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Edited by M. M. SHIKARE

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3. general survey articles, popular articles, expository papers, Book-Reviews.
4. problems and solutions of the problems,
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PROF. J. R. PATADIA: A CONSCIENTIOUS HARD WORKER

N. K. THAKARE

The period from 2004 to 2008 was full of calamities for the Indian Mathematical Society. We lost Prof. H. C. Khare, Prof. R. P. Agarwal and Prof. M. K. Singal during this period. They were the senior most office bearers of IMS who were involved in reviving and reinvigorating the IMS in eighties and nineties of the 20th century. The posts of Editors of Mathematics Student, General Secretary, Editor of Journal of IMS became vacant. The mantle of handling this grave situation fell singularly on Late Professor V. M. Shah. And Prof. Shah took the decision with the help of other colleagues to request Prof. J. R. Patadia (JRP) of the M. S. University of Baroda, Vadodara to shoulder the onerous responsibility of the editor of the Mathematics Student. I was told he was reluctant to accept the said responsibility. But being a direct student of Prof. Shah, Prof. Patadia ultimately gave in and took over the editorship of the Mathematics Student (MS) around 2004. The publication status of The Mathematics Student (MS) during those days was alarming. This is revealed by the fact that 2000 volume of MS was published in 2019 so as to bridge the gap in the process of publication of all the volumes of MS from 2000 to 2019. That was also possible solely because of the hard work of Prof. Patadia. It involved some copyright issues as the papers accepted in the said volume were not at all published before 2019 and a few authors opted for publishing their work in other periodicals. During those days, the complete procedure of publication was offline. It used to take lot of energy, much finances and inordinate delay. Everything was more or less handled through postal service which is significantly nicknamed as snail's mail. There was long backlog almost of several years in acceptance for publication of an article and in its actual publication.

In such a scenario JRP took over the editorship of MS. He worked very hard then so as to reduce the backlog of accepted articles. He almost succeeded in doing so. In the meantime I was asked by the IMS council to work as the editor of the Journal of IMS from April 1, 2007. With moral support from Prof. V. M. Shah, the then General Secretary, I decided to opt for online handling of the process of publication. After a year or two Prof. Patadia also opted for online handling of the publication of MS. This approach helped us to do away with any backlog of accepted papers. Today the situation is such that the hard copies of MS and JIMS are available much in advance; the online content of both the periodicals is certainly available much before the publication of hard copies meant for subscribers. With the insistence of Prof. Patadia many changes in the format of MS were effected. The first major change was doing away with publishing in MS of the abstracts of papers accepted for presentation in the annual conference of IMS every year. It resulted in qualitative upgradation of MS as a research journal targetted to students, teachers and upcomming researchers. In fact, I am tempted to quote what Professor Bruce Bernt wrote in the year 2013 in one of his communications to the MS:

"Quote:

Dear Professor Patadia, I have a question that I have been meaning to ask for at least 5 years. The name of the journal, Mathematics Student, would seemingly indicate that the journal is aimed at students. However, the content of the issues I have seen in the past several years demonstrate that the journal is a research level journal with no appeal to students. Since, to the best of my knowledge, the Indian Mathematical Society has no such journal for students nor is there any such journal published in India; I suggest that the Indian Mathematical Society seriously address this deficiency, if they have (not) already done so in the Year of Mathematics in India just completed."

Subsequently with the strong support of IMS council and Prof. Bruce C. Berndt he effected long lasting and fruitful changes in the format of the MS. Now MS is more or less divided into neat four to five sections. Important among them are

(i) Section of IMS Presidential Addresses, Texts of Plenary talks, Texts of Various Memorial Award Lectures named after S. Ramanujan, P. L. Bhatnagar, Hansraj Gupta, Ramaswamy Aiyer and Ganesh Prasad.

(ii) Expository Articles, Classroom Notes etc.

(iii) Research papers

(iv) Problems and Solutions (Added during the tenure of JRP, with vital support from Prof. B. Sury and the Editorial Board of the MS).

The format now looks more like an international standard. In particular, the Problems and Solutions section involves lot of hard work and involving many talented mathematicians from all over the world. But Professor Patadia left no stone unturned so as to make MS a periodical of international standard. And in this context let me recall the following development in seeking approval of UGC for inclusion of MS in their list of research periodicals. For last few years UGC prepares a list of research periodicals with a view to keep fake journals at bay. Because of some conditions inherent in the promotion of teachers from colleges and universities for higher position and grades a bountiful number of journals have come up with a view to mint money. Such journals are generally termed as paid journals. In 2016-17 unfortunately for the IMS the CARE (Consortium of Academic and Research Ethics for the creation and maintenance of Reference list of quality journals) list did not include MS. It was a great moral setback to the IMS as we do not charge page charges for publication of articles in both the periodicals. Moreover, the MS is being published online and is freely available on the website of the IMS. Online copies are also sent to life members whose e-mail addresses are available with IMS. The onerous assignment of getting MS included in the CARE list of UGC was entrusted to Professor Patadia. And he worked so relentlessly and arduously for several days and months that we are now in a happy situation that MS finds its place of honour in the CARE list of the UGC that is published on June 14, 2019. The credit for this, singularly, belongs to him.

In addition to the duties of editing of MS, Professor Patadia was instrumental in creating and maintaining IMS website at Vadodara. It involves lots of relevant work such as (i) maintaining uptodate list of life members of IMS, (ii) assigning unique number to each life member, (iii) sending online text of MS to all the life members of MS, (iv) uploading and updating IMS Library catalogue on our website, and such things. He worked very hard to get the photographs of the Founder Members and photographs of all Presidents of IMS save for one exception. He has put all this on the IMS website for use of future generations. Even the news that appeared in 1907 in the Hindu of founding the IMS was obtained by him. A noncomputer man learnt HTML (at the age of 65 or so) on his own and keeps the IMS website updated is unprecedented. We are thankful to him for all his efforts. Though he refused the editorship of MS from April 1, 2019, he is still working relentlessly so as to include MS in the list of SCOPUS, which is the largest abstract and citation database of peer-reviewed literature: scientific journals, books and conference proceedings. It is updated everyday.

We salute this Gandhian Mathematician for his yeoman services to the IMS and wish him good health and long life. Also, let me end by making note of the fact that, the IMS council had decided to dedicate volume 88, Part-II (2019) of MS in his honour which he flatly refused and ultimately IMS Council respectfully honoured his desire by concurring with his request.

Acknowledgement: The author thanks Prof. S. K. Nimbhorkar, Prof. M. M. Shikare for fruitful suggestions.

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EUCLIDEAN PLANE GEOMETRY SUFFICES TO VISUALIZE THE STANDARD DEVIATION

JYOTIRMOY SARKAR AND MAMUNUR RASHID

(Received : 10 - 11 - 2018 ; Revised : 20 - 06 - 2019)

Abstract. We present a vertical line method of equalizing the areas of two sets of rectangles in order to visualize the mean, the MD and the SD of a set of numbers or of a random variable (RV). Our method utilizes the (empirical) cumulative distribution function [(E)CDF] F of the data or of the RV, the (E)CDF G of the deviations and the (E)CDF H of the (normalized) squared deviations, all of which are constructed using only Euclidean plane geometry. We hope that our method will help readers enhance their intuition about these concepts, and approximate their values fast.

1. Introduction

The two most important concepts in probability and statistics are the measures of center and spread for a set of numbers, or for a random variable (RV). Scholars in almost all quantitative fields of study need to know these concepts. While most users can compute the mean and the SD, and many understand the mean quite well, the SD is not fully understood. We offer to help users comprehend these concepts better by letting them *see* these quantities using only two-dimensional Euclidean geometry. We also wish to inspire post-graduate students to extend these ideas to visualize other related concepts.

We first replace the guessing method to visualize the mean as a measure of center, given in [4], by an exact two-dimensional Euclidean geometric method; and then extend that method to visualize also the mean deviation (MD) and the standard deviation (SD) as measures of spread, still remaining within the scope of Euclidean plane geometry.

We use the following example for illustration.

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Key words and phrases: Arithmetic mean, geometric mean, mean proportional, root mean squared deviation, third proportional, Pythagorean theorem.

Example 1. Let X denote the number of distinct values obtained by rolling a fair, six-faced die until a value shows up a second time. This experiment was replicated five times to obtain the following values of X: 4, 1, 2, 6, 2; which after sorting became: 1, 2, 2, 4, 6. The data have mean 3, MD 1.6 and SD 2.

2. Visualizing the mean by guessing

The (arithmetic) mean or the average is the most common measure of center. The mean of a set of n numbers $\{x_1, x_2, \ldots, x_n\}$ is defined by

$$\bar{x} = (1/n) \sum_{i=1}^{n} x_i \tag{2.1}$$

See [2] for a discussion on students' understanding of the mean; and see [1] for another measure of center—the median.

Equation (2.1) is equivalent to $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$. Hence, the mean is typically interpreted as the location of a fulcrum that balances the dot plot. See [7], for example. Fig. 1 depicts the fulcrum under the dot plot of the data in Example 1.

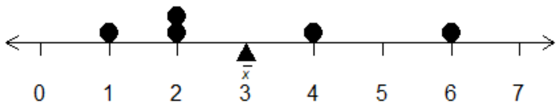


Fig. 1. The mean is a fulcrum under a dot plot that balances it.

Nonetheless, finding the location of the fulcrum without actually computing it remains only a guessing game—quite instructive, *albeit* prone to occasional mistakes. A better guessing method is given in [4], which we will recall here, since it is the starting point for developing a two-dimensional Euclidean geometric method of visualizing the mean, which involves *no* guessing at all!

Consider the empirical cumulative distribution function (ECDF) of the data given by $y = F(x) = N(x)/n$, where $N(x)$ is the number of values among the set of n numbers that are at most x. On the ECDF of the data in Example 1, shown in Fig. 2, we superimpose a vertical line so that the

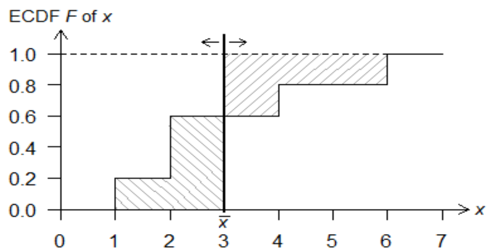
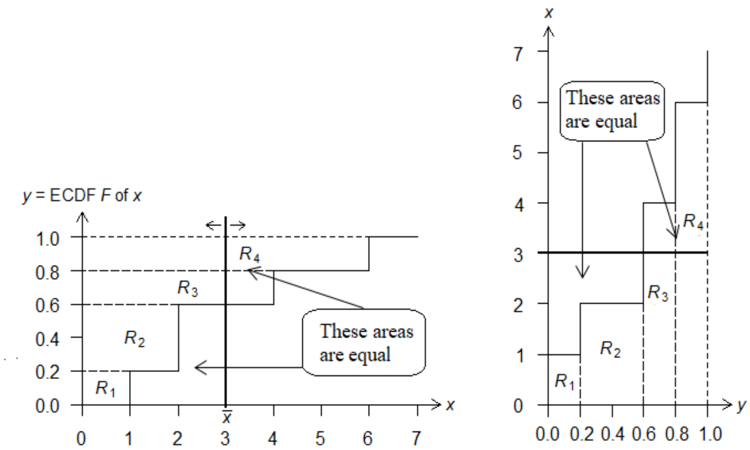


Fig. 2. The mean is a vertical line on the ECDF that equalizes the shaded areas.

area to the left of the vertical line and bounded by the ECDF and the horizontal line $y = 0$, is equal to the area to the right of the vertical line and bounded by the ECDF and the horizontal line $y = 1$. These two areas are shaded.

Why does this vertical line method work? We convert the ECDF into a collection $\mathbb{R}_F$ of at most n rectangles as shown in Fig. 3(a) for the data in Example 1. Next, we reflect them about the line $y = x$ to obtain tall rectangles in Fig. 3(b), showing the inverse-ECDF. Finally, we 'average' the heights of these tall rectangles. See details in [5].



(a) (b)

Fig. 3. The family of rectangles $\mathbb{R}_F$ formed by (a) the ECDF and (b) the inverse-ECDF, together with the mean line that equalizes the areas to its two sides.

3. Visualizing the mean without guessing

Let us return to the ECDF F of Fig. 2. Papers [4], [5], [6] advocate guessing the location of the vertical line, to be superimposed on the ECDF, in order to equalize the areas of the shaded regions. But here we want to exhibit that, in fact, no guessing is necessary! There is an exact Euclidean construction to find this vertical line, which essentially uses the following construction repeatedly.

Construction 3.1 (Euclidean Addition of Two Rectangles)**.** *To convert two left aligned and vertically stacked rectangles into a single rectangle having the same total area and having a height equal to the sum of the heights of the given rectangles.*

Proof. Let the given rectangles $PQRS$ and $STUV$ be in natural orientation (with sides parallel to the edges of the paper), left aligned and vertically stacked as in Fig. 4. on the next page. Let lines QR and UV intersect at

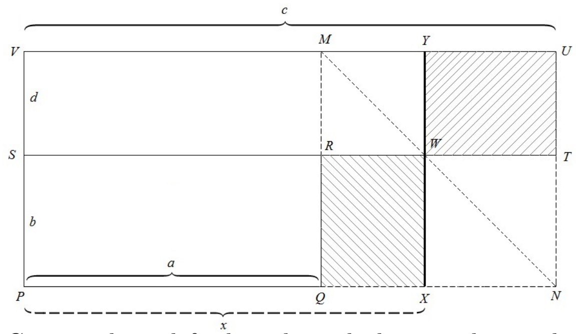


Fig. 4. Converted two left aligned, stacked rectangles $a \times b$ and $c \times d$ into a single rectangle $x \times (b + d)$ such that $ab + cd = x(b + d)$.

M so that $|RM| = |TU| = |SV| = d$; and let lines UT and PQ intersect at N so that $|TN| = |RQ| = |SP| = b$. We draw NM, cutting RT at W. Through W, we draw a line parallel to PV cutting line PQ at X and line UV at Y. Below we prove that $PXYV$ is the desired rectangle.

Triangles RWM and TWN are similar, implying that $|RW| \cdot |TN| = |TW| \cdot |RM|$, or $|RW| \cdot |RQ| = |TW| \cdot |TU|$. Hence, (the shaded) rectangles $RQXW$ and $TUYW$ have equal areas. Adding the areas of rectangles $PQRS$ and $SWYV$ to each of the shaded rectangles $RQXW$ and $TUYW$, we see that the area of rectangle $PXYV$ equals the total area of rectangles $PQRS$ and $STUV$, or $ab + cd$.

Note that the height of the combined rectangle $PXYV$ is $|UN|=|QM|=|PV|= b + d$, and the width is $|PX|=|SW|=(ab+cd)/(b+d)=x$. □

Construction 1 converts two rectangles $PQRS$ and $STUV$ into a single rectangle $PXYV$, which we will denote by $PQRS \oplus STUV$. To convert three or more left-aligned, stacked rectangles into a single rectangle having the same total area and the same total height of the given rectangles, we can apply Construction 1 repeatedly to combine pairs of adjacent rectangles (in any order). The final answer does not depend on the order in which the rectangles are joined together. We leave it to the interested reader to verify this claim.

4. Visualizing the MD

The deviations of the n given numbers from their mean are $d_i = |x_i - \bar{x}|$ for $i = 1, 2, \ldots, n$. While the average of all differences from the mean is always 0, the average of all *deviations* from the mean is called the mean deviation (MD), and is given by

$$MD = (1/n) \sum_{i=1}^{n} |x_i - \bar{x}| = (1/n) \sum_{i=1}^{n} d_i \qquad (4.1)$$

The MD, being an average, can be guessed either by the fulcrum method applied to the dot plot of the deviations; or it can be found from the ECDF G of the deviations. In fact, since the ECDF F of the given numbers is already drawn, one can simply reflect the portion of F to the left side of the vertical line $x = \bar{x}$ at the mean, about that line, with the reflection falling on the right side of this vertical line. See Fig. 5(a). Although not absolutely essential, it is a good practice to sort the rectangles in increasing order of width from bottom to top, and thereby construct the ECDF G of the deviations. See Fig. 5(b). Note that both constructions—drawing the reflection about a line and sorting the rectangles—are doable using only Euclidean plane geometric methods.

To find the MD, we superimpose on the ECDF G another vertical line that equalizes the areas to its left and to its right as shown in Fig. 5(b). This vertical line can be found either by guessing, or by using Construction 1 repeatedly: After naming the rectangles in Fig. 5(b) as S_1, S_2, S_3, we combine them in the order $S_1 \oplus (S_2 \oplus S_3)$ in Fig. 5(c).

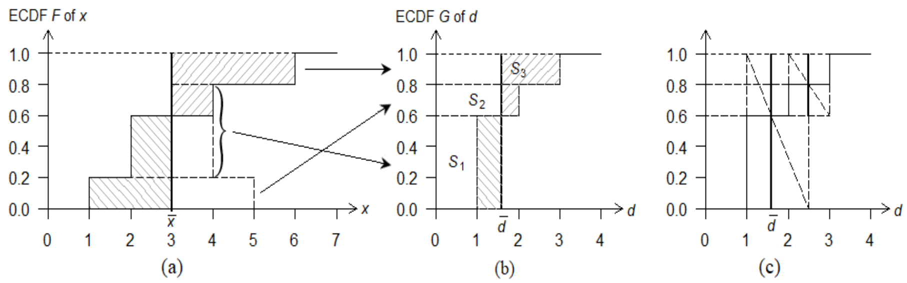


Fig. 5. (a) Deviations from the mean are obtained by reflecting the ECDF about the line $x = \bar{x}$. (b) The MD is a 'guessed' vertical line that equalizes the areas of the shaded regions, after sorting the deviations. (c) The MD line is obtained, not by guessing, but by using Construction 1 repeatedly.

5. Visualizing the SD

The mean squared deviation (MSD) from the mean for a set of numbers is defined by

$$\tilde{s}^2 = (1/n) \sum_{i=1}^{n} (x_i - \bar{x})^2 = (1/n) \sum_{i=1}^{n} d_i^2 \tag{5.1}$$

while the sample variance of the same set of numbers is defined by

$$s^2 = (1/(n-1)) \sum_{i=1}^{n} (x_i - \bar{x})^2 = (1/(n-1)) \sum_{i=1}^{n} d_i^2 \tag{5.2}$$

Taking the positive square root of (5.1) and (5.2), we obtain $\tilde{s}$ and s, respectively the root mean squared deviation (RMSD) from the mean and the sample standard deviation (SD) of a set of given numbers. See [5] for interpretations of the RMSD and the SD as the radii of cylinders, or widths of many other 3-D objects. Also, see [6] for visualizing the RMSD and the SD using the graph of a parabola, which is clearly a non-Euclidean method. We, on the other hand, present below a two-dimensional Euclidean method of visualizing the RMSD and the SD in three steps.

Step 1 (Scaled MSD). Let R be a fixed positive length: We choose R to be the largest deviation from the mean. Starting from the ECDF G of the deviations $\{d_i\}$, we construct the ECDF H of normalized squared deviations, given by $v_i = d_i^2/R$, using Euclidean Construction 2.

Construction 5.1 (Third Proportional)**.** *Given two line segments of length a and b, find a third line segment of length c such that $a : b = b : c$. In fact, $c = b^2/a$ is called the third proportional following a and b, in that order.*

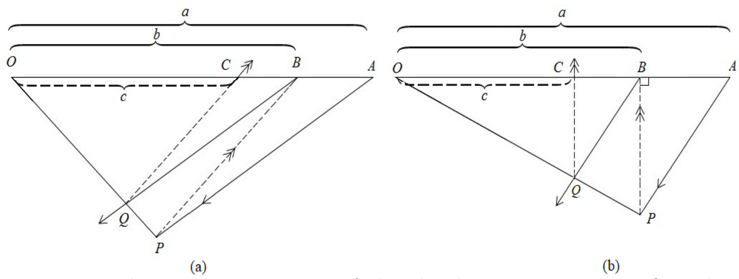


Fig. 6. For $a, b > 0$, construction of the third proportional c, after choosing the point P (a) anywhere not on OA, or (b) vertically below B.

Proof. Let O, B, A be collinear points such that $|OA| = a, |OB| = b$. We want to find another point C on line OA such that $|OC| = c$ and $a : b = b : c$. Take any point P not on line OA. Draw lines PO, PA, PB. Draw a line through B parallel to AP, cutting OP at Q. Draw a line through Q parallel to BP cutting OA at C. Then the similarity of triangles OAP and OBQ implies that $|OA| : |OB| = |OP| : |OQ|$. Next, the similarity of triangles OBP and OCQ implies that $|OB| : |OC| = |OP| : |OQ|$. Hence, $|OA| : |OB| = |OB| : |OC|$. □

Remark 5.1 (A special choice of P)**.** *Although P can be any point not on line OA, in our application below we choose P such that PB is perpendicular to OA. See Fig. 6(b).*

The ECDF G of the deviations form a collection $\mathbb{R}_G$ of rectangles whose widths equal the deviations and heights equal $1/n$ (or a multiple thereof). We repeatedly apply Euclidean Construction 2 always choosing $a = R$, the largest deviation, and choosing b equal to each deviation d_i one by one. The third proportional corresponding to each second number d_i (the first number always being R), is a normalized squared deviation $v_i = d_i^2/R$. We now construct the ECDF H of the normalized squared deviations simply by shortening the widths of the rectangles in $\mathbb{R}_G$ from d_i to $v_i = d_i^2/R$, still keeping them left aligned at 0 and without changing their heights, to obtain the rectangles in $\mathbb{R}_H$. See Fig. 7(a)-(b).

On the ECDF H, we superimpose a vertical line $v = \bar{v}$ that equalizes the areas to its left and to its right [or we apply Construction 1 repeatedly] to get the mean $\bar{v}$ of these normalized squared deviations $\{v_i\}$. Indeed, in view of (5.1), we have

$$\bar{v} = (1/n)\sum_{i=1}^{n}(d_i^2/R) = (\tilde{s}^2/R) \tag{5.3}$$

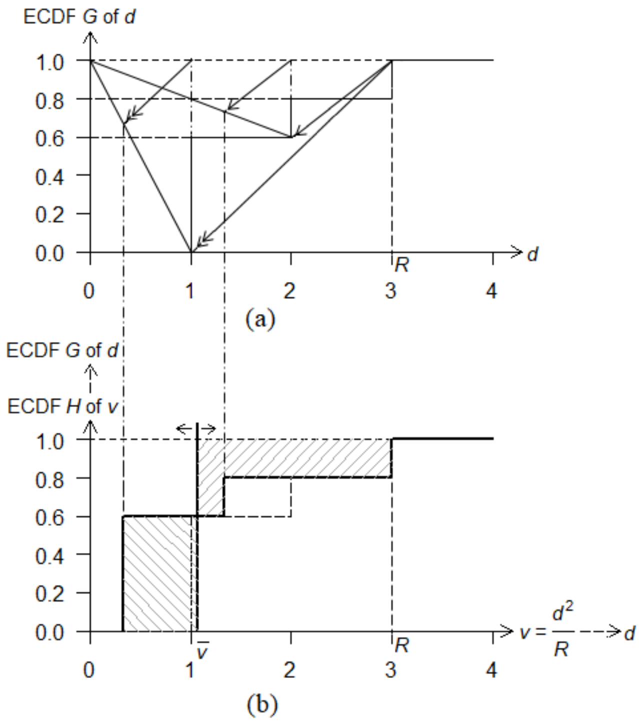


Fig. 7. (a) ECDF H of the normalized squared deviations $v = d^2/R$ for the data in Example 1 obtained using Construction 2 and starting from ECDF G of deviations. (b) The mean vertical line $v = \bar{v}$ that equalizes the shaded areas obtained thereafter.

Step 2 (Unscaled RMSD). Rewriting (5.3), we express the RMSD of a data set as

$$\tilde{s} = \sqrt{\bar{v}\,R} \tag{5.4}$$

To obtain the RMSD geometrically, we use Construction 3.

Construction 5.2 (Mean Proportional). *Given two line segments of lengths a and b, find an intermediate line segment of length g such that a : g = g : b,* *or* $g = \sqrt{ab}$. *In fact, g is called the mean proportional between a and b (or the geometric mean of a and b), in either order.*

We came across several equivalent constructions of g, all of which utilizes the Pythagorean theorem and the algebraic identity

$$ab = ((a + b)/2)^2 - ((a - b)/2))^2 \tag{5.5}$$

In Fig. 8, we present a construction that uses a small amount of space. The figure may be augmented at the bottom of ECDF H in Fig. 7, and is so done in Fig. 9(a).

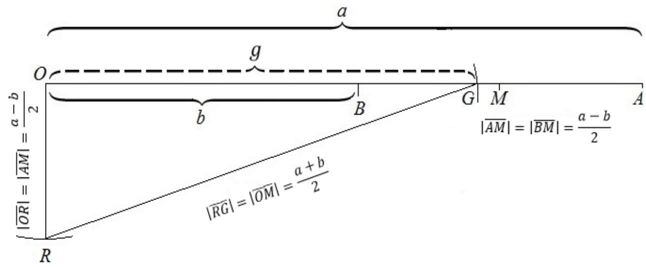


Fig. 8. Construction of the mean proportional $g = \sqrt{ab}$ between two positive numbers a and b.

Proof. Let O, B, A be collinear points such that $|OA| = a, |OB| = b$. Let M be the midpoint of segment AB. Let OR be perpendicular to OA such that $|OR| = |MA| = |a - b|/2$. Let the circle with center R and radius $|OM| = (a+b)/2$ intersect line OA at G. Since ROG is a right triangle, in view of (5.5), $|OG| = g = \sqrt{ab}$. □

Returning to the ECDF H, we obtain the geometric mean between $\bar{v}$ and R, by using Construction 3. See Fig. 9(a).

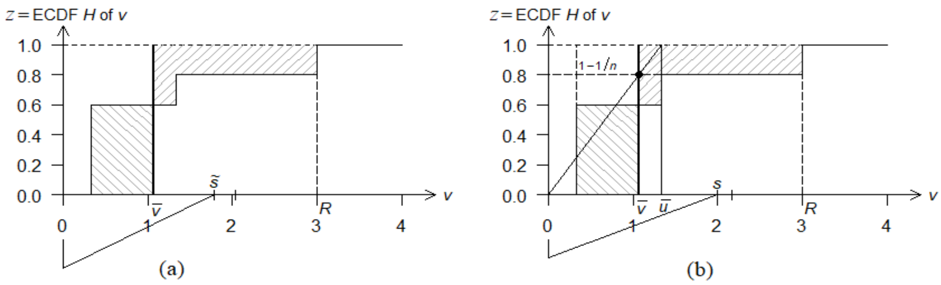


Fig. 9. (a) The RMSD $\tilde{s} = \sqrt{\bar{v}\,R}$, and (b) the SD $s = \sqrt{\bar{u}\,R}$ for the data in Example 1, obtained using Construction 3.

Step 3 (Scaled Variance and Unscaled SD). A comparison of (5.1) and (5.2) shows that the variance is linearly related to the MSD. Therefore, a simple modification of the scaled MSD $\bar{v}$ in Fig 9(a) yields the scaled variance $\bar{u}$ shown in Fig. 9(b): We merely draw a line joining the points $(0, 0)$ and $(\bar{v}, 1 - 1/n)$, and extend the line to intersect the horizontal line $z = 1$ at the point $(\bar{u}, 1)$. That is,

$$\bar{u} = \bar{v}(n/(n-1)) \tag{5.6}$$

In view of (5.2), (5.1), (5.4), (5.6), we can write

$$s = \tilde{s}\sqrt{(n/(n-1))} = \sqrt{\bar{v}R(n/(n-1))} = \sqrt{\bar{u}R}.$$

That is, the SD s is the geometric mean between $\bar{u}$ and R. Hence, we can obtain s by using Construction 3 as shown at the bottom of Fig 9(b).

Remark 5.2 (SD exceeds RMSD, which exceeds MD)**.** *Comparing (5.1) and (5.2), we know that the SD s exceeds the RMSD $\tilde{s}$. Equivalently, from (5.6), $\bar{u} \geq \bar{v}$; hence, $s = \sqrt{\bar{u}\,R} \geq \sqrt{\bar{v}\,R} = \tilde{s}$. Also, comparing (5.1) and (4.1), in view of the Cauchy-Swartz inequality, we have $\tilde{s} \geq MD$, with equality if and only if all deviations from the mean are equal.*

6. Extensions

What we have shown in this paper for a set of numbers easily extends to a discrete RV on a finite support. For instance, the random variable X in Example 1 takes on values $\{1, 2, \ldots, 6\}$ with associated probabilities $\{1/6, 5/18, 5/18, 5/27, 25/324, 5/324\}$. It has mean $\mu = 899/324 = 2.775$, MD$= 1.022$ and SD $\sigma = 1.2355$. All features of a discrete RV X are defined analogously to those of a set of numbers, except the height corresponding to each value x_i is p_i instead of $1/n$ or its multiple. Also, the SD of a RV is the same as its RMSD. Fig. 10 utilizes the Euclidean plane geometric methods to visualize the mean, the MD and the SD starting from the CDF F of X. Caution: For the sake of clarity, we stretched the horizontal scale in (c) to be larger than the horizontal scales in (a) and (b).

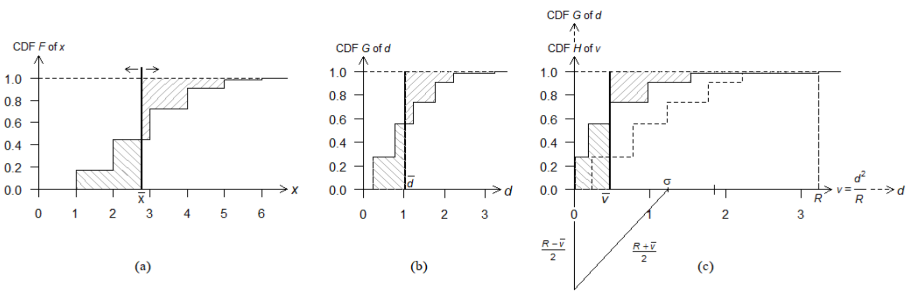


Fig. 10. Euclidean construction of (a) mean, (b) MD and (c) SD of RV X in Example 1.

Our method also extends to a continuous RV on a finite support. For example, let T follow a triangular$(0, 1; 1)$ distribution, with peak at 1; that is, T has probability density function $f(x) = 2x$ on $(0, 1)$. Fig. 11 extends the Euclidean plane geometric methods to visualize the mean 2/3, the MD 0.2 and the SD 0.236 starting from the CDF F of T. Caution: The horizontal scales in (b) and (c) are larger than the horizontal scale in (a).

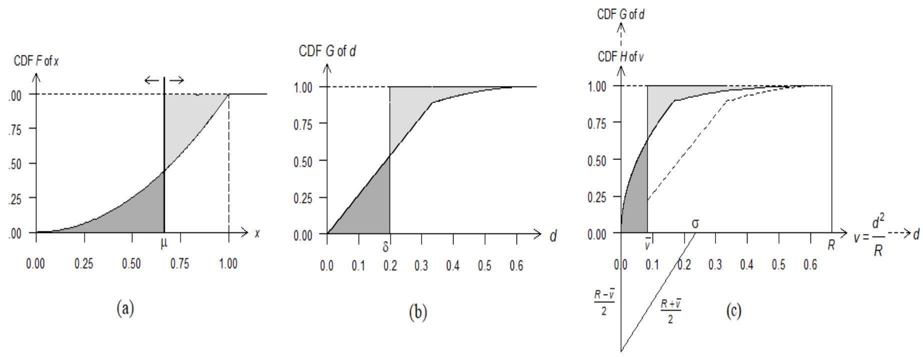


Fig. 11. Euclidean construction of (a) mean, (b) MD and (c) SD of a triangular$(0, 1; 1)$ RV.

We hope the Euclidean plane geometric methods to visualize the mean, the MD and the SD will add a little more understanding to users of probability and statistics. With some practice, the attentive user will be able to quickly approximate their values; and do so faster than obtaining the exact values through arithmetic computations. We plan to test this hypothesis in a statistically designed experiment.

7. Exercises

We invite interested readers to apply the Euclidean geometric methods of this paper to solve the following exercises:

(1) Suppose that the experiment in Example 1 is replicated four more times to yield new values of X as 4, 3, 1, 2. Visualize the mean, MD and SD of the new data. Visualize the mean, MD and SD of the combined data (of size 9) based on (1) the ECDF of the combined data, and (2) the summary statistics of the two data sets.

(2) From (5.5) or from Fig. 8, we see that the geometric mean (GM) of two numbers is at most their arithmetic mean (AM); that is, $\sqrt{ab} \leq (a+b)/2$, with equality holding if and only if $a = b$. For a variable taking on positive values, such as the combined data in Exercise (1) or for the RV X in Example 1, demonstrate that AM≥GM≥HM, where GM is given by $\sqrt[n]{\prod_i^n x_i} = \exp\{\sum_i^n \ln(x_i)/n\}$, and HM denotes the harmonic mean given by $n/\sum_i^n(1/x_i)$. You may use the graphs of $f(x) = \ln x$ and $g(x) = 1/x$ on $(0, \infty)$; and visualize the

mean before and after the transformation. Thereafter, extend your solution to visualize Jensen's inequality.

(3) In Example 1, change the phrase 'six-sided die' to (i) 'four-sided tetrahedron,' or (ii) 'eight-sided octahedron.' Draw diagrams to visualize the mean, the MD and the SD of the new discrete RV X.

(4) Draw diagrams to visualize the mean, the MD and the SD of (i) a uniform$(0, 1)$ variable, and (ii) an exponential variable with hazard rate $\alpha = 0.25$, starting from their respective CDFs.

(5) Table 1 gives the (ordered) scores of five students from each of three schools in a music competition. Did the schools perform comparably? A customary answer to this question is found by conducting a one-way analysis of variance. The answer is negative whenever the F-statistic exceeds a critical value. Extend the Euclidean methods of this paper to visualize the F-statistic $F = 7/6$.

TABLE 1. Ordered scores (out of 100 points) of contestants

School A	77, 78, 82, 88, 90
School B	72, 77, 79, 83, 84
School C	70, 75, 80, 80, 85

(6) Table 2 gives the (sorted) heights x (in centimeters) and the associated weights y (in kilograms) of seven NCC cadets. Extend the Euclidean methods of this paper to visualize (i) the correlation coefficient $r = .85$, and (ii) the least squares regression line of weight on height: $\hat{y} = -42.3 + 0.6\,x$.

TABLE 2. Physical attributes of NCC cadets

x height (cm)	145	148	150	153	153	156	159
y weight (kg)	45.5	48.6	46.7	48.2	50.0	55.0	52.9

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THE CAUCHY-SCHWARZ INEQUALITY IN MATHEMATICS, PHYSICS AND STATISTICS

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Abstract. We discuss the Cauchy-Schwarz inequality, first in the mathematical setting, and then in physics formulated as Heisenberg's uncertainty principle in quantum mechanics and then in statistics manifesting as the Cramér-Rao inequality.

1. Cauchy-Schwarz inequality in mathematics

Perhaps the most ubiquitous of inequalities in mathematics is the Cauchy-Schwarz inequality. First discovered by Cauchy in the year 1821, it states that if $a_1, ..., a_n$ and $b_1, ..., b_n$ are arbitrary real numbers, then

$$\left|\sum_{j=1}^n a_j b_j\right| \leq \left(\sum_{j=1}^n |a_j|^2\right)^{1/2} \left(\sum_{j=1}^n |b_j|^2\right)^{1/2}, \tag{1.1}$$

with equality arising if and only if there is a $\lambda \in \mathbb{R}$ such that $a_j = \lambda b_j$ for $j = 1, 2, ..., n$.

There are several immediate proofs of this. Indeed, we have

$$\sum_{j=1}^n a_j^2 \sum_{k=1}^n b_k^2 - \left(\sum_{j=1}^n a_j b_j\right)^2 = \sum_{j,k=1}^n a_j^2 b_k^2 - \sum_{j,k=1}^n a_j b_j a_k b_k$$
$$= \sum_{j=1}^n \sum_{k \geq j} (a_j b_k - a_k b_j)^2 \geq 0. \tag{1.2}$$

From this, we see that equality arises if and only if $a_j b_k = a_k b_j$ for all j, k, which is tantamount to the assertion

$$(a_1, ..., a_n) = \lambda(b_1, ..., b_n), \tag{1.3}$$

for some $\lambda \in \mathbb{R}$.

Another "proof at a glance" begins with the self-evident inequality

$$2a_j b_j \leq a_j^2 + b_j^2, \qquad 1 \leq j \leq n,$$

and noting the homogeneity of the left hand side, gets for any $\lambda \neq 0$,

$$2a_j b_j \leq \lambda^2 a_j^2 + \lambda^{-2} b_j^2, \qquad 1 \leq j \leq n.$$

Summing over j we get

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$$\sum_{j=1}^{n} 2a_j b_j \le \lambda^2 \sum_{j=1}^{n} a_j^2 + \lambda^{-2} \sum_{j=1}^{n} b_j^2.$$

Choosing

$$\lambda^2 = \left(\sum_{j=1}^{n} b_j^2\right)^{1/2} \left(\sum_{j=1}^{n} a_j^2\right)^{-1/2}$$

so as to minimize the right hand side leads immediately to the Cauchy-Schwarz inequality.

The result is easily extended to complex numbers a_i, b_i by observing that

$$\left|\sum_{j=1}^{n} a_j b_j\right| \le \sum_{j=1}^{n} |a_j||b_j| \tag{1.4}$$

and then applying (1.1) to the latter sum so as to deduce

$$\left|\sum_{j=1}^{n} a_j b_j\right| \le \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}.$$

It is not difficult to see that again, equality can arise if and only if (1.3) holds, because (1.4) is a consequence of the triangle inequality in which the case of equality is easily identified.

The corresponding inequality for integrals, namely

$$\left|\int_a^b f(x)g(x)dx\right| \le \left(\int_a^b |f(x)|^2 dx\right)^{1/2} \left(\int_a^b |g(x)|^2 dx\right)^{1/2} \tag{1.5}$$

seems to have been first stated by Buniakowsky in 1859 and later (independently) by Schwarz in 1885 and for this reason, we sometimes refer to this as the Cauchy-Buniakowsky-Schwarz inequality (see page 16 of [6]). It is easy to deduce (1.5) from (1.1) by using Riemann sums and the limiting process. It is also easy to see that (1.5) extends to improper integrals. We leave the details to the student.

All these inequalities are special cases of a more general theorem:

Theorem 1. *If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, then*

$$|(v, w)| \le ||v||||w||$$

for all $v, w \in V$ with equality if and only if v is a scalar multiple of w.

Proof. We may suppose that $w \neq 0$, for otherwise, the result is clear. We decompose v into its components parallel and perpendicular to w by writing: $v = \lambda w + (v - \lambda w)$ for some scalar λ. For $v - \lambda w$ to be perpendicular to w, we need $0 = (v - \lambda w, w) = (v, w) - \lambda ||w||^2$, that is, $\lambda = (v, w)/||w||^2$. Now, $(v - \lambda w, v - \lambda w) \ge 0$ for any λ so that $||v||^2 - \overline{\lambda}(v, w) - \lambda(w, v) + |\lambda|^2 ||w||^2 \ge 0$. With our choice of λ in particular, we deduce

$$||v||^2 - 2\frac{|(v, w)|^2}{||w||^2} + \frac{|(v, w)|^2}{||w||^4}||w||^2 \ge 0.$$

In other words, $|(v,w)| \le ||v|| ||w||$. Clearly, equality can occur if and only if $v = \lambda w$ for some λ. This completes the proof. □

Inequalities (1.1) and (1.5) are now special cases of this more general inequality using the appropriate inner product spaces such as $L^2[a,b]$.

2. A principle of duality

At the center of sieve theory and the large sieve inequality in particular, lies a fundamental principle of duality which is essentially the Cauchy-Schwarz inequality. We record this below.

Theorem 2. *Let c_{ij}, for $1 \le i \le m$, $1 \le j \le n$, be mn complex numbers. Let λ be a non-negative real number. Then, the inequality*

$$\sum_{i=1}^m \left|\sum_{j=1}^n c_{ij} a_j\right|^2 \le \lambda \sum_{j=1}^n |a_j|^2$$

holds for all complex numbers $a_1, ..., a_n$ if and only if the inequality

$$\sum_{j=1}^n \left|\sum_{i=1}^m c_{ij} b_i\right|^2 \le \lambda \sum_{i=1}^m |b_i|^2$$

holds for all complex numbers $b_1, ..., b_m$.

Proof. Let C be the $m \times n$ matrix (c_{ij}) and $\mathbf{a} = (a_1, ..., a_n)^{\mathrm{tr}}$ and $\mathbf{b} = (b_1, ..., b_m)^{\mathrm{tr}}$ be column vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. The first inequality of the theorem can then be written as $(C\mathbf{a}, C\mathbf{a}) \le \lambda(\mathbf{a}, \mathbf{a})$, and the second one as $(\mathbf{b}^{\mathrm{tr}}C, \mathbf{b}^{\mathrm{tr}}C) \le \lambda(\mathbf{b}, \mathbf{b})$. Suppose the first inequality holds and let $\mathbf{b} \in \mathbb{C}^m$. Then, by the Cauchy-Schwarz inequality, we have for all $\mathbf{a} \in \mathbb{C}^n$,

$$(\mathbf{b}^{\mathrm{tr}}C\mathbf{a}, \mathbf{b}^{\mathrm{tr}}C\mathbf{a}) \le ||\mathbf{b}||^2 ||C\mathbf{a}||^2 \le \lambda ||\mathbf{b}||^2 ||\mathbf{a}||^2$$

by our assumption. Now set $\mathbf{a} = \overline{C}^{\mathrm{tr}}\mathbf{b}$ to deduce $||\mathbf{b}^{\mathrm{tr}}C||^4 \le \lambda ||\mathbf{b}||^2 ||\mathbf{b}^{\mathrm{tr}}C||^2$ which gives the result. The converse is similarly deduced. □

The principle of duality can be used to deduce what is called the large sieve inequality which plays a fundamental role in analytic number theory. The reader can find further details in the monograph [1] as well as [3].

3. The Heisenberg uncertainty principle

The celebrated Heisenberg uncertainty principle, which is a corner stone of quantum mechanics, is an immediate consequence of the Cauchy-Schwarz inequality once we understand the dictionary that translates the concepts of physics into mathematical language. Indeed, from a mathematical standpoint, Heisenberg's uncertainty principle states that a function F and its Fourier transform $\widehat{F}$ cannot **both** have compact support. To prove this, it is convenient to introduce the following class of functions.

We define the Schwartz space (after Laurent Schwartz and no relation to the Schwarz of the Cauchy-Schwarz inequality) $\mathcal{S}$ to be the space of infinitely differentiable functions $F : \mathbb{R} \to \mathbb{C}$ such that $|x^k F^{(\ell)}(x)| \to 0$ as $|x| \to \infty$, for all non-negative integers k and ℓ.

We recall the definition of the Fourier transform $\widehat{\psi}$ of a function $\psi \in \mathcal{S}$:

$$\widehat{\psi}(x) := \int_{-\infty}^{\infty} \psi(t) e^{-2\pi i t x} dt, \quad x \in \mathbb{R}.$$

It is easily verified that $\widehat{\psi}$ is also in $\mathcal{S}$.

Theorem 3 (Heisenberg's uncertainty principle). *Let ψ be a Schwartz function satisfying $||\psi||_2 = 1$. Then,*

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} x^2 |\widehat{\psi}(x)|^2 dx \right) \geq \frac{1}{16\pi^2},$$

with equality if and only if $\psi(x) = Ae^{-Bx^2}$ for some $B > 0$ and $|A|^2 = \sqrt{\frac{2B}{\pi}}$.

Proof. Writing

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} \psi(x) \overline{\psi(x)} dx,$$

we integrate by parts to get that this is

$$= x\psi(x)\overline{\psi(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \frac{d}{dx} \left(\psi(x) \overline{\psi(x)} \right) dx.$$

Because $\psi \in \mathcal{S}$, the first term equals zero giving us

$1 = -\int_{-\infty}^{\infty} x\{\psi(x)\overline{\psi'(x)} + \psi'(x)\overline{\psi(x)}\}dx$, and hence $1 \leq 2\int_{-\infty}^{\infty} |x\psi(x)\psi'(x)|dx$.

Applying the Cauchy-Schwarz inequality, we get

$$1 \leq 2 \left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right)^{1/2} \left(\int_{-\infty}^{\infty} |\psi'(x)|^2 dx \right)^{1/2}.$$

By the Fourier inversion theorem,

$$\psi(x) = \int_{-\infty}^{\infty} \widehat{\psi}(t) e^{2\pi i t x} dt, \text{ so that } \psi'(x) = \int_{-\infty}^{\infty} (2\pi i t) \widehat{\psi}(t) e^{2\pi i t x} dt,$$

the differentiation under the integral sign being justified by the virtues of the elements of the Schwartz class $\mathcal{S}$. In other words, $\psi'(-x)$ is the Fourier transform of $(2\pi i t)\widehat{\psi}(t)$. By Parseval's formula, we deduce that the L^2-norm of ψ' is equal to $\int_{-\infty}^{\infty} 4\pi^2 t^2 |\widehat{\psi}(t)|^2 dt$. Thus,

$$1 \leq 4\pi \left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right)^{1/2} \left(\int_{-\infty}^{\infty} x^2 |\widehat{\psi}(x)|^2 dx \right)^{1/2},$$

from which the main inequality emerges. For the final part of the theorem, we note that in our application of the Cauchy-Schwarz inequality, equality can occur if and only if $\psi'(x) = \lambda x \psi(x)$, for some scalar λ. This is an ordinary differential equation which is easily solved. We find $\psi(x) = Ae^{-Bx^2}$ for certain constants A and $B > 0$ because ψ belongs to $\mathcal{S}$. The fact that the L^2-norm of ψ equals 1 gives us the final claim. □

An immediate consequence of the uncertainty principle is that ψ and $\widehat{\psi}$ cannot both be concentrated in a small neighborhood of the origin. Indeed, suppose ψ is supported in $[-M, M]$ and $\widehat{\psi}$ is supported in $[-N, N]$. Then, $M^2N^2 \geq 1/4\pi^2$. So, M and N cannot both be arbitrarily small. A manifestation of the uncertainty principle is what we mentioned earlier that both ψ and $\widehat{\psi}$ cannot have compact support. If ψ has compact support, then its Fourier transform is an entire function (see for example, pp. 371-372 of [11]). As such, the zeros of $\widehat{\psi}$ are isolated unless it is identically zero, in which case ψ is also identically zero by the inversion theorem.

In quantum mechanics, $|\psi(x)|^2 dx$ represents the probability density that a particle is near position $x \in \mathbb{R}$. Thus, the probability that such a particle lies in the interval $[a, b]$ is given by

$$\int_a^b |\psi(x)|^2 dx.$$

Our best guess for the position of the particle is given by the expectation

$$\mu := \int_{-\infty}^{\infty} x|\psi(x)|^2 dx$$

and the error (or uncertainty) involved in this guess is given by the variance

$$\int_{-\infty}^{\infty} (x - \mu)^2 |\psi(x)|^2 dx.$$

A similar analysis holds for the momentum of the particle. Indeed, the probability density that the momentum is $x \in \mathbb{R}$ is $|\widehat{\psi}(x)|^2 dx$ and so the probability that the momentum lies in $[a, b]$ is given by

$$\int_a^b |\widehat{\psi}(x)|^2 dx.$$

The expectation and variance of the momentum are defined as before. Without any loss of generality, we can normalize our functions so that the expectation of both the position and momentum are zero. With this normalization, we see that Heisenberg's uncertainty principle establishes a lower bound for the product of these variances (or errors). In other words, any attempt to lower the error in our observation of the position of a particle increases the error in determining its momentum and vice versa.

An uncertainty principle of some sort or other abounds in nature described by Fourier duality. Recently, Tao [13] discovered a discrete version of the uncertainty principle and a simple proof of this can be found in [8]. See also [7] for other variations on this theme. An uncertainty principle for the equidistribution of arithmetic sequences in arithmetic progressions and short intervals has been a topic of intense research in analytic number theory. The reader can find an exposition of this in [5].

4. The Cramér-Rao inequality

A humorous definition of statistics is that it is the converse of probability theory. Though seemingly funny, the joke contains the essential idea. In probability theory, we deal with measurable functions (also called random variables), probabilty density functions (sometimes referred to as pdfs or probability measures). By contrast, in statistics, we may not know what the pdf may be of a certain phenomenon. At best, we can take a large sample and infer from this data, the nascent pdf that describes the phenomenon.

At the dawn of the 20th century, R.A. Fisher undertook the task of laying the foundations of theoretical statistics. In his fundamental paper [4], he wrote "the object of statistical methods is the reduction of data." Expanding on this aphorism, he explained, "A quantity of data, which usually by its mere bulk is incapable of entering the mind, is to be replaced by relatively few quantities which shall adequately represent the whole." Motivated by these considerations, he was led to define (what we now call) the *Fisher information* of a random variable X with probability density function $f_\theta(x)$ attached to an unknown deterministic parameter θ as follows. The contribution of x to the information content of the random variable may be viewed to be $-\log(f_\theta(x))$ since from information theory we know that for the discrete setup an ideal lossless binary code would need roughly these many bits to represent this variable.

The rate of change of this information is then $\frac{\partial \log f_\theta}{\partial \theta}$ and it seems reasonable that the expectation

$$I(\theta) := \int_{-\infty}^{\infty} \left(\frac{\partial \log f_\theta}{\partial \theta} \right)^2 f_\theta(x) dx$$

(called *Fisher information* in statistical parlance) gives us some idea of the amount of information about θ contained in the data.

For example, if X has a Bernoulli distribution where X can have only two values "heads" or "tails" (or more precisely 0 and 1 say), with 1 having probability θ and 0 having probability $1-\theta$, with density function

$$f_\theta(x) = \theta^x(1-\theta)^{1-x},$$

then $\log f_\theta(x) = x\log\theta + (1-x)\log(1-\theta)$ and we find, after a simple calculation, that $I(\theta) = (\ \theta(1-\theta)\ )^{-1}$.

To take another example, suppose X has a normal distribution with unknown mean μ and known variance σ^2. Let us determine the Fisher information $I(\mu)$ in X. Since $f_\mu(x) = (\sqrt{2\pi}\,\sigma)^{-1}\, e^{-(x-\mu)^2/2\sigma^2}$, we see $\log f_\mu(x) = -(1/2)\log(2\pi\sigma^2) - (x-\mu)^2/2\sigma^2$, so that $\dfrac{\partial \log f_\mu}{\partial\mu} = (x-\mu)/\sigma^2$. A direct calculation now gives that $I(\mu) = 1/\sigma^2$.

In statistical problems, large amounts of data are collected to study a phenomenon. With a desire to derive a mathematical model to describe it, we may find, numerically, a function $\widetilde{\phi}$ to approximate a parameter ϕ. $\widetilde{\phi}$ is called an *unbiased estimator* of ϕ if $E(\widetilde{\phi}) = \phi$. That is,

$$\int_{-\infty}^{\infty} \widetilde{\phi} f_\theta(x)dx = \phi(\theta).$$

Here, θ and x are independent parameters. Differentiating this with respect to θ and interchanging integration and differentiation (provided of course that this is permissible) gives:

$$\int_{-\infty}^{\infty} \widetilde{\phi}(x)\frac{\partial f_\theta}{\partial\theta}(x)dx = \phi'(\theta).$$

The rate of change of information is the function

$$S(x) := \frac{\partial}{\partial\theta}\log f_\theta(x)$$

called the *score statistic*. Plainly, $S(x) = \frac{1}{f_\theta(x)}\frac{\partial f_\theta}{\partial\theta}(x)$, so that we can write

$$\int_{-\infty}^{\infty} \widetilde{\phi}(x)S(x)f_\theta(x)dx = \phi'(\theta). \tag{4.1}$$

Also, the expectation of $S(x)$ is

$$E(S(x)) = \int_{-\infty}^{\infty} S(x)f_\theta(x)dx = \int_{-\infty}^{\infty}\frac{\partial f_\theta}{\partial\theta}(x)dx = \frac{\partial}{\partial\theta}\int_{-\infty}^{\infty} f_\theta(x)dx = 0,$$

since

$$\int_{-\infty}^{\infty} f_\theta(x)dx = 1,$$

because the total probability is 1. Thus, (4.1) can be re-written as

$$\int_{-\infty}^{\infty}(\widetilde{\phi}(x) - \phi(\theta))S(x)f_\theta(x)dx = \phi'(\theta).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\phi'(\theta)^2 \le \left(\int_{-\infty}^{\infty}(\widetilde{\phi}(x)-\phi(\theta))^2 f_\theta(x)dx\right)\left(\int_{-\infty}^{\infty} S(x)^2 f_\theta(x)dx\right).$$

Writing

$$I(\theta) := \int_{-\infty}^{\infty} \left(\frac{\partial \log f_\theta}{\partial \theta}\right)^2 f_\theta(x)dx,$$

(called *Fisher information* in statistical parlance), we can write our inequality as:

Theorem 4 (The Cramér-Rao inequality). *For an unbiased estimator $\widetilde{\phi}$ of ϕ, we have*

$$\int_{-\infty}^{\infty} (\widetilde{\phi}(x) - \phi(\theta))^2 f_\theta(x)dx \geq \frac{\phi'(\theta)^2}{I(\theta)}.$$

Often, this is applied with $\phi(\theta) = \theta$ so that $\phi'(\theta) = 1$. The inequality then gives us a limitation on the accuracy of the unbiased estimator to the function θ. Somtimes it is referred to as the information inequality. It was discovered independently by C. R. Rao [10] and H. Cramér [2] in 1945 and has played a pivotal role in statistical inference. An enlightening survey of the Cramér-Rao inequality was written by K.R. Parthasarathy [9] where the reader can find discussion of Riemannian metrics to study population models.

Regarding Theorem 4, there is a lot of interest in estimators that actually achieve the Cramer-Rao lower bound. Such estimators are said to be asymptotically efficient. Under certain regularity conditions the maximum likelihood estimators are asymptotically efficient. In such cases the Fisher information about θ in the data is equal to the inverse of the variance of the estimator.

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CLASSIFICATION OF EXTRASPECIAL P-GROUPS USING QUADRATIC FORMS

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Abstract. Let p be a prime number. A p-group is said to be extraspecial p-group if its center, derived subgroup and Frattini subgroup all coincide and have order p. For every $n \in \mathbb{N}$, there are exactly two extraspecial p-groups of order p^{2n+1} up to isomorphism and no extraspecial p-group of order p^{2n}. This classification of extraspecial p-group is proved by using theory of quadratic forms and bilinear forms.

1. Introduction

The enumeration of groups of given order is one of the most interesting problems in group theory. This is a wild problem with the groups of prime power order leading to the complexity. We know that every group of prime order is cyclic. Let p be a prime number. There are exactly two groups of order p^2 and both of them are abelian groups. There are exactly five groups of order p^3 and three of them are abelian. The abelian groups of p^n corresponds to the partitions of n. The classification of all groups of order p^n is known for $n \leq 7$ [16, 7, 1, 4, 8, 9, 11]. In [5], Hall defines isoclinism of groups, which plays an important role in classification of p-groups. Higman [6] and Sims [14] showed that number of groups of order $n = p^k$ is equal to $n^{(\frac{2}{27}+o(1))k^2}$ as k tends to infinity. In [13], Pyber generalised this result for any natural number n. Berkovich and Janko have compiled the work on p-groups in six volumes of books named "Groups of prime power order".

We begin with fixing some notations. Let G be a finite group then $Z(G) := \{g \in G \mid gh = hg \ \forall\ h \in G\}$ is called *center* of G. The subgroup $G' := \langle [g, h] := g^{-1}h^{-1}gh \ : \ \forall\ g, h \in G\rangle$ is called the *derived subgroup* of

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G. The *Frattini subgroup* is the intersection of all maximal subgroups of G. It is denoted by $\phi(G)$. Moreover if G is a p-group then $\phi(G) = G'G^p := \langle g_1 g_2^p \ : \ g_1 \in G', g_2 \in G\rangle$. In this case the quotient group $G/\phi(G)$ is an elementary abelian group. Frattini Subgroup plays an important role in the theory of p-groups.

For a prime number p, a p-group G is said to be *extraspecial p-group* if $Z(G) = G' = \phi(G)$ and order of $Z(G)$ is p. For example, dihedral group $D_4 := \{a, b \ : \ a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1}\}$ and quaternion group $Q_2 := \{c, d \ : \ c^4 = 1, c^2 = d^2, d^{-1}cd = c^{-1}\}$ are extraspecial 2-groups.

The aim of this expository article is to prove that for every $n \in \mathbb{N}$ and a given prime p, there are exactly two extraspecial p-groups of order p^{2n+1} up to isomorphism and no extraspecial p-group of order p^{2n}. The extraspecial p-groups of order p^3 are fundamental units and any extraspecial p-groups of order p^{2n+1} is central product of n copies of extraspecial p-groups of order p^3. For an extraspecial p-group G, the p^{th} power map induces a map from $\dfrac{G}{Z(G)}$ to $Z(G)$, which is used in the classification of extraspecial p-groups. This map is linear when p is an odd prime number and quadratic when p is equal to 2, hence these two cases are treated separately.

The classification of extraspecial p-groups is also given in [3] using group theoretic methods. This classification using quadratic forms and bilinear forms is indicated in [15] and [2]. The classification of extraspecial 2-groups is given in [10]. This classification is sure to be known to experts but we do not know of any reference where it is mentioned in as much detail.

2. Bilinear forms and Quadratic froms

Let $\mathbb{F}$ be a field and V be n-dimensional vector space over $\mathbb{F}$. A map $b : V \times V \to \mathbb{F}$ is called $\mathbb{F}$-*bilinear* if it satisfies the following properties:

(1) $b(\alpha v_1 + \beta v_2, w) = \alpha b(v_1, w) + \beta b(v_2, w)$ for all $v_1, v_2, w \in V$ and $\alpha, \beta \in \mathbb{F}$.

(2) $b(v, \alpha w_1 + \beta w_2) = \alpha b(v, w_1) + \beta b(v, w_2)$ for all $v, w_1, w_2 \in V$ and $\alpha, \beta \in \mathbb{F}$

Two vectors $u, v \in V$ are said to be *orthogonal* if $b(u, v) = 0$. For any subset W of V, We define $W^{\perp} := \{v \in V \ : \ b(v, w) = 0 \text{ for all } w \in W\}$. The subspace $V^{\perp} := \{v \in V \ : \ b(v, v') = 0 \text{ for all } v' \in V\}$ is called *radical* of b and it is denoted by $\operatorname{rad} b$. A bilinear form b is said to be *non-degenerate* if $\operatorname{rad} b = 0$.

A map $q : V \to \mathbb{F}$ is called a *quadratic form* if

(1) $q(\alpha v) = \alpha^2 q(v)$ for all $v \in V$ and $\alpha \in \mathbb{F}$

(2) The map $b_q : V \times V \to \mathbb{F}$ given by $b_q(v, w) := q(v+w) - q(v) - q(w)$ is $\mathbb{F}$-bilinear, for all $v, w \in V$.

For a quadratic map q the map b_q is called the *polar map* of q. The pair (V, q) is called the *quadratic space* over $\mathbb{F}$. A quadratic space (V, q) is called *regular* if $\mathrm{rad}(b_q) = 0$.

Two n-dimensional quadratic spaces (V_1, q_1) and (V_2, q_2) over $\mathbb{F}$ are said to be *isometric* if there exist an $\mathbb{F}$-linear isomorphism $T : V_1 \to V_2$ such that $q_1(v) = q_2(T(v))$ for all $v \in V$. Isometry between two quadratic spaces is denoted by $(V_1, q_1) \cong (V_2, q_2)$.

A quadratic space (V, q) is called the *orthogonal sum* of (V_1, q_1) and (V_2, q_2) if $V = V_1 \oplus V_2$ and $q(v) = q_1(v_1) + q_2(v_2)$, where $v = (v_1, v_2) \in V$ is an arbitrary element of V. In this case we write $q = q_1 \perp q_2$. Conversely, let (V, q) be a quadratic space and $V_i,\ 1 \leq i \leq m$ be subspaces of V such that $V = V_1 \oplus \cdots \oplus V_m$ and $b_q(v_i, v_j) = 0$ for $v_i \in V_i, v_j \in V_j, i \neq j$. Then $q = q_1 \perp \cdots \perp q_m$ where q_i denotes the restriction of q to V_i. We also write $b_q = b_1 \perp b_2 \perp \cdots \perp b_m$, where b_i denotes the restriction of b_q to V_i.

2.1. **Quadratic forms over fields of characteristic** 2. From now onwards, $\mathrm{GF}(2^r)$ denotes the field of characteristic 2 and $\mathrm{GF}(2)$ denotes the prime field of characteristic 2. The following theorem is analogous to the diagonalisation of quadratic spaces over the field of characteristic different from 2.

Proposition 2.1. [12] *Let V be a vector space over the field $\mathrm{GF}(2^r)$. Every quadratic space (V, q) has an orthogonal decomposition $V = U \oplus \mathrm{rad}(V)$ such that $(U, q|_U)$ is regular and an orthogonal sum of 2-dimensional regular quadratic spaces whereas $(\mathrm{rad}(V), q|_{\mathrm{rad}(V)})$ is orthogonal sum of 1-dimensional quadratic spaces.*

More explicitly, there exists a basis $\{e_i, f_i, g_j, 1 \leq i \leq r, 1 \leq j \leq s\}$ of V where $2r + s = \dim(V)$ and elements $a_i, b_i, c_j \in \mathbb{F}, 1 \leq i \leq r, 1 \leq j \leq s$ such that for all $v = \sum(x_i e_i + y_i f_i) + \sum z_j g_j$, we have

$$q(v) = \sum(a_i x_i^2 + x_i y_j + b_i y_i^2) + \sum c_j z_j^2.$$

We denote above quadratic form by $q = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \cdots, c_s \rangle$. This form of a quadratic form is called *normalized form.*

It immediately follows from above theorem that every regular quadratic form over fields of characteristic two is even dimensional.

Let V be a two dimensional vector space over $\mathrm{GF}(2^r)$, then there are only two regular quadratic forms $q : V \to \mathrm{GF}(2^r)$, The normalized form of two dimensional regular quadratic forms are $[0,0]$ and $[1,1]$.

Proposition 2.2. *Let $q_1 = [0,0] \perp [0,0]$ and $q_2 = [1,1] \perp [1,1]$ be two quadratic forms over $\mathrm{GF}(2)$. Then q_1 is isometric to q_2.*

Proof. We have $q_1(w,x,y,z) = wx + yz$ and $q_2(w,x,y,z) = w^2 + wx + x^2 + y^2 + yz + z^2$. The following change of variables in q_1 converts it to the form q_2: $w \mapsto x + y + z,\quad x \mapsto w + y + z,\quad y \mapsto w + x + z,\quad z \mapsto w + x + y.$ □

Remark 2.3. *Using Prop. 2.1 and Prop.2.2 , we conclude that for any $2n$-dimensional vector space V over the field $\mathrm{GF}(2^r)$, there are exactly two regular quadratic forms with normalized forms $[0,0] \perp [0,0] \perp \cdots [0,0]$ (n copies of $[0,0]$) and $[1,1] \perp [0,0] \perp [0,0] \perp \cdots [0,0]$ ($n-1$ copies of $[0,0]$).*

2.2. **Alternating bilinear forms.** We first fix the notations, for any odd prime p, $\mathrm{GF}(p^r)$ denotes the field of characteristic p and $\mathrm{GF}(p)$ denotes the prime field of characteristic p. Let V be a vector space over the field $\mathrm{GF}(p^r)$ of characteristic p, where p is an odd prime and $b : V \times V \to \mathrm{GF}(p^r)$ be a bilinear map. Then b is called *alternating* if $b(v,v) = 0$ for all $v \in V$. It is important to note that every alternating bilinear form b, we have $b(u,v) = -b(v,u)$ for all $u, v \in V$.

Now we want to find a basis of V such that b looks as nice as possible. If b is non-zero bilinear form then there exist $u, v \in V$ such that $b(u,v) = \lambda$ for some non-zero element $\lambda \in \mathbb{F}$. We choose $e_1 = u$ and $f_1 = \lambda^{-1}v$ as the first two vectors of the basis and b satisfies the following conditions:

$$b(e_1,e_1) = b(f_1,f_1) = 0; \qquad b(e_1,f_1) = -b(f_1,e_1) = 1$$

Now restricting the bilinear form to the span of $\{e_1, f_1\}^{\perp}$, we find a basis $\{e_1, e_2, \cdots e_m, f_1, f_2, \cdots f_m\}$ such that $b(e_i,e_i) = b(f_i,f_i) = 0$ for all $1 \le i \le m$, $b(e_i,f_j) = -b(f_j,e_i) = 0$ for all $i \ne j$ and $b(e_i,f_i) = -b(f_i,e_i) = 1$ for all $1 \le i \le m$. This is called *symplectic basis* of V.

Remark 2.4.

1. *If $b : V \times V \to \mathbb{F}$ is a non-degenerate alternating bilinear form then V is even dimensional vector space.*
2. *Every $2n$ dimensional non-degenerate alternating bilinear form $b : V \times V \to \mathbb{F}$ can be written as an orthogonal sum of n two dimensional bilinear spaces. Let $\{e_1, f_1, \cdots e_n, f_n\}$ be a symplectic basis of V then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ and $b = b_1 \perp b_2 \perp \cdots \perp b_n$, where*

V_i is the subspace generated by $\{e_i, f_i\}$ and b_i is restriction of b to V_i.

(3) *Let $b : V \times V \to \mathbb{F}$ is an alternating bilinear form then the map $\hat{b} : V \to \mathrm{Hom}(V, \mathrm{GF}(p^r))$ defined by $b(x)(y) = b(x, y)$ is a vector space homomorphism. Since $\ker(\hat{b}) = V^{\perp}$, The map $\hat{b}$ is an isomorphism if and only if b is a non-degenerate alternating bilinear form.*

Proposition 2.5. *Let b be a non-degenerate alternating bilinear form on V and let $k : V \to \mathrm{GF}(p^r)$ be a linear map on V . There exists a symplectic basis $\{e_1, f_1, ..., e_n, f_n\}$ on V such that $k(e_i) = k(f_i) = k(e_n) = 0$ for all $i = 1, ..., n-1$ and $k(f_n) = 1$.*

Proof. Since k is linear, the remark 2.4(3) gives that $k(x) = \hat{b}(e) = b(x, e)$ for all $x \in V$ and some fixed $e \in V$. In particular, $k(e) = b(e, e) = 0$. Since b is non-degenerate bilinear form, there exists $v \in V$ such that $b(e, v) = \lambda$, where $0 \neq \lambda \in \mathrm{GF}(p^r)$. Now choose $f = \lambda^{-1} v \in V$, then $b(e, f) = 1$ and $k(f) = 1$. Let W be the subspace of V generated by e and f . The restriction b to W is non- degenerate, so that $V = W \oplus W^{\perp}$. For $u \in W^{\perp}$, we have $k(u) = b(e, u) = 0$, so that $W^{\perp}$ is contained in $\ker k$. Now, the restriction of b to $W^{\perp}$ is also non-degenerate so that we can choose a symplectic basis $\{e_1, f_1, ..., e_{n-1}, f_{n-1}\}$ of $W^{\perp}$ and if we let $e_n = e$ and $f_n = f$. we obtain the desired basis. □

3. Extraspecial p-groups

In this section, we associate a bilinear form and quadratic form with a extraspecial 2-group. We also associate a linear map and a bilinear map with extraspecial p-group, when p is an odd prime. We show that these maps completely determine the structure of extraspecial p-groups. These maps are very helpful in classifying the extraspecial p-groups. First, we record a lemma.

Lemma 3.1. [3] *Let G be an extraspecial p-group then*

$$[g_1 g_2, g_3] = [g_1, g_3][g_2, g_3] \text{ and } (g_1 g_2)^n = g_1^n g_2^n [g_2, g_1]^{\frac{n(n-1)}{2}}.$$

Proof. (1). The first part of lemma follows from the fact that elements of G' commute with all the elements of group G as shown below:

$$\begin{aligned}[g_1 g_2, g_3] &= (g_1 g_2)^{-1} g_3^{-1} (g_1 g_2) g_3 = g_2^{-1} g_1^{-1} g_3^{-1} g_1 g_3 g_3^{-1} g_2 g_3 \\ &= g_2^{-1} [g_1, g_3] g_3^{-1} g_2 g_3 = [g_1, g_3] g_2^{-1} g_3^{-1} g_2 g_3 = [g_1, g_3][g_2, g_3].\end{aligned}$$

(2). For second part, we denote $[g_2, g_1]$ by h. We first prove that $[g_2^i, g_1^j] = h^{ij}$ for all i, j. Since $g_2^{-1} g_1^{-1} g_2 g_1 = h$, we have $g_1^{-1} g_2^i g_1 = (g_1^{-1} g_2 g_1)^i =$

$(g_2h)^i = g_2^ih^i$ as g_2 and h commute. Conjugating by g_1 gives $g_1^{-2}g_2^ig_1^2 = g_1^{-1}g_2^ih^ig_1 = g_1^{-1}g_2^ig_1h^i = g_2^ih^ih^i = g_2^ih^{2i}$. Repeating this j times, we get $g_1^{-j}g_2^ig_1^j = g_2^ih^{ij}$, thus $[g_2^i, g_1^j] = h^{ij}$. Now we prove $(g_1g_2)^n = g_1^ng_2^nh^{\frac{n(n-1)}{2}}$ by induction on n. It holds clearly for $n = 1$. Assume result holds for $n-1$. We have $[g_2, g_1^{n-1}] = h^{n-1}$ and therefore $g_2g_1^{n-1} = g_1^{n-1}g_2h^{n-1}$. Thus $(g_1g_2)^n = (g_1g_2)(g_1g_2)^{n-1} = g_1g_2g_1^{n-1}g_2^{n-1}h^{\frac{(n-1)(n-2)}{2}} = g_1^ng_2^nh^{\frac{n(n-1)}{2}}$. □

Let G be a extraspecial p-group. Then $Z(G)$ is isomorphic to the additive group $\mathbb{F}$ of field $GF(p)$, we identify $Z(G)$ with $GF(p)$. Also, $G/Z(G)$, being elementary abelian, is isomorphic to the additive group of a vector space V over $GF(p)$. We identify $G/Z(G)$ with V. Moreover, the commutator of any two elements of G, or the p power of any element of G, lies in $Z(G)$. Also we have $[g_1, g_2h] = [g_1h, g_2] = [g_1, g_2]$ and $(g_1h)^p = h^p$ for $g_1, g_2 \in G$ and $h \in Z(G)$. Thus we have two maps

(1) $b : \frac{G}{Z(G)} \times \frac{G}{Z(G)} \to \mathbb{F}$ defined by $b(\bar{g_1}, \bar{g_2}) = [g_1, g_2]$ for all $\bar{g_1}, \bar{g_2} \in \frac{G}{Z(G)}$.
(2) $q : \frac{G}{Z(G)} \times \mathbb{F}$ defined by $q(\bar{g_1}) = g_1^p$ for all $\bar{g_1} \in \frac{G}{Z(G)}$

Now Lemma 3.1(1) shows that b is bilinear. Since $[g_1, g_1] = 1$ for all $g_1 \in G$, it is alternating. The bilinear form b is non-degenerate as $\operatorname{rad} b = Z(G)$.

We now fix some notations. From now onwards unless mention otherwise, We identify the prime field $GF(p)$ with group $\mathbb{Z}/p\mathbb{Z}$ and we write $\bar{1} = z$. We also switch freely between the multiplicative and additive notions.

Lemma 3.2. *Let V be a vector space of dimension n over the prime field $GF(p)$. Let $\{v_1, v_2, \cdots v_n\}$ be a basis of V over* $\mathrm{GF}(p)$ *. Let $b : V \times V \to \mathrm{GF}(p)$ be a non-degenerate alternating bilinear form and $q : V \to \mathrm{GF}(p)$ be a map then*
$G := \{v_1, v_2, \cdots, v_n, z | z^p = 1, v_i^p = q(v_i), [v_i, z] = 1, [v_i, v_j] = b(v_i, v_j)\, \forall i, j\}$
is an extraspecial p-group.

Proof. It is clear that $Z(G)$ is generated by z and G' is generated by $b(V \times V)$. Since $b(V \times V)$ is same as group generated by z, we have $Z(G) = G'$. Also G^p is generated by $q(V)$, which is contained in group generated by z, thus $\phi(G) = G^pG'$ is same as group generated by z. This gives $Z(G) = G' = \phi(G)$ and all have order p. This proves that G is an extraspecial p-group. □

Remark 3.3. *If G is extraspecial p-group then we have a non-degenerate alternating bilinear form $b : \frac{G}{Z(G)} \times \frac{G}{Z(G)} \to \mathrm{GF}(p)$. The remark 2.4 gives that order of $\frac{G}{Z(G)}$ is p^{2n}. Since order of $Z(G)$ is p, thus order of group G is p^{2n+1}.*

4. Classification of extraspecial p-groups

In this section, we will use the theory of quadratic forms and alternating bilinear forms developed in section 2 to classify extraspecial p-groups. We first discuss extraspecial 2-groups followed by extraspecial p-groups when p is odd prime. The need of separating these cases arises because the map $q : \frac{G}{Z(G)} \to Z(G)$ defined by $q(x) = x^p$ is a quadratic map if $p = 2$ and otherwise it is a linear map. This follows from lemma 3.1(2) as p divides $\frac{p(p-1)}{2}$, when p is odd but 2 does not divide $\frac{2(2-1)}{2} = 1$.

- **Case $p = 2$:** The lemma 3.1(2) for $n = 2$, in additive notion, gives $q(g_1 + g_2) = q(g_1) + q(g_2) + b(g_1, g_2)$, which in turn gives that q is a regular quadratic form which polarizes to b. By remarks 2.3, for any $2n$-dimensional vector space V, there are exactly two regular quadratic forms up to isometry. Thus there are exactly two extraspecial 2-groups of order 2^{2n+1} up to isomorphism.
- **Case p is odd prime:** Since p divides $\frac{p(p-1)}{2}$, The lemma 3.1(2) for $n = p$, in additive notion, gives that $q(g_1 + g_2) = q(g_1) + q(g_2)$. In other words, q is a linear map. Using remark 2.4(3), the map q can be uniquely written as $q(g) = \hat{b}(a) = b(g, a)$ for some fixed $a \in \frac{G}{Z(G)}$. If $a = 1$ then q is a zero map. Otherwise if q is a non-zero map, then by Prop. 2.5, we find a symplectic basis $\{e_1, f_1, \cdots e_n, f_n\}$ such that $q(e_n) = q(e_i) = q(f_i) = 0$ for $1 \leq i \leq n-1$ and $q(f_n) = 1$. Hence there are exactly two different linear maps up to change in variables. Thus again in this case, there are exactly two extraspecial p-groups up to isomorphism of order p^{2n+1}. One corresponding to the zero-linear map and other corresponding to the non-zero linear map.

Now we explicitly compute the extraspecial p-groups of order p^3. These small extraspecial p-groups are building blocks for extraspecial p-groups of higher orders.

4.1. **Extraspecial 2-groups of order 8.** Let V be a 2-dimensional vector space over the prime field $\mathrm{GF}(2)$. Let $\{e_1, e_2\}$ be basis of V over $\mathrm{GF}(2)$. We identify $\mathrm{GF}(2)$ by $\mathbb{Z}/2\mathbb{Z}$ and write $\bar{1} = x$.

(1) Let $q_1 : V \to \mathrm{GF}(2)$ be the regular quadratic form defined by $q_1(x,y) = xy$ for $(x,y) \in V$. Using lemma 3.2, the group associated to quadratic form q is given by

$$G_1 := \langle e_1, e_2, x \mid e_1^2 = e_2^2 = x^2 = 1, [e_1, x] = [e_2, x] = 1, [e_1, e_2] = x \rangle$$

The group G_1 is isomorphic to dihedral group $D_4 := \langle a, b \mid a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$. The map $\rho : G_1 \to D_4$ defined by $\rho(e_1) = ab, \rho(e_2) = b, \rho(x) = a^2$ is a group isomorphism.

(2) Let $q_2 : V \to \mathbb{F}_2$ be other regular quadratic form defined by $q_2(x,y) = x^2 + xy + y^2$ for $(x,y) \in V$. Using lemma 3.2, the group associated to quadratic form q is given by

$$G_2 := \langle e_1, e_2, x \mid e_1^2 = e_2^2 = x, x^2 = 1, [e_1, x] = [e_2, x] = 1, [e_1, e_2] = x \rangle$$

The group G_2 is isomorphic to the quaternion group $Q_2 := \langle c, d \mid c^4 = 1, c^2 = d^2, d^{-1}cd = c^{-1} \rangle$. The map $\psi : G_2 \to Q_2$ defined by $\psi(e_1) = c, \psi(e_2) = d, \psi(x) = c^2$ is a group isomorphism.

4.2. **Extraspecial p-groups of order p^3.** Let V be a 2-dimensional vector space over the prime field $GF(p)$. Let $\{e_1, e_2\}$ be basis of a V over $\mathrm{GF}(p)$ and identify $\mathrm{GF}(p)$ by $\mathbb{Z}/p\mathbb{Z}$ and write $\bar{1} = z$.

(1) Let $q_1 : V \to \mathrm{GF}(p)$ be zero linear map, thus group associated to q_1 has exponent p. By lemma 3.2, the group associated to zero linear map q_1 is given by

$$X_{p^3}^+ := \langle e_1, e_2, z \mid e_1^p = e_2^p = z^p = 1, [e_1, z] = [e_2, z] = 1, [e_1, e_2] = z \rangle$$

(2) Let $q_2 : V \to \mathrm{GF}(p)$ be non-zero linear map, then by Prop. 2.5 we find a symplectic basis $\{e, f\}$ of V such that $q_2(e) = 0$ and $q_2(f) = 1$. By lemma 3.2, the group associated to linear map q_2 is given by

$$X_{p^3}^- := \langle e, f, z \mid e^p = 1, f^p = z, z^p = 1, [e, z] = [f, z] = 1, [e, f] = z \rangle$$

4.3. **Central products.** We first define the central product of groups. Then we show that the extraspecial p-groups is central product of extraspecial p-groups of order p^3.

Definition 4.1. *Let G_1 and G_2 be two finite groups and θ be an isomorphism between $Z(G_1)$ and $Z(G_2)$. Let $N := \{(z^{-1}, \theta(z)) \mid z \in Z(G_1)\}$. The N is a normal subgroup of direct product $G_1 \times G_2$. The central product $G_1 \circ G_2$ of groups G_1 and G_2 with respect to θ is the factor group $\frac{G_1 \times G_2}{N}$.*

Theorem 4.2. *Let G and H be two extraspecial p-groups with respect to bilinear forms $b_1 : \frac{G}{Z(G)} \times \frac{G}{Z(G)} \to Z(G)$ and $b_2 : \frac{H}{Z(H)} \times \frac{H}{Z(H)} \to Z(H)$ respectively. Let $q_1 : \frac{G}{Z(G)} \to Z(G)$ and $q_2 : \frac{H}{Z(H)} \to Z(H)$ be maps defined by $q_1(\bar{g}) = g^p$ for $\bar{g} \in \frac{G}{Z(G)}$ and $q_2(\bar{h}) = h^p$ for $\bar{h} \in \frac{H}{Z(H)}$. Then the group associated to bilinear form $b := b_1 \oplus b_2$ is isomorphic to the central product $G \circ H$ of groups G and H.*

Proof. Let G and H be extraspecial p-groups of order p^{2n+1} and p^{2m+1} respectively. Identify the elementary abelian p-group $\frac{G}{Z(G)}$ ($\frac{H}{Z(H)}$) with $2n$-dimensional vector space V ($2m$-dimensional vector space W) over the prime field GF(p), where p is a prime number. Let $\{v_1, v_2, \cdots, v_{2n}\}$ and $\{w_1, w_2, \cdots, w_{2m}\}$ be basis of vector spaces V and W respectively. We know that the centers $Z(G)$ and $Z(H)$ are isomorphic to the additive group of field GF(p). Let $\alpha : \text{GF}(p) \to Z(G)$ and $\beta : \text{GF}(p) \to Z(H)$ be the isomorphisms and $\alpha(\bar{1}) = y$ ($\beta(\bar{1}) = x$) be generators of centers $Z(G)$ ($Z(H)$), respectively.

Using lemma 3.2, The finite presentation of group G and H is given below:

$G := \{v_1, v_2, \cdots, v_{2n}, y | y^p = 1, v_i^p = q_1(v_i), [v_i, y] = 1, [v_i, v_j] = b_1(v_i, v_j)\ \forall i, j\}$

$H := \{w_1, w_2, \cdots, w_{2m}, x | x^p = 1, w_i^p = q_2(w_i), [w_i, x] = 1, [w_i, w_j] = b_2(w_i, w_j)\, \forall i, j\}$

Now let $S := V \oplus W$ be $2n + 2m$-dimensional vector space over GF(p). Let $b := \alpha(b_1) \oplus \beta(b_2) : S \times S \to \text{GF}(p)$ be a bilinear map defined by $b((v, w), (v', w')) = \alpha(b_1(v, v')) + \beta(b_2(w, w'))$ for $(v, w), (v', w') \in S$. Let $q := \alpha(q_1) \oplus \beta(q_2) : S \to \text{GF}(p)$ be a map defined by $q(v, w) = \alpha(q_1(v)) + \beta(q_2(w))$ for $(v, w) \in S$. Recall that we identify GF(p) with $\mathbb{Z}/p\mathbb{Z}$ and write $\bar{1} = z$.

The group associated to bilinear form b and map q is following:

$$K := \{v_1, v_2, \cdots, v_{2n}, w_1, w_2, \cdots, w_{2m}, z \mid z^p = 1, v_i^p = q(v_i, 0), w_i^p = q(0, w_i), [v_i, z] = 1, [w_i, z] = 1, [w_i, w_j] = b((0, w_i), (0, w_j)), [v_i, v_j] = b((v_i, 0), (v_j, 0)), [v_i, w_j] = b((v_i, 0), (0, w_j))\ \forall\ i, j\}$$

Since $b = \alpha(b_1) \oplus \beta(b_2)$, $b((v_i, 0), (0, w_j)) = \alpha(b_1(v_i, 0)) + \beta(b_2(0, w_j)) = 0$ for all i, j. Also we have $b((v_i, 0), (v_j, 0)) = \alpha(b_1(v_i, v_j)) + \beta(b_2(0, 0)) = \alpha(b_1(v_i, v_j))$ and $b((0, w_i), (0, w_j)) = \alpha(b_1(0, 0)) + \beta(b_2(w_i, w_j)) = \beta(b_2(w_i, w_j))$. Similarly the relations $v_i^p = q(v_i, 0)$ and $w_i^p = q(0, w_i)$ becomes $v_i^p = \alpha(q_1(v_i))$ and $w_i^p = \beta(q_2(w_i))$ respectively.

Thus group K can be written as following:

$$K := \{v_1, v_2, \cdots, v_{2n}, w_1, w_2, \cdots, w_{2m}, z \mid z^p = 1, v_i^p = \alpha(q_1(v_i)), w_i^p = \beta((q_2(w_i)), [v_i, y] = 1, [w_i, y] = 1, [w_i, w_j] = \beta(b_2(w_i, w_j)), [v_i, v_j] = \alpha(b_1(v_i, v_j)), [v_i, w_j] = 1 \ \forall\ i, j\}$$

Now we define a map $\gamma : G \times H \to K$ defined by $\gamma(v_i, 1) = v_i, \gamma(1, w_j) = w_j, \gamma(y, 1) = \alpha^{-1}(y) = \bar{1} = z, \gamma(0, x) = \beta^{-1}(x) = \bar{1} = z$. It is easy to check that γ is a surjective group homomorphism. We compute the kernel of homomorphism γ.

We know that $\ker \gamma := \{(g, h) \in G \times H \mid \gamma(g, h) = 1\}$. We compute $\gamma(y^{-1}, x) = \gamma((y^{-1}, 1)(1, x)) = \gamma(y^{-1}, 1)\gamma(1, x) = z^{-1}z = 1$. Also $\beta\alpha^{-1}(z) = x$. Thus $\ker \gamma = \langle (z^{-1}, \beta\alpha^{-1}(z)) \rangle$.

The map $\beta\alpha^{-1}$ is an isomorphism between $Z(G)$ and $Z(H)$. This gives that group K is isomorphic to central product $G \circ H$. □

Theorem 4.3. *Any extraspecial p-group G of order p^{2n+1} is central product of n extraspecial p-groups of order p^3. Therefore*

(1) If $p = 2$ then G is either isomorphic to $D_4 \circ D_4 \circ \cdots \circ D_4$ (n copies of D_4) or $Q_2 \circ D_4 \circ D_4 \circ \cdots \circ D_4$ ($n - 1$ copies of D_4).

(2) If p is an odd prime then G is either isomorphic to $X_{p^3}^+ \circ X_{p^3}^+ \circ \cdots \circ X_{p^3}^+$ (n copies of $X_{p^3}^+$) or $X_{p^3}^- \circ X_{p^3}^+ \circ X_{p^3}^+ \circ \cdots \circ X_{p^3}^+$ ($n-1$ copies of $X_{p^3}^+$).

The groups D_4, Q_2, $X_{p^3}^+$, $X_{p^3}^-$ are defined in section 4.1 and 4.2.

Proof. (1). By remark 2.3, we have that for $2n$-dimensional vector space over field GF(2), there are exactly two quadratic forms up to isometry, namely orthogonal sum of n copies of quadratic form $[0, 0]$ and the orthogonal sum of quadratic form $[1, 1]$ with $n - 1$ copies of quadratic form $[0, 0]$. The section 4.1 gives that the group associated to quadratic form $[0, 0]$ is D_4 and that associated to $[1, 1]$ is Q_2. Now the theorem 4.2 gives that the group associated to $[0, 0] \perp [0, 0] \perp \cdots [0, 0]$ (n copies of $[0, 0]$) is $D_4 \circ D_4 \circ \cdots \circ D_4$ (n copies of D_4) and group associated to $[1, 1] \perp [0, 0] \perp [0, 0] \perp \cdots [0, 0]$ ($n - 1$ copies of $[0, 0]$) is $Q_2 \circ D_4 \circ D_4 \circ \cdots \circ D_4$ ($n - 1$ copies of D_4).

(2). Using remark 2.4, Every $2n$ dimensional non-degenerate alternating bilinear form $b : V \times V \to \mathrm{GF}(p)$ can be written as orthogonal sum of n two dimensional bilinear spaces. Let $\{e_1, f_1, \cdots e_n, f_n\}$ be the symplectic basis of V. Then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ and $b = b_1 \perp b_2 \perp \cdots \perp b_n$, where V_i is the subspace generated by $\{e_i, f_i\}$ and b_i is the restriction of b to V_i. Similarly the linear map $q : V \to \mathbb{F}$ can be written as $q = q_1 \perp q_2 \perp \cdots \perp q_n$, where q_i is the restriction of q to V_i. Now if q is zero map on V then each q_i is a zero map on V_i. In this case theorem 4.2 and section 4.2 gives that the group G associated to linear map q is isomorphic to $X^+_{p^3} \circ X^+_{p^3} \circ \cdots \circ X^+_{p^3}$ (n copies of $X^+_{p^3}$).

If q is a non-zero linear map then by Proposition 2.5, we can find a symplectic basis $\{e_1, f_1, \cdots, e_n, f_n\}$ of vector space V such that $q(e_n) = q(e_i) = q(f_i) = 0$ for all $1 \leq i \leq n-1$ and $q(f_n) = 1$. In this case q_i for $1 \leq i \leq n-1$ are zero linear maps on vector spaces V_i, but linear map q_n on V_n is a non-zero linear map. Now theorem 4.2 and section 4.2 gives that the group G associated to the linear map q is isomorphic to $X^-_{p^3} \circ X^+_{p^3} \circ X^+_{p^3} \circ \cdots \circ X^+_{p^3}$ ($n-1$ copies of $X^+_{p^3}$). □

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INFINITE PRODUCTS USING MULTIPLICATIVE MODULUS FUNCTION

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Abstract. The usual nonnegative modulus function is based on addition. A natural different modulus function on the set of positive reals is introduced. Arguments for results for series through the usual modulus function are transformed to arguments for results for infinite products through the new modular function for multiplication. Counterparts for Riemann rearrangement theorem and some tests for convergence are derived. These counterparts are completely new results and they are different from classical results for infinite products.

1. Introduction

After giving the definition of convergence of infinite products of complex numbers, the following two results can be derived (see, e.g., [1]). (i) For a sequence $(a_n)_{n=1}^{\infty}$ of positive reals, $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges, and if and only if $\prod_{n=1}^{\infty}(1-a_n)$ converges. (ii) For a sequence $(a_n)_{n=1}^{\infty}$ of complex numbers, $\prod_{n=1}^{\infty}(1+a_n)$ converges whenever $\prod_{n=1}^{\infty}(1+|a_n|)$ converges. The convergence of $\prod_{n=1}^{\infty}(1+|a_n|)$ in the second result is called absolute convergence of $\prod_{n=1}^{\infty}(1+a_n)$. However, our expectation is that the absolute convergence of $\prod_{n=1}^{\infty}(1+a_n)$ should mean the convergence of $\prod_{n=1}^{\infty}|1+a_n|$. This expectation is to be fulfilled by introducing a natural modulus function for multiplication. The results to be obtained for infinite products use transformed arguments of arguments used for results for infinite series. A counterpart of Riemann's rearrangement theorem is to be derived. Counterparts of tests for convergence of series are to be derived for infinite products. Tests for convergence provide necessary conditions and sufficient conditions for convergence.

Definition 1.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive reals. Let $p_n = \prod_{k=1}^{n} a_k$, for every $n = 1, 2, 3, ...$If $p_n \to p$ for some real $p > 0$ as $n \to \infty$,

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then it is said that the infinite product $\prod_{n=1}^{\infty} a_n$ converges to p and it is written as $p = \prod_{n=1}^{\infty} a_n$. Otherwise, it is said that $\prod_{n=1}^{\infty} a_n$ does not converge.

Remark 1.2. The cases $p = 0$ and $p = +\infty$ are not included for convergence. A necessary condition for convergence is $a_n \to 1$ as $n \to \infty$, but this is not sufficient. A necessary and sufficient condition is given in the following known theorem. See [1].

Theorem 1.3. *An infinite product $\prod_{n=1}^{\infty} a_n$, with $a_n > 0$, $\forall n$, converges if and only if for every $\epsilon > 0$, there exists an integer n_0 such that $|a_m a_{m+1} ... a_n - 1| < \epsilon$, whenever $n > m \geq n_0$.*

Remark 1.4. Let $1 < p(1) < p(2) <...$ be a sequence of integers such that $p(n+1) - p(n) \leq M$, for some finite $M > 0$. If $a_1 a_2 ... a_{p(1)} a_{p(2)+1} a_{p(2)+2} \cdots a_{p(3)} a_{p(4)+1} \cdots a_{p(2k)+1} \cdots a_{p(2k+1)} \cdots$ and $a_{p(1)+1} a_{p(1)+2} \cdots a_{p(2)} a_{p(3)+1} a_{p(3)+2} \cdots a_{p(4)} a_{p(5)+1} \cdots a_{p(2k+1)+1} \cdots a_{p(2k+2)} \cdots$converge to r and s, then $a_1 a_2 ...$ converges to rs. This result can be derived by using Theorem 1.3. This can be done by modifying the arguments of Theorem 8.14 in [1]. For example, if $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are sequences of positive numbers such that $\prod_{n=1}^{\infty} b_n$ and $\prod_{n=1}^{\infty} c_n$ converge, then $b_1 c_1 b_2 c_2 b_3 c_3 ...$ and $b_1 c_1^{-1} b_2 c_2^{-1} b_3 c_3^{-1} ...$ converge.

2. Modulus Function for Multiplication

For a given real number x, let

$$|x|_+ = max\{x, -x\} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

Here $|x|_+$ is the usual absolute value $|x|$. Let us call $|\ |_+$ as additive absolute value function (or additive modulus function). Observe that 0 is the additive identity in the additive group of real numbers and $-x$ is the additive inverse of x. Observe also that 1 is the multiplicative identity in the multiplicative group of positive numbers, and x^{-1} is the multiplicative inverse of $x > 0$.

Definition 2.1. For a given positive real number x, let us define

$$|x|_\times = max\{x, x^{-1}\} = \begin{cases} x & \text{if } x \geq 1 \\ x^{-1} & \text{if } x \leq 1. \end{cases}$$

Let us call $|\ |_\times$ as multiplicative absolute value function (or multiplicative modulus function). Here $|x|_\times^{-1} \leq x \leq |x|_\times$. Also, $|xy|_\times \leq |x|_\times |y|_\times$ for $x > 0$ and $y > 0$.

Positive and negative parts corresponding to multiplication can also be defined with an understanding that numbers in $(0,1)$ are multiplicative negative numbers and that numbers in $(1,+\infty)$ are multiplicative positive numbers.

Lemma 2.2. *For a given $x>0$, let $p=(|x|_\times x)^{1/2}$, $q=\left(\frac{|x|_\times}{x}\right)^{1/2}$. Then $\frac{p}{q}=x$, $pq=|x|_\times$, $p\geq 1$ and $q\geq 1$.*

Proof. Direct verification. □

The numbers p and q may be considered as multiplicative positive part and multiplicative negative part of x in Lemma 2.2.

Definition 2.3. Let $(a_n)_{n=1}^\infty$ be a sequence of positive numbers. Then $\prod_{n=1}^\infty a_n$ is said to converge m-absolutely, if $\prod_{n=1}^\infty |a_n|_\times$ converges.

Lemma 2.4. *Let $(a_n)_{n=1}^\infty$ be a sequence of positive numbers such that $\prod_{n=1}^\infty |a_n|_\times$ converges. Then $\prod_{n=1}^\infty a_n$ converges and $\left(\prod_{n=1}^\infty |a_n|_\times\right)^{-1}\leq \prod_{n=1}^\infty a_n\leq \prod_{n=1}^\infty |a_n|_\times$.*

Proof. Note that for $m>n$, $\prod_{k=n}^m \frac{1}{|a_k|_\times}\leq \prod_{k=n}^m a_k\leq \prod_{k=n}^m |a_k|_\times$. By Theorem 1.3, $\prod_{n=1}^\infty a_n$ converges and $0<\left(\prod_{n=1}^\infty |a_n|_\times\right)^{-1}\leq \prod_{n=1}^\infty a_n\leq \prod_{n=1}^\infty |a_n|_\times<\infty$. □

Lemma 2.4 may be roughly stated as "m-absolute convergence implies convergence".

Lemma 2.5. *Let $(a_n)_{n=1}^\infty$ be a sequence of positive reals converging to a positive number K. Let $(t_n)_{n=1}^\infty$ be a sequence of positive reals such that $\sum_{n=1}^\infty t_n=+\infty$. Then $(a_1^{t_1}a_2^{t_2}...a_n^{t_n})^{\frac{1}{t_1+t_2+...+t_n}}\to K$ as $n\to\infty$.*

Proof. Without loss of generality, let us assume that $K=1$. Let $\epsilon>0$ be given. Find an integer k such that $|a_n|_\times<(1+\epsilon)$, $\forall n\geq k$. Find $M>0$ such that $|a_n|_\times\leq M<\infty$, $\forall n$. Then

$$\begin{aligned}1&\leq (a_1^{t_1}a_2^{t_2}...a_n^{t_n})^{\frac{1}{t_1+t_2+...+t_n}}\\&\leq (|a_1|_\times^{t_1}...|a_k|_\times^{t_k})^{\frac{1}{t_1+t_2+...+t_n}}(|a_{k+1}|_\times^{t_{k+1}}...|a_n|_\times^{t_n})^{\frac{1}{t_1+t_2+...+t_n}}\\&\leq M^{\frac{t_1+...+t_k}{t_1+t_2+...+t_n}}(1+\epsilon)^{\frac{t_{k+1}+...+t_n}{t_1+t_2+...+t_n}}.\end{aligned}$$

Then, the right hand side tends to $1+\epsilon$ as $n\to\infty$, for every $\epsilon>0$. Thus, $a_1^{t_1}a_2^{t_2}...a_n^{t_n}\to 1$ as $n\to\infty$. This proves the result. □

3. Rearrangements

The first Theorem 3.1 is a transformed version of the classical Riemann rearrangement theorem. Let us use the word "rearrangement" in the usual sense.

Theorem 3.1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\prod_{n=1}^{\infty} a_n$ converges and $\prod_{n=1}^{\infty} |a_n|_\times$ does not converge. Suppose $0 \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\prod_{n=1}^{\infty} a'_n$ with partial products $u'_n = \prod_{k=1}^{n} a'_k$ such that $\liminf\limits_{n\to\infty} u'_n = \alpha$ and $\limsup_{n\to\infty} u'_n = \beta$.*

Proof. Let $p_n = (|a_n|_\times a_n)^{1/2}$ and $q_n = \left(\frac{|a_n|_\times}{a_n}\right)^{1/2}$, for every n. By Lemma 2.2, $\frac{p_n}{q_n} = a_n, p_n q_n = |a_n|_\times, p_n \geq 1$ and $q_n \geq 1$, for every n. Since $\prod_{n=1}^{\infty}(p_n q_n) = \prod\limits_{n=1}^{\infty} |a_n|_\times$, either $\prod_{k=1}^{n} p_k \to +\infty$ as $n \to \infty$ or $\prod_{k=1}^{n} q_k \to \infty$ as $n \to \infty$. Since $\frac{\prod_{k=1}^{n} p_k}{\prod_{k=1}^{n} q_k} = \prod_{k=1}^{n} a_k$ converges as $n \to \infty$, if $\prod_{k=1}^{n} p_k$ converges as $n \to \infty$, then $\prod_{k=1}^{n} q_k$ converges as $n \to \infty$, and similarly, if $\prod_{k=1}^{n} q_k$ converges as $n \to \infty$, then $\prod_{k=1}^{n} p_k$ converges as $n \to \infty$. Hence $\prod_{k=1}^{n} p_k \to \infty$ and $\prod_{k=1}^{n} q_k \to \infty$ as $n \to \infty$. Now, let $p'_1, p'_2, ...$ denote the factors of $\prod_{n=1}^{\infty} a_n$ which are greater than or equal to 1 in the order in which they occur, and let $q'_1, q'_2, ...$ be the multiplicative absolute values of the remaining factors of $\prod_{n=1}^{\infty} a_n$ in their original order. The products $\prod_{n=1}^{\infty} p'_n$ and $\prod_{n=1}^{\infty} q'_n$ are different from $\prod_{n=1}^{\infty} p_n$ and $\prod_{n=1}^{\infty} q_n$ only by factors 1, and therefore $\prod_{k=1}^{n} p'_k \to \infty$ and $\prod_{k=1}^{n} q'_k \to \infty$ as $n \to \infty$. Choose sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ of positive numbers such that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ as $n \to \infty, \alpha_n < \beta_n$, $\forall n$, and $\beta_1 \geq 1$. Let m_1, k_1 be the smallest integers such that $p'_1 p'_2 ... p'_{m_1} > \beta_1$, $p'_1 p'_2 ... p'_{m_1} q'^{-1}_1 q'^{-1}_2 ... q'^{-1}_{k_1} < \alpha_1$; let m_2, k_2 be the smallest integers such that $p'_1 p'_2 ... p'_{m_1} q'^{-1}_1 q'^{-1}_2 ... q'^{-1}_{k_1} p'_{m_1+1} ... p'_{m_2} > \beta_2$, $p'_1 p'_2 ... p'_{m_1} q'^{-1}_1 q'^{-1}_2 ... q'^{-1}_{k_1} p'_{m_1+1} ... p'_{m_2} q'^{-1}_{k_1+1} ... q'^{-1}_{k_2} < \alpha_2$; and let us continue in this way. Let x_n, y_n denote the partial products of the rearrangement $p'_1 p'_2 ... p'_{m_1} q'^{-1}_1 q'^{-1}_2 ... q'^{-1}_{k_1} p'_{m_1+1} ... p'_{m_2} q'^{-1}_{k_1+1} ... q'^{-1}_{k_2} p'_{m_2+1} \cdots$ whose last factors are $p'_{m_n}, q'^{-1}_{k_n}$. Then $\frac{x_n}{p'_{m_n}} \leq \beta_n$ and $\frac{y_n}{q'^{-1}_{k_n}} \geq \alpha_n$ so that $p'^{-1}_{m_n} \leq \frac{\beta_n}{x_n} \leq 1$ and $1 \leq \frac{\alpha_n}{y_n} \leq q'_{k_n}$. Since $p'_n \to 1$ and $q'_n \to 1$ as $n \to \infty$, then $x_n \to \beta$ and $y_n \to \alpha$ as $n \to \infty$. From our construction, if u'_n is the n-th partial product of the constructed rearrangement, then it is clear that $\liminf\limits_{n\to\infty} u'_n = \alpha$ and $\limsup\limits_{n\to\infty} u'_n = \beta$. □

Remark 3.2. It should be observed from this Theorem 3.1 that if every rearrangement converges then m-absolute convergence holds.

Theorem 3.3. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\prod_{n=1}^{\infty} |a_n|_\times$ converges. Then every rearrangement of $\prod_{n=1}^{\infty} a_n$ converges and all of them converge to the same sum.*

Proof. Let $M = \prod_{n=1}^{\infty} |a_n|_\times$. Let $u_n = \prod_{k=1}^{n} a_k$ and $u_n' = \prod_{k=1}^{n} a_k'$ be partial products of $\prod_{k=1}^{\infty} a_k$ and its one rearrangement $\prod_{k=1}^{\infty} a_k'$. To each n, let $A_n = \left\{k \in \{1,2,...,n\} : |a_k|_\times = |a_k'|_\times\right\}$ and $B_n = \{1,2,...,n\} \setminus A_n$. Then

$$0 \le \Big|\prod_{k=1}^{n} |a_k|_\times - \prod_{k=1}^{n} |a_k'|_\times\Big|_+ \le M\Big|\prod_{k\in B_n} |a_k|_\times - \prod_{k\in B_n} |a_k'|_\times\Big|_+ \le M\left[\Big|1 - \prod_{k\in B_n} |a_k|_\times\Big|_+ + \Big|1 - \prod_{k\in B_n} |a_k'|_\times\Big|_+\right].$$

By Theorem 1.3, the right hand side tends to zero as $n \longrightarrow \infty$, because $|a_k|_\times \ge 1$ and $|a_k'|_\times \ge 1$, $\forall k$. Therefore $\prod_{k=1}^{\infty} |a_k|_\times = \prod_{k=1}^{\infty} |a_k'|_\times$, and $\prod_{k=1}^{\infty} |a_k'|_\times$ converges. By Lemma 2.4, $\prod_{k=1}^{\infty} a_k'$ converges. In the previous arguments, for B_n defined above, let us observe that $\Big|1 - \prod_{k\in B_n} |a_k|_\times^{-1}\Big|_+ + \Big|1 - \prod_{k\in B_n} |a_k'|_\times\Big|_+$ tends to zero as $n \to \infty$. This is applicable even for subsets of B_n. So, if $C_n = \left\{k \in \{1,2,...,n\} : a_k = a_k'\right\}$ and $D_n = \{1,2,...,n\} \setminus C_n$, then $\Big|1 - \prod_{k\in D_n} a_k\Big|_+ + \Big|1 - \prod_{k\in D_n} a_k'\Big|_+ \to 0$ as $n \to \infty$, because $\prod_{k\in D_n} |a_k|_\times^{-1} \le \prod_{k\in D_n} a_k \le \prod_{k\in D_n} |a_k|_\times$ and $\prod_{k\in D_n} |a_k'|_\times^{-1} \le \prod_{k\in D_n} a_k' \le \prod_{k\in D_n} |a_k'|_\times$. Thus

$$0 \le \Big|\prod_{k=1}^{n} a_k - \prod_{k=1}^{n} a_k'\Big|_+ \le M\left[\Big|1 - \prod_{k\in D_n} a_k\Big|_+ + \Big|1 - \prod_{k\in D_n} a_k'\Big|_+\right]$$

implies that $\prod_{k=1}^{\infty} a_k = \prod_{k=1}^{\infty} a_k'$. □

Definition 3.4. A product $\prod_{k=1}^{\infty} b_k$ is called a subproduct of $\prod_{k=1}^{\infty} a_k$ if $b_k = a_k$ or 1 for all k, and $b_k = a_k$ for infinitely many k.

Theorem 3.5. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\prod_{n=1}^{\infty} |a_n|_\times$ converges. Consider a class of all products of the form $\prod_{n=1}^{\infty} a_n^{c_n}$, where $-1 \le c_n \le 1$, $\forall n$. Then all these products converge uniformly in the following sense. For every $\epsilon > 0$, there is an integer n_0, which is common for all sequences $(c_n)_{n=1}^{\infty}$ satisfying $-1 \le c_n \le 1$, $\forall n$, such that $\Big|1 - \prod_{k=n}^{m} a_k^{c_k}\Big|_+ < \epsilon$, $\forall m, n \ge n_0$ satisfying $m > n$, and so that $\Big|1 - \prod_{k=n}^{\infty} a_k^{c_k}\Big|_+ \le \epsilon$, $\forall n \ge n_0$.*

Proof. $\prod_{k=n}^{m} |a_k|_\times^{-1} \le \prod_{k=n}^{m} a_k^{c_k} \le \prod_{k=n}^{m} |a_k|_\times$, when $-1 \le c_k \le 1$, $\forall k$, and when $m > n$. Thus

$$\left|1 - \prod_{k=n}^{m} a_k^{c_k}\right|_+ \le \max\left\{\left|1 - \prod_{k=n}^{m} |a_k|_\times^{-1}\right|_+, \left|1 - \prod_{k=n}^{m} |a_k|_\times\right|_+\right\}.$$

This observation proves the result, when Theorem 1.3 is applied. □

Corollary 3.6. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\prod_{n=1}^{\infty} |a_n|_\times$ converges. Then, all subproducts $\prod_{k=1}^{\infty} b_k$ of $\prod_{k=1}^{\infty} a_k$ converge uniformly in the following sense. For every $\epsilon > 0$, there is an integer n_0, which is common for all subproducts $\prod_{k=1}^{\infty} b_k$, such that $\left|1-\prod_{k=n}^{m} b_k\right|_+ < \epsilon$, $\forall m, n \ge n_0$ satisfying $m > n$, and so that $\left|1 - \prod_{k=n}^{\infty} b_k\right|_+ \le \epsilon$, $\forall n \ge n_0$.*

Theorem 3.5 is placed in this section, because its proof uses arguments which are used in the proof of Theorem 3.3. The next section is also devoted to rearrangements in terms of unordered products.

4. Unordered Products

Let I be an infinite set. In this section, E, E_1, E_2, F, F_1, F_2 are notations to be used for finite subsets of I with or without mentioning finiteness of them. Let $(a_i)_{i\in I}$ be a collection of positive numbers. Let $I_1 = \{i \in I : a_i > 1\}$, $I_2 = \{i \in I : a_i < 1\}$ and $I_3 = \{i \in I : a_i = 1\}$. These notations are also fixed in this section. Let us say that $\prod_{i\in I} a_i$ converges, if there is a number $p \in (0, \infty)$ such that for given $\epsilon > 0$, there is a finite set $F \subseteq I$ such that $\left|\prod_{i\in E} a_i - p\right|_+ < \epsilon$, for every finite set E satisfying $F \subseteq E \subseteq I$. In this case, let us say that $\prod_{i\in I} a_i$ converges to p and let us write $\prod_{i\in I} a_i = p$. Equivalently, this convergence happens when and only when for given $\epsilon > 0$, there is a set $F \subseteq I$ such that $\left|\frac{\prod_{i\in E} a_i}{p} - 1\right|_+ < \epsilon$, for every E satisfying $F \subseteq E \subseteq I$. One may find arguments, which are necessary to prove Theorem 1.3, in the proof of the next lemma.

Lemma 4.1. *$\prod_{i\in I} a_i$ converges if and only if for given $\epsilon > 0$, there is a finite set $F \subseteq I$ such that $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I \setminus F$.*

Proof. Suppose $\prod_{i\in I} a_i$ converges to some $p > 0$. Find $F_1 \subseteq I$ such that $\left|\prod_{i\in E} a_i - p\right|_+ < \frac{p}{4}$, whenever $F_1 \subseteq E \subseteq I$, and so that $0 < \frac{3}{4}p < \prod_{i\in E} a_i < \frac{5}{4}p$. For given $\epsilon > 0$, there is a set $F_2 \supseteq F_1$ such that $\left|\prod_{i\in E} a_i - \prod_{i\in F_2} a_i\right|_+ < \frac{3}{4}p\epsilon$ whenever $E \supseteq F_2$, and so that $\left|\prod_{i\in E\setminus F_2} a_i - 1\right|_+ < \frac{3p\epsilon}{4\prod_{i\in F_2} a_i} < \epsilon$. That is, $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I \setminus F_2$.

Conversely, assume that for given $\epsilon > 0$, there is a set F such that $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I \setminus F$. In particular, there is a set F_1 such that $\left|\prod_{i\in E} a_i - 1\right|_+ < \frac{1}{4}$, whenever $E \subseteq I \setminus F_1$, and so that $\frac{3}{4} < \prod_{i\in E} a_i < \frac{5}{4}$. Therefore, $\frac{3}{4}M < \prod_{i\in E} a_i < \frac{5}{4}M$, for any $E \subseteq I$, where $M = \prod_{i\in F_1} a_i$. Fix $\epsilon > 0$. Find a set $F_2 \supseteq F_1$ such that $\left|\prod_{i\in E} a_i - 1\right|_+ < \frac{3M\epsilon}{4}$, whenever $E \subseteq I \setminus F_2$. Then, for $F_2 \subseteq E_2 \subseteq E_1$, $\left|\prod_{i\in E_1} a_i - \prod_{i\in E_2} a_i\right|_+ = \left|\prod_{i\in E_1\setminus E_2} a_i - 1\right|_+ \left|\prod_{i\in E_2} a_i\right|_+^{-1} < \epsilon$. Therefore, by triangle inequality, $\left|\prod_{i\in E_1} a_i - \prod_{i\in E_2} a_i\right|_+ < 2\epsilon$, whenever $F_2 \subseteq E_1$ and $F_2 \subseteq E_2$. Hence $\prod_{i\in I} a_i$ converges. □

The following inequality will be used at the end of the proof of the next lemma. $|1 - ab|_+ \leq |1 - a|_+ + |a|_+|1 - b|_+$.

Lemma 4.2. *$\prod_{i\in I} a_i$ converges if and only if $\prod_{i\in I_1} a_i$ converges and $\prod_{i\in I_2} a_i$ converges. (Convention: Finite products converge). Moreover, $\prod_{i\in I} a_i = \left(\prod_{i\in I_1} a_i\right)\left(\prod_{i\in I_2} a_i\right)$, in this case.*

Proof. Suppose $\prod_{i\in I} a_i$ converges. Fix $\epsilon > 0$. Then there is a set F such that $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I \setminus F$. In particular, $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I_1 \setminus (I_1 \cap F)$, and $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I_2 \setminus (I_2 \cap F)$. This proves that $\prod_{i\in I_1} a_i$ and $\prod_{i\in I_2} a_i$ converge. Since $\prod_{i\in E} a_i = \left(\prod_{i\in E\cap I_1} a_i\right)\left(\prod_{i\in E\cap I_2} a_i\right)$ for any E and multiplication is continuous in the real line, then $\prod_{i\in I} a_i = \left(\prod_{i\in I_1} a_i\right)\left(\prod_{i\in I_2} a_i\right)$. This continuity of multiplication should be observed in terms of nets over directed sets of the finite subsets of I_1, I_2, and I.

Conversely assume that $\prod_{i\in I_1} a_i$ and $\prod_{i\in I_2} a_i$ converge. Fix $\epsilon > 0$. There are F_1, F_2 such that $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I_1 \setminus F_1$, and $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon$, whenever $E \subseteq I_2 \setminus F_2$. Then $\left|\prod_{i\in E} a_i - 1\right|_+ < \epsilon + (1+\epsilon)\epsilon$, whenever $E \subseteq I \setminus F_1 \cup F_2$. This proves that $\prod_{i\in I} a_i$ converges. □

Remark 4.3. Suppose $\prod\limits_{i\in I} a_i$ converges. For each n, let $G_n = \{i \in I : a_i < 1 - \frac{1}{n}\}$ and $H_n = \{i \in I : a_i > 1 + \frac{1}{n}\}$. Then G_n and H_n are finite sets, because $\prod\limits_{i\in I_1} a_i$ and $\prod\limits_{i\in I_2} a_i$ converge. So, $I_1 = \bigcup\limits_{n=1}^{\infty} H_n$ and $I_2 = \bigcup\limits_{n=1}^{\infty} G_n$ are countable sets. Thus, let us replace I by the countable set $I_1 \cup I_2$, when there is a need for simplification in notation in connection with convergence.

On some occasions, it will be fixed as $I = \{1, 2, ...\}$. It should be observed that convergence of all these products should be explained only in terms of nets over directed sets of the finite subsets of I_1, I_2, and I.

Theorem 4.4. *The followings are equivalent.*
(a) $\prod_{i\in I} |a_i|_\times$ *converges.*
(b) $\prod_{i\in I} a_i$ *converges.*
(c) Let D *be a directed set which is a cofinal subset of the directed set* $\{E : E$ *is a finite subset of* $I\}$ *under the inclusion relation. That is, if* E *is any finite subset of* I*, then there is a set* $F \in D$ *such that* $E \subseteq F$*. (See [2] for terminology). Then the net* $\left(\prod_{i\in F} |a_i|_\times\right)_{F\in D}$ *converges for any* D *mentioned above. Moreover, all these nets converge to* $\prod_{i\in I} |a_i|_\times$*.*
(d) The net $\left(\prod_{i\in F} a_i\right)_{F\in D}$ *converges for any* D *mentioned in (c). Moreover, all these nets converge to* $\prod_{i\in I} a_i$*.*
(e) Consider I *in the form* $\{1, 2, ...\}$ *as it was mentioned in the previous Remark 4.3, on assuming (a) or (b). Write* I *in the form* $I = J_1 \cup J_2 \cup ....$ *in which* J_i *are pairwise disjoint infinite sets. Let us write each* J_i *in the ordered form* $J_i = \{i_1, i_2, ...\}$*. Then for each* i*, the product* $\prod_{j=1}^{\infty} a_{i_j}$ *converges, and then the product* $\prod_{i=1}^{\infty}\left(\prod_{j=1}^{\infty} a_{i_j}\right)$ *converges, for any decomposition* $J_1 \cup J_2 \cup ....$ *of* I*. In this case, they all converge to* $\prod_{i\in I} a_i$*.*

Note*: The statement in part (e) is given in this specific form only for the purpose of the implication: (a) or (b) implies (e). For the reverse implication, let us take* I *as a countable set.*

Proof. Suppose $\prod_{i\in I} |a_i|_\times$ converges. Then for any finite subset F of I,

$$\left(\prod_{i\in F} |a_i|_\times\right)^{-1} \le \prod_{i\in F} a_i \le \prod_{i\in F} |a_i|_\times,$$

and Lemma 4.1 implies that $\prod_{i\in I} a_i$ converges.

Conversely assume that $\prod_{i\in I} a_i$ converges. Then $\prod_{i\in I_1} a_i$ and $\prod_{i\in I_2} a_i$ converge, by Lemma 4.2. So, $\prod_{i\in I_1} |a_i|_\times$ and $\prod_{i\in I_2} |a_i|_\times$ converge. By continuity of multiplication, $\prod_{i\in I} |a_i|_\times$ converges.

Thus (a) and (b) are equivalent. From the definition for net convergence, (a), (b), (c), (d) are equivalent, and common limits for nets also do exist for (c) and (d).

To prove the equivalence of (a) and (e), suppose that $\prod_{i\in I} |a_i|_\times$ converges. By Lemma 4.1, and Theorem 1.3, $\prod_{j=1}^{\infty} |a_{i_j}|_\times$ converges for each i, when the notations given in (e) are used. Moreover,

$$1 \le \prod_{i=m}^{n} \prod_{j=1}^{\infty} |a_{i_j}|_\times \le \prod_{i\in I} |a_i|_\times,$$

for any n, m satisfying $n > m$. Again by Lemma 4.1 and Theorem 1.3, it can be observed that $\prod_{i=1}^{\infty}\left(\prod_{j=1}^{\infty}|a_{i_j}|_\times\right)$ converges. The inequalities, for $n > m$,

$$\left(\prod_{j=m}^{n}|a_{i_j}|_\times\right)^{-1} \le \prod_{j=m}^{n} a_{i_j} \le \prod_{j=m}^{n}|a_{i_j}|_\times$$

give the convergence of $\prod_{j=1}^{\infty} a_{i_j}$, by Theorem 1.3. Similarly, the inequalities, for $n > m$,

$$\left(\prod_{i=m}^{n}\left(\prod_{j=1}^{\infty}|a_{i_j}|_\times\right)\right)^{-1} \le \prod_{i=m}^{n}\prod_{j=1}^{\infty} a_{i_j} \le \prod_{i=m}^{n}\prod_{j=1}^{\infty}|a_{i_j}|_\times$$

give the convergence of $\prod_{i=1}^{\infty}\prod_{j=1}^{\infty} a_{i_j}$.

To prove the converse part, assume that $\prod_{i\in I}|a_i|_\times$ does not converge, or equivalently $\prod_{n=1}^{\infty}|a_n|_\times$ does not converge, when I is considered in the form $\{1,2,...\} = I_1 \cup I_2$ as stated in the statement for (e). Then either $\prod_{i\in I_1}|a_i|_\times = +\infty$ or $\prod_{i\in I_2}|a_i|_\times = +\infty$. For example, let us consider the case $\prod_{i\in I_1}|a_i|_\times = +\infty$. In this case, let us find an infinite subset $J_1 = \{1_1, 1_2, ...\}$ of I such that $I_1 \setminus J_1$ is infinite and such that $\prod_{j=1}^{\infty} a_{1_j}$ does not converge. This is possible in view of Remark 3.2 and Remark 1.4. Write $(I_1 \setminus J_1) \cup I_2$ in the form $J_2 \cup J_2 \cup ...$ such that each $J_i = \{i_1, i_2, ...\}$ is an infinite set and such that J_i are pairwise disjoint. Then $\prod_{i=1}^{\infty}\prod_{j=1}^{\infty} a_{i_j}$ does not converge.

It remains to establish the final part of (e). Let $p = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} a_{i_j}$ for some partition $I = J_1 \cup J_2 \cup ....$ For each n, it is possible to find integers $m(n), n_1, n_2, ..., n_{m(n)}$ such that $\left|\prod_{i=1}^{m(n)}\prod_{j=1}^{n_i} a_{i_j} - p\right|_+ < \frac{1}{n}$, and such that $k_1 > n_1, k_2 > n_2, ..., k_{m(n)} > n_{m(n)}$, $m(k) > m(n)$, whenever $k > n$. Let $D = \Big\{\{i_j : 1 \le j \le n_i, 1 \le i \le m(n)\} : n = 1, 2, ...\Big\}$. Consider D as a directed set under inclusion relation, and when I is considered as $\{i_j : j = 1, 2, ..., i = 1, 2, ...\}$. Now the final part of (d) implies that $p = \prod_{i\in I} a_i$. The proof is now completed. ☐

Remark 4.5. In the previous Theorem 4.4, (e) includes iterated convergence for double products. The part (d) includes some special types of net convergence meant for double products corresponding to the one known for double series in general forms. References for such general forms may be found in [3]. By Theorem 3.3 and Remark 3.2, it can be stated that all parts (a), (b), (c), (d),(e) are equivalent to one more part. (f): Consider I in the form $\{1, 2, ...\}$ (as it was done in (e)). Then all rearrangements $\prod_{n=1}^{\infty} a_n$ of $\prod_{i\in I} a_i$ converge. Moreover, they all converge to $\prod_{i\in I} a_i$.

5. Tests For Convergence

If a necessary condition for convergence is not satisfied then convergence fails. If a sufficient condition for convergence is satisfied then convergence is assured. All results giving conditions for convergence are classified under tests for convergence. For example, if $\limsup_{n\to\infty} |\log a_n|_+^{\frac{1}{n}} < 1$ then $\prod_{n=1}^{\infty} a_n$ converges, formally, according to the root test for series convergence. This statement can be made into a logically correct statement. But, it is not our aim. Our aim is to use necessary transformations in arguments of proofs of results. Let us consider again countable infinite products.

Proposition 5.1. *Let $(t_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ be sequences of positive numbers such that $\sum_{n=1}^{\infty} t_n$ converges and such that $\limsup_{n\to\infty} a_n^{\frac{1}{t_n}} = \beta < \infty$ and $\liminf_{n\to\infty} a_n^{\frac{1}{t_n}} = \alpha > 0$. Then $\prod_{n=1}^{\infty} a_n$ converges.*

Proof. Fix α' and β' such that $0 < \alpha' < \alpha \leq \beta < \beta' < \infty$. Then there is an integer n_0 such that $\alpha' < a_n^{\frac{1}{t_n}} < \beta'$, $\forall\ n \geq n_0$. Then, for $n > m \geq n_0$,

$$\alpha'^{\sum_{k=m}^{n} t_k} < \prod_{k=m}^{n} a_n < \beta'^{\sum_{k=m}^{n} t_k}.$$

Now, the proposition follows from Theorem 1.3. □

Corollary 5.2. *Let $(t_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} t_n$ converges. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $a_n \geq 1$, $\forall n$, and such that $\limsup_{n\to\infty} a_n^{\frac{1}{t_n}} < \infty$. Then $\prod_{n=1}^{\infty} a_n$ converges. Here "$a_n \geq 1$" and "$\limsup_{n\to\infty} a_n^{\frac{1}{t_n}} < \infty$" can be replaced by "$a_n \leq 1$" and "$\liminf_{n\to\infty} a_n^{\frac{1}{t_n}} > 0$" to get the same conclusion: $\prod_{n=1}^{\infty} a_n$ converges.*

Theorem 5.3. *Suppose $a_1 \geq a_2 \geq ... \geq 1$. Then $\prod_{n=1}^{\infty} a_n$ converges if and only if $\prod_{k=0}^{\infty} a_{2^k}^{2^k} = a_1^1 a_2^2 a_4^4 a_8^8 ....$ converges.*

Proof. Let $u_n = a_1 a_2 ... a_n$, and $v_k = a_{2^0}^{2^0} a_{2^1}^{2^1} a_{2^2}^{2^2} .... a_{2^k}^{2^k}$. For $n < 2^k$,

$$u_n \leq a_1(a_2 a_3)...(a_{2^k} a_{2^k+1} ... a_{2^{k+1}-1}) \leq a_1^1 a_2^2 a_4^4 ... a_{2^k}^{2^k} = v_k$$

so that $u_n \leq v_k$. On the other hand, if $n > 2^k$,

$$u_n \geq a_1^{\frac{1}{2}} a_2(a_3 a_4)...(a_{2^{k-1}-1} ... a_{2^k}) \geq (a_1^1 a_2^2 a_4^4 ... a_{2^k}^{2^k})^{\frac{1}{2}} = v_k^{\frac{1}{2}}$$

so that $u_n^2 \geq v_k$. These two inequalities establish the result, because $a_n \geq 1$, $\forall n$. □

Definition 5.4. Suppose $a_n > 1$, for each n. The product $a_1 a_2^{-1} a_3 a_4^{-1} a_5 a_6^{-1} ....$ is called an alternating product.

Theorem 5.5. *If $(a_n)_{n=1}^{\infty}$ is a strictly decreasing sequence converging to 1, then the alternating product $a_1a_2^{-1}a_3a_4^{-1}a_5a_6^{-1}...$ converges.*

Proof. Let $b_1 = a_1a_2^{-1}, b_2 = a_3a_4^{-1}, b_3 = a_5a_6^{-1}$,... .Then $b_n > 1$, $\forall n$. Hence, $\prod_{n=1}^{\infty} b_n$ converges if and only if $(\prod_{k=1}^{n} b_k)_{n=1}^{\infty}$ is a bounded sequence. Let us observe that $\prod_{k=1}^{n} b_k = a_1(\frac{a_2}{a_3})^{-1}(\frac{a_4}{a_5})^{-1}...(\frac{a_{2n-2}}{a_{2n-1}})^{-1}a_{2n}^{-1} < a_1$, because $(\frac{a_2}{a_3})^{-1}, (\frac{a_4}{a_5})^{-1}, ...(\frac{a_{2n-1}}{a_{2n-1}})^{-1}, a_{2n}^{-1}$ are less than 1. So, $\prod_{n=1}^{\infty} b_n$ converges. The value of $a_1a_2^{-1}a_3a_4^{-1}a_5a_6^{-1}...a_m^{(-1)^{m+1}}$ is $\prod_{k=1}^{\frac{m}{2}} b_k$ if m is even, and it is $\left(\prod_{k=1}^{\frac{m-1}{2}} b_k\right)a_m$ if m is odd. So, $a_1a_2^{-1}a_3a_4^{-1}...$ converges to $\prod_{n=1}^{\infty} b_n$, because $a_m \to 1$ as $m \to \infty$. □

Corollary 5.6. *If $(a_n)_{n=1}^{\infty}$ is a strictly decreasing sequence in $(0, 1)$ which converges to zero, then $(1 - a_1)(1 - a_2)^{-1}(1 - a_3)(1 - a_4)^{-1}...$ converges.*

Proof. Here, $((1 - a_n)^{-1})_{n=1}^{\infty}$ is a strictly decreasing sequence, which converges to 1. □

Theorem 5.7. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\prod_{n=1}^{\infty} a_n$ converges to p. Let $(t_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $t_1+t_2+... = +\infty$. Let $u_n = \prod_{k=1}^{n} a_k$, and $\sigma_n = (u_1^{t_1}u_2^{t_2}...u_n^{t_n})^{\frac{1}{t_1+t_2+...+t_n}}$, $\forall n$. Then σ_n converges to p as $n \to \infty$.*

Proof. Let $v_n = \frac{u_n}{p}$ and $\tau_n = \frac{\sigma_n}{p}$, $\forall n$. Then $(v_1^{t_1}v_2^{t_2}...v_n^{t_n})^{\frac{1}{t_1+t_2+...+t_n}} = \frac{\sigma_n}{p}$, $\forall n$. Choose a finite positive number A such that $|v_n|_\times \leq A$, $\forall n$. Given $\epsilon > 0$, choose an integer k such that $|v_n|_\times \leq 1 + \epsilon$, $\forall n > k$. Then

$$1 \leq |\tau_n|_\times = \left|\frac{\sigma_n}{p}\right|_\times \leq (|v_1|_\times^{t_1}|v_2|_\times^{t_2}...|v_k|_\times^{t_k})^{\frac{1}{t_1+t_2+...+t_n}}$$
$$(|v_{k+1}|_\times^{t_{k+1}}...|v_n|_\times^{t_n})^{\frac{1}{t_1+t_2+...+t_n}} \leq A^{\frac{t_1+t_2+...+t_k}{t_1+t_2+...+t_n}}(1 + \epsilon)^{\frac{t_{k+1}+t_2+...+t_n}{t_1+t_2+...+t_n}}.$$

Thus $1 \leq |\tau_n|_\times \leq 1 + 2\epsilon$ for sufficiently large n, because $t_1 + t_2 + ... = \infty$. This proves that $\lim_{n\to\infty} |\tau_n|_\times = 1 = \lim_{n\to\infty} \tau_n$. So, $\lim_{n\to\infty} \sigma_n = p$. □

Corollary 5.8. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\prod_{n=1}^{\infty} a_n$ converges to p. Let $u_n = \prod_{k=1}^{n} a_k$, and $\sigma_n = (u_1u_2...u_n)^{\frac{1}{n}}$, $\forall n$. Then σ_n converges to p as $n \to \infty$.*

Theorem 5.9. *Suppose $(b_n)_{n=1}^{\infty}$ be a sequence of positive reals such that $\prod_{n=1}^{\infty} \frac{b_n}{b_{n+1}}$ converges m-absolutely and such that $b_n \to 1$ as $n \to \infty$. Suppose $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $\left\{\sum_{k=n}^{m} a_k : m = 1, 2, ..., n = 1, 2, ..., n \leq m\right\}$ is a bounded set. Then $\prod_{k=1}^{\infty} b_k^{a_k}$ converges.*

Proof. Suppose $\left|\sum_{k=n}^{m} a_k\right|_+ \leq M$, $\forall\ n = 1, 2, ...$, $\forall m = 1, 2, ...$, subject to the condition $n \leq m$, for some $M > 0$. Note that, for $n > 2$,

$$b_1^{a_1} b_2^{a_2} ... b_n^{a_n} = \left(\frac{b_1}{b_2}\right)^{a_1} \left(\frac{b_2}{b_3}\right)^{a_1+a_2} ... \left(\frac{b_{n-1}}{b_n}\right)^{a_1+a_2+...+a_{n-1}} b_n^{a_1+a_2+...+a_n}$$

Then

$$\left(\left|\frac{b_1}{b_2}\right|_\times \left|\frac{b_2}{b_3}\right|_\times ... \left|\frac{b_{n-1}}{b_n}\right|_\times\right)^{-M} |b_n|_\times^{-M} \leq b_1^{a_1} b_2^{a_2} ... b_n^{a_n}$$
$$\leq \left(\left|\frac{b_1}{b_2}\right|_\times \left|\frac{b_2}{b_3}\right|_\times ... \left|\frac{b_{n-1}}{b_n}\right|_\times\right)^{M} |b_n|_\times^{M}.$$

One can derive similar inequalities for $b_m^{a_m} b_{m+1}^{a_{m+1}} ... b_n^{a_n}$, and for $m < n$, and then Theorem 1.3 can be applied to derive convergence of $\prod_{k=1}^{\infty} b_k{}^{a_k}$. □

This Theorem 5.9 gives a motivation for concepts and results to be given in Sections 6 and 7. Theorem 5.7 and Theorem 5.9 of this Section 5 are comparable with the corresponding classical summability results, Theorem 8.48 in [1] and Theorem 8.27 in [1], which are attributed to the mathematicians Cesaro and Abel, respectively.

6. Matrices and Multiplicability

Let $(0, \infty)^n$ denote the cartesian product $\prod_{i=1}^{n} X_i$ in which each $X_i = (0, \infty)$, multiplication is defined coordinatewisely, and elements are written as transpose of row vectors $(x_1, x_2, ..., x_n)$. More explicitly,

$$\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ . \\ . \\ . \\ x_n y_n \end{pmatrix}$$

or $(x_1, x_2, ..., x_n)^T (y_1, y_2, ..., y_n)^T = (x_1 y_1, x_2 y_2, ..., x_n y_n)^T$ in $(0, \infty)^n$. Let $A = (a_{ij})$ be a matrix of order $m \times n$ with real entries a_{ij}. Let us define

$$A * \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} x_1^{a_{11}} x_2^{a_{12}} ... x_n^{a_{1n}} \\ x_1^{a_{21}} x_2^{a_{22}} ... x_n^{a_{2n}} \\ . \\ . \\ . \\ x_1^{a_{m1}} x_2^{a_{m1}} ... x_n^{a_{mn}} \end{pmatrix}$$

such that A is a multiplication preserving function from $(0, \infty)^n$ to $(0, \infty)^m$. Then, for a given matrix $B = (b_{ij})$ of order $k \times m$ with real entries, it can be verified that

$$B * \left(A * \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \right) = (BA) * \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix},$$

$\forall\ (x_1, x_2, ..., x_n)^T \in (0, \infty)^n$, where BA is the usual matrix multiplication. Let C be another matrix of order $k \times m$ with real entries. Then the following is true for every $(x_1, x_2, ..., x_n)^T \in (0, \infty)^n$:

$$(C+B) * \left(A * \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \right) = \left(C * \left(A * \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \right) \right) \left(B * \left(A * \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \right) \right).$$

Also,

$$(C+B) * \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_m \end{pmatrix} = \left(C * \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_m \end{pmatrix} \right) \left(B * \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_m \end{pmatrix} \right),$$

$\forall\ (y_1, y_2, ..., y_m)^T \in (0, \infty)^m$. Let D and E be matrices of order $n \times k$ with real entries. Then it can be verified that

$$A * \left((D+E) * \begin{pmatrix} z_1 \\ z_2 \\ . \\ . \\ . \\ z_k \end{pmatrix} \right) = \left(A * \left(D * \begin{pmatrix} z_1 \\ z_2 \\ . \\ . \\ . \\ z_k \end{pmatrix} \right) \right) \left(A * \left(E * \begin{pmatrix} z_1 \\ z_2 \\ . \\ . \\ . \\ z_k \end{pmatrix} \right) \right)$$

$\forall\ (z_1, z_2, ..., z_k)^T \in (0, \infty)^k$. Thus, there are interesting matrix operations. With this familiarity of the operation $*$ for finite matrices, let us extend the same for infinite matrices or double sequences which are associated with summability. Corollary 5.8 is also a counterpart of a summability method.

Proposition 6.1. *Let $(a_{m,n})_{m,n=1}^{\infty}$ be a double sequence of positive numbers such that $\sum_{n=1}^{\infty} a_{m,n} \leq M$, for every m, for some $M > 0$, and such that $\lim_{m\to\infty} a_{m,n} = 0$, for every n. Then, for every sequence $(x_n)_{n=1}^{\infty}$ of positive reals for which $\lim_{n\to\infty} x_n = p > 0$ exists, $\lim_{m\to\infty} \prod_{n=1}^{\infty} (\frac{x_n}{p})^{a_{m,n}} = 1$. Equivalently, $\lim_{m\to\infty} \prod_{n=1}^{\infty} {x_n}^{a_{m,n}} = 1$, whenever $\lim_{n\to\infty} x_n = 1$ and $x_n > 0$, $\forall n$.*

Proof. Suppose $(x_n)_{n=1}^{\infty}$ be a sequence of positive reals such that $x_n \to 1$ as $n \to \infty$. Fix $\epsilon > 0$. Find an integer n_0 such that $|x_k|_{\times} < (1+\epsilon)^{\frac{1}{2M}}$, $\forall$ $k \geq n_0$. Then, for $l > k \geq n_0$,
$(1+\epsilon)^{\frac{-1}{2}} < \prod_{s=k}^{l} |x_s|_{\times}^{-a_{m,s}} \leq \prod_{s=k}^{l} {x_s}^{a_{m,s}} \leq \prod_{s=k}^{l} |x_s|_{\times}^{a_{m,s}} < (1+\epsilon)^{\frac{1}{2}}$,
$\forall$ m, and hence $(1+\epsilon)^{\frac{-1}{2}} \leq \prod_{n=n_0+1}^{\infty} {x_n}^{a_{m,n}} \leq (1+\epsilon)^{\frac{1}{2}}$, $\forall$ m. Find an integer m_0 such that
$(1+\epsilon)^{\frac{-1}{2}} < \prod_{k=1}^{n_0} |x_k|_{\times}^{-a_{m,k}} \leq \prod_{k=1}^{n_0} {x_k}^{a_{m,k}} \leq \prod_{k=1}^{n_0} |x_k|_{\times}^{a_{m,k}} < (1+\epsilon)^{\frac{1}{2}}$,
$\forall$ $m \geq m_0$. Thus, for $m \geq m_0$, $(1+\epsilon)^{-1} \leq \prod_{n=1}^{n_0} {x_n}^{a_{m,n}} \prod_{n=n_0+1}^{\infty} {x_n}^{a_{m,n}} \leq (1+\epsilon)$. That is, for given $\epsilon > 0$, there is an integer m_0 such that $(1+\epsilon)^{-1} \leq \prod_{n=1}^{\infty} {x_n}^{a_{m,n}} \leq (1+\epsilon)$, $\forall$ $m \geq m_0$. This proves that $\lim_{m\to\infty} \prod_{n=1}^{\infty} {x_n}^{a_{m,n}} = 1$. □

7. Power Products

Corresponding to a power series, it is possible to introduce power products.

Definition 7.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive reals. A formal power product is $a_0 {a_1}^{x} {a_2}^{x^2} {a_3}^{x^3} ...$ or $\prod_{n=0}^{\infty} {a_n}^{x^n}$.

Remark 7.2. This formal power product converges to a value, if $|x|_+ < \limsup_{n\to\infty} |\log a_n|_+^{\frac{1}{n}}$. This can be derived from the corresponding power series obtained by logarithmic transformation. This is not our method of deriving results. Let us now recall Theorem 5.9 in another version.

Theorem 7.3. *Suppose $(b_n)_{n=1}^{\infty}$ be as in Theorem 5.9. Then the power product $\prod_{k=1}^{\infty} {b_k}^{x^k}$ converges for every real $x \in [-1, 1)$.*

Proof. Let $x \in [-1, 1)$. Write $a_k = x^k$, for $k = 1, 2, ...$ Then $\Big\{ \sum_{k=n}^{m} a_k : m = 1, 2, ..., n = 1, 2, ..., n \leq m \Big\}$ is a bounded set. By Theorem 5.9, $\prod_{k=1}^{\infty} {b_k}^{x^k}$ converges. □

Corollary 7.4. *Let $(b_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive reals converging to 1. Then the power product $\prod_{k=1}^{\infty} {b_k}^{x^k}$ converges for every real $x \in [-1, 1)$.*

Proof. For $n > m$,

$$1 \leq \Big|\frac{b_m}{b_{m+1}}\Big|_\times \Big|\frac{b_{m+1}}{b_{m+2}}\Big|_\times ... \Big|\frac{b_n}{b_{n+1}}\Big|_\times = \frac{b_{m+1}}{b_m}\frac{b_{m+2}}{b_{m+1}}...\frac{b_{n+1}}{b_n} = \frac{b_{n+1}}{b_m} < \frac{1}{b_m},$$

and $\frac{1}{b_m} \to 1$ as $m \to \infty$. Therefore, by Theorem 1.3, $\prod_{n=1}^{\infty} |\frac{b_n}{b_{n+1}}|_\times$ converges. Now, the conclusion follows from Theorem 7.3. □

So, there is a need to improve Theorem 5.9. The next result may remind Cauchy product for sequences.

Theorem 7.5. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive reals such that $\prod_{n=0}^{\infty} a_n$ and $\prod_{n=0}^{\infty} b_n$ converge to A and B. For each n, let $c_n = a_0 b_n a_1 b_{n-1} ... a_n b_0$, $A_n = \prod_{k=0}^{n} a_k$, $B_n = \prod_{k=0}^{n} b_k$ and $C_n = \prod_{k=0}^{n} c_k$. Then ${A_n}^{\frac{1}{n}} \to 1$, ${B_n}^{\frac{1}{n}} \to 1$, ${C_n}^{\frac{1}{n}} \to AB$ and ${c_n}^{\frac{1}{n}} \to 1$ as $n \to \infty$.*

Proof. By Theorem 1.3, $a_n \to 1$, $b_n \to 1$ as $n \to \infty$. By Lemma 2.5, all conclusions follow, because $C_n = A_0 A_1 ... A_n B_0 B_1 ... B_n$, and $A_n \to A$, $B_n \to B$ as $n \to \infty$. □

8. Conclusion

The problem in using transformation technique lies in guessing methods for transformation. For example, transformations were not used since nineteenth century to define multiplicative modulus function from additive modulus function. One can guess that a transformation can be applied to derive multiplicative modulus function from additive modulus function, only after introducing the concept of multiplicative modulus function. So, separate techniques should be developed for infinite products. Theory of infinite products is also applicable like theory of infinite series. This is the need for development of theory of infinite products. Corollary 7.4 is simple to write examples. But it is not sufficient. So, Theorem 5.9 should be improved in all possible ways, but subject to simple conditions.

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THE WEDDERBURN FACTORIZATION THEOREM AND VALUED DIVISION ALGEBRAS

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1. Introduction

The purpose of this expository article is to describe a charming and entirely elementary theorem of Wedderburn, known as the *Wedderburn Factorization Theorem.* It describes how certain polynomials over a field F factor completely into linear factors over a division algebra D whose center is F. This theorem merits being better known than it appears to be, especially given how easy it is to prove, and our hope, indeed, is that this article will contribute to that goal.

We use this opportunity to also describe the notion of valuations on division algebras, and show how Wedderburn's theorem allows us to prove a key result about the uniqueness of such valuations. We end by giving a construction of an infinite family of division algebras, using the properties of valuations.

First recall that if K/F is a finite extension of fields, K is said to be *normal* over F if the minimal polynomial over F of every element $k \in K$ splits completely over K. Normality is thus a measure of algebraic "fullness" over F, and yields desirable characteristics. For instance, if L/F is an extension of fields containing K/F as a subextension, then any element of the group of symmetries of L/F carries K to K. As another example, the group of symmetries of K/F is as large as it can possibly be, after factoring out the elements purely inseparable over F: we have $|\mathrm{Gal}(K/F)| = [K : F^{ins}]$, where F^{ins} is the set of elements purely inseparable over F. (In general, without the normality assumption, we would only have $|\mathrm{Gal}(K/F)| \leq [K : F^{ins}]$).

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Now let D be an F-central division algebra, finite-dimensional over F. Any nonzero element $d \in D$ generates over F a commutative subring $F[d]$ of D, which as a set is just $\{f_0 + f_1 d + \cdots + f_k d^k \mid k \geq 0, f_i \in F\}$. The subring $F[d]$ is necessarily finite-dimensional as a vector space over F, since D itself is finite-dimensional over F. Hence, the set $\{1, d, d^2, \dots\}$ cannot be F-linearly independent, and exactly as in the case of field extensions, there exists some (say monic) polynomial $m_{d,F}$ of least degree with coefficients in F satisfied by d (see for instance [5, Chapter V, §1]). This polynomial is necessarily irreducible, for exactly the same reasons as in the field case: Assume that $m_{d,F} = fg$ for polynomials $f, g \in F[x]$ of lower degree than $m_{d,F}$. The evaluation map from $F[x]$ to $F[d]$ that arises from substituting $x = d$ is a ring homomorphism (this would not be true if F were a more general ring, as we will see in Remark 2.2 Part 1 ahead). Hence, $0 = m_{d,F}(d) = f(d)g(d)$. Since D is a division ring, either $f(d)$ or $g(d)$ must be zero. But this violates the minimality of $m_{d,F}$.

In analogy with the case of field extensions, it is reasonable to ask if $m_{d,F}$ factors completely in D.

Let us study a simple example, the first and most well-known example of a division algebra. This is Hamilton's Quaternions, denoted by $\mathbb{H}$. This is the four-dimensional $\mathbb{R}$-vector space with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$, so $\mathbb{H} = \{p(= p \cdot 1) + q \cdot \mathbf{i} + r \cdot \mathbf{j} + s \cdot \mathbf{k},\ p, q, r, s \in \mathbb{R}\}$. Here, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are symbols, and multiplication is given $\mathbb{R}$-bilinearly, subject to the following rules: $\mathbf{ij} = \mathbf{k}$, $\mathbf{ji} = -\mathbf{k}$, $\mathbf{i}^2 = \mathbf{j}^2 = -1$ (from which one derives the rules $\mathbf{k}^2 = -1$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$, $\mathbf{kj} = -\mathbf{i}$ and $\mathbf{ik} = -\mathbf{j}$), and, $p\mathbf{i} = \mathbf{i}p$, $p\mathbf{j} = \mathbf{j}p$ for all $p \in \mathbb{R}$ (which yields the rule $p\mathbf{k} = \mathbf{k}p$ as well). It is easy to see that this is a division algebra: we have the identity

$$(p + q\mathbf{i} + r\mathbf{j} + s\mathbf{k})(p - q\mathbf{i} - r\mathbf{j} - s\mathbf{k}) = p^2 + q^2 + r^2 + s^2 := N(d),$$

so if $d = p + q\mathbf{i} + r\mathbf{j} + s\mathbf{k}$ is nonzero, then $(p, q, r, s) \neq (0, 0, 0, 0)$, and we may divide the second factor on the left above by the nonzero real number $N(d)$ to determine the inverse of d as

$$d^{-1} = \frac{p}{N(d)} - \frac{q}{N(d)}\mathbf{i} - \frac{r}{N(d)}\mathbf{j} - \frac{s}{N(d)}\mathbf{k}.$$

In this simple example, it is easy to see that for any nonzero $d \in \mathbb{H}$, $m_{d,\mathbb{R}}$ splits completely in $\mathbb{H}$. In fact, it splits completely in $\mathbb{R}[d]$ itself. This is of course trivial if $d \in \mathbb{R}$, since $m_{d,\mathbb{R}}$ in that case is just $x - d$. For $d \notin \mathbb{R}$, first note that more generally, for any division algebra D finite-dimensional over its center F and nonzero $d \in D$, the subring $F[d]$ is necessarily a field.

For, if $m_{d,F} = x^m + f_{m-1}x^{m-1} + \cdots + f_1x + f_0$, then $f_0 \neq 0$, for otherwise, $m_{d,F}$ would admit the nontrivial factor x. We may write

$$d(d^{m-1} + f_{m-1}d^{m-2} + \cdots + f_1) = -f_0$$

and exactly as in the field case and in the case of $\mathbb{H}$ above, we may divide the second factor on the left by $-f_0$ to find that the inverse of d lies in $F[d]$ itself. Applying this same reasoning to any $e \neq 0$ in $F[d]$ shows that the inverse of e lies in $F[e] \subseteq F[d]$. Thus, $F[d]$ is a field.

Now, in our particular case of the Quaternions $\mathbb{H}$, for $d \notin \mathbb{R}$, the field $\mathbb{R}[d]$ has to be an isomorphic copy of $\mathbb{C}$, as $\mathbb{C}$ is the only nontrivial finite-dimensional field extension of $\mathbb{R}$. Thus, $m_{d,\mathbb{R}}$ has to split completely in $\mathbb{R}[d]$ as $\mathbb{C}$ is algebraically closed. (Alternatively, as $\deg(m_{d,\mathbb{R}}) = [\mathbb{C} : \mathbb{R}] = 2$ and $x - d$ is already a factor in $F[d]$, the other factor of $m_{d,\mathbb{R}}$ has to be linear.)

But what about more complicated examples of division algebras? This is where the Wedderburn Factorization theorem comes in. It shows that if F is *any* field, and if D is *any* division algebra finite-dimensional over its center F, then D/F is normal in the same sense as in field theory: $m_{d,F}$ factors completely in D for any $d \in D$! Wedderburn's proof of this rather elegant theorem is elementary. We give his original proof of this theorem, and then describe a nontrivial application of this theorem to valued division algebras due to Wadsworth [8]: we describe how this theorem can be used to show that if F has a valuation v defined on it, and if this valuation extends to D, then the extension is completely determined by the valuation on F! We give examples of valuations on division algebras extending a given valuation on the center, describing, in the process, a natural genaralization of Hamilton's quaternions that yields division algebras of dimension n^2 for any integer $n = 2, 3, \ldots$.

2. Factorization theorem

We describe the Wedderburn Factorization theorem in this section. The theorem appears in Wedderburn's paper [9], where he uses it to study division algebras whose dimension over their centers is 9. Wedderburn's proof of his theorem, which we give below, is also described in the texts [6] and [4], among others. As already noted in Section 1, the proof is quite elementary; a more conceptual proof was given later by Jacobson, and can be found in [7].

For any ring R (always with 1), let $R[t]$ denote the polynomial ring in one variable t over R, where t commutes elementwise with R. For a polynomial $f(t) = \sum_{i=0}^{n} a_i t^i$ and an element $r \in R$, we define the *evaluation of f at r* by $f(r) \coloneqq \sum_{i=0}^{n} a_i r^i \in R$. Note that $(f+g)(d) = f(d) + g(d)$ for two polynomials $f, g \in R[t]$. Note, too, that to evaluate $f = gh \in R[t]$ at $r \in R$ we must first multiply out g and h and write f in the form $\sum a_i t^i$, and then substitute r for t. (See Remark 2.2, Part 1 below.)

Definition 2.1. *An element $r \in R$ is said to be a right root of $f \in R[t]$ if $f(r) = 0$.*

Remark 2.2.
(1) *In general, if $f = gh$ it does not follow that $f(r) = g(r)h(r)$. Indeed, consider a division algebra D with center F, and $d, d_1 \in D \setminus \{0\}$ are such that $dd_1 \neq d_1 d$. Put $g(t) = (t - d)$, and $h(t) = (t - d_1)$. Then $g(d)h(d) = 0$, while*
$gh(d) = (t^2 - (d + d_1)t + dd_1)|_{t=d} = d^2 - d^2 - d_1 d + dd_1 = dd_1 - d_1 d \neq 0$.
(2) *Given f, h in $R[t]$, we say that h is a right factor of f if $f = gh$ for some $g \in R[t]$. An analogous definition holds for left factors.*
(3) *Over a field, a polynomial of degree n has at most n distinct roots. Over a division ring, this is no longer true. For instance, in the Hamiltonian division algebra $\mathbb{H}$ over $\mathbb{R}$, both $\mathbf{i}$ and $-\mathbf{i}$ satisfy $t^2 + 1$, but so do $\pm\mathbf{j}$ and $\pm\mathbf{k}$. Further, any element of $\mathbb{H}$ of the form $z\mathbf{i}z^{-1}$ with $z \neq 0$ is also a root of t^2+1. Indeed, $(z\mathbf{i}z^{-1})^2 + 1 = z\mathbf{i}^2 z^{-1} + 1 = z(\mathbf{i}^2 + 1)z^{-1} = 0$. It is easy to produce infinitely many distinct elements of the form $z\mathbf{i}z^{-1}$, so $t^2 + 1$ has infinitely many roots.*
(4) *Let D and F be as in Part 1 of this remark. If $f = \sum a_i t^i \in F[t]$, and $d \in D$ is a root of f, then any conjugate zdz^{-1} is also a root. Indeed, $zf(t)z^{-1} = \sum z a_i t^i z^{-1} = \sum a_i (ztz^{-1})^i$ (since $za_i = a_i z$). Also $zf(t)z^{-1} = f(t)$.*

For our next definition, let D be a division algebra with center F, and D^* the set of its nonzero elements. A conjugacy class in D is the set of all conjugates zdz^{-1} of a fixed nonzero element d of D, as z varies through D^*.

Definition 2.3. *We say that a conjugacy class A is algebraic over F if one (and hence all) of its element is algebraic over F.*

Lemma 2.4. *Let D be a division ring and let $f = gh \in D[t]$. Let $d \in D$ be such that $a \coloneqq h(d) \neq 0$. Then $f(d) = g(ada^{-1})h(d)$. In particular, if d is a root of f but not of h, then ada^{-1} is a root of g.*

Proof. Let $g = \sum b_i t^i$ and $h = \sum c_j t^j$. Then $f = \left(\sum b_i t^i\right)\left(\sum c_j t^j\right) = \sum\sum b_i c_j t^{i+j}$, and hence

$$f(d) = \sum\sum b_i c_j d^{i+j} = \sum b_i \left(\sum c_j d^j\right) d^i = \sum b_i \left(h(d)\right) d^i = \sum b_i a d^i$$
$$= \sum b_i a d^i a^{-1} a = \sum b_i (ada^{-1})^i a = g(ada^{-1})h(d).$$

The last statement of the lemma follows because D is a division ring and has no zero divisors. □

Lemma 2.5. *For $f(t) \in D[t]$ and $d \in D$, we have* $f(t) = q(t)(t-d)+f(d)$. *In particular, d is a root of f if and only if $t-d$ is a right factor of f.*

Proof. We proceed by induction on $\deg f(t)$. If $\deg f(t) = 1$, $f(t) = at + b$, then $f(t) = a(t-d) + ad + b$, i.e., we can take $q(t) = a$. Suppose $\deg f = n > 1$ and let $d_n \in D$ be the leading coefficient of $f(t)$. Put $g(t) = f - d_n t^{n-1}(t-d) = f - d_n t^n + d_n d t^{n-1}$, then $\deg g < \deg f$, and $g(d) = f(d) - d_n d^n + d_n d d^{n-1} = f(d)$. Thus, by the induction there exists $q_1 \in D[t]$ such that $g(t) = f(t) - d_n t^{n-1}(t-d) = q_1(t-d) + g(d)$. Hence, $f(t) = (q_1 + d_n t^{n-1})(t-d) + f(d)$. □

Proposition 2.6. *Let D be a central division algebra over F and A a conjugacy class of D which is algebraic over F with minimal polynomial $f \in F[t]$. If a polynomial $h \in D[t] \setminus \{0\}$ vanishes identically on A, then $\deg h \geq \deg f$.*

Proof. We may assume that h is monic, since if $h = e_m t^m + e_{m-1}t^{m-1} + \cdots + e_0$, then d is a root of h if and only if d is a root of $t^n + e_m^{-1}e_{m-1}t^{m-1} + \cdots + e_m^{-1}d_0$. Let $h = t^m + d_1 t^{m-1} + \cdots + d_m \in D[t]$ be such that $h(A) = 0$ and $m = \deg h < \deg f$ is as small as possible. Observe that $h \notin F[t]$, as f is the polynomial of least degree from $F[t]$ that vanishes on A. Hence some coefficient d_i is not in F, and therefore there is an $s \in D$ such that $d_i s \neq s d_i$. For any $a \in A$, $h(a) = a^m + d_1 a^{m-1} + \cdots + d_m = 0$. Hence

$$0 = (sas^{-1})^m + d_1(sas^{-1})^{m-1} + \cdots + d_m.$$

On the other hand, using $sd_j a^{m-j} s^{-1} = sd_j s^{-1} s a^{m-j} s^{-1} = (sd_j s^{-1})(sas^{-1})^{m-j}$ we also have

$$0 = (sas^{-1})^m + (sd_1 s^{-1})(sas^{-1})^{m-1} + \cdots + (sd_m s^{-1}).$$

Hence the polynomial $\sum_{j=1}^{m}(d_j - sd_j s^{-1})t^{m-j}$ vanishes on $sAs^{-1} = A$. Since $d_i s \neq s d_i$, the above polynomial is nonzero, and its degree $< m$. This contradicts the choice of m. □

Theorem 2.7 (Wedderburn Factorization Theorem)**.** *Let D be a division algebra with center F. Let A be a conjugacy class of D which is algebraic over F, with minimal (monic) polynomial $f \in F[t]$ of degree n. Then there exists $a_1, \ldots, a_n \in A$ such that $f = \prod_{i=1}^{n}(t - a_i) \in D[t]$. Also, f is product* *of the same linear factors, permuted cyclically. The element $a_1 \in A$ can be arbitrarily prescribed.*

Proof. Fix $a_1 \in A$, so a_1 is a root of f and hence $t - a_1$ is a right factor of f. Consider a factorization of f

$$f = g(t)(t - a_r) \cdots (t - a_1)$$

with $g \in D[t]$, $a_i \in A$, where r is chosen as large as possible. We claim that $h = (t - a_r) \cdots (t - a_1)$ vanishes identically on A. Indeed, for $a \in A$ we have $f(a) = 0$. If $h(a) \neq 0$ then by lemma 2.4, $g(a_{r+1}) = 0$ for a conjugate a_{r+1} of a. Then we can write $g = g_1(t)(t - a_{r+1})$ for some $g_1 \in D[t]$, and thus f has a right factor $(t - a_{r+1})(t - a_r) \cdots (t - a_1)$, which contradicts the choice of r. Hence $h(a) = 0$ for every $a \in A$. Hence by Lemma 2.6, $\deg h \geq \deg f$. Since $\deg f \geq \deg h$ by construction, we find $f(t) = \prod_{i=1}^{n}(t - a_i)$.

To prove the last assertion, we show that for $f \in F[t]$ if $f = gh \in D[t]$, then $f = hg \in D[t]$. Indeed, for $g \in D[t]$, we have $gf = fg$ since the coefficients of f are in F. Thus, $gf = fg = ghg$, i.e., $g(f - hg) = 0 \in D[t]$. Hence, $f = hg$. □

An interesting paper of Laffey and Meehan ([3]) carries Wedderburn's computations much further. For an arbitrary ring R, the authors analyze the factorization over R of an arbitrary polynomial in $Z(R)[t]$, where $Z(R)$ is the center of R. They show that under a certain existence condition, the polynomial factors into conjugate linear factors, exactly as in the Wedderburn factorization. They apply their result then to matrix rings $M_n(F)$ over arbitrary fields F of characteristic zero, and show that any polynomial of degree n with coefficients in F factors over the matrix ring into conjugate linear factors.

3. Valuation on a division algebra

In this section, we consider valuations on fields and division algebras finite-dimensional over their centers.

Recall that an abelian group Γ is *totally ordered* if it is linearly ordered as a set, and if $g \leq h$ implies $g + k \leq h + k$ for all $g, h, k \in \Gamma$.

Definition 3.1. [2, Chapter 9, Definition 9.4′] *A valuation v on a field F is a map $v\colon F \to \Gamma \cup \{\infty\}$, where Γ is a totally ordered abelian group, and ∞ a symbol such that*

$$\gamma < \infty \quad \text{and } \gamma + \infty = \infty + \infty = \infty \text{ for all } \gamma \in \Gamma, \tag{3.1}$$

subject to the following conditions: for all $x, y \in F$,

(1) $v(x) = \infty$ if and only if $x = 0$;

(2) $v(x+y) \geq \min\{v(x), v(y)\}$;

(3) $v(xy) = v(x) + v(y)$.

We denote $v\big(F \setminus \{0\}\big)$ by Γ_F. It is called the value group of F and is indeed a subgroup of Γ. Note that by Part 3 of the definition, v is a group homomorphism from $F \setminus \{0\}$ to Γ_F. In particular, $v(1) = 0$, and the relation $-1 \cdot -1 = 1$ then shows that $v(-1) = 0$. The field and its valuation are often jointly referred to as (F, v).

It is useful to observe that if $x \in F$ has positive value, then $v(1+x) = 0$. For, by Definition 3.1 Part 2, $v(1+x) \geq \min(v(1), v(x)) = 0$. If $v(1+x)$ were positive, we would find $v(1) = v\left((1+x) - x\right) \geq \min\left(v(1+x), v(-x)\right)$, and since $v(-x) = v(-1 \cdot x) = v(-1) + v(x) = v(x) > 0$, we would find $v(1) > 0$, a contradiction.

It follows from this that if $v(x) < v(y)$ then $v(x+y) = v(x)$. For, $v(x+y) = v\left(x(1+x^{-1}y)\right)$. Now, from $xx^{-1} = 1$ we find $v(x^{-1}) = -v(x)$, so $v(x^{-1}y) = v(y) - v(x) > 0$. Hence by what we just showed, $v\left(x(1+x^{-1}y)\right) = v(x) + v(1+x^{-1}y) = v(x)$.

Example 3.2. Let $F = E(x)$ be the rational function field in one variable over a field E. Note that $\mathbb{Z}$ is naturally a totally ordered abelian group. We define a valuation $v_x \colon F \to \mathbb{Z} \cup \{\infty\}$ as follows: If $0 \neq f \in E[x]$ is such that $f = a_r x^r + a_{r+1}x^{r+1} + \cdots + a_{r+s}x^{r+s}$ where $a_r \neq 0$, then define $v_x(f) = r$. For any nonzero element $f/g \in F$ we define $v(f/g) = v(f) - v(g)$, and $v(0) = \infty$.

Example 3.3. Let $F = E(x, y)$ be the rational function field over E in two variables. Note that $\mathbb{Z} \times \mathbb{Z}$ is a totally ordered abelian group under the reverse lexicographic ordering: $(a, b) > (c, d)$ if $b > d$, or else, $b = d$ and $a > c$. Define a valuation $v \colon F \to (\mathbb{Z} \times \mathbb{Z}) \cup \{\infty\}$ as follows: Let $0 \neq f \in E(x)[y]$ and write f in the form $f = f_r y^r + f_{r+1}y^{r+1} + \cdots + f_{r+s}y^{r+s}$, where $f_i \in E(x)$ and $f_r \neq 0$. Define $v(f) \coloneqq \big(v_x(f_r), r\big)$. For any nonzero element $f/g \in F$ we define $v(f/g) = v(f) - v(g)$, and $v(0) = \infty$. In particular, $v(x) = (1, 0)$ and $v(y) = (0, 1)$. Note that the zero element in $\mathbb{Z} \times \mathbb{Z}$ is $(0, 0)$ and $v(e) = (0, 0)$ for all $e \in E$.

In what follows, we restrict our attention to division algebras finite-dimensional over their centers.

Definition 3.4 (Valuation on a division algebra)**.** *Let D be a division algebra, finite-dimensional over its center. A valuation on D is a function*

$$v_D \colon D \to \Gamma \cup \{\infty\},$$

where Γ *and* ∞ *are as in Definition 3.1, subject to the following conditions: for all* $x, y \in D$,

(1) $v_D(x) = \infty$ *if and only if* $x = 0$*;*
(2) $v_D(x + y) \geq \min\{v_D(x), v_D(y)\}$*;*
(3) $v_D(xy) = v_D(x) + v_D(y)$*.*

We denote $v_D\big(D \setminus \{0\}\big)$ by Γ_D. It is called the value group of D and is indeed a subgroup of Γ. Just as in the field case, v_D is a group homomorphism from $D \setminus \{0\}$ to Γ_D, so $v_D(1) = v_D(-1) = 0$. The division algebra and its valuation are often jointly referred to as (D, v_D).

Definitions 3.4 and 3.1 are clearly very similar. Indeed, if F is the center of D, then v_D restricts to a valuation on F in the sense of Definition 3.1. In the other direction, if (F, v) is a valued field and D is a division algebra with center F, we say a valuation v_D on D extends v if $v_D|_F = v$. In general, a valuation v on F need not extend to D, but the remarkable fact is that if it does extend, it does so *uniquely*, via a simple formula that determines it completely in terms of v. The derivation of this formula, which we describe in Proposition 3.5 below, is a lovely application of the Wedderburn Factorization theorem.

To understand the ingredients in the formula, it is useful to know certain key facts about division algebras. While full proofs of these will take us deep into a study of a more general family of algebras known as simple algebras, the facts themselves can be described easily. We refer the reader to [1, Part I, §5] or [2, Chapter 4, §4.6] for more details. A key result is that if $\overline{F}$ is an algebraic closure of F, then $D \otimes_F \overline{F}$ is isomorphic to the matrix algebra $M_t(\overline{F})$ for some t. Since the $\overline{F}$ dimension of $M_t(\overline{F})$ is t^2, we find from the tensor product description that the *dimension of any division algebra over its center is a perfect square.* The square root of this dimension is called the *index* of the division algebra.

Next, given $d \in D$, we may work with its matrix avatar $M_d = d \otimes 1$ in $M_t(\overline{F})$ under the embedding

$$D \xhookrightarrow{d \mapsto d \otimes 1} D \otimes_F \overline{F} \cong M_t(\overline{F}),$$

and consider its determinant. A second key result is that $\det(M_d)$ *lies in* F, not just in $\overline{F}$! The value of this determinant is called the *reduced norm of* d, and is denoted $\mathrm{Nrd}(d)$. It can be shown to be independent of the specific isomorphism $D \otimes_F \overline{F} \cong M_t(\overline{F})$ (there can be several) and is thus intrinsic to D. (In fact, the expression $N(d) = p^2 + q^2 + r^2 + s^2$ obtained in Section

1 from the quaternion $d = p + q\mathbf{i} + r\mathbf{j} + s\mathbf{k}$ is nothing more than $\mathrm{Nrd}(d)$, a tidy formula for the reduced norm in the case of the Quaternions!)

A third key result relates the reduced norm of d to its norm from $F[d]$ to F as defined via field theory. If $[F[d] : F] = n$, the result is that

$$\mathrm{Nrd}(d) = \big(N_{F[d]/F}(d)\big)^{\mathrm{ind}\,(D)/n},$$

where $N_{F[d]/F}(d)$ stands for the field theoretic norm, and $\mathrm{ind}\,(D)$ denotes the index of D as defined above.

We can now describe the formula for the extension of v to D (assuming such an extension exists).

Proposition 3.5 (Wadsworth [8])**.** *Let D be a finite-dimensional central division algebra over a valued field (F, v). Assume that w is a valuation on D extending v. For all $a \in D^*$ we have* $w(a) = (1/\mathrm{ind}\,(D))v(\mathrm{Nrd}(a))$. *In particular, if v extends to D, then the extension is uniquely determined by v.*

Proof. Let $0 \neq a \in D$ and $f(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_0 \in F[t]$ be the minimal polynomial of a over F. Thus, $n = [F(a) : F]$, $N_{F[a]/F}(a) = (-1)^n\alpha_0$, so by the third key result described above, $\mathrm{Nrd}(a) = (-1)^{\mathrm{ind}\,(D)}\alpha_0^{\mathrm{ind}\,(D)/n}$. Hence, $v(\mathrm{Nrd}(a)) = \frac{\mathrm{ind}\,(D)}{n}v(\alpha_0)$. By the Wedderburn Factorization theorem (2.7), we may find conjugates of a, $a_i = d_i a d_i^{-1}$ $(1 \leq i \leq n)$ such that $f(t) = (t - a_1)\cdots(t - a_n)$. In particular, $\alpha_0 = (-1)^n a_1 a_2 \ldots a_n$. Now $w(a_i) = w(d_i a d_i^{-1}) = w(d_i) + w(a) + w(d^{-1})$, and the relation $d_i d_i^{-1} = 1$ shows us that $w(d_i) = -w(d_i^{-1})$. Hence, $w(a_i) = w(a)$ for all i, from which it it follows that $w(\alpha_0) = n \cdot w(a)$; hence $v(\mathrm{Nrd}(a)) = \mathrm{ind}\,(D)w(a)$. □

Example 3.6. Let E be a field containing a primitive n-th root of unity ω. Let $F = E(x, y)$ be the rational function field over E in two variables. Let v be the valuation on F defined in (Example 3.3).

Define an F-algebra generated by i, j with the following relations:

$$i^n = x, j^n = y, ij = \omega ji.$$

We denote this algebra by $(x, y)_n$. It is easy to see that an arbitrary element of this algebra can be written uniquely in the form $\sum_{k,l=0}^{n-1} c_{kl}\, i^k j^l$, where $c_{kl} \in F$, so the various powers i^k, j^l, $k, l, = 0, \ldots, n-1$ form an F basis.

Note that $\frac{1}{n}\big(\mathbb{Z} \times \mathbb{Z}\big)$ is a totally ordered abelian group under the reverse lexicographic ordering, just like its subgroup $\mathbb{Z} \times \mathbb{Z}$. Consider the mapping $v_D \colon (x, y)_n \to \frac{1}{n}\big(\mathbb{Z} \times \mathbb{Z}\big) \cup \{\infty\}$ given by

$$v_D\bigg(\sum_{k,l=0}^{n-1} c_{kl}\, i^k j^l\bigg) = \min_{k,l}\bigg(v(c_{kl}) + \frac{k}{n}(1, 0) + \frac{l}{n}(0, 1)\bigg). \tag{3.2}$$

In particular,

$$v_D(i) = \frac{1}{n}(1,0), \quad v_D(j) = \frac{1}{n}(0,1) \quad \text{and} \quad v_D(\alpha) = v(\alpha) \text{ for } \alpha \in F. \quad (3.3)$$

Notice that

$$v_D(i^k j^l) \in \Gamma_F \text{ if and only if } k \in n\mathbb{Z} \text{ and } l \in n\mathbb{Z}.$$

Furthermore, we can show, exactly as we did for valuations on fields above, that the function v_D on $(x,y)_n$ also satisfies $v_D(1+e) = 0$ if $v_D(e)$ is positive, and $v_D(d+e) = v_D(d)$ if $v_D(d) < v_D(e)$ *and* if d is invertible in $(x,y)_n$.

We first show that v_D satisfies all properties listed in Parts 1, 2, and 3 of Definition 3.4.

Suppose that $v_D\left(\sum_{k,l=0}^{n-1} c_{kl}\, i^k j^l\right) = \infty$. Then $v(c_{kl}) + \frac{k}{n}(1,0) + \frac{l}{n}(0,1) = \infty$ for every k,l, i.e., $v(c_{kl}) = \infty$ for every k,l. Since v is a valuation on F, we have $c_{kl} = 0$ for all k,l. Hence, $\sum_{k,l=0}^{n-1} c_{kl}\, i^k j^l = 0$, establishing Part 1.

Next, we have

$$\begin{aligned}
v_D\left(\sum_{k,l=0}^{n-1} c_{kl}\, i^k j^l + \sum_{k,l=0}^{n-1} d_{kl}\, i^k j^l\right) &= v_D\left(\sum_{k,l=0}^{n-1} (c_{kl} + d_{kl})\, i^k j^l\right) \\
&= \min_{k,l}\Big(v(c_{kl} + d_{kl}) + v_D(i^k j^l)\Big) \\
&\geq \min_{k,l}\Big(\min\big(v(c_{kl}), v(d_{kl})\big) + v_D(i^k j^l)\Big) \\
&= \min_{k,l}\Big(\min\big(v(c_{kl} + v_D(i^k j^l)), v(d_{kl} + v_D(i^k j^l))\big)\Big) \\
&= \min\Big(\min_{k,l}\big(v(c_{kl}) + v_D(i^k j^l), v(d_{kl}) + v_D(i^k j^l)\big)\Big) \\
&= \min\Big(v_D\big(\sum_{k,l=0}^{n-1} c_{kl}\, i^k j^l\big), v_D\big(\sum_{k,l=0}^{n-1} d_{kl}\, i^k j^l\big)\Big),
\end{aligned}$$

establishing Part 2

To establish Part 3 needs a little more work. First, let us refer to an expression such as $c_{kl} i^k j^l$ for $0 \leq k < n$, $0 \leq l < n$ as a *monomial.* Note that any monomial is invertible: this follows from the fact that the inverse of i^k is $x^{-1} i^{n-k}$ for $1 \leq k < n$, and the inverse of j^l is $y^{-1} j^{n-l}$ for $1 \leq l < n$. First, we will show that if m_1 and m_2 are two monomials, then $v_D(m_1 m_2) = v_D(m_1) + v_D(m_2)$. Note that if $m_1 = c_{kl} i^k j^l$ and $m_2 = d_{k'l'} i^{k'} j^{l'}$, then, setting $\zeta = \omega^{k'l}$ we have the following relations:

$$m_1m_2 = \begin{cases} (c_{kl}d_{k'l'}\,\zeta)i^{k+k'}j^{l+l'} & \text{if } k+k'<n \text{ and } l+l'<n \\ (c_{kl}d_{k'l'}\,\zeta x)i^{k+k'-n}j^{l+l'} & \text{if } k+k'\geq n \text{ and } l+l'<n \\ (c_{kl}d_{k'l'}\,\zeta y)i^{k+k'}j^{l+l'-n} & \text{if } k+k'<n \text{ and } l+l'\geq n \\ (c_{kl}d_{k'l'}\,\zeta xy)i^{k+k'-n}j^{l+l'-n} & \text{if } k+k'\geq n \text{ and } l+l'\geq n \end{cases}$$

Notice that from the expressions for m_1m_2 above that if m_1 and m_2 are monomials, then m_1m_2 is also a monomial.

We will show that $v_D(m_1m_2) = v_D(m_1) + v_D(m_2)$ in the second of the four cases above; the proof for the other cases is similar. We apply the formula for v_D for a single monomial, which is implicit in Definition 3.2:

$$\begin{aligned} v_D(m_1m_2) &= v(c_{kl}d_{k'l'}\,\zeta x) + \tfrac{k+k'-n}{n}(1,0) + \tfrac{l+l'}{n}(0,1) \\ &= v(c_{kl}) + v(d_{k'l'}) + v(\zeta) + v(x) + \tfrac{k+k'-n}{n}(1,0) + \tfrac{l+l'}{n}(0,1) \\ &= v(c_{kl}) + v(d_{k'l'}) + (0,0) + (1,0) + (\tfrac{k+k'}{n} - 1)(1,0) + \tfrac{l+l'}{n}(0,1) \\ &= v(c_{kl}) + \tfrac{k}{n}(1,0) + \tfrac{l}{n}(0,1) + v(d_{k'l'}) + \tfrac{k'}{n}(1,0) + \tfrac{l'}{n}(0,1) \\ &= v_D(m_1) + v_D(m_2). \end{aligned}$$

Here we have used the fact that $v(\zeta) = (0,0)$, this is because $\zeta^n = 1$, so $nv(\zeta) = v(1) = (0,0)$.

Now consider arbitrary elements $d = \sum_{k,l} c_{kl}\, i^k j^l$ and $e = \sum_{k,l} d_{kl}\, i^k j^l$ of $(x,y)_n$. Since $v_D(i^kj^l)$ and $v_D(i^{k'}j^{l'})$ are in different cosets of $\mathbb{Z}\times\mathbb{Z}$ in $\frac{1}{n}(\mathbb{Z}\times\mathbb{Z})$ if $(k,l)\neq(k',l')$, $v_D(c_{kl}i^kj^l) \neq v_D(d_{k'l'}i^{k'}j^{l'})$. Hence we find that there is a *unique* monomial m_0 in d that has the lowest value of v_D, so by definition, $v_D(d) = v_D(m_0)$. Similarly, there is a unique monomial n_0 in e that has the lowest value of v_D and $v_D(e) = v_D(n_0)$. Let us write $d = m_0 + m_1 + \cdots$, where $m_1, \cdots$ are the other monomials in d: all of these have higher value of v_D than m_0. Similarly, let us write $e = n_0 + n_1 + \cdots$. Then $m_1m_2 = m_0n_0 + \sum_{(i,j)\neq(0,0)} m_in_j$. Note that the various monomial products m_sn_t need not be in distinct cosets of $\mathbb{Z}\times\mathbb{Z}$ in $\frac{1}{n}(\mathbb{Z}\times\mathbb{Z})$, but that will not concern us. Since we have already seen that $v_D(m_im_j) = v_D(m_i) + v_D(m_j)$, we see that all the monomial products other than m_0n_0 have values of v_D higher than $v_D(m_0n_0)$. Thus we write, $m_1m_2 = m_0n_0 + w$, and it follows from Part 2 of Definition 3.4, which we have already established, that $v_D(w) > v_D(m_0n_0)$. Now m_1m_2 is invertible (since we have observed that m_1m_2 is a monomial and that every monomial is invertible), so it follows from what we have noted above that $v_D(m_1m_2 + w) = v_D(m_1m_2)$.

Thus, $v_D(de) = v_D(m_1m_2) = v_D(m_1) + v_D(m_2) = v_D(d) + v_D(e)$. This establishes Part 3.

Note that Parts 1 and 3 show that $(x, y)_n$ has no zero divisors, since if $d \neq 0$ and $e \neq 0$, then $v_D(d) \neq \infty$ and $v_D(e) \neq \infty$, so $v_D(de) = v_D(d) + v_D(e) \neq \infty$. The arguments in Section 1 which show that $m_{d,F}$ is irreducible and from this that d is invertible in $F[d]$ only relied on the finite-dimensionality of D and the fact that it has no zero divisors. Applying this to $(x, y)_n$, we find that every nonzero element d of $(x, y)_n$ is invertible, with inverse in $F[d]$, showing that $(x, y)_n$ is a division algebra.

By Proposition 3.5, v_D is the unique valuation on $(x, y)_n$ that extends the valuation v on F.

Notice that the algebras $(x, y)_n$ form a very nice generalization of $\mathbb{H}$. These algebras are clearly of index n, and thus furnish examples of division algebras of arbitrary index.

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THE LIT-ONLY σ-GAME: SOME MATHEMATICS BEHIND A PUZZLE FOR KIDS

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Abstract. The lit-only σ-game is a nondeterministic linear dynamical process whose local transition rule is specified by a digraph. The player of the game can choose among several possible transitions at each step and aim to achieve certain global phase transition. This essay tries to popularize this game to kids for their fun. We also give readers a taste for some possible mathematics behind the game. The paper basically contains no proofs and many ideas are not presented in their full details. We invite the readers to venture out into the world of the lit-only σ-game and other discrete-time graph processes.

1. Appetizer

Jaap Scherphuis maintains a webpage [1] on various puzzles. We note that a subpage there is entitled "The Mathematics of Lights Out" [2]. Then, what is this lights out game and can this puzzle really lead to any interesting mathematics?

Different players may choose different names for the same puzzle and sometimes they may adopt game rules which are not exactly the same. The lights out game and its lit-only version will be introduced as the σ-game and the lit-only σ-game in the following. Here is a rough description of the σ-game as adapted from what is told by Andries E. Brouwer [3]:

> We are given a finite graph, with a light at each vertex that can be on or off. Pressing the button at a vertex switches the state of the lights at its neighbours. The question is

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[1] http://www.jaapsch.net/puzzles/lights.htm
[2] http://www.jaapsch.net/puzzles/lomath.htm
[3] http://www.win.tue.nl/~aeb/ca/madness/madrect.html

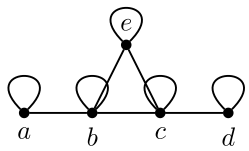


FIGURE 1.1. The Nandi (bull) graph.

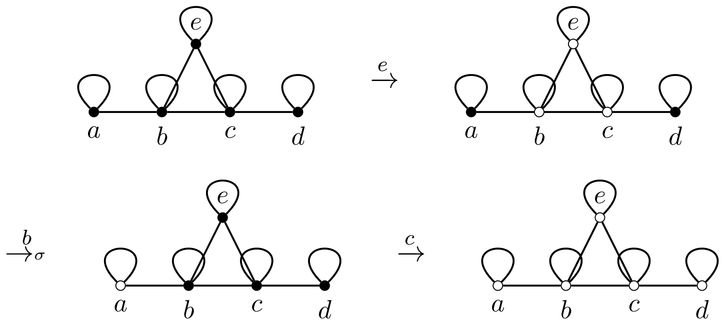


FIGURE 1.2. Playing the σ-game on the Nandi graph.

whether all lights can be switched off given an arbitrary starting position.

In Fig. 1.1, we display the Nandi (bull) graph where every vertex is attached a loop: You may think of a and d as the horns, b and c as the eyes, and e as the nose. In Fig. 1.2 we are displaying the process of moving from the all-lights-on configuration to the all-lights-off configuration by pressing vertices e, b and c in that order when playing the σ-game on the Nandi graph. Note that here we use a disk to indicate a vertex in the on state and a circle to refer to a vertex in the off state.

The lit-only σ-game on a graph is the variant of the σ-game in which one is not allowed to toggle any vertex when it is off. Among the three moves in Fig. 1.2, which are valid pushings in the σ-game, only the middle one is not allowed in the lit-only σ-game. You can tell from Fig. 1.2 that we use $\to$ for a move in the lit-only σ-game (and also in the σ-game) and we write $\to_\sigma$ for a pushing in the σ-game (which may be invalid in the lit-only σ-game). Can we go from the all-lights-on configuration to the all-lights-off configuration when playing the lit-only σ-game on the Nandi graph? To turn off all-lights, we have to press a vertex an odd number of times if it is the nose or an eye and we have to press a vertex an even number of times if it is a horn. But, after pressing any eye or the nose, the eyes and the nose will be all off and so it seems that we do not have any way to press every one of them an odd number of times. Does it imply the impossibility of

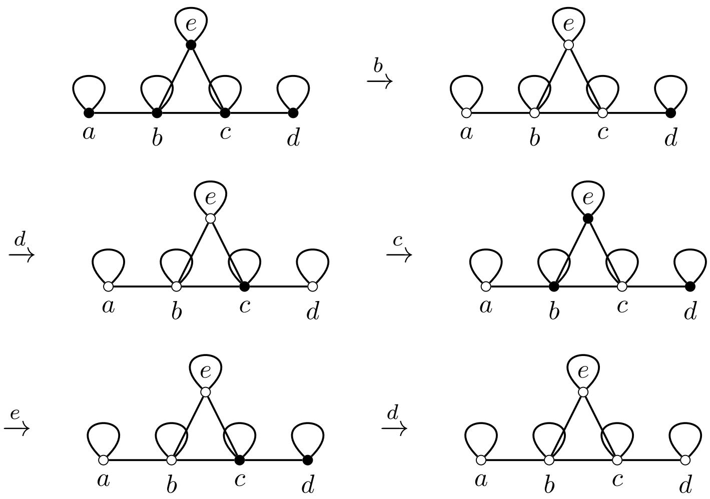


FIGURE 1.3. Playing the lit-only σ-game on the Nandi graph.

moving from the all-lights-on to the all-lights-off in the lit-only σ-game on the Nandi graph? Surely, you will not be misled by this plausible argument and, if you have a try for a little while, you will find a required path from the all-lights-on to the all-lights-off, say the one shown in Fig. 1.3.

2. Tiffin

Playing on the Nandi graph may only be interesting as a kindergarten game. To entertain elementary school students, let us share with you some more secrets about the σ-game and its lit-only relative. Before doing that, we change gears and build on a definition of our game in a more mathematical way.

A *digraph* D consists of its nonempty vertex set V_D and its arc set $\mathrm{A}_D \subseteq \mathrm{V}_D \times \mathrm{V}_D$. We say that D is *empty* when the arc set A_D is the empty set. An element $(u, v) \in \mathrm{V}_D \times \mathrm{V}_D$ will sometimes be conveniently recorded as uv. We reserve the notation L_D for the set of *loops* in D, namely $\mathrm{L}_D := \{v \in \mathrm{V}_D : vv \in \mathrm{A}_D\}$. For each $v \in \mathrm{V}_D$, we write $\mathrm{N}_D^+(v)$ for $\{w : vw \in \mathrm{A}_D\}$ and call it the set of *out-neighbours* of v in D. A digraph D is *symmetric* provided, for all $u, v \in \mathrm{V}_D$, either both uv and vu are arcs in D or none of them is an arc in D. A digraph is called a *graph* if it is symmetric and called *asymmetric* otherwise. A graph D can be thought of as a pair consisting of its vertex set V_D and edge set $\mathrm{E}_D \subseteq \binom{\mathrm{V}_D}{1} \cup \binom{\mathrm{V}_D}{2}$ where $\{u, v\} \in \mathrm{E}_D$ if and only if $uv \in \mathrm{A}_D$. For a graph D, we often simplify the notation $\mathrm{N}_D^+(v)$ as $\mathrm{N}_D(v)$ and call it the set of *neighbours* of v in D. A digraph is *strongly connected* if, for any two vertices of it, say u and v, there exist a nonnegative integer k and a sequence of vertices $u = u_0, u_1, \dots, u_k = v$ such that $u_{i-1}u_i$ is an arc of the digraph for every positive integer i not greater than k. For a graph, we often simply say that it is *connected* to mean that it is strongly connected. The set of all digraphs D on a vertex set V can be identified with the set of single variable functions β from V to $G := 2^V$ [53]: Given D, we put $\beta(v) := \mathrm{N}_D^+(v) \in G$ for all $v \in V$; Given β, we construct the digraph D by setting $\mathrm{A}_D := \{vw\colon\ w \in \beta(v)\}$. Indeed, both the map β and the digraph D can be further represented as the $V \times V$ $(0,1)$-matrix whose v, w-entry is 1 if and only if $w \in \beta(v)$, hence also standing for a linear map from the binary linear space 2^V to itself. We often freely switch among these few equivalent objects and this viewpoint explains how the binary linear algebra could come into play.

In his Presidential Address (General) to the Indian Mathematical Society Annual Conference held at University of Pune, December 2007, Ravindra Bapat [4] made the following claim:

> We are made to believe that physics, chemistry or biology are easier to explain to the general audience. $\cdots$ Their abstract concepts are presented in a way as if they are actual realities. Atoms, quarks, dark energy, strings, black holes, are all concepts but people believe in them. In comparison mathematicians are awfully shy of presenting anything for which they do not have a refereed proof. We need to be bold.

The concept of a group first appeared in the modern sense in the mind of Évariste Galois. Encouraged by the above quotation from Bapat, let us be brave enough to talk about the idea of a group without giving any definition of it here. If necessary, the reader could go to [2, 42] to check relevant definitions and see how groups measure the amount of symmetry.

Taking a group G, a set S and a map $\beta \in G^S$, the *Cayley digraph* $\Gamma(G, \beta)$ has vertex set G and arc set $\{(g, g\beta(s))\colon\ s \in S, g \in G\}$. The structure of many families of Cayley digraphs and some related mathematical objects have been widely investigated in various guises in several mathematical fields [1, 29, 36, 38]. Assume that we are travelling in a world modelled by the Cayley digraph $\Gamma(G, \beta)$. Each element $s \in S$ is a vehicle which can take us from where we are, say $g \in G$, to $g\beta(s) \in G$. The geometry of $\Gamma(G, \beta)$ is what we should concern with when we know that every vehicle $s \in S$ is available everywhere. But what if each vehicle $s \in S$ is available only at a specific set of stations? To be more precise, we now have a map α from S to 2^G, which specifies the "lit-only restriction", and the real world which we live in becomes the *lit-only Cayley digraph* $\Gamma(G,\beta,\alpha)$ that has vertex set G and arc set $\{(g,g\beta(s)):\ s\in S, g\in\alpha(s)\}$. If $\alpha(s)=G$ for all $s\in S$, surely $\Gamma(G,\beta,\alpha)$ is just $\Gamma(G,\beta)$.

Fix a set V. The power set of V, namely 2^V, forms a vector space over $\mathbb{F}_2$ where the sum $x+y$ of two elements x and y of 2^V is given by the symmetric difference $x\triangle y$ of them. The empty set $\emptyset$ is the zero element in 2^V, and it is usually denoted by $\mathbf{0}$ to emphasize that it is an element of the vector space 2^V. Take any map β from V to 2^V and let α be the map from V to 2^V such that $\alpha(v):=\{x\in 2^V:\ v\in x\}$. Recall that the map β corresponds to a digraph D with vertex set V. Playing the *σ-game* on the digraph D is just to start from any vertex of $\Gamma(2^V,\beta)$, that is, a subset of V, and to find a "good" path in $\Gamma(2^V,\beta)$ which ends at some "good" vertex. Here, a good path may refer to a path of minimum length and a good vertex may refer to a subset of V of minimum possible size. Similarly, playing the *lit-only σ-game* on the digraph D is just to start from any vertex of $\Gamma(2^V,\beta,\alpha)$ and to find a "good" path in $\Gamma(2^V,\beta,\alpha)$ which ends at some "good" vertex. We often write $\mathcal{PS}_D$ and $\mathcal{PS}^\sigma{}_D$ for $\Gamma(2^V,\beta,\alpha)$ and $\Gamma(2^V,\beta)$, respectively, and call them the *phase space of the lit-only σ-game* and the *phase space of the σ-game* on the digraph D, respectively. In the combinatorial games community, people use the concept of "game graph" [22] with the same meaning of our "phase space" here. We point the readers to [26, Appendix A] for the drawings of some concrete phase spaces of the lit-only σ-game. For $x,y\in 2^V$, we write $x\to_{\sigma D} y$ to mean that there is a path from x to y in $\mathcal{PS}^\sigma{}_D$ and we use $x\to_D y$ to indicate that there is a path in $\mathcal{PS}_D$ leading from x to y. Note that $x\to_{\sigma D} y$ follows from $x\to_D y$.

The process shown in Fig. 1.2 can be recorded as $\{a,b,c,d,e\}\xrightarrow{e}\{a,d\}$ $\xrightarrow{b}_\sigma$ $\{b,c,d,e\}\xrightarrow{c}\mathbf{0}$. We may even write directly $\{a,b,c,d,e\}\xrightarrow{ebc}_\sigma\mathbf{0}$ or $\{a,b,c,d,e\}\to_\sigma\mathbf{0}$. Similarly, the process shown in Fig. 1.3 can be represented as $\{a,b,c,d,e\}\xrightarrow{bdced}\mathbf{0}$ or $\{a,b,c,d,e\}\to\mathbf{0}$.

Among many research problems on the lit-only σ-game, let us mention a natural one here to which every lit-only σ-game player must want to know the answer.

Problem 2.1 (The lit-only σ-game reachability problem)**.** Let D be a digraph and we have as input two subsets x and y of V_D. Can we decide efficiently whether or not $x \to_D y$?

Since the size of the phase space $\mathcal{PS}_D$ is exponential in $|\,\mathrm{V}_D\,|$, it may even be unclear if the above general decision problem is in NP. Let us invite the reader to tackle some special easy cases: What about $x = \mathbf{0}$ and $y \neq \mathbf{0}$? Or, $x \neq \mathrm{L}_D$ and $y = \mathrm{L}_D$?

Let D be a graph. A classic result, called Sutner's Theorem and proved repeatedly in the literature before and after Sutner [3, 44, 45], asserts that

$$\mathrm{L}_D \to_{\sigma D} \mathbf{0}\,. \tag{2.1}$$

Goldwasser and Klostermeyer [24] wondered whether or not it indeed holds

$$\mathrm{L}_D \to_D \mathbf{0} \tag{2.2}$$

on the condition that $\mathrm{L}_D = \mathrm{V}_D$. After knowing this problem from a talk of John Goldwasser at Shanghai Jiao Tong University (SJTU) in the spring of 2008, Mr. Mu Li, an undergraduate student of SJTU then, was able to work out a short proof for a positive answer in a few days. However, after some search in the internet, Mu Li found the Lights Out webpage of Scherphuis and noticed that the same result had been announced together with a proof earlier there. Note that Scherphuis asserted on his webpage that several of his attempts at a proof had holes and it is remarkably hard to really get a proof. Inspired by the discovery of Li, Goldwasser and his two new friends in China, Xinmao Wang and Yaokun Wu, then turned to the project of understanding how much difference the lit-only restriction will make for playing the σ-game. An outcome of Goldwasser's journey to China in 2008 spring is the paper [27] in which you find the following generalization of the result of Scherphuis: Assuming that D is a connected graph with $\mathrm{V}_D = \mathrm{L}_D$ and that $x, y \in 2^{\mathrm{V}_D}$ are two configurations satisfying $x \neq \mathbf{0}$ and $y \neq \mathrm{V}_D$, then $x \to_{\sigma D} y$ if and only if $x \to_D y$. Finally, let us mention that Eq. (2.2) indeed holds for all graphs D and a proof of it comes from a simple application of the Gaussian elimination [52]. This fact just explains what you saw from Figs. 1.2 and 1.3. We hope that Sutner's Theorem (Eq. (2.1)) and its lit-only version (Eq. (2.2)) will enable you challenge and surprise elementary school students by leaving them many different digraphs, which should have a nonempty set of loops, of course.

This note will focus more on the lit-only version of the σ-game and so let us prepare one more tiffin (light meal) about it. A *tree* is a connected

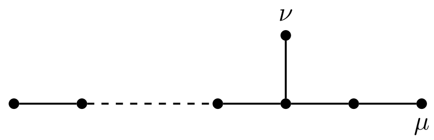


FIGURE 2.1. The ADE graphs.

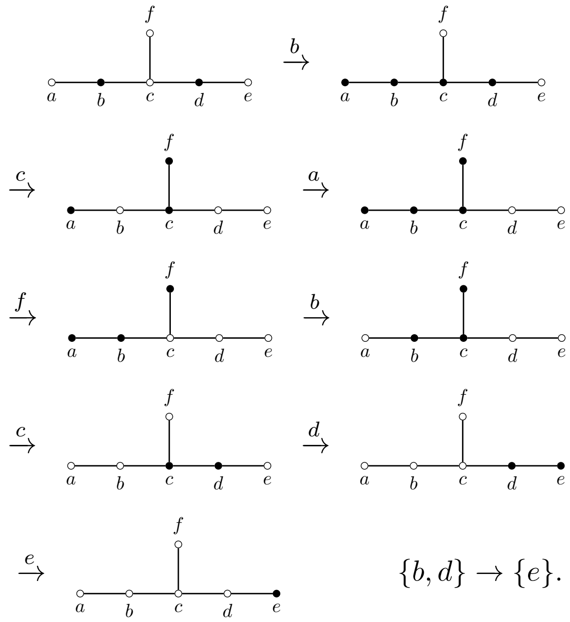


FIGURE 2.2. Playing the lit-only σ-game on E_6.

graph without cycles. Let E_n be the n-vertex tree as shown in Fig. 2.1. Let $D_{n-1} := E_n - \mu$ and let $A_{n-2} := D_{n-1} - \nu$. Note that A_n, D_n, E_6, E_7, E_8 are known as simply-laced Dynkin diagrams. In Fig. 2.2, we show an example of decreasing the size of a set from two to one in the lit-only σ-game on E_6. What is behind Fig. 2.2 is the Borel-De Siebenthal Theorem [8], which says that, whichever you start in the lit-only σ-game on a simply-laced Dynkin diagram, you always have a strategy to reach a configuration of size at most one. Going one step further, Hsin [31, Corollary 3.2] finds that a connected graph is strictly Borel-De Siebenthal if and only if it is one of the ADE graphs defined above, where a graph is called *strictly Borel-De Siebenthal* if, for the lit-only σ-game on any of its connected subgraphs, every given configuration can evolve into a one of size at most one.

A brief history of the σ-game is reported in [44]. There are vast researches on the σ-game, say [5, 19, 25, 54, 55], from which you can find out many interesting stuff for children. Due to the invention of Vogan diagram

[37] in the study of Lie algebra, some Lie theorists have been good players of the lit-only σ-game [6, 11, 12, 18, 23, 32, 33, 41, 43]. The lit-only σ-game and its close variants also appear in other interesting contexts; see [9, 16, 17, 20, 21, 34, 35].

At this point, we recommend to stop reading this note, click on the links given above or in our references and continue there or simply turn to the fundamental problem for playing the lit-only σ-game, Problem 2.1. It is normal to view Problem 2.1 as an algorithmic problem; but an algorithm is nothing but an evolution rule of a dynamics or a design of a Markov chain. So, if your parents want you prepare for your university life, say to read calculus for the study of various dynamical systems, you can still have fun with Problem 2.1 by claiming that you are working hard on a calculus-free dynamical system. In case you insist and agree with Bertrand Russell that ordinary language is totally unsuited for expressing what physics really asserts, please be ready to follow a sequel of (strange!) definitions below, which we arrive at after playing the lit-only σ-game for some while, and then, hopefully, share our excitement and curiosity on what those concepts will bring forth. Note that after providing more ideas on our tour in the land of the lit-only σ-game, in § 6 we will be able to sketch our partial solution to the algorithmic problem mentioned above.

3. A strange goose

We increase our stock of observations on the lit-only σ-game in this section, demonstrating that it is really a strange goose. We are mainly interested in the phase spaces of the game and the possible difference caused by the lit-only restriction.

3.1. **Minimum light number.** The puzzles which you can bring back home for kids from § 2 are all about how many lights can be switched off, namely the minimum light number. Instead of developing tricks on solving those puzzles faster, let us show something about the influence of the lit-only restriction from a mathematical perspective.

Take a digraph D. For any $x \in 2^{\mathrm{V}_D}$, define

$$\mathsf{ML}_D(x) := \min\{|y| : x \to_D y\} \quad \text{and} \quad \mathsf{ML}^\sigma{}_D(x) := \min\{|y| : x \to_{\sigma} {}_D\, y\}.$$

Let

$$\mathsf{ML}(D) := \max_{x \in 2^{\mathrm{V}_D}} \mathsf{ML}_D(x) \quad \text{and} \quad \mathsf{ML}^\sigma(D) := \max_{x \in 2^{\mathrm{V}_D}} \mathsf{ML}^\sigma{}_D(x).$$

For any graph G and $v \in \mathrm{V}_G$, let $\deg_G(v) := |\{e \in \mathrm{E}_G : v \in e\}|$. A *leaf* of a tree G is a vertex v with $\deg_G(v) = 1$.

On August 17, 2005, Gerard Jennhwa Chang gave a plenary talk on

Lie algebra in the Third Pacific Rim Conference on Mathematics held in Fudan University, Shanghai. Yaokun Wu tried to run across Shanghai to sit in this talk but was only able to catch the last one third of it. In the summer of 2006, Wu attended a three-week summer school on topology in Guizhou University and he met his former classmate Xinmao Wang there. They decided to do something together for fun in their spare time during the summer school. What they obtained in that cool summer in Guiyang is the following, which was inspired by a corresponding conjecture posed in the talk of Chang.

Theorem 3.1 ([47, 48]). *Let T be a tree with $\ell > 1$ leaves. Then* $\mathsf{ML}(T) \leq \lceil \frac{\ell}{2} \rceil$ *and* $\mathsf{ML}^\sigma(T) \leq \lfloor \frac{\ell}{2} \rfloor$.

Conjecture 3.2 ([26, Conjecture 14] [49, Conjecture 4, Conjecture 7]). It holds $\mathsf{ML}(G) - \mathsf{ML}^\sigma(G) \leq \frac{|\mathrm{V}_G|}{2}$ for any graph G. If G is a tree with zero or more loops attached, then $\mathsf{ML}(G) - \mathsf{ML}^\sigma(G) \leq 1$. If G is obtained from a unicyclic graph by adding zero or more loops, then $\mathsf{ML}(G) - \mathsf{ML}^\sigma(G) \leq 2$.

Conjecture 3.3 ([26, Conjecture 5] [49, Conjecture 15]). Let G be a connected graph on n vertices. Then $\max_{x \in 2^{\mathrm{V}_G}} (\mathsf{ML}_G(x) - \mathsf{ML}^\sigma{}_G(x)) \leq \max_{v \in \mathrm{V}_G} \deg_G(v) - 1$. Moreover, if $\max_{x \in 2^{\mathrm{V}_G}} (\mathsf{ML}_G(x) - \mathsf{ML}^\sigma{}_G(x)) > \frac{n}{2}$, then G is a complete m-partite graph for some positive integer $m \equiv 3 \pmod 4$.

3.2. **Reachability of the phase spaces.** Let D be a digraph and let S be the set of its strongly connected components. Let R be the set of ordered pairs (S_1, S_2) where S_1 and S_2 are different elements of S and there exists in D a path leading from a vertex in S_1 to a vertex in S_2. It is clear that R is a transitive acyclic relation on S, namely (S, R) is a poset, and this relation gives exactly the reachability relations in D. The *condensation* of the digraph D, denoted by $\mathrm{CD}(D)$, is the Hasse diagram of the poset (S, R).

The condensation operation helps us obtain a more readable picture of the phase spaces when we are only concerned with the reachability issue. Note that $\mathrm{CD}(\mathcal{PS}^\sigma{}_D)$ consists of isolated vertices as $\mathcal{PS}^\sigma{}_D$ is symmetric. Moreover, for vertices $x, y \in 2^{\mathrm{V}_D}$, $x \to_D y$ if and only if there exists a directed path from strongly connected component containing x to strongly connected component containing y in $\mathrm{CD}(\mathcal{PS}_D)$. Therefore, given $x, y \in 2^{\mathrm{V}_D}$, to test $x \to_D y$ ($x \to_{\sigma}{}_D\, y$) we can work in two steps, to determine

$\mathrm{CD}(\mathcal{PS}_D)$ ($\mathrm{CD}(\mathcal{PS}^\sigma{}_D)$) and then to determine in which strongly components of $\mathcal{PS}_D$ (components of $\mathcal{PS}^\sigma{}_D$) are x and y.

For any digraph D, $\{x \in 2^{\mathrm{V}_D} : \mathbf{0} \to_{\sigma D} x\}$ is a vector subspace of 2^{V_D}, which we call the *neighbour space* of D and denote by $\mathcal{N}_D$. We will always use rank_D to designate the dimension of $\mathcal{N}_D$ and use corank_D for $|\mathrm{V}_D| - \mathrm{rank}_D$, and call them the *rank* and *corank* of D, respectively. Note that the rank of a symmetric loopless digraph is always an even integer. We call a coset $\mathcal{M}$ of $\mathcal{N}_D$ in 2^{V_D} which is not $\mathcal{N}_D$ itself a *shifted neighbour space* of D. The concept of a neighbour space seems to be of little appearance in algebraic graph theory; in contrast, cut space and cycle space of a graph are much more popular there. However, in coding theory, neighbour spaces are well studied in the context of linear codes [14], or just binary codes if the coefficients are from the binary field as here. In the study of error-correcting code, $\mathsf{ML}^\sigma(D)$ is known as the covering radius of the binary code $\mathcal{N}_D$ [30]. The neighbour space $\mathcal{N}_D$ is of interest to the σ-game players as its cosets $\mathcal{M}$ are exactly the set of all strongly connected components of the symmetric digraph $\mathcal{PS}^\sigma{}_D$, that is, the reachability relation $\to_{\sigma D}$ coincides with $\bigcup_{\mathcal{M}}(\mathcal{M} \times \mathcal{M})$ where $\mathcal{M}$ runs through all cosets of $\mathcal{N}_D$. In other words, $\mathrm{CD}(\mathcal{PS}^\sigma{}_D)$ is the empty digraph with 2^{corank_D} vertices, each of which is a coset of $\mathcal{N}_D$.

It is clear that $\mathcal{PS}_D$ is a subgraph of $\mathcal{PS}^\sigma{}_D$ and so $x \to_D y$ could happen only when x and y come from the same coset of $\mathcal{N}_D$. So, $\mathrm{CD}(\mathcal{PS}_D)$ is the disjoint union of $\mathrm{CD}(\mathcal{PS}_D[\mathcal{M}])$ where $\mathcal{M}$ runs over all cosets of $\mathcal{N}_D$. If you have solved the easy questions mentioned after Problem 2.1, you will know that $\mathbf{0}$ and L_D are really interesting configurations for the lit-only σ-game. Indeed, $\mathbf{0}$ is a sink in $\mathcal{PS}_D$ and L_D is a source in $\mathcal{PS}_D$, and hence both $\{\mathbf{0}\}$ and $\{\mathrm{L}_D\}$, which are not necessarily distinct, form strongly connected components in $\mathcal{PS}_D$. Recall from Eq. (2.1) that $\{\mathbf{0}, \mathrm{L}_D\} \subseteq \mathcal{N}_D$. Surprising as it may first seem, essentially, this gives almost all the difference between the two reachability relationships $\to_{\sigma D}$ and $\to_D$. We clarify a bit what does it mean in the rest of this section.

To really see all possible differences caused by the lit-only restriction, we invited the computer to play the game. We say that a digraph is *loop-linked* provided every vertex can reach and be reached by a loop vertex in it. Our first try resulted in the discovery of [26, Conjecture 6], which says that, when D is a loop-linked strongly connected symmetric digraph, the differences between $\to_{\sigma D}$ and $\to_D$ can behave in two ways. When the first-named author introduced the lit-only σ-game in a course for freshmen in the winter of 2010, the second-named author was among those freshmen and thus the game was played a lot by him and his laptop then. Going beyond [26, Conjecture 6], our computer experiment suggests a classification of all strongly connected digraphs D into altogether fourteen classes, eight of them being graphs and six of them being asymmetric digraphs, such that for digraphs D from the same class the difference between $\to_{\sigma D}$ and $\to_D$ can be said to be the same. For all possible fourteen cases at most one shifted neighbour space of D is not strongly connected – we will call that possible only shifted neighbour space the *exceptional shifted neighbour space*. Moreover, $\mathrm{CD}(\mathcal{PS}_D(\mathcal{M}))$ is either an empty digraph or a directed path for exceptional shifted neighbour space $\mathcal{M}$.

A deep learning from the data generated by our computer experiment finally yields a proof of our guess for all connected graphs as well as a recognition of those eight classes of connected graphs. This means that we could have a paper-and-pencil proof for what we see for the lit-only σ-game on connected graphs and we can have a definition/characterization for those eight graph classes independent of the dynamical behaviors of the lit-only σ-game on them. These eight graph classes include two classes of loop-linked graphs and six classes of loopless graphs. Due to our structural characterization of the eight graph classes, the two loop-linked graph classes have been named loop-linked line graphs and loop-linked cuspidal graphs while the six loopless graph classes have been named opal line graphs, polished emerald line graphs, unpolished emerald line graphs, polished cuspidal graphs of type I, polished cuspidal graphs of type II, and unpolished cuspidal graphs [52]. We leave the afore-mentioned structural definition of these eight graph classes to Definitions 4.1 and 4.7. In Table 3.1, we depict $\mathrm{CD}(\mathcal{PS}_D)$ for any connected graph D according to which of the eight classes it belongs to. Indeed, the line labelled by "neighbour space" shows the digraph $\mathrm{CD}(\mathcal{N}_D)$; if there exists an exceptional shifted neighbour space $\mathcal{M}$ we will add a line labelled by "exception" to show the condensation digraph $\mathrm{CD}(\mathcal{M})$ of that exceptional shifted neighbour space. Note that the number in each ellipse refers to the size of the corresponding strongly connected component represented by that ellipse. Also note that the two singleton sets $\{\mathbf{0}\}$ and $\{\mathrm{L}_D\}$ always appear as vertices of $\mathrm{CD}(\mathcal{N}_D)$ and we simply use bullets to represent them. In Table 3.1, we do not bother to draw anything about a shifted neighbour space that is not exceptional as we know that it must be a strongly connected component of both $\mathcal{PS}_D$ and $\mathcal{PS}^{\sigma}{}_D$ of size $2^{\operatorname{rank} \mathcal{N}_D}$. With this convention, we can list a main result from [52].

Theorem 3.4 ([52])**.** *The set of all connected graphs D can be classified into eight types according to* $\mathrm{CD}(\mathcal{PS}_D)$ *as shown in Table 3.1.*

By Theorem 3.4, for any connected loop-linked graph D, $x \to_D \mathbf{0}$ is equivalent to $x \to_{\sigma D} \mathbf{0}$. According to Eq. (2.1), this fact can be said to be a generalization of Eq. (2.2). Although Eq. (2.2) possesses a short proof, we are still unaware of any simple proof of this more general result.

What we saw from computer experiment convinces us that strongly connected asymmetric digraphs can be classified into six classes, five of them being loop-linked and one of them loopless. This is summarized briefly below, where Table 3.2 should be read with the convention of reading Table 3.1.

Conjecture 3.5 ([52])**.** Strongly connected asymmetric digraphs D can be classified into six classes as shown in Table 3.2 according to $\mathrm{CD}(\mathcal{PS}_D)$.

We fail to verify Conjecture 3.5 so far. But, according to Avul Pakir Jainulabdeen Abdul Kalam, an aerospace scientist and the 11th President of India, F.A.I.L. only means "First Attempt In Learning". Our first attempt does demonstrate some small theoretical evidence for the truth of Conjecture 3.5. Let D be a strongly connected digraph. For every vertex $v \in \mathrm{V}_D$, let ϕ_v denote the set of vertices w of D for which there is a path in D from v to w such that $v \notin \mathrm{N}_D^+(u)$ for all vertices u in the path. Under the assumption that D contains a pair of its vertices v and w such that $\phi_v \cup \phi_w = \mathrm{V}_D$, we can verify Conjecture 3.5. Indeed, the phase space of such a digraph could only take the form as described in cases 6 and 8 in Table 3.1 or cases 9, 13 and 14 in Table 3.2 [52]. For a possible confirmation of Conjecture 3.5, it may be important to find, for each of the possible five classes of loop-linked asymmetric digraphs as indicated in Table 3.2, a structural characterization stated irrelevant with any specific dynamical process on the digraphs.

4. Some mathematical eggs

Let us exhibit some mathematical eggs laid by our strange goose and collected in [52]. We acknowledge that the story went deeper than we had imagined and so we do feel like the man in Fig. 4.1: are we getting paid for

Graph class	Phase space	
Six classes of loopless graphs:		
1. Opal line graphs	Neighbour space	$\mathbf{0}$ • $\binom{r+1}{2}$ $\binom{r+1}{4}$ $\cdots$ $\binom{r+1}{r-2}$ $\binom{r+1}{r}$
2. Polished emerald line graphs	Neighbour space	$\mathbf{0}$ • $\binom{r+2}{2}$ $\binom{r+2}{4}$ $\cdots$ $\binom{r+2}{r/2-1}$ $\binom{r+2}{r/2+1}$
	Exception	$\binom{r+2}{1}$ $\binom{r+2}{3}$ $\cdots$ $\binom{r+2}{r/2-2}$ $\binom{r+2}{r/2}$
3. Unpolished emerald line graphs	Neighbour space	$\mathbf{0}$ • $\binom{r+2}{2}$ $\binom{r+2}{4}$ $\cdots$ $\binom{r+2}{r/2-2}$ $\binom{r+2}{r/2}$
	Exception	$\binom{r+2}{1}$ $\binom{r+2}{3}$ $\cdots$ $\binom{r+2}{r/2-1}$ $\binom{r+2}{r/2+1}$
4. Polished cuspidal graphs, I	Neighbour space	$\mathbf{0}$ • $2^{r-1}-2^{r/2-1}$ $2^{r-1}+2^{r/2-1}-1$
5. Polished cuspidal graphs, II	Neighbour space	$\mathbf{0}$ • $2^{r-1}-2^{r/2-1}-1$ $2^{r-1}+2^{r/2-1}$
6. Unpolished cuspidal graphs	Neighbour space	$\mathbf{0}$ • $2^{r}-1$
	Exception	$2^{r-1}-2^{r/2-1}$ $2^{r-1}+2^{r/2-1}$
Two classes of loop-linked graphs:		
7. Loop-linked line graphs	Neighbour space	scriptsize $\mathbf{0} \bullet\!\leftarrow \binom{r}{1} \leftarrow \binom{r}{2} \leftarrow \cdots \leftarrow \binom{r}{r-2} \leftarrow \binom{r}{r-1} \leftarrow\!\bullet\ \mathrm{L}_D$
8. Loop-linked cuspidal graphs	Neighbour space	$\mathbf{0} \bullet\!\leftarrow 2^{r}-2 \leftarrow\!\bullet\ \mathrm{L}_D$

TABLE 3.1. The classification of all connected graphs D according to $\mathrm{CD}(\mathcal{PS}_D)$; see Theorem 3.4.

Graph class	Phase space	
9. Loopless asymmetric digraphs	Neighbour space	$\mathbf{0}\ \bullet \quad (2^r-1)$
Five classes of loop-linked asymmetric digraphs:		
10. Asymmetric digraphs, I	Neighbour space	$\mathbf{0}\bullet \leftarrow \binom{r}{1} \leftarrow \binom{r}{2} \leftarrow \cdots \leftarrow \binom{r}{r-2} \leftarrow \binom{r}{r-1} \leftarrow \bullet\ \mathrm{L}_D$
11. Asymmetric digraphs, IIA	Neighbour space	$\mathbf{0}\bullet \leftarrow \binom{r+1}{2} \leftarrow \binom{r+1}{4} \leftarrow \cdots \leftarrow \binom{r+1}{r-2} \leftarrow \binom{r+1}{r}$
	Exception	$\binom{r+1}{1} \leftarrow \binom{r+1}{3} \leftarrow \cdots \leftarrow \binom{r+1}{r-3} \leftarrow \binom{r+1}{r-1} \leftarrow \bullet\ \mathrm{L}_D$
12. Asymmetric digraphs, IIB	Neighbour space	$\mathbf{0}\bullet \leftarrow \binom{r+1}{2} \leftarrow \binom{r+1}{4} \leftarrow \cdots \leftarrow \binom{r+1}{r-3} \leftarrow \binom{r+1}{r-1} \leftarrow \bullet\ \mathrm{L}_D$
	Exception	$\binom{r+1}{1} \leftarrow \binom{r+1}{3} \leftarrow \cdots \leftarrow \binom{r+1}{r-2} \leftarrow \binom{r+1}{r}$
13. Asymmetric digraphs, IIIA	Neighbour space	$\mathbf{0}\bullet \leftarrow (2^r-2) \leftarrow \bullet\ \mathrm{L}_D$
14. Asymmetric digraphs, IIIB	Neighbour space	$\mathbf{0}\bullet \leftarrow (2^r-1)$
	Exception	$(2^r-1) \leftarrow \bullet\ \mathrm{L}_D$

TABLE 3.2. A possible classification of all strongly connected asymmetric digraphs; see Conjecture 3.5.

FIGURE 4.1. The goose that laid the golden eggs.
By Milo Winter (1888–1956).
https://www.ancient.eu/image/10121/the-goose-that-laid-the-golden-egg/

eating ice-cream? Here is a grook and a superegg from Piet Hein [4], which do suggest us taking fun in earnest:

The Eternal Twins

Taking fun as
simply fun and
earnestness in
earnest shows
how thoroughly
thou none of the
two discernest.

A superegg lamp.
https://www.piethein.com/product/superegg-300-1901/

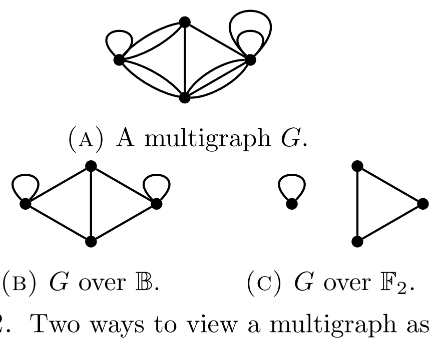
4.1. **Line graphs and cuspidal graphs.** During his visit to Shanghai in 2008 spring, John Goldwasser suggested to analyze the shape of the phase space of the lit-only σ-game on (symmetric) paths. Following this suggestion, Mu Li discovered nice inductive patterns from computer experiments on paths and then he was working with Yaokun Wu towards a paper on the very explicit description of $\mathcal{PS}_D$ for all paths D. The manuscript was not polished to a stage to be submitted anywhere, unfortunately. But, the pleasant applications of inductive arguments/descriptions in this project did direct Wu to think about the underlying geometry which controls the dynamics and then he found a connection between the lit-only σ-game and

[4] He often plays with blocks and sticks. See: https://www.piethein.com/category/games-36/.

the concept of "line graphs" in [51]. We point out that the line graph concept used in [51] is the classical one as introduced by Hassler Whitney [50] and does not coincide with our usage of line graphs in Table 3.1. Indeed, those "line graphs" as defined by Whitney correspond to our loopless ordinary line graphs in view of Definition 4.1 to be presented below.

The phase space structure as shown on the right of the last two lines of Table 3.1 has been predicted in [26, Conjecture 6]. The first significant step for us to prove that conjecture is to have a correct guess of the geometric structure which causes those two different dynamical behaviours. By trial-and-error, we divide the class of all graphs into the class of line graphs and the class of cuspidal graphs. To show you what these two graph classes are, it suffices to define line graphs.

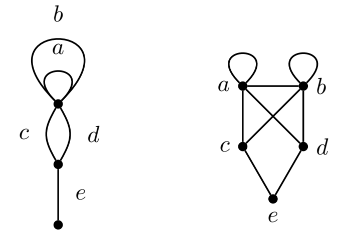
A *multigraph* G consists of a nonempty vertex set V_G, an edge set E_G, and a map ∂_G from E_G to $\binom{\mathrm{V}_G}{1} \cup \binom{\mathrm{V}_G}{2}$, called the *boundary map* of G. If $\mathrm{E}_G = \emptyset$, G is said to be an *empty* (multi)graph. When the boundary map is injective, we can identify an edge with its boundary, namely its image under the boundary map, and thus the multigraph in consideration becomes a graph (symmetric digraph). You may think of a multigraph as a symmetric nonnegative integer matrix while a graph as a symmetric $(0,1)$-matrix. There are two natural maps $\mathbb{N} \to \{0,1\}$: Viewing $\{0,1\}$ as the Boolean semifield $\mathbb{B}$, we send 0 to 0 and all positive integers to 1; Viewing $\{0,1\}$ as the binary field $\mathbb{F}_2$, we send every nonnegative integer n to $n \bmod 2$. Accordingly, as multigraphs are just nonnegative integer symmetric matrices, we have two natural maps from multigraphs to graphs. For example, the multigraph in Fig. 4.2(a) gives rise to two graphs in Fig. 4.2, 4.2(b) and 4.2(c), depending on which viewpoint we will adopt. In the literature, most definitions on line graphs and various variants use the Boolean semifield approach. We will go the other way here.



(A) A multigraph G.

(B) G over $\mathbb{B}$. (C) G over $\mathbb{F}_2$.

FIGURE 4.2. Two ways to view a multigraph as a graph.



(A) G (B) $\mathfrak{L}(G)$

FIGURE 4.3. A multigraph and its line graph.

Definition 4.1. Given a nonempty multigraph G, its *line graph*, denoted by $\mathfrak{L}(G)$, is the graph with vertex set $\mathrm{V}_{\mathfrak{L}(G)} := \mathrm{E}_G$ and edge set

$$\mathrm{E}_{\mathfrak{L}(G)} := \{\{e, f\} : |\partial_G(e) \cap \partial_G(f)| \equiv 1 \pmod 2,\ e, f \in \mathrm{E}_G\}. \tag{4.1}$$

If G has no isolated vertices, we will call G a *root multigraph* of $\mathfrak{L}(G)$. A line graph is *ordinary* if it is the line graph of a graph.

Theorem 4.2 ([7])**.** *A loopless graph is an ordinary line graph if and only if it does not contain any of the nine graphs from Fig. 4.4 as a vertex-induced subgraph.*

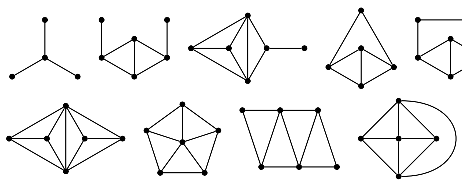


FIGURE 4.4. Nine forbidden vertex-induced subgraphs for loopless ordinary line graphs.

For an ordinary loopless line graph $\mathfrak{L}(G)$ and any $v \in \mathrm{V}_{\mathfrak{L}(G)}$, if we add several new vertices to $\mathfrak{L}(G)$ and make them adjacent with every vertex from $\mathrm{N}_{\mathfrak{L}(G)}(v)$, we will obtain a new loopless line graph with a root multigraph that can be obtained from G by adding several edges parallel to v. Note that, root multigraphs may not be unique. In Fig. 4.5, we show three forbidden vertex-induced subgraphs for loopless line graphs in Fig. 4.4 are actually line graphs. Moreover, for each of them a root multigraph is presented. In Fig. 4.5, one can also see how those parallel edges in the root multigraphs correspond to the independent set in the line graph which share the same neighbours.

Given a loopless graph G, to construct its line graph $\mathfrak{L}(G)$, which is ordinary by definition, Eq. (4.1) can be rewritten as

$$\mathrm{E}_{\mathfrak{L}(G)} := \{\{e, f\} : |\partial_G(e) \cap \partial_G(f)| > 0, \{e, f\} \in \binom{\mathrm{E}_G}{2}\}. \tag{4.2}$$

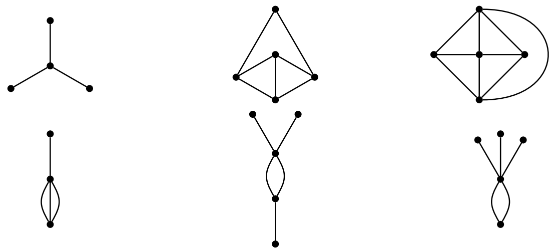


FIGURE 4.5. Three line graphs from the list of Beineke (Theorem 4.2) and their root multigraphs.

Comparing Eqs. (4.1) and (4.2), what we introduced in [52] as line graphs can be said to be a "mod two" version of the usual line graphs. Our exposure to the lit-only σ-game enables us [52] take the "mod two" definition and that has made all the difference.

A graph is called *nonsingular* if its associated matrix is nonsingular over the binary field. In all, there are 43 connected nonsingular 6-vertex graphs. They comprise of the 32 forbidden graphs from Fig. 4.6 and the 11 line graphs of 7-vertex trees from Fig. 4.7.

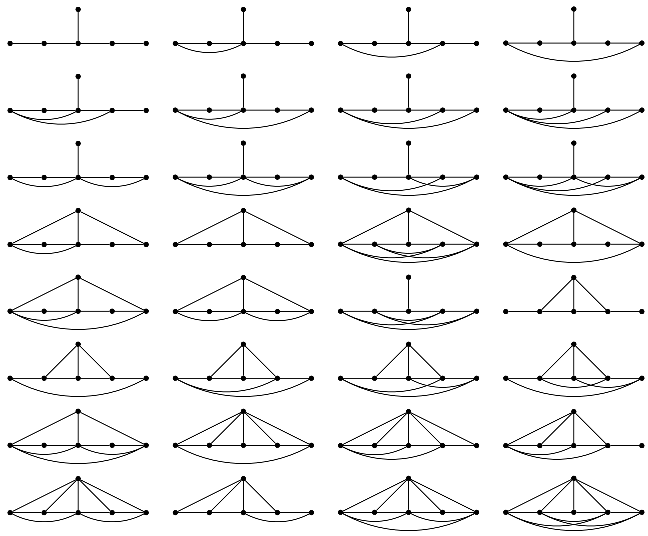


FIGURE 4.6. The 32 forbidden subgraphs for loopless line graphs.

Theorem 4.3 ([52]). *For a loopless graph G, the following statements are equivalent.*

- *The graph G is a line graph.*
- *The graph G does not contain any graph from the thirty-two graphs depicted in Fig. 4.6 as an induced subgraph.*

- *Every connected 6-vertex vertex-induced subgraph of G is a line graph.*
- *Every connected nonsingular 6-vertex vertex-induced subgraph of G is one of the eleven line graphs of 7-vertex trees as shown in Fig. 4.7.*

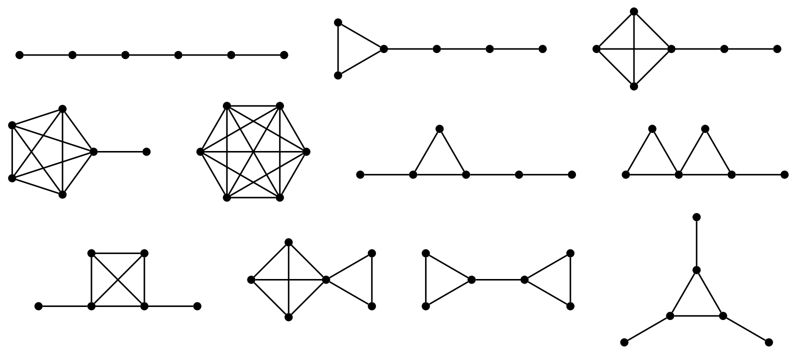


FIGURE 4.7. Eleven 6-vertex line graphs of trees.

Let G be a graph and v be a loop vertex of G. Let $G \boxminus v$ be the graph with vertex set $V_G \setminus \{v\}$ and edge set $E_G \triangle \binom{N_G(v)}{1} \triangle \binom{N_G(v)}{2}$. We say that $G \boxminus v$ is obtained from G by *melting a loop* v.

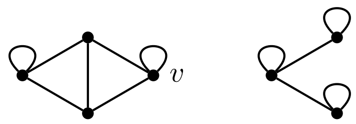


FIGURE 4.8. A graph G with a loop v, and $G \boxminus v$.

Theorem 4.4 ([52])**.** *A loop-linked graph is cuspidal if and only if it can be reduced to one of the two graphs in Fig. 4.9 by a sequence of deleting vertex operation and melting loop operation through loop-linked graphs.*

FIGURE 4.9. The two minimal (with respect to operators $-$ and $\boxminus$) loop-linked cuspidal graphs.

William Thurston made the following claim:

> The product of mathematics is clarity and understanding.
> Not theorems, by themselves.

Will Theorems 4.3 and 4.4 and other results about line graphs in [52] stimulate you to do some mathematics and yield a better understanding of some nice new structure?

4.2. **Quardratic forms and bilinear forms.** Let D be a digraph. For each $v \in \mathrm{V}_D$, let v^* be the linear functional on 2^{V_D} such that, for all $x \in 2^{\mathrm{V}_D}$,

$$v^*(x) :\equiv \begin{cases} 1, & v \in x; \\ 0, & v \notin x. \end{cases}$$

For every $v \in \mathrm{V}_D$, construct a map $\mathcal{T}_{D,v} \in \mathsf{End}(2^{\mathrm{V}_D})$:

$$x \mapsto \begin{cases} x \triangle \mathrm{N}_D^+(v), & v \in x, \\ x, & v \notin x. \end{cases}$$

Note that $\mathcal{T}_{D,v} = \mathrm{Id}_{2^{\mathrm{V}_D}} + \mathrm{N}_D^+(v)v^*$ and hence a transvection. Clearly, $\mathcal{T}_{D,v}$ is invertible if and only if $v \notin \mathrm{L}_D$. The *lit-only monoid of* D is the multiplicative monoid generated by $\mathcal{T}_{D,v}$'s:

$$\mathsf{LOM}_D := \langle \mathcal{T}_{D,v} : \ v \in \mathrm{V}_D \rangle.$$

To study the lit-only σ-game on D is to understand the action of LOM_D on 2^{V_D}. If D is loopless, the lit-only monoid of D is indeed a subgroup of $\mathsf{SL}(2^{\mathrm{V}_D})$ and we call it the *lit-only group* of D; in this case, we often write LOG_D for LOM_D accordingly. Since $\mathcal{N}_D$ is an invariant subspace under the action of LOG_D, there is a natural restriction homomorphism $\cdot|_{\mathcal{N}_D} : \mathsf{LOG}_D \to \mathsf{SL}(\mathcal{N}_D)$.

Definition 4.5. Let D be a loopless digraph. Its *topaz group*, denoted by TG_D, is the image of the natural restriction homomorphism $\cdot|_{\mathcal{N}_D}$, namely

$$\mathsf{TG}_D := \langle g|_{\mathcal{N}_D} : \ g \in \mathsf{LOG}_D \rangle,$$

which is a subgroup of $\mathsf{SL}(\mathcal{N}_D)$.

We remark that Topaz is the name of some gemstone. This name may come from "tapas", a Sanskrit (the ancient language of India) word which means "fire". Tapas is now more well-known as a wide variety of snacks in Spanish cuisine, which is designed to encourage conversation.

It is noteworthy that the classification of the phase spaces of the lit-only σ-game on loopless graphs, namely Table 3.1, is well accompanied by a classification of the lit-only groups and Topaz groups. To give more details of this, we need some facts about classical groups and so we should turn to quadratic forms and bilinear forms now.

Let $\mathcal{V}$ be a vector space over the binary field $\mathbb{F}_2$ and let $b : \ \mathcal{V} \times \mathcal{V} \to \mathbb{F}_2$ be a bilinear map. We call b a *symmetric bilinear form* on $\mathcal{V}$ if $b(x,y) \equiv b(y,x)$ for all $x, y \in \mathcal{V}$ and we call b an *alternating bilinear form* on $\mathcal{V}$ if

$b(x,x) \equiv 0$ for all $x \in \mathcal{V}$. The bilinear form b is *non-degenerate* if $x = 0$ is the only possible $x \in \mathcal{V}$ such that $b(x,y) \equiv 0$ for all $y \in \mathcal{V}$, or equivalently, the matrix for the bilinear map b with respect to any basis of $\mathcal{V}$ is nonsingular. Since we are working on a field of characteristic 2, an alternating form must be symmetric, but not vice versa. A *symplectic form* is a nondegenerate alternating bilinear form.

For any map $Q: \mathcal{V} \to \mathbb{F}_2$, we define its *polarisation* to be the map $\nabla_Q: \mathcal{V} \times \mathcal{V} \to \mathbb{F}_2$ such that $\nabla_Q(x,y) :\equiv Q(x+y) - Q(x) - Q(y)$, for all $x, y \in \mathcal{V}$. A *quadratic form* on $\mathcal{V}$ is a map $Q: \mathcal{V} \to \mathbb{F}_2$ such that ∇_Q is bilinear, and so an alternating form. We call a quadratic form Q *non-degenerate* or *non-defective* if $x = 0$ is the only $x \in \mathcal{V}$ such that $Q(x) = 0$ and $\nabla_Q(x,y) = 0$ for all $y \in \mathcal{V}$; equivalently, there is no $\mathbb{F}_2$-linear map p from $\mathcal{V}$ to one of its proper subspaces such that $Q = Q \circ p$ [13, Theorem 7.3].

Two bilinear forms b and b' on the binary linear spaces $\mathcal{V}$ and $\mathcal{V}'$ respectively are *isometric* provided there is an $\mathbb{F}_2$-linear isomorphism $\rho: \mathcal{V} \to \mathcal{V}'$ such that $b = b' \circ (\rho, \rho)$. Two quadratic forms Q and Q' on the binary linear spaces $\mathcal{V}$ and $\mathcal{V}'$ respectively are *isometric* provided there is an $\mathbb{F}_2$-linear isomorphism $\rho: \mathcal{V} \to \mathcal{V}'$ such that $Q = Q' \circ \rho$.

To illuminate the definitions above, let us pause for some simple examples. Let X be a set. All subsets of X of even size, denoted by $\binom{X}{\text{even}}$, is a vector subspace of 2^X. It is clear that $|\cdot|/2$ is a quadratic form on the vector space $\binom{X}{\text{even}}$ and its polarisation is given by $\nabla_{|\cdot|/2}(x,y) \equiv |x \cap y|$ for all $x, y \in \binom{X}{\text{even}}$. Note that the quotient space $2^X/\{\mathbf{0}, X\}$ is also a binary linear space. We often write an element $\{A, B\} \in 2^X/\{\mathbf{0}, X\}$ as $A|B$ and thus view $2^X/\{\mathbf{0}, X\}$ as the set of splits on X, the notation $A|B$ meaning that X is partitioned into two disjoint sets A and B. For each $S \in 2^X/\{\mathbf{0}, X\}$, we define $\overline{S_x}$ and S_x to be the two sets such that $x \in S_x$, $x \notin \overline{S_x}$ and $S = S_x|\overline{S_x}$. For any $x \in X$, define the quadratic form Q_x on $2^X/\{\mathbf{0}, X\}$ by setting

$$Q_x(S) :\equiv |\overline{S_x}|.$$

Note that the polarisation of Q_x vanishes everywhere. We point out that there is a natural nondegenerate pairing of $\binom{X}{\text{even}}$ and $2^X/\{\mathbf{0}, X\}$ as vector spaces [15]:

$$\begin{array}{ccc} \binom{X}{\text{even}} \times 2^X/\{\mathbf{0}, \mathrm{V}_G\} & \to & \mathbb{F}_2, \\ (Z, A|B) & \mapsto & |A \cap Z|. \end{array}$$

Consider a binary vector space $\mathcal{V}$ and a subgroup H of $\mathsf{GL}(\mathcal{V})$. We say that H *preserves* a bilinear form b on $\mathcal{V}$ if $b \circ (g, g) = b$ for all $g \in H$ and we say that H *preserves* a quadratic form q on $\mathcal{V}$ if $q \circ g = q$ for all $g \in H$.

For a digraph D, its adjacency form $\mathbb{A}_D$ is a bilinear form on 2^{V_D} given by $\mathbb{A}_D(x,y) :\equiv \sum_{uv\in \mathrm{A}_D} u^*(x)v^*(y)$. If D is loopless and symmetric, then $\mathbb{A}_D$ is an alternating form.

Lemma 4.6 ([52])**.** *Let D be a loopless strongly connected digraph. The following statements hold.*

(i) *If D is asymmetric, then TG_D does not preserve any symplectic form on $\mathcal{N}_D$.*

(ii) *If D is symmetric, then TG_D preserves a unique symplectic form ω_D on the vector space $\mathcal{N}_D$, where ω_D is defined uniquely by*
$$\omega_D \circ (\mathrm{N}_D, \mathrm{N}_D) = \mathbb{A}_D\,.$$

Take a graph G. Regarding G as a one-dimensional abstract simplicial complex, we can define its *Euler characteristic* (over $\mathbb{F}_2$), denoted by $\chi(G)$, as $|\,\mathrm{V}_G\,| - |\,\mathrm{E}_G\,| \in \mathbb{F}_2$. The *Euler form* of G (over $\mathbb{F}_2$), denoted by χ_G, is the quadratic form on 2^{V_G} given by
$$\chi_G(x) :\equiv \chi(G[x]) \equiv \sum_{v\in \mathrm{V}_G \setminus \mathrm{L}_G} v^*(x)^2 - \sum_{uv\in \mathrm{E}_G \setminus \mathrm{L}_G} u^*(x)v^*(x) \pmod 2,$$
for all $x \in 2^{\mathrm{V}_G}$, where $G[x]$ is the subgraph of G induced by $x \subseteq \mathrm{V}_G$.

Definition 4.7. A graph G is *polished* provided there exists a (unique) quadratic form q_G on $\mathcal{N}_G$ such that
$$q_G \circ \mathrm{N}_G = \chi_G. \tag{4.3}$$
A graph is *unpolished* if it is not polished.

Lemma 4.8 ([52])**.** *Let G be a loopless connected graph. The following statements hold.*

(i) *If G is unpolished, then TG_G does not preserve any non-zero quadratic form on the vector space $\mathcal{N}_G$.*

(ii) *If G is polished, then TG_G preserves a unique non-zero quadratic form q_G on vector space $\mathcal{N}_G$, where q_G is defined uniquely by Eq. (4.3).*

We use algebraic properties to define polished graphs. Is there any geometric characterization of them? Here is a small step in this direction.

Theorem 4.9 ([52])**.** *Let G be a loopless connected multigraph. Then, $\mathfrak{L}(G)$ is polished if and only if $|\,\mathrm{V}_G\,| \not\equiv 2 \pmod 4$.*

To proceed, let us review some relevant basic facts about classical groups [28, 46]. Let $\mathcal{V}$ be an $\mathbb{F}_2$-linear space. The *orthogonal group* on $\mathcal{V}$

with respect to a quadratic form q on $\mathcal{V}$, denoted by $\mathsf{O}(\mathcal{V}, q)$, is the subgroup of $\mathsf{SL}(\mathcal{V})$ preserving q, namely, $\mathsf{O}(\mathcal{V}, q) := \{g \in \mathsf{SL}(\mathcal{V}) : q \circ g = q\}$. The *symplectic group* on $\mathcal{V}$ with respect to a symplectic form ω on $\mathcal{V}$ is the subgroup of $\mathsf{SL}(\mathcal{V})$ preserving ω, namely, $\mathsf{Sp}(\mathcal{V}, \omega) := \{g \in \mathsf{SL}(\mathcal{V}) : \omega \circ (g, g) = \omega\}$. It is clear that $\mathsf{O}(\mathcal{V}, q) \leq \mathsf{Sp}(\mathcal{V}, \nabla_q)$.

Let us first consider the case that $\dim \mathcal{V} = 2k + 1$ is odd. In this case, there is no symplectic form on $\mathcal{V}$ and there is a unique nondegenerate quadratic form q_{2k+1} on $\mathcal{V}$ up to isometry. We write O_{2k+1} for $\mathsf{O}(\mathcal{V}, q_{2k+1})$, which is determined by k up to isomorphism and does not depend on the choice of q_{2k+1}.

We next turn to the case that $\dim \mathcal{V} = 2k$ is even. In this case, a quadratic form Q on $\mathcal{V}$ is non-degenerate if and only if the alternating form ∇_Q is non-degenerate, namely if and only if ∇_Q is symplectic. Up to isometry, there is a unique symplectic form on $\mathcal{V}$, hence a unique symplectic group on $\mathcal{V}$ up to isomorphism. Let $\mathsf{Sp}_{2k} := \mathsf{Sp}(\mathcal{V}, \omega)$, where ω is a symplectic form on $\mathcal{V}$, and call it the symplectic group of dimension $2k$. We remark that $\mathsf{O}_{2k+1} = \mathsf{Sp}_{2k}$ [28, Theorem 14.2] and the order of the group is given in [28, Theorem 3.12]. Up to isometry, there are exactly two non-degenerate quadratic forms on $\mathcal{V}$. If Q is one of them, i.e., if ∇_Q is symplectic, the *Arf invariant* of it, named after the Turkish mathematician Cahit Arf and referred to be the "democratic invariant" by William Browder [10], is the element of $\mathbb{F}_2$ that occurs most often among the values of Q. We can index the two nondegenerate quadratic forms on $\mathcal{V}$ by q_{2k}^+, which has Arf invariant 0, and q_{2k}^-, which has Arf invariant 1 [28, Theorem 13.14]. They indeed give rise to two nonisomorphic orthogonal groups [28, Theorem 14.48],

$$\mathsf{O}_{2k}^+ := \mathsf{O}(\mathcal{V}, q_{2k}^+) \quad \text{and} \quad \mathsf{O}_{2k}^- := \mathsf{O}(\mathcal{V}, q_{2k}^-).$$

A formula for the sizes of the two groups is reported as [28, Theorem 14.48]. We call O_{2k}^+ and O_{2k}^- the dimension-$2k$ orthogonal groups of plus type and minus type respectively. Note that $i \mapsto (-1)^i$ gives an isomorphism from the additive group $\mathbb{F}_2$ to the multiplicative group $\{+1, -1\}$ and this explains the correspondence between the Arf invariants and the types of the orthogonal groups.

A line graph is *opal* if it has a root multigraph which has an odd number of vertices, and a line graph is *emerald* if it has a root multigraph which has an even number of vertices. Let P_2 be the connected graph with two vertices and one edge. Note that P_2 is opal but not emerald. In general, a line graph may be both opal and emerald. Actually, it can be shown that loopless connected opal emerald line graphs are exactly complete 3-partite graphs and complete bipartite graphs except P_2, and all loopless connected opal emerald line graphs have rank 2. Thus, graph classes in Theorem 4.10 are disjoint and form a partition of all strongly connected loopless nonempty digraphs.

Theorem 4.10 ([52])**.** *Let D be a strongly connected loopless nonempty digraph. The topaz group TG_D and lit-only group LOG_D for symmetric digraphs are determined by the type of D as follows.*

Graph class	TG_D	LOG_D
Opal line graph	$\mathsf{Sym}_{\text{rank}_D+1}$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\text{corank}_D}$
Emerald line graph of rank ≥ 4	$\mathsf{Sym}_{\text{rank}_D+2}$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\text{corank}_D-1}$
Polished cuspidal graph	$\mathsf{O}^+(\mathcal{N}_D)$ *or* $\mathsf{O}^-(\mathcal{N}_D)$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\text{corank}_D}$
Unpolished cuspidal graph	$\mathsf{Sp}(\mathcal{N}_D)$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\text{corank}_D-1}$
Asymmetric digraph	$\mathsf{SL}(\mathcal{N}_D)$	

One day in the winter of 2012 and on the second floor of our old mathematics building at SJTU, Eiichi Bannai, one of our colleagues at SJTU then who had been sitting in several of our talks on the lit-only σ-game on different occasions, posed to us his key question: "Is there any group theory in your work"? We do not know much group theory and we even do not write down any definition of a group in § 2, although we do understand that group and symmetry are often viewed as the core of mathematics. Henceforth, what we could do is to explain to him in some way our partial work of determining the topaz group and lit-only group. Bannai firmly claimed that it should be some easy job to get a complete solution of our "group theory" question and suggested us to loop up some old work by McLaughlin [39, 40]. This really calls up our courage to reach Theorem 4.10 in a short time. We conjectured that the lit-only group for a loopless strongly connected asymmetric digraph D is $\mathsf{TG}_D \ltimes \mathcal{N}_D^{\text{corank}_D}$. Maybe, it can also be verified or disproved easily?

4.3. **Nuclei.** A *nucleus* of a digraph M is a nonsingular vertex-induced subgraph N of M fulfilling $\operatorname{rank} N = \operatorname{rank} M$. The set of all nuclei of M is denoted by $\mathfrak{N}(M)$. If M is a nonsingular digraph, $\mathfrak{N}(M)$ consists of M itself. For example, the nuclei of a complete bipartite graph coincide with all its vertex-induced P_2's.

Theorem 4.11 ([52])**.** *Let G be a connected graph and H be a nonempty connected subgraph of G. The digraph $\mathfrak{L}(H)$ is a connected nucleus of $\mathfrak{L}(G)$*

if and only if one of the following holds:

(i) $L_G \neq \emptyset$, *and* H *is the union of a spanning tree of* G *and a loop of* G*;*

(ii) $L_G = \emptyset$ *and* $|V_G|$ *is odd, and* H *is a spanning tree of* G*;*

(iii) $L_G = \emptyset$ *and* $|V_G|$ *is even, and* H *is a spanning tree of* $G - v$ *for some* $v \in V_G$*.*

A graph class is *connected-hereditary* if it is closed under taking connected vertex-induced subgraphs, is *hereditary* if it is closed under taking vertex-induced subgraphs, and is *additive* if it is closed under taking disjoint unions.

Theorem 4.12 ([52])**.** *Let* $\mathscr{G}$ *and* $\mathscr{H}$ *be two connected-hereditary graph classes. If every minimal graph in* $\mathscr{G} \setminus \mathscr{H}$ *is nonsingular, then every connected graph in* $\mathscr{H} \setminus \mathscr{G}$ *has a connected nucleus in* $\mathscr{G} \setminus \mathscr{H}$*.*

Corollary 4.13 ([52])**.** *Let* $\mathscr{G}$ *be an additive and hereditary graph class. Every (connected) graph in* $\mathscr{G}$ *has a (connected) nucleus in* $\mathscr{G}$*.*

Corollary 4.14 ([52])**.** *For each of the following graph classes, every graph in it has a connected nucleus in the same graph class.*

- $\{$*Connected loopless line graphs*$\}$=$\{$*Connected loopless line graphs*$\}$ $\setminus \emptyset$.
- $\{$*Connected loop-linked line graphs*$\}$ = $\{$*Connected line graphs*$\}\setminus$ $\{$*Loopless graphs*$\}$.
- $\{$*Connected loopless cuspidal graphs*$\}$=$\{$*Connected loopless graphs*$\}$ $\setminus$ $\{$*Line graphs*$\}$.
- $\{$*Connected loop-linked cuspidal graphs*$\}$ = $\{$*Connected graphs*$\}$ $\setminus$ $\{$*Loopless graphs or line graphs*$\}$.

Corollary 4.15 ([52])**.** *All loopless nonsingular graphs are polished. In particular, nuclei of loopless graphs are polished.*

In Fig. 4.10, we display some strongly connected digraphs which do not have any strongly connected nuclei.

Problem 4.16. When does a strongly connected digraph have a strongly connected nucleus?

5. Origin of eggs

As a player of the lit-only σ-game, you may be wondering how exactly does our goose lay those strange eggs as displayed in § 4. To convince you that the eggs are not stolen from elsewhere, let us indicate below our

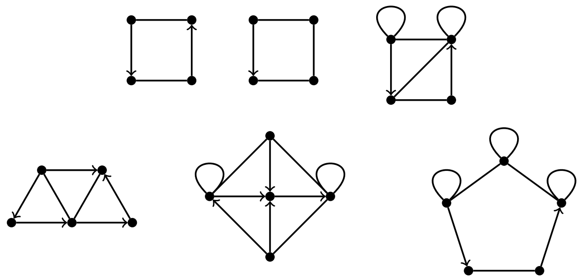


FIGURE 4.10. Some strongly connected digraphs which do not have strongly connected nuclei.

approaches in getting Theorems 3.4 and 4.10. Firstly, here is our strategy for loopless graphs.

Step 1 Understand the structures of line graphs and cuspidal graphs. § 4.1 presents some such work. Thoughout the whole process of our project, we need to apply various results established here and this step is really the foundation for everything.

Step 2 Analyze various forms attached to graphs, say symplectic form and Euler form as discussed in § 4.2. The analysis of these forms produces some rough descriptions of topaz group, lit-only group, and dynamical behavior of the lit-only σ-game.

Step 3 Classify the topaz group. The work in [40] suggests to us a list of candidates for a more careful inspection.

Step 4 Classify the phase spaces of nonsingular loopless graphs. The phase spaces for nonsingular graphs can be thought as the orbits of the topaz groups acting on the phase spaces regarded as a set. Thus, after determining all topaz groups, which are classical groups, we can get a description of the phase spaces with the help of known results about classical groups.

Step 5 Classify the phase spaces of all loopless graphs. In this step, we reduce the problem to the problem for nonsingular graphs, thanks to the concept of nuclei as introduced in § 4.3.

Step 6 Classify the lit-only group. In this step, we use the description of the phase space to prove that the lit-only group is a semidirect product of the topaz group and some abelian group.

Next, we need to deal with graphs with loops. Unfortunatenely, we do not have the blessings of groups any more. We somehow try to replace those group theoretic arguments for the previous case by some complicated combinatorial arguments and can thus study the structure of the phase space directly.

During August 12–25, 2019, the International Conference and PhD-Master Summer School on Groups and Graphs, Designs and Dynamics [5] was held in Yichang, China. The younger author of this essay gave a talk there entitled "Root systems over $\mathbb{F}_2$," which is basically to report the tour above towards Theorems 3.4 and 4.10. We assume that you have seen groups, graphs and dynamics from our story. Is design missing from the talk? No, it is not lost but already seen from the title of the talk. We mention that the connected components in the phase space of the lit-only σ-game can be thought of as root systems over $\mathbb{F}_2$. To explain briefly the connection between the usual root systems and our binary root system, let us end this section with the following dictionary.

Root system over $\mathbb{R}$	Root system over $\mathbb{F}_2$
Cartan matrix	Adjacency matrix
Simple roots	Vertices
Dynkin digram	Graph
Simple reflection	Simple transvection
Weyl group	Lit-only group
Euclidean space	Phase space
Root system	Connected component in the phase space

TABLE 5.1. A comparison between root systems over $\mathbb{R}$ and over $\mathbb{F}_2$.

6. A BITE TO GO

We will be described as a bit of an odd duck if we tell you so many stories about the lit-only σ-game but say nothing about the status of Problem 2.1. So, here is our brief answer to that problem: Yes, a polynomial time algorithm does exist for symmetric digraphs.

According to Theorem 3.4, given a connected symmetric digraph D and two subsets $x, y \subseteq \mathrm{V}_D$, we can determine whether or not $x \to_D y$ if we can

(i) find the graph class in Table 3.1 that contains D;
(ii) find the strongly connected components C_x and C_y in $\mathcal{PS}_D$ containing x and y, respectively.

The first step to determine the graph class of D is to recognize if D is a line graph or not. When D is reduced, we give a linear time algorithm to recognize line graphs and reconstruct their root graphs in [52]. When

[5] http://math.sjtu.edu.cn/conference/G2D2/

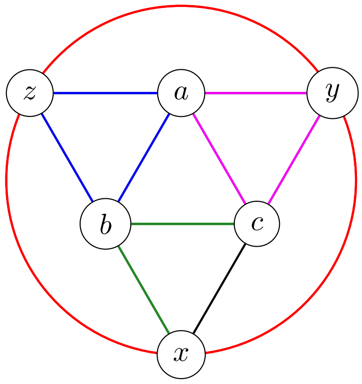


FIGURE 6.1. A line graph and its edge clique partition.

D is not reduced, we first find the modular decomposition of D, which can be constructed in linear time, and then we can read its reduced graph $\widetilde{D}$ from its modular decomposition. The graph D is a line graph if and only if its reduced graph $\widetilde{D}$ is a line graph, and the root multigraphs of D can be obtained from root graphs of $\widetilde{D}$ easily. Once we can determine whether or not D is a line graph, we can find the graph class in Table 3.1 that contains D by using corresponding linear algebraic characterization in [52], which can be done in polynomial time.

Finding the strongly connected components C_x and C_y requires explicit description of all strongly connected components in $\mathcal{PS}_D$. We refer you to [52] for more details.

To finish our short but full journey about the lit-only σ-game, let us try a small bite of some piece of the claimed polynomial time algorithm: the reconstruction of a root graph of the given graph G in Fig. 6.1. We first pick a maximal clique $\{x, y, z\}$ of G. For each vertex in the clique, say x, we choose the subgraph induced by x and its neighbourhood $\{y, z, b, c\}$ that are not in $\{x, y, z\}$. Then, we get another clique $\{x, b, c\}$. We continue this procedure for all encountered cliques and all of their vertices. At the end, we get four cliques: $\{x, y, z\}, \{x, b, c\}, \{z, a, b\}, \{y, a, c\}$. The intersection graph of these four cliques is the clique K_4, whose line graph turns out to be G. For general graphs, we can use the same idea to find good partitions of the edges sets, if they exist. A root graph can be constructed in linear time for any given such good partition.

As suggested by Friedrich Nietzsche, if you would go up high, then use your own legs! Hopefully, the bit of mathematics we have discussed so far will inspire some readers and even motivate some brave kids have an after-dinner walk further into the graden of discrete-time graph processes.

Acknowledgements: Yaokun Wu dedicates this essay to his father, Jinbao Wu (May 31st, 1936 – Jan. 24th, 2017), for his love and tolerance. When he was a kid, his father challenged him with puzzles on counterfeit coins and others; but he never succeeded in solving any such puzzles even at the end of his childhood. According to A. P. J. Abdul Kalam (again!), E.N.D means "Effort Never Dies". Seeing that many smart and kind persons have been around in different periods to help, the father of Yaokun Wu should rest assured that his son will enjoy playing many nice games as he wishes.

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HYPERGEOMETRIC BERNOULLI POLYNOMIALS AND NUMBERS OF FRACTIONAL INDEX

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Abstract. In this paper, we shall study what we referred to as Bernoulli numbers and polynomials of fractional index and prove some of their properties. While these numbers and polynomials are similar to the classical Bernoulli numbers and polynomials, they differ in many aspects. Among the properties we shall study are the recurrence formulas satisfied by these numbers and polynomials. We shall also introduce the higher order Bernoulli polynomials of fractional order and establish some of their properties. Finally, we will show their connections with Hermite polynomials.

1. Introduction

The classical Bernoulli numbers and polynomials are defined, respectively, by the generating functions

$$\frac{x}{e^x-1}=\sum\nolimits_{m=0}^{\infty} B_m(x^m/m!) \tag{1.1}$$

and

$$\frac{xe^{xz}}{e^x-1}=\sum\nolimits_{m=0}^{\infty} B_m(z)(x^m/m!), \tag{1.2}$$

where $|x|<2\pi$.

Howard [7] generalized both numbers and polynomials to:

$$\frac{x^2/2!}{e^x-1-x}=\sum\nolimits_{m=0}^{\infty}A_m(x^m/m!) \text{ and } \frac{x^2e^{xz}/2!}{e^x-1}=\sum\nolimits_{m=0}^{\infty}A_m(z)(x^m/m!), \tag{1.3}$$

respectively. In 2008 and 2010, A. Hassen and H. D. Nguyen in [5] and [6] studied the generalization of Bernoulli numbers and polynomials (that they referred to as hypergeometric Bernoulli numbers and polynomials):

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Key words and phrases: Bernoulli numbers and polynomials, Bernoulli numbers and polynomials of fractional index, higher order Bernoulli numbers and polynomials of fractional index, Hermite polynomials.

$$\frac{x^N/N!}{e^x - T_{N-1}(x)} = \sum\nolimits_{m=0}^{\infty} B_{m,N}(x^m/m!) \tag{1.4}$$

and

$$\frac{x^N e^{xz}/N!}{e^x - T_{N-1}(x)} = \sum\nolimits_{m=0}^{\infty} B_{m,N}(z)(x^m/m!), \tag{1.5}$$

where $T_N(x)$ is the Taylor polynomial of order N for the exponential function.

In this paper we shall investigate what we call *Bernoulli numbers and polynomials of fractional index* defined as in (2.1) and (2.2) respectively.

The fractional Bernoulli numbers and polynomials are introduced by Legesse and Hassen where they have made a study of the fractional hypergeometric zeta function, see [11]. However, here we have a slightly modified definition from that of theirs.

In Section 2, we shall establish some properties of the Bernoulli numbers and polynomials of fractional index and their recurrence formulas. Among other things, we will show that these numbers and polynomials satisfy the following recurrence relations for $m \in \mathbb{N}$

$$(m+1)B_{a,m+1}+2m(m+1)B_{a,m-1}+2a\sum\nolimits_{k=0}^{m}\binom{m+1}{k}B_{a,k}B_{a,m-k+1}=0, \tag{1.6}$$

and

$$(m+1)B_{a,m+1}(z) - (m+1)zB_{a,m}(z)+2m(m+1)B_{a,m-1}(z) + 2a\sum\nolimits_{k=0}^{m}\binom{m+1}{k}B_{a,k}(z)B_{a,m-k+1}=0. \tag{1.7}$$

In Section 3, we will introduce the higher order Bernoulli numbers and polynomials of fractional index and study some of their recurrence formulas and properties. Finally, in the last section we shall express the Bernoulli numbers and polynomials of fractional index using matrix determinant and investigate their connections to Hermite polynomials.

2. Fractional Bernoulli Numbers and Polynomials

In this section, we introduce the Bernoulli numbers and polynomials of fractional index and prove some of their properties.

Definition 2.1. Let a be a positive half odd integer. The *Bernoulli numbers of fractional index* are defined by the generating function

$$\frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)} = \sum\nolimits_{m=0}^{\infty} B_{a,m}(x^m/m!), \tag{2.1}$$

where $\gamma(z,\alpha)$ is the *incomplete gamma function* defined by $\gamma(z,\alpha) = \int_0^{\alpha} e^{-t}t^{z-1}dt$, for $Re\ z > 0$ and $|arg\ \alpha| < \pi$.

Using the fact that $\gamma(z,\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^{k+z}}{k!(k+z)}$, $z \neq 0, -1, -2, \cdots$, we may rewrite (2.1) to obtain

$$\begin{aligned} e^{-x^2} &= a\gamma(a,x^2)x^{-2a} \sum_{m=0}^{\infty} B_{a,m}(x^m/m!) \\ &= \left(\sum_{k=0}^{\infty} \frac{a(-1)^k x^{2k}}{k!(k+a)}\right)\left(\sum_{m=0}^{\infty} B_{a,m}(x^m/m!)\right) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}}{(k+a)k!(m-2k)!} x^m, \end{aligned}$$

Consequently,

$$\sum_{m=0}^{\infty} (-1)^m (x^{2m}/m!) = \sum_{m=0}^{\infty} \sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}}{k!(k+a)(m-2k)!} x^m.$$

Thus, we have the following recurrence formula for the Bernoulli numbers of fractional index:

$$\sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}}{k!(k+a)(m-2k)!} = \begin{cases} (-1)^{\frac{m}{2}}/(m/2)!, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

Analogues to the classical Bernoulli numbers, the Bernoulli numbers of fractional index with odd subscripts are zero. The first few nonzero Bernoulli numbers of fractional index are

$$B_{a,0} = 1, \qquad B_{a,2} = -(\ 2/(1+a)\ ), \qquad B_{a,4} = 24/\left((1+a)^2(2+a)\right),$$

$$B_{a,6} = \frac{120(-1+a)}{(1+a)^3(2+a)(3+a)}, \qquad B_{a,8} = \frac{40320(2-6a-a^2+a^3)}{(1+a)^4(2+a)^2(3+a)(4+a)},$$

$$B_{a,10} = \frac{3628800(-2+16a-11a^2-4a^3+a^4)}{(1+a)^5(2+a)^2(3+a)(4+a)(5+a)}.$$

Definition 2.2. The *Bernoulli polynomials of fractional index*, denoted by $B_{a,m}(z)$, are defined by the generating function

$$\alpha(z,x) = \frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)} = \sum_{m=0}^{\infty} B_{a,m}(z)(x^m/m!). \tag{2.2}$$

Lemma 2.1. *The Bernoulli polynomials of fractional index can be expressed in terms of the Bernoulli numbers of fractional index by the formula*

$$B_{a,m}(z) = \sum_{k=0}^{m} \binom{m}{k} B_{a,k} z^{m-k}. \tag{2.3}$$

Furthermore, the polynomials $B_{a,m}(z)$ satisfy the recurrence relation

$$\sum_{k=0}^{[m/2]} \frac{(-1)^k}{k!(m-2k)!}\left[\frac{a}{k+a} B_{a,m-2k}(z) - z^{m-2k}\right] = 0. \tag{2.4}$$

Proof. We combine (2.2) and (2.1) to get

$$\begin{aligned}\sum_{m=0}^{\infty} B_{a,m}(z)(x^m/m!) &= \frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)}e^{xz}\\ &= \left(\sum_{m=0}^{\infty} B_{a,m}(x^m/m!)\right)\left(\sum_{m=0}^{\infty} z^m(x^m/m!)\right)\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}B_{a,k}z^{m-k}(x^m/m!).\end{aligned}$$

Comparing coefficients yields (2.3). To prove (2.4), we multiply (2.2) by the denominator $a\gamma\left(a,x^2\right)$ and expand both sides of the resulting equation into power series to get

$$\begin{aligned}&\left(\sum_{m=0}^{\infty} z^m(x^m/m!)\right)\left(\sum_{k=0}^{\infty}(-1)^k(x^{2k}/k!)\right)\\ &\qquad= \left(\sum_{k=0}^{\infty}\frac{a(-1)^k x^{2k}}{k!(k+a)}\right)\left(\sum_{m=0}^{\infty} B_{a,m}(z)(x^m/m!)\right).\end{aligned}$$

Carrying out the multiplication, we obtain

$$\sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]}\frac{(-1)^k z^{m-2k}}{k!(m-2k)!}x^m = \sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]}\frac{a(-1)^k B_{a,m-2k}(z)}{(k+a)k!(m-2k)!}x^m.$$

Comparing the coefficients of x^m yields the recurrence formula of the Bernoulli polynomials fractional index. □

Here are the first few values of Bernoulli polynomials of fractional index: $B_{a,0}(z)=1$, $B_{a,1}(z)=z$, $B_{a,2}(z)=z^2-(2/(1+a))$, $B_{a,3}(z)=z^3-(6z/(1+a))$, $B_{a,4}(z)=z^4-12z^2/(1+a)+24/((1+a)^2(2+a))$, $B_{a,5}(z)=z^5-20z^3/(1+a)+120z/(\ (1+a)^2(2+a)\ )$.

Lemma 2.2. *The Bernoulli polynomials of fractional index satisfy the relation*

$$B_{a,m}(z_1+z_2)=\sum_{j=0}^{m}\binom{m}{j}B_{a,j}(z_1)z_2^{m-j}.$$

Proof. Using the fact that $\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}$, we have

$$\begin{aligned}B_{a,m}(z_1+z_2) &= \sum_{k=0}^{m}\binom{m}{k}B_{a,k}\ (z_1+z_2)^{m-k}\\ &= \sum_{k=0}^{m}\binom{m}{k}B_{a,k}\sum_{j=0}^{m-k}\binom{m-k}{j}z_1^j z_2^{m-k-j}\\ &= \sum_{k=0}^{m}\sum_{j=k}^{m}\binom{m}{k}\binom{m-k}{j-k}B_{a,k}z_1^{j-k}z_2^{m-j}\\ &= \sum_{0\le k\le j\le m}\binom{m}{j}\binom{j}{k}B_{a,k}z_1^{j-k}z_2^{m-j}\end{aligned}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \sum_{k=0}^{j} \binom{j}{k} B_{a,k} z_1^{j-k} z_2^{m-j}$$

$$= \sum_{j=0}^{m} \binom{m}{j} B_{a,j}(z_1) z_2^{m-j}$$

completing the proof. □

Remark 2.1. The following properties of the Bernoulli polynomials of fractional order can easily be verified.

(1) $B_{a,m}(0) = B_{a,m}$ and $B_{a,0}(z) = 1$.

(2) $\frac{\partial}{\partial z} B_{a,m}(z) = m B_{a,m-1}(z), \quad m \in \mathbb{N}$

(3) $\mathsf{B}_{a,m}(z+1) - B_{a,m}(z) = \sum_{k=1}^{m} \binom{m}{k} B_{a,m-k}(z)$.

(4) $B_{a,m}(0) \neq B_{a,m}(1)$.

Lemma 2.3. *The polynomials $B_{a,m}(z)$ satisfy*

$$B_{a,m}(1-z) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{k+1} B_{a,k}(z).$$

Proof. Using (2.2), we obtain

$$\sum_{m=0}^{\infty} B_{a,m}(1-z)(x^m/m!) = \frac{x^{2a} e^{x(1-z)-x^2}}{a\gamma(a,x^2)} = -\frac{(-x)^{2a} e^{-zx-x^2}}{a\gamma(a,x^2)} e^x$$

$$= -\left(\sum_{m=0}^{\infty} B_{a,m}(z)((-x)^m/m!)\right)\left(\sum_{m=0}^{\infty}(x^m/m!)\right)$$

$$= -\sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} B_{a,k}(z)(-1)^k (x^m/m!).$$

Comparing the coefficients yields the lemma. □

Theorem 2.1. *For each positive integer n, we have*

$$\sum_{j=0}^{n-1} B_{a,m}(z+(j/n)) = \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{n^{k-m}}{m+1} B_{a,k}(z)(B_{m+1-k}(n) - B_{m+1-k}), \quad (2.5)$$

where B_m and $B_m(z)$ are the classical Bernoulli numbers and polynomials, respectively.

Proof. We note that

$$\sum_{m=0}^{\infty} \left[\sum_{j=0}^{n-1} B_{a,m}(z+(j/n))\right](x^m/m!)$$

$$= \sum_{j=0}^{n-1} \left(\sum_{m=0}^{\infty} B_{a,m}(z+(j/n))(x^m/m!)\right)$$

$$= \sum_{j=0}^{n-1} \left(\frac{x^{2a} e^{x\left(z+\frac{j}{n}\right)-x^2}}{a\gamma(a,x^2)}\right) = \frac{x^{2a} e^{xz-x^2}}{a\gamma(a,x^2)} \left(\sum_{j=0}^{n-1} e^{jx/n}\right)$$

$$= \sum_{j=0}^{n-1} \left(\frac{x^{2a} e^{x\left(z+\frac{j}{n}\right)-x^2}}{a\gamma(a,x^2)} \right) = \frac{x^{2a} e^{xz-x^2}}{a\gamma(a,x^2)} \left(\sum_{j=0}^{n-1} e^{jx/n} \right)$$

$$= \frac{x^{2a} e^{xz-x^2}}{a\gamma(a,x^2)} \frac{e^x - 1}{e^{x/n} - 1} = \frac{x^{2a} e^{xz-x^2}}{a\gamma(a,x^2)} \frac{n}{x} \left\{ \frac{\frac{x}{n} e^{\frac{x}{n} n}}{e^{x/n} - 1} - \frac{x/n}{e^{x/n} - 1} \right\}$$

$$= \frac{n}{x} \left(\sum_{m=0}^{\infty} B_{a,m}(z)(x^m/m!) \right) \left(\sum_{m=0}^{\infty} [B_m(n) - B_m] \left((x/n)^m/m!\right) \right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m+1} \binom{m+2}{k} B_{a,k}(z) \frac{n^{k-m}}{m+1} (B_{m+1-k}(n) - B_{m+1-k}) \right) \frac{x^m}{m!}.$$

Therefore,

$$\sum_{j=0}^{n-1} B_{a,m}(z+(j/n)) = \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{n^{k-m}}{m+1} B_{a,k}(z)(B_{m+1-k}(n) - B_{m+1-k})$$

which completes the proof. □

If we put $n = 1$ in (2.5), we obtain

Corollary 2.1.

$$B_{a,m}(z) = (1/(m+1)) \sum_{k=0}^{m+1} \binom{m+1}{k} B_{a,k}(z)(B_{m+1-k}(1) - B_{m+1-k}).$$

Theorem 2.2. *Let p be a positive integer and $\beta_{a,m,p}$ be defined by*

$$\alpha(z,x,p) := \frac{x^{2a} e^{xz-x^2}}{(a\gamma(a,x^2))^p} = \sum_{m=0}^{\infty} \beta_{a,m,p}(z)(x^m/m!). \tag{2.6}$$

Then

$$\frac{\partial \alpha}{\partial x} = (2ax^{-1} + z - 2x)\alpha(z,x,p) - 2apx^{-1}\alpha(z,x,p)\alpha(x,1) \tag{2.7}$$

and

$$2ap \sum_{k=0}^{m+1} \binom{m+1}{k} \beta_{a,k,p}(z)\beta_{a,m-k+1,1} = (2a-m-1)\beta_{a,m+1,p}(z) + (m+1)z\beta_{a,m,p}(z) - 2(m+1)m\beta_{a,m-1,p}(z), \tag{2.8}$$

where $\alpha(x,1) = \alpha(0,x,1)$ and $\beta_{a,m,p} = \beta_{a,m,p}(0)$.

Proof. The first part is a routine computation. To see (2.8), multiply both sides of (2.7) by x to get

$$\sum_{m=0}^{\infty} m\beta_{a,m,p}(z)(x^m/m!) = 2a\alpha(z,x,p) + zx\alpha(z,x,p) - 2x^2\alpha(z,x,p) - 2ap\alpha(z,x,p)\alpha(x,1).$$

Using the series representation of $\alpha(z,x,p)$ in (2.6), we get

$$\sum_{m=0}^{\infty} m\beta_{a,m,p}(z)(x^m/m!) = 2a \sum_{m=0}^{\infty} \beta_{a,m,p}(z)(x^m/m!) + z \sum_{m=0}^{\infty} \beta_{a,m,p}(z)(x^{m+1}/m!) - 2 \sum_{m=0}^{\infty} \beta_{a,m,p}(z)(x^{m+2}/m!) - 2ap \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \beta_{a,k,p}(z)\beta_{a,m-k,1}(x^m/m!).$$

Comparing the coefficients yields (2.8). □

Remark 2.2. Note that when $p=1$, we have $\beta_{a,m,1}(z)=B_{a,m}(z)$.

Substituting equation (2.2), into the identity

$$x\frac{\partial\alpha}{\partial x}-(2a+zx-2x^2)\alpha(z,x,1)+2a\alpha(z,x)\alpha(x,1)=0,$$

where $\alpha(x,1)$ is the generating function of the Bernoulli numbers of fractional index, we obtain

$$\begin{aligned}\sum\nolimits_{m=0}^{\infty}(m-2a)B_{a,m}(z)(x^m/m!)-z\sum\nolimits_{m=0}^{\infty}B_{a,m}(z)(x^{m+1}/m!)\\ +2\sum\nolimits_{m=0}^{\infty}B_{a,m}(z)(x^{m+2}/m!)\\ +2a\sum\nolimits_{m=0}^{\infty}\sum\nolimits_{k=0}^{m}\binom{m}{k}B_{a,k}(z)B_{a,m-k}(x^m/m!)=0.\end{aligned}$$

Putting the coefficients of x^{m+1} equal to zero, we obtain (1.7):

$$\begin{aligned}(m+1)B_{a,m+1}(z)-(m+1)zB_{a,m}(z)+2m(m+1)B_{a,m-1}(z)\\ +2a\sum\nolimits_{k=0}^{m}\binom{m+1}{k}B_{a,k}(z)B_{a,m-k+1}=0,\quad m\in\mathbb{N}.\end{aligned}$$

One can also use this recurrence formula to calculate the Bernoulli polynomials of fractional index, with initial values of $B_{a,0}(z)=1$ and $B_{a,1}(z)=z$. Setting $z=0$ in (1.7) gives, for $m\in\mathbb{N}$, the following recurrence relation (see (1.6)) for the Bernoulli numbers of fractional index:

$$(m+1)B_{a,m+1}+2m(m+1)B_{a,m-1}+2a\sum\nolimits_{k=0}^{m}\binom{m+1}{k}B_{a,k}B_{a,m-k+1}=0. \quad (2.9)$$

We can now use this recurrence formula to compute the Bernoulli numbers of fractional index starting from $B_{a,0}=1$, and $B_{a,1}=0$.

We end this section with the observation that the identity $\frac{\partial\alpha}{\partial z}-x\alpha(z,x)=0$ leads to

$$\sum\nolimits_{m=0}^{\infty}B'_{a,m}(z)(x^m/m!)-\sum\nolimits_{m=0}^{\infty}mB_{a,m-1}(z)(x^m/m!)=0,$$

which in turn produces $B'_{a,m}(z)=mB_{a,m-1}(z),\ \ m\in\mathbb{N}$.

3. Higher order fractional Bernoulli polynomials and Bernoulli numbers and some of their recurrences

In this section, we shall introduce the higher order Bernoulli numbers and polynomials of fractional index and investigate some of their recurrence relations.

Definition 3.1. For $t\in\mathbb{N}$, the *higher order Bernoulli polynomials of fractional index* are defined by the generating function

$$\left(\frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)}\right)^t e^{xz}=\sum\nolimits_{m=0}^{\infty}B_{a,m}^{(t)}(z)(x^m/m!), \quad (3.1)$$

and the *higher order Bernoulli numbers of fractional index* are defined by $B_{a,m}^{(t)} = B_{a,m}^{(t)}(0)$.

Note that $B_{a,m}^{(1)}(z) = B_{a,m}(z)$, the Bernoulli polynomials of fractional index, and $B_{a,m}^{(1)}(0) = B_{a,m}$, the Bernoulli numbers of fractional index.

Theorem 3.1. *The higher order Bernoulli polynomials of fractional index satisfy, for $m \in \mathbb{N}$, the recurrence formulas*

$$(m+1)B_{a,m+1}^{(t)}(z) - (m+1)zB_{a,m}^{(t)}(z) + 2tm(m+1)B_{a,m-1}^{(t)}(z)+$$

$$2at\sum_{k=0}^{m}\binom{m+1}{k}B_{a,k}^{(t)}(z)B_{a,m-k+1} = 0, \tag{3.2}$$

and

$$(m+1)B_{a,m+1}^{(t)}+2tm(m+1)B_{a,m-1}^{(t)}+2at\sum_{k=0}^{m}\binom{m+1}{k}B_{a,k}^{(t)}B_{a,m-k+1}=0, \tag{3.3}$$

Proof. Differentiating both sides of (3.1) with respect to x and then multiplying by x, we obtain

$$2ta\sum_{m=0}^{\infty} B_{a,m}^{(t)}(z)\frac{x^m}{m!} - 2t\sum_{m=0}^{\infty} B_{a,m}^{(t)}(z)\frac{x^{m+2}}{m!} + z\sum_{m=0}^{\infty} B_{a,m}^{(t)}(z)\frac{x^{m+1}}{m!} -$$

$$2at\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}B_{a,k}^{(t)}(z)B_{a,m-k}(x^m/m!) = \sum_{m=0}^{\infty} mB_{a,m}(z)(x^m/m!).$$

Comparing the coefficients, we have the recurrence relation

$$(m+1)B_{a,m+1}^{(t)}(z) - (m+1)zB_{a,m}^{(t)}(z) + 2tm(m+1)B_{a,m-1}^{(t)}(z)+$$

$$2at\sum_{k=0}^{m}\binom{m+1}{k}B_{a,k}^{(t)}(z)B_{a,m-k+1} = 0, \quad m \in \mathbb{N}.$$

Putting $z = 0$ in (3.2) yields the recurrence relation of the higher order Bernoulli numbers of fractional index (3.3). □

Remark 3.1. For $t = 1$, (3.2) and (3.3) becomes the recurrence formulas of the Bernoulli polynomials of fractional index (1.7) and the Bernoulli numbers of fractional index (2.9), respectively. It is also clear that $B_{a,m}^{(t)}(z+1) = \sum_{k=0}^{m}\binom{m}{k}B_{a,k}^{(t)}(z)$.

Theorem 3.2. *For $m > 0$, the higher order Bernoulli polynomials of fractional index satisfy the recurrence relations*

$$B_{a,m+1}^{(t+1)}(z) = (2at)^{-1}(2at - m - 1)B_{a,m+1}^{(t)}(z) + (2at)^{-1}(m+1)zB_{a,m}^{(t)}(z)$$
$$- ma^{-1}(m+1)B_{a,m-1}^{(t)}(z).$$

Proof. Let $\alpha(a,x) = \frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)}$. Then from (3.1) we have

$$\alpha^t(a,x)e^{xz} = \sum_{m=0}^{\infty} B_{a,m}^{(t)}(z)(x^m/m!). \tag{3.4}$$

Differentiating both sides of (3.4) with respect to x, we get

$$2t(ax^{-1}-x)\alpha^t(a,x)e^{xz} - 2atx^{-1}\alpha^{t+1}(a,x)e^{xz} + z\alpha^t(a,x)e^{xz}$$
$$= \sum_{m=0}^{\infty} mB_{a,m}^{(t)}(z)(x^{m-1}/m!).$$

Multiplying these by x and using (3.4), we obtain

$$2ta\sum_{m=0}^{\infty} B_{a,m}^{(t)}(z)(x^m/m!) - 2t\sum_{m=0}^{\infty} B_{a,m}^{(t)}(z)(x^{m+2}/m!)$$
$$-\ 2ta\sum_{m=0}^{\infty} B_{a,m}^{(t+1)}(z)(x^m/m!) + z\sum_{m=0}^{\infty} B_{a,m}^{(t)}(z)\frac{x^{m+1}}{m!}$$
$$= \sum_{m=0}^{\infty} mB_{a,m}^{(t)}(z)(x^m/m!).$$

The result follows from comparing the coefficients. □

Theorem 3.3. *The higher order Bernoulli polynomials of fractional index satisfy the recurrence relations*

$$B_{a,m}^{(t)}(z_1+z_2+...+z_t) = \sum_{\substack{k_1+\cdots+k_t=m,\\ k_1,\cdots,k_t\geq 0}} \binom{m}{k_1,\cdots,k_t} B_{a,k_1}(z_1)\cdots B_{a,k_t}(z_t),$$

where $\binom{m}{k_1,\cdots,k_t}$ *is the multinomial coefficients.*

Proof. Using equation (3.1), we have

$$\sum_{m=0}^{\infty} B_{a,m}^{(t)}(z_1+z_2+\cdots+z_t)\frac{x^m}{m!} = \left(\frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)}\right)^t e^{(z_1+z_2+\cdots+z_t)x}$$
$$= \left(\sum_{m=0}^{\infty} B_{a,m}(z_1)\frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} B_{a,m}(z_2)\frac{x^m}{m!}\right)\cdots\left(\sum_{m=0}^{\infty} B_{a,m}(z_t)\frac{x^m}{m!}\right)$$
$$= \left(\sum_{m=0}^{\infty} B_a^m(z_1)\frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} B_a^m(z_2)\frac{x^m}{m!}\right)\cdots\left(\sum_{m=0}^{\infty} B_a^m(z_t)\frac{x^m}{m!}\right)$$
$$= e^{[B_a(z_1)+\cdots+B_a(z_t)]x} = \sum_{m=0}^{\infty}(B_a(z_1)+\cdots+B_a(z_t))^m\frac{x^m}{m!}$$
$$= \sum_{m=0}^{\infty}\sum_{\substack{k_1+\cdots+k_t=m\\ k_1,\cdots,k_t\geq 0}}\left[\binom{m}{k_1,\cdots,k_t}B_{a,k_1}(z_1)\cdots B_{a,k_t}(z_t)\right](x^m/m!),$$

and the proof is completed. □

Lemma 3.1. $\frac{d^k}{dz^k}B_{a,m}^{(t)}(z) = \frac{m!}{(m-k)!}B_{a,m-k}^{(t)}(z), \quad m \geq k.$

Proof. We have

$$\frac{d^k}{dz^k}\alpha^t(a,x)e^{xz} = \alpha^t(a,x)\frac{d^k}{dz^k}e^{xz} = \sum_{m=0}^{\infty}\frac{d^k}{dz^k}B_{a,m}^{(t)}(z)(x^m/m!).$$

Hence

$$\alpha^t(a,x)e^{xz}x^k = \sum_{m=0}^{\infty}\frac{d^k}{dz^k}B_{a,m}^{(t)}(z)(x^m/m!).$$

Therefore

$$\sum_{m=0}^{\infty} B_{a,m-k}^{(t)}(z)\frac{x^m}{(m-k)!} = \sum_{m=0}^{\infty}\frac{d^k}{dz^k}B_{a,m}^{(t)}(z)(x^m/m!).$$

Comparing the coefficients yields the lemma. □

Theorem 3.4. *The higher order Bernoulli polynomials of fractional index satisfy the differential equation*

$$\frac{B_{a,m}}{m!}y^m + \frac{B_{a,m-1}}{(m-1)!}y^{(m-1)} + \cdots + \left(\frac{B_{a,2}}{2!} + \frac{1}{a}\right)y^{''} - \frac{z}{2at}y' + \frac{m}{2at}y = 0.$$

Proof. By (3.2), we have

$$(m+1)B^{(t)}_{a,m+1}(z) - (m+1)zB^{(t)}_{a,m}(z) + 2tm(m+1)B^{(t)}_{a,m-1}(z) +$$
$$2at\sum\nolimits_{k=0}^{m}\binom{M+1}{k}B^{(t)}_{a,k}(z)B_{a,m-k+1} = 0,\;\; m \in \mathbb{N}.$$

Now replacing m by $m+1$, we find that

$$mB^{(t)}_{a,m}(z) - zmB^{(t)}_{a,m-1}(z) + 2tm(m-1)B^{(t)}_{a,m-2}(z)$$
$$+ 2at\sum\nolimits_{k=0}^{m-1}\binom{m}{k}B^{(t)}_{a,k}(z)B_{a,m-k} = 0. \tag{3.5}$$

Let us denote $B^{(t)}_{a,m}(z)$ by y. Then by Lemma 3.1, we have

$$B^{(t)}_{a,m-k}(z) = (\ (m-k)!/m!\ )y^{(k)},\;\; k = 1, 2, \cdots, m$$

and hence

$$B^{(t)}_{a,k}(z) = (k!/m!)y^{(m-k)},\;\; k = 0, 1, \cdots, m.$$

Substituting this in (3.5) and dividing both sides by $2at$, we get

$$(m/2at)y - (z/2at)y' + (1/a)y'' + \sum\nolimits_{k=0}^{m-1}\frac{B_{a,m-k}}{(m-k)!}y^{(m-k)} = 0,$$

from which we conclude that $(1/m!)B_{a,m}y^{(m)} + (B_{a.m-1}/(m-1)!)y^{(m-1)} + \cdots + (1/2!)B_{a,2}y'' + B_{a,1}y' + (m/2at)y - (z/2at)y' + (1/a)y'' = 0$, thereby proving the theorem. ☐

By taking $t = 1$ in the above theorem, we can get the differential equation satisfied by the Bernoulli polynomials of fractional index:

Corollary 3.1. *For $t = 1$ and $y = B_{a,m}(z)$, we have*

$$\frac{B_{a,m}}{m!}y^m + \frac{B_{a,m-1}}{(m-1)!}y^{(m-1)} + \cdots + \left(\frac{B_{a,2}}{2!} + \frac{1}{a}\right)y^{''} - \frac{z}{2a}y' + \frac{m}{2a}y = 0.$$

4. Determinant Representation of Bernoulli numbers and polynomials of fractional index and their relation with Hermite polynomials

The determinant expression of the classical Bernoulli numbers were first introduced by Glaisher in 1875, see [13], where he also discussed the Cauchy and Euler numbers. More recently, R. Booth and H.D. Nguyen in [1] established a determinant formula of Bernoulli polynomials by considering a square version of Pascal's triangle.

In this section, we shall show that the Bernoulli polynomials of fractional index also admit a determinant representation. To this end, we consider

$$f(x) = \sum_{m=0}^{\infty} c_m x^m, \quad \text{and} \quad g(x) = \sum_{m=0}^{\infty} a_m x^m,$$

and their quotient:

$$(\ f(x)/g(x)\ ) = \sum_{m=0}^{\infty} A_m x^m.$$

Clearly, from the Dirichlet product formula for power series, we have

$$c_m = \sum_{k=0}^{m} \binom{m}{k} a_k A_{m-k}.$$

As in [1], solving the above equation for A_m, we to obtain

$$A_m = (-1)^m \frac{1}{a_0^m} \begin{vmatrix} c_0 & a_0 & 0 & 0 & \cdots & 0 \\ c_1 & a_1 & a_0 & 0 & \cdots & 0 \\ c_2 & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m-1} & a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_0 \\ c_m & a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{vmatrix} \tag{4.1}$$

From the definition of the Bernoulli polynomials of fractional index, we can view the equation (2.2) as a division of two power series as follows:

$$\frac{e^{xz-x^2}}{ax^{-2a}\gamma(a,x^2)} = (\ f(x)/g(x)\ ) = \sum_{m=0}^{\infty} A_m x^m,$$

where

$$f(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{[m/2]} \frac{z^{m-2k}(-1)^k}{k!(m-2k)!} x^m \quad \text{and} \quad g(x) = \sum_{m=0}^{\infty} \frac{a(-1)^m}{m!(m+a)} x^{2m}.$$

Here

$$c_m = \sum_{k=0}^{[m/2]} \frac{z^{m-2k}(-1)^k}{k!(m-2k)!} \qquad \text{and} \qquad a_m = \begin{cases} \frac{a(-1)^{m/2}}{(\frac{m}{2})!(\frac{m}{2}+a)}, & \text{if} \quad m = \text{even}; \\ 0, & \text{if} \quad m = \text{odd}. \end{cases}$$

and $B_{a,m}(z) = m!A_m$. From (4.1), we find the following determinant formula for the Bernoulli polynomials of fractional index:

$$B_{a,m}(z) = m!(-1)^m a^m \mathbf{D} = m!(-1)^m a^m |b_{ij}|, \tag{4.2}$$

where the determinant $\mathbf{D}$ is as on the next page and

$$b_{ij} = \begin{cases} 0, & \text{if } i{+}1 < j \text{ and } i{=}j{+}2k, j \neq 1, k = 0, 1, \cdots ; \\ \sum_{k=0}^{\left[\frac{i-1}{2}\right]} \frac{z^{i-1-2k}(-1)^k}{k!(i-1-2k)!}, & \text{if } j = 1; \\ \frac{(-1)^{\left[\frac{i-j+1}{2}\right]}}{\left[\frac{i-j+1}{2}\right]!\left[\frac{i-j+1}{2}+a\right]}, & \text{if } j = i{+}1 \text{ and } i{=}j{+}k, j \neq 1, k = \text{odd}. \end{cases}$$

$$\mathbf{D} = \begin{vmatrix} 1 & \frac{1}{a} & 0 & 0 & \cdots & 0 \\ z & 0 & \frac{1}{a} & 0 & \cdots & 0 \\ \frac{z^2}{2!} - 1 & -\frac{1}{1+a} & 0 & \frac{1}{a} & \cdots & 0 \\ \frac{z^3}{3!} - z & 0 & -\frac{1}{1+a} & 0 & \cdots & 0 \\ \frac{z^4}{4!} - \frac{z^2}{2!} + \frac{1}{2} & \frac{1}{2!(2+a)} & 0 & -\frac{1}{1+a} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{[m/2]} \frac{z^{m-2k}(-1)^k}{k!(m-2k)!} & 0 & \frac{(-1)^{\left[\frac{m-1}{2}\right]}}{\left[\frac{m-1}{2}\right]!\left(\left[\frac{m-1}{2}\right]+a\right)} & 0 & \cdots & 0 \end{vmatrix}_{m+1}.$$

Finally, we put $z = 0$ in (4), get to obtain the determinant formula for the Bernoulli numbers of fractional index: $B_{a,m} = m!(-1)^m a^m \mathbf{G}$, where $\mathbf{G}$ is the determinant given by

$$\mathbf{G} = \begin{vmatrix} 1 & \frac{1}{a} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{a} & 0 & \cdots & 0 \\ -1 & -\frac{1}{1+a} & 0 & \frac{1}{a} & \cdots & 0 \\ 0 & 0 & -\frac{1}{1+a} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^{\left[\frac{m}{2}\right]}}{[m/2]!} & 0 & \frac{(-1)^{\left[\frac{m-1}{2}\right]}}{\left[\frac{m-1}{2}\right]!\left(\left[\frac{m-1}{2}\right]+a\right)} & 0 & \cdots & 0 \end{vmatrix}_{m+1}.$$

Definition 4.1. The Hermite polynomials $H_m(z)$ are defined by

$$\mathcal{H}_m(z) = (-1)^m e^{z^2} \frac{d^m}{dz^m} e^{-z^2},\ m = 0, 1, 2, ... \tag{4.3}$$

The first few Hermite polynomials are $\mathcal{H}_0(z) = 1$, $\mathcal{H}_1(z) = 2z$, $\mathcal{H}_2(z) = 4z^2 - 2$, $\mathcal{H}_3(z) = 8z^3 - 12z$, $\mathcal{H}_4(z) = 16z^4 - 48z^2 + 12$. An explicit formula for the Hermite polynomial $\mathcal{H}_m(z)$ is given by

$$\mathcal{H}_m(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k m!}{k!(m-2k)!}(2z)^{m-2k},$$

where $[\nu]$ denotes the largest integer $\leq \nu$. Furthrermore, a generating function for the Hermite polynomials is given by

$$\Omega(z,x) = e^{2zx-x^2} = \sum_{m=0}^{\infty} \frac{\mathcal{H}_m(z)}{m!} x^m, \quad |x| < \infty.$$

From equation (2.2), we find that

$$\begin{aligned} e^{xz-x^2} &= ax^{-2a}\gamma(a,x^2) \textstyle\sum_{m=0}^{\infty} B_{a,m}(z)(x^m/m!) \\ &= \left(\textstyle\sum_{k=0}^{\infty} \frac{(-1)^k a x^{2k}}{k!(k+a)}\right) \left(\textstyle\sum_{m=0}^{\infty} B_{a,m}(z)(x^m/m!)\right) \\ &= \textstyle\sum_{m=0}^{\infty} \sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}(z)}{k!(k+a)(m-2k)!} x^m. \end{aligned}$$

Thus, we have

$$\mathcal{H}_m(z) = \sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}(2z) m!}{k!(k+a)(m-2k)!}. \tag{4.4}$$

From (4.4), we obtain $\mathcal{H}_0(z) = B_{a,0}(2z)$, $\mathcal{H}_1(z) = B_{a,1}(2z)$, $\mathcal{H}_2(z) = B_{a,2}(2z) - \frac{2!a}{1+a} B_{a,0}(2z)$, $\mathcal{H}_3(z) = B_{a,3}(2z) - \frac{3!a}{1+a} B_{a,1}(2z)$, $\mathcal{H}_4(z) = B_{a,4}(2z) - \frac{4!a}{2!(1+a)} B_{a,2}(2z) + \frac{4!a}{2!(2+a)} B_{a,0}(2z)$, To express the fractional Bernoulli polynomials $B_{a,m}(z)$ in terms of Hermite polynomials $\mathcal{H}_n(z)$, we first note that

$$\frac{e^{xz-x^2}}{ax^{-2a}\gamma(a,x^2)} = \frac{\sum_{m=0}^{\infty} \mathcal{H}_m(z/2) \frac{x^m}{m!}}{\sum_{m=0}^{\infty} \frac{(-1)^m a}{(m+a)} \frac{x^{2m}}{m!}} = \sum_{m=0}^{\infty} B_{a,m}(z)(x^m/m!).$$

Using (4.1) with $c_m = (1/m!)\mathcal{H}_m(z/2)$ and

$$a_m = \begin{cases} \frac{a(-1)^{m/2}}{(m/2)!(\frac{m}{2}+a)}, & \text{if } m = even, \\ 0, & \text{if } m = \text{odd}, \end{cases},$$

we have $B_{a,m}(z) = (-1)^m m!\ \mathbf{B}$, where the determinant $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{vmatrix} \mathcal{H}_0(z/2) & 1 & 0 & 0 & \cdots & 0 \\ \mathcal{H}_1(z/2) & 0 & 1 & 0 & \cdots & 0 \\ \frac{1}{2!}\mathcal{H}_2(z/2) & \frac{-a}{1+a} & 0 & 1 & \cdots & 0 \\ \frac{1}{3!}\mathcal{H}_2(z/2) & 0 & \frac{-a}{1+a} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m!}\mathcal{H}_m(z/2) & 0 & \frac{a(-1)^{\frac{m-1}{2}}}{(\frac{m-1}{2})!(\frac{m-1}{2}+a)} & 0 & \cdots & 0 \end{vmatrix}_{m+1}.$$

Starting with the first row, if we factor $(1/(i-1)!)$ from the row i, then the Bernoulli polynomials of fractional index can be expressed as

$$((-1)^m/(1!2!3!\cdots(m-1)!))\ \mathbf{C}$$

where the determinant $\mathbf{C}$ is given by

$$\begin{vmatrix} \mathcal{H}_0(\frac{z}{2}) & 1 & 0 & 0 & \cdots & 0 \\ \mathcal{H}_1(\frac{z}{2}) & 0 & 1 & 0 & \cdots & 0 \\ \mathcal{H}_2\frac{z}{2}) & \frac{-2!a}{1+a} & 0 & 2! & \cdots & 0 \\ \mathcal{H}_2(\frac{z}{2}) & 0 & \frac{-3!a}{1+a} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{H}_{m-1}(\frac{z}{2}) & \frac{am!(-1)^{\frac{m-1}{2}}}{(\frac{m-1}{2})!(\frac{m-1}{2}+a)} & 0 & \frac{am!(-1)^{\frac{m-3}{2}}}{(\frac{m-3}{2})!(\frac{m-3}{2}+a)} & \cdots & (m-1)! \\ \mathcal{H}_m(\frac{z}{2}) & 0 & \frac{am!(-1)^{\frac{m-1}{2}}}{(\frac{m-1}{2})!(\frac{m-1}{2}+a)} & 0 & \cdots & 0 \end{vmatrix}_{m+1}.$$

Expanding this matrix with respect to the first row yields

$$B_{a,m}(z) = (\ (-1)^m/(1!2!3!\cdots(m-1)!)\ )$$
$$\times\left[\mathcal{H}_0(z/2)|b_{11}| - \mathcal{H}_1(z/2)|b_{21}| + \cdots + (-1)^m\mathcal{H}_m(z/2)|b_{m+1\ 1}|\right],$$

where b_{k1} is $m-1$ by $m-1$ matrix obtained by deleting the k^{th} row and the first column. Therefore,

$$B_{a,m}(z) = (\ (-1)^m/(1!2!3!\cdots(m-1)!)\ )\sum_{k=0}^{m}(-1)^k\mathcal{H}_k(z/2)|b_{k+1\ 1}|.$$

Here are the first few examples: $B_{a,0}(z) = \mathcal{H}_0(z/2)$, $B_{a,1}(z) = \mathcal{H}_1(z/2)$, $B_{a,2}(z) = \mathcal{H}_2(z/2) + \frac{2!a}{1+a}\mathcal{H}_0(z/2)$, $B_{a,3}(z) = \mathcal{H}_3(z/2) + \frac{3!a}{1+a}\mathcal{H}_1(z/2)$, $B_{a,4}(z) = \mathcal{H}_4(z/2) + \frac{4!a}{2!(1+a)}\mathcal{H}_2(z/2) + \left[\frac{4!a^2}{(1+a)^2} - \frac{4!a}{2!(2+a)}\right]\mathcal{H}_0(z/2)$, *and* $B_{a,5}(z) = \mathcal{H}_5(z/2) + \frac{5!a}{3!(1+a)}\mathcal{H}_3(z/2) + \left[\frac{5!a^2}{(1+a)^2} - \frac{5!a}{2!(2+a)}\right]\mathcal{H}_1(z/2)$.

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A NOTE ON A THEOREM DUE TO ABEL, PRINGSHEIM AND OLIVER

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Abstract. In this note, we improve a well-known classical Abel's theorem on positive decreasing terms of a series by imposing some new conditions on the positive terms.

1. Introduction and statements of Theorems

The following classical theorem on the positive decreasing terms of a series is due to Abel, Pringsheim and Oliver:

Theorem 1. *Let a_n be positive for all $n \geq 0$ and the sequence (a) $(a_n)_{n=0}^{\infty}$ be decreasing. Then (b) $\lim\limits_{n\to\infty} na_n = 0$ provided the series (c) $\sum_{n=0}^{\infty} a_n$ converges.*

According to Hardy [1, p. 350], this result was first discovered by Abel and then was rediscovered by Pringsheim. And according to Knopp [2, p. 124] it is due to Oliver who had discovered this result in 1827.

It seems that after 1827 no one has made any effort either to generalize this result or to relax the hypotheses of Theorem 1 till 1997 when Tong [3] replaced the condition (a) by a set of conditions imposed upon a positive function f with $f(n) = a_n$ to get (b). But his set of conditions and the condition (a) are independent. However this result of Tong provides a motivation to re-think over the hypotheses of Theorem 1.

In this note we propose to prove the following result which improves Theorem 1 by replacing the hypothesis (a) by a weaker hypothesis involving the set of two conditions on (a_n) and by replacing (c) by a necessary and sufficient condition.

Theorem 2. *Let $a_n > 0$ for all $n \geq 0$. Suppose there exists non-negative and non-increasing sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ such that*

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(a-1) $a_n = x_n - y_n$ *for all* $n \geq 0$ *and* *(a-2)* $\lim_{n\to\infty} n(y_n - y_{2n}) = 0$. *Then (b) of Theorem 1 holds if and only if (C)* $\lim_{n\to\infty} \sum_{k=n}^{2n} a_k = 0$.

Remark-1. Observe that if we take $a_n = \frac{2}{3}\frac{1}{n\ log(n+1)}$ for $n \geq 1$ then it fails to satisfy the condition (c) of Theorem 1 though the condition (b) is satisfied, and hence, in Theorem 1, the condition (c) is sufficient but not necessary for (b) to hold true. On the other hand, if we choose the sequences $x_n = \frac{1}{n\ log(n+1)}$ and $y_n = \frac{1}{3}\frac{1}{n\ log(n+1)}$, then all the conditions of Theorem 2 are satisfied and Theorem 2 is applicable to it.

The following interesting result easily follows from Theorem 2.

Theorem 3. *Let* $a_n > 0$ *for all* $n \geq 0$ *and let (A)* $(a_n)_{n=0}^{\infty}$ *be non-increasing. Then in order that condition (b) of Theorem 1 should hold true it is necessary and sufficient that the condition (C) of Theorem 2 should hold true.*

Remark-2. Consider Theorem 1 and Theorem 2. Observe that the condition (C) is not only weaker than condition (c) but it is necessary also for the condition (b) to hold true. Also, clearly the space of sequences satisfying condition (a) is contained in the space of sequences satisfying the condition (A). To see that this inclusion is strict, let

$$a_n = \frac{1}{(1+[n/2])(1+[log(n+1)])} \quad (n \in \mathbb{N}),$$

where $[x]$ denotes the integral part of $|x| \neq 0$. Then (a_n) neither satisfies (a) nor (c), but it does satisfy (A). And, since $\frac{x}{2} < 1 + [\frac{x}{2}]$ for $x > 1$, therefore we have

$$0 < \sum_{k=n}^{2n} a_k < (log(n+1))^{-1} \sum_{k=n}^{2n} 2/k, \to 0$$

as $n \to \infty$. Hence (C) holds.

2. PROOFS OF THE THEOREMS

Proof of Theorem 2. We first prove that (C) is sufficient for (b) to hold true. Using (a-1), we write $a_{k+i} = x_{k+i} - y_{k+i}$, for $i = 0, 1, 2, \cdots, k$, where $k > 0$. Since (x_n) and (y_n) are non-increasing sequences of non-negative numbers for $n \geq 0$, we get $a_{k+i} \geq x_{2k} - y_k$ $(i = 0, 1, \cdots, k)$. Hence we have

$$\sum_{m=k}^{2k} a_m \geq (k+1)(x_{2k} - y_k) = (k+1)a_{2k} - (k+1)(y_k - y_{2k})$$
$$= \left(\frac{k+1}{2k}\right) 2ka_{2k} - \left(\frac{k+1}{k}\right) k(y_k - y_{2k}),$$

by (a-1). Now letting $k \to \infty$ on both sides and using (a-2) we get

$$\lim_{k\to\infty} \sum_{m=k}^{2k} a_m \geq (1/2) \lim_{k\to\infty} 2ka_{2k} \geq 0,$$

since $a_k > 0$ for all k. In view of (C) it follows that (b) holds when $n \to \infty$ through even integers. To prove that (b) holds even when $n \to \infty$ through odd integers, observe again that we have $a_{2k+1} = x_{2k+1} - y_{2k+1} \leq x_{2k} - y_{4k} = a_{2k} + y_{2k} - y_{4k}$ using (a-1) as (x_n) and (y_n) are non increasing. Hence

$$(2k+1)a_{2k+1} \leq (1 + (1/2k))\ 2ka_{2k} + (1 + (1/2k))\ 2k(y_{2k} - y_{4k}).$$

Now, letting $k \to \infty$ on both sides and using (a-2), we get

$$0 \leq \lim_{k\to\infty} (2k+1)a_{2k+1} \leq \lim_{k\to\infty} (2k)a_{2k} = 0,$$

as (b) holds when $n \to \infty$ through even integers as proved above. The sufficiency of (C) is thus established.

To prove that (C) is necessary for (b) to hold true, proceed as above to get $a_{k+i} = x_{k+i} - y_{k+i} < x_k - y_{k+i}$ for $i = 0, 1, \cdots, k, \quad k > 0$. This gives $\sum_{i=0}^{k} a_{k+i} \leq (k+1)x_k - (k+1)y_{2k}$ which in turn gives in view of (a-1)

$$\begin{aligned} 0 \leq \sum_{m=k}^{2k} a_m &\leq (k+1)(x_k - y_k) + (k+1)(y_k - y_{2k}) \\ &= (1 + (1/k))[ka_k + k(y_k - y_{2k})]. \end{aligned}$$

Letting now $k \to \infty$ on both sides and using (b) and (a-2), we get $0 \leq \lim_{k\to\infty} \sum_{m=k}^{2k} a_m \leq 0$. This proves that condition (C) is necessary. This completes the proof of the theorem. □

Proof of Theorem 3. Let $y_n = 0$ for all $n \geq 0$ in Theorem 2. Then (a-2) obviously holds. Choose $x_n = a_n$ for all n. Then in view of hypothesis (A), (x_n) is non-increasing. Thus, we have non-negative and non-increasing sequences (x_n) and (y_n) satisfying conditions (a-1) and (a-2). Therefore, an application of Theorem 2 implies that (b) holds if and only if (C) holds. This completes the proof. □

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A CONSTRUCTIVE PROOF OF THE DERIVATION THEOREM

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Abstract. We give a new constructive proof of the well-known result that every derivation on $M_n(\mathbb{C})$ is inner, taking an approach based on fixed point theory. The proof yields an explicit formula for the matrix implementing the inner derivation (modulo translation by scalar matrices) by employing a standard averaging technique over the unitary group of $M_n(\mathbb{C})$. As a corollary, it follows that the matrix may be chosen with norm less than or equal to the norm of the derivation.

1. Introduction

For a positive integer n, let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. A derivation $\delta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a $\mathbb{C}$-linear transformation on $M_n(\mathbb{C})$ satisfying the Leibnitz rule, that is, $\delta(AB) = A\delta(B) + \delta(A)B$ for all $A, B \in M_n(\mathbb{C})$. The study of derivations is motivated by the fact that they are infinitesimal generators of (continuous) one-parameter groups of automorphisms, which arise in the study of time evolution of physical systems. Since the motivation on our part is one of mathematical curiosity and pedagogy, we limit our scope to the case of matrix algebras. For a fixed matrix $Z \in M_n(\mathbb{C})$, it is easy to verify that the linear transformation on $M_n(\mathbb{C})$ given by $A \mapsto ZA - AZ =: [Z, A]$ is a derivation. Such derivations are called *inner derivations* and we say that the matrix Z implements the inner derivation. An automorphism $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is said to be an inner automorphism if there is an invertible matrix P in $M_n(\mathbb{C})$ such that $\Phi(A) = PAP^{-1}$ for all $A \in M_n(\mathbb{C})$, that is, the automorphism is 'spatially' implemented. An inner derivation generates a one-parameter group of inner automorphisms.

In this note, we present a constructive proof of the fact that all derivations on $M_n(\mathbb{C})$ are inner (henceforth called the Derivation Theorem). For

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a derivation δ, we find an explicit formula for a matrix implementing δ as an inner derivation. As an application, we use the formula to show that it is possible to choose such a matrix with norm less than or equal to the norm of the derivation. We note that the strategy also works for a proof of the Derivation Theorem for $M_n(\mathbb{R})$.

2. Preliminaries

Definition 2.1. For a locally compact topological group G, let Σ denote a σ-algebra of subsets of G containing the Borel subsets. A Radon measure $\mu : \Sigma \to \mathbb{R}$ is said to be a left-invariant *Haar measure* on G if $\mu(gS) = \mu(S)$ for all $g \in G$ and μ-measurable subsets S of G.

It was first shown by Haar that a left-invariant Radon measure exists on any locally compact topological group and is unique upto multiplication by a positive constant. This generalizes the notion of Lebesgue measure on the Euclidean spaces $\mathbb{R}^n$. Our interest in the Haar measure stems from the averaging procedure it allows via integration over the group which, loosely speaking, results in functions or other mathematical objects that are invariant under action of the group.

Since the Haar measure is finite for a compact group, we may normalize it to a probability measure. On an n-dimensional Lie group, the measure induced by a left-invariant n-form is a Haar measure. For our discussion, the pertinent groups are the unitary group $\mathscr{U}(n)$ of $M_n(\mathbb{C})$, and the orthogonal group $\mathscr{O}(n)$ of $M_n(\mathbb{R})$, both of which are compact Lie groups.

Lemma 2.2. *Every matrix in $M_n(\mathbb{C})$ can be written as a $\mathbb{C}$-linear combination of four unitary matrices in $M_n(\mathbb{C})$.*

Proof. For a matrix A, without loss of generality we may assume that $\|A\| \le 1$ as $A = \|A\|(\frac{A}{\|A\|})$. Let $\Re(A) := \frac{A+A^*}{2}, \Im(A) := \frac{A-A^*}{2i}$, denote the real and imaginary parts, respectively, of A, so that $A = \Re(A) + i\,\Im(A)$. The matrices $\Re(A)$ and $\Im(A)$ are Hermitian, and have matrix norm less than or equal to 1. Using the spectral theorem and the continuous functional calculus for a Hermitian matrix H with $\|H\| \le 1$ (and thus, $-I \le H \le I$), we note that $\phi_1(H) := H + i\sqrt{I - H^2}$, and $\phi_2(H) := H - i\sqrt{I - H^2}$, are unitary matrices. Thus the decomposition,

$$A = \frac{1}{2}\phi_1(\Re(A)) + \frac{1}{2}\phi_2(\Re(A)) + \frac{i}{2}\phi_1(\Im(A)) + \frac{i}{2}\phi_2(\Im(A)),$$

shows that A can be written as a $\mathbb{C}$-linear combination of four unitary matrices. $\square$

Lemma 2.3. *Every matrix in $M_n(\mathbb{R})$ can be written as an $\mathbb{R}$-linear combination of finitely many orthogonal matrices in $M_n(\mathbb{R})$.*

Proof. For $n = 1$, the assertion holds trivially and hence we assume that $n \geq 2$. Note that we may represent $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as an $\mathbb{R}$-linear combination of five orthogonal matrices in $M_2(\mathbb{R})$ as follows,

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{2\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$-\frac{1}{2\sqrt{2}}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{2\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus the matrix unit $\mathcal{E}_{11}$ in $M_n(\mathbb{R})$, with 1 in the $(1,1)^{\text{th}}$ position and 0's elsewhere, can be represented as an $\mathbb{R}$-linear combination of five orthogonal matrices in $M_n(\mathbb{R})$ using the canonical embedding $\mathscr{O}(2) \hookrightarrow \mathscr{O}(n)$ given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto \left[\begin{array}{cc|c} a_{11} & a_{12} & \mathbf{0} \\ a_{21} & a_{22} & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{I}_{n-2} \end{array}\right].$$

Recall that permutation matrices are orthogonal matrices. By multiplying suitable permutation matrices to $\mathcal{E}_{11}$ (on the left or the right), we conclude that any matrix unit $\mathcal{E}_{ij}, 1 \leq i,j \leq n$, may be represented as a linear combination of five orthogonal matrices in $M_n(\mathbb{R})$. Since the n^2 matrix units $\mathbb{R}$-linearly generate $M_n(\mathbb{R})$, we note that every matrix in $M_n(\mathbb{R})$ can be written as an $\mathbb{R}$-linear combination of $5n^2$ orthogonal matrices in $M_n(\mathbb{R})$. $\square$

3. Proof of the Derivation Theorem

Theorem 3.1. *Let δ be a derivation on $M_n(\mathbb{C})$. Then there exists a matrix $Z \in M_n(\mathbb{C})$ such that $\delta(A) = ZA - AZ =: [Z, A]$ for all $A \in M_n(\mathbb{C})$. Furthermore, Z is unique upto translation by a scalar matrix. Let $\mathscr{U}(n)$ denote the compact group of unitary matrices in $M_n(\mathbb{C})$, with the left-invariant Haar probability measure on $\mathscr{U}(n)$ denoted by dU. We have the following formula,*

$$Z = \alpha I + \int_{\mathscr{U}(n)} \delta(U)U^{-1}\, dU,$$

for some $\alpha \in \mathbb{C}$.

Proof. Define a family of affine linear maps on $M_n(\mathbb{C})$, indexed by $\mathscr{U}(n)$, in the following manner:

$$\psi_U(X) := UXU^{-1} + \delta(U)U^{-1}, \text{ for } U \in \mathscr{U}(n).$$

For $U, V \in \mathscr{U}(n)$ and $X \in M_n(\mathbb{C})$, we observe that

$$\begin{aligned}\psi_U \circ \psi_V(X) &= U\psi_V(X)U^{-1} + \delta(U)U^{-1}\\ &= U\big(VXV^{-1} + \delta(V)V^{-1}\big)U^{-1} + \delta(U)U^{-1}\\ &= (UV)X(UV)^{-1} + \big(U\delta(V) + \delta(U)V\big)(UV)^{-1}\\ &= (UV)X(UV)^{-1} + \big(\delta(UV)\big)(UV)^{-1} \quad = \quad \psi_{UV}(X)\end{aligned}$$

Thus for any $U, V \in \mathscr{U}(n)$, we have $\psi_U \circ \psi_V = \psi_{UV}$.

For a fixed matrix $X_0 \in M_n(\mathbb{C})$, note that $\psi_U(X_0)$ is a continuous function in U on the compact group $\mathscr{U}(n)$. Let us define $Z := \int_{\mathscr{U}(n)} \psi_U(X_0)\, dU$. Keeping in mind the affine linearity of ψ_V and the left-invariance of the Haar probability measure on $\mathscr{U}(n)$, for every unitary matrix $V \in \mathscr{U}(n)$ we have,

$$\begin{aligned}\psi_V(Z) = \psi_V\Big(\int_{\mathscr{U}(n)} \psi_U(X_0)\, dU\Big) &= \int_{\mathscr{U}(n)} \psi_V \circ \psi_U(X_0)\, dU\\ = \int_{\mathscr{U}(n)} \psi_{VU}(X_0)\, dU &= \int_{\mathscr{U}(n)} \psi_U(X_0)\, dU \quad = \quad Z.\end{aligned}$$

Thus for all $U \in \mathscr{U}(n)$, we observe that $\psi_U(Z) = UZU^{-1} + \delta(U)U^{-1} = Z$, which implies that $\delta(U) = ZU - UZ$. Using Lemma 2.2 and the linearity of δ, we conclude that $\delta(A) = ZA - AZ = [Z, A]$ for all $A \in M_n(\mathbb{C})$. For an explicit formula for Z in terms of δ, we choose $X_0 = 0$ and note that

$$Z = \int_{\mathscr{U}(n)} \delta(U)U^{-1}\, dU.$$

If matrices Z_1, Z_2 both implement the inner derivation δ, then $[Z_1, A] = [Z_2, A]$ for all $A \in M_n(\mathbb{C})$, which implies that $Z_1 - Z_2$ is in the center of $M_n(\mathbb{C})$. Thus $Z_1 = \alpha I + Z_2$ for some $\alpha \in \mathbb{C}$. □

Remark 3.2. Note that for a unitary matrix $V \in \mathscr{U}(n)$ and $X_0 \in M_n(\mathbb{C})$, we have

$$V\Big(\int_{\mathscr{U}(n)} UX_0U^{-1} dU\Big)V^{-1} = \int_{\mathscr{U}(n)} (VU)X_0(VU)^{-1} dU = \int_{\mathscr{U}(n)} UX_0U^{-1} dU.$$

Thus $\int_{\mathscr{U}(n)} UX_0U^{-1}\, dU$ commutes with every unitary matrix and by Lemma 2.2, it must lie in the center of $M_n(\mathbb{C})$. This explains why different choices of X_0 perturb Z only by scalar matrices.

Let $\|\cdot\|$ be a norm on $M_n(\mathbb{C})$ which makes it a normed algebra ($\|AB\| \le \|A\|\|B\|$ for $A, B \in M_n(\mathbb{C})$, $\|I\| = 1$) such that $\|U\| \le 1$ for every unitary matrix $U \in M_n(\mathbb{C})$. Examples of such a norm include the Frobenius norm,

$$\|A\| = \sqrt{(1/n)\operatorname{tr}(A^*A)},$$

and the usual matrix norm,

$$\|A\| = \sup_{x \in \mathbb{C}^n \setminus \{\mathbf{0}\}} \frac{\|Ax\|_2}{\|x\|_2}.$$

For a derivation δ on $M_n(\mathbb{C})$, let $\|\delta\|$ denote the usual operator norm of δ relative to the norm $\|\cdot\|$ (with properties as described above) on $M_n(\mathbb{C})$. If $\delta(A) = ZA - AZ$ for all $A \in M_n(\mathbb{C})$, then $\|\delta(A)\| \leq \|ZA\| + \|AZ\| \leq 2\|Z\|\|A\|$, and hence $(\|\delta\|/2) \leq \|Z\|$.

Corollary 3.3. *Let δ be a derivation on $M_n(\mathbb{C})$ $(n \in \mathbb{N})$. Then we can choose $Z \in M_n(\mathbb{C})$ such that $\delta(A) = [Z, A]$ for all $A \in M_n(\mathbb{C})$ and $\|Z\| \leq \|\delta\|$.*

Proof. From Theorem 3.1, choosing $Z = \int_{\mathscr{U}(n)} \delta(U)U^{-1}\, dU$, we see that

$$\|Z\| \leq \int_{\mathscr{U}(n)} \|\delta(U)U^{-1}\|\, dU \leq \|\delta\| \Big(\int_{\mathscr{U}(n)} 1\, dU \Big) = \|\delta\|.$$

□

Theorem 3.4. *Let δ be a derivation on $M_n(\mathbb{R})$. Then there exists a matrix $Z \in M_n(\mathbb{R})$ such that $\delta(A) = ZA - AZ =: [Z, A]$ for all $A \in M_n(\mathbb{R})$. Furthermore, Z is unique upto translation by a scalar matrix. Let $\mathscr{O}(n)$ denote the compact group of orthogonal matrices in $M_n(\mathbb{R})$, with the left-invariant Haar probability measure on $\mathscr{O}(n)$ denoted by dO. We have the following formula,*

$$Z = \alpha I + \int_{\mathscr{O}(n)} \delta(O)O^{-1}\, dO, \quad \text{for some } \alpha \in \mathbb{R}.$$

Proof. The proof is almost identical to the one for Theorem 3.1 using the affine $\mathbb{R}$-linear maps $\psi_O(X) = OXO^{-1} + \delta(O)O^{-1}$, indexed by $\mathscr{O}(n)$, and using Lemma 2.3. □

4. Concluding Remarks

There are generalizations of the main result discussed in several directions. For instance, the Kadison-Sakai theorem states that all derivations of a von Neumann algebra are inner (cf. [3], [4]). Our proof is inspired by Sakai's approach which uses the Markov-Kakutani fixed point theorem in an essential way (see [5, Lemma 2.5.1]). The astute reader may have noticed that the proof of Theorem 3.1 works equally well for $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ (for positive integers $n_1, \cdots, n_k$) by considering its unitary group, and for $M_{n_1}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})$ by considering its orthogonal group.

The theory of derivations has been studied extensively in abstract algebraic settings. The approach by Hochschild (cf. [1], [2]) involves building a cohomology theory for associative algebras in which the first cohomology group corresponds to the space of outer derivations. In the context of the Derivation Theorem, one studies associative algebras that have trivial first Hochschild cohomology group. Since all (K-linear) derivations on a field K are trivial and Morita equivalent rings have isomorphic Hochschild cohomology, all derivations on $M_n(K)$ are inner. Note that this is a 'global' approach studying the space of derivations in contrast with our 'local' approach in which we directly work with the derivation under consideration.

For an account of the Hochschild cohomology of algebras, the reader may refer to [8, Chapter 9]. A survey of the (continuous) Hochschild cohomology theory of von Neumann algebras can be found in [6], [7]. The reader may also benefit from the exposition on derivations in the context of operator algebras in [5, Chapter 3,4].

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ROLE OF SLIDING CONTACT INTERFACE ON TORSIONAL WAVES

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Abstract. Boundary conditions are constructed for torsional waves in a layered media with sliding contact interface. Two different models of layered structure are considered: Model-I consists of a homogeneous layer overlaid a non-homogeneous transversely isotropic elastic half-space, and Model-II consists of a dry sandy homogeneous layer of finite thickness overlaid a non-homogeneous half-space. The layer and the half-space are assumed to be in finite sliding contact and torsional surface wave propagation has been studied. The sliding contact of the layer over the half-space has been defined through a parameter $\xi \in [0, 1]$ and general dispersion equations for the torsional wave propagation are derived. In both the models, the torsional waves are found to propagate, in general, for welded contact, finite sliding contact and smooth contact interfaces. For welded contact and finite sliding contact interfaces, the torsional wave modes are dispersive, but for smooth sliding contact interface, the fundamental mode of torsional wave is found to be non-dispersive. For a limiting value of ξ, the general dispersion equations for the considered Models are found to be in full agreement with the previously derived dispersion equations by Gupta et al. [11] and Sultana and Gupta [13] for the corresponding problems. Numerical computations have been performed to analyze the effect of sliding parameter on the dispersion curves of torsional waves and depicted graphically.

1. Introduction

The topic of surface waves in elastic medium is well known since long due to their enormous applications in diverse fields such as geophysics, engineering and seismology. The data collected via surface waves can also be used to investigate the dissipative nature of the medium. The well known books by Love [1, 4] Kolsky [15], Achenbach [14], Ewing et al. [5], Bullen and Bolt [6] among several others contain the basic literature pertaining to

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surface waves such as Rayleigh, Love and Stoneley waves. Stoneley waves [3] propagate along the interface of two elastic solids of nearly equal densities and hence also called "interfacial waves" in the literature. Apart from these surface waves, there are another kind of surface waves called *Torsional surface waves*, in which the vibrations of the medium are rotational motions around the direction of propagation of the wave and hence these waves are also called *twisting waves.* The existence of torsional surface waves was first shown by Meissner [2] in an inhomogeneous elastic half-space. Since then researchers attempted to explore torsional waves by considering various models incorporating different characteristics. Some problems notable in the last three decades are Vardoulakis [7], Dey et al. [8], Dey et al. [9] Georgiadis et al. [10], Gupta et al. [11], Kumari and Sharma [12], Sultana and Gupta [13], Kumhar et al. [21] among others.

Apart from the problems of torsional surface waves in layered media with welded contact interfaces, Vinh and Anh [16, 17, 19, 20] and Vinh et al. [18] attempted problems of Rayleigh wave propagation in layered media with smooth sliding contact interfaces. Vinh and his co-workers used effective boundary condition approach, in which the unknown occurring in the layer from the boundary conditions at the interface and on the free surface are converted in terms of the unknown of the half-space. In order to study the Rayleigh wave in the half - space, the layer thickness is made to zero in the derived frequency equation which, in turn, reduces to the frequency equation of Rayleigh wave propagating in the half-space.

In the present investigation, our focus is to understand the effect of sliding contact on torsional surface waves propagating in a layer over an inhomogeneous elastic half-space. To understand the effect of sliding contact, we have considered two different layered models: Model-I consists of a homogeneous elastic layer of uniform thickness lying over a non-homogeneous transversely isotropic elastic half-space, and Model-II consists of a dry sandy homogeneous elastic layer of uniform thickness lying over a non-homogeneous elastic half-space. In both these models, the layer and the half-space are taken to be in partially sliding contact, while the inhomogeneity of the half-space is accounted through the variations in mass density and rigidity of the half-space. To define the sliding contact between the layer and the half-space, we have introduced a sliding parameter ξ between the layer and the half-space. This sliding contact parameter ξ is enable to define smooth sliding, partial sliding and non-sliding interfaces for its different values. The general dispersion equation corresponding to both the Models considered has been derived, which involves the sliding parameter ξ. The results of some previous studies have been recovered by taking a specific limiting value of sliding parameter ξ. Numerical computations have been performed by taking numerical values of various parameters in non-dimensional form and the dispersion equation of Model-I & II has been solved using Mathematica. The results obtained have been depicted graphically and the effect of parameter ξ on the dispersion curve of torsional waves is studied.

2. Layer over a non-homogeneous transversely isotropic elastic half-space: **Model-I**

With reference to the cylindrical coordinate system (r, θ, z), we consider a homogeneous elastic layer L of uniform thickness h lying over a non-homogeneous transversely isotropic elastic solid half-space H. We take the origin O of the cylindrical coordinate system located at the interface separating the layer and the half-space such that the $z-$axis is directed vertically downwards into the half-space. Thus, $z = 0$ represents the interface between the layer and the half-space. We shall study a two-dimensional problem in $r - z$ plane such that $\frac{\partial}{\partial \theta} \equiv 0$ and the components of displacement vector $\mathbf{u}$ are taken as $u_r \equiv u = 0, u_\theta \equiv v(r, z, t)$ and $u_z \equiv w = 0$. The stress-displacement dynamical equation of motion without body force density is given by

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = \rho(z) \frac{\partial^2 v}{\partial t^2}, \tag{2.1}$$

where the expressions of stress components $\sigma_{r\theta}$ and $\sigma_{\theta z}$ are given by

$$\sigma_{r\theta} = \mu(z) \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right), \quad \sigma_{\theta z} = \mu(z) \frac{\partial v}{\partial z}. \tag{2.2}$$

Here $\mu(z)$ is the rigidity. With these relations, equation (2.1), yields

$$\mu(z) \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) + \frac{\partial}{\partial z} \left(\mu(z) \frac{\partial v}{\partial z} \right) = \rho(z) \frac{\partial^2 v}{\partial t^2}. \tag{2.3}$$

The layer is homogeneous elastic, therefore, the rigidity and the mass density respectively in the layer are taken as $\mu = \mu_0$ and $\rho = \rho_0$. The elastic half-space is transversely isotropic and non-homogeneous, therefore we take the rigidity and mass density in the half-space as $N = N_1(1 + \alpha z), P = P_1(1 + \beta z)$ and $\rho = \rho_1(1 + \delta z)$, where N, and P are the directional rigidities in the half-space; N_1, P_1 and ρ_1 are constants; α, β and δ are constants having the dimensions that are $(\text{length})^{-1}$.

For a time harmonic torsional wave propagating with speed c, the displacement can be taken in the form

$$v = V(z)J_1(kr)e^{i\omega t}, \tag{2.4}$$

where $\omega = kc$, ω being the angular frequency and k, the wavenumber.

The displacement in the elastic layer L is given by (see [11])

$$v_L \equiv v(z) = \left[A_1e^{\eta z} + A_2e^{-\eta z}\right] J_1(kr)e^{\iota\omega t}, \tag{2.5}$$

where A_1 and A_2 are arbitrary constants; $\eta = k\sqrt{1-\frac{c^2}{c_0^2}}$ and $c_0 = \sqrt{\frac{\mu_0}{\rho_0}}$ is the speed of shear wave in the layer.

Next, the displacement satisfying the radiation condition in the elastic half-space H is given by (see [11])

$$v_H \equiv v(z) = A_3\frac{J_1(kr)\sqrt{2MK_1}}{\sqrt{\beta P_1}}\left[1+(1-2R)p\right]e^{-p}\ e^{\iota\omega t} \tag{2.6}$$

where A_3 is arbitrary constant, $K_1 = k\sqrt{\frac{N_1}{P_1}}$, $\quad c_1 = \sqrt{\frac{N_1}{\rho_1}}$, $\quad p = \dfrac{MK_1(1+\beta z)}{\beta}$ and $M = \left[\dfrac{1}{\beta}\left(\alpha - \dfrac{c^2}{c_1^2}\delta\right)\right]^{1/2}$, $\quad R = \dfrac{K_1}{2M\beta}\left[\left(\dfrac{c^2}{c_1^2}-1\right)+M^2\right]$.

3. Dry sandy layer over a non-homogeneous elastic half-space: **Model-II**

Here the layer is composed of homogeneous dry sandy material, whose Young's modulus of elasticity E is defined as $E = 2\mu\zeta(1+\sigma)$, μ being the modulus of rigidity and σ is the Poisson's ratio. The quantity $\zeta > 1$ represents the sandiness parameter, which is such that the value $\zeta = 1$ signifies the elastic solid. Considering the same formulation and geometry of the problem as considered in Model-I, the stress-displacement relations in the dry sandy layer are given by

$$\sigma_{r\theta} = 2\zeta\mu(z)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right), \quad \sigma_{\theta z} = 2\zeta\mu(z)\frac{\partial v}{\partial z}. \tag{3.1}$$

With these relations, equation (2.1), yields

$$\zeta\mu(z)\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \frac{\partial}{\partial z}\left(\zeta\mu(z)\frac{\partial v}{\partial z}\right) = \rho(z)\frac{\partial^2 v}{\partial t^2}. \tag{3.2}$$

As earlier in Model-I, the displacement in the dry sandy layer for the torsional wave propagating with velocity c is given by (see Sultana and Gupta [13])

$$v_L \equiv v(z) = (A_1e^{\iota\eta_s z} + A_2e^{-\iota\eta_s z})J_1(kr)e^{\iota\omega t}, \tag{3.3}$$

where $\eta_s = k\sqrt{\dfrac{c^2}{\zeta c_0^2} - 1}$ and $c_0 = \sqrt{\dfrac{\mu}{\rho}}$ is the shear wave speed in the dry sandy layer.

Next, as in Sultana and Gupta [13], the variations in rigidity and density of the half-space H are considered as $\mu = \mu_1(1+\gamma z)^n$, $\rho = \rho_1(1+\gamma z)^n$; $n(\geq 2)$, where γ is a constant having the dimension of $(\text{length})^{-1}$. The relevant displacement in the inhomogeneous half-space is given by

$$v(z) \equiv v_H = A_3 \frac{W_{0,Q}[2(klz+m)]}{(1+\gamma z)^{\frac{n}{2}}} J_1(kr) e^{\iota\omega t}, \tag{3.4}$$

where $W_{0,Q}$ is Whittaker's function and

$$l = \sqrt{1 - \frac{c^2}{c_1^2}}, \quad m = \frac{kl}{\gamma}, \quad c_1 = \sqrt{\frac{\mu_1}{\rho_1}}, \quad Q^2 = \frac{1}{4} + \frac{n}{2}\left(\frac{n}{2} - 1\right).$$

4. Boundary conditions and dispersion equation

The layer and the half-space are in sliding contact in both the models considered. Therefore, we introduce a sliding parameter ξ $(0 \leq \xi \leq 1)$ such that (i) when $\xi = 1$, the layer and the half-space are in welded contact, (ii) when $\xi = 0$, the layer and the half-space are in smooth contact and (iii) when $0 < \xi < 1$, the layer and the half-space are considered to be in partial sliding contact. We shall set up the appropriate boundary conditions on the displacements and stresses at the interface $z = 0$ by accounting the sliding parameter ξ. We shall also assume that the top surface of the layer defined by $z = -h$ is mechanically stress free. For the considered problems in Model-I & II, the boundary conditions are constructed as

At the interface $z = 0$:

$$\sigma_{\theta z}|_L = \xi \sigma_{\theta z}|_H, \qquad [(1-\xi)\sigma_{\theta z} + \xi k F v]|_H = \xi k F v|_L. \tag{4.1}$$

At the boundary surface $z = -h$:

$$\sigma_{\theta z}|_L = 0. \tag{4.2}$$

Here F is a constant quantity having the dimension of force per unit area. Utilizing (2.2) and (2.5) into (4.2), we obtain

$$\mu\left[A_1 e^{-\eta h} - A_2 e^{\eta h}\right] J_1(kr) e^{\iota\omega t} = 0. \tag{4.3}$$

Further, putting $z = 0$ into (2.2) and (2.5), we have the effect of layer at the interface as

$$\{\sigma_{\theta z}(0), v(0)\}|_L = \{\mu\eta(A_1 - A_2), (A_1 + A_2)\} J_1(kr) e^{\iota\omega t}. \tag{4.4}$$

Next putting $z = 0$, into (2.2) and (2.6), we have the effect of the half-space at the interface as

$$\{\sigma_{\theta z}(0), v(0)\}|_H = \{S_1, S_2\} A_3, \tag{4.5}$$

where

$$S_1 = \sqrt{P_1} J_1(kr) \left(\frac{2MK_1}{\beta} \right)^{1/2} MK_1 \left[\frac{MK_1}{\beta}(2R-1) - 2R \right] e^{i\omega t} e^{\frac{-MK_1}{\beta}},$$

$$S_2 = \frac{1}{\sqrt{P_1}} J_1(kr) \left(\frac{2MK_1}{\beta} \right)^{1/2} \left[1 + \frac{MK_1}{\beta}(1-2R) \right] e^{i\omega t} e^{\frac{-MK_1}{\beta}}.$$

Solving the equations given in (4.4) for A_1 and A_2, and then substituting their expression into (4.3), we obtain

$$\tanh(\eta h) = \frac{\sigma_{\theta z}(0)|_L}{\mu \eta v(0)|_L}. \tag{4.6}$$

Now using (4.5) into the boundary conditions stated in (4.1), we can obtain the relation of the relevant components of stresses and displacement at the interface between the layer and the half-space. Using these relations into (4.6), we obtain

$$\tanh(\eta h) = \frac{F\xi S_1}{\mu\eta} \left[\frac{(1-\xi)}{\xi} S_1 + kFS_2 \right]^{-1}. \tag{4.7}$$

This is the general dispersion equation of torsional wave propagation for Model-I.

Adopting the same procedure for Model-II, the general dispersion relation is obtained as

$$\tan(\eta_s h) = \frac{F\xi\mu_1 S_3}{\zeta\mu\eta_s} \left[\frac{(1-\xi)}{\xi} \mu_1 \zeta S_3 + kF \right]^{-1}, \tag{4.8}$$

where $S_3 = \left(Q + \frac{1}{2} - \frac{n}{2}\right)\gamma - k\frac{lm}{1+m}$.

4.1. **Case of smooth contact interface.** Here we shall discuss the case of smooth contact interface between the layer and the half-space in Model-I & II. The value of sliding parameter $\xi = 0$ corresponds to smooth contact interface. For Model-I, substituting $\xi = 0$, into (4.7), the dispersion relation yields $c = c_0$. Thus in the case of smooth contact interface of Model-I, the torsional waves transform into shear wave in the layer. Similarly, for smooth contact interface in Model-II, we substitute $\xi = 0$ into (4.8) and obtain $c = \sqrt{\zeta} c_0$. Thus the torsional wave speed reduces to the speed of shear wave multiplied by square root of sandiness parameter.

4.2. **Case of welded contact interface.** Now we shall discuss the case of welded contact interface between the layer and the half-space in Model-I & II. The value of sliding parameter $\xi = 1$ corresponds to welded contact interface. For Model-I, on substituting $\xi = 1$, into (4.7), the dispersion relation reduces to the relation given in equation (29) of the manuscript by Gupta et al. [11]. Similarly, for welded contact interface in Model-II, upon substituting $\xi = 1$, into (4.8), the dispersion relation reduces to the relation derived by Sultana and Gupta [13] in equation (23).

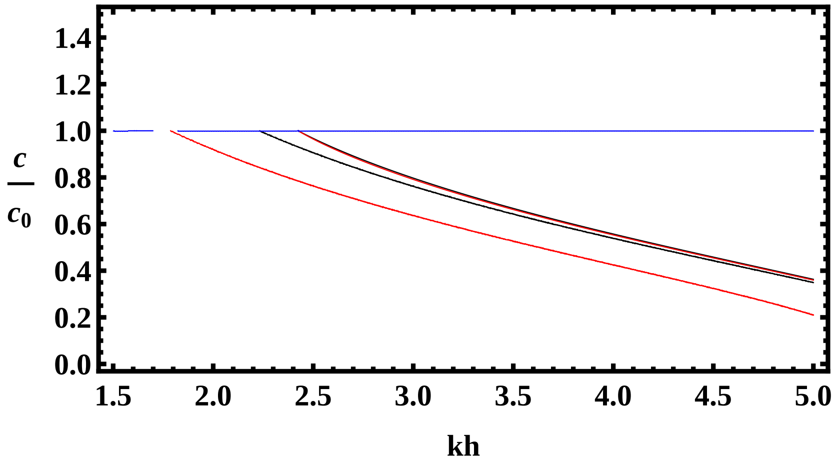

FIGURE 1. Model-I: Variation of phase speed '$\frac{c}{c_0}$' against wavenumber 'kh'.

5. Numerical Results and Discussion

Here our main aim is to investigate the effect of sliding parameter on the phase speed of torsional surface waves in the considered Models. For this purpose, we take specific values of relevant parameters in non-dimensional form. For Model-I, the numerical values of various parameters are taken as: $\alpha H = 0.8, \quad \beta H = 0.7, \quad \delta H = 0.3,$

$$\frac{N_1}{P_1} = 0.9, \quad \frac{\mu_0}{N_1} = 0.2, \quad \frac{c_0^2}{c_1^2} = 0.5, \quad \frac{F}{\mu} = 5.5 \quad \text{and} \quad \frac{F}{P_1} = 1.0.$$

while for Model-II, we take the relevant numerical data as

$$n = 2, \quad \zeta = 1.2, \quad \gamma h = 0.45, \quad \frac{c_0^2}{c_1^2} = 0.0449, \quad \frac{F}{\mu} = 1.0, \quad \text{and} \quad \frac{F}{\mu_1} = 0.075.$$

With this numerical data, the non-dimensional speed c/c_0 of torsional waves in Model-I & II have been computed from the dispersion equations (4.6) and (4.7). The main objective is to seek the effect of sliding parameter on the dispersion curves of torsional waves. To explore the hinderance of sliding parameter ξ on the dispersion curves, we have considered three different values of ξ, namely $1.0, 0.5, 10^{-9}$. The dispersion curves of torsional waves for Model-I & II have been depicted through Figures 1 & 2 respectively. It is clear from these figures that the dependence of torsional wave speed c/c_0 on the dimensionless wavenumber 'kh' is significant enough. For Model-I, the plot of dispersion curves corresponding to welded contact ($\xi = 1$, black curve), finite sliding contact ($\xi = 0.5$, red curve) and smooth contact

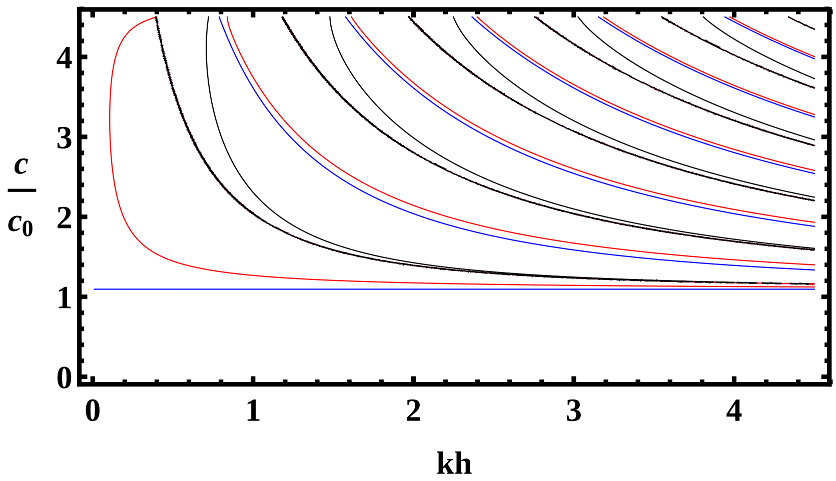

FIGURE 2. Model-II: Variation of phase speed '$\frac{c}{c_0}$' against wavenumber 'kh'.

($\xi = 0.001$, blue curve) interfaces[1] has been shown through Figure 1. The curves corresponding to welded contact and finite sliding contact yield two different behaviors. The effect of parameter ξ is found to be pronounced on the modes occurring alternatively, while those occurring in between have almost negligible effect of sliding parameter. We note that in case of smooth contact interface, the torsional wave is non-dispersive, while it is dispersive for welded contact and finite sliding contact interfaces. The non-dispersive behavior in the case of smooth contact interface is in accordance with the analytical result obtained in Section 4.1.

For Model-II, the dispersion curves for the first few modes of the torsional waves have been plotted in Figure 2. Due to occurrence of tangent function, there exist infinite modes of torsional wave propagation. In Figure 2, the dispersion curves in black, red, blue color correspond to respectively welded contact ($\xi = 1$,) finite contact ($\xi = 0.5$,) and smooth contact ($\xi = 10^{-8}$) interfaces. The effect of parameter ξ on dispersion curves can be noticed clearly. The fundamental mode of torsional wave is constant for $\xi = 10^{-8}$, indicating that the fundamental torsional wave is non-dispersive for smooth contact interface. This fact is in accordance with the theoretical result obtained in Section 4.1. The value $c/c_0 = 1.09$ is exactly equal to the

[1]The value of ξ taken here for smooth contact may not correspond to perfectly smooth contact interface. It is a small value, but not sufficiently small for declaring the perfectly smooth contact interface. Note that it was the computational compulsion to take this value

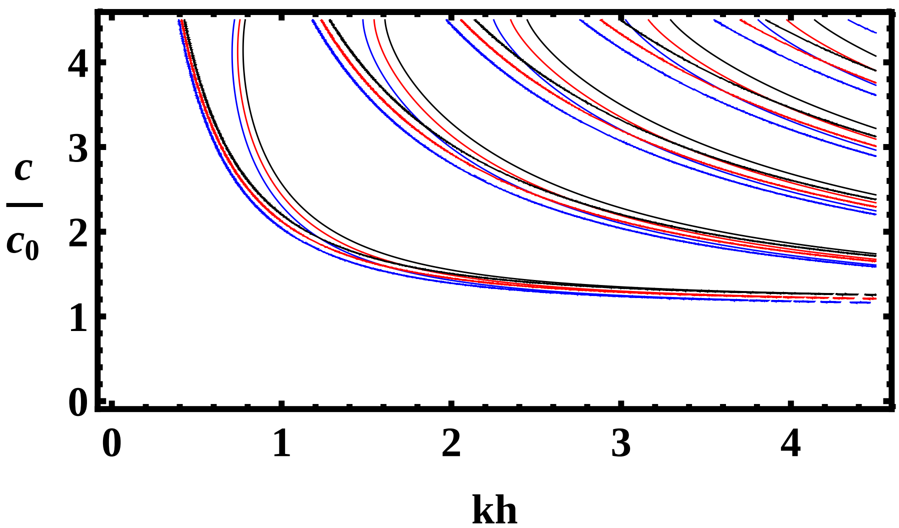

FIGURE 3. Model-II (Welded contact interface): Effect of sandiness parameter 'ζ' on the dispersion curves.

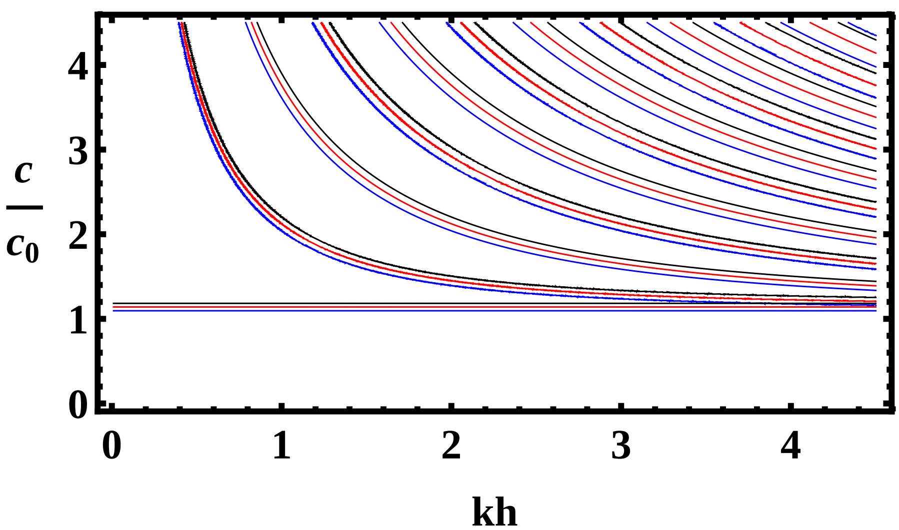

FIGURE 4. Model-II (Smooth contact interface): Effect of sandiness parameter 'ζ' on the dispersion curves.

value of $\sqrt{\zeta}$. The higher modes for smooth contact interface has evolved due to the presence of trigonometric function *tangent* in the dispersion equation (4.8).

Figures 3 and 4 present several modes of torsional wave for three different values of the sandiness parameter ζ, namely, 1.2, 1.3 and 1.4. In Figure 3, the dependency of wave velocity on sandiness parameter (ζ) is

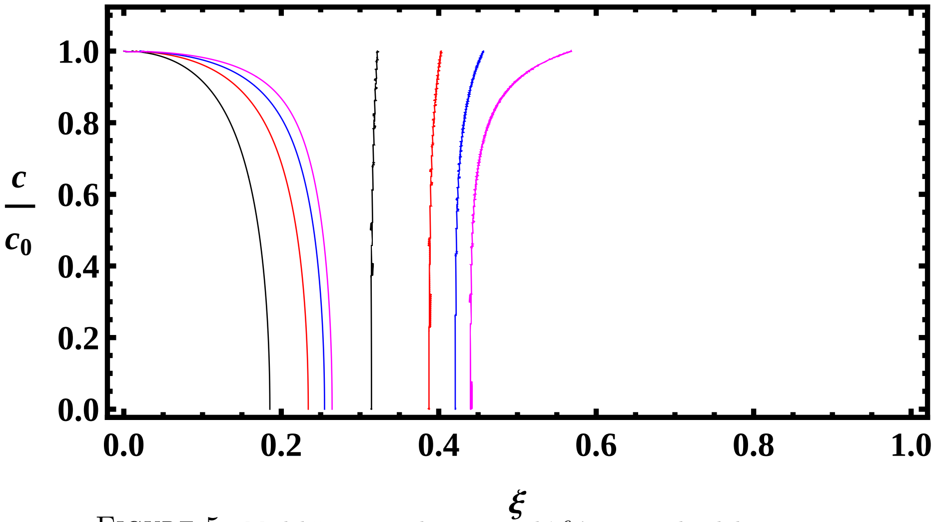

FIGURE 5. Model-I: Torsional wave speed '$\frac{c}{c_0}$' against the sliding parameter 'ξ' for different values of 'kh' ($kh = 0.5$, black curve; $kh = 1$, red curve; $kh = 1.5$, blue curve; $kh = 2$, magenta curve).

plotted for these considered values of ζ and the case of welded contact interface ($\xi = 1$) has been considered. The torsional wave speed increases with increase in the value of sandiness parameter. The fundamental mode and all the higher modes exhibit similar behavior. In the case of smooth contact interface ($\xi = 10^{-10}$), the dispersion curves are plotted in Figure 4 for different values of considered sandiness parameter ζ, which are again found to increase with increase in the value of sandiness parameter.

In Figures 5 and 6, the plots for torsional wave speed have been drawn corresponding to Model-I, against the sliding parameter ξ for different values of kh. It is important to note that the torsional wave speed exists only for peculiar range of sliding parameter. In Figure 5, we have presented the variation of phase speed against ξ for four different values of kh, namely $0.5,\ 1,\ 1.5$ and 2, (black, red, blue and magenta curves respectively). We see that phase speed of torsional wave does not exist for two ranges of ξ and these ranges depend on the value of kh chosen. Figure 6 shows the variation of torsional wave speed against the sliding parameter ξ for four different values of wavenumber kh namely $2.5,\ 3,\ 3.5$ and 4, (black, red, blue and magenta curves, respectively). The numerical results presented in Figure 6 exhibits entirely different pattern than those presented in Figure 5.

Figure 7 shows the variation of torsional wave speed against the sliding

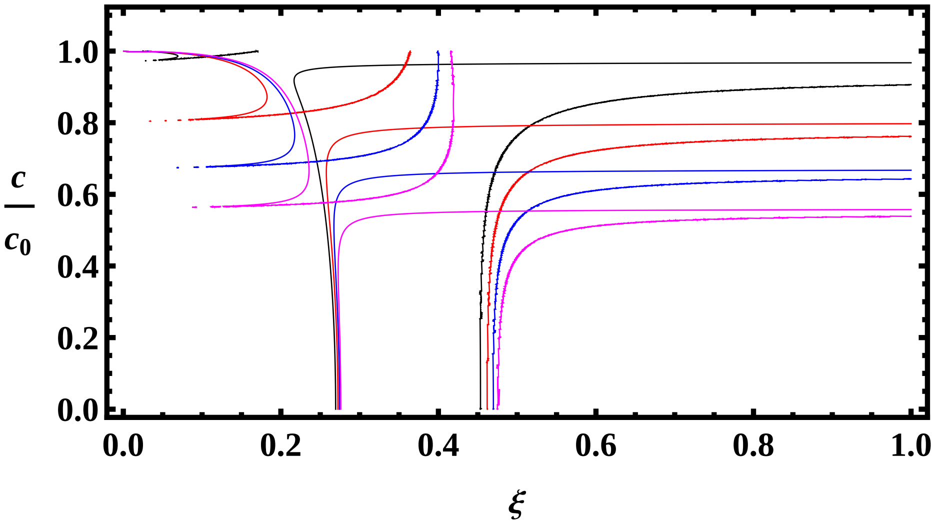

FIGURE 6. Model-I: Torsional wave speed '$\frac{c}{c_0}$' against the sliding parameter 'ξ' for different values of 'kh' ($kh = 2.5$, black curve; $kh = 3$, red curve; $kh = 3.5$, blue curve; $kh = 4$, magenta curve).

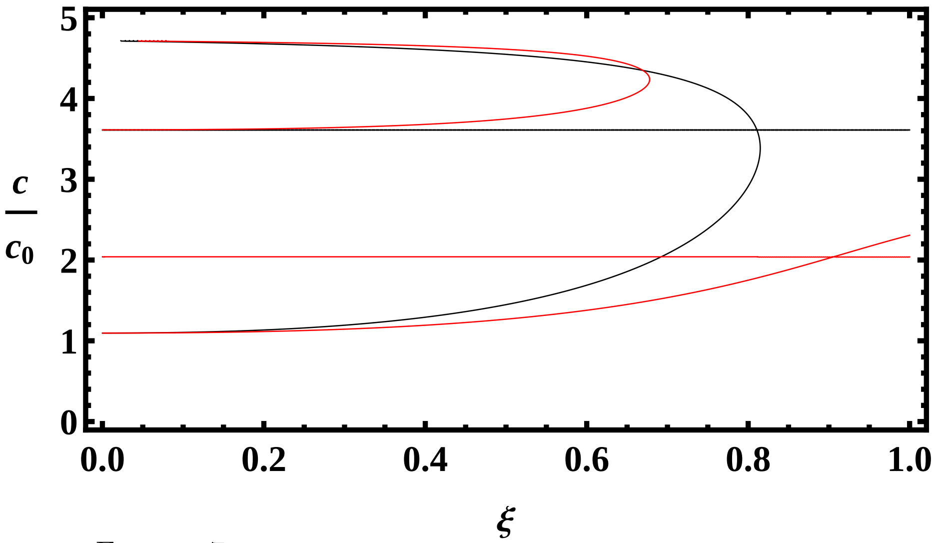

FIGURE 7. Model-II: Torsional wave speed '$\frac{c}{c_0}$' against the sliding parameter 'ξ' for different values of 'kh' ($kh = 0.5$, black curve; $kh = 1$, red curve).

parameter for different values of kh ($kh = 0.5$, black curve; $kh = 1$, red curve) and for fixed value of sandiness parameter. In Figure 8, the plot showing the variation of torsional wave speed against the sliding parameter for another set of values for wavenumber kh ($kh = 1.5$, blue curve; $kh = 2$,

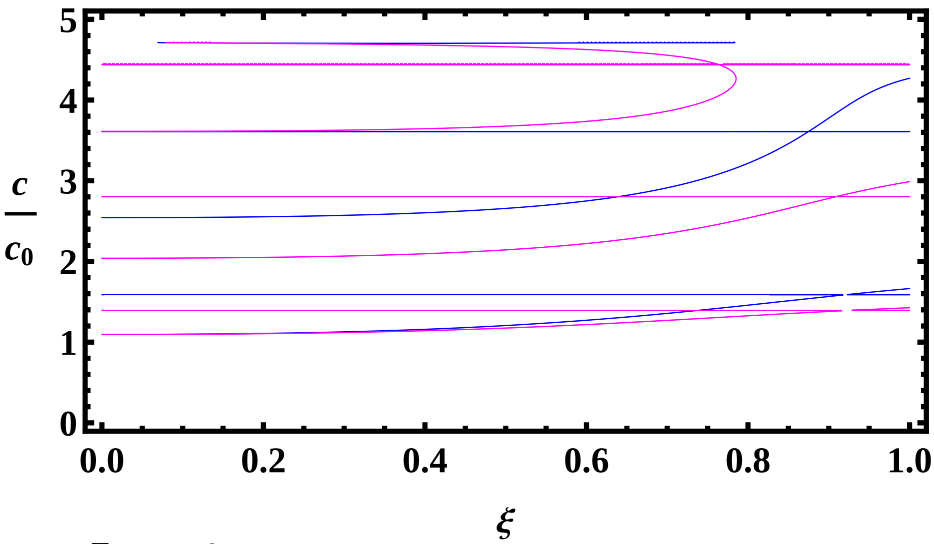

FIGURE 8. Model-II: Torsional wave speed '$\frac{c}{c_0}$' against the sliding parameter 'ξ' for different values of 'kh' ($kh = 1.5$, blue curve; $kh = 2$, magenta curve).

magenta curve) and for fixed value of sandiness parameter. It is seen that the modes of torsional waves increase with increase in the value of kh and the role of sliding parameter on the propagation of torsional wave can be noticed.

6. Conclusions

The dispersion equations of torsional surface waves for two different Models have been explicitly derived. In Model-I, the possibility of torsional surface wave in homogeneous elastic layer of uniform thickness lying over a non-homogeneous transversely isotropic elastic solid half-space is considered and in Model-II, the feasibility of torsional surface wave propagation in dry sandy layer of uniform thickness lying over a non-homogeneous elastic half-space has been explored. A general sliding contact parameter has been introduced, which can define welded contact, partially sliding contact and smooth contact interface between the layer and the half-space in the considered models. General dispersion equations have been derived from the appropriate boundary conditions at the top surface of the layer and at the interface between the layer and the half-space. The general dispersion equation enables us to provide the dispersion curves of torsional waves for welded contact, partially sliding contact and smooth contact interface by giving specific values to the sliding parameter. For considered specific models, the torsional wave speed corresponding to Model-I & II have been computed numerically, depicted graphically and discussed. Based upon the analysis and numerical results, the following main conclusions have been inferred.

- Torsional waves are found to propagate in both the models considered for welded contact, finite contact and smooth contact interfaces. However, the torsional waves are dispersive for welded contact and finite sliding contact, but non-dispersive in case of smooth contact interface. For smooth contact interface, the value of torsional wave speed c/c_0 becomes unity and $\sqrt{\zeta}$ in Model -I and II respectively.
- In Model-I, the propagation of torsional waves is not allowed in certain range of sliding parameter ξ. This range depends on the value of kh chosen. Note that the range is found numerically only for small values of kh, however for higher values of kh, the propagation of torsional modes are permitted.
- In Model-II, the influence of sandiness parameter is that the magnitude of torsional wave speed increases with increase in the value of sandiness parameter.
- In both the models, the general dispersion equations for welded contact interface reduce to those derived earlier by Gupta et al. [11] and Sultana and Gupta [13] for the corresponding problem.

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ANALOGOUS EULER SUMS VIA LOG, ARCTAN AND ARCSINE FUNCTIONS

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Abstract. We evaluate here some Euler-like sums involving harmonic numbers, odd harmonic numbers and binomial coefficients. We use expansions for the cubic powers of logarithmic function, inverse tangent function and inverse sine function.

1. Introduction

We use the following notations:

$$H_n=\sum_{k=1}^{n}\frac{1}{k};\;\; H_n^{(p)}=\sum_{k=1}^{n}\frac{1}{k^p};\;\; h_n=\sum_{k=1}^{n}\frac{1}{2k-1};\;\; h_n^{(p)}=\sum_{k=1}^{n}\frac{1}{(2k-1)^p}.$$

Two of these can be expressed in terms of the *psi* or *digamma function*:

$$h_n=\frac{1}{2}\left[\psi\left(n+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\right)\right],\; H_n^{(p)}=\frac{(-1)^{p-1}}{(p-1)!}\left[\psi^{(p-1)}(n+1)-\psi^{(p-1)}(1)\right],$$

where $\psi(z)=\frac{d}{dz}\log\Gamma(z)$ whose repeated differentiation leads to the formation of the family of *polygamma functions*

$$\psi^{(k)}(z)=\frac{d^k}{dz^k}\psi(z)=(-1)^{k+1}k!\sum_{j=0}^{\infty}\frac{1}{(z+j)^{k+1}},$$

with $z\neq 0,-1,-2,\dots$ and $k\in\mathbb{N}$.
G here denotes Catalan's constant defined by $G=\sum_0^\infty(-1)^n/(2n+1)^2$.

Euler sums have their origin in the correspondence between Goldbach and Euler during 1742–3. Their correspondence culminated in a long paper [3] by Euler wherein he recorded a *general formula* (§22, p.165) which can be written as:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^m}=\frac{m+2}{2}\zeta(m+1)-\frac{1}{2}\sum_{k=1}^{m-2}\zeta(m-k)\,\zeta(k+1),\; m=2,3,4,\dots.$$

For a detailed historical the reader may see the author's joint paper with Prof Sofo [7].

2010 Mathematics Subject Classification: 11M06, 11Y60, 33B15, 41A58, 65B10.
Key words and phrases: Euler sums, harmonic numbers, binomial coefficients, polylogarithms.

2. Sums via logarithmic function

Schwatt obtained this series expansion [5]

$$(\log x)^n = n!\sum_{\alpha_0=0}^{\infty}(-1)^{\alpha_0}\frac{x^{\alpha_0+n}}{\alpha_0+n}\left(\prod_{\beta=1}^{n-1}\sum_{\alpha_\beta=0}^{\alpha_{\beta-1}}\right)\frac{1}{\alpha_\beta+n-\beta}.$$

with a special case for $x^2 < 1$:

$$\frac{1}{6}\{\log(1+x)\}^3 = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\sum_{k=1}^{n}\frac{H_k}{k+1}\right]\frac{x^{n+2}}{n+2}. \qquad (2.1)$$

(The special case is included in [1, p.125, 6.44.13].) (2.1) straightway gives

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{n+2} = \frac{\log^3(2)}{6}. \qquad \text{Sum L1:}$$

Dividing both sides of (2.1) by x, and then integrating from -1 to 0, we get $\int_{-1}^{0}\frac{\{\log(1+x)\}^3}{6\,x}\,dx = (\pi^4/90)$, which results in :

$$\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{(n+2)^2} = \frac{\pi^4}{90} \qquad \text{Sum L2:}$$

And integrating from 0 to 1 we obtain

$$\int_0^1\frac{\{\log(1+x)\}^3}{6\ x}\,dx = \frac{\pi^4}{90} - \frac{7\zeta(3)\log(2)}{8} + \frac{\pi^2\log^2(2)}{24} - \frac{\log^4(2)}{24} - \mathrm{Li}_4\left(\frac{1}{2}\right),$$

where $\mathrm{Li}_4(z) = \sum_{n=1}^{\infty}(z^n/n^4)$, $|z| \le 1$. It yields

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{(n+2)^2} = \frac{\pi^4}{90} - \frac{7\zeta(3)\log(2)}{8} + \frac{\pi^2\log^2(2)}{24} - \frac{\log^4(2)}{24} - \mathrm{Li}_4\left(\frac{1}{2}\right). \qquad \text{Sum L3}$$

Dividing both sides of (2.1) by x^2, and then integrating from -1 to 0, we get $\int_{-1}^{0}\frac{\{\log(1+x)\}^3}{6\,x^2}\,dx = -\zeta(3)$, and integrating from 0 to 1, we obtain $\int_0^1\frac{\{\log(1+x)\}^3}{6\,x^2}\,dx = \frac{\zeta(3)}{8} - \frac{\log^3(2)}{3}$ which yield the following two sums respectively:

$$\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{(n+1)(n+2)} = \zeta(3). \qquad \text{Sum L4 :}$$

and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{(n+1)(n+2)} = \frac{\zeta(3)}{8} - \frac{\log^3(2)}{3}. \qquad \text{Sum L5}$$

Dividing both the sides of (2.1) by x^3, and then integrating from -1 to 0, we get $\int_{-1}^{0}\frac{\{\log(1+x)\}^3}{6\,x^3}\,dx = \frac{\zeta(3)}{2} + \frac{\pi^2}{12}$ so that

$$\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{n(n+2)} = \frac{\zeta(3)}{2} + \frac{\pi^2}{12}. \qquad \text{Sum L6 :}$$

Furthermore, integrating from 0 to 1, we get $\int_0^1 \frac{\{\log(1+x)\}^3}{6\,x^3}\,dx = -\frac{\zeta(3)}{16} + \frac{\pi^2}{24} - \frac{\log^2(2)}{2}$ leading to

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{n(n+2)} = -\frac{\zeta(3)}{16} + \frac{\pi^2}{24} - \frac{\log^2(2)}{2}. \qquad \text{Sum L7}$$

We deduce from Sum L1 and Sum L5:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{n+1} = \frac{\zeta(3)}{8} - \frac{\log^3(2)}{6}, \qquad \text{Sum L8}$$

while Sum L1 and Sum L7 lead to:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{n} = -\frac{\zeta(3)}{8} + \frac{\pi^2}{12} - \log^2(2) + \frac{\log^3(2)}{6}. \quad \text{Sum L9}$$

The two preceding sums can be combined to get:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{n(n+1)} = -\frac{\zeta(3)}{4} + \frac{\pi^2}{12} - \log^2(2) + \frac{\log^3(2)}{3}. \quad \text{Sum L10}$$

And now Sum L7 and Sum L10 may be combined to obtain:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{H_k}{k+1}}{n(n+1)(n+2)} = -\frac{3\zeta(3)}{16} + \frac{\pi^2}{24} - \frac{\log^2(2)}{2} + \frac{\log^3(2)}{3}. \quad \text{Sum L11}$$

3. Sums via inverse tangent function

Schwatt obtained this expansion [6] for $-\frac{\pi}{4} < x < \frac{\pi}{4}$ with $\prod_{\alpha=1}^{0} A_\alpha = 1$:

$$(\arctan x)^p = p!\sum_{k_0=1}^{\infty}(-1)^{k_0-1}\frac{x^{2k_0+p-2}}{2k_0+p-2}\prod_{\alpha=1}^{p-1}\left[\sum_{k_\alpha=1}^{k_{\alpha-1}}\frac{1}{2k_\alpha+p-\alpha-2}\right],$$

(included in [1, p.122, 6.42.3]) with a special case

$$\frac{1}{3}(\arctan x)^3 = \{1(1)\}\frac{x^3}{3} - \left\{1(1) + \frac{1}{2}\left(1+\frac{1}{3}\right)\right\}\frac{x^5}{5}$$
$$+ \left\{1(1) + \frac{1}{2}\left(1+\frac{1}{3}\right) + \frac{1}{3}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right\}\frac{x^7}{7} - \ldots$$

that is,

$$\frac{1}{3}(\arctan x)^3 = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\sum_{k=1}^{n}\frac{h_k}{k}\right]\frac{x^{2n+1}}{2n+1}. \qquad (3.1)$$

(3.1) straightway gives:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left[\sum_{k=1}^{n}\frac{h_k}{k}\right]\frac{1}{2n+1} = \frac{\pi^3}{192}. \qquad \text{Sum T1}$$

Now

$$\frac{1}{3}\int_0^1(\arctan x)^3\,dx = \frac{1}{192}\left[\pi^3 + 6\pi^2\log(2) - 48\pi G + 63\zeta(3)\right].$$

Therefore, we deduce from (2.1) the following sum:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{(2n+1)(2n+2)} = \frac{\pi^3}{192} + \frac{\pi^2\log(2)}{32} - \frac{\pi G}{4} + \frac{21\zeta(3)}{64}. \quad \text{Sum T2}$$

Dividing both sides of (3.1) by x and then integrating from 0 to 1, we get

$$\frac{1}{3}\int_0^1 \frac{(\arctan x)^3}{x}\,dx = \frac{\pi^2 G}{16} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4},$$

leading to the sum:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{(2n+1)^2} = \frac{\pi^2 G}{16} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}. \qquad \text{Sum T3}$$

Dividing both sides of (3.1) by x^2 and then integrating from 0 to 1 yields

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{2n(2n+1)} = -\frac{\pi^3}{192}+\frac{\pi^2\log(2)}{32}+\frac{\pi G}{4}-\frac{35\zeta(3)}{64}. \qquad \text{Sum T4}$$

Again, dividing both sides of (3.1) by x^3 and then integrating from 0 to 1 yields

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{(2n-1)(2n+1)} = \frac{G}{2} - \frac{\pi^2}{32} - \frac{\pi^3}{192} + \frac{\pi\log(2)}{8}. \qquad \text{Sum T5}$$

We deduce from Sum T1 and Sum T2:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{n+1} = -\frac{\pi^2\log(2)}{16} + \frac{\pi G}{2} - \frac{21\zeta(3)}{32}. \qquad \text{Sum T6}$$

Sum T2 and Sum T4 yield:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{n(n+1)} = -\frac{7\zeta(3)}{16} + \frac{\pi^2\log(2)}{8}. \qquad \text{Sum T7}$$

And the two preceding sums give:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{n} = \frac{\pi^2\log(2)}{16} + \frac{\pi G}{2} - \frac{35\zeta(3)}{32}. \qquad \text{Sum T8}$$

4. Sums via inverse hyperbolic tangent function

Using the relationship $i\,\mathrm{arctanh}\,x = \arctan(i\,x) \Rightarrow \mathrm{arctanh}\,x = -i\arctan(i\,x)$ between $\mathrm{arctanh}\,x$ and $\arctan x$, in (2.1), we obtain

$$(\mathrm{arctanh}\,x)^p = (-i)^p\,p!\sum_{k_0=1}^{\infty}(-1)^{k_0-1}\frac{(ix)^{2k_0+p-2}}{2k_0+p-2}\prod_{\alpha=1}^{p-1}\left[\sum_{k_\alpha=1}^{k_{\alpha-1}}\frac{1}{2k_\alpha+p-\alpha-2}\right]$$

that is,

$$(\mathrm{arctanh}\,x)^p = p!\sum_{k_0=1}^{\infty}\frac{x^{2k_0+p-2}}{2k_0+p-2}\prod_{\alpha=1}^{p-1}\left[\sum_{k_\alpha=1}^{k_{\alpha-1}}\frac{1}{2k_\alpha+p-\alpha-2}\right]. \qquad (4.1)$$

From (3.1), we get

$$\frac{1}{3}\{-i\arctan(i\,x)\}^3 = (-i)^3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{2n+1}\,i^{2n+1}x^{2n+1},$$

that is,

$$\frac{1}{3}(\mathrm{arctanh}\,x)^3 = \sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}\frac{h_k}{k}}{2n+1}\,x^{2n+1} \qquad (4.2)$$

Now

$$\int_0^1\{\mathrm{arctanh}(x)\}^3\,dx = \frac{9\zeta(3)}{8}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{(2n+1)(2n+2)} = \frac{3\zeta(3)}{8}. \qquad \text{Sum T9}$$

Now multiply both sides of (4.2) by x and integrate that from 0 to 1. Since $\int_0^1 x\{\operatorname{arctanh}(x)\}^3\,dx = \frac{\pi^2}{8}$, we get:

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{(2n+1)(2n+3)} = \frac{\pi^2}{24}. \qquad \text{Sum T10}$$

Dividing both sides of (4.2) by x^3 and then integrating from 0 to 1, we obtain $\int_0^1 \frac{\{\operatorname{arctanh}(x)\}^3}{x^3}\,dx = \frac{\pi^2}{4}$ which yields

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{(2n-1)(2n+1)} = \frac{\pi^2}{12}. \qquad \text{Sum T11}$$

We then deduce the sum with three linear odd factors in the denominator:

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{(2n-1)(2n+1)(2n+3)} = \frac{\pi^2}{96}. \qquad \text{Sum T12}$$

Since $\int_0^1 \frac{\{\operatorname{arctanh}(x)\}^3}{x^2}dx = \frac{3}{2}\zeta(3)$, dividing (4.2) by x^2 and integrating from 0 to 1 yields a neat sum:

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{n(2n+1)} = \zeta(3). \qquad \text{Sum T13}$$

Dividing both the sides of (4.2) by x and then integrating from 0 to 1 yields:

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{(2n+1)^2} = \frac{\pi^4}{64}. \qquad \text{Sum T14}$$

We deduce from T9 and T13

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{n(n+1)} = \frac{7\zeta(3)}{4}. \qquad \text{Sum T15}$$

The integral $\int_0^{1/3} \frac{\{\operatorname{arctanh}(x)\}^3}{x^2}dx = \frac{3\zeta(3)}{16} - \frac{\log^3(2)}{2}$ gives a fast series for $\zeta(3)$:

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{9^n\, n(2n+1)} = \frac{\zeta(3)}{8} - \frac{\log^3(2)}{3}. \qquad \text{Sum T16}$$

And the integral $\int_0^{1/3} \frac{\{\operatorname{arctanh}(x)\}^3}{x^3}\,dx = \frac{\pi^2}{8} - \frac{\log^3(2)}{2} - \frac{3\log^2(2)}{2}$ gives this fast series for π :

$$\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n} \frac{h_k}{k}}{9^{n-1}\,(2n-1)(2n+1)} = \frac{\pi^2}{8} - \frac{\log^3(2)}{2} - \frac{3\log^2(2)}{2}. \qquad \text{Sum T17}$$

These integrals also give similar sums:

$$\int_0^{1/3} x\{\operatorname{arctanh}(x)\}^3 dx = -\frac{3\log(2)\log(3)}{2} + 2\log^2(2) - \frac{\log^3(2)}{18} + \frac{3}{4}\operatorname{Li}_2\left(\frac{1}{4}\right),$$

where the polylogarithmic function (see [4]) is defined by $\mathrm{Li}_n(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^n}$.

$$\int_0^{1/3} x^2 \{\operatorname{arctanh}(x)\}^3\, dx = -\frac{\zeta(3)}{16} - \frac{\log^2(2)\log(3)}{4} - \frac{\log^2(2)}{9} + \frac{61\log^3(2)}{162}$$
$$+ \frac{5}{3}\log 2 - \log(3) + \frac{1}{4}\log(2)\,\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{1}{8}\mathrm{Li}_3\left(\frac{1}{4}\right), \quad \text{and}$$

$$\int_0^{1/3} \{\operatorname{arctanh}(x)\}^3\, dx = -\frac{3\zeta(3)}{16} - \frac{3\log^2(2)\log(3)}{4} + \frac{7}{6}\log^3(2)$$
$$+ \frac{3}{4}\log(2)\,\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{3}{8}\mathrm{Li}_3\left(\frac{1}{4}\right).$$

5. Sums via inverse sine function

We find the following series expansion in [2, p.188, Ex.1]:

$$\frac{1}{6}(\arcsin x)^3 = \frac{1}{2}(1)\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\left(1+\frac{1}{3^2}\right)\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{x^7}{7} + \dots$$

which can be written as:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)\,2^{2n-1}}\, x^{2n+1} = \frac{(\arcsin x)^3}{3}. \tag{5.1}$$

It yields with $x = 1, \frac{1}{\sqrt{2}}, \frac{1}{2}$ the following sums:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)\,2^{2n}} = \frac{\pi^3}{48}; \sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)\,2^{3n}} = \frac{\sqrt{2}\,\pi^3}{384}; \sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)\,2^{4n}} = \frac{\pi^3}{648}.$$

If integrate (5.1) using *WolframAlpha* we obtain:

$$\int (\arcsin x)^3\, dx = x\,(\arcsin x)^3 + 3\sqrt{1-x^2}\,(\arcsin x)^2$$
$$- 6x\arcsin(x) - 6\sqrt{1-x^2} + C. \tag{5.2}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)(2n+2)\,2^{2n}}\, x^{2n+2} = \frac{x}{6}(\arcsin x)^3 + \frac{1}{2}\sqrt{1-x^2}\,(\arcsin x)^2$$
$$- x\arcsin(x) - \sqrt{1-x^2} + 1. \tag{5.3}$$

It yields with $x = 1, \frac{1}{\sqrt{2}}, \frac{1}{2}$ these sums:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)(2n+2)\,2^{2n}} = \frac{\pi^3}{48} - \frac{\pi}{2} + 1. \tag{5.4}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)(2n+2)\,2^{3n+1}} = \frac{\sqrt{2}\,\pi^3}{768} + \frac{\sqrt{2}\,\pi^2}{64} - \frac{\sqrt{2}\,\pi}{8} + 1 - \frac{\sqrt{2}}{2}. \tag{5.5}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)(2n+2)\,2^{4n+2}} = \frac{\pi^3}{48} + \frac{\sqrt{3}\,\pi^2}{144} - \frac{\pi}{12} + 1 - \frac{\sqrt{3}}{2}. \tag{5.6}$$

We deduce from two of the preceding results a nice sum:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(n+1)\,2^{2n}} = \pi - 2. \tag{5.7}$$

Dividing (5.1) by x, and then integrating using *WolframAlpha* we get a series with quadratic factor in the denominator on LHS while RHS becomes:

$$\begin{aligned}\int \frac{(\arcsin x)^3}{x}\,dx &= -\frac{i}{64}\Big[48\mathrm{Li}_4\left(e^{-2i\arcsin x}\right) \\ &+96i(\arcsin x)\mathrm{Li}_3\left(e^{-2i\arcsin x}\right) - 96(\arcsin x)^2\mathrm{Li}_2\left(e^{-2i\arcsin x}\right) \\ &-16(\arcsin x)^4 + 64i(\arcsin x)^3\log\left\{1-\left(e^{-2i\arcsin x}\right)\right\} + \pi^4\Big] + C.\end{aligned} \tag{5.8}$$

Integration from 0 to 1 yields after simplification:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)^2\,2^{2n}} = \frac{\pi}{16}\left[\frac{\pi^2\,\log(2)}{3} - \frac{3\,\zeta(3)}{2}\right]. \tag{5.9}$$

Dividing (5.1) by x^2, and then integrating using *WolframAlpha* we get:

$$\begin{aligned}\int \frac{(\arcsin x)^3}{x^2}\,dx &= 6\mathrm{Li}_3\left(e^{i\arcsin x}\right) - 6\mathrm{Li}_3\left(-e^{i\arcsin x}\right) \\ &+ 6i(\arcsin x)\mathrm{Li}_2\left(-e^{i\arcsin x}\right) - 6i(\arcsin x)\mathrm{Li}_2\left(-e^{i\arcsin x}\right) \\ &- \frac{(\arcsin x)^3}{x} + 3(\arcsin x)^2\log\left\{1-\left(e^{i\arcsin x}\right)\right\} \\ &- 3(\arcsin x)^2\log\left\{1+\left(e^{i\arcsin x}\right)\right\} + C.\end{aligned} \tag{5.10}$$

Integration from 0 to 1 yields after simplification:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{2n(2n+1)\,2^{2n}} = \pi G - \frac{\pi^3}{48} - \frac{7\zeta(3)}{4}. \tag{5.11}$$

We then deduce from two of the earlier results:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{n\,2^{2n+1}} = \pi G - \frac{7\zeta(3)}{4}, \tag{5.12}$$

resulting in a new series for $\zeta(3)$.

We divide (5.3) by x and integrate it from 0 to 1. Since $\int_0^1 (\arcsin x)^3\,dx = \frac{\pi^3}{8} - 3\pi + 6$, $\int_0^1 \frac{(\arcsin x)^2\sqrt{1-x^2}}{x}\,dx = 2\pi G - \frac{7\zeta(3)}{2} - \pi + 2$, $\int_0^1 \arcsin x\,dx = \frac{\pi}{2} - 1$ and $\int_0^1 \frac{1-\sqrt{1-x^2}}{x}\,dx = 1 - \log(2)$, therefore

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(2n+1)(2n+2)^2\,2^{2n}} = \frac{\pi^3}{48} + \pi G - \frac{7\zeta(3)}{4} - \frac{3\pi}{2} - \log(2) + 4. \tag{5.13}$$

We then deduce:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n^{(2)}}{(n+1)^2\,2^{2n}} = 7\zeta(3) - 4\pi G + 4\pi + 4\log(2) - 12. \tag{5.14}$$

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ON G^*-LOCAL FUNCTION OF A SET

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Abstract. In this article, we have introduced the concept of G^*-local function of a set in a topological space with an ideal and a generalized topology on it. We have obtained some significant properties of G^*-local function and we have constructed some basic examples.

1. Introduction

The concept of ideal in Topological spaces was introduced by Kuratowski ([6]) in 1930. The notion of generalized topology was introduced by Csaszar ([2]) in 2002. Jancovic and Hamlett ([3]) have studied the concept of local function in ideal topological spaces and obtained its significant properties. Roy and Sen ([8]) have defined the notion of G^*-open set in 2013. We have studied G^*-local function of a set in an ideal topological space together with a generalized topology on it and established basic properties of it.

2. Preliminaries

We begin with the required definitions and examples.

Definition 2.1. [2] Let X be a non-empty set and let τ_G be a family of subsets of X. Then τ_G is called a *generalized topology* on X if following two conditions are satisfied viz,:

(1) $\phi, \quad X \in \tau_G$;

(2) If $G_\lambda \in \tau_G$ for $\lambda \in \wedge$ then $\bigcup_{\lambda \in \wedge} G_\lambda \in \tau_G$.

The members of the family τ_G are called *G-open sets* and their complement are called *G-closed sets* in the generalized topological space (X, τ_G).

Definition 2.2. [2] Let τ_G be a generalized topology on the set X and let $A \subseteq X$. Then *G-interior* of A is denoted by $i_G(A)$ and is defined to be the union of all G-open sets in X contained in A. The *G-closure* of A is denoted by $c_G(A)$ and is defined to be the intersection of all G-closed sets

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in X containing A.

Remark 2.1. [2] Since arbitrary union of G-open sets is a G-open set and arbitrary intersection of G-closed sets is a G-closed sets, it follows that $i_G(A)$ is a G-open set and $c_G(A)$ is a G-closed set. Hence $i_G(A)$ is the largest G-open set contained in A and $c_G(A)$ is the smallest G-closed set containing A.

Theorem 2.3. [2] *Let (X, τ) be a topological space and let τ_G be a generalized topology on X. Suppose $A \subseteq X$. Then,*

(1) A is a G-open set iff $i_G(A) = A$.

(2) A is a G-closed set iff $c_G(A) = A$.

Definition 2.4. [8] Let (X, τ) be a topological space and let τ_G be a generalized topology on X. Then a subset A of X is called G^*-open set if $A \subseteq cl(i_G(A))$.

Proposition 2.5. [8] *Let (X, τ) be a topological space with a generalized topology τ_G on X. Then each G-open set is a G^*-open set.*

Remark 2.2. [7] Converse of the above Proposition 2.5 is not necessary true. Further we observe that if $A(\neq \phi)$ is a G^*-open set then $i_G(A) \neq \phi$.

Proposition 2.6. *Let (X, τ) be a topological space with a generalized topology τ_G on X. Let $A \subseteq X$. Then A is a G^*-open set iff there exists a G-open set U in X such that $U \subseteq A \subseteq cl(U)$.*

Proof. Let A be a G^*-open set in X. Then $A \subseteq cl(i_G(A))$. Taking $U = i_G(A)$, we see U is a G-open set in X and $U \subseteq A \subseteq cl(U)$.

Conversely, let $A \subseteq X$ and let U be a G-open set in X such that $U \subseteq A \subseteq cl(U)$. From $U \subseteq A$, we have $U = i_G(U) \subseteq i_G(A)$ and since $A \subseteq cl(U)$, we find that $A \subseteq cl(U) \subseteq cl(i_G(A))$. This implies $A \subseteq cl(i_G(A))$. Thus A is a G^*-open set in X. □

Proposition 2.7. [8] *Let (X, τ) be a topological space and let τ_G be a generalized topology on X. If $\{U_\alpha\}_{\alpha \in \wedge}$ is any family of G^*-open sets in X, then $\cup_{\alpha \in \wedge} U_\alpha$ is a G^*-open set in X.*

Theorem 2.8. [7] *Let (X, τ) be a topological space with a generalized topology τ_G on X. Then A is a G^*-open set iff $cl(A) = cl(i_G(A))$.*

Theorem 2.9. [7] *Let (X, τ) be a topological space with a generalized topology τ_G on X and let $A \subseteq X$. Then A is a G^*-open set if and only if for each $x \in A$ then there exists a G^*-open set U in X such that $x \in U \subseteq A$.*

3. G^*-LOCAL FUNCTION OF A SET AND ITS PROPERTIES.

In this section we have defined the notion of G^*-local function of a set and observed its significant properties.

Definition 3.1. [6] Let I be a family of subsets of a topological space (X, τ). Then I is called an *Ideal* on X if following conditions holds:

(1) (Hereditary) If $A \in I$ and $B \subseteq A \Rightarrow B \in I$,

(2) (Finite Additivity) If $A \in I$ and $B \in I \Rightarrow A \cup B \in I$.

We see that $I = \{\emptyset\}$ and $I = \wp(X)$ are ideals on any topological space (X, τ). The Triplet (X, τ, I) is called on Ideal Topological space.

Definition 3.2. Let $(X,\ \tau,\ I)$ be an ideal topological space with a generalized topology τ_G on X and let $A \subseteq X$. The set $A^{G\star} = \{x \in X : A \cap U \notin I$ for each G^*-open set U containing $x\}$ is called G^*-*local function* of A with respect to Ideal I and generalized topology τ_G on (X, τ).

Example 3.3. Let $X = \{l,\ m,\ n\}$, and let $\tau = \{\emptyset,\ X,\ \{l, m\}, \{m\}, \{m,\ n\}\}$ be the topology on X. Let $\tau_G = \{\phi, \{l, n\},\ \{l, m\},\ X\}$ be the generalized topology on X and let $I = \{\emptyset, \{l\}, \{n\},\ \{l,\ n\}\}$ be the ideal on X. Then the family of G^*-open sets is $\{\phi,\ \{l, n\},\ \{l, m\},\ X\}$. We see that G^*-local function of each subsets of X are as follows:

(1) $\emptyset^{G\star} = \emptyset$. (2) $\{l\}^{G\star} = \emptyset$. (3) $\{m\}^{G\star} = \{m\}$.
(4) $\{n\}^{G\star} = \phi$. (5) $\{l,\ m\}^{G\star} = \{m\}$. (6) $\{l,\ n\}^{G\star} = \phi$.
(7) $\{m,\ n\}^{G\star} = \{m\}$. (8) $X^{\star} = \{m\}$.

In the following example we observe that neither a set is contained in its G^*-local function nor a G^*-local function of set is contained in the set.

Example 3.4. Let $X = \{l_1, l_2, l_3, l_4\}$ be the ideal topological space with respect to topology $\tau = \{\emptyset,\ X, \{l_2\}, \{l_1, l_2\},\ \{l_2, l_3\},\ \{l_1, l_2, l_3\}\}$ and ideal $I = \{\emptyset,\ \{l_1\},\ \{l_3\},\ \{l_1, l_3\}\}$ on X. Consider the generalized topology $\tau_G = \{\phi,\ \{l_1, l_4\},\ \{l_2, l_4\},\ \{l_1, l_2, l_4\},\ X\}$ on X. Then family of G^*-open sets is $\{\phi,\ \{l_1, l_4\},\ \{l_2, l_4\},\ \{l_1, l_2, l_4\}, \{l_2,\ l_3, l_4\},\ X\}$. Let us consider $L = \{l_1,\ l_2\}$ then $L^{G\star} = \{l_2, l_3\}$. We see that neither $L \subseteq L^{G\star}$ nor $L^{G\star} \subseteq L$.

Theorem 3.5. *Let (X, τ, I) be an ideal topological space with generalized topology τ_G on X and let $L,\ M$ be subsets of X. Then*

(1) $L \subseteq M \Rightarrow L^{G\star} \subseteq M^{G\star}$.

(2) $(L^{G\star})^{G\star} \subseteq L^{G\star}$.

(3) $(L \cup M)^{G\star} \supseteq L^{G\star} \cup M^{G\star}$.

Proof. (1) Suppose $x \in L^{G\star}$ and if possible assume that $x \notin M^{G\star}$. Then there exists a G^*-open set W in X such that $x \in W$ and $W \cap M \in I$. This implies $W \cap L \in I$, as I is an ideal on X. This is a contradiction because $x \in L^{G\star}$. Therefore we must have $x \in M^{G\star}$. Hence $L^{G\star} \subseteq M^{G\star}$.

(2) Suppose $x \in (L^{G\star})^{G\star}$. Let W be a G^*-open set containing x. Then we have $W \cap L^{G\star} \notin I$. As $\emptyset \in I$, we find that $W \cap L^{G\star} \neq \emptyset$. Let us take $y \in W \cap L^{G\star}$. Then $y \in W$ and $y \in L^{G\star}$. Now $y \in L^{G\star}$ implies that $W \cap L \notin I$. Thus for each G^*-open set containing x, we find that $W \cap L \notin I$. Therefore $x \in L^{G\star}$. Hence $(L^{G\star})^{G\star} \subseteq L^{G\star}$.

(3) Since $L \subseteq L \cup M$, from (i) we get $L^{G\star} \subseteq (L \cup M)^{G\star}$. Similarly, $M^{G\star} \subseteq (L \cup M)^{G\star}$. Thus, we get that $L^{G\star} \cup M^{G\star} \subseteq (L \cup M)^{G\star}$. □

Example 3.6. Let $X = \{l_1, l_2, l_3, l_4\}$ be the ideal topological space with respect to topology $\tau = \{\emptyset, X, \{l_1\}, \{l_2\}, \{l_1, l_2\}, \{l_2, l_3\}, \{l_1, l_2, l_3\}\}$ and ideal $I = \{\emptyset, \{l_4\}\}$ on X. Consider the generalized topology $\{\tau\}_G = \{\phi, \{l_3\}, \{l_1, l_4\}, \{l_3, l_4\}, \{l_1, l_2, l_4\}, \{l_1, l_3, l_4\}, X\}$ on X. Then collection of G^*-open sets is $\{\phi, \{l_3\}, \{l_1, l_4\}, \{l_3, l_4\}, \{l_1, l_2, l_4\}, \{l_1, l_3, l_4\}, X\}$. Let us consider $L = \{l_1\}$ and $M = \{l_3\}$ so, $N = L \cup M = \{l_1, l_3\}$. Then $L^{G\star} = \{l_1, l_2\}$, $M^{G\star} = \{l_3\}$ and $N^{G\star} = X$. We see that $L^{G\star} \neq N^{G\star}$ and $(L \cup M)^{G\star} \neq L^{G\star} \cup M^{G\star}$.

Theorem 3.7. *Let (X, τ) be a topological space with generalized topology τ_G on X and let I, J be two ideals on set X such that $I \subseteq J$. Then for any subset $L \subseteq X$, $L^{G\star}(J) \subseteq L^{G\star}(I)$.*

Proof. Let (X, τ) be a topological space with generalized topology τ_G on X and let I, J be two ideals on set X such that $I \subseteq J$. Suppose $L \subseteq X$. Let $l \in L^{G\star}(J)$ and let W be a G^*-open set containing l with the property $L \cap W \notin J$. As $I \subseteq J$ we find that $L \cap W \notin I$. This implies $l \in L^{G\star}(I)$. Hence we deduce that $L^{G\star}(J) \subseteq L^{G\star}(I)$. □

Example 3.8. Let $X = \{l_1,\ l_2,\ l_3, l_4\}$ be the topological space with respect to topology $\tau = \{\phi, \{l_1,\ l_2\},\ \{l_1, l_2,\ l_3\},\ \{l_1,\ l_2,\ l_4\},\ X\}$. Consider generalized topology $\tau_G = \{\phi,\ \{l_2,\ l_3\},\ \{l_3, l_4\},\ \{l_2, l_3, l_4\},\ X\}$. Then the family of G^*-open sets is $\{\phi,\ \{l_2,\ l_3\}, \{l_3, l_4\},\ \{l_1,\ l_2,\ l_3\},\ \{l_2, l_3, l_4\},\ X\}$. Let us consider ideals $I_1 = \{\emptyset,\ \{l_4\}\}$ and $I_2 = \{\emptyset,\ \{l_3\},\ \{l_4\},\ \{l_3,\ l_4\}\}$ on X. For $L = \{l_3\}$, we have $L^{G\star}(I_1) = X$ and $L^{G\star}(I_2) = \emptyset$. Thus we see that $L^{G\star}(I_1) \neq L^{G\star}(I_2)$

Theorem 3.9. *Let (X, τ) be a topological space with generalized topology τ_G on X and I, J be ideals on X. Then $L^{G\star}(I \cap J) \supseteq L^{G\star}(I) \cup L^{G\star}(J)$.*

Proof. Since $I \cap J \subseteq I$, from Theorem 3.7, we have $L^{G\star}(I) \subseteq L^{G\star}(I \cap J)$. Also $L^{G\star}(J) \subseteq L^{G\star}(I \cap J)$. This implies $L^{G\star}(I) \cup L^{G\star}(J) \subseteq L^{G\star}(I \cap J)$. □

Proposition 3.10. *Let (X, τ, I) be an ideal topological space with generalized topology τ_G on X. If $I = \wp(X)$ is the ideal on X, then $L^{G\star} = \emptyset$, for each subset L of X.*

Proof. Let $L \subseteq X$ and let W be an G^*-open set in X containing the point l, $l \in X$ such that $W \cap L = V$ (*say*), since $I = \wp(X)$, $V \in I$. This means $l \notin L^{G\star}$. Hence no point of X can belong to $L^{G\star}$. Thus $L^{G\star} = \emptyset$. □

Theorem 3.11. *Let (X, τ, I) be an ideal topological space with generalized topology τ_G on X. Then each non-empty G^*-open set in X can not lie in I iff $X^{G\star} = X$.*

Proof. Let (X, τ, I) be an ideal topological space with generalized topology τ_G on X and each non-empty G^*-open set in X can not lie in I. Let $l \in X$ and W be a G^*-open set containing l in X. Then $W \cap X = W$. Since $W \neq \emptyset$, we have $W \notin I$. Thus $l \in X^{G\star}$, l is arbitrary element of X. Therefore $X^{G\star} = X$.

Conversely suppose $X^{G\star} = X$. Let W be a non-empty G^*-open set. Then there exits $l \in X = X^{G\star}$, such that $l \in W$. This means W is G^*-open set containing l. Now by definition of $X^{G\star}$, $W \cap X = W \notin I$. Thus, each non-empty G^*-open set in X can not lie in I.

Let $\{X, \tau, I\}$ be an ideal topological space and let τ_G be a generalized topology on it. For $A \subseteq X$, we define $cl^{G^*}(A) = A \cup A^{G\star}$, where $A^{G\star}$ is the G^*-local function of set A. In the following result we have established some properties of the set $cl^{G^*}(A)$. □

Theorem 3.12. *Let (X, τ, I) be an ideal topological space with generalized topology τ_G on X. Let L, $M \subseteq X$,*

(1) If $L \subseteq M$ then $cl^{G^*}(L) \subseteq cl^{G^*}(M)$.
(2) $cl^{G^*}(cl^{G^*}(L)) \supseteq cl^{G^*}(L)$.
(3) $cl^{G^*}(L \cup M) \supseteq cl^{G^*}(L) \cup cl^{G^*}(M)$.
(4) $cl^{G^*}(L \cap M) \subseteq cl^{G^*}(L) \cap cl^{G^*}(M)$.

Proof. (1) Since $L \subseteq M$, from Theorem 3.5, we have $L^{G\star} \subseteq M^{G\star}$. This implies $cl^{G*}(L) = L \cup L^{G\star} \subseteq M \cup M^{G\star} = cl^{G^*}(M)$. Hence $cl^{G^*}(L) \subseteq cl^{G^*}(M)$.

(2) We have, $cl^{G^*}(cl^{G^*}(L)) = cl^{G^*}(L \cup L^{G\star}) = L \cup L^{G\star} \cup (L \cup L^{G\star})^{G\star} \supseteq L \cup L^{G\star} \cup L^{G\star} \cup (L^{G\star})^{G\star} \supseteq L \cup L^{G\star} \cup L^{G\star} \cup L^{G\star} = L \cup L^{G\star} = cl^{G^*}(L)$. Thus, $cl^{G^*}(cl^{G^*}(L)) \supseteq cl^{G^*}(L)$.

(3) Since $cl^{G^*}(L \cup M) = (L \cup M) \cup (L \cup M)^{G\star} \supseteq L \cup M \cup L^{G\star} \cup M^{G\star} = cl^{G^*}(L) \cup cl^{G^*}(M)$ (from Definition 3.1). This implies $cl^{G^*}(L \cup M) \supseteq cl^{G^*}(L) \cup cl^{G^*}(M)$.

(4) Since $L \cap M \subseteq L$ and $L \cap M \subseteq M$. From Theorem 3.5, we have, $(L \cap M)^{G\star} \subseteq L^{G\star}$ and $(L \cap M)^{G\star} \subseteq M^{G\star}$. This implies, $cl^{G^*}(L \cap M) \subseteq cl^{G^*}(L) \cap cl^{G^*}(M)$.

□

Example 3.13. Let $X = \{l_1, l_2, l_3, l_4\}$ be the ideal topological space with respect to topology $\tau = \{\emptyset,\ X, \{l_1\}, \{l_2\}, \{l_1, l_2\},\ \{l_2, l_3\},\ \{l_1, l_2, l_3\}\}$ and ideal $I = \{\emptyset,\ \{l_4\}\}$ on X. Consider the generalized topology $\tau_G = \{\phi, \{l_3\}, \{l_1, l_4\}, \{l_3, l_4\}, \{l_1, l_2, l_4\},\ \{l_1,\ l_3, l_4\},\ X\}$ on X. Then family of G^*-open sets is $\{\phi,\ \{l_3\}, \{l_1, l_4\},\ \{l_3, l_4\},\ \{l_1, l_3, l_4\}, \{l_1,\ l_2, l_4\},\ X\}$. Let us consider $L = \{l_1\}$ and $M = \{l_3\}$ so, $N = L \cup M = \{l_1,\ l_3\}$. Then $L^{G\star} = \{l_1, l_2\}$, $M^{G\star} = \{l_3\}$ and $N^{G\star} = X$. Therefore, $cl^{G\star}(L) = \{l_1, l_2\}$, $cl^{G\star}(M) = \{l_3\}$ and $cl^{G\star}(N) = X$.

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$*$-RICCI SOLITONS ON THREE DIMENSIONAL TRANS-SASAKIAN MANIFOLDS

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Abstract. In the present paper $*$-Ricci tensor on a three-dimensional trans-Sasakian manifold has been studied. A necessary and sufficient condition for a three-dimensional trans-Sasakian manifold to be ϕ-Einstein has been deduced. Three-dimensional trans-Sasakian manifolds admitting $*$-Ricci tensor have also been considered. An illustrative example has been given.

1. Introduction

The revolutionary concept of Ricci flow was introduced by Hamilton [13]. The notion of Ricci flow was used by Perelman[17] to prove the famous 'Poincare Conjecture'. A self similar solution of Ricci flow is known as Ricci soliton. The study of Ricci solitons in the context of almost contact or contact geometry was first done by Sharma [19]. In the theory of Ricci solitons the Ricci curvature plays pivotal role. Generalizing the concept of Ricci tensor, Tachibana [20] introduced the notion of $*$-Ricci tensor. $*$-Ricci tensor has also been studied in the paper [12].

A manifold is called Einstein if its Ricci tensor is a constant multiple of the metric tensor. A Ricci soliton is a generalization of Einstein manifold. A manifold is called Ricci soliton if for a vector field V and a constant λ its Ricci tensor satisfies

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0,$$

where $\pounds$ is Lie derivative, X, Y are arbitrary vector fields on the manifold, S is Ricci tensor, g is metric tensor. Using the $*$-Ricci tensor, instead of Ricci tensor, analogus concept of $*$-Ricci soliton has been defined. For details see [5], [15], [23]. Using the concept of $*$-Ricci tensor, A. Ghosh

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and D. S. Patra [9] studied $*$-Ricci soliton on Sasakian and (κ, μ) contact manifolds. In the papers [21] and [22] further studies on $*$-Ricci tensor and $*$-Ricci soliton have been done.

The theory of trans-Sasakian structure was developed in [2] and [16]. Trans-Sasakian manifolds have been further studied in the papers [6] and [14]. In the present paper we would like to study $*$-Ricci soliton on trans-Sasakian manifolds.

The present paper is organized as follows. After the introduction we mention some required preliminaries in Section 2. In Section 3, we deduce some characteristic properties of $*$-Ricci tensor of three dimensional trans-Sasakian manifolds. In this section we also study weakly ϕ-Einstein trans-Sasakian manifolds. In Section 4, we obtain some interesting results on three dimensional trans-Sasakian manifolds admitting $*$-Ricci soliton. The last section contains an example.

2. Preliminaries

Let (M, ϕ, ξ, η, g) be a three dimensional almost contact metric manifold, where ϕ is a $(1,1)$-tensor field, ξ a unit vector field, η a smooth 1-form and g is a compatible Riemannian metric satisfying [1]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called a trans-Sasakian structure [16] if $(M \times \mathbb{R}, J, G)$ belongs to the class $\mathcal{W}_4$ [10], where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{2.4}$$

for any $X \in \mathfrak{X}(M)$, f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.5}$$

for smooth functions α and β on M, and the trans-Sasakian structure is said to be of type (α, β).

From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta\{X - \eta(X)\xi\}, \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

From [7] we know that for a three dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi(\alpha) = 0, \tag{2.8}$$

$$S(X,\xi) = \{2(\alpha^2 - \beta^2) - \xi\beta\}\eta(X) - X\beta - (\phi X)\alpha, \tag{2.9}$$

$$S(X,Y) = \left\{\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\right\} g(X,Y) - \left\{\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2)\right\} \eta(X)\eta(Y) - \{Y\beta + (\phi Y)\alpha\}\eta(X) - \{X\beta + (\phi X)\alpha\}\eta(Y), \tag{2.10}$$

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2-\beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\ &- (X\alpha)\,\phi Y + (Y\alpha)\,\phi X + (X\beta)Y - (X\beta)\eta(Y)\xi \\ &- (Y\beta)X + (Y\beta)\eta(X)\xi, \end{aligned} \tag{2.11}$$

where S is the Ricci tensor, R is the curvature tensor and r is the scalar curvature of the manifold M.

From (2.11) we have

$$\begin{aligned} \eta(R(X,Y)Z) &= (\alpha^2-\beta^2)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\} \\ &+2\alpha\beta\{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)\} \\ &+(X\alpha)\,g(\phi Y,Z)-(Y\alpha)\,g(\phi X,Z)-(X\beta)g(Y,Z) \\ &+(X\beta)\eta(Y)\eta(Z)+(Y\beta)g(X,Z)-(Y\beta)\eta(X)\eta(Z). \end{aligned} \tag{2.12}$$

3. ∗-Ricci tensor on trans-Sasakian manifolds

The ∗-Ricci tensor and ∗-scalar curvature on an almost contact manifold (M,ϕ,ξ,η,g) of dimension $(2n+1)$ are respectively defined by

$$S^*(X,Y) = \sum_{i=1}^{2n+1} R(X,e_i,\phi e_i,\phi Y), \quad r^* = \sum_{n=1}^{2n+1} S^*(e_i,e_i), \tag{3.1}$$

for all $X,Y \in \chi(M)$, where $e_1,e_2,\cdots,e_{2n+1}$ is an orthonormal basis of tangent space TM and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

From (3.1) it can be written

$$S^*(X,Y) = \sum_{i=1}^{2n+1} R(\phi Y,\phi e_i,e_i,X). \tag{3.2}$$

Lemma 3.1. *In a trans-Sasakian manifold, the following relations hold*

i). $R(X,Y)\phi Z = \phi(R(X,Y)Z) + (X\alpha)\{g(Y,Z)\xi - \eta(Z)Y\} - (Y\alpha)\{g(X,Z)\xi - \eta(Z)X\} + (X\beta)\{g(\phi Y,Z)\xi - \eta(Z)\phi Y\} - (Y\beta)\{g(\phi X,Z)\xi - \eta(Z)\phi X\} + (\alpha^2-\beta^2)\{g(X,Z)\phi Y - g(Y,Z)\phi X + g(\phi X,Z)Y - g(\phi Y,Z)X\} + 2\alpha\beta\{g(Y,Z)X-g(X,Z)Y-g(\phi Y,Z)\phi X+g(\phi X,Z)\phi Y\}.$ (3.3)

ii). $R(\phi X,\phi Y)Z = R(X,Y)Z + (Z\alpha)\{\eta(Y)\phi X - \eta(X)\phi Y\} + (D\alpha)\{g(\phi Y,Z)\eta(X)-g(\phi X,Z)\eta(Y)\}+(Z\beta)\{\eta(Y)X-\eta(X)Y\} +(D\beta)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}+$

$$(\alpha^2-\beta^2)\{g(X,Z)Y-g(Y,Z)X-g(\phi X,Z)\phi Y+g(\phi Y,Z)\phi X\}$$
$$+2\alpha\beta\{g(Y,Z)\phi X-g(X,Z)\phi Y+g(\phi Y,Z)X-g(\phi X,Z)Y\}, \qquad (3.4)$$

where D is the gradient operator.

Proof. From (2.5) we have

$$\nabla_Y(\phi Z)=\alpha\{g(Y,Z)\xi-\eta(Z)Y\}+\beta\{g(\phi Y,Z)\xi-\eta(Z)\phi Y\}+\phi(\nabla_Y Z). \quad (3.5)$$

Differentiating covariantly (3.5) along the vector field X and using (2.6) we get

$$\begin{aligned}\nabla_X\nabla_Y(\phi Z) &= (X\alpha)\{g(Y,Z)\xi-\eta(Z)Y\}+\alpha(\nabla_X g(Y,Z))\xi\\ &+ g(Y,Z)(-\alpha\phi X-\beta\phi^2 X)-(\nabla_X\eta(Z))Y-\eta(Z)\nabla_X Y\}\\ &+ (X\beta)\{g(\phi Y,Z)\xi-\eta(Z)\phi Y\}+\beta\{g(\nabla_X\phi Y,Z)\xi\\ &+ g(\phi Y,\nabla_X Z)\xi+g(\phi Y,Z)(-\alpha\phi X-\beta\phi^2 X)\\ &- (\nabla_X\eta(Z))\phi Y-\eta(Z)\nabla_X(\phi Y)\}+\alpha\{g(X,\nabla_Y Z)\xi\\ &-\eta(\nabla_Y Z)X\}+\beta\{g(\phi X,\nabla_Y Z)\xi-\eta(\nabla_Y Z)\phi X\}+\phi(\nabla_X\nabla_Y Z). \quad (3.6)\end{aligned}$$

Interchanging X and Y in the above equation, we obtain

$$\begin{aligned}\nabla_Y\nabla_X(\phi Z) &= (Y\alpha)\{g(X,Z)\xi-\eta(Z)X\}+\alpha\{(\nabla_Y g(X,Z))\xi\\ &+ g(X,Z)(-\alpha\phi Y-\beta\phi^2 Y)-(\nabla_Y\eta(Z))X-\eta(Z)\nabla_Y X\}\\ &+ (Y\beta)\{g(\phi X,Z)\xi-\eta(Z)\phi X\}+\beta\{g(\nabla_Y\phi X,Z)\xi\\ &+ g(\phi X,\nabla_Y Z)\xi+g(\phi X,Z)(-\alpha\phi Y-\beta\phi^2 Y)\\ &- (\nabla_Y\eta(Z))\phi X-\eta(Z)\nabla_Y(\phi X)\}+\alpha\{g(Y,\nabla_X Z)\xi\\ &- \eta(\nabla_X Z)Y\}+\beta\{g(\phi Y,\nabla_X Z)\xi-\eta(\nabla_X Z)\phi Y\}+\phi(\nabla_Y\nabla_X Z). \quad (3.7)\end{aligned}$$

From (3.5), we have

$$\begin{aligned}\nabla_{[X,Y]}(\phi Z) &= \alpha\{g(\nabla_X Y,Z)\xi-g(\nabla_Y X,Z)\xi-\eta(Z)\nabla_X Y\\ &+ \eta(Z)\nabla_Y X\}+\beta\{g(\phi\nabla_X Y,Z)\xi-g(\phi\nabla_Y X,Z)\xi\\ &- \eta(Z)\phi\nabla_X Y+\eta(Z)\phi\nabla_Y X\}+\phi(\nabla_{[X,Y]}Z). \qquad (3.8)\end{aligned}$$

From (3.6), (3.7) and (3.8) we have

$$\begin{aligned}R(X,Y)\phi Z &= \phi(R(X,Y)Z)+(X\alpha)\{g(Y,Z)\xi-\eta(Z)Y\}-\\ &\quad (Y\alpha)\{g(X,Z)\xi-\eta(Z)X\}+(X\beta)\{g(\phi Y,Z)\xi-\eta(Z)\phi Y\}\\ &= (Y\beta)\{g(\phi X,Z)\xi-\eta(Z)\phi X\}+(\alpha^2-\beta^2)\times\\ &\quad \{g(X,Z)\phi Y-g(Y,Z)\phi X+g(\phi X,Z)Y-g(\phi Y,Z)X\}\\ &+ 2\alpha\beta\,\{g(Y,Z)X-g(X,Z)Y-g(\phi Y,Z)\phi X+g(\phi X,Z)\phi Y\}.\end{aligned}$$

Now, from (3.3) we have

$$\begin{aligned}
g(R(X,Y)\phi Z,\phi W) = {} & g(\phi(R(X,Y)Z),\phi W) - (X\alpha)\eta(Z)g(Y,\phi W) \\
& + (Y\alpha)\eta(Z)g(X,\phi W) - (X\beta)\eta(Z)g(\phi Y,\phi W) \\
& + (Y\beta)\eta(Z)g(\phi X,\phi W) \\
& + (\alpha^2-\beta^2)\{g(X,Z)g(\phi Y,\phi W) - g(Y,Z)g(\phi X,\phi W) \\
& + g(\phi X,Z)g(Y,\phi W) - g(\phi Y,Z)g(X,\phi W)\} \\
& + 2\alpha\beta\,\{g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W) \\
& - g(\phi Y,Z)g(\phi X,\phi W) + g(\phi X,Z)g(\phi Y,\phi W)\}. \qquad (3.9)
\end{aligned}$$

Using (2.12), in the above equation, we get

$$\begin{aligned}
g(R(\phi Z,\phi W)X,Y) &= g(R(Z,W)X,Y) \\
&+ (X\alpha)\{g(Y,\phi Z)\eta(W) - g(Y,\phi W)\eta(Z)\} \\
&+ (Y\alpha)\{g(X,\phi W)\eta(Z) - g(X,\phi Z)\eta(W)\} \\
&+ (X\beta)\{\eta(W)g(Z,Y) - \eta(Z)g(W,Y)\} \\
&+ (Y\beta)\{g(W,X)\eta(Z) - g(Z,X)\eta(W)\} \\
&+ (\alpha^2-\beta^2)\{g(Z,X)g(W,Y) - g(W,X)g(Z,Y) \\
&- g(\phi Z,X)g(\phi W,Y) + g(\phi W,X)g(\phi Z,Y)\} \\
&+ 2\alpha\beta\,\{g(W,X)g(\phi Z,Y) - g(Z,X)g(\phi W,Y) \\
&+ g(\phi W,X)g(Z,Y) - g(\phi Z,X)g(W,Y)\}.
\end{aligned}$$

From the above equation we get (3.4). □

Theorem 3.1. *In a 3-dimensional trans-Sasakian manifold M, ∗-Ricci tensor is given by*

$$\begin{aligned}
S^*(X,Y) &= S(X,Y) + \eta(Y)\{(\phi X)\alpha + X\beta\} + (\xi\beta)\,g(X,Y) \\
&- (\alpha^2-\beta^2)\{g(X,Y) + \eta(X)\eta(Y)\}. \qquad (3.10)
\end{aligned}$$

Proof. Contracting the equation (3.4) we get

$$\begin{aligned}
\sum_{i=1}^{3} R(\phi X,\phi e_i,e_i,W) &= S(W,X) + \eta(X)\{(\phi W)\alpha + W\beta\} \\
&+ (\xi\beta)g(X,W) + (\xi\alpha + 2\alpha\beta)g(\phi X,W) \\
&- (\alpha^2-\beta^2)\{g(W,X) + \eta(W)\eta(X)\}, \quad (3.11)
\end{aligned}$$

where $\{e_i\}, i = 1,2,3$ is an orthonormal basis of tangent space with $e_3 = \xi$, at any point of the manifold. From (2.8) and (3.2) we obtain

$$\begin{aligned}
S^*(W,X) &= S(W,X) + \eta(X)\{(\phi W)\alpha + W\beta\} + (\xi\beta)g(W,X) \\
&- (\alpha^2-\beta^2)\{g(W,X) + \eta(W)\eta(X)\},
\end{aligned}$$

which proves the theorem. □

Definition 3.1. An almost contact metric manifold M is said to be *weakly ϕ-Einstein* [3] *if* $S^{\phi}(X,Y) = \gamma g^{\phi}(X,Y)$, for all vector fields X, Y on M, for some function γ on M. Here $S^{\phi}(X,Y)$ denotes the symmetric part of $S^{*}(X,Y)$, that is,

$$S^{\phi}(X,Y) = (1/2)\{S^{*}(X,Y) + S^{*}(Y,X)\}, \tag{3.12}$$

and $g^{\phi}(X,Y) = g(\phi X, \phi Y)$, for all vector fields X, Y on M.

Theorem 3.2. *Let M be a three dimensional trans-Sasakian manifold of type (α,β). Then M is $*$-η-Einstein and weakly ϕ-Einstein, provided $grad\,\beta = \phi(grad\,\alpha)$.*

Proof. Suppose grad $\beta = \phi(\text{grad}\,\alpha)$. Then $(\phi X)\alpha + X\beta = 0$, for all vector fields X on M. Consequently $\xi\beta = 0$. Putting (2.10) in (3.10) and using $(\phi X)\alpha + X\beta = 0$ and $\xi\beta = 0$, we obtain

$$S^{*}(X,Y) = \gamma g(X,Y) - \gamma\eta(X)\eta(Y), \tag{3.13}$$

where $\gamma = \frac{r}{2} - 2(\alpha^2 - \beta^2)$ is a smooth function, showing that M is $*$-η-Einstein. Next, from (2.2) we have $S^{*}(X,Y) = \gamma g(\phi X, \phi Y)$. Also $S^{\phi} = S^{*}$. Hence M is weakly ϕ-Einstein. □

Theorem 3.3. *Let $M(\phi, \xi, \eta, g)$ be a three dimensional trans-Sasakian manifold of type (α,β). Then M is weakly ϕ-Einstein if and only if $grad\,\beta - \phi(grad\alpha) = (\xi\beta)\xi$.*

Proof. Suppose M is weakly ϕ-Einstein. From (3.10) and (3.12), we have

$$S(X,Y) + (1/2)\eta(Y)\{(\phi X)\alpha + X\beta\} + (1/2)\eta(X)\{(\phi Y)\alpha + Y\beta\} + (\xi\beta)\, g(X,Y) - (\alpha^2 - \beta^2)\{g(X,Y) + \eta(X)\eta(Y)\} = \gamma g(\phi X, \phi Y), \tag{3.14}$$

for some function γ on M. Substituting $Y = \xi$ and using (2.9), we get $(\phi X)\alpha + X\beta - (\xi\beta)\eta(X) = 0$. This implies, grad $\beta - \phi(\text{grad}\alpha) = (\xi\beta)\xi$. Conversely from (3.10) and (2.10) we get

$$\begin{aligned} S^{*}(X,Y) &= \left\{\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\right\} g(X,Y) \\ &- \left\{\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\right\} \eta(X)\eta(Y). \end{aligned}$$

Also $S^{*} = S^{\phi}$. From above we get $S^{\phi}(X,Y) = \delta g(\phi X, \phi Y)$ for all vector fields X, Y on M, where $\delta = \frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)$ is a smooth function on M. Hence M is ϕ-Einstein. □

From [24], it is known that on a trans-Sasakian 3-manifold of type (α,β), following three conditions are equivalent :

i) The Reeb vector field is minimal or harmonic.

ii) The following equation holds: $\phi(\text{grad}\alpha) - \text{grad}\,\beta + (\xi\beta)\xi = 0$.

iii) The Reeb vector field is an eigen vector field of the Ricci operator.

Hence from Theorem 3.3, we get the following corollary.

Corollary 3.1. *If $M(\phi, \xi, \eta, g)$ is a three dimensional trans-Sasakian manifold of type (α, β), then the following conditions are equivalent :*

i) M is weakly ϕ-Einstein.

ii) The Reeb vector field is minimal or harmonic.

iii) The following equation holds : $\phi(grad\alpha) - grad\,\beta + (\xi\beta)\xi = 0$.

iv) The Reeb vector field is an eigen vector field of the Ricci operator.

4. Three dimensional trans-Sasakian manifolds admitting ∗-Ricci solitons

A Riemannian metric g, defined on a smooth manifold M is called *Ricci soliton* if there exists a vector field V and a constant λ such that

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \tag{4.1}$$

where $\pounds$ is Lie derivative and X, Y are arbitrary vector fields on the manifold.

The notion of ∗-Ricci soliton was introduced by George and Konstantina[8], in 2014, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor in (4.1) with ∗-Ricci tensor.

Definition 4.1. A Riemannian metric g, defined on a smooth manifold M, is called ∗*-Ricci soliton* if there exists a vector field V and a constant λ such that

$$(\pounds_V g)(X,Y) + 2S^*(X,Y) + 2\lambda g(X,Y) = 0 \tag{4.2}$$

for all vector fields X, Y on M.

If λ in the equation (4.2) is a smooth function, then we say that it is an *almost ∗-Ricci soliton*. Moreover, if $V = Df$, for some smooth function on M, then we say that it is a *gradient almost ∗-Ricci soliton*.

Lemma 4.1. *Let $M(\phi, \xi, \eta, g)$ be a three dimensional trans-Sasakian manifold. If g is a ∗-Ricci solition then,*

$$(\phi X)\alpha + X\beta = (\xi\beta)\eta(X). \tag{4.3}$$

Proof. Interchanging X and Y in (4.2), we have

$$(\pounds_V g)(Y,X) + 2S^*(Y,X) + 2\lambda g(Y,X) = 0. \tag{4.4}$$

Subtracting (4.4) from (4.2) and using (3.10), we obtain

$$\eta(Y)\{(\phi X)\alpha + X\beta\} - \eta(X)\{(\phi Y)\alpha + Y\beta\} = 0.$$

Putting $Y = \xi$ in the above equation, we get the result. □

Definition 4.2. In an almost contact Riemannian manifold, if the Ricci tensor S satisfies $S = ag + b\eta \otimes \eta$, for some smooth functions a and b, then it is called *η-Einstein manifold.*

Theorem 4.1. *If the metric g of a three dimensional trans-Sasakian manifold $M(\phi, \xi, \eta, g)$ is a $*$-Ricci soliton with potential vector field V, then (i) M is an η-Einstein manifold, (ii) V is Jacobi along geodesic of ξ.*

Proof. Putting (4.3) in (2.10), we get

$$\begin{aligned} S(X,Y) &= \left\{(r/2) + \xi\beta - (\alpha^2 - \beta^2)\right\} g(X,Y) \\ &- \left\{(r/2) + 3\xi\beta - 3(\alpha^2 - \beta^2)\right\} \eta(X)\eta(Y). \end{aligned} \tag{4.5}$$

This implies that M is η-Einstein manifold. From (3.10), (4.5) and (4.3), the $*$-Ricci soliton equation (4.2) can be expressed as

$$(\pounds_V g)(X,Y) = -2f\{g(X,Y) - \eta(X)\eta(Y)\} - 2\lambda g(X,Y) \tag{4.6}$$

for all vectors field X,Y on M, where $f = (r/2) + 2\xi\beta - 2(\alpha^2 - \beta^2)$ is a smooth function. Differentiating (4.6) covariantly along an arbitrary vector field Z on M and then using (2.7) we obtain

$$\begin{aligned} (\nabla_Z \pounds_V g)(X,Y) = & -2(Zf)\{g(X,Y) - \eta(X)\eta(Y)\} + \\ & 2f\big(\alpha\{g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X)\} + \\ & \beta\{g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}\big). \end{aligned} \tag{4.7}$$

From Yano [25], we know the following well known commutation formula :

$$\begin{aligned} (\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) &= -g((\pounds_V \nabla)(X,Y), Z) \\ &- g((\pounds_V \nabla)(X,Z), Y). \end{aligned}$$

The above equation gives

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y), Z) + g((\pounds_V \nabla)(X,Z), Y). \tag{4.8}$$

As $\pounds_V \nabla$ is a symmetric, from (4.8) it follows that

$$\begin{aligned} g((\pounds_V \nabla)(X,Y), Z) &= (1/2)(\nabla_X \pounds_V g)(Y,Z) + (1/2)(\nabla_Y \pounds_V g)(X,Z) \\ &- (1/2)(\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{4.9}$$

Using (4.7) in (4.9)

$$\begin{aligned} g((\pounds_V \nabla)(X,Y), Z) &= (Zf)\{g(X,Y) - \eta(X)\eta(Y)\} \\ &- (Xf)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &- (Yf)\{g(Z,X) - \eta(Z)\eta(X)\} \\ &+ 2f\big(\alpha\{g(X,\phi Z)\eta(Y) + g(Y,\phi Z)\eta(X)\} \\ &+ \beta\{g(X,Y) - \eta(X)\eta(Y)\}\eta(Z)\big). \end{aligned}$$

Substituting $X = Y = \xi$ in the above equation, we have

$$(\pounds_V\nabla)(\xi,\xi)=0. \tag{4.10}$$

Now putting $X=Y=\xi$ in the identity $(\pounds_V\nabla)(X,Y)=R(V,X)Y+\nabla_X\nabla_Y V-\nabla_{\nabla_X Y}V$ and using (4.10), we obtain $R(V,\xi)\xi+\nabla_\xi\nabla_\xi V=0$. This proves (ii) and the theorem is proved. □

Theorem 4.2. *Let $M(\phi,\xi,\eta,g)$ be a three dimensional compact trans-Sasakian manifold. If g is a $*$-Ricci solition then, it is a α-Sasakian manifold.*

Proof. By previous Lemma, we have

$$X\beta+g(D\alpha,\phi X)-(\xi\beta)\eta(X)=0. \tag{4.11}$$

Differentiating (4.11) covariantly along Y, we get

$$\begin{aligned}Y(X\beta)-(\nabla_Y X)\beta \quad &+\quad g(\nabla_Y D\alpha,\phi X)+g(D\alpha,(\nabla_Y\phi)X)\\ &-\quad (\xi\beta)(\nabla_Y\eta)X-\eta(X)(Y(\xi\beta))=0.\end{aligned}$$

By antisymmetrization with respect to X and Y in the above equation, we have

$$\begin{aligned}&g(\nabla_Y D\alpha,\phi X)-g(\nabla_X D\alpha,\phi Y)+g(D\alpha,(\nabla_Y\phi)X)-g(D\alpha,(\nabla_X\phi)Y)\\ &-\quad (\xi\beta)\{(\nabla_Y\eta)X-(\nabla_X\eta)Y\}-\eta(X)(Y(\xi\beta))+\eta(Y)(X(\xi\beta))=0.\end{aligned} \tag{4.12}$$

Let $\{e_1,e_2,e_3\}$ be an orthonormal ϕ-basis where $\phi e_1=-e_2,\phi e_2=e_1$ and $e_3=\xi$. Taking $X=e_1$ and $Y=e_2$ in (4.12), we obtain

$$g(\nabla_{e_1}D\alpha,e_1)+g(\nabla_{e_2}D\alpha,e_2)=2\alpha(\xi\beta)+2\beta(\xi\alpha). \tag{4.13}$$

From (2.8), $g(D\alpha,\xi)=-2\alpha\beta$. Differentiating covariantly along ξ and using $\nabla_\xi\xi=0$, we get

$$g(\nabla_\xi D\alpha,\xi)=-2\alpha(\xi\beta)-2\beta(\xi\alpha). \tag{4.14}$$

From (4.13) and (4.14), we have $\Delta\alpha=0$, where Δ is the Laplacian operator. Since M is compact, α is constant. If $\alpha\neq 0$, (2.8) implies $\beta=0$. This prove that M is α-Sasakian. □

5. Example

We Consider three dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$, where (x,y,z) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields $e_1=z^2\frac{\partial}{\partial x}$, $e_2=z^2\frac{\partial}{\partial y}$, $e_3=\frac{\partial}{\partial z}$ are linearly independent at each point of M. Define the almost contact structure (ϕ,ξ,η) on M by

$$\phi(e_1)=e_2,\quad \phi(e_2)=-e_1,\quad \phi(e_3)=0,\xi=e_3,\quad \eta=dz.$$

Let g be the Riemannian metric such that $\{e_1,e_2,e_3\}$ is a orthonormal frame. Then (ϕ,ξ,η,g) defines an almost contact metric structure on M.

The Riemannian connection ∇ is given by the Koszul's formula:

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y).$$

By Koszul's formula

$$\nabla_{e_1} e_1 = \tfrac{2}{z} e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\tfrac{2}{z} e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = \tfrac{2}{z} e_3, \quad \nabla_{e_2} e_3 = -\tfrac{2}{z} e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From above we see that $(\nabla_X \phi)Y = -\frac{2}{z}\{g(\phi X, Y)\xi - \eta(Y)\phi X\}$, for all $X, Y \in \chi(M)$. Hence M is a trans-Sasakian manifold of type $(0, -\frac{2}{z})$. Here $\alpha = 0, \beta = -\frac{2}{z}$.

The components of curvature tensor $R(X,Y)Z$ are

$$R(e_1,e_2)e_1 = \tfrac{4}{z^2} e_2, \quad R(e_1,e_2)e_2 = -\tfrac{4}{z^2} e_1, \quad R(e_1,e_2)e_3 = 0,$$
$$R(e_1,e_3)e_1 = \tfrac{6}{z^2} e_3, \quad R(e_1,e_3)e_2 = 0, \quad R(e_1,e_3)e_3 = -\tfrac{6}{z^2} e_1,$$
$$R(e_2,e_3)e_1 = 0, \quad R(e_2,e_3)e_2 = \tfrac{6}{z^2} e_3, \quad R(e_2,e_3)e_3 = -\tfrac{6}{z^2} e_2.$$

From the above expressions of curvature tensor we obtain

$S(e_1,e_1) = -\frac{10}{z^2}$, $S(e_2,e_2) = -\frac{10}{z^2}$, $S(e_3,e_3) = -\frac{12}{z^2}$, and $S(e_1,e_2) = S(e_2,e_3) = S(e_3,e_1) = 0$.

Then using linearity of S we have $S(X,Y) = -\frac{10}{z^2} g(X,Y) - \frac{2}{z^2}\eta(X)\eta(Y)$ for all $X, Y \in \chi(M)$. The Ricci curvature is given by

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = -(32/z^2).$$

From the expressions of curvature tensor

$$S^*(e_1,e_1) = g(R(e_1,e_1)\phi e_1, \phi e_1) + g(R(e_1,e_2)\phi e_2, \phi e_1) = -(4/z^2).$$

Similarly, we have

$$S^*(e_1,e_1) = -(4/z^2), \quad S^*(e_2,e_2) = -(4/z^2), \quad S^*(e_3,e_3) = 0$$

and $S^*(e_i,e_j) = 0$ for $i \neq j$ and $i,j = 1,2,3$. From above

$$S^*(X,Y) = -(4/z^2)g(X,Y) + (4/z^2)\eta(X)\eta(Y). \tag{5.1}$$

Also from (3.10), we get the same $S^*(X,Y)$. For all $X \in \chi(M)$, $(\phi X)\alpha + X\beta = (\xi\beta)\eta(X)$ holds. That is, $\operatorname{grad}\beta - \phi(\operatorname{grad}\alpha) = (\xi\beta)\xi$. Also, $S^* = S^\phi$. From (5.1), $S^\phi(X,Y) = -\frac{4}{z^2} g(\phi X, \phi Y)$ and M is weakly ϕ-Einstein. Hence the Theorem 3.3 is verified.

Let $V = -\frac{4}{z}\frac{\partial}{\partial z}$. By direct computations we obtain

$$[V, e_1] = -(8/z^2)e_1, \quad [V, e_2] = -(8/z^2)e_2, \quad [V, e_3] = -(4/z^2)e_3.$$

we see that $(\pounds_V g)(e_i,e_j) + 2S^*(e_i,e_j) + 2\lambda g(e_i,e_j) = 0$ holds for $\lambda = -\frac{4}{z^2}$, for all $i,j = 1,2,3$. Hence

$$(\pounds_V g)(X,Y) + 2S^*(X,Y) + 2\lambda g(X,Y) = 0$$

for all $X, Y \in \chi(M)$. This implies that g is an almost $*$-Ricci soliton.

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PROBLEM SECTION

In the last issue of the Math. Student Vol. **88**, Nos. 1-2, January-June (2019), we had invited solutions from the floor to the remaining one problem *12* of the MS, 87, 3-4, 2018 as well as to the eight new problems *1, 2, 3, 4, 5, 6, 7* and *8* presented therein, till October 31, 2019.

The status of the only remaining problem 12 of MS, 87, 3-4, 2018.

No solution is received from the floor to this problem and hence we are to provide in this issue the Proposer's solution to this problem. However, due to unavoidable circumstances, the Proposer's solution could not be received in time, and hence it will be published in the next issue. In the meantime, the readers can try to solve it till April 30, 2020.

The status of the new eight problems of MS, 88, 1-2, 2019.

1. We received from the floor **one correct** solution to the problem *2* and we publish it here. We also publish Proposer's solution to this problem as it is solved by a different approach.

2. We received from the floor **two correct** solutions to the problem *4* and we publish one of them here.

3. We did receive one solution to each of the problems *5, 6, 7* and *8*, but none of them was correct. *No solutions* were received from the floor to the problems *1* and *3.* There was some error in the Problem *7* so we now present here the *correct version* of this problem. The Readers can try to solve these problems till April 30, 2020.

In this issue we first present **eight new problems**. Solutions to these problems as also to the remaining problems *1, 3, 5, 6, 8* and the corrected problem *7* of MS, 88, 1-2, 2019, as also to the remaining one problem *12* of the MS, 87, 3-4, 2018 received from the floor till April 30, 2020 will be published in the MS 89, 1-2, 2020 if approved by the Editorial Board.

The following problems are proposed by Prof. B. Sury, Stat-Math Unit, ISI, Benguluru.

MS-2019, Nos. 3-4: Problem-1

Let the sequence $\{E_n\}$ be defined by $\tan(t) + \sec(t) = \sum_{n\geq 0} E_n t^n/n!$. Show that E_n's are positive integers. Find an interpretation of E_n as counting permutations of $1, \cdots, n$ of a particular type.

MS-2019, Nos. 3-4: Problem-2

Prove that a cubic integer polynomial $ax^3 + bx^2 + cx + d$ where ad is odd and bc is even must have an irrational root.

MS-2019, Nos. 3-4: Problem-3

Let u_n denote the number of essentially different ways to tile the $2n \times 2n$ chess board with dominos (1x2 pieces). For instance $u_1 = 2$. Prove

$$u_n = 2^{2n^2} \prod_{r=1}^{n} \prod_{s=1}^{n} \left(\cos^2 \frac{r\pi}{2n+1} + \cos^2 \frac{s\pi}{2n+1} \right).$$

MS-2019, Nos. 3-4: Problem 4

Find the number a_n of permutations σ of $1, \cdots, n$ such that there is NO triple $i < j < k$ with $\sigma(j) < \sigma(i) < \sigma(k)$. For instance, $a_2 = 2$ while $a_3 = 5$. Further, find all n for which a_n is odd.

MS-2019, Nos. 3-4: Problem 5

The mixed fraction $91\frac{5742}{638}$ equals $91 + 9 = 100$. It involves all the digits 1 to 9 once. Find all such mixed fraction expressions with the value 100. Further, find mixed fractions for $20, 40$ and 72 also.

MS-2019, Nos. 3-4: Problem 6 (Proposed by Prof. Ram Murty)

Prove that $\int_0^1 \frac{x-1}{log\ x}\ dx$ is a transcendental number.

MS-2019, Nos. 3-4: Problem 7 (Proposed by Prof. J. R. Patadia)

If a set $E = \{n_k : k = 1, 2, \cdots\}$ of natural numbers satisfy the condition $\frac{n_{k+1}}{n_k} \geq q > 1$ for all k, then show that $Sup_{n\in\mathbb{N}}\ R_2(E, n) < \infty$, where $R_2(E, n) =$ number of different representations of n as $n = \pm n_r \pm n_s,\ n_r, n_s \in E$

MS-2019, Nos. 3-4: Problem 8 (Proposed by Prof. R. P. Pakshirajan, formerly of Mysore University, Mysore, Karnataka, through M. M. Shikare)

> Let f be an absolutely continuous function defined on [0, 1], $f(0) = 0$ with Lebesgue derivative f'. Let $(f')^2$ be Lebesgue integrable. Then show that $lim_{n\to\infty}\ 2^n\sum_{k=1}^{2^n}\{f(\frac{k}{2^n}) - f(\frac{k-1}{2^n})\}^2$ exists and is equal to $\int_0^1 (f'(t))^2\, dt$.

The *corrected version* of the **Problem 7, from MS, 88, 1-2, 2019**.

> For a positive integer n, a prime $p \le n$, and any $0 \le k < p$, show that the power of p dividing $\sum_{r\equiv k\ mod\ p} \binom{n}{r}(-1)^r$ is at least the greatest integer $\le (n-1)/(p-1)$.

Solution from the floor: MS-2019, Nos. 1-2, Prob.-2: Prove that $\limsup_{n\to\infty} \sin n = 1$ and $\liminf_{n\to\infty} \sin n = -1, \quad n \in \mathbb{N}$.

Solution. (Submitted on 02 - 07 - 2019 by **Supriya Saha**; Ramkrishna Mission's Vivekanand Educational Research Institute, Belur Math, Howrah - 711 202, West Bengal; E-mail: sahasupriya2020@outlook.com).

Let us denote $\lfloor n\rfloor_{2\pi} = x$, where $0 \le x < 2\pi$. Let $S = \{x : x = \lfloor n\rfloor_{2\pi}, n \in \mathbb{N}\}$. If we can find a sequence $\{a_n\}$ in S converging to $\frac{\pi}{2}$, then the sequence $\{sin(a_n)\}$ will converge to 1. I will first prove the following.

Observation 1: If n_1, n_2 are natural numbers and $n_1 \ne n_2$, then $\lfloor n_1\rfloor_{2\pi} \ne \lfloor n_2\rfloor_{2\pi}$.

Let $n_1 \ne n_2$ and $\lfloor n_1\rfloor_{2\pi} = \lfloor n_2\rfloor_{2\pi} = x$, $0 \le x < 2\pi$. Then $n_1 = 2k_1\pi + x$ and $n_2 = 2k_2\pi + x$ for some distinct natural numbers k_1, k_2. But then, $\frac{n_1-n_2}{2k_1-2k_2} = \pi$, implying π a rational number. So, $\lfloor n_1\rfloor_{2\pi} \ne \lfloor n_2\rfloor_{2\pi}$.

Observation 2: If $m, n, b \in \mathbb{N}$ and $0 \le b < m$ then $\lfloor n\rfloor_{2\pi} \ne 2\pi\frac{b}{m}$.
Let us assume, for some $m, n \in \mathbb{N}$, we have $\lfloor n\rfloor_{2\pi} = \frac{2\pi b}{m}$. Then $n = 2k\pi + \frac{2\pi b}{m}$, $k \in \mathbb{N}$. So we have, $nm = 2km\pi + 2\pi b$, $\frac{nm}{2km+2b} = \pi$. Again, this implies π is a rational number. So $\lfloor n\rfloor_{2\pi} \ne 2\pi\frac{b}{m}$.

Next, for an $m \in \mathbb{N}$ divide the interval $[0, 2\pi]$ in m equal partitions. $[0, 2\pi] = [0, \frac{2\pi}{m}]\cup[\frac{2\pi}{m}, \frac{4\pi}{m}]\cup...\cup[\frac{2\pi.(m-1)}{m}, 2\pi]$. Let us name the subintervals as $[0, \frac{2\pi}{m}] = I_1$, $[\frac{2\pi}{m}, \frac{4\pi}{m}] = I_2$,..., $[\frac{2\pi.(m-1)}{m}, 2\pi] = I_m$. Each of these subintervals is of length $\frac{2\pi}{m}$.

Claim 1: For each $m \in \mathbb{N}$, we can find $n \in \mathbb{N}$ such that $0 < \lfloor n\rfloor_{2\pi} < \frac{2\pi}{m}$.

Consider the first (m+1) numbers of the set S. Each of them must belong to one of the m subintervals. By pigeon-hole-principle, one of the sub-intervals must contain more than one element from the first $(m+1)$ elements of S. Let us say, $\lfloor n_1 \rfloor_{2\pi} = x_1, \lfloor n_2 \rfloor_{2\pi} = x_2 \in I_j$ for some $j < m$. Without lose of generality, let $n_1 > n_2$. If $x_1 > x_2$, then $n_3 = (n_1 - n_2)$ is our required n. Then we have $\lfloor n_3 \rfloor_{2\pi} = (x_1 - x_2) < \frac{2\pi}{m}$.
If $x_2 > x_1$, $\lfloor n_3 \rfloor_{2\pi} = 2\pi - (x_2 - x_1)$, $|(x_2 - x_1)| < \frac{2\pi}{m}$. Let $|x_2 - x_1| = x_3$. $Z = \{z \in \mathbb{N} : zx_3 \in I_m\}$. The set Z is not empty. Also, the set has a maximum element. Let the maximum element be p_m. Then, $\frac{2\pi(m-1)}{m} < p_m x_3 < 2\pi$ and $(p_m + 1)x_3 > 2\pi$. Now consider the natural number $p_m n_3$. $\lfloor p_m n_3 \rfloor_{2\pi}$=$\lfloor p_m(2\pi - x_3)\rfloor_{2\pi}$=$\lfloor 2\pi(p_m - 1) + 2\pi - p_m x_3 \rfloor_{2\pi}$= $(2\pi - p_m x_3)$. Now, $0 < (2\pi - p_m x_3) < \frac{2\pi}{m}$. So, $\lfloor p_m n_3 \rfloor_{2\pi} \in I_1$.

The above result holds for any $m \in \mathbb{N}$.

claim 2: For a each m and each I_k where $k < m$, we can find at least one natural number n, such that, $\lfloor n \rfloor_{2\pi} \in I_k$.

Choose $m \in \mathbb{N}$. Now pick $n \in \mathbb{N}$ such that $\lfloor n \rfloor_{2\pi} = x < \frac{2\pi}{m}$. $\lfloor n \rfloor_{2\pi} \in I_1$. Then $n = 2k\pi + x$ for some $k \in \mathbb{N}$. Consider the set $Z_2 = \{z \in \mathbb{N} : zx < \frac{2\pi}{m}\}$. The set Z_2 is non-empty, it has a maximum element, say p_2. That means $p_2 x < \frac{2\pi}{m}$ and $(p_2 + 1)x > \frac{2\pi}{m}$. Also $1, 2, 3..., p_2 \in Z_2$. Now, $(p_2 + 1)x = p_2 x + x < \frac{2\pi}{m} + \frac{2\pi}{m} = \frac{4\pi}{m}$. This implies $\lfloor (p_2 + 1)n \rfloor_{2\pi} \in I_2$.
Similarly, $Z_3 = \{z \in \mathbb{N} : zx < \frac{4\pi}{m}\}$. It is non-empty. Then let us say it has a maximum element say p_3. $\frac{4\pi}{m} < (p_3 + 1)x = p_3 x + x < \frac{4\pi}{m} + \frac{2\pi}{m} = \frac{6\pi}{m}$.
So by continuing this way, we establish that for any $m \in \mathbb{N}$ each subinterval of length $\frac{2\pi}{m}$ contains some element from the set $\{\lfloor n \rfloor_{2\pi}\}$.

Claim 3: The set $\{\lfloor n \rfloor_{2\pi}\}$ is dense in $[0, 2\pi]$.
Every open set U in $[0, 2\pi]$ contains an interval of the form (a, b). We can choose a point $\frac{2\pi p}{q}$ with q sufficiently large, so that $(\frac{2\pi p}{q}, \frac{2\pi(p+1)}{q}) \subset (a, b)$. Then $U \cap \{\lfloor n \rfloor_{2\pi}\} \neq \phi$. So $\{\lfloor n \rfloor_{2\pi}\}$ is dense in $[0, 2\pi]$.
So, we can construct a sequence $\{a_n\}$ converging to $\frac{\pi}{2}$. So $\limsup_{n\to\infty} sin(n) = 1$. Similarly, we can construct a sequence $\{b_n\}$ converging to $\frac{3\pi}{2}$. Then we will have $\liminf_{n\to\infty} sin(n) = -1$. □

Prof. Ram Murty who proposed the problem gave another solution to the problem. The solution provided by Prof. Murty follows.

Proposer's Solution. Clearly, the problem reduces to the study of special rational approximations to $\pi/2$ and $3\pi/2$. It is convenient to first prove a general equidistribution result. Let θ be an irrational number. Then, the sequence of fractional parts $\{(4k+1)\theta\}$ is uniformly distributed modulo 1. This is evident from Weyl's criterion. Recall that this criterion states that a sequence of numbers x_k is uniformly distributed modulo 1 if and only if for every natural number m,

$$\sum_{k=1}^{N} e^{2\pi i m x_k} = o(N),$$

as $N \to \infty$. In particular, this can be applied to $\theta = \pi/2$ to deduce that given any ϵ with $0 < \epsilon < 1$, for infinitely many k, we have

$$0 < (4k+1)\pi/2 - [(4k+1)\pi/2] < \epsilon,$$

where the notation $[x]$ denotes the greatest integer less than or equal to x. Letting $n = [(4k+1)\pi/2]$, we see upon using the familiar addition formulas for the trigonometric functions, $\sin n > \cos \epsilon$ for infinitely many n. The lim sup result is now immediate. The proof of the lim inf assertion is similar.

Solution from the floor: MS-2019, Nos. 1-2, Prob.-4: Prove that

$$xe^x = \sum_0^{\infty} \left(e^x - \sum_{j=0}^{n} (x^j/j!) \right).$$

Solution. (Submitted on 23 - 10 - 2019 by **Prajnanaswaroopa S.**; Amrita University, Coimbatore - 641 112, Tamilnadu, India; E-mail: sntrm4@rediffmail.com).

By straight expansion, we have the right hand side as

$$\begin{aligned}
&(e^x - 1) + (e^x - 1 - x) + (e^x - 1 - \frac{x^2}{2!}) + \ldots \\
&= (\sum_{n=1}^{\infty} \frac{x^n}{n!} + \sum_{n=2}^{\infty} \frac{x^n}{n!} + \sum_{n=3}^{\infty} \frac{x^n}{n!} + \ldots) \\
&= x(\sum_{n=1}^{1} \frac{1}{1!}) + x^2(\sum_{n=1}^{2} \frac{1}{2!}) + x^3(\sum_{n=1}^{3} \frac{1}{3!}) + \ldots \\
&= \sum_{n=1}^{\infty} x^n (\frac{n}{n!}) = \sum_{n=1}^{\infty} x^n \frac{1}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = xe^x
\end{aligned}$$

which is the left hand side. □

Correct solution was also received from the floor from:

Subhash Chand Bhoria; on 28 - 09 - 2019; Village bakali, near Ladwa, Kurukshetra - 136 132, Haryana, India; E-mail: scbhoriya@yahoo.com

The solution provided by him is more or less similar to the one given above.

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(See Rule 8)

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