets. Further $A - f_0 \subset A - f_0 + S_\varepsilon$. Thus $A - f_0 + S_\varepsilon$ is an ε- neighbourhood of the closed set $A - f_0$. Also $\lim\limits_{\varepsilon \downarrow 0} \{A - f_0 + S_\varepsilon\} = A - f_0$. Now,

$\nu_j(A) = \mu(A - f_j) \leq \mu(A - f_0 + S_\varepsilon)$ for all large j

$\leq \mu(A - f_0)$ (setting $\varepsilon = \frac{1}{n}$ and letting $n \to \infty$) $= \nu_0(A)$

This is equivalent to saying $\nu_j \overset{w}{\to} \nu_0$. By this we have proved that every sequence of elements in $(\mathcal{M}_\tau)$ contains a π-convergent subsequence, converging to an element in $\mathcal{M}_\tau$. That $\mathcal{M}_\tau$ is a compact set follows now.

Remark

The following arguments fall a little short of proving the claim but shows that $\mathcal{M}_\tau$ is sequentially relatively compact.

Since $(\mathbf{C}, \ \rho)$ is a complete and separable metric space, measure μ is tight (ref. p44, Theorem 1.9.4, [5]). Hence given $\varepsilon > 0$ there is a compact set $K \subset \mathbf{C}$ such that $\mu(K) > 1 - \varepsilon$. Since K is a compact set and since $\mathbf{F}_\tau$ is a compact set (ref. (3)), it follows

$$\mathcal{K} = K \oplus \mathbf{F}_\tau = \{g + f \ : \ g \in K \text{ and } f \in \mathbf{F}_\tau\} \text{ is a compact set.}$$

For $f \in \mathbf{F}_\tau, \ \nu_f(\mathcal{K}) \geq \nu_f(K \oplus \{f\}) = \nu_f(K + f)$

$= \mu(K + f - f) = \mu(K) > 1 - \varepsilon$. Thus $\mathcal{M}_\tau$ is a tight family. This implies the claim, again using the separability and completeness of $(\mathbf{C}, \ \rho)$. (ref. [3])

(ii) This follows immediately from the general result quoted at (5). Even so it is felt desirable to establish $Q \perp P$ by steps special for the problem on hand.

$Q \perp P$ if $\prod\limits_1^\infty \int\limits_C \left(\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu}(x)\right)^{\frac{1}{2}} \mathrm{d}\mu(x) = 0$. i.e., if $\prod\limits_1^\infty e^{-\frac{1}{4}v_f} \int\limits_C e^{\frac{u_f(x)}{2}} \mathrm{d}\mu(x) = 0$.

i.e., if $\prod\limits_1^\infty e^{-\frac{1}{4}v_f} \int\limits_{-\infty}^{\infty} e^{\frac{y}{2}} \frac{1}{\sqrt{v_f 2\pi}} e^{-\frac{1}{2}\frac{y^2}{v_f}} \mathrm{d}y = 0$

i.e., if $\prod\limits_1^\infty e^{-\frac{1}{8}v_f} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{v 2\pi}} e^{-\frac{1}{2v_f}\{y - v_f\}^2} \mathrm{d}y = 0$ i.e., if $\prod\limits_1^\infty e^{-\frac{v_f}{8}} = 0$

which is obviously true, since $v_f > 0$ and hence $e^{-\frac{v_f}{8}} < 1$.

(iii) It has already been noted that, under measure P, the U_ns form an independent sequence of normal variables with $\mathbb{E}U_n = 0$ and $\mathbb{E}U_n^2 = v_n$.

MEMBER'S COPY

Hence $\mathbb{E}e^{it\sum\limits_{j=1}^{n}U_j} = \prod\limits_{j=1}^{n}\mathbb{E}e^{itU_j} = \prod\limits_{j=1}^{n}e^{-\frac{1}{2}v_n t^2} = e^{-\frac{1}{2}q_n t^2}$. Hence $\sum\limits_{n=1}^{\infty}U_n$ converges in distribution if and only if q_n (which is a monotonic increasing sequence) has a finite limit. i.e., if and only if (2) holds.

Suppose now $\sum\limits_{n=1}^{\infty}U_n$ converges in distribution. Hence, by what has just been proved, $q_n \to v < \infty$. This implies $\prod\limits_{n=1}^{\infty}\mathbb{E}e^{itU_n} = e^{-\frac{1}{2}vt^2} > 0$ for all values of t. Now appeal to Theorem 2.7(i) converse part, p115 [2] and conclude that $\sum\limits_{n=1}^{\infty}U_n$ converges with P-measure 1. Proof in the reverse direction is trivial since convergence $wp1$ implies convergence in distribution.

$$
\begin{aligned}
\text{(iv)} \int\limits_{\mathbf{C}}\left(\tfrac{\mathrm{d}\nu_n}{\mathrm{d}\mu}(x)\right)^{\frac{1}{2}}\mathrm{d}\mu(x) &= \int\limits_{\mathbf{C}}\left(e^{-\frac{1}{2}v_n+u_n(x)}\right)^{\frac{1}{2}}\mathrm{d}\mu(x)\\
&= e^{-\frac{1}{4}v_n}\int\limits_{\mathbf{C}}e^{\frac{1}{2}u_n(x)}\,\mathrm{d}\mu(x)\\
&= e^{-\frac{1}{4}v_n}\int\limits_{-\infty}^{\infty}e^{\frac{1}{2}y}\frac{1}{\sqrt{v_n 2\pi}}e^{-\frac{y^2}{2v_n}}\,\mathrm{d}y\\
&= e^{-\frac{v_n}{8}}\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{v_n 2\pi}}e^{-\frac{1}{2v_n}(y^2-yv_n+\frac{1}{4}v_n^2)}\,\mathrm{d}y = e^{-\frac{1}{8}v_n}
\end{aligned}
$$

leading to $\prod\limits_{n=1}^{\infty}\int\limits_{\mathbf{C}}\left(\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu}(x)\right)^{\frac{1}{2}}\mathrm{d}\mu(x) = e^{-\frac{1}{8}\sum\limits_{n=1}^{\infty}v_n} > 0$ appealing to (2).
That $Q \sim P$ follows now.

(v) Let $E \in \mathscr{C}^{(\infty)}$ be arbitrary. Let $A_n = (E \cap \mathbf{C}^{(n)}) \times \mathbf{C}_{n+1} \times \mathbf{C}_{n+2}......$ and note $A_n \downarrow E$. We have :

$$
\begin{aligned}
Q(E) = \lim_{n\to\infty}Q(A_n) = \lim_{n\to\infty}\tilde{Q}_n(A_n) &= \lim_{n\to\infty}\int\limits_{\mathbf{C}^{(\infty)}}\chi_{A_n}(\mathbf{x})\frac{\mathrm{d}\tilde{Q}_n}{\mathrm{d}\tilde{P}_n}(\mathbf{x})\,\mathrm{d}P_n(\mathbf{x})\\
&= \lim_{n\to\infty}\int\limits_{\mathbf{C}^{(\infty)}}\chi_{A_n}(\mathbf{x})e^{-\frac{1}{2}q_n+\sum\limits_{j=1}^{n}U_j(\mathbf{x})}\,\mathrm{d}P_n(\mathbf{x})\\
&= \lim_{n\to\infty}\int\limits_{\mathbf{C}^{(\infty)}}\chi_{A_n}(\mathbf{x})e^{-\frac{1}{2}q_n+\sum\limits_{j=1}^{n}U_j(\mathbf{x})}\,\mathrm{d}P(\mathbf{x}) \qquad (11)
\end{aligned}
$$

Since, under P, $\sum\limits_{j=1}^{n}U_j$ is normally distributed, taking the limit under the integral sign in (11) would be justified if it is justified in the case of $\int\limits_{-\infty}^{\infty}e^{y}\frac{1}{\sqrt{q_n 2\pi}}e^{-\frac{y^2}{2q_n}}\,\mathrm{d}y$. i.e., if justified in the case $\int\limits_{-\infty}^{\infty}e^{|y|\sqrt{q_n}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,\mathrm{d}y$. Justification in this case is available since $q_n \to v < \infty$. Hence, taking limit

MEMBER'S COPY

under the integral sign,

$$Q(E) = \int\limits_{E} e^{-\frac{1}{2}v+U(\mathbf{x})}\,\mathrm{d}P(\mathbf{x})$$

This being true for every $E \in \mathscr{C}^{(\infty)}$, it follows that $\frac{\mathrm{d}Q}{\mathrm{d}P}(\mathbf{x}) = e^{-\frac{1}{2}v+U(\mathbf{x})}$ $\square$

REFERENCES

[1] R. H. Cameron and W. T. Martin, *Transformation of Wiener integrals under translations*, Ann. of Math. **45**, 386 - 396 (1944).

[2] J. L. Doob, *Stochastic Processes*, John Wiley & Sons, Inc., New York (1952).

[3] S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math., **49**, 214 - 224 (1948).

[4] MIT OpenCourseWare, Fall 2013, 11/27/2013, Therem 2, p2

[5] R. P. Pakshirajan, *Probability Theory*, Hindustan Book Agency, New Delhi, (2013).

[6] R. Vaidyanathaswamy, *Treatise On Set Topology.* The Indian Mathematical Society, 1947.

R. P. Pakshirajan
Formerly at Mysore University, Mysore .
E-mail : vainatheyarajan@yahoo.in

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July-December (2020), 171–176

A NECESSARY AND SUFFICIENT CONDITION FOR 2 TO BE A PRIMITIVE ROOT OF $2P+1$

V. P. RAMESH, R. THANGADURAI, M. MAKESHWARI AND SASWATI SINHA
(Received : 02 - 03 - 2020 ; Revised : 26 - 08 - 2020)

Abstract. Let p be an odd prime such that $2p+1$ is a prime or prime power. Then, in this article, we prove that 2 is a primitive root of $2p+1$ if and only if $p \equiv 1 \pmod 4$.

1. Introduction

Gauss proved that the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ is cyclic if and only if $n = 2, 4, p^k$ or $2p^k$ for all odd primes p and for all positive integers k. For such integers n, the generators are called *primitive roots* of n. Indeed, while studying the periods of rational numbers of the form $1/p$ for a prime $p \neq 2$ or 5, Gauss proved the above result and he conjectured that 10 *is a primitive root of p for infinitely many primes p*. Later E. Artin generalized this conjecture and gave a heuristic argument for a quantitative form of this conjecture and nowadays, it is well-known as *Artin's primitive root conjecture* [4]. Due to these conjectures there are many efforts leading to discoveries around primitive roots of n, to list a few [1, 4, 5, 6].

We will first set up some notations. For any $x \in \mathbb{R}$, $[x]$ denotes the *greatest integer function* i.e., the largest integer less than or equal to x. A prime p is said to be a *Sophie Germain prime* [2] if $2p+1$ is also a prime. It is expected that there is an infinitude of such primes. Let σ be an element of the symmetric group S_n. It is easy to observe that the following relation is an equivalence relation. For $i, j \in \{1, 2, 3, \ldots, n\}$, we say $i \sim j$ if there exists $k \in \mathbb{Z}$ such that $\sigma^k(i) = j$. The equivalence classes of this relation are called orbits of σ. Furthermore, $\sigma \in S_n$ is said to be a *cycle* of length ℓ,

2010 Mathematics Subject Classification: 11A07, 11A41, 20F05
Key words and phrases: primitive root, Sophie Germain prime, permutation, orbits of permutation

© *Indian Mathematical Society, 2020*.

MEMBER'S COPY

if one of its orbits has ℓ elements and rest of them have only one element.

In this article, we prove the following results.

Theorem 1.1. *Let p be an odd prime such that $2p+1$ is a prime or prime power. Then 2 is a primitive root of $2p+1$ if and only if $p \equiv 1 \pmod 4$.*

Lemma 1.2. *Let p be an odd prime such that $2p+1 = q^k$ for some prime q and some integer $k \geq 2$. Then $q = 3$, k is a prime number and $p \equiv 1 \pmod 4$.*

Lemma 1.3. *For any natural number k, we have*

$$\left[\frac{2^{\phi(3^k)}}{3^k}\right] \equiv 1 \pmod 3,$$

where ϕ is the Euler's totient function.

Corollary 1.4. *For any natural number ℓ,*

$$\left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right] \textit{ divides } \left[\frac{2^{\phi(3^{\ell+1})}}{3^{\ell+1}}\right].$$

From Gauss we know that *"For a prime p, if a is a primitive root of p and p^2, then a is a primitive root of p^ℓ for all $\ell \geq 3$".* We consider a special case of this statement, namely for $a = 2$, $p = 3$ and in this article we present the following result which is a stronger result for this special case.

Lemma 1.5. *For any $k \in \mathbb{N}$, 2 is a primitive root of 3^k.*

Though, Lemma 1.5 can be proved using the above result of Gauss, in this article we have invoked Lemma 1.3 to give a self-contained proof of this lemma. It is to be noted that these lemmas are useful while proving Theorem 1.1.

In 1969, D. J. Aulicino and Morris Goldfeld [1] have studied the permutation $(n!)$ defined as $(n!) = \prod_{k=0}^{n-1}(1, 2, \ldots, (n-k))$, i.e., the product of first n cycles. They observed a connection between a primitive root of $2n+1$ and the permutation $(n!)$ having only one orbit (which is called as a *transitive permutation*) and proved that *for any natural number n, the permutation $(n!)$ is transitive if and only if $2n+1$ is a prime for which 2 is a primitive root* [1]. Therefore, we have the following natural corollary from Theorem 1.1.

MEMBER'S COPY

Corollary 1.6. *Let p be an odd prime. Then the permutation $(p!)$ is transitive if and only if $2p+1$ is prime and $p \equiv 1 \pmod 4$.*

We performed a few computations with primes up to 3×10^6 and observed that about 4.515% of primes in the above range are such that $2p+1$ is also prime with 2 as a primitive root. Furthermore, the primes $13, 1093$ and 797161 are the only primes in the above range for which 2 is a primitive root and $2p+1$ is not prime. It is easy to observe that for the above listed primes, $2p+1$ is an odd power of 3, namely $27 = 3^3, 2187 = 3^7$ and $1594323 = 3^{13}$. We have also estimated that for the prime $p = 6957596529882152968992225251835887181478451547013$, $2p+1 = 3^{103}$ with 2 as a primitive root. It is worth mentioning here that the powers of 3 in the representations of $2p+1$ are also primes.

We state the following lemma (see Theorem 2 of [3]) which will be used while proving Theorem 1.1.

Lemma 1.7. *Let p be an odd prime such that $2p+1$ is also a prime. Then, we have*

(1) $2p+1$ *divides* $2^p - 1$, *if* $p \equiv 3 \pmod 4$;
(2) $2p+1$ *divides* $2^p + 1$, *if* $p \equiv 1 \pmod 4$.

2. Proofs of Lemmas 1.2, 1.3 and 1.5

Proof of Lemma 1.2. Let p be an odd prime such that $2p+1 = q^k$ for some prime q and for some integer $k \geq 2$. Clearly, $q \geq 3$. Therefore,

$$2p = q^k - 1 = (q-1)(1 + q + q^2 + \cdots + q^{k-1}).$$

Since $q \geq 3$, by the unique factorization in integers, we conclude that $2 = q - 1$ and $p = 1 + q + q^2 + \cdots + q^{k-1}$. Thus, we get

$$q = 3 \text{ and } p = 1 + 3 + 3^2 + \cdots + 3^{k-1}.$$

Since $3^{2m} \equiv 1 \pmod 4$ and $3^{2m+1} \equiv -1 \pmod 4$, we see that k must be an odd integer. For otherwise, we get $p \equiv 0 \pmod 4$, a contradiction to p being prime. Since k is an odd integer, we get $p \equiv 1 \pmod 4$.

Now, suppose k is not prime, equivalently $k = mn$ for some $1 < m, n < k$, then $3^m - 1$ and $3^n - 1$ are factors of $3^k - 1$ since

$$3^k - 1 = (3^m - 1)(1 + 3^m + 3^{2m} + \cdots + 3^{(n-1)m})$$

MEMBER'S COPY

which is a contradiction. □

Proof of Lemma 1.3. Now we prove Lemma 1.3 by induction on k. When $k = 1$, it is clearly true. We shall assume the result for $k = \ell$ and we prove for $\ell + 1$. Since $2^{\phi(3^\ell)} \equiv 1 \pmod{3^\ell}$, we get

$$2^{\phi(3^\ell)} = \left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right] 3^\ell + 1. \tag{2.1}$$

Taking the 3-rd power both sides and since $3 \cdot \phi(3^\ell) = \phi(3^{\ell+1})$ we get

$$2^{\phi(3^{\ell+1})} = \left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right]^3 3^{3\ell} + \left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right]^2 3^{2\ell+1} + \left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right] 3^{\ell+1} + 1.$$

On simplification, we get,

$$\left[\frac{2^{\phi(3^{\ell+1})}}{3^{\ell+1}}\right] = \left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right] \left(\left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right]^2 3^{2\ell-1} + \left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right] 3^\ell + 1\right).$$

And, by induction hypothesis, the lemma follows. □

Proof of Lemma 1.5. Now, we prove Lemma 1.5 by induction on k. Since 2 is a primitive root of 3, we shall assume that 2 is a primitive root of 3^ℓ for some integer $\ell \geq 2$ and we prove that 2 is a primitive root of $3^{\ell+1}$.

Let the order of 2 modulo $3^{\ell+1}$ be d. Then, $d \mid \phi(3^{\ell+1}) = 2 \cdot 3^\ell$. Since 2 is a primitive root of 3^ℓ, we get $\phi(3^\ell) \mid d$ and therefore it is clear that $d = 2 \cdot 3^{\ell-1}$ or $2 \cdot 3^\ell$. By Lemma 1.3, we see that

$$3 \nmid \left[\frac{2^{\phi(3^\ell)}}{3^\ell}\right] \iff 3^{\ell+1} \nmid 2^{2\cdot 3^{\ell-1}} - 1 \text{ (from (2.1))}.$$

Hence, we get $d \neq 2 \cdot 3^{\ell-1}$ and $d = 2 \cdot 3^\ell$. And therefore 2 is a primitive root of $3^{\ell+1}$. □

3. Proof of Theorem 1.1

Proof. Let p be an odd prime such that $2p + 1 = q^k$ for some odd prime q and for some natural number k.

Case 1. $k = 1$, i.e. both p and $2p + 1$ are primes.

Let us assume that 2 be a primitive root of $2p + 1$ and we prove that $p \equiv 1 \pmod 4$. Suppose, $p \not\equiv 1 \pmod 4$, then $2^p \equiv 1 \mod 2p + 1$ from Lemma 1.7 which is a contradiction to 2 being a primitive root of $2p + 1$. Conversely, if $p \equiv 1 \pmod 4$, then again from Lemma 1.7, we have $2^p \equiv -1$

MEMBER'S COPY

mod $2p+1$ which implies $2^p \not\equiv 1 \mod 2p+1$ and hence 2 is a primitive root of $2p+1$.

Case 2. $k > 1$, i.e. $2p+1 = q^k$ for some odd prime q and for some natural number $k \geq 2$.

Now, by Lemma 1.2, we conclude that $q = 3$, k is an odd integer and $p \equiv 1 \pmod 4$. Conversely, from Lemma 1.5, it follows that 2 is a primitive root of $2p+1$. □

Acknowledgement: We thank Professor M. Ram Murty for carefully going through this manuscript and also for various comments improving the presentation and results. We also thank the reviewers of *JRMS* and *The Mathematics Students* for various comments improving the presentation.

References

[1] Aulicino, D. J., and Goldfeld, M., A New Relation Between Primitive Roots and Permutations, *The American Mathematical Monthly*, **76** (1969), no. 6, 664-666.

[2] Burton, D., Elementary Number Theory, 7th ed. Tata McGraw-Hill, 2012.

[3] Jaroma, J. H. and Reddy, K. N., Classical and alternative approaches to the Mersenne and Fermat numbers, *The American Mathematical Monthly*, **114** (2007), no. 8, 677-687.

[4] Ram Murty, M., Artin's conjecture for primitive roots, *The Mathematical Intelligencer*, **10** (1988), 59-67.

[5] Ramesh, V. P., Thangadurai, R. and Thatchaayini, R., A Note on Gauss's Theorem on Primitive Roots, *The American Mathematical Monthly*, **126** (2019), no. 3, 252-254.

[6] Yuan, Y. and Wenpeng, Z., On the distribution of primitive roots modulo a prime, *Publicationes Mathematicae Debrecen*, **61** (2002), no. 3-4, 383-391.

V. P. Ramesh
Department of Mathematics
Central University of Tamil Nadu
Thiruvarur
Tamilnadu - 610 005.
E-mail: vpramesh@gmail.com

R. Thangadurai
Harish-Chandra Research Institute, HBNI
Chhatnag Road, Jhunsi

MEMBER'S COPY

Allahabad - 211019
E-mail: thanga@hri.res.in

M. Makeshwari
Department of Mathematics
Central University of Tamil Nadu
Thiruvarur
Tamilnadu - 610 005.
E-mail: makheswarim@gmail.com

Saswati Sinha
Department of Mathematics
Central University of Tamil Nadu
Thiruvarur
Tamilnadu - 610 005.
E-mail: saswati.sinha1994@gmail.com

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 3-4, July - December (2020), 177–192

PROBLEM SECTION

In Volume 89 (1-2) 2020 of the Mathematics Student, we had invited solutions from the floor, till September 30, 2020 to the old Problems 1, 2, 3, 4, 5 and 7 mentioned in the Math. Student Vol. 88 (3-4) 2019 as well as solutions to the eight new problems which were proposed in the Math. Student Vol. 89 (1-2) 2020.

As regards to solutions to Problems mentioned in MS 88 (3-4) 2019, we received three correct solutions to Problem 2 and one solution to Problem 1 that was not complete. One solution to problem 4 was received but it was not correct. We did not receive solutions to other problems so we are printing below the solutions provided by the proposers of the Problems. Problem 3 of this Volume is interesting and we give some more time to researchers to submit their solutions to this problem.

As far as solutions to Problems in MS 89(1-2) 2020 are concerned, we received two correct solutions to Problem 1, one correct solution to Problem 2, two correct solutions to Problem 3 and two correct solutions to Problem 4. We publish these solution in this section.

We first present **nine new problems**. We invite Solutions to these problems, solutions to the remaining problems of MS 89 (1-2) 2020 and solution to Problem 3 of MS 88 (3-4) 2019 from the researchers till March 15, 2021. Correct solutions received from the researchers by March 15, 2021 will be published in the MS 90 (1-2) 2021.

The following five problems have been proposed by Prof. B. Sury, Indian Statistical Institute, Bangalore. Prof. Sury is the President of the IMS for the year 2019-20.

MS 89 (3-4) 2020 : Problem 1. Note that $(x, y) \mapsto (y, -x)$ is a bijection of the plane. In contrast, show that there is no bijection f from $\mathbb{R}^3$ to itself such that for any two points $P \neq Q$, the line joining P and Q is perpendicular to the line joining $f(P)$ and $f(Q)$.

© *Indian Mathematical Society, 2020.*

MEMBER'S COPY

MS 89 (3-4) 2020 : Problem 2. Find all positive integral solutions of $a^3 = b^2 + p^3$ where p is a prime and 3 and p do not divide b.

MS 89 (3-4) 2020 : Problem 3. A positive integer is written in each square of an 100 by 100 chess board. The difference between the numbers in any two adjacent squares that share an edge is at the most 10. Prove that at least six squares must contain the same number. Generalize?

MS 89 (3-4) 2020 : Problem 4. In a meeting, n participants sit around a table and are served drinking water in the following manner. First, any one of the participants is selected and served. Then, moving clockwise, one participant is skipped and the next is served. The next two are skipped, and the next one is served; the next 3 are skipped and the next person is served and so on. After a while, everyone has been served water at least once. Prove that if $n > 1$, it cannot be odd.

MS 89 (3-4) 2020 : Problem 5. For any continuous function f on $[-1/2, 3/2]$, prove that

$$\int_{-1/2}^{3/2} f(3x^2 - 2x^3)dx = 2\int_0^1 f(3x^2 - 2x^3)dx;$$

$$\int_{-1/2}^{3/2} xf(3x^2 - 2x^3)dx = 2\int_0^1 xf(3x^2 - 2x^3)dx;$$

$$\int_{-1/2}^{3/2} x^2 f(3x^2 - 2x^3)dx = 2\int_0^1 (3x/2 - x^2/2)f(3x^2 - 2x^3)dx.$$

The following problem has been proposed by Dr. Siddhi Pathak, Department of Mathematics, Pennsylvania State University, State College, PA 16802, USA.

MS 89 (3-4) 2020 : Problem 6. Let $\mathbb{Q}^+$ denote the set of positive rational numbers, and $P : \mathbb{Q}^+ \to \mathbb{N}$ be defined as $P(m/n) = mn$ for $\gcd(m, n) = 1$. Show that

$$\sum_{q \in \mathbb{Q}^+} \frac{1}{P(q)^2} = \frac{5}{2}.$$

MEMBER'S COPY

The following two problems are proposed by Dr. Anup Dixit, Department of Mathematics, Chennai Mathematical Institute, Chennai. Earlier he was associated with Queen's University, Ontario, Canada.

MS 89 (3-4) 2020: Problem 7. For a real number x, let $\lfloor x \rfloor$ denotes the largest integer $\leq x$. Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{4n+1} \rfloor}}{n(n+1)}.$$

MS 89 (3-4) 2020: Problem 8. Let $a_1, a_2, \cdots, a_n$ be positive real numbers all ≥ 1. Let G denote its geometric mean given by $G = (a_1 a_2 \cdots a_n)^{1/n}$. Show that

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \geq \frac{n}{1+G}.$$

Prof. Ram Murty, Queen's University, Canada proposed the following problem.

MS 89 (3-4) 2020: Problem 9. Prove that

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \alpha^n$$

is transcendental for every non-zero algebraic α satisfying $|\alpha| < 1/e$.

Solutions to the Old Problems

MS 88 (3-4) 2019: Problem 1. (Proposed by Prof. B. Sury) Let the sequence $\{E_n\}$ be defined by $\tan(t) + \sec(t) = \sum_{n \geq 0} E_n t^n / n!$. Show that E_n's are positive integers. Find an interpretation of E_n as counting permutations of $1, \cdots, n$ of a particular type.

MEMBER'S COPY

Solution (by Prof. B. Sury).
Call a sequence of distinct positive integers $\{a_1, a_2, \cdots, a_n\}$ alternating if

$$a_1 > a_2 < a_3 > a_4 < \cdots$$

Call a sequence reverse alternating, if

$$a_1 < a_2 > a_3 < a_4 > \cdots$$

One may talk of alternating and reverse alternating permutations in S_n. For instance, the 5 reverse alternating permutations of $1, 2, 3, 4$ are

$$2143, 3142, 3241, 4132, 4231.$$

Clearly, the number of alternating permutations of $1, 2, \cdots, n$ equals the number of reverse alternating permutations via the bijection given by

$$a_i \mapsto n + 1 - a_i.$$

Call this common number e_n; we will show that $e_n = E_n$ in the problem. For this, we will obtain a recursion satisfied by the e_n's as follows. For any k-element subset S of $\{1, 2, \cdots, n\}$, choose a reverse alternating permutation

$$a_1 < a_2 > a_3 < \cdots a_k$$

of S and a reverse alternating permutation

$$b_1 < b_2 > b_3 < \cdots b_{n-k}$$

of the complement of S. Clearly the permutation

$$a_k, a_{k-1}, \cdots, a_2, a_1, n + 1, b_1, b_2, \cdots, b_{n-k}$$

of $1, 2, \cdots, n + 1$ is alternating or reverse alternating according as k is even or odd. Moreover, for each of the $\binom{n}{k}$ choices for the SET $S_k = \{a_1, \cdots, a_k\}$, e_k choices of a reverse alternating permutation $a_1, a_2, \cdots, a_k$ of S_k and e_{n-k} choices of a reverse alternating permutation of the complement $\{1, 2, \cdots, n, n + 1\} \setminus S_k$, we get each alternating permutation and each reverse alternating permutation of $\{1, 2, \cdots, n+1\}$ once each. In other words, we have the recursion

$$2e_{n+1} = \sum_{k=0}^{n} \binom{n}{k} e_k e - n - k$$

MEMBER'S COPY

where we have put $e_0 = 1$ for convenience. We form a generating function by multiplying the recursion by $\frac{t^{n+1}}{(n+1)!}$ and summing for $n \geq 0$, we have

$$g(t) = \sum_{n \geq 0} e_n \frac{t^n}{n!}$$

satisfying $2g'(t) = 1 + g(t)^2$ and $g(0) = 1$.
Solving this initial value problem, we obtain

$$g(t) = tan(t) + sec(t)$$

which proves that $e_n = E_n$, and hence, our assertion.

MS 88 (3-4) 2019 : Problem 2 (Proposed by B. Sury).
Prove that a cubic integer polynomial $ax^3 + bx^2 + cx + d$ where ad is odd and bc is even must have an irrational root.

Mr. Shivam Jadhav, IIT, Delhi provided the solution to this problem on June 2, 2020. The solution is given below.

Solution.

$$f(x) = ax^3 + bx^2 + cx + d \quad a, b, c, d \in \mathbf{Z} \tag{1}$$

We will prove the statement by contradiction. Assume that $f(x)$ has no irrational roots. Also we know that ad is odd and bc is even . Then we can write

$$f(x) = (\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)(\alpha_3 x + \beta_3) \quad \alpha_i, \beta_i \in \mathbf{Z} \ \ \forall \ \ i = 1, 2, 3. \tag{2}$$

$\implies$

$$a = \alpha_1 \alpha_2 \alpha_3 \tag{3}$$

$$b = \alpha_1 \alpha_2 \beta_3 + \alpha_1 \beta_2 \alpha_3 + \beta_1 \alpha_2 \alpha_3 \tag{4}$$

$$c = \alpha_1 \beta_2 \beta_3 + \beta_1 \beta_2 \alpha_3 + \beta_1 \alpha_2 \beta_3 \tag{5}$$

$$d = \beta_1 \beta_2 \beta_3 \tag{6}$$

MEMBER'S COPY

Case 1 :
If ad is odd $\implies$ both a, d must be odd , which further $\implies$ all α_i, β_i must be odd . Which means that b, c are odd and bc is odd since product of three odd integers is odd and sum of three odd integers is odd.But it's not true .
Case 2 :
If bc is even either one of them is even or both of them . In either scenario for that to happen at least one α_i and one β_i must be even . Which implies that ad is even. But it's not true .

Hence the assumption that $f(x)$ has no irrational roots is false. So $f(x)$ must have an irrational root.

bf Mr. Prajnanaswaroopa, Amity University, Coimbtore and Mr. Prithwijit De, HBCSE, Mumbai have also provided correct solutions to this problem.

MS 88 (3-4) 2019 : Problem 4 (Proposed by Prof. B. Sury).
Find the number a_n of permutations σ of $1, \cdots, n$ such that there is NO triple $i < j < k$ with $\sigma(j) < \sigma(i) < \sigma(k)$. For instance, $a_2 = 2$ while $a_3 = 5$. Further, find all n for which a_n is odd.

Solution 4 (by Prof. B. Sury).
The study of the number a_n of "321-avoiding" permutations is due to R. Stanley. It is convenient to define for any permutation σ, a new function $\overline{\sigma}$ by

$$\overline{\sigma}(i) := |\{1 \leq j \leq i : \sigma(j) \geq \sigma(i)\}|.$$

Now, if there is NO triple $i < j < k$ with $\sigma(j) < \sigma(i) < \sigma(k)$, then every $i < j$ satisfying $\sigma(i) > \sigma(j)$ also satisfies $\sigma(i) > \sigma(j+1)$. Hence,

$$\overline{\sigma}(j) \leq \overline{\sigma}(j+1).$$

That is, $\overline{\sigma}$ is a monotonic function.
Conversely, if $\overline{\sigma}$ is monotonic, we claim there is NO triple $i < j < k$ with $\sigma(j) < \sigma(i) < \sigma(k)$.
If there were such a triple, consider the sequence

$$\sigma(j), \sigma(j+1), \cdots, \sigma(k).$$

MEMBER'S COPY

Then, there must be two consecutive terms $\sigma(l), \sigma(l+1)$ in this sequence such that

$$\sigma(l) < \sigma(i) < \sigma(i+1).$$

In this case, any $m < l+1$ such that $\sigma(m) > \sigma(l+1)$ also satisfies $m < l$ and $\sigma(m) > \sigma(l)$. Also, $i < l$ and $\sigma(i) > \sigma(l)$ whereas $\sigma(i) < \sigma(l+1)$. Therefore, we would have $\overline{\sigma}(l)$ is strictly larger than $\overline{\sigma}(l+1)$, a contradiction.
Thus, the number a_n sought is exactly the number of monotonic functions θ from $\{1, 2 \cdots, n\}$ to itself such that $\theta(i) \leq i$ for all $i \leq n$.
Counting these is of independent interest and it is usually done by counting lattice paths (Dyck words) on the integer lattice from the lower left corner to the right upper corner of an $n \times n$ square grill where each step is either to the right ot upwards and the path never goes above the diagonal. Calling R and U as unit steps to the right and upwards respectively, we have words in R and U. Counting the number of all words, and subtracting the number of those words which correspond to going above the diagonal, we have the number

$$a_n = \binom{2n-1}{n-1} = \binom{2n-1}{n-2} = \frac{1}{n+1}\binom{2n}{n}.$$

These are called the Catalan numbers.
As the power of 2 dividing a binomial coefficient $\binom{n}{r}$ is given by the number of carry-overs when the binarye xpansion of r and $n-r$ are added (this is due to Kummar and Legendre), it is easy to see that the number of carry-overs when the binarye xpansion of n is added with itself can equal the power of 2 dividing $n+1$ (equivalent to the oddness of a_n) if, and only if, $n+1$ is a power of 2.

MS 88 (3-4) 2019 : Problem 5 (Proposed by Prof. B. Sury).
The mixed fraction $91\frac{5742}{638}$ equals $91 + 9 = 100$. It involves all the digits 1 to 9 once. Find all such mixed fraction expressions with the value 100. Further, find mixed fractions for $20, 40$ and 72 also.

Solution (by Prof. B. Sury).
This problem is due to the famous puzzlist Henry Dudeney. As he mentions, it can be "done by patient trial, and there is a singular pleasure in discovering each correct arrangement."
To understand how to rule about many possibilities, look at an expression like $100 = 91\frac{5742}{638} = 91 + 9$ as given in the problem. Similarly, consider

MEMBER'S COPY

$100 = w + f$ where w is a whole number like 91 above and the fraction f has an integer value (like 9 above). If we were to have a 5 in the whole number w, we would be forced to have 5 or 0 in the numerator or denominator of f; so, 5 is ruled out in w. If the whole number w ends in 9, then the fraction is like 21 or 31 etc. But then the last digits of numerator and denominators of f are the same which is not allowed. An expression like $100 = 98 + 2$ with 2 equal to a fraction involving the 7 digits $1, 2, 3, 4, 5, 6, 7$ is checked to be impossible, Note that the fraction involves either 7 digits or 8 digits totally in its numerator and denominator because the whole number w has 1 or 2 digits. Moreover, we can argue to show the following concerning the "digital root" (its value modulo 9 but with 0 taken as 9; that is, the number obtained by adding all the digits and repeating this process until we get a one-digit number) of the numerator and the denominator of the fraction f. As he digital root of all 9 digits is 9, for a given whole number w, the digital root of the numerator and of the denominator can have only one choice (if there is a choice at all).
For example, $100 = 96 + f$ means the fraction $f = n/d = 4$ and the digital roots of d and of $n = 4d$ add up to the digital root of $1+2+3+4+5+7+8$ (the digits other than those in 96) which is 3. Hence, the digital root of $n + d = 5d$ which is 3 gives that $5d$ modulo 9 is 3 and hence d mod 9 is 6. In this manner, we may restrict possibilities. The final answer is that there are exactly 11 mixed fractions of the form for 100. These are

$$96\frac{2148}{537}, 96\frac{1752}{438}, 96\frac{1428}{357}, 94\frac{1578}{263},$$

$$91\frac{7524}{836}, 91\frac{5823}{647}, 91\frac{5742}{638}, 82\frac{3546}{197},$$

$$81\frac{7524}{396}, 81\frac{5643}{297}, 3\frac{69258}{714}.$$

For $20, 40, 72$ we have mixed fractions

$$20 = 6\frac{13258}{947}, 40 = 27\frac{5148}{396}, 72 = 59\frac{3614}{278}.$$

MS 88 (3-4) 2019 : Problem 7 (Proposed by Prof. J. R. Patadia).
If a set $E = \{n_k : k = 1, 2, \cdots\}$ of natural numbers satisfy the condition $\frac{n_{k+1}}{n_k} \geq q > 1$ for all k, then show that $Sup_{n\in\mathbb{N}}\ R_2(E, n) < \infty$, where $R_2(E, n)$ = number of different representations of n as $n = \pm n_r \pm$

MEMBER'S COPY

$n_s,\ n_r, n_s \in E$

Solution (by Prof. J. R. Patadia).
Observe that $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots < n_{k+s} < \cdots$ for all $k, s \in \mathbb{N}$. Since $\frac{n_{k+1}}{n_k} \geq q > 1$ for all k, we have, for any $k, s \in \mathbb{N}$,

$$\frac{n_{k+s}}{n_k} = \frac{n_{k+s}}{n_{k+s-1}} \cdot \frac{n_{k+s-1}}{n_{k+s-2}} \cdots \frac{n_{k+1}}{n_k} \geq q \cdot q \cdot q \cdots q = q^s.$$

Thus,

$$n_s \text{ increases at least as rapidly as } q^s \text{ for every } s \in \mathbb{N}. \qquad (*)$$

Now consider any $n \in \mathbb{N}$.
Suppose $n = n_s + n_r$ or $n = n_s - n_r,\ \ 1 \leq r < s,\ r, s \in \mathbb{N}$, and $n_r, n_s \in E$
In case $n = n_s + n_r$,

$$\frac{n}{2} = \frac{n_s + n_r}{2} < \frac{2n_s}{2} = n_s < n_s + n_r = n, \text{ that is, } \frac{n}{2} < n_s < n. \qquad (7)$$

In case $n = n_s - n_r$, obviously $n < n_s$. On the other hand, $n > n_s - \frac{n_s}{q} = n_s \frac{q-1}{q}$, as $\frac{n_s}{n_r} \geq q$. It follows that

$$n < n_s < \frac{nq}{q-1}. \qquad (8)$$

Now, n is arbitrary, and in view of (*), number of n_s satisfying (1) or (2) is bounded, the bound depending only on q, and hence the conclusion.

MS 89 (1-2) 2020 : Problem 1 (Proposed by Prof. Ram Murty, Queen's University, Canada).
Let p be a prime number and suppose that $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}$ is a non-zero function. We may view f as a function on the natural numbers by setting $f(n)$ to be $f(a)$ when $n \equiv a \bmod p$. Suppose that $\sum_{a=1}^{p} f(a) = 0$. Show that

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!}$$

is transcendental.

Dr. Siddhi Pathak, Department of Mathematics, Pennsylvania State University, State College, PA 16802, USA provided the solution to this problem. In fact, she proves a more general statement. Her solution is presented below.

MEMBER'S COPY

Solution. We will prove the following more general statement: Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be a non-zero periodic function, with period $q \geq 2$. Then the series

$$S_f(z) := \sum_{n=0}^{\infty} \frac{f(n)}{n!} z^n$$

is transcendental for every non-zero algebraic value of z.

The convergence of the above series is immediate from the Stirling's formula, namely,

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \text{ as } n \to \infty,$$

and the fact that $|f(n)|$ is bounded.

One can view f as a function on $\mathbb{Z}/q\mathbb{Z}$. The Fourier transform of f can then be defined as

$$\widehat{f}(a) := \frac{1}{q} \sum_{b=1}^{q} f(b) e^{-2\pi i ab/q},$$

and the Fourier inversion formula gives

$$f(b) = \sum_{a=1}^{q} \widehat{f}(a) e^{2\pi i ab/q}.$$

Replacing the above expression for $f(n)$ in the given series leads to

$$S_f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{a=1}^{q} \widehat{f}(a) e^{2\pi i an/q} = \sum_{a=1}^{q} \widehat{f}(a) \sum_{n=0}^{\infty} \frac{(z\,\zeta_q^a)^n}{n!},$$

where $\zeta_q = e^{2\pi i/q}$. Using the series expansion of the exponential function, we obtain that

$$S_f(z) = \sum_{a=1}^{q} \widehat{f}(a) \exp(z\,\zeta_q^a). \tag{9}$$

Let $\{\alpha_1, \cdots, \alpha_r\}$ be a maximal $\mathbb{Q}$-linearly independent subset of the set $\{\zeta_q^a : 1 \leq a \leq q\}$. Thus, for each $1 \leq a \leq q$, we can write

$$\zeta_q^a = \sum_{j=1}^{r} c_j(a)\, \alpha_j, \quad c_j(a) \in \mathbb{Q}.$$

On expressing $c_j(a)$ in their reduced form with positive denominators, let M be the maximum of their denominators. Thus, $M\, c_j(a) \in \mathbb{Z}$. Inserting

MEMBER'S COPY

this back in (9), we get

$$S_f(z) = \sum_{a=1}^{q} \widehat{f}(a) \prod_{j=1}^{m} \exp\left(z\,\alpha_j\right)^{c_j(a)} = \sum_{a=1}^{q} \widehat{f}(a) \prod_{j=1}^{m} \exp\left(\frac{z\,\alpha_j}{M}\right)^{M\,c_j(a)}. \quad (10)$$

Suppose that $c_j(a) \geq 0$ for all $1 \leq a \leq q$ and $1 \leq j \leq m$. Then the above equation expresses $S_f(z)$ as a polynomial in $\exp\left(z\,\alpha_j/M\right)$ over $\overline{\mathbb{Q}}$. Moreover, if $\{\alpha_j : 1 \leq j \leq m\}$ are $\mathbb{Q}$-linearly independent, then so are the numbers

$$\left\{\frac{z\,\alpha_j}{M} : 1 \leq j \leq m\right\},$$

for a non-zero algebraic number z. Hence, by the Lindemann-Weierstrass theorem, the numbers $\exp\left(z\,\alpha_j/M\right)$, for $1 \leq j \leq m$ are algebraically independent. Therefore, if $\widehat{f}(a) \not\equiv 0$ (that is $f \not\equiv 0$), then $S_f(z)$ is transcendental.

Now assume that $c_j(a) < 0$ for some $1 \leq a \leq q$ and $1 \leq j \leq m$. Further suppose that $S_f(z) = \beta \in \overline{\mathbb{Q}}$. Define

$$d_j := \begin{cases} 0 & \text{if } c_j(a) \geq 0 \text{ for } 1 \leq a \leq q, \\ \max_{1\leq a\leq q} \; -c_j(a) & \text{otherwise.} \end{cases}$$

Multiplying both sides of (10) with

$$\prod_{j=1}^{m} \exp(z\,\alpha_j/M)^{M\,d_j},$$

gives

$$\beta \prod_{j=1}^{m} \exp\left(\frac{z\,\alpha_j}{M}\right)^{M\,d_j} - \sum_{a=1}^{q} \widehat{f}(a) \prod_{j=1}^{m} \exp\left(\frac{z\,\alpha_j}{M}\right)^{M\,c_j(a)+M\,d_j} = 0.$$

Thus, the numbers $\exp(z\,\alpha_j/M)$ satisfy a polynomial over $\overline{\mathbb{Q}}$. By the Lindemann-Weierstrass theorem, this polynomial should be identically zero. This is possible only if all monomials in the above polynomial are of the same form, that is, $c_j(a) = 0$ for all $1 \leq a \leq q$ and $1 \leq j \leq m$. This is clearly a contradiction. Hence, $S_f(z)$ is transcendental.

We also received a solution to this problem by **Prof. B. Sury**.

MEMBER'S COPY

MS 89 (1-2) 2020 : Problem 2 (Proposed by Prof. Ram Murty, Queen's University, Canada).
Prove that for $n \geq 1$,

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{1}{k^2} = \sum_{k=1}^{n}\frac{H_k}{k}$$

where $H_n = 1 + 1/2 + \cdots + 1/n$.

Prof. B. Sury provided the solution to this problem. The solution by Prof. Sury is given below.

Solution. It is well known and easy to check (the binomial transform) that if

$$a_n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}b_r,$$

then

$$b_n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}a_r.$$

In view of this, proving the asserted equality

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{1}{k^2} = \sum_{k=1}^{n}\frac{H_k}{k}$$

where $H_k = \sum_{r=1}^{k}\frac{1}{r}$ becomes equivalent to showing that

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\sum_{r=1}^{k}\frac{H_r}{r} = \frac{1}{n^2}.$$

Note that $\sum_{r=1}^{k}\frac{H_r}{r} = \sum_{1\leq i\leq j\leq k}\frac{1}{ij}$.
So, we need to prove

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\sum_{1\leq i\leq j\leq k}\frac{1}{ij} = \frac{1}{n^2}.$$

In this form, this was posed as problem 11164 in the American Mathematical Monthly in the year 2005 by Dias-Barrero. Instead of recalling the solution, we describe below a generalization due to W. Chu and Q. L. Yan which proves much more:

MEMBER'S COPY

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\sum_{1\le i_1\le\cdots\le i_r\le k}\frac{1}{i_1 i_2\cdots i_r}=\frac{1}{n^r}.$$

In fact, this is itself a consequence of the formal identity of rational functions:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1}\sum_{0\le i_1\le\cdots\le i_r\le k}\frac{1}{(x+i_1)\cdots(x+i_r)}=\frac{x}{(x+n)^{r+1}}\cdots(\spadesuit)$$

evaluated at $x=1$ (and taking $n-1$ in place of n). We give Chu's proof of this now.

Use the relation

$$\binom{n}{k}\binom{x+k}{k}^{-1}=\binom{x+n}{n-k}\binom{x+n}{n}^{-1}.$$

Denoting by L, the left hand side of $(\spadesuit)$, we have

$$\binom{x+n}{n}L=\sum_{0\le i_1\le\cdots\le i_r\le n}\frac{1}{(x+i_1)\cdots(x+i_r)}\sum_{k=i_r}^{n}(-1)^k\binom{x+n}{n-k}.$$

Let us now use the well known binomial identity

$$\sum_{k=i_r}^{n}(-1)^k\binom{x+n}{n-k}=(-1)^{i_r}\binom{x+n}{n-i_r}\frac{x+i_r}{x+n}\cdots(A)$$

We have then

$$\binom{x+n}{n}L=\frac{1}{x+n}\sum_{0\le i_1\le\cdots\le i_{r-1}\le n}\frac{1}{(x+i_1)\cdots(x+i_{r-1})}\sum_{i_r=i_{r-1}}^{n}(-1)^{i_r}\binom{x+n}{n-i_r}.$$

We use (A) once again to the rightmost sum; that is,

$$\sum_{i_r=i_{r-1}}^{n}(-1)^{i_r}\binom{x+n}{n-i_r}=(-1)^{i_{r-1}}\binom{x+n}{n-i_{r-1}}\frac{x+i_{r-1}}{x+n}.$$

We obtain

$$\binom{x+n}{n}L=$$

$$\frac{1}{(x+n)^2}\sum_{0\le i_1\le\cdots\le i_{r-1}\le n}\frac{1}{(x+i_1)\cdots(x+i_{r-2})}\sum_{i_{r-1}=i_{r-2}}^{n}(-1)^{i_{r-1}}\binom{x+n}{n-i_{r-1}}.$$

MEMBER'S COPY

Repeating this a total of r times, we finally obtain

$$\binom{x+n}{n}L = \frac{1}{(x+n)^r}\binom{x+n}{n}\frac{x}{x+n}$$

which immediately gives the asserted identity

$$L = \frac{x}{(x+n)^{r+1}}.$$

MS 89 (1-2) 2020 : Problem 3 (Proposed by Dr. Siddhi Pathak).
Let $f : \mathbb{R}\setminus\{1\} \to \mathbb{R}$ be an integrable function satisfying the condition that

$$f(x)f\left(1-\frac{1}{x}\right) = -x$$

for all $x \in \mathbb{R}\setminus\{0,1\}$. Suppose that $f(x) > 0$ for $x > 1$. Then show that

$$\int_2^\infty (f(x)-1)^2 dx = 1.$$

Mr. Prithwijit De, HBCSE, Mumbai gave the correct solution to the problem. The solution by him is presented below.
Solution. Observe that each of the transformations

(1) $x \to 1/x$,
(2) $x \to 1-x$,
(3) $x \to \dfrac{1}{1-x}$

is a bijection on $\mathbb{R}\setminus\{0,1\}$. Making the first and the second transformations in succession leads to

$$f\left(\frac{1}{1-x}\right)f(x) = -\frac{1}{1-x} \tag{11}$$

for all $x \in \mathbb{R}\setminus\{0,1\}$. The third one transforms the given functional equation to

$$f\left(1-\frac{1}{x}\right)f\left(\frac{1}{1-x}\right) = \frac{1-x}{x} \tag{12}$$

for all $x \in \mathbb{R}\setminus\{0,1\}$.

$$\begin{aligned}\left\{f(x)f\left(1-\frac{1}{x}\right)\right\}.\left\{f\left(\frac{1}{1-x}\right)f(x)\right\} &= (f(x))^2\left\{f\left(1-\frac{1}{x}\right)f\left(\frac{1}{1-x}\right)\right\}\\ &= \frac{x}{1-x} \qquad (13)\end{aligned}$$

MEMBER'S COPY

whence by virtue of (2) we obtain

$$(f(x))^2 = \left(\frac{x}{1-x}\right)^2 \tag{14}$$

for all $x \in \mathbb{R} \setminus \{0,1\}$. As $f(x) > 0$ for $x > 1$, $f(x) = \dfrac{x}{x-1}$ on $(2,\infty)$ and

$$\int_2^\infty (f(x)-1)^2 dx = \int_2^\infty \frac{dx}{(x-1)^2} = 1. \tag{15}$$

It is easy to see that $f(x) = \dfrac{x}{x-1}$ satisfies the given functional equation for all $x \in \mathbb{R} \setminus \{0,1\}$.

MS 89 (1-2) 2020 : Problem 4 (Proposed by Dr. Anup Dixit).
Let a, b, c be positive real numbers. Show that

$$\frac{3a}{4b+5c} + \frac{3b}{4c+5a} + \frac{3c}{4a+5b} \geq 1. \tag{1}$$

Dr. Yagub N. Aliyev, ADA University, School of IT and Engineering, Baku, Azerbaijan provided the solution to the problem on June 24, 2020. The solution by him is presented below. **Mr. Desari Naga Vijay Krishna**, Machilipatnam, A. P. also submitted the correct solution to the problem.

Solution. The following inequality is well known and follows directly from Cauchy-Schwarz's inequality:

$$\frac{\alpha^2}{x} + \frac{\beta^2}{y} + \frac{\gamma^2}{z} \geq \frac{(\alpha+\beta+\gamma)^2}{x+y+z}. \tag{2}$$

Let us write our inequality as

$$\frac{3a^2}{4ab+5ac} + \frac{3b^2}{4bc+5ab} + \frac{3c^2}{4ac+5bc} \geq 1. \tag{3}$$

By (2),

$$\frac{3a^2}{4ab+5ac} + \frac{3b^2}{4bc+5ab} + \frac{3c^2}{4ac+5bc} \geq \frac{3(a+b+c)^2}{9ab+9bc+9ac}. \tag{4}$$

It remains to prove that

$$\frac{(a+b+c)^2}{3ab+3bc+3ac} \geq 1, \tag{5}$$

MEMBER'S COPY

which is obvious.

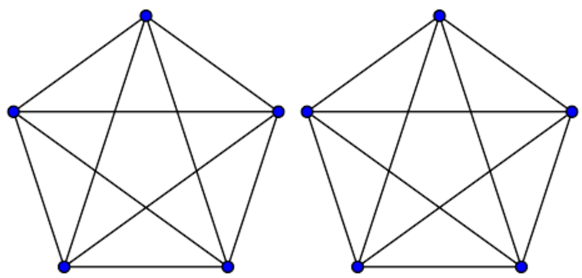
MS 89 (1-2) 2020 : Problem 6 (Proposed by Prof. M. M. Shikare).
Let G be a graph with 10 vertices. Among any three vertices of G, at least two are adjacent. Find the least number of edges that G can have. Find a graph with this property.

Mr. Shivam Jadhav, IIT, Delhi provided solution to this problem on June 4, 2020. Solution provided by him is produced below.

Solution. The answer is 20 edges. The graph satisfying the property asked in the question is a pair of K_5. The details are given below.

Consider a graph G satisfying the properties mentioned in the question. Let the v be one of the vertex of G. Let S be the set of vertices not adjacent to v and S does not contain v.
Case 1 : $deg(v) \leq 2$. We have $|S| \geq 7$.
Then in order to satisfy the property the sub-graph formed by S must be complete. So the number of edges is more than 20.
Case 2 : $deg(v) = 3$. Then $|S| = 6$.
In order to satisfy the property the sub-graph formed by S must be complete. There will be 3 edges more so that the property is not violated . Let $p,\ q,\ r$ be adjacent to v. There is an edge in each of $(p,\ q,\ x)\,(p,\ r,\ y)$ and $(q,\ r,\ z)$, where $x,\ y,\ z$ are three distinct vertices in S. So the tota l number of edges is $15 + 3 + 3 = 21$ which is greater than 20.

Thus every vertex of G is of degree at least 4 and hence G has at least 20 edges (by Handshaking Lemma).



The graph G

The graphs G consists of a pair of two copies of K_5.

MEMBER'S COPY

MEMBER'S COPY

FORM IV
(See Rule 8)

1.	Place of Publication:	PUNE
2.	Periodicity of publication:	QUARTERLY
3.	Printer's Name: Nationality: Address:	DINESH BARVE INDIAN PARASURAM PROCESS 38/8, ERANDWANE PUNE-411 004, INDIA
4.	Publisher's Name: Nationality: Address:	SATYA DEO INDIAN GENERAL SECRETARY THE INDIAN MATHEMATICAL SOCIETY HARISH CHANDRA RESEARCH INSTITUTE CHHATNAG ROAD, JHUNSI ALLAHABAD-211 019, UP, INDIA
5.	Editor's Name: Nationality: Address:	M. M. SHIKARE INDIAN CENTER FOR ADVANCED STUDY IN MATHEMATICS, S. P. PUNE UNIVERSITY PUNE-411 007, MAHARASHTRA, INDIA
6.	Names and addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital:	THE INDIAN MATHEMATICAL SOCIETY

I, Satya Deo, the General Secretary of the IMS, hereby declare that the particulars given above are true to the best of my knowledge and belief.

SATYA DEO

Dated: October 6, 2020 Signature of the Publisher

Published by Prof. Satya Deo for the Indian Mathematical Society, type set by M. M. Shikare, "Krushnakali", Survey No. 73/6/1, Gulmohar Colony, Jagtap Patil Estate, Pimple Gurav, Pune 411061 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune 411 041 (India). Printed in India

MEMBER'S COPY

The Mathematics Student ISSN: 0025-5742
Vol. 89, Nos. 1-2, January-June (2020)

EDITORIAL BOARD

M. M. Shikare (Editor-in-Chief)
Center for Advanced Study in Mathematics
S. P. Pune University, Pune-411 007, Maharashtra, India
E-mail : msindianmathsociety@gmail.com

Bruce C. Berndt
Dept. of Mathematics, University of Illinois 1409 West Green St. Urbana, IL 61801, USA
E-mail : berndt@illinois.edu

George E. Andrews
Dept. of Mathematics, The Pennsylvania State University, University Park PA 16802, USA
E-mail : gea1@psu.edu

M. Ram Murty
Queens Research Chair and Head
Dept. of Mathematics and Statistics
Jeffery Hall, Queens University
Kingston, Ontario, K7L3N6, Canada
E-mail : murty@mast.queensu.ca

N. K. Thakare
C/o :
Center for Advanced Study in Mathematics, Savitribai Phule Pune University, Pune-411007, India
E-mail : nkthakare@gmail.com

Satya Deo
Harish-Chandra Research Institute
Chhatnag Road, Jhusi
Allahabad-211019, India
E-mail : sdeo94@gmail.com

Gadadhar Misra
Dept. of Mathematics
Indian Institute of Science
Bangalore-560012, India
E-mail : gm@iisc.ac.in

B. Sury
Theoretical Stat. and Math. Unit
Indian Statistical Institute
Bangalore-560059, India
E-mail : surybang@gmail.com

A. S. Vasudeva Murthy
TIFR Centre for Applicable Mathematics
P. B. No. 6503, GKVK Post Sharadanagara
Chikkabommasandra, Bangalore-560065, India
E-mail : vasu@math.tifrbng.res.in

S. K. Tomar
Dept. of Mathematics, Panjab University
Sector-4, Chandigarh-160014, India
E-mail : sktomar@pu.ac.in

Krishnaswami Alladi
Dept. of Mathematics, University of Florida, Gainesville, FL32611, USA
E-mail : alladik@ufl.edu

Clare D'Cruz
Dept. of Mathematics, CMI
IT Park Padur P.O., Siruseri
Kelambakkam-603103, T.N., India
E-mail : clare@cmi.ac.in

L. Sunil Chandran
Dept. of Computer Science&Automation
Indian Institute of Science
Bangalore-560012, India
E-mail : sunil.cl@gmail.com

J. R. Patadia
5, Arjun Park, Near Patel Colony, Behind Dinesh Mill, Shivanand Marg, Vadodara-390007, Gujarat, India
E-mail : jamanadaspat@gmail.com

T. S. S. R. K. Rao
Theoretical Stat. and Math. Unit
Indian Statistical Institute
Bangalore-560059, India
E-mail : tss@isibang.ac.in

Kaushal Verma
Dept. of Mathematics
Indian Institute of Science
Bangalore-560012, India
E-mail : kverma@iisc.ac.in

C. S. Aravinda
TIFR Centre for Applicable Mathematics
P. B. No. 6503, GKVK Post Sharadanagara
Chikkabommasandra, Bangalore-560065, India
E-mail : aravinda@math.tifrbng.res.in

Indranil Biswas
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Rd., Mumbai-400005, India
E-mail : indranil29@gmail.com

Timothy Huber
School of Mathematics and statistical Sciences
University of Texas Rio Grande Valley, 1201 West Univ. Avenue, Edinburg, TX78539 USA
E-mail : timothy.huber@utrgv.edu

Atul Dixit
AB 5/340, Dept. of Mathematics
IIT Gandhinagar, Palaj, Gandhinagar-382355, Gujarat, India
E-mail : adixit@iitg.ac.in

MEMBER'S COPY

THE INDIAN MATHEMATICAL SOCIETY
Founded in 1907
Registered Office: Center for Advanced Study in Mathematics
Savitribai Phule Pune University, Pune - 411 007

COUNCIL FOR THE SESSION 2020-2021

PRESIDENT: B. Sury, Theoretical Stat. and Math. Unit, Indian Statistical Institute, Bangalore-560 059 , Karnataka, India

IMMEDIATE PAST PRESIDENT: S. Arumugam, Kalaslingam University, Anand Nagar, Krishnankoil-626190, T. N., India

GENERAL SECRETARY: Satya Deo, Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad-211 019, UP, India

ACADEMIC SECRETARY: Peeyush Chandra, Professor (Retired), Department of Mathematics & Statistics, I. I. T. Kanpur-208 016, Kanpur (UP), India

ADMINISTRATIVE SECRETARY: B. N. Waphare, Center for Advanced Study in Mathematics, S. P. Pune University, Pune-411 007, Maharashtra, India

TREASURER: S. K. Nimbhorkar, (Formerly of Dr. B. A. M. University, Aurangabad), C/O Dr. Mrs. Prachi Kulkarni, Ankur Hospital, Tilaknagar, Aurangabad-431 001, Maharashtra, India

EDITOR: **J. Indian Math. Society**: Sudhir Ghorpade, Department of Mathematics, I. I. T. Bombay-400 056, Powai, Mumbai (MS), India

EDITOR: **The Math. Student**: M. M. Shikare, Center for Advanced Study in Mathematics, S. P. Pune University, Pune-411 007, Maharashtra, India

LIBRARIAN: Sushma Agrawal, Director, Ramanujan Inst. for Advanced Study in Mathematics, University of Madras, Chennai-600 005, Tamil Nadu, India

OTHER MEMBERS OF THE COUNCIL:

Asma Ali: Dept. of Mathematics, Aligarh Muslim University, Aligarh-200 202, UP, India
G. P. Rajasekhar: Dept. of Maths., IIT Kharagpur. Kharagpur-700 019, WB, India
Sanjib Kumar Datta: Dept. of Mathematics, Univ. of Kalyani, Kalyani-741 235, WB, India
S. Sreenadh: Dept. of Mathematics, Sri Venkateswara Univ., Tirupati-517 502, A.P., India
Nita Shah: Dept. of Mathematics, Gujarat University, Ahmedabad-380 009, Gujarat, India
S. Ahmad Ali: 292, Chandralok, Aliganj, Lucknow-226 024 (UP), India
A. K. Das: Dept. of Mathematics, SMVDU, Katra, Jammu and Kashmir, India
G. P. Youvaraj: Ramanujan Institute, Uni. of Madras, Chennai-600 005, T. N., India
N. D. Baruah: Dept. of Mathematical Sciences, Tezpur University, Assam, India

Back volumes of our periodicals, except for a few numbers out of stock, are available.

Edited by M. M. Shikare and published by Satya Deo, the General Secretary of the IMS, for the Indian Mathematical Society.

Typeset by M. M. Shikare, “Krushnakali”, Survey No. 73/6/1, Gulmohar Colony, Jagtap Patil Estate, Pimple Gurav, Pune 411061 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No.129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune - 411 041, Maharashtra, India. Printed in India

Copyright ©The Indian Mathematical Society, 2020.

MEMBER'S COPY