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Edited by J. R. PATADIA

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KNUTH'S MOVES ON TIMED WORDS

AMRITANSHU PRASAD

Abstract. We give an exposition of Schensted's algorithm to find the length of the longest increasing subword of a word in an ordered alphabet, and Greene's generalization of Schensted's results using Knuth equivalence. We announce a generalization of these results to timed words.

1. Introduction

The theory of Young tableaux lies at the cross-roads of modern combinatorics, the theory of symmetric functions, enumerative problems in geometry, and representation theory (see [Ful97, Man98, Pra15, Pra18a]). Young tableaux are named after Alfred Young, who introduced them in his study of the representation theory of symmetric groups, which he called *substitutional analysis* [You00]. Young tableaux played an important role in the proof of the Littlewood-Richardson rule. This is a rule for computing the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$, which arise as (for details, see [Ful97, Man98]):

- the multiplicity of the irreducible polynomial representation W_λ of $GL_n(\mathbf{C})$ in a tensor product $W_\mu \otimes W_\nu$.
- the coefficient of the Schur polynomial s_λ in the expansion of a product $s_\mu s_\nu$ of Schur polynomials.
- the number of points of intersection of Schubert varieties X_μ, X_ν and $X_{\check{\lambda}}$ in general position.

Robinson [Rob38] outlined an approach to the Littlewood-Richardson rule based on Young tableaux, which was perfected in the work of Lascoux and Schützenberger [LS78] forty years later. In this expository article, our point of departure is Schensted's observation [Sch61] that Robinson's construction of the insertion tableau of a word can be used as an algorithm to determine the longest increasing subword of a word in an ordered language. Schensted used this to give a formula for the number of words with longest increasing subword of a given length. Schensted's

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results were generalized by Greene [Gre74] using relations which were introduced by Knuth [Knu70] to characterize the class of words with a given insertion tableau.

In a different context, Alur and Dill [AD94] introduced timed words as a part of their description of timed automata. Timed automata are generalizations of finite automata, and are used for the formal verification of real-time systems. The author has extended Greene's theorem to timed words [Pra18b], with the goal of providing a framework to study piecewise linear versions of bijective correspondences involving Young tableau, such as the ones studied by Berenstein and Kirillov [Kir11]. The salient features of this extension are outlined here. Detailed proofs, technical details, and applications to piecewise-linear bijections will appear in [Pra18b].

2. Schensted's Algorithm

2.1. **Words.**

Let A_n denote the set $\{1, \ldots, n\}$, which we regard as an *ordered alphabet.* A *word* in A_n is a finite sequence $w = c_1 \cdots c_k$ of elements of A_n. The set of all words in A_n is denoted A_n^*. A *subword* of $w = c_1 \cdots c_k$ is defined to be a word of the form

$$w' = c_{i_1} \cdots c_{i_m}, \text{ where } 1 \leq i_1 < \cdots < i_m \leq k.$$

The subword w' is said to be *weakly increasing* if $c_{i_1} \leq \cdots \leq c_{i_m}$.

Consider the following computational problem:

> Given a word $w \in A_n^*$, determine the maximal length of a weakly increasing subword of w.

2.2. **Tableaux.**

Schensted [Sch61] gave an elegant algorithm to solve the preceding computational problem. His algorithm makes one pass over the word. At each stage of its running, it stores a combinatorial object called a *semistandard Young tableau* (see Section 2.2.1). This tableau is modified as each successive letter of the word is read. The length of the longest increasing subword can be read off from the tableau (see Sections 2.5 and 2.6) obtained when all of w has been read.

Definition 2.2.1 (Semistandard Young Tableau)**.** A semistandard Young tableau in A_n is a finite arrangement of integers from A_n in rows and columns so that the numbers increase weakly along rows, strictly along columns, so that there is an element in the first row of each column, there is an element in the first column of each row, and there are no gaps between numbers.

Let l be the number of rows in the tableau, and for each $i = 1, \ldots, l$, let λ_i be the length of the ith row. Then $\lambda = (\lambda_1, \ldots, \lambda_l)$ is called the *shape* of the tableau.

Example 2.2.2. The arrangement

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

is a semistandard Young tableau of shape $(3, 2, 1)$ in A_5.

The notion of a semistandard Young tableau is a generalization of *Young tableau*, which was introduced by Young [You00, p. 133]. In Young's version, each element of A_n occurs exactly once in the tableau. For brevity, we shall henceforth use the term *tableau* to refer to a semistandard Young tableau.

2.3. **Row Insertion.** A word $c_1 c_2 \cdots c_k$ in A_n^* is called a *row* if $c_1 \leq \cdots \leq c_k$. Each row of a tableau is a row in the sense of this definition. For each row $u = a_1 \cdots a_k \in A_n^*$, define the *row insertion* of a into u by:

$$\mathrm{RINS}(u, a) = \begin{cases} (\emptyset, a_1 \cdots a_k a) & \text{if } a_k \leq a, \\ (a_j, a_1 \cdots a_{j-1} a a_{j+1} \cdots a_k) & \text{otherwise, with} \\ & j = \min\{i \mid a < a_i\}. \end{cases}$$

Here $\emptyset$ should be thought of as an empty word of length zero.

Example 2.3.1. $\mathrm{RINS}(115, 5) = (\emptyset, 1155)$, $\mathrm{RINS}(115, 3) = (5, 113)$.

It is clear from the construction that, for any row $u \in A_n^*$ and $a \in A_n$, if $(a', u') = \mathrm{RINS}(u, a)$, then u' is again a row. For convenience set $\mathrm{RINS}(u, \emptyset) = (\emptyset, u)$.

2.4. **Tableau Insertion.** Let t be a tableau with rows $u_1, u_2, \ldots, u_l$. Then $\mathrm{INSERT}(t, a)$, the insertion of a into t, is defined as follows: first a is inserted into u_1; if $\mathrm{RINS}(u_1, a) = (a_1', u_1')$, then u_1 is replaced by u_1'. Then a_1' is inserted into u_2; if $\mathrm{RINS}(u_2, a_1') = (a_2', u_3)$, then u_2 is replaced by u_2', a_2' is inserted into u_3, and so on. This process continues, generating $a_1', a_2', \ldots, a_k'$ and $u_1', \ldots, u_k'$. The tableau $t' = \mathrm{INSERT}(t, a)$ has rows $u_1', \ldots, u_k'$, and a last row (possibly empty) consisting of a_k'. It turns out that $\mathrm{INSERT}(t, a)$ is a tableau [Knu70].

Example 2.4.1. For t as in Example 2.2.2, we have

$$\mathrm{INSERT}(t, 3) = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array},$$

since $\mathrm{RINS}(115, 3) = (5, 113)$, $\mathrm{RINS}(24, 5) = (\emptyset, 245)$.

2.5. **Insertion Tableau of a Word.**

Definition 2.5.1. The insertion tableau $P(w)$ of a word w is defined recursively as follows:

$$P(\emptyset) = \emptyset \tag{2.1}$$

$$P(c_1 \cdots c_k) = \mathrm{INSERT}(P(c_1 \cdots c_{k-1}), c_k). \tag{2.2}$$

Example 2.5.2. Take $w = 3421153$. Sequentially inserting the terms of w into the empty tableau $\emptyset$ gives the sequence of tableaux:

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array},$$

and finally, the insertion tableau $P(w) = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$.

2.6. **Schensted's Theorem.** Schensted [Sch61] proved the following:

Theorem 2.6.1. *The length of the longest increasing subword of any* $w \in A_n^*$ *is the length of the first row of* $P(w)$.

In other words, the algorithm for constructing the insertion tableau of w solves the computational problem posed in Section 2.1.

The proof of Schensted's theorem is not very difficult, and the reader is invited to attempt it. The proof is by induction on k, and uses the observation is that the last entry of the first row of $P(a_1 \cdots a_k)$ is the *least last element* of all maximal length weakly increasing subword of $a_1 \cdots a_k$.

2.7. **Greene's Theorem.** The insertion tableau $P(w)$ obtained from a word w seems to contain a lot more information than just the length of the longest weakly increasing subword. For example, what do the lengths of the remaining rows of $P(w)$ signify? The answer to this question was given by Greene [Gre74]. We say that subwords $c_{1_1} \cdots c_{i_r}$ and $c_{j_1} \cdots c_{j_s}$ of $c_1 \cdots c_k$ are *disjoint* if the subsets $\{i_1, \dots, i_r\}$ and $\{j_1, \dots, j_s\}$ are disjoint.

Definition 2.7.1 (Greene Invariants)**.** The rth Greene invariant of a word $w \in A_n^*$ is defined to be the maximum cardinality of a union of r pairwise disjoint weakly increasing subwords of w.

Example 2.7.2. For $w = 3421153$ from Example 2.5.2, the longest weakly increasing subwords have length 3 (for example, 113 and 345). The subwords 345 and 113 are disjoint, and no pair of disjoint weakly increasing subwords of w can have cardinality greater than 6. However, the entire word w is a union of three disjoint weakly increasing subwords (for example 345, 23 and 15). So the Greene invariants of w are $a_1(w) = 3$, $a_2(w) = 6$, and $a_3(w) = 7$.

Theorem 2.7.3 (Greene)**.** *For any* $w \in A_n$, *if* $P(w)$ *has shape* $\lambda = (\lambda_1, \dots, \lambda_l)$, *then for each* $r = 1, \dots, l$, $a_r(w) = \lambda_1 + \cdots + \lambda_l$.

Example 2.7.2 is consistent with Greene's theorem as the shape of $P(w)$ is $(3, 3, 1)$ and the Greene invariants are 3, $6 = 3 + 3$ and $7 = 3 + 3 + 1$, respectively.

2.8. **Knuth Equivalence.** Greene's proof of Theorem 2.7.3 is based on the notion of Knuth equivalence. Knuth [Knu70] identified a pair of elementary moves on words:

$$xzy \equiv zxy \text{ if } x \leq y < z, \tag{$K1$}$$

$$yxz \equiv yzx \text{ if } x < y \leq z. \tag{$K2$}$$

For example, in the word 4213443, the segment 213 is of the form yxz, with $x < y \leq z$. A Knuth move of type $(K2)$ replaces this segment by yzx, which is

231. Thus a Knuth move of type ($K2$) transforms 4**213**443 into 4**231**443. Knuth equivalence is the equivalence relation on A_n^* generated by Knuth moves:

Definition 2.8.1 (Knuth Equvalence)**.** Words $w, w' \in A_n^*$ are said to be Knuth equivalent if w can be transformed into w' by a series of Knuth moves ($K1$) and ($K2$). If this happens, we write $w \equiv w'$.

Example 2.8.2. The word 3421153 is Knuth equivalent to 3245113:

$$\mathbf{342}1153 \equiv_{K2} 3\mathbf{241}153 \equiv_{K2} 32\mathbf{141}53 \equiv_{K1} 3241\mathbf{153} \equiv_{K1} 324\mathbf{151}3 \equiv_{K1} 3245113.$$

At each stage, the letters to which the Knuth moves will be applied to obtain the next stage are highlighted.

2.9. **Reading Word of a Tableau.** Given a tableau, its reading word is obtained by reading its rows from left to right, starting with the bottom row, and moving up to its first row.

Example 2.9.1. The reading word of the tableau:

$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&4&5\\\hline 3\\\cline{1-1}\end{array}$$

is 3245113.

2.10. **Proof of Greene's Theorem.** The proof of Greene's theorem is based on three observations, all fairly easy to prove:

1. If w is the reading word of a tableau of shape $\lambda = (\lambda_1, \dots, \lambda_l)$, then $a_r(w) = \lambda_1 + \cdots + \lambda_r$ for $r = 1, \dots, l$.
2. Every word is Knuth equivalent to the reading word of its insertion tableau.
3. Greene invariants remain unchanged under Knuth moves.

We illustrate these points with examples (for detailed proofs, see Lascoux, Leclerc and Thibon [LLT02], or Fulton [Ful97]). For the first point, in Example 2.9.1 the first k rows of the tableau

$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&4&5\\\hline 3\\\cline{1-1}\end{array}$$

are indeed disjoint weakly increasing subwords of its reading word of maximal cardinality. For the second point, observe that the sequence of Knuth moves in Example 2.8.2 transform 3421153 to the reading word of its insertion tableau. For the third point, consider the case of the Knuth move ($K1$). A word of the from $w = uxzyv$ is transformed into the word $w' = uzxyv$. The only issue is that a weakly increasing subword g of w may contain both the letters x and z. Then it no longer remains a weakly increasing subword of w'. However, the subword, being weakly increasing, cannot contain y, so the z can be swapped for a y. This could be a problem if y is part of another weakly increasing subword g' in a collection of pairwise disjoint weakly increasing subwords. In that case, we have $g = g_1xzg_2$

and $g' = g_1'yg_2'$. We may replace them with g_1xyg_2' and $g_1'zg_2$, which would still be weakly increasing, and would have the same total length as g and g'.

2.11. **Characterization of Knuth Equivalence.** Knuth equivalence can be characterized in terms of Greene invariants (see [LS78, Theorem 2.15]).

Theorem 2.11.1. *Two words w and w' in A_n^* are Knuth equivalent if and only if $a_r(uwv) = a_r(uw'v)$ for all words u and v in A_n^*, and all $r \geq 1$.*

3. Timed Words

3.1. **From Words to Timed Words.** Words, in the sense of Section 2, play an important role in computer science, specifically in the formal verification of systems. Each letter of the alphabet is thought of as an event. A sequence of events it then nothing but a word in A_n^*. The system is modeled as an automaton having a starting state, and each time an event occurs, its state changes, depending both, on its current state, and the event that has occurred. Following the groundbreaking work of Rabin and Scott [RS59], finite state automata are widely used to model and formally verify the integrity of systems.

For many real-time systems, such as controllers of washing machines, industrial processes, and air or railway traffic control, the *time gaps* between the occurrences of the events modeled by words are as important as the events themselves.

To deal with real-time systems, Alur and Dill [AD94] developed the theory of *timed automata*. A timed automaton responds to a sequence of events that come with time stamps for their occurrence. They represented a sequence of events with time stamps by timed words. We introduce a finite variant of the notion of timed word that they used:

Definition 3.1.1 (Timed Word)**.** A timed word in A_n is a sequence of the form:

$$w = c_1^{t_1} c_2^{t_2} \cdots c_k^{t_k}, \tag{3.1}$$

where $c_1, \ldots, c_k \in A_n$, and $t_1, \ldots, t_k$ are positive real numbers, and $c_i \neq c_{i+1}$ for $i = 1, \ldots, k-1$. The length of the timed word w above is $l(w) = t_1 + \cdots + t_k$.

A sequence (3.1) where $c_i = c_{i+1}$ also represents a timed word; segments of the form $c^{t_1}c^{t_2}$ are replaced by $c^{t_1+t_2}$ until all consecutive pairs of terms have different letters. The timed word w in (3.1) may also be regarded as a piecewise constant left-continuous function $\mathbf{w} : [0, l(w)) \to A_n$, where

$$\mathbf{w}(t) = c_i \text{ if } t_1 + \cdots + t_{i-1} \leq t < t_1 + \cdots + t_i.$$

The function $\mathbf{w} : [0, l(w)) \to A_n$ is called the *function associated to the timed word* w. We say that the timed word w is a *timed row* if $c_1 < \cdots < c_k$. Timed words form a monoid under concatenation. The monoid of timed words in the alphabet $\{1, \ldots, n\}$ is denoted $A_n^\dagger$. The map:

$$a_1 \cdots a_k \mapsto a_1^1 a_2^1 \cdots a_k^1$$

defines an embedding of A_n^* in $A_n^\dagger$ as a submonoid.

Example 3.1.2. An example of a timed word in $A_6^\dagger$ of length 7.19 is:

$$w = 3^{0.82}5^{0.08}2^{0.45}6^{0.64}5^{0.94}1^{0.15}5^{0.09}1^{0.52}4^{0.29}1^{0.59}3^{0.97}4^{0.42}2^{0.61}1^{0.07}4^{0.55}$$

Using a color-map to represent the integers 1 to 6,

the timed word w can be visualized as a colored ribbon:

.

3.2. Subwords of Timed Words.

Definition 3.2.1 (Time Sample)**.** A *time sample* of a word w is a subset of $[0, l(w))$ of the form:

$$S = [a_1, b_1) \cup \cdots \cup [a_k, b_k),$$

where $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq l(w)$. The length of the time sample S is $\sum_i (b_i - a_i)$, the Lebesgue measure $\mu(S)$ of S.

Given a time sample $S \subset [0, l(w))$, and $0 \leq t \leq l(S))$, the set

$$\{\tilde{t} \mid \mu(S \cap [0, \tilde{t})) = t\}$$

is a closed interval $[a_t, b_t] \subset [0, l(S))$. This happens because the function $t' \mapsto \mu(S \cap [0, t'))$ is a piecewise-linear continuous function on $[0, l(w)]$ which takes value 0 at $t' = 0$, and $l(S)$ at $t' = 1$.

Definition 3.2.2 (Subword of a Timed Word)**.** The subword of a timed word with respect to a time sample $S \subset [0, l(w))$ is the timed word w_S of length $\mu(S)$ whose associated function is given by:

$$\mathbf{w}_S(t) = \mathbf{w}(b_t) \text{ for } 0 \leq t < \mu(S),$$

where b_t is the largest number in $[0, l(w))$ such that $\mu(S \cap [0, \tilde{t})) = t$.

3.3. Timed Tableau.

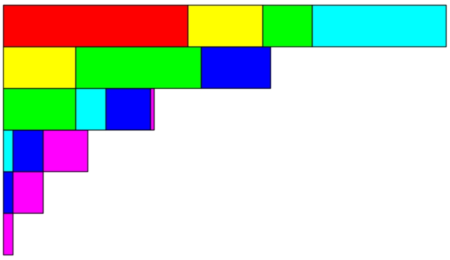
Definition 3.3.1 (Timed Tableau)**.** A timed tableau is a collection $u_1, u_2, \ldots, u_l$ of timed words such that

(1) Each u_i is a timed row (in the sense of Section 3.1).
(2) For each $i = 1, \ldots, l-1$, $l(u_i) \geq l(u_{i+1})$.
(3) For each $i = 1, \ldots, l-1$ and $0 \leq t < l(u_{i+1})$, $u_i(t) < u_{i+1}(t)$.

Example 3.3.2. A timed tableau of shape $(3.20, 1.93, 1.09, 0.61, 0.29, 0.07)$ is :

$$\begin{aligned} t = & 1^{1.33}2^{0.54}3^{0.36}4^{0.97} \\ & 2^{0.52}3^{0.91}5^{0.50} \\ & 3^{0.52}4^{0.22}5^{0.32}6^{0.03} \\ & 4^{0.07}5^{0.22}6^{0.32} \\ & 5^{0.07}6^{0.22} \\ & 6^{0.07} \end{aligned}$$

In using the color-map from Section 3.1, it can be visualized as:



The three properties of Definition 3.3.1 are easily perceived from the figure.

Definition 3.3.3 (Reading Word of a Timed Tableau)**.** The reading word of a timed tableau with rows $u_1, \dots, u_l$ is the timed word

$$u_l u_{l-1} \cdots u_1.$$

Example 3.3.4. The reading word of the timed tableau in Example 3.3.2 is $6^{0.07}\ 5^{0.07}6^{0.22}\ 4^{0.07}5^{0.22}6^{0.32}\ 3^{0.52}4^{0.22}5^{0.32}6^{0.03}\ 2^{0.52}3^{0.91}5^{0.50}\ 1^{1.33}2^{0.54}3^{0.36}4^{0.97}$.

3.4. **Timed Insertion.** Given a timed word w and $0 \leq a < b \leq l(w)$, according to Definition 3.2.2, $w_{[a,b)}$ is the timed word of length $b-a$ such that:

$$w_{[a,b)}(t) = w(a+t) \text{ for } 0 \leq t < b-a.$$

Definition 3.4.1 (Timed Row Insertion)**.** Given a timed row w, define the insertion $\mathrm{RINS}(w, c^{t_c})$ of c^{t_c} into w as follows: if $w(t) \leq c$ for all $0 \leq t < l(u)$, then

$$\mathrm{RINS}(w, c^{t_c}) = (\emptyset, wc^{t_c}).$$

Otherwise, there exists $0 \leq t < l(w)$ such that $\mathbf{w}(t) > c$. Let

$$t_0 = \min\{0 \leq t < l(w) \mid \mathbf{w}(t) > c\}.$$

Define

$$\mathrm{RINS}(w, c^{t_c}) = \begin{cases} (w_{[t_0, t_0+t_c)}, w_{[0,t_0)} c^{t_c} w_{[t_0+t_c, l(w))}) & \text{if } l(w) - t_0 > t_c, \\ (w_{[t_0, l(u))}, w_{[0,t_0)} c^{t_c}) & \text{if } l(w) - t_0 \leq t_c. \end{cases}$$

It is obvious that the above definition is compatible with the definition of RINS from Section 2.4 when u is a row in A_n^*, and $t_c = 1$. If $u = c_1^{t_1} \cdots c_l^{t_l}$ is a timed word, define $\mathrm{RINS}(w, u)$ by induction on l as follows: Having defined $(v', w') = \mathrm{RINS}(w, c_1^{t_1} \cdots c_{l-1}^{t_{l-1}})$, let $(v'', w'') = \mathrm{RINS}(w', c_l^{t_l})$. Then define

$$\mathrm{RINS}(w, u) = (v'v'', w'').$$

Example 3.4.2. $\mathrm{RINS}(1^{1.4}2^{1.6}3^{0.7}, 1^{0.7}2^{0.2}) = (2^{0.7}3^{0.2}, 1^{2.1}2^{1.1}3^{0.5})$.

Definition 3.4.3 (Timed Tableau Insertion)**.** Let w be a timed tableau with row decomposition $u_l \dots u_1$, and let v be a timed row. Then $\mathrm{INSERT}(w, v)$, the insertion of v into w is defined as follows: first v is inserted into u_1. If $\mathrm{RINS}(u_1, v) = (v_1', u_1')$, then v_1' is inserted into u_2; if $\mathrm{RINS}(u_2, v_1') = (v_2', u_2')$, then v_2' is inserted in u_3, and so on. This process continues, generating $v_1', \dots, v_l'$ and $u_1', \dots, u_l'$. $\mathrm{INSERT}(t, v)$ is defined to be $v_l' u_l' \cdots u_1'$. Note that it is quite possible that $v_l' = \emptyset$.

Example 3.4.4. Take

$$\begin{aligned} w =& 1^{1.4}2^{1.6}3^{0.7} \\ & 3^{0.8}4^{1.1}, \end{aligned}$$

a timed tableau in A_5 of shape $(3.7, 1.9)$. Then

$$\begin{aligned} \mathrm{INSERT}(w, 1^{0.7}2^{0.2}) =& 1^{1.7}2^{3}3^{0.2} \\ & 2^{0.3}3^{1.2}4^{0.4} \\ & 3^{0.3}4^{0.7} \end{aligned}$$

of shape $(4.9, 1.9, 1.0)$.

3.5. Insertion Tableau of a Timed Word.

Definition 3.5.1 (Insertion Tableau of a Timed Word)**.** The insertion tableau $P(w)$ of a timed word w is defined recursively by the rules:

1. $P(\emptyset) = \emptyset$,
2. $P(wc^t) = \mathrm{INSERT}(P(w), c^t)$.

Example 3.5.2. The tableau in Example 3.3.2 is the insertion tableau of the timed word in Example 3.1.2.

3.6. Greene Invariants for Timed Words.

Definition 3.6.1 (Greene Invariants for Timed Words)**.** The rth Greene invariant for a timed word w is defined as:

$$a_r(w)=\sup\left\{\mu(S_1)+\cdots+\mu(S_r)\,\middle|\,\begin{matrix} S_1,\ldots,S_r \text{ are pairwise disjoint time samples} \\ \text{of } w \text{ such that } w_{S_i} \text{ a timed row for each } i\end{matrix}\right\}.$$

3.7. **Greene's Theorem for Timed Words.** All the ingredients are now in place to state Greene's theorem for timed words:

Theorem 3.7.1 (Greene's Theorem for Timed Words)**.** *Let $w \in A_n^\dagger$ be a timed word. Suppose that $P(w)$ has shape $\lambda = (\lambda_1, \ldots, \lambda_l)$, then the Greene invariants of w are given by:*

$$a_r(w) = \lambda_1 + \cdots + \lambda_r \textit{ for } r = 1, \ldots, l.$$

For the word w from Example 3.1.2, the insertion tableau has shape

$$(3.20, 1.93, 1.09, 0.61, 0.29, 0.07),$$

(given in Example 3.3.2) so the Greene invariants are given by:

$$\begin{aligned} a_1(w) &= 3.20 \\ a_2(w) &= 3.20 + 1.93 = 5.13 \\ a_3(w) &= 3.20 + 1.93 + 1.09 = 6.22 \\ a_4(w) &= 3.20 + 1.93 + 1.09 + 0.61 = 6.83 \\ a_5(w) &= 3.20 + 1.93 + 1.09 + 0.69 + 0.29 = 7.12 \\ a_6(w) &= 3.20 + 1.93 + 1.09 + 0.69 + 0.29 + 0.07 = 7.19 \end{aligned}$$

3.8. **Knuth Moves on Timed Words.** As explained in Section 2.10, the proof of Greene's theorem in [Gre74] uses Knuth moves to reduce to the case of reading words of tableau. The main difficulty in generalizing his theorem to timed words is to identify the analogues of Knuth relations $(K1)$ and $(K2)$. These relations need to be simple enough so that it can we shown that if two words differ by such a relation, then they have the same Knuth invariants. At the same time, they need to be strong enough to reduce any timed word to its insertion tableau.

Consider the relations:

$$xzy \equiv zxy \text{ when } xyz \text{ is a timed row, } l(z) = l(y), \text{ and } \lim_{t \to l(y)^-} \mathbf{y}(t) < \mathbf{z}(0), \quad (\kappa_1)$$

$$yxz \equiv yzx \text{ when } xyz \text{ is a timed row, } l(x) = l(y), \text{ and } \lim_{t \to l(x)^-} \mathbf{x}(t) < \mathbf{y}(0). \quad (\kappa_2)$$

Example 3.8.1. We have:

$$w = 5^{1.10}3^{2.19}4^{0.89}5^{1.20}1^{0.32}2^{0.44} \equiv w' = 5^{1.10}3^{2.19}4^{0.62}1^{0.32}2^{0.41}4^{0.27}5^{1.20}2^{0.03},$$

because we may write

$$w = 5^{1.10}3^{2.08}yzx2^{0.03},$$
$$w' = 5^{1.10}3^{2.08}yxz2^{0.03},$$

where $x = 1^{0.32}2^{0.41}$, $y = 3^{0.11}4^{0.62}$, and $z = 4^{0.27}5^{1.20}$, so w and w' differ by a Knuth move of the form (κ_2).

We say that two timed words w and w' are Knuth equivalent (denoted $w \equiv w'$) if w can be obtained from w' by a sequence of Knuth moves of the form (κ_1) and (κ_2).

With these definitions, we have the following results, which suffice to complete the proof of Theorem 3.7.1:

1. if w is the reading word of a timed tableau of shape $\lambda = (\lambda_1, \dots, \lambda_l)$, then $a_r(w) = \lambda_1 + \dots + \lambda_r$ for $r = 1, \dots, l$.
2. for every $w \in A_n^\dagger$, $w \equiv P(w)$,
3. if $w \equiv w'$, then $a_r(w) = a_r(w')$ for all r.

3.9. **Characterization of Knuth Equivalence.** Finally, it turns out that Knuth equivalence for timed words is characterized by Greene invariants, just as in the classical setting (Section 2.11):

Theorem 3.9.1. *Given timed words $w, w' \in A_n^\dagger$, $w \equiv w'$ if and only if, for all $u, v \in A_n^\dagger$,*

$$a_r(uwv) = a_r(uw'v) \text{ for all } r > 0.$$

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CLASSIFICATION OF MANIFOLDS, DIFFERENT SHADES

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Abstract. Classifications of 1 and 2 dimensional manifolds are classical. Classifications of $3, 4$ and higher (> 4) dimensional manifolds present techniques of different shades. This expository survey article gives an overview of these developments.

1. Introduction

1.1. **Problem.** (*Classification problem of manifolds*): Classification problem of manifolds involve:

(1) producing a list of all equivalence classes of manifolds (under some suitable equivalence like, isometry, topological, PL or smooth equivalence etc.), and

(2) developing usable means to determine where a given manifold fits in the list (recognizing a manifold).

1.2. *Remark* (see eg. [56]). The problem of recognizing a manifold is not easy, the manifolds may be given in any form. For example

(i) $M = \{(x,y) \in \mathbb{C} \mid x^3 + y^3 = 1\}$ represents a torus minus three points, where as

(ii) $N = \{[x,y,z] \in \mathbb{C}P^2 \mid x^3 + y^3 = z^3\}$ represents a torus.

One should be able to deal with all sorts of description of manifolds.

1.3. **Definition.** (i) An *n-manifold* M is a Hausdorff topological space with a countable base such that for every point $x \in M$ there is an open neighbourhood U of x in M and a homeomorphism $h : U \to V$, V is an open subset of $\mathbb{R}^n$ containing

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(2). This expository article is an expanded version of the 28^{th} V. Ramaswami Memorial Award lecture delivered at the 83^{rd} Annual Conference of the Indian Mathematical Society-An international Meet held at Sri Venkateswara University, Tirupati - 517 502, Andhra Pradesh, India during December 12 - 15, 2017. For the benefit of the interested reader, material from many sources have been adapted and used, many technical terms, definitions and statements of important theorems are included, along with the reference of their sources.

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$\bar{0}$, such that $h(x) = \bar{0}$. That is, the n-manifold locally looks like the Euclidean n-space. The pair (U, h) is called a chart or a coordinate system at x.

(ii) An *n-manifold with boundary*, $(M, \partial M)$, is a Hausdorff topological space M with a countable base such that for every point $x \in M$ there is an open neighbourhood U of x in M and a homeomorphism $h : U \to V$, V is an open subset of $\mathbb{H}^n$, the closed half n-space. That is the n-manifold locally looks like the euclidean half n-space. The pair (U, h) is called a chart or a coordinate system at x. For points $x \in M \setminus \partial M$ a chart can be chosen as in part (i) with $h(x)$ an interior point of $\mathbb{H}^n$ and for the points $x \in \partial M$, V looks like a semi-open-n-ball with center $\bar{0}$, such that $h(x) = \bar{0}$.

Boundary ∂M is well defined by invaiance of domain.

2. Classification of curves (1-manifolds)

2.1. **Theorem.** *Let M be a connected 1-manifold. Then M is diffeomorphic either to* $[0, 1], [0, 1), (0, 1)$, *or* S^1.

Here we only give an intuitive idea of how to proceed, the formal proof can be found in any of the following sources:

(a) Classification of 1-manifolds -
http://www.math.northwestern.edu e$_m$*urphynotes*4.*pdf*,

(b) THE CLASSIFICATION OF 1 DIMENSIONAL MANIFOLDS -
https://fenix.tecnico.ulisboa.ptdownloadFile3779577337825classif1manifs.pdf,

(c) Classification of 1-Manifolds -
http://www.math.boun.edu.trinstructorswdgillam1manifolds.pdf,

(d) Classification of 1-manifolds -
http://math.mit.educlasses18.9662014SP965class.pdf,

(e) the book of Guillemin and Pollack [24]. Or

(f) the book of Milnor [65].

Using compactness and connectedness arguments one can see that all the above manifolds are diffeomorphically distinct, two of them are with boundary and two of them are without boundary. Given a 1-manifold M, take a point $x \in M$, let (U, h) be a chart at x. Then V will either be of the type $(-\epsilon, \epsilon)$, or of the type $[0, \epsilon)$, or of the type $(-\epsilon, 0]$. Move to the right of x in M as far as possible. Only three things can happen: either you have to stop at a point and you can not move any further (closed end point), or you keep on moving forever (open end point), or you end up coming back to x where you started from the left side (like a circle).

Do the same by moving to the left of x in M as far as possible. This leaves us with only the above four possibilities for M upto diffeomorphism.

3. Classification of surfaces (2-manifolds)-19 the Century

It has been proved in the 19^{th} century that

(i) Any compact orientable surface is homeomorphic to a sphere or a connected sum of tori (see figures - “Connected sum” and “Compact orientable surfaces”).

(ii) Any compact non-orientable surface is homeomorphic to the connected sum of real projective planes and a compact orientable surface (possibly empty), see the figure - "Compact nonorientable surface - Klein's bottle".

"Euler characteristic" (a surface can be written upto topological equivalence as a union of triangles joined along their common edges and the Euler characteristic is given by $(V - E + F)$, where V, E, F are respectively the number of vertices, edges and faces; this definition has been generalized further in the literature) together with "orientability" (roughly a two sided surface is orientable and a one sided surface is not orientable) provide complete set of invariants (of topological equivalence).

Main contributors of these results have been, among others, Möbius 1861, Jordan 1866, Dyck 1888, Dehn and Heegaard 1907, Alexander 1915, Brahana 1921.

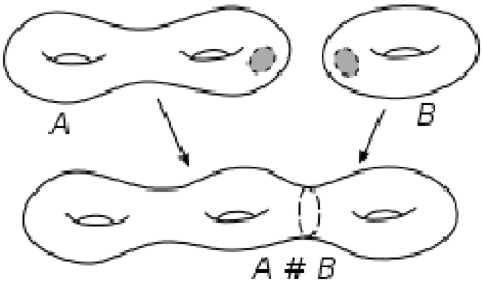


Figure 1: Connected sum.

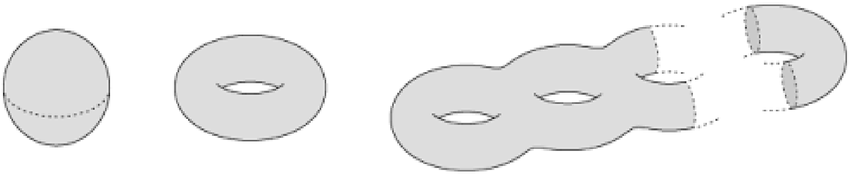


Figure 2: Compact orientable surfaces.

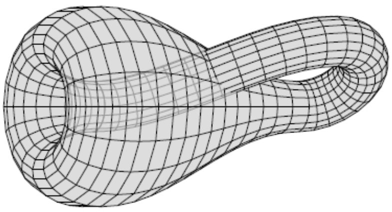


Figure 3: Compact nonorientable surface-Klein's bottle.

Refer to "A guide to classification theorem for compact surfaces" - by Jean Gallier, Dianna Xu, 2013.

3.1. **The technique.** The technique of cutting and pasting has been used, which involve the following steps, see [2]:
(i) one starts with an arbitrary compact orientable surface S which in general has Euler characteristic $\chi(S) \leq 2$. One uses the following characterisation of the 2-sphere, S^2 ($\cong$ will mean topological equivalence):

$$S \cong S^2 \Leftrightarrow \chi(S) = 2 \Leftrightarrow \text{every simple cloesd curve on } S \text{ separates it}$$

(The Jordan curve theorem).

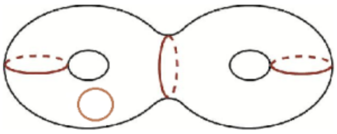


Figure 4: Separating and nonseparating curves.

(ii) If S is not equivalent to S^2, there must be at least one simple closed curve which will not separate S (see the figure - "Separating and nonseparating curves").

(iii) Take such a nonseparating curve C on S, thicken it (i.e. take a tubular or a regular neighbourhood of C in S) to get N, which will be a cylinder with two circle boundary components (see the figure - "Cylinder and Möbius band"), if the surface is orientable or two sided (like a torus), or a Möbius band, with one circle boundary (see the figure - "Cylinder and Möbius band"), if the surface is nonorientable or one sided (like Klein's bottle).

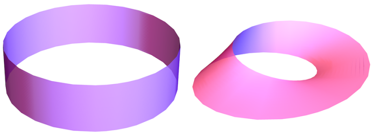
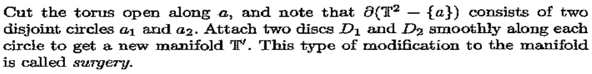


Figure 5: Cylinder and Möbius band.

(iv) Remove the interior of N and glue two copies (or one copy) of 2-disks "suitably" along the bounding pair of circles (or the bounding circle). The resulting surface S^* will remain a compact orientable (or nonorientable) surface.



Cut the torus open along a, and note that $\partial(\mathbb{T}^2 - \{a\})$ consists of two disjoint circles a_1 and a_2. Attach two discs D_1 and D_2 smoothly along each circle to get a new manifold $\mathbb{T}'$. This type of modification to the manifold is called *surgery*.

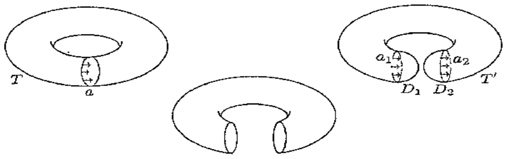


Figure 6: Surgery along a nonseparating curve a.

One says that S^* is obtained from S by a *surgery along* C (see the figure - 6 "Surgery along a nonseparating curve"); S^* is "cobordant" to S, that is, S^*

together with S forms the boundary of a 3-manifold with boundary, and $\chi(S^*) > \chi(S)$.

After doing finite number of such surgeries one gets $\chi(S^*) = 2$, i.e., $S^* \cong S^2$.

By doing the reverse surgeries on S^2 one therefore recovers S in finitely many steps (see the figure - "Reverse surgery"). So S is a sphere in which finitely many hollow handles (or cross caps) are attached.

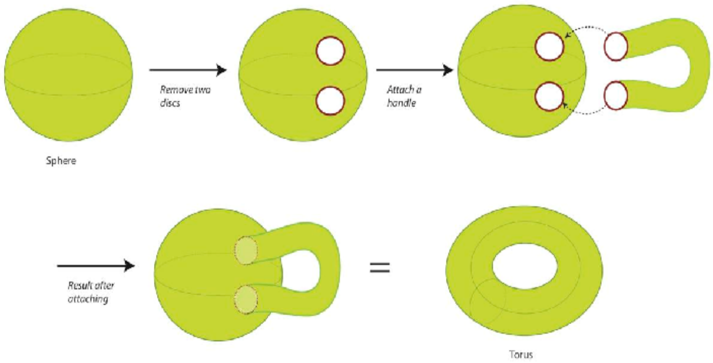


Figure 7: Reverse surgery.

This idea has been successfully employed to different classification problems of higher dimensional manifolds of various shades.

3.1. *Remark.* On a surface (i) any simple closed curve C can always be thickened, (ii) surgery along C changes the fundamental group and the Euler characteristics. Each of these is a complete invariant for compact surfaces without boundaries.

4. Classifications of 3-manifolds
Early 20 th Century - the beginning

In the beginning of the 20th century a number of different approaches were taken to address the problem of classification of 3-manifolds, see [56].

(i) (Combinatorial approach of Moise) (see [70, 56]): All 3-manifolds can be constructed by gluing tetrahedra along their faces.

(ii) (Heegaard Gluing approach) (see [32, 56]): S^3 is obtained by gluing two solid 3-balls (or solid tori) along their boundaries (see the figure on the next page - "Heegaard-gluing: 3-sphere (i) is the union of two solid 3-balls glued along their common boundary (the equator) (ii) is also the union of two solid tori glued along their boundaries"):

Consider the three dimensional sphere S^3 and view it as the union $S^3 = E^3_+ \cup E^3_-$ of two solid 3-balls identified along the boundaries; these are the two hemispheres of S^3 and the equator of the sphere is the common boundary of the two 3-balls.

Remove a solid torus T^3_1 from the interior of S^3. The figure ("Heegaard-gluing: 3-sphere (i) is the union of two solid 3-balls glued along their common boundary (the equator) (ii) is also the union of two solid tori glued along their boundaries") shows the effect of this on the two solid 3-balls E^3_+, E^3_-, creating canals on their surfaces. Then the remaining part is glued back, which again gives a solid torus T^3_2.

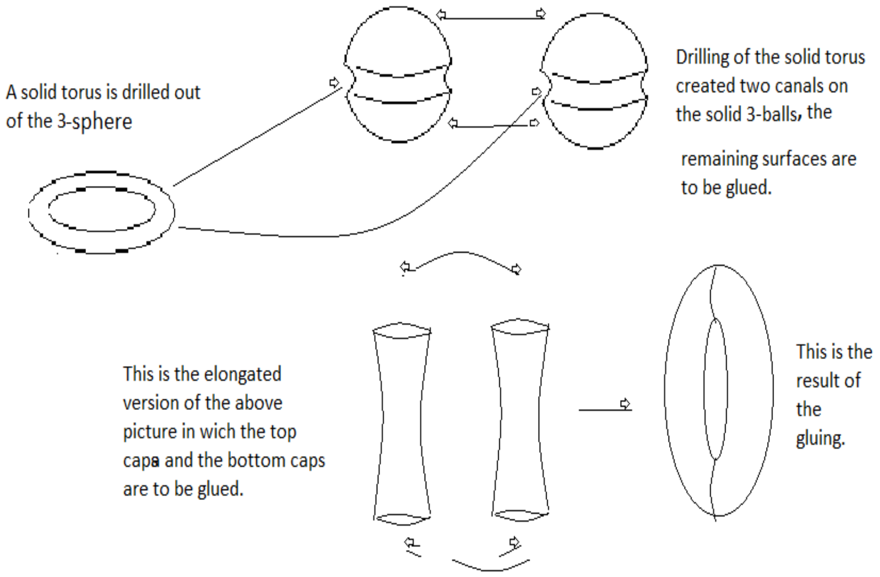


Figure 8: Heegaard-gluing: 3-sphere (i) is the union of two solid 3-balls glued along their common boundary (the equator) (ii) is also the union of two solid tori glued along their boundaries.

In general any 3-manifold is obtained by gluing two solid n-holed tori along their boundary, called a **Heegaard gluing**. Since gluing along boundaries can be done in many different ways, one can ask which of the manifolds obtained in these manner are homeomorphic.

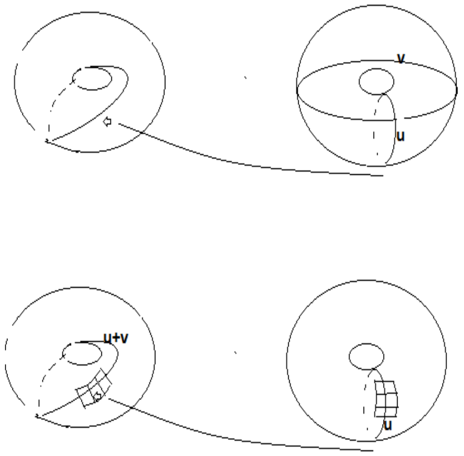
(iii) (Dehn Surgery approach)(see [14, 56]): As described in the figure - "Process of Dehn surgery" on the next page.

Remove a solid torus T^3_1 from S^3.

Call the curve labelled as u as the meridian and the curve labelled as v as the longitude of the boundary ∂T^3_1 of the solid torus T^3_1.

We now glue the removed torus T^3_1 back into S^3 in such a way that the meridian curve u is identified to the curve labelled $u + v$ on the boundary ∂T^3_1, in a manner that the small rectangle drawn on the surface on the right is identified with the small rectangle drawn on the surface on the left.

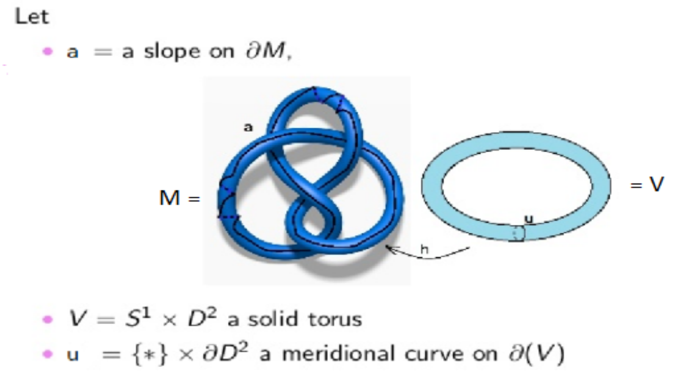
This gives rise to a 3-manifold $M = (S^3 \setminus \mathrm{int}T^3_1) \cup_\sim T^3_1$, where $\sim$ is the identification along the boundary torus ∂T^3_1 as described above and is independent of the way in which the rest of the torus is identified. M is denoted by $(S^3 \backslash K)_{(1,1)}$.



Gluing back the removed solid torus in the following different way: Gluing the meridian circle **u** to the curve drawn in the left picture, that is **u+v**, where **v** stands for the logitudinal curve

Figure 9: Process of Dehn surgery.

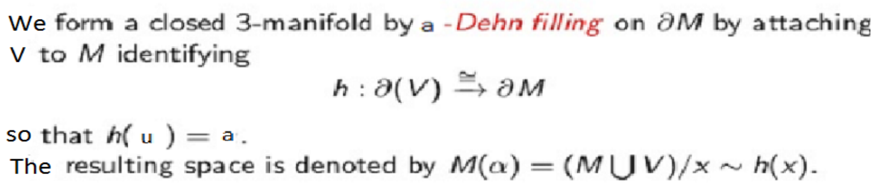
Instead of removing an ordinary solid torus one can remove a solid torus neighbourhood of an arbitrary knotted circle, M, from S^3 and then obtain a new 3-manifold by gluing back the solid torus neighbourhood by sending u into a given curve a drawn on the boundary ∂M of M. See the figures - "Dehn surgery" and "Dehn surgery - contd.".


Let

- a = a slope on ∂M,

- $V = S^1 \times D^2$ a solid torus
- u $= \{*\} \times \partial D^2$ a meridional curve on $\partial(V)$

Figure 11: Dehn surgery.


We form a closed 3-manifold by a -*Dehn filling* on ∂M by attaching V to M identifying

$$h : \partial(V) \xrightarrow{\cong} \partial M$$

so that $h($ u $) =$ a .
The resulting space is denoted by $M(\alpha) = (M \bigcup V)/x \sim h(x)$.

12: Dehn surgery - contd.

This process of removing a solid torus (possibly knotted) from S^3 and gluing it back in a different way to get a new 3-manifold is known as **Dehn Surgery**.

Example : (i) Dehn constructed the **Poincaré homology 3-sphere** $(S^3 \setminus K)_{(2,3)}$ from the standard 3-sphere S^3 by performing Dehn surgery along the trefoil knot (see the figure - "Trefoil knot") embedded in S^3.

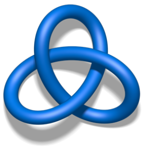


Figure 10: Trefoil knot.

(ii) $(S^3 \backslash K)_{(1,0)}$ is the standard 3-sphere S^3.

(iii) There are infinitely many homotopy types of homology 3-spheres.

4.1. **Theorem** (Likorish, Wallace):(see [88, 49, 50, 118]). *All closed, orientable 3-manifolds can be obtained by performing Dehn surgery on links (a link is a collection of knots which do not intersect, but which may be linked (or knotted) together) in the standard 3-sphere.*

4.2. *Remark.* This result is same in spirit as the theorem about surfaces, and in fact can classify all "Seifert fibered spaces" (these are 3-manifolds together with a "nice" decomposition as a disjoint union of circles) (see [95]), but we are not in so comfortable a position to ascertain when two such manifolds are homeomorphic (This is an open question even today.)

5. Classification of 3-manifolds- 1950's and 1960's

We begin by stating the following:

5.1. **Theorem** (Dehn's lemma, 1910; [14, 22]). *A piecewise-linear map of a disk into a 3-manifold, with the map's singularity set in the disc's interior, implies the existence of another piecewise-linear map of the disc which is an embedding and is identical to the original on the boundary of the disc.*

5.2. *Remark.* This theorem was thought to be proven by Max Dehn (1910), but Hellmuth Kneser (1929) (see [47]) found a gap in the proof.

The status of Dehn's lemma remained in doubt until **Christos Papakyriakopoulos** (1957) (see [79, 80]) proved it using his "tower construction" (constructing a tower of covering spaces). In 1958, Arnold Shapiro and J.H.C. Whitehead gave a substantially simpler proof, and an extension of Dehn's lemma (see [97]).

Papakyriakopoulos also proved the loop and sphere theorems.

5.3. **Theorem** (Loop Theorem). *(see* [79, 80, 22, 32]*) If there is a map* $f : (D^2, \partial D^2) \to (M, \partial M)$ *with* $f|_{\partial D^2}$ *not nullhomotopic in* ∂M*, then there is an embedding with the same property.*

5.4. **Theorem** (Sphere Theorem). *(see* [79, 80, 22, 32]*) Let* M *be an orientable 3-manifold such that* $\pi_2(M)$ *is not a trivial group. Then there exists a non-zero element in* $\pi_2(M)$ *having representative that is an embedding* $S^2 \hookrightarrow M$*.*

5.5. **Definition.** (see [22, 32]) An incompressible surface in a 3-manifold is a two sided embedded surface of genus ≥ 1 whose fundamental group maps injectively into the fundamental group of the manifold (something like non separating curve in a surface of genus ≥ 1).

5.6. *Remark.* Wolfgang Haken showed (see [25, 22, 32]) that if a 3-manifold contains an incompressible surface, the manifold can be simplified by cutting along the surface (something like doing surgery on a surface along a non separating curve to simplify the surface).

5.7. **Definition.** A 3-manifold is called a *Haken manifold* if it contains a (properly) embedded incompressible surface.

5.8. *Remark.* (i) Haken sketched out a proof of an algorithm to check if two Haken manifolds were homeomorphic or not. His outline was filled in by Waldhausen, Johannson, Hemion, Matveev, et al. (see [112, 39, 31, 55])

(ii) Since there is an algorithm to check if a 3-manifold is Haken (cf. Jaco-Oertel [36]), the basic problem of recognition of 3-manifolds can be considered to be solved for Haken manifolds.

6. Classification of n-manifolds and algebraic topology

As we have seen earlier that in the classification of surfaces fundamental group, Euler characteristics and of course orientability gave complete invariants. These are algebraic topological invariants of the surfaces. As we go for manifolds of higher dimensions more and more algebraic topological invariants will come into play. Further development on the problem of classification of manifolds therefore depended on the development of algebraic topology which was going on side by side, specifically,

(Co)homolgy theory and cohomology operations has been developed by Alexander, Cěck, Steenrod, Whitney among others, (see eg. [100, 30, 108, 102, 103]),

Vector bundles and characteristic classes has been developed by Hopf, Pontrjagin, Steenrod, Stiefel, Whitney among others, (see eg. [69, 104, 33]),

Morse theory and homotopy theory has been developed by Morse and Whitehead, (see eg. [60]).

7. Classification of n-manifolds, $n > 4$ - 1950's

Pontrjagin's and Thom's cobordism theory (see [109, 84]) gave rise to a new shade of classification problem and a new (co)homology theory.

7.1. **Definition.** Two compact manifolds of same dimension are *cobordant* if they together form the boundary of a compact manifold of one dimension higher.

Cobordism classes form a group under addition defined by disjoint union of manifolds and a ring under addition together with a multiplication defined by cartesion product of two manifolds, barring some technical details. Pontrjagin-

Thom studied cobordism classification by converting cobordism into a homotopy problem and analyzing the latter (see [109, 105]).

Cobordism classes were characterized by algebraic invariants like Stiefel-Whitney numbers, Chern numbers, Pontrjagin numbers, index, etc. (see [109, 105]).

J. P. Serre's determination of homotopy groups of spheres by using the machinary of spectral sequences of a fibration (see [96, 100, 108, 30]) gave a big boost to the computation of framed cobordism ring of framed manifolds which by virtue of the Thom-Pontrjagin isomorphism is isomorphic to the stable homotopy group of spheres (see [65, 84, 105]).

Further development of homotopy groups of unitary and orthogonal groups (Bott's periodicity theorem) (see [6, 7, 8, 33]) helped in the determination of more general cobordism rings by Thom, Milnor, Wall etc. (see [105]).

Some sample results are as follows (see [109, 61, 115, 105]):

Oriented cobordism ring modulo torsion (that is tensor product of the ring with rationals) is a polynomial algebra generated by the cobordism classes of complex projective spaces.

Full oriented cobordism ring, unoriented cobordism ring, etc. have also been completely determined as graded algebras with generators which are cobordism classes of real projective spaces, Dold manifolds, Milnor manifolds, Wall manifolds (will be defined a little later).

Stong has given an exhaustive survey of the cobordism classification of different classes of manifolds in [105].

8. H-COBORDISM AND THE BREAKTHROUGH

Stephen Smale and others considered h-cobordism (see [98, 62]):

8.1. **Definition.** Two manifolds are *h-cobordant* if they are cobordant and each is deformation retract of the third manifold.

8.2. **Theorem** (h-cobordism theorem, proved by Smale). *(see* [98, 62]*) Two simply connected smooth manifolds of dimension* ≥ 5 *are h-cobordant if and only if they are diffeomorphic (in fact the cobordism is a cylinder).*

Technique of Morse theory or handle decomposition of the h-cobordism and simplification of this decomposition lead to the proof of the theorem.

This theorem lead to a proof of higher dimensional ≥ 5 Poincaré's conjecture: Any homotopy n-sphere is homeomorphic to the standard n-sphere (see [99, 101, 76]).

The crux of Smale's proof was

8.3. **Theorem** (Simply-connected Whitney's lemma). *(see e,g,* [22, 90]*)* $P^p, Q^q \subset M^m$, $p + q = m$ *oriented submanifolds,* P, Q *intersects transversally in a finite number of points. Let* $x, y \in P \cap Q$ *with opposite algebraic intersection numbers*

$(P,Q)_x = -(P,Q)_y$. *Then, if* $p \geq 3, q \geq 3$ *and* $\pi_1(M) = 0$, *or* $p = 2, q \geq 3$ *and* $\pi_1(M \setminus Q) = 0$, *there is an isotopy of* M *carrying* P *to* P' *which also intersects* Q *transversally in a finite number of points such that* $P' \cap Q = P \cap Q \setminus \{x, y\}$. *The isotopy has support in a compact set* K *which does not meet other intersection points (i.e. the isotopy keeps* $M \setminus K$ *fixed) (See the figure - "Whitney trick to remove pair of points with opposite signs").*

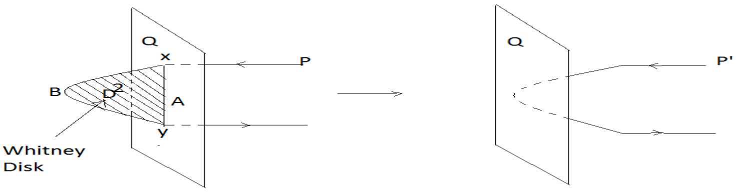


Figure 13: Whitney trick to remove pair of points with opposite signs.

Whitney's lemma depends on the following:

8.4. **Theorem** (Existence of embedded disks, dim M $\geq$ 5)**.** *Suppose* $f : D^2 \to M^n$, $n \geq 5$ *is a smooth map such that* $\{x \in D^2 \mid f^{-1}f(x) \neq \{x\}\} \cap \partial D^2 = \emptyset$. *Then there is a smooth embedding* $f' : D^2 \to M^n$ *with* $f' |_{\partial D^2} = f |_{\partial D^2}$ *and* f' *is homotopic to* f *rel* ∂D^2.

8.5. **Theorem** (Existence of immersed disks, dim M = 4)**.** *Suppose* $f : D^2 \to M^4$ *is a smooth map such that* $\{x \in D^2 \mid f^{-1}f(x) \neq \{x\}\} \cap \partial D^2 = \emptyset$. *Then there is a smooth immersion* $f' : D^2 \to M^n$ *with* $f' |_{\partial D^2} = f |_{\partial D^2}$ *and* f' *is homotopic to* f *rel* ∂D^2 *and* f' *has only double points.*

8.6. *Remark.* (i) The above theorem says that an embedded circle $S^1 \hookrightarrow M^n$, $n \geq 5$, bounds a smooth embedded disk if and only if it is homotopic to a constant map.

(ii) The last theorem on existence of immersed disks in dim 4 helped Casson and Freedman to build a Smale type theorem in dimension 4 which we will mention later.

(iii) If $n = 3$ this kind of freedom of movement is not available, for example, in the figure - "Failure of Whitney trick in dimension 3", the embedded (blue, knotted) circle in ($S^3 \setminus$ red, unknotted circle) can be shrunk to a point in ($S^3 \setminus$ red, unknotted circle) (the bounding disk overlaps itself), but it does not bound an embedded disk in ($S^3 \setminus$ red, unknotted circle) (see [66]).

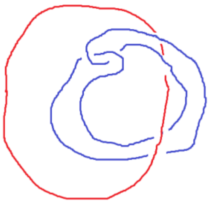


Figure 14: Failure of Whitney trick in dimension 3.

9. CLASSIFICATION OF n-MANIFOLDS, $n > 4$ - 1960'S

By a result of A. A. Markov (and S.P. Novikov) one cannot classify all manifolds of dimension ≥ 4 upto homeomorphism (and hence upto diffeomorphism and pl-homeomorphism) (see Markov's result [52] in Russian and S.P. Novikov's result in the appendix of [111]).

If one can give a construction which to any finite presentation $< S|P >$ of a group associates a n-manifold $M(S, P)$, $n \geq 4$, in such a way that $\pi_1(M(S, P))$ is isomorphic to the group defined by the presentation $< S|P >$, and two such manifolds are homeomorphic if and only if they have isomorphic fundamental groups, then one can construct a class of n-manifolds, $n \geq 4$, for which the homeomorphism problem is equivalent to the isomorphism problem for finitely presented groups, and is therefore unsolvable. Consequently classification of higher dimensional manifolds is not to classify all the manifolds but subclasses of manifolds. In fact, one fixes a manifold X and considers the class $hT(X)$ of all manifolds which are of the same homotopy type as X, and then classifies manifolds belonging to $hT(X)$, upto homeomorphism (diffeomorphism and pl-homeomorphism).

Classification problems as mentioned above have been initiated in 1960's by Milnor with the discovery of "Exotic diffentiable structure of S^7" (see [67]). Subsequent work by Milnor-Kervaire on groups of homotopy spheres led to "Classification of homotopy spheres" (see [44, 68, 67]).

9.1. Classification of homotopy spheres by Milnor, Milnor-Kervaire.

9.1. **Definition** (see [44, 68, 67])**.** Let $hT_{Diff}(S^n)$ be the set of all oriented diffeomorphism classes of closed smooth homotopy n-spheres. $hT_{Diff}(S^n)$ forms a commutative monoid under the connected sum operation. This monoid is actually a finite abelian group except possibly when n = 4. Let $hT_{Diff}(S^n)^{bp} \subset hT_{Diff}(S^n)$ be the subgroup represented by homotopy spheres that bound "parallelizable" manifolds (manifolds with trivial tangent bundles).

This subgroup fits in a left short exact sequence:

$$0 \to hT_{Diff}(S^n)^{bp} \to hT_{Diff}(S^n) \to \pi_n^S/imJ, \tag{1}$$

$J : \pi_n(SO) \to \pi_n^S$ being the stable Whitehead J-homomorphism. (see Whitehead, George W., Elements of homotopy theory, GTM, Springer,(1978).)

9.2. *Remark.* (see [44, 68, 67]) $hT_{Diff}(S^n)^{bp}$ is the best understood part of the group $hT_{Diff}(S^n)$.

9.3. **Theorem.** *(see* [44, 68, 67]*) For $n \neq 4$ the group $hT_{Diff}(S^n)^{bp}$ is finite cyclic with an explicitly known generator. In fact this group is:*

(i) trivial when n is even,

(ii) either trivial or cyclic of order two when $n = 4k - 3$, and

(iii) cyclic of order $2^{2k-2}(2^{2k-1} - 1)$ numerator $\left(\frac{4B_k}{k}\right)$ when $n = 4k - 1 > 3$.

9.4. *Remark* (see [44, 68, 67]). This last number depends on the computation of the order $|imJ_{4k-1}|$ of the image of J_{4k-1}. B_k stands for the kth Bernouli's number.

9.5. *Remark* (see [44, 68, 67]). If $n = 2q - 1$, an explicit generator for the $hT_{Diff}(S^{2q-1})^{bp}$ can be constructed using one basic building block, namely the total space E^{2q} of tangent disk q-bundle of the q-sphere S^q, which are parallelizable $2q$-dimensional manifolds with boundary, by plumbing construction of the following type, where in $\Box$ represents *plumbing* of two copies of E^{2q}, constructed by pasting across each other, so that their central q-spheres intersect transversally with intersection number $+1$.

1. $E_1^{2q} \;\Box\; E_2^{2q}$
2. $$\begin{array}{ccccccccccccc} E_1^{2q} & \Box & \mathrm{E}_2^{2q} & \Box & \mathrm{E}_3^{2q} & \Box & \mathrm{E}_4^{2q} & \Box & \mathrm{E}_5^{2q} & \Box & \mathrm{E}_7^{2q} & \Box & \mathrm{E}_8^{2q} \\ &&&&&&&& \Box &&&& \\ &&&&&&&& \mathrm{E}_6^{2q} &&&& \end{array}$$

9.6. *Remark* (see [44, 68, 67]). The result of plumbing is a smooth parallelizable manifold with corners. After straightening these corners we obtain a smooth manifold X^{2q} with smooth boundary.

For q odd, one uses the first diagram, and for q even one uses the second diagram.

In either case, if $q \neq 2$, the resulting smooth boundary ∂X^{2q} will be a homotopy sphere representing the required generator of $hT_{Diff}(S^{2q-1})^{bp}$.

The case $q = 2$ is exceptional since ∂X^4 has only the *homology* of the 3-sphere, S^3.

In all other cases where $hT_{Diff}(S^n)^{bp}$ is trivial, the boundary will be diffeomorphic to the standard n-sphere, S^n.

The left exact sequence (1) can be complemented in the following cases:

9.7. **Theorem** (see [44, 68, 67]). *For $n \not\equiv 2(mod\, 4)$, every element of the stable n^{th} homotopy group of sphere, π_n^S, can be represented by a topological sphere. Hence the left exact sequence (1) takes the more precise form*

$$0 \to hT_{Diff}(S^n)^{bp} \to hT_{Diff}(S^n) \to \pi_n^S/imJ \to 0 \tag{2}$$

However, for $n = 4k - 2$, it extends to an exact sequence

$$0 \to hT_{Diff}(S^{4k-2})^{bp} \to hT_{Diff}(S^{4k-2}) \to \pi_{4k-2}^S/imJ \xrightarrow{\Phi_k} \mathbb{Z}/2 \to hT_{Diff}(S^{4k-3})^{bp} \to 0. \tag{3}$$

9.2. The procedure of Surgery in higher dimensions.

9.8. **Definition** (see [59, 44, 68, 67]). Let M^n be a closed manifold. Let $S^p \times D^{n-p} \subset M$ be an embedding. Note that $\partial(D^{p+1} \times D^{n-p}) = S^p \times D^{n-p} \bigcup D^{p+1} \times S^{n-p-1}$ and $\partial(S^p \times D^{n-p}) = S^p \times S^{n-p-1} = \partial(D^{p+1} \times S^{n-p-1})$.

If we remove from M the interior of $S^p \times D^{n-p}$ and attach $D^{p+1} \times S^{n-p+1}$ along the boundary $S^p \times S^{n-p-1}$ then we obtain a new manifold $M' = M \backslash int(S^p \times D^{n-p}) \bigcup_{S^p \times S^{n-p-1}} D^{p+1} \times S^{n-p-1}$.

M' is said to be *obtained from* M by a *surgery of type* $(p+1,\, n-p)$.

9.9. *Remark* (see [59, 44, 68, 67]). It follows clearly that M is *obtained* from M' by a surgery of type $(n-p,\;\, p+1)$. M and M' are cobordant; the cobordism being $W = M \times I \bigcup\, D^{p+1} \times D^{n-p}$, where the *handle* $D^{p+1} \times D^{n-p}$ is attached to $M \times \{1\}$ along $S^p \times D^{n-p}$ (see figures “solid handle”, and “attaching 1-handles and 2-handles”).

By handle decomposition of a cobordism (see [67]) one also knows that if two manifolds are cobordant then one can be obtained from the other by a sequence of surgeries.

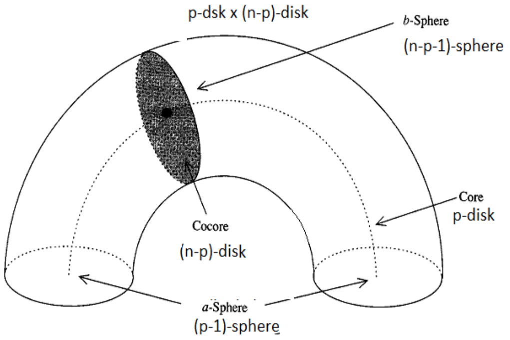


Figure 15: solid handle.

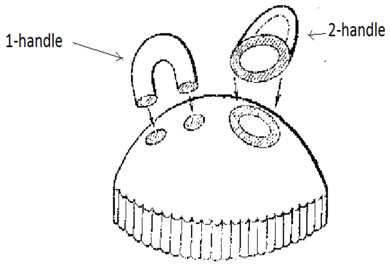


Figure 16: attaching-1-handles-and-2-handles.

Suppose $\alpha \in \pi_p(M)$ and $\alpha = [i]$ with $i : S^p \to M^n$ is the restriction of an embedding $S^p \times D^{n-p} \hookrightarrow M$ to $S^p \times 0$. Let M' be obtained from M by a surgery of type $(p+1,\, n-p)$ performed on this $S^p \times D^{n-p}$. Then

9.10. **Proposition** ([59]). *For $n \geq 2p+2$ we have*

$$\pi_q(M') \cong \begin{cases} \pi_q(M) & \text{if } q < p \\ \pi_p(M)/H & \text{if } H \leq \pi_p(M), \text{ and } \alpha \in H \end{cases}$$

9.11. *Remark* ([59]). Thus performing a surgery of type $(p+1,\ n-p)$ on M with $p+1 \leq \frac{n}{2}$ kills the class $\alpha\varepsilon\pi_p(M)$ represented by $S^p \times D^{n-p}$.

9.12. **Question.** Given an arbitrary element $\alpha \in \pi_p(M^n)$ with $2(p+1) \leq n$, when can it be killed by surgery ?

In other words we want to know when can α be represented as a restriction to $S^p \times 0$ of an embedding $S^p \times D^{n-p} \subset M^n$. The answer lies in the following:

9.13. **Theorem** ([59]). *If* $n \geq 2p+1$ *, then* $\alpha = [i] \in \pi_p(M)$ *can be killed by surgery if and only if* $i^*\tau_M$ *is trivial.*

9.14. *Remark* ([59]). Not every homotopy class (below middle dimension) could be killed for an arbitrary manifold.

9.15. **Example** ([59]). Let $M = \mathbf{CP}^{2m}$. Then $w_2^m[\mathbf{CP}^{2m}] \neq 0$. where as for any two-connected manifold $N \quad w_2^m[N] = 0$. So $\mathbf{CP}^{2m}$ is not cobordant to a 2-connected manifold (X is called n-connected if all k th homotopy groups $1 \leq k \leq n$, are zero).

9.16. **Definition** (see [59, 44, 68, 67]). A manifold M^n is called *S-parallelizable* (or a π-*manifold*) if $\tau_M \oplus O_M^1$ is a trivial bundle.

9.17. **Example.** Every sphere is S-parallelizable.

9.18. *Remark.* M^n is S-parallelizable if and only if its normal bundle in $\mathbb{R}^{n+k}$, $k > n$ is trivial.

The following theorem gives a positive answer to the question of killing homotopy groups below middle dimension by surgery.

9.19. **Theorem** (see [59, 44, 68, 67]). *(Surgery below middle dimension). Let* M^n *be an S-parallelizable manifold of dimention* $n \geq 2p+1$. *Then every homotopy class* $\alpha \in \pi_p(M)$ *is represented by some embedding* $i : S^p \times D^{n-p} \hookrightarrow M$ *such that the manifold* M' *obtained by surgery on* M *of type* $(p+1,\ n-p)$ *is also S-parallelizable.*

9.20. **Corollary** (see [59, 44, 68, 67]). *Any compact S-parallelizable manifold* M^n *is cobordant to a* $[\frac{n}{2}-1]$*-connected S-parallelizable manifold.*

9.21. *Remark* (see [59]). (i) Every S-parallelizable manifold is a boundary (i.e cobordant to a sphere).

(ii) Converse of the above statement is not true.

9.22. **Example** (see [59]). For example $\mathbf{CP}^2 \cup (-\mathbf{CP}^2)$ is a boundary but is not S-parallelizable.

9.23. *Remark.* (i) Theorem 9.19 and Corollary 9.20 tell us that one can perform *surgery below middle dimension* successfully. If one succeeds to do *surgery in the middle dimension* to change the given S-parallelizable manifold M^n upto cobordism to a $[\frac{n}{2}]$- connected S-parallelizable manifold, then the resulting manifold becomes, by Poicaré duality, a homologically trivial manifold.

(ii) Surgery in the middle dimension is however not always possible and the main hard work in surgery theory lies in the determination of the obstruction to do so.

(iii) The work of Milnor and Kervaire ([59, 44, 68, 67]) can be consulted for further details.

9.3. **Surgery in higher dimensions - surgery on maps.** To extend the work initiated by Milnor and Kervaire ([59, 44, 68, 67]) for manifolds other than spheres a more general set-up was employed. We very briefly introduce some of these set-up and state the main results obtained using these. In what follows let $f : M^n \to A$ be a map of degree one between compact oriented n-dimensional manifolds.

9.24. **Definition** (see [11]). An element $\alpha \in \pi_{p+1}(f) := \pi_{p+1}$(mapping cone of f) is represented by a pair of maps (i, i') appearing in the following diagram :

$$\begin{tikzcd}
S^p \arrow[r, "i"] \arrow[d, phantom, "\cap"] & M^n \arrow[d, "f"] \\
D^{p+1} \arrow[r, "i'"] & A
\end{tikzcd}$$

9.25. **Definition** (see [11]). A *normal map* is a pair (f, b) given by the commutative diagram:

$$\begin{tikzcd}
\nu_M \arrow[r, "b"] \arrow[d] & \xi \arrow[d] \\
M \arrow[r, "f"] & A.
\end{tikzcd}$$

In this diagram the left vertical map is the "stable normal bundle over M", the right vertical map is the "a vector bundle over A", and the top horizontal map is "the map of bundles covering the degree one map f" (see [104, 33, 69] for bundle theory).

9.26. **Definition** (see [11]). A *normal cobordism* from (f, b) to another (f', b') is a commutative diagram:

$$\begin{tikzcd}
\nu_W \arrow[r, "B"] \arrow[d] & \xi \times I \arrow[d] \\
W \arrow[r, "F"] & A \times I,
\end{tikzcd}$$

W being the cobordism between M and M', such that $F|M = f$, $B|\nu_M = b$, $F|M' = f'$, $B|\nu_{M'} = b'$.

9.27. **Theorem** (see [11, 116]). *Let (f, b) be a normal map with target A having a vector bundle ξ over it. Let $\alpha \in \pi_{p+1}(f)$ with $2p \leq n$. Then α determine a regular homotopy class of immersions of $S^p \times D^{n-p}$ in int M. We can do surgery on α so as to obtain (f', b') normally cobordant to (f, b) if this class contains an embedding.*

9.28. **Corollary** (see [11, 116]). *If $2p < n$. we can do surgery on any $\alpha \in \pi_{p+1}(f)$.*

9.29. **Theorem** (see [11, 116]). *If $2p \leq n$ we can make f p-connected by a finite number of surgeries on homotopy classes α in dimension $\leq p$.*

9.30. *Remark.* Recall the remark made earlier that surgery in the middle dimension is not possible unless the obstruction to do the surgery vanishes. In cases when the manifolds concerned are simply connected the obstruction to do surgery lie either in the groups $\mathbb{Z}$ or $\mathbb{Z}/2$ and are detected by "the index" which is an integer, or "the Arf-Kervaire invariant" which is an integer modulo 2.

The following theorem give a description of what can be achieved by surgery on simply connected manifolds.

9.31. **Theorem** (see [11, 116]). *(Simply-connected surgery obstructions) Let (f, b) be a normal map, $f : (M^n, \partial M^n) \to (X, Y)$, $b : \nu \to \xi$ as usual, such that $f|\partial M$ induces an isomorphism on homology. There is defined an invariant $\sigma(f, b)$, called surgery obstruction for (f, b),*

$$\sigma(f,b)\begin{cases} = 0 & \textit{if } n \textit{ is odd,} \\ \in \mathbb{Z} & \textit{if } n = 4k, \textit{ and} \\ \in \mathbb{Z}/2 & \textit{if } n = 4k+2, \end{cases}$$

and such that $\sigma(f, b) = 0$ if (f, b) is normally cobordant to a map inducing isomorphism on homology.

9.32. **Theorem** (see [11]). *[Fundamental theorem of simply-connected surgery] Let (f, b) be a normal map, $f : (M^n, \partial M^n) \to (X, Y)$, $b : \nu \to \xi$ as usual and suppose*

1. *$f|\partial M$ induces isomorphism in homology,*
2. *X is 1-connected,*
3. *$n \geq 5$.*

Then,

(a) if n is odd then (f, b) is normally cobordant rel Y to (f', b') with $f' : M' \to X$ a homotopy equivalence,

(b) if $n = 2k, (f, b)$ is normally cobordant rel Y to (f', b') with $f' : M' \to X$ a homotopy equivalence if and only if $\sigma(f, b) = 0$

9.33. **Theorem** (see [11]). *[Plumbing theorem] Let $(X, Y) = (D^n, S^{n-1})$. If $n = 2k > 4$, then there are normal maps $(g, c), g : (M, \partial M) \to (D^n, S^{n-1}), c : \nu^k \to \varepsilon^k =$ trivial bundle, with $g|\partial M$ a homotopy equivalence and with $\sigma(g, c)$ taking on any desired value.*

It is proved by a technique called "plumbing" as mentiobed earlier.

10. Classification of n-manifolds, $n > 4$ - 1970's & Further development

The development made by Kervaire-Milnor (see [44]) and Browder-Novikov (see [11, 77, 78]) paved the way to develop a comprehensive technology of classifying those CAT (= DIFF, PL or TOP) manifolds upto CAT equivalence which are "simple homotopy equivalent" to a given fixed CAT manifold by Browder-Novikov-Sullivan-Wall-Kirby-Siebenmann (see [11, 106, 116, 45]). Roughly, two manifolds are simple homotopy equivalent if one can be obtained from the other by a finite sequence of collapses and extensions (see [120, 63, 13]).

The technology of Surgery theory analyzes the CAT-equivalence classes of manifolds simple homotopy equivalent to a given manifold by relating it to groups which are more computable ("normal cobordism group") and analyzing this relation more closely (in some suitable sense).

10.1. **Definition.** A *homotopy CAT-structure* of the manifold X is a pair (M, f), where $(M, \partial M)$ is a CAT-manifold with boundary and $f : (M, \partial M) \to (X, \partial X)$ is a simple homotopy equivalence with $f\mid_{\partial M} : \partial M \to \partial X$ a CAT-equivalence.

(M, f) and (M', f') are equivalent if there is a CAT-equivalence $h : (M, \partial M) \to (M', \partial M')$ for which the maps $f' \circ h$ and f are homotopic relative to the boundary ∂M. $hT_{CAT}(X)$ denotes the set of these equivalence classes.

$hT_{CAT}(X)$ is called **the Structure Set**, or **the Set of homotopy CAT-Structures on** X. This is a pointed set with base point (X, id_X), and is the set which we intend to determine.

10.1. **Method of Determination of $hT_{CAT}(X)$, The Surgery Exact Sequence.** Determination of $hT_{CAT}(X)$ involves fitting it into an exact sequence, the Sullivan-Wall-Kirby-Siebenmann *surgery exact sequences* (see [116, 45]) involving more algebraic and computable objects:

$(S-E-S): \ \to L_{n+1}(\mathbb{Z}\pi_1(X)) \xrightarrow{\delta} hT_{CAT}(X) \xrightarrow{\eta} [X, G/CAT] \xrightarrow{\theta} L_n(\mathbb{Z}\pi_1(X)),$

and then:

STEP 1. : To determine the *Normal invariants* $\mathcal{N}^n(\mathrm{X}) \cong [X, G/CAT]$.

$\mathcal{N}^n(X) =$ (Normal) cobordism classes of triple (M, f, Fr), where $f : M \to X$ be a map of degree one, i.e. $f_*([M]) = [X]$, and such that if ξ is the CAT-tangent bundle over X, $\tau_M \oplus f^*\xi$ is trivial, and Fr is a choice of trivialization of $\tau_M \oplus f^*\xi$ (equivalently, normal cobordism classes of normal maps (f, b) as described earlier).

$[X, G/CAT]$ = Equivalence classes of stable CAT-bundle over X which is "fibre homotopically trivial".

10.2. **Theorem** (see [106, 116, 45])**.** *If $K(A, n)$ denote "Eilenberg-MacLane spaces" (see e.g. Spanier* [100]*), then cohomology classes L and K defined by Sullivan gives the following isomorphism :*

$$G/TOP_{(2)} \cong \prod K(\mathbb{Z}/2, 4i-2) \times K(\mathbb{Z}_{(2)}, 4i), i \geq 1.$$

Similarly, the "Pontrjagin character" (see e.g. Husemoller [33]*) gives the following isomorphism :*

$$G/TOP[1/2] \cong BO[1/2].$$

Here $X_{(2)}$ means "localization" of X at the prime 2 and $X[1/2]$ means "localization" of X away from the prime 2. (for localization of spaces see e.g. Arkowitz [1], Sullivan [107]).

As a consequence of this the normal invariants for a manifold X can be calculated using the following "fibre square":

$$\begin{tikzcd} G/TOP \arrow[r] \arrow[d] & G/TOP_{(2)} \arrow[d] \\ G/TOP[1/2] \arrow[r] & G/TOP_{(0)} \end{tikzcd}$$

which gives by definition, the following exact sequence:

$$0 \to [X, G/TOP] \to KO^0(X)[1/2] \times$$

$$\bigoplus H^{4i-2}(X;\mathbb{Z}/2) \otimes H^{4i}(X;\mathbb{Z}_{(2)}) \to \bigoplus H^{4i}(X;\mathbb{Q}) \to 0.$$

Using this exact sequence and "Atiya-Hirzebruch spectral sequence" (see e.g. Switzer(1975)) to calculate $KO^0(X)[1/2]$ one can compute the normal invariants.

STEP 2 : To determine Wall's **surgery obstruction groups** $L_n(\mathbb{Z}\pi_1(X))$ (see [116]).

Before defining these groups we recall the remark made earlier that surgery in the middle dimension is not possible unless the obstruction to do the surgery vanishes. The Wall surgery obstruction groups are the the groups in which these obstructions lie when the manifolds concerned are not necessarily simply connected. For simply connected manifolds these groups coincide with the simply connected surgery obstruction groups described earlier in the Theorem 9.31 and Theorem 9.32, see also [11].

Let $K_k(M) := \pi_{k+1}(f)$, called the (surgery) *kernel complex.* It is a $\mathbb{Z}\pi_1(X)$-module, where $\mathbb{Z}\pi_1(X)$ is a ring with involution. The homology intersection form, λ, and self intersection form, μ, determine a triple $(K_k(M), \lambda, \mu)$ which is a $(-1)^k$-*hermitian form* on $K_k(M)$ over $\mathbb{Z}\pi_1(X)$.

Case I : Dimension of X *is* $n = 2k$ *:*

Wall's *even dimensional surgery obstruction group* $L_{2k}(\mathbb{Z}\pi_1(X))$ is the "Witt group" of stable isomorphism classes of $(-1)^k$-hermitian forms over $\mathbb{Z}\pi_1(X)$, where stability is with respect to addition of "hyperbolic forms" (see [116, 64]).

Case II : Dimension of X *is* $n = 2k + 1$ *:*

Wall's *odd dimensional surgery obstruction group* $L_{2k+1}(\mathbb{Z}\pi_1(X))$ is stable unitary group of automorphisms of hyperbolic $(-1)^k$-forms over $\mathbb{Z}\pi_1(X)$.

The even dimensional Wall's groups are analogous to the algebraic $K_0(\mathbb{Z}\pi_1(X))$-groups, and odd dimensional Wall's groups are analogous to the algebraic $K_1(\mathbb{Z}\pi_1(X))$-groups (see [3]), where instead of $(\mathbb{Z}\pi_1(X))$-modules we take Hermitian forms (i.e. $(\mathbb{Z}\pi_1(X))$-modules with Hermitian forms described above)

We give below a selected list of Wall's surgery obstruction L-groups, in which π^+ is used for the orientable case and π^- for the non-orientable case. These groups are 4-periodic (see [116, 117]).

$\pi^{\pm}$	L_0	L_1	L_2	L_3
1^+	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0
$(\mathbb{Z}/2)^+$	$8\mathbb{Z} \oplus 8\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$(\mathbb{Z}/2)^-$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
$(\mathbb{Z}/2 \oplus \mathbb{Z}/2)^+$	$4(8\mathbb{Z})$	0	$\mathbb{Z}/2$	$3(\mathbb{Z}/2)$
$(\mathbb{Z}/2 \oplus \mathbb{Z}/2)^-$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

$\pi^{\pm}$	L_0	L_1	L_2	L_3
$(\mathbb{Z})^+$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$(\mathbb{Z})^-$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}^+ \oplus \mathbb{Z}/2^+$	$\mathbb{Z} + \mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\mathbb{Z}^+ \oplus \mathbb{Z}/2^-$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}^- \oplus \mathbb{Z}/2^-$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}^- \oplus \mathbb{Z}/2^+$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\mathbb{Z}^+ \oplus \mathbb{Z}^-$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

STEP 3.: To determine the maps in the Sullivan-Wall-Kirby-Siebenmann exact sequence (S-E-S): $\eta : hT_{CAT}(X) \to [X, G/CAT]$ is essentially the forgetful map or the "Thom-Pontryagin map".

$\theta : N^n(X) = [X, G/CAT] \to L_n(\mathbb{Z}\pi_1(X))$, "the surgery obstruction map", which associates to each triple $[M, f, Fr]$ in $[X, G/CAT]$ the (surgery) obstruction in $L_n(\mathbb{Z}\pi_1(X))$ to make f a simple homotopy equivalence.

$\delta : L_{n+1}(\mathbb{Z}\pi_1(X)) \to hT_{CAT}(X)$ is given by the following theorem:

10.3. **Theorem.** *(see [116]) Given $x \in L_{n+1}(\mathbb{Z}\pi_1(X))$, $n+1 \geq 6$, there is a map of cobordism*

$$g : (W, \partial_- W, \partial_+ W) \to (X \times I, (X \times 0 \cup \partial X \times I), X \times 1)$$

of degree one and a trivialization Gr of $\tau_W \oplus g^*(\nu_{X\times I})$ such that $\theta([W, g, Gr]) = x$, $g\mid_{\partial_- W}$ is identity, and $g\mid_{\partial_+ W}$: $\partial_+ W \to X \times 1 \equiv X$ is a simple homotopy equivalence.

In the notation of the statement of the above theorem of Wall, define $\delta(x) = [g\mid_{\partial_+ W}]$. If $H^k = B^i \times B^{n-k}$ denote a k-handle, then in the theorem of Wall above one constructs the manifold $(W, \partial W)$ by attaching handles on $X \times I$ along $X \times 1$,

$$W^{n+1} = \begin{cases} X \times I \cup \cup H^i & \text{if } n+1 = 2i \\ X \times I \cup \cup H^i \cup H^{i+1} & \text{if } n+1 = 2i+1 \end{cases}$$

where the intersection and self intersection of the attaching maps are determined by the given element $x \in L_{n+1}(\mathbb{Z}\pi_1(X))$, $n+1 \geq 6$

10.2. **Classification of simply connected n-manifolds $n > 4$ - revisited.** For CAT = PL or TOP the classification for simply-connected manifolds X turned out to be quite manageable (see e.g. [11, 116]). From the above table of Wall groups we note the following: If $\pi_1(X) = 0$, then the Wall's surgery obstruction groups are given by

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \bmod 4 \\ 0 & \text{if } n \equiv 1 \bmod 4 \\ \mathbb{Z}/2 & \text{if } n \equiv 2 \bmod 4 \\ 0 & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

Therefore, for odd dimensional manifolds, $hT_{CAT}(X) = [X, G/CAT]$.
For manifolds having dimension of the type $4k$, $[X, G/CAT] = hT_{CAT}(X) \oplus \mathbb{Z}$.

The determination of $hT_{CAT}(X)$ involve the extension problem :
(E) $$0 \to hT_{CAT}(X) \to [X, G/CAT] \to \mathbb{Z}/2 \to 0,$$
when the dimension of the manifold involved is of type $4k + 2$.

11. A Survey of Classification theorems - Higher (> 4) dimensional manifolds

11.1. **Classification theorems for higher (> 4) dimensional simply connected manifolds - continued.** Let CAT = PL or TOP.

11.1. **Theorem** ([45]). *$hT_{TOP}(S^n \times D^k, \partial) = 0$, if $n + k \geq 5$.*

11.2. **Theorem** ([45]). *$hT_{TOP}(S^n \times S^m) = L_n(\mathbb{Z}) \oplus L_m(\mathbb{Z})$.*

11.3. **Theorem** ([106, 116]). *If X = Complex projective space $\mathbb{C}P^n$, $n > 2$, then*

$$hT_{PL}(X) = \oplus_i H^{4i}(X; \mathbb{Z}) \times \oplus_i H^{4i-2}(X; \mathbb{Z}/2), \text{ if } n \text{ is odd,}$$

$$hT_{PL}(X) \oplus \mathbb{Z} = \oplus_i H^{4i}(X; \mathbb{Z}) \times \oplus_i H^{4i-2}(X; \mathbb{Z}/2), \text{ if } n \equiv 0 (mod 4)$$

$$hT_{PL}(X) \oplus \mathbb{Z}/2 = \oplus_i H^{4i}(X; \mathbb{Z}) \times \oplus_i H^{4i-2}(X; \mathbb{Z}/2), \text{ if } n \equiv 2 (mod 4).$$

11.2. **Classification theorems for higher (> 4) dimensional non simply connected manifolds. (i)** Lopéz de Medrano [51], and Wall [116] have determined $hT_{CAT}(X)$ for X = Real projective space, $\mathbb{R}P^n$, $n > 4$.

11.4. **Theorem** ([51]). *The structure set $hT_{PL}(\mathbb{R}P^n)$ is given by*

$$hT_{PL}(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} \oplus (2l-2)\mathbb{Z}/2, & \text{if } n = 4l+1, \\ \mathbb{Z} \oplus (2l-2)\mathbb{Z}/2, & \text{if } n = 4l+2, \\ \mathbb{Z} \oplus \mathbb{Z} \oplus (2l-2)\mathbb{Z}/2, & \text{if } n = 4l+3, \\ \mathbb{Z} \oplus (2l-1)\mathbb{Z}/2, & \text{if } n = 4l+4. \end{cases}$$

(ii) Kharshiladze [42] has determined $hT_{CAT}(X)$ for X = Product of real projective spaces, $\mathbb{R}P^m \times \mathbb{R}P^n$, $m + n > 4$, $m, n > 0$.

11.5. **Theorem** (([42]). *The structure set $hT_{PL}(X)$, where $X = \mathbb{R}P^m \times \mathbb{R}P^n$, is give by*

$$hT_{PL}(X) = \begin{cases} \Sigma_{i=1}^{r-1} H^{4i}(X) \oplus \Sigma_{i>0}^{4i+2}(X; \mathbb{Z}/2), & m+n = 4r, \\ & m, n = odd, \\ \Sigma_{i \geq 0} H^{4i}(X) \oplus \Sigma_{i \geq 0} H^{4i+2}(X; \mathbb{Z}/2), & m+n = 4r+2, \\ & m, n = odd. \end{cases}$$

11.6. **Theorem** ([42]).

$$hT_{PL}(X) = \begin{cases} \Sigma_{i>0} H^{4i}(X) \oplus \Sigma_{i>0} H^{4i+2}(X;\mathbb{Z}/2), & m = even, \\ & n = odd, \\ \Sigma_{i>0} H^{4i}(X) \oplus \Sigma_{i=0}^{r-1} H^{4i+2}(X;\mathbb{Z}/2), & m+n = 4r+2, \\ & m,n = even, \\ \Sigma_{i=1}^{r-1} H^{4i}(X) \oplus \Sigma_{i=1}^{r-2} H^{4i+2}(X;\mathbb{Z}/2), & m+n = 4r, \\ & m,n = even. \end{cases}$$

(iii) Wall [116] and Kirby-Siebenmann [45] have determined $hT_{CAT}(X)$ for X = Tori $= T^n = S^1 \times ... \times S^1$, n times.

11.7. **Theorem** ([116, 45]).

$$hT_{TOP}(T^n \times D^k, \partial) = 0, \;\; for \quad n+k \geq 5,$$

$$hT_{PL}(T^n \times D^k, \partial) = \begin{cases} 0, & if\, k \neq 3, \\ \mathbb{Z}/2, & if\, k = 3. \end{cases}$$

(iv) The author [73] has determined $hT_{CAT}(X)$, CAT = PL and TOP, for X = **Dold manifolds** $= P(r,s)$, defined as the quotient $(S^r \times \mathbb{C}P^s)/\sim$, where $(x,y) \sim (x',y')$ if and only if $x' = -x$, and $y' = \bar{y}$.

A Dold manifold can also be written as the total space of a fibre bundle over $\mathbb{R}P^r$ with fibre $\mathbb{C}P^s$:

$$(*) \qquad \mathbb{C}P^s \xrightarrow{incl} P(r,s) \xrightarrow{proj} \mathbb{R}P^r.$$

These manifolds form a set of generators of the unoriented cobordism group of closed smooth manifolds.

(v) The author [75] has determined $hT_{CAT}(X)$, CAT = PL and TOP, for X = **Wall's manifold** $Q(r,s)$, defined as the mapping torus of some homeomorphism $A : P(r,s) \to P(r,s)$, of the Dold manifolds.

These manifolds are of importance to cobordism groups of manifolds, and give rise to and intermediate cobordism group between unoriented and oriented cobordism groups and was used by Wall for determination of oriented cobordism ring.

(vi) The author [74] has determined $hT_{CAT}(X)$, CAT = PL and TOP, for X = **Real Milnor manifolds** $= \mathbb{R}H_{r,s}$, defined as the codimension 1 submanifold of $\mathbb{R}P^r \times \mathbb{R}P^s$ given in terms of the homogeneous coordinates of the real projective spaces as

$$\mathbb{R}H_{r,s} \overset{def}{=} \{([z_0, z_1, ..., z_r], [w_0, w_1, ..., w_s]) \mid z_0.w_0 + z_1.w_1 + ... + z_s.w_s = 0\},$$

assuming that $r \geq s$.

A real Milnor manifold can be written as the total space of a fibre bundle $\mathbb{R}P^{r-1} \xrightarrow{incl} \mathbb{R}H_{r,s} \xrightarrow{proj} \mathbb{R}P^s$ with fibre $\mathbb{R}P^{r-1}$. This is actually the projective bundle of the vector bundle $\gamma_r^\perp : \mathbb{R}^r \to E^\perp \to \mathbb{R}P^s$, which is the orthogonal complement in $\mathbb{R}P^s \times \mathbb{R}^{r+1}$ of the line bundle $\gamma : \mathbb{R} \to E \to \mathbb{R}P^s$, $E = \{(x,y) \in \mathbb{R}P^s \times \mathbb{R}^{r+1} \mid y \in x\}$.

These manifolds also form a set of generators of the unoriented cobordism group of closed smooth manifolds.

12. Statements of selected classification results by the author

I am giving statements only for the topological classification, CAT = TOP, for brevity. The results for PL classification, CAT = PL, can be seen from the referred papers.

12.1. Homotopy Dold's manifolds.

12.1. **Theorem.** [73] *(Dold manifolds; TOP Class. theorem* $(4k+1)$*).* $P(r,s), r, s > 1, r+2s = 4k+j,\ j = 1, or\ 2, or\ 3, or\ 4.$ *Then for* $k \geq 1$ *(Coefficients of integral cohomologies are dropped)*

$$hT_{TOP}(P(r,s)^{4k+1}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(P(r,s)).$$

12.2. **Theorem.** [73] *(Dold's manifold; TOP Class. theorem* $(4k+2)$*).* *If* $r \geq 4,\ s \geq 2$ *then*

$$hT_{TOP}(P(r,s)^{4k+2}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(P(r,s)).$$

12.3. **Theorem.** [73] *(Dold's manifold; TOP Class. theorem* $(4k+3)_+$*).*

$$hT_{TOP}(P(r,s)_+^{4k+3}) \cong \mathbb{Z} \oplus \sum_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2) \oplus \mathbb{Z}/2 \oplus \sum_{i=2}^{k} H^{4i}(P(r,s)).$$

12.4. **Theorem.** [73] *(Dold's manifold; TOP Class. theorem* $(4k+3)_-$*).*

$$hT_{TOP}(P(r,s)_-^{4k+3}) \cong \sum\nolimits_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(P(r,s)).$$

12.5. **Theorem.** [73] *(Dold's manifold; TOP Class. theorem* $(4k+4)_+$*).*

$$hT_{TOP}(P(r,s)_+^{4k+4}) \cong \sum\nolimits_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(P(r,s)).$$

12.6. **Theorem.** [73] *(Dold's manifold; TOP Class. theorem* $(4k+1)_-$*).*

$$hT_{TOP}(P(r,s)_-^{4k+4}) \cong \sum_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2) \oplus (\mathbb{Z}/2)^2 \oplus \sum_{i=2}^{k+1} H^{4i}(P(r,s)).$$

12.2. Homotopy real Milnor's manifolds.

12.7. **Theorem.** [74] *(Real Milnor manifolds; TOP Class. theorem* $(4k+1)$*).* $\mathbb{R}H_{r,s},\ r \geq s > 2,\ r+s-1 = 4k+j,\ j = 1,\ or\ 2,\ or\ 3,\ or\ 4.$ *Then for* $k \geq 1$, *(Coefficients of integral cohomologies are dropped for typographic convenience)*

$$hT_{TOP}(\mathbb{R}H_{r,s}^{4k+1}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(\mathbb{R}H_{r,s};\mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(\mathbb{R}H_{r,s}).$$

12.8. **Theorem.** [74] *(Real Milnor's manifold; TOP Class. theorem* $(4k+2)$*).*

$$hT_{TOP}(\mathbb{R}H_{r,s}^{4k+2}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(\mathbb{R}H_{r,s};\mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(\mathbb{R}H_{r,s}).$$

12.9. **Theorem.** [74] *(Real Milnor's manifold; TOP Class. theorem* $(4k{+}3)_+$*).*

$$hT_{TOP}(\mathbb{R}H^{4k+3}_{r,s}{}_{+}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \sum_{i=2}^{k} H^{4i-2}(\mathbb{R}H_{r,s}; \mathbb{Z}/2) \oplus \sum_{i=2}^{k} H^{4i}(\mathbb{R}H_{r,s}).$$

12.10. **Theorem.** [74] *(Real Milnor's manifold; TOP Class. theorem* $(4k{+}3)_-$*).*

$$hT_{TOP}(\mathbb{R}H^{4k+3}_{r,s}{}_{-}) \cong \sum\nolimits_{i=2}^{k+1} H^{4i-2}(\mathbb{R}H_{r,s}; \mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(\mathbb{R}H_{r,s}).$$

12.11. **Theorem.** [74] *(Real Milnor's manifold; TOP Class. theorem* $(4k+4)$*).*

$$hT_{TOP}(\mathbb{R}H^{4k+4}_{r,s}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(\mathbb{R}H_{r,s}; \mathbb{Z}/2) \oplus (\mathbb{Z}/2)^2 \oplus \sum\nolimits_{i=2}^{k+1} H^{4i}(\mathbb{R}H_{r,s}).$$

12.3. Homotopy Wall's manifolds.

12.12. **Theorem.** [75] *(Wall's manifolds; TOP Class. theorem* $(4k+1)$*).* $Q(r,s), r,s > 1, r+2s+1 = 4k+j, j = 1, or\ 2, or\ 3, or\ 4$. *Then for* $k \geq 1$, *(Coefficients of integral cohomologies are dropped)*

$$hT_{TOP}(Q(r,s)^{4k+1}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(Q(r,s); \mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(Q(r,s)).$$

12.13. **Theorem.** [75] *(Wall's manifolds; TOP Class. theorem* $(4k+2)$*).*

$$hT_{TOP}(Q(r,s)^{4k+2}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(Q(r,s); \mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(Q(r,s)).$$

12.14. **Theorem.** [75] *(Wall's manifolds; TOP Class. theorem* $(4k+3)_{-+}$*).*

$$hT_{TOP}(Q(r,s)^{4k+3}_{-+}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(Q(r,s); \mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(Q(r,s)).$$

12.15. **Theorem.** [75] *(Wall's manifolds; TOP Class. theorem* $(4k+3)_{--}$*).*

$$hT_{TOP}(Q(r,s)^{4k+3}_{--}) \cong \sum_{i=2}^{k} H^{4i-2}(Q(r,s); \mathbb{Z}/2) \oplus \mathbb{Z}/2 \oplus \sum_{i=2}^{k} H^{4i}(Q(r,s)).$$

12.16. **Theorem.** [75] *(Wall's manifolds; TOP Class. theorem* $(4k+4)_{-+}$*).*

$$hT_{TOP}(Q(r,s)^{4k+4}_{-+}) \cong \sum\nolimits_{i=2}^{k} H^{4i-2}(Q(r,s); \mathbb{Z}/2) \oplus \mathbb{Z}/2 \times \mathbb{Z}/2 \oplus \sum\nolimits_{i=2}^{k} H^{4i}(Q(r,s)).$$

12.17. **Theorem.** [75] *(Wall's manifolds; TOP Class. theorem* $(4k+4)_{--}$*).*

$$hT_{TOP}(Q(r,s)^{4k+4}_{--}) \cong \sum\nolimits_{i=2}^{k+1} H^{4i-2}(Q(r,s); \mathbb{Z}/2) \oplus \sum\nolimits_{i=2}^{k} H^{4i}(Q(r,s)).$$

12.18. *Remark.* Classification of homotopy Grassmann manifolds and other such manifolds are worthwhile problems. Calculation for the Grassmannian manifolds is under way.

13. APPLICATION OF SURGERY EXACT SEQUENCE IN TOPOLOGICAL RIGIDITY THEOREM

Before closing the survey on higher dimensional manifolds, we would like to mention that there have been reformulations of the surgery exact sequence in terms of algebraic and geometric "$\mathbb{L}$-theory spectra" (see [86, 87, 58]), in which the surgery obstruction map has been replaced by "$\mathbb{L}$-theory assembly maps"

$$A_* : \mathbb{H}_*(X, \mathbb{L}^h_*(pt)) \to \mathbb{L}^h_*(X).$$

Using this reformulation works of Farrell, Jones and many others resulted in proving many special cases of topological rigidity theorem:

13.1. **Theorem.** *(*[19, 20]*) Any homotopy equivalance* $h : (N, \partial N) \to (X \times I^k, \partial(X \times I^k))$ *of compact pairs, which is homeomorphism when restricted to boundaries, is homotopic rel* ∂ *to a homeomorphism. (Here* k *is an integer, and* $I =$ *[0,1]).*

A comprehensive survey "Topological Rigidity Problems" is given in https://arxiv.org/pdf/1510.04139, by R Kasilingam (2015).

14. CLASSIFICATION OF 4-MANIFOLDS IN 1980'S

M. Freedman and S. Donaldson made fundamental advances in the knowledge about 4-manifolds (see [21, 22, 16]).

As mentioned earlier the success of determination of structure set and (limited) classification of higher dimensional manifolds was due to the availability of the Whitney trick, particularly the existence of embedded Whitney 2-discs along which one can isotope to remove pairs of intersection points with opposite signs. This was not so complicated for manifolds of dimension more that 4. However for manifolds of dimension 4 it is an uphill task, as while removing pairs of intersection points with opposite signs it introduces two further such intersection points, and this process continues indefinitely and one gets a tower called "Casson handle" (see e.g. [22]). The credit of Freedman lies in successfully using Casson handles and doing surgery for 4-manifolds .

Freedman proved:

14.1. **Theorem.** *(see* [21, 22]*) Homeomorphism classes of simply connected closed 4-manifolds are in one to one correspondence with the set of pairs*

$$\{([\omega], \alpha) \mid \textit{if } \omega \textit{ is even type, then } Signature(\omega)/8 \equiv \alpha (mod 2)\}$$

where $[\omega]$ *is the isomorphism class of unimodular symmetric bilinear forms on finitely generated free abelian groups, and* α *is the Kirby-Siebenmann invariant (for a* M^4 *as above* $\alpha(M) \in \mathbb{Z}/2$ *such that* $\alpha(M) = 0$ *if* $M \times S^1$ *is smoothable; and* $\alpha(M) = 1$ *otherwise.)*

For $\underline{n = 3, 4}$, one defines a stable structure set $h\widetilde{CAT}(X)$ as follows (see [46]): $\widetilde{hCAT(X)}^{0} = hCAT(X)$, and inductively

$$\widetilde{hCAT(X)}^{r} = \{f \in hCAT(X \# r(S^2 \times S^2)) \mid \eta(f) \in Im\, p^*_X\},$$

where $p_X : X\#r(S^2 \times S^2)/\partial X \to X/\partial X$.

There is a natural map $\widetilde{hCAT(X)}^{r} \to \widetilde{hCAT(X)}^{r+1}$, giving one an inductive system and $\widetilde{hCAT(X)}$ is the inductive limit of this system.

For n = 3, 4, this structure set fits into a Surgery exact sequence:

$$\to L_{n+1}(\mathbb{Z}\pi_1(X)) \xrightarrow{\delta} \widetilde{hCAT(X)} \xrightarrow{\eta} [X, G/CAT] \xrightarrow{\theta} L_n(\mathbb{Z}\pi_1(X)),$$

The failure of surgery is the failure of the natural map $\psi_{CAT} : hCAT(X) \to \widetilde{hCAT(X)}$ to become bijective.

14.2. *Remark.* Freedman has proved that for simply connected manifolds ψ_{TOP} is bijective and so is the case for manifolds with reasonable fundamental groups, e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}/2 * \mathbb{Z}/2$ etc. (see [21, 22]).

Donaldson has shown that ψ_{DIFF} is neither injective (surgery fails) nor surjective (s-cobordism theorem fails) (see [16]).

15. Classification of 3-manifolds in 1960 - 2006

With the proofs of Dehn's lemma and sphere theorem by Papakyriacopoulos the foundation of further development was laid. Based on this Haken [25], Waldhausen [112], Jaco [37, 35], Shalen [37], Johanson [39], gave what is known as the JSJ decomposition of a closed 3-manifold:

Given a closed 3-manifold, one can cut it along embedded essential 2-spheres into finitely many irreducible pieces (irreducible means every embedded 2-sphere bounds a 3-ball).

These irreducible pieces can further be cut along embedded incompressible tori (incompressible means the inclusion map induces an injection on π_1) into finite collection of compact 3-manifolds with toral boundary each of which is either a Seifert fibered space or is simple.

Thurston showed that the simple pieces have geometric structure (The Geometrization conjecture) and proved the conjecture for Haken manifolds (see [110, 94]).

Most of the geometric manifolds were well understood then, except the hyperbolic manifolds (that is manifolds admitting Riemannian metrics of constant negative sectional curvature).

Thurston's work (see [110]) has indicated that (see [56])

(1) a typical 3-manifold is either topologically simple (topological classification exits, like Seifert fibre spaces) or possess a hyperbolic structure.

(2) all but finite number of 3-manifolds obtained by performing Dehn surgery on a given hyperbolic knot (i.e. a knot in S^3 having hyperbolic complement, i.e. neither a torus nor a satelite knot) possess hyperbolic structure.

The consequence of Thurston's work is that hyperbolic 3-manifolds are the most abundant, the most complicated and the most important class of 3-manifolds. So an approach to understand 3-manifolds topologically is to restrict attention to hyperbolic manifolds. The topological invariants like *Euler characteristic, fundamental group, homology and cohomology groups* are inadequate. So the hyperbolic structure was needed by Thurston, Meyerhoff, Ruberman, Jeffry Weeks, Colin Adams and many other mathematicians to define hyperbolic invariants, like *The volume, the Chern-Simon invariant, and the η-invariant* for the classification problem (see [56]). The following theorem of Mostow ensures that these hyperbolic invariants are topological invariants as well.

15.1. **Theorem.** *(Mostow)(see* [72, 18]*) If a closed, orientable 3-manifold possess a hyperbolic structure, then that structure is unique (upto isometry).*

The above invariants individually or collectively are still inadequate to provide complete invariants for the classification of hyperbolic 3-manifolds.

Hamilton and Perelman brought in technique of a completely different shade not thought of until then using geometric analysis, “Ricci flow with surgery” (some thing like heat flow which uniformize the temperature after a certain time, like wise Ricci flow uniformize the Ricci curvature after a certain time) and have given a proof of the geometrization conjecture of which Poincaré’s conjecture is a consequence (see [81, 82, 83, 71]). Perelman received the Fileds medal for this work.

16. Classification of 3-manifolds - current state of affairs

Hamilton-Perelman’s analytic and geometric approach could dispose off topological classification of 3-manifolds homotopy equivalent to 3-sphere (it is a singleton set), the famous Poincaré’s conjecture and the Thurston’s geometrization conjecture.

However, existence and uniqueness of hyperbolic structure on 3-manifold give little information about the structure itself. Given a combinatorial description of a 3-manifold that admits a hyperbolic structure, what can be said about the geometry of that structure ?

Masur, Minsky [53, 54, 9] and many other mathematicians have been working and greatly contributing to the connection between geometric, topological and combinatorial descriptions of hyperbolic 3-manifolds.

Combinatorial objects like surface mapping class groups, curve complexes and their various relatives like Pants complex, Hatcher Thurston complex, homology curve complex etc. (see [17, 34, 93, 92, 4, 5, 34, 38]), play very important role in this project.

Author and Ninthoujam Jiban Singh have contributed in this direction in the form of the paper “Homology Curve Complex” [38]

The main results of the paper “Homology Curve Complex” are:

16.1. **Theorem.** [38] *Given a closed, connected, orientable 3-manifold M and a Heegaard splitting $M = V \cup_g V$ of genus $g > 1$, there is an algorithm, which runs in polynomial time, to decide whether M contains a nonseparating, 2-sided, closed incompressible surface.*

16.2. **Theorem.** [38] *For $g > 1$, there is an explicit algorithm which takes as input Σ_g , a canonical ordered basis $B = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ for the homology $H_1(\Sigma_g; \mathbb{Z})$, a pair of vertices α, β of $HC(\Sigma_g)$ and returns the distance between them.*

16.3. **Theorem.** [38] *Let M be a closed connected orientable 3-manifold with a Heegaard splitting $(V, V'; \Sigma_g)$ of genus $g > 1$. Then for any pair of complete meridian systems L =$\{D_1, \ldots, D_g\}$, L' = $\{D'_1, \ldots, D'_g\}$ for the respective handlebodies V and V', the following statements are equivalent: 1) M contains a non-separating, two-sided, closed incompressible surface; 2) $H_1(M)$ is infinite; 3) The matrix $A = (a_{ij})$, where a_{ij} is the algebraic intersection number of D_i and $\partial D'_j$, is singular; 4) $d^\infty(\Sigma_g) = 0$; 5) $d_H(V, V') = 0$.*

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MODULAR-TYPE TRANSFORMATIONS AND INTEGRALS INVOLVING THE RIEMANN Ξ-FUNCTION

ATUL DIXIT

In memory of the great mathematician Hansraj Gupta

Abstract. A survey of various developments in the area of modular-type transformations (along with their generalizations of different types) and integrals involving the Riemann Ξ-function associated to them is given. We discuss their applications in Analytic Number Theory, Special Functions and Asymptotic Analysis.

1. Introduction

The Jacobi theta function $\theta(z) := \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}$ is one of the most important special functions of Mathematics. At the beginning of the last chapter on theta functions in his book [26, p. 314], Rainville remarks *'It seems safe to say that no topic in Mathematics is more replete with beautiful formulas than that on which we now embark'.* In Mathematics theta functions are encountered in Special Functions, Partial Differential Equations, Number Theory, and, in general, in Science in Heat Conduction, Electrical Engineering, Physics etc.

For $z \in \mathbb{H}$ (upper half plane), the famous theta transformation formula is given by [5, p. 12]

$$\theta\left(-1/4z\right) = \sqrt{-2iz}\,\theta(z),$$

or, equivalently,

$$\sum_{n=-\infty}^{\infty} \exp\left(\pi n^2/2iz\right) = \sqrt{-2iz}\sum_{n=-\infty}^{\infty} \exp\left(2\pi i n^2 z\right). \tag{1.1}$$

This implies [5, p. 12]

$$\theta\left(z/(4z+1)\right) = \sqrt{4z+1}\,\theta(z).$$

Along with the obvious fact $\theta(z+1) = \theta(z)$, this implies that for any $\gamma \in \Gamma_0(4)$,

$$\theta^2(\gamma z) = \chi_{-1}(d)(cz+d)\theta^2(z),$$

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where χ_{-1} is the Dirichlet character modulo 4 defined by $\chi_{-1}(n) = \left(\frac{-1}{n}\right) = (-1)^{(n-1)/2}$. Thus $\theta \in M_{1/2}(\Gamma_0(4), \chi_{-1})$, that is, the theta function is a weight $1/2$ modular form on $\Gamma_0(4)$ twisted by the Dirichlet character χ_{-1}. Even though Eisenstein, and later Hardy, anticipated the theory of modular forms of half integral weight $k/2$, where k is an odd positive integer, a systematic study of such a theory commenced with a seminal paper by Shimura [30].

Letting $z = i\alpha^2/2$ and $\beta = 1/\alpha$, one can easily write (1.1) in a symmetric form, namely, for $\mathrm{Re}(\alpha^2) > 0$, $\mathrm{Re}(\beta^2) > 0$,

$$\sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum\nolimits_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\right) = \sqrt{\beta}\left(\frac{1}{2\beta} - \sum\nolimits_{n=1}^{\infty} e^{-\pi\beta^2 n^2}\right). \tag{1.2}$$

Hardy [20] obtained an integral representation for the left-hand side of (1.2), namely for $\mathrm{Re}(\alpha^2) > 0$,

$$\sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum\nolimits_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\right) = \frac{2}{\pi}\int_0^{\infty} \frac{\Xi(t/2)}{1+t^2}\cos\left(\frac{1}{2}t\log\alpha\right)dt, \tag{1.3}$$

and used (1.2) and (1.3) to prove that infinitely many zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line. Note that the integral in (1.3) is invariant if we replace α by β for $\alpha\beta = 1$. Hence, (1.3) also gives (1.2).

Even though the transformation (1.2) is associated with the modularity of the theta function $\theta(z)$, not all transformations of such type are known to be associated with modular forms. We begin with the following beautiful example from page 220 of Ramanujan's Lost Notebook [28].

Theorem 1.1. *Define* $\lambda(x) := \psi(x) + \frac{1}{2x} - \log x$, *where* $\psi(x)$ *is the logarithmic derivative of the gamma function. Let the Riemann* ξ*-function be defined by*

$$\xi(s) = (1/2)s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s),$$

and let

$$\Xi(t) := \xi(1/2 + it)$$

be the Riemann Ξ*-function. If* α *and* β *are positive numbers such that* $\alpha\beta = 1$, *then*

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\lambda(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\lambda(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt, \tag{1.4}$$

where γ *denotes Euler's constant.*

Note that [1, p. 259, formula 6.3.18] for $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots.$$

This implies that $\lambda(x) = O(x^{-2})$, and hence the series $\sum_{n=1}^{\infty}\lambda(n\alpha)$ and $\sum_{n=1}^{\infty}\lambda(n\beta)$ converge.

This formula was first proved in [2] where the authors gave two proofs. Later in [7], [8], it was obtained as a special case of a more general result which we will soon discuss. A yet another proof was given in [6].

A transformation of the form $\mathfrak{F}(z) = \mathfrak{F}(-1/z), z \in \mathbb{H}$, can be equivalently written in the form $F(\alpha) = F(\beta)$, where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $\alpha\beta = 1$. Indeed, if $\mathrm{Im}(z) > 0$, then letting $\alpha = -iz$ gives $\mathrm{Re}(\alpha) > 0$. Thus, if $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$ and $\alpha\beta = 1$, then $-1/z = i\beta$, so that $\mathrm{Re}(\beta) > 0$. Now let $g(w) = h(e^{2\pi i w})$ so that $g(-1/z) = g(z)$ is equivalent to $h(e^{-2\pi\beta}) = h(e^{-2\pi\alpha})$. Now for $x > 0$, let $F(x) = h(e^{-2\pi x})$, so that $F(\alpha) = F(\beta)$. The process can also be reversed so that the transformation $\mathfrak{F}(z) = \mathfrak{F}(-1/z), z \in \mathbb{H}$, is actually equivalent to $F(\alpha) = F(\beta)$, where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\alpha\beta = 1$.

By a *modular-type transformation*, we mean a relation of the form $F(\alpha) = F(\beta), \alpha\beta = 1$. The word 'modular-type' is used to indicate that there may be some such transformations which cannot be made 'modular' in the sense that they may not be associated to a modular form on $\mathrm{SL}_2(\mathbb{Z})$ or its congruence subgroups. There are umpteen examples of modular-type transformations in Ramanujan's Notebooks [29] as well as in his Lost Notebook [28]. He preferred writing them in the form $F(\alpha) = F(\beta)$ over $\mathfrak{F}(z) = \mathfrak{F}(-1/z)$, such as the one in (1.4), and even though he always considered α, β to be positive real numbers, by analytic continuation, one can almost always extend his identities for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

In this survey, we will also discuss more general modular-type transformations of the form $F(z, \alpha) = F(z, \beta), F(w, \alpha) = F(iw, \beta)$, and $F(z, w, \alpha) = F(z, iw, \beta)$, where $\alpha\beta = 1$ and $i = \sqrt{-1}$.

Using the theory of Mellin transforms and residue calculus, or some ad-hoc techniques from special functions, the integrals involving the Riemann Ξ-function such as the ones in (1.3) and (1.4) can be respectively evaluated to one of the two expressions in a modular-type transformation such as the ones in (1.2) and (1.4) and then the corresponding modular-type transformations can be established through the invariance of the integrals upon replacing α by β. For the results obtained through this approach, see [2], [3], [6], [7], [8], [9] and [13]. Alternatively, one might first establish a modular-type transformation and then link it to an integral involving the Riemann Ξ-function. An indispensable part of this latter approach is the theory of reciprocal functions, and of self-reciprocal functions. Since the results obtained through the former approach are already surveyed in [10], we concentrate on the latter in this survey.

2. MODULAR-TYPE TRANSFORMATIONS AND INTEGRALS OF $\Xi(t)$ THROUGH THE THEORY OF RECIPROCAL FUNCTIONS

We first begin with a generalization of integrals of the type $\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos(\frac{1}{2}t\log\alpha)\,dt$. where $f(t)$ is of the form $f(t) = g(it)g(-it)$ with g analytic in t, in which the cosine is replaced by a more general class of functions [14].

Let $\phi(x)$ and $\psi(x)$ be two integrable functions on the real line. The functions

ϕ and ψ are said to be reciprocal in the Fourier cosine transform if

$$\phi(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \psi(u)\cos(2ux)du \quad \text{and} \quad \psi(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \phi(u)\cos(2ux)du.$$

Define $Z_1(s)$ and $Z_2(s)$ by

$$\Gamma\left(\frac{s}{2}\right)Z_1(s) := \int_0^\infty x^{s-1}\phi(x)dx, \quad \Gamma\left(\frac{s}{2}\right)Z_2(s) := \int_0^\infty x^{s-1}\psi(x)dx,$$

each valid in a specific vertical strip in the complex s-plane. Note that in case of a non-empty intersection of the two corresponding vertical strips, the Mellin inversion theorem gives

$$\phi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{s}{2}\right)Z_1(s)x^{-s}ds, \quad \psi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{s}{2}\right)Z_2(s)x^{-s}ds,$$

where $\mathrm{Re}(s) = c$ lies in the intersection. Here and throughout this paper, by $\int_{(c)}$ we mean $\int_{c-i\infty}^{c+i\infty}$. Let

$$\Theta(x) := \phi(x) + \psi(x) \quad \text{and} \quad Z(s) := Z_1(s) + Z_2(s) \tag{2.1}$$

so that

$$\Gamma\left(\frac{s}{2}\right)Z(s) = \int_0^\infty x^{s-1}\Theta(x)dx$$

for values of s in the intersection of the two strips.

Let $0 < \omega \leq \pi$ and $\lambda < \frac{1}{2}$. If $f(z)$ is such that

i) $f(z)$ is analytic with $z = re^{i\theta}$, regular in the angle defined by $r > 0$, $|\theta| < \omega$,

ii) $f(z)$ satisfies the bounds

$$f(z) = \begin{cases} O(|z|^{-\lambda-\varepsilon}) & \text{if } |z| \text{ is small,} \\ O(|z|^{-b-\varepsilon}) & \text{if } |z| \text{ is large,} \end{cases}$$

for every $\varepsilon > 0$ and $b > \lambda$, and uniformly in any angle $\theta < \omega$, then we say that f belongs to the class K and write $f(z) \in K(\omega, \lambda, b)$.

With this set-up, the following result was proved in [14, Theorem 1.2].

Theorem 2.1. *Let $b > 1$ and $\phi, \psi \in K(\omega, 0, b)$ and let Θ and Z be defined in* (2.1). *Then we have*

$$\int_0^\infty \frac{\Xi(t)}{t^2 + 1/4}Z(1/2 + it)dt = (\pi/2)Z(1) - (\pi/2)\sum_{n=1}^\infty \Theta(n\sqrt{\pi}).$$

This not only gives (1.3) as a special case but also the following general theta transformation along with a general integral involving $\Xi(t)$ [14, Corollary 1.2].

For $\alpha\beta = 1$, $\mathrm{Re}(\alpha^2) > 0$, $\mathrm{Re}(\beta^2) > 0$, and $w \in \mathbb{C}$,

$$\sqrt{\alpha}\left((e^{-\frac{w^2}{8}}/2\alpha) - e^{\frac{w^2}{8}}\sum_{n=1}^\infty e^{-\pi\alpha^2n^2}\cos(\sqrt{\pi}\alpha nw)\right)$$
$$= \sqrt{\beta}\left((e^{\frac{w^2}{8}}/2\beta) - e^{-\frac{w^2}{8}}\sum_{n=1}^\infty e^{-\pi\beta^2n^2}\cosh(\sqrt{\pi}\beta nw)\right)$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\Xi(t/2)}{1+t^2}\nabla\left(\alpha, w, (1+it)/2\right)\, dt, \tag{2.2}$$

where

$$\nabla(x, w, s) := \rho(x, w, s) + \rho(x, w, 1-s),$$

$$\rho(x, w, s) := x^{\frac{1}{2}-s}e^{-\frac{w^2}{8}}{}_1F_1\left((1-s)/2; 1/2; w^2/4\right),$$

with ${}_1F_1(a; c; z)$ being the confluent hypergeometric function.

Though the first equality in (2.2) is known since Jacobi, the integral involving $\Xi(t)$ in (2.2) was first found in [9]. In fact the first equality in (2.2) was obtained by first evaluating this integral to the expression on far left and then utilizing the fact that the integral is invariant under the simultaneous replacement of α by β and w by iw. This is one among the three examples of the generalized modular-type transformation of the form $F(w, \alpha) = F(iw, \beta)$ studied in [9], the other two being generalizations of some results of Ferrar [18] and Hardy [21].

In the last section of his paper [27], Ramanujan considered the integral

$$\mathfrak{I}_1(z, x) = \int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos(\frac{t}{2}\log x)dt}{(z+1)^2+t^2}, \tag{2.3}$$

$x > 0$, and obtained alternate integral representations for it in the regions[1] $\mathrm{Re}(s) > 1$, $-1 < \mathrm{Re}(s) < 1$, $-3 < \mathrm{Re}(s) < -1$. In [7, Theorem 1.4], [8, Theorem 1.5], it was shown that this integral generalizes Ramanujan's result (1.4), thereby giving a generalized modular-type transformation of the type $F(z, \alpha) = F(z, \beta), \alpha\beta = 1$. This result is given below.

Theorem 2.2. *Let* $-1 < \mathrm{Re}(z) < 1$. *Let* $\lambda(z, x) = \zeta(z+1, x) - \frac{1}{2}x^{-z-1} + \frac{x^{-z}}{-z}$, *where* $\zeta(z, x)$ *is the Hurwitz zeta function. Let* $\mathfrak{I}_1(z, x)$ *be defined in* (2.3). *Then for* α, $\beta > 0$, $\alpha\beta = 1$,

$$\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\mathfrak{I}_1(z, \alpha) = \alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^\infty \lambda(z, n\alpha) - \frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{\zeta(z)}{\alpha z}\right)$$

$$= \beta^{\frac{z+1}{2}}\left(\sum_{n=1}^\infty \lambda(z, n\beta) - \frac{\zeta(z+1)}{2\beta^{z+1}} - \frac{\zeta(z)}{\beta z}\right).$$

The integral $\mathfrak{I}_1(z, \alpha)$ involves a product of the Riemann Ξ-function at two different arguments, namely $\Xi(\frac{t+iz}{2})\Xi(\frac{t-iz}{2})$. An integral of a similar type, namely,

$$\mathfrak{I}_2(z, x) := \int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log x\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\, dt \tag{2.4}$$

was studied first in [8]. It is associated to the famous Ramanujan-Guinand formula that will be discussed in the next section.

These examples motivate us, and indeed as will be seen in the next section, it is extremely fruitful to consider a more general integral where the cosine is replaced by a general class of functions. This was done in [15]. We

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[1] Each of the representations for $\mathrm{Re}(s) > 1$ and $-3 < \mathrm{Re}(s) < -1$ involves an extra expression which should not be present. See [7, Theorem 1.2] for the corrected version.

provide below the set-up given in [15], albeit with one extra parameter w, for reasons to be clear soon. However, we first note that while the appropriate kernel with respect to which we study the reciprocal functions for studying integrals of the form $\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)Z\left(\frac{1+it}{2}\right)\,dt$ is the cosine function, the one while studying integrals of the form $\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2}\right)\,dt$ turns out to be $\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})$, where $M_z(x):=\frac{2}{\pi}K_z(x)-Y_z(x)$, with $J_z(x), Y_z(x)$ being the Bessel functions of the first and second kinds respectively and $K_z(x)$ being the modified Bessel function of the second kind.

Let the functions φ and ψ be related by

$$\varphi(x,z,w)=2\int_0^\infty\psi(t,z,w)\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)\,dt,$$

$$\psi(x,z,w)=2\int_0^\infty\varphi(t,z,w)\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)\,dt.$$

Let the normalized Mellin transforms $Z_1(s,z,w)$ and $Z_2(s,z,w)$ of the functions $\varphi(x,z,w)$ and $\psi(x,z,w)$ be defined by

$$\Gamma\left((s-z)/2\right)\Gamma\left((s+z)/2\right)Z_1(s,z,w)=\int_0^\infty x^{s-1}\varphi(x,z,w)\,dx,$$

$$\Gamma\left((s-z)/2\right)\Gamma\left((s+z)/2\right)Z_2(s,z,w)=\int_0^\infty x^{s-1}\psi(x,z,w)\,dx,$$

where each equation is valid in a specific vertical strip in the complex s-plane. Set

$$Z(s,z,w)=Z_1(s,z,w)+Z_2(s,z,w)\quad\text{and}\quad\Theta(x,z,w)=\varphi(x,z,w)+\psi(x,z,w),\tag{2.5}$$

so that

$$\Gamma\left((s-z)/2\right)\Gamma\left((s+z)/2\right)Z(s,z,w)=\int_0^\infty x^{s-1}\Theta(x,z,w)\,dx$$

for values of s which lie in the intersection of the two vertical strips.

We now define a class of functions which will be used in the theorem below. Let $0<\omega\leq\pi$ and $\eta>0$. For fixed z and w, let $u(s,z,w)$ be such that

(i) $u(s,z,w)$ is an analytic function of $s=re^{i\theta}$ regular in the angle defined by $r>0$, $|\theta|<\omega$,

(ii) $u(s,z,w)$ satisfies the bounds

$$u(s,z,w)=\begin{cases}O_{z,w}(|s|^{-\delta}) & \text{if } |s|\leq 1,\\ O_{z,w}(|s|^{-\eta-1-|\mathrm{Re}(z)|}) & \text{if } |s|>1,\end{cases}$$

for every positive δ and uniformly in any angle $|\theta|<\omega$. Then we say that u belongs to the class $\lozenge_{\eta,\omega}$ and write $u(s,z,w)\in\lozenge_{\eta,\omega}$.

With this set-up, the following result was obtained in [15, Theorem 1.2] (see also [11, Equation (1.18)].

Theorem 2.3. *Let $\eta>1/4$ and $0<\omega\leq\pi$. Suppose that $\varphi,\psi\in\lozenge_{\eta,\omega}$, are reciprocal in the Koshliakov kernel, and that $-1/2<\mathrm{Re}(z)<1/2$. Let $Z(s,z,w)$ and $\Theta(x,z,w)$ be defined in* (2.5)*. Let $\sigma_{-z}(n)=\sum_{d|n}d^{-z}$. Then,*

$$\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2},w\right)\frac{dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}$$
$$=\sum\nolimits_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\Theta\left(\pi n, z/2, w\right)-R(z,w),$$

where

$$R(z,w){=}\pi^{z/2}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)Z\left(1{+}\frac{z}{2},\frac{z}{2},w\right){+}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)Z\left(1{-}\frac{z}{2},\frac{z}{2},w\right).$$

This results in the following corollary.

Corollary 2.4. *Let* $-1<\mathrm{Re}(z)<1$. *Let* $\mathfrak{I}_2(z,x)$ *be defined in* (2.4). *Then*

$$\mathfrak{I}_2(z,\alpha)=-(\pi\sqrt{\alpha}/32)\big(\alpha^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\tfrac{z}{2}\right)\zeta(z)+\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\tfrac{-z}{2}\right)\zeta(-z)$$
$$-4\sum\nolimits_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\big). \qquad (2.6)$$

Further integrals of the type $\mathfrak{I}_1(z,x), \mathfrak{I}_2(z,x)$ are studied in [13] and [3, Theorem 15.6]. A companion to Theorem 2.3, which evaluates a generalization of $\mathfrak{I}_1(z,x)$, is also studied in [15, Theorem 1.4].

3. Applications of modular-type transformations and the integrals of $\Xi(t)$ linked to them

Here we discuss three different applications of modular-type transformations and the integrals of $\Xi(t)$ associated to them.

3.1. Theory of the generalized modified Bessel function $K_{z,w}(x)$ and the generalized modular-type transformations $F(z,w,\alpha)=F(z,iw,\beta)$, where $\alpha\beta=1$.

The theta transformation (1.2) can be simply derived by invoking the Poisson summation formula and the Laplace integral evaluation

$$e^{-\alpha^2x^2}=\frac{2}{\alpha\sqrt{\pi}}\int_0^{\infty}e^{-u^2/\alpha^2}\cos(2ux)\,du. \qquad (3.1)$$

In the similar vein, using a generalization of (3.1), namely

$$e^{-\alpha^2x^2}\cos(wx)=\frac{2e^{-w^2/(4\alpha^2)}}{\alpha\sqrt{\pi}}\int_0^{\infty}e^{-u^2/\alpha^2}\cosh(wu/\alpha^2)\cos(2ux)\,du \qquad (w\in\mathbb{C}), \qquad (3.2)$$

one gets the general theta transformation in (2.2). Since the inverse Mellin transform of $\Gamma(s)$ is essentially e^{-x^2}, one may want to ask if one can obtain an integral identity similar to (3.1), which renders $K_0(x)$ as a self-reciprocal function in a kernel, since $K_0(x)$ is essentially the inverse Mellin transform of $\Gamma^2(s)$. More generally one may ask the same question for $K_z(x)$. This was already solved by Koshliakov [23, Equation (8)] who obtained the following remarkable identity for $-1/2<z<1/2$ [2],

$$2\int_0^{\infty}K_z(2t)\left(\cos(\pi z)M_{2z}(4\sqrt{xt})-\sin(\pi z)J_{2z}(4\sqrt{xt})\right)\,dt=K_z(2x). \qquad (3.3)$$

---

[2] It is easy to see that this identity actually holds for $-1/2<\mathrm{Re}(z)<1/2$.

For this reason, the kernel $\cos(\pi z)M_{2z}(4\sqrt{xt}) - \sin(\pi z)J_{2z}(4\sqrt{xt})$ is called the *Koshliakov kernel* in [3] and [15].

Now it is natural to ask if there exists a pair of functions *reciprocal* in the Koshliakov kernel, and which gives (3.3) as a special case, similar to how (3.2) subsumes (3.1). This question was answered in [11]. The interesting thing here is, while generalizing (3.1) to (3.2) still involves elementary functions, namely $e^{-\alpha^2x^2}\cos(wx)$ and $e^{-\alpha^2x^2}\cosh(wx)$, generalizing (3.3) involves a *new* special function $K_{z,w}(x)$, which we call the generalized modified Bessel function. It is defined for $z, w \in \mathbb{C}$, $x \in \mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$ and c=Re$(s) > \pm$ Re(z) by an inverse Mellin transform [11], namely,

$$K_{z,w}(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma((s-z)/2)\,\Gamma((s+z)/2)$$
$$ {}_1F_1\Big((s-z)/2; 1/2; -w^2/4\Big)\, {}_1F_1\Big((s+z)/2; 1/2; w^2/4\Big) 2^{s-2}x^{-s} ds. \quad (3.4)$$

Note that if we let $w = 0$, the generalized modified Bessel function reduces to the modified Bessel function $K_z(x)$. It is shown in [11] that $K_{z,w}(x)$ satisfies a rich and a beautiful theory like its special case $K_z(x)$. The generalization of (3.3) is then given in the following theorem [11, Theorem 1.1].

Theorem 3.1. *Let* $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$. *Let* $w \in \mathbb{C}$ *and* $x > 0$. *Let* α *and* β *be two positive numbers such that* $\alpha\beta = 1$. *The functions* $e^{-\frac{w^2}{2}}K_{z,iw}(2\alpha x)$ *and* $\beta\, K_{z,w}(2\beta x)$ *form a pair of reciprocal functions in the Koshliakov kernel, that is,*

$$e^{-\frac{w^2}{2}}K_{z,iw}(2\alpha x) = 2\int_0^\infty \beta\, K_{z,w}(2\beta t)\Big(\cos(\pi z)M_{2z}(4\sqrt{xt}) - \sin(\pi z)J_{2z}(4\sqrt{xt})\Big)\, dt,$$

$$\beta\, K_{z,w}(2\beta x) = 2\int_0^\infty e^{-\frac{w^2}{2}}K_{z,iw}(2\alpha t)\Big(\cos(\pi z)M_{2z}(4\sqrt{xt}) - \sin(\pi z)J_{2z}(4\sqrt{xt})\Big)\, dt.$$

However, we emphasize here that we stumbled upon this interesting generalization of the modified Bessel function while seeking a generalization of a formula of Ramanujan [28, p. 253] rediscovered by Guinand [19]. For $\alpha\beta = \pi^2$, this formula is given by

$$\sqrt{\alpha}\sum\nolimits_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2n\alpha) - \sqrt{\beta}\sum\nolimits_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{z}{2}\right)\zeta(z)\{\beta^{(1-z)/2}-\alpha^{(1-z)/2}\}+\frac{1}{4}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\{\beta^{(1+z)/2}-\alpha^{(1+z)/2}\}. \quad (3.5)$$

This formula can be written symmetrically in α and β [8, Theorem 1.4], and is, in this latter form, an example of the generalized modular-type transformation of the type $F(z, \alpha) = F(z, \beta)$. As discussed in [4, p. 23], this identity is equivalent to the functional equation of the non-holomorphic Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$. In [8], (3.5) was derived from (2.6) whereas in [15], Theorem 2.3 and (3.5) are used to obtain (2.6).

The elegant generalization of the Ramanujan-Guinand formula, symmetric in α and β, that was established in [11, Theorem 1.5] is now given.

Theorem 3.2. *Let* $w \in \mathbb{C}$, $z \in \mathbb{C}\backslash\{-1,1\}$. *For* $\alpha, \beta > 0$ *such that* $\alpha\beta = 1$,

$$\sqrt{\alpha}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2n\pi\alpha)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)$$

$$-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\Bigg)$$

$$=\sqrt{\beta}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2n\pi\beta)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\beta^{\frac{z}{2}-1}$$

$${}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right)-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\beta^{-\frac{z}{2}-1}{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)\Bigg). \tag{3.6}$$

This is an example of a generalized modular-type transformation of the form $F(z,w,\alpha) = F(z,iw,\beta)$, where $\alpha\beta = 1$. Indeed, (3.5) follows at once from (3.6) by letting $w = 0$.

Let $\nabla_2(x,z,w,s)$ be defined by

$$\nabla_2(x,z,w,s) := \rho(x,z,w,s) + \rho(x,z,w,1-s), \tag{3.7}$$

where

$$\rho(x,z,w,s) := x^{\frac{1}{2}-s}{}_1F_1\left(\frac{1-s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{1-s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right).$$

Using the reciprocal pair $(e^{-w^2/2}K_{z,iw}(2\alpha x), \beta\, K_{z,w}(2\beta x)), \alpha\beta = 1$, in Theorem 2.3 along with (3.6), the integral involving $\Xi(t)$ corresponding to the expressions in (3.6) was obtained [11, Theorem 1.3] as shown below.

Theorem 3.3. *Let* $w \in \mathbb{C}$ *and* $-1 < \mathrm{Re}(z) < 1$. *Let* $K_{z,w}(x)$ *and* $\nabla_2(x,z,w,s)$ *be defined in* (3.4) *and* (3.7) *respectively. If* α *and* β *are positive integers satisfying* $\alpha\beta = 1$, *then*

$$\frac{16}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\nabla_2\left(\alpha,\frac{z}{2},w,\frac{1+it}{2}\right)\,dt}{(t^2+(z+1)^2)\,(t^2+(z-1)^2)}$$

$$= e^{-\frac{w^2}{4}}\sqrt{\alpha}\Bigg\{4\sum\nolimits_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2n\pi\alpha)$$

$$-\Gamma(z/2)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1((1-z)/2;1/2;w^2/4)$$

$$-\Gamma(-z/2)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}\,{}_1F_1((1+z)/2;1/2;w^2/4)\Bigg\}.$$

3.2. **A far-reaching generalization of Hardy's theorem on infinitude of zeros of $\zeta(s)$ on the critical line.**

This sub-section illustrates an application of a modular-type transformation associated with an integral involving $\Xi(t)$, this time the general theta transformation (2.2), in analytic number theory.

As mentioned in the introduction, Hardy [20] proved in 1914 that infinitely many zeros of $\zeta(s)$ lie on the critical line using (1.2) and (1.3). Let

$$\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \quad \text{and} \quad \rho(t) := \eta(1/2+it).$$

In [14], we generalized Hardy's result by showing that infinitely many zeros of an infinite series whose summands involve the completed zeta function $\rho(t)$ on bounded vertical shifts lie on the critical line too. The precise theorem is now given.

Theorem 3.4. *Let* $\{c_j\}$ *be a sequence of non-zero real numbers so that* $\sum_{j=1}^{\infty}|c_j| < \infty$. *Let* $\{\lambda_j\}$ *be a bounded sequence of distinct real numbers that attains its bounds. Then the function* $F(s) = \sum_{j=1}^{\infty} c_j\eta(s+i\lambda_j)$ *has infinitely many zeros on the critical line* $\mathrm{Re}(s) = 1/2$.

The above theorem also uses (1.2) and (1.3). Hardy's result is simply its special case when all but one c_j's are zero and the remaining non-zero c_j is 1.

Now a natural question arises - can one generalize the above theorem where one uses the general theta transformation (2.2) rather than (1.2) and (1.3)? Indeed, this can be done. It led to the following result that appeared in [12, Theorem 2].

Theorem 3.5. *Let* $\{c_j\}$ *be a sequence of non-zero real numbers so that* $\sum_{j=1}^{\infty}|c_j| < \infty$. *Let* $\{\lambda_j\}$ *be a bounded sequence of distinct real numbers such that it attains its bounds. Let* $\mathfrak{D}$ *denote the region* $|\mathrm{Re}(w) - \mathrm{Im}(w)| < \sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}}\,\mathrm{Re}(w)\mathrm{Im}(w)$ *in the* w*-complex plane. Then for any* $w \in \mathfrak{D}$, *the function*

$$F_w(s)=\sum_{j=1}^{\infty} c_j\eta(s+i\lambda_j)\left\{{}_1F_1\left(\frac{1-(s+i\lambda_j)}{2};\frac{1}{2};\frac{w^2}{4}\right)+{}_1F_1\left(\frac{1-(\bar{s}-i\lambda_j)}{2};\frac{1}{2};\frac{\bar{w}^2}{4}\right)\right\}$$

has infinitely many zeros on the critical line $\mathrm{Re}(s) = 1/2$.

3.3. **Asymptotic expansion of an integral involving** $\Xi(t)$**.** The advantage of having an alternate representation for an expression, that is, an identity, is that it may give more information about the expression thereby enhancing our understanding of it. This sub-section bears a testimony to an instance of such a phenomenon.

In [13, Theorem 6.3], the integral $\mathfrak{I}_1(z,x)$, defined in (2.3), was expressed as a Laplace transform:

Theorem 3.6. *Assume* $-1 < \mathrm{Re}(z) < 1$. *Define* $\Omega(x,z)$ *by*

$$\Omega(x,z) = 2\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\left(e^{\pi iz/4}K_z(4\pi e^{\pi i/4}\sqrt{nx}) + e^{-\pi iz/4}K_z(4\pi e^{-\pi i/4}\sqrt{nx})\right),$$

where $\sigma_{-z}(n) = \sum_{d|n} d^{-z}$. *Then for* $\alpha,\beta > 0, \alpha\beta = 1$,

$$\begin{aligned}\frac{1}{2\pi^{(z+5)/2}}\mathfrak{I}_1(z,\alpha) &= \alpha^{(z+1)/2}\int_0^{\infty} e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx\\ &= \beta^{(z+1)/2}\int_0^{\infty} e^{-2\pi\beta x}x^{z/2}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx.\end{aligned}$$

Applying Watson's lemma to the first expression for $\mathfrak{I}_1(z,\alpha)$ involving α led us to its following asymptotic expansion [17, Theorem 1.10]:

Theorem 3.7. *Fix* z *such that* $-1 < \mathrm{Re}\ z < 1$. *As* $\alpha \to \infty$,

$$\frac{1}{\pi^{(z+3)/2}}\mathfrak{I}_1(z,\alpha) \sim -\frac{\Gamma(z)\zeta(z)\alpha^{\frac{z-1}{2}}}{(2\pi)^z} - \frac{\Gamma(z+1)\zeta(z+1)}{2\alpha^{\frac{z+1}{2}}(2\pi)^z}$$

$$+ 2\alpha^{\frac{z+1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2\pi\alpha)^{2m+z+2}} \Gamma(2m+2+z)\zeta(2m+2)\zeta(2m+z+2).$$

Oloa's asymptotic expansion[3] [24, Equation 1.5] of $\mathfrak{I}_1(0,\alpha)$, namely, as $\alpha \to \infty$,

$$\frac{1}{\pi^{3/2}}\mathfrak{I}_1(0,\alpha) \sim \frac{1}{2}\frac{\log\alpha}{\sqrt{\alpha}} + \frac{1}{2\sqrt{\alpha}}(\log 2\pi - \gamma) + \frac{\pi^2}{72\alpha^{3/2}} - \frac{\pi^4}{10800\alpha^{7/2}} + \cdots,$$

can be readily obtained by letting $z \to 0$ in (3.7).

4. CONCLUDING REMARKS AND FURTHER QUESTIONS

We hope to have demonstrated the usefulness of modular-type transformations along with the associated integrals involving $\Xi(t)$. It would be remarkable if one is able to associate at least some of them to modular forms.

While it may seem from the variety of examples considered here that one can always associate an integral involving $\Xi(t)$ to a modular-type transformation, there are some conjectured modular-type transformations for which there are no such integral representations. For example, consider the following remarkable conjecture of Hardy and Littlewood [22, p. 158, Equation (2.516)] suggested to them by work of Ramanujan.

Conjecture 4.1. *Let $\mu(n)$ denote the Möbius function. Let α and β be two positive numbers such that $\alpha\beta = 1$. Assume that the series $\sum_\rho \left(\Gamma((1-\rho)/2)/\zeta'(\rho)\right) a^\rho$ converges, where ρ runs through the non-trivial zeros of $\zeta(s)$ and a denotes a positive real number, and that the non-trivial zeros of $\zeta(s)$ are simple. Then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}(\mu(n)/n)e^{-\pi\alpha^2/n^2} - \frac{1}{4\sqrt{\pi}\sqrt{\alpha}}\sum_\rho \frac{\Gamma((1-\rho)/2)}{\zeta'(\rho)}\pi^{\frac{\rho}{2}}\alpha^\rho$$
$$= \sqrt{\beta}\sum_{n=1}^{\infty}(\mu(n)/n)e^{-\pi\beta^2/n^2} - \frac{1}{4\sqrt{\pi}\sqrt{\beta}}\sum_\rho \frac{\Gamma((1-\rho)/2)}{\zeta'(\rho)}\pi^{\frac{\rho}{2}}\beta^\rho.$$

A generalization of this conjecture was obtained in [9, Theorem 1.6] which led to a Riesz-type criterion for the Riemann Hypothesis in [16, Theorem 1.1].

Let erf(w) and erfi(w) denote the error function and the complementary error function respectively. In view of the remark made before the conjecture (4.1), we do like to point out that there is a modular-type transformation obtained in [17, Equation (1.18)], namely

$$\sqrt{\alpha}e^{\frac{w^2}{8}}\left(\operatorname{erf}\left(\frac{w}{2}\right) + 4\int_{-\infty}^{0}\frac{e^{-\pi\alpha^2x^2}\sin(\sqrt{\pi}\alpha xw)}{e^{2\pi x}-1}\,dx\right)$$
$$= \sqrt{\beta}e^{\frac{-w^2}{8}}\left(\operatorname{erfi}\left(\frac{w}{2}\right) + 4\int_{-\infty}^{0}\frac{e^{-\pi\beta^2x^2}\sinh(\sqrt{\pi}\beta xw)}{e^{2\pi x}-1}\,dx\right), \qquad (4.1)$$

---

[3] There is a slight misprint in this asymptotic expansion given in Oloa's paper. The minus sign in front of the second expression on the right-hand side there should be a plus. This has been corrected here.

whose expressions, we believe, are equal to an integral involving $\Xi(t)$. However, we are unable to find this integral. If it exists, it would be significant, as it would enable us to find an integral involving $\Xi(t)$ for the modular-type transformation corresponding to an integral analogue of the Jacobi theta function. See [17, p. 32] for a discussion on this topic.

In [17, Section 7], two questions were posed regarding the exact evaluation of

$$\int_0^\infty \frac{xe^{-\pi x^2}}{e^{2\pi x}-1} {}_1F_1(-2k; \tfrac{3}{2}; 2\pi x^2)\, dx$$

for $k \in \mathbb{Z}^+ \cup \{0\}$, and an exact evaluation of, or at least an approximation to

$$\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1} {}_1F_1\left(-2k-1; \frac{3}{2}; 2\alpha x^2\right) dx$$

when $\alpha \neq \pi$ is a positive real number and $k \in \mathbb{Z}^+ \cup \{0\}$. These integrals resulted from differentiating some modular type transformations of the form $F(w, \alpha) = F(iw, \beta), \alpha\beta = 1$, involving the error functions. These questions were recently solved partially by Paris [25] who obtained approximations of the integrals to within exponentially small accuracy when k is large and $\alpha = O(1)$.

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ON THE TOPOLOGY OF CERTAIN MATRIX GROUPS

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Abstract. We illustrate how some basic algebraic properties of certain real and complex classical matrix groups have a significant say in the analysis of their topological structures.

1. Introduction

The groups of invertible square matrices over the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, also known as the *general linear groups*, play an important role in different branches of mathematics like linear algebra, field theory, Lie groups, differential geometry, representation theory, harmonic analysis, operator algebras and non commutative geometry, to name a few. Various subgroups of the general linear groups, *e.g.*, the *special linear groups, the orthogonal and special orthogonal groups, the unitary and special unitary groups* and the *symplectic groups* have attracted the attention of some of the best minds in the world of mathematics for decades, and have also found significance in physics. These subgroups are usually referred to as the *classical linear groups.* Operator algebraists have in fact developed the quantum versions of most of these classical groups.

Interestingly, apart from their applications to different areas, studying the topological properties of matrix groups is itself quite significant and occupies prominent space in mathematics. In this short article, we make an attempt to show how the linear algebraic results that we learn at undergraduate level turn out to provide deep implications towards the analysis of the topological structures of these classical groups.

2. Preliminaries

Throughout this article, $\mathbb{K}$ will denote either the field $\mathbb{R}$ or the field $\mathbb{C}$ with the usual metric $d : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ given by $d(x, y) = |x - y|$ and $M_n(\mathbb{K})$ will denote the space of $n \times n$ matrices with entries from $\mathbb{K}$.

2.1. **Topology on the matrix groups.** Recall that there is a natural identification between $M_n(\mathbb{K})$ and $\mathbb{K}^{n^2}$ via the canonical map

$$M_n(\mathbb{K}) \ni [a_{ij}] \mapsto (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots a_{nn}) \in \mathbb{K}^{n^2}.$$

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Under this identification, $M_n(\mathbb{K})$ becomes a metric space with the usual metric. There are some natural continuous maps from and into $M_n(\mathbb{K})$. For instance, if $p_{rs} \in \mathbb{K}[x_{11}, x_{12}, \ldots, x_{mm}]$, $1 \leq r, s \leq n$, is a collection of n^2 polynomials in m^2 variables, then the map $M_m(\mathbb{K}) \ni [a_{ij}] \mapsto [p_{rs}(a_{11}, a_{12}, \ldots, a_{mm})] \in M_n(\mathbb{K})$ is continuous. And, if X is a metric space and $\varphi_{rs} : X \to \mathbb{K}$, $1 \leq r, s \leq n$ is a collection of n^2 continuous functions, then the map $X \ni x \mapsto [\varphi_{rs}(x)] \in M_n(\mathbb{K})$ is continuous. In particular the determinant function, being a polynomial function of the entries of a matrix, is continuous.

We aim to study the compactness and connectedness of the multiplicative group of invertible square matrices

$$GL(n, \mathbb{K}) := \{A \in M_n(\mathbb{K}) : A \text{ is invertible}\} \qquad \text{(general linear group)},$$

and its subgroups

$$\begin{aligned}
SL(n,\mathbb{K}) &= \{A \in GL(n,\mathbb{K}) : \det(A) = 1\}, && \text{(special linear group)}\\
O(n) &= \{A \in GL(n,\mathbb{R}) : AA^T = I_n = A^T A\}, && \text{(orthogonal group)}\\
SO(n) &= \{A \in O(n) : \det(A) = 1\}, && \text{(special orthogonal group)}\\
U(n) &= \{U \in GL(n,\mathbb{C}) : UU^* = I_n = U^*U\}, && \text{(unitary group)}\\
SU(n) &= \{U \in U(n) : \det(U) = 1\}, && \text{(special unitary group)}\\
Sp(n,\mathbb{K}) &= \{A \in GL(2n,\mathbb{K}) : A^T J_n A = J_n\}, && \text{(symplectic group)},
\end{aligned}$$

where A^T and U^* denote the transpose of A and conjugate transpose of U, respectively, and $J_n = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$. It follows from their definitions that $O(n)$ and $U(n)$ are closed under multiplication as well as inversion; and hence, they form multiplicative groups. Further, for a matrix A with determinant 1, multiplicativity of the determinant function yields $\det(A^{-1}) = 1$. In particular, $SL(n, \mathbb{K}), SO(n)$ and $SU(n)$ are all multiplicative groups. That the symplectic matrices form a multiplicative group will be shown in Section 5.

2.2. **Some gems from the world of linear algebra.**

We now recall some algebraic properties of the matrix algebra and its subsets (mostly without proof) which will be used in analyzing the topological structures of multiplicative groups of invertible matrices.

The Euclidean spaces admit natural inner products given by $\langle v, w\rangle = w^T v$ for all $v, w \in \mathbb{R}^n$ and $\langle v, w\rangle = w^* v$ for all $v, w \in \mathbb{C}^n$, where we have treated the vectors v and w as column vectors. We would require the following definition of a positive semidefinite matrix and the subsequent results, the details of which may be found in any standard text of linear algebra, see [1, 3] for instance.

Definition 2.1. *A square matrix $P \in M_n(\mathbb{K})$ is said to be positive semidefinite if $\langle Px, x\rangle \geq 0$ for all $x \in \mathbb{K}^n$.*

Remark. For any $A \in M_n(\mathbb{C})$, A^*A is positive semidefinite and likewise for any $A \in M_n(\mathbb{R})$, $A^T A$ is positive semidefinite and it is a fact that these are the only positive semidefinite matrices possible.

Proposition 2.2. *Let* $P \in M_n(\mathbb{R})$ *be a positive semidefinite matrix. Then* P *is symmetric,* $\det(P) \geq 0$ *and there exists a unique postive semidefinite matrix* $P^{1/2} \in M_n(\mathbb{R})$ *such that* $(P^{1/2})^2 = P$ *and* $P^{1/2}$ *is invertible if and only if* P *is so.*

We now prove the following important result also known as the *polar decomposition* of determinant one real matrices.

Theorem 2.3. *Let* $A \in SL(n, \mathbb{R})$. *Then there exists a matrix* $R \in SO(n)$ *and a real, symmetric and positive semidefinite matrix* $P \in SL(n, \mathbb{R})$ *such that* $A = RP$.

Proof. There is an obvious candidate for P, namely $P = (A^T A)^{1/2}$ and this forces R to be defined as $R = AP^{-1}$. Clearly, P is real, symmetric and positive semidefinite matrix. Further,

$$RR^T = AP^{-1}P^{-1}A^T = AP^{-2}A^T = A(A^TA)^{-1}A^T = I_n$$

and similarly $R^TR = I_n$ - implying that R is orthogonal. Hence, $1 = \det(R^TR) = \det(R^T)\det(R) = \det(R)^2$ so that $\det(R) = 1$ or -1. Now, $1 = \det(A) = \det(R)\det(P)$ and, by Proposition 2.2 and the fact that P is invertible, we have $\det(P) > 0$. Therefore, we must have $\det(R) = 1$, i.e., $R \in SO(n)$. ☐

We now state a result that justifies the name *rotation matrices* for the elements of $SO(n)$ - a proof of which can be found in [1, Theorem 6.39].

Proposition 2.4. *Any matrix in* $SO(n)$ *is orthogonally similar to a block diagonal of the form* $A_1 \oplus A_2 \oplus \cdots \oplus A_r$, *where each* A_i *is* (1) *or a* 2×2 *rotation matrix of the type* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A(\theta)$ *say, for some* $\theta \in \mathbb{R}$.

3. Real Classical groups

In this section, we discuss the topological properties of the real general linear group $GL(n, \mathbb{R})$ and its subgroups $SL(n, \mathbb{R}), O(n)$ and $SO(n)$.

Observe that $GL(1, \mathbb{R}) \cong (-\infty, 0) \cup (0, \infty)$ is open, non-compact and disconnected. Interestingly, the same properties hold in higher dimensions as well.

Proposition 3.1. *(1).* $GL(n, \mathbb{R})$ *is open and unbounded. (2).* $GL(n, \mathbb{R})$ *is not connected.*

Proof. (1). The complement of $GL(n, \mathbb{R})$ in $M_n(\mathbb{R})$ is the set $\{A \in M_n(\mathbb{R}) : \det A = 0\}$. Since determinant is a continuous function and $\{0\}$ is closed in $\mathbb{R}$, $M_n(\mathbb{R}) \setminus GL(n, \mathbb{R})$ is closed and hence $GL(n, \mathbb{R})$ is open in $M_n(\mathbb{R})$. Also, $kI_n \in GL(n, \mathbb{R})$ for all $k > 0$. Therefore $GL(n, \mathbb{R})$ is unbounded - implying that $GL(n, \mathbb{R})$ is not compact.

(2). Note that $\det : GL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is a surjective continuous map and $\mathbb{R} \setminus \{0\}$ is not connected. Since a continuous image of a connected set must be connected, $GL(n, \mathbb{R})$ cannot be connected. ☐

We shall, in fact, show that $GL(n, \mathbb{R})$ has precisely two (path) components, namely, $GL_+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) > 0\}$ and $GL_-(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) < 0\}$. However, in order to achieve this, we will first have to analyze the topological properties of some of its subgroups.

Proposition 3.2. *The groups $O(n)$ and $SO(n)$ are compact.*

Proof. Write any matrix $A \in O(n)$ as $\Big(v_1, v_2, \cdots v_n\Big)^T$, where each v_i is a row matrix. Then from the identity $AA^T = I_n$, we get $v_i v_i^T = 1$ for all $1 \leq i \leq n$. This implies that A is inside the closed ball of radius $\sqrt{n}$ of $\mathbb{R}^{n^2}$. Therefore $O(n)$ is a bounded subset of the Euclidean space $\mathbb{R}^{n^2}$. Let $\{A_k\}$ be any sequence in $O(n)$ and suppose $A_k \to A$ in $M_n(\mathbb{R})$. Taking limit as $k \to \infty$ in the relation $A_k A_k^T = A_k^T A_k = I_n$, by continuity of multiplication, we get $AA^T = A^T A = I_n$ proving that $A \in O(n)$. Thus $O(n)$ is closed too. Hence, by the Heine-Borel theorem, $O(n)$ is compact. If, in addition, each of the matrices A_k above have determinant 1 then by continuity of the determinant, we also see that $\det A = 1$, which shows that $SO(n)$ is closed in $O(n)$ and hence compact. ☐

Theorem 3.3. *$O(n)$ is not connected whereas $SO(n)$ is path connected.*

Proof. Let $M \in O(n)$. Then, as seen in Theorem 2.3, $\det(M) \in \{1, -1\}$. Let

$$O_{\pm}(n) := \{M \in O(n) : \det(M) = \pm 1\} = GL_{\pm}(n, \mathbb{R}) \cap O(n).$$

Then $O_+(n)$, which is the same as the subgroup $SO(n)$, and $O_-(n)$ are open in the subspace topology and they form a disconnection of $O(n)$, implying that $O(n)$ is not connected.

Note that in order to show that $SO(n)$ is path connected, using the reverse of a path and concatenation of two paths, it is enough to show that any matrix in $SO(n)$ is joined to I_n by a path. Since $SO(1) = \{(1)\}$, let us assume that $n \geq 2$. Let $R \in SO(n)$. Then, by Proposition 2.4, there exists an orthogonal matrix $M \in O(n)$ such that

$$MRM^T = A_1 \oplus A_2 \oplus \cdots \oplus A_r,$$

where each A_i is (1) or a 2×2 rotation matrix of the type $A(\theta)$ for some $\theta \in \mathbb{R}$. Without loss of generality, assume that, for some $k \leq r$, for $1 \leq i \leq k$, A_i=$A(\theta_i)$ for some $\theta_i \in \mathbb{R}$, and that $A_i = (1)$ for $k < i \leq r$. We can now look for an appropriate path. For each $1 \leq i \leq k$, consider the map $\varphi_i : [0,1] \to SO(2)$ given by $\varphi_i(t) = A(t\theta_i)$. Then each φ_i is a path in $SO(2)$ with end points I_2 and $A(\theta_i)$. Therefore, the map $\varphi : [0,1] \to SO(n)$ given by

$$\varphi(t) = M^T\big(\varphi_1(t) \oplus \varphi_2(t) \oplus \cdots \oplus \varphi_k(t) \oplus I_{n-2k}\big)M$$

is a path in $SO(n)$ with end points $\varphi(0) = I_n$ and $\varphi(1) = R$. ☐

Corollary 3.4. *$O(n)$ has precisely two path components, namely, $O_+(n)$ and $O_-(n)$.*

Proof. By Theorem 3.3, $O_+(n) = SO(n)$ is path connected. Now, let $A, B \in O_-(n)$ and fix a $C \in O_-(n)$. Then $AC, BC \in O_+(n)$ and, therefore, there exists a path φ in $O_+(n)$ joining AC and BC. Consider the map $\tilde{\varphi} : [0,1] \to O_-(n)$ given by $\tilde{\varphi}(t) = \varphi(t)C^{-1}$. Then $\tilde{\varphi}$ is a path in $O_-(n)$ joining A and B.

Also, we know that $O(n)$ is a disjoint union of $O_+(n)$ and $O_-(n)$, so these are the only two path components of $O(n)$. ☐

Note that $SL(1,\mathbb{R}) = \{(1)\}$ is clearly path connected and compact. However, in higher dimensions compactness takes a back seat.

Corollary 3.5. *$SL(n,\mathbb{R})$ is closed, path connected and is not compact for $n \geq 2$.*

Proof. Since $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous and $SL(n,\mathbb{R}) = \det^{-1}(\{1\})$, it is closed. It is not bounded as it contains $SL(2,\mathbb{R})$ which contains the matrices $\begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$ for all $r > 0$.

For path connectedness, it is again enough to show that I_n is connected by a path to any other matrix in $SL(n,\mathbb{R})$. Let $A \in SL(n,\mathbb{R})$. Then, by Theorem 2.3, there exists an $R \in SO(n)$ and a real, symmetric and positive semidefinite matrix $P \in SL(n,\mathbb{R})$ such that $A = RP$. By Theorem 3.3, $SO(n)$ is path connected, so there exists a path $\varphi : [0,1] \to SO(n) \subset SL(n,\mathbb{R})$ with end points $\varphi(0) = I_n$ and $\varphi(1) = R$. Then the map $\tilde{\varphi} : [0,1] \to SL(n,\mathbb{R})$ given by $\tilde{\varphi}(t) = \varphi(t)P$ is a path in $SL(n,\mathbb{R})$ with end points $\tilde{\varphi}(0) = P$ and $\tilde{\varphi}(1) = RP = A$.

It now suffices to show that there exists a path in $SL(n,\mathbb{R})$ with end points I_n and P for then a path from I_n to A would be obtained by concatenating the paths from I_n to P and from P to A.

Since P is a symmetric matrix, there exists an orthogonal matrix Q such that QPQ^{-1} equals the diagonal matrix $D := \mathrm{diag}(r_1, r_2, \ldots, r_n)$, where $r_1, r_2, \ldots, r_n$ are the eigenvalues of P. Since P is positive semidefinite and invertible, $r_i > 0$, so that $1 + t(r_i - 1) > 0$, for all $1 \leq i \leq n$ and $0 \leq t \leq 1$. The map $\psi : [0,1] \to GL_+(n,\mathbb{R})$ given by

$$\psi(t) = \mathrm{diag}\Big(1 + t(r_1 - 1), 1 + t(r_2 - 1), \ldots, 1 + t(r_n - 1)\Big)$$

is a path with end points $\psi(0) = I_n$ and $\psi(1) = D$. Then, $(1/\sqrt[n]{\det(\psi(t))}\,)Q\psi(t)Q^{-1} \in SL(n,\mathbb{R})$ for all $0 \leq t \leq 1$, so that the map

$$[0,1] \ni t \mapsto (1/\sqrt[n]{\det(\psi(t))}\,)Q\psi(t)Q^{-1} \in SL(n,\mathbb{R})$$

is a path in $SL(n,\mathbb{R})$ joining I_n and P. □

We now have the required tools to show that $GL(n,\mathbb{R})$ has precisely two components, namely, $GL_+(n,\mathbb{R})$ and $GL_-(n,R)$.

Corollary 3.6. *$GL_+(n,\mathbb{R})$ and $GL_-(n,\mathbb{R})$ are path connected and these are the only two path components of $GL(n,\mathbb{R})$.*

Proof. Let $A \in GL_+(n,R)$. Then $\tilde{A} = (\ 1/\sqrt[n]{\det(A)}\ )A \in SL(n,\mathbb{R})$. So, by path connectedness of $SL(n,\mathbb{R})$, there exists a path φ in $SL(n,\mathbb{R}) \subset GL_+(n,\mathbb{R})$ with end points I_n and $\tilde{A}$. Also, there is an obvious path in $GL_+(n,\mathbb{R})$ with end points $\tilde{A}$ and A, namely, $[0,1] \ni t \to \Big((1-t)/\sqrt[n]{\det(A)} + t\Big)\, A \in GL_+(n,\mathbb{R})$. Therefore, $GL_+(n,\mathbb{R})$ is path connected.

The fact that $GL_-(n,\mathbb{R})$ is also path connected follows on the lines of the proof of path connectedness of $O_-(n)$ as in Corollary 3.4.

Also, we know that $GL(n,\mathbb{R})$ is a disjoint union of $GL_+(n,\mathbb{R})$ and $GL_-(n,\mathbb{R})$ so these are the only two path components of $GL(n,\mathbb{R})$. □

4. Complex Classical groups

In this section, we take up the complex general linear group $GL(n,\mathbb{C})$ and its subgroups $SL(n,\mathbb{C})$, $U(n)$ and $SU(n)$. The elements of $U(n)$ are called unitary matrices and satisfy the equivalent angle preserving property:

$$\langle Uv, Uw\rangle = (Uw)^*(Uv) = w^*U^*Uv = w^*v = \langle v, w\rangle, \text{ for all } v, w \in \mathbb{C}^n$$

Proposition 4.1. *(1) $GL(n,\mathbb{C})$ is open and unbounded. (2) $GL(n,\mathbb{C})$ is path connected.*

Proof. (1) Since $\det : M_n(\mathbb{C}) \to \mathbb{C}$ is continuous and $A \in GL(n,\mathbb{C})$ if and only if $\det(A) \neq 0$, we see that $GL(n,\mathbb{C}) = \det^{-1}(\mathbb{C}\setminus\{0\})$ is open. Since $GL(n,\mathbb{R}) \subset GL(n,\mathbb{C})$, $GL(n,\mathbb{C})$ is unbounded.

(2) Let $A \in GL(n,\mathbb{C})$ with distinct (non-zero) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ and multiplicites $m_1, m_2, \ldots, m_r$, respectively. Then, $m_1 + m_2 + \cdots + m_r = n$. Since $\mathbb{C}$ is alebraically closed, A possesses a Jordan canonical form ([1, Corollary 2, p. 291]), that is, there exists a $P \in GL(n,\mathbb{C})$ such that $PAP^{-1} = A_1 \oplus A_2 \oplus \cdots A_r$, where each A_i is a block diagonal matrix of the form

$$A_i = J_{m_{1,i}}(\lambda_i) \oplus J_{m_{2,i}}(\lambda_i) \oplus \cdots \oplus J_{m_{k_i,i}}(\lambda_i),$$

with $m_{1,i} + m_{2,i} + \cdots + m_{k_i,i} = m_i$, where for each j, $J_{m_{j,i}}(\lambda_i)$ is the $m_j \times m_j$ Jordan block

$$J_{m_{j,i}}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}.$$

Note that for each $0 \neq \lambda \in \mathbb{C}$, we can easily find a path $\psi_\lambda : [0,1] \to \mathbb{C}$ with end points $\psi_\lambda(0) = 1$ and $\psi(1) = \lambda$ such that ψ_λ does not pass through the origin of $\mathbb{C}$. As a consequence, for each Jordan block $J_m(\lambda)$ with $\lambda \neq 0$, the map $\varphi_{:}[0,1] \to M_m(\mathbb{C})$ given by

$$\varphi_{m,\lambda}(t) = \begin{pmatrix} \psi_\lambda(t) & t & & & \\ & \psi_\lambda(t) & t & & \\ & & \ddots & \ddots & \\ & & & \psi_\lambda(t) & t \\ & & & & \psi_\lambda(t) \end{pmatrix}$$

is a path because each component of $\varphi_{m,\lambda}$ is continuous, and has end points $\varphi_{m,\lambda}(0) = I_m$ and $\varphi_{m,\lambda}(1) = J_m(\lambda)$. Also, since 0 does not lie in the range of ψ_λ, we see that $\varphi_{m,\lambda}(t) \in GL(m,\mathbb{C})$ for all $t \in [0,1]$. Therefore, the map $\varphi : [0,1] \to GL(n,\mathbb{C})$ given by

$\varphi(t) =$

$$P^{-1}\Big(\varphi_{m_{1,1},\lambda_1}(t) \oplus \varphi_{m_{2,1},\lambda_1}(t) \oplus \cdots \oplus \varphi_{m_{k_1,1},\lambda_1}(t) \oplus \varphi_{m_{1,2},\lambda_2}(t) \oplus \varphi_{m_{2,2},\lambda_2}(t) \oplus$$
$$\cdots \oplus \varphi_{m_{k_2,2},\lambda_2}(t) \oplus \cdots \varphi_{m_{1,r},\lambda_r}(t) \oplus \varphi_{m_{2,r},\lambda_r}(t) \oplus \cdots \oplus \varphi_{m_{k_r,r},\lambda_r}(t)\Big)P$$

is a path in $GL(n, \mathbb{C})$ with end points $\varphi(0) = I_n$ and $\varphi(1) = A$ and we are done. $\square$

Note that $SL(1, \mathbb{C}) = \{(1)\}$ is compact and path connected. However, there is slight difference in compactness property in higher dimensions.

Corollary 4.2. *$SL(n, \mathbb{C})$ is closed, path connected and is not compact for $n \geq 2$.*

Proof. Let $A \in SL(n, \mathbb{C})$. Then, by path connectedness of $GL(n, \mathbb{C})$, there exists a path $\varphi : [0, 1] \to GL(n, \mathbb{C})$ such that $\varphi(0) = I_n$ and $\varphi(1) = A$. Note that components of the path φ are all paths in $\mathbb{C}$ and suppose they are given by $\varphi(t) = [\varphi_{ij}(t)]$. Then, if $\theta : [0, 1] \to \mathbb{C}\backslash\{0\}$ is given by $\theta(t) = \det(\varphi(t))^{-1}$, by n-linearity of the determinant function, we see that the matrix $\begin{pmatrix} \theta(t)\varphi_{11}(t) & \dots & \theta(t)\varphi_{1n}(t) \\ \varphi_{21}(t) & \dots & \varphi_{2n}(t) \\ \vdots & \ddots & \vdots \\ \varphi_{n1}(t) & \dots & \varphi_{nn}(t) \end{pmatrix}$ has determinant 1 for all $0 \leq t \leq 1$. This suggests us to consider the map $\tilde{\varphi} : [0, 1] \to SL(n, \mathbb{C})$ given by $\tilde{\varphi}(t) = \begin{pmatrix} \theta(t)\varphi_{11}(t) & \dots & \theta(t)\varphi_{1n}(t) \\ \varphi_{21}(t) & \dots & \varphi_{2n}(t) \\ \vdots & \ddots & \vdots \\ \varphi_{n1}(t) & \dots & \varphi_{nn}(t) \end{pmatrix}$. Since θ is continuous, it is easily seen that $\tilde{\varphi}$ is a path in $SL(n, \mathbb{C})$ with end points $\tilde{\varphi}(0) = I_n$ and $\tilde{\varphi}(1) = A$. Hence $SL(n, \mathbb{C})$ is path connected.

Since $SL(n, \mathbb{C}) = \det^{-1}(\{1\})$ and det is continuous, $SL(n, \mathbb{C})$ is a closed subset of $M_n(\mathbb{C})$. However, $SL(n, \mathbb{C})$ is not bounded as it contains $SL(n, \mathbb{R})$ which is unbounded, as seen in Corollary 3.5. $\square$

Theorem 4.3. *$U(n)$ is compact and path connected.*

Proof. Since the map $M_n(\mathbb{C}) \ni A \mapsto A^*A \in M_n(\mathbb{C})$ is continuous and $U(n)$ is the inverse image of the singleton closed set $\{I_n\}$ under this map, we see that $U(n)$ is a closed subset of $M_n(\mathbb{C})$. Also, it is easily seen that $U(n)$ lies in the closed ball of radius $\sqrt{n}$ of $\mathbb{C}^{n^2}$ under its usual metric. Hence, by the Heine-Borel theorem, $U(n)$ is compact.

Again, it is enough to show that any unitary matrix is connected to I_n by a path in $U(n)$. If $A \in U(n)$, then A is normal and therefore unitarily diagonalizable ([3, Corollary to Theorem 21, Chapter 8]), i.e., there exists a $U \in U(n)$ such that U^*AU is diagonal. Note that, for each $U \in U(n)$, the operation $Ad(U) : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $Ad(U)(X) = U^*XU$ is a homeomorphism; so, it is enough to prove that every diagonal unitary matrix is connected to I_n by a path in $U(n)$. Indeed, if $D = U^*AU$ is a diagonal unitary matrix, and φ is a path in $U(n)$ connecting

I_n to D, then the map $\Phi : [0,1] \to U(n)$ given by $\Phi(t) = U\varphi(t)U^*$ is a path with $\Phi(0) = I_n$ and $\Phi(1) = U\varphi(1)U^* = A$.

Let $D = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ be a diagonal unitary matrix, where $\theta_i \in \mathbb{R}$ for all $1 \le i \le n$. Consider the path $\varphi : [0,1] \to U(n)$ given by $\varphi(t) = \operatorname{diag}(e^{it\theta_1}, e^{it\theta_2}, \ldots, e^{it\theta_n})$. Clearly, φ is a path in $U(n)$ with $\varphi(0) = I_n$ and $\varphi(1) = D$. This proves our assertion. ☐

Note that $U(1)$ has the obvious metric space structure, namely, it equals the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Also, $GL(1, \mathbb{C})$ is just the punctured complex plane at the origin. It is usually not possible to visualize the metric space structure of higher matrix group, except for the following beautiful metric space realization of $SU(2)$.

Proposition 4.4. *The group $SU(2)$ is homeomorphic to the real Euclidean sphere*

$$S^3 := \{(x, y, w, z) \in \mathbb{R}^4 : x^2 + y^2 + w^2 + z^2 = 1\}.$$

Proof. Note that for $A = (a_{ij}) \in SU(2)$, its inverse is given by $A^{-1} = A^*$. Since $A^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$ by direct computation and $A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}$ by definition, comparing entries, we obtain $A = \begin{pmatrix} a_{11} & a_{12} \\ -\overline{a_{12}} & \overline{a_{11}} \end{pmatrix}$. If $a_{rs} = x_{rs} + \imath y_{rs}$, $x_{rs}, y_{rs} \in \mathbb{R}$ for $r, s = 1, 2$, we obtain $1 = \det(A) = |a_{11}|^2 + |a_{12}|^2 = x_{11}^2 + y_{11}^2 + x_{12}^2 + y_{12}^2$. This induces the map

$$SU(2) \ni \begin{pmatrix} a_{11} & a_{12} \\ -\overline{a_{12}} & \overline{a_{11}} \end{pmatrix} \mapsto (x_{11}, y_{11}, x_{12}, y_{12}) \in S^3.$$

Since $SU(2)$ is compact, being a closed subset of $U(2)$, this map is a continuous bijection from the compact space $SU(2)$ to the Haudorff space S^3. Hence, it is a homeomorphism ([5, Theorem 26.6]). ☐

This obviously tells us that $SU(2)$ is path connected. We now show that the same holds in higher dimensions as well.

Theorem 4.5. *$SU(n)$ is compact and path connected.*

Proof. By definition, $SU(n)$ is a closed subset of the compact space $U(n)$ and hence is compact.

One is tempted to think that connectedness of $SU(n)$ can be deduced from that of $U(n)$ on the lines of Corollary 4.2. However, that trick does not provide us with a path in $SU(n)$. We actually try to imitate the proof of Theorem 4.3. Indeed, if $A \in SU(n)$, then, there exists a $U \in U(n)$ such that $U^*AU = D$ is diagonal. If $D = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ for some $\theta_i \in \mathbb{R}$, we get $1 = \det(A) = \det(D) = e^{i\sum_{i=1}^{n}\theta_i}$, so that $e^{-i\sum_{i=1}^{n-1}\theta_i} = e^{i\theta_n}$. Consider the map $\varphi : [0,1] \to SU(n)$ given by

$$\varphi(t) = U \operatorname{diag}(e^{it\theta_1}, e^{it\theta_2}, \ldots, e^{it\theta_{n-1}}, e^{-it\sum_{i=1}^{n-1}\theta_i})\, U^*.$$

Clearly, φ is a path in $SU(n)$ with end points $\varphi(0) = I_n$ and $\varphi(1) = UDU^* = A$ because $e^{-i\sum_{i=1}^{n-1}\theta_i} = e^{i\theta_n}$. The above trick was motivated by the proof of [2, Proposition 1.10]. ☐

5. Symplectic groups

Before discussing the topological properties of symplectic groups, let us quickly get a short overview of some of their algebraic properties.

The matrix J_n (see page-2) belongs to $Sp(n, \mathbb{K})$, it satisfies the equalities $J_n^T = -J_n = J_n^{-1}$ and has determinant 1. Also, for any $A \in Sp(n, \mathbb{K})$, the defining equation $A^T J_n A = J_n$ yields $\det(A)^2 = 1$ so that $\det(A) = \pm 1$. We show in the following that $Sp(n, \mathbb{K}) \subseteq SL(2n, \mathbb{K})$. Notice that J_n induces a bilinear form $B : \mathbb{K}^{2n} \times \mathbb{K}^{2n} \to \mathbb{K}$ given by

$$B(x, y) = x^T J_n y = \sum\nolimits_{i=1}^{k} (x_i y_{n+i} - x_{n+i} y_i) \tag{5.1}$$

which is easily seen to be non-degenerate (i.e., $B(x, y) = 0$ for all $y \in \mathbb{K}^{2n}$ implies $x = 0$) and skew-symmetric (i.e., $B(x, y) = -B(y, x)$). We observe that

$$Sp(n, \mathbb{K}) = \{A \in M_{2n}(\mathbb{K}) : B(Ax, Ay) = B(x, y) \text{ for all } x, y \in \mathbb{K}^{2n}\}. \tag{5.2}$$

Lemma 5.1. *$Sp(n, \mathbb{K})$ is a group that is closed under transposition. Also, for $A \in Sp(n, \mathbb{K})$, we have $A^T = -J_n A^{-1} J_n$ and $A^{-1} = -J_n A^T J_n$.*

Proof. From (5.2), we see that $Sp(n, \mathbb{K})$ is multiplicatively closed, i.e., $AB \in Sp(n, \mathbb{K})$ for all $A, B \in Sp(n, \mathbb{K})$. Also, for $A \in Sp(n, \mathbb{K})$, we have $B(A^{-1}x, A^{-1}y) = B(AA^{-1}x, AA^{-1}y) = B(x, y)$ for all $x, y \in \mathbb{K}^{2n}$. Thus, $Sp(n, \mathbb{K})$ is a subgroup of $GL(2n, \mathbb{K})$. Then, for $A \in Sp(n, \mathbb{K})$, its defining condition yields $A^T = J_n A^{-1} J_n^{-1}$ which implies that $Sp(n, \mathbb{K})$ is closed under transposition and it also provides the desired expressions for A^T and A^{-1}. ☐

Lemma 5.2. *Let $A \in Sp(n, \mathbb{K})$ and $p(\lambda)$ be its characteristic polynomial. Then, the following hold:*
(1). $p(\lambda) = \pm\lambda^{2n} p(1/\lambda)$. *(2). If λ is an eigenvalue of A, then so is $1/\lambda$.*
(3). A and A^{-1} have same eigenvalues.
Moreover, if $\mathbb{K} = \mathbb{C}$ and λ is an eigenvalue of A, then so are $\bar{\lambda}$ and $\left(\bar{\lambda}\right)^{-1}$.
In particular, $Sp(n, \mathbb{K}) \subseteq SL(2n, \mathbb{K})$.

Proof. (2) and (3) follow from (1); and (1) follows from the following:

$$\begin{aligned} p(\lambda) = \det\left(A - \lambda I_{2n}\right) = \det\left(A^T - \lambda I_{2n}\right) &= \det\left(-J_n A^{-1} J_n - \lambda I_{2n}\right) \\ &= \det\left(-J_n A^{-1} J_n + \lambda J_n J_n\right) \\ &= \det\left(-A^{-1} + \lambda I_{2n}\right) \\ &= \det(A^{-1}) \det\left(-I_{2n} + \lambda A\right) \\ &= \det(A^{-1}) \lambda^{2n} \det\left(-\lambda^{-1} I_{2n} + A\right) \\ &= \pm\lambda^{2n} p(1/\lambda). \end{aligned}$$

Since complex roots of a real polynomial always occur in conjugate pairs we are done. ☐

Lemma 5.3. *$Sp(1,\mathbb{K}) = SL(2,\mathbb{K})$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{K})$. Then $A \in Sp(1,\mathbb{K}) \Longleftrightarrow A^T J_1 A = J_1$. Since $A^T J_1 A = \begin{pmatrix} 0 & -bc+ad \\ -ad+bc & 0 \end{pmatrix}$, we have $A \in Sp(1,\mathbb{K}) \Longleftrightarrow \det(A)=1$. ☐

However, for $n \geq 2$, it easily seen that $Sp(n,\mathbb{K}) \subsetneq SL(2n,\mathbb{K})$. For instance,

$$D_r = \mathrm{diag}\big(r, r, ..., r, 1/r^{2n-1}\big) \in SL(2n,\mathbb{K}) \setminus Sp(n,\mathbb{K})$$

for every $r > 1$ because, unlike r, $1/r$ is not an eigenvalue of D_r.

We now analyse the topological properties of symplectic groups with the help of linear algebra. For that, we recall an important class of symplectic matrices called *symplectic transvections*. For each nonzero $u \in \mathbb{K}^{2n}$ and $\lambda \in \mathbb{K}$, consider the linear map $\tau = \tau_{u,\lambda} : \mathbb{K}^{2n} \to \mathbb{K}^{2n}$ given by $\tau_{u,\lambda}(v) = v + \lambda B(v,u)u$, $v \in \mathbb{K}^{2n}$. Let $W = \{v \in \mathbb{K}^{2n} : B(u,v) = 0\}$. Then, it is easily seen that W is a hyperplane, i.e., $\dim\big(\mathbb{K}^{2n}/W\big) = 1$, $\tau_{|W} = Id_W$ and $\tau(v) - v \in W$ for all $v \in V$. A linear map of the form $\tau_{u,\lambda}$ is called a *symplectic transvection*.

Note that $B(\tau_{u,\lambda}(x), \tau_{u,\lambda}(y)) = B(x,y)$ for all $x, y \in \mathbb{K}^{2n}$; so that $\tau_{u,\lambda} \in Sp(n,\mathbb{K})$ for all $u \in \mathbb{K}^{2n}$ and $\lambda \in \mathbb{K}$. Also, $\tau_{u,0} = I_{2n}$ for all $u \in \mathbb{K}^{2n}$. Interestingly, the symplectic transvections generate the symplectic groups - a proof of which can be found, for instance, in [4, § 6.9].

Theorem 5.4. *$Sp(n,\mathbb{K})$ is generated by the symplectic transvections.*

The symplectic groups share some topological properties with the special linear groups.

Proposition 5.5. *$Sp(n,\mathbb{K})$ is closed and is not compact for all $n \in \mathbb{N}$.*

Proof. If $\{X_m\}$ is a sequence in $Sp(n,\mathbb{K})$ converging to some X in $M_{2n}(\mathbb{K})$, then $J_n = (X_m)^T J_n X_m \to X^T J_n X$ as $m \to \infty$, and hence $X^T J_n X = J_n$ implying that $X \in Sp(n,\mathbb{K})$. So, $Sp(n,\mathbb{K})$ is closed in $M_{2n}(\mathbb{K})$.

For each $r > 0$, consider the block diagonal matrix $A_r = B_r \oplus B_{\frac{1}{r}}$, where $B_r = \mathrm{diag}(r, \frac{1}{r}, 1, 1, \ldots, 1) \in SL(n,\mathbb{K})$. It is easily seen that $(A_r)^T J_n A_r = J_n$, i.e., $A_r \in Sp(n,\mathbb{K})$ for all $r > 0$ and $\{A_r : r > 0\}$ is not bounded in $M_{2n}(\mathbb{K})$. Hence, $Sp(n,\mathbb{K})$ is not compact. ☐

Exercise. Let G be a subgroup of $GL(n,\mathbb{K})$ generated by a set S. If each element of S can be joined by a path in G to the identity matrix I_n, then show that G is path connected.

Proposition 5.6. *$Sp(n,\mathbb{K})$ is path connected for all $n \in \mathbb{N}$.*

Proof. By Theorem 5.4, every symplectic matrix is a (finite) product of symplectic transvections. So, it is enough to show that every symplectic transvection can be connected to the identity matrix by a path in $Sp(n,\mathbb{K})$. Consider a symplectic transvection $\tau_{u,\lambda}$ and define $\gamma : [0,1] \to Sp(n,\mathbb{K})$ by

$$\gamma(t) = \tau_{u,(1-t)\lambda}, t \in [0,1].$$

It is an easy exercise to show that γ is a path in $Sp(n, \mathbb{K})$ with end points $\gamma(0) = \tau_{u,\lambda}$ and $\gamma(1) = \tau_{u,0} = I_{2n}$. $\square$

Although both real and complex symplectic groups have similar basic topological properties, they are topologically different as they are known to have different "fundamental groups", a notion studied in "algebraic topology".

There is also a compact version of (complex) symplectic groups given by

$$Sp(n) = Sp(n, \mathbb{C}) \cap U(2n).$$

It is clearly compact and is known to be path-connected. However, a proof of it requires some advanced mathematics ("Lie group theory") which is out of the reach of this discussion.

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ON DETERMINATION OF GALOIS GROUP OF QUARTIC POLYNOMIALS

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Abstract. Determining the Galois group of a polynomial is one of the major problems of Algebra. In general, it is a difficult problem but for a polynomial of degree less than or equal to 4, it is completely determined. Almost all standard textbooks of Algebra, such as [1], [2], [3], give methods to deal with it. But these methods are tedious with hands-on computation. This article is an attempt to combine the methods given in these books with the method developed in [7] and [6], to simplify the computation of the Galois group of polynomials up to degree four.

1. Introduction and preliminaries

The main aim of this article is to give a method for computing the Galois group of an irreducible, separable quartic polynomial over a field K of characteristic not equal to 2. In the first section, we recall the general theory needed for finding the Galois group of polynomials of any degree. In order to keep the article self-contained we will define all the terms required here. For the sake of completeness, in the second section we will compute the Galois group of quadratic and cubic polynomials even though it is easy to determine. The third section is the main part of the article where we determine the Galois group of an irreducible separable quartic polynomials. In the last section, we will see an application to determine the Galois group of a quartic polynomial which is the minimal polynomial of elements of the form $\sqrt{a+b\sqrt{d}}$. We also compute the splitting fields of irreducible polynomials whenever feasible.

Let $L|K$ be a field extension. We know that the set of all automorphisms of L fixing K forms a group. When order of this group is same as the degree $[L : K]$ of the extension $L|K$ then the extension is called the Galois extension and the group is called the Galois group of L over K denoted by $Gal(L|K)$. If $f(x)$ is a separable polynomial of degree $n \geq 1$ over K and $L|K$ is the splitting field of $f(x)$, then $L|K$ is a Galois extension and then the Galois group $Gal(L|K)$ is referred to as the Galois group of $f(x)$, and we denote it by G_f. Our aim is to determine G_f, when

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$f(x)$ is an irreducible polynomial. In order to study the Galois group of $f(x)$ it is enough to consider $f(x)$ to be monic. Henceforth we assume $f(x)$ to be a monic polynomial.

Let α be any root of $f(x)$. We know that for any $\sigma \in G_f$, $\sigma(\alpha)$ is also a root of $f(x)$. Hence, if $r_1, r_2, \cdots, r_n$ are the roots of $f(x)$ then for any $\sigma \in G_f$, $\sigma(r_i) = r_j$ for some $j = 1, 2, \cdots n$. Thus, G_f acts on the set $\{r_1, r_2, \cdots, r_n\}$ and any K-automorphism σ of L induces a permutation of $r_1, r_2, \cdots, r_n$. As $r_1, r_2, \cdots, r_n$ generate L over K, these K-automorphisms can be uniquely determined by the permutations which they induce. Hence, we can view G_f as a subgroup of S_n, the symmetric group on n symbols.

The following theorem gives a necessary and sufficient condition for the above action to be transitive (i.e., for any two roots r_i, r_j of $f(x)$, there exists $\sigma \in G_f$ such that $\sigma(r_i) = r_j$).

Theorem 1.1. *Let $f(x)$ be a separable polynomial of degree n over a field K. Then its Galois group G_f acts transitively on the set of all roots of $f(x)$ if and only if $f(x)$ is irreducible over K.*

Proof. Suppose $f(x)$ is irreducible polynomial over K. Since $f(x)$ is separable over K of degree n, the set of all roots of $f(x)$ may be assumed to be $\{r_1, r_2, \cdots, r_n\}$. Let $L|K$ be the splitting field of $f(x)$. Since any r_i, r_j $(1 \leq i, j \leq n)$ are roots of the same irreducible polynomial $f(x)$, there exists an isomorphism $\sigma : K(r_i) \to K(r_j)$ such that $\sigma(r_i) = r_j$. This isomorphism can be extended to a K-automorphism of L (refer to Theorem 13.27 in [4]). This proves that G_f acts transitively on the set of all roots of $f(x)$.

To prove the converse, suppose $f(x)$ is not irreducible. Assume that $g(x)$ and $h(x)$ are any two distinct irreducible factors of $f(x)$ and r_g, r_h are the roots of $g(x)$ and $h(x)$ respectively. Since G_f acts transitively on the roots of $f(x)$, there exists $\sigma \in G_f$, such that $\sigma(r_g) = r_h$. But this is not possible as any K- automorphism of L maps r_g to a root of $g(x)$. □

Note that if α is any root of an irreducible separable polynomial $f(x)$ of degree n over a field K, then $K(\alpha)$ is a subfield of L such that $[K(\alpha) : K] = n$. Hence, by the fundamental theorem of Galois theory, G_f has a subgroup of index n - implying that the order of G_f is divisible by n. Since we have seen that G_f can be viewed as a subgroup of S_n, we have the following theorem in view of the above theorem.

Theorem 1.2. *Let K be a field and $f(x)$ be an irreducible separable polynomial of degree n over K. Then the Galois group G_f of $f(x)$ is a transitive subgroup of S_n whose order is divisible by n.*

From the above theorems, in order to determine G_f one needs to look at only the transitive subgroups of S_n of order divisible by n. Since for a large n the number of such subgroups of S_n is large, there are many possibilities for G_f. Let us give one more criterion which enables us to determine G_f. For this, we require the notion of the discriminant of a polynomial.

Definition 1.1. *Let $f(x)$ be any polynomial of degree $n \geq 1$ over a field K. Let $r_1, r_2, \cdots, r_n$ be the roots of $f(x)$. Then the discriminant $\triangle$ of $f(x)$ is defined as*

$$\triangle = \prod_{1 \leq i < j \leq n} (r_i - r_j)^2$$

Theorem 1.3. *Let $f(x)$ be an irreducible separable polynomial of degree n over a field K, $r_1, r_2, \cdots, r_n$ be the roots of $f(x)$ and $\triangle$ be the discriminant of $f(x)$. Then $\triangle \in K$. Further, if characteristic of $K \neq 2$, then $\sqrt{\triangle} \in K$ if and only if $G_f \subseteq A_n$, where A_n denotes the alternating group on n symbols.*

Proof. To prove $\triangle \in K$ it is enough to prove $\sigma(\triangle) = \triangle$ for every $\sigma \in G_f$. Since every permutation is a product of transpositions and every transposition fixes $\triangle$, $\triangle$ is fixed by every element of G_f. Therefore $\triangle \in K$.

Consider $\sqrt{\triangle} = \prod_{1 \leq i < j \leq n} (r_i - r_j)$. If $\tau = (r_i, r_j)$ is any transposition then τ changes the sign of the factor $r_i - r_j$ and leaves other factors unchanged. Thus, for any $\sigma \in G_f$

$$\sigma(\sqrt{\triangle}) = \begin{cases} \sqrt{\triangle}, & \text{if } \sigma \in A_n; \\ -\sqrt{\triangle}, & \text{otherwise.} \end{cases}$$

But $\sqrt{\triangle} \neq -\sqrt{\triangle}$ as characteristic of $K \neq 2$. This proves that $\sigma(\sqrt{\triangle}) = \sqrt{\triangle}$ for all $\sigma \in G_f$ if and only if $G_f \subseteq A_n$. ☐

We will prove another lemma which will be used later.

Lemma 1.1. *Let $f(x)$ be an irreducible separable polynomial of degree n over a field K of characteristic $\neq 2$ and L be the splitting field of $f(x)$ over K. Suppose $G_f = S_n$. Then $K(\sqrt{\triangle})$ is the unique quadratic extension of K contained in L.*

Proof. Since $G_f = S_n$, the discriminant $\triangle$ of $f(x)$ is not a square in K. Therefore $K(\sqrt{\triangle})$ is a quadratic extension of K contained in L. The alternating group A_n is the unique subgroup of S_n, of index 2. Hence by the fundamental theorem of Galois theory L has a unique subfield of degree 2, which has to be $K(\sqrt{\triangle})$. ☐

2. Galois group of Quadratic and Cubic Polynomials

Determining the Galois groups of quadratic and cubic polynomials is easy. It depends totally on the discriminant of the polynomial. So let us start by looking at the discriminant of a quadratic polynomial. If $f(x) = x^2 + bx + c$ is a quadratic polynomial with roots r_1, r_2 over a field K then its discriminant $\triangle = (r_1 - r_2)^2 = (r_1 + r_2)^2 - 4r_1r_2 = b^2 - 4c$. Clearly, if $f(x)$ is an irreducible polynomial then $\triangle$ is not a square in K and hence $K(\sqrt{\triangle})$ is a quadratic extension of K. If characteristic of $K \neq 2$, we may find the discriminant by substituting $x = y - b/2$. This gives the polynomial $g(y) = y^2 - b^2/4 + c = y^2 - \triangle/4$. Clearly the discriminant of

$g(y) = \triangle$, the discriminant of $f(x)$. The roots of $g(y)$ are simply $\pm\sqrt{\triangle}/2$. Since the roots of $f(x)$ and $g(y)$ differ by the constant $b/2$, both have the same splitting field. But the splitting field of $g(y)$ is $K(\sqrt{\triangle})$, hence the splitting field of $f(x)$ is also $K(\sqrt{\triangle})$. Therefore, if $f(x)$ is irreducible then $G_f = S_2$.

Next, consider the cubic polynomial $f(x) = x^3 + ax^2 + bx + c$. Substituting $x = y - a/3$, $f(x)$ reduces to $g(y) = y^3 + py + q$, where $p = b - a^2/3$ and $q = (1/27)\left(2a^3 - 9ab + 27c\right)$ (this substitution is valid only when the characteristic of field $K \neq 3$).[1] Note that the roots of $f(x)$ and $g(y)$ differ by the constant $a/3$; and hence, both polynomials have the same splitting field and the same discriminant. If we assume y_1, y_2, y_3 are the roots of $g(y)$, then $g(y) = (y - y_1)(y - y_2)(y - y_3)$ and its derivative $g'(y) = (y - y_2)(y - y_3) + (y - y_1)(y - y_3) + (y - y_1)(y - y_2)$. Hence we get

$$g'(y_1)g'(y_2)g'(y_3) = -\triangle. \tag{2.1}$$

Since $g'(y) = 3y^2 + p$, from (2.1) we get

$$\triangle = -4p^3 - 27q^2. \tag{2.2}$$

Substituting back the values of p and q in terms of a, b, c we get

$$\triangle = a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc \tag{2.3}$$

For detailed computation, one can refer to [4]. Once we have the discriminant, the following theorem determines the Galois group of $f(x)$.

Theorem 2.1. *Let $f(x)$ be an irreducible separable polynomial of degree 3 over K (characteristic $K \neq 2$). Then the Galois group G_f of $f(x)$ will be A_3 or S_3 depending, respectively, on whether the discriminant $\triangle$ of $f(x)$ is a square in K or not.*

Proof. By theorem (1.2), G_f is a transitive subgroup of S_3 of order divisible by 3. Hence the only possibilities for G_f are A_3 or S_3. By Theorem 1.3, $G_f = A_3$ if and only if $\sqrt{\triangle} \in K$, otherwise $G_f = S_3$. □

Remark 2.1. *Let L be the splitting field of $f(x)$ over K. If $\sqrt{\triangle} \in K$, then $[L : K] = |G_f| = 3$. Hence $L = K(r_1)$, where r_1 is any root of $f(x)$. If $\sqrt{\triangle} \notin K$ then $L = K(r_1, \sqrt{\triangle})$.*

3. The Galois group of Quartic Polynomials

We continue to assume characteristic of $K \neq 2$. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be an irreducible separable polynomial over K, r_1, r_2, r_3, r_4 be its roots and $\triangle$ be its discriminant. As in the case of cubic polynomials, substituting $x = y - a/4$, we get a polynomial $g(y) = y^4 + b_1y^2 + c_1y + d_1$. Both $f(x)$ and $g(y)$ have a same splitting field and the same discriminant as their roots differ by the constant $a/4$. By a procedure similar to the cubic polynomials we can find the discriminant $\triangle$ of $f(x)$ as

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[1] Discriminant can be found for any polynomial over any field. This method for finding discriminant requires the characteristic of $K \neq 3$.

$$\begin{aligned}\triangle &= a^2b^2c^2 - 4a^2b^3d - 6a^2c^2d - 128b^2d^2 - 4a^3c^3 + 16b^4d - 4b^3c^2 \\ &\quad -27c^4 - 27a^4d^2 + 18abc^3 + 144a^2bd^2 - 192acd^2 + 144bc^2d + 256d^3 \\ &\quad -80ab^2cd + 18a^3bcd. \qquad (3.1)\end{aligned}$$

A detailed computation is given in [4].

Let $L|K$ be the splitting field of $f(x)$ and G_f be its Galois group. By Theorem 1.2, G_f is a transitive subgroup of S_4 of order divisible by 4. The transitive subgroups of S_4 of order divisible by 4 are: S_4, A_4, three conjugate subgroups of order 8 isomorphic to the dihedral group (we denote it by D_4), three cyclic subgroups of order 4 (we denote it by C_4) and one subgroup isomorphic to the Klein-4 group which we denote by V_4 (we take $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$ because out of total 3 subgroups of order 4 isomorphic to the Klein-4 group only V_4 is transitive). Therefore we get

$$G_f = S_4,\ A_4,\ D_4,\ C_4 \text{ or } V_4. \qquad (3.2)$$

Note that V_4 is the unique normal subgroup of S_4 of order 4. Therefore $G_f \cap V_4$ is a normal subgroup of G_f. If F is the fixed field of $G_f \cap V_4$, then by the fundamental theorem of Galois theory, $F|K$ is a Galois extension with Galois group $G_f/(G_f \cap V_4)$. As G_f is a subgroup of S_4, this means that $G_f/(G_f \cap V_4)$ is a subgroup of S_4/V_4 which is isomorphic to S_3. Thus $F|K$ is a Galois extension whose Galois group is isomorphic to a subgroup of S_3. In the following subsection, we will show that F is the splitting field of a cubic polynomial over K.

3.1. **Resolvent Cubic.** Consider the following partially symmetric functions.

$$\alpha = r_1r_2 + r_3r_4,\ \beta = r_1r_3 + r_2r_4,\ \text{and } \gamma = r_1r_4 + r_2r_3. \qquad (3.3)$$

Observe that α, β, γ are invariant under V_4. We prove the following theorem.

Theorem 3.1. *Let* α, β, γ *be as in* (3.3) *and* F *be the fixed field of* $G_f \cap V_4$. *Then* $F = K(\alpha, \beta, \gamma)$.

Proof. Being invariant under V_4, α, β, γ are fixed by $G_f \cap V_4$ and hence $K(\alpha, \beta, \gamma) \subseteq F$. To prove $F \subseteq K(\alpha, \beta, \gamma)$ we proceed as follows. Denote the three transitive subgroup of S_4 of order 8, isomorphic to the dihedral group, by $D_4^{(1)}, D_4^{(2)}, D_4^{(3)}$. Assume $D_4^{(1)}$ is generated by $\sigma = (1324)$ and $\tau = (12)$. Then $\sigma(\alpha) = \alpha$ and $\tau(\alpha) = \alpha$, and hence α is fixed by generators of $D_4^{(1)}$. In fact, $D_4^{(1)}$ is the stabilizer of α in S_4. Similarly, the other two conjugates $D_4^{(2)}$ and $D_4^{(3)}$ are the stabilizers of β and γ respectively.

One can easily check that $D_4^{(1)} \cap D_4^{(2)} \cap D_4^{(3)} = V_4$. Therefore, $D_4^{(1)} \cap D_4^{(2)} \cap D_4^{(3)} \cap G_f = G_f \cap V_4$. Hence the subgroup of G_f which stabilizes α, β, γ is $G_f \cap V_4$, and hence $Gal(L|K(\alpha, \beta, \gamma)) = G_f \cap V_4$. Thus F, the fixed field of $G_f \cap V_4$, is contained in $K(\alpha, \beta, \gamma)$. □

The polynomial $c(x)$ having roots α, β and γ is called the *resolvent cubic* of $f(x)$. Using the relations between roots and coefficients of a polynomial one can verify that

$$c(x) = x^3 - bx^2 + (ac - 4d)x - (a^2d + c^2 - 4bd). \tag{3.4}$$

As we will soon see, the resolvent cubic plays an important role in determining the Galois group of the quartic polynomial.

Let us now consider the discriminants of $f(x)$ and $c(x)$. Let L be the splitting field of $f(x)$ over K and assume that $Gal(L|K)= S_4$. Then, by Theorem 3.1, $K(\alpha, \beta, \gamma)|K$ is the Galois extension with the Galois group S_4/V_4 which is isomorphic to S_3. We denote $K(\alpha, \beta, \gamma)$ by M for convenience. By Lemma 1.1, L contains a unique quadratic subfield $K(\sqrt{\triangle})$, where $\triangle$ is the discriminant of the quartic polynomial $f(x)$. By the same lemma, if D is the discriminant of $c(x)$ then $K(\sqrt{D})$ is the unique quadratic subfield of M(of L also). Then by uniqueness, these subfields must be same. This means that the discriminant of $f(x)$ and $c(x)$ differ by a multiple of a square in K. By direct computation, one can verify that these discriminants are same even if $\mathrm{Gal}(L|K) \neq S_4$.

One can also consider the partially symmetric functions $\alpha' = (r_1 + r_2)(r_3 + r_4)$, $\beta' = (r_1 + r_3)(r_2 + r_4)$ and $\gamma' = (r_1 + r_4)(r_2 + r_3)$. We will see that the cubic polynomial having roots α', β' and γ' occurs naturally when one tries to solve the quartic using geometric ideas. Let us discuss it in the following subsection.

3.2. **A geometric approach to the resolvent cubic.** As mentioned in the beginning of section 3, it is enough to consider the polynomial $f(x) = x^4 + bx^2 + cx + d$. By putting $y = x^2$, the polynomial reduces to $y^2 + by + cx + d$. Hence in order to find the roots of $f(x)$, one needs to solve the equations $f_1 : x^2 - y = 0$ and $f_2 : y^2 + by + cx + d = 0$ simultaneously. For this it is enough to find λ, so that $f_2 + \lambda f_1$ is a product of a pair of lines $y - m_1x + c_1, y - m_2x + c_2$. Suppose r_1, r_2, r_3, r_4 are the roots of $f(x)$. Then the four points (x, y^2) of intersection of f_1 and f_2 are given by

$$(r_1, r_1^2), (r_2, r_2^2), (r_3, r_3^2), (r_4, r_4^2). \tag{3.5}$$

One can join these four points in pairs to get six lines and the equations of these lines can be suitably multiplied in pairs to get λ. For example, consider $m_1 = (r_1^2 - r_2^2)/(r_1 - r_2) = r_1 + r_2$, $m_2 = (r_3^2 - r_4^2)/(r_3 - r_4) = r_3 + r_4$; then $f_2 + \lambda f_1 = (y - m_1x + c_2)(y - m_2x + c_2)$ and by comparing the coefficient of x^2, we get $\lambda = (r_1 + r_2)(r_3 + r_4)$. By considering different pairs of points from (3.5), we get two more values of λ. The list of these three values is, say,

$$\alpha' = (r_1 + r_2)(r_3 + r_4), \ \beta' = (r_1 + r_3)(r_2 + r_4), \ \gamma' = (r_1 + r_4)(r_2 + r_3). \tag{3.6}$$

Now, by plane geometry, $f_2 + \lambda f_1 = y^2 + by + cx + d + \lambda(x^2 - y)$ represents a pair of straight lines if the determinant

$$\begin{vmatrix} \lambda & 0 & \frac{c}{2} \\ 0 & 1 & \frac{b-\lambda}{2} \\ \frac{c}{2} & \frac{b-\lambda}{2} & d \end{vmatrix} = 0$$

Simplifying this, we get the following cubic equation in λ :

$$c'(\lambda) = \lambda^3 - 2b\lambda^2 + (b^2 - 4d)\lambda + c^2 = 0. \tag{3.7}$$

Since r_1, r_2, r_3, r_4 are the roots of $f(x) = x^4 + bx^2 + cx + d$, we have

$$r_1 + r_2 + r_3 + r_4 = 0, \quad b = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4,$$

$$c = r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4, \quad d = r_1r_2r_3r_4.$$

Using these relations one can easily verify that α', β', γ' are the roots of the cubic polynomial $c'(x)$. Hence while finding roots of $f(x)$, one naturally gets the polynomial $c'(x)$ as in (3.7) having roots as in (3.6). This polynomial $c'(x)$ is also called the resolvent cubic of $f(x)$. We refer to [9] for more precise statements and detailed account in this regard.

Let L be the splitting field of $f(x)$. By the arguments similar to Theorem 3.1, we can see that $K(\alpha', \beta', \gamma')$ is fixed field of $G_f \cap V_4$. Hence both the fields $K(\alpha, \beta, \gamma)$ and $K(\alpha', \beta', \gamma')$ are fixed fields of the same subgroup of G_f. Therefore both polynomials $c(x)$ and $c'(x)$ give rise to the same splitting field. Also note that $\alpha + \beta = \gamma', \alpha + \gamma = \beta'$ and $\beta + \gamma = \alpha'$. Hence, now onwards we will take $c(x)$ as the resolvent cubic.

3.3. **Determination of the Galois Group.** Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be an irreducible separable polynomial over a field K of characteristic $\neq 2$. As pointed out earlier the resolvent cubic plays an important role in the determination of G_f. In fact, consider the resolvent cubic $c(x)$ and its roots α, β, γ as defined in (3.4) and (3.3) respectively. By Theorem 3.1, the splitting field $K(\alpha, \beta, \gamma)$ of the resolvent cubic is a Galois extension of K with Galois group $G_f/(G_f \cap V_4)$. Therefore, if one can determine the Galois group of the resolvent cubic, then it becomes easy to determine G_f.

Theorem 3.2. *Let $f(x)$ be an irreducible separable polynomial of degree 4 over a field K of characteristic $\neq 2$. If $\triangle, c(x), \alpha, \beta$ and γ are as defined above, then the Galois group G_f of $f(x)$ is either A_4 or S_4 if and only if $c(x)$ is irreducible over K. Further, the Galois group of $f(x)$ is A_4 if $\triangle$ is a square in K and is S_4 if $\triangle$ is not a square in K*

Proof. Let us assume $c(x)$ is irreducible over K. As disussed in the para after (3.4), the discriminant of $c(x)$= the discriminant of $f(x)$=$\triangle$, and the splitting field of $c(x)$ over K is $K(\alpha, \beta, \gamma)$. By theorem 3.1, $K(\alpha, \beta, \gamma)$ is the fixed field of $G_f \cap V_4$ - a normal subgroup of G_f. So, by Galois theory, the Galois group of $c(x)$ is $G_f/(G_f \cap V_4)$. Now, by Theorem 2.1, the Galois group of $c(x)$ is either S_3 or A_3; and it is S_3 if and only if $\triangle$ is not a square in K. Then, in this case, the order of G_f is divisible by 4 and 6 (Theorem 1.2 and the fact that $G_f \cap V_4$ is normal in G_f and $|G_f| = |G_f/(G_f \cap V_4)||G_f \cap V_4|$. Hence G_f is either A_4 or S_4; but by Theorem 1.3, $G_f = S_4$. If $\triangle$ is a square in K, then obviously $G_f = A_4$.

Conversely, let us assume G_f is either S_4 or A_4. We have to prove that $c(x)$ is irreducible over K. If $c(x)$ is reducible over K, then either $c(x)$ splits completely over K or has an irreducible quadratic factor over K. In the first case, the order of $G_f/(G_f \cap V_4)$ is 1 and in the second case, the order is 2. As the order of $G_f \cap V_4 = 1, 2$, or 4, we get the order of $G_f = 1, 2, 4$ or 8, which is not possible as $G_f = S_4$ or A_4. Hence $c(x)$ is irreducible over K. □

Theorem 3.3. *Let $f(x), L, K, c(x), \triangle, G_f$ be as in the previous theorem. Let us assume $c(x)$ is reducible over K, then we have*

- *(i) $c(x)$ splits completely over K if and only if $G_f = V_4$.*
- *(ii) If $c(x)$ has an irreducible quadratic factor then*
  - *(a) $G_f = D_4$ if and only if $f(x)$ is irreducible over $K(\sqrt{\triangle})$.*
  - *(b) $G_f = C_4$ if and only if $f(x)$ is reducible over $K(\sqrt{\triangle})$.*

Proof. Let us assume that $c(x)$ splits completely over K, then $\alpha, \beta, \gamma \in K$, hence $K(\alpha, \beta, \gamma) = K$. Therefore the field extension $L|K(\alpha, \beta, \gamma)$ and $L|K$ are same. So their Galois groups $G_f \cap V_4$ and G_f are same. Hence we get $G_f = V_4$.

Conversely, if $G_f = V_4$ then $G_f \cap V_4 = V_4$. Hence $G_f/(G_f \cap V_4)$ is trivial group. Therfore $K(\alpha, \beta, \gamma) = K$ which means $\alpha, \beta, \gamma \in K$. It follows that $c(x)$ splits completely over K. This proves (i).

Assume now that $c(x)$ has an irreducible quadratic factor. Notice that in view of (3.2), Theorem 3.2 and (i) above, the only possibilities left out for G_f are C_4 or D_4. Let us assume that $c(x)$ has a unique root say α in K. Then $K(\alpha, \beta, \gamma) = K(\sqrt{\triangle})$ is a quadratic extension of K. We can view L as the splitting field of $f(x)$ over $K(\sqrt{\triangle})$ and $L|K(\sqrt{\triangle})$ is a Galois extension with Galois group $G_f \cap V_4$. If $G_f = D_4$ then $G_f \cap V_4 = V_4$, which is a transitive subgroup of S_4. Therefore V_4 acts transitively on roots of $f(x)$ over $K(\sqrt{\triangle})$. But by the Theorem 1.1 this is possible if and only if $f(x)$ is irreducible over $K(\sqrt{\triangle})$. Where as if $G_f = C_4$, then $G_f \cap V_4$ is non transitive subgroup of order 2 in V_4. Therefore by Theorem 1.1, $f(x)$ is reducible over $K(\sqrt{\triangle})$. This proves (ii). □

Combining the theorems (3.2) and (3.3) we get complete classification of the Galois group of an irreducible separable quartic polynomial over a field K of characteristic not equal to 2.

Example 3.1. *Consider the polynomial $x^4 - 5$ over $\mathbb{Q}$. By Eisenstein's criteria, it is easy to see that this is irreducible over $\mathbb{Q}$. The discriminant $\triangle = -256 \times 5^3$, and $c(x) = x^3 + 20x$. So $\mathbb{Q}(\sqrt{\triangle}) = \mathbb{Q}(\sqrt{-5})$. As $c(x)$ has unique root in $\mathbb{Q}$, so G_f can be C_4 or D_4 depending upon whether $f(x)$ is reducible over $\mathbb{Q}(\sqrt{-5})$ or not. The roots of $f(x)$ over $\mathbb{C}$ are $\pm\sqrt[4]{5}$ and $\pm i\sqrt[4]{5}$. None of these or their combinations are in $\mathbb{Q}(\sqrt{-5})$, and hence none of the quadratic factors of $f(x)$ are in $\mathbb{Q}(\sqrt{-5})[x]$. It follows that $f(x)$ is irreducible over $\mathbb{Q}(\sqrt{-5})$, and therefore $G_f = D_4$.*

In general it is tedious to determine whether $f(x)$ is irreducible over $K(\sqrt{\triangle})$ or not. Kappe and Warren [7] have proved that instead of checking irreducibility of the quartic polynomial it is sufficient to check irreducibility of two quadratic polynomials over $K(\sqrt{\triangle})$. We discuss this here.

Let us go back to the case where $c(x)$ has unique root say $\alpha = r_1r_2 + r_3r_4$ in K and $G_f = C_4$ or D_4 (Theorem 3.3, (ii)). We will take $D_4 = \langle(1324), (12)\rangle$. The reason is, if we take $\sigma = (1324)$ and $\tau = (12)$ then $\sigma(\alpha) = \alpha$ and $\tau(\alpha) = \alpha$. Since the generators of D_4 fix α, therefore α is fixed by each of the elements of D_4. For the same reason, we take $C_4 = \langle\sigma\rangle$ or $\left\langle\sigma^{-1}\right\rangle$.

Consider the elements $r_1 + r_2$, $r_3 + r_4$, r_1r_2 and r_3r_4 of L, where r_1, r_2, r_3, r_4 are the roots of an irreducible separable polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$. Let $g(x)$ and $h(x)$ be the polynomials having roots $r_1 + r_2$, $r_3 + r_4$, and r_1r_2, r_3r_4 respectively. Then

$$\begin{aligned} g(x) &= (x - (r_1 + r_2))(x - (r_3 + r_4)) = x^2 + ax + b - \alpha, \\ h(x) &= (x - r_1r_2)(x - r_3r_4) = x^2 - \alpha x + d. \end{aligned} \tag{3.8}$$

Since $\alpha \in K$, $g(x)$ and $h(x)$ are polynomials over K.

Theorem 3.4 (Kappe, Warren). *Let $f(x)$ be an irreducible separable polynomial of degree 4 over a field K of characteristic $\neq 2$ and assume that its resolvent cubic $c(x)$ has an irreducible quadratic factor. Then the Galois group f is C_4 if and only if $g(x)$ and $h(x)$, as defined in (3.8), are reducible over $K(\sqrt{\triangle})$.*

Proof. Suppose $G_f = C_4 = \langle\sigma\rangle$, where $\sigma = (1324)$. Then $L|K$ is a cyclic extension of degree 4 containing a unique quadratic extension of K. This field must be $K(\sqrt{\triangle})$ with $\mathrm{Gal}\big(K(\sqrt{\triangle})/K\big) = \left\langle\sigma^2\right\rangle$, where $\sigma^2 = (12)(34)$. Note that the elements $r_1 + r_2,\ r_3 + r_4,\ r_1r_2$ and r_3r_4 are fixed by σ^2. Therefore they belong to $K(\sqrt{\triangle})$. Hence $g(x)$ and $h(x)$ both split over $K(\sqrt{\triangle})$.

Conversely let us assume that $g(x), h(x)$ both split over $K(\sqrt{\triangle})$. Then $r_1 + r_2,\ r_3 + r_4,\ r_1r_2$ and r_3r_4 belong to $K(\sqrt{\triangle})$. We shall prove $G_f = C_4$. Observe that $c(x)$ splits completely over $K(\sqrt{\triangle})$ as one root of $c(x)$ is already in K and $c(x)$ has a quadratic irreducible factor over K which splits over $K(\sqrt{\triangle})$. So $\beta, \gamma \in K(\sqrt{\triangle})$.

Consider the polynomial $k(x) = x^2 - (r_1 + r_2)x + r_1r_2$, having roots r_1, r_2. Note that $k(x)$ is a polynomial over $K(\sqrt{\triangle})$. Let F be the splitting field of $k(x)$ over $K(\sqrt{\triangle})$. Then F is a quadratic extension of $K(\sqrt{\triangle})$. So we have $K(\sqrt{\triangle}) \subseteq F \subseteq L$. As $r_1, r_2, r_1 + r_2, r_3 + r_4, r_1r_2, r_3r_4, \beta, \gamma \in F$, $\beta - \gamma = -(r_1 - r_2)(r_3 - r_4) \in F$. Therefore $r_3 - r_4 \in F$ and hence $r_3, r_4 \in F$. This means $F = L$, and hence L is a quadratic extension of $K(\sqrt{\triangle})$ which is itself a quadratic extension of K. Therefore $[L : K] = 4$ and L contains a unique quadratic subfield. Hence $L|K$ is a cyclic extension and $G_f = C_4$. □

Finally, we give a proof of the main theorem of this article.

Theorem 3.5 (Conrad, Keith [6]). *Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be an irreducible quartic polynomial over a field K of characteristic $\neq 2$. Suppose $c(x) = x^3 - bx^2 + (ac - 4d)x - (a^2d + c^2 - 4bd)$ is the resolvent cubic of $f(x)$ with roots*

$\alpha = r_1r_2 + r_3r_4,\ \beta = r_1r_3 + r_2r_4$ and $\gamma = r_1r_4 + r_2r_3$ and $\triangle$ is the discriminant of $f(x)$. The following holds true.

cases	*c(x) is*	*$\triangle$ in K is*	$(\alpha^2 - 4d)\triangle,\ (a^2 - 4(b-\alpha))\triangle$	$G_f =$
case-1	*irreducible*	*Not square*		S_4
case-2	*irreducible*	*Square*		A_4
case-3	*reducible*	*Square*		V_4
case-4	*reducible*	*Not square*	*square in K*	C_4
case-5	*reducible*	*Not square*	*one of them is not square in K*	D_4

Proof. case-1, case-2 and case-3, when $G_f = S_4, A_4$, and V_4, are clear. Only case-4 and case-5, when $G_f = C_4$ or D_4, are required to be discussed. Consider case-4. Suppose $G_f = C_4 =< \sigma >$, where $\sigma = (1324)$. Let $g(x)$ and $h(x)$ be as defined in (3.8). If the root $r_1 + r_2$ of $g(x)$ belongs to K, then its second root $r_3 + r_4 = \sigma(r_1 + r_2) = r_1 + r_2$ - showing that $g(x)$ has a double root; and hence its discriminant $d_1 = a^2 - 4(b - \alpha)$ is zero. By the same argument, if $h(x)$ is reducible over K then its discriminant $d_2 = \alpha^2 - 4d$ is also zero. Hence we get $d_1\triangle$ and $d_2\triangle$ are squares in K. If $g(x)$ and $h(x)$ are irreducible over K then their discriminants, d_1 and d_2 respectively, are non-squares in K. So their splitting fields are $K(\sqrt{d_1})$ and $K(\sqrt{d_2})$ respectively. But as we saw, the splitting field of $g(x)$ and $h(x)$ is $K(\sqrt{\triangle})$, so both the splitting fields must be the same. This is possible if and only if $d_1\triangle$ and $d_2\triangle$ are squares in K. Hence, in any case, we get $G_f = C_4$ if and only if $(\alpha^2 - 4d)\triangle$ and $(a^2 - 4(b - \alpha))\triangle$ are squares in K.
In view of the case-4 discussed just now, clearly the case-5, when $G_f = D_4$, is true if and only if not both of $d_1\triangle$ and $d_2\triangle$ are squares in K simultaneously, that is, at least one of them is not a suare in K. ☐

Let us determine the Galois group of $x^4 - 5$ over $\mathbb{Q}$ using above theorem. Here $c(x) = x(x^2 + 20)$, having a root in $\mathbb{Q}$. The discriminant $\triangle$ of the polynomial, is -32000. $g(x) = x^2 - 5$ and $h(x) = x^2$. The discriminant d_1 of $g(x)$ is 20, hence $d_1\triangle < 0$, which not a square in $\mathbb{Q}$. So $G_f = D_4$.

Let us have some examples. We consider irreducible polynomials over $\mathbb{Q}$. We will have the following conventions.

(a) The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients is irreducible polynomial over $\mathbb{Q}$ and has roots $r_i, 1 \leq i \leq 4$.
(b) The resolvent cubic $c(x)$ is as in (3.4) and $\triangle$ is the discriminant of $f(x)$.
(c) α, β, γ defined as in (3.3) are the roots of $c(x)$, where we assume $\alpha \in \mathbb{Q}$, whenever $c(x)$ has unique root in $\mathbb{Q}$.
(d) To check $c(x)$ is irreducible or not over $\mathbb{Q}$, we use the following fact. Since $c(x)$ is monic with integer coefficients, hence all rational roots of $c(x)$ are integers. So the only possible roots of $c(x)$ in $\mathbb{Q}$ are divisors of the constant term.

(e) The polynomials $g(x) = x^2 - \alpha x + d$ and $h(x) = x^2 + ax + b - \alpha$ are as in (3.8). We will check whether $(\alpha^2 - 4d)\triangle$ and $(a^2 - 4(b - \alpha))\triangle$ are squares or not in $\mathbb{Q}$

$f(x)$	$c(x)$ and $\triangle$	$(\alpha^2 - 4d)\triangle$	$(a^2 - 4(b - \alpha))\triangle$	G_f
$x^4 + 3x^3 - 3x - 2$	$x^3 - x + 9$, irreducible $\triangle = -2183$	-	-	S_4
$x^4 + 4x - 1$	$(x - 2)(x^2 + 2x + 8)$. $\triangle = -16^2 2^2 7$, $\alpha = 2$,	$-16^2 2^5 7$ non-square	$-16^2 2^5 7$ non-square	D_4
$x^4 + 8x + 12$	$x^3 - 48x - 64$, irreducible $\triangle = 8^4 3^4$, square	-	-	A_4
$x^4 - 4x^2 + 5$	$(x + 4)(x^2 - 20)$, $\alpha = -4$, $\triangle = 2^7 10$,	$-2^9 10$ non square	0	D_4
$x^4 + 8x + 14$	$(x - 8)(x^2 + 8x + 8)$ $\alpha = 8$, $\triangle = 2^{11} 7^2$,	$2^{14} 7^2$	$2^{16} 7^2$	C_4
$x^4 + 3x + 3$	$(x + 3)(x^2 - 3x - 3), \alpha = -3$, $\triangle = 3^3 5^2 7$,	$-3^4 5^2 7$	$-2^2 3^4 5^2 7$	D_4
$x^4 + 4$	$x(x-4)(x+4)$	-	-	V_4

4. APPLICATION

We now give an application of Theorem 3.5 by determining the Galois group of the quartic polynomial $f(x) = x^4 + bx^2 + d$ over a field K of characteristic $\neq 2$. Note that if α is a root of $f(x)$, then $-\alpha$ is also a root. Hence we assume that the roots of $f(x)$ are $\pm\alpha, \pm\beta$. If we put $x^2 = t$, we get $f(t) = t^2 + bt + d$ which is reducible over K if and only if $b^2 - 4d$ is a square in K.

Let us assume that $f(x)$ is irreducible separable over K. If $c(x)$ is the resolvent cubic of $f(x)$ then $c(x) = (x-b)(x^2-4d)$ and its discriminant is $\triangle = 16d(b^2-4d)^2$. Hence the Galois group G_f of $f(x)$ can be V_4, C_4 or D_4. The Galois group $G_f = V_4$ if and only if $c(x)$ splits completely over K, that is, if and only if $\sqrt{d} \in K$ (equivalently $\alpha\beta \in K$). So now we assume that $\sqrt{d} \notin K$ and $c(x)$ has a unique root d in K. The polynomial $g(x)$ and $h(x)$ as in (3.8) are $x^2 - bx + d$ and x^2. By Theorem 3.5, $G_f = C_4$ if and only if $(b^2 - 4d)\triangle$ is a square in K, which is equivalent to $d(b^2 - 4d)$ is a square in K. Therefore we have the following theorem.

Theorem 4.1. *Let $f(x) = x^4 + bx^2 + d$ be an irreducible separable quartic polynomial over a field K of characteristic $\neq 2$. Then $c(x) = (x - b)(x^2 - 4d)$ is the resolvent cubic and $\triangle = 16d(b^2 - 4d)^2$ is the discriminant of $f(x)$. Let G_f denote the Galois group of this f then*

- *(i) $G_f = V_4$ if and only if $\sqrt{d} \in K$, or equivalently, if and only if $\alpha\beta \in K$.*
- *(ii) If $\sqrt{d} \notin K$ then*
  - *(a) $G_f = C_4$ if and only if $d(b^2 - 4d)$ is a square in K.*
  - *(b) $G_f = D_4$ if and only if $d(b^2 - 4d)$ is not a square in K.*

As an application, we will determine the Galois group of the minimal polynomial of $\alpha = \sqrt{a + b\sqrt{d}}$ over $\mathbb{Q}$, where $a, b, d \in \mathbb{Z}$, $\gcd(a, b)$ is a square free integer and d is non-square in $\mathbb{Z}$. We will assume that $a + b\sqrt{d}$ is not a square in $\mathbb{Q}[\sqrt{d}]$. We note the following few points regarding such an α.

- (a) The minimal polynomial of α over $\mathbb{Q}$ is $f(x) = x^4 - 2ax^2 + a^2 - b^2d$.
- (b) The roots of $f(x)$ are $\pm\alpha,\ \pm\alpha'$, where $\alpha' = \sqrt{a - b\sqrt{d}}$.
- (c) The discriminant $\triangle$ of $f(x)$ is $256b^4d^2(a^2 - b^2d)$ and resolvent cubic $c(x) = (x + 2a)(x^2 - 4(a^2 - b^2d))$.

By Theorem 4.1, we can make the following conclusion regarding the Galois group G_f of this $f(x)$.

- (i) $G_f = V_4$ if and only if $c(x)$ splits completely over $\mathbb{Q}$, that is, $a^2 - b^2d$ is a square in $\mathbb{Z}$. We can also express the Galois extension L of $f(x)$ explicitly. If $a^2 - b^2d = j^2$ for some $j \in \mathbb{Z}$ then $L = \mathbb{Q}\left(\sqrt{2(a - j)}, \sqrt{2(a + j)}\right)$ (refer to [8] for a proof).
- (ii) In case $a^2 - b^2d$ is not a square in $\mathbb{Z}$ then
  - (a) $G_f = C_4$ if and only if $d(a^2 - b^2d)$ is square in $\mathbb{Q}$. In this case, the splitting field L of $f(x)$ is a cyclic extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$, i.e., $L = \mathbb{Q}(\alpha)$.
  - (c) $G_f = D_4$ if and only if $d(a^2 - b^2d)$ is not a square in $\mathbb{Q}$. In this case the splitting field L of $f(x)$ contains a quadratic field $\mathbb{Q}(\sqrt{\triangle}) = \mathbb{Q}(\sqrt{a^2 - b^2d})$, so $L = \mathbb{Q}(\alpha, \sqrt{a^2 - b^2d})$.

Here are some examples:

α	G_f
$\sqrt{4 + \sqrt{7}}$	V_4
$\sqrt{5 + 2\sqrt{5}}$	C_4
$\sqrt{3 + 2\sqrt{5}}$	D_4

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PARSEVAL'S IDENTITY AND VALUES OF ZETA FUNCTION AT EVEN INTEGERS

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Abstract. Historically known as the Basel problem, evaluating the value of the Riemann zeta function $\zeta(2)$ has resulted in numerous proofs, many of which have been generalized to compute the function's values at even positive integers. We apply Parseval's identity to the Bernoulli polynomials to find such values.

1. Introduction

The search for the sum of all the reciprocal squares,

$$\sum\nolimits_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \tag{1.1}$$

is considered to have begun with Pietro Mengoli (1626-1686), who posed the challenge in [13] in 1650. Eventually it became known as the Basel problem, largely due to the attention given it by University of Basel professor Jakob Bernoulli (1654-1705). Bernoulli is reported to have written of it, "If somebody should succeed in finding what till now withstood our efforts and communicate it to us we shall be much obliged to him."†

Bernoulli's words convey the difficulty of the Basel problem, but his statement is even more interesting given that Bernoulli himself discovered the key to solving it. Without knowing the full significance of them, Bernoulli had derived formulae which gave the numbers that would become known as the Bernoulli numbers. These formulae were published in 1713, in his posthumous text, *Ars Conjectandi*, but it would be Leonhard Euler (1707-1783) who would use these numbers to finally answer Mengoli's challenge.

Euler was made aware of the Basel problem by Johann Bernoulli (1667-1748), his mentor and Jakob's younger brother; in papers presented from 1731-36, Euler

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† The origin of this statement is attributed to *Tractatus de Seriebus Infinitis*, a collection Bernoulli made of his own work on infinite series that was published in 1689.

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expounded an original method of approximation to achieve the exact value of the series and proved that the sum

$$\sum\nolimits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In addition, Euler discovered a second technique, which uses infinite products and partial fraction decomposition for approximating values of infinite series (see [4] for details of these publications). Euler eventually refined his methods to precisely determine the sums of reciprocal series raised to even powers. Arising in the computations are the numbers that Bernoulli discovered. Denoted as B_{2k}, we see Bernoulli numbers in the general formula

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}2^{2k-1}}{(2k)!} B_{2k}, \quad k \geq 1. \tag{1.2}$$

Up to this day no one knows the exact values of series of reciprocals raised to odd powers. [1]

The Basel problem and the study of infinite series underwent its next significant transformation due to Bernhard Riemann (1826-1866). In 1859, Riemann wrote a fundamental paper [16] studying the function represented by the series

$$\sum\nolimits_{n=1}^{\infty} \frac{1}{n^s}$$

in the region $\Re(s) > 1$. He showed this function admits an analytic continuation to the entire complex plane, except at $s = 1$ where it has a simple pole. This extended function, denoted $\zeta(s)$, is called the Riemann zeta function. In his historic paper, Riemann indicated how the study of the distribution of prime numbers is intertwined with the study of $\zeta(s)$.

Following the success of Euler and with the importance Riemann imparted on it, interest in the Riemann zeta function has continued; different approaches to the Basel problem have led to several elementary methods for finding values of $\zeta(2)$ and $\zeta(2k)$, where k is a positive integer [8, 21, 2, 10]. These approaches are the result of seeing the problem from different perspectives furnished by various branches of mathematics.

We will consider one of these methods from Fourier analysis. To evaluate $\sum \frac{1}{n^2}$, Parseval's identity applied to $f(x) = x$ is a common textbook technique (for examples, see [18, p.198] and [20, p.440]). To apply the same approach for even integer values $\zeta(2k)$, for all $k \geq 1$, one requires the appropriate function whose absolute value of n^{th} Fourier coefficient is $\frac{1}{n^k}$. We found that the Bernoulli polynomials are appropriate functions for obtaining values of $\zeta(2k)$.

The history of the Basel problem is much richer than we've been able to present here. We encourage the reader to consider the resources for this paper's introduction, in particular [4] and [11].

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[1] It is known that $\zeta(3)$ is an irrational number; this is due to Roger Apery [1]. Furthermore, T. Rivoal in [17] proved that infinitely many of $\zeta(2k+1), k = 2, 3, \cdots$ are irrational.

We have structured the paper in four sections. In section two, by following the original work of Bernoulli, Bernoulli numbers and polynomials are introduced and some of their properties are studied. The next section includes a theory of Fourier coefficients and Parseval's identity. Then a brief geometrical interpretation of this identity is discussed by means of an introductory approach to Hilbert spaces. In the last section, using properties of Bernoulli polynomials, their Fourier coefficients are computed. Then Parseval's identity is applied and the values of the zeta function at even integers are computed (Theorem 3). The last section is concluded by some remarks on our proof and related works in the literature. All sections are written to be accessible to undergraduate math students and we have tried to keep with the historical order.

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2. Bernoulli Numbers and Polynomials

The starting point of the Bernoulli polynomials goes back to the sum of powers of integer numbers. By the 6th Century B.C.E. the Pythagoreans knew how to find the sum of the first natural numbers,

$$\sum\nolimits_{n=1}^{m-1} n = \frac{1}{2}m(m-1) = \frac{m^2}{2} - \frac{m}{2}. \tag{2.1}$$

Archimedes (c.287-212 B.C.E.) discovered how to calculate the sum of squares: [11]

$$\sum\nolimits_{n=1}^{m-1} n^2 = \frac{1}{6}m(m-1)(2m-1) = \frac{1}{3}m^3 - \frac{1}{2}m^2 + \frac{1}{6}m. \tag{2.2}$$

Finding sums of other powers began to reach its climax in the 17th Century, with mathematicians such as Pierre de Fermat and Blaise Pascal coming closer to the objective. Then, Jacob Bernoulli discovered the right way of looking at the problem.

Let us first fix a notation[2]

$$S_p(m) := \sum_{n=1}^{m-1} n^p. \tag{2.3}$$

In the study of binomial coefficients, Bernoulli found the following identity[3]

$$\sum\nolimits_{n=0}^{m-1} \binom{n}{p} = \binom{m}{p+1}.$$

Note that when expanded, the summand of the left side, $\frac{1}{p!}n(n-1)\cdots(n-p+1)$, will give a polynomial of degree p in n. The sums of each term of this polynomial gives some $S_k(m)$. Using this identity and by induction, Bernoulli found values of $S_p(m)$ for $p = 1, \cdots, 10$ [6]. Furthermore, by an attentive examination of the

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[2] Bernoulli in [6] looks for sums of first m numbers rather than $m-1$. This will introduce some slight differences between what we will find and what is available in Bernoulli's notes.

[3] A very good exercise for the interested reader would be to attempt a combinatorial proof for this identity.

these formulae, he discovered the pattern for coefficients of S_p. This pattern is the main focus of the following theorem.

Theorem 1. *Let S_p be the quantity defined by* (2.3). *Then $S_p(m)$'s are polynomials of order $p+1$ in m and there is a sequence of rational numbers $\{B_j\}_{j=0}^{\infty}$ such that*

$$S_p(t) = \frac{1}{p+1} \sum\nolimits_{j=0}^{p} B_j \binom{p+1}{j} t^{p-j+1}, \qquad p > 1. \tag{2.4}$$

These numbers satisfy the following recursive relation

$$B_j = -\frac{1}{j+1} \sum\nolimits_{l=0}^{j-1} B_l \binom{j+1}{l}, \quad B_0 = 1. \tag{2.5}$$

Proof. Let's first find a recursive formula for S_p. To do so we will apply a simple trick which is the change of index of summation in the definition of S_p from n to $n-1$:

$$\begin{aligned} S_{p+1}(m+1) - 1 = \sum_{n=2}^{m} n^{p+1} &= \sum_{n=1}^{m-1} (n+1)^{p+1} \\ &= \sum_{n=1}^{m-1} \sum_{k=0}^{p+1} \binom{p+1}{k} n^k = \sum_{k=0}^{p+1} \binom{p+1}{k} S_k(m). \end{aligned}$$

The above equality can be used to write

$$(p+1)S_p(m) = S_{p+1}(m+1) - 1 - S_{p+1}(m) - \sum_{k=0}^{p-1} \binom{p+1}{k} S_k(m).$$

Using the simple fact that $m^{p+1} = S_{p+1}(m+1) - S_{p+1}(m)$, we find the recursive formula

$$S_p(m) = \frac{1}{p+1} \left(m^{p+1} - 1 - \sum_{k=0}^{p-1} \binom{p+1}{k} S_k(m) \right), \tag{2.6}$$

where the initial value is given by $S_0(m) = m - 1$. A direct result of this recursive formula is that S_p is a polynomial of order $p+1$ for every p, with rational coefficients. From now on we will change the integer variable m to the general real variable t.

To prove (2.4) we will use induction on p and construct B_j as induction proceeds. As for the base case, we set $B_0 = 1$ and $B_1 = -\frac{1}{2}$, then it is easily seen that $S_1(t)$ is of the form given by (2.4).

Then, as the induction hypothesis, assume that there are rational numbers $\{B_j\}_{j=0}^{p-1}$ such that for all $k < p$ we have

$$S_k(t) = \frac{1}{k+1} \sum_{l=0}^{k} B_l \binom{k+1}{l} t^{k-l+1}, \qquad p > 1 \tag{2.7}$$

One can readily see that $S_k(1) = 0$ and using the induction hypothesis (2.7), the constants B_k, for $k < p$, satisfy the equality

$$B_k = -\frac{1}{k+1} \sum_{l=0}^{k-1} B_l \binom{k+1}{l}, \quad 1 \le k < p. \tag{2.8}$$

Now, substituting from (2.7) the formula for S_k into the recursive formula (2.6) and noticing that while doing so we should use $S_0(t) = t - 1 = B_0(t) - 1$, we get

$$S_p(t) = \frac{1}{p+1}\left(t^{p+1} - \sum_{k=0}^{p-1}\binom{p+1}{k}\frac{1}{k+1}\sum_{l=0}^{k} B_l\binom{k+1}{l}t^{k-l+1}\right).$$

By setting a new variable $j = p - k + l$ we can find the coefficient of t^{p-j+1} for any $1 \leq j \leq p$ given by

$$\frac{-1}{p{+}1}\sum_{l=0}^{j-1}\binom{p{+}1}{p{-}j{+}l}\binom{p{-}j\ {+}l{+}1}{l}\frac{B_l}{p{-}j\ {+}l{+}1} = \frac{\binom{p+1}{j}}{p{+}1}\left(\frac{-1}{j{+}1}\sum_{l=0}^{j-1}\binom{j{+}1}{l}B_l\right).$$

Now we can use (2.8) for any $j < p$; however, for $j = p$ we let B_p be defined by

$$B_p = -\frac{1}{p+1}\sum\nolimits_{l=0}^{p-1}\binom{p+1}{l}B_l.$$

Therefore we have

$$S_p(t) = \frac{1}{p+1}\left(t^{p+1} + \sum_{j=1}^{p}\binom{p+1}{j}B_j t^{p-j+1}\right).$$

Noting that $B_0 = 1$, the induction step is complete. Observe that while proving the induction step, we constructed the sequence B_p inductively such that the relation (2.5) is satisfied. ☐

Definition 1. The constant B_j, obtained in the above theorem, is called the j^{th} *Bernoulli number.*[4]

In the early 1730s, while proving his summation formula, Euler also discovered these numbers [9]. Among the many of his discoveries was a recursive formula for finding the Bernoulli numbers, and a generating function. Here we shall use the recursive formula (2.5) to show how the generating function can be computed.

Let $G(x)$ be the generating function of the Bernoulli numbers, i.e. formally $G(x) = \sum_{j=0}^{\infty}\frac{B_j}{j!}x^j$. Then we have

$$\begin{aligned} G(x) = \sum_{j=0}^{\infty}\frac{B_j}{j!}x^j &= 1 - \sum_{j=1}^{\infty}\frac{1}{(j+1)!}\sum_{l=0}^{j-1}B_l\binom{j+1}{l}x^j \\ &= 1 - \sum_{j=1}^{\infty}\sum_{l=0}^{j-1}\frac{B_l}{(j+1-l)!l!}x^j \\ &= 1 - \sum_{l=0}^{\infty}\frac{B_l}{l!}\sum_{j=l+1}^{\infty}\frac{1}{(j+1-l)!}x^j \\ &= 1 - \sum_{l=0}^{\infty}\frac{B_l}{l!}\sum_{j=2}^{\infty}\frac{1}{j!}x^{j+l-1} \end{aligned}$$

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[4] Bernoulli originally denoted B_2 by A, and B_3 by B, so on and so forth.

$$
\begin{aligned}
&= 1-\frac{1}{x}\left(\sum\nolimits_{l=0}^{\infty}\frac{B_l}{l!}x^l\right)\left(\sum\nolimits_{j=2}^{\infty}\frac{1}{j!}x^j\right)\\
&= 1-(1/x)G(x)\left(e^x-x-1\right).
\end{aligned}
$$

Therefore $G(x)=1-(1/x)\left(e^x-x-1\right)G(x)$, which implies that

$$G(x)=x/(e^x-1). \tag{2.9}$$

From (2.9) one can find

$$B_0=1,\, B_1=-\frac{1}{2},\, B_2=\frac{1}{6},\, B_3=0,\, B_4=-\frac{1}{30},\, B_5=0,\, B_6=\frac{1}{42},\; B_7=0,\cdots.$$

Note that $G(x)-(-\frac{1}{2}x)=\frac{x(e^x+1)}{2(e^x-1)}$ is an even function. This implies that all the odd Bernoulli numbers, B_{2k+1} for $k\geq 1$, are zero (B_1 is the exception).

Definition 2. The derivative of the polynomial $S_p(t)$ is called the p^{th} *Bernoulli polynomial* and we denote it by $B_p(t)$.

Bernoulli polynomials are monic polynomials and they can be written in terms of Bernoulli numbers as follows (derive (2.4)):

$$B_p(t):=\sum\nolimits_{k=0}^{p}B_k\binom{p}{k}t^{p-k},\quad k\geq 0. \tag{2.10}$$

Using (2.10) and (2.9), one can easily find the generating function of the Bernoulli polynomials

$$G(x,t)=\sum\nolimits_{p=0}^{\infty}B_p(t)(x^p/p!)=xe^{tx}/(e^x-1). \tag{2.11}$$

Examples of the first few Bernoulli polynomials are

$$B_0(t)=1,\; B_1(t)=t-(1/2),\; B_2(t)=t^2-t+(1/6),\; B_3(t)=t^3-(3t^2/2)+(t/2).$$

By differentiating (2.10) we have

$$B_p'(t)=pB_{p-1}(t),\quad p\geq 1. \tag{2.12}$$

As a result, we have $S_p'(t)=B_{p+1}'(t)/(p+1)$, which can be used to write the sums of powers in terms of Bernoulli polynomials

$$S_p(m)=S_p(m)-S_p(0)=\int_0^m S_p'(t)dt=\int_0^m\frac{B_{p+1}'(t)}{p+1}=\frac{1}{p+1}(B_{p+1}(m)-B_{p+1}(0)). \tag{2.13}$$

Additionally, (2.10) readily shows $B_p(0)=B_p$. Moreover, by (2.13) we have

$$0=S_p(1)=(1/(p+1))(B_{p+1}(m)-B_{p+1}(0)).$$

Therefore,

$$B_p(1)=B_p(0)=B_p,\quad p\geq 2. \tag{2.14}$$

The reader can refer to [3] for more details and more identities involving Bernoulli numbers and polynomials.

3. Fourier Series and Parseval's Identity

In this section, we will introduce Fourier coefficients and Parseval's identity which play a central role in our strategy to find values of the zeta function at even integers. Fourier analysis, at its original form, is concerned with the decomposition of functions as infinite sums of trigonometric functions.

This branch of mathematical inquiry arose following Joseph Fourier (1768-1830) who, motivated by a need for formulae that could model the conduction of heat, used this technique to find real-valued solutions of functions; they are also used to measure frequencies of vibrations.

Parseval's identity is named for Marc-Antoine Parseval (1755-1836), and deals with summability of the Fourier coefficients. From a different perspective, both of these tools are among the first versions of more abstract theory, that is the theory of Hilbert spaces. We have chosen the latter to present the topic here; however, to avoid the technical complications we shall not include proofs and instead show similar results in the finite case to help the reader develop the right intuition for the topic.

Definition 3. Let f be an integrable function on $[0,1]$ then the n^{th} Fourier coefficient $c_n(f)$ of f is defined by

$$c_n(f) := \langle f, e_n \rangle = \int_0^1 f(t)e^{-2\pi int}dt, \quad n \in \mathbb{Z}.$$

To understand a geometric meaning of the Fourier coefficients we need to see them in the general setting of Hilbert spaces, which are complex vector spaces equipped with a Hermitian inner product with complete topology. Further introduction to Fourier analysis in Hilbert space can be found in [19]. Let's first see the finite dimensional versions of such spaces $V = \mathbb{C}^k$ with the inner product given by

$$\langle (v_1, \cdots, v_k), (w_1, \cdots, w_k) \rangle = \sum\nolimits_{j=1}^{k} v_j \overline{w_j}.$$

Here $\overline{w_j}$ denotes the complex conjugate of the complex number w_j. On such vector space we can define the length of vectors by

$$\|v\| := \sqrt{\langle v, v \rangle}, \qquad v \in V. \tag{3.1}$$

Let e_n, for all $1 \leq n \leq k$, denote the vector with 1 in the n^{th} component and zero in all other components. The set of vectors $\{e_n\}_{n=1}^m$ form a so-called orthonormal basis for V; that is,

(1) they are orthonormal: $\langle e_n, e_m \rangle = \delta_{nm} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$;

(2) every vector in V can be written as a linear combination of e_n's.

The important property of an orthonormal basis, in particular $\{e_n\}$, is that the coefficients of e_n in the linear combination which gives the vector $v \in V$ is given by the inner product. In other words, if $v = \sum_{m=1}^{k} c_m e_m$ then

$$\langle v, e_n \rangle = \langle \sum\nolimits_{m=1}^{k} c_m e_m, e_n \rangle = \sum\nolimits_{m=1}^{k} c_m \langle e_m, e_n \rangle = \sum\nolimits_{m=1}^{k} c_m \delta_{nm} = c_n.$$

Moreover, the length of a vector can also be computed using its inner product by e_n as follows

$$\|v\|^2 = \| \sum\nolimits_{m=1}^{k} \langle v, e_m \rangle e_m \|^2 = \sum\nolimits_{n=1}^{k} |\langle v, e_n \rangle|^2. \tag{3.2}$$

This equality is nothing but the Pythagorean theorem in higher dimensional Hermitian spaces.

To come back to our case, where the Fourier coefficients can be obtained, we need infinite dimensional complete inner product spaces, called Hilbert spaces. Consider the linear space

$$H = \left\{ f : [0,1] \to \mathbb{C} |\ f \text{ is measurable and } \int_0^1 |f(t)|^2 dt < \infty \right\}.$$

Here the functions are going to play the role of vectors and the inner product is given by

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

The norm of a function (called L^2-norm) is defined as (3.1) using this inner product. Unlike V, H is infinite dimensional, meaning that we need infinitely many elements $\{e_n\}_{j=1}^{\infty}$ to form a basis. Also, we may encounter infinite sums while trying to write linear combinations of elements, so a notion of convergence of linear combinations will be needed. In particular, the second criteria in the definition of orthonormal basis should be replaced by

(2'). Every vector in H is the limit of (possibly infinite) linear combinations of e_j's.

As an example, the functions $e_n(t) := e^{2\pi int}, \quad n \in \mathbb{Z}$, form an orthonormal basis for H (see examples in [18, p.187]). With all these in hand it is obvious that $c_n(f) = \langle f, e_n \rangle$. Moreover, an infinite dimensional version of the computation (3.2) can be performed and the result is known as Parseval's identity (for a proof see [18, p.191]).

Theorem 2 (Parseval's identity)**.** *Suppose f is a Riemann-integrable function. Then*

$$\int_0^1 |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n(f)|^2. \tag{3.3}$$

Similar to (3.2), Parseval's identity can be considered as the generalization of the Pythagorean theorem in infinite dimensional space H, where the absolute value of the Fourier coefficients $|c_n(f)|$ play the role of length of the orthogonal sides of (an infinite dimensional) right triangle, and the sum of their squares is equal to the square of length of the function (hypotenuse) given by the integral.

Remark 1. At the beginning of the 20th Century David Hilbert (1862-1943) introduced abstract inner product spaces to embrace existing theories of function spaces, such as Fourier analysis, and to develop new tools to study such notions as integral operators. In particular, these abstract spaces, known as Hilbert spaces, allow for the manipulation of functions which otherwise would not meet the conditions of convergence and continuity required to perform such manipulations.

4. The Main Theorem

In this section we show how the values of $\zeta(2k)$ are obtained by applying Parseval's identity to the Fourier coefficients of Bernoulli polynomials. We first find the Fourier coefficients of Bernoulli polynomials in Lemma 1, and then their L^2 norm. In both the following lemmas, the main idea lies in the following simple computation for any differential function f on $[0,1]$ which one can easily obtain by integration by parts and (2.12) and (2.14):

$$\int_0^1 B_k(t)f'(t)dt = (f(1)-f(0))B_k - k\int_0^1 B_{k-1}(t)f(t)dt, \quad k \geq 2,$$

while for $k=1$ we have $\int_0^1 B_1(t)f'(t)dt = (f(1)+f(0))B_1 - \int_0^1 f(t)dt$.

Lemma 1. *For all integers $k \geq 1$,*

$$c_n(B_k) = \begin{cases} \frac{-k!}{(2\pi in)^k}, & n \neq 0; \\ 0, & n = 0. \end{cases}.$$

Proof. Observe that by the definition of the Bernoulli polynomials in terms of S_p we have $\int_0^1 B_k(t)dt = S_p(1) - S_p(0) = 0$, so that $c_0(B_k) = 0$, $k \geq 1$. Let $n \neq 0$. By integration by parts

$$c_n(B_k) = B_k(t)\frac{e^{-2\pi int}}{-2\pi in}\bigg]_0^1 - \int_0^1 B_k'(t)\frac{e^{-2\pi int}}{-2\pi in}\,dt,$$

in which the first term vanishes for $k \geq 2$ because of (2.14); and applying (2.12) it reduces to

$$c_n(B_k) = \frac{k}{2\pi in}\int_0^1 B_{k-1}(t)e^{-2\pi int}\,dt = \frac{k}{2\pi in}c_n(B_{k-1}). \tag{4.1}$$

Now, for $k=1$ this gives $c_n(B_1) = -1/2\pi in$ and combining this with (4.1) we recursively get $c_n(B_k) = -k!/(2\pi in)^k$ for all k. ☐

Remark 2. Another interesting approach to find the Fourier coefficients of Bernoulli polynomials is to use their generating function (2.11). Being careful with the convergence conditions, one needs to see that

$$\int_0^1 G(x,t)e^{-2\pi int}dt = \sum\nolimits_{p=0}^{\infty} c_n(B_p)\frac{x^p}{p!}.$$

To obtain L^2 norm of Bernoulli polynomials, we first shall find a recursive expression for the integration of products of two Bernoulli polynomials.

Lemma 2. *For all integers $1 \leq k \leq l$,*

$$\int_0^1 B_k(t)B_l(t)dt = \frac{(-1)^{k-1}l!k!B_{l+k}}{(l+k)!}.$$

Proof. Denoting the left side by $A_{k,l}$, using (2.12) and integrating by parts we get

$$A_{k,l} = \int_0^1 B_k(t)\frac{B_{l+1}'(t)}{l+1}\,dt = B_k(t)\frac{B_{l+1}(t)}{l+1}\bigg]_0^1 - \int_0^1 B_k'(t)\frac{B_{l+1}(t)}{l+1}dt$$

in which the first term on the right vanishes for $k \geq 2$ because of (2.14); and applying (2.12) it reduces to

$$A_{k,l} = \frac{-k}{(l+1)} \int_0^1 B_{k-1}(t) B_{l+1}(t) dt = \frac{-k}{(l+1)} A_{k-1,l+1}. \tag{4.2}$$

Now, for $k = 1$ we have

$$A_{1,l} = \frac{B_{l+1}(0)}{l+1} \left(B_1(1) - B_1(0) \right) - \int_0^1 \frac{B_{l+1}(t)}{l+1} dt = \frac{B_{l+1}}{l+1}, \tag{4.3}$$

and hence the desired result $A_{k,l} = ((-1)^{k-1} l!\ k!\ B_{l+k})/(l+k)!, \quad 0 < k \leq l$ is obtained recursively by combining (4.2) and (4.3). □

Finally, we have the main theorem.

Theorem 3. *For any positive integer $k \geq 1$, we have*

$$\zeta(2k) = \frac{(-1)^{k-1} \pi^{2k} 2^{2k-1}}{(2k)!} B_{2k}. \tag{4.4}$$

Proof. Applying Parseval's identity to B_k, $k \geq 1$, we have

$$\int_0^1 |B_k(t)|^2 dt = \sum_{-\infty}^{\infty} |c_n(B_k)|^2.$$

The value of the left side, given by Lemma 2, is equal to

$$\int_0^1 B_k(t) B_k(t) dt = ((-1)^{k-1} (k!)^2 B_{2k})/(2k)!. \tag{4.5}$$

The sum on the right side, provided by Lemma 1, gives us

$$\sum_{-\infty}^{\infty} |c_n(B_k)|^2 = \sum_{n \neq 0} \left| \frac{-k!}{(2\pi i n)^k} \right|^2 = 2 \frac{(k!)^2}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}. \tag{4.6}$$

From (4.5) and (4.6) we get (4.4). □

We would like to finish this section with a few remarks on our proof and other related works.

Remark 3. Our work is not the first one which extends a method to evaluate the zeta function at two, to a general method to find $\zeta(2k)$, and in it to bring in Bernoulli polynomials; for example, see [7] where a telescoping series technique to find $\zeta(2)$, offered in [5], is generalized to find $\zeta(2k)$ using Bernoulli polynomials.

Remark 4. Despite the very central role of Bernoulli polynomials in our work, there is nothing that made them unique in this process. In fact, there are infinitely many families of functions f_k that can do the job. In fact, every function f_k whose Fourier coefficients are different than that of Bernoulli polynomials by a phase factor, $c_n(f_k) = e^{i\theta_{n,k}} c_n(B_k)$ with $\theta_{n,k} \in [0, 2\pi]$, can be used here. On the other hand, a closer look at our proof reveals that property (2.12) is critical to it. For example, $f_k(x) = x^k$ is another family of functions with the same property.[5]

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[5] While preparing this paper, we became aware of the recently-posted paper [12] where Parseval's identity is applied on x^k to find $\zeta(2k)$.

Remark 5. Another technique in evaluating the zeta function at even integers involves the pointwise convergence of Fourier series $\sum_{n\in\mathbb{Z}} c_n(f)e^{2\pi int}$ to $f(t)$. For a beautiful instance of this technique see [15].

Remark 6. The functional equation

$$\zeta(s) = 2^s\pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(1-s)\ \zeta(1-s) \tag{4.7}$$

of the Riemann zeta function, where Γ is the gamma function, plays a very important role in the study of the Riemann zeta function. The functional equation relates the value of the zeta function at s to its value at $1-s$. Hence we can now see that

$$\zeta(-2k+1) = -B_{2k}/2k, \quad k \geq 1.$$

In fact, this is true for all negative integers and the odd Bernoulli numbers being zero gives the trivial zeros of the Riemann zeta function at negative even integers. An interesting study, investigating the relation between the values of zeta at negative integers and functions S_p can be found in [14].

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ON THE ORIENTABILITY OF COMPACT HYPERSURFACES IN EUCLIDEAN SPACE

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Abstract. In this expository article we discuss the concept of transversality and use it to explain why a compact hypersurface in $\mathbb{R}^n$ is orientable. In order to keep the discussion as elementary and nontechnical as possible we have taken the liberty of placing our emphasis on illustrative examples and geometric ideas involved rather than complete formal proofs.

1. Introduction

The notion of a smooth (or differentiable) manifold arose naturally from the study of curves and surfaces in three-dimensional Euclidean space and ranks among the most fundamental concepts of modern mathematics. Precise definitions will be given below but let us first take an informal look. Roughly speaking, a smooth manifold X of dimension k in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which, for the purposes of differential calculus, may be locally regarded as an open subset of $\mathbb{R}^k$ (possibly in several different ways). If we take for instance the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, the upper hemisphere $\{(x, y, z) \in \mathbb{S}^2 : z > 0\}$ may be identified with an open disc in the plane $Z = -1$ by orthogonally projecting it onto the plane as shown in Figure 1(A) on the next page. In a similar fashion the lower hemisphere (Figure 1(B) on the next page), or the left and the right hemispheres, may all be regarded as open discs by orthogonally projecting them onto appropriate planes. Thus, while the whole sphere at once cannot be regarded as an open subset of a plane, any small enough piece of it may be so regarded.

To study smooth manifolds it is necessary to extend the basic concepts and methods of calculus to manifolds. This can be done by using the local identification of the manifold as open sets in $\mathbb{R}^k$. It is, for example, clear how to make sense of a smooth map between two manifolds: locally any such map can be identified with a map between open sets in Euclidean spaces and we will say that the original map is smooth if this later map is smooth (in the ordinary sense). A straightforward application of the chain rule ensures that this definition of smoothness of maps is independent of the local identification chosen. A very important construction for the study of smooth manifolds is that of the *tangent space* at each point of the

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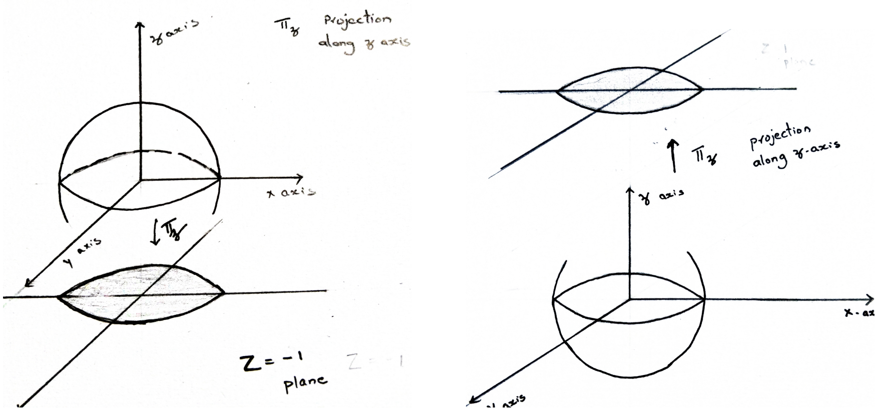


(A) (B)

Figure 1. Sphere

manifold which is the infinitesimal *linear approximation* to the manifold near that point. This allows us to make use of the concepts from linear (and multilinear) algebra in an essential way in the study of smooth manifolds. In particular, it is possible to linearize smooth maps and extend all the key *local* results of calculus like inverse and implicit function theorems to the manifold setting.

A manifold of dimension $n-1$ in $\mathbb{R}^n$ is called a *hypersurface* in $\mathbb{R}^n$. Thus a curve (one-dimensional manifold) in the plane or a sphere in three-dimensional Euclidean space are some examples of hypersurfaces. It is sometimes necessary to give a direction or *orientation* to a hypersurface. For example, the familiar Stokes' formula in vector calculus uses such a notion. If X is a hypersurface in $\mathbb{R}^n$ and x is a point in X, the tangent space T_xX is a $n-1$ dimensional linear subspace of $\mathbb{R}^n$. Therefore there are exactly two unit normal vectors to the hypersurface X at x. We say that a *loop* $\alpha : [0,1] \to X$ is *orientation reversing* if there exists a unit normal vector to X at $\alpha(0)$ which, when transported continuously along α while keeping it unit normal to X, comes back to the opposite unit normal at $\alpha(1) = \alpha(0)$. By definition, X is *orientable* if there does not exist any orientation reversing loop in X. A moment's thought will easily convince the reader that a plane or a sphere in $\mathbb{R}^3$ are some examples of orientable surfaces and so is a circular cylinder or a torus. However not all surfaces in $\mathbb{R}^3$ are orientable. The simplest such example is provided by the so called *Mobius band.* It is possible to write down equations for this set but it is much easier to construct a paper model of it by taking a rectangular sheet of paper and identifying a pair of opposite edges after giving it a half-twist. See Figure 2 on the next page. A little experimentation with the paper model will convince the reader that the central circle is an orientation reversing loop.

In this example note that the boundary circle is not part of the Mobius band

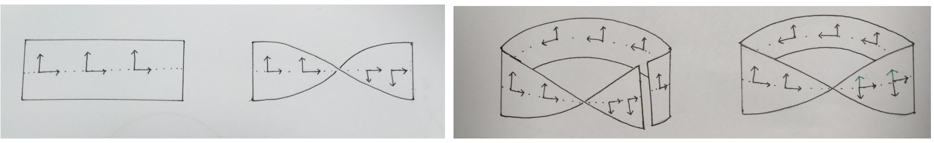


FIGURE 2. Mobius Strip

(otherwise it will *not* be a manifold in our sense) and thus the Mobius band is not compact. The purpose of this article is to explain the following

Theorem. *Every compact hypersurface in* $\mathbb{R}^n$ *must be orientable.*

A proof due to Hans Samelson of this theorem (see [6]) based on transversality theory will be sketched in section 3. A simple but interesting consequence of the theorem is given after the proof. For the proof of the theorem it is necessary to consider a generalisation of the notion of manifolds called manifolds-with-boundary. Just as a manifold is locally modelled on open subsets in some Euclidean space, a manifold-with-boundary is modelled on open subsets of the half-space $\mathbb{H}^k = \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_k \geq 0\}$.

For a manifold-with-boundary, each point has a (relative) neighborhood that can be identified either with an open subset in $\operatorname{int} \mathbb{H}^n$ (in which case we say that the point in question is an *interior point*) or with an open subset of $\mathbb{H}^n$ in such a way that the point is identified with some point in $\partial\mathbb{H}^n$ (in which case we say that the point is a *boundary point.*). The set of all boundary points constitutes the *boundary* of the manifold, which is itself a (boundaryless) manifold of dimension one less than that of the original manifold. This is illustrated in Figure 3. Most of the concepts that we have discussed for manifolds extend to manifolds-with-boundary.

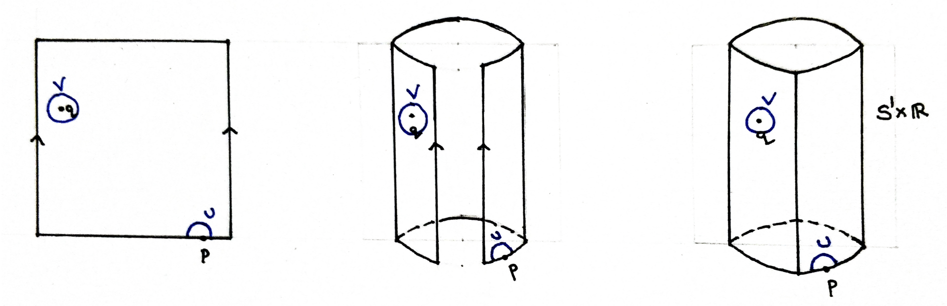


FIGURE 3. Cylinder $S^1 \times \mathbb{R}$

We will now give some definitions.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let U and V be two open subsets in $\mathbb{R}^n$. A map $f : U \to V$ is called *smooth* if it has continuous partial derivatives of all orders on U. For our discussion below, it is necessary to be able to talk about smoothness for maps that are defined on arbitary subsets of $\mathbb{R}^n$. Let $X \subseteq \mathbb{R}^n$. A map $f : X \to \mathbb{R}^m$ is said to be smooth if for every $x \in X$ there exist an open set $U \subseteq \mathbb{R}^n$ and a smooth map $F : U \to \mathbb{R}^m$ such that $F \mid_{U \cap X} = f$. Thus a map defined on an arbitrary subset is smooth if it can be locally extended to a smooth map around each point of the subset. A map $f : X \to Y$ is called a diffeomorphism if f is bijective and both f and f^{-1} are smooth maps. We are now in a position to define manifolds precisely.

A subset X in $\mathbb{R}^n$ is called a smooth n-dimensional manifold if each $x \in X$ admits a (relative) neighborhood $U \subseteq X$ and a diffeomorphism $\phi : U \to V$ where V is an open subset of $\mathbb{R}^n$. Such a pair (U, ϕ) is called a *chart* about x. An *atlas* for X is a collection of charts $\{(U_i, \phi_i)\}$ such that $\cup_i U_i = X$. The maps $\phi_j \circ \phi_i^{-1}$, which are diffeomorphisms between certain open subsets in $\mathbb{R}^n$, are called *transition functions* of the atlas. An atlas for a manifold thus provides a means for studying the manifold in a piece-by-piece manner.

Let us see some examples of smooth manifolds.

(i) Any open subset of $\mathbb{R}^n$ is a manifold of dimension n.

(ii) Spheres: The n-dimensional Sphere $\mathbb{S}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$ is a smooth n-dimensional manifold. In order to see this, let $x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{S}^n$. There exists an i such that $1 \leq i \leq n+1$ and $x_i \neq 0$. If $x_i > 0$, let $U = \{y = (y_1, y_2, \ldots, y_{n+1}) \in \mathbb{S}^n : y_i > 0\}$; otherwise let $U = \{y = (y_1, y_2, \ldots, y_{n+1}) \in \mathbb{S}^n : y_i < 0\}$. If $D = \{v \in \mathbb{R}^n : \|v\| < 1\}$ and $\phi : U \to D$ is defined as $\phi(y_1, y_2, \ldots, y_{n+1}) = (y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1})$, then (U, ϕ) provides a chart about x, as can be easily checked. This construction has already been illustrated for the special case $n = 2$ in Figure 1.

(iii) Our next example is a general construction of building new examples out of the old ones. If M and N are two smooth manifolds then their cartesian product $X \times Y$ is a smooth manifold of dimension equal to $\dim M + \dim N$. If $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$ are atlases for X and Y then the collection $\{(U_i \times V_j, \phi_i \times \psi_j)\}$ provides an atlas for $X \times Y$. An immediate consequence of this construction is that the n-dimensional Torus $\mathbb{T}^n = S^1 \times S^1 \times \ldots \times S^1 \subseteq \mathbb{R}^{2n}$ is a smooth manifold of dimension n. See Figure 4 for the 2-dimensional case.

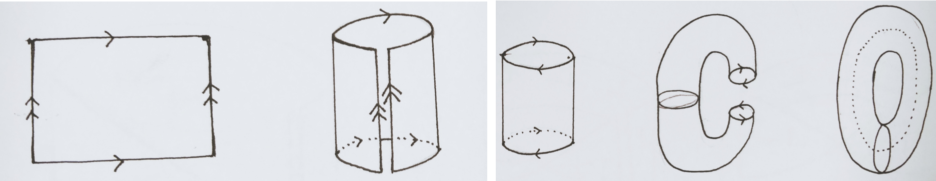


FIGURE 4. Torus

Let $f : X \to Y$ be a map between smooth manifolds. We say that f is smooth at $x \in X$ if the map $\psi \circ f \circ \phi^{-1} : U \to V$ is smooth at 0. Here (U, ϕ) is a chart about x in X and (V, ψ) is a chart about $f(x)$ in Y. This is clearly independent of the choice of the charts chosen. The map is called smooth if it is smooth at every point of the domain.

Let X be a smooth manifold and $x \in X$. Let $T_xX = \{\sigma'(0) |\ \sigma : (-\epsilon, \epsilon) \to X$ smooth and $\sigma(0) = x\}$. Thus, T_xX is the set of all tangent vectors to curves on X which pass through the point x. It is not difficult to show that $T_xX = D\phi^{-1}(0)(\mathbb{R}^k)$ for any chart ϕ about x with $\phi(x) = 0$. Thus the set T_xX is a k-dimensional linear subspace of $\mathbb{R}^n$, which we call the tangent space to X at x.

Every smooth map $f : X \to Y$ induces (for any $x \in X$) a linear map $Df(x) : T_xX \to T_{f(x)}Y$ which is defined as follows:

$$Df(x)(v) = (f \circ \sigma)'(0),$$

where $\sigma : (-\epsilon, \epsilon) \to X$ is any smooth curve with $\sigma(0) = x$ and $\sigma'(0) = v$.

Certain maps play a special role in differential topology. A map $f : X \to Y$ is called a *submersion* if its differential map $Df(x) : T_xX \to T_{f(x)}Y$ is surjective for every $x \in X$. A point $y \in Y$ is called a *regular value* of f if the map $Df(x) : T_xX \to T_yY$ is surjective for each $x \in f^{-1}(y)$. A fundamental theorem of topology (Sard's theorem) asserts that the set of regular values of any smooth map is dense.

Let X be a smooth k-dimensional manifold. We say that a subset Z of X is a submanifold of X of dimension l if for any point $z \in Z$, there exist a chart (U, ϕ) about z in X such that $\phi_{l+1} = \phi_{l+2} = \ldots = \phi_k = 0$ on $U \cap Z$. In this case, Z itself inherits the structure of a smooth l-manifold with the collection $\{(U \cap Z, (\phi_1|_{U\cap Z}, \phi_2|_{U\cap Z}, \ldots, \phi_l|_{U\cap Z}))\}$ serving as an atlas for Z. The codimension of Z in X is the number $k - l$. Submanifolds of codimension 1 are called hypersurfaces.

The importance of regular values comes from the following property which is proved using the inverse function theorem from calculus:

Let $f : X \to Y$ be a smooth map and let $y \in Y$ be a regular value of f. Then the inverse image $Z = f^{-1}(y)$ is a submanifold X with codimension $\dim Y$. *Moreover, $T_zZ = \ker Df(z) : T_xX \to T_yY$ for any $z \in Z$.*

Let $\mathbb{H}^k = \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_k \geq 0\}$. We call $\mathbb{H}^k$ the k-dimensional half-space. A subset X of $\mathbb{R}^n$ is called a k-dimensional smooth manifold-with-boundary if each $x \in X$ admits a (relative) neighbourhood U and a diffeomorphism $\phi : U \to V$ where V is an open subset of the half-space $\mathbb{H}^k$. Concepts like tangent space and submanifolds that were discussed for manifolds readily extend to manifolds-with-boundary. In the proof of the theorem we will also make use of the following classification theorem for compact one dimensional manifolds-with-boundary. We refer the reader to [1] for a proof at the expository level.

Fact 0. *(Classification of compact one dimensional manifolds-with-boundary.) Any compact and connected one dimensional manifold-with-boundary is diffeomorphic to either the circle S^1 (which has empty boundary) or the closed interval* $[0, 1]$.

2. Transversality

In this section we will introduce and explain the concept (originally due to the French mathematician René Thom) that will play a key role in the proof of our theorem. We should mention that this important idea has many other applications in differential topology. For a thorough discussion of the concept of transversality (which is merely outlined in this article) and its many applications in topology we refer the interested reader to the excellent book [3].

Given a vector space and two of its subspaces, we say that these subspaces intersect transversally (or, are in general position) if the sum of the subspaces equals the original space. Note that we do *not* require that the vector space be the direct sum of the subspaces, only that the subspaces together span the space. As an example, observe that any two coordinate planes in $\mathbb{R}^3$ intersect transversally. We now extend this notion to manifolds and maps. Unless otherwise mentioned all our manifolds below are boundaryless.

Definition. Let X and Y be manifolds and $f : X \to Y$ be a smooth map. Let $Z \subset Y$ be a submanifold. We say that f is *transversal* to the submanifold Z (written $f \pitchfork Z$) if for every $x \in f^{-1}(Z)$ the subspaces $Df(x)(T_xX)$ and $T_{f(x)}Z$ of $T_{f(x)}Y$ intersect transversally, i. e. ,

$$Df(x)(T_xX) + T_{f(x)}Z = T_{f(x)}Y.$$

If Z and W are two submanifolds of a manifold Y we say that they intersect *transversally* (written $W \pitchfork Z$) if the inclusion map $i : W \to Y$ is transversal to Z. In other words, $T_wW + T_wZ = T_wY$ for every $w \in W \cap Z$. Observe that this later condition is symmetrical in W and Z. See the pictures in the following pages for examples of transversal and non-transversal intersections.

Consider now a special case. Let y_0 be a point in Y and let $Z = \{y_0\}$. Then $f \pitchfork Z$ if and only y_0 is a regular value of f. Thus the notion of transversality may be viewed as a generalisation of the notion of regular values. In the later case we have seen that the level set $f^{-1}(y_0)$ is a submanifold of X of codimension equal to $\dim Y$. This raises the following question:

If $f : X \to Y$ is a map and Z is a submanifold of Y and if $f \pitchfork Z$, is it true that $f^{-1}(Z)$ is a submanifold of X?

The answer is yes and can be seen as follows. Let $x \in f^{-1}(Z)$ and $y = f(x)$. Assume that Z is a submanifold of Y of codimension l. Then, in some (relative) neighborhood of the point y, Z can be expressed as $Z = g^{-1}(0)$ for some submersion g from some neighborhood of y in Y into $\mathbb{R}^l$. Therefore, in a neighborhood of x, $f^{-1}(Z)$ equals $f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$. Thus $f^{-1}(Z)$ is (locally) a submanifold of X of codimension l if 0 is a regular value of $g \circ f$. This latter condition is that $D(g \circ f)(x)(T_xX) = \mathbb{R}^l$, which is equivalent to $Dg(f(x))(Df(x)(T_xX)) = \mathbb{R}^l$. Since g is a submersion, $Dg(f(x))(T_{f(x)}Y) = \mathbb{R}^l$ and $\mathrm{Ker}Dg(f(x)) = T_{f(x)}Z$. Therefore, if $Df(T_xX) + T_{f(x)}Z = T_{f(x)}Y$, then the required condition is certainly satisfied. This, however, is just the transversality condition $f \pitchfork Z$.

One immediate consequence of the above is that the intersection $Z \cap W$ of two submanifolds Z and W of a manifold Y is again a submanifold provided the intersection is transversal. In this case the codimension of $Z \cap W$ in Y is the sum of the codimensions of Z and W in Y.

The notion of transversality can be extended to manifolds-with-boundary using the same definition given above. For a manifold-with-boundary X and a smooth map $f : X \to Y$, we denote by ∂f the restriction map $f|_{\partial X} : \partial X \to Y$. An argument similar to the one given above establishes the following important result for manifolds with boundary.

Fact 1. *Let X be a manifold-with-boundary, Y, Z be (boundaryless) manifolds and Z is a submanifold of Y. Assume that the maps $f : X \to Y$ and $\partial f : \partial X \to Y$ are transversal to Z. Then $f^{-1}(Z)$ is a submanifold with boundary of X of codimension* $\dim Y - \dim Z$ *and*

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X.$$

Observe that under the stated hypothesis the boundary of $f^{-1}(Z)$ is contained in that of X. In fact, this condition is what motivates the hypothesis of above result.

See Figure 5 for examples of manifolds intersecting transversally. Observe that in each of these cases the intersection is again a manifold.

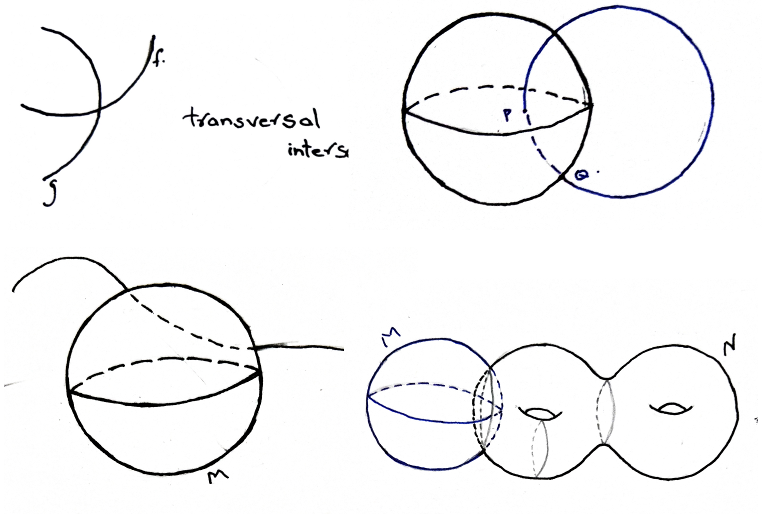
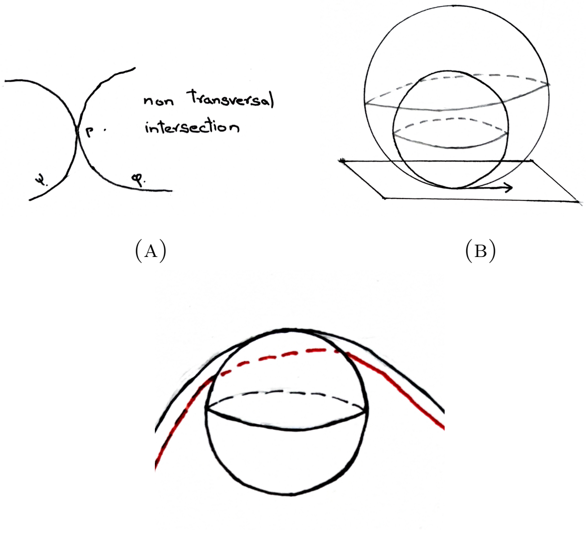


FIGURE 5. Transversal Intersections

Below we give some examples of non-transversal intersections (see Figure-6 on the next page). Observe that an arbitrarily small perturbation of one of the manifolds will make the intersection transversal.

We will need another result from transversality theory which we call transversal extension property.

Fact 2. *Let X be a manifold-with-boundary, Y, Z be (boundaryless) manifolds and Z is a submanifold of Y which is also closed as a subset of Y. If for the map $f : X \to Y$ the boundary map $\partial f : \partial X \to Y$ is transversal with respect to Z, then there exists a map $g : X \to Y$ such that $\partial g = \partial f$ and g is transversal to Z.*



(A) (B)

(C)

Figure 6. Non-transversal Intersections

The main idea behind the proof of *Fact* 2 is the fact that transversality is a *generic property.* To make this idea precise we need the notion of deformations of maps. Suppose that X is a manifold with boundary and Y is a (boundaryless) manifold. Two smooth maps $f, g : X \to Y$ are said to be (smoothly) *homotopic* if there exists a smooth map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for every $x \in X$. In this case we say that H is a (smooth) *homotopy* between the maps f and g. If we think of the parameter t as time variable then this definition conforms to our intuitive idea of smoothly deforming one map into another. It turns out that if $f : X \to Y$ is a smooth map and if Z is a (boundaryless) submanifofd of Y then there exists a smooth map $g : X \to Y$ which is homotopic to f and is transversal to Z. Thus any smooth map can be perturbed slightly so as to make it transversal with respect to any prescribed (boundaryless) submanifold. The crucial ingredient in the proof of this fact is Sard's theorem (which was mentioned earlier in section 2) which says that regular values are generic for any smooth map. (Sard's theorem has many other important applications and may be considered as a foundational result in topology.) Moreover, when Z is a *closed* subset of Y, the map f can be perturbed to a smooth map g without disturbing it on the boundary ∂X in such a way that $g \pitchfork Z$. (See [3], p. 67-73, for complete details of the proof.)

3. Proof of the theorem

We will now outline the proof of the theorem. We will actually prove the following slightly more general result:

Let Z be a smooth (boundaryless) hypersurface in $\mathbb{R}^n$ which, as a subset of

$\mathbb{R}^n$*, is closed. Then* Z *must be orientable.*

Proof is by contradiction. Assume that Z is nonorientable and take any *orientation-reversing* loop $\alpha : [0,1] \to Z$. Thus, any unit normal vector to Z at $\alpha(0)$, when transported continuously along α while keeping it unit normal to Z, comes back to the opposite unit normal at $\alpha(1) = \alpha(0)$. Fix a (sufficiently small) positive number ϵ. By choosing a point along each unit normal at $\alpha(t)$ which is at a distance ϵ from the base point we obtain a smooth curve in $\mathbb{R}^n$ and, finally, by connecting the ends of this curve by a straight line segment we obtain a closed curve which intersects Z at exactly one point. This curve is not smooth but it can be smoothened to obtain a smooth closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ which intersects Z at precisely one point where the intersection is transversal.

Since $\mathbb{R}^n$ is simply connected the curve γ can be smoothly deformed to a point. Thus there exists a smooth homotopy $H : S^1 \times [0,1] \to \mathbb{R}^n$ such that $H(x,0) = \gamma(x)$ and $H(x,1) = y$ for every $x \in \mathbb{S}^1$. (Here y is some point in $\mathbb{R}^n$.)

Consider now the smooth map $F : S^1 \times [0,1] \to D^2$ defined by $F(x,t) = (1-t)x$. This is a smooth quotient map which is injective on $S^1 \times [0,1)$ and maps $S^1 \times \{1\}$ into $\{0\}$. Since H is constant on $S^1 \times \{1\}$, F induces a smooth map $f : D^2 \to Z$ such that $f|_{\partial D^2} = \gamma$. This is the stage where transversality enters the proof. By the transversal extension theorem there exists a smooth map $g : D^2 \to \mathbb{R}^n$ transversal to Z and $g|_{\partial D^2} = \gamma$. Since $g^{-1}(Z)$ is a smooth one-dimensional submanifold-with-boundary of D^2, it must consist of a disjoint union of a finite number of circles in the interior of D^2 and arcs whose ends lie on ∂D^2. Clearly, the total number of such end points must be even. This, however, contradicts the fact that $\partial(g^{-1}(Z)) = g^{-1}(Z) \cap \partial D^2$ is just one point.

Remarks.

1. The above proof works verbatim in the case where $\mathbb{R}^n$ is replaced by any smooth manifold which is *simply-connected.* Thus any (boundaryless) hypersurface in any simply-connected (boundaryless) manifold must be orientable provided it is a closed subset of the manifold.

2. The definition of orientability of a hypersurface that we have given can be shown to be equivalent to the following purely intrinsic property (which makes sense for manifolds of *arbitrary* codimension):

There exists an atlas for which all the transition maps are orientation preserving, i. e., det$(\phi_j \circ \phi_i^{-1}) > 0$ *for any two charts* ϕ_i *and* ϕ_j *in the atlas.*

Using this later definition one can show that the following 2-manifolds are nonorientable:

Real Projective Plane $\mathbb{R}P^2$. This is the image of $\mathbb{S}^2$ in $\mathbb{R}^4$ under the mapping $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by $f(x,y,z) = (x^2 - y^2, xy, xz, yz)$.

Klein Bottle $\mathbb{K}$. This is the image of $\mathbb{R}^2$ in $\mathbb{R}^4$ under the map $g : \mathbb{R}^2 \to \mathbb{R}^4$ given by $g(x, y) = ((\cos y + 2) \cos x, (\cos y + 2) \sin x, \sin y \cos x/2, \sin y \sin x/2)$.
Both $\mathbb{R}P^2$ and $\mathbb{K}$ are compact. Since they are nonorientable, it follows immediately from the theorem that *they can not be realised as surfaces in* $\mathbb{R}^3$.
3. The examples of Klein bottle and the real projective plane above show that, in general, a k-manifold cannot be realised as a manifold in $\mathbb{R}^{2k-1}$. If the manifold is assumed to be compact and orientable then it is known that it can be realised as a manifold in $\mathbb{R}^{2k-1}$. We refer the interested reader to [4], [2] for the details of proof.

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GENERALIZED KONHAUSER MATRIX POLYNOMIAL AND ITS PROPERTIES

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Abstract. We propose a generalized Konhauser matrix polynomial and obtain its properties such as the differential equation, inverse series relation and certain generating function relations involving Mittag-Leffler matrix function.

1. Introduction and Notations

Many of the Special Functions and most of their properties can be derived from the theory of Group representations [12]. Their matrix analogues often occur in Statistics, Number theory and in Lie Group theory [1, 5, 11]. In [6, 7, 10], are studied matrix differential equations and Frobenius method for the Laguerre, Hermite and Gegenbauer matrix polynomials. Interestingly in [10] is studied the quadrature matrix integration process with the help of matrix Laguerre polynomial. It is well known that the Konhauser polynomial

$$Z_m^{\alpha}(x;r) = \frac{\Gamma(rm+\alpha+1)}{\Gamma(m+1)} \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{x^{rn}}{\Gamma(rn+\alpha+1)}, \quad (\Re(\alpha) > -1)$$

is the biorthogonal polynomial for the distribution function of the Laguerre polynomial [14]. This can also be viewed as a generalization of the Laguerre polynomial. In 2014, the above Konhauser polynomial $Z_m^{\alpha}(x;r)$ was further generalized by Prajapati, Ajudia and Agarwal in the form [13, Eq.(5), p.640]:

$$L_{[\frac{m}{q}]}^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha m+\beta+1)}{m!} \sum_{n=0}^{[\frac{m}{q}]} \frac{(-m)_{qn}}{\Gamma(\alpha n+\beta+1)} \frac{z^n}{n!}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{C}, m, q \in \mathbb{N}, \Re(\beta) > -1$ and $[\frac{m}{q}]$ denotes the integral part of $\frac{m}{q}$. Here, we define a matrix analogue of this polynomial and derive certain properties of it. In what follows, the following definitions and notations will be used. Throughout, we shall let A to be a matrix in $C^{p\times p}$ and $\sigma(A)$ to be the set of all eigenvalues of A. The matrix A is said to be positive stable matrix if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$. If $A_0, A_1, A_2, ...A_n$ are elements of $C^{p\times p}$ and $A_n \neq 0$ then

$$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + ... + A_1 x + A_0$$

is a matrix polynomial of degree n in x.

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The 2-norm of the matrix A, denoted by $\|A\|$, is defined by

$$\| A \| = \sup_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} = max\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\},$$

where for a vector $x \in \mathbb{C}^p$, $\|x\|_2 = \left(x^T x\right)^{1/2}$ is Euclidean norm of x, and A^* denotes the transposed conjugate of A.

If $f(z)$ and $g(z)$ are holomorphic functions of a complex variable z which are defined on an open set Ω of the complex plane and if $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [3] it follows that

$$f(A)g(A) = g(A)f(A).$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = [\Gamma(z)]^{-1} = \frac{1}{\Gamma(z)}$ is an entire function of complex variable z [4, p. 253] and thus for any matrix A in $C^{n\times n}$, the functional calculus [3] shows that $\Gamma^{-1}(A)$ is a well defined matrix function. If I denotes identity matrix of order p and $A + nI$ is invertible for every integer $n \geq 0$ then [8, Eq. (6) and (7), p.206]

$$(A)_n = \Gamma(A + nI)\Gamma^{-1}(A).$$

For positive stable matrices $C, D \in C^{p\times p}$, the Beta matrix function is denoted and defined by [8, Eq.(9), p.207] (also [9])

$$B(C, D) = \int_0^1 t^{C-I}(1-t)^{D-I} dt. \tag{1.2}$$

Further, if $CD = DC$ and if $C + nI$, $D + nI$ and $C + D + nI$ are invertible for all nonnegative integers n then [8, Theorem 2, p. 209]

$$B(C, D) = \Gamma(C)\Gamma(D)\Gamma^{-1}(C + D). \tag{1.3}$$

For $A(k, n), B(k, n) \in C^{p\times p}$, $n, k \geq 0$ and $m \in \mathbb{N}$, there holds the double series identities (cf. [16, Eq.(1.7), p.606])

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + mk) \tag{1.4}$$

and (cf. [2, Eq.(8), p.324])

$$\sum_{i=0}^{mn} \sum_{j=0}^{[i/m]} B(i, j) = \sum_{j=0}^{n} \sum_{i=0}^{mn-mj} B(i + mj, j). \tag{1.5}$$

For any matrix A in $C^{p\times p}$ and for $|x| < 1$, the following series expansion holds [8].

$$(1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n.$$

Also, we have the formula [16, Eq.(2.23), p.616]

$$(A)_{mk} = m^{mk} \prod_{i=1}^{m} \left(\frac{A + (i-1)I}{m}\right)_k = \Delta(m; A). \tag{1.6}$$

In particular, for non negative integer n,

$$(-nI)_{mk} = (-1)^{mk} \frac{n!}{(n-mk)!} I = m^{mk} \prod_{i=1}^{m} \left(\frac{-n+i-1}{m} I\right)_k. \tag{1.7}$$

We shall denote the zero matrix by O.

2. Generalized Konhauser matrix polynomial

We propose the extension of (1.1) as follows.

Definition 2.1. For the matrix A in $C^{p\times p}$

$$Z_{m^*}^{(A,\lambda)}(x^k;r)=\frac{\Gamma(A+rmI+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}(-mI)_{sn}\Gamma^{-1}(A+rnI+I)\frac{(\lambda x^k)^n}{n!}, \quad (2.1)$$

where $r,\lambda,\mu\in\mathbb{C},\ k\in\mathbb{R}_{>0},\ s\in\mathbb{N},\ m\in\mathbb{N}\cup\{0\},\ \Re(\lambda)>0,\ \Re(\mu)>-1$ for all eigen values $\mu\in\sigma(A)$ and the floor function $\lfloor u\rfloor = floor\ u$, represents the greatest integer $\leq u$.

It may be seen that when $r=k\in\mathbb{N}$ and $s=1$, this polynomial reduces to

$$Z_m^{(A,\lambda)}(x;k)=\Gamma(kmI+A+I)\sum_{n=0}^{m}\frac{(-1)^n(\lambda x)^{nk}}{(m-n)!n!}\ \Gamma^{-1}(knI+A+I)$$

studied by Varma, Çekim, and Taşdelen [18]. Further, if $k=1$ then this reduces to the Laguerre matrix polynomial [6]:

$$L_m^{(A,\lambda)}(x)=\sum_{n=0}^{m}\frac{(-1)^n}{n!(m-n)!}\ (A+I)_m\ [(A+I)_n]^{-1}\ (\lambda x)^n.$$

For the polynomial (2.1), we derive the differential equation and inverse series relation. Also, we show the relation of (2.1) with Mittag-Leffler matrix function which will be used in the generating function relations derived here. At last, the Euler(Beta) matrix transform is applied on this polynomial.

3. Differential Equations

If $\{A_i; i=1,2,\ldots,p\}$ and $\{B_j; j=1,2,\ldots,q\}$ are matrices in $C^{n\times n}$ and B_j+nI are invertible for all $n=0,1,2,\ldots$, then it is known that the generalized hypergeometric matrix function [16, Eq. (2.2), p. 608]:

$$\begin{aligned}&{}_pF_q(A_1,A_2,\ldots,A_p;B_1,B_2,\ldots,B_q;z)\\ &\quad=\sum_{k=0}^{\infty}(A_1)_k(A_2)_k\cdots(A_p)_k\ [(B_1)_k]^{-1}[(B_2)_k]^{-1}\cdots[(B_q)_k]^{-1}\frac{z^k}{k!}\end{aligned} \quad (3.1)$$

satisfies the matrix differential equation [16, Eq. (2.10), p. 610]:

$$\left[\theta\prod_{j=1}^{q}(\theta I+B_j-I)-z\prod_{i=1}^{p}(\theta I+A_i)\right]\ {}_pF_q(z)=O, \quad (3.2)$$

where $\theta=zd/dz$ and O is the zero matrix of order n. Here, if we express the polynomial (2.1) in ${}_pF_q$ form then the equation (3.2) will readily yield the differential equation corresponding to the polynomial (2.1). In fact, assuming that the matrices occurring here commute with one another, we have, for $r,\ s\in\mathbb{N}$,

$$\begin{aligned}Z_{m^*}^{(A,\lambda)}(x^k;r)&=\frac{\Gamma(A+rmI+I)}{m!}\Gamma^{-1}(A+I)\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-mI)_{sn}(A+I)_{rn}^{-1}\ (\lambda x^k)^n}{n!}\\ &=\frac{\Gamma(A+rmI+I)}{m!}\Gamma^{-1}(A+I)\sum_{n=0}^{\lfloor m/s\rfloor}\left\{\prod_{i=1}^{s}\left(\frac{-m+i-1}{s}I\right)_n\right\}\end{aligned}$$

$$\times \left\{ \prod_{j=1}^{r} \left(\frac{A+jI}{r} \right)_n^{-1} \right\} \frac{1}{n!} \left(\frac{\lambda x^k s^s}{r^r} \right)^n .$$

Hence, in (3.1), setting $p = s,\ q = r,\ A_i = (-m+i-1)I/s,\ B_j = (A+jI)/r,\ z = \lambda s^s x^k / r^r$, the equation immediately leads us to the differential equation for (2.1) of order $max.\{r+1,\ s\}$. This is stated in

Theorem 3.1. *If $r, s \in \mathbb{N}$ and the operator Θ is defined by $\Theta f(x) = \frac{x}{k}\frac{d}{dx} f(x)$ then $U = Z_{m^*}^{(A,\lambda)}(x^k; r)$ satisfies the equation*

$$\left[\left\{ \Theta \prod_{j=1}^{r} \left(\Theta\ I + \frac{A+jI}{r} - I \right) \right\} - \left(\frac{s^s}{r^r} \right) \ \lambda\ x^k \ \left\{ \prod_{i=1}^{s} \left(\Theta\ I + \frac{-m+i-1}{s} I \right) \right\} \right] U = O.$$

4. Inverse series relations

For deriving the inverse series of the matrix polynomial (2.1), the following lemma will be used.

Lemma 4.1. *If $\{P_n\}$ and $\{Q_n\}$ are finite sequences of matrices in $C^{n\times n}$, then*

$$Q_n = \sum_{j=0}^{n} \frac{(-nI)_j}{j!}\ P_j \ \Leftrightarrow\ P_n = \sum_{j=0}^{n} \frac{(-nI)_j}{j!}\ Q_j .$$

Proof. Let us denote the right hand side of second series by T_n, then

$$\begin{aligned} T_n = \sum_{k=0}^{n} \frac{(-nI)_k}{k!}\ Q_k \quad &= \quad \sum_{k=0}^{n} \frac{(-1)^k n!}{k!\ (n-k)!}\ I \sum_{j=0}^{k} \frac{(-kI)_j}{j!}\ P_j \\ &= \quad \sum_{k=0}^{n} \frac{(-1)^k n!}{k!\ (n-k)!} I\ \sum_{j=0}^{k} \frac{(-1)^j\ k!}{j!\ (k-j)!}\ P_j \\ &= \quad \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} (-1)^k\ \binom{n-j}{k}\ P_j \\ &= \quad P_n + \sum_{j=0}^{n-1}\ \binom{n}{j}\ \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k}\ P_j . \end{aligned}$$

Thus, $T_n = P_n$ and hence, first series implies the second series. Here we have used the simple fact that the inner sum vanishes being equal to $(1+a)^{n-j} P_j$ with $a = -1$. The converse part is similar hence its proof is omitted. □

Using this lemma, we now establish the inverse series relation in the next theorem.

Theorem 4.2. *For a matrix $A \in C^{p\times p}$, $r, \lambda \in \mathbb{C}$, $s \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$,*

$$Z_{m^*}^{(A,\lambda)}(x^k; r) = \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/s \rfloor} (-mI)_{sj} \Gamma^{-1}(A + rjI + I) \frac{(\lambda x^k)^j}{j!} \qquad (4.1)$$

if and only if

$$\frac{(\lambda x^k)^m}{m!}I = \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{(A,\lambda)}(x^k;r), \quad (4.2)$$

and for $m \neq sl,\ l \in \mathbb{N}$,

$$\sum_{j=0}^{m}(-mI)_j\ \Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r) = O. \quad (4.3)$$

Proof. We first show that the series (4.1) implies both (4.2) and (4.3). The proof of (4.1) implies (4.2) runs as follows. Denoting the right hand side of (4.2) by matrix Ξ_m, substituting the series expression for $Z_{j^*}^{(A,\lambda)}(x^k;r)$ from (4.1) and then using the double series relation (1.5), we get

$$\begin{aligned}
\Xi_m &= \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r)\\
&= \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}\frac{(-msI)_j}{j!}\sum_{i=0}^{\lfloor j/s\rfloor}(-jI)_{si}\ \Gamma^{-1}(A+riI+I)\ \frac{(\lambda x^k)^i}{i!}\\
&= \sum_{j=0}^{ms}\sum_{i=0}^{\lfloor j/s\rfloor}\frac{\Gamma(A+rmI+I)\ (-1)^{j+si}\ \Gamma^{-1}(A+riI+I)}{(ms-j)!\ (j-si)!\ i!}(\lambda x^k)^i\\
&= \sum_{i=0}^{m}\sum_{j=0}^{ms-si}\frac{\Gamma(A+rmI+I)\ (-1)^{j}\ \Gamma^{-1}(A+riI+I)}{(ms-si-j)!\ j!\ i!}(\lambda x^k)^i\\
&= \frac{(\lambda x^k)^m}{m!}I + \sum_{i=0}^{m-1}\frac{\Gamma(A+rmI+I)\ \Gamma^{-1}(A+riI+I)}{(ms-si)!\ i!}(\lambda x^k)^i\\
&\quad \times \sum_{j=0}^{ms-si}(-1)^j\binom{ms-si}{j}.
\end{aligned}$$

Here the inner sum in the second term on the right hand side vanishes being equal to $(1+a)^{ms-si}$ with $a=-1$. Consequently, we arrive at $\Xi_m = \frac{(\lambda x^k)^m}{m!}I$. Next, to show further that (4.1) also implies (4.3), let us substitute the series expression for $Z_{j^*}^{(A,\lambda)}(x^k;r)$ from (4.1) to the left hand side of (4.3). Then in view of (1.5), we get

$$\begin{aligned}
&\sum_{j=0}^{m}(-mI)_j\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r)\\
&= \sum_{j=0}^{m}\frac{(-1)^j m!}{(m-j)!}I\sum_{i=0}^{\lfloor j/s\rfloor}\frac{(-1)^{si}\ \Gamma^{-1}(A+riI+I)}{(j-si)!\ i!}(\lambda x^k)^i\\
&= \sum_{i=0}^{\lfloor m/s\rfloor}\frac{m!\ \Gamma^{-1}(A+ri+I)}{(m-si)!\ i!}(\lambda x^k)^i\sum_{j=0}^{m-si}(-1)^j\binom{m-si}{j} \ = \ O
\end{aligned}$$

if $m \neq sl,\ l \in \mathbb{N}$. This completes the proof of the first part. The proof of converse part which uses the technique due to Dave and Dalbhide [2], runs as follows. In order to show that the series (4.2) and the condition (4.3) together imply the series (4.1), we use Lemma 4.1 with

$$P_j = j!\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r),$$

and consider one sided relation in the lemma, that is, the series on the left hand side implies the series on the right hand side. Then

$$Q_m = \sum_{j=0}^{m}(-mI)_j\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r) \tag{4.4}$$

implies

$$Z^{(A,\lambda)}_{m^*}(x^k; r) = \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{m} \frac{(-mI)_j}{j!}\ Q_j. \tag{4.5}$$

Since the condition (4.3) holds, $Q_m = O$ for $m \neq sl,\ l \in \mathbb{N}$, whereas

$$Q_{ms} = \sum_{j=0}^{ms}(-msI)_j\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r).$$

Also the series (4.2) holds true, whence it follows that

$$\begin{aligned} Q_{ms} &= \sum_{j=0}^{ms}(-msI)_j\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r) \\ &= \frac{(ms)!\ \Gamma^{-1}(A + rmI + I)}{m!}(\lambda x^k)^m. \end{aligned}$$

Consequently, the inverse pair (4.4) and (4.5) assume the form:

$$\begin{aligned} \frac{(\lambda x^k)^m}{m!} I &= \frac{\Gamma(A + rmI + I)}{(ms)!} \sum_{j=0}^{ms}(-msI)_j\ \Gamma^{-1}(A + rjI + I) \\ &\quad \times Z^{(A,\lambda)}_{j^*}(x^k; r) \end{aligned}$$

from which it follows that

$$\begin{aligned} Z^{(A,\lambda)}_{m^*}(x^k; r) &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/s \rfloor} \frac{(-mI)_{sj}}{(sj)!}\ Q_{sj} \\ &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/s \rfloor} \frac{(-mI)_{sj}\ \Gamma^{-1}(A + rjI + I)}{j!}(\lambda x^k)^j, \end{aligned}$$

subject to the condition (4.3). □

5. Mittag-Leffler Matrix Function

In 2007, Shukla and Prajapati [17] introduced a generalization of the Mittag-Leffler function in the form:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{5.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C},\ \Re(\alpha, \beta, \gamma) > 0,\ q \in (0, 1) \cup \mathbb{N}$. Here we allow q to take value 0 in which case the series retains convergence behavior. Also, if α is allowed to assume value 0 then with $q = 0$ and $\beta = 1$, the reducibility of (5.1) to the exponential function e^z occurs. Thus, with $q \geq 0,\ \Re(\alpha) \geq 0,\ \Re(\beta, \gamma) > 0$ and $z \in \mathbb{C}$, (5.1) yields an instance

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)\ n!}. \tag{5.2}$$

We define here the matrix analogues of (5.1) and (5.2) as follows.

Definition 5.1. For $A, B \in C^{p\times p}$, $\Re(\mu) > -1$ for all eigen values $\mu \in \sigma(A), r \in \mathbb{C}$ and $s \in \mathbb{N}$,

$$E_{rI,A+I}^{B,sI}(z) = \sum_{n=0}^{\infty} (B)_{sn} \Gamma^{-1}(A + rnI + I) \frac{z^n}{n!}. \tag{5.3}$$

Definition 5.2. For $A \in C^{p\times p}$, $r \in \mathbb{C}$, $\Re(\mu) > -1$ for all eigen values $\mu \in \sigma(A)$,

$$E_{rI,A+I}(z) = \sum_{n=0}^{\infty} \Gamma^{-1}(A + rnI + I) \frac{z^n}{n!}. \tag{5.4}$$

Putting $B = -mI$, where $m \in \mathbb{N}$ and $z = \lambda x^k$ in (5.3), and comparing it with the defined function (2.1), we obtain the relation:

$$E_{rI,A+I}^{-mI,sI}(\lambda x^k) = m!\ \Gamma^{-1}(A + rmI + I) Z_{m^*}^{(A,\lambda)}(x^k; r).$$

The functions (5.3) and (5.4) will be used in the generating function relations derived in the following section.

6. Generating Function relations

We derive the generating function relations for the matrix polynomial $Z_{m^*}^{(A,\lambda)}(x^k; r)$ in the form of Theorems 6.1, 6.3 and 6.5.

Theorem 6.1. *Let $r \in \mathbb{C}$, $s \in \mathbb{N}$ and A, B be the matrices in $C^{p\times p}$, $\Re(\mu) > -1$ for all eigenvalues $\mu \in \sigma(A)$, then for $|t| < 1$,*

$$\sum_{m=0}^{\infty} (B)_m\ \Gamma^{-1}(A + rmI + I)\ Z_{m^*}^{(A,\lambda)}(x^k; r)\ t^m$$
$$= \quad (1-t)^{-B}\ E_{rI,A+I}^{B,sI}\left(\lambda x^k (-t)^s (1-t)^{-sI}\right).$$

Proof. Observe that on substituting the series for $Z_{m^*}^{(A,\lambda)}(x^k; r)$ from (2.1) on the left hand side and using (1.4), we get

$$\sum_{m=0}^{\infty} (B)_m\ \Gamma^{-1}(A + rmI + I)\ Z_{m^*}^{(A,\lambda)}(x^k; r)\ t^m$$
$$= \sum_{m=0}^{\infty} (B)_m \Gamma^{-1}(A+rmI+I) \frac{\Gamma(A+ rmI+I)}{m!} \sum_{n=0}^{\lfloor m/s \rfloor} \frac{m!(-1)^{sn} I \Gamma^{-1}(A+ rnI+I)}{n!(m-sn)!}$$
$$\times (\lambda x^k)^n t^m$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor m/s \rfloor} \frac{(-1)^{sn} (B)_m \Gamma^{-1}(A + rnI + I)}{n!\ (m - sn)!}\ (\lambda x^k)^n t^m$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{sn} (B)_{m+sn} \Gamma^{-1}(A + rnI + I)}{n!\ m!}\ (\lambda x^k)^n t^{m+sn}$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(B + snI)_m t^m}{m!} \frac{(-1)^{sn} (B)_{sn} \Gamma^{-1}(A + rnI + I)}{n!}\ (\lambda x^k)^n t^{sn}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}(1-t)^{-B-snI}\,\frac{(-1)^{sn}(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!}\,(\lambda x^k)^n t^{sn}\\
&= (1-t)^{-B}\sum_{n=0}^{\infty}\frac{(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!}\,(\lambda x^k(-t)^s(1-t)^{-sI})^n \qquad (6.1)\\
&= (1-t)^{-B}\;E_{rI,A+I}^{B,sI}\left(\lambda x^k(-t)^s(1-t)^{-sI}\right).
\end{aligned}$$

This completes the proof. □

Corollary 6.2. *If $r \in \mathbb{N}$, then for $s \le r$ or $s = r+1$,*

$$\sum_{m=0}^{\infty}(B)_m\;(A+I)_{rm}^{-1}\;Z_{m^*}^{(A,\lambda)}(x^k;r)\;t^m \;=\; (1-t)^{-B}\;\times$$
$$ {}_sF_r\left(\frac{B}{s},\frac{B+I}{s},\dots,\frac{B+(s-1)I}{s};\frac{A+I}{r},\frac{A+2I}{r},\dots,\frac{A+rI}{r};\frac{s^s}{r^r}\lambda x^k R^s\right),$$

where $R = (-t)(1-t)^{-I}$.

Proof. For $r \in \mathbb{N}$, the infinite series on the right hand side in (6.1) assumes the form

$$(1-t)^{-B}\Gamma^{-1}(A+I)\sum_{n=0}^{\infty}(B)_{sn}(A+I)_{rn}^{-1}\frac{(\lambda x^k R^s)^n}{n!}.$$

In view of the formula (1.6) and the matrix function (3.1), this leads us to the corollary. □

If $(B)_m$ is dropped from the left hand side of this theorem, then it takes the following form.

Theorem 6.3. *In the usual notations and meaning, there holds the generating function relation:*

$$\sum_{m=0}^{\infty}\Gamma^{-1}(A+rmI+I)\;Z_{m^*}^{(A,\lambda)}(x^k;r)\;t^m = e^t\;E_{rI,A+I}\left(\lambda x^k(-t)^s\right).$$

Proof. The proof follows in a straight forward manner. In fact, by using the double series relation (1.4), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\Gamma^{-1}(A+rmI+I)\;Z_{m^*}^{(A,\lambda)}(x^k;r)\;t^m\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!\;(m-sn)!}(\lambda x^k)^n t^m\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!\;m!}(\lambda x^k)^n t^{m+sn}\\
&= \sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!}(\lambda x^k)^n t^{sn}\\
&= e^t\;E_{rI,A+I}\left(\lambda x^k(-t)^s\right).
\end{aligned}$$

□

Again, we have the following corollary. (cf. [16, Eq. (3.5), p. 619])

Corollary 6.4. *For $r \in \mathbb{N}$,*

$$\begin{aligned}
&\sum_{m=0}^{\infty} (A+I)^{-1}_{rm}\ Z^{(A,\lambda)}_{m^*}(x^k;r)\ t^m \\
= \quad & e^t\ {}_0F_r\left(--;\frac{A+I}{r},\frac{A+2I}{r},\ldots,\frac{A+rI}{r};\frac{\lambda x^k(-t)^s}{r^r}\right).
\end{aligned}$$

The proof follows by proceeding as in corollary 6.2. Next, in the notations and meaning of Theorem 6.1, we have

Theorem 6.5. *Let a and b be complex constants which are not zero simultaneously, then there holds the generating function relation*

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z^{(A,\lambda)}_{n^*}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\ \Gamma^{-1}(A+rnI+I)\ t^n \\
= \quad & e^{ax}\ (1-bte^{bx})^{-1}\ E_{rI,A+I}(\lambda x^k(-t)^s e^{bsx}).
\end{aligned}$$

Proof. Beginning with the left hand side, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z^{(A,\lambda)}_{n^*}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\Gamma^{-1}(A+rnI+I)\ t^n \\
= \quad & \sum_{n=0}^{\infty}\sum_{j=0}^{\lfloor n/s\rfloor}\frac{(-1)^{sj}\ \Gamma^{-1}(A+rjI+I)(\lambda x^k)^j}{(n-sj)!\ j!}(a+bn)^{n-sj}t^n \\
= \quad & \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{((-t)^s\lambda x^k)^j\Gamma^{-1}(A+rjI+I)}{j!}\frac{(a+bn+bsj)^n}{n!}t^n. \qquad (6.2)
\end{aligned}$$

We use here the Lagrange expansion formula [15, Eq. (18), p. 146]:

$$\frac{f(x)}{1-tg'(x)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left[D^n f(x)(g(x))^n\right]_{x=0}, \quad (t = x/g(x))$$

by taking $f(x) = e^{(a+bsj)x}$ and $g(x) = e^{bx}$. Then we find that

$$\frac{e^{(a+bsj)x}}{1-bte^{bx}} = \sum_{n=0}^{\infty}(a+bsj+bn)^n\ \frac{t^n}{n!}.$$

Thus (6.2) simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z^{(A,\lambda)}_{n^*}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\Gamma^{-1}(A+rnI+I)\ t^n \\
= \quad & \sum_{j=0}^{\infty}\frac{\Gamma^{-1}(A+rjI+I)}{j!}\ ((-t)^s\lambda x^k)^j\ \frac{e^{(a+bsj)x}}{1-bte^{bx}}.
\end{aligned}$$

In view of (5.4), this yields the desired form. □

We again have the following corollary. (cf. [16, Eq. (3.14), p. 621])

Corollary 6.6. *For $r \in \mathbb{N}$, there holds the matrix generating function relation:*

$$\sum_{n=0}^{\infty} Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\ (A+I)_{rn}^{-1}\ t^n\ =\ e^{ax}\ (1-bte^{bx})^{-1}$$
$$\times\ {}_0F_r\left(--;\frac{A+I}{r},\frac{A+2I}{r},\dots,\frac{A+rI}{r};\frac{\lambda x^k(-t)^s e^{bsx}}{r^r}\right).$$

7. Matrix Integral transform

Using the integral formula (7.1), we define Euler (Beta) matrix transform as follows.

Definition 7.1. For the matrices $P, Q \in C^{p\times p}$, a Beta matrix transform may be defined as

$$\mathfrak{B}\left\{f(x):P,Q\right\} = \int_0^1 x^{P-I}(1-x)^{Q-I}f(x)\ dx. \tag{7.1}$$

We apply this transform to the polynomial (2.1) in the following theorem.

Theorem 7.2. *If $A, P, Q \in C^{p\times p}$, P, Q are positive stable matrices, for $q = 0, 1, 2, \dots,$ the matrices $P + qI$, Q are commutative, $P + qI, Q + qI$, $P + Q + qI$ are invertible and $k, r, s, m \in \mathbb{N}$, then*

$$\mathfrak{B}\left\{Z_{m^*}^{(A,\lambda)}(tx^k;r):P,Q\right\} = \frac{(A+I)_{rm}}{m!}\ \Gamma(Q)\Gamma^{-1}(P)\Gamma^{-1}(P+Q)$$
$$\times\ {}_{s+k}F_{r+k}\left[\begin{array}{ccc}\Delta(s;-mI), & \Delta(k;P); & \dfrac{s^s}{r^r}t\\ \Delta(r;A+I), & \Delta(k;P+Q); & \end{array}\right],$$

where the notation $\Delta(j;C)$ carries the meaning as in (1.6).

Proof. From (7.1),

$$\mathfrak{B}\left\{Z_{m^*}^{(A,\lambda)}(tx^k;r):P,Q\right\}$$
$$=\int_0^1 x^{P-I}(1-x)^{Q-I}Z_{m^*}^{(A,\lambda)}(tx^k;r)dx$$
$$=\int_0^1 x^{P-I}(1-x)^{Q-I}\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)(tx^k)^n dx$$
$$=\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)t^n\int_0^1 x^{knI+P-I}(1-x)^{Q-I}dx$$
$$=\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)\ t^n\ \mathfrak{B}(knI+P,Q)$$
$$=\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)\ t^n\ \Gamma(knI+P)\ \Gamma(Q)$$

$$\times\Gamma^{-1}(knI+P+Q)$$

$$=\frac{(A+I)_{rm}}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}(-m)_{sn}(A+I)_{rn}^{-1}(P)_{kn}(P+Q)_{kn}^{-1}\Gamma(Q)\Gamma(P)\Gamma^{-1}(P+Q)\frac{t^n}{n!}$$

$$=\frac{(A+I)_{rm}\ \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q)}{m!}$$

$$\times\ {}_{s+k}F_{r+k}\left[\begin{array}{lll}\Delta(s;-mI), & \Delta(k;P); & \dfrac{s^s}{r^r}t\\ \Delta(r;A+I), & \Delta(k;P+Q); & \end{array}\right].$$

□

This theorem reduces to the Euler (Beta) transform given in [13, Theorem 9.4, p. 649] when the P, Q, A are scalars.

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ASYMPTOTIC EVALUATION OF EULER ϕ SUMS OF VARIOUS RESIDUE CLASSES

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Abstract. This note contains some asymptotic formulas for the sums of various residue classes of Euler's ϕ-function.

1. Introduction

The *ϕ-function* was introduced by Euler in connection with his generalization of Fermat's Theorem. It occurs without the functional notation in his 1759 paper *Theoremata arithmetica nova methodo demonstrata* [6]. In §3 of his 1775 paper [7], Euler denotes by πD "the multitude of numbers less than D, and which have no common divisor with it" and then provides a table of πD for $D = 1$ to 100 writing $\pi 1 = 0$. Gauss introduced the symbol ϕ in §38 of his *Disquitiones Arithmeticae*(1801) with $\phi(1) = 1$. The function $\phi(n)$ denotes the number of positive integers not exceeding n which are relatively prime to n. Clearly, for p prime, we have $\phi(p) = p - 1$.

As Euler observed (Theorem 3, pp.81–82), if p is a prime, the positive integers $\leq p^k$ that are not relatively prime to p^k are the p^{k-1} multiples of p : $p, 2p, 3p, \ldots, p^{k-1} \cdot p$. So $\phi(p^k) = p^k - p^{k-1} = p^k(1 - \frac{1}{p}) = p^{k-1}(p-1)$, and $\sum_{j=0}^{k} \phi(p^j) = (p-1)[1+p+p^2+\cdots+p^{k-1}] = p^k$. Furthermore, if $(a, b) = 1$, then $\phi(a\,b) = \phi(a)\,\phi(b)$. Thus if m has the prime factorization $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, then $\phi(m) = p_1^{r_1-1} p_2^{r_2-1} \cdots p_k^{r_k-1}(p_1-1)(p_2-1)\cdots(p_k-1)$. And, $\phi(m^k) = m^{k-1}\phi(m)$. Also, if $(a, b) = d$, then $\phi(a\,b) = \phi(a)\,\phi(b)(d/\phi(d))$. As Gauss showed:

$$\sum\nolimits_{d|n} \phi(d) = \sum \phi(n/d) = n.$$

The value of $\phi(n)$ fluctuates as n varies. Since averages sooth out fluctuations, it may be fruitful to study the arithmetic mean $(\Phi(n)/n)$, where $\Phi(n) = \sum_{m=1}^{n} \phi(m)$.

In 1874, Mertens obtained [3, p.122][11] an asymptotic value for $\Phi(N)$ for large N. He employed the function $\mu(n)$ and proved that

$$\sum_{m=1}^{G} \phi(m) = \frac{1}{2} \sum_{n=1}^{G} \mu(n) \left\{ \left[\frac{G}{n}\right]^2 + \left[\frac{G}{n}\right] \right\} = \frac{3}{\pi^2} G^2 + \Delta$$

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with $|\Delta| < G(\frac{1}{2}\ln G + \frac{1}{2}\gamma + \frac{5}{8}) + 1$, where γ is Euler's constant and $\mu(n)$ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if} \, n = 1, \\ (-1)^r & \text{if } n \text{ is product of } r \text{ distinct prime numbers,} \\ 0 & \text{if } n \text{ has one or more repeated prime factors.} \end{cases}$$

If $(a,\, b) = 1,\ \mu(a\,b) = \mu(a)\,\mu(b)$. Further, $\sum_{d|n} \mu(d) = 0 \quad (n > 1)$.
For any positive integer n, we have[1, pp.78–80]:

$$\phi(n) = \sum\nolimits_{d|n} \frac{n}{d}\,\mu(d) = \sum\nolimits_{d|n} d\,\mu\left(\frac{n}{d}\right).$$

It is shown in [8, p.268 Theorem 330][2, pp.61-62] that

$$\Phi(n) = (3n^2/\pi^2) + O(n\,\ln n). \tag{1.1}$$

To prove (1.1), we may recall here Euler's *zeta function* and identity:

$$\zeta(s) = \sum\nolimits_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p-prime} \left(1 - \frac{1}{p^s}\right)^{-1},\ \Re(s) > 1.$$

Since for $s > 1$,

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - p^{-s}\right) = \prod \{1 + \mu(p)p^{-s} + \mu(p^2)p^{-2s} + \ldots\} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \text{ and}$$

$\phi(n) = n \sum_{d|n} (\mu(d)/d)$, hence we have

$$\Phi(n) = \sum_{m=1}^{n} \phi(m) = \sum_{m=1}^{n} m \sum_{d|m} \frac{\mu(d)}{d} = \sum_{dd' \le n} d'\mu(d) = \sum_{d=1}^{n} \mu(d) \sum_{d'=1}^{\lfloor \frac{n}{d} \rfloor} d'.$$

That is,

$$\Phi(n) = \sum\nolimits_{d=1}^{n} \mu(d) \left\{ \frac{1}{2} \left\lfloor \frac{n}{d} \right\rfloor \left(\left\lfloor \frac{n}{d} \right\rfloor + 1 \right) \right\} = \frac{1}{2} \sum\nolimits_{d=1}^{n} \mu(d) \left\{ \frac{n^2}{d^2} + O\left(\frac{n}{d}\right) \right\},$$

leading to

$$\begin{aligned} \Phi(n) = \frac{n^2}{2} \sum_{d=1}^{n} \frac{\mu(d)}{d^2} + O\left(n \sum_{d=1}^{n} \frac{1}{d}\right) &= n^2 \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - n^2 \sum_{d=n+1}^{\infty} \frac{\mu(d)}{d^2} + O(n \ln n). \\ &= \frac{n^2}{2\zeta(2)} + O\left(n^2 \sum_{d=n+1}^{\infty} \frac{1}{d^2}\right) + O(n \ln n). \end{aligned}$$

Or,
$\Phi(n) = (n^2/2\zeta(2)) + O(n) + O(n \ln n) = (3n^2/\pi^2) + O(n \ln n)$. □

Lehmer studied sums of $\phi(n)$ in [9] and revisited in [10]. I seek here an extension of Lehmer's formula occurring in [10] by using his argument.

2. Asymptotic summation of $\phi(pn)$

Since $\phi(2^k) = 2^{k-1}$, one has $\phi(4m + 2) = \phi(2m + 1)$; $\phi(4m) = 2\phi(2m)$. Denoting $\Phi_e(n) = \sum_{m \le n;\, m\, even} \phi(m)$, $\Phi_o(n) = \sum_{m \le n;\, m\, odd} \phi(m)$ and, using the relation

$$\Phi_e(n) = \Phi_o(n/2) + 2\Phi_e(n/2) = \Phi(n/2) + \Phi_e(n/2),$$

Lehmer [10] deduced: $\Phi_e(n) = \sum_{\lambda=1}^{\ell} \Phi_e(n/2)$ $(\ell = [\ln n/\ln 2])$ and then used the formula (1.1) to derive

$$\Phi_e(n) = (n/\pi)^2 + O(n \ln n); \qquad \Phi_o(n) = 2(n/\pi)^2 + O(n \ln n). \tag{2.1}$$

Let $\Phi r_i(n) = \sum_{k=1}^{m} \phi(kp-i)$, with fixed $i = 0, 1, 2, \ldots, p-1$ and $(mp-i) \leq n$. Then

$$\Phi_{r_0}(n) = (p-1) \sum_{i=1}^{p-1} \Phi_{r_i}(n/p) + p\,\Phi_{r_0}(n/p) \quad \text{which gives}$$

$$\Phi r_0(n) = (p-1)\,\Phi(n/p) + \Phi r_0(n/p).$$

Mimicking Lehmer's proof, we see that for any prime p

$$\begin{aligned}\Phi r_0(n) &= (p-1)\,3\pi^{-2}\,n^2 \sum_{\lambda=1}^{q} p^{-2\lambda} + O(n \log n) \quad (q = [\ln n/\ln p]) \\ &= \frac{3(p-1)}{p^2-1}\pi^{-2}\,n^2 + O\left(n^2 \int_q^{\infty} (p^{-2})^t\,dt\right) + O(n \log n) \\ &= \frac{3}{p+1}\pi^{-2}\,n^2 + O(n \log n).\end{aligned}$$

□

The last asymptotic formula implies the following theorem:

Theorem 1. *For any prime p, we have:*

$$\lim_{m\to\infty} \left(\sum_{k=1}^{m} \phi(pk)/(pm)^2\right) = (3/(p+1)\pi^2). \tag{2.2}$$

If the set $\mathbb{N}$ is partitioned into p residue classes modulo p, we will have one class consisting of composite numbers of the form pm while the remaining $p-1$ classes contain nearly an equal number of prime numbers, and the ratio of the cumulative sums of the two types of classes will be $p : (p-1)$. The rationale behind the first part of the statement is found in Dirichlet's famous theorem relating to primes in arithmetic progressions: *every arithmetic progression, with the first member and the difference being coprime, will contain infinitely many primes.* In other words, if $k > 1$ is an integer and $(k, \ell) = 1$, then there are infinitely many primes of the form $kn + \ell$, where n runs over the positive integers. If k is a prime p, then ℓ is one of the numbers $1, 2, \ldots, p-1$.

Let us recall here the arithmetic function known as the *Mangoldt function* which is defined as:

$$\Lambda(n) = \begin{cases} \ln p, \text{ if } n = p^m \text{ for some prime } p \text{ and positive integer } m, \\ 0 \text{ otherwise.} \end{cases}$$

This function has an important role in elementary proofs of the prime number theorem which states that if $\pi(n)$ denotes the number of primes $\leq n$, then $\pi(n) \sim (n/\ln n)$. We have ([8, pp.253-254]) for $n \geq 1$:

$$\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \ln d = \sum_{d|n} \mu(d) \ln\left(\frac{n}{d}\right) = -\sum_{d|n} \mu(d) \ln d, \text{ and } \sum_{d|n} \Lambda(d) = \ln n.$$

Further [8, p.348][2, p.89], $\sum_{n\leq x}(\Lambda(n)/n) = \ln x + O(1)$, whence

$$\sum_{p\leq x}(\ln p/p) = \ln x + O(1). \tag{2.3}$$

This related result is well-known[2, p.148]:

$$\sum_{p\leq x;\ p\equiv \ell\,(mod\,k)} (\ln p/p) = (1/\phi(k))\ln x + O(1), \tag{2.4}$$

where the sum is extended over those primes $p \leq x$ which are congruent to $\ell\ (\bmod k)$. Since $\ln x \to \infty$ as $x \to \infty$ this relation implies that there are infinitely many primes $p \equiv \ell (\bmod k)$, hence infinitely many in the progression $kn + \ell$. Since the principal term on the right hand side in (2.4) is independent of ℓ, therefore it not only implies Dirichlet's theorem but it also shows [2, p. 148] that the primes in each of the $\phi(k)$ reduced residue classes $(\bmod k)$ make the same contribution to the principal term in (2.3), that is, the *primes are equally distributed among* $\phi(k)$ *reduced residue classes* $(mod k)$. We thus have a prime number theorem for arithmetic progressions [2, p. 154]: If $\pi_\ell(x)$ counts the number of primes $\leq x$ in the progression $kn + \ell$, then

$$\pi_\ell(x) \sim (\pi(x)/\phi(k)) \sim (1/\phi(k))(x/\ln x).$$

Hence, as $m \to \infty$, $\Phi_{r_i}(m) \sim \Phi_{r_j}(m)$, $i, j \neq 0$. And therefore, we deduce from (1.1) and our Theorem 1 the following result:

Theorem 2. *For any prime* p, *we have for each* $i = 1, 2, 3, \dots, p-1$,

$$\lim_{m\to\infty} \left(\sum\nolimits_{k=1}^{m} \phi(pk-i)/(pm)^2\right) = (3p/(p^2-1)\pi^2). \tag{2.5}$$

We will now obtain asymptotic evaluation of the sums of residue classes modulo p for the ϕ-function.

Since $\phi(4m-2) = \phi(2m-1)$; $\phi(4m) = 2\,\phi(2m)$ and as $n \to \infty, \Phi(2n-1) = \sum_{m=1}^{n} \phi(2m-1) = 2\Phi(2n) = 2\sum_{m=1}^{n} \phi(2m)$, hence we have

$$\lim_{n\to\infty} (\Phi(4n-2)/(4n)^2) = \lim_{n\to\infty} (\Phi(4n)/(4n)^2) = (1/2\pi^2).$$

Further, as $n \to \infty$; $\Phi(2n-1) = \Phi(4n-3) + \Phi(4n-1) = 2\,\Phi(2n) = 2\,\Phi(4n-2) + 2\,\Phi(4n)$ and the two forms $4k-3,\ 4k-1$ yield almost equal number of primes, therefore we have:

$$\lim_{n\to\infty} \frac{\Phi(4n-3)}{(4n)^2} = \lim_{n\to\infty} \frac{\Phi(4n-1)}{(4n)^2} = \frac{1}{\pi^2}; \quad \text{again}$$

$$\lim_{n\to\infty} \frac{\Phi(6n-4}{(6n)^2} + \lim_{n\to\infty} \frac{\Phi(6n-2)}{(6n)^2} + \lim_{n\to\infty} \frac{\Phi(6n)}{(6n)^2} = \frac{1}{\pi^2} \quad \text{and}$$

$$\lim_{n\to\infty} \frac{\Phi(6n-5)}{(6n)^2} + \lim_{n\to\infty} \frac{\Phi(6n-3)}{(6n)^2} + \lim_{n\to\infty} \frac{\Phi(6n-1)}{(6n)^2} = \frac{2}{\pi^2}; \quad \text{further}$$

$$\lim_{n\to\infty} \frac{\Phi(6n-4)}{(6n)^2} = \lim_{n\to\infty} \frac{\Phi(6n-2)}{(6n)^2} = \frac{3}{2} \lim_{n\to\infty} \frac{\Phi(6n)}{(6n)^2} \quad \text{and}$$

$$\lim_{n\to\infty} \frac{\Phi(6n-5)}{(6n)^2} = \lim_{n\to\infty} \frac{\Phi(6n-1)}{(6n)^2} = \frac{3}{2} \lim_{n\to\infty} \frac{\Phi(6n-3)}{(6n)^2}; \quad \text{and still further}$$

$$\lim_{n\to\infty} (\Phi(3(2n-1))/(6n)^2) = 2 \lim_{n\to\infty} (\Phi(3(2n))/(6n)^2).$$

This helps deduce the following results:

$$\lim_{n\to\infty}\frac{\Phi(6n-4)}{(6n)^2}=\lim_{n\to\infty}\frac{\Phi(6n-2)}{(6n)^2}=\frac{3}{8\pi^2},$$
$$\lim_{n\to\infty}\frac{\Phi(6n-3)}{(6n)^2}=\frac{1}{2\pi^2};\qquad \lim_{n\to\infty}\frac{\Phi(6n)}{(6n)^2}=\frac{1}{4\pi^2},$$
$$\lim_{n\to\infty}\frac{\Phi(6n-5)}{(6n)^2}=\lim_{n\to\infty}\frac{\Phi(6n-1)}{(6n)^2}=\frac{3}{4\pi^2}.$$

In fact, we have the following general theorem based on two facts: (i) the sum of all odd residue classes equals twice the sum of all even classes, and (ii) the ratio of residue classes modulo p containing primes to the class having only composite numbers is $\frac{p}{(p-1)}:1$.

Theorem 3. *For an odd prime* p,

$$\lim_{n\to\infty}\frac{\Phi(2pn-(2p-2))}{(2pn)^2}=\lim_{n\to\infty}\frac{\Phi(2pn-2)}{(2pn)^2}=\lim_{n\to\infty}\frac{\Phi(2pn-(2p-4))}{(2pn)^2}=$$
$$\lim_{n\to\infty}\frac{\Phi(2pn-4)}{(2pn)^2}=\cdots$$
$$\lim_{n\to\infty}\frac{\Phi(2pn-(p+1))}{(2pn)^2}=\lim_{n\to\infty}\frac{\Phi(2pn-(p-1))}{(2pn)^2}=\frac{p}{(p^2-1)\pi^2};$$
$$\lim_{n\to\infty}\frac{\Phi(2pn)}{(2pn)^2}=\frac{1}{(p+1)\pi^2};\quad \textit{and}$$

$$\lim_{n\to\infty}\frac{\Phi(2pn-(2p-1))}{(2pn)^2}=\lim_{n\to\infty}\frac{\Phi(2pn-1)}{(2pn)^2}=$$
$$\lim_{n\to\infty}\frac{\Phi(2pn-(2p-3))}{(2pn)^2}=\lim_{n\to\infty}\frac{\Phi(2pn-3)}{(2pn)^2}=\cdots$$
$$\lim_{n\to\infty}\frac{\Phi(2pn-(p+2))}{(2pn)^2}=\lim_{n\to\infty}\frac{\Phi(2pn-(p-2))}{(2pn)^2}=\frac{2p}{(p^2-1)\pi^2};$$
$$\lim_{n\to\infty}\frac{\Phi(2pn-p)}{(2pn)^2}=\frac{2}{(p+1)\pi^2}.$$

Remark. *If* $m < p$, *then* m *cannot divide* p. *Also,* p *cannot divide* $2m$ *and* $2m-1$ *simultaneously; it may not divide either. So gcd* $(p, 2m) = 1$ *or* p *and gcd* $(p, 2m-1) = p$ *or* 1. *Hence,* $\phi(p\,(2m)) = (p-1)\,\phi(2m)$ *or* $p\,\phi(2m)$; *and* $\phi(p(2m-1)) = p\,\phi(2m-1)$ *or* $(p-1)\,\phi(2m-1)$ *depending on* m. *Lehmer proved that* $\lim_{n\to\infty}(\Phi(2n-1)/\Phi(2n)) = 2$. *Hence,* $\lim_{n\to\infty}\left(\sum_{m=1}^{n}\phi(p\,(2m-1))/\sum_{m=1}^{n}\phi(p\,(2m)\right) = 2.$

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A PROBLEM RELATED TO PRIME NUMBERS

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Abstract. In this short note, we solve a generalized problem related to prime numbers using techniques of linear algebra and elementary number theory. We discuss further generalizations of the same problem.

1. Introduction

The problem "*Find a positive integer n such that $n/2$ is a square, $n/3$ is a cube and $n/5$ is a fifth power*", the (smallest) solution for which is $2^{15} \cdot 3^{10} \cdot 5^{6}$, is stated on the page 29 in [3]. This leads to the following generalized problem:

Can one find a positive integer n such that $n/2$ is a square, $n/3$ is a cube, $n/5$ is a fifth power, $n/7$ is a seventh power ... n/p_k is a p_k-th power ?

Here we give solution of this problem by two different methods.

2. Solution

2.1. **Method I.** In this method, we use only the concepts of basic number theory. One can look in [1] for general references.

To begin with, let us note that the smallest such positive integer n should be of the form:

$$n = 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdots p_k^{n_k}, \tag{2.1}$$

where each n_i should satisfy the following two conditions:

(1) Each n_i should be divisible by p_j for $j = 1, \cdots, k$ and $j \neq i$.
(2) Each n_i should satisfy the congruence relation $n_i \equiv 1 (\text{mod } p_i)$.

Condition (1) would imply that each n_i can be written as $m_i r_i$, where

$$m_i = \prod_{j=1\ j\neq i}^{k} p_j.$$

As m_i should be the lcm (p_j), where $j = 1, \cdots, k$; $j \neq i$ and gcd $(p_j) = 1$, therefore condition (2) can be written as

$$m_i r_i \equiv 1 \ (\text{mod } p_i). \tag{2.2}$$

If we let $m_i \equiv l_i (\text{mod } p_i)$ then (2.2) reduces to

$$l_i r_i \equiv 1 \ (\text{mod } p_i).$$

Further, $(m_i, p_i) = 1$ implies $(l_i, p_i) = 1$.

We know that $ax \equiv b (\text{mod } m)$ has a solution if $(a, m) = 1$, therefore we get

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$$
\begin{aligned}
r_i &\equiv l_i^{(\phi(p_i)-1)} \pmod{p_i} && \text{(using Euler–Fermat Theorem)}\\
&\equiv l_i^{(p_i-2)} \pmod{p_i}.
\end{aligned}
$$

Now, we put the value of r_i in equation (2.2) and then substitute the value of each n_i in equation (2.1) to get the value of n.

Before we move to the second method, let us make two useful observations.

Remark 1. The calculation process of r_i can be reduced if we find the exponent of $l_i < (\phi(p_i) - 1)$.

2. The value of n is not unique. Another n can be found by increasing each n_i by $\prod_{j=1}^{k} p_j$.

2.2. **Method II.** In this method, we use the concepts of linear algebra and we refer [2] for general references. Let the solution of the problem be n. Therefore n can be written as

$$
\begin{aligned}
n &= 2 \cdot 2^{2n_{1,1}} \cdot 3^{2n_{1,2}} \cdot 5^{2n_{1,3}} \cdots p_k^{2n_{1,k}}\\
n &= 3 \cdot 2^{3n_{2,1}} \cdot 3^{3n_{2,2}} \cdot 5^{3n_{2,3}} \cdots p_k^{3n_{2,k}}\\
n &= 5 \cdot 2^{5n_{3,1}} \cdot 3^{5n_{3,2}} \cdot 5^{5n_{3,3}} \cdots p_k^{5n_{3,k}}\\
&\vdots\\
n &= p_k \cdot 2^{p_k n_{k,1}} \cdot 3^{p_k n_{k,2}} \cdot 5^{p_k n_{k,3}} \cdots p_k^{p_k n_{k,k}}.
\end{aligned}
$$

On comparing the exponent of $2, 3, 5, \cdots, p_k$, we get

$$
\begin{aligned}
&2n_{1,1} + 1 = 3n_{2,1} = 5n_{3,1} = \cdots = p_k n_{k,1}\\
&2n_{1,2} = 3n_{2,2} + 1 = 5n_{3,2} = \cdots = p_k n_{k,2}\\
&2n_{1,3} = 3n_{2,3} = 5n_{3,3} + 1 = \cdots = p_k n_{k,3}\\
&\vdots\\
&2n_{1,k} = 3n_{2,k} = 5n_{3,k} = \cdots = p_k n_{k,k} + 1,
\end{aligned}
$$

which is same as the k sets of equations, where each set has $k - 1$ equations in k unknowns.

$$
\begin{gathered}
2n_{1,1} + 1 = 3n_{2,1}, \quad 2n_{1,1} + 1 = 5n_{3,1}, \quad \cdots \quad 2n_{1,1} + 1 = p_k n_{k,1}, \text{ i.e.,}\\
-2n_{1,1} + 3n_{2,1} = 1, \quad -2n_{1,1} + 5n_{3,1} = 1, \quad \cdots \quad -2n_{1,1} + p_k n_{k,1} = 1.
\end{gathered}
$$

Now, we can write these equations in matrix form $AX = B$, where

$$
A = \begin{bmatrix}
-2 & 3 & 0 & 0 & 0 & \cdots & 0\\
-2 & 0 & 5 & 0 & 0 & \cdots & 0\\
-2 & 0 & 0 & 7 & 0 & \cdots & 0\\
-2 & 0 & 0 & 0 & 11 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-2 & 0 & 0 & 0 & 0 & \cdots & p_k
\end{bmatrix}, \quad
X = \begin{bmatrix} n_{1,1}\\ n_{2,1}\\ \vdots\\ n_{k,1} \end{bmatrix}, \quad
B = \begin{bmatrix} 1\\ 1\\ \vdots\\ 1 \end{bmatrix}.
$$

The row reduced echelon form of the augmented matrix $A|B$ is as follows:

$$A|B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -p_k/2 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & -p_k/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & -p_k/5 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & -p_k/7 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -p_k/p_{k-1} & 0 \end{bmatrix}$$

which gives the equations

$$2n_{1,1} - p_k n_{k,1} = -1, \quad 3n_{2,1} - p_k n_{k,1} = 0, \quad \cdots \quad p_{k-2}n_{k-2,1} - p_k n_{k,1} = 0,$$
$$p_{k-1}n_{k-1,1} - p_k n_{k,1} = 0.$$

Now, we will solve this system of equations by back substitution method. The last equation implies $p_{k-1}n_{k-1,1} = p_k n_{k,1}$ that is $n_{k,1}$ should be a multiple of p_{k-1} and $n_{k-1,1}$ should be a multiple of p_k.

Similarly, the second last equation implies $p_{k-2}n_{k-2,1} = p_k n_{k,1}$ that is $n_{k,1}$ should be a multiple of p_{k-2} and $n_{k-2,1}$ should be a multiple of p_k.

Continuing in this manner, second equation implies $3n_{2,1} = p_k n_{k,1}$ that is $n_{k,1}$ should be a multiple of 3 and $n_{2,1}$ should be a multiple of p_k.

Therefore, $n_{k,1}$ is a multiple of $3, 5, 7, \cdots, p_{k-1}$. The smallest possible such value of $n_{k,1}$ is $3 \cdot 5 \cdot 7 \cdots p_{k-1}$. After getting the value of $n_{k,1}$, we use above equations to find the value of $n_{1,1}, n_{2,1}, n_{3,1}, \cdots, n_{k-1,1}$.

Similar procedure is used to find the exponents of $3, 5, 7, \cdots, p_k$. After calculating all the k^2 exponents $n_{i,j}$ where $1 \leq i, j \leq k$, we get the value of n.

Note 1. It is not necessary that we should take only prime numbers for the problem statement. If we take any set of k distinct natural numbers $a_1, a_2, a_3, ..., a_k$ such that $(a_i, a_j) = 1,$ where $1 \leq i, j \leq k, i \neq j$, then similar problem can be solved.

3. Extension of the problem

Further natural question would be:

Is it necessary that all the k natural numbers should be mutually coprime? Suppose we have the set of k natural numbers $a_1, a_2, a_3, ..., a_k$. We will consider two cases.

Case 1: We assume that there exists at least one prime p which is not a factor of each number $a_1, a_2 \cdots a_k$. Let a_i and a_j be two numbers which are not coprime. Suppose a_i and a_j has prime factorization

$$a_i = p_1^{x_1} \cdot p_2^{x_2} \cdot p_3^{x_3} \cdots p_l^{x_l}, \qquad a_j = q_1^{y_1} \cdot q_2^{y_2} \cdot q_3^{y_3} \cdots q_m^{y_m}.$$

We assume that there exists a prime p which is not common in both a_i and a_j. Without loss of generality, we can assume that p is a factor of a_i but not of a_j.

Suppose the solution exists and its value is N and the exponent of p in N be n_p. We will apply above mentioned conditions on n_p.

From condition (1) and (2) we conclude that $n_p \equiv 1(\text{mod } a_i)$ and $n_p \equiv 0(\text{mod } a_j)$. Further, let there exist a common prime p' in a_i and a_j. Therefore, the above congruence relations lead to $n_p \equiv 1(\text{mod } p')$ and $n_p \equiv 0(\text{mod } p')$, which is a contradiction.

Thus, we conclude that all the k numbers should be mutually coprime.

Case 2: We will give an example to show that similar problem can be solved if $(a_i, a_j) \neq 1$, where $1 \leq i, j \leq k, i \neq j$. Let a_i and a_j be of the form

$$a_i = p_1^{x_1} \cdot p_2^{x_2} \cdot p_3^{x_3} \cdots p_l^{x_l} \quad \text{and} \quad a_j = p_1^{y_1} \cdot p_2^{y_2} \cdot p_3^{y_3} \cdots p_l^{y_l},$$

where $x_k \neq y_k$ at least for one k between $1, 2, \cdots, l$. In particular, consider $a_1 = 2 \cdot 3, \quad a_2 = 2 \cdot 3^7$. The solution is $n = 2^{2(3^7)+1} \cdot 3^{2(3^7)+7}$.

We can thus conclude that the condition of all k natural numbers $a_1, a_2, a_3, ..., a_k$ being mutually coprime is not necessary.

Remark 2. We have discussed two methods of finding the solution of the problem. One can compare both the algorithms and find out which algorithm is better.

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REALS IN THE UNIT INTERVAL AS AVERAGE OF TWO REALS IN THE CANTOR'S MIDDLE THIRD SET

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Abstract. Using the ternary representation of real numbers between 0 and 1, a proof of (a slight generalization of) the fact "for any real $y \in [0, 1]$ there exist not necessarily distinct numbers c_1 and c_2 in the Cantor middle third set such that $y = (c_1 + c_2)/2$" is outlined in Paul Halmos's book [2]. In this note another proof is provided which does not ostensibly use the ternary representation of the Cantor middle third set.

1. Introduction

Given a particular real number $x \in [0, 1]$, there are reals a, b ($b \geq a$) belonging to the Cantor middle-third set so that $b - a = x$ (see [1] [3]). From this it is easy to conclude, as illustrated in [1], that for any real $y \in [0, 1]$ there exist not necessarily distinct numbers c_1 and c_2 in the Cantor middle third set such that $y = (c_1 + c_2)/2$.

A novel geometric proof of a slight generalisation of this result is found in [4]. Using the ternary representation of real numbers between 0 and 1, the now well known proof is outlined in Paul Halmos's book [2]. This relies on using the bijection between the elements of the Cantor middle third set and the set of infinite ternary decimal sequences with digits 0 and 2. Here another proof is provided, which does not ostensibly use the ternary representation of the Cantor middle third set.

2. The Theorem

Theorem 2.1. *Every real $y \in [0, 1]$ is the average of two not necessarily distinct real numbers each belonging to the Cantor middle third set C.*

Proof. Take an arbitrary real number $y \in [0, 1]$. In the process of constructing the Cantor set from $[0, 1]$ by deleting the middle thirds, after a finite number (k_0, say) of steps, y would fall in the interior of, or on the boundary of, an open set that is

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* If y falls in the interior of some cut out middle third set, it corresponds to the first time we have a 1 in the ternary expansion of y

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cut out for the first time. The length of the interval cut out at this k_0^{th} iteration is $(1/3^{k_0})^*$. Now, perform following two steps.

Step (i). Let the closest end point to y at this stage on the right be $a_1 \in [0,1]$ at a distance $r_1 = |a_1 - y|$, and that on the left be $b_1 \in [0,1]$ at a distance $l_1 = |b_1 - y|$ (we have $a_1, b_1 \in C$). Consider the unique $k_1 > 0$ so that $1/3^{k_1+1} < |l_1 - r_1| \leq 1/3^{k_1}$. We have $k_1 \geq k_0$. We may assume without loss of generality that $r_1 \geq l_1$. [1]

Step (ii). To the left of b_1, we further iterate and remove successive middle thirds so that eventually there is a point $b_2 \in C$ to the left of b_1 with $l_2 - l_1 = 2/3^{k_1+1}$, where $l_2 = |b_2 - y|$. At this stage, take $a_2 = a_1,$ and $r_2 = r_1$.

We have: $1/3^{k_1+1} - 2/3^{k_1+1} = -1/3^{k_1+1} < r_2 - l_2 = (r_2 - l_1) - (l_2 - l_1) \leq (1/3^{k_1} - 2/3^{k_1+1}) = 1/3^{k_1+1}$, and so $|r_2 - l_2| \leq 1/3^{k_1+1}$. Thus we can find a unique $k_2 > k_1$ such that $1/3^{k_2+1} < |r_2 - l_2| \leq 1/3^{k_2}$.

Now we perform steps exactly analogous to the steps (i) and (ii) above. It may happen that at the k'th stage, we have $l_k > r_k$. In this case, corresponding to (ii), we would find a point a_{k+1} to the right of a_k, while keeping $b_{k+1} = b_k$. The sequence $s_k = |r_k - l_k|$ in the k'th iterative step is bounded by a higher power of 1/3, and so $s_k \to 0$ as $k \to \infty$.

Here $\{a\}_{i=1}^{\infty}$ (resp. $\{b\}_{i=1}^{\infty}$) is bounded within [0,1], is non-decreasing (resp. non-increasing) and thus converges to a limit point a_∞ (resp. b_∞) that also belongs to the Cantor set itself - Cantor set being closed. In the limit, we thus have within the Cantor set, two points that are equidistant from y (with $r_\infty = |a_\infty - y| = |b_\infty - y| = l_\infty$), and this proves our assertion. □

For an example, consider at random, $y = \frac{2}{5}$. In this case, we have $a_1 = \frac{2}{3}, b_1 = \frac{1}{3}$, and $r_1 = \frac{4}{15}, l_1 = \frac{1}{15}$. Following our algorithm, we get $l_2 = l_1 + \frac{2}{3^2} = \frac{13}{45}$, and $r_2 = r_1 = \frac{12}{45}$. In this case, we have $a_2 = \frac{2}{3}, b_2 = \frac{1}{9}$. Further on, we would find that $b_3 = \frac{1}{9}, a_3 = \frac{56}{81}$, $a_4 = \frac{56}{81}, b_4 = \frac{79}{729}$, and $a_5 = \frac{4538}{6561}, b_5 = \frac{79}{729}$. The sequences a_k, b_k can be continued with a code, and the limit behavior of either sequence studied.

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[1] In case $l_1 > r_1$, the proof follows in an analogous way.

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A PROBLEM RELATED TO PRIME POWERS

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Abstract. Using elementary means, we solve a problem which is a variant of a result proved in [1].

The following result is recently proved by Gupta and Chakraborty [1]:

If p_k *denotes the* k*-th prime for any positive integer* k*, then there exist infinitely many positive integers* n *such that* $\frac{n}{2}$ *is a square,* $\frac{n}{3}$ *is a cube, ... ,* $\frac{n}{p_k}$ *is a* p_k*-th power.*

Note that in the above result the division by primes is considered. This naturally intrigues one to inquire what happens if instead of division, multiplication by primes is considered. In this short note, we consider this question and we prove the following result.

Theorem. *Let* k *be a positive integer and let* $p_1, p_2, \ldots, p_k$ *be* k *distinct prime numbers. Then there exist infinitely many positive integers* n *such that* np_1 *is a* p_1*-th power,* np_2 *is a* p_2*-th power, ...,* np_k *is a* p_k*-th power.*

Observe that the Theorem is obvious when $k = 1$ as one can take $n = p_1^a$ for any integer a such that $a + 1$ is divisible by p_1.

Proof. Let $p_1, p_2, \ldots, p_k$ be the given distinct prime numbers and put

$$S = \{n \in \mathbb{N} : n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \text{ with } a_i \in \mathbb{N}\} \quad \text{and}$$

$$T = \{n \in S : np_i = m_i^{p_i} \text{ for some integer } m_i \text{ and for all } i = 1, 2, \ldots, k\}.$$

The theorem will be proved if we show that T is an infinite subset of S.

Note that $n \in S$ can be written as $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and it will be in T if a_i's are so chosen that the following condition is satisfied:

For any $j \in \{1, 2, \ldots, k\}$,

$$p_j \mid a_i \text{ for all } i \in \{1, 2, \ldots, k\} \backslash \{j\} \quad \text{and} \quad p_j \mid (a_j + 1). \tag{0.1}$$

For this, let

$$s_i = \prod_{r=1\ ;\ r \neq i}^{k} p_r \text{ for all } i = 1, 2, \ldots, k.$$

Then, for a given $j \in \{1, 2, \ldots, k\}$, in view of (0.1), we need to choose a_i's in such a way that for all $i \in \{1, 2, \ldots, k\} \backslash \{j\}$

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$$a_i = s_i q_i \text{ for some integer } q_i, \tag{0.2}$$

and simultaneously

$$a_j = s_j q_j \equiv -1 \pmod{p_j} \text{ for some integer } q_j. \tag{0.3}$$

As s_i's are given and $gcd(s_i, p_i) = 1$ for all $i = 1, 2, \ldots, k$, we consider the simultaneous congruences

$$\begin{aligned} s_1 X &\equiv -1 \pmod{p_1}; \\ s_2 X &\equiv -1 \pmod{p_2}; \\ &\vdots \\ s_k X &\equiv -1 \pmod{p_k}. \end{aligned}$$

Since p_i's are all distinct prime numbers, by Chinese remainder theorem, there exist infinitely many integers q satisfying the above congruences. Then, by choosing $a_i = s_i q$ for all $i = 1, 2, \ldots, k$ we see that each a_i satisfies (0.2) and (0.3). Hence $n = p_1^{s_1 q} p_2^{s_2 q} \ldots p_k^{s_k q}$ lies in T for each q. But there are infinitely many such q's, therefore we see that T is an infinite subset of S and hence the theorem. ☐

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PARTITION FUNCTIONS
USING RECURSIVE RELATIONS

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Abstract. The study of partitions of numbers forms an important area in number theory. The theoretical developments usually make use of the properties of generating functions. The recurrence relations for the development of partition theory can be developed independently. The theme of this paper is to explore the application of recursive relations for partition functions.

1. Introduction

A *partition* of a positive integer n is its representation as a sum of natural numbers, called parts or summands. The order of the summands is irrelevant. For example, $4+2+1$, $2+2+1+1+1$ are partitions of the number 7. Since the order is irrelevant, $4+2+1$ is the same partition as $2+4+1$. The number of partitions of an integer n is denoted by $p(n)$. For example, the partitions of 5 are

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1$$

Thus, $p(5) = 7$. The reader can easily verify that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(6) = 11$, and $p(7) = 15$.

The partition function is a part of additive number theory, and good references for this topic are [1] and [2]. A partition of a number is referred to as *a restricted partition* if one puts some conditions on the summands such as requiring an odd number of parts or restricting the smallest part, etc.; otherwise it is referred to as *a partition* or *an unrestricted partition*. We shall consider here restricted partitions.

In this line of exposition, we wish to mention that the paper [3] by Hansraj Gupta is an excellent reference. In particular, there is an impressive list of references given in this paper. The book [4] by the same author is also a good reference for further reading. We thank the referee for making us aware of these two references and for the valuable comments that improved the article. For general reading on number theory, we recommend [5]. Finally, for an easy read on the partition function, we suggest [6].

We begin our discussion with an example. In the following table, we list the 15 partitions of the number 7 in a special grouping which we ask the reader to study

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now. Note that in the last column, $p(m, 7)$ stands for the number of partitions of 7 obtained using the largest part equal to m.

m	Partitions of 7 obtained using the largest part equal to m.	$p(m, 7)$
7	7	1
6	6 + 1	1
5	5 + 2, 5 + 1 + 1	2
4	4 + 3, 4 + 2 + 1, 4 + 1 + 1 + 1	3
3	3 + 3 + 1, 3 + 2 + 2, 3 + 2 + 1 + 1, 3 + 1 + 1 + 1 + 1	4
2	2 + 2 + 2 + 1, 2 + 2 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1 + 1	3
1	1 + 1 + 1 + 1 + 1 + 1 + 1	1

It is observed from the table that $\sum_{m=1}^{7} p(m, 7) = 1{+}1{+}2{+}3{+}4{+}3{+}1 = 15 = p(7)$, the total number of partitions of 7. This *suggests* that for any n in general, the partition function $p(n)$ can be given by

$$p\,(n) = \sum\nolimits_{m=1}^{n} p\,(m, n),$$

where $p(m, n)$ *denotes the number of partitions of* n *obtained using the largest part equal to* m, and this set is referred to as *the set representing* $p(m, n)$. Assuming that this is true, it is clear that we then need to find a method to compute $p(m, n)$ for any n and for $m = 1, \cdots, n$. One way of doing this is to express $p(m, n)$ in terms of $p(x, y)$, where $x \leq m(\leq n)$ and $y \leq n$. In this way, the calculation of $p(m, n)$ can possibly be made easier. Consider the 7 partitions that arise when we calculate $p(3, 9)$. These can be expressed as a union of two sets as:

$$\left\{ \begin{array}{l} 3+2+2+2 \\ 3+2+2+1+1 \\ 3+2+1+1+1+1 \\ 3+1+1+1+1+1+1 \end{array} \right\} \bigcup \left\{ \begin{array}{l} 3+3+3 \\ 3+3+2+1 \\ 3+3+1+1+1 \end{array} \right\}.$$

The first set consists of all those partitions of 9 in which 3 appears just once. If we replace this 3 by 2, we get the partitions whose count yields exactly $p(2, 8)$. The second set consists of all those partitions of 9 in which 3 appears more than once. If we remove one of the 3's in each partition, we get the set of partitions whose count yields exactly $p(3, 9-3) = p(3, 6)$. Thus, we get $p(3, 9) = p(2, 8) + p(3, 6)$. Let us consider another example. Let $p'(m, n)$ *denote the number of partitions of* n *with largest part less than or equal to* m. Consider $p'(3, 7)$. The list of the partitions of $p'(3, 7)$ is: $3+3+1$, $3+2+1+1$, $3+1+1+1+1$, $2+2+2+1$, $2+2+1+1+1$, $2+1+1+1+1+1$, $1+1+1+1+1+1+1$. (Note, for example, that the partition $2+5$ of 7 can't appear in this list). Thus $p'(3, 7) = 7$. To find the recurrence relation, we split these 7 partitions into two sets as follows:

$$\left\{\begin{array}{l} 2+2+2+1 \\ 2+2+1+1+1 \\ 2+1+1+1+1+1 \\ 1+1+1+1+1+1+1 \end{array}\right\} \bigcup \left\{\begin{array}{l} 3+3+1 \\ 3+2+1+1 \\ 3+1+1+1+1 \end{array}\right\}$$

The first set consists of all those partitions of 7 whose greatest part is less than or equal to 2. There are $4 = p'(2,7) = p'(3-1,7)$ partitions in this set. The second set consists of all those partitions of 7 in which the largest part is 3. If this 3 is removed from each partition, we get the set of partitions whose count yields exactly $3 = p'(3,4) = p'(3,7-3)$. Thus we see that $p'(3,7) = p'(2,7) + p'(3,4)$.

In the next two sections, we shall prove in general the recurrence relations we discussed in the above two examples. We will also consider other restricted partitions. In Section 3, we will consider unrestricted partitions and give a recurrence relation that expresses $p(n)$ in terms of restricted partitions. More specifically, we will show that

$$p(n) = \sum\nolimits_{m=1}^{n} \sum\nolimits_{k=m}^{m(m+1)/2} C(m,k) p(n-k),$$

where $C(m,n)$ denotes the difference between the number of distinct partitions of n starting with the largest number m and with odd number of parts, and the number of distinct partitions of n with the largest number m and with even number of parts. (See Theorem 3.2.) We then use this result to prove the celebrated formula for $p(n)$:

$$p(n) = \sum\nolimits_{k=1}^{\infty} (-1)^{k+1} \left\{p\left(n - k(3k-1)/2\right) + p\left(n - k(3k+1)/2\right)\right\}.$$

(See Theorem 3.3 below.)

2. Restricted Partitions and Recurrence Relations

We now state and prove the general recurrence formula for $p(m,n)$

Theorem 2.1. *With $p(m,n)$ as considered earlier, we have a recurrence relation for it given by*

$$p(m,n) \;=\; p(m-1,n-1) + p(m,n-m) \tag{2.1}$$

Before we give the proof, let us offer an example that illustrates this recurrence by considering $p(4,10)$. Here $m = 4, n = 10$ and one observes that the set {4+4+2, 4+4+1+1, 4+3+3, 4+3+2+1, 4+3+1+1+1, 4+2+2+2, 4+2+2+1+1, 4+2+1+1+1+1, 4+1+1+1+1+1+1} represents $p(4,10)$ and hence $p(4,10) = 9$. One can easily see from this set that $p(4,10-4) = 2, p(3,6) = 3, p(2,6) = 3$, $p(1,6) = 1$; and their sum is $p(4,10)$.

Proof. Assume that we have computed the values of $p(m,n-m)$, $p(m-1,n-m)$, ..., $p(1,n-m)$. Then the set representing $p(m,n)$ is obtained from the sets representing each of $p(m,n-m)$, $p(m-1,n-m)$, ..., $p(1,n-m)$ by adding m to each member of each of these sets and taking their union. Thus

$$p(m,n) = p(m,n-m) + p(m-1,n-m) + \cdots + p(1,n-m). \tag{2.2}$$

Replace n by $n-1$ and m by $m-1$ in (2.2) to get

$$p(m-1,n-1) = p(m-1,n-m) + p(m-2,n-m) + \cdots + p(1,n-m). \quad (2.3)$$

Subtracting (2.3) from (2.2) and solving for $p(m,n)$ we get (2.1). ☐

Theorem 2.2. *If $h(m,n)$ denotes the number of those partitions of n in which the least part is equal to m, then $h(m,n)$ satisfies the following recurrence relation*

$$h(m,n) = h(m-1,n-1) - h(m-1,n-m). \quad (2.4)$$

Before we give the proof, let us offer an example that illustrates this recurrence by considering $h(2,8)$. Note that the set representing $h(1,7)$ can be expressed as the union of two sets as follows:

$$\left\{\begin{array}{l} 6+1 \\ 4+2+1 \\ 3+3+1 \\ 2+2+2+1 \end{array}\right\} \bigcup \left\{\begin{array}{l} 5+1+1 \\ 4+1+1+1 \\ 3+2+1+1 \\ 3+1+1+1+1 \\ 2+2+1+1+1 \\ 2+1+1+1+1+1 \\ 1+1+1+1+1+1+1 \end{array}\right\}$$

The first set consists of all those partitions of 7 in which the least part (equal to 1) appears just once. The second set consists of all those partitions of 7 in which the least part appears more than once. Observe that if the least part of each member in the first set is incremented by 1, it results in a set of partitions of 8 whose least part is 2; and this represents $h(2,8)$. The other set cannot be dealt similarly because to increment one least part by 1 in each member of the second set would require 7 to be incremented by at least 2 (why?). Further, observe that if one least part is removed from each partition of each set, the set of new partitions will represent $h(1,6)$ - the fact which is left to the reader to verify. Thus $h(2,8) = h(1,7) - h(1,6)$.

We now give the proof of Theorem 2.2.

Proof. The central idea of the proof is same as that of the proof of Theorem 2.1. The main difference is that we now assume here that the values $h(m,n-m)$, $h(m+1,n-m)$, ..., $h(n-m,n-m)$ are obtained. Then the set representing $h(m,n)$ is obtained from the sets representing each of $h(m,n-m)$, $h(m+1,n-m)$, ..., $h(n-m,n-m)$ by adding m to each member of each of these sets and taking their union. Hence

$$h(m,n) = h(m,n-m) + h(m+1,n-m) + \cdots + h(n-m,n-m). \quad (2.5)$$

Replace n by $n-1$ and m by $m-1$ in (2.5) to get

$$h(m-1,n-1) = h(m-1,n-m) + h(m,n-m) + \cdots + h(n-m,n-m) \quad (2.6)$$

Subtracting (2.6) from (2.5) and solving for $h(m,n)$ we get (2.4). ☐

The methods of proofs and the results of the Theorems 2.1 and 2.2 have several consequences. We prove some of these in the following corollaries.

Corollary 2.3. *With $p'(m,n)$ considered as earlier, we have a recurrence relation for it given by*

$$p'(m,n) = p'(m-1,n) + p'(m,n-m).$$

Proof. We have $p'(m,n) = p'(m-1,n) + p(m,n)$. From (2.2) we can easily see that $p(m,n) = p'(m,n-m)$, and the corollary follows. □

Corollary 2.4. *If $h'(m,n)$ denotes the number of partitions of n which have the least part m or larger, then the following recurrence relation holds*

$$h'(m,n) = h'(m+1,n) + h'(m,n-m). \tag{2.7}$$

Proof. We note that $h'(m,n) = h'(m+1,n) + h(m,n)$. From (2.5), we have

$$h(m,n) = h(m+1,n-m)+h(m+2,n-m)+\cdots+h(n-m,n-m) = h'(m+1,n-m).$$

This proves (2.7). □

Corollary 2.5. *If $\pi(m,n)$ denotes the number of partitions of n into m parts then*

$$\pi(m,n) = p(m,n).$$

Let us first illustrate this corollary through an example using Ferrers diagrams. Conside $m = 3$ and $n = 8$. The set $\{6+1+1,\ 5+2+1,\ 4+3+1,\ 4+2+2,\ 3+3+2\}$ representing $\pi(3,8)$ and the set $\{3+1+1+1+1+1,\ 3+2+1+1+1,\ 3+2+2+1,\ 3+3+2\}$ representing $p(3,8)$ are illustrated simultaneouly as follows.

3 1 1 1 1 1	3 2 1 1 1	3 2 2 1	3 3 1 1	3 3 2
6 ● ● ● ● ● ●	5 ● ● ● ● ●	4 ● ● ● ●	4 ● ● ● ●	3 ● ● ●
1 ●	2 ● ●	3 ● ● ●	2 ● ●	3 ● ● ●
1 ●	1 ●	1 ●	2 ● ●	2 ● ●

Obviously, in an intuitive way, a look at the diagrams from one direction gives idea of one set of partitions and a look at it from the other direction gives idea of the other set. The one-to-one relationship is clear. Observe that $n - m = 5$ and the set $\{1+1+1+1+1\}$ represents $p(1,5)$, the set $\{2+1+1+1,\ 2+2+1\}$ represents $p(2,5)$, the set $\{3+1+1,\ 3+2\}$ represents $p(3,5)$ and $\pi(3,8) = 5 = p(1,5) + p(2,5) + p(3,5) = p(3,8)$. Now, we offer the brief proof of the corollary.

Proof. Let us separate m units from n to represent the m parts. Then $\pi(m,n)$ is precisely the sum of the number of partitions of the remaining number $n - m$ obtained with the largest part equal to m, $m-1$, $m-2$, ... and 1. This leads to the same recurrence as defined in Theorem 2.1, with $\pi(1,k) = 1 = p(1,k)$. □

Theorem 2.6. *Let $p_k(m,n)$ denote the number of partitions of n with the largest part equal to m and using parts differing from m by a multiple of k. Then the following recurrence relation holds*

$$p_k(m,n) = p_k(m-k,n-k) + p_k(m,n-m). \tag{2.8}$$

Let us consider a simple example to illustrate this result. Taking $k = 2, m = 4, n = 10$, consider $p_2(4, 10)$ which consists of two partitions $4 + 4 + 2$ and $4 + 2 + 2 + 2$. These partitions are separated into two sets, one consisting of those partitions which have only one part equal to $m = 4$ (clearly, $4 + 2 + 2 + 2$ in this case) and the other set comprised of the rest (clearly, $4 + 4 + 2$ in this case). Then, for the partitions of the first set having only one part equal to $m = 4$, this part is decremented by $k = 2$, giving a partition (namely, $2 + 2 + 2 + 2$ in this case) of $n - k$ (=8) with largest part $m - k = 2$ and still only using parts which are multiples of $k = 2$, thus belonging to $p_k(m-k, n-k)$ (=$p_2(2, 8)$ in this case). For the partitions of the other set, one part equaling $m = 4$ can be removed from each partition giving a partition (namely, $4 + 2$ in this case) with the largest part still as $m = 4$, but now partitioning $n - m = 6$ and still using multiples of $k = 2$ as other parts, thus belonging to $p_k(m, n - m)$ (=$p_2(4, 6)$ in this case).

We now give the proof of Theorem 2.6

Proof. If $p_k(m,\ n-m)$, $p_k(m-k,\ n-m)$, $p_k(m-2k,\ n-m)$, $p_k(m-3k,\ n-m), \cdots$ are obtained then the set representing $p_k(m, n)$ is obtained from the sets representing each of $p_k(m,\ n-m)$, $p_k(m-k,\ n-m)$, $p_k(m-2k,\ n-m)$, $p_k(m-3k,\ n-m), \cdots$ by adding m to each member of each of these sets and taking their union. Thus

$p_k(m,n)=p_k(m,n-m)+p_k(m-k,n-m)+p_k(m-2k,n-m)+p_k(m-3k,n-m)+\cdots$.

In this expression, replace m by $m-k$ and n by $n-k$ to get

$p_k(m-k,n-k) = p_k(m-k,n-m)+p_k(m-2k,n-m)+ p_k(m-3k,n-m)+\cdots$.

Subtracting these equations and solving for $p_k(m, n)$ completes the proof of the theorem. ☐

Corollary 2.7. *If $d_k(m, n)$ denotes the number of partitions of n with the largest part equal to m and using parts differing from m by a multiple of k, then the following recurrence relations holds*

$$d_k(m, n) = d_k(m - k, n - k) + d_k(m - k, n - k).$$

Proof. The proof follows proceeding as in the proof of Theorem 2.6. ☐

Theorem 2.8. *If $d(m, n)$ denotes the number of partitions of n using distinct parts and with the largest part equal to m, then the recurrence relation for $d(m, n)$ is given by*

$$d(m, n) = d(m - 1, n - 1) + d(m - 1, n - m). \tag{2.9}$$

By convention, $d(m, n) = 0$, if $m(m + 1)/2 < n$, and $d(n, n) = 1$.

Proof. If $d(m-1, n-m)$, $d(m-2, n-m)$, ..., $d(1, n-m)$ are obtained then $d(m, n)$ is obtained by prefixing m to each member of these. Thus

$$d(m, n) = d(m - 1, n - m) + d(m - 2, n - m) + \cdots + d(1, n - m).$$

The rest of the argument is exactly the same as that of Theorem 2.1. ☐

Corollary 2.9. *Let $d'(m, n)$ denote the partition of n using numbers no greater than m, then the following recurrence relation holds*

$$d'(m, n) \ = \ d'(m - 1, n) \ + \ d'(m - 1, n - m). \tag{2.10}$$

Note that $d\,(n) = d'(n, n)$.

Proof. We have $d'(m, n) \; = d'(m-1, n) + d(m, n)$. Arguing as in the proof of Theorem 2.8, we have $d(m, n) \; = d'(m-1, n-m)$. (2.10) follows from (2.9). $\square$

Remark 2.10. Note that if $d(n)$ is a partition of n into distinct summands, then

$$d(n) = d'(n, n) = \sum\nolimits_{m=1}^{n} d(m, n).$$

Theorem 2.11. *Let $\theta(m, n)$ denote the number of partitions of n into m distinct parts. Then the following recurrence relation holds for $\theta(m, n)$.*

$$\theta\,(m, n) = \theta\,(m, n-m) + \theta\,(m-1, n-m)\,. \tag{2.11}$$

Furthermore,

$$\theta\,(m, n) = p\,(m, n - m\,(m-1)\,/2)\,. \tag{2.12}$$

Proof. To prove the first recurrence, we note that if $n = a_1 + a_2 + \cdots + a_m$ is a partition of n into m distinct parts, then

$$n - m = (a_1 - 1) + (a_2 - 1) + \cdots + (a_m - 1)\,.$$

If $a_1 = 1$, then it is a representation of $n - m$ with $m - 1$ distinct parts and if $a_1 > 1$, it is a representation of $n - m$ into m distinct parts. This proves (2.11). Note that

$$n - (0 + 1 + 2 + \cdots + m - 1) = (a_1 - 0) + (a_2 - 1) + \cdots + (a_m - (m-1))$$

is a presentation of $n - m(m-1)/2$. We now apply Corollary 2.5 to conclude (2.12). $\square$

3. The Partition Function Theorem

Theorem 3.1. *Let $C(m, n)$ denote the difference between the number of distinct partitions of n with largest part m and with odd number of parts, and the number of distinct partitions of n with largest part m and with even number of parts, then the following recurrence relation holds.*

$$C\,(m, n) = C\,(m-1, n-1) - C\,(m-1, n-m)$$

with $C(1, 1) = 1$. Furthermore,

$$\sum_{m=1}^{n} C\,(m, n) = \begin{cases} 1, & \text{if } k \text{ is odd and } n = k(3k \pm 1)/2, \\ -1, & \text{if } k \text{ is even and } n = k(3k \pm 1)/2, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Proof. By Theorem 2.8 we have

$$d\,(m, n) = d\,(m-1, n-1) + d\,(m-1, n-m)\,.$$

Observe that each member of the set representing the second term $d(m-1, n-m)$ of the above relation is a partition of $n - m$ using distinct parts and with largest part $m - 1$; and hence, if m is added to it we get a partion of n again using distinct parts (but now with largest part m). Similarly, each member of the set representing the first term $d(m-1, n-1)$ of the above relation is a partition of $n - 1$ using distinct parts and with largest part $m - 1$; and hence if 1 is added it we get a partition of n again using distinct parts. In both the cases, number of partitions does not change, only its type gets changed - odd becomes even and even becomes odd as one part is addded. If we identify the number of odd partitions as a positive number, starting with $C(1, 1) = 1$, and the number of even partitions as a negative number, then the left side may be replaced by $C(m,n)$ and the right two terms may be replaced by $C(m-1,n-1)$, and $-C(m-1,n-m)$.

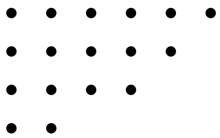
To prove (3.1), we note that $C(m,n) = p_0(m,n) - p_e(m,n)$, where $p_o(m,n)$ denotes the number of partitions of n using odd number of distinct parts and with largest part m, and $p_e(m,n)$ denotes the number of partitions of n using distict even number of parts and with largest part m. It follows that

$$\begin{aligned}\sum\nolimits_{m=1}^{n} C(m,n) &= \sum\nolimits_{m=1}^{n} (p_o(m,n) - p_e(m,n)) \\ &= \sum\nolimits_{m=1}^{n} p_o(m,n) - \sum\nolimits_{m=1}^{n} p_e(m,n) \quad = \quad p_o(n) - p_e(n).\end{aligned}$$

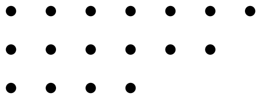
Thus, it suffices to prove

$$p_0(n) - p_e(n) = \begin{cases} 1, & \text{if } k \text{ is odd and } n = k(3k \pm 1)/2, \\ -1, & \text{if } k \text{ is even and } n = k(3k \pm 1)/2, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

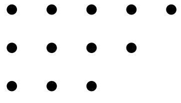
Before giving the general proof of (3.2), let us consider an example with $n = 17$. We make use of Franklin's argument using Ferrers diagrams to establish the relationship. The diagram below illustrates one partition of 17 with $k = 4$ distinct parts, namely $17 = 6 + 5 + 4 + 2$.



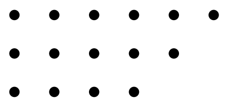
In such a diagram, let N_B denote the number of dots in the bottom row and N_S denote the number of dots along the slant line on the right in the northwest direction from the extreme right upper corner. We look for the possibility of the bottom dots to be placed on the right or the right dots to be placed at the bottom to get a valid distinct partition. In this case, we have $N_B = 2$ and $N_S = 3$. Note that the bottom two dots can be placed on the right as shown in the diagram below to yield $17 = 7 + 6 + 4$.



We say the odd partition of 17 corresponding to the even partition $6+5+4+2$ is $7+6+4$ and the even partition of 17 corresponding to the odd partition $7+6+4$ is $6+5+4+2$. The correspondence is unique. This correspondence does not exist for two cases for every k. Consider, for example, $n = 12$ and $k = 3$. Note that this is the case of $n = k(3k-1)/2$. As can be seen from the diagram below, we have $N_B = 3$ and $N_S = 3$.



If we now take $n = 15$ and $k = 3$ then we have $n = k(3k+1)/2$, $N_B = 4$ and $N_S = 3$, as shown in the diagram below.



It is clear from each diagram that in both of the above cases we can neither move N_B dots nor N_S dots to get distinct partitions of n from the given partition.

We now give the general proof of (3.2). Consider a partition of n given by

$$n = a_1 + a_2 + \cdots + a_j + \cdots + a_k \quad \text{with} \tag{3.3}$$

$$a_1 > a_2 > a_3 > \cdots > a_k,\ a_i = a_{i+1} + 1 \ for\ i = 1, 2, \cdots j-1 \text{ and } a_j > a_{j+1} + 1.$$

We now consider four cases:

Case 1. $a_k \le j < k$ or $a_k < j \le k$.

In this case, let $a_k = r$, then replace r in (3.3) by adding 1 to the first r terms in the sum to get

$$n = (a_1 + 1) + (a_2 + 1) + \cdots (a_r + 1) + a_{r+1} + \cdots + a_j + \cdots + a_{k-1}. \tag{3.4}$$

Clearly there is a one-to-one correspondence between representations of n in (3.3) and (3.4). Thus $p_o(n) - p_e(n) = 0$.

Case 2. $a_k > j$.

In this case, define $a_{k+1} = j$, then subtract 1 from the first j terms of (3.3) and add a_{k+1} to get

$$n = (a_1 - 1) + (a_2 - 1) + \cdots (a_j - 1) + a_{j+1} + \cdots + a_k + a_{k+1}. \tag{3.5}$$

Clearly, there is a one-to-one correspondence between representations of n in (3.3) and (3.5). Hence $p_o(n) - p_e(n) = 0$.

Case 3. $j = k = a_k - 1$. In this case, the terms in (3.3) are

$$a_k = k+1,\ a_{k-1} = k+2,\ a_{k-2} = k+3,\ \cdots,\ \ a_1 = k + (k+1)$$

and hence

$$n = 2k + (2k-1) + \cdots + (k+1) = k(3k+1)/2.$$

Thus, if k is odd, then we cannot write n as distinct parts with even number of terms and hence $p_o(n) - p_e(n) = 1$. Similarly, if k is even, we have $p_o(n) - p_e(n) = -1$.

Case 4. $j = k = a_k$. In this case, the terms in (3.3) are

$$a_k = k,\ a_{k-1} = k+1,\ a_{k-2} = k+2,\ \cdots,\ \ a_1 = k + (k-1)$$

and hence

$$n = (2k-1) + \cdots + (k+1) + k = k(3k-1)/2.$$

We arrive at the same conclusion as in Case 3. □

Theorem 3.2. *The partition $p(n)$ of n is given by*

$$p(n) = \sum_{m=1}^{n} \sum_{k=m}^{m(m+1)/2} C(m,k)\ p(n-k),$$

with $p(0) = 1$.

Proof. Note that

$$p\,(n) = \sum\nolimits_{m=1}^{n} p\,(m,n).$$

Using the recurrence relation of $p(m,n)$ given in Theorem 2.1, we can write

$$p(n)=\sum_{m=1}^{n} p(m-1,n-1)+\sum_{m=1}^{\lfloor n/2\rfloor} p(m,n-m)=p\,(n-1)+\sum_{m=1}^{\lfloor n/2\rfloor} p(m,n-m), \tag{3.6}$$

where $\lfloor n/2\rfloor$ is the floor function, which equals the integer part of $n/2$. This gives

$$\sum\nolimits_{m=1}^{\lfloor n/2\rfloor} p(m,n-m) = p(n) - C(1,1)p(n-1), \tag{3.7}$$

with $C(1,1) = 1$. Again using the recurrence of Theorem 2.1 on $p(m,n-m)$, we have

$$\sum\nolimits_{m=1}^{\lfloor n/2\rfloor} p(m,n-m) = \sum\nolimits_{m=2}^{\lfloor n/2\rfloor} p(m-1,n-2-(m-1))+\sum\nolimits_{m=1}^{\lfloor n/3\rfloor} p(m,n-2m).$$

Since

$$\begin{aligned}\sum\nolimits_{m=1}^{\lfloor n/2\rfloor} p(m-1,n-1-m) &= \sum\nolimits_{m=2}^{\lfloor n/2\rfloor -1} p(m,n-2-m)\\ &= \sum\nolimits_{m=2}^{\lfloor (n-2)/2\rfloor} p(m,n-2-m) = p(n-2) - p(n-3),\end{aligned}$$

where we have used (3.6) with n replaced by $n-2$, the previous equation can be expressed as

$$\sum\nolimits_{m=1}^{\lfloor n/2\rfloor} p(m,n-m) = p(n-2) - p(n-3) + \sum\nolimits_{m=1}^{\lfloor n/3\rfloor} p(m,n-2m). \tag{3.8}$$

Combining (3.7) and (3.8), and noting that $C(1,0) = 0$, $C(1,2) = 0$, $C(2,2) = C(1,1) - C(1,0)$, and $C(2,3) = C(1,2) - C(1,1)$, we get

$$\sum\nolimits_{m=1}^{\lfloor n/3\rfloor} p(m,n-2m) = p(n) - C(1,1)p(n-1) - \sum\nolimits_{k=2}^{3} C(2,k)p(n-k). \tag{3.9}$$

As before, we use the recurrence formula of Theorem 2.1 on $p(m,n-2m)$ to write

$$\sum_{m=1}^{\lfloor n/3\rfloor} p(m,n-2m) = \sum_{m=1}^{\lfloor n/3\rfloor} p(m-1,n-2m-1) + \sum_{m=1}^{\lfloor n/4\rfloor} p(m,n-3m). \tag{3.10}$$

But, the first term on the right can be expressed as

$$\begin{aligned}\sum\nolimits_{m=1}^{\lfloor n/3\rfloor} p(m-1,n-2m-1) &= \sum\nolimits_{m=2}^{\lfloor n/3\rfloor} p(m-1,n-3-(m-1))\\ &= \sum\nolimits_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-3-2m).\end{aligned}$$

We now use the recurrence of Theorem 2.1 twice to get

$$\begin{aligned}p(m,n-3) &= p(m-1,n-4) + p(m,n-3-m)\\ &= p(m-1,n-4) + p(m-1,n-4-m) + p(m,n-3-2m).\end{aligned}$$

Solving for $p(m,n-3-2m)$ and summing over m, we obtain

$$\begin{aligned}\sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-3-2m)&=\sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-3)-\sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m-1,n-4)\\ &\quad -\sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m-1,n-4-m) - \sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-3-2m).\end{aligned} \tag{3.11}$$

We note that $\sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-3) = p(n-3)$ and $\sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-4) = p(n-4)$. From the recurrence of Theorem 2.1, we also have

$$\sum\nolimits_{m=1}^{\lfloor (n-3)/3\rfloor} p(m-1,n-4-m)$$
$$= \sum\nolimits_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-5) - \sum\nolimits_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-6) = p(n-5) - p(n-6).$$

Thus, (3.11) becomes

$$\sum_{m=1}^{\lfloor (n-3)/3\rfloor} p(m,n-3-2m) = p(n-3) - p(n-4) - p(n-5) + p(n-6). \quad (3.12)$$

Combining (3.9), (3.10), (3.12), and the recurrence formula for $C(m,n)$ given in Theorem 3.1, we obtain

$$\sum\nolimits_{m=1}^{\lfloor n/4\rfloor} p(m,n-3m) = p(n) - C(1,1)p(n-1) - \sum\nolimits_{k=2}^{3} C(2,k)p(n-k)$$
$$- \sum\nolimits_{i=3}^{6} C(3,j)p(n-j). \quad (3.13)$$

To summarize what we have so far, let us define

$$Q_1(n) = p(n) - C(1,1)p(n-1), \quad Q_2(n) = Q_1(n) - \sum\nolimits_{k=2}^{3} C(2,k)p(n-k)$$
$$\text{and } Q_3(n) = Q_2(n) - \sum\nolimits_{i=3}^{6} C(3,j)p(n-j).$$

Then (3.13) can be expressed as $\sum\nolimits_{m=1}^{\lfloor n/4\rfloor} p(m,n-3m) = Q_3(n)$.

Let us also look closely at the process that produced the equation in (3.13). The last sum in this formula was obtained after applying the recurrence of Theorem 2.1 four times. Hence the four terms in the summation. It is also worth noting that this is the third time we were using the recurrence and hence the sum from $j=3$ to $j=6=3(3+1)/2$. This leads us to conjecture that

$$\sum\nolimits_{m=1}^{\lfloor n/k\rfloor} p(m,n-km) = Q_{k-1}(n) = \sum\nolimits_{j=k}^{k(k+1)/2} C(k,j)p(n-j). \quad (3.14)$$

When $k=n$, we obtain

$$0 = \sum\nolimits_{m=1}^{\lfloor n/n\rfloor} p\left(m,n-nm\right) = p(n) - \sum\nolimits_{m=1}^{n} \sum\nolimits_{j=m}^{n(n+1)/2} C(m,j)p(n-j).$$

The result follows by solving for $p(n)$. □

Theorem 3.3. (The Partition Function Theorem). *The partition of n is given by*

$$p(n) = \sum\nolimits_{k=1}^{\infty} (-1)^{k+1}\left\{p\left(n-k(3k-1)/2\right) + p\left(n-k(3k+1)/2\right)\right\}.$$

with $p(0)=1$, and $p(j)=0$ if $j<0$.

Proof. From the Theorem 3.2, we have

$$p(n) = \sum\nolimits_{m=1}^{n} \sum\nolimits_{k=m}^{m(m+1)/2} C(m,k)p(n-k).$$

Noting that $C(m,k)$ is zero when $k<m$ and $p(n-k)$ is zero when $k>n$, we can put the above equation in the form

$$p(n) = \sum\nolimits_{k=1}^{n} p(n-k) \sum\nolimits_{m=1}^{n} C(m,k).$$

The result then follows from (3.1). □

Theorem 3.4. *The distinct partitions $d(n)$ of n is given by*

$$d\,(n) = \sum\nolimits_{m=1}^{r} \sum\nolimits_{k=m}^{m(m+1)/2} C\,(m,k)\,d\,(n-k) + \delta_r,$$

where

$$r(r-1)/2 \leq n \leq r(r+1)/2 \text{ and } \delta_k = \begin{cases} 1, & \text{if } \frac{k(k+1)}{2} = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We make use of the relation $d\,(n) = \sum_{m=1}^{n} d\,(m,n)$. For $n = 1$, $d(1) = 1$. Using the recurrence relation given in Theorem 2.8, we have

$$\begin{aligned} d(n) &= \sum\nolimits_{m=1}^{n} d(m-1,n-1) + \sum\nolimits_{m=2}^{\lfloor (n+1)/2 \rfloor} d(m-1,n-m) \\ &= d(n-1) + \sum\nolimits_{m=2}^{\lfloor (n+1)/2 \rfloor} d(m-1,n-m). \end{aligned}$$

The second term vanishes when $n = 2$, and is equal to 1 (or $d(1)$) when $n = 3$. We rewrite this in the form

$$\sum\nolimits_{m=2}^{\lfloor (n+1)/2 \rfloor} d(m-1,n-m) = d(n) - C(1,1)d(n-1). \tag{3.15}$$

Upon expanding, we get

$$\begin{aligned} \sum\nolimits_{m=3}^{\lfloor (n+1)/2 \rfloor} d(m-2,n-m-1) + \sum\nolimits_{m=3}^{\lfloor (n+3)/3 \rfloor} d(m-2,n-2m+1) \\ = d(n) - C(1,1)d(n-1) \end{aligned}$$

On substituting from (3.15),

$$\sum_{i=3}^{\lfloor (n+3)/3 \rfloor} d\,(i-2,n-2i+1) = d\,(n) - C\,(1,1)\,d\,(n-1) - \sum_{j=2}^{3} C\,(2,j)\,d\,(n-j).$$

At step k, we get

$$\begin{aligned} \sum\nolimits_{m=k}^{\left\lfloor \frac{n+k(k-1)/2}{k} \right\rfloor} d\,(m-k+1,n-(k-1)m+(k-1)(k-2)/2) \\ = d(n) - \sum\nolimits_{m=1}^{k} \sum\nolimits_{j=m}^{m(m+1)/2} C(m,j)d(n-j). \end{aligned}$$

The left side has a value of 1 when the upper limit is exactly k. The result follows from this equation. ☐

Theorem 3.5. *The partition function $d(n)$ can be expressed as*

$$d\,(n) = \sum_{k=1}^{\infty} (-1)^{k+1} \left(d(n-k(3k-1)) + d(n-k(3k+1))\right) + \delta,$$

where $d(0) = 1$ *and* $\delta = \begin{cases} 1 \text{ if } n \text{ is a triangular number,} \\ 0 \text{ otherwise.} \end{cases}$

Proof. The result follows from the arguments in the proof of Theorem 3.4. ☐

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PROBLEM SECTION

In the last issue of the Math. Student Vol. **87**, Nos. 1-2, January-June (2018), we had invited solutions from the floor to the remaining problems *6, 8, 9,* correctly stated *10* and *11* of the MS, 86, 3-4, 2017 as well as to the six new problems *1, 2, 3, 4, 5* and *6* presented therein till October 31, 2018.

The status regarding the remaining problems of MS, 86, 3-4, 2017 is as under.

1. We received from the floor **one correct** solution to the problems *9* and *10* which we publish here.

2. Complete and correct solutions were not received from the floor for the problems *6, 8* and *11* and hence we provide in this issue the Proposer's solution to these problems.

The status regarding the problems of MS 87, 1-2, 2018 is as under.

1. We received from the floor **one correct** solution to problem *3* which we publish here.

2. **No solutions** were received from the floor to the remaining problems *1, 2, 4, 5, 6 and 7*. Readers can try their hand on these till April 30, 2019.

In this issue we first present **six new problems**. Solutions to these problems as also to the remaining problems *1, 2, 4, 5, 6 and 7* of MS 87, 1-2, 2018, received from the floor till April 30, 2019, if approved by the Editorial Board, will be published in the MS 88, 1-2, 2019.

Problem proposed by M. Ram Murty. MS-2018, Nos. 3-4: Problem-7:

Prove that

$$\sum_{m,n=1}^{\infty} \frac{(m,n)}{m^2n^2} = \frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n^3},$$

where (m,n) denotes the greatest common divisor of m and n.

Problems proposed by B. Sury

MS-2018, Nos. 3-4: Problem-8:

If $\lceil a \rceil$ denotes the smallest integer greater than or equal to a, prove:

(i) the closest integer to a is $\lceil a - 1/2 \rceil$.

(ii) Further, prove

$$\sum\nolimits_{k=1}^{n} d(2k-1) = \sum\nolimits_{r=1}^{n} \lceil n/(2r-1) - 1/2 \rceil.$$

MS-2018, Nos. 3-4: Problem-9:

Determine with proof the set of all polynomials with complex coefficients that take rational values at all rational numbers and irrational values at all irrational numbers.

MS-2018, Nos. 3-4: Problem-10:

Let $a_1 < a_2 < \cdots$ be an infinite sequence of pairwise coprime positive integers which are all composites. Prove that $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$.

MS-2018, Nos. 3-4: Problem-11:

For any positive integer n and odd prime p, prove that the following congruence holds modulo p:

$$(n^p - n)/p \equiv -\sum_{r=1}^{p-1}(1^r + 2^r + \cdots + n^r)/r.$$

Here, two rational numbers s and t are said to be congruent modulo p if their difference $\frac{u}{v}$ written in the lowest terms satisfies $p|u$.

MS-2018, Nos. 3-4: Problem-12:
proposed by N. Tejaswi, The Netherlands, through B. Sury:

A spider moving along the integer lattice can move from any point (u, v) to either one of the two points $(u + v, v)$ or $(u, u + v)$. Show that no matter which point (u, v) it starts from for any integers u, v, the spider can reach a point of the form (m^2, n^2) in a finite number of steps.

Solution from the floor: MS-2017, Nos. 3-4: Problem 9: Pick and fix your favourite number α. Consider the $n \times n$ matrix with entries $\alpha + i; 1 \le i \le n^2$ written in a spiral fashion clockwise starting with $\alpha + 1$ in the $(1, 1)$-position. For example, the matrix for $n = 3$ is

$$\begin{pmatrix} \alpha+1 & \alpha+2 & \alpha+3 \\ \alpha+8 & \alpha+9 & \alpha+4 \\ \alpha+7 & \alpha+6 & \alpha+5 \end{pmatrix}.$$

Find the determinant of this matrix for general n.

(Solution submitted on 16-06-2018 by **Dasari Naga Vijay Krishna**, Machilipatnam, Andhra Pradesh-521001; *Vijay9290009015@gmail.com*).

Solution. Consider a spiral $n \times n$ matrix in α as under:

$$V_n(\alpha) = \begin{pmatrix} \alpha+1 & \alpha+2 & \rightarrow & \alpha+n-1 & \alpha+n \\ \alpha+4n-4 & \rightarrow & \alpha+5n-5 & \alpha+5n-6 & \alpha+n+1 \\ \alpha+4n-3 & \rightarrow & \rightarrow & \downarrow & \downarrow \\ \uparrow & \uparrow & \leftarrow & \leftarrow & \alpha+2n-2 \\ \alpha+3n-2 & \alpha+3n-3 & \leftarrow & \alpha+2n & \alpha+2n-1 \end{pmatrix}.$$

First consider $n = 3$ and $\alpha = 0$ so that we have the matrix $V_3(0)$. We find its determinant in two different ways and then we will generalize the process to find the determinant of $V_n(\alpha)$. Clearly

$$\det(V_3(0)) = \begin{vmatrix} 1 & 2 & 3 \\ 8 & 9 & 4 \\ 7 & 6 & 5 \end{vmatrix} = 8 \begin{vmatrix} 1 & 1 & 1 \\ 8 & 9 & 4 \\ 7 & 6 & 5 \end{vmatrix} \quad (R_1 \to R_1 + R_3)$$

$$= 8 \begin{vmatrix} 1 & 1 & 1 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{vmatrix} \quad (R_i \to R_i - R_1, i = 2, 3)$$

$$= \frac{8}{6} \begin{vmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{vmatrix} \quad (R_1 \to 6R_1 - R_3) \quad = \quad \det V_3(-1)$$

as can be verified. Thus $6\det(V_3(0)) = 8\ det(V_3(-1))$. Now, applying the same operations in the case of the general spiral matrix $V_n(0)$, we get

$$\det(V_n(0)) = \begin{vmatrix} 1 & \cdots & n-1 & n \\ 4n-4 & \cdots & 5n-6 & n+1 \\ \vdots & \cdots & \vdots & \vdots \\ 3n-2 & \cdots & 2n & 2n-1 \end{vmatrix}$$

$$= (3n-1) \begin{vmatrix} 1 & \cdots & 1 & 1 \\ 4n-4 & \cdots & 5n-6 & n+1 \\ \vdots & \cdots & \vdots & \vdots \\ 3n-2 & \cdots & 2n & 2n-1 \end{vmatrix} \quad (R_1 \to R_1 + R_n)$$

$$= (3n-1) \begin{vmatrix} 1 & \cdots & 1 & 1 \\ 4n-5 & \cdots & 5n-7 & n \\ \vdots & \cdots & \vdots & \vdots \\ 3n-3 & \cdots & 2n-1 & 2n-2 \end{vmatrix} (R_i \to R_i - R_1,\ i = 2 \cdots n)$$

$$= \frac{(3n-1)}{(3n-3)} \begin{vmatrix} 0 & \cdots & n-1 & 2n-1 \\ 4n-5 & \cdots & 5n-7 & n \\ \vdots & \cdots & \vdots & \vdots \\ 3n-3 & \cdots & 2n-1 & 2n-2 \end{vmatrix} (R_1 \to (3n-3)R_1 - R_n)$$

$$= \frac{(3n-1)}{(3n-3)} V_n(-1),$$

as can be verified, and hence $(3n-3)\det(V_n(0) = (3n-1)\det(V_n(-1))$. Since $\det(V_n(\alpha))$ is obviously a polynomial in α of degree 1, say $\det(V_n(\alpha)) = A_n\alpha + B_n$, so that $\det(V_n(0)) = B_n$ and $\det(V_n(-1)) = B_n - A_n$, it follows that $(3n-3)B_n = (3n-1)(B_n - A_n)$, that is,

$$A_n = 2/(3n-1)B_n. \tag{0.1}$$

Observe that, as seen above

$$B_3 = \det(V_3(0)) = 8\begin{vmatrix}1 & 1 & 1\\ 8 & 9 & 4\\ 7 & 6 & 5\end{vmatrix}, \text{ so that } 7B_3(0) = 8\begin{vmatrix}7 & 7 & 7\\ 8 & 9 & 4\\ 7 & 6 & 5\end{vmatrix};$$

but also $B_3 = \begin{vmatrix}1 & 2 & 3\\ 8 & 9 & 4\\ 7 & 6 & 5\end{vmatrix} = \begin{vmatrix}-7 & -7 & -1\\ 8 & 9 & 4\\ 7 & 6 & 5\end{vmatrix}$ $(R_1 \to R_1 - R_2)$.

Therefore, adding these we get

$15\ B_3 = 8\begin{vmatrix}0 & 0 & 6\\ 8 & 9 & 4\\ 7 & 6 & 5\end{vmatrix} = 8\times 6\begin{vmatrix}8 & 9\\ 7 & 6\end{vmatrix} = (8\times 6\times 3)\begin{vmatrix}1 & 2\\ 4 & 3\end{vmatrix} = (8\times 6\times 3)\ B_2$, that is, $B_3 = \frac{8\times 6}{5}B_2$. To obtain expression for B_n in terms of B_{n-1}, one can apply the same operations and procced ahead. Thus

$$B_n = \det(V_n(0)) = \begin{vmatrix}1 & \cdots & n-1 & n\\ 4n-4 & \cdots & 5n-6 & n+1\\ \vdots & \cdots & \vdots & \vdots\\ 3n-2 & \cdots & 2n & 2n-1\end{vmatrix}.$$

We first subtract the second row from the first, then add $(4n-5)(111\cdots 11)$ and finally we rebuild $JV_{n-1}(0)J$ in the lower left part of the matrix, where J is the anti-Identity matrix to get

$$\begin{aligned}B_n &= ((3n-1)/(3n-4))\begin{vmatrix}0 & \cdots & 0 & 4n-6\\ . & . & . & \vdots\\ . & JV_{n-1}(0)J & . & \vdots\\ . & . & . & \vdots\end{vmatrix}\\ &= (-1)^{n+1}((3n-1)/(3n-4))(4n-6)\det(V_{n-1}(0))\\ &= (-1)^{n+1}((3n-1)/(3n-4))(4n-6)B_{n-1}.\end{aligned}$$

This recursively gives

$$\begin{aligned}B_n &= (-1)^{\frac{n(n-1)}{2}}2^{n-2}(3n-1)(1.3.5\cdots(2n-3))\\ &= (-1)^{\frac{n(n-1)}{2}}\left(\frac{3n-1}{2}\right)\left(\frac{(2n-2)!}{(n-1)!}\right).\end{aligned}$$

Since $\det(V_n(\alpha)) = A_n\alpha + B_n$ and $A_n = (2/(3n-1))B_n$, therefore

$$\det(V_n(\alpha))=\left(\frac{2\alpha+3n-1}{3n-1}\right)B_n=\left(\frac{2\alpha+3n-1}{2}\right)(-1)^{\frac{n(n-1)}{2}}\left(\frac{(2n-2)!}{(n-1)!}\right). \tag{0.2}$$

This is the expression for the determinant of the $n\times n$ spiral matrix.

Verification: Taking $n = 2$ in the expression we get $V_2(\alpha) = \frac{2\alpha+5}{2}(-1)\frac{2!}{1!} = -(2\alpha+5)$. Also, from the definition, $V_2(\alpha) = \begin{vmatrix}\alpha+1 & \alpha+2\\ \alpha+4 & \alpha+3\end{vmatrix} = -(2\alpha+5)$.

Take now $n = 3$ in the above expression. We get $V_3(\alpha) = \frac{2\alpha+8}{2}(-1)\frac{4!}{2!} = -(12\alpha + 48)$. On the other hand by definition, $V_3(\alpha) = \begin{vmatrix} \alpha+1 & \alpha+2 & \alpha+3 \\ \alpha+4 & \alpha+8 & \alpha+9 \\ \alpha+7 & \alpha+6 & \alpha+5 \end{vmatrix} = (\alpha + 1)(4\alpha + 21) - (\alpha + 2)(2\alpha + 12) + (\alpha + 3)(-2\alpha - 15) = (-12\alpha - 48)$. The above expression is thus verified for $n = 2$ and $n = 3$.

Solution from the floor: MS-2017, Nos. 3-4: Problem 10: Show that there does not exist a square matrix over rational numbers whose characteristic polynomial is $X^2 - 3$.

(Solution submitted on 25-05-2018 by **Dasari Naga Vijay Krishna**, Machilipatnam, Andhra Pradesh-521001; *Vijay9290009015@gmail.com*).

Solution. We first prove some lemmas related to divisibility which are needed in solving the problem.

Lemma 0.1. *3 divides sum of two perfect squares iff 3 divides each of them.*

Proof. If $3|x$ and $3|y$ then $x = 3x_1$ and $y = 3x_2$ for some integers x_1, x_2. Then $x^2 + y^2 = 3(3x_1^2 + 3x_2^2)$ and hence $3|(x^2 + y^2)$. To prove the converse we use congruency. Suppose $3|(x^2 + y^2)$ so that $x^2 + y^2 \cong 0(mod\ 3)$. We know that $x, y \cong 0, 1, -1(mod\ 3)$ and $x^2, y^2 \cong 0, 1(mod\ 3)$. Hence $x^2 + y^2 \cong 0 + 0, 0 + 1, 1 + 0, 1 + 1(mod\ 3) = 0, 1, 2(mod\ 3)$. It follows that $x^2 + y^2 \cong 0(mod\ 3)$ only when $x, y \cong 0(mod\ 3)$ which implis that x, y are divisible by 3. ☐

Lemma 0.2. *The only integer solution for the equation $x^2 + y^2 = 3z^2$ is $(0, 0, 0)$.*

Proof. Obviously $3|3z^2$. By the above Lemma, we then have

$$\begin{aligned} 3|3z^2 &\Rightarrow 3|(x^2 + y^2) \Rightarrow 3|x, 3|y \Rightarrow x = 3x_0, y = 3y_0, \text{ for some } x_0, y_0 \in \mathbb{Z} \\ &\Rightarrow 3z^2 = x^2 + y^2 = 9x_0^2 + 9y_0^2 \Rightarrow z^2 = 3x_0^2 + 3y_0^2 \Rightarrow 3|z^2 \Rightarrow 3|z \\ &\Rightarrow z = 3z_0 \text{ for some } z_0 \in \mathbb{Z} \Rightarrow x_0^2 + y_0^2 = 3z_0^2. \end{aligned}$$

This process continues infinitely. Hence, by Fermat principle of infinite decent, the only integer solution for this equation is $(0, 0, 0)$. ☐

We now give solution to the problem. Consider any symmetric square matrix of order 2, say, $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. (we need to consider only 2×2 symmetric matrix because our characteristic polynomial x^2-3 is of second degree). The characteristic polynomial of A is $x^2 - tr(A)x + det(A) = x^2 - (a + c)x + (ac - b^2)$. Suppose A is the matrix over $\mathbb{Q}$ whose characteristic polynomial is $x^2 - 3$. Then comparing these two polynomials, we get $a + c = 0, ac - b^2 = -3$. This gives $c = -a$ and hence $a^2 + b^2 = 3$. If a, b are rational numbers with least common denominator z then $x = za$ and $y = zb$ are integers satisfying $x^2 + y^2 = 3z^2$. But by the second Lemma, we know that $(0, 0, 0)$ is the only solution. Hence $x = 0, y = 0$, and therefore $a = 0, b = 0$ - implying that characteristic polynomial of A is x^2, a contradiction as the given polynomial is $x^2 - 3$. This completes the solution.

Solution from the floor: MS-2018, Nos. 1-2: Problem 3: Consider a convex polygon of n sides. Draw $n-3$ diagonals which do not intersect inside. Then, the polygon is broken into $n-2$ triangles. Let a_i be the number of those triangles formed which have P_i as vertex. For instance, for $n=6$, one may draw diagonals P_1P_3, P_1P_4 and P_1P_5. in this case, $(a_1, a_2, a_3, a_4, a_5, a_6) = (4, 1, 2, 2, 2, 1)$. Notice that $4 - \frac{1}{1-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}} = 0$. Prove that in general

$$a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\cdots - \cfrac{1}{a_{n-1}}}}} = 0. \tag{0.3}$$

(Solution submitted on 16-06-2018 by **Dasari Naga Vijay Krishna**, Machilipatnam, Andhra Pradesh-521001; *Vijay9290009015@gmail.com*).

Solution. For simplicity, let us write the left side of (0.3) as $\langle a_1, a_2, \cdots, a_{n-1}\rangle$, which we prove to be equal to 0 by induction on n. More precisely, for $n \geq 3$, we will show that $\langle a_k, a_{k+1}, a_{k+2}, \cdots, a_{n-1}\rangle$ is positive for $1 < k < n$ and $\langle a_1, a_2, a_3, \cdots, a_{n-1}\rangle = 0$. For induction base $n = 3$, clearly $(a_1, a_2) = (1, 1)$ and hence $\langle a_1, a_2\rangle = 1 - \frac{1}{1} = 0$. Thus (0.3) is true for $n = 3$. Now assume $n > 3$ and that the statement holds for $n - 1$.

Write E for the collection of edges of the n-gon and Δ for the collection of triangles in the triangulation. Each $x \in E$ belongs to exactly one $\Delta \in T$ and this induces a mapping $\phi : E \to T$. We have $|E| = n, |T| = n - 2$ and $|\phi^{-1}(\Delta)| \leq 2$ for all $\Delta \in T$, hence there exist 4 distinct edges $x, y, z, t \in E$ with $\phi(x) = \phi(y) = \Delta_1$ and $\phi(z) = \phi(t) = \Delta_2$. Since x, y and z, t have a vertex in common we can write $\Delta_1 = P_iP_{i+1}P_{i+2}, \Delta_2 = P_jP_{j+1}P_{j+2}$, where $1 \leq i, j \leq n$ are distinct. All bases are to be taken modulo n. we may assume $i \neq n-1$. Then we can verify that $\Delta = \Delta_1$ as a cap - meaning that we obtain a triangulation of an $(n-1)$-gon if we delete Δ from T. Also $i \neq n-1$ guarantees that P_n is not the middle vertex of this cap. We consider three cases.

Case 1. $n \notin \{i, i+1, i+2\}$. Then we have $(a_1, a_2, \cdots, a_{n-1}) = \{\sigma, \alpha, 1, \beta, \tau\}$, where σ stands for $a_1, a_2, \cdots, a_{i-1}$, τ stands for $a_{i+3}, a_{i+4}, \cdots, a_{n-1}$, (both possibly empty) and $\alpha, \beta > 1$. If we delete Δ from T, we obtain the sequence $(\sigma, \alpha - 1, \beta - 1, \tau)$ and to this form we may apply the induction hypothesis. All forms corresponding to suffixes $(\beta - 1, \tau)$ are positive by induction hypothesis, hence the same holds for the suffixes of the sequence (β, τ). Write $x = \langle \beta, \tau\rangle$, then $x > 1$, $\langle 1, \beta, \tau\rangle = 1 - 1/x > 0$ and $\langle \alpha, 1, \beta, \tau\rangle = \langle \sigma, 1, x\rangle = \alpha - \frac{1}{1-\frac{1}{x}} = \alpha - \frac{x}{x-1} = \alpha - 1 - \frac{1}{x-1} = \langle \alpha - 1, x - 1\rangle = \langle \alpha - 1, \beta - 1, \tau\rangle$.

If σ is non- empty, this value is positive by the induction hypothesis. For each suffix ϕ of σ we have $\langle \phi, \alpha, 1, \beta, \tau\rangle = \langle \phi, \alpha - 1, \beta - 1, \tau\rangle$ and the induction hypothesis for n follows. The other cases are easier.

case-2. $i = n$. Then we have $(a_1, a_2, \cdots, a_{n-1}) = (1, \alpha, \sigma,)$ and deleting Δ we obtain $(\alpha - 1, \sigma)$. By the induction hypothesis we have $\alpha - 1 = 1/\sigma$, hence $\langle \alpha, \sigma \rangle = 1$ and $\langle 1, \alpha, \sigma \rangle = \langle 1, 1 \rangle = 0$. The induction hypothesis for n follows.

Case-3. $i = n - 2$. We then have $(a_1, a_2, \cdots, a_{n-1}) = (\sigma, \alpha, 1)$ and deleting Δ we obtain $(\sigma, \alpha - 1)$. The induction hypothesis for n is obvious in this case, and we are done.

Solution by the Proposer S. K. Tomar: MS-2017, Nos. 3-4: Problem 6: Using the techniques of calculus of variation, find the minimum distance between the yolk and ellipsoidal shell of an egg. You may take the yolk as a sphere and the shell as ellipsoidal both centered at origin, that is, $x^2 + y^2 + z^2 = 4$ and $\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$.

Solution. The problem is to find the minimum distance between the yolk and the ellipsoidal shell of an egg. The yolk of an egg is of spherical shape having certain radius, and the shell of the egg is of ellipsoidal shape, so let us take their equations as

$$x^2 + y^2 + z^2 = 4, \quad \text{and} \tag{0.4}$$

$$(x^2/25) + (y^2/16) + (z^2/9) = 1. \tag{0.5}$$

Let A (x_0, y_0, z_0) be any point on the surface of the yolk and B (x_1, y_1, z_1) be any point of the surface of the shell. We can draw number of curves joining these points A and B. The functional is of the form

$$I[y, z] = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx, \tag{0.6}$$

where the end points x_0 and x_1 are the movable on the surface of (0.4) and (0.5) respectively. The problem is to find an extremal of (0.6) having movable end points. The geometrical sketch of the problem is given in Figure-1. In the moving end points problems, if one end point of the extremal moves along the surface $z = \phi(x, y)$ and the other end point moves along the surface $z = \psi(x, y)$, then the

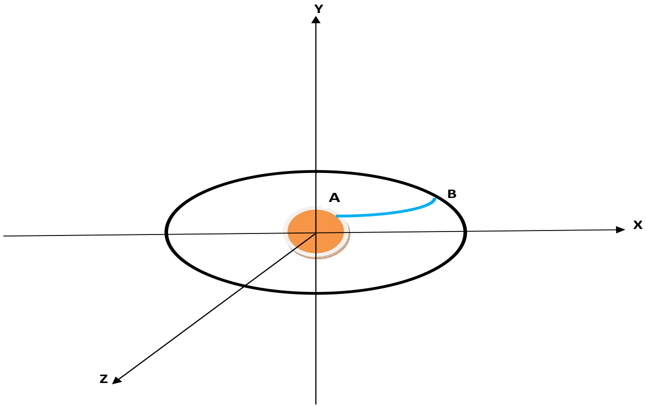

FIGURE 1. Rough Sketch

arbitrary constants occurring in the solution of Euler's equation are determined by the following transversality conditions:

$$\begin{aligned} &[F - y'F_{y'} + (\phi'_x - z')F_{z'}]_{x=x_0} = 0, \qquad [F_{y'} + \phi'_y F_{z'}]_{x=x_0} = 0, \\ &[F - y'F_{y'} + (\psi'_x - z')F_{z'}]_{x=x_1} = 0, \qquad [F_{y'} + \psi'_y F_{z'}]_{x=x_1} = 0. \end{aligned} \tag{0.7}$$

In the present problem

$F= \sqrt{1 + y'^2 + z'^2}$, $\phi(x, y)= \sqrt{4 - x^2 - y^2}, \psi(x, y) = 3\sqrt{1 - (x^2/25) - (y^2/16)}.$

The extremal of the given functional (0.6) are the solution of Euler's equations:

$$F_y - \frac{d}{dx}F'_y = 0, \qquad F_z - \frac{d}{dx}F'_z = 0, \tag{0.8}$$

given by

$$y = A_1x + B_1, \qquad z = A_2x + B_2, \tag{0.9}$$

where A_1, B_1, A_2 and B_2 are arbitrary constants. Thus the extremal is a line determined by (0.9). Now, it can be seen that the form of the functional (0.6) is such that the transversality conditions reduce to orthogonality conditions (see Example-1 on page 76 of Elsgolc, L. E., Calculus of Variations). Thus the line represented by (0.10) must be orthogonal to the sphere as well as to the ellipsoid. Any line orthogonal to the sphere must be diameter of the sphere. Hence, let us take the straight line represented by (0.9) as

$$(x - 0)/a = (y - 0)/b = (z - 0)/c, \tag{0.10}$$

where $< a, b, c >$ are the direction ratios of the line. If this line is extremal then it must pass through the point (x_0, y_0, z_0) as well as through the point (x_1, y_1, z_1). Thus we have

$$x_0/a = y_0/b = z_0/c, \quad \text{and} \quad x_1/a = y_1/b = z_1/c. \tag{0.11}$$

This line is already orthogonal to the sphere as it is passing through the center of the sphere. At (x_1, y_1, z_1), the line must be orthogonal to the ellipsoid. The equation of tangent plane to the ellipsoid at the point (x_1, y_1, z_1) is given by

$$xx_1/25 + yy_1/16 + zz_1/9 = 1. \tag{0.12}$$

The direction ratios of the normal to this plane must match with those of the extremal (0.10). Therefore we have

$$x_1/25a = y_1/16b = z_1/9c. \tag{0.13}$$

This is possible in the following three cases:

(i) Case - I: $a = b = 0$ and $c \neq 0$, (ii) Case-II: $b = c = 0$, $a \neq 0$ and (iii) Case - III: $c = a = 0$, and $b \neq 0$.

In (i), the extremal is $z-$axis and the point A (x_0, y_0, z_0) is $(0, 0, \pm 2)$ and point B (x_1, y_1, z_1) is $(0, 0, \pm 3)$.

In (ii), the extremal is $x-$axis and the point A (x_0, y_0, z_0) is $(\pm 2, 0, 0)$ and point B (x_1, y_1, z_1) is $(\pm 5, 0, 0)$.

In (iii), the extremal is $y-$axis and the point A (x_0, y_0, z_0) is $(0, \pm 2, 0)$ and point B (x_1, y_1, z_1) is $(0, \pm 4, 0)$.

The distance between the sphere and the ellipsoid is minimum when extremal is $z-$axis and minimum distance is 1, while the distance is maximum when the extremal is $x-$axis and it is equal to 3.

Solution by the Proposer B. Sury: MS-2017, Nos. 3-4: Problem 8: For every positive integer $n > 3$, prove that there is a prime factor of $2^n + 1$ which does not divide $2^m + 1$ for any $m < n$.

Solution. Now, $x^n - 1 = \prod_{k=1}^{n}(x - e^{2i\pi k/n}) = \prod_{d|n} \Phi_d(x)$, where the d-th cyclotomic polynomial $\Phi_d(x) = \prod_{(r,d)=1}(x - e^{2i\pi r/d})$ is an irreducible integer polynomial. For any integer a and for any natural number n, one has

$$a^n - 1 = \prod_{d|n} \Phi_d(a)$$

which is a product of integers. Thus, if p is any prime dividing $a^n - 1$ for some a then p divides $\Phi_d(a)$ for some $d|n$.

Now, we make the following interesting assertion:

Let $n > 2$. If p is a prime dividing $\Phi_n(a)$ for some integer $a > 1$ then p divides $a^n - 1$, and in this case n is the smallest natural number such that p divides $a^n - 1$ unless $p|n$ in which case the smallest number is of the form n/p^i for some $i \geq 1$. In the latter case, p is the largest prime dividing n. Finally, given $a > 1$ and $n > 2$, if there are no primes p for which a has order n mod p, then $\Phi_n(a)$ is a prime.

We apply the above result to prove what the problem asserts.

For each $n > 3$, if we find a prime p such that the order of 2 mod p is $2n$, then from $(2^{2n} - 1) = (2^n - 1)(2^n + 1)$, we would have $p|(2^n + 1)$ because p does not divide $2^n - 1$. Also, if p divided $2^m + 1$ for some $m < n$, then it would divide $2^{2m} - 1$ which would contradict the fact that the order of 2 mod p is $2n$.

In order to get a prime p such that 2 has order $2n$ mod p, we need to get a prime p dividing $\Phi_{2n}(2)$ and not dividing $2n$. If there is no such prime, then as we saw above, we must have that $\Phi_{2n}(2) = p$ is the largest prime dividing n, and that p is odd. Write $2n = p^i d$ with d dividing $p - 1$. Now

$$|\Phi_{2n}(2)| = \frac{|\Phi_d(2^{p^i})|}{|\Phi_d(2^{p^{i-1}})|} = \frac{\prod_{r=1}^{\phi(d)} |b^p - \zeta_r|}{\prod_{r=1}^{\phi(d)} |b - \zeta_r|} > \left(\frac{b^p - 1}{b + 1}\right)^{\phi(d)},$$

where $b = 2^{p^{i-1}}$ and ζ_r are the $\phi(d)$ primitive d-th roots of unity (the roots of $\Phi_d(x)$). As $b^p - 1 \geq b^{p-2}(b^2 - 1)$, the right side above is $> b^{(p-2)\phi(d)}(b-1)^{\phi(d)}$. As $b \geq 2$, this last expression is at least 2^{p-2}. Therefore, we have

$$p = \Phi_{2n}(2) > 2^{p-2}$$

which is possible only if $p = 3$. In that case we must also have $2n = 6$ which we rule out. In other words, when $n > 3$, then there does exist a prime divisor p of $\Phi_{2n}(2)$ which does not divide n; the above discussion then shows that n is the smallest natural number for which p divides $2^n + 1$. This complets the solution.

Let us now prove the above assertion that we used.

Note that p divides $a^n - 1$ since $\Phi_n(a)$ divides $a^n - 1$. Also, if p divides $a^m - 1$ for some $m < n$ as well, then p divides $a^{(m,n)} - 1$ where (m, n) is the GCD of m and n. Therefore, if n were not the smallest for which p divides $a^n - 1$, we would have a factor d of n such that $d < n$ and $p|(a^d - 1)$. As $d < n$ and $d|n$, there is some prime q such that $qd|n$. Thus, d divides n/q and so p divides $a^{n/q} - 1$. Writing $b = a^{n/q}$, we see that b is congruent to 1 modulo p. So, we have

$$\frac{a^n - 1}{a^{n/q} - 1} = \frac{b^q - 1}{b - 1} = 1 + b + b^2 + \cdots + b^{q-1} \equiv q$$

modulo p. On the other hand, the left hand side is a multiple of $\Phi_n(a)$ which is a multiple of p. Thus, we must have that $p = q$ and that it divides n.

This also shows that $a^{n/q} - 1$ is not a multiple of p for any prime divisor $q \neq p$ of n. Thus, the order of a mod p is either n or of the form n/p^i for some $i \geq 1$.

When the order of a mod p is $< n$, we have seen that it is of the form n/p^i. Thus, n/p^i divides $p - 1$ by Fermat's little theorem, which means every other prime divisor of n is $< p$. This proves all the assertions excepting the last statement.

To see that the last statement also holds true, consider $n > 2$, $a > 1$ and a prime divisor p of $\Phi_n(a)$. Under the hypothesis that there are no primes modulo which a has order n, we have seen that p is the largest prime dividing n and that $\Phi_n(a) = p^k$ for some $k \geq 1$. We assert that $k = 1$. Observe that $\Phi_n(a)$ divides

$$\frac{a^n - 1}{a^{n/p} - 1} = 1 + a^{n/p} + a^{2n/p} + \cdots + a^{(p-1)n/p}.$$

As $a^{n/p} = 1 + pb$ for some b, the right hand side above is

$$1 + (1 + pb) + \cdots + (1 + pb)^{(p-1)} = p + p(b + 2b + \cdots + (p - 1)b) + p^2c = p + p^2d$$

for some c, d if $p > 2$. Therefore, p^2 does not divide $\Phi_n(a)$; hence $\Phi_n(a) = p$.

When $p = 2$, the argument is again easy remembering that $n > 2$ is a power of 2 as p is the largest prime divisor of n. This proves the assertion completely.

Solution by the Proposer B. Sury: MS-2017, Nos. 3-4: Problem 11: Consider a group G generated by $\{x_1, x_2, x_3, \cdots\}$ and relations $x_2x_1x_2^{-1} = x_3x_2x_3^{-1} = x_4x_3x_4^{-1} = \cdots\cdots$ Prove that G is not finitely generated.

Note. *The source of this problem was the American Manthematical Monthly, but we are unable to pinpoint the concerned volume for problem or solution.*

Solution. The idea is to show that if G were finitely generated, it would be free with both $\{x_1, x_2\}$ and $\{x_2, x_3\}$ as bases. But then the surjective homomorphism $\pi : G \to \mathbb{Z}$ given by $\pi(x_2) = 0, \pi(x_3) = 1$ satisfies $\pi(x_1) = 0$ as $x_1 = x_2^{-1}(x_3x_2x_3^{-1})x_2$. As $G =< x_1, x_2 >$, this is a contradiction of surjectivity of π. We give the proof (of finite generation implying freeness) via three simpler observations.

Observation 1. *For each r, x_r and x_{r+1} do not commute in G.*

Indeed, the map $x_r \mapsto (1, 2), x_{r+1} \mapsto (1, 3)$ gives an onto homomorphism onto S_3 (the defining relations of G hold in S_3 clearly) and $(1, 2), (1, 3)$ do not commute.

Observation 2. *For each n, G has generators x_r and relations $x_{r+1}x_rx_{r+1}^{-1} = x_{r+2}x_{r+1}x_{r+2}^{-1}$ for $r \geq n$.*

Indeed, first note that $x_1 = x_2^{-1}(x_3x_2x_3^{-1})x_2$ and the other relations do not involve x_1 which means that G is generated by x_r and the relations $x_{r+1}x_rx_{r+1}^{-1} = x_{r+2}x_{r+1}x_{r+2}^{-1}$ for $r \geq 2$. Proceeding in this manner inductively, the assertion follows for any n, and it follows that $\theta_n : x_r \mapsto x_{r+n}$ for all r gives an automorphism of G.

Observation 3. *If G is finitely generated, then it is free of rank 2 with basis $\{x_1, x_2\}$ as well as the basis $\{x_2, x_3\}$.*

Suppose G is generated by x_r for $r \leq n$. As done above, we may write x_r in terms of x_{r+1}, x_{r+2} for $r \leq n-2$ and deduce that G is generated by x_{n-1}, x_n. As θ_{n-2} is an automorphism, it follows that G is generated by $\theta_{n-2}^{-1}(x_{n-1}) = x_1, \theta_{n-2}^{-1}(x_n) = x_2$ as well as by $\phi_1(x_1) = x_2, \phi_2(x_2) = x_3$.

Now, consider the abstract group F generated by $x_1, \cdots, x_n$ and with defining relations $x_2x_1x_2^{-1} = x_3x_2x_3^{-1} = \cdots = x_{n-1}x_{n-2}x_{n-1}^{-1} = x_nx_{n-1}x_n^{-1}$.

The above argument shows that F is generated by x_{n-1}, x_n and has no relations; so, F is freely generated by x_{n-1}, x_n. As our group G is generated by x_1, x_2 which do not commute, G is a free subgroup of rank 2 on the basis $\{x_1, x_2\}$. As ϕ_1 is an automorphism taking this basis to the set $\{x_2, x_3\}$, G is free also on $\{x_2, x_3\}$. Thus, we obtain a contradiction.

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