ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 86, Numbers 3-4, July-December (2017)
(Issued: November, 2017)

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
www.indianmathsociety.org.in

THE MATHEMATICS STUDENT

Edited by J. R. PATADIA

In keeping with the current periodical policy, THE MATHEMATICS STUDENT will seek to publish material of interest not just to mathematicians with specialized interest but to the postgraduate students and teachers of mathematics in India. With this in view, it will ordinarily publish material of the following type:

1. the texts (written in a way accessible to students) of the Presidential Addresses, the Plenary talks and the Award Lectures delivered at the Annual Conferences.
2. general survey articles, popular articles, expository papers, Book-Reviews.
3. problems and solutions of the problems,
4. new, clever proofs of theorems that graduate / undergraduate students might see in their course work,
5. research papers of interest to larger readership, and
6. articles that arouse curiosity and interest for learning mathematics among readers and motivate them for doing mathematics.

Articles of the above type are invited for publication in THE MATHEMATICS STUDENT. Manuscripts intended for publication should be submitted online in the LaTeX and .pdf file including figures and tables to the Editor J. R. Patadia on E-mail: *msindianmathsociety@gmail.com*

Manuscripts (including bibliographies, tables, *etc.*) should be typed double spaced on A4 size paper with 1 inch (2.5 cm.) margins on all sides with font size 10 pt. in LaTeX. Sections should appear in the following order: Title Page, Abstract, Text, Notes and References. Comments or replies to previously published articles should also follow this format with the exception of abstracts. In LaTeX the following preamble be used as is required by the Press:

\ documentclass[10 pt,a4paper,twoside,reqno]{amsart}
\ usepackage {amsfonts, amssymb, amscd, amsmath, enumerate, verbatim, calc}
\ renewcommand{\ baselinestretch}{1.2}
\ textwidth=12.5 cm
\ textheight=20 cm
\ topmargin=0.5 cm
\ oddsidemargin=1 cm
\ evensidemargin=1 cm
\ pagestyle{plain}

The details are available on Indian Mathematical Society website: *www.indianmath society.org.in*

Authors of articles / research papers printed in the the Mathematics Student as well as in the Journal shall be entitled to receive a *soft copy (PDF file with watermarked "Author's copy")* of the paper published. There are no page charges. However, if author(s) (whose paper is accepted for publication in any of the IMS periodicals) *is (are) unable to send the LaTeX file of the accepted paper, then a charge Rs. 100 (US $ 10) per page will be levied for LaTeX typesetting charges.*

All business correspondence should be addressed to S. K. Nimbhorkar, Treasurer, Indian Mathematical Society, Dept. of Mathematics, Dr. B. A. M. University, Aurangabad - 431 004 (Maharashtra), India. E-mail: *sknimbhorkar@gmail.com*

In case of any query, the Editor may be contacted.

ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 86, Numbers 3-4, July-December, (2017)
(Issued: November, 2017)

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
www.indianmathsociety.org.in

ISSN: 0025-5742

Published by Prof. N. K. Thakare for the Indian Mathematical Society, type set by J. R. Patadia at 5, Arjun Park, Near Patel Colony, Behind Dinesh Mill, Shivanand Marg, Vadodara - 390 007 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune 411 041 (India). Printed in India.

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December, (2017)

CONTENTS

****

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 01-06

PROFESSOR H. P. DIKSHIT - AN INSTITUTIONAL BUILDER (27.12.1941-29.04.2017)

SATYA DEO

Prof. Hanuman Prasad Dikshit (Popularly known as H. P. Dikshit) breathed his last on April 29, 2017 after a brief illness of only a fortnight. It was a great shock to anyone who heard this news, especially those who were close to him during any point of his life. He was the Vice Chancellor of many universities like the R. D. University, Jabalpur, Himachal Pradesh University, Shimla, M. P. Bhoj Open University at Bhopal, Indira Gandhi National Open University, New Delhi. His passing away is a great loss to his family-wife and a daughter, to the institutions which he built, to the educational system of our country and to various friends and colleagues. We all will always miss him as a man of action, remarkable promptness, administrative alertness and the visionary decisiveness.

A humble Start at the University of Allahabad

Dr. Dikshit started his career as a lecturer in mathematics at the University of Allahabad in the year 1964 after completing his D. Phil. under Dr. T. Pati from the same university. His Ph. D. work was on summability theory and Fourier analysis, which was the area of Dr. Pati's research interest. At that time this area of research, led by Prof B. N. Prasad, was the most favored subject in the whole of India. With the arrival of Prof. R. S. Mishra at Allahabad University, new courses like set theory, topology, functional analysis, group theory etc. had all been started in the department, but there were no faculty members in these areas. Dr. Dikshit gave a course on group theory and also on topology. He followed the book of A. G. Kurosh on Group theory, studied the subject himself and tried hard to do justice. One can imagine the difficulties which he may have faced in teaching these courses. Misconceptions in the new subjects were almost common and were correctly understood only after some time. That was the case with all other new courses also. That was the time when several other teachers were also teaching courses like set theory, functional analysis etc. following some books only, and having no previous exposures. They were all self-taught. Even then some of these teachers were so brilliant that nobody could see or feel this shortcoming. However, it was clear to Dr. Dikshit that the subjects were really interesting and fundamental. This attitude of Dr. Dikshit about good mathematics remained intact with him throughout his life. While he was quite active in his area of research, he always respected and frankly admired all those who understood the new subjects and contributed in those areas.

Firmly established at Jabalpur University

Prof Pati, his supervisor, moved to the Jabalpur University in 1964, and brought Dr. Dikshit to Jabalpur as a Reader in mathematics in 1969. The time came when Prof Pati moved back to the University of Allahabad as Professor and Head, and as a result Dr. Dikshit became Reader and Head of the Department of Mathematics at Jabalpur University. He deserved Professorship, but there was no post of Professor in the Department. So he had to move to Gwalior. After a short stay at Jiwaji University, Gwalior as Professor, he returned back to Jabalpur University as Professor and Head of the Department and tried to build that department as a good center for mathematics teaching and research. It is here that he understood the importance of administration in a university. While he performed his administrative duties, he always kept on building the department of mathematics in terms new courses and the infrastructure. There was a natural instinct in him to do something everlasting for the department. He took up new research projects in modern areas of mathematics and completed them nicely. His expertise in writing research proposals to various funding agencies was amazing- he could write a nicely drafted perfect proposal within a matter of only few minutes!

He worked very sincerely for the development of the department and became well-known on the campus at Jabalpur for his dedication and commitment to the subject. He used to run seminars for his research scholars and trained them in the new area called 'splines' in approximation theory which he learnt from his association with Prof. Ambikeshwar Sharma of Edmonton, Canada. This was easy for him since he had a sound training in the convergence of sequences and series. These splines are simply piecewise polynomials on real intervals which play very important role in approximating the solutions of differential equations in applicable mathematics. He then naturally went toward the computer aide geometric designs (CAGD), a field which is directly applicable in the design of cars, aeroplanes etc. Many students joined research under his supervision and a compact group of researchers in computational mathematics was created at Jabalpur. Punctuality and hard work were implanted amongst the research scholars of the department. Prof. Pati was a model for everyone in the department and there hung a nice photograph of Prof. Pati in his office. That has always been there since then onwards. He published a number of quality research papers on splines and CAGD along with his students. By now the name of Jabalpur University had been changed to Rani Durgawati Vishwavidyalay, Jabalpur (RDVV, Jabalpur).

Vice Chancellor at R. D. University, Jabalpur

All of the above qualities of Prof Dikshit attracted the attention of an old Chief Minister of M. P., Sri D. P. Mishra living in the city, who supported him to be the Vice Chancellor of Jabalpur University at the young age of only 42 ignoring the seniority claims of other professors of the university. Prof. Dikshit was smart enough to run the university nicely by developing contacts in Education Department of the state government at Bhopal and UGC, New Delhi. Very soon he was in command and full control of the university because of his sound understanding of the people around the university and the educational system of our country. He started many new departments at R. D. University, Jabalpur, brought outstanding faculty to various departments and turned the university in one of the best universities in the state of M. P. I was one of those among many others who were offered Professorship at Jabalpur University with several increments as an attraction to come to Jabalpur. I accepted to join the university when he was the vice chancellor. I had never met him personally, but he knew me through my publications in good journals and my stay at Calgary, Canada. This was the case with many new faculty members who were brought in other departments. He appointed some of his Ph. D. students as faculty in the department of mathematics and expanded the activities in the department. He was totally committed to the development of the university using all the skills that he had acquired by that time.

It is at Jabalpur that Prof. Dikshit made impressive contacts with leading mathematicians of the country. He became a member of the National Board of Higher Mathematics, member of several state level committees and the UGC committees, having an impact on the entire higher education department of our country. There was always a visible impact on Prof. Dikshit of his teacher Prof. T. Pati who was a good mathematician, a great orator, a sound intellectual and a hard task master. Very often Prof. Dikshit used to confide to me that whatever he has achieved in his career is mainly because of the training and inspiration obtained from Prof. T. Pati. I can definitely testify to this fact since I also had the privilege of working with Prof. Pati at Allahabad University. Prof. Dikshit had an excellent quality of dealing with people and taking decisions. He could impress the bureaucrats through his personality, his contacts and his remarkable positive attitude. Having a wonderful memory he would win them over using his contacts of the University of Allahabad and his own ingenious methods. He knew exactly what the other person wanted. He obliged many people in exchange for bringing good programmes in the university. He had a hobby of building institutions and spotting talents to assign various jobs. As a result, he would help the honest and meritorious scholars by all means at his disposal and will not allow a mediocre to have his way except only in a few rare exceptions.

Vice Chancellor at H. P. University, Shimla and M. P. Bhoj University, Bhopal

When he completed his term as Vice Chancellor at the Jabalpur University, he came back to the Department of Mathematics for a while, but was soon appointed Vice Chancellor of Himachal University, Shimla. He worked there only briefly and then came back to the Jabalpur University as professor of Mathematics. I could see that he was not very comfortable in the department and was always looking for a better opportunity somewhere else. This was a stage in his lifetime when 'going up in the ladder' had become his mission. Mathematics was an ambition for him, but the goal was to build an institution. This is exactly where he differed from his mentor Prof. T. Pati for whom attaining intellectual heights was the goal, not the administration. Then, suddenly an offer came from the State Government when he was appointed vice chancellor of an Open University called, M. P. Bhoj Open University at Bhopal. That university was started by Prof. Dikshit from scratch and built into a good open university by starting several important and relevant courses including the department of computer science. Using his brilliant administrative abilities, he became an important vice chancellor amongst all vice chancellors of other regular and better known M. P. Universities. The state Education Minister and the Chancellor of the M. P. university had very high opinion about him as an educationist. I myself attended several meetings as a Vice Chancellor of APS University, Rewa and a member of the coordination committee of all M. P. vice chancellors along with him. He was always tactful and very experienced to offer pragmatic suggestions without hurting anyone.

Pro Vice Chancellor of IGNOU, New Delhi

Being the vice chancellor of M. P. Bhoj Open University, he came in contact with the H. R. D. Ministry officials in New Delhi and was picked up as one of the three Pro Vice-Chancellors of the Indira Gandhi National Open University (IGNOU), New Delhi. The IGNOU was indeed a huge university of the central government having impact all over India and even abroad. Every state of India had an open university and each one of them was supervised and controlled by the IGNOU. This offered an excellent opportunity to Prof. Dikshit to understand the system of open universities and work for them. It is through his hard work that he came in contact with the eminent educationists of the whole country.

Vice Chancellor of Indira Gandhi National Open University, New Delhi

When the post of Vice Chancellor of IGNOU fell vacant, Prof. Dikshit was a natural choice of the HRD ministry and he was appointed VC of IGNOU. This appointment brought Prof. Dikshit at the highest level of academic engagement and he enjoyed his work immensely. By virtue of his position as VC of IGNOU he became members of the important national committees like MPCOST, AICTE, USEFI and a number of other academic organizations. He was the chairman of the Distance Education Cell of the MHRD. It is worth mentioning that while he performed his duties of vice chancellor at many universities, he always kept in touch with his subject Mathematics, and kept on organizing national and international conferences in Delhi and elsewhere. He invited eminent people working in various foreign universities, even Fields Medalists, to visit India and start collaboration in mathematics. It is here that he was awarded many honoris causa degrees from various open universities and became member of Board of Governors of important institutions like I. I. T.s and I.I.Sc. etc. He was a member of selection committees of several vice chancellors. In fact, at this point of time, he became a 'Committee Man' in the words of Prof. K. R. Parthasarthy of I. S. I., New Delhi. This word was once used by Prof. K. R. Parthasarathy for me when he suddenly saw me in the corridors of I. S. I., New Delhi, but it is really more suitable for Prof. Dikshit and other stalwarts like him! No one could say that Prof. Dikshit was backed by any party or any man in power - there was no stamp on his personality. He was picked up for the highest administrative positions by governements of any party in power. He remained careful about his independent identity as an academician which is normally a difficult situation. He was, perhaps, one of the best known mathematician vice chancellors from India as compared to anyone else.

Director General of Good Governance and Policy Planning of M. P. Government

After a successful tenure as the Vice Chancellor of IGNOU, Prof. Dikshit retired from IGNOU as Vice Chancellor, and came back to the state of M. P. The habit of working hard with some mission did not allow him to relax and enjoy life as many people do after retirement. He accepted an offer from the state Government of M. P. to be the Director General of Good Governance and Policy Planning. No one could imagine that he will accept such an appointment and will be of concrete help to the M. P. government. But he did it and worked with distinction helping the government in various ways. He found it very easy as compared to any other thing that he had done earlier. Even at this time he continued to remain in touch with mathematics whatever he had been doing. He had earned most of the usual awards that a professor of university aspires. He was elected Fellow of the National Academy of Sciences, India, the President of the Indian Mathematical Society, Sectional President of the Indian Science Congress in Mathematics including Statistics and so on. After attaining the age of 70, he did not accept any regular assignment of any kind, but kept on serving the important national and state level committees for important works. He also continued to give mathematical talks in various conferences, seminars, workshops. Whenever he was invited to inaugurate an event, he would prepare his talk meticulously and will justify his presence as the chief guest. I met him last on March 22, 2017 at his home in Jabalpur when I was visiting Jabalpur University for a few lectures. When I was departing from his house, he came outside the gate of his house and said, “Doctor Saheb, take care of your health, that is more important than anything else!”. He was aware that I am having some health problems myself and so this is what he used to say to me every time whenever I was leaving him or he was leaving me. Strange are the ways of Almighty which no one understands!

Satya Deo
Harish-Chnadra Research Institute
Chhatnag Road, Jhusi, Allahabad-211 019, (UP), India
E-mail: *sdeo@hri.res.in*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 07-19

SOME PROBLEMS ON POLYNOMIAL RINGS*

NEENA GUPTA

Abstract. In this survey article we describe some of the fundamental problems on polynomial rings, highlighting recent developments and the interrelationships among them.

1. Introduction

Let k be a field. The polynomial ring $k[X_1, \ldots, X_n]$ is one of the oldest rings that mathematicians have attempted to investigate. Polynomial rings have been a central theme of research in algebra and algebraic geometry for more than a century, at least from the time of David Hilbert. The famous "Hilbert Basis Theorem" on polynomial rings over k led to a breakthrough by Hilbert in the "theory of invariants" in the early 1890's. The "Nullstellensatz" (Zero Theorem), a celebrated result of Hilbert on polynomial rings, built a bridge between algebra and geometry. Consequently, one can make an algebraic study of the geometric space $\mathbb{C}^n$ utilising the rich theory of commutative rings, just as coordinate geometry in high school makes use of the algebra of quadratic equations to study problems in the geometry of conics.

Although polynomial rings are of fundamental importance in algebra and algebraic geometry, they are still far from being understood. There are several interesting problems in the area, which are easy to state but difficult to tackle. The most celebrated problems on polynomial rings include the Jacobian Problem (first raised by Ott-Heinrich Keller in 1939), the Zariski Cancellation Problem, the Epimorphism or Embedding Problem of Abhyankar-Sathaye, the Affine Fibration Problem of Dolgačev-Veĭsfeĭler, the Linearization Problem of Kambayashi, the problem of Characterisation of Polynomial Rings by a few chosen properties, and the problem of determining the automorphism groups of polynomial rings. Some of these problems involve examining whether a certain given ring is a polynomial ring over some base field or to determine whether a certain polynomial $F \in k[X_1, \ldots, X_n]$ is a coordinate in the ring $k[X_1, \ldots, X_n]$, i.e., whether there

---

* The article is based on the text of the 27th Hansraj Gupta Memorial Award Lecture delivered at the 82^{nd} Annual Conference of the Indian Mathematical Society held at the University of Kalyani, Kalyani-741 235, Nadia, West Bengal, India during December 27 - 30, 2016.

2010 Mathematics Subject Classification : Primary: 14R10, 14R25. Secondary: 13F20.
Key words and phrases: Polynomial Ring, Jacobian Conjecture, Cancellation Problem, Affine Fibration, Linearization Problem, Epimorphism Problem.

exist polynomials $F_2, \ldots, F_n$ such that the polynomial ring $k[X_1, \ldots, X_n]$ gets generated by the n polynomials $F, F_2, \ldots, F_n$.

The articles of H. Kraft [29], G. Freudenburg and P. Russell [40] and M. Miyanishi [36] give a survey on problems in Affine Algebraic Geometry; the monograph of A. van den Essen [16] gives an account of Polynomial Automorphisms and the Jacobian Conjecture; the Epimorphism Problem and its offshoots are discussed in the articles by P. Russell and A. Sathaye [43], and by A.K. Dutta and N. Gupta [15]; the problems on Affine Fibrations are discussed by S.M. Bhatwadekar and A.K. Dutta in [6] and [13] and the Cancellation Problem is discussed by the author in [23].

In this article, we shall briefly discuss these problem and their interrelationships. Throughout the article, our rings will be assumed to be commutative with unity. The notation k will denote a field. We shall use the notation $R^{[n]}$ for a polynomial ring in n variables over a commutative ring R. Thus, $E = R^{[n]}$ will mean that $E = R[t_1, \ldots, t_n]$ for some elements $t_1, \ldots, t_n$ in E which are algebraically independent over R. Unless otherwise stated, capital letters like $X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_m, X, Y, Z, T$ will be used as variables of polynomial rings. We first start with the Jacobian Conjecture.

2. The Jacobian Conjecture

The Jacobian problem first appeared in a paper by Ott-Heinrich Keller ([28]) in 1939. This appears as one of the open problems in S. Smale's list ([49]) of 18 famous open problems for the 21st century.

Before stating the problem, we make a simple observation. Let $F, G \in \mathbb{C}[X, Y]$ be two polynomials in X and Y. Then the Jacobian of F and G is defined to be the determinant of the matrix

$$\begin{pmatrix} \dfrac{\partial(F)}{\partial(X)} & \dfrac{\partial(F)}{\partial(Y)} \\ \dfrac{\partial(G)}{\partial(X)} & \dfrac{\partial(G)}{\partial(Y)} \end{pmatrix}$$

and is denoted by $J(F, G)$. Now suppose there exist $P, Q \in \mathbb{C}[X, Y]$ such that

$$X = P(F, G) \text{ and } Y = Q(F, G).$$

Then, by the chain rule, we have

$$\begin{pmatrix} \dfrac{\partial(P)}{\partial(F)} & \dfrac{\partial(P)}{\partial(G)} \\ \dfrac{\partial(Q)}{\partial(F)} & \dfrac{\partial(Q)}{\partial(G)} \end{pmatrix} \begin{pmatrix} \dfrac{\partial(F)}{\partial(X)} & \dfrac{\partial(F)}{\partial(Y)} \\ \dfrac{\partial(G)}{\partial(X)} & \dfrac{\partial(G)}{\partial(Y)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking determinant on both sides, it follows that $J(F, G)$ is a nonzero complex number. The Jacobian Conjecture for two variables asserts that the converse is also true, i.e.,

Jacobian Conjecture (for two variables): *Suppose F and G are two polynomials* *such that the Jacobian of F and G is a nonzero complex number. Then there exist two polynomials $P, Q \in \mathbb{C}[X, Y]$ such that $X = P(F, G)$ and $Y = Q(F, G)$.*

This conjecture can be easily stated to a high school student. It can be generalised to the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$ as stated below. For $F_1, \ldots, F_n \in \mathbb{C}[X_1, \ldots, X_n]$, $J(F_1, \ldots, F_n)$ is defined as the determinant of the $n \times n$ matrix whose ij^{th} entry is $\dfrac{\partial(F_i)}{\partial(X_j)}$.

Jacobian Conjecture (for n variables): *Let $F_1, \ldots, F_n \in \mathbb{C}[X_1, \ldots, X_n]$ be such that $J(F_1, \ldots, F_n)$ is a nonzero complex number. Then there exist $P_1 \ldots, P_n \in \mathbb{C}[X_1, \ldots, X_n]$ such that*

$$X_i = P_i(F_1, \ldots, F_n) \ \forall\ 1 \leq i \leq n.$$

There are several equivalent formulations of the problem; some of them have inspired rich theories which are important in their own right. For example, the following equivalent version of the Jacobian Conjecture for the ring $\mathbb{C}[X, Y]$ is formulated as a problem in locally nilpotent derivations (defined in Section 7):

Q. Suppose $J(F, G)$ is a nonzero complex number for some $F, G \in \mathbb{C}[X, Y]$. Does it follow that the derivation D defined on the ring $\mathbb{C}[X, Y]$ by $D := \dfrac{\partial(F)}{\partial(Y)}\dfrac{\partial}{\partial(X)} - \dfrac{\partial(F)}{\partial(X)}\dfrac{\partial}{\partial(Y)}$ is necessarily a locally nilpotent derivation?

Though this formulation has not yet led to a solution of the Jacobian Conjecture itself, results and techniques on locally nilpotent derivations have turned out to be very effective in solving other problems on polynomial rings. For more discussions on Jacobian Conjecture see [16].

We now discuss the Zariski Cancellation Problem. Recall that an affine variety is the set of zeros of a system of polynomial equations over a field k, together with certain topology called "Zariski topology" and a certain sheaf structure. The space k^n is an affine variety, being the zero locus of the zero polynomial.

3. The Zariski Cancellation Problem

It is important in mathematics to ascertain whether elements in a structure or the structures themselves have cancellative properties. We have seen the enormous usefulness of the cancellation of the type $ax = ay \implies x = y$. It is used all the time right from school mathematics. But we have to be careful about the validity of such cancellations. For instance, zero is not cancellative. Not all matrices are cancellative. In fact, the well known algebraic structure called "Group" is defined in such a way that all its elements are cancellative.

We have been referring to cancellation of *elements*. A very important theme in mathematics is the cancellation property of *sets* with certain specified structures. For example, over a field k, k^n is cancellative as a k vector space, i.e., if for any k-vector space V, $V \times k \cong k^{n+1}$ as vector spaces, then V itself has to be isomorphic to k^n as a vector space.

Now let $\mathbb{V}$ be an affine variety over k. The famous Zariski Cancellation Problem asks whether k^n is cancellative as an affine variety, i.e.,

Suppose $\mathbb{V} \times k \cong k^{n+1}$ as affine varieties. Does it follow that the affine variety V is necessarily isomorphic to the affine n-space k^n?

When k is algebraically closed, the above problem is equivalent to asking whether the polynomial ring $k^{[n]}(= k[X_1, \ldots, X_n])$ is cancellative, i.e.,

Suppose the ring A is such that $A[W] \cong k[X_1, \ldots, X_{n+1}]$ as k-algebras. Does it follow that the ring A is itself isomorphic to the polynomial ring $k[X_1, \ldots, X_n]$ as k-algebras?

Note that the polynomial ring in $n+1$ variables can be expressed as a polynomial ring in one variable over its various subrings in infinitely many different ways. For example, the polynomial ring $k[X, Y]$ is a polynomial ring in one variable over its various subrings $k[X]$, $k[X+Y]$, or $k[X+Y^2]$ and soon. These rings are all distinct as sets but are all isomorphic to $k[X]$. The Zariski Cancellation Problem asks whether all such coefficient rings are necessarily isomorphic.

During 1970's the affine line was proved to be cancellative by Abhyankar-Eakin-Heinzer ([1]) and the affine plane was shown to be cancellative by Miyanishi-Sugie ([34]) and Fujita ([18]) over fields of characteristic zero and by Russell ([41]) over perfect fields. This was later extended to all fields by Bhatwadekar-Gupta ([7]).

However after Russell's work in 1981, the cancellation problem remained open in higher dimension for more than three decades. In [20], the author showed that when characteristic $k > 0$, the affine 3-space k^3 is not cancellative. Subsequently in [22], the author showed that when characteristic $k > 0$, the affine n-space k^n is not cancellative for any $n \geq 3$. Thus, over a field of positive characteristic, the Zariski Cancellation Problem has been completely answered in all dimensions. However, over a field of characteristic zero, the problem still remains open for $n \geq 3$. For more discussions, one may see [23].

We now discuss the Epimorphism Problem. For convenience, we call a polynomial f in two variables X, Y to be a *nontrivial line* if $k[X,Y]/(f) = k^{[1]}$ but $k[X,Y] \neq k[f]^{[1]}$.

4. Epimorphism Problem

The Epimorphism Problem or the Embedding Problem for hyperplanes asks whether any hyperplane in $k^{[n]}$ is a coordinate, i.e.,

Suppose $\frac{k[X_1,\ldots,X_n]}{(F)} = k^{[n-1]}$. Does it follow that $k[X_1, \ldots, X_n] = k[F]^{[n-1]}$?

The celebrated Epimorphism Theorem of Abhyankar-Moh ([2]) (also proved independently by Suzuki ([50])) shows that if k is a field of characteristic zero, and $f(Z,T) \in k[Z,T]$ is a polynomial such that $k[Z,T]/(f) = k^{[1]}$ then $k[Z,T] = k[f]^{[1]}$. However, as early as in 1957, Segre ([46]) had exhibited an example of a nontrivial line over any field of positive characteristic. Later Nagata gave a family of examples ([37]).

Example 4.1. *Let k be a field of characteristic $p(>0)$ and $e, s \in \mathbb{N}$ be integers such that $e, s > 1$ and $p \nmid s$. Then*

$$f = Z^{p^e} + T + T^{sp} \in k[Z, T]$$

is a nontrivial line.

The Abhyankar-Sathaye Conjecture envisages an affirmative solution to the Epimorphism Problem over any field of characteristic zero. However for $n > 2$, no complete solution is known to this problem. The special case of linear planes in $k^{[3]}$ was considered by A. Sathaye ([44]) in characteristic zero and by P. Russell ([40]) in general; they showed that linear planes are coordinates. More precisely, they have shown:

Theorem 4.2. *Let k be any field. Let $F \in k[X, Y, Z]$ be such that $F = aZ - b$, where $a(\neq 0), b \in k[X, Y]$, and $k[X, Y, Z]/(F) = k[U, V](= k^{[2]})$. Then there exist $X_0, Y_0 \in k[X, Y]$ such that $k[X, Y] = k[X_0, Y_0]$ with $a \in k[X_0]$ and $k[X, Y, Z] = k[X_0, F]^{[1]}$. Moreover, setting $R := k[X_0]/(a)$, and denoting the image in $k[X, Y]/(a)$ of any polynomial $h \in k[X, Y]$ by $\bar{h}$, we have $R[\bar{Y_0}] = R[\bar{b}]$.*

In fact, the above description characterises all variables in $k[X, Y, Z]$ which are linear in Z and hence describes all linear planes in $k[X, Y, Z]$. We can ask a similar question about linear hyperplanes in dimension 4. We have the following partial solution to this problem ([21]):

Theorem 4.3. *Let k be any field and*

$$A = k[X, Y, Z, W]/(X^m W - b(X, Y, Z)), \text{ where } m > 1.$$

Set $f(Y, Z) := b(0, Y, Z)$ and $F := X^m W - b(X, Y, Z)$. Then the following statements are equivalent:

(i) *$f(Y, Z)$ is a variable in $k[Y, Z]$.*
(ii) *$A = k[x]^{[2]}$, where x denotes the image of X in A.*
(iii) *$A = k^{[3]}$.*
(iv) *F is a variable in $k[X, Y, Z, W]$.*
(v) *F is a variable in $k[X, Y, Z, W]$ along with X.*

There are some other partial results by D. Wright ([52]), P. Das and A. K. Dutta ([10]), S. Kaliman ([26]), P. Russell and A. Sathaye ([43]) for the case $n = 3$. For more details on the Epimorphism Problem see [15]. Let us now discuss the Affine Fibration Problem.

5. Affine Fibration Problem

In the area of affine fibrations, mathematicians try to extract information about an S-algebra A from information on its fibre rings. A finitely generated flat S-algebra A is said to be an $\mathbb{A}^n$-*fibration* (over S) if $A \otimes_S k(P) = k(P)^{[n]}$ for each prime ideal P of S. A central problem in this area, posed by Dolgačev-Veĭsfeĭler in 1974, asks if every $\mathbb{A}^n$-fibration over a regular local ring S is necessarily a polynomial ring over S.

Any $\mathbb{A}^1$ fibration over a regular local ring S was shown to be $\mathbb{A}^1$ by T. Kambayashi and M. Miyanishi ([27]). In fact, A.K. Dutta ([12]) showed that in the case $n = 1$, it is enough to assume conditions only on the generic and codimension-one fibres (rather than all fibres). A nontrivial theorem of A. Sathaye ([45]) showed that *any $\mathbb{A}^2$-fibration over a PID S containing $\mathbb{Q}$ must be isomorphic to $S^{[2]}$*. This result was proved as an application of expansion techniques which were developed by Abhyankar and Moh to prove the Epimorphism Theorem and which were further extended by Sathaye. On the other hand, it is not known whether an $\mathbb{A}^2$-fibration over a two-dimensional regular local ring containing $\mathbb{Q}$ is a polynomial ring.

In [3], Asanuma made the next major breakthrough in the Affine Fibration Problem. He proved a stable structure theorem for any affine fibration over a regular local ring. In the same paper, Asanuma also constructed a counter-example to the $\mathbb{A}^2$-fibration problem over a PID in positive characteristic. We present below a version of Asanuma's example (cf. [3, Theorem 5.1]).

Example 5.1. *Let k be a field of characteristic $p(> 0)$ and let $R = k[X,Y,Z,T]/(X^mY + Z^{p^e} + T + T^{sp})$, where m, e, s are positive integers such that $p^e \nmid sp$ and $sp \nmid p^e$. Let x denote the image of X in R. Then $k[x] \subset R$ and the following properties are satisfied by R:*

(1) $R \otimes_{k[x]} k(P) = k(P)^{[2]}$ *for all* $P \in \operatorname{Spec} k[x]$.
(2) $R^{[1]} = k[x]^{[3]} = k^{[4]}$.
(3) $R \neq k[x]^{[2]}$.

In his construction of R, Asanuma made use of the non-trivial line of Segre-Nagata to show that R is an $\mathbb{A}^2$ fibration over $k[x]$, a stably polynomial ring over $k[x]$ but not a polynomial ring over $k[x]$.

The ring R was soon to acquire a wider significance. In a subsequent paper ([4, Theorem 2.2]), using the ring R, Asanuma constructed non-linearizable algebraic torus actions on $\mathbb{A}^n_k$ over any infinite field k of positive characteristic when $n \geq 4$ (cf. Section 6). He then asked whether R is a polynomial ring and explained the significance of his question as follows ([4, Remark 2.3]):

> *"If R is a polynomial ring then it will give an example of a non-linearizable torus action on k^3 in positive characteristic. On the other hand if R is not a polynomial ring then it will clearly give a counter-example to the Cancellation Problem."*

Thus either way one would answer a major problem in Affine Algebraic Geometry. This dichotomy has been popularised by P. Russell as "Asanuma's Dilemma" (cf. [42, Problem 2, p 9]).

In [20], using techniques of A. Crachiola and L. Makar-Limanov (cf. [8]), the author has shown that R is not a polynomial ring over k at least when $m \geq 2$. Consequently, it follows that the Zariski Cancellation Problem does not have an affirmative answer in positive characteristic. In a subsequent paper ([21]), the author made further investigations on the Asanuma ring (see Theorem 2.1).

A survey of known results on $\mathbb{A}^n$-fibrations can be obtained in [6], [9], [14] and [13].

6. Linearization Problem

The Linearization Conjecture first posed by Kambayashi in 1979 asserts that any algebraic action of the reductive group k^* on $k^{[n]}$ is linearizable. Algebraically this would mean that given any $\mathbb{Z}$-algebra grading on $k^{[n]}$, there exists a homogeneous set of coordinates.

It was known even in the 1970's that the conjecture has an affirmative answer for the affine plane in any characteristic. But it took around two decades to get an affirmative answer for the affine three-space in characteristic zero culminating the efforts of several mathematicians. T. Asanuma showed that when characteristic of $k = p > 0$, then Example 5.1 leads to a non-linearizable k^*-action on $k^{[n]}$ for $n \geq 4$. It is still not known whether in Example 5.1 $R = k^{[3]}$ when $m = 1$. If so, then Example 5.1 would lead to a non-linearizable k^*-action on $k^{[3]}$. Asanuma, in fact, described a general method for constructing candidate counterexamples to the linearization conjecture using non-trivial embeddings in affine spaces ([5]). Using the non-trivial embedding of the trefoil knot in $\mathbb{R}^3$ given by A.R. Shastri ([47]), Asanuma constructed non-linearizable $\mathbb{R}^*$-actions on $\mathbb{R}^{[n]}$ for $n \geq 5$. The Linearisation Conjecture is still an open problem over an algebraically closed field of characteristic zero for $n \geq 4$ and over a field of positive characteristic for $n = 3$. For more discussions on this see [29].

The Threefold $x^2y + x + z^2 + t^3 = 0$

An important example in the history of the Linearisation Problem is the Russell-Koras threefold A defined by $x^2y + x + z^2 + t^3 = 0$ in $\mathbb{C}^4$. It was an open problem for sometime whether this ring A is isomorphic to $\mathbb{C}^{[3]}$. There was lot of excitement when it was constructed as A shares several properties of polynomial ring $\mathbb{C}^{[3]}$: A is a UFD; A is contained in a polynomial ring in three variables; A is smooth, contractible and diffeomorphic to $\mathbb{R}^6$. For a long time no known invariant could distinguish A from the polynomial ring $\mathbb{C}^{[3]}$. If A happened to be isomorphic to the polynomial ring $\mathbb{C}^{[3]}$, then it would have led to a counterexample to the Linearisation Problem for $\mathbb{C}^{[3]}$ as well as the Epimorphism Problem for $\mathbb{C}^{[3]}$.

However, Makar Limanov ([32]) showed that A is not a polynomial ring by discovering an invariant which is now known after him (cf. Section 7). This led Russell and Koras to a major theorem confirming the Linearisation Conjecture for the polynomial ring $\mathbb{C}^{[3]}$. The Epimorphism Problem is still open. It is not known whether $A^{[1]}$ is a polynomial ring. If yes, then it would lead to a counterexample to the Zariski Cancellation Problem in characteristic zero.

The similarity between the Asanuma threefold and the Russell Koras three fold led the author to investigate the more general threefold $x^m y - F(x, z, t) = 0$ over *any* field of any characteristic and she discovered an equivalence of 10 statements ([21], Theorem 3.8) given below. For a ring R, the Derksen invariant of R, denoted by $\mathrm{DK}(R)$, is the subring of R generated by the ring of invariants of all $\mathbb{G}_a$ actions of R. A k-algebra R is said to be *geometrically factorial*, if for any algebraic extension $\tilde{k}$ of k, $A \otimes_k \tilde{k}$ is a factorial domain.

Theorem 6.1. *Let k be a field and*

$$A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T)), \text{ where } m > 1.$$

Let x, y, z and t denote, respectively, the images of X, Y, Z and T in A. Set $f(Z, T) := F(0, Z, T)$ and $G := X^m Y - F(X, Z, T)$. Then the following statements are equivalent:

(i) $A = k^{[3]}$.
(ii) $A = k[x]^{[2]}$.
(iii) $k[Z, T] = k[f]^{[1]}$.
(iv) $k[X, Y, Z, T] = k[G]^{[3]}$.
(v) $k[X, Y, Z, T] = k[X, G]^{[2]}$.
(vi) $A^{[\ell]} = k^{[\ell+3]}$ *for some integer* $\ell \geq 0$ *and* $\mathrm{DK}(A) \neq k[x, z, t]$.
(vii) A *is an* $\mathbb{A}^2$*-fibration over* $k[x]$ *and* $\mathrm{DK}(A) \neq k[x, z, t]$.
(viii) $k[Z, T]/(f) = k^{[1]}$ *and* $\mathrm{DK}(A) \neq k[x, z, t]$.
(ix) A *is geometrically factorial over* k, $\mathrm{DK}(A) \neq k[x, z, t]$ *and* $\mathbf{K}_1(A) = k^*$.
(x) A *is geometrically factorial over* k, $\mathrm{DK}(A) \neq k[x, z, t]$ *and* $(A/xA)^* = k^*$.

This gave a unified treatment of several apparently different-looking questions including the Cancellation, Epimorphism and Affine Fibration problems and the non-triviality of Asanuma and Russell threefolds. The theorem relates them with diverse concepts like coordinates, stably polynomial rings, fibrations, lines, Derksen invariants of $\mathbb{G}_a$-actions, Whitehead group K_1, group of units of A/xA and geometric factoriality. This theorem is independent of the characteristic and this generalisation has led to new results as well as simplification of the earlier proofs.

In [21, Theorem 4.2], it has been shown that the ring A is a stably polynomial ring if $F(0, Z, T)$ is a line and now by the equivalence of (i) and (iii) in Theorem 6.1, it follows that A is not a polynomial ring if $F(0, Z, T)$ is not a coordinate of $k[Z, T]$. Thus, if $F(0, Z, T)$ is a nontrivial line in $k[Z, T]$, then A is a stably polynomial ring but not a polynomial ring. This gives a recipe for constructing counter-examples to the Zariski Cancellation Problem. It also explains the nontriviality of the Russell-Koras threefold. In the Russel-Koras threefold $F(X, Z, T) = -X - Z^2 - T^3$, so that $F(0, Z, T) = -Z^2 - T^3$ which is not a coordinate. Similarly one can see the nontriviality of Example 5.1 for $m > 1$.

7. Characterisation Problem

The Characterisation Problem seeks useful characterisations of the affine space in terms of its algebraic, geometric or topological properties. Interesting in its own right, the Characterisation Problem is closely related to other challenging problems on the affine space like the Cancellation Problem. Algebraic characterisations have been particularly fruitful in solving problems on affine spaces.

The characterisations of the affine line are quite well-known and certainly very useful. For instance, if k is an algebraically closed field, then $k^{[1]}$ is the only one-dimensional UFD with non-trivial units. This is an algebraic characterisation. This immediately solves the cancellation problem: $A^{[1]} = k^{[2]} \implies A = k^{[1]}$. If $k = \mathbb{C}$, then the affine line $\mathbb{A}^1_{\mathbb{C}}$ is the only acyclic normal curve, a topological characterisation.

In his attempt to solve the cancellation problem $A^{[1]} = \mathbb{C}^{[3]} \implies A = \mathbb{C}^{[2]}$, C.P. Ramanujam ([39]) obtained a beautiful topological characterisation of the affine plane over $\mathbb{C}$. He proved that $\mathbb{C}^2$ is the only smooth contractible surface which is simply connected at infinity, in particular, $\mathbb{C}^2$ is the only normal affine surface which is homeomorphic to $\mathbb{C}^2$. Subsequently, M. Miyanishi ([33]) obtained an algebraic characterisation of the affine plane over an algebraically closed field of characterisation zero. This algebraic characterisation enabled T. Fujita, M. Miyanishi and T. Sugie ([18], [34]) to solve the Cancellation Problem for the affine plane. Later, R.V. Gurjar ([24]) gave another proof of the cancellation theorem for the affine plane using Ramanujam's topological characterisation.

M. Miyanishi ([35]) and S. Kaliman ([26]) also gave characterisations of the affine three space. These characterisations involve some topological invariants. Nikhilesh Dasgupta and the author ([11]) have given a new purely algebraic characterisation of the affine two space and three space using some invariant, which they have called F-ML invariant, as described below.

Let k be an algebraically closed field of characteristic zero and R be a k algebra. A k-linear derivation D of R is said to be a locally nilpotent if for each $a \in R$, there exists an integer $n \geq 0$ (depending on a), such that $D^n(a) = 0$. The Makar-Limanov invariant of the R, denoted by $ML(R)$, is defined to be the intersection of the kernels of all locally nilpotent derivations of R. We say that the Makar-Limanov invariant is trivial if $ML(R) = k$. The F-ML invariant of the ring R is defined to be the intersections of the kernels of all locally nilpotent derivations *with slices* (i.e., derivations in which 1 lies in the image of with D). If there does not exist any locally nilpotent derivation with slice, then F-ML invariant is defined to be R. The triviality of the Makar-Limanov invariant itself gives a characterisation of $k^{[1]}$ (i.e., for an affine k-domain B with $\operatorname{tr.deg}_k B = 1$, $B = k^{[1]}$ if and only if $ML(B) = k$). However, when $\operatorname{tr.deg}_k B = 2$ (or 3) the invariant

$ML(B)$ alone does not characterise $k^{[2]}$ (or $k^{[3]}$). The results of Dasgupta and the author show that under the additional hypothesis of the existence of a locally nilpotent derivation D with slice, the triviality of the Makar-Limanov invariant does suffice to characterise the affine 2-space and the affine 3-space. More precisely, they have shown that $k^{[2]}$ is the only two-dimensional affine domain with trivial F-ML invariant and $k^{[3]}$ is the only three-dimensional affine UFD with trivial F-ML invariant ([11]). They have also given an example to show that such a characterisation does not extend to a four dimensional affine factorial domain. No characterisations for $k^{[n]}$ is known for $n \geq 4$.

8. The Automorphism Problem

Let ϕ be an automorphism of the polynomial ring $k[X]$. One can easily see that $\phi(X) = \lambda X + \mu$, where $\lambda \in k^*$ and $\mu \in k$. This determines the automorphism group of the polynomial ring in one variable.

The automorphism group of the polynomial ring $k[X, Y]$ is also known. Before describing it, we first recall some definitions.

Let $\mathrm{GA}_n(k)$ denote the group of all automorphisms of the polynomial ring $k[X_1, \ldots, X_n]$. An automorphism $\phi \in \mathrm{GA}_n(k)$ is called *affine* if the degree of $\phi(X_i) \leq 1$, for each i, $1 \leq i \leq n$ and *triangular* if $\phi(X_1) \in k^*$ and for each i, $2 \leq i \leq n$, $\phi(X_i) = \lambda_i X_i + q_i$, where $\lambda_i \in k^*$ and $q_i \in k[X_1, \ldots, X_{i-1}]$. The subgroup of $\mathrm{GA}_n(k)$ generated by all the affine and triangular automorphisms is called the tame subgroup and is denoted by $\mathrm{TA}_n(k)$. Any automorphism $\phi \in \mathrm{TA}_n(k)$ is called an tame automorphism. An automorphism $\phi \in \mathrm{GA}_n(k)$ is called *wild* if $\phi \in \mathrm{GA}_n(k) \setminus \mathrm{TA}_n(k)$.

We have the following theorem due to the works of H.W.E. Jung for ch $k = 0$ ([25]) and by W. van der Kulk for arbitrary characteristic ([51]).

Theorem 8.1. $\mathrm{GA}_2(k) = \mathrm{TA}_2(k)$.

It was an open problem for long time whether there exists wild automorphisms of $k^{[3]}$. In 1972, Nagata ([38]) constructed the following automorphism for the polynomial ring $k[X, Y, Z]$

$\phi(X) = X$, $\phi(Y) = Y + X(XZ + Y^2)$, and $\phi(Z) = Z - 2Y(XZ + Y^2) - X(XZ + Y^2)^2$

and conjectured that ϕ is an wild automorphism of $k[X, Y, Z]$. In 2004, Shestakov-Umirbaev ([48]) solved this longstanding open conjecture of Nagata affirmatively. Remarkable simplifications and further developments on the work of Shestakov and Umirbaev were done by S. Kuroda ([30]). Now many wild automorphisms of $k[X, Y, Z]$ are known, but still we do not know a description of $\mathrm{GA}_3(k)$. A description of $\mathrm{GA}_n(k)$, for $n \geq 4$, seems completely out of reach.

A major difficulty in solving problems about the affine space, for $n \geq 3$, lies in the fact that very little is known about $\mathrm{GA}_n(k)$ for $n \geq 3$. For more discussions on this see [16], [29] and [31].

Acknowledgements. The author thanks Amartya K. Dutta for carefully going through the draft and suggesting improvements.

References

[1] Abhyankar, S. S., Eakin, P. and Heinzer, W., On the uniqueness of the coefficient ring in a polynomial ring, *J. Algebra*, **23** (1972), 310–342.

[2] Abhyankar, S. S. and Moh, T. T., Embeddings of the line in the plane, *J. Reine Angew. Math.*, **276** (1975), 148–166.

[3] Asanuma, T., Polynomial fibre rings of algebras over Noetherian rings, *Invent. Math.*, **87** (1987), 101–127.

[4] Asanuma, T., Non-linearizable Algebraic Group Actions on $\mathbb{A}^n$, *J. Algebra*, **166(1)**, (1994) 72–79.

[5] Asanuma, T., Non-linearizable algebraic k^*- actions on affine spaces, *Invent. Math.*, **138(2)** (1999), 281–306.

[6] Bhatwadekar, S. M. and Dutta, A. K., On affine fibrations, *Commutative Algebra*, (ed. Simis, A., Trung, N. V. and Valla, G.), World Sc. (1994) 1–17.

[7] Bhatwadekar, S. M. and Gupta, Neena, A Note on the Cancellation Property of $k[X, Y]$, *J. Algebra and its Applications, (special issue in honour of Prof. Shreeram S. Abhyankar)*, **14(9)**, (2015), 1540007.

[8] Crachiola, A. J., The hypersurface $x+x^2y+z^2+t^3$ over a field of arbitrary characteristic, *Proc. Amer. Math. Soc.*, **134 (5)** (2005), 1289–1298.

[9] Daigle, D. and Freudenburg, G., Families of affine fibrations, Symmetry and spaces (ed. Campbell, H. E. A., et al.), *Progr. Math.*, **278** Birkhäuser (2010), 35–43.

[10] Das, P. and Dutta, A. K., Planes of the form $b(X, Y)Z^n - a(X, Y)$ over a DVR, *J. Commut. Algebra*, **3(4)** (2011), 491–509.

[11] Dasgupta, N. and Gupta, Neena, An algebraic characterisation of the affine 3-space, work in progress.

[12] Dutta, A. K., On $\mathbb{A}^1$-bundles of affine morphisms, *J. Math. Kyoto Univ.*, **35(3)** (1995), 377–385.

[13] Dutta, A. K., *Some results on affine fibrations*, Advances in Algebra and Geometry (ed. Musili, C.), Hindustan Book Agency, India (2003), 7–24.

[14] Dutta, A. K., Some results on subalgebras of polynomial algebras, Commutative Algebra and Algebraic Geometry (ed. Ghorpade, S., et al.), *Contemp. Math.*, **390** (2005), 85–95.

[15] Dutta, A. K. and Gupta, Neena, The Epimorphism Theorem and its generalisations, *J. Algebra and its Applications, (special issue in honour of Prof. Shreeram S. Abhyankar)*, **14(9)** (2015), 1540010.

[16] Van den Essen, A., Polynomial Automorphisms and the Jacobian Conjecture, *Progr. Math.*, **190** Birkhäuser (2000).

[17] Freudenburg, G. and Russell, P., Open problems in affine algebraic geometry, Affine Algebraic Geometry (ed. J. Gutierrez et al.), *Contemp. Math.*, **369** (2005), 1–30.

[18] Fujita, T., On Zariski problem, *Proc. Japan Acad.*, **55A** (1979), 106–110.

[19] Ganong, R., The pencil of translates of a line in the plane, Affine Algebraic Geometry (ed. D. Daigle et al.), *CRM Proc. Lec. Notes*, **54** AMS (2011), 57–71.

[20] Gupta, Neena, On the Cancellation Problem for the Affine Space $\mathbb{A}^3$ in characteristic p, *Invent. Math.*, **195** (2014), 279–288.

[21] Gupta, Neena, On the family of affine threefolds $x^m y = F(x, z, t)$, *Compositio Math.*, **150(6)** (2014), 979–998.

[22] Neena Gupta, *On Zariski's Cancellation Problem in positive characteristic*, Adv. Math. **264** (2014), 296–307.

[23] Gupta, Neena, A survey on Zariski Cancellation Problem, *Indian J. of Pure Appl. Math.*, **46(6)** (2015), 865–877.

[24] Gurjar, R. V., A topological proof of cancellation theorem for $\mathbb{C}^2$, *Math. Z.*, **240(1)** (2002), 83–94.

[25] Jung, H. W. E., Über ganze birationale Transformationen der Ebene, *J. Reine Angew. Math.*, **184** (1942), 161–174.

[26] Kaliman, S., Polynomials with general $\mathbb{C}^{[2]}$-fibers are variables, *Pacific J. Math.*, **203** (2002), no. 1, 161–190.

[27] Kambayashi, T. and Miyanishi, M., On flat fibrations by the affine line, *Illinois J. Math.*, **22(4)** (1978), 662–671.

[28] Keller, O. H., Ganze Cremona-Transformationen, *Monatsh. Math. Phys.*, **47** (1939), 299–306.

[29] Kraft, H., Challenging problems on affine n-space, *Séminaire Bourbaki*, **802** (1995), 295–317.

[30] Kuroda, S., Shestakov-Umirbaev reductions and Nagata's conjecture on a polynomial automorphism, *Tohoku Math. J.*, (2) **62(1)** (2010), 75–115.

[31] Kuroda, S., Recent developments in polynomial automorphisms: the solution of Nagata's conjecture and afterwards, *Sugaku Expositions*, **29(2)** (2016), 177–201.

[32] Makar-Limanov, L., On the hypersurface $x+x^2y+z^2+t^3=0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$, *Israel J. Math.*, **96(B)** (1996), 419–429.

[33] Miyanishi, M., An algebraic characterization of the affine plane, *J. Math. Kyoto Univ.*, **15** (1975), 169–184.

[34] Miyanishi, M. and Sugie, T., Affine surfaces containing cylinderlike open sets, *J. Math. Kyoto Univ.*, **20** (1980), 11–42.

[35] Miyanishi, M., An algebro-topological characterization of the affine space of dimension three, *Amer. J. Math.*, **106** (1984), 1469–1486.

[36] Miyanishi, M., Recent developments in affine algebraic geometry: from the personal viewpoints of the author, *Affine algebraic geometry (ed. Hibi, T.), Osaka Univ. Press*, (2007), 307–378.

[37] Nagata, M., A theorem on valuation rings and its applications, *Nagoya Mathematical Journal*, **29** (1967), 85–91.

[38] Nagata, M., *On automorphism group of $k[x,y]$*, Department of Mathematics, Kyoto University, Lectures in Mathematics, **5**, Kinokuniya Book-Store Co., Ltd., Tokyo, 1972.

[39] Ramanujam, C. P., A topological characterization of the affine plane as an algebraic variety, *Annals of Math.*, **94** (1971), 69–88.

[40] Russell, P., Simple birational extensions of two dimensional affine rational domains, *Compositio Math.*, **33(2)** (1976), 197–208.

[41] Russell, P., On Affine-Ruled rational surfaces, *Math. Ann.*, **255** (1981), 287–302.

[42] Russell, P., Open problems in affine algebraic geometry, Affine algebraic geometry, *Contemp. Math.*, **369** (2005), 1–30.

[43] Russell, P. and Sathaye, A., Forty years of the epimorphism theorem, *Eur. Math. Soc. Newsl.*, **90** (2013), 12–17.

[44] Sathaye, A., On linear planes, *Proc. Amer. Math. Soc.* **56** (1976), 1–7.

[45] Sathaye, A., Polynomial ring in two variables over a DVR: a criterion, *Invent. Math.*, **74(1)** (1983), 159–168.

[46] Segre, B., Sur un problème de M. Zariski, *Colloque international d'algèbre et de thérie des nombres*, (Paris 1949), 135–138, C.N.R.S., Paris 1950.

[47] Shastri, A. R., Polynomial representations of knots, *Tohoku Math. J.*, (2) **44(1)** (1992), 11–17.

[48] Shestakov, I. P. and Umirbaev, U. U., The tame and the wild automorphisms of polynomial rings in three variables, *J. American Math. Soc.*, **17(1)** (2004), 197–227.

[49] Smale, S., Mathematical Problems for the Next Century, *Mathematical Intelligencer*, **20(2)** (1998), 7–15.

[50] Suzuki, M., Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbb{C}^2$, *J. Math. Soc. Japan*, **26** (1974), 241–257.

[51] Van der Kulk, W., On polynomial rings in two variables, *Nieuw Arch. Wisk.*, **1 (3)** (1953), 33–41.

[52] Wright, D., Cancellation of variables of the form $bT^n - a$, *J. Algebra*, **52(1)** (1978), 94–100.

Neena Gupta
Statistics and Mathematics Unit, Indian Statistical Institute,
203 B.T. Road, Kolkata 700 108, India
E-mail: *neenag@isical.ac.in, rnanina@gmail.com*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 21-28

TUMOR GROWTH-CHEMO-MECHANICAL MODELING AND EXISTENCE THEORY*

G. P. RAJA SEKHAR

Abstract. There are many experimental models to understand tumor growth in a bio-reactor. It will be important and quite challenging to put these results in a theoretical framework. Mathematical modeling is one of the best tools to understand the experimental results in terms of physical and bio-chemical mechanisms. This talk introduces some of the developments to model the mechanical behavior of tumor at the macro scale using porous media approach. Our purpose is to focus on the well-posed-ness of these mathematical models which are governed by partial differential equations. In most of the general situations it is difficult to find a classical solution. Therefore one can break the existence problem into two subparts: (i) find a weak or generalized solution (means less smoother than the required solution) (ii) and then show that this weak solution is smooth via suitable regularity techniques. We study related existence and uniqueness results for the hydrodynamic problem inside a tumor, however, support these existence results in limited cases with series solution for the combined hydrodynamic and nutrient transport. We explain important results on necrosis formation in a tumor. Further, in specific cases, we show correlations with clinical data.

1. Introduction

Unusual growth of tissue forms a mass and is called a tumor. Solid tumors are generally a mixture of several cell populations. These cells are attached to the extracellular matrix (ECM) which is wetted by an extracellular fluid. In general, two fluid constituents of a solid tumor are intravascular (mainly blood plasma) and interstitial fluids. Depending on the types of cell, the three constituents would be: (i) normal host cells (ii) viable tumor cells (which are able to proliferate) (iii) dead cells. The three types of cell and ECM components of tumor tissue would be treated as single solid phase [12, 7]. The ECM gives structure and rigidity to the cells. The blood vessel distribution within a solid tumor is not regular. As a result, the nutrient concentration may not be uniform. Based on the nutrient concentration, a tumor may be classified as follows: in the outer most region, cells grow rapidly due to adequate nutrients concentration. On the other hand,

---

* The article is based on the text of the 30th P. L. Bhatnagar Memorial Award Lecture delivered at the 82^{nd} Annual Conference of the Indian Mathematical Society held at the University of Kalyani, Kalyani-741 235, Nadia, West Bengal, India during December 27 - 30, 2016.

2010 Mathematics Subject Classification : 76, 76T99.
Key words and phrases: tumor, mixture theory, porous media, necrosis.

in the inner most region cells die due to lack of nutrients or starvation [7]. The rate of growth of tumor cells depends on the supply of nutrients inside a patient body. Significant growth of a tumor and formation of new tumors result reduction in the quality of life. Our study concerns tumor growth, transport of drugs and blood borne solute macromolecules inside interstitial space of a solid tumor. Due to lack of nutrient concentration one can observe necrosis inside a tumor. We identify critical issues like necrosis formation inside a solid tumor based on various parameters involved in the mathematical model.

Evolution of avascular tumor (an early stage of tumor) totally depends on the surrounding environment. We focus on the mathematical modeling and analysis corresponding to such tumors with small percentage of vasculature which are not in a stage of metastasis. Growth has been estimated using the nutrient concentration obtained from the diffusion advection equation. This study is a prerequisite to understand the solid tumor growth. The solid phase (tumor cell+ECM) velocity is useful to understand the hydromechanics of tissue deformation towards extracellular fluid and convective solute transport based on the biphasic mixture theory. Detailed literature survey shows that the concept of biphasic mixture theory is useful to describe the mechanics of interstitial fluid flow, solute transport and growth inside a solid tumor [5].

Our attempt is to establish the well-posed ness of poroelastodynamics inside an isolated non growing solid tumor in the weak sense using mixture theory model. We focus on the existence of unique weak solution of the model using Lax Milgram theorem and inf-sup condition. Further, we comment on the stability estimates of the solution. An analytical treatment on the one dimensional spherical symmetry model gives an idea of necrosis formation (from the solute concentration profile) and hydromechanics of solid phase deformation.

2. Mathematical Formulation

The present model considers an isolated tumor which behaves as a homogeneous deformable porous medium. Deformable solid phase of the tumor is assumed to be constituted by an ensemble of interstitial, vascular regions and cell population and fluid phase is constituted by extracellular fluid in which tumor cells can survive. Fluid in the extracellular matrix is viscous and contains various nutrients, drug molecules, electrolytes and plasma proteins. The movement of the extracellular fluid within the interstitial space causes the solid phase deformation. Suppose $\Omega \subset \mathbb{R}^d$, $d = 2,\ 3$, denotes a bounded, Lipschitz domain in which tumor is occupied. Let $\mathbf{V}_f$ and $\mathbf{V}_s$ denote the extracellular fluid velocity and velocity of the solid phase respectively. The apparent densities of the fluid and solid phases are denoted by $\tilde{\rho}_f$ and $\tilde{\rho}_s$, respectively and ϕ_f and ϕ_s are the respective volume fractions. Following are the mass conservation equations for each phase in a generic form. Thus for $x \in \Omega$,

$$\frac{\partial(\tilde{\rho}_f \phi_f)}{\partial t} + \nabla \cdot [(\tilde{\rho}_f \phi_f)\mathbf{V}_f] = \tilde{\rho}_f S_f(x,t), \tag{2.1}$$

$$\frac{\partial(\tilde{\rho}_s \phi_s)}{\partial t} + \nabla \cdot [(\tilde{\rho}_s \phi_s)\mathbf{V}_s] = \tilde{\rho}_s S_s(x,t), \tag{2.2}$$

where $S_f(x,t)$ and $S_s(x,t)$ represent the fluid phase and solid phase generation (mainly the growth of cell population) respectively. For the saturated mixture $\phi_f + \phi_s = 1$ and we have the following combined from of (2.1)-(2.2) as (when the apparent densities of each phases are constant)

$$\nabla \cdot (\phi_f \mathbf{V}_f + \phi_s \mathbf{V}_s) = S_f(x,t) + S_s(x,t). \tag{2.3}$$

When the mixture is closed, i.e., $S_f(x,t) + S_s(x,t) = 0$, equation (2.3) becomes

$$\nabla \cdot (\phi_f \mathbf{V}_f + \phi_s \mathbf{V}_s) = 0. \tag{2.4}$$

When there is no generation of new tumor cells and fibrous skeleton during perfusion of solutes, that is when the tumor is not a closed mass, equation (2.4) becomes

$$\nabla \cdot (\phi_f \mathbf{V}_f + \phi_s \mathbf{V}_s) = S_f(x,t). \tag{2.5}$$

Typically, the fluid source $S_f(x,t)$ is assumed to be driven by the average transmural pressure (for details one can visit [12, 7]).

Momentum balance equation for each of the constituent phases (solid and fluid) in the binary mixture of cellular phase (solid) and extracellular fluid are given by

$$\rho_j \left(\frac{\partial \mathbf{V}_j}{\partial t} + (\mathbf{V}_j \cdot \nabla)\mathbf{V}_j \right) = \nabla \cdot \mathbf{T}_j + \mathbf{\Pi}_j + \mathbf{b}_j \text{ for } j \in \{f, s\}, \tag{2.6}$$

where $\rho_j = \phi_j \tilde{\rho}_j$ true mass density of the j^{th} phase of the mixture. $\mathbf{T}_j$ stress tensor for the j^{th} phase of the mixture. $\mathbf{\Pi}_j$ is resultant of the forces acting on the j^{th} phase of mixture due to the interactions with other phase. $\mathbf{b}_j$ body force of the j^{th} phase of mixture. Constitutive relations for the corresponding stress tensors for each phase are

$$\mathbf{T}_f = -\phi_f P\mathbf{I} + \lambda_f (\nabla \cdot \mathbf{V}_f)\mathbf{I} + \mu_f (\nabla \mathbf{V}_f + (\nabla \mathbf{V}_f)^T), \tag{2.7}$$

$$\mathbf{T}_s = -\phi_s P\mathbf{I} + \chi(\phi_s)(\nabla \cdot \mathbf{U}_s)\mathbf{I} + \mu_s(\phi_s)(\nabla \mathbf{U}_s + (\nabla \mathbf{U}_s)^T). \tag{2.8}$$

Here P is hydrodynamic pressure, λ_f and μ_f are first and second coefficients of viscosity corresponding to interstitial fluid respectively, $\mathbf{U}_s$ is displacement vector, $\mathbf{V}_s := \partial \mathbf{U}_s / \partial t$ is the velocity vector of solid phase, χ and μ_s are elastic parameters corresponding to the solid phase. Moreover, $\chi = \nu_p \mathcal{Y}/(1+\nu_p)(1-2\nu_p)$ and $\mu_s = \mathcal{Y}/2(1+\nu_p)$, $\mathcal{Y}$ and ν_p are Young's modulus and Poisson ratio respectively.

The interaction forces take the following form

$$-\mathbf{\Pi}_s = \mathbf{\Pi}_f = \frac{1}{K}(\mathbf{V}_s - \mathbf{V}_f) + (\nabla \phi_f)P. \tag{2.9}$$

The term (1/K) is known as the drag coefficient. The equation (2.9) shows that interaction forces satisfy Newton's third law. For details one may refer [12].

2.1. **Assumptions on the present model.** The above formulation is very generic and appears very difficult to handle straight away. Hence, we propose to follow a step by step simplification. To this extent, we consider a simplified model based

on the following assumptions.
(i) Nutrient proliferation rate is much faster than the tumor cell growth (static perfused model is assumed).
(ii) Infinitesimal deformation of solid phase.
(iii) Motion of interstitial fluid flow and solid phase deformation are slow (i.e. we can neglect inertial terms compared to stress terms).
Under these assumptions, by substituting constitutive relations (2.7), (2.8) and (2.9) in the momentum equation (2.6), we get the following system of equations along with the mass conservation equation,

$$-\nabla \cdot (2\mu_f D(\mathbf{V}_f) + \lambda_f(\nabla \cdot \mathbf{V}_f)\mathbf{I} - \phi_f P\mathbf{I}) + \frac{1}{K}\mathbf{V}_f = \mathbf{b}_f \text{ in } \Omega, \tag{2.10}$$

$$-\nabla \cdot (2\mu_s D(\mathbf{U}_s) + \chi(\nabla \cdot \mathbf{U}_s)\mathbf{I} - \phi_s P\mathbf{I}) - \frac{1}{K}\mathbf{V}_f = \mathbf{b}_s \text{ in } \Omega, \tag{2.11}$$

$$\nabla \cdot (\phi_f \mathbf{V}_f) = S_f \text{ in } \Omega. \tag{2.12}$$

In order to close the above system of equations, we need to support with suitable boundary conditions. The model at hand is in-vitro analogous, which is typically done while maintaining suitable ambient pressure or stress conditions during experiments. Most of the studies involving in vitro model have considered pressure conditions (see [2, 12, 5, 7]). However, it may be noted that pressure being the pore pressure, it relates directly to the fluid phase of the tissue. On the other hand ambient solid phase stress condition relates to the tissue deformation. Hence, the stress boundary condition is expected to show significant impact on the mechanical behavior of the tissue. Accordingly, we propose the following ambient conditions.

2.2. **Boundary conditions.**

$$\mathbf{T}_s \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \tag{2.13}$$

$$\mathbf{T}_f \cdot \mathbf{n} = \mathbf{T}_\infty \text{ on } \partial\Omega, \tag{2.14}$$

where $\mathbf{n}$ is the normal unit vector on the boundary $\partial\Omega$, and $D(\cdot)$ denotes the deviatoric matrix which is defined as

$$D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t),$$

$(\nabla\mathbf{u})^t$ denotes the transpose of the matrix $\nabla\mathbf{u}$.

2.3. **Existence and uniqueness.** For $d = 2, 3$, set $\mathbf{X} = H^1(\Omega)^d$, $\mathcal{H} = H^1(\Omega)^d/\mathbb{R}^d$ and $M = L^2(\Omega)$. Assume that $\mathbf{b}_f,\ \mathbf{b}_s \in L^2(\Omega)^d$, and $\mathbf{T}_\infty \in L^2(\partial\Omega)^d$. Also assume that the physical parameters (μ_f, λ_f, ϕ_f, K, μ_s, χ, ϕ_s) involve in the models are constant functions. Using some standard theorems (Lax-Milgram, Babuska-Brezzi (see [4, 8])) we have shown that the above system (2.10-2.14) has a unique weak solution $(\mathbf{V}_f, \mathbf{U}_s, P) \in X \times \mathcal{H} \times M$ and solution satisfies the following stability estimates.

(i) In case of a proliferating (not closed) mixture,

$$\frac{\alpha}{2}||\mathbf{V}_f||^2_{\mathbf{X}}+\frac{a_0}{2}||P||^2_M\leq\frac{1}{2\alpha}\left(||\mathbf{b}_f||_{L^2(\Omega)^d}+t_r||\mathbf{T}_\infty||_{L^2(\partial\Omega)^d}\right)^2+\frac{1}{2a_0}||a_1||^2_{L^2(\Omega)} \quad (2.15)$$

(ii) For a closed mixture,

$$||\mathbf{V}_f||_{\mathbf{X}}+||P||_M\leq c_0\left(||\mathbf{b}_f||_{L^2(\Omega)^d}+||\mathbf{T}_\infty||_{L^2(\partial\Omega)^d}\right) \quad (2.16)$$

and

$$||\mathbf{U}_s||_{\mathcal{H}}\leq\frac{c_0}{2\mu_s}\left(c_1\phi_s||P||_{L^2(\Omega)}+\frac{c_2}{K}||\mathbf{V}_f||_{L^2(\Omega)^d}+c_2||\mathbf{b}_s||_{L^2(\Omega)^d}\right) \quad (2.17)$$

where α, a_0, t_r, c_0, c_1, and c_2 are some positive constants.

3. Macroscopic transport and metabolism of blood borne chemical macromolecules

The present study deals with the macro-molecular solute transport inside a tumor. The micro-vessels release chemical macromolecules (viz. nutrients, growth factor, chemotactic factors, drug molecules, and so on) within the interstitial space carrying within it. Chemicals have to cross the micro-vessel wall to reach the interstitial space by convective-diffusive transport across the wall. On reaching the interstitial space, chemical macromolecules have to be transported via convection or diffusion or both through the tumor cells and the interstitial space by the extracellular fluid to reach the living tumor cells [10, 6]. Tumor cells consume chemicals at a specific rate. We assume that the metabolic rate is first order reversible in nature. This guarantees creation of no starvation zone within the tumor [13] during transport of nutrients. Since, permeability is low near the tumor core, advection becomes negligibly small there. Therefore, for all sorts of molecules, the central region becomes diffusion dominated.

We consider that a single type of solute macromolecule say nutrient "$\mathcal{N}$" gets transported to the interstitial space within Ω. Let, $C_j(\mathbf{x},t)$, $j\in\{\mathrm{s},\ \mathrm{f}\}$ be its concentration per unit volume within j^{th} constituent of the mixture. Next we assume that C_j remain the same across all constituents. For homogeneous and isotropic diffusivity D of $\mathcal{N}$, C satisfies

$$\frac{\partial C}{\partial t}+\nabla.(C\mathbf{V}_{\text{com}})=D\nabla^2C+S_{\text{sol}}-K_{\text{rec}}C,\ \ (\mathbf{x}\text{ in }\Omega). \quad (3.1)$$

In general, the solute source term S_{sol}, is found to follow Patlak equation and can be expressed as

$$S_{\text{sol}}=(1-\sigma_f)C_pS_V+\frac{P_cA}{V}(C_p-C)\left(\frac{Pe_{\text{v}}}{\exp(Pe_{\text{v}})-1}\right)-S_LC\,. \quad (3.2)$$

This equation basically governs the transport across blood vessel and drain through the lymph vessel within Ω. One can consider low / high Pe_{v} to get some asymptotic cases.

Convection is negligible towards the solute transport except at the peripheral region where significant interstitial fluid pressure (IFP) can be noted ([3, 10, 9]), which drives solutes out of the tumor. High diffusive macromolecules can only overcome this barrier. In general, diffusion depends on the concentration gradient and small molecules exhibit diffusion dominated transport. But, for the larger molecules, diffusion dominated transport becomes insufficient. In this situation, we get motivated from the study of ([14]) to obtain the solution of the Eq. (3.1) for those solutes which possess much smaller diffusion time compared to the time of advection (i.e., molecules of $\mathcal{N}$ possesses high diffusivity).

4. Triggering of Necrosis

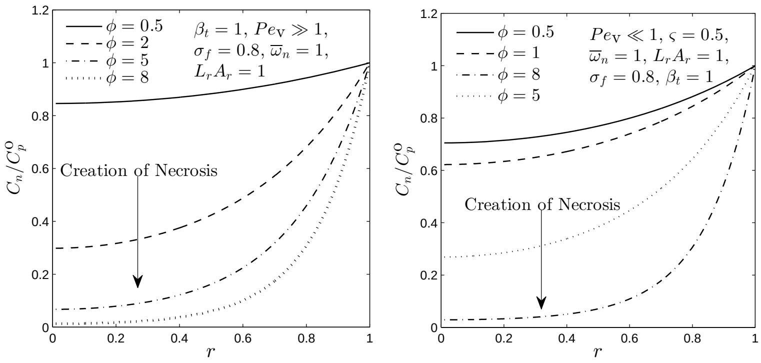
Figs. 1(a) and 1(b) highlight the variation of solute macromolecule concentration with Theile modulus $\left(\phi = \sqrt{K_{\text{rec}}R^2/D}\right)$ in case of a rich and poor vascularized tumor interior, respectively. Theile modulus which signifies the ratio of interstitial diffusion time to the reversible nutrient ($\mathcal{N}$) metabolism by the tumor cells. When $\phi < 1$, except the peripheral region, the concentration of macromolecules of $\mathcal{N}$ maintains nearly a uniform value throughout the interstitium. In general, for this range of ϕ, rate of solute metabolism by the tumor cells is dominated by the rate of diffusion through the interstitial space. In other words, diffusion through tumor interstitium is much faster than the metabolic rate. On the other hand, increase in ϕ assists the metabolic rate of solute macromolecules. As a result, the concentration level decreases with increase in ϕ within the tumor interstitium. Since, solute macromolecules are metabolized by the cell population at a first order reversible rate, there is no chance of the starvation zone to occurs inside the tumor (starvation zone can occur for zeroth order consumption kinetics increasing values of $\phi > 1$. This suggests that those cells are present in the peripheral region receive sufficient nutrients in order to survive in contrary to the core region. This behavior clearly indicates the creation of necrosis within the tumor interior due to insufficient levels of nutrient. The concentration of solute nutrient decreases towards the center and the minimum value decreases with increase in ϕ.

5. Comparison with the clinical study

We refer [11] for mathematical modeling and experimental results corresponding to concentration data of injected Doxorubincin with tumor islet of radii 75, $105, 150$ μm at two different time levels 2hrs and 24hrs. We made suitable comparisons with these studies and show a good qualitative agreement in [7]. We wish to present in brief some of those results to give the reader an idea of how mathematical modeling could be compared to experimental results.

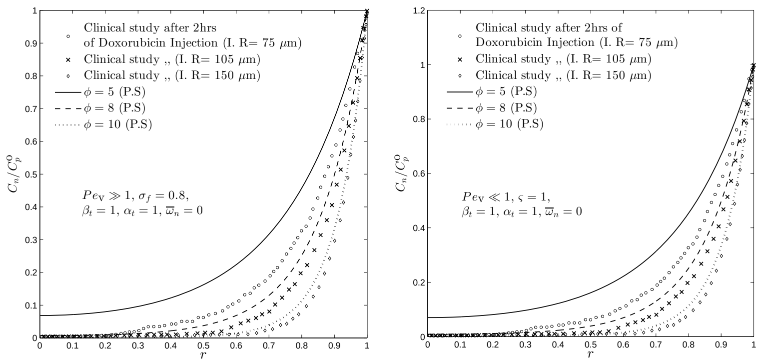
6. Remarks on this study

This talk introduces some aspects while modeling the hydrodynamics of solute macromolecules transport inside a tumor in the light of biphasic mixture theory. The presenter and his group are recently working on the applications of deformable porous media to biological tissues. One may refer [7] for details. We have briefed



(a) (b)

FIGURE 1. Dimensionless interstitial solute concentration ($C_n/C_{\mathrm{p}}^{\mathrm{o}}$) versus dimensionless radial distance (r) for different ϕ when (a) $Pe_{\mathrm{v}} \gg 1$ (b) $Pe_{\mathrm{v}} \ll 1$. Source: [7].



(a) cm (b)

FIGURE 2. Comparison between present study (P.S) and the clinical study done by [11] corresponding to (a) $Pe_{\mathrm{v}} \gg 1$ and (b) $Pe_{\mathrm{v}} \ll 1$ for different values of tumor islet radius (I.R). Source: [7].

here a specific model on characterizing necrosis formation which has potential applications in cancer treatment. A more detailed version is available at [1].

Acknowledgements. The author thanks the Academic Committee, 82^{nd} Annual Conference of IMS, for giving this opportunity. The author also thanks Dr Bibaswan Dey and Meraj Alam of the Department of Mathematics, IIT Kharagpur for the help they extended in creating this document out of my lecture.

REFERENCES

[1] Alam, Meraj, Dey, Bibaswan and Raja Sekhar, G. P., Well-posedness of poroelastodynamics inside a solid tumor, *Applicable Analysis*, 2017 : submitted.

[2] Baxter, Laurence T. and Jain, Rakesh K., Transport of uid and macromolecules in tumors, i. role of interstitial pressure and convection, *Microvascular research*, **37** (1) (1989), 77–104.

[3] Boucher, Yves, Baxter, Laurence T. and Jain, Rakesh K., Interstitial pressure gradients in tissue-isolated and subcutaneous tumors: implications for therapy, *Cancer Research*, **50** (15) (1990), 4478–4484.

[4] Brezzi, Franco and Fortin, Michel, *Mixed and hybrid nite element methods*, Vol. **15**, Springer Science & Business Media, 2012.

[5] Byrne, Helen and Preziosi, Luigi, Modelling solid tumour growth using the theory of mixtures, *Mathematical Medicine and Biology*, **20** (4) (2003), 341–366.

[6] Chauhan, Vikash P., Stylianopoulos, Triantafyllos, Boucher, Yves and Jain, Rakesh K., Delivery of molecular and nanoscale medicine to tumors: transport barriers and strategies, *Annual Review of Chemical and Biomolecular Engineering*, **2** (2011), 281–298.

[7] Dey, Bibaswan and Raja Sekhar, G. P., Hydrodynamics and convection enhanced macromolecular uid transport in soft biological tissues: Application to solid tumor, *Journal of theoretical biology*, **395** (2016), 62–86.

[8] Evans, Lawrence Craig, *Partial Dierential Equations*, Vol. **19**, American Mathematical Society, Providence, 2010.

[9] Jain, Aviral, Jain, Abhishek, Gulbake, Arvind, Hurkat, Pooja and Jain, Sanjay K., Solid tumors: A review, *Int J Pharm Pharm Sci.*, **3** (5) (2011), 45–51.

[10] Kuszyk, Brian S., Corl, Frank M., Franano, F. Nicholas, Bluemke, David A., Hofmann, Lawrence V., Fortman, Brian J. and Fishman, Elliot K., Tumor transport physiology implications for imaging and imagingguided therapy, *American Journal of Roentgenology*, **177** (4) (2001), 747–753.

[11] Lankelma, Jan, Luque, Rafael Fernandez, Dekker, Henk, Schinkel, Wim and Pinedo, Herbert M., A mathematical model of drug transport in human breast cancer, *Microvascular Research*, **59** (1) (2000), 149–161.

[12] Netti, Paolo A., Baxter, Laurence T., Boucher, Yves, Skalak, Richard and Jain, Rakesh K., Macro-and microscopic uid transport in living tissues: Application to solid tumors, *AIChE journal*, **43** (3) (1997), 818–834.

[13] Prakash, Jai, Raja Sekhar, G. P., De, Sirshendu and Böhm, Michael, Convection, diusion and reaction inside a spherical porous pellet in the presence of oscillatory ow, *European Journal of MechanicsB/Fluids*, **29** (6) (2010), 483–493.

[14] Preziosi, Luigi and Tosin, Andrea, Multiphase modelling of tumour growth and extracellular matrix interaction: mathematical tools and applications, *Journal of mathematical biology*, **58** (4) (2009), 625–656.

G. P. Raja Sekhar
Department of Mathematics
Indian Institute of Technology Kharagpur
Kharagpur-721 302, West bengal, India
E-mail *rajas@iitkgp.ac.in*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 29-43

GEODESIC CONJUGACIES OF SURFACES*

C. S. ARAVINDA

Abstract. A Geodesic conjugacy between two closed manifolds is a homeomorphism between their unit tangent bundles taking geodesic flow orbits of one to that of the other in a time-preserving manner. Recalling some of the essential notions as an introduction to the uninitiated, this expository article intends to provide a quick overview of results on closed surfaces – the most important illustrative case where a clear picture is available.

1. Background

Starting point in Riemannian Geometry is the metric tensor or the Riemannian metric on a smooth manifold from which arise several geometric invariants. It is an interesting prospect to understand the extent of influence that the geometric invariants cast on the Riemannian manifold itself. In other words: Are there invariants that determine the metric uniquely up to some canonical modification, at least in certain specific contexts?

In this article we look at one such invariant called the geodesic flow, explore what it means for it to determine the underlying smooth manifold under the notion of geodesic conjugacy and discuss some of the available answers.

Let us begin by crisply but succinctly going through these notions, essentially capturing their meaning without pretending to be precise with all the finer nuances involved.

In an informal sense, a manifold is usually described as an object that locally resembles an euclidean space. That is, if it is a 1-dimensional object such as a circle, then a sufficiently small neighbourhood around each of its points looks like an open interval in $\mathbb{R}$. Or, if it is 2-dimensional like the surface of a sphere, then around each of its points there is a neighbourhood that resembles an open disc in $\mathbb{R}^2$.

---

* This article is of an expository nature based on the 27^{th} V. Ramaswami Aiyar Memorial Award Lecture delivered at the 82^{nd} Annual Conference of the Indian Mathematical Society held at the University of Kalyani, Kalyani - 741 235, Nadia, West Bengal, India during December 27 - 30, 2016. The author expresses his deep gratitude to the IMS for the invitation to deliver this lecture.

2010 Mathematics Subject Classification : Primary 43A85; Secondary 22E30.
Key words and phrases: Geodesic flow, closed surfaces, geodesic conjugacy.

For example, to an explorer looking around on the surface of our planet earth trying to perceive its shape, it would be impossible to imagine that it is actually a sphere. The following graphic description, by Laura Ingalls Wilder in her book 'Little House on the Prairie', beautifully captures this sentiment:

> Kansas was an endless flat land covered with tall grass blowing in the wind. Day after day they travelled in Kansas, and saw nothing but rippling grass and the enormous sky. In a perfect circle the sky curved down to the level land, and the wagon was in the circle's exact middle.
>
> All day long Pet and Patty went forward, trotting and walking and trotting again, but they couldn't get out of the middle of that circle. When the sun went down, the circle was still around them and the edge of the sky was pink. Then slowly the land became black. The wind made a lonely sound in the grass. The camp fire was small and lost in so much space. But large stars hung from the sky, glittering so near that Laura felt she could almost touch them.
>
> Next day the land was the same, the sky was the same, the circle did not change. Laura and Mary were tired of them all. There was nothing new to do and nothing new to look at. . . .

Getting back to business, a smooth manifold is one on which the resemblance of being locally Euclidean is sharp enough to permit all the essential features of differential calculus. For example, it will be possible to talk about directional derivatives, smooth curves on these objects as well as vectors tangent to smooth curves.

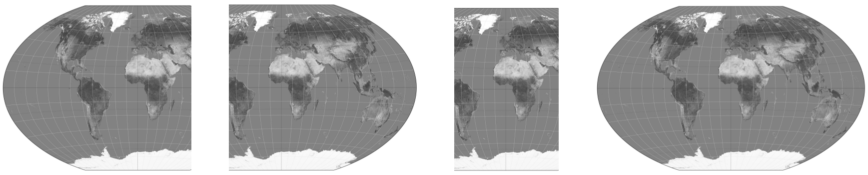
The prescription of a Riemannian metric on a smooth manifold further allows one to bring in metric considerations like lengths of tangent vectors and angles between tangent vectors, lengths of smooth curves, distance between two points and the all important notion of curvatures.

Formally, a *smooth n-dimensional manifold*, $n \in \mathbb{N}$, is a second countable, Hausdorff topological space M together with a family $\mathcal{U} = \{(\varphi_\alpha, U_\alpha)\}$ of homeomorphisms $\varphi_\alpha : U_\alpha \to M$ of open balls U_α in $\mathbb{R}^n$ into M, called coordinate charts, such that

(1) $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$
(2) For each pair of indices α, β with $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = W(\neq \emptyset)$, the sets $\varphi_\alpha^{-1}(W)$ and $\varphi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and the maps $\varphi_\beta^{-1} \circ \varphi_\alpha$ and $\varphi_\alpha^{-1} \circ \varphi_\beta$, called *transition maps*, are smooth; that is, are differentiable of any order.
(3) The family $\mathcal{U} = \{(\varphi_\alpha, U_\alpha)\}$ is maximal, in the sense that if (ψ, V) is any pair, with $\psi : V \to M$ a homeomorphism of an open ball V in $\mathbb{R}^n$ into M, compatible with conditions (1) and (2), then the pair (ψ, V) must also belong to the family $\mathcal{U}$.

Clearly, the condition (1) is a formal statement of the above description that, around each point $p \in M$ there exists a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$. It is the condition (2) which empowers us to do differential calculus on manifolds.

The Figure 1 below serves to illustrate the conditions (1) and (2), that a manifold is the union of the individual charts and that the intersecting regions of any two local charts are homeomorphic via appropriate maps.



$\varphi_\alpha(U_\alpha)$ $\varphi_\beta(U_\beta)$ W $\varphi_\alpha(U_\alpha) \cup \varphi_\beta(U_\beta)$

Figure 1

Conditions (1) and (2) also help in making sense of the smoothness of a map $f : M \to N$ from an m-dimensional manifold M to an n-dimensional manifold N by establishing appropriate maps from open sets in $\mathbb{R}^n$ to open sets in $\mathbb{R}^m$, and requiring them to be smooth. For example, the map f is smooth at a point $p \in M$ if the function $\psi^{-1} \circ f \circ \phi : U \to V$ is smooth at the point $\phi^{-1}(p)$, where (ϕ, U) is any chart containing p and (ψ, V) is any chart containing $f(p)$. The condition (2) that the transition maps are smooth ensures that the smoothness of f at p is independent of the choice of the coordinate charts. In particular, one can introduce the notion of smooth curves and vectors tangent to them at each of its points.

Condition (3) is a technical one needed for completeness of the collection $\{(\varphi_\alpha, U_\alpha)\}$, of compatibility with respect to the conditions (1) and (2). It follows from a simple application of the Zorn's lemma that any family $\{(\varphi_\alpha, U_\alpha)\}$ satisfying conditions (1) and (2) can always be extended to a maximal one with respect to the two conditions; hence, we do not make heavy weather of the condition (3).

Thus, a family $\{(\varphi_\alpha, U_\alpha)\}$ satisfying conditions (1) and (2) can be called a *differentiable structure*. A set M together with a differentiable structure is simply called a *differentiable manifold*.

If $c : (-1, 1) \to M$ is any smooth curve on a smooth n-manifold M with $c(0) = p$, then the vector tangent to the curve c at $c(0)$ is called a *tangent vector to M at the point $p \in M$*. In a natural way, a smooth structure equips the set T_pM of all tangent vectors at every point $p \in M$ with the structure of an n-dimensional vector space called the *tangent space at the point p*. A *vector field* V on M is an object that assigns a tangent vector $V(p)$ to each point $p \in M$; in a very definite sense, one can also talk about the smoothness of this assignment.

The metric tensor or the Riemannian metric is the prescription of a smoothly varying family of inner products on its tangent spaces. That is, each tangent space T_pM carries an inner product $\langle , \rangle_p$ that varies smoothly, in the sense that for every pair of smooth vector fields V, W on M the real-valued function $p \mapsto \langle V(p), W(p) \rangle_p$ is a smooth function on M. One can then look at the collection of all unit tangent vectors at all points of M, which is succinctly referred to as the unit tangent bundle, or the sphere bundle of M, and denoted by SM. In a natural way, the unit tangent bundle SM of an n-manifold M is itself a smooth manifold of dimension $2n-1$ and the map taking a $v \in SM$ to its foot point $p \in M$ is a smooth map from SM to M.

2. Geodesics on closed surfaces

Geodesics are special curves on a Riemannian manifold that minimize distance between two points locally in general, and globally in some specific situations.

On a flat plane surface, or the so called Euclidean plane, the straight line joining a pair of points realizes the distance between them. In other words, the straight lines are the geodesics on a plane surface.

On the surface of a sphere arcs of great circles are the geodesics. For example, on the surface of our planet earth, all the longitudes are geodesics whereas the equator is the only latitude that is also a great circle and hence a geodesic. From the north pole to a point on the equator, the corresponding longitudinal arc is the unique distance minimizing geodesic joining them; whereas, from the north pole to the south pole any of the infinitely many longitudinal segments minimizes the distance between them.

Figure 2a

Figure 2b

One studies in high school geometry that the sum of the interior angles of a triangle on the Euclidean plane is equal to 180^o. On the surface of a sphere, it turns out that the sum of the interior angles of a geodesic triangle is bigger than 180^o.

The celebrated discoveries of Lobachevsky and Bolyai, and the seminal work of Gauss on curved surfaces during the first half of the nineteenth century clearly demonstrated the existence of geometries other than the Euclidean and spherical geometries – the so called (noneuclidean) hyperbolic geometries – in which the sum of the interior angles of a geodesic triangle is less than 180^o.

One of the beautiful theorems of Gauss on geodesic triangles, called the Excess theorem, showed that the difference between 180° and the sum of interior angles of a geodesic triangle in either the spherical or the hyperbolic geometry is, in fact, proportional to the area of the geodesic triangle in that geometry. To illustrate this remarkable fact, let us do a small calculation from our knowledge of high school geometry.

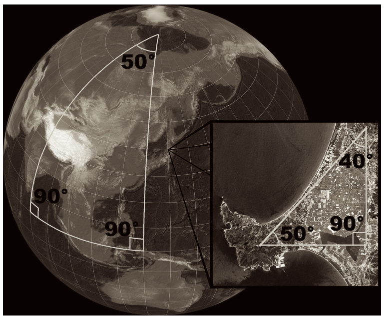
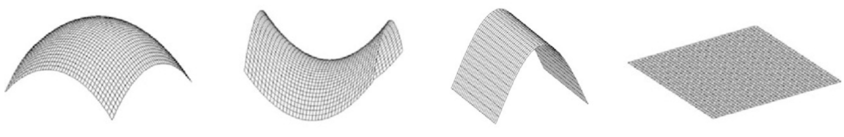


Figure 3

Consider a geodesic triangle on the unit sphere, similar to the one in Figure 3, but also making an angle of 90° or $\pi/2$ radians at the north pole. The sum of the interior angles of this geodesic triangle then is $3\pi/2$ which is $\pi/2$ in excess of π radians or 180°. And the surface area of this geodesic triangle, which is one-eighth of the entire sphere, will then be one-eighth of 4π which is exactly equal to $\pi/2$! If sphere were to be of radius r, then the angle excess would still be $\pi/2$ but the area of the triangle would be $\pi r^2/2$, i.e., proportional to $\pi/2$. Those who may have gotten very curious by now, should go ahead and check a few more cases to verify that this is not just an anomaly but indeed an amazing fact. One may also look up [11] for a detailed discussion.

The notion of Gaussian curvature is a very important geometric invariant that gives crucial information of the different ways the uneven terrain of a surface can bend around. To each point p on the surface, Gaussian curvature associates a certain real number $K(p) \in \mathbb{R}$, and depending on the sign of the Gaussian curvature $K(p)$ at p, whether it is positive, zero or negative, one can extract information about the shape of the surface near the point p. Figure 4 below illustrates the bending nature of a surface depending on the sign of Gaussian



$K(p) > 0$ $K(p) < 0$ $K(p) = 0$

Figure 4

curvature.

The surface of a torus metrically embedded in $\mathbb{R}^3$, that is, as one perceives it in the 3-dimensional space around us, is a classic example where all three signs of curvatures are present at different points.

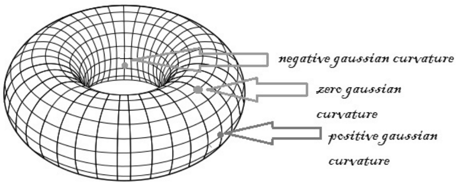


Figure 5

The pathbreaking work of Gauss triggered a brisk activity simultaneously at several places. One of them is the work of Bonnet who, in his paper of 1848 *Mémoire sur la théorie générale des surfaces*, proved a far-reaching generalization of Gauss's Excess theorem on geodesic triangles where, he considered polygons whose sides are not necessarily geodesic segments. Thus, for a closed surface S in $\mathbb{R}^3$, one was able to neatly assemble the local versions together and prove the celebrated global *Gauss-Bonnet formula*

$$\int_S K \; dA = 2\pi\chi(S),$$

where the integrand K is the Gaussian curvature function on the surface S and $\chi(S)$ is the well-known shape invariant, namely, the *Euler number* (also known as the Euler characteristic) of the surface S; and the symbol dA, like its one-variable version dx, is the *area element* or the so called area form. The Gauss-Bonnet formula reveals a deep and subtle interplay between geometry and topology of the underlying surface. We shall elaborate a bit on what this means.

A landmark theorem, essentially proven around the end of nineteenth century, about the classification of closed surfaces in $\mathbb{R}^3$ says that such a surface is necessarily homeomorphic to one of the sphere, torus or a surface of higher genus.

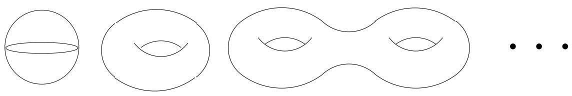


Figure 6

It turns out that the Euler number of the sphere is 2, while that of the torus is 0; and the surface with 2 holes, as in the figure above, has Euler number equal to -2. More generally, the Euler number of a surface with g number of holes is $2-2g$; that is, the Euler number of a surface of genus bigger than or equal to 2 is negative.

The sign of the Euler number of a closed surface in $\mathbb{R}^3$ has a bearing on the possible geometries it can admit. For example, the Gaussian curvature at every point on the unit sphere in $\mathbb{R}^3$ is equal to 1. Although one can deform the sphere into different shapes creating certain points where the Gaussian curvature is either 0 or negative, the Gauss-Bonnet formula forces the total curvature $\int_S K\ dA$ of any deformation of the sphere to be positive always; that is $\int_S K\ dA$ can only be positive because the right hand side of the Gauss-Bonnet formula says so. In other words, one will never be able to deform the unit sphere to a case where the Gaussian curvature at all of its points is simultaneously 0 or negative.

Thinking along in a similar fashion, one can essentially conclude that although one can deform the torus to situations where different points carry different signs of Gaussian curvature, the Gauss-Bonnet formula again shows that one can never come across a situation in which all of its points simultaneously carry either positive or negative Gaussian curvatures. However, it is potentially possible to think that one may be able to prescribe a Riemannian metric on the torus such that all of its points have Gaussian curvatures equal to 0. Does torus really admit such a metric? If so, how does one visualize the same?

A standard procedure to mathematically construct a torus, or the surface of an inflated bicycle tube, is illustrated by Figure 7 below:

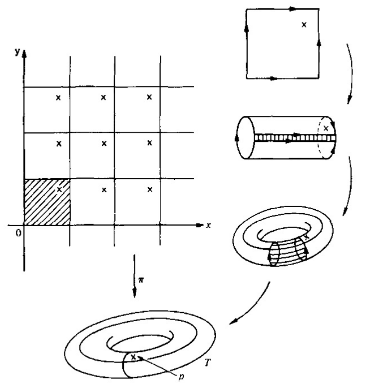


Figure 7

What the above sequence of pictures essentially conveys is that each point on the surface of the torus T in the figure can naturally be identified with an appropriate unique point in the shaded square representative set of points on the Euclidean plane. This is indicated by showing a sample identification of the point p on the torus T with the point marked $*$ in the shaded square region in the xy-coordinate plane. Stretching our imagination further, one can think of the tangent space at the point p on T as the Euclidean space whose origin is at the point marked $*$ in shaded square in the xy-coordinate plane.

In other words, we have been able to prescribe a Riemannian metric on the torus T, the inner product prescriptions being the same Euclidean inner product on the plane with only a change in the origin of reference. One can now further imagine that the Gaussian curvature at every point of the torus T with this specific Riemannian metric prescription is the same as that of a point in Euclidean plane which we know is the all important zero! What makes this particular procedure work beautifully is that the maps employed in enabling the identifications illustrated in Figure 7 are also Euclidean isometries, that is, maps preserving the Euclidean distances between any two points of the Eucldean plane. However, one must remark here that the torus equipped with this flat metric does not metrically embed in $\mathbb{R}^3$.

Does the above success give us a clue to similarly construct the most natural Riemannian metrics of constant negative curvature, known as the Hyperbolic metrics, on a surface of genus bigger than or equal to 2?

Before we have embarked on this next journey, we must first see at least one concrete model to understand the noneuclidean geometry of Bolyai and Lobachevsky, and see what its geodesics are.

One such beautiful model is the so called Poincaré disc in which the geodesics joining any two points in the disc are segments of either a diameter or that of a semi-circular arc which intersects the boundary of the disc at right angles. Figure 8 below illustrates some of these geodesics of Hyperbolic geometry.

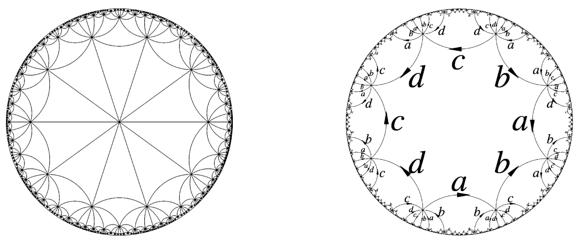


Figure 8

It is called the Poincaré disc because it was Poincaré who discovered this model. His discovery of this unit disk model is one of the most well-known accounts of a mathematical discovery. He wrote:

> At that moment I left Caen where I then lived, to take part in a geological expedition organized by the École des Mines. The circumstances of the journey made me forget my mathematical work; arriving at Coutances we boarded an omnibus for I don't know what journey. At the moment when I put my foot on the step the idea came to me, without anything in my previous thoughts having prepared me for it; that the transformations I had made use of to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify this, I did not have time for it, since scarcely had I sat down in the bus than I resumed the conversation

> already begun, but I was entirely certain at once. On returning to Caen I verified the result at leisure to salve my conscience.

A captivating art work of the brilliant dutch artist M.C. Escher inspired by how objects look in this geometry, and a picture from the eternal charm of nature below (see Figure 9) may serve to display a surprising presence of the beauty of Hyperbolic geometry in different walks of life.

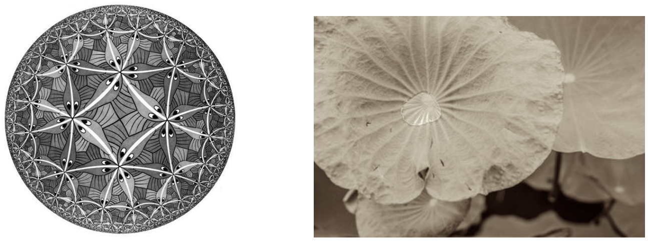


Figure 9

We are now ready to describe how closed surfaces of genus bigger than or equal to 2 naturally admit Hyperbolic geometries, in a manner very similar to how we saw that a torus admits a metric of zero Gaussian curvature or a Flat metric; that is, as a quotient space $\mathbb{E}^2/\mathbb{Z}^2$ of the Euclidean plane $\mathbb{E}^2$ where $\mathbb{Z}^2$ is the group generated by the two identifying Euclidean isometry maps marked by the two pairs a, b of opposite edges of the square in Figure 10 below. The only difference is that while we worked with the Euclidean plane $\mathbb{E}^2$ and identifications by Euclidean isometries in the case of torus, we will now work with the Hyperbolic plane $\mathbb{H}^2$ and identifications by Hyperbolic isometries for closed surfaces of genus bigger than 1.

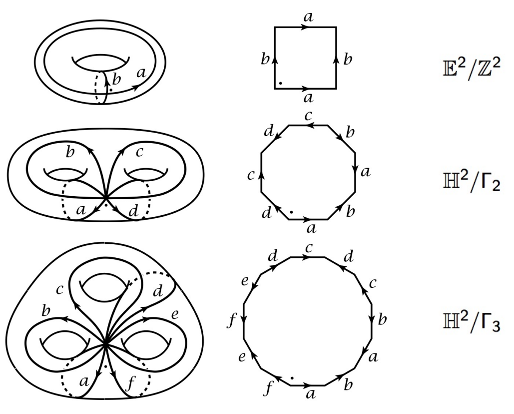


Figure 10

On that note, the middle picture above in Figure 10, together with the right side picture in Figure 8 (marked by identifying side specifications with a, b, c and d), may serve to graphically convey the idea of prescribing a Hyperbolic metric on a surface of genus 2 as a quotient space $\mathbb{H}^2/\Gamma_2$. The appropriate groups generated by the identifying Hyperbolic isometry maps are denoted by Γ_2 and Γ_3, with the subscripts indicating the number of holes present in the quotient spaces $\mathbb{H}^2/\Gamma_2$ and $\mathbb{H}^2/\Gamma_3$.

Proceeding in exactly a similar manner, it turns out to be possible that one can prescribe a Hyperbolic metric on a surface of genus $g > 1$ by realizing it as a quotient space $\mathbb{H}^2/\Gamma_g$ of the Hyperbolic plane $\mathbb{H}^2$, where Γ_g is the group generated by the appropriate identifying Hyperbolic isometry maps of a $4g$-sided Hyperbolic polygon. Once again, as remarked earlier for the flat torus, surfaces of higher genus equipped with Hyperbolic metrics do not metrically embed in $\mathbb{R}^3$.

3. The Geodesic flow, conjugate points and cut points

Given a point and a direction, geodesics through the given point and in the given direction exist uniquely as solutions of second order non-linear ODEs. In particular, their existence is guaranteed at least on some short time intervals. One may consult [4] as a general further reference for this as well as the next section.

However, for the case of closed surfaces that we are presently working with, it is a well-known fact that all geodesic solutions exist for all time, with only certain of its segments realizing the distance between any two given points. It is important to also mention here that the lengths of tangent vectors of a geodesic are always equal; in particular, if the length of any one of them is equal to 1 then so are the lengths of all its tangent vectors.

For purposes of our discussion in this article, it is more convenient to think of geodesics as orbits of a certain flow called the *Geodesic flow*. That is, as the orbits of a 1-parameter group ϕ_t, with $t \in \mathbb{R}$, of diffeomorphisms acting on the unit tangent bundle SM of M.

Given a point $p \in M$ and a unit direction v based at p, if γ_v is the unique geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$, then for each $t \in \mathbb{R}$, the diffeomorphism $\phi_t : SM \to SM$ is defined by

$$\phi_t((p, v)) = (\gamma_v(t), \gamma_v'(t)).$$

It follows easily that $\phi_0 = id_{SM}$, and that $\phi_s \circ \phi_t = \phi_{s+t} = \phi_t \circ \phi_s$, for all $s, t \in \mathbb{R}$.

There is an important consideration about geodesics on closed surfaces that we now discuss, and which will be relevant later in this article. That is, although all of the geodesics of the closed surfaces under consideration are defined on all of the real line $\mathbb{R}$, it is not quite true that lengths of all geodesic segments from $\gamma_v(0)$ to $\gamma_v(t)$ minimize the distance between the points $\gamma_v(0)$ and $\gamma_v(t)$ for all $t \in \mathbb{R}$.

Let us look at two typical situations where this happens. First of these arises in the case of the standard unit sphere S^2. On S^2, if γ_v is any geodesic starting from the north pole $\mathcal{N}$ in the direction of a unit vector v based at $\mathcal{N}$ then the length of any segment from $\gamma_v(0)$ to $\gamma_v(t)$ for $t > \pi$ is always bigger than the distance between $\gamma_v(0)$ and $\gamma_v(t)$. This is because the point $\gamma_v(\pi)$ will be nothing but the south pole $\mathcal{S}$, and the distance between any two points on S^2 is always less than or equal to π. Said differently, the south pole is the farthest possible point from the north pole. Furthermore, between the north pole and the south pole there is, in fact, a smooth 1-parameter family of geodesic segments all of which are of lengths equal to π, as indicated in the first of the three images in Figure 12.

In situations as above when, between a certain pair of points p, q in a Riemannian manifold M, there is a smooth 1-parameter family of geodesic segments joining p and q, the point q is said to be *conjugate to* p and vice-versa. In other words, one says that such a Riemannian manifold has conjugate points.

A second typical situation, different from the above, where a geodesic segment γ_v between a pair of points does not minimize the distance beyond a certain point from its starting point $\gamma_v(0)$, arises in the case of the torus $T^2 = \mathbb{E}^2/\mathbb{Z}^2$ equipped with the Flat metric described in the previous section. In Figure 11 below, the dark meridian and longitudinal circles on the torus are some of its typical geodesics. Moreover, even though the longitudinal circles appear to be of different lengths all of them have the same lengths with respect to the Flat metric under consideration.

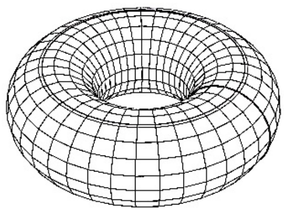


Figure 11

Indeed, the geodesics on the torus T^2 are quite simply images of geodesics on the Euclidean plane $\mathbb{E}^2$, under the quotient map from the Euclidean plane $\mathbb{E}^2$ to the torus $T^2 = \mathbb{E}^2/\mathbb{Z}^2$. It turns out that starting from any point on any of the meridian or longitudinal circles in the picture above, the particular geodesic stops minimizing the distance exactly half-way through on that circle. Furthermore, unlike as in the case of the unit sphere S^2, it turns out that there can be no smooth 1-parameter family of geodesic segments between two such points at half lengths on any of the meridian and longitudinal circles. Or for that matter, no such thing is possible between any pair of points on T^2. In other words, one says that there are no conjugate points on the Flat torus T^2.

The situation that the lengths of geodesic segments from the point $p = \gamma_v(0)$ to the point $\gamma_v(t)$ minimize the distance between them only for $t \in [0, t_v]$, where $t_v \in \mathbb{R}^+$ depends on the point $p \in M$ and $v \in T_pM$, hold for all geodesics γ_v, for all $p \in M$ and for all unit vectors $v \in T_pM$, is typically the case whenever M is a closed manifold. The point $\gamma_v(t_v)$ is called the *cut point* of p along the geodesic γ_v. The following important lemma from Riemannian geometry, that we only state here without a proof, gives a very useful characterization of cut points.

Lemma 3.1. *$\gamma_v(t_v)$ is the cut point of $p = \gamma_v(0)$ along γ_v if and only if one of the following holds for $t = t_v$ and neither holds for any smaller value of t:*

(1) $\gamma_v(t_v)$ is conjugate to p along γ_v, or

(2) there is a geodesic $\sigma \neq \gamma_v$ from p to $\gamma(t_v)$ such that $L(\sigma) = L(\gamma_v)$.

4. Geodesic conjugacies

If ϕ_t^M and ϕ_t^N denote the geodesic flows of two closed Riemannian manifolds M and N respectively, then a *time-preserving geodesic conjugacy* between M and N is a homeomorphism $F : SM \to SN$ such that the relation

$$F \circ \phi_t^M = \phi_t^N \circ F$$

holds for $t \in \mathbb{R}$.

Examples: *(i)* $N = M$, and $F = \phi_{t_0}$ for any $t_0 \in \mathbb{R}$. That is, any one of the diffeomorphism maps from the 1-parameter family of the geodesic flow ϕ_t^M of M.

(ii) $F = Df$ where $f : M \to N$ is any isometry, and Df is the differential map of f.

(iii) Compositions of *(i)* and *(ii)*.

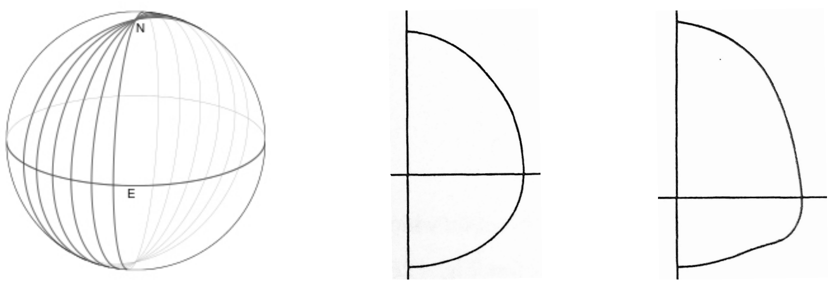
The following two questions below seem natural to explore:

Question 1: *If F is a time-preserving geodesic conjugacy between two closed manifolds M and N, are they necessarily isometric?*

Question 2: *If the answer to Question 1 is yes, then should F be necessarily of the form mentioned in the examples (i) to (iii)?*

The answer, in general, to both the questions is NO. However, in certain specific cases the answers are affirmative as we shall see.

4.1. **Answer to Question 1:** As we have seen before, all geodesics of the unit sphere S^2 are great circles with lengths equal to 2π. In other words, they are all closed and of same length.



Meridian profiles of examples of Zoll surfaces of revolution

Figure 12

A *Zoll surface* is a surface diffeomorphic to the unit sphere S^2, equipped with a Riemannian metric all of whose geodesics are closed and are of length equal to 2π. Otto Zoll, a student of David Hilbert, discovered the first non-trivial examples in his doctoral thesis [13] in 1903. The ones he constructed were surfaces of revolution around a fixed axis. While the unit sphere S^2 obviously has this property, it also has a family of geometrically distinct deformations that give rise to Zoll surfaces. In particular, Zoll surfaces other than the unit sphere do not have constant curvature and, hence, are not isometric to S^2. Figure 12 illustrates some examples.

It is a standard result in the geometry of surfaces that any two Riemannian metrics on S^2 are *conformal*. Two smooth inner product prescriptions g_0 and g_1 on a manifold M are said to be conformal if they are related by a smooth function $\lambda : M \to (0, \infty)$; that is, $g_1 = \lambda g_0$. Since all metrics on S^2 are conformal to the canonical round metric g_0, any smooth metric g on S^2 is of the form $g = \exp(\rho) g_0$ where $\rho : S^2 \to \mathbb{R}$ is a smooth function.

In 1913, Funk, another student of Hilbert, gave a necessary condition (see [6]) that if $g_t = \exp(\rho_t) g_0$ is a 1-parameter family of Zoll deformations on S^2, with $\rho_0 = 0$ and $(d\rho_t/dt)_{t=0} = \dot{\rho}$, then $\dot{\rho}$ must be an odd function with respect to antipodal map τ; that is, $\dot{\rho}(\tau p) = -\dot{\rho}(p)$ for every $p \in S^2$.

Essentially, what this also shows is the non-existence of Zoll deformations if the functions ρ_t are even; recall that when the functions ρ_t are even, the metrics g_t on S^2 induce metrics on the Real projective space $\mathbb{R}P^2$ since S^2 is a two-sheeted covering space of $\mathbb{R}P^2$. In other words, infinitesimally near the standard metric on $\mathbb{R}P^2$, there are no Zoll deformations. The non-existence of any non-trivial Zoll deformation on $\mathbb{R}P^2$ was later proved by L.W. Green in [7] in 1963.

It was not until 1976 that the above condition of the oddity of the initial derivative $\dot{\rho}$ of the 1-parameter family of functions ρ_t on S^2 was shown by V. Guillemin in [8] to be sufficient as well. That is, making use of this condition he proved the existence of many Zoll metrics on S^2 near the standard round metric. In his proof, Guillemin employed certain powerful tools from analysis like the Radon transform, the Nash-Moser implicit function theorem and the theory of Fourier Integral Operators.

In his paper [12] around the same time, A. Weinstein showed that if g_0 and g_1 are two Zoll metrics on S^2, then their geodesic flows are conjugate by a certain explicit diffeomorphism. Thus, the existence of non-trivial Zoll metrics on S^2 show that the answer to our Question 1 is NO in general.

4.2. **Answer to Question 2 :** First consider the case $M = T^2$, the 2-torus equipped with a flat metric, and let N be any closed surface geodesically conjugate to M. Croke showed [5] in 1990 that N is isometric to M, thereby giving an affirmative answer to Question 1 when M is a flat torus.

Recall that there are no conjugate points for the flat metric on the torus M. With the hypotheses as above, the proof involves showing first that the metric on N has no conjugate points. And, it is a classic theorem of E. Hopf (see [9]) that any metric on the torus which has no conjugate points must necessarily be flat.

The final conclusion is then drawn from the comparison of the so called marked length spectrums of the two surfaces M and N.

But as far as Question 2 is concerned, Croke constructed such geodesic conjugacies F of ST^2 to itself which are not of the form described in examples (i) to (iii), thus showing that answer to Question 2 is NO, in general. Basically, he constructed homeomorphisms F of ST^2 to itself that did not preserve fibers of ST^2.

BRIEF SKETCH OF THE CONSTRUCTION: Consider the flat torus $T^2 = \mathbb{E}^2/\mathbb{Z}^2$, and let (x, y) be the standard coordinates for $\mathbb{E}^2$ and θ the angle from the x-axis. It is a standard fact from topology that the unit tangent bundle ST^2 of T^2 is homeomorphic to the product bundle $T^2 \times S^1$. Thus, for the flat torus T^2 under consideration, $ST^2 = \{(x, y, \theta) \in \mathbb{E}^2/\mathbb{Z}^2 \times \mathbb{R}/2\pi\}$.

The tangent vector at the point $p = (x_0, y_0, \theta_0) \in ST^2$ of the geodesic flow vector field on ST^2 can be seen to be $\cos\theta_0 \frac{\partial}{\partial x}|_p + \sin\theta_0 \frac{\partial}{\partial y}|_p$.

Hence, when $a, b : S^1 \to \mathbb{R}$ are smooth functions of angle θ such that $a(0) = b(0) = 0$ and $(a(2\pi), b(2\pi)) \in \mathbb{Z}^2$, the maps $F : (x, y, \theta) \to (x + a(\theta), y + b(\theta), \theta)$ induce diffeomorphisms of ST^2 taking geodesic flow orbits to geodesic flow orbits. In other words, the maps F are also conjugacies of the geodesic flow.

Making appropriate choices for the functions a and b, such geodesic conjugacies F can be constructed that are not homotopic to a fiber preserving map. However, since the geodesic conjugacies considered in the examples *(i)* to *(iii)* are necessarily homotopic to fiber preserving maps, the aforementioned geodesic conjugacies F serve to provide, in general, otherwise examples to Question 2. □

Finally, when M is a closed surface of genus bigger than 1 equipped with a negatively curved metric, such as the ones described in section 2, Croke showed, in the same paper, that for any closed surface N geodesically conjugate to M the answers to both the Questions 1 and 2 are affirmative. Similar results were also obtained independently, around the same time, by J.-P. Otal in [10]. Sketching a proof of this result is, presently, beyond the scope of this article.

Both their proofs made essential use of the Gauss-Bonnet theorem, and the argument does not have an obvious generalization to the case of closed negatively curved manifolds of dimensions bigger than 2.

The best available result so far, in dimensions bigger than 2, is the theorem of Besson, Courtois and Gallot in [3], when one of the manifolds M is a closed, rank-1 locally symmetric space. They showed that the answer to Question 1 in such a case is affirmative.

In a joint work with V. Aithal and R. Binoy, we showed that the answer to Question 2 under the BCG setup is also affirmative.

References

[1] Aithal, V. T., Aravinda, C. S. and Binoy, R., A note on geodesic conjugacies, (unpublished preprint).

[2] Besse, A., *Manifolds all of geodesics are closed*, Ergebnisse Grenzgeb. Math., No. 93, Springer, Berlin 1978.

[3] Besson, G., Courtois, G. and Gallot, S., Entropies et rigidités des espaces localement symétriques de courbure strictement négative, *G.A.F.A.*, **5** (5) (1995), 731–799.

[4] Burns, K. and Gidea, M., *Differential Geometry and Topology: With a View to Dynamical Systems*, Studies in Advanced Mathematics, Chapman and Hall/CRC, 2005.

[5] Croke, C. B., Rigidity for surfaces of nonpositive curvature, *Comm. Math. Helv.*, **65** (1990), 150–169.

[6] Funk, P., Über Fläschen mit lauter geschlossenen geodätischen Linien, *Mathematische Annalen* **74** (1913), 278–300.

[7] Green, L. W., Auf Wiedersehensflächen, *Ann. Math.* **78** (1963), 289–299.

[8] Guillemin, V., The Radon transform on Zoll surfaces, *Advances in Mathematics*, **22** (1976), 85–119.

[9] Hopf, E., Closed surfaces without conjugate points, *Proc. Nat. Acad. Sci. U.S.A.*, **34** (1948), 47–51.

[10] Otal, J. -P., Le spectre marqué des longueurs des surfaces à courbure négative, *Ann. of Math.*, **131** (1990), 151–162.

[11] Pressley, A., *Elementary Differential Geometry*, Springer Undergraduate Mathematics Series, Indian Edition, 2001.

[12] Weinstein, A., *Fourier Integral Operators, Quantization and the Spectra of Riemannian Manifolds*, In Colloque International de Géométrie Symplectique et Physique Mathématique CNRS Aix (Juin 1974). Paris: Publication CNRS 1976.

[13] Zoll, O., Über Flächen mit Scharen geschlossener geodätischer Linien, *Math. Ann.*, **57** (1903), no. 1, 108–133.

C. S. Aravinda
TIFR Centre for Applicable Mathematics
P.O. Box 6503, Sharadanagar, GKVK Post
Bangalore - 560065, Karnataka, India
E-mail: *aravinda@math.tifrbng.res.in*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 45-50

A NOTE ON REEB FLOW SYMMETRY ON 3-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS

SRIMAYEE SAMUI

(Received : 28 - 07 - 2016; Revised : 07 - 11 - 2017)

Abstract. The object of this paper is to study 3-dimensional normal almost contact metric manifolds whose Ricci operator is invariant along Reeb flow.

1. Introduction

A $(2n+1)$-dimensional differentiable manifold M is called almost contact manifold if there is an almost contact structure $(\phi,\ \xi,\ \eta)$ consisting of an $(1,1)$ tensor field ϕ, a vector field ξ and an 1-form η satisfying

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta\circ\phi = 0. \tag{1.1}$$

We call ξ the Reeb vector field. An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\mathrm{M}^{(2n+1)} \times \mathbb{R}$ defined by

$$\mathrm{J}(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{1.2}$$

is integrable [1], where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $\mathrm{M}^{(2n+1)} \times \mathbb{R}$.

The condition for being normal is equivalent to vanishing of the torsion tensor

$$[\phi,\phi] + 2d\eta\otimes\xi,$$

where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ defined by

$$[\phi,\phi](X,Y) = [\phi X,\phi Y] + \phi^2[X,Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$$

for all vector fields X, Y.

Let g be a compatible Riemannian metric with (ϕ,ξ,η), that is,

$$g(X,Y) = g(\phi X,\phi Y) + \eta(X)\eta(Y), \tag{1.3}$$

or, equivalently,

$$g(X,\xi) = \eta(X), g(\phi X,Y) = -g(X,\phi Y) \tag{1.4}$$

for all vector fields X,Y. Then (M; ϕ, ξ, η, g) is called an almost contact metric manifold.

2010 Mathematics Subject Classification: 53C15, 53C25.
Key words and phrases: 3-dimensional normal almost contact metric manifolds, Reeb flow, β-Sasakian manifolds.

The fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X,\phi Y)$. An almost contact metric structure becomes a contact metric structure if

$$g(X,\phi Y) = \Phi(X,Y) = d\eta(X,Y) \tag{1.5}$$

holds for all vector fields X,Y.
3-dimensional normal almost contact metric manifolds have been studied by Welyczko, De and Mondal, De, Yildiz, and Yaliniz ([8], [3], [4], [5]) and many others. In [2] the authors studied Reeb flow symmetry on Kenmotsu three manifolds and almost Kenmotsu three manifolds.
Motivated by the above study, we study Reeb flow symmetry on 3-dimensional normal almost contact metric manifolds, that is, the manifolds which satisfy $L_\xi Q = 0$, where Q is the Ricci operator and obtain some results. Finally, we construct an example and verify the results.

2. Preliminaries

For a normal almost contact metric structure $(\phi,\ \xi,\ \eta,\ g)$ on M, we have [7]

$$(\nabla_X\phi)(Y) = g(\phi\nabla_X\xi, Y) - \eta(Y)\phi\nabla_X\xi, \tag{2.1}$$

$$\nabla_X\xi = \alpha\{X - \eta(X)\xi\} - \beta\phi X, \tag{2.2}$$

where $2\alpha = div\xi$ and $2\beta = tr(\phi\nabla\xi)$, $div\xi$ is the divergence of ξ defined as $div\xi = trace\{X \longmapsto \nabla_X\xi\}$ and $tr(\phi\nabla\xi) = trace\{X \longmapsto \phi\nabla_X\xi\}$. If $\alpha, \beta \in \mathbb{R}$ and $\alpha = 0$, the manifold reduces to a β-Sasakian manifold [6].
In normal almost contact metric manifolds, the following relations hold:

$$\begin{aligned} R(X,Y)\xi &= \{Y\alpha + (\alpha^2-\beta^2)\eta(Y)\}\phi^2X - \{X\alpha + (\alpha^2-\beta^2)\eta(X)\}\phi^2Y \\ &+ \{Y\beta + 2\alpha\beta\eta(Y)\}\phi X - \{X\beta + 2\alpha\beta\eta(X)\}\phi Y, \end{aligned} \tag{2.3}$$

$$S(Y,\xi) = -Y\alpha - (\phi Y)\beta - \{\xi\alpha + 2(\alpha^2-\beta^2)\}\eta(Y), \tag{2.4}$$

$$\xi\beta + 2\alpha\beta = 0, \tag{2.5}$$

where R denotes the curvature tensor and S is the Ricci tensor.
If $\alpha, \beta = constant$, from (2.5) it follows that the manifold is either α-Kenmotsu or β-Sasakian or cosympletic.
On the other hand, the curvature tensor in a 3-dimensional Riemannian manifold always satisfies

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= g(X,W)S(Y,Z) - g(X,Z)S(Y,W) + g(Y,Z)S(X,W) \\ &- g(Y,W)S(X,Z) - \frac{r}{2}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)], \end{aligned} \tag{2.6}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and r is the scalar curvature.
From (2.3) we have

$$\tilde{R}(\xi,Y,Z,\xi) = -(\xi\alpha + \alpha^2 - \beta^2)g(\phi Y,\phi Z) - (\xi\beta + 2\alpha\beta)g(Y,\phi Z). \tag{2.7}$$

By (2.3), (2.6) and (2.7) we obtain for $\alpha = constant$, $\beta = constant$ and

$$S(Y,Z) = (\frac{r}{2} + \alpha^2 - \beta^2)g(Y,Z) - (\frac{r}{2} + 3(\alpha^2-\beta^2))\eta(Y)\eta(Z). \tag{2.8}$$

Applying (2.8) in (2.6) we get

$$R(X,Y)Z = (\frac{r}{2} + 2(\alpha^2 - \beta^2))\{g(Y,Z)X - g(X,Z)Y\}$$
$$+g(X,Z)\{(\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(Y)\xi\} - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(Y)\eta(Z)X$$
$$-g(Y,Z)\{(\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(X)\xi\} + (\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)Y. \quad (2.9)$$

3. 3-dimensional normal almost contact metric manifolds with Reeb flow symmetry

Theorem 3.1. *A 3-dimensional normal almost contact metric manifold M satisfies $L_\xi Q = 0$ if and only if either M is a β-Sasakian manifold or scalar curvature $r = -6(\alpha^2 - \beta^2)$, provided $\alpha, \beta = constant$.*

Proof. Let M be a 3-dimensional normal almost contact metric manifold.
From (2.8) we have

$$S(X,Z) = (\frac{r}{2} + \alpha^2 - \beta^2)g(X,Z) - \big(\frac{r}{2} + 3(\alpha^2 - \beta^2)\big)\eta(X)\eta(Z), \quad (3.1)$$

where $\alpha = constant$ and $\beta = constant$.
It is known that $S(X,Z) = g(QX,Z)$, where Q is the Ricci operator.
From (3.1) we obtain

$$Q(X) = (\frac{r}{2} + \alpha^2 - \beta^2)X - \big(\frac{r}{2} + 3(\alpha^2 - \beta^2)\big)\eta(X)\xi. \quad (3.2)$$

From (3.2) we have

$$Q(\phi X) = \phi Q(X). \quad (3.3)$$

Now $(\nabla_Y Q)X = \nabla_Y QX - Q(\nabla_Y X)$. Hence using (3.2) we get

$$(\nabla_Y Q)X = \frac{1}{2}(Yr)(X - \eta(X)\xi) - \big(\frac{r}{2} + 3(\alpha^2 - \beta^2)\big)[g(\nabla_Y\xi, X)\xi + \eta(X)\nabla_Y\xi], \quad (3.4)$$

where α, β are constant. Using (2.2) in (3.4) we have

$$(\nabla_Y Q)X = \frac{1}{2}(Yr)(X - \eta(X)\xi) - \big(\frac{r}{2} + 3(\alpha^2 - \beta^2)\big)$$
$$[\alpha g(Y,X)\xi - 2\alpha\eta(Y)\eta(X)\xi - \beta g(\phi Y, X)\xi - \beta\eta(X)\phi Y + \alpha\eta(X)Y]. \quad (3.5)$$

It is known that

$$Xr = 2\sum_i (\nabla_{e_i} S)(e_i, X), \quad (3.6)$$

for any orthonormal basis $\{e_i\}$ of M.
From (3.1) and (3.6) we obtain

$$\xi r = -4\alpha\big(3(\alpha^2 - \beta^2) + \frac{r}{2}\big). \quad (3.7)$$

Let us consider M satisfies $(L_\xi Q)X = 0$; it then implies that

$$L_\xi QX - Q(L_\xi X) = 0. \quad (3.8)$$

From (3.8) we can state

$$\nabla_\xi QX - \nabla_{QX}\xi - Q(\nabla_\xi X - \nabla_X \xi) = 0. \quad (3.9)$$

Using (2.2) in (3.9) we get

$$\nabla_\xi QX - Q(\nabla_\xi X) - \alpha Q(X) + \alpha\eta(QX)\xi +$$
$$\beta\phi(QX) + \alpha Q(X) - \alpha\eta(X)Q(\xi) - \beta Q(\phi X) = 0. \tag{3.10}$$

From (3.3) and (3.10) we get

$$(\nabla_\xi Q)X = \alpha\eta(X)Q(\xi) - \alpha\eta(QX)\xi. \tag{3.11}$$

Using (3.2) the equation (3.7) turns into

$$(\nabla_\xi Q)X = 0. \tag{3.12}$$

From (3.12) and (3.5) we obtain

$$\xi r = 0. \tag{3.13}$$

Again from (3.13) and (3.7) we have either $\alpha = 0$, or, $r = -6(\alpha^2 - \beta^2)$. Hence, either M is a β-Sasakian manifold, or, $r = -6(\alpha^2 - \beta^2)$.

Conversely, let M be a β-Sasakian manifold. From the properties of β-Sasakian manifold we obtain

$$Q(X) = (\frac{r}{2} - \beta^2)X - \big(\frac{r}{2} - 3\beta^2\big)\eta(X)\xi \tag{3.14}$$

and

$$\nabla_X\xi = -\beta\phi X. \tag{3.15}$$

Taking covariant derivative of (3.14) we have

$$(\nabla_Y Q)X = \frac{1}{2}(Yr)(X - \eta(X)\xi) - \big(\frac{r}{2} - 3\beta^2\big)\big[-\beta g(\phi Y, X)\xi - \beta\eta(X)\phi Y\big]. \tag{3.16}$$

From (3.7) we get

$$\xi r = 0. \tag{3.17}$$

Again, from the definition of Lie derivative and (3.17) we can state the following:

$$\begin{aligned}(L_\xi Q)X = L_\xi QX - Q(L_\xi X) &= \nabla_\xi QX - \nabla_{QX}\xi - Q(\nabla_\xi X - \nabla_X\xi)\\ &= \nabla_\xi QX - Q(\nabla_\xi X) + \beta\phi(QX) - \beta Q(\phi X)\\ &= (\nabla_\xi Q)X = \frac{1}{2}(\xi r)(X - \eta(X)\xi) = 0. \end{aligned}\tag{3.18}$$

Therefore M satisfies $L_\xi Q = 0$. If $r = -6(\alpha^2 - \beta^2)$, it trivially follows that M satisfies $L_\xi Q = 0$. This completes the proof. □

4. Example

Let us consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Then consider the frame field $\{e_1, e_2, e_3\}$ of M defined by

$$e_1 = e^z\frac{\partial}{\partial x},\ e_2 = e^z\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Let us consider the Riemannian metric g on M w.r.t which $\{e_1, e_2, e_3\}$ becomes an orthogonal frame field on M. Define g by

$$g(e_i, e_j) = 1, \qquad for\ i = j, \qquad \text{and} \qquad = 0, \qquad for\ i \neq j.$$

Here i and j run from 1 to 3. Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any vector field Z tangent to M. Considering ϕ to be the $(1,1)$ tensor field defined by $\phi e_1 = -e_2,\ \phi e_2 = -e_1,\ \phi e_3 = 0$ and using the linearity property of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any vector field Z, W. Then for $e_3 = \xi$ the structure (ϕ, ξ, η, g) defines an almost metric structure on M.

Let ∇ be the Levi-Civita connection on M with respect to the metric g. Then we have

$$[e_1, e_2] = e^z \frac{\partial}{\partial x}(e^z \frac{\partial}{\partial y} - e^z \frac{\partial}{\partial y}(e^z \frac{\partial}{\partial x}) = e^z e^z \frac{\partial^2}{\partial x \partial y} - e^z e^z \frac{\partial^2}{\partial x \partial y} = 0.$$

Similarly

$$[e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2, \quad [e_2, e_1] = 0, \quad [e_3, e_1] = e_1, \quad [e_3, e_2] = e_2,$$

From Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{4.1}$$

Using (4.1) we have

$$\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3,$$

$$\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above relations we can state that the manifold satisfies (2.2) for $\alpha = -1$, $\beta = 0$ and $\xi = e_3$. So, the manifold is a normal almost contact metric manifold with $\alpha,\ \beta = \mathit{constant}$ and the manifold is not a β-Sasakian manifold.

It is known that Riemannian curvature tensor is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{4.2}$$

Using (4.2) we obtain

$$R(e_1, e_2)e_3 = 0, \quad R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_2)e_1 = e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_2, e_3)e_1 = 0,$$

$$R(e_1, e_3)e_3 = -e_1, \quad R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_1 = -e_3.$$

From the above results we have Ricci tensor

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2,$$

and Ricci operator

$$Q(e_1) = -2e_1, Q(e_2) = -2e_2, Q(e_3) = -2e_3. \tag{4.3}$$

Hence, from (4.3) we obtain $L_\xi Q = 0$. It is known that

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6 = -6(\alpha^2 - \beta^2),$$

and thus the Theorem 3.1 is verified.

References

[1] Blair, D. E., *Contact manifolds in Riemannian geemetry*, Lecture note in mathematics, vol. 509, Springer-Verlag, Berlin-New York, 1976.

[2] Cho, J. T. and Kimura, M., Reeb flow symmetry on almost contact three-manifolds, *Differential geometry and its applications*, **35**(2014), 266–273.

[3] De, U. C. and Mondal, A. K., On 3-dimensional normal almost contact metric manifolds satisfying certain curvature conditios, *Commun. Korean Math. Soc.*, **24**(2009), 265–275.

[4] De, U. C. and Mondal, A. K., The structure of some classes of 3-dimensional normal almost contact metric manifolds, *Bull. Malays. Math. Sci. Soc.*, **36** (2013), 501–509.

[5] De, U. C., Yildiz, A. and Yaliniz, A. F., Locally ϕ-symmetric normal almost contact metric manifolds of dimension 3, *Applied Mathematics Letters*, **22** (2009), 723–727.

[6] Janssens, D. and Vanheke, L., Almost contact structures and curvature tensors, *Kodai Math. J.*, **4** (1981), 1–27.

[7] Olszak, Z., Normal almost contact manifolds of dimension three, *Annales Pol. Math.*, **XLVII** (1986),41–50.

[8] Welyczko, J., On Legendre curves in 3-dimensional normal almost contact metric manifolds, *Soochow J. Math.*, **33** (2007), 929–937.

Srimayee Samui,
Department of Mathematics, Umeschandra college
13, Surya Sen Street, Kolkata-700012, West Bengal, India.
E-mail: *srimayee.samui@gmail.com*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 51-53

A NOTE ON THE IRREDUCIBILITY 0F POLYNOMIALS[1]

ANUJ JAKHAR

(Received : 28 - 02 - 2017; Revised : 11 - 07 - 2017)

Abstract. Let n be a positive integer and let $a_0, a_1, \cdots, a_n$ be arbitrary integers with $\gcd(a_i, n) = 1$ for all i. Then it is proved that $a_0 + a_1x + \frac{a_2x^2}{2!} + \cdots + \frac{a_nx^n}{n!}$ is irreducible over the field of rational numbers. As an application, some examples of irreducible polynomials are given.

1. Introduction

For $n \in \mathbb{N}, a_i \in \mathbb{Z}$ for all i, $0 \leq i \leq n$, let $f_n(x)$ denote the polynomial $a_0 + a_1x + \frac{a_2x^2}{2!} + \cdots + \frac{a_nx^n}{n!}$. In 1929, Schur proved the following result (cf. [1, Theoem 1], [2, Theorem 1]):

Theorem 1.A *Let n be a positive integer and let $a_0, a_1, \cdots, a_n$ be arbitrary integers with $|a_0| = |a_n| = 1$. Then $a_0 + a_1x + \frac{a_2x^2}{2!} + \cdots + \frac{a_nx^n}{n!}$ is irreducible.*

In 1996, M. Filaseta[1] used Newton polygons to establish the irreducibility of $f_n(x)$ in the case $|a_0| = 1, 0 < |a_n| < n$ and $n > 10$. He also noted that T. Y. Lam observed that $f_n(x)$ is irreducible if $\gcd(a_0a_n, n!) = 1$ and $n \geq 2$ (see [1, Theorem 4]). In this paper, using the theory of Newton Polygons, we provide a different condition on the coefficients of $f_n(x)$ which implies the irreducibility of $f_n(x)$. More precisely, we prove:

Theorem 1.1. *For $n \in \mathbb{N}$, let $a_0, a_1, \cdots, a_n$ be arbitrary integers with $\gcd(a_i, n) = 1$ for all i. Then $f_n(x) = a_0 + a_1x + \frac{a_2x^2}{2!} + \cdots + \frac{a_nx^n}{n!}$ is irreducible over $\mathbb{Q}$.*

The following result is an immediate consequence of the above theorem.

Corollary 1.2. *Let p be a prime number and let $a_0, a_1, \cdots, a_{p^k-1}, a_{p^k}$ be arbitrary integers not divisible by p with $k \geq 1$. Then $f_{p^k}(x) = a_0 + a_1x + \frac{a_2x^2}{2!} + \cdots + \frac{a_{p^k}x^{p^k}}{p^k!}$ is irreducible over $\mathbb{Q}$.*

As an application of the above results, some examples of irreducible polynomials are given in the end of the paper (see Examples 3.1 and 3.2).

---

[1] The financial support from IISER Mohali is gratefully acknowledged by the author.
2010 Mathematics Subject Classification : 12E05.
Key words and phrases : Polynomial Irreducibility, truncated exponential expansion.

2. Preliminary results

Let p be a prime number. For any non-zero integer r, $v_p(r)$ denote the highest power of p which divides r. For a polynomial $f(x)=\sum\limits_{j=0}^{n} b_jx^j \in \mathbb{Q}[x]$ with $b_0b_n \neq 0$, we associate a point $(i, v_p(b_i))$ to each non-zero term b_ix^i and form the set $S=\{(j, v_p(b_j)) | 0\leq j\leq n, b_j\neq 0\}$. As in [4, Chapter 9], the Newton polygon of $f(x)$ with respect to p is defined to be the polygonal path formed by the lower edges along the convex hull of points of S.

As usual for a real number λ, $[\lambda]$ stands for the largest integer not exceeding λ. Using the well-known equality $v_p(m!)=\sum\limits_{j=1}^{\infty}\left[\frac{m}{p^j}\right]$ for any $m\in\mathbb{N}$, the following simple result (cf. [3]) can be easily obtained. Its proof is omitted here.

Lemma 2.A. For any positive integer r with p a prime and $r=a_0+a_1p+\cdots+a_lp^l$, $0\leq a_i<p$, one has $v_p(r!)=\frac{r-(a_0+\cdots+a_l)}{p-1}$.

The following already known result will be used in the proof of Theorem 1.1 (see [5, Theorem 5.1.F]).

Theorem 2.B. *Let $l_1,\cdots,l_m$ denote the lengths of horizontal projections of successive edges and $\lambda_1<\cdots<\lambda_m$ the slopes of the edges of the Newton polygon of a polynomial $h(x)\in\mathbb{Z}[x]$ with respect to a prime p. Let $\tilde{v}_p$ denote the unique extension of the $p-adic$ valuation v_p on the field $\mathbb{Q}_p$ of p-adic numbers to the algebraic closure of $\mathbb{Q}_p$. Then $h(x)$ factors over $\mathbb{Q}_p$ as $h_1(x)h_2(x)\cdots h_m(x)$ where the degree of $h_i(x)$ is l_i, $i=1,2,\ldots,m$ and all the roots of $h_i(x)$ in the algebraic closure of $\mathbb{Q}_p$ have $\tilde{v}_p$ valuation λ_i.*

3. Proof of Theorem 1.1

Let p be a prime factor of n. Write $n=c_1p^{n_1}+c_2p^{n_2}+\cdots+c_kp^{n_k}$ where $n_1>n_2>\cdots>n_k>0$ and $0<c_i<p$. For $1\leq x_i\leq n$, let $x_i=c_1p^{n_1}+c_2p^{n_2}+\cdots+c_ip^{n_i}$. From Lemma 2.A and the fact that $\gcd(a_i,n)=1$ for all i, it can be easily seen that the vertices of the Newton polygon of $f_n(x)$ are $(x_i,-v_p(x_i!))$, $1<i<k$. So, the slopes λ_i of $f_n(x)$ are $(v_p(x_{i-1}!)-v_p(x_i!))/(x_i-x_{i-1})$. Again using Lemma 2.A, we see that

$$\lambda_i=\frac{-(p^{n_i}-1)}{p^{n_i}(p-1)}. \tag{3.1}$$

Let $n=\prod\limits_{j=1}^{t}p_j^{k_j}$ be the prime factorization of n. Since $p_j^{k_j}$ divides n, observe that for each j, $k_j\leq n_k<n_{k-1}<\cdots<n_1$. Therefore in view of (1), $p_j^{k_j}$ divides λ_i for each i. Now let $g(x)$ be any irreducible factor of $f_n(x)$ and α be a root of $g(x)$ in the algebraic closure of $\mathbb{Q}_p$, then by Theorem 2.B, $p_j^{k_j}$ divides the denominator of the valuation of α. It follows that $p_j^{k_j}$ divides the index of ramification[1] of the

[1]If $\widetilde{K}$ is an algebraic extension of a field K and $\tilde{v}$ is an extension of the valuation v from K to $\widetilde{K}$, then the index $[\tilde{v}(\widetilde{K}):v(K)]$ is called the ramification index of v in $\widetilde{K}/K$.

extension $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ which divides the degree of the extension which equals the degree of $g(x)$. Thus $p_j^{k_j}$ divides the degree of each factor of $f_n(x)$ over $\mathbb{Q}$ for all j, $1 \leq j \leq t$. Therefore $f_n(x)$ is irreducible over $\mathbb{Q}$.

Example 3.1. *Let* $f(x) = 9 + 3x + \frac{5x^2}{2} + \frac{7x^3}{3!} + \frac{11x^4}{4!}$, *then in view of Corollary* 1.2, $f(x)$ *is irreducible over* $\mathbb{Q}$.

Example 3.2. *Let* k, l *are positive integers. Assume that* $p_1, \cdots, p_{q^l}$ *are any prime numbers which are distinct with the prime number* q. *Then* $f(x) = \sum_{j=0}^{q^l} \frac{(\prod_{i=1}^{j} p_i^k)x^j}{j!}$ *is irreducible over* $\mathbb{Q}$ *by virtue of Corollary* 1.2.

References

[1] Filaseta, M., A generalization of an irreducibility theorem of I. Schur, *Analytic number theory*, **138** of Progr. Math., 371–396, Birkhäuser Boston, Boston, MA, 1996.

[2] Finch, C. and Saradha, N., On the irreducibility of certain polynomials with coefficients as product of terms in an arithmetic progression, *Acta Arith.*, **143** (2010), 211–226.

[3] Legendre, A. M., *Théorie des Nombres*, Firmin Didot Freres, Paris, 1830.

[4] Murty, M. Ram, *Introduction to p-adic analytic number theory*, Amer. Math. Soc., International Press, 2002.

[5] Ribenboim, P., *The Theory of Classical Valuation*, Springer-Verlag, New York, 1999.

Anuj Jakhar
Department of Mathematical Sciences
Indian Institute of Science Education and Research (IISER)
Mohali, Sector - 81, S.A.S. Nagar, Mohali - 140306, Punjab, India
Email: *anujjakhar@iisermohali.ac.in*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 55-58

FINITE SETS OF RATIONALS ARE ACHIEVABLE AS BERNOULLI PROBABILITIES

ARINDAM SENGUPTA

(Received : 02 - 03 - 2017; Revised : 12 - 08 - 2017)

Abstract. We continue investigating finite subsets A of [0,1] admitting a value $p \in (0,1)$ such that all the elements of A can be obtained as probabilities of events when a coin turning up heads with probability p is tossed a finite number of times and prove a result implying that any set of rational numbers lying in [0,1] is one such.

In an earlier article (Sengupta, [4]) in this journal, a partial answer to the question of characterizing (finite) subsets of the unit interval which could possibly arise as a set of probabilities of events that may occur when a coin is tossed a finite number of times, was given. Such sets were termed *achievable*, and a consequence (Corollary 2.1) of the theorem proved there was that all finite sets of rational numbers bounded above by $1/e$ are achievable. In this note, we prove a more general theorem that allows us to drop the bound $1/e$, thus concluding that all finite sets of rationals in [0,1] are achievable. This is stated as Corollary 1.

It is pertinent to briefly recall the essential definitions of Sengupta [4]. For $p \in (0,1)$ and $k \in \mathbb{N}$, $R_{p,k}$ stands for the range of the probability $P_p = P_{p,k}$ defined on subsets of the sample space $\Omega_k := \{H,T\}^k$ assigning probability $p^i(1-p)^{k-i}$ to any outcome in which H appears i times and T appears $(k-i)$ times; and R_p the (increasing) union of these sets over k. Specifically,

$$R_{p,k} = \left\{ \sum_{i=0}^{k} a_i p^i (1-p)^{k-i} : a_0, a_1, \ldots, a_k \in \mathbb{Z}^+, 0 \le a_i \le \binom{k}{i}, \, i = 1, 2, \ldots, k \right\};$$

and $R_p = \uparrow \cup_{k=1}^{\infty} R_{p,k}$. A finite subset $A = \{\alpha_1, \ldots, \alpha_m\}$ of [0,1] is called *achievable* if it is contained in an R_p or equivalently, an $R_{p,k}$; that is, if $\exists\, p \in (0,1)$, $k \in \mathbb{N}$ and events $E_1, \ldots, E_m \subseteq \Omega_k$ such that $P_p(E_j) = \alpha_j$ for $1 \le j \le m$.

We note in passing that denseness of R_p in [0,1] for any $p \in (0,1)$, as proved in Theorem 3 of Sengupta [4], is also a consequence of the de Moivre-Laplace central limit theorem (see e.g. Feller [1], Section VII.3).

Here, we establish

2010 Mathematics Subject Classification: 60C05, 60E05
Key words and phrases : Bernoulli trials, achievability, Poisson approximation.

Theorem 1. *If $m \geq 1$, $0 < \alpha_1 < \ldots < \alpha_m < 1 - 1/e$ and their pairwise ratios are rational then $\{\alpha_1, \ldots, \alpha_m\}$ is achievable.*

Before presenting the proof, we record the elementary point that the criterion of pairwise rational ratios is in no way necessary for achievability; for instance the set $\{\frac{1}{\sqrt{2}}, \frac{1}{2}\}$ is evidently achievable with $\frac{1}{\sqrt{2}}$ as a choice for p.

Proof. Define, for $n \geq 1$ and $k \geq n$, the function $g_{n,k} : [0,1] \to [0,1]$ as:

$$g_{n,k}(p) = k \sum_{i=1}^{n} \left\lfloor \frac{\binom{k}{i}}{k} \right\rfloor p^i (1-p)^{k-i}.$$

Let us first prove that for a fixed n, $g_{n,k}(k^{-1}) \to e_n := e^{-1}[1 + \frac{1}{2} + \cdots + \frac{1}{n!}]$ as $k \to \infty$. For this, we write $g_{n,k}(k^{-1})$ as $\sum_{i=1}^{n} t_{i,k}$, where

$$t_{i,k} := k \left\lfloor \frac{\binom{k}{i}}{k} \right\rfloor \left(\frac{1}{k}\right)^i \left(1 - \frac{1}{k}\right)^{k-i} \quad \text{for } 1 \leq i \leq k,$$

and claim that $t_{i,k} \to \frac{e^{-1}}{i!}$ as $k \to \infty$ for every fixed i.

To prove the claim, compare $t_{i,k}$ with $s_{i,k} := \binom{k}{i} \left(\frac{1}{k}\right)^i \left(1 - \frac{1}{k}\right)^{k-i}$. Clearly, $t_{1,k} = s_{1,k}$ and for $2 \leq i \leq n$, from the easily verified fact that the numbers

$$a_{i,k} := \frac{\binom{k}{i}}{k} = \frac{(k-1)\cdots(k-i+1)}{i!}$$

satisfy $a_{i,k} \to \infty$ as $k \to \infty$, we get

$$\frac{t_{i,k}}{s_{i,k}} = \frac{\lfloor a_{i,k} \rfloor}{a_{i,k}} \to 1.$$

But since $\binom{k}{i}$ is a polynomial in k with leading term $k^i/i!$, we have

$$s_{i,k} = \frac{\binom{k}{i}}{k^i} \left(1 - \frac{1}{k}\right)^k \left(1 - \frac{1}{k}\right)^{-i} \to \frac{e^{-1}}{i!},$$

establishing the claim.

It may be easily observed by readers familiar with the standard Poisson approximation to binomial probabilities (see e.g. Hoel, Port and Stone [2], Section 3.4.2) that $s_{i,k}$ is exactly the probability that a binomial variable with parameters k and $\frac{1}{k}$ takes the value i and as such, converges to the corresponding probability for a Poisson variable with parameter 1, as $k \to \infty$.

Evidently, e_n strictly increases to $1 - 1/e$ as $n \to \infty$. Hence it is possible to find n such that $e_n > \alpha_m$. Having obtained such an n, find $K \geq n$ such that $\forall\, k \geq K$, $g_{n,k}(k^{-1}) > \alpha_m$.

Now, the hypothesis implies the existence of $\gamma \in (0,1)$ and positive integers $k_1, k_2, \ldots, k_m$ such that $\alpha_j = k_j \gamma$, $j = 1, 2, \ldots, m$. Clearly, there is no loss of generality in assuming that $k_m \geq K$.

It follows that $g_{n,k_m}(k_m^{-1}) > \alpha_m$. But g_{n,k_m} being a polynomial, and hence a continuous function, taking $0 = g_{n,k_m}(0)$ also as a value, by the intermediate value theorem (see e.g. Rudin [3], Theorem 4.23) there exists $p \in (0, k_m^{-1})$ such that $g_{n,k_m}(p) = \alpha_m$. This means $\gamma = \alpha_m/k_m = \sum_{i=1}^{n} \left\lfloor \frac{\binom{k_m}{i}}{k_m} \right\rfloor p^i(1-p)^{k_m-i}$.

Consider now the experiment of tossing a coin k_m times, with this p as the probability of a head. The number of elementary events with probability $p^i(1-p)^{k_m-i}$ each, is $\binom{k_m}{i}$. Name these events $\{E_{i,l} : 1 \le l \le \binom{k_m}{i}\}$.

Now, since for each $j = 1, \ldots, m$ and $i = 1, \ldots, n$, the numbers

$$k_{j,i} := k_j \left\lfloor \frac{\binom{k_m}{i}}{k_m} \right\rfloor, \; 1 \le i \le n$$

satisfy $k_{j,i} \le \binom{k_m}{i}$, there are $k_{j,i}$ elementary events, e.g. $E_{i,1}, \ldots, E_{i,k_{j,i}}$, the probability of whose union $E_j^{(i)}$ is $k_{j,i}p^i(1-p)^{k_m-i}$.

Observe further that for $1 \le i_1 \ne i_2 \le n$, any two E_{i_1,l_1} and E_{i_2,l_2} are also clearly disjoint. Thus, if we set

$$E_j := \cup_{i=1}^{n} E_j^{(i)} = \cup_{i=1}^{n} \cup_{l=1}^{k_j} E_{i,l},$$

then

$$P_{p,k_m}(E_j) = \sum_{i=1}^{n} k_{j,i}\, p^i(1-p)^{k_m-i} = \frac{k_j}{k_m}\alpha_m = \alpha_j$$

for each $1 \le j \le m$. □

It is worth recording from the proof that p can be chosen arbitrarily close to 0; that is, given any $\epsilon > 0$, one could actually choose $p \in (0, \epsilon)$ to achieve the given set A. Also, in case k_m can be chosen from the set of prime numbers, the argument simplifies somewhat since for every $i \le n < k_m$, $\binom{k_m}{i}$ then itself becomes an integral multiple of k_m so that we have $t_{i,k_m} = s_{i,k_m}$. To illustrate this, let us make any choice of $m \le 19$ and integers $1 \le l_1 \le \cdots \le l_{m-1} \le 18$; then with $A = \{\frac{l_1}{\sqrt{911}}, \frac{l_2}{\sqrt{911}}, \ldots \frac{l_{m-1}}{\sqrt{911}}, \frac{19}{\sqrt{911}}\}$, one could choose $k_m = 19$ and $k_j = l_j$ for $1 \le j \le m-1$. In this case taking $n = 5$ suffices for the method described in the proof to apply since $\alpha_m = \frac{19}{\sqrt{911}} \approx 0.6295$ and $e_5 \approx 0.6315$. It turns out that $g_{5,19}(19^{-1}) \approx .6417$. Actually, $g_{4,19}(19^{-1}) \approx 0.6395 > \alpha_m$ already, so taking $n = 4$ would work as well. A specific case is afforded by the example $m = 2$ and $A = \{\frac{12}{\sqrt{911}}, \frac{19}{\sqrt{911}}\}$, say.

It may be noted however that both choosing an arbitrarily small p and retaining k_m to be prime cannot be effected simultaneously; for the former objective entails dividing γ by an arbitrarily large integer M, thereby inflating each k_j by the same M, and solving for p in $g_{n,Mk_m}(p) = a_m = \gamma k_m = (M \cdot k_m) \cdot \gamma/M$. Of course $M \cdot k_m$ is no longer a prime even if k_m were so.

Corollary 1. *Any finite set of rational numbers in* $[0, 1]$ *is achievable.*

Proof. By looking at complementary events if necessary, there is no loss of generality in assuming that the largest element in the set is at most $1/2 < 1 - 1/e$. The result follows. □

The special case of Corollary 1 appeared as Problem No. 11866 in the Problems and Solutions section of the American Mathematical Monthly, November 2015 issue (Vol. 122, No. 9, page 899).

Remark. The upper bound $1 - 1/e$ imposed on the set A in Theorem 1 can be removed. In short, the argument runs as follows: taking only $\alpha_m < 1$, it is possible to find $\lambda > 0$ such that $1 - e^{-\lambda} > \alpha_m$. Now, since the Poisson distribution with parameter λ can be approximated by a binomial distribution with parameters k and λ/k for k large enough, one considers $g_{n,k}(\lambda/k)$ rather than $g_{n,k}(1/k)$ for appropriate n and $k > max\{\lambda, n\}$, and argues as in the proof of Theorem 1 to obtain a k_m and a p such that $g_{n,k_m}(p) = \alpha_m$. The remainder of the proof goes exactly as in Theorem 1. We can therefore conclude

Theorem 2. If $m \geq 1,\ 0 < \alpha_1 < \cdots < \alpha_m < 1$ and their pairwise ratios are rational then $\alpha_1, \cdots, \alpha_m$ is achievable.

Acknowledgement. The suggestions made by an anonymous referee are gratefully acknowledged.

References

[1] Feller, W,. *An Introduction to Probability Theory and Its Applications, Vol. 1*, 3rd Ed (Revised Printing), John Wiley & Sons, (2000).

[2] Hoel, P. G., Port, S. C. and Stone, C. J., *Introduction to Probability Theory*, (1971). Indian Reprint (2000). UBPSD.

[3] Rudin, Walter, *Principles of Mathematical Analysis*, 3rd Ed (Indian). (2013). UBPSD.

[4] Sengupta, Arindam, The range of Bernoulli probabilities. *The Mathematics Student*. **82**, nos. 1–4 (2013), 183–189.

Arindam Sengupta
Department of Statistics, University of Calcutta
35 Ballygunge Circular Road
Kolkata-700 019, West Bengal, India.
E-mail: *arindamstat@gmail.com*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 59-65

SUBCLASS OF $C^{\infty}-$VECTORS OF AN OPERATOR MATRIX

M. P. SHEKHAWAT AND A. B. PATEL

(Received : 19 - 03 - 2017, Revised : 03 - 06 - 2017)

Abstract. We find bounded vectors, analytic vectors, quasi-analytic vectors and Stieltjes vectors for operator matrices having unbounded entries, and develop criterion for the self-adjointness of certain operator matrices.

1. Introduction

Let A be an operator, not necessarily bounded, in a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A)$, a dense subspace of $\mathcal{H}$. A vector $x \in C^{\infty}(A) = \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$ is called

(1) *bounded vector* of A, if there exists a constant $b_x > 0$ such that $||A^n x|| \leq b_x^n$ for each $n \in \mathbb{N}$,

(2) *analytic vector* of A, if $\sum_{n=1}^{\infty} \frac{t^n}{n!} ||A^n x|| < \infty$, for some $t > 0$,

(3) *quasi-analytic* vector of A, if $\sum_{n=1}^{\infty} ||A^n x||^{-\frac{1}{n}} = \infty$ and

(4) *Stieltjes* vector of A, if $\sum_{n=1}^{\infty} ||A^n x||^{-\frac{1}{2n}} = \infty$.

Denote by $\mathcal{D}^b(A)$, $\mathcal{D}^a(A)$, $\mathcal{D}^{qa}(A)$ and $\mathcal{D}^s(A)$, the collection of bounded vectors, analytic vectors, quasi-analytic vectors and Stieltjes vectors of A, respectively. It is clear that $\mathcal{D}^b(A) \subset \mathcal{D}^a(A) \subset \mathcal{D}^{qa}(A) \subset \mathcal{D}^s(A)$, and $\mathcal{D}^b(A)$ as well as $\mathcal{D}^a(A)$ are linear subspace of $C^{\infty}(A)$. Throughout in this paper, by a bounded operator B commuting with an operator A (not necessarily bounded), we mean $BA \subset AB$ (i.e $\mathcal{D}(BA) \subset \mathcal{D}(AB)$ and $B(Ax) = A(Bx)$ for all $x \in \mathcal{D}(BA)$, where $\mathcal{D}(BA) = \mathcal{D}(A)$ and $\mathcal{D}(AB) = \{x \in \mathcal{H} \ / \ Bx \in \mathcal{D}(A)\}$). Let $(\mathcal{X}, \ \|.\|_1)$ be a Banach space. For equivalent norms $\|.\|_2$ and $\|.\|_1$, x is an analytic vector, respectively bounded vector, quasi-analytic vector, Stieltjes vector of A with respect to $\|.\|_1$ if and only if x is an analytic vector, respectively bounded vector, quasi-analytic vector, Stieltjes vector of A with respect to $\|.\|_2$. Since $||(x, \ y)||_1 = ||x|| + ||y||$ and

The research work is supported by UGC-SAP-DRS-III.
2010 Mathematics Subject Classification: 47B25.
Keywords and Phrases: Operator matrices, analytic vector, quasi analytic vector, Stieltjes vector.

$||(x,\ y)||_2 = \sqrt{||x||^2 + ||y||^2}$ for $x,\ y \in \mathcal{H}$ are equivalent, we consider $\|.\|_1$ norm to prove the results.

We recall that a densely defined operator A on $\mathcal{H}$ is said to be symmetric if $\langle Ax,\ y\rangle = \langle x,\ Ay\rangle$ for all $x,\ y \in \mathcal{D}(A)$. An operator A on $\mathcal{H}$ is said to be self-adjoint, if A is densely defined and $A = A^*$. An operator A is closed if and only if following holds: If $\{x_n\}$ is a sequence in $\mathcal{D}(A)$ that is convergent in $\mathcal{H}$ and the sequence $\{A(x_n)\}$ is convergent in $\mathcal{H}$, then we have $\lim\limits_{n\to\infty} x_n \in \mathcal{D}(A)$ and $A(\lim\limits_{n\to\infty} x_n) = \lim\limits_{n\to\infty} A(x_n)$. We also note that for a symmetric operator A, if $x \in C^\infty(A)$ with $||x|| = 1$, then $\{\|A^n x\|^{\frac{1}{n}}\}$ is an increasing sequence [4].

The presence of these vectors in the domain of unbounded operator indicate some kind of nice property of the operator. They have provided a powerful tool for self-adjointness criterian. We investigate these classes of $C^\infty-$vectors for certain operator matrices.

2. $C^\infty-$Vectors of Operator Matrices

Theorem 2.1. *Let x and y be analytic vectors of A, and B be a bounded operator which commutes with A. Then X is an analytic vector of $\mathcal{M}$, where $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathcal{M} = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$.*

Proof. Clearly $\mathcal{D}(\mathcal{M}) = \mathcal{D}(A) \times \mathcal{D}(A)$. Then $\mathcal{M}^n X = \begin{pmatrix} A^n x + nBA^{n-1}y \\ A^n y \end{pmatrix}$.

Thus $\|\mathcal{M}^n X\| = \|A^n x + nBA^{n-1}y\| + \|A^n y\| \leq \|A^n x\| + \|A^n y\| + n\|B\|\|A^{n-1}y\|$. Since x and y are analytic vectors of A, there exists $t > 0$ such that

$$\sum_{n=1}^{\infty} \frac{t^n}{n!}\|A^n x\| < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{t^n}{n!}\|A^n y\| < \infty.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{t^n}{n!}\|\mathcal{M}^n X\| &\leq \sum_{n=1}^{\infty} \frac{t^n}{n!}\|A^n x\| + \sum_{n=1}^{\infty} \frac{t^n}{n!}\|A^n y\| + \sum_{n=1}^{\infty} n\frac{t^n}{n!}\|B\|\|A^{n-1}y\| \\ &= \sum_{n=1}^{\infty} \frac{t^n}{n!}\|A^n x\| + \sum_{n=1}^{\infty} \frac{t^n}{n!}\|A^n y\| + t\|B\| \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}\|A^{n-1}y\| \\ &< \infty.\end{aligned}$$

It follows that X is an analytic vector of $\mathcal{M}$. □

Theorem 2.2. *Let x be a quasi-analytic (respectively Stieltjes) vector of A, and B be a bounded operator which commutes with A. Then Y is a quasi-analytic (respectively Stieltjes) vector of $\mathcal{M}$, where $Y = \begin{pmatrix} x \\ 0 \end{pmatrix}$.*

Proof. Observe that $\|\mathcal{M}^n Y\| = \|A^n x\|$. If x is a quasi- analytic vector of A, then $\sum\limits_{n=1}^{\infty} \|A^n x\|^{-\frac{1}{n}} = \infty$. Therefore $\sum\limits_{n=1}^{\infty} \|\mathcal{M}^n Y\|^{-\frac{1}{n}} = \sum\limits_{n=1}^{\infty} \|A^n x\|^{-\frac{1}{n}} = \infty$. Thus Y is a quasi-analytic vector of $\mathcal{M}$. Proof for the case when x is Stieltjes vector is similar. □

Lemma 2.3. *Let x be an analytic vector of a symmetric operator A, and B be a bounded operator which commutes with A, then Y is an analytic vector of $\mathcal{K}$, where* $\mathcal{K}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}$.

Proof. It is enough to prove the result for $\|x\| = 1$. Observe that $\mathcal{K}Y = \begin{pmatrix} Ax \\ Bx \end{pmatrix}$ and $\mathcal{K}^2Y = \begin{pmatrix} A^2x + B^2x \\ BAx + ABx \end{pmatrix} = \begin{pmatrix} A^2x + B^2x \\ 2BAx \end{pmatrix}$. Hence

$$\mathcal{K}^3Y = \begin{pmatrix} A^3x + AB^2x + 2B^2Ax \\ BA^2x + B^3x + 2ABAx \end{pmatrix} = \begin{pmatrix} A^3x + 3B^2Ax \\ B^3x + 3BA^2x \end{pmatrix};$$

$$\mathcal{K}^4Y = \begin{pmatrix} A^4x + 3AB^2Ax + B^4x + 3B^2A^2x \\ BA^3x + 3B^3Ax + AB^3x + 3ABA^2x \end{pmatrix} = \begin{pmatrix} A^4x + 6B^2A^2x + B^4x \\ 4BA^3x + 4B^3Ax \end{pmatrix}.$$

Continuing in this way, we have

$$\mathcal{K}^nY = \begin{pmatrix} A^nx + \begin{pmatrix} n \\ 2 \end{pmatrix} B^2A^{n-2}x + \begin{pmatrix} n \\ 4 \end{pmatrix} B^4A^{n-4}x + \ldots. \\ \begin{pmatrix} n \\ 1 \end{pmatrix} BA^{n-1}x + \begin{pmatrix} n \\ 3 \end{pmatrix} B^3A^{n-3}x + \begin{pmatrix} n \\ 5 \end{pmatrix} B^5A^{n-5}x + \ldots. \end{pmatrix}.$$

Thus

$$\begin{aligned}
\|\mathcal{K}^nY\| &\le \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \|B\|^k \|A^{n-k}x\| \\
&= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \|B\|^{n-k} \|A^{k}x\| \\
&\le \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \|B\|^{n-k} \|A^{n}x\|^{\frac{k}{n}} \\
&= \left(\|B\| + \|A^nx\|^{\frac{1}{n}}\right)^n = \|A^nx\| \left(1 + \frac{\|B\|}{\|A^nx\|^{\frac{1}{n}}}\right)^n \le \|A^nx\| \left(1 + \frac{\|B\|}{\|Ax\|}\right)^n.
\end{aligned}$$

Since x is an analytic vector of A, there exists $t > 0$ such that $\sum\limits_{n=1}^{\infty} \frac{t^n}{n!}\|A^nx\| < \infty$. Let $t_1 = t/(1 + (\|B\|/\|Ax\|)$. Then $\sum\limits_{n=1}^{\infty} \frac{{t_1}^n}{n!} \|\mathcal{K}^nY\| \le \sum\limits_{n=1}^{\infty} \frac{t^n}{n!}\|A^nx\| < \infty$. Thus Y is an analytic vector of $\mathcal{K}$. □

Using similar arguments, we can also prove the following result.

Lemma 2.4. *Let y be an analytic vector of a symmetric operator A, and B be a bounded operator which commutes with A, then* $\begin{pmatrix} 0 \\ y \end{pmatrix}$ *is an analytic vector of $\mathcal{K}$.*

Since the set of analytic vectors of an operator forms a linear subspace, Lemma 2.3 and Lemma 2.4 implies the following result.

Theorem 2.5. *Let x and y are analytic vectors of a symmetric operator A, and B be a bounded operator which commutes with A. Then X is an analytic vector of $\mathcal{K}$.*

Theorem 2.6. *Let x be a quasi-analytic (respectively Stieltjes) vector of a symmetric operator A, and B be a bounded operator which commutes with A, then Y is a quasi-analytic (respectively Stieltjes) vector of $\mathcal{K}$.*

Proof. As in the proof of Lemma 2.3, we have $\|\mathcal{K}^n Y\| \leq \|A^n x\| \left(1 + \frac{\|B\|}{\|Ax\|}\right)^n$. So $\sum\limits_{n=1}^{\infty} \|\mathcal{K}^n Y\|^{-\frac{1}{n}} \geq \left(1 + \frac{\|B\|}{\|Ax\|}\right)^{-1} \sum\limits_{n=1}^{\infty} \|A^n x\|^{-\frac{1}{n}}$. Thus Y is a quasi-analytic vector of $\mathcal{K}$ if x is a quasi-analytic vector of A. □

Theorem 2.7. *Let A be a symmetric operator and B be a bounded operator which commutes with A, then $\mathcal{D}^s(A) \subset \mathcal{D}^s(A+B)$ and $\mathcal{D}^{qa}(A) \subset \mathcal{D}^{qa}(A+B)$.*

Proof. Let $x \in \mathcal{D}^s(A)$. We may assume that $Ax \neq 0$ and $||x|| = 1$. Then $\{||A^n x||^{\frac{1}{n}}\}$ is an increasing sequence. Now

$$\begin{aligned} ||(A+B)^n x|| &\leq \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} ||B||^{n-k} ||A^n x||^{\frac{k}{n}} \\ &= \left(||A^n x||^{\frac{1}{n}} + ||B||\right)^n \\ &= ||A^n x|| \left(1 + ||B||\, ||A^n x||^{-\frac{1}{n}}\right)^n \ \leq\ ||A^n x|| \left(1 + ||B||\, ||Ax||^{-1}\right)^n. \end{aligned}$$

Therefore $\sum\limits_{n=1}^{\infty} ||(A+B)^n x||^{-\frac{1}{2n}} \geq \left(1 + ||B||\, ||Ax||^{-1}\right)^{-\frac{1}{2}} \sum\limits_{n=0}^{\infty} ||A^n x||^{-\frac{1}{2n}} = \infty$.

Hence $\mathcal{D}^s(A) \subset \mathcal{D}^s(A+B)$. Proof of the second result is similar. □

Theorem 2.8. *Let A and D be symmetric operators and B be a bounded operator such that $BD \subset AB$, $B^*A \subset DB^*$. If x is a quasi-analytic vector of A and y is a bounded vector of D, then X is a quasi-analytic vector of $\begin{pmatrix} A & B \\ B^* & D \end{pmatrix} = \mathcal{S}$, say.*

Proof. Since x is a quasi-analytic vector of A and y is a bounded vector of D, there exists $b_y > 0$ such that $||D^n y|| \leq b_y^n$ for all n and $\sum\limits_{n=1}^{\infty} ||A^n x||^{-\frac{1}{n}} < \infty$. Let $\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Then $\mathcal{T}^n X = \begin{pmatrix} A^n x \\ D^n y \end{pmatrix}$ and hence

$$\|\mathcal{T}^n X\| = ||A^n x|| + ||D^n y|| \leq ||A^n x|| + b_y^n$$

so that

$$\sum_{n=1}^{\infty} \|\mathcal{T}^n X\|^{-\frac{1}{n}} \geq \sum_{n=1}^{\infty} \left(||A^n x|| + b_y^n\right)^{-\frac{1}{n}}.$$

Now let $E = \{n/\ ||A^n x|| \leq b_y^n\}$. If E is infinite set, then

$$\begin{aligned}\sum_{n=1}^{\infty} \|\mathcal{T}^n X\|^{-\frac{1}{n}} &\geq \sum_{n\in E} \left(2b_y^n\right)^{-\frac{1}{n}} + \sum_{n\notin E} \left(||A^n x|| + b_y^n\right)^{-\frac{1}{n}} \\ &\geq \frac{1}{2b_y} \sum_{n\in E} 1 + \sum_{n\notin E} \left(||A^n x|| + b_y^n\right)^{-\frac{1}{n}} \quad = \quad \infty.\end{aligned}$$

If E is a finite set, then there exists $k > 0$ such that $||A^n x|| > b_y^n$ for all $n \geq k$. Therefore, x being a quasi-analytic vector of A, we have

$$\sum_{n=1}^{\infty} \|\mathcal{T}^n X\|^{-\frac{1}{n}} \geq \sum_{n=1}^{k-1} \left(||A^n x|| + b_y^n\right)^{-\frac{1}{n}} + 2^{-\frac{1}{k}} \sum_{n=k}^{\infty} (||A^n x||)^{-\frac{1}{n}} = \infty,$$

It follows that X is a quasi-analytic vector of $\mathcal{T}$, and by Theorem 2.7, it is a quasi-analytic vector of $\mathcal{S}$. □

Corollary 2.9. *Let A be a symmetric operator, and B be a bounded operator which commutes with A. If x is a quasi-analytic vector of A and y is a bounded vector of A, then X is a quasi-analytic vector of* $\begin{pmatrix} A & B \\ B^* & A \end{pmatrix}$.

Theorem 2.10. *Let x and y be analytic vectors of A, and B be a bounded operator such that B and B^* commutes with A. Then X is an analytic vector of* $\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix} = \mathcal{U}$, *say.*

Proof. Observe that $\mathcal{U}X = \begin{pmatrix} Ax + By \\ B^*x \end{pmatrix}$, $\|\mathcal{U}X\| \leq ||Ax|| + ||B||(||x|| + ||y||)$;

$$\mathcal{U}^2 X = \begin{pmatrix} A^2x + ABy + BB^*x \\ B^*Ax + B^*By \end{pmatrix},$$

$$\left\|\mathcal{U}^2 X\right\| \leq (||A^2x|| + 2||B||\ ||Ax|| + ||B||^2||x||) + ||B||(||Ay|| + ||B||\ ||y||);$$

$$\mathcal{U}^3 X = \begin{pmatrix} A^3x + A^2By + ABB^*x + BB^*Ax + BB^*By \\ B^*A^2x + B^*ABy + B^*BB^*x \end{pmatrix},$$

$$\begin{aligned}\left\|\mathcal{U}^3 X\right\| &= ||A^3x + A^2By + ABB^*x + BB^*Ax + BB^*By|| \\ &\quad + ||B^*A^2x + B^*ABy + B^*BB^*x|| \\ &\leq (||A^3x|| + 3||B||\ ||A^2x|| + 3||B||^2||Ax|| + ||B||^3||x||) \\ &\quad + ||B||(||A^2y|| + 2||B||\ ||Ay|| + ||B||^2||y||).\end{aligned}$$

In general, we have

$$\begin{aligned}\|\mathcal{U}^n X\| &\leq \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} ||A^k x||\ ||B||^{n-k} + ||B|| \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} ||A^k y||\ ||B||^{n-1-k} \\ &\leq \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} ||A^n x||^{\frac{k}{n}} ||B||^{n-k} + ||B|| \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} ||A^{n-1} y||^{\frac{k}{n-1}} ||B||^{n-1-k}\end{aligned}$$

$$= \left(||A^n x||^{\frac{1}{n}} + ||B||\right)^n + ||B|| \left(||A^{n-1}y||^{\frac{1}{n-1}} + ||B||\right)^{n-1}$$

$$= ||A^n x|| \left(1 + \frac{||B||}{||A^n x||^{\frac{1}{n}}}\right)^n + ||B|| \, ||A^{n-1}y|| \left(1 + \frac{||B||}{||A^{n-1}x||^{\frac{1}{n-1}}}\right)^{n-1}$$

$$\leq ||A^n x|| \left(1 + \frac{||B||}{||Ax||}\right)^n + ||B|| \, ||A^{n-1}y|| \left(1 + \frac{||B||}{||Ay||}\right)^{n-1}.$$

Thus

$$\|\mathcal{U}^n X\| \leq M^n ||A^n x|| + M^{n-1}||B|| \, ||A^{n-1}y|| \leq M^n \left(||A^n x|| + ||B|| \, ||A^{n-1}y||\right),$$

where $M = max\{1 + ||B||/||Ax||, \ 1 + ||B||/||Ay||\}$. Since x and y are analytic vectors of A, there exists $t_1 > 0$ such that $\sum_{n=1}^{\infty} \frac{{t_1}^n}{n!} ||A^n x|| < \infty$ and $\sum_{n=1}^{\infty} \frac{{t_1}^n}{n!} ||A^n y|| < \infty$. It follows that for $t = t_1/M$, we have

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \|\mathcal{U}^n X\| \leq \sum_{n=1}^{\infty} \frac{{t_1}^n}{n!} ||A^n x|| + ||B|| t_1 \sum_{n=1}^{\infty} \frac{{t_1}^{n-1}}{(n-1)!} ||A^{n-1}y|| < \infty,$$

and hence X is an analytic vector of $\mathcal{U}$. □

3. Self-adjointness of Operator Matrices

C. Tretter [5, Proposition 2.3.6] has established a criterion for a symmetric operator matrix in Hilbert space to be essentially self-adjoint. Here we give another criterian for a closed symmetric operator matrix in a Hilbert space to be self-adjoint.

Lemma 3.1. *Let A and D be selfadjoint operators and B be a bounded operator. Then $\mathcal{S}$ is densely defined closed symmetric operator.*

Proof. Since A, D are densely defined operators and $\mathcal{D}(\mathcal{S}) = \mathcal{D}(A) \times \mathcal{D}(D)$, $\mathcal{S}$ is also densely defined operator. Now, $\left\langle \mathcal{S} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\rangle = \langle Ax + By, \ x_1 \rangle + \langle B^* x + Dy, \ y_1 \rangle = \langle x, \ Ax_1 \rangle + \langle y, \ B^* x_1 \rangle + \langle x, \ By_1 \rangle + \langle y, \ Dy_1 \rangle = \langle x, \ Ax_1 + By_1 \rangle + \langle y, \ B^* x_1 + Dy_1 \rangle = \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \mathcal{S} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\rangle$, where $(x, y), (x_1, y_1) \in \mathcal{D}(\mathcal{S})$. Thus $\mathcal{S}$ is a symmetric operator. Now let $\left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}$ be a sequence in $\mathcal{D}(S)$ converging to X in $\mathcal{H} \times \mathcal{H}$ such that $\mathcal{S}\left(\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right) \to \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Therefore $\begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Thus $Ax_n + By_n \to z_1$ and $B^* x_n + Dy_n \to z_2$. Since B is bounded, $Ax_n \to z_1 - By$ and $Dy_n \to z_2 - B^* x$. Since A and D are selfadjoint, we have $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(B)$ such that $z_1 - By = Ax$ and $z_2 - B^* x = Dy$. Therefore $\mathcal{S}X = \begin{pmatrix} Ax + By \\ B^* x + Dy \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Hence $\mathcal{S}$ is closed. □

Theorem 3.2. *Let A and D be selfadjoint operators. If B is a bounded operator such that such that $BD \subset AB$, then $\mathcal{S}$ is selfadjoint.*

Proof. By Theorem 3.1, $\mathcal{S}$ is closed densely defined symmetric operator. Since $BD \subset AB$, by (Theorem 4.19, [6]) $B^*A \subset DB^*$. Thus by Theorem 2.7, $\mathcal{D}^a(A) \times \mathcal{D}^a(D) \subset \mathcal{D}^s \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \subset \mathcal{D}^s\mathcal{S}$. Since A and D are selfadjoint operators, by (Lemma 5.1, [3]), $\mathcal{D}^a(A) \times \mathcal{D}^a(D)$ is dense in $\mathcal{H} \times \mathcal{H}$. Hence $\mathcal{D}^s(\mathcal{S})$ is dense in $\mathcal{H} \times \mathcal{H}$. Thus $\mathcal{S}$ is closed densely defined symmetric operator with $\mathcal{D}^s(\mathcal{S})$ dense. Hence $\mathcal{S}$ is a selfadjoint operator. □

References

[1] Chernoff, P. R., Some remarks on quasi-analytic vectors, *Trans. of Amer. Math. Soc.*, **167** (1972), 105–113.

[2] Chernoff, P. R., Quasi-analytic vectors and quasi-analytic functions, *Bull. Amer. Math. Soc.*, **81** (1975), 637–646.

[3] Nelson, E., Analytic vectors, *Ann. of Math.*, **70** (1959), 572–615.

[4] Nussbaum, A. E., A note on quasi-analytic vectors, *Studia Math.*, **33** (1969), 305–309.

[5] Tretter, C., *Spectral theory of block operator matrices and applications*, Imperial College Press, (2008).

[6] Weidmann, J., *Linear operators in Hilbert spaces*, Springer-Verlag, New York Inc., (1980).

M. P. Shekhawat
Department of Mathematics, M. P. Shah Arts & Science College
Surendranagar-363001, Gujarat, India
E-mail: *mahaveer.maths@gmail.com*

A. B. Patel
Department of Mathematics, Sardar Patel University
Vallabh Vidyanagar-388 120, Gujarat, India
E-mail: *abp1908@gmail.com*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 67-89

TRANSMISSION OF WAVES THROUGH A PLANE INTERFACE BETWEEN TWO DISSIMILAR THERMO-VISCOELASTIC HALF SPACES WITH VOIDS

JAI BHAGWAN AND S. K. TOMAR

(Received : 29 - 04 - 2017, Revised : 24 - 05 - 2017)

Abstract. Reflection and transmission phenomena of plane waves impinging obliquely at a perfect plane interface between two dissimilar thermo-viscoelastic half-spaces with voids have been investigated. A general formulation is constructed and two cases of incidences have been discussed: (a) when a set of coupled dilatational waves is made incident; (b) when a shear wave is made incident. Using Iesan's theory of thermo-viscoelastic material with voids, the appropriate boundary conditions at the interface are formulated and the equations yielding the amplitude ratios corresponding to various reflected and transmitted waves have been presented. The expressions of energy ratios corresponding to each reflected and transmitted waves are also presented. For a specific model, the variation of amplitude and energy ratios have been shown graphically against the angle of incidence. It is observed that the amplitude and energy ratios are influenced by the void and thermal parameters. At the interface, the incident energy is found to be distributed among interactional energies, reflected and transmitted energies at each angle of incidence.

1. Introduction

For the inspection of geological materials, the phenomenon of reflection and transmission of seismic waves from discontinuities is very helpful. In particular, the transmission of seismic waves has been proved a viable tool to explore the valuable materials like oils, waters, and hydrocarbons etc, buried beneath the earth surface. Thus, the seismic wave techniques are most reliable exploration method for oil/mineral industries. Knott [14] was perhaps the first who gave explicit formulae for reflection and transmission coefficients due to incidence of a plane wave at a welded contact interface between two distinct elastic half-spaces. Since then numerous problems concerning the transmission of plane waves have been attempted

2010 Mathematics Subject Classification: 74A35 · 74D05 · 74F05 · 74J05 · 74J20
Keywords and Phrases: Thermo-viscoelastic, void, plane wave, reflection, transmission, amplitude, energy.

by several researchers in the past. Some pioneer works in the pertinent area of research can be had in the books by Ewing et al. [9], Achenbach [1], Borcherdt [4], Aki and Richards [2], Sheriff and Geldart [18] among several others.

An elastic solid material with voids is a material whose skeleton is elastic solid and voids are small vacuous pores containing nothing of mechanical or energetic significance. These materials may be called porous material in the literature. Cowin and his coworker [17, 8] developed the theory of elastic material with voids, in which the bulk density of the material is written as a product of density of the matrix material and the void volume fraction field. This representation of bulk density introduces an extra kinematic variable in the theory corresponding to change in void volume fraction. Thus the deformation in the elastic material with voids is characterized by the displacement field and the change in void volume fraction field. The load transmitted through a surface element of the material is not only given by a stress force but also by an equilibrated force. Tomar [19] presented a review of the theory of elastic material with voids and briefly discussed the notable works done by several researchers.

Iesan [12] extended Cowin and Nunzaito's theory to include thermal effects and presented a theory of thermo-elastic material with voids. Here, Iesan did not consider the mechanical dissipation of the material and hence dropped the term corresponding to rate of change of void volume fraction. Some problems of reflection/transmission in thermo-elastic material with voids have been studied by Singh and Tomar [24], Kumar and Kumar [15], Singh [25, 26], Singh [27] etc. Iesan [13] further extended his theory and developed a linear theory of thermo-viscoelastic materials with voids, in which the set of independent constitutive variables includes the time derivative of strain tensor, void volume fraction field and gradient of void volume fraction field. In this way, Iesan generalizes Cowin and Nunziato theory [8] and his own earlier theory [12]. Following Iesan [13] theory, Tomar et al. [20] explored the possibility of plane wave propagation and studied the reflection phenomena of dilatational and shear waves at the free plane boundary surface of a thermo-viscoelastic half-space with voids. They found that four basic waves may travel with different speeds consisting of three sets of coupled dilatational waves and an independent shear wave. All these waves are found to be dispersive and attenuating. Chirita [6] studied the spatial behavior of the amplitude of steady-state vibrations in a thermo-viscoelastic porous medium. Later, Chirita and Dansecu [7] studied surface waves in a linear thermo-viscoelastic half-space with voids. Bhagwan and Tomar [3] studied the reflection and transmission phenomena due to incidence of a set of coupled dilatational waves striking obliquely at the welded contact plane interface between a uniform elastic solid half-space and a thermo-viscoelastic solid half-space with voids. Recently, few similar problems have been investigated by taking different media, e.g., Tomar and Goyal [21] explored the propagation of time harmonic waves in swelling porous elastic medium of infinite extent and investigated the reflection of body waves from the stress free boundary surface surface of a swelling porous half-space. Later Goyal and Tomar [10, 11] extended their reflection problem and investigated the reflection/refraction due to incidence of a dilatational wave at plane interface between elastic/porous and porous/porous media. They showed that due to the presence of swelling phenomena, there exist an additional shear wave and the reflection/transmission from the boundary surface of the considered models is affected significantly by the swelling character of the porous media. The work by Sharma and Kumar [23] is worth notable, in which they extended Iesan's equations to incorporate Lord-Shulman theory of thermo-elasticity [16] and studied propagation of plane waves.

The layout of the present paper is as follows: Section-1 contains the introductory remarks and some works done on wave propagation in the pertinent area of research. Section-2, recapitulates the governing equations and constitutive relations of linear homogeneous thermo-viscoelastic material with voids and a brief account of wave propagation in this material. Section-3 contains the general problem formulation for the reflection and transmission of an obliquely incident plane wave (dilatational or shear) at the plane interface between two dissimilar thermo-viscoelastic half-spaces with voids. Appropriate boundary conditions at the interface are set up and the equations yielding the amplitude ratios of various reflected and transmitted waves are presented. Bulk energies carried along incident wave, reflected waves, transmitted waves and the interactional energies between dissimilar pairs of waves are given in matrix form. In Section-4, the problems already studied by Tomar et al. [20] have been recovered from the present formulation. For a specific model composed of copper materials, the numerical computations have been performed to compute amplitude ratios, interactional energies and energy ratios carried along each of the reflected and refracted waves at the interface. The computed results have been displayed graphically and discussed in Section-5. Important conclusions drawn from the present study have been mentioned in Section-6.

2. Basic relations and equations

Following Iesan [13], for a uniform thermo-viscoelastic material with voids, the constitutive relations are given by

$$t_{ij} = (\lambda_0 e_{rr} + b_0\phi - \beta\theta)\,\delta_{ij} + 2\mu_0 e_{ij}, \tag{2.1}$$

$$H_i = \alpha_0\phi_{,i} + \tau^*\theta_{,i}, \tag{2.2}$$

$$g = -\nu_0 e_{rr} - \xi_0\phi + m\theta, \tag{2.3}$$

$$\rho_0\eta = \beta e_{rr} + a\theta + m\phi, \tag{2.4}$$

$$Q_i = k_t\theta_{,i} + \zeta\dot{\phi}_{,i}, \tag{2.5}$$

and the governing equations are given by

$$\mu_0\nabla^2 u_i + (\lambda_0 + \mu_0)u_{j,ji} + b_0\phi_{,i} - \beta\theta_{,i} = \rho_0\ddot{u}_i, \tag{2.6}$$

$$\alpha_0\nabla^2\phi - \nu_0 u_{j,j} - \xi_0\phi + \Gamma_1\theta = \rho_0\kappa\ddot{\phi}, \tag{2.7}$$

$$k_t\nabla^2\theta - \beta T_0\dot{u}_{j,j} - c\dot{\theta} + \Gamma_2\phi = 0, \tag{2.8}$$

where t_{ij} is the force stress tensor, H_i is the equilibrated stress vector, g is the equilibrated body force, η is the entropy, Q_i is the heat flux vector, and

$$\mu_0 = \mu + \mu^*\frac{\partial}{\partial t}, \quad \lambda_0 = \lambda + \lambda^*\frac{\partial}{\partial t}, \quad b_0 = b + b^*\frac{\partial}{\partial t}, \quad \alpha_0 = \alpha + \alpha^*\frac{\partial}{\partial t}, \quad c = aT_0,$$
$$\nu_0 = b + \gamma^*\frac{\partial}{\partial t}, \quad \xi_0 = \xi + \xi^*\frac{\partial}{\partial t}, \quad \Gamma_1 = \tau^*\nabla^2 + m, \quad \Gamma_2 = (\zeta\nabla^2 - mT_0)\frac{\partial}{\partial t},$$

λ and μ are well known Lame's parameters; b, α, ξ and ξ^* are void parameters; β, τ^*, m, k_t, ζ and a are thermal parameters; λ^*, μ^*, b^*, α^* and γ^* are viscoelastic parameters; ρ_0 is density of the medium, $u_i(x_i, t)$ are the components of displacement vector $\mathbf{u}$, ϕ is the change in void volume fraction, θ is the small change in temperature from the constant ambient temperature T_0. Other symbols have their usual meaning and borrowed from [20]. In writing the governing equations, we have neglected the body force and heat source densities. For the stability of thermodynamic state, the following restrictions on material moduli must hold

$$3\lambda^* + 2\mu^* \geqslant 0, \quad \mu^* \geqslant 0, \quad \alpha^* \geqslant 0, \quad k_t \geqslant 0, \quad \xi^* \geqslant 0, \quad T_0\left(\tau^* + \frac{1}{T_0}\zeta\right)^2 \leqslant 4\alpha^* k_t.$$

Introducing standard length (l_0), standard velocity (c_0) and the following non-dimensional quantities

$$x_i' = \frac{x_i}{l_0}, \quad u_i' = \frac{u_i}{l_0}, \quad t' = \frac{c_0 t}{l_0}, \quad \omega' = \frac{l_0\omega}{c_0}, \quad T' = \frac{\theta}{T_0}, \tag{2.9}$$

the field equations (2.6)-(2.8) can be written as (after suppressing the primes)

$$\mu_1\nabla^2 u_i + (\lambda_1 + \mu_1)u_{j,ji} + b_1\phi_{,i} - \beta_1 T_{,i} + \mu_1^*\nabla^2\dot{u}_i + (\lambda_1^* + \mu_1^*)\dot{u}_{j,ji} + b_1^*\dot{\phi}_{,i} = \ddot{u}_i, \tag{2.10}$$

$$\alpha_1\nabla^2\phi - b_1 u_{j,j} - \xi_1\phi + m_1 T + \alpha_1^*\nabla^2\dot{\phi} + \tau_1^*\nabla^2 T - \gamma_1^*\dot{u}_{j,j} - \xi_1^*\dot{\phi} = r_1\ddot{\phi}, \tag{2.11}$$

$$k_1\nabla^2 T - (1/\tau)\left(\beta_1\dot{u}_{j,j} + m_1\dot{\phi}\right) - \dot{T} + \zeta_1\nabla^2\dot{\phi} = 0, \tag{2.12}$$

where

$$\mu_1 = \frac{\mu}{\rho_0 c_0^2}, \quad \lambda_1 = \frac{\lambda}{\rho_0 c_0^2}, \quad b_1 = \frac{b}{\rho_0 c_0^2}, \quad \beta_1 = \frac{T_0\beta}{\rho_0 c_0^2}, \quad \mu_1^* = \frac{\mu^*}{\rho_0 l_0 c_0},$$
$$\lambda_1^* = \frac{\lambda^*}{\rho_0 l_0 c_0}, \quad b_1^* = \frac{b^*}{\rho_0 l_0 c_0}, \quad \alpha_1 = \frac{\alpha}{\rho_0 l_0^2 c_0^2}, \quad \xi_1 = \frac{\xi}{\rho_0 c_0^2}, \quad m_1 = \frac{T_0 m}{\rho_0 c_0^2},$$
$$\alpha_1^* = \frac{\alpha^*}{\rho_0 l_0^3 c_0}, \quad \tau_1^* = \frac{T_0\tau^*}{\rho_0 l_0^2 c_0^2}, \quad \gamma_1^* = \frac{\gamma^*}{\rho_0 l_0 c_0}, \quad \xi_1^* = \frac{\xi^*}{\rho_0 l_0 c_0}, \quad r_1 = \frac{\kappa}{l_0^2},$$
$$k_1 = \frac{k_t}{l_0 c_0 c}, \quad \tau = \frac{cT_0}{\rho_0 c_0^2}, \quad \zeta_1 = \frac{\zeta}{cT_0 l_0^2}.$$

2.1. **Wave propagation.** By means of Helmholtz decomposition theorem, the displacement vector $\mathbf{u}$ can be expressed as

$$\mathbf{u} = \nabla p + \nabla \times \mathbf{q}, \quad \nabla \cdot \mathbf{q} = 0, \tag{2.13}$$

where p is the scalar potential corresponding to dilatational wave and $\mathbf{q}$ is the vector potential corresponding to shear wave. Tomar et al. [20] have shown that by inserting (2.13) into equations (2.10)-(2.12), one obtains two sets of equations: one corresponding to dilatational wave potentials and other corresponding to shear wave potential. On solving these sets of equations, it has been shown that there exist four basic waves consisting of three coupled dilatational waves, namely, P_i-waves ($i = 1, 2, 3$) and a lone shear wave, namely, $S-$wave propagating with distinct speeds v_i and v_4 respectively in an infinite thermo-viscoelastic material with voids. The expressions of speeds v_i and v_4 are given by

$$v_i^2 = -\omega^2 z_i^{-1} \quad \text{and} \quad v_4^2 = \frac{\mu_1 + \iota\omega\mu_1^*}{\rho_0}, \tag{2.14}$$

where z_i are the roots of the following cubic equation

$$AZ^3 + \omega^2 BZ^2 + \omega^4 CZ^2 + \omega^6 D = 0. \tag{2.15}$$

The expression of various coefficients occurring in (2.15) are given in Tomar et al. [20]. They have also obtained the following expressions

$$G_i = \frac{A_2 - B_2 v_i^2}{A_1 - B_1 v_i^2 + C_1 v_i^4}, \qquad F_i = \frac{A_3 - B_3 v_i^2}{A_1 - B_1 v_i^2 + C_1 v_i^4}, \tag{2.16}$$

representing the coupling coefficient between temperature, void and displacement potentials. Note that all the waves explored here are dispersive and attenuating.

3. Reflection/transmission phenomena

Consider a cartesian co-ordinate system $oxyz$ such that x and $y-$axes are along the horizontal directions and $z-$axis is pointing vertically downwards. Let H^1 and H^2 be the two half-spaces filled with distinct thermo-viscoelastic material with voids and separated by the plane interface $z = 0$. Thus the regions of half-spaces are given by: $H^1 = \{(x, y, z) : -\infty < x, y < \infty,\ 0 < z < \infty\}$, and $H^2 = \{(x, y, z) : -\infty < x, y < \infty,\ -\infty < z < 0\}$. We denote that the quantities with prime ($'$) correspond to the half-space H^2, while those without prime correspond to the half-space H^1. We assume that these half-spaces are in welded contact at the interface $z = 0$. Owing to (2.13) and using the potentials (p, p') and ($\mathbf{q}$, $\mathbf{q}'$) corresponding to vectors ($\mathbf{u}$, $\mathbf{u}'$), the relevant components of displacements in two-dimensional $x - z$ plane are given by

$$\{u,\ u'\} = \left\{\frac{\partial p}{\partial x} - \frac{\partial q}{\partial z},\ \frac{\partial p'}{\partial x} - \frac{\partial q'}{\partial z}\right\}, \quad \{w,\ w'\} = \left\{\frac{\partial p}{\partial z} + \frac{\partial q}{\partial x},\ \frac{\partial p'}{\partial z} + \frac{\partial q'}{\partial x}\right\}. \tag{3.1}$$

To investigate the reflection and transmission phenomena due to (a) incidence of a set of coupled dilatational waves propagating with speed v_1, (b) incidence of a shear wave propagating with speed v_4, we shall construct a general formulation of the incidence of these waves as follows.

Let an inhomogeneous plane wave (P_1- wave or $SV-$ wave) propagating through the lower half-space (H^1) having amplitude A_0 be striking the plane interface $z = 0$ at an angle θ_0 with the normal to the interface. For this incidence wave, let the propagation vector, attenuation vector and the angle between these vectors form a triplet $(\mathbf{P}_0, \mathbf{A}_0, \gamma_0)$. In order to satisfy the boundary conditions, we postulate that the incident wave gives rise to: (i) three distinct sets of reflected coupled dilatational waves (P_i) in H^1 having triplet $(\mathbf{P}_{Ri}, \mathbf{A}_{Ri}, \gamma_{Ri})$ propagating with speed v_i, amplitude A_{Ri} and making an angle θ_{Ri} with the normal to the interface; (ii) three distinct sets of transmitted coupled dilatational waves (P_i') in H^2 having triplet $(\mathbf{P'}_{Ti}, \mathbf{A'}_{Ti}, \gamma'_{Ti})$ propagating with speed v_i', amplitude A'_{Ti} and making an angle θ'_{Ti} with the normal to the interface. (iii) a reflected shear wave (SV) in H^1 having triplet $(\mathbf{P}_{R4}, \mathbf{A}_{R4}, \gamma_{R4})$ propagating with speed v_4, amplitude A_{R4} and making an angle θ_{R4} with the normal to the interface; (iv) a transmitted shear wave (SV') in H^2 having triplet $(\mathbf{P'}_{T4}, \mathbf{A'}_{T4}, \gamma'_{T4})$ propagating with speed v_4', amplitude A'_{T4} and making an angle θ'_{T4} with the normal to the interface. The subscripts Ri and Ti ($i = 1, 2, 3$) stand for reflected and transmitted waves respectively. A rough geometry of the problem is shown in Figure 1.

Now, we shall write the relevant potential functions for incident, reflected and transmitted waves in the two half-spaces. As in [20], in the half-space H^1, we have

$$p = A_0 \exp\left[\iota(\omega t - \mathbf{K}_0 \cdot \mathbf{r})\right] + \sum_{i=1}^{3} A_{Ri} \exp\left[\iota(\omega t - \mathbf{K}_{Ri} \cdot \mathbf{r})\right], \tag{3.2}$$

$$\phi = A_0 F_1 \exp\left[\iota(\omega t - \mathbf{K}_0 \cdot \mathbf{r})\right] + \sum_{i=1}^{3} A_{Ri} F_i \exp\left[\iota(\omega t - \mathbf{K}_{Ri} \cdot \mathbf{r})\right], \tag{3.3}$$

$$T = A_0 G_1 \exp\left[\iota(\omega t - \mathbf{K}_0 \cdot \mathbf{r})\right] + \sum_{i=1}^{3} A_{Ri} G_i \exp\left[\iota(\omega t - \mathbf{K}_{Ri} \cdot \mathbf{r})\right], \tag{3.4}$$

$$q = A_0 \exp\left[\iota(\omega t - \mathbf{K}_0 \cdot \mathbf{r})\right] + A_{R4} \exp\left[\iota(\omega t - \mathbf{K}_{R4} \cdot \mathbf{r})\right]. \tag{3.5}$$

and in the half-space H^2, we have

$$p' = \sum_{i=1}^{3} A'_{Ti} \exp\left[\iota\left(\omega t - \mathbf{K}'_{Ti} \cdot \mathbf{r}\right)\right], \tag{3.6}$$

$$\phi' = \sum_{i=1}^{3} A'_{Ti} F'_i \exp\left[\iota\left(\omega t - \mathbf{K}'_{Ti} \cdot \mathbf{r}\right)\right], \tag{3.7}$$

$$T' = \sum_{i=1}^{3} A'_{Ti} G'_i \exp\left[\iota\left(\omega t - \mathbf{K}'_{Ti} \cdot \mathbf{r}\right)\right], \tag{3.8}$$

$$q' = A'_{T4} \exp\left[\iota\left(\omega t - \mathbf{K}'_{T4} \cdot \mathbf{r}\right)\right], \tag{3.9}$$

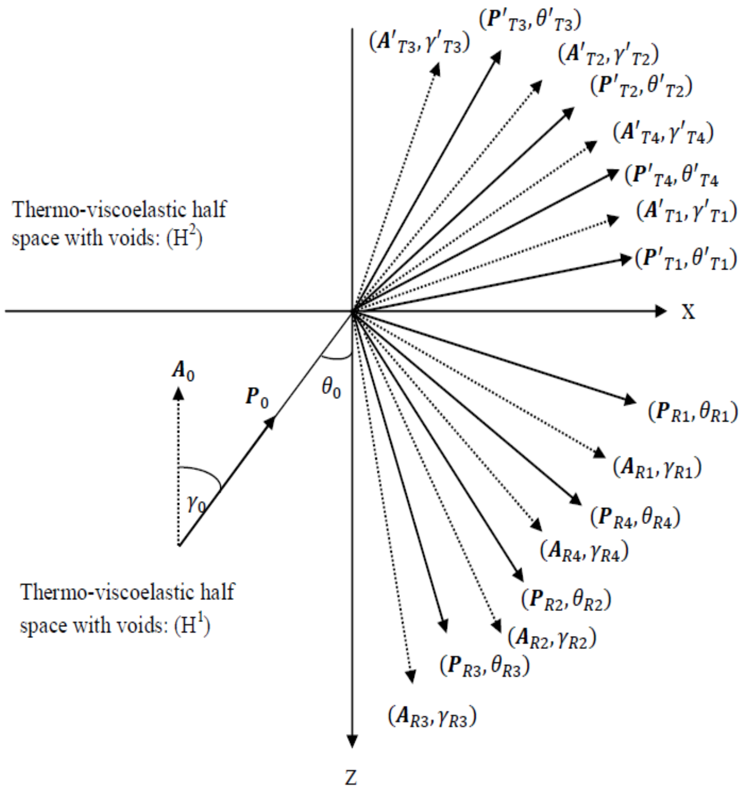


Fig. 1. Geometry of the problem.

where $\mathbf{r}\left(= x\hat{\mathbf{i}} + z\hat{\mathbf{k}}\right)$ is the position vector. F_i and G_i are coupling coefficients in H^1 defined earlier, while F_i' and G_i' are corresponding coupling coefficients in H^2, whose expressions can be written likewise. And

$$\mathbf{K}_0 = \mathbf{P}_0 - \iota\mathbf{A}_0, \quad \mathbf{K}_{Rj} = \mathbf{P}_{Rj} - \iota\mathbf{A}_{Rj} \quad \mathbf{K'}_{Tj} = \mathbf{P'}_{Tj} - \iota\mathbf{A'}_{Tj}, \quad (j = 1, 2, 3, 4)$$

are the complex wave vectors. Following Borcherdt [5], the complex wavenumber k written in terms of given propagation angle θ_0 and attenuation angle γ_0 for an incident inhomogeneous wave is expressed as

$$k = |\mathbf{P}_0| \sin\theta_0 - \iota|\mathbf{A}_0| \sin(\theta_0 - \gamma_0), \tag{3.10}$$

where $\gamma_0 = \gamma_{R1}$ (or γ_{R4}) for the case of incidence of $P-$ (or $SV-$) wave and the vectors $\mathbf{A}_0$ and $\mathbf{P}_0$ have been given in Tomar et al. [20]. For reflected and transmitted waves, the complex wavenumbers k and k' are given by

$$\begin{aligned} k &= |\mathbf{P}_{Rj}| \sin\theta_{Rj} - \iota|\mathbf{A}_{Rj}| \sin\left(\theta_{Rj} - \gamma_{Rj}\right), \\ k' &= |\mathbf{P}'_{Tj}| \sin\theta'_{Tj} - \iota|\mathbf{A}'_{Tj}| \sin\left(\theta'_{Tj} - \gamma'_{Tj}\right), \end{aligned} \tag{3.11}$$

and the vectors $\mathbf{P}_{Rj}$, $\mathbf{A}_{Rj}$, $\mathbf{P'}_{Tj}$ and $\mathbf{A'}_{Tj}$ are given by

$$\left\{|\mathbf{P}_{Rj}|^2, |\mathbf{A}_{Rj}|^2\right\} = \frac{1}{2}\left\{\left\{\Re\left(K_{pj}^2\right), -\Re\left(K_{pj}^2\right)\right\} + \sqrt{\left(\Re\left(K_{pj}^2\right)\right)^2 + \frac{\left(\Im\left(K_{pj}^2\right)\right)^2}{\cos^2\gamma_{Rj}}}\right\},$$

$$\left\{|\mathbf{P}'_{Tj}|^2, |\mathbf{A}'_{Tj}|^2\right\} = \frac{1}{2}\left\{\left\{\Re\Big(K'^2_{pj}\Big), -\Re\Big(K'^2_{pj}\Big)\right\} + \sqrt{\big(\Re(K'^2_{pj})\big)^2 + \frac{\big(\Im(K'^2_{pj})\big)^2}{\cos^2\gamma'_{Tj}}}\right\},$$

$$K^2_{pj} = (\omega^2/v_j^2), \qquad K'^2_{pj} = (\omega^2/v'^2_j).$$

Note that for an incident P_1- wave, we shall take $A_0 = 0$ in equation (3.5) and

$$\mathbf{P}_0 = \Re(k)\hat{\mathbf{i}} - \Re(d_{v_1})\hat{\mathbf{k}}, \quad \mathbf{A}_0 = -\Im(k)\hat{\mathbf{i}} + \Im(d_{v_1})\hat{\mathbf{k}},$$

while for an incident $SV-$ wave, we shall take $A_0 = 0$ in equations (3.2)-(3.4) and

$$\mathbf{P}_0 = \Re(k)\hat{\mathbf{i}} - \Re(d_{v_4})\hat{\mathbf{k}}, \quad \mathbf{A}_0 = -\Im(k)\hat{\mathbf{i}} + \Im(d_{v_4})\hat{\mathbf{k}}.$$

For reflected and transmitted waves, we take

$$\mathbf{P}_{Rj} = \Re(k)\hat{\mathbf{i}} + \Re(d_{v_j})\hat{\mathbf{k}}, \quad \mathbf{A}_{Rj} = -\Im(k)\hat{\mathbf{i}} - \Im(d_{v_j})\hat{\mathbf{k}},$$

$$\mathbf{P'}_{Tj} = \Re(k')\hat{\mathbf{i}} - \Re(d'_{v_j})\hat{\mathbf{k}}, \quad \mathbf{A'}_{Tj} = -\Im(k')\hat{\mathbf{i}} + \Im(d'_{v_j})\hat{\mathbf{k}},$$

such that $\Re(k) \geq 0$ and $\Re(k') \geq 0$ in order to ensure the propagation of various waves in the positive $x-$direction. And

$$\left\{d_{v_j}, d'_{v_j}\right\} = \text{principal value}\left\{\sqrt{\frac{\omega^2}{v_j^2} - k^2}, \quad \sqrt{\frac{\omega^2}{v'^2_j} - k'^2}\right\}.$$

The principal value of the square root implies that $d_{v_j},\ d'_{v_j} \geq 0$ (see [5]).

3.1. **Boundary conditions.** The half-spaces H^1 and H^2 are considered to be in perfect contact. Therefore, the force stress, equilibrated stress, displacement and heat flux vectors across the interface must be continuous at the common boundary surface. These boundary conditions can be written through mathematical equations as: At $z = 0$,

$$\begin{aligned} &(i)\ t_{zz} = t'_{zz}, \quad (ii)\ t_{zx} = t'_{zx}, \quad (iii)\ u = u', \quad (iv)\ w = w', \\ &(v)\ Q_z = Q'_z, \quad (vi)\ Q_x = Q'_x, \quad (vii)\ H_z = H'_z, \quad (viii)\ H_x = H'_x. \end{aligned} \tag{3.12}$$

Using relations (2.1), (2.2), (2.5) and (3.1), these boundary conditions can be written in terms of potentials as

$$\begin{aligned} &\left(\lambda_1 + \lambda_1^*\frac{\partial}{\partial t}\right)\nabla^2 p + 2\left(\mu_1 + \mu_1^*\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 q}{\partial x \partial z}\right) + \left(b_1 + b_1^*\frac{\partial}{\partial t}\right)\phi - \beta_1 T \\ &= \frac{\rho'_0}{\rho_0}\left\{\left(\lambda'_1 + \lambda'^*_1\frac{\partial}{\partial t}\right)\nabla^2 p' + 2\left(\mu'_1 + \mu'^*_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 p'}{\partial z^2} + \frac{\partial^2 q'}{\partial x \partial z}\right)\right. \\ &\left.+\left(b'_1 + b'^*_1\frac{\partial}{\partial t}\right)\phi' - \beta'_1 T'\right\}; \quad \left(\mu_1 + \mu_1^*\frac{\partial}{\partial t}\right)\left(2\frac{\partial^2 p}{\partial x \partial z} + \frac{\partial^2 q}{\partial x^2} - \frac{\partial^2 q}{\partial z^2}\right) \\ &= \frac{\rho'_0}{\rho_0}\left(\mu'_1 + \mu'^*_1\frac{\partial}{\partial t}\right)\left(2\frac{\partial^2 p'}{\partial x \partial z} + \frac{\partial^2 q'}{\partial x^2} - \frac{\partial^2 q'}{\partial z^2}\right); \quad \frac{\partial p}{\partial x} - \frac{\partial q}{\partial z} = \frac{\partial p'}{\partial x} - \frac{\partial q'}{\partial z}; \end{aligned}$$

$$\frac{\partial p}{\partial z}+\frac{\partial q}{\partial x}=\frac{\partial p'}{\partial z}+\frac{\partial q'}{\partial x};\quad k_1\frac{\partial T}{\partial z}+\zeta_1\frac{\partial^2\phi}{\partial t\partial z}=\frac{a'T_0'^2}{aT_0^2}\left\{k_1'\frac{\partial T'}{\partial z}+\zeta_1'\frac{\partial^2\phi'}{\partial t\partial z}\right\};$$
$$k_1\frac{\partial T}{\partial x}+\zeta_1\frac{\partial^2\phi}{\partial t\partial x}=\frac{a'T_0'^2}{aT_0^2}\left\{k_1'\frac{\partial T'}{\partial x}+\zeta_1'\frac{\partial^2\phi'}{\partial t\partial x}\right\};$$
$$\tau_1^*\frac{\partial T}{\partial z}+\left(\alpha_1+\alpha_1^*\frac{\partial}{\partial t}\right)\frac{\partial\phi}{\partial z}=\frac{\rho_0'}{\rho_0}\left\{\tau_1'^*\frac{\partial T'}{\partial z}+\left(\alpha_1'+\alpha_1'^*\frac{\partial}{\partial t}\right)\frac{\partial\phi'}{\partial z}\right\};$$
$$\tau_1^*\frac{\partial T}{\partial x}+\left(\alpha_1+\alpha_1^*\frac{\partial}{\partial t}\right)\frac{\partial\phi}{\partial x}=\frac{\rho_0'}{\rho_0}\left\{\tau_1'^*\frac{\partial T'}{\partial x}+\left(\alpha_1'+\alpha_1'^*\frac{\partial}{\partial t}\right)\frac{\partial\phi'}{\partial x}\right\}. \tag{3.13}$$

Inserting the potentials given by (3.2) to (3.9) into (3.13), we see that the arguments of exponential terms are not equal throughout and hence the terms cannot be cancelled out. To cancel these exponential terms, their arguments must be equal for all values of x and t at the interface $z = 0$, for which the wavenumbers on either side of the interface must be equal. Hence, the considered form of the above potentials yields an extension of Snell's law for reflection-transmission problem as

$$\Re(k)=|\mathbf{P}_0|\sin\theta_0=|\mathbf{P}_{Ri}|\sin\theta_{Ri}=|\mathbf{P'}_{Ti}|\sin\theta_{Ti}', \tag{3.14}$$

$$-\Im(k)=|\mathbf{A}_0|\sin(\theta_0-\gamma_0)=|\mathbf{A}_{Ri}|\sin(\theta_{Ri}-\gamma_{Ri})=|\mathbf{A'}_{Ti}|\sin(\theta_{Ti}'-\gamma_{Ti}'), \tag{3.15}$$

where $i = 1, 2, 3, 4$. The set of equations so obtained from (3.13) can be written as

$$\sum_{j=1}^{8}a_{ij}Z_j=h_i,\quad (i=1,2,...,8). \tag{3.16}$$

The expressions of a_{ij} and h_i are given in Appendix-A. Note that the quantities h_i will be different for different incident waves. And

$$Z_j=(A_{Rj}/A_0),\qquad Z_{j+4}=(A_{Tj}'/A_0),\quad (j=1,2,3,4). \tag{3.17}$$

represent the reflection and transmission coefficients respectively. Solving equations in (3.16) using Cramer's rule, one can obtain

$$Z_\ell=(\triangle_\ell/\triangle),\qquad (\ell=1,2,...,8), \tag{3.18}$$

where $\triangle$ and $\triangle_\ell$ are the usual determinants of the matrix of the coefficients of (3.16). Looking at the elements of coefficient matrix, it is clear that the various amplitude ratios Z_ℓ are functions of the angle of incidence, frequency and material properties of both the half-spaces. In order to check the validity of the problems, we shall verify the energy balance among the incident and various reflected and transmitted waves at the interface. For this purpose, we analyze the distribution of incident energy at the interface in the following section.

3.2. **Energy partitioning.** For partitioning of incident energy among various reflected and transmitted waves at the interface, we consider a surface element of unit area at the interface. The rate at which the energy communicated per unit area of the surface P^E is given by the scalar product of the surface traction and the particle velocity. The average of P^E over the complete period denoted by

$\langle P^E \rangle$ represents the average energy transmission per unit area per unit time. The average energy intensities of the waves on the surface are given by (see Achenbach, [1])

$$\langle P_{ij}^E \rangle = \frac{1}{2}\Re\Big(\big(t_{zz}^i\big)\left(\overline{\dot{w}}^j\right)\Big) + \frac{1}{2}\Re\Big(\big(t_{zx}^i\big)\left(\overline{\dot{u}}^j\right)\Big) + \frac{1}{2}\Re\Big(\big(H_z^i\big)\left(\overline{\dot{\phi}}^j\right)\Big), \quad (3.19)$$

where $i, j = 0, 1, 2, 3, 4$ and overbar denotes the complex conjugate. The quantity having index 0 corresponds to the incident wave, and those having indices $1, 2, 3, 4$ correspond to reflected P_1, P_2, P_3 and $SV-$ waves respectively. Keep in mind that when $i = j$, the energy transmission $\langle P_{ij}^E \rangle$ represents the bulk energy, while for $i \neq j$, the energy transmission $\langle P_{ij}^E \rangle$ represents the interactional energy. On similar lines, the energy transmission of transmitted waves can be computed. For example, to find out the energy transmission along the transmitted $P_1'-$ wave, we shall use the expression of the potentials corresponding to the transmitted $P_1'-$ wave and so on.

In both the cases of incident wave, the half-space H^1 supports five waves, i.e., one incident wave and four reflected wave, while the half-space H^2 supports four transmitted waves. Apart from the bulk energy of these waves, there will be interactional energy flux across the interface $z = 0$, due to the interaction between dissimilar pairs of incident-reflected waves, reflected- reflected waves and transmitted- transmitted waves. The bulk and the interactional energies can be put together through a square matrix called energy matrix E of order five for the lower half-space H^1, while for the upper half-space H^2, the energy matrix E' will be of order four. For the case of incidence P_1-(or SV-) wave, the elements of energy matrices E and E' are given respectively as

$$E_{ij} = -\frac{\Re\big(F_{ij} Z_i \overline{Z}_j\big)}{\Re(F_{00})}, \quad (i, j = 0, 1, 2, 3, 4), \quad (3.20)$$

and

$$E'_{ij} = \frac{\Re\big(F'_{ij} Z_{i+4} \overline{Z}_{j+4}\big)}{\Re(F_{00})}, \quad (i, j = 1, 2, 3, 4). \quad (3.21)$$

The explicit expressions of the elements of the matrices F_{ij} and F'_{ij} are given as:

a. For an incident P_1-wave:

$$F_{i0} = M_i \overline{d}_{v_1} - 2M|k|^2 d_{v_i} - \left(A^\star F_i + \tau_1^* G_i\right) \overline{F}_1 d_{v_i}, \quad (i = 1, 2, 3),$$

$$F_{0i} = -\left[M_1 \overline{d}_{v_i} - 2M|k|^2 d_{v_1} - \left(A^\star F_1 + \tau_1^* G_1\right) \overline{F}_i d_{v_1}\right], \quad F_{00} = -F_{11},$$

$$F_{04} = -\left(2M_1 \overline{k} + 2kM d_{v_1} \overline{d}_{v_4}\right), \quad F_{40} = M\left[2k d_{v_4} \overline{d}_{v_1} - \left(k^2 - d_{v_4}^2\right) \overline{k}\right].$$

b. For an incident $SV-$ wave:

$$F_{i0} = -\left[M_i \overline{k} + 2Mk d_{v_i} \overline{d}_{v_4}\right], \quad F_{0i} = M\left[2k d_{v_4} \overline{d}_{v_i} - \left(k^2 - d_{v_4}^2\right) \overline{k}\right], \quad F_{00} = -F_{44},$$

$$F_{04} = M\left[2|k|^2 d_{v_4} + (k^2 - d_{v_4}^2) \overline{d}_{v_4}\right], \quad F_{40} = -M\left[2|k|^2 d_{v_4} + \left(k^2 - d_{v_4}^2\right) \overline{d}_{v_4}\right].$$

The remaining quantities common to both type of incidences are given by

$$F_{ij} = -\left[M_i\overline{d}_{v_j} + 2M|k|^2 d_{v_i} + (A^\star F_i + \tau_1^* G_i)\,\overline{F}_j d_{v_i}\right], \quad (i,j=1,2,3)$$
$$F_{4j} = -\left[2kMd_{v_4}\overline{d}_{v_j} + M\left(k^2 - d_{v_4}^2\right)\overline{k}\right], \quad F_{i4} = -\left(M_i\overline{k} - 2kMd_{v_i}\overline{d}_{v_4}\right),$$
$$F_{44} = -M\left[2|k|^2 d_{v_4} - \left(k^2 - d_{v_4}^2\right)\overline{d}_{v_4}\right],$$
$$F'_{ij} = \left(\frac{\rho'_0}{\rho_0}\right)\left[M'_i\overline{d'}_{v_j} + 2M'|k'|^2 d'_{v_i} + (A'^\star F'_i + \tau_1'^* G'_i)\,\overline{F'}_j d'_{v_i}\right],$$
$$F'_{4j} = -M'\left(\frac{\rho'_0}{\rho_0}\right)\left[2k'd'_{v_4}\overline{d'}_{v_j} + \left(k'^2 - d'^2_{v_4}\right)\overline{k'}\right],$$
$$F'_{i4} = -\left(\frac{\rho'_0}{\rho_0}\right)\left(M'_i\overline{k'} + 2k'M'd'_{v_i}\overline{d'}_{v_4}\right),$$
$$F'_{44} = M'\left(\frac{\rho'_0}{\rho_0}\right)\left[2|k'|^2 d'_{v_4} - \left(k'^2 - d'^2_{v_4}\right)\overline{d'}_{v_4}\right],$$
$$M_i = \left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_i}^2\right) + 2Md_{v_i}^2 - (b_1 + \iota\omega b_1^*)\,F_i + \beta_1 G_i\right],$$
$$M'_i = \left[(\lambda'_1 + \iota\omega\lambda_1'^*)\left(k'^2 + d'^2_{v_i}\right) + 2M'd'_{v_i} - (b'_1 + \iota\omega b_1'^*)\,F'_i + \beta'_1 G'_i\right],$$
$$A^\star = \alpha_1 + \iota\omega\alpha_1^*, \quad M = \mu_1 + \iota\omega\mu_1^*, \quad A'^\star = \alpha'_1 + \iota\omega\alpha_1'^*, \quad M' = \mu'_1 + \iota\omega\mu_1'^*.$$

Let E_{IR} denotes the total interactional energy ratio due to interaction of incident wave with four reflected wave, E_{RR} denotes the total interactional energy ratio due to interaction among the four dissimilar pairs of reflected waves and E'_{TT} represents total interactional energy ratio among the four dissimilar pairs of transmitted waves. Thus, the net interactional energy ratio is equal to the sum of all interactional energy ratios, given by

$$E_{IR}+E_{RR}+E'_{TT} = \sum_{i=1}^{4}(E_{0i}+E_{i0}) + \sum_{i=1}^{4}\left(\sum_{j=1}^{4}E_{ij} - E_{ii}\right) + \sum_{i=1}^{4}\left(\sum_{j=1}^{4}E'_{ij} - E'_{ii}\right). \tag{3.22}$$

According to the energy balance law at the interface, we must have

$$\sum_{r=1}^{4}E_{rr} + \sum_{r=1}^{4}E'_{rr} + E_{IR} + E_{RR} + E'_{TT} = -E_{00} = 1. \tag{3.23}$$

This relation must hold at the interface for each angle of incidence during reflection/transmission phenomena.

4. Reflection from free surface (a reduced case)

If we neglect the presence of upper half-space H^2 from the model, then we shall be left with the problem of reflection of a set of coupled dilatational waves/SV-wave from the free boundary surface of a thermo-viscoelastic half-space with voids. For this, setting the relevant material parameters to zero into (3.16), we obtain

$$\left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_1}^2\right) + 2\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_1}^2 - \left(b_1 + \iota\omega b_1^*\right)F_1 + \beta_1 G_1\right]Z_1$$
$$\left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_2}^2\right) + 2\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_2}^2 - \left(b_1 + \iota\omega b_1^*\right)F_2 + \beta_1 G_2\right]Z_2$$
$$\left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_3}^2\right) + 2\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_3}^2 - \left(b_1 + \iota\omega b_1^*\right)F_3 + \beta_1 G_3\right]Z_3$$
$$+ 2k\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_4}Z_4$$

$$= -\left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_1}^2\right) + 2\left(\mu_1 + \iota\omega\mu_1^*\right) d_{v_1}^2 - \left(b_1 + \iota\omega b_1^*\right) F_1 + \beta_1 G_1\right], \tag{4.1}$$

$$2kd_{v_1} Z_1 + 2kd_{v_2} Z_2 + 2kd_{v_3} Z_3 + \left(k^2 - d_{v_4}^2\right) Z_4 = 2kd_{v_1}, \tag{4.2}$$

$$(k_1 G_1 + \iota\omega\zeta_1 F_1)\, d_{v_1} Z_1 + (k_1 G_2 + \iota\omega\zeta_1 F_2)\, d_{v_2} Z_2 + (k_1 G_3 + \iota\omega\zeta_1 F_3)\, d_{v_3} Z_3 = (k_1 G_1 + \iota\omega\zeta_1 F_1)\, d_{v_1}, \tag{4.3}$$

$$[(\alpha_1 + \iota\omega\alpha_1^*) F_1 + \tau_1^* G_1]\, d_{v_1} Z_1 + [(\alpha_1 + \iota\omega\alpha_1^*) F_2 + \tau_1^* G_2]\, d_{v_2} Z_2 + [(\alpha_1 + \iota\omega\alpha_1^*) F_3 + \tau_1^* G_3]\, d_{v_3} Z_3 = [(\alpha_1 + \iota\omega\alpha_1^*) F_1 + \tau_1^* G_1]\, d_{v_1}, \tag{4.4}$$

Note that the boundary conditions corresponding to the continuity of displacement components, tangential components of heat and equilibrated stress are no more required for this reduced case as the boundary surface is free from mechanical and thermal stresses. It can be seen that equations (4.1)-(4.4) are same as the equations in (58) of Tomar et al. [20] apart from the notations of the amplitudes for the corresponding problem of P_1-wave incidence. On similar lines, it can be

Symbols	*Values*	*Unit*
(λ, λ')	$(7.76, 8.2) \times 10^{10}$	N m^{-2}
(λ^*, λ'^*)	$(0.45, 0.45) \times 10^{9}$	N s m^{-2}
(μ, μ')	$(3.86, 4.2) \times 10^{10}$	N m^{-2}
(μ^*, μ'^*)	$(0.1, 0.1) \times 10^{9}$	N s m^{-2}
(b, b')	$(1.139, 0.2023) \times 10^{10}$	N m^{-2}
(b^*, b'^*)	$(1.0, 1.0) \times 10^{10}$	N s m^{-2}
(a, a')	$(0.5, 0.8) \times 10^{2}$	N $m^{-2}K^{-2}$
(ξ^*, ξ'^*)	$(3.0, 1.03) \times 10^{9}$	N s m^{-2}
(α, α')	$(1.688, 1.388) \times 10^{-5}$	N
(α^*, α'^*)	$(1.0, 1.0) \times 10^{-4}$	N s
(β, β')	$(5.518, 0.1518) \times 10^{6}$	N $m^{-2}K^{-1}$
(γ^*, γ'^*)	$(0.2, 0.21) \times 10^{9}$	N s m^{-2}
(ξ, ξ')	$(1.475, 1.275) \times 10^{10}$	N m^{-2}
(τ^*, τ'^*)	$(0.01, 0.01) \times 10^{6}$	NK^{-1}
(m, m')	$(0.2, 0.23) \times 10^{6}$	N $m^{-2}K^{-1}$
(κ, κ')	$(1.75, 0.17) \times 10^{-15}$	m^2
(k_t, k_t')	$(0.386, 0.113) \times 10^{3}$	N $s^{-1}K^{-1}$
(ζ, ζ')	$(1.5, 0.12) \times 10^{6}$	N
(ρ_0, ρ_0')	$(8.954, 8.950) \times 10^{3}$	Kg m^{-3}
(T_0, T_0')	$(0.293, 0.300) \times 10^{3}$	K
c_0	1×10^{3}	m s^{-1}
l_0	1.0	m

Table-1. Values of various quantities.

verified that the problem of reflection due to $SV-$wave incidence earlier studied by Tomar el al. [20] can also be recovered from the present formulation.

5. Numerical results and discussion

Here, we shall compute the amplitude and energy ratios numerically for a specific model. This specific model consists of two different copper materials occupying the half-spaces H^1 and H^2. The numerical values of relevant parameters in H^1 and H^2 have been taken from Bhagwan and Tomar [3] and Sharma [22] respectively. Values of parameters involved in the model are given in the Table 1 on the previous page.

A MatLab program is developed to compute the amplitude and energy ratios corresponding to various reflected and transmitted waves from formulae (3.18), (3.20) and (3.21) at three different values of frequency ω, namely, $2\pi \times 50\ Hz$, $2\pi \times 100\ Hz$ and $2\pi \times 150\ Hz$ and at different angles of incidence ranging from 0^o to 90^o. The nature of dependence of these ratios on the angle of incidence has been depicted through the following Figures 2 to 7.

Figures 2 and 3 depict the variation of modulus of reflection/transmission coefficients and corresponding energy ratios with angle of incidence of P_1- wave, while Figures 4 and 5 depict the same in case of incidence of $SV-$wave. For the incidence of P_1 and SV waves, the net interactional and bulk energies is shown in Figure 6. The effect of void parameter (b) on various amplitude ratios for the case of P_1-wave incidence only has been depicted through Figure 7.

From Figure 2(a-h), it can be noticed that the reflection coefficient $|Z_1|$ corresponding to the reflected coupled dilatational P_1- wave dominates heavily over the other reflection coefficients at all the three frequencies chosen. This coefficient starts with a lowest value at normal incidence and increases gradually with angle of incidence and attains the maximum value, which is different for different frequency. The corresponding energy ratio E_{11} is also found to follow similar pattern of variation. The reflection coefficients $|Z_2|$, $|Z_3|$ and the corresponding energy ratios E_{22}, E_{33} are of very small magnitudes and follow similar pattern of variation. These reflection coefficients increase with increase of frequency, while the corresponding energy ratios decrease with increase of frequency, at each angle of incidence. The reflection coefficient $|Z_4|$ for the reflected shear wave initially increases up to $\theta_0 \approx 45^o$ and after that it decreases as θ_0 goes to 89^0. The corresponding energy ratio E_{44} is even much smaller. It is also observed from these figures that all the reflection coefficients and corresponding energy ratios are functions of frequency.

Figure 3(a-h) depicts the variation of modulus of transmission coefficients and corresponding energy ratios with respect to the angle of incidence of (P_1-)wave at different frequencies. It can be seen that the coefficient $|Z_5|$ and the corresponding energy ratio E'_{11} decreases gradually as the angle of incidence goes from normal

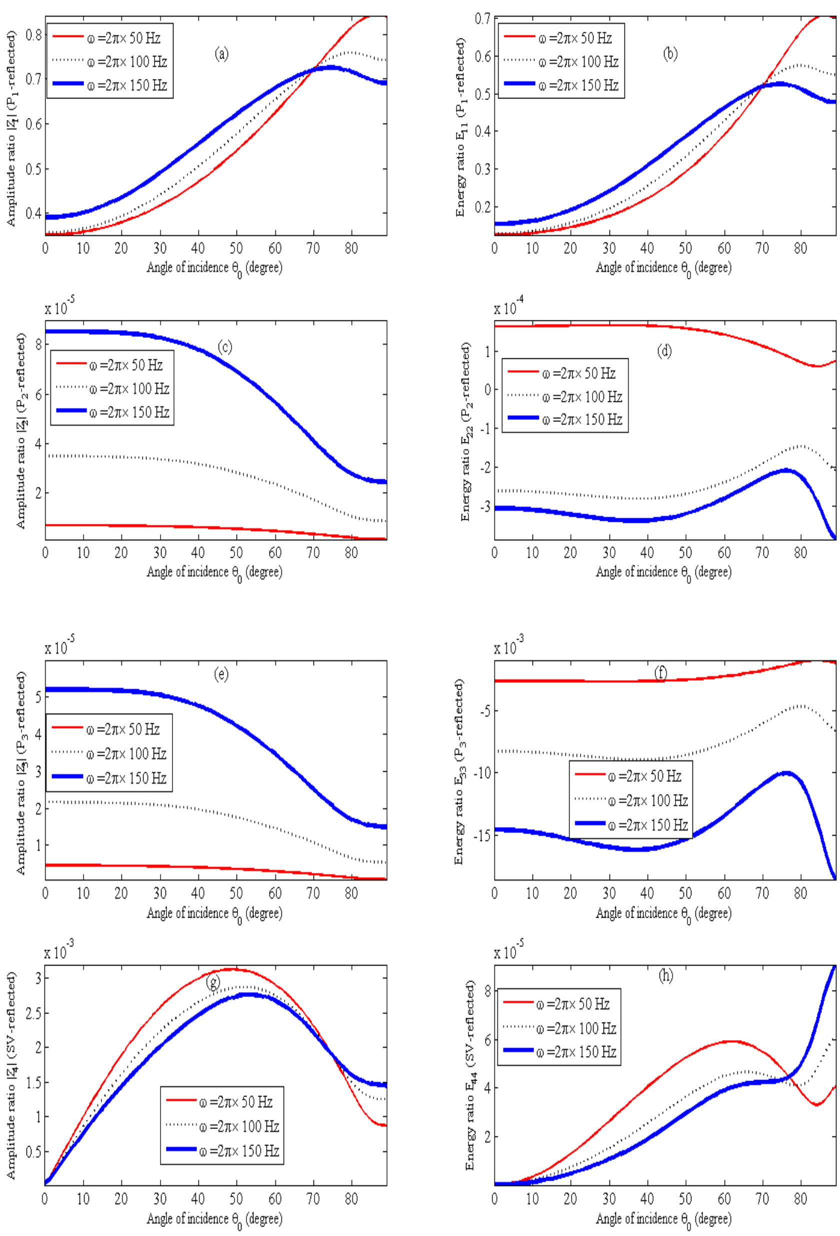


FIG. 2. (Incidence of a set of coupled dilatational waves -P_1) Variation of reflection coefficients and corresponding energy ratios.

to grazing incidence. The coefficient $|Z_6|$ and the corresponding energy ratio E'_{22}, both have small magnitudes and vary with angle of incidence and frequencies. The coefficient $|Z_7|$ and the corresponding energy ratio E'_{33} are still very small in magnitude. The value of amplitude ratio $|Z_8|$ corresponding to transmitted

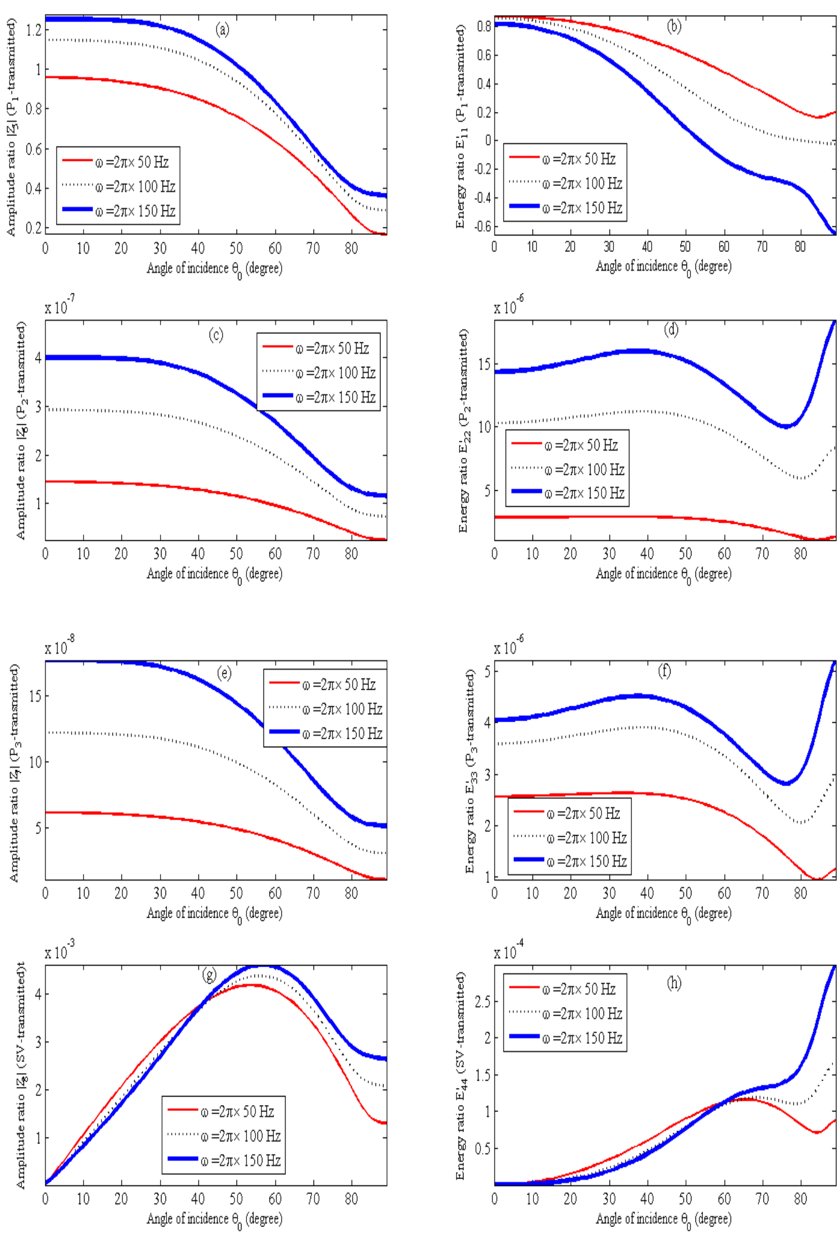


FIG. 3. (Incidence of a set of coupled dilatational waves -P_1) Variation of transmission coefficients and corresponding energy ratios.

shear wave increases with the angle of incidence, achieving its maximum value at $\theta_0 = 50^0$ angle of incidence and then decreases with further increase of the angle of incidence, while the corresponding energy ratio E'_{44} is small but varies with angle of incidence at different value of frequency near grazing incidence. From

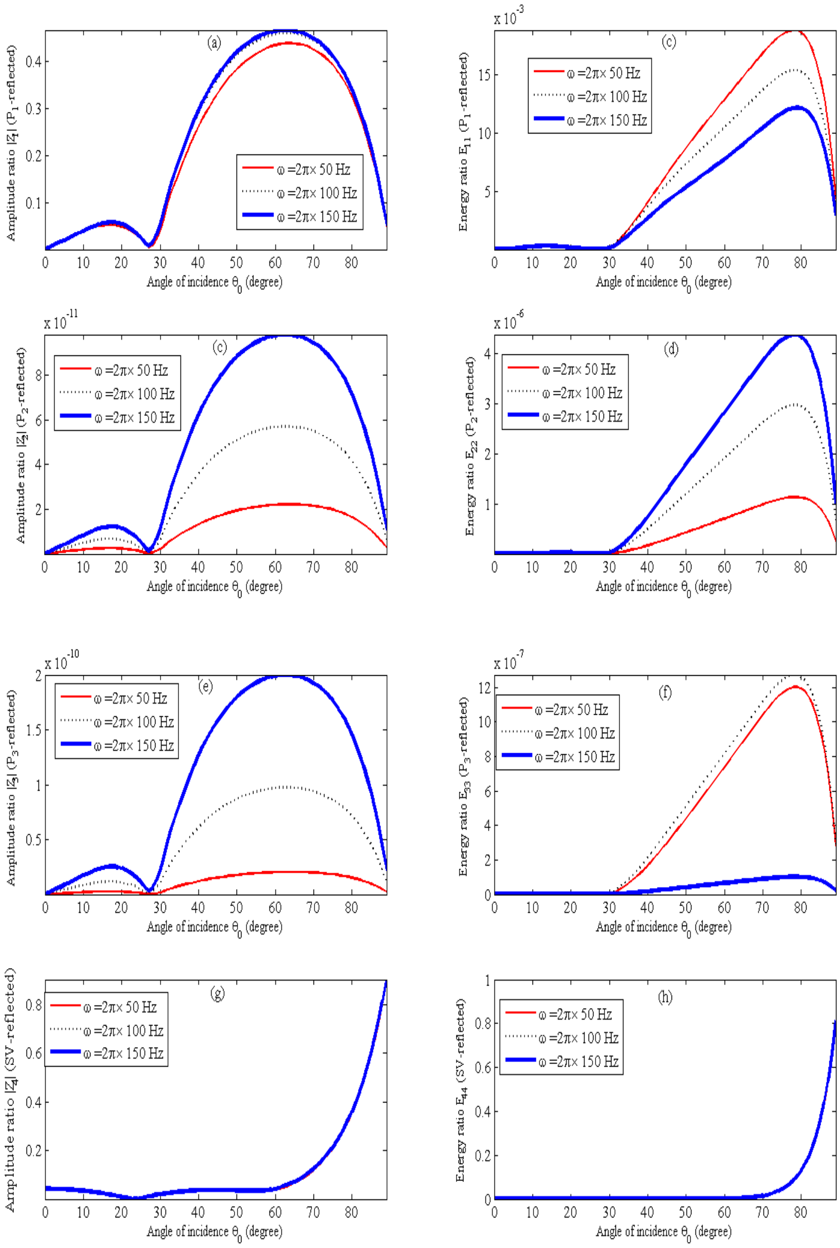


FIG. 4. (Incidence of SV-wave) Variation of reflection coefficients and corresponding energy ratios.

Figures 3(a), it can be noted that the transmission coefficient corresponding to transmitted P_1- wave is stronger in comparison to that of in [3] given in Figure 3(a). The reason is that the thermo-viscoelastic half-space with voids allows the transmission of P_1 wave more easily than that of in classical elastic half-space.

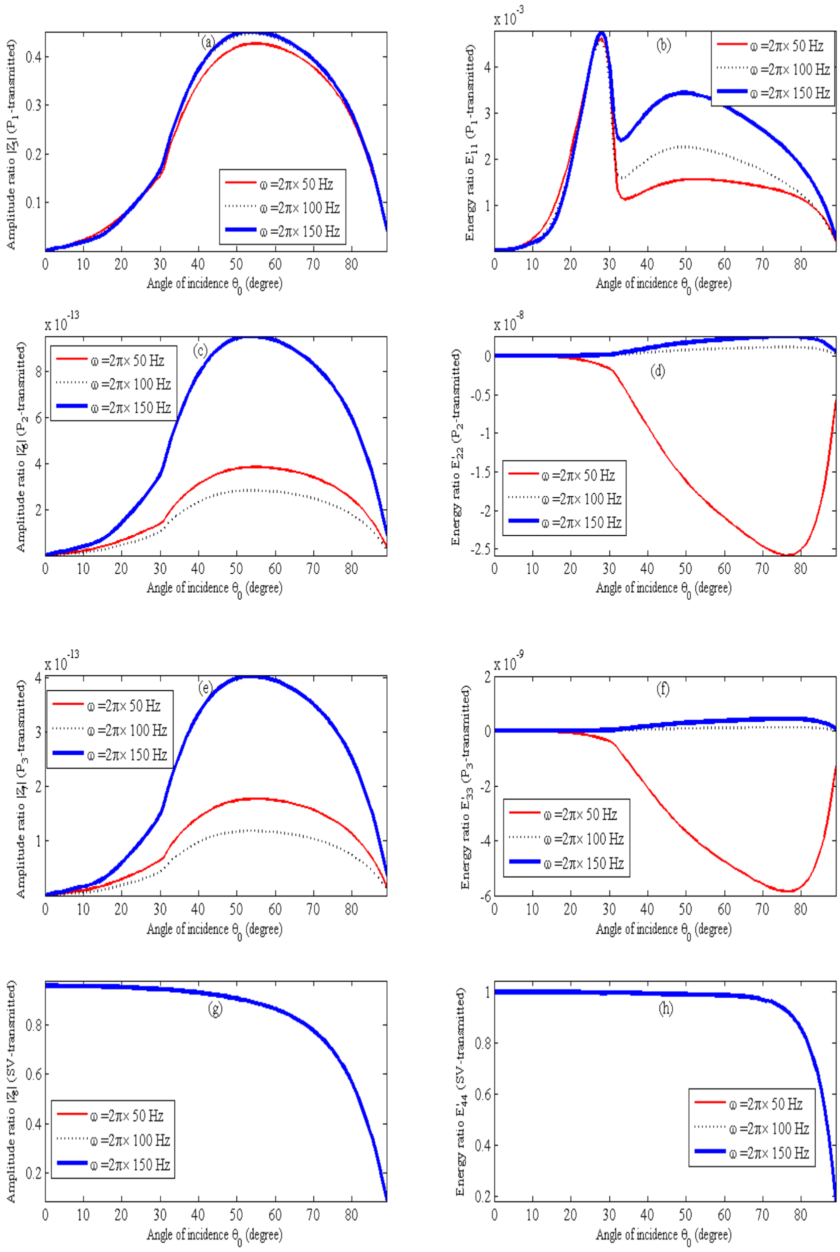


FIG. 5. (Incidence of SV-wave) Variation of transmission coefficients and corresponding energy ratios.

Since the presence of voids makes the material more compressive. Figure 4 (a-h) depicts the variation of modulus of reflection coefficients and corresponding energy ratios versus angle of incidence of $SV-$ wave at different frequencies. From this figure, it can be observed that the reflection coefficients $|Z_2|$, $|Z_3|$ and their

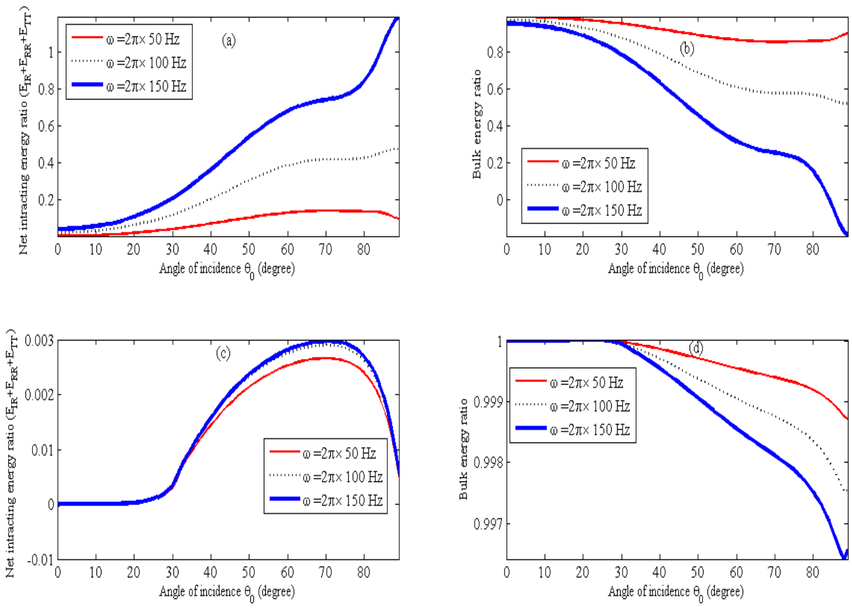


FIG. 6. Variation of interacting energy and bulk energies. (a) and (b) - Incidence of coupled dilatational wave, (c) and (d) Incidence of SV-wave.

corresponding energy ratios E_{22}, E_{33} are extremely low as compared to the other reflection coefficients and their corresponding energy ratios. While the reflection coefficient $|Z_4|$ and corresponding energy ratio E_{44} of the reflected $SV-$wave dominates over all the other amplitude ratios of reflected dilatational waves and does not depend on frequency parameter. In general, the dependency of any of the reflection coefficients and corresponding energy on frequency parameter is hardly seen, however each one is a function of angle of incidence for the corresponding problem of P_1-wave incidence. From Figure 5(a-h), we see that the value of amplitude ratio $|Z_5|$ and energy ratio E'_{11} corresponding to transmitted P_1- wave increases and then decreases with angle of incidence. The coefficients $|Z_6|$, $|Z_7|$ and the corresponding energy ratio E'_{22}, E'_{33} are found to be very small in magnitude, while the amplitude ratio $|Z_8|$ and corresponding energy ratio E'_{44} of transmitted $SV-$wave dominates. It starts around the value 1 at normal incidence and then decreases first very slowly and then quickly and finally vanishes as $\theta_0 \to 89^0$. Note that this coefficient does not depend on the frequency at all, similar to Z_4. The other coefficients also do not heavily depend on frequency parameter but depend on angle of incidence.

Figure 6(a-d) depicts the graph of net energy ratio with respect to the incident angle θ_0 at three different frequencies for both the cases of incidence. It is seen that at each angle of incidence, the net energy ratio is approximately unity, that

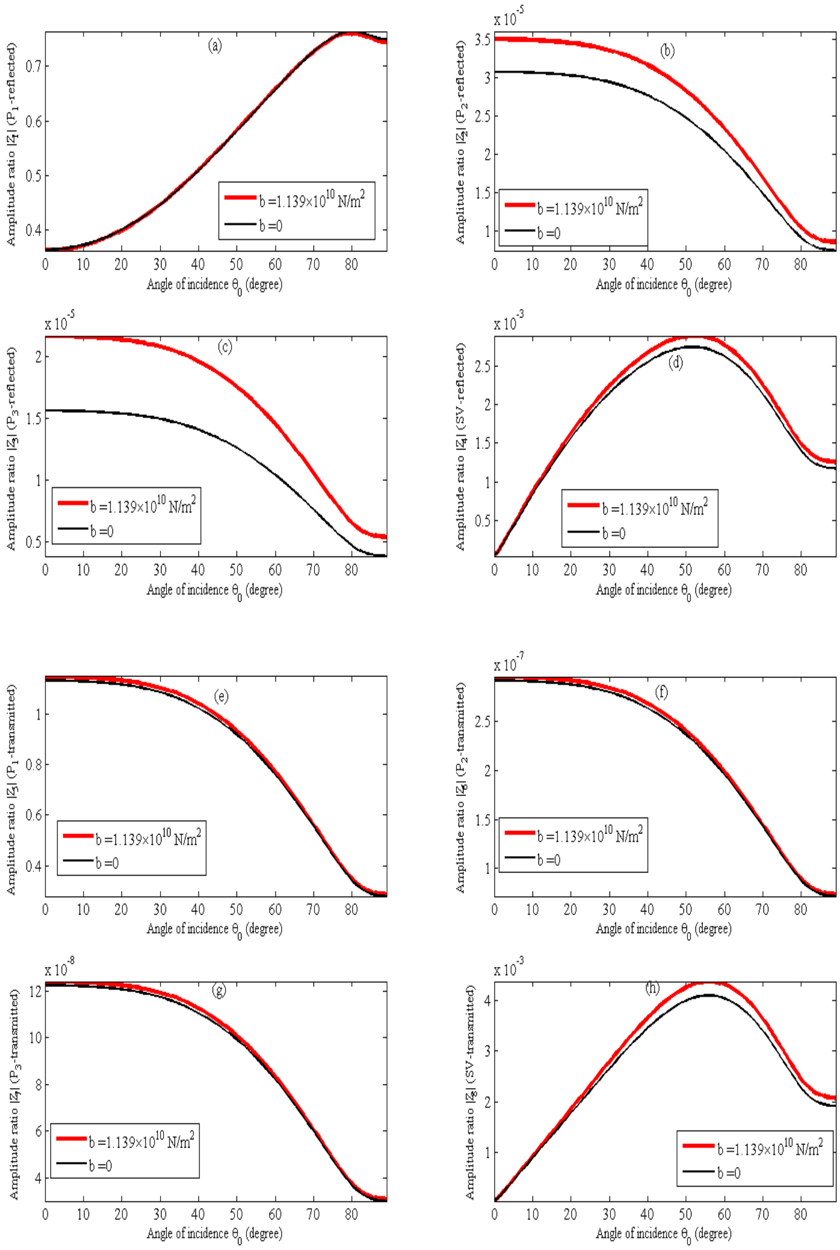


FIG. 7. (Incidence of coupled dilatational wave-P_1) Effect of void parameter (b) on reflection/transmission coefficients.

is, the law $E_{IR} + E_{RR} + E'_{TT} + \sum_{r=1}^{4} E_{rr} + \sum_{r=1}^{4} E'_{rr} \approx 1$ is verified.

Figure 7(a-h) depicts the effect of void parameter b on various amplitude ratios. It can be seen that in the absence of void parameter b, the amplitude ratios corresponding to all the reflected waves at each angle of incidence are slightly less in magnitude than those of in the presence of b, except the amplitude ratio corresponding to reflected P_1 wave, which is slightly higher. While, the amplitude ratios corresponding to all the transmitted waves at each angle of incidence are slightly less in magnitude than those of in the presence of b. We have also noticed that the thermal parameters also do affect the amplitude and corresponding energy ratios, but differently than that of void parameters. The graphs showing the effect of thermal parameters have not been included here.

6. Conclusions

In this paper, the equations giving the amplitude ratios and expressions of energy ratios have been derived when a dilatational or shear wave is made incident at the interface of two dissimilar thermo-viscoelastic half-spaces with voids. For a particular model, the variation of these amplitude and energy ratios corresponding to various reflected/transmitted waves versus angle of incidence has been investigated numerically. The important inferences drawn from this study are as follows

1. The reflection/transmission phenomena is significantly affected by the incidence angle and frequency of the impinging wave.
2. The amplitude and energy ratios of that reflected and refracted waves, which are propagating at the same speed as that of the incident wave, dominate over that of the other reflected/transmitted waves.
3. At each angle of incidence, the sum of energy ratios of various reflected and transmitted waves together with interactional energies is found to be nearly unity. This verifies that there is no dissipation of energy at the interface during transmission phenomena.
4. It is observed that voids and thermal present in the half-spaces do affect the amplitude ratios.

Acknowledgment. One of the authors, Jai Bhagwan is thankful to the Department of Mathematics, Panjab University, Chandigarh for providing computational and library facilities for completion of this work.

Appendix-A.

$$a_{11} = \left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_1}^2\right) + 2\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_1}^2 - \left(b_1 + \iota\omega b_1^*\right)F_1 + \beta_1 G_1\right],$$

$$a_{12} = \left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_2}^2\right) + 2\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_2}^2 - \left(b_1 + \iota\omega b_1^*\right)F_2 + \beta_1 G_2\right],$$

$$a_{13} = \left[(\lambda_1 + \iota\omega\lambda_1^*)\left(k^2 + d_{v_3}^2\right) + 2\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_3}^2 - \left(b_1 + \iota\omega b_1^*\right)F_3 + \beta_1 G_3\right],$$

$$a_{14} = 2k\left(\mu_1 + \iota\omega\mu_1^*\right)d_{v_4},$$

$$a_{15} = -\frac{\rho_0'}{\rho_0}\left[(\lambda_1' + \iota\omega\lambda_1'^*)\left(k'^2 + d_{v_1}'^2\right) + 2\left(\mu_1' + \iota\omega\mu_1'^*\right)d_{v_1}'^2 - \left(b_1' + \iota\omega b_1'^*\right)F_1' + \beta_1' G_1'\right],$$

$$a_{16} = -\frac{\rho_0'}{\rho_0}\left[(\lambda_1' + \iota\omega\lambda_1'^*)\left(k'^2 + d_{v_2}'^2\right) + 2\left(\mu_1' + \iota\omega\mu_1'^*\right)d_{v_2}'^2 - \left(b_1' + \iota\omega b_1'^*\right)F_2' + \beta_1' G_2'\right],$$

$$a_{17} = -\frac{\rho_0'}{\rho_0}\left[(\lambda_1' + \iota\omega\lambda_1'^*)\left(k'^2 + d_{v_3}'^2\right) + 2\left(\mu_1' + \iota\omega\mu_1'^*\right)d_{v_3}'^2 - \left(b_1' + \iota\omega b_1'^*\right)F_3' + \beta_1' G_3'\right],$$

$$a_{18} = \left(\frac{\rho_0'}{\rho_0}\right) 2k' \left(\mu_1' + \iota\omega\mu_1'^*\right) d_{v_4}', \qquad a_{21} = 2k \left(\mu_1 + \iota\omega\mu_1^*\right) d_{v_1},$$

$$a_{22} = 2k \left(\mu_1 + \iota\omega\mu_1^*\right) d_{v_2},\ a_{23} = 2k \left(\mu_1 + \iota\omega\mu_1^*\right) d_{v_3},\ a_{24} = \left(\mu_1 + \iota\omega\mu_1^*\right)\left(k^2 - d_{v_4}^2\right),$$

$$a_{25} = \left(\frac{\rho_0'}{\rho_0}\right) 2k' \left(\mu_1' + \iota\omega\mu_1'^*\right) d_{v_1}', \quad a_{26} = \left(\frac{\rho_0'}{\rho_0}\right) 2k' \left(\mu_1' + \iota\omega\mu_1'^*\right) d_{v_2}',$$

$$a_{27} = \left(\frac{\rho_0'}{\rho_0}\right) 2k' \left(\mu_1' + \iota\omega\mu_1'^*\right) d_{v_3}', \quad a_{28} = -\left(\frac{\rho_0'}{\rho_0}\right)\left(\mu_1' + \iota\omega\mu_1'^*\right)\left(k'^2 - d_{v_4}'^2\right),$$

$$a_{31} = a_{32} = a_{33} = k, \quad a_{34} = -d_{v_4}, \quad a_{35} = a_{36} = a_{37} = -k', \quad a_{38} = -d_{v_4}',$$

$$a_{41} = d_{v_1}, \quad a_{42} = d_{v_2}, \quad a_{43} = d_{v_3}, \quad a_{44} = k, \quad a_{45} = d_{v_1}', \quad a_{46} = d_{v_2}',$$

$$a_{47} = d_{v_3}', \ \ a_{48} = -k', \ \ a_{51} = (k_1 G_1 + \iota\omega\zeta_1 F_1) d_{v_1}, \ a_{52} = (k_1 G_2 + \iota\omega\zeta_1 F_2) d_{v_2},$$

$$a_{53} = (k_1 G_3 + \iota\omega\zeta_1 F_3) d_{v_3}, \quad a_{54} = 0, \quad a_{55} = \left(\frac{a' T_0'^2}{a T_0^2}\right)(k_1' G_1' + \iota\omega\zeta_1' F_1') d_{v_1}',$$

$$a_{56} = \frac{a' T_0'^2}{a T_0^2}(k_1' G_2' + \iota\omega\zeta_1' F_2') d_{v_2}', \quad a_{57} = \frac{a' T_0'^2}{a T_0^2}(k_1' G_3' + \iota\omega\zeta_1' F_3') d_{v_3}', \quad a_{58} = 0,$$

$$a_{61} = (k_1 G_1 + \iota\omega\zeta_1 F_1) k, \quad a_{62} = (k_1 G_2 + \iota\omega\zeta_1 F_2) k, \quad a_{63} = (k_1 G_3 + \iota\omega\zeta_1 F_3) k,$$

$$a_{64} = 0, \quad a_{65} = -\frac{a' T_0'^2}{a T_0^2}(k_1' G_1' + \iota\omega\zeta_1' F_1') k', \quad a_{66} = -\frac{a' T_0'^2}{a T_0^2}(k_1' G_2' + \iota\omega\zeta_1' F_2') k',$$

$$a_{67} = -\frac{a' T_0'^2}{a T_0^2}(k_1' G_3' + \iota\omega\zeta_1' F_3') k', \ a_{68} = 0, \ a_{71} = \left[\left(\alpha_1 + \iota\omega\alpha_1^*\right) F_1 + \tau_1^* G_1\right] d_{v_1},$$

$$a_{72} = \left[\left(\alpha_1 + \iota\omega\alpha_1^*\right) F_2 + \tau_1^* G_2\right] d_{v_2}, \ a_{73} = \left[\left(\alpha_1 + \iota\omega\alpha_1^*\right) F_3 + \tau_1^* G_3\right] d_{v_3}, \ a_{74} = 0,$$

$$a_{75} = \frac{\rho_0'}{\rho_0}\left[\left(\alpha_1' + \iota\omega\alpha_1'^*\right) F_1' + \tau_1'^* G_1'\right] d_{v_1}', \quad a_{76} = \frac{\rho_0'}{\rho_0}\left[\left(\alpha_1' + \iota\omega\alpha_1'^*\right) F_2' + \tau_1'^* G_2'\right] d_{v_2}',$$

$$a_{77} = \frac{\rho_0'}{\rho_0}\left[\left(\alpha_1' + \iota\omega\alpha_1'^*\right) F_3' + \tau_1'^* G_3'\right] d_{v_3}', \ a_{78} = 0, \ a_{81} = \left[\left(\alpha_1 + \iota\omega\alpha_1^*\right) F_1 + \tau_1^* G_1\right] k,$$

$$a_{82} = \left[\left(\alpha_1 + \iota\omega\alpha_1^*\right) F_2 + \tau_1^* G_2\right] k, \quad a_{83} = \left[\left(\alpha_1 + \iota\omega\alpha_1^*\right) F_3 + \tau_1^* G_3\right] k, \quad a_{84} = 0,$$

$$a_{85} = -\frac{\rho_0'}{\rho_0}\left[\left(\alpha_1' + \iota\omega\alpha_1'^*\right) F_1' + \tau_1'^* G_1'\right] k', \ a_{86} = -\frac{\rho_0'}{\rho_0}\left[\left(\alpha_1' + \iota\omega\alpha_1'^*\right) F_2' + \tau_1'^* G_2'\right] k',$$

$$a_{87} = -\frac{\rho_0'}{\rho_0}\left[\left(\alpha_1' + \iota\omega\alpha_1'^*\right) F_3' + \tau_1'^* G_3'\right] k', \quad a_{88} = 0.$$

For an incident P_1-wave:

$$h_1 = -a_{11}, \qquad h_2 = a_{21}, \qquad h_3 = -a_{31}, \qquad h_4 = a_{41},$$

$$h_5 = a_{51}, \qquad h_6 = -a_{61}, \qquad h_7 = a_{71}, \qquad h_8 = -a_{81}.$$

For an incident $SV-$ wave:

$$h_1 = a_{14}, \qquad h_2 = -a_{24}, \qquad h_3 = a_{34}, \qquad h_4 = -a_{44},$$

$$h_5 = a_{54}, \qquad h_6 = a_{64}, \qquad h_7 = a_{74}, \qquad h_8 = a_{84}.$$

References

[1] Achenbach, J. D., *Wave Propagation in Elastic Solids*, North Holland, Amsterdam, 1976.

[2] Aki, K. and Rechards, P. G., *Quantitative Seismology*, University Science Books, 1980.
[3] Bhagwan, J. and Tomar, S. K., Reflection and transmission of plane dilatational wave at a plane interface between an elastic solid half-space and a thermo-viscoelastic solid half-space with voids, *J. Elasticity*, **121** (2015), 69–88.
[4] Borcherdt, R. D., Reflection and refraction of type-II S waves in elastic and anelastic media, *Bull. Seismol. Soc. Am.*, **67** (1977), 43–67.
[5] Borcherdt, R. D., *Viscoelastic Waves in Layered Media*, Cambridge University Press, New York, 2009.
[6] Chirita, S., On the spatial behavior of the steady-state vibrations in thermoviscoelastic porous materials, *J. Therm. Stress.*, **38** (2015), 96–109.
[7] Chirita, S. and Danescu, A., Surface waves problem in a thermoviscoelastic porous half-space, *Wave Motion*, **54** (2015), 100–114.
[8] Cowin, S. C. and Nunziato, J. W., Linear elastic materials with voids, *J. Elasticity*, **13** (1983), 125–147.
[9] Ewing, W. M., Press, F. and Jardetzky, W. S., *Elastic Waves in Layered Media*, Mc-Graw Hill Book Co., 1957.
[10] Goyal, S. and Tomar, S. K., Reflection/refraction of a dilatational wave at a plane interface between uniform elastic and swelling porous half-spaces, *Transp. Porous Med.*, **109** (2015), 609–632.
[11] Goyal, S. and Tomar, S. K., Reflection and transmission of inhomogeneous waves at the plane interface between two dissimilar swelling porous half-spaces, *STRPM - An Int. J.*, **6** (2015), 51–69.
[12] Iesan, D., A theory of thermoelastic materials with voids, *Acta Mech.*, **60** (1986), 67–89.
[13] Iesan, D., On a theory of thermo-viscoelastic materials with voids, *J. Elasticity*, **104** (2011), 369–384.
[14] Knott, C. G., Reflection and refraction of elastic waves with seismological applications, *Phil. Mag., 5th Ser.*, **48** (1899), 64–97.
[15] Kumar, R. and Kumar, R., Reflection and refraction of elastic waves at the interface of an elastic half-space and initially stressed thermoelastic with voids half-space, *Multi. Mod. Mat. Struct.*, **8** (2012), 355–379.
[16] Lord, H. W. and Shulman, Y., A generalized dynamical theory of thermoelasticity, *J. Mech. Phys. Solids*, **15** (1967), 299–309.
[17] Nunziato, J. W. and Cowin, S. C., A non-linear theory of elastic material with voids, *Arch. Ration. Mech. Anal.*, **72** (1979), 174–200.
[18] Sheriff, R. E. and Geldart, L. P., *Exploration Seismology*, Cambridge University Press, New York, 1995.
[19] Tomar, S. K., Elastic wave propagation in materials with voids-A review, *The Math. Student*, (2007), 241–260.
[20] Tomar, S. K., Bhagwan, J. and Steeb, H., Time harmonic waves in thermo-viscoelastic material with voids, *J. Vib. Cont.*, **20** (2014), 1119–1136.
[21] Tomar, S. K. and Goyal, S., Elastic waves in swelling porous media, *Transp. Porous. Med.*, **100** (2013) 39–68. [Erratum, *ibid*, **103** (2014), 315–324].
[22] Sharma, J. N., Some considerations on the Rayleigh-Lamb wave propagation in visco-thermoelastic plates, *J. Vib. Cont.*, **11** (2005), 1311–1335.
[23] Sharma, K. and Kumar, P., Propagation of plane waves and fundamental solution in thermoviscoelastic medium with voids, *J. Therm. Stress.*, **36** (2013), 91–111.

[24] Singh, J. and Tomar, S. K., Plane waves in thermo-elastic material with voids, *Mech. Mat.*, **39** (2007), 932–940.

[25] Singh, S. S., Reflection and transmission of couple longitudinal waves at a plane interface between two dissimilar half-spaces of thermo-elastic materials with voids, *Appl. Math. Comp.*, **218** (2011), 3359–3371.

[26] Singh, S. S., Transverse wave at a plane interface in thermo-elastic materials with voids, *Meccanica*, **48** (2013), 617–630.

[27] Singh, J., Transmission of a transverse wave across a generalized porous thermo-elastic solid interface, *J. Vib. Cont.*, **22** (2016), 2938–2945.

Jai Bhagwan
Department of Mathematics, Government College
Saha (Ambala), Haryana, India
E-mail: *jaipuchd@gmail.com*

S. K. Tomar
Department of Mathematics, Panjab University
Chandigarh, India
E-mail: *sktomar66@gmail.com*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 91-96

GENERALIZING A PARTITION THEOREM OF ANDREWS

S. FU AND D. TANG

(Received : 14 - 05 - 2017; Revised : 29-05-2017)

Abstract. Motivated by Andrews' recent work related to Euler's partition theorem, we consider the set of partitions of an integer n where the set of even parts has exactly j elements, versus the set of partitions of n where the set of repeated parts has exactly j elements. These two sets of partitions turn out to be equinumerous, and this naturally encloses Euler's theorem and Andrews' theorem as two special cases. We give two proofs, one using generating function, and the other is a direct bijection that builds on Glaisher's bijection.

1. Introduction

A *partition* [1] π of a positive integer n is a finite sequence of weakly decreasing positive integers $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_r > 0$ such that $\sum_{i=1}^{r} \pi_i = n$. The π_i's are called the *parts* of π. Sometimes it is useful to use a notation that makes explicit the number of times that a particular integer occurs as a part and we write

$$\pi = (1^{m_1} 2^{m_2} 3^{m_3} \cdots),$$

where exactly m_j of the π_i's are equal to j.

In 1748, Euler [3] gave the generating function proof of the following partition theorem:

Theorem 1.1. *The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

When odd numbers are viewed as numbers not divisible by 2, the following theorem of Glaisher [1, Corollary 1.3] can be viewed as a natural generalization of Theorem 1.1.

Theorem 1.2. *The number of partitions of n into parts not divisible by d equals the number of partitions of n into parts each occuring no more than $d-1$ times.*

Partitions into parts not divisible by d are usually called *d-regular partitions*, and this is the term we use in the sequel.

In order to solve two conjectures of Beck posted on the OEIS (see [4] and [5]), Andrews [2] established the following

2010 Mathematics Subject Classification : 05A17, 11P83
Key words and phrases : Partition; Euler's partition theorem; Glashier's theorem.

Theorem 1.3 (Theorem 1 in [2]). *If $a(n)$ denotes the number of partitions of n such that the set of even parts has only one element (possibly repeated), $b(n)$ counts the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n, and $c(n)$ denotes the number of partitions of n where exactly one part is repeated. Then $a(n) = b(n) = c(n)$ for all $n \geq 1$.*

In particular, the assertion $a(n) = b(n)$ verifies Beck's conjecture listed in [4]. In this note, we give two generalizations of Andrews' theorem. Let $O_{j,k}(n)$ denote the number of partitions of n such that there are exactly j different parts $\equiv 0 \pmod{k}$ (possibly repeated), and $D_{j,k}(n)$ denote the number of partitions of n such that there are exactly j different parts repeated at least k times. We obtain the first generalization of Theorem 1.3.

Theorem 1.4. *For all $j, n \geq 0$ and $k \geq 1$, $O_{j,k}(n) = D_{j,k}(n)$.*

Evidently, $j = 0, k = 2$ gives us Theorem 1.1, $j = 0, k \geq 2$ produces Theorem 1.2, and $j = 1, k = 2$ leads us back to the $a(n) = c(n)$ part of Theorem 1.3.

Next, in order to state another generalization of Theorem 1.3 involving $b(n)$, let $E_k(n)$ denote the excess of the number of parts $\equiv 1 \pmod{k}$ in the k-regular partitions of n, over the number of distinct parts in the partitions of n with no parts being repeated more than $k-1$ times, then we have the following theorem, which is precisely Theorem 1.3 when we set $k = 2$.

Theorem 1.5. *For all $n \geq 0$ and $k \geq 2$, $O_{1,k}(n) = D_{1,k}(n) = E_k(n)$.*

2. Proof of Theorems

In this section, we first give a generating function proof as well as a bijective proof of Theorem 1.4. This partially answers to Andrews' request [2]. Then we follow Andrews' method and give a generating function proof of Theorem 1.5.

Generating function proof of Theorem 1.4.

First, define the following two bivariate generating functions

$$\mathcal{O}_k(z,q) := \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} O_{j,k}(n) z^j q^n, \qquad \mathcal{D}_k(z,q) := \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} D_{j,k}(n) z^j q^n.$$

Obviously, we have

$$\begin{aligned}\mathcal{O}_k(z,q) &= \prod_{m=1}^{\infty}\left(1 + zq^{km} + zq^{2km} + zq^{3km} + \cdots\right) \prod_{\substack{n=1 \\ n\not\equiv 0 \pmod{k}}}^{\infty} \frac{1}{1-q^n} \\ &= \prod_{m=1}^{\infty}\left(1 + \frac{zq^{km}}{1-q^{km}}\right)\prod_{n=1}^{\infty}\frac{1-q^{kn}}{1-q^n}, \qquad (2.1)\end{aligned}$$

and

$$\mathcal{D}_k(z,q) = \prod_{n=1}^{\infty}\left(1 + q^n + q^{2n} + \cdots + q^{(k-1)n} + zq^{kn} + zq^{(k+1)n} + \cdots\right)$$

$$= \prod_{n=1}^{\infty} \left(1 + q^n + q^{2n} + \cdots + q^{(k-1)n}\right) \left(1 + zq^{kn} + zq^{2kn} + \cdots\right)$$
$$= \prod_{n=1}^{\infty} \frac{1-q^{kn}}{1-q^n} \prod_{m=1}^{\infty} \left(1 + \frac{zq^{km}}{1-q^{km}}\right). \tag{2.2}$$

Equating the coefficients of $z^j q^n$ for $\mathcal{O}_k(z,q)$ and $\mathcal{D}_k(z,q)$ completes the proof. $\square$

Bijective proof of Theorem 1.4.
We construct a family of bijections $\{\varphi_k\}_{k\geq 1}$ on $\mathcal{P}$, the set of all partitions. For $k=1$, we simply take φ_1 to be the identity map since in this case the two sets of partitions in question coincide with each other. For general $k \geq 2$, let $\pi = \pi_1 + \cdots + \pi_r$ be a partition. We discuss two cases, depending on whether the part π_i is divisible by k or not.

- $\pi_i \equiv 0 \pmod{k}$: Denote by $\overline{\pi}$ the partition composed of all parts $\equiv 0 \pmod{k}$ in π. In this case, we simply map the part π_i to the k-copies of equal parts
$$\underbrace{\frac{\pi_i}{k} + \frac{\pi_i}{k} + \cdots + \frac{\pi_i}{k}}_{k \text{ copies}}.$$
Collect all parts converted this way and denote by $\overline{\lambda}$ the partition composed of them. Consequently, the parts in $\overline{\lambda}$ are all repeated at least k times, which means we can identify them when we apply the inverse map. Actually the multiplicity of every part in $\overline{\lambda}$ must be a multiple of k.
- $\pi_i \not\equiv 0 \pmod{k}$: Denote by $\tilde{\pi}$ the partition composed of all parts $\not\equiv 0 \pmod{k}$ in π. Assume $\widetilde{\pi}$ has the form
$$\underbrace{q_1 + q_1 + \cdots + q_1}_{m_1 \text{ times}} + \underbrace{q_2 + q_2 + \cdots + q_2}_{m_2 \text{ times}} + \cdots + \underbrace{q_s + q_s + \cdots + q_s}_{m_s \text{ times}}.$$
We now consider two cases again, depending on whether m_i is no less than k or not. If $m_i \geq k$, then we split k copies off of the part q_i and combine (add) these k parts of equal size into a new part, and we keep splitting k copies off of the remaining parts (if any) and combining these parts of equal size into a new parts until there are no more parts repeated more than $k-1$ times, then we eventually obtain a partition $\widetilde{\lambda}$ where all parts occur less than k times. This is essentially Glaisher's map for proving Theorem 1.2.

  Finally let $\varphi_k(\pi) := \lambda = \overline{\lambda} \bigcup \widetilde{\lambda}$, where the union $\overline{\lambda} \bigcup \widetilde{\lambda}$ is the partition consisting of all parts in $\overline{\lambda}$ and $\widetilde{\lambda}$ (arranged in nonincreasing order). We observe that the number of different parts $\equiv 0 \pmod{k}$ in partition π is the same as the number of parts that are repeated more than $k-1$ times in λ via the above map φ_k.

Table 1. One-to-one correspondence in Example 2.1

$\equiv 0 \pmod 4$	$f_i \geq 4$
$(12, 4)$	$(3, 3, 3, 3, 1, 1, 1, 1)$
$(8, 4, 4)$	$(2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1)$
$(8, 4, 3, 1)$	$(3, 2, 2, 2, 2, 1, 1, 1, 1, 1)$
$(8, 4, 2, 2)$	$(2, 2, 2, 2, 2, 2, 1, 1, 1, 1)$
$(8, 4, 2, 1, 1)$	$(2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1)$
$(8, 4, 1, 1, 1, 1)$	$(4, 2, 2, 2, 2, 1, 1, 1, 1)$

The inverse map should mostly be clear, with the only caveat being that when we decompose λ as $\overline{\lambda}\bigcup\widetilde{\lambda}$, $\overline{\lambda}$ consists of those parts in λ being repeated, say $m \geq k$ times, but in general not all m copies of this part are included in $\overline{\lambda}$. More precisely, we can uniquely write $m = tk + l$, with $0 \leq l < k$, then exactly tk copies of this part are included in $\overline{\lambda}$, with the remaining l copies included in $\widetilde{\lambda}$. □

Example 2.1. Table 1 lists all partitions of 16 enumerated by $O_{2,4}(16)$ and $D_{2,4}(16)$ respectively, with one-to-one correspondence in each row via φ_4.

Proof of Theorem 1.5. The proof of Theorem 1.5 relies on generating functions and differentiation. According to (2.1) and (2.2), by extracting the coefficients of z in both generating functions we have

$$\sum_{n=0}^{\infty} O_{1,k}(n)q^n = \sum_{n=0}^{\infty} D_{1,k}(n)q^n = \sum_{m=1}^{\infty} \frac{q^{km}}{1-q^{km}} \prod_{n=1}^{\infty} \frac{1-q^{kn}}{1-q^n}. \tag{2.3}$$

Moreover, in the infinite product

$$\prod_{n=1}^{\infty} \frac{1}{(1-zq^{k(n-1)+1})(1-q^{k(n-1)+2})\cdots(1-q^{k(n-1)+k-1})},$$

the coefficient of $z^M q^N$ is the number of k-regular partitions of N with exactly M parts $\equiv 1 \pmod k$, while in the infinite product

$$\prod_{n=1}^{\infty} \left(1 + zq^n + zq^{2n} + \cdots + zq^{(k-1)n}\right),$$

the coefficient of $z^M q^N$ is the number of partitions of N into M different parts, each part repeated less than k times.

Thus if we differentiate each of these functions with respect to z we will then be counting each partition with M designated parts with weight M. Consequently

$$\sum_{n=0}^{\infty} D_k(n)q^n = \frac{\partial}{\partial z}\bigg|_{z=1} \left(\prod_{n=1}^{\infty} \frac{1}{(1-zq^{k(n-1)+1})(1-q^{k(n-1)+2})\cdots(1-q^{k(n-1)+k-1})} - \prod_{n=1}^{\infty} \left(1 + zq^n + zq^{2n} + \cdots + zq^{(k-1)n}\right)\right)$$

$$
\begin{aligned}
&=\sum_{n=1}^{\infty}\frac{q^{k(n-1)+1}}{(1-q^{k(n-1)+1})^2(1-q^{k(n-1)+2})\cdots(1-q^{k(n-1)+k-1})}\times\\
&\prod_{\substack{m=1\\ m\neq n}}^{\infty}\frac{1}{(1-q^{k(m-1)+1})(1-q^{k(m-1)+2})\cdots(1-q^{k(m-1)+k-1})}\\
&-\sum_{n=1}^{\infty}\left(q^n+q^{2n}+\cdots+q^{(k-1)n}\right)\prod_{\substack{m=1\\ m\neq n}}^{\infty}\left(1+q^m+q^{2m}+\cdots+q^{(k-1)m}\right)\\
&=\prod_{m=1}^{\infty}\frac{1-q^{km}}{1-q^m}\sum_{n=1}^{\infty}\frac{q^{k(n-1)+1}}{1-q^{k(n-1)+1}}-\sum_{n=1}^{\infty}\frac{q^n-q^{kn}}{1-q^{kn}}\prod_{m=1}^{\infty}\frac{1-q^{km}}{1-q^m}\\
&=\sum_{n=1}^{\infty}\frac{q^{kn}}{1-q^{kn}}\prod_{m=1}^{\infty}\frac{1-q^{km}}{1-q^m},
\end{aligned}
$$

where the last equality follows from

$$
\sum_{n=1}^{\infty}\frac{q^{k(n-1)+1}}{1-q^{k(n-1)+1}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q^{m(k(n-1)+1)}=\sum_{m=1}^{\infty}\frac{q^m}{1-q^{km}}.
$$

Comparing the final expression with (2.3) completes the proof. □

Example 2.2. Table 2 lists all four types of partitions of 8, i.e., partitions with exactly one part divisible by 3, partitions with exactly one part repeated at least 3 times, 3-regular partitions and partitions all of whose parts are repeated at most 2 times. One checks that $O_{1,3}(8) = D_{1,3}(8) = 9, E_3(8) = 37 - 28 = 9$.

TABLE 2. Three-way correspondence in Example 2.2

$\equiv 0 \pmod 3$	$f_i \geq 3$	3-regular partitions	$f_i \leq 2$
$(6,2)$	$(2,2,2,2)$	(8)	(8)
$(6,1,1)$	$(2,2,2,1,1)$	$(7,1)$	$(7,1)$
$(3,3,2)$	$(2,1,1,1,1,1,1)$	$(2,2,2,2)$	$(6,2)$
$(3,3,1,1)$	$(1,1,1,1,1,1,1,1)$	$(2,2,2,1,1)$	$(6,1,1)$
$(5,3)$	$(5,1,1,1)$	$(5,1,1,1)$	$(5,3)$
$(4,3,1)$	$(4,1,1,1,1)$	$(5,2,1)$	$(5,2,1)$
$(3,2,2,1)$	$(2,2,1,1,1,1)$	$(4,4)$	$(4,4)$
$(3,2,1,1,1)$	$(3,2,1,1,1)$	$(4,1,1,1,1)$	$(4,3,1)$
$(3,1,1,1,1,1)$	$(3,1,1,1,1,1)$	$(4,2,2)$	$(4,2,2)$
		$(4,2,1,1)$	$(4,2,1,1)$
		$(2,1,1,1,1,1,1)$	$(3,3,2)$
		$(1,1,1,1,1,1,1,1)$	$(3,3,1,1)$
		$(2,2,1,1,1,1)$	$(3,2,2,1)$

3. Conclusion

Several questions present themselves from this study. We collect them here to motivate further work.

1) The bijection between partition sets enumerated by $a(n)$ and $b(n)$, or between $c(n)$ and $b(n)$ is still unclear.
2) A more ambitious task than 1) is to "complete the puzzle" and find the family of partitions enumerated by certain $E_{j,k}(n)$ such that $O_{j,k}(n) = D_{j,k}(n) = E_{j,k}(n)$ for all $j \geq 1$ and our Theorem 1.5 becomes the special case of $j = 1$. The differentiation technique suggests the complexity of the computation. Therefore, reverse engineering of our proof to track down this third family becomes less optimistic.
3) Theorem 1.4 sets the classical Euler's partition theorem at the center of a natural family of partition theorems. On a general note, can we try this new perspective with other classical partition theorems? More precisely, for a partition theorem that links those partitions of n without certain parts of "type A" (even parts in the case of Euler's theorem), with those partitions of n without certain parts of "type B" (repeated parts in the case of Euler's theorem), we can consider what happens if we require that there are exactly j different parts of type A (resp. type B) in the first (resp. the second) set of partitions.

Acknowledgement. Both authors were supported by the Fundamental Research Funds for the Central Universities (No. CQDXWL-2014-Z004) and the National Science Foundation of China (No. 11501061).

References

[1] Andrews, G. E., *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications, Vol. **2** (G.-C. Rota, ed.), Addison-Wesley, Reading, 1976 (Reprinted: Cambridge Univ. Press, London and New York, 1984).

[2] Andrews, G. E., Euler's partition identity and two problems of George Beck, *Preprint*.

[3] Euler, L., *Introductio in Analysin Infinitorum*, vol. **2**, MM Bousquet, Hanson (1748).

[4] The On-Line Encyclopedia of Integer Sequences, *http://oeis.org/A090867* oeis: A090867.

[5] The On-Line Encyclopedia of Integer Sequences, *http: //oeis.org/A265251* oeis: A265251.

Shishuo Fu
College of Mathematics and Statistics, Chongqing University
Huxi Campus LD506, Chongqing 401331, P.R. China
E-mail: *fsshuo@cqu.edu.cn*

Dazhao Tang
College of Mathematics and Statistics, Chongqing University
Huxi Campus LD206, Chongqing 401331, P.R. China
E-mail: *dazhaotang@sina.com*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 97-110

EXTENDING THE BEAUTY OF RAMANUJAN'S NUMBER

B. SAGAR

(Received : 28 - 05 - 2017; Revised : 21 - 08 - 2017)

Abstract. Using the four digits of famous Ramanujan's number 1729, namely $1, 7, 2$ and 9, *in that order*, we obtain interesting expressions to express all integers from -529 to $+529$ using mathematical operations, functions (such as square root, exponential, integral part, factorial, logarithm etc.) and formulas (such as summation formulae, etc.)

1. Introduction

The number 1729 is well known as the Ramanujan's number. In fact, according to a well-known story, Hardy rode a taxicab bearing this number on his way to visit an ailing Ramanujan. He remarked to him that his cab number didn't seem to be particularly interesting, to which Ramanujan immediately responded [1], contrariwise, that it is a very interesting number indeed. It is the smallest natural number that can be written in two different ways as the sum of two cubes of natural numbers:

$$1729 = 10^3 + 9^3 \quad = \quad 12^3 + 1^3$$

In reference to this story, Diophantine equations (that is, equations with integer coefficients for which solutions are sought in integers or rational numbers) involving sums of two cubes are sometimes dubbed as taxicab equations. Interestingly, Ramanujan knew the following formula for the sum of cubes [5], namely

$$(x^2 + 9xy - y^2)^3 + (12x^2 - 4xy + 2y^2)^3 = (9x^2 - 7xy - y^2)^3 + (10x^2 + 2y^2)^3$$

with 1729 as the first instance by putting $x = 1$ and $y = 1$.

To mention some more beautiful facts about the Ramanujan number, it may be noted that Masahiko Fujiwara showed [2, 3] that 1729 is one of the four positive integers (the others being $81, 1458$, and the trivial case 1) which, when its digits are added together, produces a sum which, when multiplied by its reversal, yields the original number: $1 + 7 + 2 + 9 = 19$ and $19 \times 91 = 1729$. It is also known that

$$1729 = 55^2 - 36^2 = 73^2 - 60^2 = 127^2 - 120^2 = 865^2 - 864^2.$$

Here, we extend the beauty of Ramanujan's number by representing all integers from -529 to 529 through interesting and innovative expressions using the

---

2010 Mathematics Subject Classification: 00A08, 97R80.
Key words and phrases: Ramanujan's number, Exponential, factorial, integral part, pi, square root.

four digits $1, 7, 2,$ and 9, *in that order*, and using mathematical operations, functions (such as square root, exponential, integral part, factorial, logarithm etc.) and formulas (such as summation formulae, etc).

2. Representation of integers using 1, 7, 2, 9.

$0 = 1(7 + 2 - 9)$

$1 = 1 + 7 + 2 - 9$

$2 = 1 + (7 + 2) \div 9$

$3 = -1 - 7 + 2 + 9$

$4 = -1(7) + 2 + 9$

$5 = 1 - 7 + 2 + 9$

$6 = 17 - 2 - 9$

$7 = (1 + 7)2 - 9$

$8 = 1^7 \times 2^{\sqrt{9}}$

$9 = 1^7 + 2^{\sqrt{9}}$

$10 = 17 + 2 - 9$

$11 = -1(7) + 2(9)$

$12 = -17 + 29$

$13 = -1 + 7 - 2 + 9$

$14 = 1(7) - 2 + 9$

$15 = 1 + 7 - 2 + 9$

$16 = 1 + 7 + 2^{\sqrt{9}}$

$17 = -1 + 7 + 2 + 9$

$18 = 1 + (7 \times 2) + \sqrt{9}$

$19 = 1 + 7 + 2 + 9$

$20 = 1 \times 7 \times 2 + (\sqrt{9})!$

$21 = -(1 + 7) + 29$

$22 = -(1 \times 7) + 29$

$23 = 1 - 7 + 29$

$24 = -1 + 7 + (2 \times 9)$

$25 = 1 \times 7 + (2 \times 9)$

$26 = 1 + 7 + (2 \times 9)$

$27 = [(-1 + 7) \div 2] \times 9$

$28 = (-1)^7 + 29$

$29 = 1^7 \times 29$

$30 = 1^7 + 29$

$31 = (17 \times 2) - \sqrt{9}$

$32 = 1 + 7 + (-2 + (\sqrt{9})!)!$

$33 = (-1 + 7 - 2)! + 9$

$34 = 17 \times (\sqrt{-2 + (\sqrt{9})!})$

$35 = 17 + (2 \times 9)$

$36 = 1 \times 7 + 29$

$37 = 1 + 7 + 29$

$38 = -1 + {}^7P_2 - \sqrt{9}$

$39 = -1 + 7^2 - 9$

$40 = 1 \times (7^2) - 9$

$41 = 1 + 7^2 - 9$

$42 = -1 + 7^2 - (\sqrt{9})!$

$43 = 17 \times 2 + 9$

$44 = 1 + 7^2 - (\sqrt{9})!$

$45 = -1 + 7^2 - \sqrt{9}$

$46 = 17 + 29$

$47 = 1 + 7^2 - \sqrt{9}$

$48 = -1 + 7(-2 + 9)$

$49 = 1 \times 7(-2 + 9)$

$50 = 1 + 7(-2 + 9)$

$51 = -1 + 7^2 + \sqrt{9}$

$52 = 1 \times 7^2 + \sqrt{9}$

$53 = 1 + 7^2 + \sqrt{9}$

$54 = 1 \times (7 + 2) \times (\sqrt{9})!$

$55 = -1 + 7(2^{\sqrt{9}})$

$56 = 1 \times 7(2^{\sqrt{9}})$

$57 = 1 + 7(2^{\sqrt{9}})$

$58 = 1 \times 7^2 + 9$

$59 = 1 + 7^2 + 9$

$60 = (1 + 7 + 2) \times (\sqrt{9})!$

$61 = (1 + 7)^2 - \sqrt{9}$

$62 = -1 + 72 - 9$

$63 = 1 \times 72 - 9$

$64 = 1 + 72 - 9$

$65 = -1 + 72 - (\sqrt{9})!$

$66 = 1 \times 72 - (\sqrt{9})!$

$67 = 1 + 72 - (\sqrt{9})!$

$68 = -1 + 72 - \sqrt{9}$

$69 = 1 \times 72 - \sqrt{9}$

$70 = 1 + 72 - \sqrt{9}$

$71 = 1 \times 7 + 2^{(\sqrt{9})!}$

$72 = 1 + 7 + 2^{(\sqrt{9})!}$

$73 = (1 + 7)^2 + 9$

$74 = -1 + 72 + \sqrt{9}$

$75 = 1 \times 72 + \sqrt{9}$

$76 = 1 + 72 + \sqrt{9}$

$77 = -1 + 72 + (\sqrt{9})!$

$78 = 1 \times 72 + (\sqrt{9})!$

$79 = 1 + 72 + (\sqrt{9})!$

$80 = -1 + 72 + 9$

$81 = 1 \times 72 + 9$

$82 = 1 + 72 + 9$

$83 = -1 + 7 \times 2 \times (\sqrt{9})!$

$84 = 1 \times 7 \times 2 \times (\sqrt{9})!$

$85 = 1 + 7 \times 2 \times (\sqrt{9})!$

$86 = 1 + [(7 - 2)^e] + (\sqrt{9})!$

$87 = |\sum_{n=-17}^{(2+9)} n|$

$88 = (1 + 7) \times (2 + 9)$

$$89 = -1 + \sum_{n=-(7+2)}^{9} |n|$$
$$90 = (1 + 7 + 2) \times 9$$
$$91 = 1 + \sum_{n=-(7+2)}^{9} |n|$$
$$92 = (\sum_{n=1}^{7} n) + 2^{(\sqrt{9})!}$$
$$93 = (\sum_{n=-1}^{7} |n|) + 2^{(\sqrt{9})!}$$
$$94 = \sum_{n=-(1\times7)}^{(2+9)} |n|$$
$$95 = (\sum_{n=-1}^{(7\times2)} n) - 9$$
$$96 = (\sum_{n=1}^{(7\times2)} n) - 9$$
$$97 = (\sum_{n=-1}^{(7\times2)} |n|) - 9$$
$$98 = (\sum_{n=-1}^{(7\times2)} n) - (\sqrt{9})!$$
$$99 = (\sum_{n=1}^{(7\times2)} n) - (\sqrt{9})!$$
$$100 = \sum_{n=-(1+7+2)}^{9} |n|$$
$$101 = (\sum_{n=-1}^{(7\times2)} n) - \sqrt{9}$$
$$102 = 17 \times 2 \times (\sqrt{9})$$
$$103 = \sum_{n=-1}^{(7\times2)} |n| - \sqrt{9}$$
$$104 = \sum_{n=-1}^{(7-2+9)} n$$
$$105 = \sum_{n=1}^{(7-2+9)} n$$
$$106 = \sum_{n=-1}^{(7-2+9)} |n|$$
$$107 = (\sum_{n=-1}^{(7\times2)} n) + \sqrt{9}$$
$$108 = (\sum_{n=1}^{(7\times2)} n) + \sqrt{9}$$
$$109 = (\sum_{n=-1}^{(7\times2)} |n|) + \sqrt{9}$$
$$110 = -1 + (7 - 2)! - 9$$
$$111 = 1 \times (7 - 2)! - 9$$
$$112 = 1 + (7 - 2)! - 9$$
$$113 = -1 + (7 - 2)! - (\sqrt{9})!$$
$$114 = 1 \times (7 - 2)! - (\sqrt{9})!$$
$$115 = 1 + (7 - 2)! - (\sqrt{9})!$$
$$116 = -1 + (7 - 2)! - \sqrt{9}$$
$$117 = 1 \times (7 - 2)! - \sqrt{9}$$
$$118 = 1 + (7 - 2)! - \sqrt{9}$$
$$119 = -1 + (7 \times 2 - 9)!$$
$$120 = 1 \times (7 \times 2 - 9)!$$
$$121 = 1 + (7 \times 2 - 9)!$$
$$122 = -1 + (7 - 2)! + \sqrt{9}$$
$$123 = 1 \times (7 - 2)! + \sqrt{9}$$
$$124 = 1 + (7 - 2)! + \sqrt{9}$$
$$125 = -1 + 7 \times 2 \times 9$$
$$126 = 1 \times 7 \times 2 \times 9$$
$$127 = 1 + 7 \times 2 \times 9$$
$$128 = -1 + (7 - 2)! + 9$$
$$129 = 1 \times (7 - 2)! + 9$$

$$130 = 1 + (7-2)! + 9$$

$$131 = [(-1+7)^e] - 2 + \sqrt{9}$$

$$132 = \sum_{n=-17}^{(2-9)} |n|$$

$$133 = \sum_{n=-(1+7)\times 2}^{-\sqrt{9}} |n|$$

$$134 = [(-1+7)^e] - 2 + (\sqrt{9})!$$

$$135 = (1 + 7 \times 2) \times 9$$

$$136 = 17 \times 2^{\sqrt{9}}$$

$$137 = [(-1+7)^e] - 2 + 9$$

$$138 = \sum_{n=-17}^{(-2\times\sqrt{9})} |n|$$

$$139 = [(1+7-2)^e] + 9$$

$$140 = \sum_{n=1}^{7} n^2 + [\log\sqrt{9}]$$

$$141 = \sum_{n=-17}^{2} (-n) - 9$$

$$142 = -1 + \sum_{n=-7}^{2\times 9} n$$

$$143 = 1 \times \sum_{n=-7}^{2\times 9} n$$

$$144 = 1 + \sum_{n=-7}^{2\times 9} n$$

$$145 = [(-1+7)^e] + [2^e] + 9$$

$$146 = -1 + 7^2 \times \sqrt{9}$$

$$147 = 1 \times 7^2 \times \sqrt{9}$$

$$148 = 1 + 7^2 \times \sqrt{9}$$

$$149 = -1 + \sum_{n=-7\times 2}^{9} |n|$$

$$150 = 1 \times \sum_{n=-7\times 2}^{9} |n|$$

$$151 = 1 + \sum_{n=-7\times 2}^{9} |n|$$

$$152 = 1 + {}^7C_2 + [(\sqrt{9})!^e]$$

$$153 = (\sum_{n=-17}^{2} |n|) - \sqrt{9}$$

$$154 = (\sum_{n=-17}^{-2} |n|) + [\ln 9]$$

$$155 = (\sum_{n=-17}^{-2} |n|) + \sqrt{9}$$

$$156 = (\sum_{n=-17}^{2} (|n|) + [log(9)]$$

$$157 = \sum_{n=-(1+7)\times 2}^{(\sqrt{9})!} |n|$$

$$158 = (\sum_{n=-17}^{-2} |n|) + (\sqrt{9})!$$

$$159 = (\sum_{n=-17}^{2} |n|) + \sqrt{9}$$

$$160 = [(-1+7)! \times 2] \div 9$$

$$161 = (\sum_{n=-17}^{-2} |n|) + 9$$

$$162 = (\sum_{n=-17}^{2} |n|) + (\sqrt{9})!$$

$$163 = 172 - 9$$

$$164 = 17 \times 2 + [(\sqrt{9})!^e]$$

$$165 = \sum_{n=-17}^{2} |n| + 9$$

$$166 = 172 - (\sqrt{9})!$$

$$167 = -(1+7) + \sum_{n=[2^e]}^{[(\sqrt{9})^e]} n$$

$$168 = (\sum_{n=1}^{7} n) \times 2 \times \sqrt{9}$$

$$169 = 172 - \sqrt{9}$$

$$170 = 1 + [7^e] - 29$$

$171 = (17 + 2) \times 9$

$172 = 1 + (7 + 2) \times [(\sqrt{9})^e]$

$173 = [(-1 + 7 - 2)^e] + [(\sqrt{9})!^e]$

$174 = (-1 + 7) \times 29$

$175 = 172 + \sqrt{9}$

$176 = 1^7 + \sum_{n=[2^e]}^{[(\sqrt{9})^e]} n$

$177 = (\sum_{n=-17}^{[2^e]} |n|) + \sqrt{9}$

$178 = 172 + (\sqrt{9})!$

$179 = -1 + [7^e] - 2 \times 9$

$180 = 1 \times [7^e] - 2 \times 9$

$181 = 172 + 9$

$182 = -1 + [7^e] - [2^e] - 9$

$183 = 1 \times [7^e] - [2^e] - 9$

$184 = 1 + [7^e] - [2^e] - 9$

$185 = -1 + (\sum_{n=-7}^{2} |n|) \times (\sqrt{9})!$

$186 = 1 \times (\sum_{n=-7}^{2} |n|) \times (\sqrt{9})!$

$187 = 1 + (\sum_{n=-7}^{2} |n|) \times (\sqrt{9})!$

$188 = -1 + {}^7C_2 \times 9$

$189 = 1 \times {}^7C_2 \times 9$

$190 = 1 + {}^7C_2 \times 9$

$191 = 1 \times [7^e] + 2 - 9$

$192 = 1 + [7^e] + 2 - 9$

$193 = 1 \times [7^e] - 2 - \sqrt{9}$

$194 = 1 + [7^e] - 2 - \sqrt{9}$

$195 = 1 + [7^e] + 2 - (\sqrt{9})!$

$196 = -1 + [7^e] + 2 - \sqrt{9}$

$197 = 1 \times [7^e] + 2 - \sqrt{9}$

$198 = 1 + [7^e] + 2 - \sqrt{9}$

$199 = 1 \times \sum_{n=-7}^{(2\times 9)} |n|$

$200 = 1 + \sum_{n=-7}^{(2\times 9)} |n|$

$201 = -1 + [7^e] - 2 + (\sqrt{9})!$

$202 = 1 \times [7^e] - 2 + (\sqrt{9})!$

$203 = 1 + [7^e] - 2 + (\sqrt{9})!$

$204 = -1 + [7^e] - 2 + 9$

$205 = 1 \times [7^e] - 2 + 9$

$206 = 1 + [7^e] - 2 + 9$

$207 = 1 + [7^e] + 2 + (\sqrt{9})!$

$208 = -1 + [7^e] + 2 + 9$

$209 = 1 \times [7^e] + 2 + 9$

$210 = 1 + [7^e] + 2 + 9$

$211 = 1 + [7^e] + 2 \times (\sqrt{9})!$

$212 = -1 + [7^e] + [2^e] + 9$

$213 = 1 \times [7^e] + [2^e] + 9$

$214 = 1 + [7^e] + [2^e] + 9$

$215 = -1 + [7^e] + 2 \times 9$

$216 = 1 \times [7^e] + 2 \times 9$

$217 = 1 + [7^e] + 2 \times 9$

$218 = -1 + [7^e] + 2 + [(\sqrt{9})^e]$

$219 = 1 \times [7^e] + 2 + [(\sqrt{9})^e]$

$220 = 1 + [7^e] + 2 + [(\sqrt{9})^e]$

$221 = [(1 + 7)^e] - 2^{(\sqrt{9})!}$

$222 = 1 \times [7^e] + \sum_{n=-2}^{(\sqrt{9})!} |n|$

$223 = 1 + [7^e] + \sum_{n=-2}^{(\sqrt{9})!} |n|$

$224 = -1 + [7^e] + \sum_{n=-[2^e]}^{\sqrt{9}} |n|$

$225 = (\sum_{n=-1}^{7} n) + [(-2 + 9)^e]$

$226 = -1 + [7^e] + 29$

$227 = 1 \times [7^e] + 29$

$228 = 1 + [7^e] + 29$

$229 = 1 - ([-(7^e)]) + 29$

$230 = (-1 + 7)! \div 2 - [(\sqrt{9})!^e]$

$231 = [(1 + 7)^e] - [2^e] \times 9$

$232 = 17 \times [2^e] + [(\sqrt{9})!^e]$

$233 = -1 + [7^e] + [2^e] \times (\sqrt{9})!$

$234 = 1 \times [7^e] + [2^e] \times (\sqrt{9})!$

$235 = 1 + [7^e] + [2^e] \times (\sqrt{9})!$

$236 = 1 \times [7^e] + 2 \times [(\sqrt{9})^e]$

$237 = 1 + [7^e] + 2 \times [(\sqrt{9})^e]$

$238 = 1 - ([-(7^e)]) + 2 \times [(\sqrt{9})^e]$

$239 = -1 + [7^e] + \sum_{n=-2}^{9} n$

$240 = 1 \times [7^e] + \sum_{n=-2}^{9} n$

$241 = 1 + [7^e] + \sum_{n=-2}^{9} n$

$242 = 1 \times [7^e] + \sum_{n=2}^{9} n$

$243 = [(1 + 7)^e] - \sum_{n=-2}^{9} n$

$244 = 1 + [7^e] + \sum_{n=[\log 2]}^{9} n$

$245 = -1 + [7^e] + \sum_{n=-2}^{9} |n|$

$246 = 1 \times [7^e] + \sum_{n=-2}^{9} |n|$

$247 = 1 + [7^e] + \sum_{n=-2}^{9} |n|$

$248 = 1 + (7 + [2^e]) \times [(\sqrt{9})^e]$

$249 = -1 + (7 - 2)! + [(\sqrt{9})!^e]$

$250 = 1 \times (7 - 2)! + [(\sqrt{9})!^e]$

$251 = 1 + (7 - 2)! + [(\sqrt{9})!^e]$

$252 = 1 \times [7^e] + [2^e] \times 9$

$253 = 1 + [7^e] + [2^e] \times 9$

$254 = [(-1 + 7)^e] \times 2 - (\sqrt{9})!$

$255 = [(1 + 7)^e] - \sum_{n=[2^e]}^{9} n$

$256 = [(1 + 7)^e] - 29$

$257 = [(-1 + 7)^e] \times 2 - (\sqrt{9})$

$258 = [(-1 + 7)^e] - 2 + [(\sqrt{9})!^e]$

$259 = -1^7 + 2 \times [(\sqrt{9})!^e]$

$260 = [(-1+7)^e] \times 2 + [\log\sqrt{9}]$

$261 = -1 + [7^e] + 2^{(\sqrt{9})!}$

$262 = 1 \times [7^e] + 2^{(\sqrt{9})!}$

$263 = 1 + [7^e] + 2^{(\sqrt{9})!}$

$264 = -[(-1+7)^e] + 2 + [9^e]$

$265 = [(1+7)^e] - \sum_{n=2}^{(\sqrt{9})!} n$

$266 = [(-1+7)^e] \times 2 + (\sqrt{9})!$

$267 = [(1+7)^e] - 2 \times 9$

$268 = 1 + 7 + 2 \times [(\sqrt{9})!^e]$

$269 = [(-1+7)^e] \times 2 + 9$

$270 = [(1+7)^e] - [2^e] - 9$

$271 = -1 - (7-2)! + [9^e]$

$272 = -1 \times (7-2)! + [9^e]$

$273 = 1 - (7-2)! + [9^e]$

$274 = [(1+7)^e] - 2 - 9$

$275 = [(1+7)^e] + [-\log\sqrt{2}] - 9$

$276 = [(1+7)^e] - \sum_{n=(-2)}^{\sqrt{9}} |n|$

$277 = 17 + 2 \times [(\sqrt{9})!^e]$

$278 = [(1+7)^e] + 2 - 9$

$279 = [(1+7)^e] - 2 \times \sqrt{9}$

$280 = [(1+7)^e] - 2 - \sqrt{9}$

$281 = [(1+7)^e] + 2 - (\sqrt{9})!$

$282 = [(1+7)^e] - \sum_{n=(-2)}^{\sqrt{9}} n$

$283 = [(1+7)^e] - 2 + [\log\sqrt{9}]$

$284 = [(1+7)^e] + 2 - \sqrt{9}$

$285 = [(1+7)^e] + [\log(2+\sqrt{9})]$

$286 = [(1+7)^e] - 2 + \sqrt{9}$

$287 = [(1+7)^e] + 2 + [\log\sqrt{9}]$

$288 = [(1+7)^e] + \sum_{n=(-2)}^{\sqrt{9}} n$

$289 = [(1+7)^e] - 2 + (\sqrt{9})!$

$290 = [(1+7)^e] + 2 + \sqrt{9}$

$291 = [(1+7)^e] + 2 \times \sqrt{9}$

$292 = [(1+7)^e] - 2 + 9$

$293 = -1 + 7^2 \times (\sqrt{9})!$

$294 = 1 \times 7^2 \times (\sqrt{9})!$

$295 = 1 + 7^2 \times (\sqrt{9})!$

$296 = [(1+7)^e] + 2 + 9$

$297 = [(1+7)^e] + 2 \times (\sqrt{9})!$

$298 = (17)^2 + 9$

$299 = \sum_{n=17}^{29} n$

$300 = (1 + 7^2) \times (\sqrt{9})!$

$301 = -[(-(1+7)^e)] + [2^e] + 9$

$302 = 172 + [(\sqrt{9})!^e]$

$303 = [(1+7)^e] + 2 \times 9$

$304 = [(1+7)^e] + [\log 2] + [(\sqrt{9})^e]$

$305 = [(1+7)^e] + \sum_{n=2}^{(\sqrt{9})!} n$

$306 = [(1+7)^e] + \sum_{n=[\log 2]}^{(\sqrt{9})!} n$

$$307 = -1 + 7 \times \sum_{n=2}^{9} n$$

$$308 = 1 \times 7 \times \sum_{n=2}^{9} n$$

$$309 = 1 + 7 \times \sum_{n=2}^{9} n$$

$$310 = -[(1+7)^{\pi}] + 2 + [9^{\pi}]$$

$$311 = -1 + [-(7-2)^{e}] + [9^{e}]$$

$$312 = -1 - [(7-2)^{e}] + [9^{e}]$$

$$313 = -1 \times [(7-2)^{e}] + [9^{e}]$$

$$314 = [(1+7)^{e}] + 29$$

$$315 = 1 \times 7 \times \sum_{n=[\log 2]}^{9} n$$

$$316 = 1 + 7 \times \sum_{n=[\log 2]}^{9} n$$

$$317 = [(-1+7) \times 2^{e}] + [(\sqrt{9})!^{\pi}]$$

$$318 = -1 + 7^{\pi} - 2 - [(\sqrt{9})!^{e}]$$

$$319 = -1 - 72 + [9^{e}]$$

$$320 = -1 \times 72 + [9^{e}]$$

$$321 = 1 - 72 + [9^{e}]$$

$$322 = 1 \times 7 \times \sum_{n=[-\log 2]}^{9} |n|$$

$$323 = 1 + 7 \times \sum_{n=[-\log 2]}^{9} |n|$$

$$324 = \sum_{n=-17}^{2\times 9} |n|$$

$$325 = 1 - [7^{e}] \times 2 + ((\sqrt{9})!)!$$

$$326 = (1 - [7^{e}]) \times 2 + ((\sqrt{9})!)!$$

$$327 = [(1+7)^{e}] + \sum_{n=-2}^{9} n$$

$$328 = -(1+7)^{2} + [9^{e}]$$

$$329 = [(1+7)^{e}] + \sum_{n=2}^{9} n$$

$$330 = [(17 \div 2)^{e}] - (\sqrt{9})!$$

$$331 = -[-(1+7)^{e}] + \sum_{n=[\log 2]}^{9} n$$

$$332 = 1 - [-(7^{e})] + 2 + [(\sqrt{9})!^{e}]$$

$$333 = [(17 \div 2)^{e}] - \sqrt{9}$$

$$334 = [(17 \div 2)^{e}] - [\ln 9]$$

$$335 = -1 + 7 \times \sum_{n=-2}^{9} |n|$$

$$336 = 1 \times 7 \times \sum_{n=-2}^{9} |n|$$

$$337 = 1 + 7 \times \sum_{n=-2}^{9} |n|$$

$$338 = [(17 \div 2)^{e}] + [\ln 9]$$

$$339 = [(17 \div 2)^{e}] + \sqrt{9}$$

$$340 = -[-(17 \div 2)^{e}] + \sqrt{9}$$

$$341 = -1 \times 7 \times [-(2)^{\pi}] + [(\sqrt{9})!^{\pi}]$$

$$342 = -1 - 7^{2} + [9^{e}]$$

$$343 = -1 \times 7^{2} + [9^{e}]$$

$$344 = 1 - 7^{2} + [9^{e}]$$

$$345 = [(17 \div 2)^{e}] + 9$$

$$346 = 1 - ([-\ln 7]) - ([-(2)^{e}])^{\sqrt{9}}$$

$$347 = (-\sum_{n=1}^{(7+2)} n) + [9^{e}]$$

$$348 = (-\sum_{n=-1}^{(7+2)} n) + [9^{e}]$$

$$349 = -1 - 7 \times [2^{e}] + [9^{e}]$$

$$350 = -1 \times 7 \times [2^{e}] + [9^{e}]$$

$351 = 1 - 7 \times [2^e] + [9^e]$

$352 = 1 - 7 \times [2^e] - ([-(9)^e])$

$353 = -[(-1 + 7) \times 2^e] + [9^e]$

$354 = (-1 + 7)! \div 2 - (\sqrt{9})!$

$355 = (-\sum_{n=-1}^{7} |n|) - [2^\pi] + [9^e]$

$356 = (-\sum_{n=1}^{7} n) - [2^\pi] + [9^e]$

$357 = (-\sum_{n=-1}^{7} |n|) - [2^e] + [9^e]$

$358 = -17 \times 2 + [9^e]$

$359 = -1 + (7 - 2)! \times \sqrt{9}$

$360 = 1 \times (7 - 2)! \times \sqrt{9}$

$361 = 1 + (7 - 2)! \times \sqrt{9}$

$362 = (-\sum_{n=1}^{7} n) - 2 + [9^e]$

$363 = -(\sum_{n=-1}^{7} n) - 2 + [9^e]$

$364 = (-\sum_{n=1}^{7} n) + [\log 2] + [9^e]$

$365 = -(\sum_{n=-1}^{7} |n|) + 2 + [9^e]$

$366 = (-\sum_{n=1}^{7} n) + 2 + [9^e]$

$367 = 1 \times \sum_{n=-7}^{2} n + [9^e]$

$368 = 1 + \sum_{n=-7}^{2} n + [9^e]$

$369 = -17 - [2^e] + [9^e]$

$370 = -(\sum_{n=1}^{7} n) + [2^e] + [9^e]$

$371 = -(\sum_{n=-1}^{7} n) + [2^e] + [9^e]$

$372 = 1 - {}^7C_2 + [9^e]$

$373 = -17 - 2 + [9^e]$

$374 = -17 + [-\log 2] + [9^e]$

$375 = -17 + [\log 2] + [9^e]$

$376 = -(1 + 7) \times 2 + [9^e]$

$377 = -1 - 7 \times 2 + [9^e]$

$378 = -1 \times 7 \times 2 + [9^e]$

$379 = 1 - 7 \times 2 + [9^e]$

$380 = (1 - 7) \times 2 + [9^e]$

$381 = -17 + [2^e] + [9^e]$

$382 = -1 - 7 - 2 + [9^e]$

$383 = -1 \times 7 - 2 + [9^e]$

$384 = 1 - 7 - 2 + [9^e]$

$385 = -1 \times 7 + [\log 2] + [9^e]$

$386 = -1 - 7 + 2 + [9^e]$

$387 = -1 \times 7 + 2 + [9^e]$

$388 = 1 - 7 + 2 + [9^e]$

$389 = -1^7 - 2 + [9^e]$

$390 = -1^7 \times 2 + [9^e]$

$391 = 1^7 - 2 + [9^e]$

$392 = (-1^7 + 2) \times [9^e]$

$393 = (-1^7 + 2) + [9^e]$

$394 = 1^7 \times 2 + [9^e]$

$395 = 1^7 + 2 + [9^e]$

$396 = -1 + 7 - 2 + [9^e]$

$397 = 1 \times 7 - 2 + [9^e]$

$398 = 1 + 7 - 2 + [9^e]$

$399 = 1 \times 7 + [\log 2] + [9^e]$

$400 = -1 + 7 + 2 + [9^e]$

$401 = 1 \times 7 + 2 + [9^e]$

$402 = 1 + 7 + 2 + [9^e]$

$403 = 17 - [2^e] + [9^e]$

$404 = (-1 + 7) \times 2 + [9^e]$

$405 = -1 + 7 \times 2 + [9^e]$

$406 = 1 \times 7 \times 2 + [9^e]$

$407 = 1 + 7 \times 2 + [9^e]$

$408 = (1 + 7) \times 2 + [9^e]$

$409 = 17 + [\log 2] + [9^e]$

$410 = 17 - [-\log 2] + [9^e]$

$411 = 17 + 2 + [9^e]$

$412 = -1 + {}^7C_2 + [9^e]$

$$413 = (\sum_{n=-1}^{7} n) - [2^e] + [9^e]$$

$$414 = (\sum_{n=1}^{7} n) - [2^e] + [9^e]$$

$415 = 17 + [2^e] + [9^e]$

$$416 = -1 - (\sum_{n=-7}^{2} n) + [9^e]$$

$$417 = -1 \times (\sum_{n=-7}^{2} n) + [9^e]$$

$$418 = 1 - (\sum_{n=-7}^{2} n) + [9^e]$$

$$419 = (\sum_{n=-1}^{7} |n|) - 2 + [9^e]$$

$$420 = (\sum_{n=1}^{7} n) + [\log 2] + [9^e]$$

$$421 = (\sum_{n=-1}^{7} n) + 2 + [9^e]$$

$$422 = (\sum_{n=1}^{7} n) + 2 + [9^e]$$

$$423 = (\sum_{n=1}^{7} n) + 2 - [-(9)^e]$$

$$424 = (\sum_{n=-1}^{7} |n|) + 2 - [-(9)^e]$$

$$425 = \sum_{n=-[\sqrt{1}7]}^{29} n$$

$426 = 17 \times 2 + [9^e]$

$$427 = \sum_{n=-1}^{7} |n| + [2^e] + [9^e]$$

$$428 = -1 + \sum_{n=-[(7\div 2)^e]}^{-9} |n|$$

$$429 = 1 \times \sum_{n=-[(7\div 2)^e]}^{-9} |n|$$

$$430 = 1 + \sum_{n=-[(7\div 2)^e]}^{-9} |n|$$

$431 = -1 + 72 \times (\sqrt{9})!$

$432 = 1 \times 72 \times (\sqrt{9})!$

$433 = 1 + 72 \times (\sqrt{9})!$

$$434 = -1 + \sum_{n=[\log 7]}^{29} n$$

$$435 = \sum_{n=1^7}^{29} n$$

$436 = 1 + \sum_{n=[\log 7]}^{29} n$

$437 = -1 + [7 \times (2^e)] + [9^e]$

$438 = 1 \times [7 \times (2^e)] + [9^e]$

$439 = 1 + [7 \times (2^e)] + [9^e]$

$440 = -1 + 7^2 + [9^e]$

$441 = 1 \times 7^2 + [9^e]$

$442 = 1 + 7^2 + [9^e]$

$443 = 1 + 7^2 - [-(9)^e]$

$444 = [(1 + 7) \times 2^e] + [9^e]$

$445 = [(1 + 7) \times 2^e] - [-(9)^e]$

$446 = (\sum_{n=-1}^{7} n) \times 2 + [9^e]$

$447 = -1 + 7 \times 2^{(\sqrt{9})!}$

$448 = 1 \times 7 \times 2^{(\sqrt{9})!}$

$449 = 1 + 7 \times 2^{(\sqrt{9})!}$

$450 = (\sum_{n=-1}^{7} |n|) \times 2 + [9^e]$

$451 = (\sum_{n=-1}^{7} |n|) \times 2 - [-(9)^e]$

$452 = 1 \times [7^\pi] - 2 + \sqrt{9}$

$453 = 1 + [7^\pi] - 2 + \sqrt{9}$

$454 = -1 + [7^\pi] - 2 + (\sqrt{9})!$

$455 = -1 + [7^\pi] + 2 + \sqrt{9}$

$456 = 1 \times [7^\pi] + 2 + \sqrt{9}$

$457 = 1 + [7^\pi] + 2 + \sqrt{9}$

$458 = 1 \times [7^\pi] - 2 + 9$

$459 = 1 + [7^\pi] - 2 + 9$

$460 = -[(-1 + 7)^e] \times 2 + ((\sqrt{9})!)!$

$461 = -1 + 7 \times \sum_{n=-[(2)^e]}^{9} |n|$

$462 = -1 + \sum_{n=-7}^{29} |n|$

$463 = 1 \times \sum_{n=-7}^{29} |n|$

$464 = 1 + \sum_{n=-7}^{29} |n|$

$465 = -1 + [7^\pi] + [2^e] + 9$

$466 = 1 \times [7^\pi] + [2^e] + 9$

$467 = 1 + [7^\pi] + [2^e] + 9$

$468 = -1 + [7^\pi] + 2 \times 9$

$469 = 1 \times [7^\pi] + 2 \times 9$

$470 = -1 + [(7 - 2)^e] + [9^e]$

$471 = \sum_{n=-(1+7)}^{29} |n|$

$472 = 1 + [(7 - 2)^e] + [9^e]$

$473 = 1 + [(7 - 2)^e] - [-(9^e)]$

$474 = 1 - [-(7 - 2)^e] - [-(9)^e]$

$475 = -1 + [7^\pi] + [2^e] + [(\sqrt{9})^e]$

$476 = 1 \times [7^\pi] + [2^e] + [(\sqrt{9})^e]$

$477 = -1 + [-(7^e)] + [(2 + 9)^e]$

$478 = -1 - [7^e] + [(2 + 9)^e]$

$479 = -1 \times [7^e] + [(2 + 9)^e]$

$480 = (-1 + 7) \times ([2^e]! \div 9)$

$481 = 1 + [7^\pi] + 29$

$482 = -1 + [7^e] + [(2^{\sqrt{9}})^e]$

$483 = -(\sum_{n=-1}^{7} |n|) + 2^9$

$484 = (-\sum_{n=1}^{7} |n|) + 2^9$

$485 = (-\sum_{n=-1}^{7} n) + 2^9$

$486 = -1 + [7^\pi] + [2^e] \times (\sqrt{9})!$

$487 = -1 + [7^\pi] + [2^e] + [(\sqrt{9})^\pi]$

$488 = 1 \times [7^\pi] + [2^e] + [(\sqrt{9})^\pi]$

$489 = 1 + [7^\pi] + [2^e] + [(\sqrt{9})^\pi]$

$490 = 1 \times [7^\pi] + [2^\pi] + [(\sqrt{9})^\pi]$

$491 = 1 + [7^\pi] + [2^\pi] + [(\sqrt{9})^\pi]$

$492 = -1 + [7^\pi] + \sum_{n=-2}^{9} n$

$493 = 1 \times [7^\pi] + \sum_{n=-2}^{9} n$

$494 = 17 \times [2^e] + [9^e]$

$495 = -17 + 2^9$

$496 = 1 + [7^\pi] + \sum_{n=2}^{9} n$

$497 = 1 + [7^\pi] + \sum_{n=[\log 2]}^{9} n$

$498 = -1 + [7^\pi] + \sum_{n=-2}^{9} |n|$

$499 = [-(1+7)^e] + [2 \times (9^e)]$

$500 = -[(1+7)^e] + [2 \times (9^e)]$

$501 = -[(1+7)^e] - [-(2 \times (9^e))]$

$502 = [(1+7+2)^e] + [-(\sqrt{9})^e]$

$503 = [(1+7+2)^e] - [(\sqrt{9})^e]$

$504 = -1 + [7^\pi] + [2^e] \times 9$

$505 = 1 \times [7^\pi] + [2^e] \times 9$

$506 = 1 + [7^\pi] + [2^e] \times 9$

$507 = \log(1) - [\ln(7^e)] + 2^9$

$508 = 1 - [\ln(7^e)] + 2^9$

$509 = [-\ln(17)] + 2^9$

$510 = -[\ln(17)] + 2^9$

$511 = -1^7 + 2^9$

$512 = 1^7 \times 2^9$

$513 = 1^7 + 2^9$

$514 = -1 + [7^\pi] + 2^{(\sqrt{9})!}$

$515 = 1 \times [7^\pi] + 2^{(\sqrt{9})!}$

$516 = 1 + [7^\pi] + 2^{(\sqrt{9})!}$

$517 = 1 - [-(7)^\pi] + 2^{(\sqrt{9})!}$

$518 = -[-(-1+7)^e] - [2^e] - [-(9)^e]$

$519 = [(1+7+2)^e] - \sqrt{9}$

$520 = -[-(1+7+2)^e] - \sqrt{9}$

$521 = [(1+7+2)^e] + [-\log(\sqrt{9})]$

$522 = [(1+7+2)^e] - [\log(\sqrt{9})]$

$523 = 1 \times [7^\pi] + [2^\pi] \times 9$

$524 = 1 + [7^\pi] + [2^\pi] \times 9$

$525 = [(1+7+2)^e] + \sqrt{9}$

$526 = -[-(1+7+2)^e] + \sqrt{9}$

$527 = [(1+7+2)^e] + [\ln(9^e)]$

$528 = [(1+7+2)^e] + (\sqrt{9})!$ $529 = 17 + 2^9.$

Now, putting minus sign(-) before all the representations of integers from 1 through 529, one gets required expressions for integers -1 to -529 as well.

It may be noted that

1. the integers beyond ± 529 may also be expressed in this way.
2. the expressions for above integers are not necessarily unique. For example:

$$466 = 1 + \sum_{n=-[(7\div 2)^e]}^{[\log\sqrt{9}]} |n| \quad = \quad 1 \times [7^\pi] + [2^e] + 9.$$

3. Larger and larger integers can be represented in this manner, for example: $(1729)!$, $(17)!^{2^{9!}}$, $(\sum_{n=1}^{7!} n)^{2^{9!}}$.

Similarly putting minus sign(-) before these expressions one can represent smaller and smaller integers in this manner.

It will be interesting to know

1. the smallest positive integer that can not be expressed in this manner;
2. the largest positive integer that can be expressed in this manner.

Acknowledgements. The author is thankful to the referee for his valuable remarks and suggestions leading to the improvement of the article. The author is also thankful to his family and Smt. Y. Dhanalakshmi, Ms. Manasi Mahapatra and Rajesh Kumar Palei for their cooperation.

References

[1] Hardy, G. H., The Indian Mathematician Ramanujan, *American Math. Monthly*, **44** no. 3 (1937), 137–155.

[2] *http://www.sankalpindia.net/book/facts-1729.*

[3] *http://en.wikipedia.org/wiki/1729*

[4] Kanigel, R., *The Man Who Knew Infinity: A Life of the Genius Ramanujan*, New York: Washington Square Press, (1991), p. 312.

[5] Piezas, T., Ramanujan and the cubic equation $3^3 + 4^3 + 5^3 = 6^3$, *http://www.geocities.com/titus piezas/ramanujan page9.html.*

B. Sagar
M.Sc. 2^{nd} year Student
Department of mathematics
Ravenshaw Univesity, Cuttack, Odisha-753003, India.
Email- *calculusboysagar@gmail.com*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December (2017), 111-122

SYMBOLIC BLOWUP ALGEBRAS OF MONOMIAL CURVES IN A^3 DEFINED BY AN ARITHMETIC SEQUENCE

CLARE D'CRUZ

(Received : 04 - 08 - 2017; Revised : 05-11-2017)

Abstract. In this paper, we consider monomial curves in $\mathbb{A}_k^3$ parameterized by $t \to (t^{2q+1}, t^{2q+1+m}, t^{2q+1+2m})$ where $gcd\ (2q+1, m) = 1$. The symbolic blowup algebras of these monomial curves is Gorenstein ([10], [11]). From the results in [17], it follows that the defining ideal of $R_s(\mathfrak{p})$ is given by the Pfaffians of a skew-symmetric matrix. In this paper we give a proof which mainly involves the use of Gröebner basis. We are also able to describe all the symbolic powers $\mathfrak{p}^{(n)}$ for all $n \geq 1$.

1. Introduction

Let $\mathbb{k}$ be a field and let $\mathbb{A}_{\mathbb{k}}^n$ (or $\mathbb{A}^n$) be the affine n-space over $\mathbb{k}$. One well known question is: Is every affine irreducible curve Y in $\mathbb{A}_{\mathbb{k}}^n$ a set theoretic complete intersection of $(n-1)$ hypersurfaces? In other words, if $T = k[x_1, \ldots, x_n]$ and $I(Y) = \{f \in T | f(a) = 0 \text{ for all } a \in Y\}$, then does there exist $n-1$ elements $f_1, \ldots, f_{n-1} \in T$ such that $I(Y) = \sqrt{(f_1, \ldots, f_{n-1})}$? The answer to this question is quite difficult and depends upon the characteristic of the field $\mathbb{k}$. The oldest known result in this direction is a result of L. Kronecker where he showed that $I(Y)$ can be generated set theoretically by $(n+1)$-equations [18].

Later, several researchers showed that there exist interesting examples of algebraic subsets $Y \subseteq \mathbb{A}^n$ which can be set theoretically defined by $(n-1)$-equations. One remarkable result which appeared in 1978 was by R. Cowsik and M. Nori. They showed that if $\mathbb{k}$ is a perfect field such that char$(\mathbb{k}) = p$ and if I is a radical ideal of codimension one, then I is a set theoretic complete intersection [3, Theorem 1]. Later in 1979, for char$(\mathbb{k}) = 0$, H. Bresinsky showed that all monomial curves in $\mathbb{A}_{\mathbb{k}}^3$ are set theoretic complete intersection [1]. In [2] the author extended his ideas to monomial curves in $\mathbb{A}^4$. In [25] the author extends the ideas in [24] to

2010 Mathematics Subject Classification : 13A30, 1305, 13H15, 13P10
Key words and phrases : Symbolic Rees algebra, Cohen-Macaulay, Gorenstein.
The author was partially funded by a grant from Infosys Foundation.

study symbolic powers of monomial curves in $\mathbb{A}^4$. In 1990, D. Patil gave more general class of monomial curves in $\mathbb{A}^n$ which are set theoretic complete intersection [23, Theorem 1.1].

In 1981, R. Cowsik gave a new direction to this problem. He showed that if $(R, \mathfrak{m})$ is a regular local ring and $\mathfrak{p}$ is a prime ideal such that $\dim(R/\mathfrak{p}) = 1$, then Noetherianness of the symbolic Rees algebra $R_s(\mathfrak{p}) := \oplus_{n\geq 0}\mathfrak{p}^{(n)}$ implies that $\mathfrak{p}$ is a set theoretic complete intersection [4]. However, the converse need not be true [12]. Motivated by Cowsik's result, in 1987, Huneke gave necessary and sufficient conditions for $\mathcal{R}_s(\mathfrak{p})$ to be Noetherian when dim $R = 3$ [16]. Huneke's result was generalised in 1991 for dim $R \geq 3$ by M. Morales [21]. All these results paved a new way to study the famous problem of set theoretic complete intersection. In the last twenty-five years several researchers have worked on the symbolic Rees algebra $\mathcal{R}_s(\mathfrak{p})$. However, there are still many unanswered questions related to both the problems, i.e., set theoretic complete intersection and Noetherianness of symbolic Rees algebra. A few problems will be listed at the end of this paper. A good survey article with some open questions on set theoretic complete intersection is by G. Lyubeznik [19].

In this paper, we consider the monomial curve in $\mathbb{A}^3$ parameterized by $(t^{n_1}, t^{n_2}, t^{n_3})$ where $n_i = 2q + 1 + (i-1)m$ and $gcd(n_1, m) = 1$. Throughout this paper $R = k[[x_1, x_2, x_3]]$, $S = k[[t]]$ and $\mathfrak{p} = \ker(\phi)$, where ϕ is the homomorphism defined by $\phi(x_i) = t^{n_i}$ for $1 \leq i \leq 3$. We say that $\mathfrak{p}$ is the defining ideal of the monomial curve parameterized by $(t^{n_1}, t^{n_2}, t^{n_3})$. It is well known that these curves are a set-theoretic complete intersection ([1], [27], [23]). The symbolic powers of curves are also of interest ([7], [15], [17], [24], [29]). In this paper we describe all the symbolic powers $\mathfrak{p}^{(n)}$ (Theorem 5.9).

The symbolic Rees algebra of these monomial curves have been studied by several authors in the past. For example see [27], [14], [10], [17] and [11]. The Cohen-Macaulay and Gorenstein property of these monomial curves has also been studied [10] and [11]. In [17], the authors are able to explicitly describe when $R_s(\mathfrak{p})$ is Cohen-Macaulay and when it is Gorenstein.

We give a different and simple proof for the Cohen-Macaulayness and Gorensteinness of the blowup algebras $R_s(\mathfrak{p})$ and $G_s(\mathfrak{p}) := \oplus_{n\geq 0}\mathfrak{p}^{(n)}/\mathfrak{p}^{(n+1)}$. One crucial point here is that though the ideals we are interested in are binomial ideals in $\mathbb{k}[[x_1, x_2, x_3]]$, we can get our main result, by dealing with the corresponding monomial ideals in the polynomial ring $\mathbb{k}[x_2, x_3]$. These ideals are defined in Section 2. Powers and quotients of these monomial ideals are much easier to compute and do not depend on any sophisticated results. Hence, any student with a good knowledge of basic Commutative Algebra can follow the proof. Our proofs may give a different approach and help to answer some questions related to symbolic powers. All basic results used in this paper can be found in [20].

We now describe the organisation of this paper. In Section 2, we state some basic results. We also state a result which gives a way to compute lengths of modules in the polynomial ring $k[x_2, x_3]$. In Section 3, we prove the Cohen-Macaulayness of the filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ where the ideals $I_n \subseteq \mathbb{k}[x_2, x_3]$ are defined in Section 2. In Section 4, we compute length of $I_{n-1}/(I_n : x_3^q)$ which is useful in the computations in the next section. In Section 5, we compute lengths various modules over $\mathbb{k}[[x_1, x_2, x_3]]$ which are needed to prove the Cohen-Macaulay and Gorenstein property of the blowup algebras $R_s(\mathfrak{p})$ and $G_s(\mathfrak{p})$. In Section 6, we prove our main results. In Section 7 we suggest a list of few questions which might be of interest to the reader.

2. Preliminaries

It is well known that the generators for $\mathfrak{p}$ are the 2×2 minors of the matrix $\begin{pmatrix} x_1 & x_2 & x_3^q \\ x_2 & x_3 & x_1^{m+q} \end{pmatrix}$ [13]. In particular, if $g_1 = x_1^{m+q}x_2 - x_3^{q+1}$, $g_2 = x_1^{m+q+1} - x_2x_3^q$ and $g_3 = x_2^2 - x_1x_3$, then $\mathfrak{p} = (g_1, g_2, g_3)$.

The following result is well known ([11, Corollary 4.4]). We state it for the sake of completion.

Lemma 2.1. *$R_s(\mathfrak{p})$ is a Noetherian ring.*

Proof. Let $f_1 := g_3 = x_2^2 - x_2x_3 \in \mathfrak{p}$ and $f_2 = -x_1^{2(m+q)+1} - x_1^{m+q-1}x_2^3x_3^{q-1} + 3x_1^{m+q}x_2x_3^q - x_3^{2q+1}$. Then

$$x_3 \cdot f_2 = -g_1^2 + x_1^{m+q-1}g_2g_3 \in \mathfrak{p}^2 \subseteq \mathfrak{p}^{(2)}.$$

As x_3^n is nonzerodivisor on $R/\mathfrak{p}^{(2)}$ for all n, $f_2 \in \mathfrak{p}^{(2)}$. Moreover,

$$\ell\left(\frac{R}{(x_1, f_1, f_2)}\right) = \ell\left(\frac{R}{(x_1, x_2^2, x_3^{2q+1})}\right) = 2(2q+1) = 2 \cdot e(x_1; R/\mathfrak{p}).$$

By Huneke's criterion [16, Theorem 3.1], $R_s(\mathfrak{p})$ is Noetherian. □

Let $T = \mathbb{k}[x_1, x_2, x_3]$. The following lemma gives us a way to compute the length of an R-module in terms of the length of the corresponding T-module.

Lemma 2.2. [6, Lemma 2.8] *Let $\mathfrak{m} = (x_1, x_2, x_3)T$ and M a finitely generated T-module such that* $\operatorname{Supp}(M) = \{\mathfrak{m}\}$. *Then*

$$\ell_R(M \otimes_T R) = \ell_T(M).$$

Let $\mathcal{J}_1 = \{g_1, g_2, g_3\}$ and $\mathcal{J}_2 = \{f_2\}$. We define

$$\mathcal{J}_1T := (g_1, g_2, g_3)T, \quad \mathcal{J}_2T := (f_2)T, \quad \mathcal{I}_nT := \sum_{a_1+2a_2=n} (\mathcal{J}_1^{a_1}T)(\mathcal{J}_2^{a_2}T). \quad (2.3)$$

As R is a flat T-module, $\mathcal{I}_nR = \mathcal{I}_nT \otimes_T R$.

Let $T' = k[x_2, x_3] \cong T/x_1T$. For $i = 1, 2$ put

$$J_1 := \mathcal{J}_1 T' = (x_2^2, x_2x_3^q, x_3^{q+1}), \quad J_2 := \mathcal{J}_2 T' = (x_3^{2q+1}), \quad \textit{and}$$

$$I_n := \mathcal{I}_n T' = \sum_{a_1+2a_2=n} J_1^{a_1} J_2^{a_2}. \tag{2.4}$$

Proposition 2.5. *Let $n \geq 1$. Then*

(1) $\mathcal{I}_n R \subseteq \mathfrak{p}^{(n)}$.

(2) $(\mathcal{I}_n + (x_1))T$ is an homogeneous ideal.

(3) $(\mathcal{I}_n + (x_1))T$ is an $\mathfrak{m}$-primary ideal.

Proof. (1) As $\mathcal{J}_1 \mathfrak{p}$ and $J_2 = (f_2) \subseteq \mathfrak{p}^{(2)}$, for all $a_1, a_2 \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{J}_1^{a_1} \mathcal{J}_2^{a_2} \subseteq \mathfrak{p}^{a_1} (\mathfrak{p}^{(2)})^{a_2} \subseteq \mathfrak{p}^{(a_1+2a_2)}. \tag{2.6}$$

Summing over all $a_1 + 2a_2 = n$ and applying (2.6) to (2.3) we get (1).

(2) As $(\mathcal{J}_1 + (x_1))T = (J_1, (x_1))T$ and $(\mathcal{J}_2 + (x_1))T = (J_2, x_1)T$ are homogenous ideals and $(\mathcal{I}_n + (x_1))T = (\sum_{a_1+2a_2=n} J_1^{a_1} J_2^{a_2}, x_1)T$ we get (2).

(3) By (2.3), $\mathcal{J}_1^n T \subseteq \mathcal{I}_n T$ and $(\mathcal{J}_1^n + (x_1))T = ((x_2^2, x_2x_3^q, x_3^{q+1})^n, x_1)T$ which implies that $\mathfrak{m}T = (\sqrt{\mathcal{J}_1^n + (x_1)})T \subseteq (\sqrt{\mathcal{I}_n + (x_1)})T \subseteq \mathfrak{m}T$. □

We state a result on monomial ideals which follows from [8, Proposition 1.14] and will be consistently used in all the proofs which involve monomial ideals.

Proposition 2.7. *Let $I = (u_1, \ldots, u_r)$ and $J = (v)$ be monomial ideals in a polynomial ring over a field $\Bbbk$. Then $I : J = (\{u_i/gcd(u_i, v) : i = 1, ..., r\})$.*

3. The associated graded ring corresponding to the filtration $\mathcal{F} := \{I_n\}_{n\geq 0}$.

Throughout this section we will work with the ring $T' = \Bbbk[x_2, x_3]$ and the ideals J_1, J_2 and I_n. Our goal is to compute $\ell(T'/(I_n + (x_2)^2))$ and $\ell(T'/(I_n + (x_2^2, x_3^{2q+1})))$. To attain our goal, we show that x_2^2, x_3^{2q+1} is a regular sequence in $G(\mathcal{F})$, where $G(\mathcal{F}) := \oplus_{n\geq 0} I_n/I_{n+1}$ is the associated graded ring corresponding to the filtration $\mathcal{F} := \{I_n\}_{n\geq 0}$.

Theorem 3.1. *$G(\mathcal{F})$ is Cohen-Macaulay.*

Proof. We first show that $(I_n : x_2^2) = I_{n-1}$ for all $n \geq 2$. Clearly $x_2^2 I_{n-1} \subseteq J_1 I_{n-1} \subseteq I_n$. Write $J_1^{a_1} = \sum_{i=0}^{a_1-1} x_2^{2(a_1-i)} x_3^{qi} (x_2, x_3)^i + x_3^{qa_1}(x_2, x_3)^{a_1}$. Then

$$\begin{aligned}
(I_n : x_2^2) &= \left(\left(\sum_{a_1+2a_2=n} J_1^{a_1} J_2^{a_2} \right) : x_2^2 \right) = \sum_{a_1+2a_2=n} (J_1^{a_1} J_2^{a_2} : x_2^2) \\
&= \sum_{a_1+2a_2=n} \sum_{i=0}^{a_1-1} (x_2^{2(a_1-i)} x_3^{qi+(2q+1)a_2} (x_2, x_3)^i : x_2^2) \\
&\qquad + ((x_3^{qa_1+(2q+1)a_2} (x_2, x_3)^{a_1}) : x_2^2) \\
&= \sum_{a_1+2a_2=n} \sum_{i=0}^{a_1-1} (x_2^{2(a_1-i-1)} x_3^{qi+(2q+1)a_2} (x_2, x_3)^i) \\
&\qquad + (x_3^{qa_1+(2q+1)a_2} (x_2, x_3)^{a_1-1}) \\
&\subseteq I_{n-1}.
\end{aligned} \tag{3.2}$$

Let — denote the image in $R/(x_2^2)$. Then

$$\frac{G(\mathcal{F})}{((x_2^2)^\star)} \cong \bigoplus_{n\geq 0} \frac{I_n}{I_{n+1}+x_2^2 I_{n-1}} = G(\overline{\mathcal{F}}).$$

To show that $\overline{x_3^{2q+1}}$ is a regular element in $G(\overline{\mathcal{F}})$, we need to verify that

$$((I_{n+2}+x_2^2 I_{n+1}):(x_3^{2q+1})) = I_n + x_2^2 I_{n-1}. \tag{3.3}$$

Let $m \geq 1$. Then

$$\begin{aligned}
&(J_1^{m+2}:x_3^{2q+1})\\
&= \sum_{i=0}^{2}(x_2^{2(m+2-i)}x_3^{qi}(x_2,x_3)^i : x_3^{2q+1}) + \sum_{i=3}^{m+2}(x_2^{2(m+2-i)}x_3^{qi}(x_2,x_3)^i : x_3^{2q+1})\\
&= x_2^{2m}(x_2,x_3) + \sum_{i=3}^{m+2}(x_2^{2(m+2-i)}x_3^{q(i-3)}(x_2,x_3)^{i-3})(x_2^3x_3^{q-1}, x_2^2x_3^q, x_2x_3^{q+1}, x_3^{q+2})\\
&\subseteq ((x_2^{2m})) + \sum_{i=3}^{m+2}(x_2^{2(m+2-i)}x_3^{q(i-3)+1}(x_2,x_3)^{i-3})(x_2^2, x_2x_3^q, x_3^{q+1})\\
&= J_1^m.
\end{aligned}\tag{3.4}$$

Hence

$$\begin{aligned}
&((I_{n+2}+x_2^2I_{n+1}):(x_3^{2q+1}))\\
&= \sum_{a_1+2a_2=n+2}(J_1^{a_1}J_2^{a_2}:x_3^{2q+1}) + \sum_{a_1+2a_2=n+1}(x_2^2J_1^{a_1}J_2^{a_2}:(x_3^{2q+1}))\\
&\subseteq \sum_{a_1+2a_2=n+2;a_2\neq 0}(J_1^{a_1}J_2^{a_2-1}) + (J_1^{n+2}:x_3^{2q+1})\\
&\qquad + \sum_{a_1+2a_2=n+1;a_2\neq 0}(x_2^2J_1^{a_1}J_2^{a_2-1}) + (x_2^2J_1^{n+1}:x_3^{2q+1})\\
&\subseteq \sum_{a_1+2a_2=n+2;a_2\neq 0}(J_1^{a_1}J_2^{a_2-1}) + J_1^n + \sum_{a_1+2a_2=n+1;a_2\neq 0}(x_2^2J_1^{a_1}J_2^{a_2-1}) + x_2^2J_1^{n-1}\\
&\subseteq I_n + x_2^2 I_{n-1},
\end{aligned}\tag{3.5}$$

last but one inclusion in view of (3.4). The other inclusion is easy to verify. □

We are now ready to prove the main result of this section.

Proposition 3.6. *For all $n \geq 1$,*

$$\begin{aligned}
\ell\left(\frac{T'}{(I_n+(x_2^2))T'}\right) &= \ell\left(\frac{T'}{(I_n)T'}\right) - \ell\left(\frac{T'}{(I_{n-1})T'}\right),\\
\ell\left(\frac{T'}{(I_n+(x_2^2,x_3^{2q+1}))T'}\right) &= \ell\left(\frac{T'}{(I_n)T'}\right) - \ell\left(\frac{T'}{(I_{n-1})T'}\right)\\
&\qquad - \ell\left(\frac{T'}{(I_{n-2})T'}\right) + \ell\left(\frac{T'}{(I_{n-3})T'}\right).
\end{aligned}$$

Proof. The proof follows from [6, Proposition 2.4] and Theorem 3.1. □

4. The inductive step

In this section we describe the generators of I_{n-1} modulo $(I_n : x_3^q)$. This will be used in our computations in the next section.

Lemma 4.1. *For all $n \geq 1$,*

(1) $(I_n : x_3^q) \subseteq I_{n-1}$.

(2) $$I_{n-1} = \begin{cases} \displaystyle\sum_{a_2=0}^{\frac{n-2}{2}} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1)a_2}(x_2, x_3^q) + (I_n : x_3^q), & \text{if } 2 \not| (n-1); \\ \left(x_3^{(2q+1)\left(\frac{n-1}{2}\right)}\right) + \\ \displaystyle\sum_{a_2=0}^{\frac{n-3}{2}} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1)a_2}(x_2, x_3^q) + (I_n : x_3^q), & \text{if } 2|(n-1). \end{cases}$$

Proof. (1) One can verify that

$$\begin{aligned} (I_n : x_3^q) &= \sum_{a_1+2a2=n; a_2 \neq 0} (J_1^{a_1} J_2^{a_2} : x_3^q) + (J_1^n : x_3^q) \\ &= \sum_{a_1+2a2=n; a2 \neq 0} (x_3^{q+1}) J_1^{a_1} J_2^{a_2-1} + (x_2^{2n}) + \sum_{i=1}^{n} (x_2^{2(n-i)} x_3^{q(i-1)} (x_2, x_3^q)^i) \\ &\subseteq I_{n-1}. \end{aligned}$$

(2) As $I_1 = J_1 = x_2(x_2, x_3^q) + (x_3^{q+1}) \subseteq x_2(x_2, x_3^q) + (I_2 : x_3^q)$, (2) is true for $n = 1$. Let $n > 1$. For all $a_1 \geq 1$,

$$\begin{aligned} J_1^{a_1} &= J_1 J_1^{a_1-1} \\ &\subseteq (x_2(x_2, x_3^q), x_3^{q+1}) \left(x_2^{2a_1-3}(x_2, x_3^q) + (I_{a_1} : x_3^q)\right) \text{ [by induction hypothesis]} \\ &= x_2^{2a_1-1}(x_2, x_3^q) + (x_2^{2a_1-2} x_3^{2q}) + x_2(x_2, x_3^q)(I_{a_1} : x_3^q) \\ &+ x_3^{q+1} x_2^{2a_1-3}(x_2, x_3^q) + x_3^{q+1}(I_{a_1} : x_3^q) \\ &= x_2^{2a_1-1}(x_2, x_3^q) + (I_{a_1+1} : x_3^q) \end{aligned} \tag{4.2}$$

as

$$\begin{aligned} (x_2^{2a_1-2} x_3^{2q}) x_3^q &= x_2^{2(a_1-1)} x_3^{2q+1} x_3^{q-1} \subseteq J_1^{a_1-1} J_2^{a_2} \subseteq I_{a_1-1+2} = I_{a_1+1} \\ \left(x_3^{q+1} x_2^{2a_1-3}(x_2, x_3^q)\right) x_3^q &= x_2^{2(a_1-2)} \cdot x_2(x_2, x_3^q) \cdot x_3^{2q+1} \\ &\subseteq J_1^{a_1-1} J_2^{a_2} \subseteq I_{a_1-1+2} = I_{a_1+1} \\ x_3^{q+1}(I_{a_1} : x_3^q) &\subseteq (I_{a_1+1} : x_3^q) \end{aligned}$$

Hence

$$I_n = \begin{cases} \displaystyle\sum_{a_2=0}^{\frac{n-2}{2}} J_1^{n-1-2a_2} J_2^{a_2}, & \text{if } 2 \not| (n-1); \\ J_2^{(n-1)/2} + \displaystyle\sum_{a_2=0}^{\frac{n-3}{2}} J_1^{n-1-2a_2} J_2^{a_2}, & \text{if } 2|(n-1); \end{cases}$$

$$\subseteq \begin{cases} A + x_3^{(2q+1)a_2}(I_{n-2a_2} : x_3^q), & \text{if } 2 \not| (n-1); \\ x_3^{(2q+1)(\frac{n-1}{2})} + B + x_3^{(2q+1)a_2}(I_{n-2a_2} : x_3^q), & \text{if } 2|(n-1); \end{cases}$$

$$\subseteq \begin{cases} A + (I_n : x_3^q), & \text{if } 2 \not| (n-1); \\ x_3^{(2q+1)(\frac{n-1}{2})} + B + (I_n : x_3^q), & \text{if } 2|(n-1); \end{cases}$$

in view of (4.2), wherein

$$A = \sum_{a_2=0}^{\frac{n-2}{2}} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1)a_2}(x_2, x_3^q),\ B = \sum_{a_2=0}^{\frac{n-3}{2}} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1)a_2}(x_2, x_3^q).$$

This implies that $I_n \subseteq RHS$. The other inclusion follows from (1) and checking element-wise. □

Proposition 4.3. *For all $n \geq 1$, $\dim_{\Bbbk}\left(\frac{I_{n-1}}{(I_n : x_3^q)}\right) = n$.*

Proof. Put $\mathfrak{m}' = (x_2, x_3)$. Then $x_3^q \mathfrak{m}' I_{n-1} \subseteq J_1 I_{n-1} \subseteq I_n$. Hence by Lemma 4.1(2) we get

$$\dim_{\Bbbk}\left(\frac{I_{n-1}}{(I_n : x_3^q)}\right) \leq \begin{cases} 2(n/2), & \text{if } 2 \not| (n-1); \\ 1 + [2(n-1)/2], & \text{if } 2|(n-1); \end{cases} = n.$$

To complete the proof we need to show that we have n linearly independent elements. As all the elements in $I_{n-1}/(I_n : x_d)$ has the degree of x_2 different, they form a linearly independent set. □

5. Cohen-Macaulayness of $R/(\mathfrak{p}^{(n)} + (\mathbf{f}_k))$

Let $(\mathbf{f}_1) := (f_1)$ and $(\mathbf{f}_2) := (f_1, f_2)$, where f_1 and f_2 are defined in Lemma 2.1. The main step in proving the Cohen-Macaulayness of $\mathcal{R}_s(\mathfrak{p})$ is to show that for all $n \geq 1$ and $k = 1, 2$, the rings $R/(\mathfrak{p}^{(n)} + (\mathbf{f}_k))$ are Cohen-Macaulay [9]. This was proved by Goto for $q = 1$ and $n \leq 3$ ([9, Proposition 7.6]). We prove it for all $q \geq 1$ and for all n. As a consequence, we prove that $\mathfrak{p}^{(n)} = \mathcal{I}_n R$ and $(\mathfrak{p}^{(n)} T') = I_n T'$ for all $n \geq 1$.

We first compute the length of the T'/I_n. Next, we prove an interesting result which gives the the equality of the lengths of the various modules (Theorem 5.2, Theorem 5.6).

Proposition 5.1. *For all $n \geq 1$, $\ell\left(\frac{T'}{I_n}\right) = (2q+1)\binom{n+1}{2}$.*

Proof. We prove induction on n. If $n = 1$, then

$$\ell\left(\frac{T'}{I_1}\right) = \ell\left(\frac{k[x_2, x_3]}{(x_2^2, x_2 x_3^q, x_3^{q+1})}\right) = 2q + 1.$$

Now let $n > 1$. From the exact sequence

$$0 \longrightarrow \frac{T'}{(I_n : x_3^q)} \xrightarrow{.x_3^q} \frac{T'}{I_n} \longrightarrow \frac{T'}{I_n + (x_3^q)} \longrightarrow 0$$

we get

$$\begin{aligned}\ell\left(\frac{T'}{I_n}\right) &= \ell\left(\frac{T'}{I_n+(x_3^q)}\right)+\ell\left(\frac{T'}{(I_n:x_3^q)}\right)\\ &= \ell\left(\frac{T'}{(x^{2n},x_3^q)}\right)+\ell\left(\frac{T'}{I_{n-1}}\right)+\ell\left(\frac{I_{n-1}}{(I_n:x_3^q)}\right) \text{ [by Lemma 4.1(1)]}\\ &= 2qn+(2q+1)\binom{n}{2}+n \quad = \quad (2q+1)\binom{n+1}{2},\end{aligned}$$

the last but one equality by induction hypothesis & Proposition 4.3. □

Theorem 5.2. *For all $n \geq 1$,*

$$\begin{aligned}e\left(x_1;\frac{R}{\mathfrak{p}^{(n)}}\right) &= \ell\left(\frac{R}{\mathfrak{p}^{(n)}+(x_1)}\right)=\ell_R\left(\frac{R}{(\mathcal{I}_n,x_1)R}\right)\\ &= \ell_{T'}\left(\frac{T'}{\mathcal{I}_nT'}\right) \quad = \quad \ell\left(\frac{T'}{I_n}\right) \quad = \quad (2q+1)\binom{n+1}{2}.\end{aligned}$$

Proof. By Proposition 2.5(1), $\mathcal{I}_nR \subseteq \mathfrak{p}^{(n)}$. Hence, as $R/\mathfrak{p}^{(n)}$ is Cohen-Macaulay,

$$e\left(x_1;\frac{R}{\mathfrak{p}^{(n)}}\right)=\ell_R\left(\frac{R}{\mathfrak{p}^{(n)}+(x_1)}\right)\leq \ell_R\left(\frac{R}{(\mathcal{I}_n,x_1)R}\right). \tag{5.3}$$

By Proposition 2.5(3), for any prime $\mathfrak{q} \neq \mathfrak{m}$, $((\mathcal{I}_n,x_1)T)_{\mathfrak{q}} = T$. Hence $\operatorname{Supp}_T\left(\frac{T}{(\mathcal{I}_n,x_1)T}\right)=\{\mathfrak{m}\}$ and we get

$$\begin{aligned}\ell_R\left(\frac{R}{(\mathcal{I}_n,x_1)R}\right) &= \ell_{T'}\left(\frac{T'}{\mathcal{I}_nT'}\right) && \text{by [Lemma 2.2]}\\ &= \ell_{T'}\left(\frac{T'}{I_n}\right) && \text{by [(2.4)]}\\ &= (2q+1)\binom{n+1}{2} && \text{by [Proposition 5.1]}\\ &= e\left(x_1;\frac{R}{\mathfrak{p}}\right)\ell_{R_\mathfrak{p}}\left(\frac{R_\mathfrak{p}}{\mathfrak{p}^nR_\mathfrak{p}}\right)\\ &= e\left(x_1;\frac{R}{\mathfrak{p}}\right)\ell_{R_\mathfrak{p}}\left(\frac{R_\mathfrak{p}}{\mathfrak{p}^{(n)}R_\mathfrak{p}}\right) && \text{[since } \mathfrak{p}^{(n)}R_\mathfrak{p}=\mathfrak{p}^nR_\mathfrak{p}]\\ &= e\left(x_1;\frac{R}{\mathfrak{p}^{(n)}}\right) && \text{by [20, Theorem 14.7].}\end{aligned} \tag{5.4}$$

Thus equality holds in (5.3) and (5.4) which proves the theorem. □

Notation 5.5. Let $\mathbf{f}_1 = f_1$, $\mathbf{f}_2 = f_1, f_2$, $(\mathbf{x_2}) = (x_2^2)$ and $(\mathbf{x_3}) = (x_2^2, x_3^{2q+1})$.

Theorem 5.6. *Let $k = 1, 2$. Then for all $n \geq 1$,*

$$\begin{aligned}e\left(x_1;\frac{R}{\mathfrak{p}^{(n)}+(\mathbf{f}_k)}\right) &= \ell_R\left(\frac{R}{\mathfrak{p}^{(n)}+(x_1,\mathbf{f}_k)}\right)\\ &= \ell_{T'}\left(\frac{T'}{(\mathcal{I}_n+\mathbf{f}_k)T'}\right)=\ell\left(\frac{T'}{I_n+(\mathbf{x_{k+1}})}\right).\end{aligned}$$

In particular, $R/(\mathfrak{p}^{(n)} + (\mathbf{f}_k))$ *is Cohen-Macaulay.*

Proof. From Proposition 2.5(1) $(\mathcal{I}_n, x_1, \mathbf{f}_k)R \subseteq (\mathfrak{p}^{(n)}, x_1, \mathbf{f}_k)R$. Hence

$$e\left(x_1; \frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_k)}\right) \leq \ell_R\left(\frac{R}{\mathfrak{p}^{(n)} + (x_1, \mathbf{f}_k)}\right) \leq \ell_R\left(\frac{R}{(\mathcal{I}_n, x_1, \mathbf{f}_k)R}\right) \tag{5.7}$$

in view of [20, Theorem 14.10]. Since $(\mathcal{I}_n, x_1)T \subseteq (\mathcal{I}_n, \mathbf{f}_k, x_1)T$ by Proposition 2.5(3), for any prime $\mathfrak{q} \neq \mathfrak{m}$, $((\mathcal{I}_n, \mathbf{f}_k, x_1)T)_{\mathfrak{q}} = T$. This implies that $\operatorname{Supp}_T\left(\frac{T}{(\mathcal{I}_n, \mathbf{f}_k, x_1)T}\right) = \{\mathfrak{m}\}$. Hence we get

$$\begin{aligned}
&\ell_R\left(\frac{R}{(\mathcal{I}_n, x_1, \mathbf{f}_k)R}\right)\\
&= \ell_T\left(\frac{T}{(\mathcal{I}_n, x_1, \mathbf{f}_k)T}\right) && \text{[Lemma 2.2]}\\
&= \ell_{T'}\left(\frac{T'}{I_n + (\mathbf{x_{k+1}})}\right)\\
&= \begin{cases} \ell\left(\frac{T'}{(I_n)T'}\right) - \ell\left(\frac{T'}{(I_{n-1})T'}\right), & \text{if } k=1;\\ \ell\left(\frac{T'}{(I_n)T'}\right) - \ell\left(\frac{T'}{(I_{n-1})T'}\right) - \ell\left(\frac{T'}{(I_{n-2})T'}\right) + \ell\left(\frac{T'}{(I_{n-3})T'}\right), & \text{if } k=2; \end{cases} && \text{[Prop. 3.6]}\\
&= \begin{cases} \ell\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}^n R_{\mathfrak{p}}}\right) - \ell\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}^{n-1} R_{\mathfrak{p}}}\right), & \text{if } k=1;\\ \ell\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}^n R_{\mathfrak{p}}}\right) - \ell\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}^{n-1} R_{\mathfrak{p}}}\right) - \ell\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}^{n-2} R_{\mathfrak{p}}}\right) + \ell\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}^{n-3} R_{\mathfrak{p}}}\right), & \text{if } k=2; \end{cases} && \text{[Thm. 5.2]}\\
&= e\left(x_1; \frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_k)}\right), && \text{[6, Corollary 2.6].}
\end{aligned} \tag{5.8}$$

Hence equality holds in (5.7) and (5.8) which proves the theorem. □

We end this section by explicitly describing the generators of $\mathfrak{p}^{(n)}$ for all $n \geq 1$.

Theorem 5.9. *For all* $n \geq 1$, $\mathfrak{p}^{(n)} = \mathcal{I}_n R$; $\mathfrak{p}^{(n)} = \sum_{a_1 + 2a_2 = n} \mathfrak{p}^{a_1}(\mathfrak{p}^{(2)})^{a_2}$ *and* $\mathfrak{p}^{(n)} T' = I_n = \mathcal{I}_n T'$.

Proof. By Theorem 5.2 we get $\mathfrak{p}^{(n)} + (x_1) = \mathcal{I}_n R + (x_1)$. Hence $\mathfrak{p}^{(n)} = \mathcal{I}_n R + x_1(\mathfrak{p}^{(n)} : (x_1))$. As x_1 is a nonzerodivisor on $R/\mathfrak{p}^{(n)}$, $(\mathfrak{p}^{(n)} : (x_1)) = \mathfrak{p}^{(n)}$. By Nakayama's lemma, $\mathfrak{p}^{(n)} = \mathcal{I}_n R$.

For all $n \geq 3$, applying Proposition 2.5(1) we get

$$\mathfrak{p}^{(n)} = \mathcal{I}_n R = \sum_{a_1 + 2a_2 = n} \mathcal{J}_1^{a_1} \mathcal{J}_2^{a_2} R \subseteq \sum_{a_1 + 2a_2 = n} \mathfrak{p}^{a_1}(\mathfrak{p}^{(2)})^{a_2} \quad \subseteq \quad \mathfrak{p}^{(n)}.$$

Hence equality holds. The last equality follows from Theorem 5.2. □

6. Cohen-Macaulayness and Gorensteinness of symbolic blowup algebras

In this section we show that both symbolic blowup algebras $G_s(\mathfrak{p})$ and $\mathcal{R}_s(\mathfrak{p})$ are Cohen-Macaulay and Gorenstein. From Proposition 3.7 in [11] it follows that $G_s(\mathfrak{p})$ is Cohen-Macaulay. We use the results in this paper to prove it.

Theorem 6.1. *(1) $G_s(\mathfrak{p})$ is Cohen-Macaulay; and (2). $G_s(\mathfrak{p})$ is Gorenstein.*

Proof. (1) By [11, Corollary 3.2], it is enough to show that $x_1^\star, f_1^\star, f_2^\star$ is a regular sequence in $G_s(\mathfrak{p})$. For all $n \geq 1$, x_1 is a nonzerodivisor on $R/\mathfrak{p}^{(n)}$. Hence $x_1^\star$ is a regular element in $G_s(\mathfrak{p})$ and

$$G_s\left(\frac{\mathfrak{p}+x_1R}{x_1R}\right) \cong \bigoplus_{n\geq 0} \frac{\mathfrak{p}^{(n)}+x_1R}{\mathfrak{p}^{(n+1)}+x_1R} \cong \bigoplus_{n\geq 0} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^{(n+1)}+x_1\mathfrak{p}^{(n)}} \cong \frac{G_s(\mathfrak{p})}{x_1^\star G_s(\mathfrak{p})}.$$

To show that $f_1^\star$ is a regular sequence we need to show that $((\mathfrak{p}^{(n)}+x_1R) : f_1)$ $= \mathfrak{p}^{(n)}+x_1R$. Now

$$\begin{aligned}
\ell\left(\frac{R}{((\mathfrak{p}^{(n+1)},x_1):(f_1))}\right) &= \ell\left(\frac{R}{(\mathfrak{p}^{(n+1)},x_1)}\right) - \ell\left(\frac{R}{(\mathfrak{p}^{(n+1)},x_1,f_1)}\right) \\
&= \ell\left(\frac{T'}{I_{n+1}}\right) - \ell\left(\frac{T'}{I_{n+1}+(x_2^2)}\right) \quad \text{[Theorem 5.6]} \\
&= \ell\left(\frac{T'}{(I_{n+1}:x_2^2)}\right) \\
&= \ell\left(\frac{T'}{I_n}\right) \quad \text{[proof of Theorem 3.1]} \\
&= \ell\left(\frac{R}{(\mathfrak{p}^{(n)},x_1)}\right) \quad \text{[Theorem 5.6].}
\end{aligned}$$

Similarly, one can show that $((\mathfrak{p}^{(n+1)},x_1,f_1):(f_2)) = (\mathfrak{p}^{(n-1)},x_1,f_1)$ which will imply that $f_2^\star$ is a nonzerodivisor on $G_s(\mathfrak{p})/(x_1^\star, f_1^\star)$. This proves (1).

(1) As $G(\mathfrak{p}R_\mathfrak{p})$ is a polynomial ring, it is Gorenstein. Hence the result follows from Theorem 5.6 and [9, Corollary 5.8]. □

Theorem 6.2. *(*[11, Theorem 4.1], [17, Theorem 2]

(1) $\mathcal{R}_s(\mathfrak{p}) = R[\mathfrak{p}t, \mathcal{J}_2t^2] = R[\mathfrak{p}t, f_2t^2]$.
(2) $\mathcal{R}_s(\mathfrak{p})$ is Cohen-Macaulay.
(3) $\mathcal{R}_s(\mathfrak{p})$ is Gorenstein.

Proof. (1) The proof follows from Theorem 5.9.

(2) By Theorem 5.6, $\dfrac{R}{\mathfrak{p}^{(n)}+(\mathbf{f}_2)}$ is Cohen-Macaulay for all $n \geq 1$. Hence, by [9, Theorem 6.7], $\mathcal{R}_s(\mathfrak{p})$ is Cohen-Macaulay.

(3) By [9, Lemma 6.1], the a-invariant of $(G_s(\mathfrak{p}))$, $a(G_s(\mathfrak{p})) = -(2)$. By [9, Theorem 6.6], and Theorem 6.1, $\mathcal{R}_s(\mathfrak{p})$ is Gorenstein. □

7. A few questions

In this section we state a few related questions which are of interest.

Question 7.1. In [22] S. Goto and M. Morimoto studied the monomial curves $(t^{n^2+2n+2}, t^{n^2+2n+1}, t^{n^2+n+1})$. They showed that for $n = p^r$ $(r \geq 1)$, $R_s(\mathfrak{p})$ is Noetherian but not Cohen-Macaulay if char $\Bbbk = p$. In [12], the authors raised the following question. If char $\mathbb{k} = p' \neq p$, is $R_s(\mathfrak{p})$ Noetherian? In [28], W. Vasconcleos showed that if char $\mathbb{k} = 0$ and $n = 2$, then $R_s(\mathfrak{p})$ Noetherian. If p is a prime and $n = p > 2$, then is $R_s(\mathfrak{p})$ Noetherian?

Question 7.2. Let $\mathbb{k}$ be a field and let $\mathfrak{p}$ be the prime ideal defining the monomial curve (t^a, t^b, c^c) in $\mathbb{A}^3$ where a, b and c are pairwise coprime. In [5], D. Cutkosky gave a geometric meaning to symbolic primes. He found some interesting examples of monomial curves for which $R_s(\mathfrak{p})$ is finitely generated. Using the criteria in D. Cutkosky's paper, H. Srinivasan showed that if $a = 6$, then $R_s(\mathfrak{p})$ is finitely generated [26]. In [16], C. Huneke showed that $R_s(\mathfrak{p})$ is finitely generated if $a = 4$. Can we find all possible (a, b, c) such that $R_s(\mathfrak{p})$ is finitely generated?

References

[1] Bresinsky, H., Monomial space curves in $\mathbb{A}^3$ as set-theoretic complete intersections, *Proc. Amer. Math. Soc.*, **75** (1979), no. 1, 23–24.

[2] Bresinsky, H., Monomial Gorenstein curves in A^4 as set-theoretic complete intersections, *Manuscripta Math.*, **27** (1979), no. 4, 353–358.

[3] Cowsik, R. C. and Nori, M. V., Affine curves in characteristic $\mathfrak{p}$ are set theoretic complete intersections, *Invent. Math.*, **45** (1978), no. 2, 111–114.

[4] Cowsik, R. C., *Symbolic powers and number of defining equations.* Algebra and its applications (New Delhi, 1981), 13–14, Lecture Notes in Pure and Appl. Math., **91**, Dekker, New York, 1984.

[5] Cutkosky, D., Symbolic algebras of monomial primes, *Journal für die reine und angewandte Mathematik*, **416** (1991), 71–90.

[6] D'Cruz, C. and Masuti, S., *Cohen-Macaulayness and Gorensteinness of symbolic blowup algebras of certain monomial curves.* http://arxiv.org/abs/1610.03658`arXiv:1610.03658`

[7] Eliahou, S., *Courbes monomiales et algebre de Rees Symbolique*, Theèse, Universitè de Genève, 1983.

[8] Ene, V. and Herzog, J., *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics, **130** American Mathematical Society, (2012).

[9] Goto, S., The Cohen-Macaulay symbolic Rees algebras for curve singularities, The Cohen-Macaulay and Gorenstein Rees algebras associated to filtrations, *Mem. Amer. Math. Soc.*, **110** (1994), no. 526, 1–68.

[10] Goto, S. Shiro; Nishida, K. and Shimoda, Y., Topics on symbolic Rees algebras for space monomial curves, *Nagoya Math. J.*, **124** (1991), 99–132.

[11] Goto, S. Shiro; Nishida, K. and Shimoda, Y., The Gorensteinness of symbolic Rees algebras for space curves, *J. Math. Soc. Japan*, **43** (1991), no. 3, 465–481.

[12] Goto, S. Shiro; Nishida, K. and Watanabe, K., Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question., *Proc. Amer. Math. Soc.*, **120** (1994), no. 2, 383–392.

[13] Herzog, J., Generators and relations of abelian semigroups and semigroup rings, *Manuscripta Math.*, **3** (1970), 175–193.

[14] Herzog, J. and Ulrich, B., Self-linked curve singularities, *Nagoya Math. J.*, **120** (1990), 129–153.

[15] Huneke, C., The Primary components of and Integral Closures of Ideals in 3-Dimensional Regular Local Rings, *Mathematische Annalen*, **275** (1986), 617–636

[16] Huneke, C., Hilbert functions and symbolic powers, *Michigan Math. J.*, **34** (1987), 293–318.

[17] Knödel, G., Schenzel, P. and Zonsarow, R., Explicit computations on symbolic powers of monomial curves in affine space, *Comm. Algebra*, **20** (1992), no. 7, 2113–2126.

[18] Kronecker, L., Grundzüge einer arithmetischen Theorie der algebraische Grössen, *J. Reine Angew. Math.*, **92** (1882), 1–122.

[19] Lyubeznik, G., *A survey of problems and results on the number of defining equations. Commutative algebra* (Berkeley, CA, 1987), 375–390, Math. Sci. Res. Inst. Publ., **15**, Springer, New York, 1989.

[20] Matsumura, H., *Commutative ring theory*, translated from the Japanese by Reid, M., Second edition, Cambridge Studies in Advanced Math., **8** Cambridge University Press, (1989).

[21] Morales, M., Noetherian symbolic blow-ups, *J. Algebra*, **140** (1991), 12–25.

[22] Morimoto, M. and Goto, S., Non-Cohen-Macaulay Symbolic Blow-ups for Space Monomial Curves, *Proc. Amer. Math. Soc.*, **116**, No. 2 (1992), 305–311

[23] Patil, D., Certain monomial curves are set-theoretic complete intersections, *Manuscripta Math.*, **68** (1990), 399–404.

[24] Schenzel, P., Examples of Noetherian symbolic blow-up rings, *Rev. Roumaine Math. Pures Appl.*, **33** (1988), no. 4, 375–383.

[25] Solčan, *Š*., On the computation of symbolic powers of some curves in $\mathbb{A}^4$, *Acta Math. Univ. Comenian. (N.S.)*, 69 (2000), no. 1, 85–95.

[26] Srinivasan, H., On finite generation of symbolic algebras of monomial primes, *Communications in Algebra*, **19** (1991), 2557–2564.

[27] Valla, G., On determinantal ideals which are set-theoretic complete intersections, *Compositio Mathematica*, **42** (1980), 3–11.

[28] Vasconcelos, W. V., *Symmetric algebras and factoriality*. Math. Sci. Res. Inst. Publ., **vol. 15**, Springer, New York, 1989, pp. 467-496.

[29] Vasconcelos, W. V., *Arithmetic of Blowup Algebras*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge (1994)

Clare D'Cruz
Chennai Mathematical Institute, Plot H1 SIPCOT IT Park
Siruseri, Kelambakkam 603103, Tamil Nadu, India.
E-mail: *clare@cmi.ac.in*

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December, (2017), 123-134

PROBLEM SECTION

In the last issue of the Math. Student Vol. **86**, Nos. 1-2, January-June (2017), we had invited solutions from the floor to the remaining problems *9, 10* and *12* of the MS, 85, 3-4, 2016 as well as to the five new problems *1, 2, 3, 4* and *5* presented therein till October 31, 2017.

No solution was received from the floor to the remaining problems *9, 10* and *12* of the MS, 85, 3-4, 2016 and hence we provide in this issue the Proposer's solution to these problems.

For the problems of MS, 86, 1-2, 2017, we received from the floor **two** correct solutions to problem *3* and publish here the well written solution; **three** solutions to the Problem *1* out of which one was much better but incomplete partial solution, one more was partial but incomplete solution and the third was also partially correct and partially incorrect, and hence we provide here the Proposer's two solutions. We also received **one** solution each to problems *2, 5*, but the solutions were incorrect. For the problem *4*, we received **one** correct solution to only part (1), and hence we provide here the proposer's solution to this probem. Readers can try their hand on the remaining problems *2* and *5* till April 30, 2018.
In this issue we first present **seven new problems**. Solutions to these problems as also to the remaining problems *2* and *5* of MS, 86, 1-2, 2017, received from the floor till April 30, 2018, if approved by the Editorial Board, will be published in the MS, 87, 1-2, 2018.

MS-2017, Nos. 3-4: Problem-6:
Proposed by **S. K. Tomar.**

> Using the techniques of calculus of variation, find the minimum distance between the yolk and ellipsoidal shell of an egg. You may take the yolk as a sphere and the shell as ellipsoidal both centered at origin, that is, $x^2+y^2+z^2 = 4$ and $\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$.

MS-2017, Nos. 3-4: Problem-7:
Proposed by **M. Ram Murty.**

> Evaluate
> $$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{3}}}.$$

Problems proposed by B. Sury

MS-2017, Nos. 3-4: Problem-8:

For every positive integer $n > 3$, prove that there is a prime factor of $2^n + 1$ which does not divide $2^m + 1$ for any $m < n$.

MS-2017, Nos. 3-4: Problem-9:

Pick and fix your favourite number α. Consider the $n \times n$ matrix with entries $\alpha + i; 1 \leq i \leq n^2$ written in a spiral fashion clockwise starting with $\alpha + 1$ in the $(1,1)$-position. For example, the matrix for $n = 3$ is

$$\begin{pmatrix} \alpha+1 & \alpha+2 & \alpha+3 \\ \alpha+8 & \alpha+9 & \alpha+4 \\ \alpha+7 & \alpha+6 & \alpha+5 \end{pmatrix}.$$

Find the determinant of this matrix for general n.

MS-2017, Nos. 3-4: Problem-10:

Show that there does not exist a square matrix over rational numbers whose characteristic polynomial is $X^2 - 3$.

MS-2017, Nos. 3-4: Problem-11:

Consider a group G generated by $\{x_1, x_2, x_3, \cdots\}$ and relations $x_2 x_1 x_2^{-1} = x_3 x_2 x_3^{-1} = x_4 x_3 x_4^{-1} = \cdots\cdots$ Prove that G is not finitely generated.

MS-2017, Nos. 3-4: Problem-12:

Show that

$$\prod_{n \geq 2} \left(\left(1 - \frac{1}{n^2}\right)^{2(n^2-1)} \left(\frac{1+1/n}{1-1/n}\right)^n \right) = \pi.$$

Solution by the Proposer S. K. Tomar: MS-2016, Nos. 3-4: Problem 9: Equation of small motion in a uniform elastic solid medium is given as

$$(\lambda + \mu) \bigtriangledown (\vec{\bigtriangledown} \cdot \vec{u}) + \mu \bigtriangledown^2 \vec{u} = \rho \ddot{\vec{u}}, \tag{0.1}$$

where λ and μ are Lame's parameters; ρ and $\vec{u}$ represent respectively the density and displacement vector. Superposed dot represents the temporal derivative and other symbols have their usual meanings.

Using Lie symmetry method, show that (0.1) leads to two waves propagating with distinct speeds.

Solution. The displacement vector equation of small motion is given by

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^{\mathbf{2}}\mathbf{u}=\rho\ddot{\mathbf{u}}, \tag{0.2}$$

$$\text{or}\quad (\lambda+2\mu)\nabla^2\mathbf{u}-\mu\nabla\times(\nabla\times\mathbf{u})=\rho\ddot{\mathbf{u}}, \tag{0.3}$$

Employing the time harmonic solution proportional to $\exp(-\iota\omega t)$, (ω being the angular frequency and t the time variable) into (0.2), we have

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{U})+\mu\nabla^2\mathbf{U}=-\omega^2\rho\mathbf{U}. \tag{0.4}$$

where $\mathbf{U}=\mathbf{U}(x,y,z)$. Lie's method ([1], [2], [3]) is the simplest method among group theoretic techniques. We shall apply Lie classical method to obtain similarity variables of equation (0.4) in 2-Dimension. Using similarity variables, we will reduce the equation (0.4) into ordinary differential equations (ODE). The solution of reduced ODE will help in determining the complete solution of equation (0.2). For the special choice of arbitrary constants in this solution, we shall show that equation (0.2) leads to two waves propagating with distinct speeds. Let U_1 and U_2 be the components of $\mathbf{U}$ along x- and y- directions respectively, then the vector equation (0.4) yields

$$(\lambda+2\mu)U_{1xx}+(\lambda+\mu)U_{2xy}+\mu U_{1yy}+\rho\omega^2U_1=0,$$
$$(\lambda+2\mu)U_{2yy}+(\lambda+\mu)U_{1xy}+\mu U_{2xx}+\rho\omega^2U_2=0. \tag{0.5}$$

Consider the Lie group of point transformations given by

$$x^*=x+\epsilon\xi_1(x,y,U_1,U_2)+o(\epsilon^2), \tag{0.6}$$
$$y^*=y+\epsilon\xi_2(x,y,U_1,U_2)+o(\epsilon^2), \tag{0.7}$$
$$U_1^*=U_1+\epsilon\eta_1(x,y,U_1,U_2)+o(\epsilon^2), \tag{0.8}$$
$$U_2^*=U_2+\epsilon\eta_2(x,y,U_1,U_2)+o(\epsilon^2), \tag{0.9}$$

with $\epsilon << 1$(see [3]). For determining the symmetry group of (0.4), one has to find the infinitesimals ξ_1,ξ_2,η_1 and η_2. Herein, on invoking invariance criterion, the following relations from the coefficient of first degree of ϵ are deduced.

$$(\lambda+2\mu)\eta_1^{xx}+(\lambda+\mu)\eta_2^{xy}+\mu\eta_1^{yy}+\rho\omega^2\eta_1=0,$$
$$(\lambda+2\mu)\eta_2^{yy}+(\lambda+\mu)\eta_1^{xy}+\mu\eta_2^{xx}+\rho\omega^2\eta_2=0, \tag{0.10}$$

where $\eta_1^{xx},\eta_1^{xy},\eta_1^{yy},\eta_2^{xx},\eta_2^{xy}$ and η_2^{yy} are extended (prolonged) infinitesimals acting on an enlarged space corresponding to $U_{1xx},U_{1xy},U_{1yy},U_{2xx},U_{2xy}$ and U_{2yy} respectively. Using the procedure as explained in ([3; page no. 153-155]), one can obtain the expressions for η_1^{xx}, η_1^{yy}, η_1^{xy}, η_2^{xy}, η_2^{xx}, η_2^{yy}. Inserting these expressions into equations in (0.10) and equating the coefficients of different differentials to zero, we get a number of PDEs in ξ_1,ξ_2,η_1 and η_2, that need to be satisfied. The special solution of this large system helps us to obtain the infinitesimals ξ_1,ξ_2,η_1 and η_2, as follows:

$$\begin{aligned}\xi_1 &= a_1+ya_3, \qquad \xi_2=a_2-xa_3,\\ \eta_1 &= a_3U_2+a_4\cos\left(\omega\sqrt{\frac{\rho}{\mu}}y\right)+a_5\sin\left(\omega\sqrt{\frac{\rho}{\mu}}y\right)+a_6\cos\left(\omega\sqrt{\frac{\rho}{\lambda+2\mu}}x\right)\end{aligned}$$

$$+a_7 \sin\left(\omega\sqrt{\frac{\rho}{\lambda+2\mu}}x\right), \quad \text{and} \quad \eta_2 = -a_3U_1, \tag{0.11}$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 are arbitrary constants. To obtain the symmetry reductions of equations in (0.5), we have to solve the characteristic equation

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dU_1}{\eta_1} = \frac{dU_2}{\eta_2}, \tag{0.12}$$

where ξ_1, ξ_2, η_1 and η_2 are given by (0.11). For $a_3 = 0$ and using the characteristic equation (0.12), we obtain following similarity variables for equation (0.5) as

$$\begin{aligned} U_1 &= \frac{1}{a_2\omega}\sqrt{\frac{\mu}{\rho}}\left[a_4 \sin\left(\omega\sqrt{\frac{\rho}{\mu}}y\right) - a_5\cos\left(\omega\sqrt{\frac{\rho}{\mu}}y\right)\right] \\ &\quad + \frac{1}{a_1\omega}\sqrt{\frac{\lambda+2\mu}{\rho}}\left[a_6\sin\left(\omega\sqrt{\frac{\rho}{\lambda+2\mu}}x\right) - a_7\cos\left(\omega\sqrt{\frac{\rho}{\lambda+2\mu}}x\right)\right] + F(\xi), \\ U_2 &= G(\xi), \qquad (0.13) \\ \xi &= a_2x - a_1y, \qquad (0.14) \end{aligned}$$

where ξ is new independent variable; $F(\xi)$ and $G(\xi)$ are new dependent variables. Using similarity variables (0.14) and (0.13) into system of equations (0.5), we can obtain a system of ODEs, whose complete solution is given by

$$\begin{aligned} F(\xi) = C_1\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right) + C_2\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right) + C_3\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right) \\ + C_4\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right), \end{aligned} \tag{0.15}$$

$$\begin{aligned} G(\xi) = \frac{1}{a_1a_2}\left[a_2^2C_1\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right) - a_1^2C_4\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right)\right. \\ \left. + a_2^2C_2\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right) - a_1^2C_3\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right)\right], \end{aligned} \tag{0.16}$$

where $a = \sqrt{a_1^2 + a_2^2}$; C_1, C_2, C_3 and C_4 are arbitrary constants.

Corresponding solution of system of equations in (0.5) is given by

$$\begin{aligned} U_1 &= \frac{1}{a_2\omega}\sqrt{\frac{\mu}{\rho}}\left[a_4 \sin\left(\omega\sqrt{\frac{\rho}{\mu}}y\right) - a_5\cos\left(\omega\sqrt{\frac{\rho}{\mu}}y\right)\right] \\ &\quad + \frac{1}{a_1\omega}\sqrt{\frac{\lambda+2\mu}{\rho}}\left[a_6\sin\left(\omega\sqrt{\frac{\rho}{\lambda+2\mu}}x\right) - a_7\cos\left(\omega\sqrt{\frac{\rho}{\lambda+2\mu}}x\right)\right] \\ &\quad + C_1\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right) + C_2\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right) \\ &\quad + C_3\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right) + C_4\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right), \\ U_2 &= \frac{1}{a_1a_2}\left[a_2^2C_1\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right) - a_1^2C_4\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right)\right. \end{aligned}$$

$$+a_2^2C_2\cos\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{\mu}}\right)-a_1^2C_3\sin\left(\frac{\omega\sqrt{\rho}\xi}{a\sqrt{2\,\mu+\lambda}}\right)\Bigg], \tag{0.17}$$

where ξ is given by (0.14). Thus the solution of equation (0.2) can be written as

$$\mathbf{u}=(U_1\hat{\imath}+U_2\hat{\jmath})\exp\left(-i\omega t\right), \tag{0.18}$$

where expressions for U_1 and U_2 are given by (0.17). Now let us consider two cases:

Case (i) For $a_6=a_7=C_3=C_4=0$, it can be seen that

$$\nabla\cdot\mathbf{u}=0, \qquad \text{and} \qquad \nabla\times\mathbf{u}\neq 0. \tag{0.19}$$

With these equation (0.2) reduces to the wave equation

$$\nabla^2\mathbf{u}=\frac{1}{c_t^2}\frac{\partial^2\mathbf{u}}{\partial t^2}, \qquad c_t^2=\frac{\mu}{\rho}. \tag{0.20}$$

This represents a wave propagating with speed c_t and called *distorsional wave*.

Case (ii) For $a_4=a_5=C_1=C_2=0$, it can be seen that

$$\nabla\cdot\mathbf{u}\neq 0, \qquad \text{and} \qquad \nabla\times\mathbf{u}=0. \tag{0.21}$$

With these, equation (0.3) reduces to the wave equation

$$\nabla^2\mathbf{u}=\frac{1}{c_l^2}\frac{\partial^2\mathbf{u}}{\partial t^2}, \qquad c_l^2=\frac{\lambda+2\mu}{\rho}. \tag{0.22}$$

This represents a wave propagating with speed c_l and called *dilatational wave*.

Acknowledgement. In solving equation (0.12), the proposer was helped out by Sachin Kumar, center for Mathematics and Statistics, school of Basic and Applied Sciences, central University of Punjab, bhatinda 151001, Punjab, India; E-mail: *sachin@cup.ac.in*

References

[1] Olver, P. J., *Applications of Lie Groups to Differential Equations*, Graduate Text in Mathematics, Vol. 107, Springer-Verlag, New York, 1993.

[2] Bluman, G. W. and Kumei, S., *Symmetries and Differential Equations*, Springer, Berlin, 1989.

[3] Bluman, G. W. and Cole, J. D., *Similarity Methods for Differential Equations*, Springer, Berlin, 1974.

Solution by the Proposer Purushottam Rath: MS-2016, Nos. 3-4: Problem 10: Let $n\geq 5$ be a natural number. How many distinct topologies, not necessarily Hausdorff, can be given to the alternating group A_n so that it becomes a topological group?

Solution. Each topology generates a normal subgroup whose cosets form a base. Hence there are as many topological group structures as the number of normal subgroups.

Solution by the Proposers George Andrews and Emeric Deutsch: MS-2016, Nos. 3-4: Problem 12: Show that the number of parts having odd multiplicities in all partitions of n is equal to the difference between the number of odd parts in all partitions of n and the number of even parts in all partitions of n.

Example: $n = 5$. The partitions $5',\ 4'1',\ 3'2',\ 3'11,\ 221',\ 2'1'11,\ 1'1111$ have 10 parts with odd multiplicities (marked with $'$). On the other hand, the number of odd parts is 15

$$(5, 1, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

while the number of even parts is 5

$$(4,\ 2,\ 2,\ 2,\ 2).$$

Solution [1] By standard procedure one can show that

(i) the generating function for the statistic "number of parts having odd multiplicity" is

$$G_1(t,x) = \prod_{j\geq 1} \frac{1+tx^j}{1-x^{2j}};$$

(ii) the generating function for the statistic "number of odd parts" is

$$G_2(t,x) = \prod_{j\geq 1} \frac{1}{(1-tx^{2j-1})(1-x^{2j})};$$

(iii) the generating function for the statistic "number of even parts" is

$$G_3(t,x) = \prod_{j\geq 1} \frac{1}{(1-x^{2j-1})(1-x^{2j})};$$

The values of these statistics summed over all partitions of n have generating functions $g_i(x) = \frac{dG_i}{dt}|_{t=1}$. Straightforward computation, using logarithmic dierentiation, gives

$$g_1(x) = P\sum_{j\geq 1} \frac{x^j}{1+x^j}, \qquad g_2(x) = P\sum_{j\geq 1} \frac{x^{2j-1}}{1-x^{2j-1}}, \qquad g_3(x) = P\sum_{j\geq 1} \frac{x^{2j}}{1-x^{2j}},$$

where $P = \prod_{j\geq 1} \frac{1}{1-x^j}$. Now we have

$$\sum_{j\geq 1} \frac{x^{2j-1}}{1-x^{2j-1}} - \sum_{j\geq 1} \frac{x^{2j}}{1-x^{2j}} = \sum_{j\geq 1} \frac{(-1)^{j-1}x^j}{1-x^j} = \sum_{j\geq 1} (-1)^{j-1} \sum_{k\geq 1} x^{kj}$$

$$= \sum_{k\geq 1}\sum_{j\geq 1} (-1)^{j-1}x^{kj} = \sum_{k\geq 1} \frac{x^k}{1+x^k}.$$

The equality between the left most and right most expressions yields $g_2(x) - g_3(x) = g_1(x)$.

Solution from the floor: MS-2017, Nos. 1-2: Problem 3: If there is a painting of dimensions 40 inches by 100 inches, what is the smallest square frame which can cover it completely? Answer the same question when the painting has dimensions 40 inches by 90 inches.

(Solution submitted on 27-10-2017 by **Siddhi Pathak and François Séguin**,

---

[1] *The example $n = 5$ may be replaced by a shorter one.*

Example. $n = 3$. *The partitions $3', 2'1, 1'11$ have 4 parts with odd multiplicities marked with '. On the other hand, the number of odd parts is 5 $(3, 1, 1, 1, 1)$ while the number of even parts is $1, (2)$.*

I do not take $n = 4$ because I like to have a part with odd multiplicity 3.

Graduate Students, Department of Mathematics and Statistics, Room 232, Jeffery Hall, Queen's University, Kingston, Ontario K7K 3T2, Canada; E-mails: *14ssp2@queensu.ca* , *francois.seguin@queensu.ca*).

Solution.

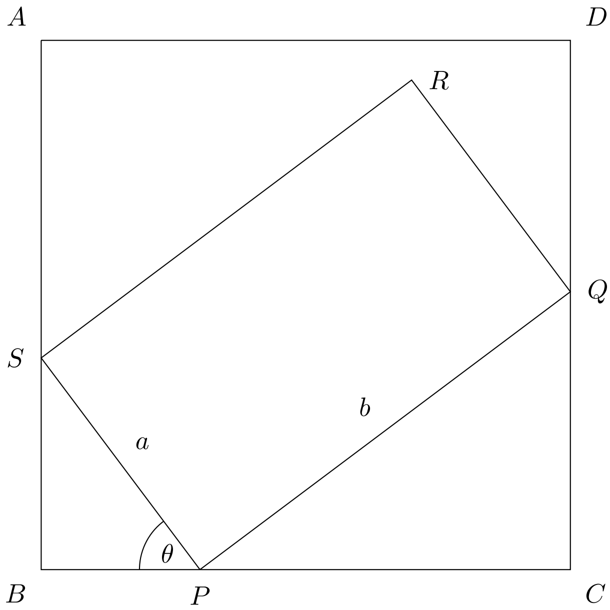


FIGURE 1. Framed painting

We consider a general painting of dimensions a inches by b inches, with $b \geq a$. For simplicity of exposition, we label the corners of the painting as P,Q, R and S and the corners of the square frame as A,B,C and D. Without loss of generality, we can assume that at least 3 corners of the painting touch the frame (see Figure 1). Suppose that $\angle SPB = \theta$, with $0 \leq \theta \leq \pi/2$. Then,

$$f(\theta) := \text{length(BC)} = a\cos(\theta) + b\sin(\theta).$$

Similarly, we calculate the length of side CD as

$$\text{length(CD)} = b\cos(\theta) + a\sin(\theta) + x,$$

where x is the distance between the corner R and the side AD. We want to minimize $f(\theta)$ under the constraint that $x \geq 0$ and $0 \leq \theta \leq \pi/2$. Since the frame ABCD is a square,

$$b\cos(\theta) + a\sin(\theta) + x = a\cos(\theta) + b\sin(\theta).$$

Thus, the constraint $x \geq 0$ amounts to

$$0 \leq (b-a)(\sin\theta - \cos\theta).$$

As $b \geq a$, the above constraints reduce to $\sin\theta \geq \cos\theta$ with $0 \leq \theta \leq \pi/2$. This is true when $\pi/4 \leq \theta \leq \pi/2$. Therefore, we have to minimize $f(\theta)$ with $\pi/4 \leq \theta \leq \pi/2$. We note that

$$f'(\theta) = b\cos\theta - a\sin\theta,$$

which is zero precisely when $\tan\theta = b/a \geq 1$. Thus, the critical values of f are at $\theta = \pi/4, \pi/2$ and $\arctan(b/a)$. Hence, the minimum value of $f(\theta)$ is

$$\min\left\{\frac{a+b}{\sqrt{2}},\ \sqrt{a^2+b^2},\ b\right\}.$$

Since $\sqrt{a^2+b^2} \geq b$, the minimum actually is

$$\min\left\{\frac{a+b}{\sqrt{2}},\ b\right\}.$$

Therefore, when $a \leq (\sqrt{2}-1)b$, the smallest square frame is $(a+b)/\sqrt{2}$ inches by $(a+b)/\sqrt{2}$ inches, i.e, the painting is placed diagonally across the frame. On the other hand, if $a \geq (\sqrt{2}-1)b$, the smallest square frame is b inches by b inches, i.e., the painting is placed upright. Hence, for $a = 40$ and $b = 100$, length of the smallest square frame is $70\sqrt{2}$ inches and when $a = 40$ and $b = 90$, the minimum length is 90 inches.

Correct solution was also received from the floor from:
Prajanaswaroopa S.; (Bangalore, India; E-mail: *sntrm4@rediffmail.com*, received on 31-10-2017)

Solutions by the Proposers Zoltán Boros and Árpád Száz: MS-2017, Nos. 1-2: Problem 1: For all $a, b \in \mathbb{R}$ determine

$$F(a,b) = \inf_{n\in\mathbb{N}}\left(an + \frac{b}{n}\right),$$

where $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all natural and real numbers respectively.

Solution 1. Putting $f(n) = an+b/n$ for all $n \in \mathbb{N}$, we have $F(a,b) = \inf_{n\in\mathbb{N}} f(n)$. Observe that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty}(an+b/n) = \lim_{n\to\infty} an = (\operatorname{sgn} a)\infty$, in view of the usual conventions that $-1\,\infty = -\infty$, $0\,\infty = 0$ and $1\,\infty = \infty$. Thus

$$a < 0 \implies \lim_{n\to\infty} f(n) = -\infty \quad \textit{and} \quad F(a,b) = -\infty;$$

and

$$a = 0 \leq b \implies f(n) = b/n \geq 0\ (n \in \mathbb{N}) \text{ and } \lim_{n\to\infty} f(n) = 0 \text{ giving } F(a,b) = 0.$$

If we now put $d(n) = f(n+1) - f(n)$ for all $n \in \mathbb{N}$, then for any $n \in \mathbb{N}$, we have

$$f(n+1) \leq f(n) \iff d(n) \leq 0 \qquad \text{and} \qquad f(n) \leq f(n+1) \iff 0 \leq d(n).$$

Moreover, we can also note that

$$d(n) = a(n+1) + b/(n+1) - \big(an + b/n\big) = a - b/(n(n+1))$$

for all $n \in \mathbb{N}$. Therefore, if $0 \leq a$ and $b \leq 2a$, we see that

$$0 \leq 2a - b \leq an(n+1) - b, \qquad \text{and hence} \qquad 0 \leq a - b/n(n+1) = d(n)$$

for all $n \in \mathbb{N}$. Therefore, the sequence $(f(n))$ is increasing, and hence

$$F(a,b) = \inf_{n\in\mathbb{N}} f(n) = \min_{n\in\mathbb{N}} f(n) = f(1) = a + b.$$

Now, it remains only to consider the case when $0 < a$ and $2a < b$. Clearly

$$\lim_{n\to\infty} d(n) = \lim_{n\to\infty}(a - b/n(n+1)) = a > 0.$$

Therefore, $d(n) > 0$ for all sufficiently large natural numbers n, and hence in particular $m = \min\{n \in \mathbb{N} : d(n) \geq 0\}$ exists. Moreover, since $d(1) = a - b/2 < 0$, we note that $m > 1$.

Furthermore, since $b > 0$, we also note that

$$d(n) = a - b/n(n+1) < a - b/(n+1)(n+2) = d(n+1)$$

for all $n \in \mathbb{N}$, and thus in particular the sequence $(d(n))$ is increasing. Therefore, for any $n \in \mathbb{N}$, we can state that

$$d(n) < 0 \quad \text{if} \quad n < m \qquad \text{and} \qquad 0 \leq d(n) \quad \text{if} \quad n \geq m,$$

and hence

$$f(n) > f(n+1) \quad \text{if} \quad n < m \qquad \text{and} \qquad f(n) \leq f(n+1) \quad \text{if} \quad n \geq m.$$

Clearly therefore

$$f(1) > f(2) > \cdots > f(m-1) > f(m) \leq f(m+1) \leq f(m+2) \leq \cdots,$$

and hence

$$F(a,b) = \inf_{n\in\mathbb{N}} f(n) = \min_{n\in\mathbb{N}} f(n) = f(m) = am + b/m.$$

Now, to determine m, we note that

$$\begin{aligned} 0 \leq d(n) &\iff & 0 \leq a - b/n(n+1) &\iff 0 \leq a(n^2+n) - b \\ &\iff & b/a \leq n^2 + n &\iff b/a + 1/4 \leq (n+1/2)^2 \\ &\iff & \sqrt{b/a+1/4} \leq n + 1/2 &\iff -1/2 + \sqrt{b/a+1/4} \leq n. \end{aligned}$$

Hence

$$m = \min\{n \in \mathbb{N} : 0 \leq d(n)\} = \min\{n \in \mathbb{N} : \lambda \leq n\},$$

where $\lambda = -1/2 + \sqrt{b/a + 1/4}$.

Summarizing all these observations, we obtain that

$$F(a,b) = \begin{cases} -\infty, & \text{if} \quad a < 0; \\ 0 & \text{if} \quad a = 0 \leq b; \\ a+b & \text{if} \quad 0 \leq a \quad \text{and} \quad b \leq 2a; \\ am + b/m & \text{if} \quad 0 < a \quad \text{and} \quad 2a < b; \end{cases}$$

where $m = \min\{n \in \mathbb{N} : \lambda \leq n\}$ with λ as above.

Solution 2. Putting $f(x) = ax + b/x$ for all $x \in \mathbb{R}_+$, we again have $F(a,b) = \inf_{n\in\mathbb{N}} f(n)$; and hence, as in **Solution 1**, for $a < 0$, we get $F(a,b) = -\infty$.

Now, f is differentiable and $f'(x) = a - b/x^2$ for all $x \in \mathbb{R}_+$. Hence

$$f'(x) \leq 0 \iff ax^2 \leq b \qquad \text{and} \qquad f'(x) \geq 0 \iff ax^2 \geq b.$$

Therefore, if $a \leq 0 \leq b$ then the function f, and hence in particular the sequence $(f(n))$, is decreasing and hence $F(a,b) = \operatorname{sgn}(a)\infty$.

Similarly, if $b \leq 0 \leq a$ then the function f, and hence in particular the sequence $(f(n))$, is increasing and hence $F(a,b) = f(1) = a + b$.

Consider now the case when $a, b > 0$. Let $r = (\sqrt{b}/\sqrt{a})$. Clearly, $ax^2 \leq b$ if $0 < x \leq r$ and $b \leq ax^2$ if $r \leq x < \infty$; and hence $f'(x) \leq 0$ if $0 < x \leq r$ and $0 \leq f'(x)$ if $r \leq x < \infty$.

Therefore f is decreasing on $]0, r]$ and increasing on $[r, \infty[$. Thus $f(r) = 2\sqrt{ab}$ is the global minimum of f. Hence, if $b \leq a$, i. e., $r \leq 1$, then the sequence $(f(n))$ is increasing, and in view of the earlier observation

$$F(a, b) = a + b.$$

If $a < b$, i. e., $1 < r$, then putting $k = [r]$, the integral part of r, and noting that $k \leq r < (k+1)$ we get

$$F(a, b) = \min\{f(k), f(k+1)\}.$$

It is easy to see that $f(k+1) - f(k) = a - b/k(k+1)$, and hence

$$f(k) \leq f(k+1) \iff b/a \leq k(k+1) \iff r^2 \leq k(k+1)$$

and

$$f(k+1) \leq f(k) \iff k(k+1) \leq b/a \iff k(k+1) \leq r^2.$$

Summerizing these observations, we can state that

$$F(a,b) = \begin{cases} -\infty, & \text{if } a < 0; \\ 0, & \text{if } a = 0 \leq b; \\ a+b & \text{if } b \leq 0 \leq a \text{ or } 0 < b \leq a. \end{cases}$$

Moreover, if $0 < a < b$ then we have

$$F(a, b) = a[r] + b/[r] \ \ if \ \ r^2 \leq [r]([r] + 1)$$

and

$$F(a, b) = a([r] + 1) + b/([r] + 1) \ \ if \ \ [r]([r] + 1) \leq r^2.$$

Remark. The observations applied in the two solutions can be sharpened by considering the strict montonicity properties of the functions.

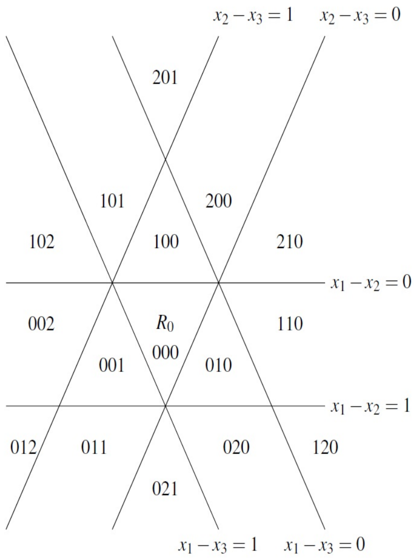
Note. **1**. Partially correct and better, but incomplete solutions to this problem were received on 10-10-2017 from **Diyath Nelaka Pannipitiya**, (IT Unit-2, Faculty of Science, University of Colombo, Shrilanka; E-mail: *diyathnp@yahoo.com*.

2. Still less partially correct and incomplete solution to this problem was received on 09-10-2017 from **Dasari Naga Vijay Krishna** (Machilipatnam, Andhra Pradesh-521001; *Vijay9290009015@gmail.com*) .

3. A still lesser partially correct and partially incorrect solutions to this problem were received on 31-10-2017 from **Prajnanaswroopa S.** (Bangalore, Karnataka, India; *sntrm4@rediffmail.com*) .

Solution by the Proposer B. Sury: MS-2017, Nos. 1-2: Problem 4: There is a one way street which has n parking spaces numbered 1 to n. Suppose there are n cars $C_1, \cdots, C_n$. Each car C_i has a certain favourite parking slot a_i $(1 \leq a_i \leq n)$. Each car enters the street and, if its favourite slot is empty, it parks there. If not, it proceeds ahead and parks at the next available slot. If no slot is available, the car has to leave the street. A sequence $(a_1, \cdots, a_n)$ is a parking sequence if every car can park. For example, $(1, 1), (1, 2), (2, 1)$ are parking sequences when $n = 2$.

(i) Find the number of parking sequences.
(ii) Find a natural bijection between the parking sequences and the number of regions formed in $\mathbf{R}^n$ by the hyperplanes $x_i - x_j = 0, x_i - x_j = 1$ for $1 \leq i < j \leq n$.
Solution. (i) It is convenient to order the parking slots as $0, 1, \cdots, n-1$. If a_i is the preferred space for car C_i and the cars enter the one-way street in the order $C_1, \cdots, C_n$ then clearly a sequence $(a_1, \cdots, a_n)$ is a parking sequence if and only if it contains at least r terms less than r. In other words, there is a permutation σ of $\{1, 2, \cdots, n\}$ so that $0 \leq a_{\sigma(r)} < r$ for all $r = 1, \cdots, n$. The number of such sequences turns out to be $(n+1)^{n-1}$ originally proved by analytic methods. Later, Pollak came up with the following beautiful argument.
Add an additional slot n and arrange the slots $0, 1, \cdots, n$ in (clockwise) circular order. Once again, the n cars $C_1, \cdots, C_n$ enter in order and each C_i has a preferred parking slot a_i among these $n+1$ slots. If a car finds its preferred slot occupied, it moved further along the circle to occupy the next empty slot available. Each occupation leaves one slot empty and a parking sequence is obtained when slot n is empty. By symmetry, the number of sequences leaving a slot r empty is the same for each r in $0 \leq r \leq n$. Hence, the number of sequences leaving any given particular empty (in particular, slot n empty) is $(n+1)^n/(n+1) = (n+1)^{n-1}$. This is the number of parking sequences; note that this ia also in bijection with the number of labelled trees on $n+1$ vertices by a famous theorem of Cayley.
(ii) Now, we come to the second part of the problem.



Consider the arrangement of hyperplanes $x_i - x_j = 0$ or 1 for $1 \leq i < j \leq n$. We give a labelling of the regions formed to the n-tuples which are naturally in bijection with parking functions.

Now, the unique 'base' region R_0 contained between all pairs of parallel hyperplanes is the region

$$R_0 : x_1 > x_2 > \cdots > x_n; x_1 - x_n < 1.$$

Label R_0 by the label $\lambda(R_0) = (0, 0, \cdots, 0)$. Now, if R_1 is a region that has already been labelled, and if R_2 is yet to be labelled, and which is separated from R_1 by a hyperplane, give the label

$$\lambda(R_2) = \lambda(R_1) + e_i \quad \text{or} \quad \lambda(R_2) = \lambda_(R_1) + e_j$$

according as to whether the separating hyperplane is $x_i - x_j = 0$ or $x_i - x_j = 1$. This labelling is well-defined and independent of the ordering of the hyperplanes. See the attached figure for the case $n = 3$. Note that if a region R has labelling $\lambda(R) = (a_1, \cdots, a_n)$, then the number of hyperplane separating it from R_0 is $a_1 + a_2 + \cdots + a_n$. The labelling λ gives a bijection from the regions above to the parking sequences of length n.

Note. Correct solution, but only of part (i), of this problem was received on 31-10-2017 from **Prajnanaswroopa S.** (Bangalore, Karnataka, India; *sntrm4@rediffmail.com*).

Note. Correct solutions of the MS-2017, Nos. 1-2: Problems **1 & 3** were also received on 19-11-2017, *late by few days after the issue was sent to press for printing and publishing*, from **Subhash Chand Bhoria** (Village & P.O.: Bakali-136132, near Ladwa, Distt.- Kurukshetra, Haryana, India; *scbhoria89@gmail.com*). Therefore, **this note will not appear in the Print version** of The Math. Student, Vol. **86** Nos. 3-4 (2017).

FORM IV
(See Rule 8)

1.	Place of Publication:	PUNE
2.	Periodicity of publication:	QUARTERLY
3.	Printer's Name: Nationality: Address:	DINESH BARVE INDIAN PARASURAM PROCESS 38/8, ERANDWANE PUNE-411 004, INDIA
4.	Publisher's Name: Nationality: Address:	N. K. THAKARE INDIAN GENERAL SECRETARY THE INDIAN MATHEMATICAL SOCIETY c/o:CENTER FOR ADVANCED STUDY IN MATHEMATICS, S. P. PUNE UNIVERSITY PUNE-400 007, MAHARASHTRA, INDIA
5.	Editor's Name: Nationality: Address:	J. R. PATADIA INDIAN (DEPARTMENT OF MATHEMATIC, THE M. S.UNIVERSITY OF BARODA) 5 , ARJUN PARK, NEAR PATEL COLONY BEHIND DINESH MILL, SHIVANAND MARG VADODARA - 390 007, GUJARAT, INDIA
6.	Names and addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital:	THE INDIAN MATHEMATICAL SOCIETY

I, N. K. Thakare, hereby declare that the particulars given above are true to the best of my knowledge and belief.

N. K. THAKARE
Signature of the Publisher

Dated: 27^{th} November 2017

Published by Prof. N. K. Thakare for the Indian Mathematical Society, type set by J. R. Patadia at 5, Arjun Park, Near Patel Colony, Behind Dinesh Mill, Shivanand Marg, Vadodara - 390 007 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune 411 041 (India). Printed in India

The Mathematics Student ISSN: 0025-5742
Vol. 86, Nos. 3-4, July-December, (2017)

EDITORIAL BOARD

Back volumes of our periodicals, except for a few numbers out of stock, are available.

Edited by J. R. Patadia and published by N. K. Thakare
for the Indian Mathematical Society.

Type set by J. R. Patadia at 5, Arjun Park, Near Patel Colony, Behind Dinesh Mill, Shivanand Marg, Vadodara-390 007 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No.129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune – 411 041, Maharashtra, India. Printed in India