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Edited by J. R. PATADIA

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TWISTED CONJUGACY IN CERTAIN PL-HOMEOMORPHISM GROUPS*

P. SANKARAN

Abstract. The aim of these notes is to establish the R_∞-property for certain classes of PL-homeomorphism groups which include certain generalizations of the Richard Thompson group F. We shall describe the Bieri-Neumann-Strebel invariant $\Sigma(G)$ associated to a group G and how it is applied to decide on the R_∞-property. We shall construct a family $\mathcal{G}$ of finitely generated PL-homeomorphism groups of the interval $[0,1]$ each of which has the R_∞-property, such that $\mathcal{G}$ has cardinality the continuum, and, members $\mathcal{G}$ are pairwise non-isomorphic.

1. Introduction

Let $\phi : G \to G$ be an endomorphism of an infinite group G. We say that two elements $x, y \in G$ are ϕ-twisted conjugates if there exists a $g \in G$ such that $y = g.x.\phi(g^{-1})$. This is an equivalence relation on G and the equivalence classes are called ϕ-twisted conjugacy classes. When ϕ equals the identity, the twisted conjugacy relation is the same as the usual conjugacy relation in G. The cardinality $\#\mathcal{R}(\phi)$ of the set $\mathcal{R}(\phi)$ of all ϕ-twisted conjugacy classes is called the Reidemeister number of ϕ. If the Reidemeister number of every automorphism of G is infinite, one says that G has the property R_∞ or that G is an R_∞-group.

The notion of the Reidemeister number arose in fixed point theory. The notion of twisted conjugacy arises in other areas, such as representation theory, number theory, besides topology. The problem of determining which (finitely generated) infinite groups has property R_∞ was first formulated by Fel'shtyn and Hill [15]. It was shown by Levitt and Lustig [23] that non-elementary torsion free word hyperbolic groups have the R_∞-property. (They in fact proved a stronger property which readily implies the property R_∞.) This was extended to all non-elementary hyperbolic groups by Fel'shtyn. This prompted Fel'shtyn and Hill to conjecture that groups with exponential growth have the property R_∞. This was resolved in

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the negative by Gonçalves and Wong [20]. They also showed that the property R_∞ is not a geometric property, by showing that it is not inherited by finite index subgroups. For example, the fundamental group of the Klein bottle has the property R_∞ but not $\mathbb{Z}^2$. The Fel'shtyn-Hill problem of classifying groups according as whether or not they have the R_∞-property has now become an active research area. Depending on the class of groups under consideration the resolution of the problem draws results and techniques from several branches of mathematics, such as geometric group theory [13], combinatorial group theory [21], C^*-algebra [17], Lie groups and algebraic groups [26], and, very recently, number theory [19].

In this largely expository article, our aim is to show the following result, obtained by D. L. Gonçalves, P. Sankaran, and R. Strebel.

Theorem 1.1. *([19]) There exists a family $\mathcal{G}$ of PL-homeomorphism groups of the interval $[0,1]$ such that: (i) members of $\mathcal{G}$ are pairwise non-isomorphic, (ii) each member of $\mathcal{G}$ is an R_∞-group, and, (iii) the cardinality of $\mathcal{G}$ equals that of $\mathbb{R}$.*

We begin §2 with a brief discussion on the R_∞ property and give well-known examples of groups which have and those which do not have the R_∞-property. We introduce in §3 certain classes of PL-homeomorphism groups. In §4 we shall recall the Bieri-Neumann-Strebel invariant and explain its relevance to the property R_∞. In §5 we shall recall the classical Gelfond-Schneider theorem and show how it can be used to show certain PL-homeomorphism groups, which are generalizations of Richard Thompson's group F, have the R_∞-property. We shall prove Theorem 1.1 in the §6.

2. R_∞-groups

Recall from §1 that a group G has the R_∞-property if the Reidemeister number of every automorphism of G is infinite.

Suppose that

$$1 \to N \overset{i}{\hookrightarrow} G \overset{\eta}{\to} H \to 1 \tag{1}$$

is an exact sequence of groups and that ϕ is an endomorphism of G such that $\phi(N) \subset N$. Denote by $\bar{\phi} : H \to H$ the surjective endomorphism of H that ϕ induces and by ϕ' the restriction of ϕ to N We have the following commuting diagram of groups and their homomorphisms with exact rows:

$$\begin{array}{ccccccccc} 1 & \to & N & \hookrightarrow & G & \overset{\eta}{\to} & H \to & 1 \\ & & \downarrow \phi' & & \downarrow \phi & & \downarrow \bar{\phi} & \\ 1 & \to & N & \hookrightarrow & G & \overset{\eta}{\to} & H \to & 1 \end{array} \tag{2}$$

Note that if ϕ is an automorphism and if $\phi(N) = N$, then $\bar{\phi}$ is an automorphism of H.

Lemma 2.1. *Let $\phi : G \to G$ be an endomorphism of a group such that $\phi(N) \subset N$. With the above notations, if $R(\bar{\phi}) = \infty$, then $R(\phi) = \infty$.* □

Proof follows from the fact that if $x, y \in G$ are ϕ-twisted conjugates, then $\eta(x), \eta(y) \in H$ are $\bar{\phi}$-twisted conjugates.

Note that, in the above lemma, if H is isomorphic to $\mathbb{Z}^n$ for some positive integer n, and if $\bar{\phi} = id$, for *every* $\phi \in Aut(G)$, then G is an R_∞ group.

We shall now list some classes of groups which are known to have, or not have, the R_∞-property.

Example 2.2. (1) The infinite cyclic group $\mathbb{Z}$ does not have the R_∞ property. Indeed $R(-id) = 2$. More generally, if $\phi : \mathbb{Z}^n \to \mathbb{Z}^n$ is an automorphism of $\mathbb{Z}^n$, it is not difficult to show that $R(\phi)$ equals the index of $Im(id - \phi)$ in $\mathbb{Z}^n$. Thus $R(\phi) < \infty$ if and only if 1 is not an eigenvalue of ϕ, that is, if and only if $Fix(\phi) = 0$.

(2) The lamplighter group L_m defined as the restricted wreath product $(\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z} = (\mathbb{Z}/m\mathbb{Z})^\infty \rtimes \mathbb{Z}$, is an R_∞-group if and only if $\gcd(6, m) \neq 1$. This result is due to [20].

(3) The Baumslag-Solitar groups $B(m, n)$ with $\gcd(m, n) = 1, m, n > 0$, defined in terms of the presentation $\langle x, y \mid xy^m x^{-1} = y^n \rangle$, is an R_∞-group if $m, n > 1$. This was established by Fel'shtyn and Gonçalves [14].

(4) Fel'shtyn, Leonov, and Troitsky [16] showed that a class of saturated weakly branch groups that includes the Grigorchuk group and the Gupta-Sidki group have the R_∞-property.

(5) Fel'shtyn and Troitsky [17] showed that a finitely generated residually finite group which is non-amenable has the R_∞-property.

(6) Irreducible lattices in semisimple Lie groups of real rank at least 2 have the R_∞-property [26]. When lattice is residually finite, this result follows from the previous example. Although any lattice in a semisimple *linear* Lie group is residually finite, it is known that there are irreducible lattices in certain simple Lie groups which are *not* residually finite.

(7) The groups $SL(n, R), GL(n, R), n > 2$, have the R_∞-property when R is an infinite integral domain with trivial automorphism group (for example, $R = \mathbb{Q}$) or when R is an integral domain of characteristic zero and for which $Aut(R)$ is a torsion group (for example, $K = \mathbb{Z}$). This result is due to Nasybullov [27]. Recently this result has been extended to certain Chevalley groups of Lie type over fields of characteristic zero [28].

3. PL-homeomorphisms of the interval

Denote by $PL_o(\mathbb{R})$ the group of all orientation preserving piecewise linear (PL) self-homeomorphisms of the reals. We denote by $PL_o(I)$ the group of orientation preserving PL-homeomorphisms of an interval $[0, b], b > 0$. It is regarded as the subgroup of $PL_o(\mathbb{R})$ consisting of those homeomorphisms of $\mathbb{R}$ which are identity outside I. A linear isomorphism $[0, 1] \cong [0, b]$ induces an isomorphism

$PL_o(I) \cong PL_o([0,1])$. However, it would be convenient to allow b to be an arbitrary positive real number. We shall be particularly interested in subgroups of $PL_o(I)$ consisting of elements whose slopes (that is, the finitely many values of its derivatives at smooth points) and whose break-points (that is, the finitely many points of $(0,b)$ where the element fails to be differentiable) would be suitably restricted. An example of such a group is the Richard Thompson's group $F \subset PL_o([0,1])$ consisting of elements whose slopes are in the multiplicative group generated by $2 \in \mathbb{R}^{\times}_{>0}$ and whose break points are dyadic rationals $m/2^n, m, n \in \mathbb{N}$.

The group F, as well as the similarly defined group T of homeomorphisms of the circle $\mathbb{S}^1 = [0,1]/\{0,1\}$ that contains F, were discovered by Richard Thompson [34] in the course of his research on some problems in logic. See [24]. The group T is the first known example of a finitely presented infinite simple group. Since its discovery, the group F has been observed to arise very naturally in many different branches of mathematics. It is a long standing open problem to decide whether F is an amenable group or not. For a survey of basic properties of these groups, see [12].

The construction of the groups F and T have been generalized by G. Higman [22], R. Bieri and R. Strebel [3], [4], and K. S. Brown [10].

Let $P \subset \mathbb{R}^*_{>0}$ be a non-trivial subgroup of the multiplicative group of the positive reals, and let $A \subset \mathbb{R}$ be a non-zero subgroup of the additive group of real numbers. Suppose that $\lambda.a \in A$ for all $\lambda \in P, a \in A$. Thus A is a $\mathbb{Z}[P]$-module and A is dense in $\mathbb{R}$. Bieri and Strebel [3] introduced the group $G(I; A, P)$ of all PL-homeomorphisms $f : I \to I$ which have break-points in $A \cap I$ and slopes in P. Although they considered the cases where the interval was allowed to be the whole of $\mathbb{R}$ or $[0, \infty)$, we shall only consider the case when $I = [0, b], b > 0$. We will assume that $b \in A$. We shall be particularly interested in the slope of elements of $f \in G(I; A, P)$ at the end-points $0, b$, where it is understood that slope at an end point of I refers to the value of the appropriate one-sided derivative of f at that point. The groups that we shall consider here will be finitely generated subgroups of the group $G(I; A, P)$ for appropriate I, A, and P. It is an open problem to classify which of the groups $G(I; A, P)$ are finitely generated when $I = [0, b]$. Bieri and Strebel [4] showed that the following conditions are necessary: (i) $b \in A$, (ii) P is finitely generated, (iii) A is finitely generated as a $\mathbb{Z}P$-module and the $\mathbb{Z}P$-module $A/(IP.A)$ is finite, where IP is the kernel of the augmentation $\mathbb{Z}P \to \mathbb{Z}$. It appears that, besides Example 3.1(i) below, only two other examples of $G(I; A, P)$ are known to be finitely generated. See [4] for details as well as the results when I is the whole of $\mathbb{R}$ or a ray.

We observe that there are homomorphisms $\lambda_0, \lambda_1 : PL_o(I) \to \mathbb{R}^{\times}_{>0}$, defined as $\lambda_0(f) = \lim_{t\to 0^+} f'(t) =: f'(0)$, $\lambda_1(f) = \lim_{t \to b^-} f'(t) =: f'(b)$ which are referred to as the slopes at $0, b$ respectively of f. Note that the restrictions of these homomorphisms, again denoted by the same symbols λ_0, λ_1, to $G(I; A, P)$ have images equal to P. Composing with the homomorphism $\log : \mathbb{R}^{\times}_{>0} \to \mathbb{R}$ (where log denotes the natural logarithm) we obtain homomorphisms $\chi_j = \log \circ \lambda_j : PL_o(I) \to \mathbb{R}, j = 0, 1$, into the additive group of the reals. We shall denote their restrictions to any subgroup of $PL_o(I)$ also by the same symbols $\chi_j, j = 0, 1$.

Example 3.1. (i) Let $I = [0, 1], P = \langle n_1, \dots, n_k \rangle, A = \mathbb{Z}[1/n]$ where $n_1, \dots, n_k > 1$ are integers and $n = n_1 \dots n_k$. It is known that the group $G(I; A, P)$ is finitely presented; see [30]. In this case $Im(\chi_0) = \sum_{1 \leq j \leq k} \mathbb{Z} \log n_j = Im(\chi_1) \subset \mathbb{R}$.

(ii) Let G be generated by a finite set of elements $S = \{f_1, \dots, f_m\} \subset P_o(I), I = [0, 1]$ such that $\cup_{1 \leq j \leq m} \mathrm{supp}(f_j) = (0, b)$, where each f_j is identity near at least one of the end points $0, 1$. (Here $\mathrm{supp}(f) = \{x \in I \mid f(x) \neq x\}$ is the support of f.) We order the generators $f_j, 1 \leq j \leq m$ so that the first k of them are identity near b and the remaining generators are identity near 0. Then $Im(\chi_0) = \sum_{1 \leq j \leq k} \mathbb{Z} \log f'_j(0), Im(\chi_1) = \sum_{k < j \leq m} \mathbb{Z} \log f'_j(1)$.

(iii) Let $I = [0, 1]$. For $s \in \mathbb{R}$, $s \neq 0$, define $f_s : I \to I$ be the unique piecewise linear homeomorphism which has the property that the slopes $f'(0) = \exp(s), f'(1) = \exp(-s)$ and f has exactly one break-point in $(0, 1)$. Let S be a finite set of real numbers and let G denote the subgroup of $PL_o(I)$ generated by $f_s, s \in S$. In this case $Im(\chi_0) = Im(\chi_1) = \sum_{s \in S} \mathbb{Z}s$. Observe that $f_s^{-1} = f_{-s}$. It is easily seen that conjugation by the homeomorphism $t \to 1 - t$ of the interval I induces an involutive automorphism $\theta : G \to G$ where $\theta(f_s) = f_{-s} = f_s^{-1} \ \forall s \in S$. It follows that $\theta \circ \chi_j = \chi_{1-j} = -\chi_j$, $j = 0, 1$.

4. The Σ-invariant and its generalizations

Let G be a finitely generated group with infinite abelianization $G/[G, G]$. (Here $[G, G]$ denotes the commutator subgroup of G.) Consider the real vector space $\mathrm{Hom}(G, \mathbb{R})$. An element of $\mathrm{Hom}(G, \mathbb{R})$ is called a *character*. One has an equivalence relation on $\mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}$ where $\chi \sim \chi'$ if there exists a positive real number r such that $\chi = r\chi'$. The equivalence classes are called *character classes*. The set of character classes is denoted $S(G)$. Since any character class $[\chi]$ is a ray $\mathbb{R}_{>0}\chi$, it is clear that $S(G)$ is homeomorphic to the sphere $\mathbb{S}^{d-1}$, where $d = \dim_{\mathbb{R}} \mathrm{Hom}(G, \mathbb{R})$ which is also the rank of $G/[G, G]$. Note that if $f : G \to H$ is a homomorphism of groups, then we have the induced homomorphism $f^* : \mathrm{Hom}(H, \mathbb{R}) \to \mathrm{Hom}(G, \mathbb{R})$. If f is an isomorphism of groups, then f^* is an isomorphism of real vector spaces. In this case f^* induces a homeomorphism of the character spheres $f^* : S(H) \to S(G)$ where $f^*([\chi]) = [\chi \circ f]$. This yields a (right) action of $Aut(G)$ on $S(G)$.

Given a non-zero character $\chi : G \to \mathbb{R}$, one has a monoid $G_\chi := \{g \in G \mid \chi(g) \geq 0\}$. Note that G_χ depends only on $[\chi]$. Bieri, Neumann, and Strebel introduced the following invariant for a finitely generated group G:

$$\Sigma(G) = \{[\chi] \in S(G) \mid [G,G] \text{ is finitely generated over a finitely generated submonoid of } G_\chi\}.$$

Note that if $[G,G]$ is finitely generated, then $\Sigma(G) = S(G)$, on the other hand, for a non-abelian free group the invariant is the empty set. The invariant $\Sigma(G)$ originated in the work of Bieri and Strebel [2]. Brown [11] gave a characterization of $\Sigma^c(G) := S(G) \setminus \Sigma(G)$ that involves actions of G on $\mathbb{R}$-trees and is applicable to groups which are not necessarily finitely generated. It was shown in [1] that $\Sigma(G)$ is an open set.

Perhaps a definition that helps to visualise what constitutes $\Sigma(G)$ is the one in terms of the Cayley graph, which we now recall. Let $S \subset G$ be a finite generating set. Let $\mathcal{C} = \mathcal{C}(G,S)$ denote the Cayley graph of G with respect to $S \cup S^{-1}$. Thus the vertices of $\mathcal{C}$ are elements of G and two distinct vertices u, v form an edge if $u^{-1}v \in S \cup S^{-1}$. Given a non-zero character $\chi : G \to \mathbb{R}$ consider the full subgraph $\mathcal{C}_\chi \subset \mathcal{C}$ with vertices $\chi^{-1}([0,\infty))$. It turns out that if $\mathcal{C}' = \mathcal{C}(G,S')$ is the Cayley graph of G with respect to another finite generating set S', then $\mathcal{C}_\chi$ is connected if and only if $\mathcal{C}'_\chi$ is. One has $\Sigma(G) = \{[\chi] \in S(G) \mid \mathcal{C}_\chi \text{ is connected.}\}$ See [32] for further details.

There are also higher Σ-invariants that are homotopical and homological generalizations of $\Sigma(G)$, denoted $\Sigma^m(G) \subset S(G), m \geq 1$, with $\Sigma(G) = \Sigma^1(G)$; see [5].

The paper of Bieri, Neumann, and Strebel [1] contains a wealth of examples, although, in general, it is difficult to compute $\Sigma(G)$. The following theorem describes it when G is a PL-homeomorphism group of $I = [0,b]$ satisfying two properties. A group $G \subset PL_0(I)$ is called *irreducible* if there is no G-fixed point in the open interval $(0,b)$. Thus, if G is not irreducible, then it is a subgroup of a product $G_1 \times G_2$ where $G_1 \subset PL_o([0,b_0]), G_2 \subset PL_o([b_0,b])$ for some $0 < b_0 < b$.

Definition 4.1. *Two characters $\chi, \eta : G \to \mathbb{R}$ of a finitely generated group G are said to be independent if $\eta(\ker(\chi)) = \eta(G), \chi(\ker(\eta)) = \chi(G)$.*

It is clear that independent characters are $\mathbb{R}$-linearly independent in the vector space $\mathrm{Hom}(G,\mathbb{R})$. However, taking $G = \mathbb{Z}^2$, very simple examples can be constructed to show that the converse is not true.

Recall the characters $\chi_0, \chi_1 : G \to \mathbb{R}$ defined earlier where $G \subset PL_0(I)$.

Theorem 4.2. *(Bieri-Neumann-Strebel* [1, Theorem 8.1]*) Let $I = [0,b], b > 0$. Let G be a finitely generated subgroup of $PL_0(I)$ which is irreducible. Suppose that χ_0, χ_1 are independent. Then $\Sigma^c(G) := S(G) \setminus \Sigma(G)$ equals $\{[\chi_0], [\chi_1]\}$.*

The R_∞-property for the Thompson group F was first established by Bleak, Felsh'tyn and Gonçalves [6] using the structure of its automorphism group from the work of [7], [9]. The same result was obtained by Gonçalves and Kochloukova [18] using Σ-theory without using the knowledge of $Aut(F)$. They applied their approach to many different classes of groups, including the groups $F_{n,0} := G([0, n-1]; \mathbb{Z}[1/n], \langle n \rangle)$.

We outline the main ideas involved in applying Σ-theory to the twisted conjugacy problem. First one shows that $\Sigma^c(G)$ or $\Sigma^1(G)$ admits a fixed point under the action of the automorphism group $Aut(G)$ of G. This happens, for example, when $\Sigma^c(G)$ is a finite set contained in an open hemisphere of $S(G)$; see [18]. If $\chi : G \to \mathbb{R}$ represents such a fixed point $[\chi] \in S(G)$, and if $Im(\chi)$ is cyclic, then $\chi \in \mathrm{Hom}(G, \mathbb{R})$ itself is fixed under the action of $Aut(G)$ on $\mathrm{Hom}(G, \mathbb{R})$. Applying Lemma 2.1 to the short exact sequence $1 \to \ker(\chi) \hookrightarrow G \to Im(\chi) \to 1$, we see that for any $\phi \in Aut(G)$, the induced homomorphism $\bar{\phi} : Im(\chi) \to Im(\chi)$ (in the commutative diagram (2)) is the identity map. (Note that $\ker(\chi) = \ker(r\chi)$ $\forall r \in \mathbb{R}_{>0}$.) Consequently $R(\phi) = \infty$ and we conclude that G is an R_∞-group.

5. Rigid subgroups and transcendental characters

In order to successfully apply Σ-theory to the R_∞ problem, it does not suffice to know that the induced map on the character sphere of an automorphism $\phi \in Aut(G)$ admits a fixed point $[\chi] \in S(G)$. One needs to know that the character χ itself is fixed under the induced isomorphism ϕ^* on $\mathrm{Hom}(G, \mathbb{R})$. There is a class of finitely generated abelian subgroups of $\mathbb{R}$ such that if $Im(\chi)$ belongs to that class and if $\phi^*([\chi]) = [\chi]$, then $\chi \circ \phi = \chi \in \mathrm{Hom}(G, \mathbb{R})$ for any automorphism ϕ of G. This is the class of *rigid groups* introduced in [19], which we now define.

Definition 5.1. *Let $A \subset \mathbb{R}$ be a non-trivial finitely generated abelian group. Let $U(A)$ denote the group $\{r \in \mathbb{R} \setminus 0 \mid rA = A\}$ (under multiplication). We call $U(A)$ the* unit group *of A. We say that A is* rigid *if $U(A) = \{1, -1\}$.*

Any infinite cyclic subgroup contained $\mathbb{R}$ is rigid. Also if A is rigid, so is $\lambda A = \{\lambda a \mid a \in A\}$ for any non-zero real number λ.

Example 5.2. (i) Let $A = \mathbb{Z}[\sqrt{2}]$. Then $U(A)$ is the group of units in the *ring* A, which is the ring of integers of the number field $\mathbb{Q}(\sqrt{2})$. In particular it contains the cyclic group generated by $(\sqrt{2} + 1)$.
(ii) Let $A = \mathbb{Z} + \mathbb{Z}\sqrt{2} + \mathbb{Z}\sqrt{3}$. Then A is rigid.
(iii) Let $A \subset \mathbb{R}$ is the group with $\mathbb{Z}$-basis $1, \alpha, \ldots, \alpha^{n-1}$ where $\alpha \in \mathbb{R}$ is transcendental. Then A is rigid.

We have the following proposition whose proof we omit, referring the reader to [19]. Recall that an algebraic unit is an algebraic number r such that $1/r$ is also an algebraic number. (Thus r is a unit in the ring of all algebraic numbers.)

Proposition 5.3. *Let A be a non-trivial finitely generated abelian subgroup of $\mathbb{R}$. Let $r \in U(A)$. Then $r = a/b$ for some $a, b \in A$. Also, r is an algebraic unit of degree at most $rank(A)$.*

Definition 5.4. *Let A be a non-trivial finitely generated subgroup of $\mathbb{R}$. We say that A is transcendental if $a, b \in A$ are non-zero, then a/b is rational or is transcendental. We say that a (non-zero) character $\chi : G \to \mathbb{R}$ is transcendental if $Im(\chi)$ is transcendental.*

In view of the above proposition, if χ is transcendental, then $Im(\chi)$ is rigid.

Example 5.5. If $s_1, \ldots, s_n$ is $\overline{\mathbb{Q}}$-linearly independent, then $A = \sum_{1\le j\le n} \mathbb{Z}s_j$ is transcendental. Indeed if $r = s/t$ where $s = \sum a_j s_j, t = \sum b_j s_j \in A \setminus \{0\}$ and $r \in \overline{\mathbb{Q}}$ then $rt - s = 0$ contradicting our hypothesis that $s_1, \ldots, s_n$, are $\overline{\mathbb{Q}}$-linearly independent unless $rb_j = a_j$ for *every* j, in which case $r \in \mathbb{Q}$.

Lemma 5.6. *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be positive algebraic numbers. Then $A = \mathbb{Z}\log\alpha_1 + \ldots + \mathbb{Z}\log\alpha_n$ is transcendental.*

Proof. This is immediate from the famous theorem discovered independently by A. O. Gelfond and T. Schneider [29], which we now recall. Suppose that α, γ are algebraic numbers, γ is not rational, and $0 \ne \alpha \ne 1$. Then the Gelfond-Schneider theorem asserts that (any value of) $\alpha^\gamma = \beta$ is transcendental.

Let $a = \sum_j a_j \log\alpha_j \in A, a \ne 0$. Thus $a = \log\alpha$ where $\alpha := \prod_j \alpha_j^{a_j}$. Since $a \ne 0, \alpha \ne 1$. As the α_j are algebraic, so is α. Let $b \in A, b \ne 0$, so that $b = \log\beta$, with β algebraic. Then $\gamma := b/a = \log\beta/\log\alpha = \log_\alpha\beta$, which implies that $\beta = \alpha^\gamma$. As both α and β are algebraic, by the Gelfond-Schneider theorem, either $\gamma = b/a$ is rational or is transcendental. □

We have the following theorem. Recall the definition of $G(I; A, P)$.

Theorem 5.7. *Let P the multiplicative group of positive algebraic numbers, and let $A = \overline{\mathbb{Q}}$. Let $b \in A$ be positive and let $I = [0, b]$. Let $G \subset G([0,1], A, P)$ be a subgroup generated by elements $f_1, \ldots, f_n$ such that (i) $\cup_{1\le j\le n} f_j = (0, b)$, and, (ii) for some positive integer $k < n$, $f_1, \ldots, f_k$ are identity in a neighbourhood of b, and $f_{k+1}, \ldots, f_n$ are identity in a neighbourhood of 0. Then G has the R_∞-property.*

Proof. Our hypotheses (i) and (ii) imply that G is irreducible and that the characters χ_0, χ_1 are independent. Hence by Theorem 4.2, $\Sigma^c(G) = \{[\chi_0], [\chi_1]\}$. Let $\alpha_j = f_j'(0), \beta_j = f_j'(1)$. Then $Im(\chi_0) = \sum_{1\le j\le k} \mathbb{Z}\log\alpha_j$ and $Im(\chi_1) = \sum_{k<j\le n} \mathbb{Z}\log\beta_j$. Since $\alpha_j, \beta_j, 1 \le j \le n$ are positive algebraic numbers, by the above lemma χ_0, χ_1 are transcendental.

Suppose that $\phi \in Aut(G)$. If $\phi^*([\chi_0]) = [\chi_0]$, then we must have $\phi^*(\chi_0) = \chi_0$ by transcendence of χ_0. If $\phi^*([\chi_0]) = [\chi_1]$, then $[\chi_0 \circ \phi] = [\chi_1]$ and $[\chi_1 \circ \phi] = [\chi_0]$ since ϕ^* is a homeomorphism of $\Sigma^c(G)$. It follows that $\chi_0 \circ \phi = r\chi_1, \chi_1 \circ \phi = s\chi_0$ for some positive reals r, s. Hence $\chi_0 = rs\chi_0$ which implies that $r = 1/s$ by transcendence of χ_0. Now it is clear that $\phi^*(\chi_0 + r\chi_1) = \chi_0 + r\chi_1 =: \chi$. Independence of χ_0, χ_1 implies that $\chi \ne 0$. So by Lemma 2.1 we conclude that $R(\phi) = \infty$. This completes the proof. □

The groups $G(I; A, P)$ considered in Example 3.1(i) are finitely generated and irreducible. Moreover, by Lemma 5.6, χ_0, χ_1 are transcendental characters. By the above theorem we conclude that these groups have the R_∞-property.

6. Proof of Theorem 1.1

Fix integers k, l such that $k > l \geq 1$ and let $S := (S_0, S_1)$, where S_0, S_1 are subsets of $\mathbb{R}_{>0}$, $\#S_0 = k, \#S_1 = l$ with S_0 being $\overline{\mathbb{Q}}$-linearly independent. Let $G_S \subset PL_0([0,1])$ be the group generated by elements $f_s, s \in S_0$ and $g_t, t \in S_1$ defined uniquely by the following requirements: (i) $f_s'(0) = \exp(s), g_t'(1) = \exp(t), \lim_{x\nearrow 3/4} f'(x) = \exp(-s), \lim_{x\searrow 1/4} g_t'(x) = \exp(-t)$, (ii) $\mathrm{supp}(f_s) = (0, 3/4), \mathrm{supp}(g_t) = (1/4, 1)$, and, (iii) f_s (resp. g_t) has exactly one break point in $(0, 3/4)$ (resp. in $(1/4, 1)$). We observe that G_S is irreducible since the union of supports of $f_s, s \in S_0$, and $g_t, t \in S_1$, equals $(0, 1)$. The characters $\chi_0, \chi_1 : G_S \to \mathbb{R}$ are independent since $Im(\chi_0)$ is generated by $\chi_0(f_s) = s, s \in S_0$ and $\chi_1(f_s) = 0, \ \forall s \in S_0$ and $Im(\chi_1)$ is generated by $\chi_1(g_t) = t, t \in S_1$, $\chi_0(g_t) = 0, \ \forall t \in S_1$. Thus, by Theorem 4.2, $\Sigma^c(G_S) = \{[\chi_0], [\chi_1]\}$. Since $rank(Im(\chi_0)) = k > l = rank(Im(\chi_0))$, there does not exist any real number $r > 0$ such that $Im(\chi_0) = Im(r\chi_1)$. It follows that for any automorphism $\phi : G_S \to G_S$, $\phi^*(\chi_0) = \chi_0 \circ \phi \neq r\chi_1$ for any $r \in \mathbb{R}$. Hence ϕ induces the identity map of $\Sigma^c(G_S)$ and we must have $\phi^*([\chi_0]) = [\chi_0]$. Our hypothesis on S_0 implies that $Im(\chi_0)$ is transcendental and hence rigid by Proposition 5.3. So $\phi^*(\chi_0) = \chi_0$ and we conclude G_S is an R_∞ group.

We need the following lemma. When $k = 1$, the lemma asserts that $tr.deg_{\mathbb{Q}}\mathbb{R} = \#\mathbb{R}$. The proof would have been a lot simpler if only we had used the continuum hypothesis.

Lemma 6.1. *Let k be a positive integer. There exists a collection $\mathcal{T}$ of k-elements subsets $A \subset \mathbb{R}$ such that the following properties hold: (i) The subfield $\mathbb{Q}(A)$ of $\mathbb{R}$ is transcendental over $\mathbb{Q}$ of transcendence degree $tr.deg_{\mathbb{Q}}\mathbb{Q}(A) = k$. (ii) If $A, B \in \mathcal{T}$ are distinct, then $tr.deg_{\mathbb{Q}}\mathbb{Q}(A \cup B) = 2k$, (iii) $\#\mathcal{T} = \#\mathbb{R}$.*

Proof. It is evident that there is a non-empty collection $\mathcal{T}$ satisfying conditions (i), (ii). If $\mathcal{T}_{\alpha \in J}$ is an increasing chain of such collections indexed by an ordered set J, then $\mathcal{T}_J = \cup_{\alpha \in J}\mathcal{T}_\alpha$ also satisfies conditions (i) and (ii). Hence by the maximum principle, there exists a maximal such collection $\mathcal{T}$. We claim that (iii) holds for this $\mathcal{T}$. Clearly $\mathcal{T}$ is infinite. Suppose that $\#\mathcal{T} < \#\mathbb{R}$. Consider the subfield $F := \mathbb{Q}(\cup_{A \in \mathcal{T}} A) \subset \mathbb{R}$. Since $\mathcal{T}$ and $\cup_{A \in \mathcal{T}} A$ have the same cardinality, the field F also has the same cardinality as $\mathcal{T}$. It follows that F is a *proper* subfield of $\mathbb{R}$. Let $\overline{F} \subset \mathbb{C}$ be the algebraic closure F and let $K := \overline{F} \cap \mathbb{R}$. Then $\#\mathcal{T} = \#F = \#\overline{F} = \#K$. If $tr.deg_K\mathbb{R} \leq \#\mathcal{T}$, then $\mathbb{R}$ has a K-vector space basis having cardinality at most equal to $\#\mathcal{T}$ and so $\#\mathbb{R} = \#K = \#\mathcal{T}$, a contradiction. So $tr.deg_K\mathbb{R} > \#\mathcal{T}$. Therefore $\mathbb{R}$ must contain infinitely many elements which are algebraically independent over K. We let A_0 be a set of consisting of k such elements. Then $\mathcal{T} \cup \{A_0\}$ is a strictly bigger collection than $\mathcal{T}$ and satisfies (i) and (ii), contradicting the maximality of $\mathcal{T}$. Hence we must have $\#\mathcal{T} = \#\mathbb{R}$. $\square$

Proof of Theorem 1.1: Fix integers $k > l \geq 1$. Fix a set $S_1 \subset \mathbb{R}$ of cardinality l. Consider the collection $\mathcal{A}$ of all pairs $A = (A_0, S_1)$ with $A_0 \in \mathcal{T}$ with $\mathcal{T}$ as in the above lemma. By the discussion preceding the lemma we know that G_A is an R_∞-group for all $A \in \mathcal{A}$.

Let $A = (A_0, S_1), B = (B_0, S_1) \in \mathcal{A}$ be distinct; thus $A_0, B_0 \in \mathcal{T}$ are distinct. We claim that G_A and G_B are not isomorphic. Suppose, on the contrary, $f : G_A \to G_B$ is an isomorphism. Then f induces a homeomorphism $f^* : \Sigma^c(G_A) \to \Sigma^c(G_B)$. Write $\lambda_0, \lambda_1 : G_A \to \mathbb{R}$ and $\eta_j : G_B \to \mathbb{R}$ for the restrictions of $\chi_j : PL_o(I) \to \mathbb{R}$. We must have $f^*([\eta_j]) = [\lambda_j], j = 0, 1$, since otherwise $[\lambda_1] = [\eta_0 \circ f] = f^*([\eta_0])$ which is impossible since $rank(\lambda_1) = l < k = rank(\eta_0) = rank(\eta_0 \circ f)$. Now $[\eta_0 \circ f] = [\lambda_0]$ implies that $\eta_0 \circ f = r\lambda_0$ for some $r > 0$. So $Im(\eta_0 \circ f) = r.Im(\lambda_0)$. As $Im(\eta_0 \circ f) = Im(\eta_0)$ is the free abelian group with basis B_0 whereas $Im(\lambda_0)$ is free abelian with basis A_0 we must have $rA_0 \subset Im(\eta_0)$. Let $a_1, a_2 \in A_0$ be two distinct elements. (Here we are using the hypothesis that $k \geq 2$.) Then $a_2/a_1 = c/d$ for some $c, d \in Im(\eta_0)$. This means that $a_2 \in \mathbb{Q}(B_0)(a_1)$. Varying a_2 we see that $A_0 \subset \mathbb{Q}(B_0 \cup \{a_1\})$. Hence the transcendence degree of $\mathbb{Q}(A_0 \cup B_0)$ is at most $k + 1 < 2k$. This contradicts property 6.1 (ii) of $\mathcal{T}$. This completes the proof of Theorem 1.1. $\square$

Remark 6.2. (i) The above proof is slightly different from the one given in [19], although the construction of the groups G_S is the same.
(ii) If, in the construction of G_S we let $k = l = 1$, then G_S is isomorphic to the Thompson group F. This follows from the main theorem of [8].
(iii) I do not know if the groups in Example 3.1(iii) have the R_∞-property, except when G is infinite cyclic (in which case the group does not have the R_∞-property).

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A CURIOUS BANACH ALGEBRA: BANACH ALGEBRA OF FUNCTIONS WITH LIMITS*

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Abstract. The space $\mathcal{L}t(X)$ consisting of all complex valued functions f on a compact Hausdorff space X such that $\lim\limits_{t\to x} f(t)$ exists for every $x \in X$ is a commutative supnorm C^*-algebra containing the algebra of all continuous functions on X; contained in the algebra of all bounded functions on X; whose Gel'fand space is the disjoint union of X and the set of all the nonisolated points of X; and the Gel'fand topology of which is not the usual topology of the disjoint union. This together with analogous algebras of differentiable functions, results into several new Banach algebras of functions presumably unexplored.

The present note is addressed to M.Sc./M.Phil. students, as well as to college teachers, demanding familiarity with compact Hausdorff spaces and weak topology; and is aimed at illuminating an uncharted terrain. It is based on the concept of limit of real or complex valued functions defined on a topological space, in particular on $A \subset \mathbb{R}$. We consider functions with limits (a class larger than continuous functions) which was evolved while one of the authors was engaged in [4]. The central idea, not usually highlighted in teaching of Calculus, is that boundedness of continuous functions on a closed interval is essentially due to existence of limits, and not involving continuity at all. We aim to highlight some aspects of functional analytic nature that arise from this and that are believed to be unexplored. We shall deal with complex valued functions defined on a compact Hausdorff space X. It would suffice to consider X to be a closed and bounded interval $[a, b] \subset \mathbb{R}$.

We define the limit of a function at a point in a topological space as follows. For a topological space X, $x \in X$, $\mathcal{U}_x$ is the set of all neighbourhoods of x in X. If $\{x\} \in \mathcal{U}_x$, then we say that x is an *isolated point* of X. For $\{x\} \neq U \in \mathcal{U}_x$, $U^{(x)}$ denotes the deleted neighbourhood $U \setminus \{x\}$ of x. Let X, Y be topological spaces, $x \in X$, $y \in Y$ and $f : X \to Y$ be a function. We define $\lim\limits_{t\to x} f(t)$, denoted by $\ell_x(f)$, as follows. If x is an isolated point of X, then $\lim\limits_{t\to x} f(t) = f(x)$. If x is

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not an isolated point, then $\lim\limits_{t \to x} f(t) = y$ if for every $V \in \mathcal{U}_y$, there exists $U \in \mathcal{U}_x$ such that $f(U^{(x)}) \subset V$. It is easy to see that when $X \subset \mathbb{R}$, in particular when $X = [a, b]$, this coincides with the usual $\epsilon - \delta$ definition of limit. We also refer to section 14 in reference [5] for this consideration of limit values of functions. A function $f : X \to \mathbb{C}$ is continuous on X if it is continuous at every $x \in X$; i.e., if for every $x \in X$, $\lim\limits_{t \to x} f(t) = f(x)$ as done in Calculus. This is equivalent to the statement that $f^{-1}(G)$ is open in X for all open sets $G \subset \mathbb{C}$ as done in Topology. Let $C(X)$ (respectively $B(X)$) be the set of all functions $f : X \to \mathbb{C}$ that are continuous on X (respectively bounded on X). Note that $\lim\limits_{t \to x} f(t)$ depends on the domain in which it is computed. For example, the limits of the characteristic function of $\mathbb{Z}$, at an integer, are different when computed in $\mathbb{Z}$ and in $\mathbb{R}$. However, for a continuous function this is never a case because at the point of continuity of a function, the limit of the function at the point coincides with the value of the function at that point.

An *algebra* $\mathcal{A}$ is a vector space over $\mathbb{C}$ which is also a ring such that for all $x, y \in \mathcal{A}$ and the scalar α, $(\alpha x)y = \alpha(xy) = x(\alpha y)$. For simplicity, we assume that $\mathcal{A}$ contains the unit element (= the multiplicative identity). Let $\mathcal{L}t(X)$ consist of all functions $f : X \to \mathbb{C}$ for which $\lim\limits_{t \to x} f(t)$ exists for all $x \in X$. It is easily seen (by working out "algebra of limits") that $\mathcal{L}t(X)$ is an algebra with pointwise operations and complex conjugation as the involution and $C(X)$ is a subalgebra of $\mathcal{L}t(X)$.

Let $Y = \{x \in X : x \text{ is not an isolated point of } X\}$. If $Y = \emptyset$, then X is finite because of its compactness. In this case, one easily observes that $B(X) = \mathcal{L}t(X) = C(X) = \mathbb{C}^n$, where n is the number of points in X. To avoid this triviality, we assume now onwards that $Y \neq \emptyset$ so that X is not a finite set. Let

$$\mathcal{F}(Y) = \{f : Y \to \mathbb{C} : \text{ support of } f \text{ is finite}\};$$

$$\mathcal{F}_Y(X) = \{f : X \to \mathbb{C} : \text{ support of } f \text{ is finite and is contained in } Y\}.$$

As the algebras $\mathcal{F}(Y) \cong \mathcal{F}_Y(X)$ under the map $f \in \mathcal{F}_Y(X) \to f|_Y \in \mathcal{F}(Y)$. Let $f \in \mathcal{F}_Y(X)$. We claim that the set of discontinuities of f is $\mathrm{supp}(f)$. For this, let $x \in \mathrm{supp}(f)$. Being finite, $\mathrm{supp}(f)$ is closed giving $\mathrm{supp}(f) = \{y \in Y : f(y) \neq 0\}$. Since X is Hausdorff, there is $U \in \mathcal{U}_x$ with $U \cap \mathrm{supp}(f) = \{x\}$. Now let $\varepsilon = |f(x)|/2$ and $V = \{y \in X : \frac{|f(x)|}{2} < |f(y)|\}$. Clearly $U \cap V = \{x\}$, which is not open in X as x is not an isolated point of X. Thus V cannot be open and hence f cannot be continuous at x. Now let $x \notin \mathrm{supp}(f)$. If x is an isolated point of X, then f is continuous at x. Now let $x \in Y$. Since $\mathrm{supp}(f)$ is finite, there is $U \in \mathcal{U}_x$ such that $U \cap \mathrm{supp}(f) = \emptyset$. Hence for any $\varepsilon > 0$, $f(U) \subset \{z \in \mathbb{C} : |f(x)| < \varepsilon\}$ giving the continuity of f at x. Thus $\mathrm{supp}(f)$ is the set of discontinuities of f. It also follows that $\mathcal{F}_Y(X) \cap C(X) = \{0\}$. Thus $\mathcal{F}_Y(X) \oplus C(X) \subset \mathcal{L}t(X)$. The following example shows that this inclusion is proper.

Example 1. Let $X = [0,1]$. Then $Y = X$. Define $f : [0,1] \to \mathbb{C}$ by $f(t) = \sum_{n=1}^{\infty} \frac{1}{n}\chi_{\{\frac{1}{n}\}}$, where $\chi_{\{\frac{1}{n}\}}$ denotes the characteristic function of $\{\frac{1}{n}\}$. It is easily seen that $f \in \mathcal{L}t([0,1])$ but $f \notin \mathcal{F}_Y(X) \oplus C(X)$.

Recall [1] that an *algebra norm* on an algebra $\mathcal{A}$ is a norm $\|\cdot\|$ on $\mathcal{A}$ making it a normed linear space and satisfying the norm inequality $\|xy\| \le \|x\|\|y\|$ for all $x, y \in \mathcal{A}$. Presence of a norm on $\mathcal{A}$ makes it a metric space with the metric $d(x,y) = \|x - y\|$. $\mathcal{A}$ is a *Banach algebra* if $(\mathcal{A}, d)$ is complete. In a Banach algebra, the algebraic structure and the topological (metric) structure are intertwined through norm inequality and completeness. It is worth noting that though every vector space admits a norm making it a normed linear space, not every algebra can be made into a normed algebra with an algebra norm. This points to a strong interplay between Algebra and Topology in Banach algebras which constitute a fascinating aspect of the subject [2], [3]. Though interesting, but apparently difficult, is the problem of algebraically characterizing an algebra that is a Banach algebra under some norm. A Banach algebra $\mathcal{A}$ with an involution $x \in \mathcal{A} \mapsto x^* \in \mathcal{A}$ satisfying $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda} x^*$ and $(xy)^* = y^* x^*$ for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ is called a *Banach *-algebra.* A Banach *-algebra $\mathcal{A}$ is called a *C^*-algebra* if its norm satisfies $\|x^*x\| = \|x\|^2$. For a bounded function $f : X \to \mathbb{C}$, we define the supnorm $\|f\|_\infty = \sup_{x \in X} |f(x)|$. A continuous function on a compact space is bounded resulting in $C(X) \subset B(X)$. The following refines this.

Proposition 2. *For a compact Hausdorff topological space X, $\mathcal{L}t(X)$ is a commutative C^*-algebra with $\mathbf{1}$; $C(X)$ is a closed *-subalgebra of $\mathcal{L}t(X)$ and $(\mathcal{L}t(X), \|\cdot\|_\infty)$ is a closed *-subalgebra of $(B(X), \|\cdot\|_\infty)$.*

Proof. Let $x \in X$ be a nonisolated point of X. Then there is $U_x \in \mathcal{U}_x$ such that $|f(t) - \ell_x(f)| < 1$ (giving $|f(t)| < 1 + |\ell_x(f)|$) for all $t \in U_x^{(x)}$. Consequently, $|f(t)| < 1 + |\ell_x(f)| + |f(x)| = C_x$, (say), for all $t \in U_x$. If x is an isolated point of X, we set $U_x = \{x\}$ and $C_x = |f(x)|$. Appealing to the compactness of X, we find $x_1, x_2, \ldots, x_n \in X$ such that $\bigcup_{i=1}^{n} U_{x_i} = X$. This forces $|f(y)| \le \max\{C_{x_i} : i = 1, 2, \ldots, n\}$ for all $y \in X$. Thus $f \in B(X)$.

Let $f \in B(X)$ be a limit point of $\mathcal{L}t(X)$. Let $x \in X$. If x is an isolated point of X, then $\lim_{t \to x} f(t)$ exists and equals $f(x)$. So, we assume that x is not an isolated point of X. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, choose $f_n \in \mathcal{L}t(X)$ such that $\|f - f_n\|_\infty < \frac{1}{n}$. Let $n_1 \in \mathbb{N}$ be such that $\frac{1}{n_1} < \frac{\varepsilon}{4}$. Then for any fixed $n, m \ge n_1$, and $t \in X$,

$$
\begin{aligned}
&|\ell_x(f_n) - \ell_x(f_m)| \\
&\le |\ell_x(f_n) - f_n(t)| + |f_n(t) - f_m(t)| + |f_m(t) - \ell_x(f_m)| \\
&\le |\ell_x(f_n) - f_n(t)| + |f_n(t) - f(t)| + |f(t) - f_m(t)| + |f_m(t) - \ell_x(f_m)|
\end{aligned}
$$

$$
\begin{aligned}
&\leq |\ell_x(f_n) - f_n(t)| + \|f_n - f\|_\infty + \|f_m - f\|_\infty + |f_m(t) - \ell_x(f_m)| \\
&< |\ell_x(f_n) - f_n(t)| + \frac{\varepsilon}{2} + |f_m(t) - \ell_x(f_m)|. \qquad (1)
\end{aligned}
$$

Since $f_n, f_m \in \mathcal{L}t(X)$, we obtain $U \in \mathcal{U}_x$ such that $|\ell_x(f_i) - f_i(t)| < \frac{\varepsilon}{4}$, for $t \in U^{(x)}$ for $i = m, n$. By this and (1), $|\ell_x(f_n) - \ell_x(f_m)| < \varepsilon$ for all $n, m \geq n_1$. Thus $\ell_x(f_n)$ is Cauchy in $\mathbb{C}$. Let $l = \lim\limits_{n\to\infty} \ell_x(f_n)$. Fix an integer $n_2 > n_1$ such that $|\ell_x(f_n) - l| < \frac{\varepsilon}{3}$ for all $n \geq n_2$. Fix $V \in \mathcal{U}_x$ such that $|f_{n_2}(t) - \ell_x(f_{n_2})| < \frac{\varepsilon}{3}$ for all $t \in V^{(x)}$. This gives,

$$
\begin{aligned}
|f(t) - l| &\leq |f(t) - f_{n_2}(t)| + |f_{n_2}(t) - \ell_x(f_{n_2})| + |\ell_x(f_{n_2}) - l| \\
&\leq \|f - f_{n_2}\|_\infty + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad \text{for all } t \in V^{(x)} \\
&< \varepsilon.
\end{aligned}
$$

Hence $\lim\limits_{t\to x} f(x) = l$, proving that $f \in \mathcal{L}t(X)$. Thus $\mathcal{L}t(X)$ is closed in $B(X)$. □

For each $x \in X$, the limit $\lim\limits_{t\to x} f(x) = \ell_x(f)$, $(f \in \mathcal{L}t(X))$ can be reinterpreted exhibiting a flavour typical of Functional Analysis. For $f \in \mathcal{L}t(X)$, define a function $\ell(f) : X \to \mathbb{C}$, $\ell(f)(x) = \ell_x(f)$. Then $\ell(f) \in C(X)$. This defines $\ell : f \in \mathcal{L}t(X) \mapsto \ell(f) \in C(X)$. The map $\ell : \mathcal{L}t(X) \to C(X)$ is surjective algebra homomorphism; and $f \in C(X)$ if and only if $\ell(f) = f$. For any $f \in \mathcal{L}t(X)$, $\|\ell(f)\|_\infty \leq \|f\|_\infty$ showing that ℓ is continuous. Thus $E = \ker \ell$ is a closed ideal, and $\mathcal{L}t(X) = C(X) \oplus E$. The map ℓ is not a (strict) contraction, and has a multitude of fixed points.

A *multiplicative linear functional on* a Banach algebra $\mathcal{A}$ is a linear function $f : \mathcal{A} \to \mathbb{C}$ such that $f(xy) = f(x)f(y)$, for all $x, y \in \mathcal{A}$. The *Gel'fand space* $\triangle(\mathcal{A})$ is the set of all nonzero multiplicative linear functionals on $\mathcal{A}$. Notice that a multiplicative linear functional on $\mathcal{A}$ is necessarily continuous; and this is just a tip of the iceberg. Automatic continuity of natural maps from $\mathcal{A}$ or into $\mathcal{A}$ is an exciting aspect of Banach Algebra Theory exhibiting deep connection between Algebra and Topology in $\mathcal{A}$ [2]. For $x \in \mathcal{A}$, define $\widehat{x} : \triangle(\mathcal{A}) \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$, $(f \in \triangle(\mathcal{A}))$. The *Gel'fand topology* on $\triangle(\mathcal{A})$ is the weakest topology making the functions in the family $\{\widehat{x} : x \in \mathcal{A}\}$ continuous. It is the relative topology on $\triangle(\mathcal{A})$ from the product topology on $\mathbb{C}^{\mathcal{A}}$; and is also the restriction of the weak *-topology on the Banach space dual $\mathcal{A}^*$ of $(\mathcal{A})$. The Gel'fand space is a compact Hausdorff space [3]. The map $x \in \mathcal{A} \mapsto \widehat{x} \in C(\triangle(\mathcal{A}))$ is called the *Gel'fand transform of* x. The Gel'fand-Naimark Theorem states that if $\mathcal{A}$ is a commutative C^*-algebra with identity, then $x \in \mathcal{A} \mapsto \widehat{x} \in C(\triangle(\mathcal{A}))$ is an isometric *-isomorphism between $\mathcal{A}$ and $C(\triangle(\mathcal{A}))$. Thus $\mathcal{L}t(X) \cong C(\triangle(\mathcal{L}t(X)))$; and it is important to recognize the Gel'fand space $\triangle(\mathcal{L}t(X))$.

For $x \in X$, we define $\ell_x, \nu_x : \mathcal{L}t(X) \to \mathbb{C}$ by $\ell_x(f) = \lim\limits_{t \to x} f(t)$ and $\nu_x(f) = f(x)$, $(f \in \mathcal{L}t(X))$. Then ℓ_x, ν_x are multiplicative linear functionals on $\mathcal{L}t(X)$. We set $X_\nu = \{\nu_x : x \in X\}; X_\ell = \{\ell_x : x \in Y\}$, $\widetilde{X} = X_\nu \cup X_\ell$. Let x, y be two distinct points of X. Since $C(X)$ separates points of X, we get $f \in C(X)$ such that $f(x) \neq f(y)$. Thus $\nu_x(f) \neq \nu_y(f)$. Also $\ell_x(f) = \nu_x(f) = f(x)$ and $f(y) = \nu_y(f) = \ell_y(f)$. Hence $\ell_x(f) \neq \ell_y(f)$. Therefore, all elements of X_ν are distinct and all elements of X_ℓ are distinct. Now we show that $\ell_x \neq \nu_x$ for any $x \in Y$. So, let $x \in Y$. Then x is not an isolated point of X. Let f be the characteristic function of $\{x\}$. Then f is not continuous, $\ell_x(f) = 0 \neq 1 = \nu_x(f)$. Finally, we show that $\ell_y \neq \nu_x$ for any $x \in X, y \in Y$. For that let $x \in X, y \in Y$ be two distinct points. Since the characteristic function f of $\{x\}$ is continuous at y, $\ell_y(f) = 0$. But $\nu_x(f) = 1$. Thus $X_\nu \cap X_\ell = \emptyset$, guaranteing that $\widetilde{X}$ is a well-defined set.

Identifying X_ν and X_ℓ with subsets of X, the *disjoint union topology* on $\widetilde{X}$ is the weakest topology in which the inclusion maps $X_\nu \to \widetilde{X}$ and $X_\ell \to \widetilde{X}$ are continuous.

Theorem 3. *For a compact Hausdorff topological space X, the Gel'fand topology on $\triangle(\mathcal{L}t(X)) = \widetilde{X}$ is strictly stronger than the disjoint union topology.*

Proof. As observed earlier, $\widetilde{X} \subset \triangle(\mathcal{L}t(X))$. Let M be a maximal ideal of $\mathcal{L}t(X)$. Assume that there is no x in X such that $M = \ker(\nu_x)$ or $M = \ker(\ell_x)$. By the maximality of M, $M \not\subset \ker(\nu_x)$ and also $M \not\subset \ker(\ell_x)$ for any $x \in X$. Hence for each $x \in X$, we fix $f_x, g_x \in M$ such that $f_x \notin \ker(\nu_x)$, $g_x \notin \ker(\ell_x)$. If necessary, multiplying each by its complex conjugate, we assume that $f_x, g_x \geq 0$. Since $\lim\limits_{t \to x} g_x(t) = \ell_x(g) > 0$, there is $U_x \in \mathcal{U}_x$ such that $g_x(t) > 0$ for all $t \in U_x^{(x)}$. But then $h_x = f_x + g_x > 0$ on U_x. If necessary, replacing U_x by a smaller neighbourhood of x, we assume that $h_x(t) > \frac{\ell_x(h_x)}{2}$ for all $t \in U_x$. But then $h_x(t) \neq 0 \neq \ell_t(h_x)$ for all $t \in U_x$. By the compactness of X, there exists $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup\limits_{i=1}^{n} U_{x_i}$. But then $h = \sum\limits_{i=1}^{n} h_{x_i} > 0$ on X and $h \in \mathcal{L}t(X)$. Hence $\ell_x(\frac{1}{h})$ exists for all $x \in X$. This shows that h is invertible in $\mathcal{L}t(X)$. But by the construction of h, it is clear that $h \in M$ and so, M cannot be a proper ideal, a contradiction. Thus we assert that there exists $x \in X$ such that $M = \ker(\nu_x)$ or $M = \ker(\ell_x)$. The following shows that the Gel'fand topology is strictly stronger than the disjoint union topology.

Claim:

(1) Each $\{\nu_x\} \subset X_\nu$ is open in $\widetilde{X}$. Consequently, X_ν is discrete and X_ℓ is compact.
(2) $X_\ell \cup \{\nu_x : x \text{ is an isolated point of } X\}$ is homeomorphic to X.

(1) Fix $\nu_x \in X_\nu$. Let $f = \chi_{\{x\}}$, the characteristic function of x. Consider the open set $N(\nu_x, f, \frac{1}{2}) = \{\varphi \in \triangle(\mathcal{L}t(X)) : |\varphi(f) - \nu_x(f)| < \frac{1}{2}\} = \{\varphi \in \triangle(\mathcal{L}t(X)) : |\varphi(f) -$

$1| < \frac{1}{2}\}$ in $\triangle(\mathcal{L}t(X))$. Then $\nu_x \in N(\nu_x, f, \frac{1}{2})$ and $\ell_y(f) = 0$ for all $y \in X$ with $y \neq x$. Hence $\ell_y \notin N(\nu_x, f, \frac{1}{2})$. If x is not an isolated point, then $\ell_x(f) = 0$. So, $\ell_x \notin N(\nu_x, f, \frac{1}{2})$. If x is an isolated point, then $\ell_x \notin X_\ell$. Thus $\ell_y \notin N(\nu_x, f, \frac{1}{2})$ for any $y \in X$. Clearly, for any $y \in X$ with $y \neq x$, $|\nu_y(f) - \nu_x(f)| = 1$. Consequently, $\nu_y \notin N(\nu_x, f, \frac{1}{2})$. Hence, $N(\nu_x, f, \frac{1}{2}) = \{\nu_x\}$, completing the proof.
(2) Without loss of generality, we assume that X has no isolated point. This case reduces the proof to show that X_ℓ is homeomorphic to X. Define $\iota : X \to X_\ell$ by $\iota(x) = \ell_x$. Since X is compact Hausdorff, its topology is the weak topology given by $C(X)$. To consider its basic open set, let $f \in C(X)$ and $G \subset \mathbb{C}$ be open in $\mathbb{C}$. Clearly, $f \in \mathcal{L}t(X)$. Denoting the Gel'fand transform of $f \in \mathcal{L}t(X)$, by $\hat{f}$, we see using the continuity of f that $f(x) = \nu_x(f)(= \hat{f}(\nu_x)) = \ell_x(f)(= \hat{f}(\ell_x))$. Thus $\nu_x \in \hat{f}^{-1}(G) \Leftrightarrow \ell_x \in \hat{f}^{-1}(G) \Leftrightarrow x \in f^{-1}(G)$. Consequently, $\iota(f^{-1}(G)) = \{\ell_x : x \in \hat{f}^{-1}(G)\}$ is open in X_ℓ. Thus ι is a homeomorphism. □

This leaves open the problem of intrinsically (using pointset topology only, without a reference to $\mathcal{L}t(X)$) describing the Gel'fand topology on $\widetilde{X}$. The next obvious question is the problem of determination of closed ideals of $\mathcal{L}t(X)$. We may recall [2] that $\mathcal{I}$ is a closed ideal of $C(X)$ if and only if $\mathcal{I} = \mathcal{I}_K$ for a closed $K \subset X$, where $\mathcal{I}_K = \{f \in C(X) : f(x) = 0 \text{ for all } x \in K\}$.

Remark 4.

(1) As the subspaces of $\widetilde{X}$, X_ℓ is homeomorphic to $X \setminus \{x : x \text{ is an isolated point of } X\}$ and X_v is discrete. Also $\hat{f}$ redefines each $f \in \mathcal{L}t(X)$ on X_ℓ to reclaim the continuity of f at points $\ell_x(= \iota(x))$, where f is not continuous.

(2) One obtains the nonunital analogue of $\mathcal{L}t(X)$ by considering X to be locally compact Hausdorff and replacing $\mathcal{L}t(X)$ by $\mathcal{L}t_0(X)$, the set of all $f \in \mathcal{L}t(X)$ vanishing at infinity. Also, $\mathcal{L}t(X^*)$ is the unitization of $\mathcal{L}t_0(X)$, where X^* is the one point compactification of X.

(3) $\mathcal{L}t(X)$ also suggests the investigation of its C^1-analogue, viz., the algebra $\mathcal{L}t^1[0,1] = \{f : [0,1] :\to \mathbb{C} : f' \text{ exists on } [0,1] \text{ and } f' \in \mathcal{L}t[0,1]\}$ with $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$. Then $C^1[0,1] \subset \mathcal{L}t^1[0,1] \subset C[0,1]$. The inclusions are proper as $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), \ x \in (0,1] \\ 0, \text{ otherwise.} \end{cases}$ is in $\mathcal{L}t^1[0,1]$ but not in $C^1[0,1]$ and $g(x) = |x|$ is not in $\mathcal{L}t^1[0,1]$. One can analogously define $\mathcal{L}t^n[0,1]$ containing $C^n[0,1]$. For the algebra $B^1[0,1] = \{f : [0,1] \to \mathbb{C} : f' \text{ exists on } [0,1] \text{ and } f' \text{ is bounded}\}$, $\mathcal{L}t^1[0,1] \subset B^1[0,1]$; and $B^1[0,1]$ is contained in the Banach algebra of Lipschitz functions

$$Lip[0,1] := \{f \in C[0,1] : f' \text{ exists a.e. on } [0,1] \text{ and } f' \in L^\infty[0,1]\}.$$

Thus one gets a large collection of new normed algebras of functions. Note that $\bigcap_{n=1}^{\infty} \mathcal{L}t^n[0,1] = \bigcap_{n=1}^{\infty} C^n[0,1] = C^\infty[0,1]$.

(4) The Banach algebra $C(X)$ for a compact Hausdorff space X, enjoys a special status in Functional Analysis. It manifests its intrinsic beauty through several noble theorems like Riesz Representation Theorem, Gel'fand-Naimark Theorem, Banach-Stone Theorem, Stone-Weierstrass Theorem and Ascoli-Arzela Theorem. It would be interesting to search for analogues of these celebrated theorems for $\mathcal{L}t(X)$, it being a half brother of $C(X)$.

(5) If $X = [a, b] \subset \mathbb{R}$, then $f \in \mathcal{L}t[a, b]$ if and only if for all $x \in [a, b]$, both the left limit $f(x-)$ as well as the right limit $f(x+)$ exist and are equal. One may consider algebras $\mathcal{L}t_-([a, b])$ and $\mathcal{L}t_+([a, b])$ consisting of functions having left limits and right limits respectively. Each of these algebras contains monotonic functions as well as functions of bounded variations; and $\mathcal{L}t[a, b] \subset \mathcal{L}t_+[a, b] \cap \mathcal{L}t_-[a, b]$. Though the functions therein need not be bounded, the determination of their Gel'fand spaces makes sense.

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INFINITE PRODUCTS FOR e^n VIA CATALAN'S PRODUCT

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Abstract. In this note we present a decomposition of Catalan's infinite product for e leading to a new infinite product for e. Further, a generalization of Catalan's result is also obtained that yields products for e^n.

1. Introduction

The number e, often called the Euler number, sought entry into mathematics unobtrusively through a table of logarithms in the appendix to the *Mirifici logarithmorum canonis constructio* (1619), a posthumous work of John Napier (1550–1617). However, as Cajori [1] observed, the notion of a "base" is inapplicable to his system and it was not recognised at that time that these logarithms were based on e. Napier used $1-10^{-1}$ as his 'given' number which raised to power 10^7 is very nearly e^{-1}.

It was Leonhard Euler (1707–1783) who identified the number using the notation e in his letter dated November 25, 1731 to Goldbach wherein he wrote "e denotes that number whose hyperbolic logarithm is equal to 1". He used e again five years later in *Mechanica* (1736) which was its first appearance in print. He gave full treatment to e in his *Introductio in Analysin Infinitorum* (Book I, 1748) where he established the equality of the limit definition and the series representation: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{\infty}\frac{1}{m!}$, and evaluated e to twenty three decimal places (Ch. VII, §122) and gave a continued fraction representation (Ch. XVIII) for $\frac{e-1}{2}$.

2. Decomposition of Catalan's product

Eugène Charles Catalan (1814–94) discovered the following infinite product [2].

$$e = \frac{2}{1}\left(\frac{4}{3}\right)^{1/2}\left(\frac{6\cdot 8}{5\cdot 7}\right)^{1/4}\left(\frac{10\cdot 12\cdot 14\cdot 16}{9\cdot 11\cdot 13\cdot 15}\right)^{1/8}\cdots \tag{1}$$

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which can also be written as

$$e = \lim_{m\to\infty} \frac{2^{m+1}}{\prod_{k=1}^{2^m-1}(2k+1)^{\frac{1}{2^m}}}.$$

We will prove a generalization of this formula at the end of this note.

Pippenger [3] obtained the following product in 1980.

$$\frac{e}{2} = \left(\frac{2}{1}\right)^{1/2}\left(\frac{2\cdot 4}{3\cdot 3}\right)^{1/4}\left(\frac{4\cdot 6\cdot 6\cdot 8}{5\cdot 5\cdot 7\cdot 7}\right)^{1/8}\cdots. \tag{2}$$

Since $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 1$, we have $2 = 2^{1/2}2^{1/4}2^{1/8}\cdots$. So multiplying the last product by 2, we get Catalan's product for e.

In 2010, Sondow and Yi [4] derived a few Catalan-type products including

$$\frac{e}{4} = \left(\frac{2}{3}\right)^{1/2}\left(\frac{4\cdot 6}{5\cdot 7}\right)^{1/4}\left(\frac{8\cdot 10\cdot 12\cdot 14}{9\cdot 11\cdot 13\cdot 15}\right)^{1/8}\cdots. \tag{3}$$

The following two products provide a decomposition of Catalan's product — one having terms with factors of the type $4m$ and the other with factors $4m+2$.

$$\frac{e}{2} = \left(\frac{4}{3}\right)^{1/3}\left(\frac{8\cdot 8}{5\cdot 7}\right)^{1/6}\left(\frac{12\cdot 12\cdot 16\cdot 16}{9\cdot 11\cdot 13\cdot 15}\right)^{1/12}\cdots. \tag{4}$$

$$\frac{e}{4} = \frac{2}{3}\left(\frac{6\cdot 6}{5\cdot 7}\right)^{1/2}\left(\frac{10\cdot 10\cdot 14\cdot 14}{9\cdot 11\cdot 13\cdot 15}\right)^{1/4}\cdots. \tag{5}$$

The formula (4) is believed to be new while a variant form of (5) occurs in [4].

To verify that (4) $\times$ (5) indeed yields Catalan's product, we multiply both sides of (5) by 2, take the cube root of the two sides and then multiply the two products getting

$$\left(\frac{e}{2}\right)^{4/3} = \left(\frac{4^2}{3^2}\right)^{1/3}\left(\frac{6^2\cdot 8^2}{5^2\cdot 7^2}\right)^{1/6}\left(\frac{10^2\cdot 12^2\cdot 14^2\cdot 16^2}{9^2\cdot 11^2\cdot 13^2\cdot 15^2}\right)^{1/12}\cdots,$$

that is,

$$\left(\frac{e}{2}\right)^{1/3} = \left(\frac{4}{3}\right)^{1/6}\left(\frac{6\cdot 8}{5\cdot 7}\right)^{1/12}\left(\frac{10\cdot 12\cdot 14\cdot 16}{9\cdot 11\cdot 13\cdot 15}\right)^{1/24}\cdots$$

which on raising to the third power and then multiplying by 2 results in Catalan's product. Hence, proving one product will validate the other.

Now (5) can be written as

$$\frac{e}{4} = \left(2\left(\frac{1}{3}\right)\right)\left(2\left(\frac{3\cdot 3}{5\cdot 7}\right)^{1/2}\right)\left(2\left(\frac{5\cdot 5\cdot 7\cdot 7}{9\cdot 11\cdot 13\cdot 15}\right)^{1/4}\right)\cdots.$$

We observe that the numerator of a factor equals the denominator of the preceding factor so that all factors except the last reduce to 2 after cancellation of the common factors. So the nth partial product is given by

$$\frac{2^n}{[(2^n+1)(2^n+3)(2^n+5)\cdots(2^{n+1}-1)]^{\frac{1}{2^{n-1}}}} = \frac{2^n}{\prod_{k=2^{n-1}}^{2^n-1}(2k+1)^{\frac{1}{2^{n-1}}}}.$$

Hence we have to prove that

$$e = \lim_{n\to\infty} \frac{2^{n+2}}{\prod_{k=2^{n-1}}^{2^n-1}(2k+1)^{\frac{1}{2^{n-1}}}}. \tag{6}$$

Replacing 2^{n-1} by m, it is sufficient to prove that $e = \lim_{m\to\infty} \frac{8m}{\prod_{k=m}^{2m-1}(2k+1)^{\frac{1}{m}}}$.

Using Stirling's formula $n! \sim (n/e)^n \sqrt{2\pi n}$ $(n \to \infty)$ and the following asymptotic formula for the central binomial coefficients

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \quad (n\to\infty)$$

we find

$$\lim_{m\to\infty} \frac{8m}{\prod_{k=m}^{2m-1}(2k+1)^{\frac{1}{m}}} = \lim_{m\to\infty} \frac{16m}{\sqrt[m]{m!\binom{4m}{2m}}} = e$$

from which (4) follows.

3. Generalization of Catalan's product

Sondow and Yi gave this product and conjectured that one can generalize it and Catalan's product to a product for a power of $e^{1/K}$, $K \in \mathbb{N}\setminus\{1\}$

$$\frac{e^{2/3}}{\sqrt{3}} = \left(\frac{3}{2}\right)^{1/3}\left(\frac{3\cdot 6\cdot 6\cdot 9}{4\cdot 5\cdot 7\cdot 8}\right)^{1/9} \left(\frac{9\cdot 12\cdot 12\cdot 15\cdot 15\cdot 18\cdot 18\cdot 21\cdot 21\cdot 24\cdot 24\cdot 27}{10\cdot 11\cdot 13\cdot 14\cdot 16\cdot 17\cdot 19\cdot 20\cdot 22\cdot 23\cdot 25\cdot 26}\right)^{1/27}\cdots.$$

Since $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \frac{1}{2}$, we have $\sqrt{3} = 3^{1/3}3^{1/9}3^{1/27}\cdots$. So multiplying the last product by $\sqrt{3}$, multiplying the numerators/denominators by the same numbers and then raising the two sides to the third power, we obtain

$$e^2 = \frac{3^3}{2\cdot 3}\left(\frac{6^3 9^3}{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}\right)^{1/3}\left(\frac{12^3 15^3 18^3 21^3 24^3 27^3}{10\cdot 11\cdot 12\cdots 27}\right)^{1/9}\cdots. \tag{7}$$

This is the case $n=3$ of the following general formula that yields products for e^{n-1}, $n\in\mathbb{N}$

$$e^{n-1} = \frac{n^n}{n!}\prod_{m=1}^{\infty}\left(\frac{\prod_{k=1}^{(n-1)n^{m-1}}(n^m+kn)^n}{\prod_{k=1}^{(n-1)n^m}(n^m+k)}\right)^{\frac{1}{n^m}}. \tag{8}$$

Cancelling the common factors, we can rewrite the partial product of (8) as

$$\begin{aligned}
&\frac{n^n}{n!}\prod_{m=1}^{p}\left(\frac{\prod_{k=1}^{(n-1)n^{m-1}}(n^m+kn)^n}{\prod_{k=1}^{(n-1)n^m}(n^m+k)}\right)^{\frac{1}{n^m}}\\
&\quad= \frac{n^n}{n!}\prod_{m=1}^{p}\left(\frac{\prod_{k=1}^{(n-1)n^{m-1}}(n^{m-1}+k)^n n^n}{\prod_{k=1}^{(n-1)n^m}(n^m+k)}\right)^{\frac{1}{n^m}}\\
&\quad= \frac{n^n}{n!}\prod_{m=1}^{p}\left(\frac{n^{(n-1)n^m}\prod_{k=1}^{(n-1)n^{m-1}}(n^{m-1}+k)^n}{\prod_{k=1}^{(n-1)n^m}(n^m+k)}\right)^{\frac{1}{n^m}}
\end{aligned}$$

$$= \frac{n^n}{n!} n^{p(n-1)} \prod_{m=1}^{p} \frac{\prod_{k=1}^{(n-1)n^{m-1}} (n^{m-1}+k)^{\frac{1}{n^{m-1}}}}{\prod_{k=1}^{(n-1)n^m} (n^m+k)^{\frac{1}{n^m}}}$$

$$= \frac{n^n}{n!} n^{p(n-1)} \frac{n!}{\prod_{k=1}^{(n-1)n^p} (n^p+k)^{\frac{1}{n^p}}} = \frac{n^n n^{p(n-1)}}{\prod_{k=1}^{(n-1)n^p} (n^p+k)^{\frac{1}{n^p}}}.$$

Now we want to evaluate the limit of this expression (with n fixed) as $p \to \infty$. Note that this is the special case with $\ell = n^p$ of

$$\frac{n^n \ell^{n-1}}{\prod_{k=1}^{(n-1)\ell} (\ell+k)^{\frac{1}{\ell}}} = \frac{n^n \ell^{n-1}}{\left(\frac{(n\ell)!}{\ell!}\right)^{1/\ell}}.$$

Using Stirling's formula, we find that

$$\lim_{\ell\to\infty} \frac{n^n \ell^{n-1}}{\left(\frac{(n\ell)!}{\ell!}\right)^{1/\ell}} = \lim_{\ell\to\infty} \frac{e^{n-1}}{n^{\frac{1}{2\ell}}} = e^{n-1}.$$

(8) is now an immediate consequence and (1) a special case ($n = 2$). We conclude with the formula for $n = 4$ that

$$e^3 = \frac{4^4}{2\cdot 3\cdot 4} \left(\frac{8^4 12^4 16^4}{5\cdot 6\cdot 7\cdots 15\cdot 16}\right)^{1/4} \left(\frac{20^4 24^4 \cdots 64^4}{17\cdot 18\cdot 19\cdots 63\cdot 64}\right)^{1/16} \cdots . \qquad (9)$$

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FATOU'S LEMMA AND INTEGRALITY CRITERIA IN NUMBER FIELDS

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Abstract. While Fatou's lemma in measure theory is well known, his lemma on the power series ring over integers is perhaps not so well known. In this note we describe this lemma, present its generalisation to Dedekind domains and finally apply it to integrality questions on number fields.

1. Introduction

We begin with some preliminaries. An algebraic number field K is a subfield of the field of complex numbers $\mathbb{C}$ which is a finite extension of $\mathbb{Q}$. A complex number α which satisfies a non-zero polynomial with integer coefficients is called an algebraic number. For $\alpha \in \mathbb{C}$, if there exists a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$ then α is called an algebraic integer.

Let K be a number field and $\alpha \in K$. The trace of α in K, denoted by $Tr_{K/\mathbb{Q}}(\alpha)$, is defined to be the sum $\sum_{i=1}^{n} \sigma_i(\alpha)$, where σ_i's are the distinct field embeddings of K in $\mathbb{C}$ fixing $\mathbb{Q}$. It is an standard result that the trace of an algebraic number is an element of $\mathbb{Q}$ and the trace of an algebraic integer is an element in $\mathbb{Z}$. However, the converse is not true, i.e. an algebraic number with integral trace need not be an algebraic integer. One such example is $\alpha = \frac{1}{\sqrt{2}}$.

During a number theory course by David Goss, John Lame asked the following question.

Question 1. Suppose α is an algebraic number such that $Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^i) \in \mathbb{Z}$ for all $i \geq 1$. Does it imply that α is an algebraic integer?

The answer to this question is given positively by many mathematicians. In fact, this has been known since 1915, due to George Pólya, where he proved it using Fatou's lemma.

In this regard, we first recall Fatou's lemma. Any rational function $\frac{p(x)}{q(x)}$ where $p(x), q(x) \neq 0 \in \mathbb{Q}[x]$, can be written as a Laurent series with coefficients in $\mathbb{Q}$. Fatou's lemma gives a sufficient condition for such a rational function to be a rational function over $\mathbb{Z}$ with the denominator having constant term 1. More precisely,

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Theorem 1.1 (Fatou's lemma). *If $f(x) \in \mathbb{Z}[[x]] \cap \mathbb{Q}(x)$, then $f(x)$ can be written as $\frac{g(x)}{h(x)}$ where $g(x), h(x)$ are in $\mathbb{Z}[x]$ having no common factors and $h(0) = 1$.*

When we say $f(x) \in \mathbb{Z}[[x]] \cap \mathbb{Q}(x)$, we mean that it is a rational function which admits a formal power series expansion with coefficients in $\mathbb{Z}$. To illustrate, consider the rational function $\frac{1}{1-x} \in \mathbb{Q}(x)$. It can also be viewed as a formal power series in $\mathbb{Z}[[x]]$, i.e. there is an element $f(x) \in \mathbb{Z}[[x]]$ such that $(1-x)f(x) = 1$ in $\mathbb{Z}[[x]]$. One can check that $f(x) = \sum_{n=0}^{\infty} x^n$.

In this article, we present an analogue of Fatou's lemma which is in general true for any Dedekind domain (see §3 for definition and other properties).

Theorem 1.2 (Fatou's lemma for Dedekind domains). *Let A be a Dedekind domain and K be its field of fractions. If $f(x) \in A[[x]] \cap K(x)$ then $f(x)$ can be written as $\frac{g(x)}{h(x)}$ where $g(x)$,$h(x) \in A[x]$, having no common factor in $K[x]$ and $h(0) = 1$.*

2. Fatou's lemma

We begin with the definition of the order and the content of an element $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$.

Definition 2.1. The order of $f \neq 0$ is the smallest integer n such that $a_n \neq 0$. We denote it by $\omega(f)$. By definition, $\omega(0) = +\infty$.

Note that the order can be defined for formal power series with coefficients in any ring.

Proposition 2.1. *Let A be an integral domain. Then for $f, g \in A[[x]]$,*

$$\omega(fg) = \omega(f) + \omega(g).$$

Proof. Let

$$f := \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g := \sum_{n=0}^{\infty} b_n x^n.$$

Let us assume $\omega(f) = i, \omega(g) = j$. Thus we can write

$$f = \sum_{n=0}^{\infty} a_{n+i} x^{n+i} \quad \text{and} \quad g = \sum_{n=0}^{\infty} b_{n+j} x^{n+j},$$

where $a_i, b_j \neq 0$. Hence

$$fg = a_i b_j x^{i+j} + \sum_{m=1}^{\infty} \left(\sum_{k=0}^{m} a_{k+i} b_{m-k+j} \right) x^{m+i+j}.$$

Clearly, $\omega(fg) = i + j = \omega(f) + \omega(g)$. $\square$

Remark 2.1. If A is not an integral domain then $\omega(fg) \geq i + j = \omega(f) + \omega(g)$.

Definition 2.2. For $f \in \mathbb{Z}[[x]]$, the content of f is defined to be the non-negative generator of the ideal generated by coefficients $\{a_i | i \geq 0\}$. We denote it by $c(f)$.

Note that the ideal generated by the coefficients of f is generated by a single element, since $\mathbb{Z}$ is a principal ideal domain, and therefore the content of a power series with coefficients in $\mathbb{Z}$ is well defined.

Lemma 2.1 (Gauss lemma)**.** *Let $f, g \in \mathbb{Z}[[x]]$. Then $c(fg) = c(f)c(g)$.*

Proof. If $f = 0$ or $g = 0$, then the equality is clear. Hence we will assume that f and $g \neq 0$. We can write $f = c(f)\widetilde{f}$ and $g = c(g)\widetilde{g}$, where $\widetilde{f}, \widetilde{g} \in \mathbb{Z}[[x]]$. Hence $fg = c(f)c(g)\widetilde{f}\widetilde{g}$. We can take $c(f)$, $c(g) = 1$ and we have to show that $c(fg) = 1$.

Suppose not. Then there exists some prime p such that $p|c(fg)$. We have a natural surjection $\Phi : \mathbb{Z}[[x]] \longrightarrow \mathbb{Z}/p\mathbb{Z}[[x]]$, $\sum_{n=0}^{\infty} a_n x^n \mapsto \sum_{n=0}^{\infty} \bar{a_n} x^n$. Since $p|\mathrm{c}(fg)$, $\Phi(fg) = 0$ whereas $\Phi(f), \Phi(g) \neq 0$, because $c(f), c(g) = 1$. We obtain a contradiction as $\mathbb{Z}/p\mathbb{Z}[[x]]$ is a integral domain. $\square$

Proof of Theorem 1.1 We write $f(x) = c(f)\widetilde{f}(x)$, where $c(\widetilde{f}) = 1$. We will prove the theorem for $\widetilde{f}(x)$ and observe that the theorem will also hold for $f(x)$. So we can assume that $c(f) = 1$. Let $f(x) = \frac{g_1(x)}{h_1(x)} = \frac{a\widetilde{g}(x)}{b\widetilde{h}(x)}$, where $g_1(x)$ and $h_1(x)$ $\in \mathbb{Q}[x]$ are co-prime, $\widetilde{g}(x)$ and $\widetilde{h}(x) \in \mathbb{Z}[x]$ and a and b are integers. We have $b\widetilde{h}(x)f(x) = a\widetilde{g}(x)$. Upon applying Lemma 2.1

$$bc(\widetilde{h})c(f) = ac(\widetilde{g}) \implies bc(\widetilde{h}) = ac(\widetilde{g}).$$

Therefore

$$f(x) = \frac{a\widetilde{g}(x)}{b\widetilde{h}(x)} = \frac{ac(\widetilde{g})g(x)}{bc(\widetilde{h})h(x)} = \frac{g(x)}{h(x)}.$$

Here $g(x)$ and $h(x) \in \mathbb{Z}[x]$, have no common factor and $c(g) = c(h) = 1$. We observe that $h(0) \neq 0$, because if $h(0) = 0$ then by Proposition 2.1 we have

$$\omega(hf) = \omega(h) + \omega(f) = \omega(g) \implies \omega(g) > 0 \implies g(0) = 0$$

which contradicts the fact that g and h have no common factor. Since $g(x)$ and $h(x)$ are co-prime in $\mathbb{Q}[x]$, there exists $u(x)$ and $v(x) \in \mathbb{Q}[x]$ such that

$$u(x)g(x) + v(x)h(x) = 1.$$

This equation can be rewritten as

$$du(x)\frac{g(x)}{h(x)} + dv(x) = \frac{d}{h(x)},$$

where d is a non zero integer such that $du(x)$ and $dv(x) \in \mathbb{Z}[x]$. Thus $\frac{d}{h(x)} \in \mathbb{Z}[[x]]$. We set $\widetilde{P}(x) := \frac{d}{h(x)}$. Applying Lemma 2.1 we get

$$h(x)\widetilde{P}(x) = d \implies c(h)c(\widetilde{P}) = c(d) \implies c(\widetilde{P}) = d.$$

Therefore $\frac{d}{h(x)} = \widetilde{P}(x) = d\widetilde{Q}(x)$ for some $\widetilde{Q}(x) \in \mathbb{Z}[[x]]$. We get $\widetilde{Q}(x) = \frac{1}{h(x)}$ and $\widetilde{Q}(0) = \frac{1}{h(0)} \in \mathbb{Z}$, hence $h(0) = \pm 1$. We then write $f(x) = \frac{h(0)g(x)}{h(0)h(x)}$ to obtain the result as stated in the theorem. $\square$

Remark 2.2. The phrases 'two polynomials $g(x), h(x)$ have no common factor' and 'two polynomials $g(x), h(x)$ are co-prime' are not equivalent. It depends on the ring. For example the polynomials $g(x) = x$ and $h(x) = x^2 + 2$ have no common factor in $\mathbb{Z}[x]$ but they are not co-prime in $\mathbb{Z}[x]$. This is because the ideal $(x, x^2+2) = (2, x)$ does not generate the whole ring $\mathbb{Z}[x]$. However we note that these two polynomials are co-prime in $\mathbb{Q}[x]$.

3. Gauss lemma for Dedekind domains

We begin by recalling some definitions.

Definition 3.1. A ring in which every ideal is finitely generated is called noetherian ring. An equivalent definition is that any increasing chain of ideals becomes stationary at a finite stage.

Example 3.1. $\mathbb{Z}$, $k[x,y]$ where k is a field are examples of noetherian rings. An example of a non noetherian ring is $\mathbb{Z}[S]$ where $S = \{x_i | i \in \mathbb{N}\}$.

Definition 3.2 (Dedekind domains)**.** Let A be an integral domain. We say that A is a Dedekind domain if

(1) A is noetherian.
(2) Every non zero prime ideal of A is maximal.
(3) A is integrally closed in its field of fractions K (i.e any element of K satisfying a monic polynomial with coefficients in A is an element of A)

Here are some examples of Dedekind domains.

Example 3.2. $\mathbb{Z}, \mathbb{Z}\left[\sqrt{3}\right], \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right], \mathbb{Z}\left[\sqrt{-5}\right]$.

From now on, we consider A to be a Dedekind domain. We state an important property for A which will be used in the subsequent sections.

Theorem 3.1. *Any non zero ideal I of A can be written as a product of powers of prime ideals of A.*

For a proof of the above Theorem, one can refer [1], Chapter 3, Page 57-60. The above theorem can be seen as a unique factorization property for the non zero ideals of A. In the proofs the following elementary fact is used.

For some non zero prime ideal $\mathcal{P}$ and an ideal $\mathcal{I}$ of A $\mathcal{P} \mid \mathcal{I} \iff \mathcal{I} \subset \mathcal{P}$.

Throughout the section we denote the i-th coefficient of $f \in A[[x]]$ as $a_i^{(f)}$. Similar notation is used for the i-th coefficient of $g \in A[[x]]$.

Definition 3.3. The content of f is the ideal in A generated by $\left\{a_i^{(f)} | i \geq 0\right\}$. We denote it by $c(f)$.

Note that in Definition 2.2 we took the non negative generator of the ideal when $A = \mathbb{Z}$.

Lemma 3.1 (Gauss lemma for Dedekind domains)**.** *Let $f, g \in A[[x]]$. Then $c(fg) = c(f)c(g)$.*

Proof. If $f = 0$ or $g = 0$, then the equality is clear. Hence we can assume that $f, g \neq 0$. By Theorem 3.1 we can write $c(f)$ and $c(g)$ as product of powers of prime ideals, say as $c(f) = \prod_{i=1}^{k} \mathcal{P}_i^{e_i}$, $c(g) = \prod_{i=1}^{k} \mathcal{P}_i^{f_i}$, where $e_i, f_i \geq 0$ $e_i + f_i > 0$. We write the factorization in this way so as to have the same indexing set.

For any non zero prime ideal $\mathcal{P}$, consider the natural surjection $\Psi_{\mathcal{P}} : A[[x]] \to A/\mathcal{P}[[x]]$. If $f \in \mathrm{Ker}(\Psi_{\mathcal{P}})$, then $a_i \in \mathcal{P}$ for all i, thus $c(f) \subset \mathcal{P}$, that is $\mathcal{P}|c(f)$. We claim that $\mathcal{P}_1, \dots \mathcal{P}_k$ are the only divisors of $c(fg)$. In fact, $\Psi_{\mathcal{P}}(fg) = 0$ then

$$\Psi_{\mathcal{P}}(f)\Psi_{\mathcal{P}}(g) = 0 \implies \Psi_{\mathcal{P}}(f) = 0 \text{ or } \Psi_{\mathcal{P}}(g) = 0 \implies \mathcal{P}|c(f) \text{ or } \mathcal{P}|c(g).$$

The factorisation of $c(f)$ and $c(g)$ ensures the above claim. Also, for all $1 \leq i \leq k$, $a_j^{(f)} \in \mathcal{P}_i^{e_i}$ and $a_j^{(g)} \in \mathcal{P}_i^{f_i}$ for all j. Therefore,

$$\sum_{j+j'=n} a_j^{(f)} a_{j'}^{(g)} \in \mathcal{P}_i^{e_i+f_i} \quad \text{for all } n \implies c(fg) \subset \mathcal{P}_i^{e_i+f_i} \quad \text{for all} \quad 1 \leq i \leq k$$
$$\implies \prod_{i=1}^{k} \mathcal{P}_i^{e_i+f_i} | c(fg).$$

It remains to show that $\mathcal{P}_i^{e_i+f_i+1} \nmid c(fg)$. Assume the contrary. Let j, m be the smallest index of the coefficients of f and g such that $a_m^{(f)} \in \mathcal{P}_i^{e_i} \setminus \mathcal{P}_i^{e_i+1}$ and $a_j^{(g)} \in \mathcal{P}_i^{f_i} \setminus \mathcal{P}_i^{f_i+1}$. Set $k = m + j$ and consider the k^{th} coefficient of the product fg. By our assumption this is an element of $\mathcal{P}_i^{e_i+f_i+1}$. Observe that

$$\sum_{n=0}^{k} a_n^{(f)} a_{k-n}^{(g)} \in \mathcal{P}_i^{e_i+f_i+1} \implies a_m^{(f)} a_j^{(g)} + \sum_{n=0, n\neq m}^{k} a_n^{(f)} a_{k-n}^{(g)} \in \mathcal{P}_i^{e_i+f_i+1}$$
$$\implies a_m^{(f)} a_j^{(g)} \in \mathcal{P}_i^{e_i+f_i+1}.$$

We have a contradiction to the choice of m and j. Hence the theorem is proved. $\square$

4. Fatou's lemma for Dedekind domains

We now proceed to prove Theorem 1.2 by modifying the proof of Theorem 1.1. A proof of this theorem can also be found in [2], Page 15-16.

Proof of Theorem 1.2 Let $f(x) = \frac{g_1(x)}{h_1(x)} = \frac{a\widetilde{g}(x)}{b\widetilde{h}(x)}$ where $g_1(x)$ and $h_1(x) \in K[x]$ are co-prime, $a\widetilde{g}(x)$ and $b\widetilde{h}(x) \in A[x]$ for some a and $b \in A$. We can write $f(x) = \frac{g(x)}{h(x)}$ with $g(x)$ and $h(x) \in A[x]$ after cancellation of common factors and let $h(x) := \sum_{i=0}^{r} h_i x^i$. As in the proof of Theorem 1.1 we observe that $h(0) \neq 0$ and there exists $d \in A$ such that $\frac{d}{h(x)} \in A[[x]]$. Let $\widetilde{P}(x) := \frac{dx^{r-1}}{h(x)} = \sum_{n=0}^{\infty} b_n x^n$. By comparing the coefficients of the formal power series $\widetilde{P}(x)h(x)$ with dx^{r-1}, we get

$$h_i b_k = -\sum_{\substack{j=0 \\ j\neq i}}^{r} h_j b_{k-j+i} \quad = -\left(\sum_{j=0}^{i-1} h_j b_{k-j+i} + \sum_{i+1}^{r} h_j b_{k-j+i} \right), \tag{1}$$

for all $1 \leq i \leq r$, $k \geq r - 1$. We claim that $c(h) = (h_0)$. Suppose not. Let $\mathcal{P}$ be a non zero prime ideal such that $\mathcal{P}c(h)|(h_0)$. If

$$\mathcal{P}c(h)|(h_i) \tag{2}$$

for all $1 \leq i \leq r$, then we get a contradiction as this implies $\mathcal{P}c(h)|c(h)$, proving the claim. So assuming the contrary, it is enough to prove (2) and we prove it by induction.

Let $k \geq r-1$ be such that $\mathcal{P}c(\widetilde{P}) \nmid (b_k)$. In order to prove (2) for each i in the induction step, it is enough to prove that

$$\mathcal{P}c(\widetilde{P})c(h)|(h_i b_k). \tag{3}$$

We write $(h_i) = c(h)A_i$ and $(b_k) = c(\widetilde{P})B_k$, where A_i and B_k are ideals of A. By Lemma 3.1, we get $\mathcal{P} \mid (A_i B_k)$ and since $\mathcal{P} \nmid (b_k)$, $\mathcal{P} \mid A_i$ therefore proving (2) in the induction step.

We now prove the equation (2) for all $1 \leq i \leq r$ by induction. We denote

$$S_{<i} := \sum_{j=0}^{i-1} h_j b_{k-j+i} \quad \text{and} \quad S_{>i} := \sum_{j=i+1}^{r} h_j b_{k-j+i}.$$

Therefore equation (1) can be re-written as $h_i b_k = -(S_{<i} + S_{>i})$. For $i = 1$, we have $S_{<1} = h_0 b_k \in \mathcal{P}c(h)c(\widetilde{P})$ and from the minimality of k, $S_{>1} \in \mathcal{P}c(h)c(\widetilde{P})$. Therefore, $h_1 b_k \in \mathcal{P}c(h)c(\widetilde{P})$. Hence (2) is true for i=1. Suppose, (2) is true for $i < t$ where $t \leq r$. For $i = t$, we get $S_{<t} \in \mathcal{P}c(\widetilde{P})c(h)$ by the induction step $\mathcal{P}c(h)|(h_i)$ for $i < t$ and $S_{>t} \in \mathcal{P}c(\widetilde{P})c(h)$ by the minimality of k, i.e., $\mathcal{P}c(\widetilde{P})|(b_{k-j})$ for positive integer $j > 0$. Therefore, $h_i b_k \in \mathcal{P}c(h)c(\widetilde{P})$. Hence (2) is true for $i = t$. By induction, (2) is thus true for all $1 \leq i \leq r$. Therefore $c(h) = (h_0)$. Now we can write $h(x) = h_0 q(x) = h_0 \sum_{i=0}^{r} q_i x^i$ with q_0=1. It remains to show that $h_0 \in A^*$. Observe that

$$h(x)f(x) = g(x) \implies h_0 q(x) f(x) = g(x) \implies h_0 | g(x).$$

Since $g(x)$ and $h(x)$ have no common factors, this implies that h_0 is an invertible element in A. Writing $f(x) = \frac{h_0^{-1} g(x)}{h_0^{-1} h(x)} = \frac{h_0^{-1} g(x)}{q(x)}$, we get the theorem. □

We next proceed to answer Question 1.

5. Application of Fatou's lemma

A conjugate of an algebraic number α is a root of the minimal polynomial of α. In the following theorem, we call α, an algebraic number and let $K := \mathbb{Q}(\alpha)$. For theorems about splitting field and Galois Theory, one can refer [3].

Theorem 5.1. *If $Tr_{K/\mathbb{Q}}(\alpha^i) \in \mathbb{Z}$ for all $i \geq 1$, then α is an algebraic integer.*

Proof. Let $[K : \mathbb{Q}] = n$ and $f(x) := \prod_{i=1}^{n} (1 - \alpha_i x)$ where α_i are the distinct conjugates of α over $\mathbb{Q}$ with $\alpha_1 = \alpha$. Let L be the splitting field of the polynomial $f(x)$ over $\mathbb{Q}$. Consider the generating function of trace of powers $h(x) := \sum_{i=0}^{\infty} \mathrm{Tr}_{K/\mathbb{Q}}(\alpha^i) x^i$. We have $Tr_{K/\mathbb{Q}}(\alpha^i) = \sum_{j=1}^{n} \alpha_j^i$. Substituting this in the equation we get

$$h(x) = \sum_{i=0}^{\infty} \left(\sum_{j=1}^{n} \alpha_j^i \right) x^i = \sum_{j=1}^{n} \sum_{i=0}^{\infty} \alpha_j^i \; x^i.$$

When viewed this as an element of $L(x)$ we get

$$h(x) = \sum_{j=1}^{n} \frac{1}{1-\alpha_j x} = \frac{\sum\limits_{i=1}^{n} \prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (1-\sigma_j(\alpha)x)}{f(x)} = \frac{g(x)}{f(x)},$$

where σ_j are the distinct field embeddings of K in $\mathbb{C}$. The coefficients of the polynomial $g(x)$ is invariant for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ (as σ maps α_j to its conjugates) and hence is an element of $\mathbb{Q}[x]$. Also $g(x)$ and $f(x)$ have no common factors. It follows from Theorem 1.1 that $f(x)$ and $g(x) \in \mathbb{Z}[x]$ with $f(0) = 1$. Consider the reciprocal polynomial $x^n f(\frac{1}{x}) = \prod_{i=1}^{n}(x - \alpha_i)$. Clearly it is the minimal monic polynomial of α with coefficients in $\mathbb{Z}[x]$. Hence α is an algebraic integer. $\square$

6. Complementary modules

Before starting this section, we recall that any submodule of a finitely generated module over a Principal ideal domain is finitely generated. The statement along with a few basic properties of complementary modules will help us prove Theorem 5.1. Let A be a Dedekind domain, K its field of fractions, E be a finite extension of K. The trace of an element β in E is the trace of the linear transformation $Tr_\beta : E \mapsto E$, $Tr_\beta(x) := \beta x$. It is denoted by $Tr_K^E(\beta)$.

Remark 6.1. When $A = \mathbb{Z}$, both the definitions of Trace are equivalent.

Definition 6.1 (Complimentary module)**.** Let L be an additive subgroup of E which is also an A module. We define the complementary module L' as $L' = \{x \in E \mid \mathrm{Tr}_K^E(xL) \subset A\}$.

Note that L' is also an A module.

Proposition 6.1. *Let $\alpha_1 \ldots \alpha_k$ be an basis of E over K. If $L = A\alpha_1 + \cdots + A\alpha_k$ then $L' = A\alpha_1' + \cdots + A\alpha_k'$ where α_i' are dual basis with respect to the trace.*

The proof of this proposition can be found in [4], Chapter 3, Page 57-58. Along with this we also mention an equivalent condition for checking, whether a number is algebraic integer.

Proposition 6.2. *The following are equivalent for a complex number α*

(1) *α is an algebraic integer.*
(2) *$\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$ module.*

Refer [1], Chapter 2, Page 15-16 for the proof. Now we present a proof of Theorem 5.1.

Proof of Theorem 5.1. Let $L = \mathbb{Z}\alpha + \cdots + \mathbb{Z}\alpha^{n-1}$ where $n = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. From the hypothesis, the complementary module L' contains $\mathbb{Z}[\alpha]$. By Proposition 6.1, L' is finitely generated as a $\mathbb{Z}$ module, therefore we see that $\mathbb{Z}[\alpha]$ is finitely generated as a $\mathbb{Z}$ module and hence by Proposition 6.2 we prove that α is an algebraic integer. $\square$

This was proved independently by Hendrik Lenstra and Paul Ponomarev. The above theorem was improved by Bart De Smit who showed that it is enough to check the Traces of algebraic number α for the powers till $n + n\log_2 n$, where $n = \mathbb{Q}(\alpha)$, to find out whether α is an algebraic integer. We mention the precise statement below.

Let A be a Dedekind domain with quotient field K of characteristic 0. For each prime $\mathbf{p}$ of K, let $\mathbf{v_p} : K^* \mapsto \mathbb{Z}$ be the normalised $\mathbf{p}$ adic valuation.

Theorem 6.1. *Let n be a positive integer. Define b to be the maximum of the numbers $m + m\mathbf{v_p}(\mathbf{m})$ where $\mathbf{p}$ ranges over all the primes of K and $m \in \{1, 2, \ldots n\}$. Let L be a K module of dimension n and $\alpha \in L$. If there exists $a \geq b$ such that $\mathrm{Tr}_K^L(\alpha^i) \in A$ for all i with $a - n \leq i \leq a$, then α is integral over A.*

The proof uses results from Local fields which is beyond the scope of the present article. More details about the proof can be found in [5].

Remark 6.2. We note that the bound is strict, in the sense that a smaller bound does not guarantee that α is integral. This can be shown by taking $\alpha = \frac{1}{\sqrt{2}}$. We see that α is not an algebraic integer as the minimal polynomial of α over $\mathbb{Q}$ is f(x) $= x^2 - \frac{1}{2}$. Here the upper bound is four and we observe that $\mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}(\alpha)}(\alpha^i)$ is an integer for $1 \leq i \leq 3$.

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IN EULER'S FOOTSTEPS: EVALUATION OF SOME SERIES INVOLVING $\zeta(2)$ AND G

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Abstract. We present here some series obtained by leaving out terms from the $\zeta(2)$ series and changing the sign scheme in the original series. Some of Euler's results are used for establishing our formulas. Similar series involving Catalan's constant G are treated at the end.

1. Introduction

Leonhard Euler (1707–1783) gained instant fame by solving the Basel problem which asked for the exact sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. First posed in 1644 by Pietro Mengoli, the problem figured in Jacob (also known as James or Jacques) Bernoulli's 1689 treatise *Tractatus de seriebus infinitis* that formed pages 241–306 of his posthumous work *Ars Conjectandi* (1713) published in Basel, the hometown of Euler and the Bernoullis. Stating the problem, Bernoulli wrote on p.254:

> at, quod notatu dignum, quando ſunt puri Quadrati, ut in ſerie $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$ &c. difficilior eſt, quàm quis expectaverit, ſummæ perveſtigatio, quam tamen finitam eſſe, ex altera, qua manifeſto minor eſt, colligimus: Si quis inveniat nobisque communicet, quod induſtriam noſtram eluſit hactenus, magnas de nobis gratias feret.

The last sentence translated into English (by the author) reads as: "*If anyone should find and communicate to us, what has so far eluded our efforts, we shall be highly grateful to him.*"

Stirling [13, p.29] computed $\zeta(2)$ to nine (8 correct) decimal places but could not find its closed form. Jacob Bernoulli's younger brother Johann Bernoulli tried but failed and suggested it to Euler who began initially by computing numerical approximations of some of the partial sums of the series and found various expressions for the desired sum (definite integrals, other series) but these did not help. He eventually cracked the problem in 1734 by analogy using an audacious approach — extending the logic valid for a polynomial of finite degree to an infinite series treated as a polynomial of infinite degree. Taking the Taylor series

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expansion of the sine function, he divided its two sides by the argument to get:

$$\frac{\sin x}{x} = \frac{x^0}{1!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

Treating the right hand side as a polynomial in x, and using the fact that the zeros of the left hand side are precisely $\pm\pi$, $\pm 2\pi$, $\pm 3\pi, \dots$, he wrote:

$$\begin{aligned} \frac{\sin x}{x} &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots \qquad (1) \\ &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\cdots \end{aligned}$$

Therefore,

$$\frac{x^0}{1!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\cdots$$

Setting $x^2 = y$, the polynomial reduces to a polynomial in y whose roots are $\pi^2, 4\pi^2, 9\pi^2, \cdots$. Since the constant term in the polynomial is $+1$, the negative of the linear term, namely $\frac{1}{3!}$, equals the sum of the reciprocals of all roots. Hence,

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots = \frac{1}{6},$$

that is,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

But Euler's fruitful method was open to serious objection. Granting that $\frac{\sin x}{x}$ can be expressed as an infinite product of linear factors corresponding to the non-zero roots of the equation $\sin x = 0$, there is a lurking possibility that all roots may not be real; in that case the whole exercise would be invalid. Realizing this, Euler worked to validate his method. Sandifer [12] remarks that Euler gave four distinct solutions to the Basel problem — three in [5] and a fourth in [6]. Many more have appeared since then. We evaluate $\zeta(2)$ by using Euler's formula given in [7, Ch.X, §178]

$$\frac{\pi}{n}\frac{\cos\frac{m\pi}{n}}{\sin\frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \cdots.$$

Multiplying both sides of the formula by n and setting $\frac{m}{n} = x$ gives

$$\pi\cot(\pi x) = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \dots . \qquad (2)$$

Note that this equation can be deduced from (1): After multiplying by x, we take the logarithmic derivative of both sides and replace x by πx.

Differentiating (2) with respect to x yields

$$\pi^2\csc^2(\pi x) = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \dots$$

for all $x \neq k\pi$ with k integer (for a rigorous proof, see [9]).

Setting $x = \frac{1}{2}$ in the above expansion, we get

$$\pi^2 = \frac{4}{1} + \frac{4}{1} + \frac{4}{3^2} + \frac{4}{3^2} + \frac{4}{5^2} + \frac{4}{5^2} + \cdots \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

We now get $\zeta(2)$ by using the relation $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$.

In the rest of the paper, we will evaluate series whose general terms consist of reciprocals of two squares and the sums, involving $\zeta(2)$ and Catalan's constant, can be written with only real radicals.

2. Non-alternating series

Recall the trigamma function denoted by $\psi'(z)$, $z \neq 0, -1, -2, \dots$:

$$\psi'(z) = \frac{d}{dz}\psi(z);\ \psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+z}\right),$$

and defined by the infinite series [11, p.144, 5.15.1]: $\psi'(z) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+z)^2}$.

Some special values are: $\psi'(1) = \dfrac{\pi^2}{6}$, $\psi'\left(\dfrac{1}{2}\right) = \dfrac{\pi^2}{2}$, $\psi'\left(\dfrac{1}{4}\right) = \pi^2 + 8G$, where $G = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ is Catalan's constant.

The trigamma function satisfies the reflection formula [11, p.144, 5.15.6]

$$\psi'(1-z) + \psi'(z) = -\pi\frac{d}{dz}\cot(\pi z) = \frac{\pi^2}{\sin^2(\pi z)}.$$

For $1 \le \ell < m$ and $gcd(\ell, m) = 1$, setting $z = \frac{\ell}{m}$ $(m = 2, 3, 4, \dots)$ yields

Theorem 1.

$$\sum_{k=0}^{\infty}\left[\frac{1}{(mk+\ell)^2} + \frac{1}{(mk+(m-\ell))^2}\right] = \left(\frac{\pi/m}{\sin(\ell\pi/m)}\right)^2.$$

Now $m = 2, 3, 4, 6$ give these four formulas (also deducible from Euler's $\zeta(2)$):

$$\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)^2} + \frac{1}{(2k+1)^2}\right] = \frac{\pi^2}{4}, \tag{3}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(3k+1)^2} + \frac{1}{(3k+2)^2}\right] = \frac{4\pi^2}{27}, \tag{4}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(4k+1)^2} + \frac{1}{(4k+3)^2}\right] = \frac{\pi^2}{8}, \tag{5}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(6k+1)^2} + \frac{1}{(6k+5)^2}\right] = \frac{\pi^2}{9}. \tag{6}$$

Further, $m = 5, 8, 10, 12, 15, 16, 20, 24, 30$ result in the following.

$$\sum_{k=0}^{\infty}\left[\frac{1}{(5k+1)^2}+\frac{1}{(5k+4)^2}\right]=\frac{2(5+\sqrt{5})\,\pi^2}{125}, \tag{7}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(5k+2)^2}+\frac{1}{(5k+3)^2}\right]=\frac{2(5-\sqrt{5})\,\pi^2}{125}. \tag{8}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(8k+1)^2}+\frac{1}{(8k+7)^2}\right]=\frac{(2+\sqrt{2})\,\pi^2}{32}, \tag{9}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(8k+3)^2}+\frac{1}{(8k+5)^2}\right]=\frac{(2-\sqrt{2})\,\pi^2}{32}. \tag{10}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(10k+1)^2}+\frac{1}{(10k+9)^2}\right]=\frac{(3+\sqrt{5})\,\pi^2}{50}, \tag{11}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(10k+3)^2}+\frac{1}{(10k+7)^2}\right]=\frac{(3-\sqrt{5})\,\pi^2}{50}. \tag{12}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(12k+1)^2}+\frac{1}{(12k+11)^2}\right]=\frac{(2+\sqrt{3})\,\pi^2}{36}, \tag{13}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(12k+5)^2}+\frac{1}{(12k+7)^2}\right]=\frac{(2-\sqrt{3})\,\pi^2}{36}. \tag{14}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(15k+1)^2}+\frac{1}{(15k+14)^2}\right]=\frac{2\left(4+\sqrt{5}+\sqrt{3(5+2\sqrt{5})}\right)\pi^2}{225}, \tag{15}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(15k+2)^2}+\frac{1}{(15k+13)^2}\right]=\frac{2\left(4-\sqrt{5}+\sqrt{3(5-2\sqrt{5})}\right)\pi^2}{225}, \tag{16}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(15k+4)^2}+\frac{1}{(15k+11)^2}\right]=\frac{2\left(4+\sqrt{5}-\sqrt{3(5+2\sqrt{5})}\right)\pi^2}{225}, \tag{17}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(15k+7)^2}+\frac{1}{(15k+8)^2}\right]=\frac{2\left(4-\sqrt{5}-\sqrt{3(5-2\sqrt{5})}\right)\pi^2}{225}. \tag{18}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(16k+1)^2}+\frac{1}{(16k+15)^2}\right]=\frac{(2+\sqrt{2})(2+\sqrt{2+\sqrt{2}})\,\pi^2}{128}, \tag{19}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(16k+3)^2}+\frac{1}{(16k+13)^2}\right]=\frac{(2-\sqrt{2})(2+\sqrt{2-\sqrt{2}})\,\pi^2}{128}, \tag{20}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(16k+5)^2}+\frac{1}{(16k+11)^2}\right]=\frac{(2-\sqrt{2})(2-\sqrt{2-\sqrt{2}})\,\pi^2}{128}, \tag{21}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(16k+7)^2}+\frac{1}{(16k+9)^2}\right]=\frac{(2+\sqrt{2})(2-\sqrt{2+\sqrt{2}})\,\pi^2}{128}. \tag{22}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(20k+1)^2}+\frac{1}{(20k+19)^2}\right]=\frac{(3+\sqrt{5})(4+\sqrt{10+2\sqrt{5}})\pi^2}{400}, \tag{23}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(20k+3)^2}+\frac{1}{(20k+17)^2}\right]=\frac{(3-\sqrt{5})(4+\sqrt{10-2\sqrt{5}})\pi^2}{400}, \tag{24}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(20k+7)^2}+\frac{1}{(20k+13)^2}\right]=\frac{(3-\sqrt{5})(4-\sqrt{10-2\sqrt{5}})\pi^2}{400}, \tag{25}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(20k+9)^2}+\frac{1}{(20k+11)^2}\right]=\frac{(3+\sqrt{5})(4-\sqrt{10+2\sqrt{5}})\pi^2}{400}. \tag{26}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(24k+1)^2}+\frac{1}{(24k+23)^2}\right]=\frac{(2+\sqrt{3})(4+\sqrt{2}+\sqrt{6})\pi^2}{288}, \tag{27}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(24k+5)^2}+\frac{1}{(24k+19)^2}\right]=\frac{(2-\sqrt{3})(4-\sqrt{2}+\sqrt{6})\pi^2}{288}, \tag{28}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(24k+7)^2}+\frac{1}{(24k+17)^2}\right]=\frac{(2-\sqrt{3})(4+\sqrt{2}-\sqrt{6})\pi^2}{288}, \tag{29}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(24k+11)^2}+\frac{1}{(24k+13)^2}\right]=\frac{(2+\sqrt{3})(4-\sqrt{2}-\sqrt{6})\pi^2}{288}. \tag{30}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(30k+1)^2}+\frac{1}{(30k+29)^2}\right]=$$
$$\frac{(7+3\sqrt{5})\left((9-\sqrt{5})+\sqrt{3(10+2\sqrt{5})}\right)\pi^2}{1800}, \tag{31}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(30k+7)^2}+\frac{1}{(30k+23)^2}\right]=$$
$$\frac{(7-3\sqrt{5})\left((9+\sqrt{5})+\sqrt{3(10-2\sqrt{5})}\right)\pi^2}{1800}, \tag{32}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(30k+11)^2}+\frac{1}{(30k+19)^2}\right]=$$
$$\frac{(7+3\sqrt{5})\left((9-\sqrt{5})-\sqrt{3(10+2\sqrt{5})}\right)\pi^2}{1800}, \tag{33}$$

$$\sum_{k=0}^{\infty}\left[\frac{1}{(30k+13)^2}+\frac{1}{(30k+17)^2}\right]=$$

$$\frac{(7-3\sqrt{5})\left((9+\sqrt{5})-\sqrt{3(10-2\sqrt{5})}\right)\pi^2}{1800}. \tag{34}$$

Further series with $m = 2^n$, $m = 3 \cdot 2^n$ and $m = 5 \cdot 2^n$ can also be evaluated by using the three general formulae:

$$\cos\frac{\pi}{2^{n+1}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{2}}}}}}{2},$$

where 2 under the square root occurs n times, with $n \in \mathbb{N}$.

$$\cos\frac{\pi}{3\cdot 2^{n}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{3}}}}}}{2},$$

where 2 under the square root occurs $n-1$ times, with $n \in \mathbb{N}$.

$$\cos\frac{\pi}{5\cdot 2^{n}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\frac{\sqrt{10+2\sqrt{5}}}{2}}}}}}{2},$$

where 2 under the square root occurs $n-1$ times, with $n \in \mathbb{N}$.

3. Alternating series

Again, for $1 \le \ell < m$ and $gcd(\ell, m) = 1$ $(m = 2, 3, 4, \ldots)$, we have

Theorem 2.

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(mk+\ell)^2}-\frac{1}{(mk+(m-\ell))^2}\right] = \left(\frac{\pi/m}{\sin(\ell\pi/m)}\right)^2\cos\frac{\ell\pi}{m}.$$

Proof. We give a proof of Theorem 2 for $\ell = 1$. Analogous proofs can be devised for other values as well. Observe that Euler gave this partial fraction expansion in [7, Ch.X, §178]:

$$\frac{\pi}{n\sin\frac{m\pi}{n}} = \frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}-\cdots$$

which on putting $m = 1$ and replacing $\frac{\pi}{n}$ by z gives

$$\frac{z}{\sin z} = 1+\frac{z}{\pi-z}-\frac{z}{\pi+z}-\frac{z}{2\pi-z}+\frac{z}{2\pi+z}+\frac{z}{3\pi-z}-\frac{z}{3\pi+z}-\cdots$$

that is,

$$\frac{1}{\sin z} = \sum_{k=-\infty}^{+\infty}\frac{(-1)^k}{z-k\pi}. \tag{35}$$

Taking the derivative of the two sides of (35) with respect to z yields

$$\frac{\cos z}{\sin^2 z} = \sum_{k=-\infty}^{+\infty}\frac{(-1)^k}{(z-k\pi)^2}$$

for $z \neq n\pi$ with n integer (see [9] for a rigorous proof), which on putting $z = -\frac{\pi}{m}$ and then multiplying both sides by $\left(\frac{\pi}{m}\right)^2$ becomes

$$\left(\frac{\pi/m}{\sin(\pi/m)}\right)^2 \cos\frac{\pi}{m} = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(km+1)^2}.$$

We may rewrite the right hand side as

$$\sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(km+1)^2} = \sum_{k=-\infty}^{-1} \frac{(-1)^k}{(km+1)^2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(km+1)^2}$$

and

$$\sum_{k=-\infty}^{-1} \frac{(-1)^k}{(km+1)^2} = \sum_{k=1}^{\infty} \frac{1}{(-1)^k(-km+1)^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(km-1)^2}$$

which equals the second term in the left hand side of Theorem 2 because

$$-\sum_{k=0}^{\infty} \frac{(-1)^k}{(mk+(m-1))^2} = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(m(k-1)+(m-1))^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(mk-1)^2}.$$

□

For $m = 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30$ we get these sums

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(3k+1)^2} - \frac{1}{(3k+2)^2}\right] = \frac{2\pi^2}{27}. \tag{36}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(4k+1)^2} - \frac{1}{(4k+3)^2}\right] = \frac{\sqrt{2}\,\pi^2}{16}. \tag{37}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(5k+1)^2} - \frac{1}{(5k+4)^2}\right] = \frac{\pi^2}{25}\left(\frac{3}{\sqrt{5}}+1\right), \tag{38}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(5k+2)^2} - \frac{1}{(5k+3)^2}\right] = \frac{\pi^2}{25}\left(\frac{3}{\sqrt{5}}-1\right). \tag{39}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(6k+1)^2} - \frac{1}{(6k+5)^2}\right] = \frac{\sqrt{3}\,\pi^2}{18}. \tag{40}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(8k+1)^2} - \frac{1}{(8k+7)^2}\right] = \frac{\pi^2(2+\sqrt{2})^{\frac{3}{2}}}{64}, \tag{41}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(8k+3)^2} - \frac{1}{(8k+5)^2}\right] = \frac{\pi^2(2-\sqrt{2})^{\frac{3}{2}}}{64}. \tag{42}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(10k+1)^2} - \frac{1}{(10k+9)^2}\right] = \frac{\pi^2}{10\sqrt{10}}\sqrt{5+\frac{11}{\sqrt{5}}}, \tag{43}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(10k+3)^2} - \frac{1}{(10k+7)^2}\right] = \frac{\pi^2}{10\sqrt{10}}\sqrt{5-\frac{11}{\sqrt{5}}}. \tag{44}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(12k+1)^2} - \frac{1}{(12k+11)^2}\right] = \frac{\pi^2\sqrt{2}}{288}\left(\sqrt{3}+1\right)^3, \tag{45}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(12k+5)^2}-\frac{1}{(12k+7)^2}\right]=\frac{\pi^2\sqrt{2}}{288}\left(\sqrt{3}-1\right)^3. \tag{46}$$

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k&\left[\frac{1}{(15k+1)^2}-\frac{1}{(15k+14)^2}\right]\\&=\frac{\left(\sqrt{6}(5+\sqrt{5})^{3/2}+(3\sqrt{5}\ +1)+3\sqrt{10(7+3\sqrt{5})}\right)\pi^2}{900},\end{aligned} \tag{47}$$

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k&\left[\frac{1}{(15k+2)^2}-\frac{1}{(15k+13)^2}\right]\\&=\frac{\left(\sqrt{6}(5-\sqrt{5})^{3/2}+(3\sqrt{5}\ -1)+3\sqrt{10(7-3\sqrt{5})}\right)\pi^2}{900},\end{aligned} \tag{48}$$

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k&\left[\frac{1}{(15k+4)^2}-\frac{1}{(15k+11)^2}\right]\\&=\frac{\left(\sqrt{6}(5+\sqrt{5})^{3/2}-(3\sqrt{5}\ +1)-3\sqrt{10(7+3\sqrt{5})}\right)\pi^2}{900},\end{aligned} \tag{49}$$

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k&\left[\frac{1}{(15k+7)^2}-\frac{1}{(15k+8)^2}\right]\\&=\frac{\left(\sqrt{6}(5-\sqrt{5})^{3/2}-(3\sqrt{5}\ -1)-3\sqrt{10(7-3\sqrt{5})}\right)\pi^2}{900}.\end{aligned} \tag{50}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(16k+1)^2}-\frac{1}{(16k+15)^2}\right]=\frac{\pi^2(2+\sqrt{2})}{256}\left(\sqrt{2+\sqrt{2+\sqrt{2}}}\right)^3, \tag{51}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(16k+3)^2}-\frac{1}{(16k+13)^2}\right]=\frac{\pi^2(2-\sqrt{2})}{256}\left(\sqrt{2+\sqrt{2-\sqrt{2}}}\right)^3, \tag{52}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(16k+5)^2}-\frac{1}{(16k+11)^2}\right]=\frac{\pi^2(2-\sqrt{2})}{256}\left(\sqrt{2-\sqrt{2-\sqrt{2}}}\right)^3, \tag{53}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(16k+7)^2}-\frac{1}{(16k+9)^2}\right]=\frac{\pi^2(2+\sqrt{2})}{256}\left(\sqrt{2-\sqrt{2+\sqrt{2}}}\right)^3. \tag{54}$$

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k&\left[\frac{1}{(20k+1)^2}-\frac{1}{(20k+19)^2}\right]\\&=\left(\frac{\pi(1+\sqrt{5})}{80}\right)^2\left(\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}\right)^3,\end{aligned} \tag{55}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(20k+3)^2}-\frac{1}{(20k+17)^2}\right] = \left(\frac{\pi(\sqrt{5}-1)}{80}\right)^2\left(\sqrt{3-\sqrt{5}}+\sqrt{5+\sqrt{5}}\right)^3, \tag{56}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(20k+7)^2}-\frac{1}{(20k+13)^2}\right] = \left(\frac{\pi(\sqrt{5}-1)}{80}\right)^2\left(\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}\right)^3, \tag{57}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(20k+9)^2}-\frac{1}{(20k+11)^2}\right] = \left(\frac{\pi(\sqrt{5}+1)}{80}\right)^2\left(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}\right)^3. \tag{58}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(24k+1)^2}-\frac{1}{(24k+23)^2}\right] = \left(\frac{\pi(1+\sqrt{3})}{48}\right)^2\sqrt{2}\left(4+\sqrt{2}+\sqrt{6}\right)^{\frac{3}{2}}, \tag{59}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(24k+5)^2}-\frac{1}{(24k+19)^2}\right] = \left(\frac{\pi(\sqrt{3}-1)}{48}\right)^2\sqrt{2}\left(4-\sqrt{2}+\sqrt{6}\right)^{\frac{3}{2}}, \tag{60}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(24k+7)^2}-\frac{1}{(24k+17)^2}\right] = \left(\frac{\pi(\sqrt{3}-1)}{48}\right)^2\sqrt{2}\left(4+\sqrt{2}-\sqrt{6}\right)^{\frac{3}{2}}, \tag{61}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(24k+11)^2}-\frac{1}{(24k+13)^2}\right] = \left(\frac{\pi(\sqrt{3}+1)}{48}\right)^2\sqrt{2}\left(4-\sqrt{2}-\sqrt{6}\right)^{\frac{3}{2}}. \tag{62}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(30k+1)^2}-\frac{1}{(30k+29)^2}\right] = \frac{\left(\sqrt{3}(11+5\sqrt{5})+\sqrt{2(65+29\sqrt{5})}\right)\left((9-\sqrt{5})+\sqrt{3(10+2\sqrt{5})}\right)\pi^2}{7200}, \tag{63}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(30k+7)^2}-\frac{1}{(30k+23)^2}\right] = \frac{\left(\sqrt{3}(-11+5\sqrt{5})+\sqrt{2(65-29\sqrt{5})}\right)\left((9+\sqrt{5})+\sqrt{3(10-2\sqrt{5})}\right)\pi^2}{7200}, \tag{64}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(30k+11)^2}-\frac{1}{(30k+19)^2}\right]=$$
$$\frac{\left(\sqrt{3}(11+5\sqrt{5})-\sqrt{2(65+29\sqrt{5})}\right)\left((9-\sqrt{5})-\sqrt{3(10+2\sqrt{5})}\right)\pi^2}{7200}, \tag{65}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(30k+13)^2}-\frac{1}{(30k+17)^2}\right]=$$
$$\frac{\left(\sqrt{3}(11-5\sqrt{5})+\sqrt{2(65-29\sqrt{5})}\right)\left((9+\sqrt{5})-\sqrt{3(10-2\sqrt{5})}\right)\pi^2}{7200}. \tag{66}$$

Remark. Choi [4] gives a long proof (involving integrals) of (37) while (41), (45), (51), (55) occur differently as formulas (10)–(13) without proofs.

Note that (39) can easily be established by using earlier results:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(5k+2)^2}-\frac{1}{(5k+3)^2}\right]\\
&\quad=\sum_{k=0}^{\infty}\left((-1)^{2k}\left[\frac{1}{(10k+2)^2}-\frac{1}{(10k+3)^2}\right]\right.\\
&\qquad\qquad\left.+(-1)^{2k+1}\left[\frac{1}{(10k+7)^2}-\frac{1}{(10k+8)^2}\right]\right)\\
&\quad=\sum_{k=0}^{\infty}\left[\frac{1}{(10k+2)^2}-\frac{1}{(10k+3)^2}-\frac{1}{(10k+7)^2}+\frac{1}{(10k+8)^2}\right]\\
&\quad=\frac{1}{4}\sum_{k=0}^{\infty}\left[\frac{1}{(5k+1)^2}+\frac{1}{(5k+4)^2}\right]-\sum_{k=0}^{\infty}\left[\frac{1}{(10k+3)^2}+\frac{1}{(10k+7)^2}\right]\\
&\quad=\frac{(5+\sqrt{5})\,\pi^2}{250}-\frac{(3-\sqrt{5})\,\pi^2}{50}=\frac{(3\sqrt{5}-5)\,\pi^2}{125}.
\end{aligned}$$

4. Series involving Catalan's constant

Catalan's constant was introduced in 1865 by the Franco-Belgian mathematician Eugène Charles Catalan (1814-94). He used the symbol G to denote the constant and defined it on page 20 of [2] by means of an alternating series and on page 33 by means of an integral

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\int_0^1\frac{\arctan x}{x}dx.$$

Mark the close connection of the defining formula G and $\frac{\pi^2}{8}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. The constant G is important in enumerative combinatorics and commonly appears in combinatorial problems. It also occurs occasionally in certain classes of sums and definite integrals. Finch finds “a fascinating and unexpected occurrence” of Catalan's constant [8, pp.5-6]. The constant also pops up frequently in relation to the Clausen function and in the values of the trigamma function at fractional arguments. It is also known to surface in 3-manifold geometric topology where it is a rational multiple of the volume of an ideal hyperbolic octahedron, and therefore of the hyperbolic volume of the complement of the Whitehead link. The hyperbolic volume of the complement of the Whitehead link is 4 times Catalan's constant. It is "arguably the most basic constant whose irrationality and transcendence (though strongly suspected) remain unproven."[1, p.849] The irrationality of G would imply that the volume of the unique hyperbolic structure on the Whitehead link complement is irrational. Till date, it is not known that any hyperbolic 3-manifold has irrational volume.

Catalan investigated G again in 1883 [3]. The Inverse tangent integral was studied by Glaisher and in particular by Ramanujan. Mathematicians including Ramanujan looked for rapidly converging series for G. The first author worked on a series of Ramanujan in [10]. Both authors have recently found some fast converging series (not yet published) for this constant.

All the series given above are of this form

$$1\left\{\begin{matrix}+\\-\\+0\cdot\end{matrix}\right\}\frac{1}{2^2}\left\{\begin{matrix}+\\-\\+0\cdot\end{matrix}\right\}\frac{1}{3^2}\left\{\begin{matrix}+\\-\\+0\cdot\end{matrix}\right\}\frac{1}{4^2}\left\{\begin{matrix}+\\-\\+0\cdot\end{matrix}\right\}\cdots\left\{\begin{matrix}+\\-\\+0\cdot\end{matrix}\right\}\frac{1}{n^2}\left\{\begin{matrix}+\\-\\+0\cdot\end{matrix}\right\}\cdots$$

where for each term a choice is made: a plus sign, a minus sign or skip the next term. (The series defining Catalan's constant is also of this form.) For instance, for the series (36)

$$\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}-\frac{1}{10^2}+\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{14^2}-\frac{1}{16^2}+\frac{1}{17^2}+\ldots$$

the choice made is a repetition of the pattern $-\,(+0\cdot)\,-\,+\,(+0\cdot)\,+$, since the series is obtained from the original Basel series by changing the sign of the second term, omitting the next term, again changing the sign of the next one and so on.

Using a similar trick as the one used in the introduction, we can find the sum of other similar series. For instance, it is easy to see that

$$\begin{aligned}S &= 1+\frac{1}{3^2}-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}-\frac{1}{13^2}-\frac{1}{15^2}+\frac{1}{17^2}+\cdots\\ &= 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots-2\left(\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{13^2}+\frac{1}{15^2}+\cdots\right)\\ &= \frac{\pi^2}{8}-2\sum_{k=0}^{\infty}\left[\frac{1}{(8k+5)^2}+\frac{1}{(8k+7)^2}\right].\end{aligned}$$

Note that we can rearrange S also as follows

$$S=1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots\ +2\left(\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{11^2}-\frac{1}{13^2}+\frac{1}{19^2}-\frac{1}{21^2}+\cdots\right)$$

$$= G + 2\sum_{k=0}^{\infty}\left[\frac{1}{(8k+3)^2} - \frac{1}{(8k+5)^2}\right].$$

Equating the two forms of S we can derive

$$\sum_{k=0}^{\infty}\left[\frac{1}{(8k+3)^2} + \frac{1}{(8k+7)^2}\right] = \frac{1}{2}\left(\frac{\pi^2}{8} - G\right) \tag{67}$$

which is equivalent to

$$\sum_{k=0}^{\infty}\frac{1}{(4k+3)^2} = \frac{1}{2}\left(\frac{\pi^2}{8} - G\right) \quad \Leftrightarrow \quad \psi'\left(\frac{3}{4}\right) = \pi^2 - 8G.$$

Formula (67) combined with (3) then yields

$$\sum_{k=0}^{\infty}\left[\frac{1}{(8k+1)^2} + \frac{1}{(8k+5)^2}\right] = \frac{1}{2}\left(\frac{\pi^2}{8} + G\right) \tag{68}$$

or

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)^2} = \frac{1}{2}\left(\frac{\pi^2}{8} + G\right) \quad \Leftrightarrow \quad \psi'\left(\frac{1}{4}\right) = \pi^2 + 8G.$$

We now use (67) and (10) to deduce

$$\sum_{k=0}^{\infty}\left[\frac{1}{(8k+5)^2} - \frac{1}{(8k+7)^2}\right] = \frac{1}{2}\left(G - \frac{\pi^2\sqrt{2}}{16}\right). \tag{69}$$

Formula (68) combined with (10) gives

$$\sum_{k=0}^{\infty}\left[\frac{1}{(8k+1)^2} - \frac{1}{(8k+3)^2}\right] = \frac{1}{2}\left(G + \frac{\pi^2\sqrt{2}}{16}\right). \tag{70}$$

Similarly, using (67) we obtain

$$\begin{aligned}
T &= 1 + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} - \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{19^2} - \cdots \\
&= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots - 2\left(\frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{19^2} + \cdots\right) \\
&= \frac{\pi^2}{8} - 2\sum_{k=0}^{\infty}\left[\frac{1}{(12k+7)^2} + \frac{1}{(12k+11)^2} + \frac{1}{9(8k+3)^2} + \frac{1}{9(8k+7)^2}\right] \\
&= \frac{\pi^2}{8} - 2\sum_{k=0}^{\infty}\left[\frac{1}{(12k+7)^2} + \frac{1}{(12k+11)^2}\right] - \frac{1}{9}\left(\frac{\pi^2}{8} - G\right).
\end{aligned}$$

But we also have

$$\begin{aligned}
T &= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots + 2\left(\frac{1}{3^2} - \frac{1}{9^2} + \frac{1}{15^2} - \frac{1}{21^2} + \cdots\right) \\
&= G + \frac{2}{9}G = \frac{11}{9}G.
\end{aligned}$$

Equating both results one gets

$$\sum_{k=0}^{\infty}\left[\frac{1}{(12k+7)^2}+\frac{1}{(12k+11)^2}\right]=\frac{1}{18}(\pi^2-10G) \tag{71}$$

and, using (13) and (14) one gets

$$\sum_{k=0}^{\infty}\left[\frac{1}{(12k+1)^2}+\frac{1}{(12k+5)^2}\right]=\frac{1}{18}(\pi^2+10G). \tag{72}$$

In terms of ψ' this can be expressed as

$$\psi'\left(\frac{1}{12}\right)+\psi'\left(\frac{5}{12}\right)=8\pi^2+80G, \quad \psi'\left(\frac{7}{12}\right)+\psi'\left(\frac{11}{12}\right)=8\pi^2-80G. \tag{73}$$

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A NEW CLASS OF FUNCTIONS SUGGESTED BY THE Q-HYPERGEOMETRIC FUNCTIONS

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Abstract. In the present work, we introduce a class of functions which extends the well-known basic hypergeometric function ${}_1\phi_1[z]$ with the aid of a q-analogue of entire function $f(z) = \sum_{n\geq 1} \frac{z^n}{n!^n}$ considered by P. C. Sikkema [Differential Operators and Equations, Djakarta, P. Noordhoff N. V.,1953]. For this function, the difference equation, eigen function property and the contiguous functions relations are derived. The work characterizes the q-Exponential function and hence the trigonometric and hyperbolic functions.

1. Introduction

Let $0 < q < 1$. A q-analogue of the factorial function

$$(a)_n = \begin{cases} 1, & \text{if } n = 0 \\ (a+1)(a+2)...(a+n-1), & \text{if } n \in \mathbb{N}, \end{cases} \tag{1}$$

is defined by

$$(q^a;q)_n = \begin{cases} 1, & \text{if } n = 0 \\ (1-q^a)(1-q^{a+1})...(1-q^{a+n-1}), & \text{if } n \in \mathbb{N}. \end{cases} \tag{2}$$

Also, in the notation [3, Eq.(1.2.29), p.6]

$$(q^a;q)_\infty = \prod_{n\geq 0}(1-q^{a+n}),$$

we have [3, Eq.(1.2.30), p.6]

$$(q^a;q)_n = \frac{(q^a;q)_\infty}{(q^{a+n};q)_\infty}$$

for arbitrary n. The q-Gamma function defined by [3, Eq.(1.10.1). p.16]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\ (1-q)^{1-x},$$

admits an alternative expression:

$$(q^a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}(1-q)^n \tag{3}$$

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which verifies as $q \to 1^-$, the well known relation: $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$. As a justification to call $(q^a;q)_n$ a q-analogue of the factorial, it is easy to see that

$$\begin{aligned}\frac{(q^\alpha;q)_n}{(1-q)^n} &= \frac{(1-q^\alpha)}{(1-q)}\ \frac{(1-q^{\alpha+1})}{(1-q)}\ \frac{(1-q^{\alpha+2})}{(1-q)}\cdots\ \frac{(1-q^{\alpha+n-1})}{(1-q)} \\ &\to\ \alpha\ (\alpha+1)\ (\alpha+2)\cdots\ (\alpha+n-1)\end{aligned}$$

as $q \to 1$ and this equals $n!$ when $\alpha = 1$.

In these terminologies, it is known that a basic hypergeometric series [3, Eq. (1.2.22), p. 4]

$${}_1\phi_1\left[\begin{array}{lll} a; & q, & z \\ b; & & \end{array}\right] = \sum_{n\geq 0}\frac{(q^a;q)_n}{(q^b;q)_n}\ \left[(-1)^n\ q^{\binom{n}{2}}\right]\ \frac{z^n}{(q;q)_n} \tag{4}$$

defines an entire function of z where $b \neq 0, -1, -2, \ldots$.
Let us consider the series

$$\sum_{n\geq 1}\frac{z^n}{(q;q)_n^n} \tag{5}$$

as a q-form of the series $\sum\limits_{n\geq 1}\frac{z^n}{n!^n}$ representing an entire function due to Sikkema [8, p. 6]. The series in (5) leads us to an extension of (4) which we define as follows.

Definition 1.1. For $\Re(\ell) \geq 0, a, z \in \mathbf{C}$, and $b, c \in \mathbf{C} \setminus \{0, -1, -2, \ldots\} = \mathbf{C_1}$, define the function Ψ by

$$\Psi\left[\begin{array}{lll} a; & q, & z \\ b; & (c:\ell); & \end{array}\right] = \sum_{n\geq 0}\frac{(q^a;q)_n}{(q^b;q)_n}\ \frac{(-1)^n\ q^{\binom{n}{2}}}{(q^c;q)_n^{\ell n}}\ \frac{z^n}{(q;q)_n}. \tag{6}$$

We shall call this function to be q-ℓ-Ψ hypergeometric function and in brief, the q-ℓ-Ψ function. As $q \to 1$, this q-ℓ-Ψ function approaches to our *ordinary* analogue, the ℓ-Hypergeometric function [2, Def.1]:

$$H\left[\begin{array}{lll} a; & & z \\ b; & (c:\ell); & \end{array}\right] = \sum_{n\geq 0}\frac{(a)_n}{(b)_n\ (c)_n^{\ell n}}\frac{z^n}{n!}, \tag{7}$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, $a, \ell, z \in \mathbf{C}$ with $\Re(\ell) \geq 0$, and $b, c \in \mathbf{C} \setminus \{0, -1, -2, \ldots\}$.

2. Main Results

In this section, we prove certain properties of q-ℓ-Ψ function (6) namely, difference equation, eigen function property and contiguous function relations. We first investigate the convergence behavior of the series in (6). Throughout the work, we take

$$\frac{(q^a;q)_n\ \ (-1)^n\ q^{n(n-1)/2}}{(q^b;q)_n\ (q^c;q)_n^{\ell n}\ (q;q)_n} = \xi_n.$$

2.1. **Convergence.** The absolute convergence of series in (6) is however evident due to the presence of the factor $q^{\binom{n}{2}}$, we verify that the series indeed represents the entire function.

Theorem 2.1. *If* $0 < q < 1$, $\Re(\ell) \geq 0$ *then* q-ℓ-Ψ *function is an entire function of* z.

Proof. In view of the formula (3), we have

$$\begin{aligned} |\xi_n|^{\frac{1}{n}} &= \left| \frac{(q^a;q)_n \; q^{n(n-1)/2}}{(q^b;q)_n \; (q^c;q)_n^{\ell n} \; (q;q)_n} \right|^{\frac{1}{n}} \\ &= \left|\frac{\Gamma_q(b)}{\Gamma_q(a)}\right|^{\frac{1}{n}} \left|\frac{\Gamma_q(a+n)}{\Gamma_q(b+n)}\right|^{\frac{1}{n}} \; \left|\frac{\Gamma_q(c)\;(1-q)^{-n}}{\Gamma_q(c+n)}\right|^{\ell} \frac{q^{(n-1)/2}\;(1-q)^{-1}}{\Gamma_q^{\frac{1}{n}}(n+1)}. \end{aligned}$$

Here applying the Stirling's formula of q-Gamma function [7, Eq.(2.25), p. 482]:

$$\Gamma_q(z) \sim (1+q)^{\frac{1}{2}} \; \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-z} \; e^{\frac{\theta q^z}{1-q-q^z}}, \text{ for large } |z|, \quad 0 < \theta < 1, \tag{8}$$

we further have for large n,

$$\begin{aligned} |\xi_n|^{\frac{1}{n}} \quad \sim \quad & \left|\frac{\Gamma_q(b)}{\Gamma_q(a)}\right|^{\frac{1}{n}} \left| \frac{(1+q)^{\frac{1}{2}} \; \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(a+n)} \; e^{\frac{\theta \; q^{a+n}}{1-q-q^{a+n}}}}{(1+q)^{\frac{1}{2}} \; \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(b+n)} \; e^{\frac{\theta \; q^{b+n}}{1-q-q^{b+n}}}} \right|^{\frac{1}{n}} \\ & \times \frac{\left|\Gamma_q^{\ell}(c)\;(1-q)^{-\ell n}\right|}{\left|(1+q)^{\frac{1}{2}} \; \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(c+n)} \; e^{\frac{\theta \; q^{c+n}}{1-q-q^{c+n}}}\right|^{\ell}} \\ & \times \frac{q^{(n-1)/2} \; (1-q)}{\left|(1+q)^{\frac{1}{2}} \; \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(n+1)} \; e^{\frac{\theta \; q^{n+1}}{1-q-q^{n+1}}}\right|^{\frac{1}{n}}}. \end{aligned}$$

Hence using the Cauchy-Hadamard formula, we find

$$\begin{aligned} \frac{1}{R} &= \lim_{n\to\infty} \sup \sqrt[n]{|\xi_n|} \\ &= \lim_{n\to\infty} \frac{\Gamma_q^{\ell}(c)\;(1-q)^{c\ell - \frac{\ell}{2}+1}}{(1+q)^{\frac{\ell}{2}} \; \Gamma_{q^2}^{\ell}\left(\frac{1}{2}\right)} \; q^{(n-1)/2} \;=\; 0, \end{aligned}$$

provided $\Re(\ell) \geq 0$ and $0 < q < 1$. □

Remark 2.2. Here if $\ell = 0$ then q-ℓ-Ψ function reduces to the basic hypergeometric function ${}_1\phi_1(a;b;q,z), z \in \mathbf{C}$.

2.2. q**-Difference Equation.** The difference equation of q-ℓ-Ψ function occurs for $\ell \in \mathbf{N} \cup \{0\}$, which is obtained by introducing the following operator.

Definition 2.3. Let $f(z) = \sum_{n\geq 1} a_{n,q} \; z^n$, $0 \neq z \in \mathbf{C}$, $p \in \mathbf{N} \cup \{0\}$ and $\alpha \in \mathbf{C}$. Define

$${}_p\Delta_{\alpha}^{\theta_q} f(z) = \begin{cases} \sum_{n\geq 1} a_{n,q} \; (q^{\alpha};q)_{n-1}^{p} \; (q^{\alpha-1} \; \theta_q - q^{\alpha-1}+1)^{pn} \; z^n, & \text{if } p \in \mathbf{N} \\ f(z), & \text{if } p = 0 \end{cases}, \tag{9}$$

where the difference operator θ_q is defined by

$$\theta_q f(z) = f(z) - f(zq). \tag{10}$$

The operator defined in (9) leads us to construct an infinite order differential operator as given below.

Definition 2.4. Let $f(z) = \sum_{n\geq 1} a_{n,q}\ z^n$, $0 \neq z \in \mathbf{C},\ p \in \mathbf{N}\cup\{0\}$ and $\alpha, \beta \in \mathbf{C}$. We define the operator

$$_{\beta}\Lambda^{\theta_q}_{(\alpha,p)} f(z) = \left[\left\{\ _{p}\Delta^{\theta_q}_{\alpha}\right\}\left\{q^{\beta-1}\theta_q - q^{\beta-1} + 1\right\}\theta_q\right]\ (-q)\ f\left(\frac{z}{q}\right), \tag{11}$$

where $_{p}\Delta^{\theta_q}_{\alpha}$ is as defined in (9).

Note 2.5. The operators $_{\ell}\Delta^{\theta_q}_{c}$ and θ_q *do not* commute.

By means of above operator, we construct difference equation of q-ℓ-Ψ function.

Theorem 2.6. *For $\ell \in \mathbf{N}\cup\{0\}, a, z \in \mathbf{C}$, and $b, c \in \mathbf{C_1}$, the function* $w = \Psi\left[\begin{matrix} a; & q, & z \\ b; & (c:\ell); & \end{matrix}\right]$ *satisfies the difference equation*

$$\left[\ _{b}\Lambda^{\theta_q}_{(c,\ell)} - z(q^a\theta_q - q^a + 1)\right] w = 0, \tag{12}$$

where the operator $_{b}\Lambda^{\theta_q}_{(\alpha,p)}$ *is as defined in (11).*

Unlike the finite order operators that act on operand in straightforward manner, here the operator (9) acts on w subject to a condition; which is stated and proved as

Lemma 2.7. *If $\ell \in \mathbf{N}\cup\{0\}, a, z \in \mathbf{C}$, $b, c \in \mathbf{C_1}$, and*

$$w = \Psi\left[\begin{matrix} a; & q, & z \\ b; & (c:\ell); & \end{matrix}\right] = \sum_{n\geq 0} \xi_n\ z^n$$

then the operator $_{b}\Lambda^{\theta_q}_{(c:\ell)} := \sum_{n\geq 0} f_{n,q}(a, b, (c\ :\ \ell); z)$ *is applicable to the q-ℓ-Ψ function provided that the series*

$$\sum_{n\geq 0} \xi_n\ f_{n,q}(a, b, (c:\ell); z)$$

converges (cf. [8, Definition 11, p.20]*).*

Proof. Let us put

$$\frac{(q^a;q)_n\ (-1)^{n+1} q^{(n-1)(n-2)/2}}{(q^b;q)_n\ (q^c;q)^{\ell n}_n} = A_n, \tag{13}$$

then

$$\begin{aligned}
{}_b\Lambda^{\theta_q}_{(c,\ell)} w &= \left[\left\{{}_\ell\Delta^{\theta_q}_c\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\theta_q\right]\sum_{n\geq 0}\xi_n\ (-q)\left(\frac{z}{q}\right)^n \\
&= \left[\left\{{}_\ell\Delta^{\theta_q}_c\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\theta_q\right]\sum_{n\geq 0}A_n\ \frac{z^n}{(q;q)_n} \\
&= \left[\left\{{}_\ell\Delta^{\theta_q}_c\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\right]\sum_{n\geq 0}A_n\ \frac{(1-q^n)}{(q;q)_n}z^n \\
&= \left[\left\{{}_\ell\Delta^{\theta_q}_c\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\right]\sum_{n\geq 1}A_n\ \frac{z^n}{(q;q)_{n-1}} \\
&= \left\{{}_\ell\Delta^{\theta_q}_c\right\}\sum_{n\geq 1}\frac{A_n}{(q;q)_{n-1}}\left(q^{b-1}(z^n - z^n q^n) - z^n(q^{b-1}-1)\right) \\
&= \left\{{}_\ell\Delta^{\theta_q}_c\right\}\sum_{n\geq 1}\frac{A_n}{(q;q)_{n-1}}\ (1-q^{b+n-1})\ z^n \\
&= \sum_{n\geq 1}\frac{A_n}{(q;q)_{n-1}}\ (1-q^{b+n-1})\ (q^c;q)^{\ell}_{n-1}\ \left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell n}\ z^n.
\end{aligned}$$

Here a little computations provide us

$$\left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell n}\ z^n = (1-q^{n+c-1})^{\ell n}\ z^n.$$

We thus have

$$\begin{aligned}
{}_b\Lambda^{\theta_q}_{(c,\ell)} w &= \sum_{n\geq 1}\frac{A_n\ (q^c;q)^{\ell}_{n-1}}{(q;q)_{n-1}}\ (1-q^{b+n-1})(1-q^{n+c-1})^{\ell n}\ z^n \\
&= \sum_{n\geq 1}\frac{(q^a;q)_n\ (-1)^{n+1}q^{(n-1)(n-2)/2}\ z^n}{(q^b;q)_{n-1}\ (q^c;q)^{\ell n-\ell}_{n-1}\ (q;q)_{n-1}} \qquad (14)\\
&= \sum_{n\geq 0}\xi_n\ (1-q^{a+n})\ z^{n+1} \qquad (15)\\
&= \sum_{n\geq 0}f_{n,q}(a,b,(c:\ell);z), \quad \text{say.}
\end{aligned}$$

To complete the proof of lemma, it suffices to show that

$$\sum_{n\geq 0}a_{n,q}\ f_{n,q}(a,b,(c:\ell);z) = \sum_{n\geq 0}\frac{(q^a;q)^2_n\ (1-q^{a+n})\ (-1)^n q^{n(n-1)/2}\ z^{n+1}}{(q^b;q)^2_n\ (q^c;q)^{2\ell n}_n\ (q;q)^2_n}$$

is convergent. For that, take

$$\begin{aligned}
|\mu_n| &= |a_{n,q}\ f_{n,q}(a,b,(c:\ell);z)| = \left|\frac{(q^a;q)^2_n\ (1-q^{a+n})\ (-1)^n q^{n(n-1)/2}\ z^{n+1}}{(q^b;q)^2_n\ (q^c;q)^{2\ell n}_n\ (q;q)^2_n}\right| \\
&= \left|\left(\frac{\Gamma_q(a+n)}{\Gamma_q(a)}\right)^2\left(\frac{\Gamma_q(b)}{\Gamma_q(b+n)}\right)^2\left(\frac{\Gamma_q(c)\ (1-q)^{-n}}{\Gamma_q(c+n)}\right)^{2\ell n}\right| \\
&\qquad\times\left|\left(\frac{q^{n(n-1)/2}\ (1-q)^{-1}}{\Gamma_q(n+1)}\right)^2\ (1-q^{a+n})\ z^{n+1}\right|
\end{aligned}$$

$$\sim \left|\frac{\Gamma_q(b)}{\Gamma_q(a)}\right|^2 \ |\Gamma_q^{2\ell n}(c)\ z^{n+1}| \left|\frac{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-a-n}\ e^{\theta\frac{q^{a+n}}{1-q-q^{a+n}}}}{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-b-n}\ e^{\theta\frac{q^{b+n}}{1-q-q^{b+n}}}}\right|^2$$
$$\times\frac{q^{n(n-1)}\ (1-q)^{-2\ell n^2}}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-c-n}\ e^{\theta\frac{q^{c+n}}{1-q-q^{c+n}}}\right|^{2\ell}}$$
$$\times\frac{(1-q^{a+n})\ (1-q)^{-2}}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-n-1}\ e^{\theta\frac{q^{n+1}}{1-q-q^{n+1}}}\right|^2}.$$

Then

$$\lim_{n\to\infty}|\mu_n|^{\frac{1}{n}} \quad\sim\quad \lim_{n\to\infty}\frac{|\Gamma_q^{2\ell}(c)|\ |z|\ (1-q^{a+n})^{\frac{1}{n}}}{|(1+q)^{\ell}|\ \Gamma_{q^2}^{2\ell}\left(\frac{1}{2}\right)\ (1-q)^{\ell-2\ell c}}\ q^{(n-1)} \quad=\quad 0,$$

whenever $\ell\in\mathbf{N}\cup\{0\}$ as $0<q<1$. □

Proof. (of Theorem 2.6). From (15) we get

$$\begin{aligned}{}_b\Lambda_{(c,\ell)}^{\theta_q}w &= \sum_{n\geq 0}\xi_n\ (1-q^{a+n})\ z^{n+1}\\ &= z\sum_{n\geq 0}\xi_n\ (q^a z^n - q^a z^n q^n - q^a z^n + z^n)\\ &= z\sum_{n\geq 0}\xi_n\ (q^a\theta_q - q^a + 1)\ z^n \quad=\quad z\ (q^a\theta_q - q^a + 1)\ w,\end{aligned}$$

and hence (12) follows. □

2.3. **Eigen function property.** In order to obtain q-ℓ-Ψ function as an eigen function, we need to define yet another operator using following definition.

Definition 2.8. Let $f(z)=\sum\limits_{n\geq 1}a_{n,q}\ z^n,\ |z|<R, z\neq 0, R>0$ and $\alpha\in\mathbf{C}$ with $\Re(\alpha)>0$. Define

$$I_q^{\alpha}f(z)=\frac{z^{-\alpha}}{1-q}\ I_q\left(z^{\alpha-1}f(z)\right), \tag{16}$$

where the q-integral of function is given by [3, Eq.(1.11.1), p. 19]

$$I_q f(z)=\int_0^z f(t)d_qt = z\ (1-q)\sum_{n\geq 0}f(zq^n)\ q^n. \tag{17}$$

Definition 2.9. Let $f(z)=\sum\limits_{n\geq 0}a_{n,q}\ z^n,\ |z|<R, z\neq 0, R>0$. Define

$${}_{\alpha}\mathcal{E}_{\beta}^{(\gamma:p)}f(z)=\left[I_q^{\alpha}\ z^{-1}\ {}_{\beta}\Lambda_{(\alpha,p)}^{\theta_q}\right]f(z), \tag{18}$$

where ${}_{\beta}\Lambda_{(\alpha,p)}^{\theta_q}$ and I_q^{α} are as defined in (11) and (16) respectively.
The following theorem establishes the eigen function property with respect to the operator (18).

Theorem 2.10. *If $\ell \in \mathbf{N} \cup \{0\}$ then the q-ℓ-Ψ function is an eigen function with respect to the operator ${}_a\mathcal{E}_b^{(c:\ell)}$ defined in (18). That is,*

$$ {}_a\mathcal{E}_b^{(c:\ell)}\left(\Psi\left[\begin{array}{ccc} a; & q, & \lambda z \\ b; & (c:\ell); & \end{array}\right]\right) = \lambda\ \Psi\left[\begin{array}{ccc} a; & q, & \lambda z \\ b; & (c:\ell); & \end{array}\right], \quad \lambda \in \mathbf{C}. \tag{19} $$

Proof. The applicability of this operator to the q-ℓ-Ψ function follows from Lemma 2.7. Now for $z \neq 0$,

$$ {}_a\mathcal{E}_b^{(c:\ell)}\left(\Psi\left[\begin{array}{ccc} a; & q, & \lambda z \\ b; & (c:\ell); & \end{array}\right]\right) = \left[I_q^a\ z^{-1}\ {}_b\Lambda_{(c,\ell)}^{\theta_q}\left(\sum_{n\geq 0}\lambda^n\ \xi_n\ z^n\right)\right]. $$

For each n, putting

$$ \frac{\lambda^n\ (q^a;q)_n\ (-1)^{n+1}\ q^{(n-1)(n-2)/2}}{(q^b;q)_{n-1}\ (q^c;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} = B_n, $$

in view of (14) one gets

$$ \begin{aligned} {}_a\mathcal{E}_b^{(c:\ell)}\left(\Psi\left[\begin{array}{ccc} a; & q, & \lambda z \\ b; & (c:\ell); & \end{array}\right]\right) &= I_q^a\ z^{-1}\left[\sum_{n\geq 1} B_n\ z^n\right] \\ &= \frac{z^{-a}}{1-q}\ I_q\left[\sum_{n\geq 1} B_n\ z^{a+n-2}\right] \\ &= \frac{z^{-a}}{1-q}\ \sum_{n\geq 1} B_n\ z(1-q)\ \sum_{k\geq 0}(zq^k)^{a+n-2}\ q^k \\ &= \frac{z^{-a}}{1-q}\ \sum_{n\geq 1} B_n\ z(1-q)\ z^{a+n-2}\ \sum_{k\geq 0} q^{k(a+n-1)} \\ &= \sum_{n\geq 1} B_n\ \frac{z^{n-1}}{1-q^{a+n-1}} = \lambda \sum_{n\geq 0}\lambda^n\ \xi_n\ z^n, \end{aligned} $$

and thus the theorem is proved. □

2.4. **Contiguous function relations.** The contiguous function relations for basic hypergeometric series have been derived by Swarttouw [9]. Our attempt in this direction led us to following two identities:

$$ (q^{b-1} - q^a)\Psi = q^{b-1}(1-q^a)\Psi(a+) - q^a(1-q^{b-1})\Psi(b-) \tag{20} $$

and

$$ (1-q^b)\Psi = (1-q^a)\Psi(a+,b+) - (q^b - q^a)\Psi(b+, zq), \tag{21} $$

in which the function notations carry the following meaning. We take

$$ \Psi = \Psi\left[\begin{array}{ccc} a; & q, & z \\ b; & (c:\ell); & \end{array}\right], $$

then

$$\Psi(a+) := \Psi\begin{bmatrix} aq; & q, & z \\ b; & (c:\ell); & \end{bmatrix},\ \Psi(a-) := \Psi\begin{bmatrix} aq^{-1}; & q, & z \\ b; & (c:\ell); & \end{bmatrix},$$
$$\Psi(a+,b+) := \Psi\begin{bmatrix} aq; & q, & z \\ bq; & (c:\ell); & \end{bmatrix},\ \Psi(b+,zq) := \Psi\begin{bmatrix} a; & q, & zq \\ bq; & (c:\ell); & \end{bmatrix}.$$

The functions $\Psi(b+), \Psi(b-), \Psi(c+), \Psi(c-)$ etc. are defined similarly. Now, with

$$\xi_n = \frac{(q^a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(q^b;q)_n\ (q^c;q)_n^{\ell n}\ (q;q)_n}$$

as before, we have $\Psi = \sum_{n\geq 0} \xi_n\ z^n$ whence the following series representations are evident.

$$\begin{aligned} \Psi(a+) &= \sum_{n\geq 0} \frac{1-q^{a+n}}{1-q^a}\ \xi_n\ z^n, & \Psi(a-) &= \sum_{n\geq 0} \frac{1-q^{a-1}}{1-q^{a+n-1}}\ \xi_n\ z^n, \\ \Psi(b+) &= \sum_{n\geq 0} \frac{1-q^b}{1-q^{b+n}}\ \xi_n\ z^n, & \Psi(b-) &= \sum_{n\geq 0} \frac{1-q^{b+n-1}}{1-q^{b-1}}\ \xi_n\ z^n, \\ \Psi(c+) &= \sum_{n\geq 0} \frac{(1-q^c)^{\ell n}}{(1-q^{c+n})^{\ell n}}\ \xi_n\ z^n, & \Psi(c-) &= \sum_{n\geq 0} \frac{(1-q^{c+n-1})^{\ell n}}{(1-q^{c-1})^{\ell n}}\ \xi_n\ z^n. \end{aligned} \tag{22}$$

If $\Theta_q = zD_q$, where D_q is as defined in (33) then,

$$\Theta_q \Psi = \sum_{n\geq 0} \frac{(1-q^n)}{(1-q)} \xi_n\ z^n. \tag{23}$$

From (23), we have

$$\begin{aligned} \left(q^a \Theta_q + \frac{1-q^a}{1-q}\right)\Psi &= q^a \sum_{n\geq 0} \frac{(1-q^n)}{(1-q)} \xi_n z^n + \frac{1-q^a}{1-q} \sum_{n\geq 0} \xi_n z^n \\ &= \frac{1}{1-q} \sum_{n\geq 0} (1-q^{a+n})\ \xi_n\ z^n = \frac{1-q^a}{1-q} \Psi(a+). \end{aligned} \tag{24}$$

Also,

$$\begin{aligned} \left(q^{b-1} \Theta_q + \frac{1-q^{b-1}}{1-q}\right)\Psi &= q^{b-1} \sum_{n\geq 0} \frac{(1-q^n)}{(1-q)} \xi_n z^n + \frac{1-q^{b-1}}{1-q} \sum_{n\geq 0} \xi_n z^n \\ &= \frac{1}{1-q} \sum_{n\geq 0} (1-q^{b+n-1})\ \xi_n z^n = \frac{1-q^{b-1}}{1-q} \Psi(b-). \end{aligned} \tag{25}$$

Now, elimination of Θ_q from (24) and (25) yields (20). Next, from (13) and by using the technique adopted in the proof of Lemma 2.7, we have

$$\begin{aligned} \{\ {}_\ell\Delta_c^{\theta_q}\} \theta_q\ (-q)\ \Psi(z/q) &= \left\{{}_\ell\Delta_c^{\theta_q}\right\} \theta_q \left(\sum_{n\geq 0} A_n\ \frac{z^n}{(q;q)_n}\right) \\ &= \left\{{}_\ell\Delta_c^{\theta_q}\right\} \left(\sum_{n\geq 0} A_n\ \frac{1-q^n}{(q;q)_n}\ z^n\right) \end{aligned}$$

$$
\begin{aligned}
&= \left\{{}_\ell\Delta_c^{\theta_q}\right\}\left(\sum_{n\geq 1} A_n\ \frac{z^n}{(q;q)_{n-1}}\right)\\
&= \sum_{n\geq 1}\frac{A_n}{(q;q)_{n-1}}(q^c;q)_{n-1}^{\ell}\left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell n} z^n\\
&= \sum_{n\geq 1}\frac{A_n\ (q^c;q)_{n-1}^{\ell}}{(q;q)_{n-1}}(1-q^{n+c-1})^{\ell n} z^n.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\{\ {}_\ell\Delta_c^{\theta_q}\}\theta_q\ (-q)\ \Psi(z/q) &= \sum_{n\geq 1}\frac{(q^a;q)_n\ (-1)^{n+1}q^{(n-1)(n-2)/2}\ z^n}{(q^b;q)_n\ (q^c;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\\
&= \sum_{n\geq 0}\frac{(q^a;q)_{n+1}\ (-1)^n q^{n(n-1)/2}\ z^{n+1}}{(q^b;q)_{n+1}\ (q^c;q)_n^{\ell n}\ (q;q)_n}\\
&= z\sum_{n\geq 0}\frac{1-q^{a+n}}{1-q^{b+n}}\ \xi_n\ z^n \qquad (26)\\
&= z\ \frac{1-q^a}{1-q^b}\ \Psi(a+,b+). \qquad (27)
\end{aligned}
$$

Now putting

$$\frac{1-q^{a+n}}{1-q^{b+n}} = 1+\frac{q^{b+n}-q^{a+n}}{1-q^{b+n}}$$

in (26) we find that

$$
\begin{aligned}
\{\ {}_\ell\Delta_c^{\theta_q}\}\theta_q\ (-q)\ \Psi(z/q) &= z\sum_{n\geq 0}\left(1+\frac{q^{b+n}-q^{a+n}}{1-q^{b+n}}\right)\xi_n z^n\\
&= z\left[\sum_{n\geq 0}\xi_n z^n + \sum_{n\geq 0}\frac{q^b-q^a}{1-q^b}\,\frac{1-q^b}{1-q^{b+n}}\ \xi_n\ (zq)^n\right]\\
&= z\left[\Psi+\frac{q^b-q^a}{1-q^b}\Psi(b,zq)\right]. \qquad (28)
\end{aligned}
$$

The relation (21) now follows from (27) and (28).

3. Particular q-ℓ-Ψ functions

3.1. q-ℓ-Ψ Exponential Function. In (6), if the parameters a and b are absent and z is replaced by $-z$ then

$$ {}_0\Psi_0^1\left[\begin{array}{ccc} -; & q, & -z\\ -; & (c:\ell); & \end{array}\right] = \sum_{n\geq 0}\frac{q^{n(n-1)/2}\ z^n}{(q^c;q)_n^{\ell n}\ (q;q)_n},\quad (\Re(\ell)\geq 0). \qquad (29)$$

This function characterizes the q-Exponential function. In fact, for $c=1$, $q^c=q$ hence the above series would reduce to (cf. [1, Def.5, p.249])

$$ {}_0\Psi_0^1\left[\begin{array}{ccc} -; & q, & z\\ -; & (q:\ell); & \end{array}\right] = \sum_{n\geq 0}\frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}}$$

which enables us to define the q-ℓ-Ψ Exponential function as follows.

Definition 3.1. The q-ℓ-Ψ Exponential function is denoted and defined by

$$E_{\Psi}^{\ell}(z;q) = \sum_{n\geq 0} \frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}}, \tag{30}$$

for all $z \in \mathbf{C}$ and $\Re(\ell) \geq 0$.

Remark 3.2. Obviously, $E_{\Psi}^{0}(z;q) = E_q(z)$, and $E_{\Psi}^{\ell}(0;q) = 1$.

Eventually, when the parameters a and b are absent, the difference equation obtained in Theorem 2.6 would reduce to the form

$$\left(\Lambda_{(1,\ell)}^{\theta_q} - z\right) w = 0, \tag{31}$$

which is satisfied by $w = E_{\Psi}^{\ell}(z;q)$. This can be verified as follows.

$$\begin{aligned}
\left(\Lambda_{(1,\ell)}^{\theta_q}\right) w = \{{}_{\ell}\Delta_1^{\theta_q}\}\ \theta_q\ q\ E_{\Psi}^{\ell}\left(\frac{z}{q};q\right) &= \{{}_{\ell}\Delta_1^{\theta_q}\}\ \theta_q \sum_{n\geq 0} \frac{q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n+1}}\ z^n \\
&= \{{}_{\ell}\Delta_1^{\theta_q}\} \sum_{n\geq 0} \frac{q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n+1}}\ (1-q^n) z^n \\
&= \sum_{n\geq 1} \frac{q^{(n-1)(n-2)/2}\ (q;q)_{n-1}^{\ell}\ (\theta_q)^{\ell n}\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= \sum_{n\geq 1} \frac{q^{(n-1)(n-2)/2}\ (q;q)_{n-1}^{\ell}\ (1-q^n)^{\ell n}\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= \sum_{n\geq 1} \frac{q^{(n-1)(n-2)/2}\ (q;q)_{n-1}^{\ell}\ z^n}{(q;q)_{n-1}^{\ell n}\ (q;q)_{n-1}} \\
&= \sum_{n\geq 1} \frac{q^{(n-1)(n-2)/2}\ z^n}{(q;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \quad = \quad z\ w.
\end{aligned}$$

Note 3.3. The case $\ell = 0$ in (31) yields the equation $(\Delta_q - z)\ w = 0$, where $\Delta_q := \Lambda_{(1,0)}^{\theta_q}$ and $w = E_q(z)$.

In order to derive eigen function property for generalized q-ℓ-Ψ exponential function, we introduce one more differential operator below.

Definition 3.4. Let $f(z;q) = \sum_{n=0}^{\infty} a_{n,q}\ z^n$, $|z| < R, R > 0$, $p \in \mathbf{N} \cup \{0\}$. Define an operator

$${}_p\Omega_{\alpha}^{\mathbf{D}_q^z} = \begin{cases} \sum_{n\geq 1} a_{n,q}\ (q^{\alpha};q)_{n-1}^{p}\ ((1-q)\ \mathbf{D}_q^z)^{pn}\ z^n, & \text{if} \quad p \in \mathbf{N} \\ f(z;q), & \text{if} \quad p = 0 \end{cases}, \tag{32}$$

where $\mathbf{D}_q^z$ is the q-hyper-Bessel type operator given by (see e.g. [4, 5, 6]).

$$(\mathbf{D}_q^z)^n = \underbrace{D_q z D_q \ldots D_q z D_q}_{n \text{ derivatives}}$$

in which D_q is the usual q-Derivative operator given by [3, Ex.(1.12), p. 22]

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}. \tag{33}$$

Definition 3.5. Let $f(z;q) = \sum\limits_{n=0}^{\infty} a_{n,q}\ z^n$, $|z| < R, R > 0$, $p \in \mathbf{N} \cup \{0\}$. Define an operator

$${}_p\mathcal{D}_M^q f(z;q) = \left(\left\{ {}_p\Omega_1^{\mathbf{D}_q^z} \right\}\ \theta_q\ q \right) f\left(\frac{z}{q}; q \right), \tag{34}$$

where ${}_p\Omega_1^{\mathbf{D}_q^z}$ is as defined in (32).

In view of this, we have

Theorem 3.6. *If $\ell \in \mathbf{N} \cup \{0\}$ then the q-ℓ-Ψ Exponential function is an eigen function of the operator ${}_\ell\mathcal{D}_M^q$. That is,*

$${}_\ell\mathcal{D}_M^q \left(E_\Psi^\ell(\lambda z; q) \right) = \lambda\ E_\Psi^\ell(\lambda z; q),\ \lambda \in \mathbf{C}. \tag{35}$$

Proof. Observe that

$$\begin{aligned}
&{}_\ell\mathcal{D}_M^q \left(E_\Psi^\ell(\lambda z; q) \right) \\
&= \left(\left\{ {}_\ell\Omega_1^{\mathbf{D}_q^z} \right\}\ \theta_q \right) \left[\sum_{n \geq 0} \frac{\lambda^n\ \ q^{(n-1)(n-2)/2}\ \ z^n}{(q;q)_n^{\ell n+1}} \right] \\
&= \left\{ {}_\ell\Omega_1^{\mathbf{D}_q^z} \right\} \left[\sum_{n \geq 1} \frac{\lambda^n\ \ q^{(n-1)(n-2)/2}\ \ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \right] \\
&= \sum_{n \geq 1} \frac{\lambda^n\ \ q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} (q;q)_{n-1}^\ell\ \left((1-q)\mathbf{D}_q^z\right)^{\ell n} z^n,
\end{aligned}$$

and, for $n \in \mathbf{N}$,

$$(1-q)^{\ell n} \left(\mathbf{D}_q^z \right)^{\ell n}\ \ z^n = (1-q^n)^{\ell n}\ z^{n-1}. \tag{36}$$

Hence

$$\begin{aligned}
{}_\ell\mathcal{D}_M^q \left(E_\Psi^\ell(\lambda z; q) \right) &= \sum_{n \geq 1} \frac{\lambda^n\ \ q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} (q;q)_{n-1}^\ell\ (1-q^n)^{\ell n}\ z^{n-1} \\
&= \sum_{n \geq 1} \frac{\lambda^n\ \ q^{(n-1)(n-2)/2}\ \ z^{n-1}}{(q;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \\
&= \sum_{n \geq 0} \frac{\lambda^{n+1}\ \ q^{n(n-1)/2}\ \ z^n}{(q;q)_n^{\ell n+1}}\ =\ \lambda\ E_\Psi^\ell(\lambda z; q),
\end{aligned}$$

which completes the proof. □

3.2. **q-ℓ-Ψ Trigonometric Functions.** We consider the q-ℓ-ψ Exponential function with arguments $\pm iz$ and observe that

$$\begin{aligned}
&\frac{1}{2} \left[E_\Psi^\ell(iz;q) + E_\Psi^\ell(-iz;q) \right] \\
&= \frac{1}{2} \left[\sum_{n \geq 0} \frac{q^{n(n-1)/2}\ (iz)^n}{(q;q)_n^{\ell n+1}} + \sum_{n \geq 0} \frac{q^{n(n-1)/2}\ (-iz)^n}{(q;q)_n^{\ell n+1}} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[1 - \frac{iz}{((q;q)_1)^{\ell+1}}\frac{q^{2(2-1)/2}z^2}{(q;q)_2^{2\ell+1}} + \ldots + 1 + \frac{iz}{(q;q)_1^{\ell+1}}\frac{q^{2(2-1)/2}z^2}{(q;q)_2^{2\ell+1}} - \ldots\right]\\
&= \frac{1}{2}\left[2\left(1 - \frac{q^{2(2-1)/2}\ z^2}{(q;q)_2^{2\ell+1}} + \frac{q^{4(4-1)/2}\ z^4}{(q;q)_4^{4\ell+1}} - \ldots\right)\right]\\
&= \sum_{n\geq 0}\frac{q^{2n(2n-1)/2}\ z^{2n}}{(q;q)_{2n}^{2\ell n+1}}, \qquad (37)
\end{aligned}$$

and likewise,

$$\frac{1}{2i}\left[E_\Psi^\ell(iz;q) - E_\Psi^\ell(-iz;q)\right] = \sum_{n\geq 0}\frac{q^{2n(2n+1)/2}\ z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}. \qquad (38)$$

Then the series on the right hand side of (37) and (38) enable us to extend the q-cosine and q-sine functions respectively which are denoted here by $\mathrm{Cos}_\Psi^\ell(z;q)$ and $\mathrm{Sin}_\Psi^\ell(z;q)$. In fact, for any $z \in \mathbf{C}$,

$$\frac{1}{2}\left[E_\Psi^\ell(iz;q) + E_\Psi^\ell(-iz;q)\right] := \mathrm{Cos}_\Psi^\ell(z;q), \qquad (39)$$

and

$$\frac{1}{2i}\left[E_\Psi^\ell(iz;q) - E_\Psi^\ell(-iz;q)\right] := \mathrm{Sin}_\Psi^\ell(z;q), \qquad (40)$$

which imply that

$$E_\Psi^\ell(iz;q) = \mathrm{Cos}_\Psi^\ell(z;q) + i\ \mathrm{Sin}_\Psi^\ell(z;q). \qquad (41)$$

Remark 3.7. It is noteworthy that $\mathrm{Cos}_\Psi^0(z;q) = \mathrm{Cos}_q(z)$, and $\mathrm{Sin}_\Psi^0(z;q) = \mathrm{Sin}_q(z)$. Further, simple calculations show that

$$\begin{aligned}
\mathrm{Cos}_\Psi^\ell(0;q) &= \frac{1}{2}\left[E_\Psi^\ell(0;q) + E_\Psi^\ell(0;q)\right] = 1,\\
\mathrm{Sin}_\Psi^\ell(0;q) &= \frac{1}{2i}\left[E_\Psi^\ell(0;q) - E_\Psi^\ell(0;q)\right] = 0.
\end{aligned}$$

Note 3.8. If $f(z;q) = \sum_{n=0}^{\infty} a_{n,q}\ z^n$ and $g(z;q) = \sum_{n=0}^{\infty} b_{n,q}\ z^n$, $|z| < R$ then for $\alpha, \beta \in \mathbf{R}$ and $p \in \mathbf{N}\cup\{0\}$,

$${}_p\mathcal{D}_M^q\left(\alpha\ f(z;q) + \beta\ g(z;q)\right) = \alpha\ {}_p\mathcal{D}_M^q(f(z;q)) + \beta\ {}_p\mathcal{D}_M^q(g(z;q)). \qquad (42)$$

Remark 3.9. The operator (34) when acts on (39) and (40), yields **(1).** ${}_\ell\mathcal{D}_M^q(\mathrm{Cos}_\Psi^\ell(z;q)) = -\ \mathrm{Sin}_\Psi^\ell(z;q)$ and **(2).** ${}_\ell\mathcal{D}_M^q(\mathrm{Sin}_\Psi^\ell(z;q)) = \mathrm{Cos}_\Psi^\ell(z;q)$ respectively.

Just as the functions $\sin z$ and $\cos z$ are solutions of equation $\frac{d^2y}{dz^2} + y = 0$, these generalized functions are also solutions of a differential equation shown in the following theorem.

Theorem 3.10. *The generalized q-ℓ-Ψ Cosine and Sine functions are solutions of the differential equation*

$$\left({}_\ell\mathcal{D}_M^q\right)^2\nu + \nu = 0.$$

Proof. Note first that from (35),

$$_{\ell}\mathcal{D}_M^q(E_\Psi^\ell(iz;q)) = i(E_\Psi^\ell(iz;q))).$$

Hence,

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2(E_\Psi^\ell(iz;q)) = {}_{\ell}\mathcal{D}_M^q({}_{\ell}\mathcal{D}_M^q(E_\Psi^\ell(iz;q))) = {}_{\ell}\mathcal{D}_M^q(i(E_\Psi^\ell(iz;q))) = -E_\Psi^\ell(iz;q).$$

In view of (41), this may be written as

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2(\mathrm{Cos}_\Psi^\ell(z;q) + i\ \mathrm{Sin}_\Psi^\ell(z;q)) = -(\mathrm{Cos}_\Psi^\ell(z;q) + i\ \mathrm{Sin}_\Psi^\ell(z;q)).$$

By making an appeal to the property (42) and comparing real and imaginary parts, we find that

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2\ \mathrm{Cos}_\Psi^\ell(z;q) + \mathrm{Cos}_\Psi^\ell(z;q) = 0,$$

and

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2\ \mathrm{Sin}_\Psi^\ell(z;q) + \mathrm{Sin}_\Psi^\ell(z;q) = 0. \qquad \square$$

3.3. **q-ℓ-Ψ Hyperbolic Functions.** From the Definition 3.1,

$$\begin{aligned}
&\frac{1}{2}\left[E_\Psi^\ell(z;q) + E_\Psi^\ell(-z;q)\right] \\
&= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}} + \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ (-z)^n}{(q;q)_n^{\ell n+1}}\right] \\
&= \frac{1}{2}\left[1 + \frac{z}{(q;q)_1^{\ell+1}} + \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} + \ldots - 1 - \frac{z}{(q;q)_1^{\ell+1}} - \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} - \ldots\right] \\
&= \frac{1}{2}\left[2\left(1 + \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} + \frac{q^{4(4-1)}\ z^4}{(q;q)_4^{4\ell+1}} + \ldots\right)\right] \\
&= \sum_{n=0}^{\infty}\frac{q^{2n(2n-1)}\ z^{2n}}{(q;q)_{2n}^{2\ell n+1}}, \qquad (43)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\left[E_\Psi^\ell(z;q) - E_\Psi^\ell(-z;q)\right] \\
&= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}} - \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ (-z)^n}{(q;q)_n^{\ell n+1}}\right] \\
&= \frac{1}{2}\left[1 + \frac{z}{(q;q)_1^{\ell+1}} + \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} + \ldots - 1 - \frac{-z}{(q;q)_1^{\ell+1}} - \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} - \ldots\right] \\
&= \frac{1}{2}\left[2\left(\frac{z}{(q;q)_1^{\ell+1}} + \frac{q^{3(3-1)}\ z^3}{(q;q)_3^{3\ell+1}} + \frac{q^{5(5-1)}\ z^5}{(q;q)_5^{5\ell+1}} + \ldots\right)\right] \\
&= \sum_{n=0}^{\infty}\frac{q^{2n(2n+1)}\ z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}. \qquad (44)
\end{aligned}$$

From the first and second series of the right hand side of (43) and (44), we extend the hyperbolic q-Cosine and q-Sine functions which are denoted here by

$\mathrm{Cosh}_{\Psi}^{\ell}(z;q)$ and $\mathrm{Sinh}_{\Psi}^{\ell}(z;q)$ respectively. In fact, for any $z \in \mathbb{C}$,

$$\frac{1}{2}\left[E_{\Psi}^{\ell}(z;q) + E_{\Psi}^{\ell}(-z;q)\right] := \mathrm{Cosh}_{\Psi}^{\ell}(z;q), \tag{45}$$

and

$$\frac{1}{2}\left[E_{\Psi}^{\ell}(z;q) - E_{\Psi}^{\ell}(-z;q)\right] := \mathrm{Sinh}_{\Psi}^{\ell}(z;q), \tag{46}$$

which together imply that

$$E_{\Psi}^{\ell}(z;q) = \mathrm{Cosh}_{\Psi}^{\ell}(z;q) + \mathrm{Sinh}_{\Psi}^{\ell}(z;q). \tag{47}$$

Remark 3.11. $\mathrm{Cosh}_{\Psi}^{0}(z;q) = \mathrm{Cosh}_{q}(z)$, and $\mathrm{Sinh}_{\Psi}^{0}(z;q) = \mathrm{Sinh}_{q}(z)$.
In particular, from Remark 3.2

$$\begin{aligned} \mathrm{Cosh}_{\Psi}^{\ell}(0;q) &= \frac{1}{2}\left[E_{\Psi}^{\ell}(0;q) + E_{\Psi}^{\ell}(0;q)\right] = 1, \\ \mathrm{Sinh}_{\Psi}^{\ell}(0;q) &= \frac{1}{2}\left[E_{\Psi}^{\ell}(0;q) - E_{\Psi}^{\ell}(0;q)\right] = 0. \end{aligned}$$

Parallel to Theorem 3.10, we have

Theorem 3.12. *The hyperbolic q-ℓ-Ψ Cosine and Sine functions are solutions of the differential equation*

$$\left({}_{\ell}\mathcal{D}_{M}^{q}\right)^{2}\nu - \nu = 0.$$

Proof. It is seen from (45), (42) and (35) that

$$\begin{aligned} &\left({}_{\ell}\mathcal{D}_{M}^{q}\right)^{2}\mathrm{Cosh}_{\Psi}^{\ell}(z;q) - \mathrm{Cosh}_{\Psi}^{\ell}(z;q) \\ &= \left({}_{\ell}\mathcal{D}_{M}^{q}\right)^{2}\left(\frac{E_{\Psi}^{\ell}(z;q) + E_{\Psi}^{\ell}(-z;q)}{2}\right) - \left(\frac{E_{\Psi}^{\ell}(z;q) + E_{\Psi}^{\ell}(-z;q)}{2}\right) \\ &= \frac{1}{2}\left[E_{\Psi}^{\ell}(z;q) + E_{\Psi}^{\ell}(-z;q) - E_{\Psi}^{\ell}(z;q) - E_{\Psi}^{\ell}(-z;q)\right] = 0. \end{aligned}$$

Likewise

$$\left({}_{\ell}\mathcal{D}_{M}^{q}\right)^{2}\mathrm{Sinh}_{\Psi}^{\ell}(z;q) - \mathrm{Sinh}_{\Psi}^{\ell}(z;q) = 0$$

follows. □

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AN IRREDUBILITY CRITERION FOR INTEGER POLYNOMIALS*

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Abstract. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with coefficients from the ring $\mathbb{Z}$ of integers satisfying either (i) $0 < a_0 \leq a_1 \leq \cdots \leq a_{k-1} < a_k < a_{k+1} \leq \cdots \leq a_n$ for some k, $0 \leq k \leq n-1$; or (ii) $|a_n| > |a_{n-1}| + \cdots + |a_0|$ with $a_0 \neq 0$. In this paper, it is proved that if $|a_n|$ or $|f(m)|$ is a prime number for some integer m with $|m| \geq 2$ then the polynomial $f(x)$ is irreducible over $\mathbb{Z}$.

1. Introduction

One of the major themes in the development of number theory is the relationship between prime numbers and irreducible polynomials. There are better methods to determine prime numbers than to determine irreducible polynomials over the ring $\mathbb{Z}$ of integers. In 1874, Bouniakowsky [1] made a conjecture that if $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial for which the set of values $f(\mathbb{Z}^+)$ has greatest common divisor 1, then $f(x)$ represents prime numbers infinitely often. Bouniakowsky conjecture is true for polynomials of degree 1 in view of the well known Dirichlet's Theorem for primes in arithmetic progression's; however it is still open for higher degree polynomials. The converse of the Bouniakowsky conjecture, viz. if $f(x)$ is an integer polynomial such that the set $f(\mathbb{Z}^+)$ has infinitely many prime numbers, then $f(x)$ is irreducible over $\mathbb{Z}$, is true; because if $f(x) = g(x)h(x)$ where $g(x)$, $h(x) \in \mathbb{Z}[x]$, then at least one of the polynomials $g(x)$, $h(x)$, say $g(x)$ takes the values ± 1 infinitely many times which is not possible, since the polynomials $g(x)+1$ and $g(x)-1$ can have at most finitely many roots over the field of complex numbers. In this paper, we give a proof (using elementary methods) of a similar version of the converse for a special class of polynomials with a weaker hypothesis. Indeed we prove the following theorem.

Theorem 1.1. *Let* $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ *be a polynomial which satisfies one of the following conditions:*
(i) $0 < a_0 \leq a_1 \leq \cdots \leq a_{k-1} < a_k < a_{k+1} \leq \cdots \leq a_n$ *for some* k, $0 \leq k \leq n-1$*;*

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(ii) $|a_n| > |a_{n-1}| + \cdots + |a_0|$ *with* $a_0 \neq 0$.
Suppose that $|a_n|$ *is a prime number or* $|f(m)|$ *is a prime for some integer* m *with* $|m| \geq 2$. *Then* $f(x)$ *is irreducible in* $\mathbb{Z}[x]$. *Further if* $|m|$ *is the* r^{th} *power of some integer, then* $f(x^r)$ *is irreducible in* $\mathbb{Z}[x]$.

It may be pointed out that a similar result is proved in [3, Theorem 1] with a different hypothesis.

2. Preliminary Results

Let the set $\{z \in \mathbb{C} : |z| < 1\}$ be denoted by $\mathcal{C}$, where $\mathbb{C}$ denotes the complex numbers. The following proposition proved in [2, Proposition 2.3] will be used in the sequel. We omit its proof.

Proposition 2.A. Let $f(x) = \sum\limits_{l=0}^{n} a_l x^l \in \mathbb{Q}[x]$ and suppose that $a_i \neq 0$ and $a_j \neq 0$ for some $0 \leq i < j \leq n$. Suppose further that

$$\sum_{0 \leq l \leq n; l \neq t} |a_l| \leq q^t |a_t| \tag{1}$$

for some $0 \leq t \leq n$, with $t \neq i$ and $t \neq j$, and some $q \in \mathbb{R}$ with $0 < q \leq 1$. If $f(x)$ has a zero $\alpha \in \{z \in \mathbb{C} | q \leq |z| \leq 1\}$, then equality holds in (1) and $\alpha^{2(j-i)} = 1$.

Now we prove two elementary lemmas which are of independent interest as well.

Lemma 2.1. *Let* $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ *be a polynomial such that* $0 < a_0 \leq a_1 \leq \cdots \leq a_{k-1} < a_k < a_{k+1} \leq \cdots \leq a_{n-1} \leq a_n$ *for some* k, $0 \leq k \leq n-1$. *Then* $f(x)$ *has all zeros in the set* $\mathcal{C}$.

Proof. We first show that $f(x)$ has all zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$. Suppose to the contrary that $f(x)$ has a zero α with $|\alpha| > 1$. Then α is a root of $F(x) = (x-1)f(x) = a_n x^{n+1} + (a_{n-1} - a_n)x^n + \cdots + (a_0 - a_1)x - a_0$. Therefore in view of the hypothesis and the assumption $|\alpha| > 1$, we have

$$\begin{aligned} |a_n \alpha^{n+1}| &\leq a_0 + (a_1 - a_0)|\alpha| + \cdots + (a_n - a_{n-1})|\alpha|^n \\ &< a_0|\alpha|^n + (a_1 - a_0)|\alpha|^n + \cdots + (a_n - a_{n-1})|\alpha|^n = |a_n \alpha^n| \end{aligned}$$

which is a contradiction as $a_n > 0$. Now we show that $|\alpha| < 1$. Assume that $|\alpha| = 1$. Observe that the coefficients of x^k and x^{k+1} in $F(x)$ are negative and other coefficients except a_n are non-positive. Thus the hypothesis of Proposition 2.A is satisfied for $t = n+1, i = k, j = k+1$ and $q = 1$. By this proposition, we have $\alpha^2 = 1$, which is impossible as $f(1)$ and $f(-1)$ are easily seen to be non-zero using the hypothesis. □

Lemma 2.2. *Let* $f(x) \in \mathbb{Z}[x]$ *be a polynomial having all its zeros in the set* $\mathcal{C}$. *If there exists an integer* m *with* $|m| \geq 2$ *such that* $|f(m)|$ *is a prime number, then* $f(x)$ *is irreducible in* $\mathbb{Z}[x]$.

Proof. Suppose to the contrary that $f(x) = g(x)h(x)$, where $g(x), h(x) \in \mathbb{Z}[x]$ and neither $g(x)$ nor $h(x)$ is a unit in $\mathbb{Z}[x]$. In view of the hypothesis, at least one of $|g(m)|, |h(m)|$ equals to 1. Say $|g(m)| = 1$. It follows that either $g(x)$ is a unit in

$\mathbb{Z}[x]$ or $\deg(g(x)) \geq 1$. If $\deg(g(x)) \geq 1$, then we can write $g(x) = c\prod_{i=1}^{k}(x-\alpha_i)$, where $\alpha_i \in \mathbb{C}$ and $k \geq 1$. Keeping in mind that $|\alpha_i| < 1$ and $|m| \geq 2$, we have

$$|g(m)| = |c\prod_{i=1}^{k}(m-\alpha_i)| \geq |c|\prod_{i=1}^{k}(|m|-|\alpha_i|) > |c|\prod_{i=1}^{k}(|m|-1) \geq 1,$$

which gives $|g(m)| > 1$, leading to a contradiction. So we have that $g(x)$ is a unit in $\mathbb{Z}[x]$, which is again a contradiction and hence the lemma is proved. □

Arguing as in the above lemma, the following lemma can be easily proved.

Lemma 2.3. *Let $f(x) \in \mathbb{Z}[x]$ be a polynomial having all its zeros in the set $\{z \in \mathbb{C} : |z| > 1\}$. If $|f(0)|$ is a prime number, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.*

3. Proof of the Theorem 1.1.

Note that if $f(x)$ satisfies condition (i) of the theorem, then in view of Lemma 2.1, it has all its zeros inside the unit circle $\mathcal{C} = \{z \in \mathbb{C} : |z| < 1\}$. Observe that if (ii) holds and $|\alpha| \geq 1$, then

$$|f(\alpha)| \geq |a_n||\alpha|^n - \sum_{j=0}^{n-1}|a_j||\alpha|^j \geq |\alpha|^n\left(|a_n| - \sum_{j=0}^{n-1}|a_j|\right) \geq 1,$$

so $f(\alpha) \neq 0$. Thus, in case $f(x)$ satisfies condition (ii) of the theorem then all its roots lie in $\mathcal{C}$. As $a_0 \neq 0$ by hypothesis, it now follows that all roots of the polynomial $x^n f(\frac{1}{x}) = g(x)$(say) lie in the set $\{z \in \mathbb{C} : |z| > 1\}$. Therefore when $|a_n| = |g(0)|$ is a prime number then $g(x)$ is irreducible over $\mathbb{Z}$ by virtue of Lemma 2.3 and hence is $f(x)$. If there exists an integer m with $|m| \geq 2$ such that $|f(m)|$ is prime then $f(x)$ is irreducible over $\mathbb{Z}$ in view of Lemma 2.2. It only remains to be shown that if $|m|$ is a r^{th} power of some integer then $f(x^r)$ is irreducible over $\mathbb{Z}$. Since all the roots of $f(x)$ lie in $\mathcal{C}$, therefore so do the roots of $f(\pm x^r)$. As $|f(m)|$ is a prime number, the desired assertion follows from Lemma 2.2.

The following examples are quick applications of Theorem 1.1.

Example 3.1. Let $2^{2^n}+1$ be a prime number for some $n \in \mathbb{N}$. Then the polynomial $F(x) = (2^{2^n}+1)x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with $1 \leq a_0 \leq \cdots \leq a_{k-1} < a_k < a_{k+1} \leq \cdots \leq a_{n-1} \leq 2^{2^n}+1$ for some $1 \leq k \leq n-1$ is irreducible by Theorem 1.1.

Example 3.2. Let $F(x) = 10x^5 + 3x^4 + 2x^3 + x^2 + 1$ be a polynomial. Since $F(2) = 389$ is a prime, $F(x)$ is irreducible in view of Theorem 1.1.

Example 3.3. Let $f(x) = 2^{51}x^{10} + 2^{10}x^9 - 2^{11}x^8 + 2^{15}x^7 - 2^{16}x^6 - 1$ be a polynomial. One can check that $f(2) = 2^{61}-1$ is a prime number (Mersenne prime) and $f(x)$ satisfies condition (ii) of the Theorem 1.1. Therefore $f(x)$ is irreducible in $\mathbb{Z}[x]$ in view of Theorem 1.1.

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THREE FUNDAMENTAL THEOREMS OF FUNCTIONAL ANALYSIS REVISITED

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Abstract. We give an overview of the three fundamental theorems, i.e., Open Mapping Theorem, Closed Graph Theorem and Uniform Boundedness Principle of functional analysis via the Zabreiko's lemma.

1. Introduction

We state the three fundamental theorems of functional analysis, namely, the open mapping theorem, the closed graph theorem and the uniform boundedness principle and give the details of proofs of these theorems by using the Zabreiko's lemma [9]. We also provide some comments, observations and applications of these results. The developments discussed here could not be seen in many books of functional analysis including [5, 4]. All the linear spaces considered here are over the same field of scalars, real or complex. The Hahn-Banach Theorem implies that for a normed linear space X, the dual space X^* (the normed linear space of the continuous linear functionals on X) is nonempty (as one can easily construct a continuous linear functional on one dimensional spaces) and moreover, there are sufficiently many bounded linear functionals to separate points of X, i.e., for any two points $x_1, x_2 \in X$ there is a $x' \in X^*$ such that $x'(x_1) = 0$ and $x'(x_2) = 1$. The Banach space $X^{**} = (X^*)^*$ is called the second dual of X. Every $x \in X$ can be identified as an element of X^{**} by the evaluation formula

$$x(x') = x'(x),$$

i.e., X can be viewed as a subspace of X^{**}. To indicate that there is some symmetry between X, X^* and X^{**} we write

$$x'(x) = \langle x', x\rangle.$$

In general $X \neq X^{**}$. Spaces for which $X = X^{**}$ are called reflexive spaces. Standard examples of reflexive spaces are: finite dimensional Banach spaces, Hilbert spaces, l^p spaces, $1 < p < \infty$ (with usual norms). However, l^1 and

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l^∞ (with usual norms) are nontrivial spaces which fail to be reflexive. Uniform boundedness principle is perhaps the strongest Banach space result having important applications. For example,

1. in deriving global (uniform) estimates from pointwise estimates.

2. in showing convergence of quadrature formula, in proving divergence of Fourier series of continuous functions at points and in various summability methods.

3. in a way, it says: to show a subset B of a Banach space is bounded, it suffices to 'look' at B through the bounded linear functionals.

If a mapping T from a set X to a set Y is bijective then there exists a map T^{-1} from Y to X; if X and Y are linear spaces and T is additionally a linear map then so is T^{-1}; if X and Y are Banach spaces and T is additionally bounded also, then T^{-1} is bounded as well - the fact that is known as the 'open mapping theorem' (formally stated in section 3) and is used to study perturbation technique in the theory of numerical differential equations.

For Banach spaces X and Y, one defines the two algebraic operations on $X \times Y$ in the usual pointwise manner and can define norm on it by $||(x,y)|| = ||x|| + ||y||$, $x \in X$, $y \in Y$. For a linear transformation $T : X \to Y$, its graph is then given by $\{(x, T(x)) | x \in X\}$ and is denoted by $G(T)$. Obviously, it is a subset of $X \times Y$ and we say that the graph of T is closed if $G(T)$ is a closed subset of $X \times Y$.

It is easy to see [8, p. 132] that the graph of any continuous map from a topological space E to a Hausdorff space F is closed in the product space $E \times F$. In the present context, clearly therefore, graph of a continuous linear transformation from a Banach space X to a Banach space Y is closed. The fact that the converse of this is also true is the well known closed graph theorem (formally stated in section 3).

The third theorem is the uniform boundedness principle which says that a pointwise bounded family of continuous linear transformations from a Banach space to a normed linear space is uniformly bounded, the formal statement of which is given in the section 3.

To our knowledge Zabreiko's work [9] is not cited in many functional analysis books except [3, 6]. The present article addresses the proofs of said fundamental theorems in very simple and lucid manner. We hope that these proofs will reach to a large audience in the field of functional analysis. For more details see [9, 3]. Earlier Helson [1, 2], using an absolutely original approach, proved a better result than the uniform boundedness principle via probabilistic methods ([1], MR0461090 (57#1075), by D. A. Herrero).

2. Seminorm and Countably Subadditive Property

We restrict ourselves to real vector spaces, define a seminorm on such a vector space and derive its several properties.

Definition 1. *A seminorm (or a prenorm) on a real vector space X is a real valued function p on X such that the following conditions are satisfied.*
(i) p is positively homogeneous, i.e., $p(\alpha x) = |\alpha| p(x)$, $x \in X$, α real;
(ii) p is subadditive, i.e., $p(x+y) \leq p(x) + p(y)$ $\forall x, y \in X$.

For example, if X and Y are normed spaces and T is a linear transformation from X to Y, then T induces a seminorm on X through the function $x \mapsto ||Tx||$ from X to $\mathbb{R}$ as can be seen from the following. Indeed,
(i) $\alpha x \mapsto ||T(\alpha x)|| = ||\alpha T(x)|| =| \alpha | \, ||Tx||, \quad x \in X$, α real;
(ii) $x + y \mapsto ||T(x+y)|| = ||Tx + Ty|| \leq ||Tx|| + ||Ty||, \quad \forall x, y \in X$.
Of course a norm is a seminorn if we take $T = I$ - the identity operator on the space X $(= Y)$. Suppose p is a seminorm on a vector space. Then $p(0) = 0$, since $p(0) = p(0.0) = 0p(0) = 0$. It is also nonnegative as $0 = p(0) = p(x + (-x)) \leq p(x) + p(-x) = p(x) + |-1|p(x) = 2p(x)$, i.e., $p(x) \geq 0$.

Thus a seminorm is a function on a vector space that satisfies the definition of a norm except that the value of the seminorm of a nonzero vector is allowed to be zero.
Let (x_n) be a sequence in a normed linear space X and

$$s_m = \sum_{n=1}^{m} x_n, \quad m = 1, 2, \cdots .$$

The series $\sum\limits_{n=1}^{\infty} x_n$ is said to be convergent in X if the sequence s_m of its partial sums converges in X, i.e., there exists $x \in X$ such that $s_m \to x$ as $m \to \infty$; we then write $x = \sum_{n=1}^{\infty} x_n$ and call it the sum of the series.

Definition 2. A nonnegative function f on a normed linear space X is said to be countably subadditive if

$$f\Big(\sum_{n=1}^{\infty} x_n\Big) \leq \sum_{n=1}^{\infty} f(x_n)$$

for each convergent series $\sum\limits_{n=1}^{\infty} x_n$ in X.

For example, a norm of a normed linear space is always countably subadditive. More generally, a continuous seminorm on a normed linear space is countably subadditive. The following lemma due to Zabreiko [9] is a converse of this when X is a Banach space. See [9, 3] for its proof.
Lemma 1. [Zabreiko's lemma] *Every countably subadditive seminorm on a Banach space is continuous.*

Note that the Zabreiko's lemma does not extend to all the normed linear spaces. In this regard we have the following counterexample.

Example 1. Let Y be the space of finitely nonzero sequences under pointwise operations and equipped with the maximum norm. It can be checked that this Y

is not a Banach space. The function $p : Y \to \mathbb{R}$ defined by $p(y) = \sum_{i=1}^{\infty} |y_i|,\ y = (y_1, y_2, y_3, ...)$ can be checked to be a seminorm on Y. However p is not continuous on Y as $x^{(n)} = (1, 1, ...1, 0, 0, 0, ...)$, with first n-components 1, belongs to Y, $||x^{(n)}||_\infty = 1$, for all $n = 1, 2, 3, ...$ but $p(x^{(n)}) = n$ for all n. □

3. Fundamental Theorems

Theorem 1. (Open Mapping Theorem)[8, 9, 3]. *If X and Y are Banach spaces and T is a continuous linear transformation of X onto Y, then T is an open mapping, i.e., if U is open in X then $T(U)$ is open in Y.*

Proof. First assume that the image of $U = B(0, 1) = \{x \in X|\ ||x|| < 1\}$ by T is open in Y. Let V be an open subset of X. If $x \in V$ then $B(x, r) = x + rU \subseteq V$ for some positive integer r. Hence, T being linear,

$$T(x) + rT(U) \subseteq T(V),$$

i.e., $T(V)$ includes the neighborhood $T(x) + rT(U)$ of $T(x)$. It follows that $T(V)$ is open. Thus the theorem will be proved once it is shown that $T(U)$ is open in Y.

Now for each $y \in Y$, let $p(y) = \inf\ \{||x|| : x \in X,\ Tx = y\}$. Then p is well defined as T is onto. We will show that p is countably subadditive seminorm on Y. In fact, if $y \in Y$ and α is a nonzero scalar then $\{x\ :\ x \in X,\ Tx = \alpha y\} = \{\alpha x\ :\ x \in X,\ Tx = y\}$, and hence
$p(\alpha x) = \inf\ \{||\alpha x|| : x \in X,\ Tx = y\}\ = |\alpha|\inf\{||x|| : x \in X,\ Tx = y\}\ = |\alpha|\ p(y)$.
Since $p(0y) = 0 = |0|p(y)$ whenever $y \in Y$, it follows that

$$p(\alpha y) = |\alpha|\ p(y)\ \forall \text{ scalar } \alpha \text{ and each } y \in Y.$$

Next, let $\sum_{n=1}^{\infty} y_n$ be a convergent series in Y. We show that $p(\sum_{n=1}^{\infty} y_n) \leq \sum_{n=1}^{\infty} p(y_n)$. If $\sum_{n=1}^{\infty} p(y_n) = \infty$, the result follows. Assume that $\sum_{n=1}^{\infty} p(y_n) < \infty$. Take any positive ϵ and fix it. Let (x_n) be a sequence in X such that $Tx_n = y_n$ and $||x_n|| < p(y_n) + \frac{\epsilon}{2^n}$ for each n. Then $\sum_{n=1}^{\infty} ||x_n|| \leq \sum_{n=1}^{\infty} p(y_n) + \epsilon$, a finite number. Thus $\sum_{n=1}^{\infty} \|x_n\|$ converges. Since X is a Banach space, $\sum_{n=1}^{\infty} x_n$ converges. Now, $T(\sum_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} T(x_n) = \sum_{n=1}^{\infty} y_n$; and since $p(y) \leq ||x||$, we get

$$p(\sum_{n=1}^{\infty} y_n) \leq ||\sum_{n=1}^{\infty} x_n|| \leq \sum_{n=1}^{\infty} ||x_n|| < \sum_{n=1}^{\infty} p(y_n) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that p is a countably subadditive. Setting $y_n = 0$ for $n \geq 3$, finite additivity of p obviously follows. p being countably subadditive seminorm in Y, it is continuous by Zabreiko's lemma. Now observe that

$$T(U) = \{y \in Y : y = Tx \text{ for some } x \in U\} = \{y \in Y : p(y) < 1\} = p^{-1}(-\infty, 1),$$

and hence $T(U)$ is open. This completes the proof. $\square$

A one-to-one map from one topological space onto another topological space is a homeomorphism iff it is both continuous and open. Combining this with the above theorem we get the following result.

Corollary 1. *Every one-one and onto bounded linear operator on a Banach space is a homeomorphism.*

Note that this corollary is also named as the bounded inverse theorem.

Theorem 2. (Closed graph theorem)[8, p.238],[9, 3]. *If X and Y are Banach spaces and $T : X \to Y$ is a linear transformation then T is continuous if its graph is closed.*

Proof. Let us consider the map $p(x) = ||Tx||$ for all $x \in X$. We prove that p defines a countably subadditive seminorm. Observe that $p(x+y) = ||T(x+y)|| \leq ||Tx|| + ||Ty|| = p(x) + p(y)$, and $p(\alpha x) = ||T(\alpha x|| = |\alpha|||Tx|| = |\alpha|p(x)$, i.e, p is a seminorm on X. Next we show that p is countably subadditive.

Let $\sum\limits_{n=1}^{\infty} x_n$ be a convergent series in X. To show that $p(\sum\limits_{n=1}^{\infty} x_n) \leq \sum\limits_{n=1}^{\infty} p(x_n)$, i.e, $||T(\sum\limits_{n=1}^{\infty} x_n)|| \leq \sum\limits_{n=1}^{\infty} ||T(x_n)||$, which is obviously true if $\sum\limits_{n=1}^{\infty} ||T(x_n)|| = \infty$. Hence, assume that $\sum\limits_{n=1}^{\infty} ||T(x_n)|| < \infty$. Then Y being Banach space, it follows that $\sum\limits_{n=1}^{\infty} Tx_n$ converges. Since $\sum\limits_{n=1}^{m} x_n \to \sum\limits_{n=1}^{\infty} x_n$, $T(\sum\limits_{n=1}^{m} x_n) = \sum\limits_{n=1}^{m} Tx_n \to \sum\limits_{n=1}^{\infty} Tx_n$ as $m \to \infty$, $\left(\sum\limits_{n=1}^{m} x_n, \sum\limits_{n=1}^{m} Tx_n\right) \in G(T)$ for each m and $G(T)$ is closed, it follows that $T(\sum\limits_{n=1}^{\infty} x_n) = \sum\limits_{n=1}^{\infty} Tx_n$. Therefore

$$||T(\sum_{n=1}^{\infty} x_n)|| = ||\sum_{n=1}^{\infty} Tx_n|| \leq \sum_{n=1}^{\infty} ||Tx_n||,$$

which shows that p is countably subadditive. Hence p is continuous by Zabreiko's lemma, i.e., $\forall \epsilon > 0$, $\exists a\ \delta > 0$, such that $|p(x) - p(0)| < \epsilon$ whenever $||x|| < \delta$, i.e, $||Tx|| < \epsilon$ whenever $||x|| < \delta$. Hence T is continuous. $\square$

Theorem 3. (Uniform Boundedness Principle)[8, p.239],[9, 3]. *If $F = \{T_i\}$ is a family of continuous linear transformations from a Banach space X into a normed linear space Y with the property that $\{T_i(x)\}$ is a bounded subset of Y for each x in X, i.e., $\sup\{||T_i(x)|| : T_i \in F\}$ is finite for each x in X, then $\{||T_i||\}$ is a bounded set of scalars, i.e., $\sup\{||T_i|| : T_i \in F\}$ is finite.*

This is also known as the **Banach-Steinhaus Theorem**.

Proof. Let $p(x) = sup\ \{||Tx||\ :\ T \in \mathcal{F}\}$ for each $x \in X$, then p is finite valued. We will show that p defines a countably subadditive seminorm. Observe that

$$p(\alpha x) = sup\ \{\ \|T(\alpha x)\|\ :\ T \in \mathcal{F}\} = |\alpha|\ sup\ \{\|T(x)\|\ :\ T \in \mathcal{F}\} = |\alpha|\ p(x)$$

for each $x \in X$ and each scalar α. If $\sum_{n=1}^{\infty} x_n$ is a convergent series in X and $T \in \mathcal{F}$, then T being continuous we have

$$\|T\left(\sum_{n=1}^{\infty} x_n\right)\| = \|\sum_{n=1}^{\infty} Tx_n\| \leq \sum_{n=1}^{\infty} \|Tx_n\| \leq \sum_{n=1}^{\infty} p(x_n),$$

$$\text{i.e.,} \quad p(\sum_{n=1}^{\infty} x_n) \leq \sum_{n=1}^{\infty} p(x_n).$$

In particular, letting $x_n = 0$ when $n \geq 3$, one gets $p(x_1+x_2) \leq p(x_1)+p(x_2)$, i.e, p is countably subadditive seminorm on X. Therefore p is continuous by Zabreiko's lemma. Hence $\exists\ \delta > 0$ such that $p(x) \leq 1$ whenever $||x|| \leq \delta$. It follows that for $x \neq 0$, $||\delta \frac{x}{||x||}|| = \delta \Rightarrow p(\delta \frac{x}{||x||}) \leq 1$, which means $\delta p(y) \leq 1$ for all $y \in\ B(0,1)$. Thus $p(y) \leq \delta^{-1}$ for all $y \in\ B(0,1)$. Therefore $\|T(x)\| \leq \delta^{-1}$ for all $T \in \mathcal{F}$ and for all $x \in B(0,1)$. Therefore $\|T\| \leq \delta^{-1}$ for all $T \in \mathcal{F}$. This completes the proof. □

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MANIFOLD TOPOLOGY: A PRELUDE

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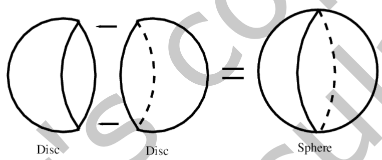
Abstract. In this expository article we describe some of the fundamental questions in Manifold Topology and the obstruction groups which give answers to the questions.

1. Introduction and some basic concepts

In Topology we study spaces and properties of spaces which are invariant under 'deformation'. For example, we do not distinguish between a solid rubber ball and a solid rubber cube because one can be deformed into another without tearing. Next, we can build new spaces by gluing 'simple' pieces. For example, take two 2-discs and glue them along their boundaries, we get a sphere. See the following picture.



This way one constructs very complicated useful spaces and then, we ask when are any two such objects same under deformation. To answer such questions we need to define invariants of spaces which remain same under deformations. Many a time this method is useful to give answers in negative and also sometime one finds a complete set of invariants to give a positive answer.

Note that, there are mainly two kinds of deformations (or equivalences) we deal with in this subject; one is 'homeomorphism' and the other is 'homotopy equivalence'. The first one only 'stretches' the underlying space without tearing and the second one 'squeezes' or 'thickens' the underlying space continuously.

There are various other kinds of deformations we come across, e.g., *weak homotopy equivalence*, *simple homotopy equivalence*, *diffeomorphism* and *PL-homeomorphism*.

We urge the reader to look at the reference [16], or any other text book on Algebraic Topology for the remainder of this section.

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1.1. **Homotopy Theory.**

Definition 1.1.1. Two topological spaces X and Y are called *homeomorphic* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$, here id_* denotes the identity map. When this happens we call f (or g) a *homeomorphism.*

Definition 1.1.2. Two continuous maps $f, g : X \to Y$ are called *homotopic* if there is a continuous map $F : X \times I \to Y$, such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$. If $x_0 \in X$ and $y_0 \in Y$ are given such that $f(x_0) = y_0$ and $g(x_0) = y_0$, then f and g are called *homotopic relative to base point* if, in addition, $F(x_0, t) = y_0$ for all $t \in I$.

Definition 1.1.3. Two topological spaces X and Y are called *homotopy equivalent* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. When this happens we call f (or g) a *homotopy equivalence.*

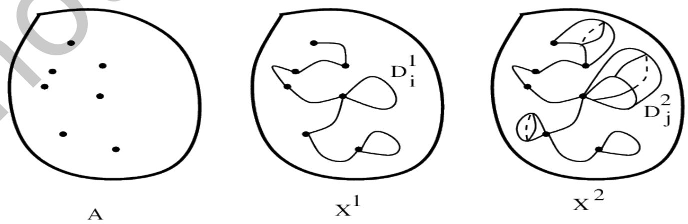
Obviously, homeomorphism is stronger than homotopy equivalence, but there are times when homotopy equivalence implies homeomorphism. These kinds of instances are major breakthroughs in Topology.

In this article we will be considering spaces which are 'CW-complexes' and 'manifolds'. CW-complexes are spaces which are built from the following subspaces of $\mathbb{R}^n$.

$$\mathbb{D}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{i=n} x_i^2 \leq 1\}.$$

Here, $\mathbb{D}^n$ is called an *n-disc* and n its *dimension.* The boundary $\partial\mathbb{D}^n$ is defined as $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{i=n} x_i^2 = 1\}$, it is also called the $(n-1)$-sphere $\mathbb{S}^{n-1}$. These subspaces are the simple pieces we referred above.

To avoid unnecessary hypothesis, we define a *finite CW-complex.* We begin with a finite set A with discrete topology. Then, consider finitely many maps $\phi_i^1 : \mathbb{S}^0 \to A$, for $i = 1, 2, \ldots, j_1$. Construct the quotient space X^1 of the disjoint union $A \cup_{i=1}^{i=j_1} \mathbb{D}_i^1$, where each $\mathbb{D}_i^1$ is a copy of $\mathbb{D}^1$, under the relation: $\phi_i^1(x)$ is identified with x where $x \in \mathbb{S}^0$. Suppose we have constructed X^{n-1}. Next, consider continuous maps $\phi_i^n : \mathbb{S}^{n-1} \to X^{n-1}$, for $i = 1, 2, \ldots, j_n$, and construct the quotient space



CW–complex construction

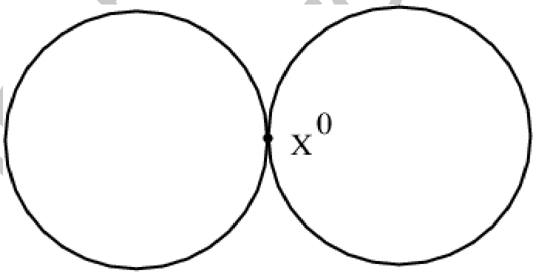
$X := X^n$ of X^{n-1} disjoint union with j_n copies of $\mathbb{D}^n$, in a similar way as defined in the case $n = 1$. X is called a *finite CW-complex* and X^k is called its *k-skeleton*. Above we show the construction of a CW-complex pictorially. Also, we say X^k is obtained from X^{k-1} by attaching j_k k-discs by the *characteristic maps* ϕ_i^k, for $i = 1, 2, \ldots, j_k$. Note that each $\mathbb{D}_i^k - \mathbb{S}^{k-1}$ is embedded in X and is called a *k-cell*. The maximum value of k, for which X has a k-cell but no cells of higher dimensions, is called the *dimension* of the CW-complex. We call X finite as there are only finitely many cells in X. We can similarly define an infinite CW-complex which has finitely many cells in each dimension. To define CW-complex with arbitrary number of cells in each dimension is tricky and we avoid it here.

A subspace Y of a CW-complex X is called a *subcomplex* of X if Y is closed in X and is a union of cells of X. We call the pair (X, Y) a *CW-pair*. The dimension of the pair (X, Y) is defined considering the cells in X which are not in Y, and the k-skeleton of the pair (X, Y) is defined as $Y \cup X^k$. Therefore, the dimension of the pair (X, Y) could be less than the dimension of Y.

There is yet another class of spaces called *polyhedra* which is a special class of CW-complexes, in this case the attaching maps are injective.

In this article by a 'complex' we will always mean a CW-complex.

Example 1.1.1. Let $X^0 = \{*\}$ be a singleton and let $\phi_1^1, \phi_2^1 : \mathbb{S}^0 \to X^0$ be the obvious maps. Then X^1 is the one point union (or *wedge*) of two circles, called the *figure eight*. It is a finite complex. More generally, a wedge of finitely many spheres of (different dimensions) is also a finite complex.

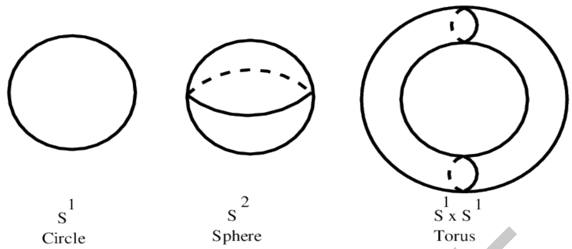


X^1 = Figure eight

Next, we define manifolds and give examples.

Definition 1.1.4. A Hausdorff second countable space M is called a *topological manifold with boundary* if any point $x \in M$ has a neighborhood V homeomorphic to an open set in $\mathbb{R}_+^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$, for some n. V is called an *Euclidean neighborhood* of x. n is called the *dimension* of M if the same n works for all points of M. The points on M which corresponds to the points in $\mathbb{R}_+^n$ with $x_n = 0$ are called the *boundary* points. A compact manifold with empty boundary is called *closed*. Similarly, one defines *smooth* or *differentiable* manifolds. There is yet another class of manifolds called *piecewise linear* or PL-manifolds.

By 'invariance of domain' (see Theorem 1.2.2), if M is connected then the dimension of M is well-defined. Below we give examples of closed manifolds of dimensions 1 and 2. An example of a manifold with boundary is $\mathbb{D}^n$. One can construct many other examples of manifolds with boundary by removing Euclidean neighborhoods of points from a manifold. The figure eight is a space which is not a manifold, the 0-cell is the troubling point.



Examples of manifolds

For some basics on manifold theory see the book [6] or any other book on manifolds. But, we recall the concept of tangent space which is intuitive. In the case of differentiable manifolds, by Whitney embedding theorem, a compact manifold M of dimension n can be embedded in the Euclidean space $\mathbb{R}^N$, for $N \geq 2n$. Now, we consider the set TM of disjoint union of all lines in $\mathbb{R}^N$ which are tangent to the manifold at some point. The collection of these lines form a manifold of dimension $2n$. And the set of all lines tangent to M at a given point is the translate of some n-dimensional vector subspace of $\mathbb{R}^N$. Therefore, there is a map $TM \to M$ which becomes a vector bundle projection of rank n (see Definition 1.2.1), called the *tangent bundle* of M. For example, one can show that for $\mathbb{S}^1$ the tangent bundle is, in fact, the cylinder $\mathbb{S}^1 \times \mathbb{R}$. Also, the tangent bundle of $\mathbb{R}^n$ is the trivial bundle $\mathbb{R}^n \times \mathbb{R}^n$.

Next, we recall a very crucial and important theorem in Homotopy Theory, called the *Whitehead theorem* which is used to study many fundamental questions in Manifold Topology, we will see in the next section. There are algebraic invariants called *homotopy groups* associated to topological spaces. It is defined as the homotopy classes of base point preserving maps from the n-sphere $\mathbb{S}^n$ with a base point to the topological space X with a base point, say x_0. It is denoted by $\pi_n(X, x_0)$. For $n \geq 1$ this set has a group structure and for $n = 0$ it is the set of all path components of X. For $n = 1$ it is called the *fundamental group* of the space. The Whitehead theorem says the following.

Theorem 1.1.1. *Let X and Y be complexes and $f : (X, x_0) \to (Y, f(x_0))$ be a map. If the induced homomorphisms $f_*^q : \pi_q(X, x_0) \to \pi_q(Y, f(x_0))$ are isomorphisms for all $x_0 \in X$ and $q \geq 1$ and a bijection for $q = 0$, then, f is a homotopy equivalence.*

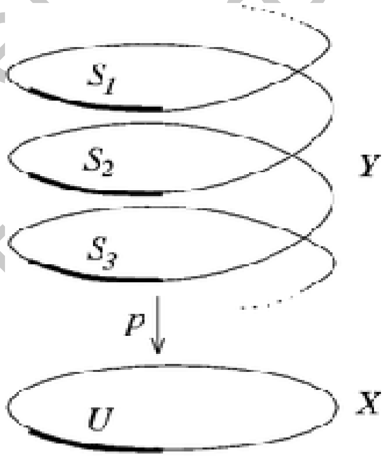
When the hypothesis of the theorem is satisfied, f is called a *weak homotopy equivalence*. See Theorem 7.5.9 and Corollary 7.6.24 in [16] for a proof and related materials.

For finite dimensional complexes, even a stronger statement is true. If X and Y are k-dimensional and $f : X \to Y$ induces isomorphisms on homotopy groups for all $q < n$, for some $n > k$ and induces a surjective homomorphism for $q = n$, then f is a homotopy equivalence. This implies that although a homotopy equivalence induces isomorphisms on homotopy groups in all dimensions; for finite complexes, the converse is true with a weaker assumption. That is, we only need to assume that f induces isomorphism in homotopy groups in low dimensions, although for most finite complexes there are nonzero homotopy groups in arbitrary high dimensions. This fact is very crucial in the proofs of many theorems in Manifold Topology as we highlighted above.

At this point we inform the reader about the significance of the theory of covering spaces, which is intimately related to fundamental group.

Definition 1.1.5. Given a topological space X, a *covering space* is defined to be a topological space Y together with a continuous map $p : Y \to X$, called *covering map*, which has the following property: given any point $x \in X$, there is a neighborhood U of x in X, such that $p^{-1}(U)$ is a disjoint union of open sets S_i, $i \in I$, of Y and $p|_{S_i} : S_i \to U$ is a homeomorphism for all $i \in I$. Here I is an indexing set.

For example, any homeomorphism is a covering map. An example of a covering



Non-trivial covering projection

map which is not a homeomorphism is the exponential map $\mathbb{R} \to \mathbb{S}^1$. A pictorial view of this covering map is given above.

One then defines, given a fixed space X, its universal covering space $\tilde{X}$ to be the maximal element in the category of all covering spaces of X. For technical reason, we need to assume that the space X is connected, locally path connected and semilocally simply connected, to ensure existence of the universal cover of X. Let us call such a space a *nice space*. Complexes are nice spaces.

The study of covering spaces helps us to understand the fundamental group well, through group action on spaces. (We recall here that, we say a group G *acts* on a space if there is a group homomorphism from G to the group of homeomorphisms of the space.) In fact, if X has a universal covering space, then the fundamental group G of X acts on $\tilde{X}$, so that the quotient $\tilde{X}/G = X$. This is the first instance one sees a connection between topological spaces and groups. This also transfers a topological problem into a problem in group theory and solves topological problems with the help of group theory and vice versa. For example, using covering space theory it becomes easy to prove that subgroup of a free group is free. Also, one checks using the above mentioned example that the fundamental group of the circle is infinite cyclic, which in turn proves the following Brouwer fixed point theorem in dimension 2.

Theorem 1.1.2. *For any map $f : \mathbb{D}^2 \to \mathbb{D}^2$, there is a point $x \in \mathbb{D}^2$ such that $f(x) = x$.*

In fact, this is true for $\mathbb{D}^n$ also, which requires the concept of homology theory.

1.2. **Homology Theory.** Homology theory is basically 'formal sum' of maps or of subspaces of a space, and then introducing 'natural' relations among elements of these sums to define invariants of the space. Initially, the subject is difficult to appreciate, but after certain amount of work one sees beautiful applications, which otherwise are impossible to see. One indication we have already given in the Brouwer fixed point theorem. Another advantage of this subject, in contrast to homotopy theory, is that the invariants, although defined in a difficult way, are not very difficult to compute.

The very first example is $H_0(X)$, of a topological space X. It is by definition, the free abelian group generated by the path components of X. Note that, if two spaces are homotopy equivalent then they have isomorphic $H_0(-)$.

Let us define this group differently, which will give the motivation for the definition of higher dimensional invariants.

Let $C_0(X)$ be the free abelian group generated by the points of X, and $C_1(X)$ be the free abelian group generated by all continuous maps $\sigma_1 : [0,1] \to X$. Define $\partial_1 : C_1(X) \to C_0(X)$ by $partial_1(\sigma_1) = \sigma_1(1) - \sigma_1(0)$, for all $\sigma_1 \in C_1(X)$. Then, ∂_1 is a homomorphism of abelian groups. It can be shown that $H_0(X)$ is isomorphic to $C_0(X)/\partial_1(C_1(X))$. Let ∂_q be the convex hull of the vectors $e_i \in \mathbb{R}^n$, for $i = 0, 1, 2, \ldots, n$, where $e_0 = (0, 0, \ldots, 0)$ and $e_i = (0, 0, \ldots, 1, \ldots, 0)$, where 1 is at the i-th place. Note that $\partial_1 = [0,1]$. Now, more generally, define $C_q(X)$ to be the free abelian group generated by all continuous maps $\sigma_q : \partial_q \to X$. Next, define the map (called *boundary map*) $\partial_q : C_q(X) \to C_{q-1}(X)$ by

$$\partial_q(\sigma_q) = \sigma_q^0 - \sigma_q^1 \ldots + (-1)^q \sigma_q^q.$$

Here, σ_q^i evaluated at $(x_1, x_2, \ldots, x_{q-1}) \in \partial_{q-1}$ is equal to σ_q evaluated at $(x_1, x_2, \ldots, 0(i-th), \ldots, x_{q-1}) \in \partial_q$. σ_q^i is said to be the *i-th face* of σ_q. One now checks that $\partial_{q-1} \circ \partial_q = 0$ for all q. Define $C_i(X) = 0$ if $i \leq -1$. We call $\{C_*(X), \partial_*\}$ the *singular chain complex* of X. It is easily checked that the kernel Z_q of ∂_q contains the image B_q of ∂_{q+1}. Define the q-the *singular homology group* of X as the quotient group Z_q/B_q and it is denoted by $H_q(X, \mathbb{Z})$. For a complex X, there is a similar object called *cellular chain complex*, denoted by $\{S_*(X), \partial_*\}$. The group $S_q(X)$ is defined as the free abelian group generated by the q-cells of X. In this particular case the boundary map is defined using singular homology theory. But, one can show that the singular and cellular homology (that is the homology of the chain complex $S_*(X)$) are isomorphic for complexes.

One can define the homology $H_*(X, A; \mathbb{Z})$ of pairs (X, A) by defining the associated chain complex as the quotient chain complex $\{C_*(X, A), \partial_*\}$, where $C_i(X, A) = C_i(X)/C_i(A)$.

Also, one constructs a dual class of groups $\{C^*(X)\}$ whose elements $C^i(X)$ are defined as $hom(C_i(X), \mathbb{Z})$ and define *coboundary* maps $\delta^i : C^i(X) \to C^{i+1}(X)$. Now, renaming $C^i(X)$ as $C_{-i}(X)$ one gets a chain complex and the homology groups of this chain complex are called the *(integral) cohomology groups* of X, and is denoted by $H^i(X, \mathbb{Z})$.

Below we give couple of well known examples of homology computation.

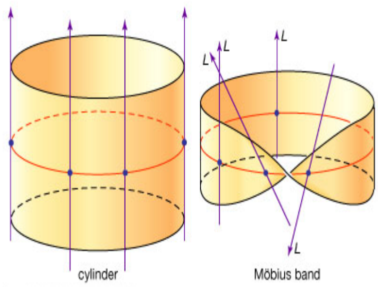
Example 1.2.1. $H_n(\mathbb{R}^m, \mathbb{Z}) = 0$ for $n \neq 0$ and $H_0(\mathbb{R}^m, \mathbb{Z}) = \mathbb{Z}$. This is also true for any contractible space. A space is *contractible* if the identity map is homotopic to the constant map.

Example 1.2.2. $H_n(\mathbb{S}^m, \mathbb{Z}) = 0$ if $n \neq m$ and $n, m \neq 0$. $H_0(\mathbb{S}^m, \mathbb{Z}) = H_m(\mathbb{S}^m, \mathbb{Z})$ $= \mathbb{Z}$ if $m \neq 0$. And $H_0(\mathbb{S}^0, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$, $H_n(\mathbb{S}^0, \mathbb{Z}) = 0$ for $n \neq 0$.

Now, we motivate the reader to the notion of orientability and few more concepts we need.

In Topology sometimes it is helpful to embed a space X into another space Y and look at its 'surroundings' or the 'complement' to understand X and Y. Probably, the first such situation one observes is in General Topology, where, one can show that a space X is Hausdorff if and only if the image of the embedding $X \to X \times X$, sending $x \mapsto (x, x)$ is closed with respect to the product topology on $X \times X$.

We want to study such surroundings of a manifold and more generally, for complexes later. We begin with one example. Consider the following picture of a cylinder and a Möbius band. Both of these spaces are obtained as quotients of the square $[0, 1] \times [0, 1]$ under certain identifications of its boundary.



In the cylinder case $(0, t)$ is identified with $(1, t)$ and in the Möbius band case $(0, t)$ is identified with $(1, 1-t)$ for all $t \in [0, 1]$. The middle line $\{(\frac{1}{2}, t) \| \ t \in [0, 1]\}$ goes to the mid circles in the cylinder and in the Möbius band. Therefore we have got embedding of the circle in the two different spaces. We leave it an exercise to check that the cylinder and the Möbius bands are not homeomorphic.

There is yet another aspect to these two examples. Replace $[0, 1] \times [0, 1]$ by $[0, 1] \times \mathbb{R}$ and do the same identifications, one gets the open cylinder and the open Möbus band. Now, over each point on the mid circle in each of the example, there lies the real line $\mathbb{R}$. This gives the impression of a bundle of real lines on the circle. But we get two different spaces although they look the same locally, that is around any point of the mid circle the two spaces look like $(\epsilon, 1 - \epsilon) \times \mathbb{R}$. One can show these are the only two different $\mathbb{R}$-bundle spaces over the circle. This motivates the following definition.

Definition 1.2.1. A surjective continuous map $p : E \to B$ is called a *fiber bundle projection*, E is called the *total space* and B is called the *base space* if the following are satisfied. Each point $b \in B$ has a neighborhood U_b, such that $p^{-1}(U_b)$ is homeomorphic to $p^{-1}(b) \times U_b$ by a homeomorphism f with $p|_{p^{-1}(U_b)} = \pi_2 \circ f$. Here π_2 denotes the second projection. It is called a *vector bundle* of rank n if the fibers $p^{-1}(x) = \mathbb{R}^n$ for all $x \in B$ and the restriction of f to each fiber is a linear isomorphism. It is called a *trivial bundle* if U_b can be taken to be the whole space B.

We now see an important aspect associated to the Möbius band and cylinder, which is crucial in Manifold Topology. Consider the mid circles in the two examples. Draw a line L at some point, say c, on the mid circle and perpendicular to the circle and give a direction to the line. Now start moving the line, keeping it perpendicular to the circle, along the circle in a fixed direction. When we reach the point c, the direction of the line gets reversed in the case of Möbius band and remain the same in the cylinder case. This gives the fundamental concept of *orientability* of manifolds. This is also intimately related to the concept of homology of manifolds. Let us try to understand this phenomenon differently. Consider another line T at the point c which is tangent to the mid circle of the Möbius band. Then, L and T form a basis of the tangent space of the manifold at c. Now, as we move the frame $\{L, T\}$ along the circle maintaining the position and direction of the lines and come back to c, we see the basis of the tangent space changed to another one with the change matrix having determinant negative in the case of Möbius band. But there is no change in the cylinder case.

We state a theorem, whose proof can be found in any Differential Topology book.

Theorem 1.2.1. *Let M be a closed manifold of dimension n. Then the following are equivalent.*

1. $H_n(M, \mathbb{Z}) = \mathbb{Z}$.

2. The following statement holds for the image C of any embedding of the circle $\mathbb{S}^1$ in the manifold. At any point on C, take any framing F_1 (that is, a basis of the tangent space of M at the point), and then move along the circle with the framing, when we come back to the point again, we get another framing F_2. Then, the determinant of the matrix which sends the framing F_1 to F_2 is positive.

If one of the conditions in the statement of the theorem is satisfied then, we call the manifold *orientable*. In such a situation one of the generators of $H_n(M, \mathbb{Z})$, denoted $[M]$, is called the *fundamental class* or *the orientation class* of M. Once an orientation class is fixed, the manifold is called *oriented*. One can show that the top homology is either trivial or infinite cyclic for any closed manifold.

In this context we make a definition which we will require later.

Definition 1.2.2. A map $f : M \to N$ between two n-dimensional closed oriented manifolds is said to be of degree k if $f_*([M]) = k[N]$.

For example, the map $\mathbb{S}^1 \to \mathbb{S}^1$ defined by $z \mapsto z^k$ has degree k.

Now, take two Möbius bands and attach their boundaries together to get the Klein bottle or attach a disc to the boundary of the Möbius band to get the real projective plane. By the above theorem and the remark following it, both these manifolds are not orientable and their top homology is trivial. Furthermore, note that the open Möbius band is embedded in both the examples, this embedding is called the *normal bundle* of the mid circle in the embedding. Similarly the open cylinder is embedded in the torus, and in this case also this embedded object is called the normal bundle of the mid circle. Therefore, we see examples where the normal bundles can be trivial or non-trivial depending on the ambient manifold. In fact, one can show that if the submanifold (in our case the mid circle) and the ambient manifold are both orientable, the normal bundle of the submanifold is always trivial.

We recall Poincaré duality theorem which gives an important relationship between homology and cohomology of an oriented manifold.

Theorem 1.2.2. *Let M be a closed oriented manifold of dimension n. There is an isomorphism, called the duality isomorphism, obtained by taking cap product with the fundamental class*

$$\cap[M] : H^k(M, \mathbb{Z}) \to H_{n-k}(M, \mathbb{Z})$$

for all k.

Using homology theory one can also prove the following very important *invariance of domain* theorem.

Theorem 1.2.3. *If an open set in* $\mathbb{R}^n$ *is homeomorphic to an open set in* $\mathbb{R}^m$*, then* $m = n$.

This theorem is needed to ensure the well-definedness of dimension of a connected manifold.

1.3. **Algebra.** Here we recall some basic Algebra used in this article.

Let R be a ring with unity. An (left) *R-module* M is an abelian group together with an action of R, that is a map $R \times M \to M$, sending (r, m) to an element, denoted by rm, of M satisfying the following properties. For all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$, $(r_1 + r_2)m = r_1 m + r_2 m$, $r(m_1 + m_2) = rm_1 + rm_2$, $(r_1 r_2)m = r_1(r_2 m)$ and $1m = m$. Equivalently, an R-module structure on an abelian group M is nothing but a ring homomorphism from R to the endomorphism ring $End(M)$ of M. For example, any abelian group is a $\mathbb{Z}$-module.

In this article we mostly consider the (integral) group ring $\mathbb{Z}[G]$ of a group G. The group ring, by definition, consists of the formal finite sums $\Sigma_{i=1}^k r_{g_i} g_i$, where $g_i \in G$ and $r_{g_i} \in \mathbb{Z}$, for $i = 1, 2, \ldots, k$. Equivalently, the group ring can be defined as the set of maps $f : G \to \mathbb{Z}$, such that f takes zero value on all but finitely many elements in G, and the operations are defined by the following. Given f and g in $\mathbb{Z}[G]$, $f + g : G \to \mathbb{Z}$ denotes the map $(f + g)(\alpha) = f(\alpha) + g(\alpha)$ and fg denotes the map $(fg)(\alpha) = \Sigma_{\alpha=uv} f(u)g(v)$. The summation is well defined since there are only finitely many elements of G on which f or g takes nonzero values.

We now give a special example which we require in this article. Let X be a nice space and $\tilde{X}$ be its universal cover. Then, one knows that the fundamental group G of X acts on $\tilde{X}$ as a group of covering transformations. That is, we have a map $G \times \tilde{X} \to \tilde{X}$, so that the induced map from G to the group of homeomorphisms of $\tilde{X}$ is a homomorphism. It is easy to check that this induces a homomorphism from G to the group of isomorphisms of $C_i(\tilde{X})$ for each i. It now follows that each $C_i(\tilde{X})$ becomes a free $\mathbb{Z}[G]$-module. When X is a complex then also one can get such a module structure, by using the induced complex structure on the universal cover. For example, one gives a complex structure on $\mathbb{R}$ lifting a complex structure on $\mathbb{S}^1$ using the exponential map and then make $S_i(\mathbb{R})$, a free $\mathbb{Z}[\mathbb{Z}]$-module.

We now recall the definition and some examples of a particular class of R-modules, which are useful in Topology.

Definition 1.3.1. A *projective R-module* is by definition a direct summand of a free R-module. That is, an R-module P is called projective, if there exists another R-module Q such that $P \oplus_R Q$ is isomorphic to a free R-module.

Of course, free R-modules are projective. We now give an example of an R-module which is projective, but not free.

Example 1.3.1. Let $R = \mathbb{Z}_6$ and $M = \mathbb{Z}_3 \oplus_R \mathbb{Z}_2$, then $M = R$, and $\mathbb{Z}_3$ and $\mathbb{Z}_2$ are R-modules and hence they are projective. Obviously, they are not free.

Acknowledgement. I would like to thank C. S. Aravinda for inviting me to write this article and for many suggestions. Also, thanks to the referee for carefully reading the article and for critical comments and suggestions.

2. Some questions

To motivate the reader we begin with some of the fundamental questions in this subject.

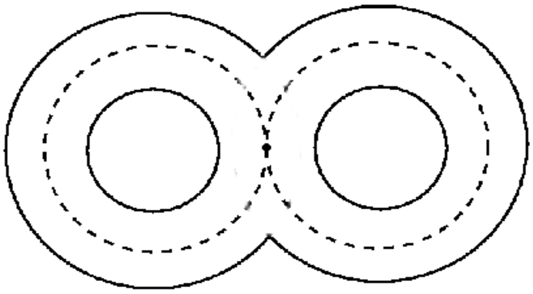
- When is a topological space homotopy equivalent to a finite complex?
- When is a finite complex homotopy equivalent to a compact manifold?
- Are two homotopy equivalent closed manifolds homeomorphic? That is, we are asking about the uniqueness of the manifold in the previous question.

The examples given below, which show that the answer to the above questions are in general no, follow the same order as the questions.

- Note that, the necessary conditions the space should satisfy are those known properties of a finite complex which are invariant under homotopy equivalence. We may not need to assume all these properties at a time. At some point we will see that only some of the properties are enough or we will hit on an obstruction.

We begin with the number of path components. We know this number is homotopy invariant, that is, if two spaces are homotopy equivalent then they have the same number of path components. Therefore, if we want our finite complex to be path connected, then we have to assume that the space we started with should also be path connected. Compactness need not be preserved under homotopy equivalence, so we do not consider this. The most basic homotopy invariant we study in Algebraic Topology is the fundamental group. A finite complex has finitely generated fundamental group. Therefore, the wedge of infinitely many circles or the Euclidean plane with all points whose both coordinates are integers deleted, can never be homotopy equivalent to a finite complex, as they have infinitely generated fundamental groups. The further conditions we need to put on the space are discussed in the next section.

- First, note that any finite complex is homotopy equivalent to a compact manifold with nonempty boundary. This can be obtained by first taking a compact polyhedron homotopy equivalent to the finite complex, and then taking the regular neighborhood of the polyhedron after embedding it in some Euclidean space. We see this with an example. Consider the figure eight embedded in $\mathbb{R}^2$. The regular neighborhood is shown in the picture below, which is a 2-dimensional manifold with three boundary components.



Regular neighborhood of Figure eight.

Therefore, the question is to ask for a closed manifold. Next, recall that a manifold has vanishing homology groups in higher dimensions, therefore, the complex should not have cells of arbitrary high dimensions. For example, the wedge of spheres $\mathbb{S}^n$ for $n = 1, 2, \ldots$ can never be homotopy equivalent to any closed manifold. Hence the complex should be assumed to be finite dimensional. Furthermore, homological properties are homotopy invariants, for example, the homology groups must satisfy Poincaré duality (Theorem 1.2.2). There are further conditions required which are described in the next section.

We now give examples of finite complexes which are not homotopy equivalent to closed manifold as they do not satisfy Poincaré duality.

Example 2.0.1. Let X be the figure eight. Then $H_i(X, \mathbb{Z}) = 0$ for $i \geq 2$, since X is an one-dimensional complex. Therefore, if there is any manifold homotopy equivalent to X, then it must be one-dimensional. Note that, $H_1(X, \mathbb{Z})$ has rank 2, but an orientable closed one-dimensional manifold must have first homology of rank 1 by Poincaré duality. For a similar reason (taking homology with coefficient in $\mathbb{Z}_2$) it follows that X is not homotopy equivalent to any closed non-orientable manifold. Similarly, one shows that the wedge of two spheres of the same dimension also gives an example of a finite complex not homotopy equivalent to any closed manifold.

• There are homotopy equivalent manifolds which are not homeomorphic. For example, the lens spaces $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent but not homeomorphic. There is another kind of equivalence called *simple homotopy equivalence* which lies in between homotopy equivalence and homeomorphism. We will study this later in this article. It can be shown that the above two spaces are not even simple homotopy equivalent.

These questions were well studied over the last several decades. The Whitehead group $Wh(G)$, reduced projective class group $\tilde{K}_0(\mathbb{Z}[G])$, and the surgery L-groups $L_n(\mathbb{Z}[G])$ of the fundamental group G contain the answers to the above and many more questions. Mainly, the breakthrough on classifying manifolds started during 1960 to 1970. Then, there were results proved about computing the above obstruction groups for small classes of groups. Several conjectures were formulated which still remain open. The second breakthrough happened with works which use geometry and controlled topology. Apart from geometry and controlled topology, Frobenius induction technique was also found to be very useful. Finally, in 1993 Farrell and Jones formulated an important conjecture, now popularly known as *Farrell-Jones Isomorphism conjecture* ([5]), which captures the subject in a single statement and implies all the previous conjectures. Since then, an enormous amount of work has been done by many authors and it is still an ongoing front line area of research to prove the Isomorphism conjecture for different classes of groups.

In this article we describe the above obstruction groups and see how they answer the questions. Also, we recall some of the classical results. In a future article we plan to review the development that took place during the last two decades.

The subject we are exposing in this article is enormous and, therefore, we urge the reader to look at the sources in the reference list to know more about it.

3. In more detail

In this section we see exactly the conditions needed and how the obstruction groups appear in solving the questions in the previous section. We have already recalled many of the basics required in this section. There are times when we will use some new concepts, which we do not recall due to technical reason. We refer the reader to look at the corresponding sources. But this lacking will not prevent the reader from understanding the core ideas behind the proofs.

3.1. **Reduced projective class group $\tilde{K}_0(-)$.** Recall the definition of homotopy equivalence. There are two conditions which need to be satisfied. We start with the following definition.

Definition 3.1.1. A space X is said to be *dominated* by another space Y if there are maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f \simeq id_X$. And X is called *finitely dominated* if in addition Y is a finite complex.

Note that this is half of saying that X is homotopy equivalent to Y. It is also known that if X is finitely dominated then it is homotopy equivalent to a countable complex, which can be checked using some facts from [12] (or see Theorem 3.9 of [19] or Exercise G6 in Chapter 7 of [16]). In fact, the *mapping telescope* of $f \circ g$ becomes homotopy equivalent to X. Therefore, to investigate whether a space X is homotopy equivalent to a finite complex, we can assume that X is a complex. But, to show whether a complex is homotopy equivalent to a finite complex, one needs more difficult work. This is answered in a very important and celebrated series of papers by C.T.C. Wall (see [22] and [23]). This leads to an obstruction which lies in an abelian group called the *reduced projective class* group and denoted by $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$. Interestingly, the answer to this question depends only on the fundamental group of X.

Before going further let us define reduced projective class group of a ring. Let R be a ring with unity. Let $\mathcal{P}$ denote the free abelian group generated by the set of all isomorphism classes of finitely generated projective R-modules. Let $K_0(R)$ be the quotient of $\mathcal{P}$ by the relations $[P] + [Q] = [P \oplus Q]$ where $[P], [Q] \in \mathcal{P}$. Note that, $K_0(R)$ is a covariant functor from the category of rings with unity to the category of abelian groups. Thus the homomorphism $\mathbb{Z} \to R$ sending 1 to the unity of R induces a map $i : K_0(\mathbb{Z}) \to K_0(R)$. The *reduced projective class* group $\tilde{K}_0(R)$ is by definition the quotient $K_0(R)/i(K_0(\mathbb{Z}))$. By abusing notation we will denote elements of $\tilde{K}_0(R)$ by the same notation $[P]$. For details on the projective class groups see [15] or any standard book on Algebraic K-theory. Below we give some examples.

Example 3.1.1. Let $R = \mathbb{Z}$, the ring of integers. Then one knows that, since $\mathbb{Z}$ is a principal ideal domain, any finitely generated projective module over $\mathbb{Z}$ is free. Therefore, we have $\tilde{K}_0(\mathbb{Z}) = 0$. Same conclusion holds if R is a field.

In this subject one encounters only integral group ring $\mathbb{Z}[G]$ of a group G. See Section 4 for more on this. Although, more abstract coefficients are considered for the purpose of a general framework and applications in other areas of Mathematics.

Recall that, any finitely dominated space is homotopically equivalent to a (countable) complex, and hence up to homotopy such a space is nice.

Let X be a nice space and $\tilde{X}$ be its universal cover. Then, recall that $C_*(\tilde{X})$ is a chain complex of free $R(=\mathbb{Z}[G])$-module. In general $C_i(\tilde{X})$ is a very large R-module. On the other hand if X were a finite complex then it follows that $S_*(\tilde{X})$ is a finite chain complex of finitely generated free R-module. For an arbitrary nice space we can make the situation better by assuming that X is finitely dominated. C.T.C. Wall proves that in this case $S_*(\tilde{X})$ is chain equivalent to a finite chain complex of finitely generated projective R-modules, say $P_*(X)$. Consider the object $\chi(P_*(X)) = \Sigma_i(-1)^i[P_i(X)] \in \tilde{K}_0(R)$. $\chi(P_*(X))$ is called the *Wall finiteness obstruction* of the space X.

Theorem 3.1.1 (C.T.C. Wall)**.** *Let X be a finitely dominated space and $R = \mathbb{Z}[\pi_1(X)]$. Then X is homotopy equivalent to a finite complex if and only if $\chi(P_*(X)) = 0$ in $\tilde{K}_0(R)$. Furthermore, given an element ω in $\tilde{K}_0(R)$ there is a finitely dominated space X' with fundamental group isomorphic to $\pi_1(X)$, such that $\omega = \chi(P_*(X'))$.*

This gives a complete solution to the first problem. But computing the reduced projective class group of a group is another story. We will talk about it in the next section.

3.2. **Surgery groups $L_*(-)$.** If we want a finite complex to be homotopy equivalent to a closed (orientable) manifold there are several more necessary conditions needed. One such condition is that the homology of the complex must satisfy

Poincaré duality as mentioned in Section 1. Such a complex is called a *Poincaré complex*. We give the precise definition below.

Definition 3.2.1. Let X be a connected finite complex and for some n, $H_n(X,\mathbb{Z}) \simeq \mathbb{Z}$. Let $[X] \in H_n(X,\mathbb{Z})$ be a generator such that the cap product with $[X]$ gives an isomorphism $H^q(X,\mathbb{Z}) \to H_{n-q}(X,\mathbb{Z})$ for all q. Then X is called a *Poincaré complex* with *fundamental class* $[X]$ and dimension n.

The second condition needed is the existence of a bundle over the complex, which has properties similar to the normal bundle of a manifold embedded in some Euclidean space. We describe this below.

Let M be a closed connected oriented smooth manifold of dimension n. By the Whitney Embedding Theorem we can embed this manifold in $\mathbb{R}^{n+k} \subset \mathbb{S}^{n+k}$ for $k \geq n$. Let ν_M be the normal bundle of M in $\mathbb{S}^{n+k}$. Let τ_M be the tangent bundle of M. Then $\tau_M \oplus \nu_M$ is the product bundle as $M \subset \mathbb{S}^{n+k} - \{\infty\} = \mathbb{R}^{n+k}$. Let N be the subset of ν_M consisting of vectors of length $< \epsilon$, with respect to some Riemannian metric, for some $\epsilon > 0$. Then N is an open neighborhood of M in $\mathbb{S}^{n+k}$ and in fact diffeomorphic to the total space $E(\nu_M)$ of ν_M. The one point compactification of $E(\nu_M)$ is called the *Thom space* of ν_M. On the other hand the one point compactification N^* of N is homeomorphic to $\mathbb{S}^{n+k}/\mathbb{S}^{n+k} - N$. Hence, we get a map $\alpha : \mathbb{S}^{n+k} \to N^* \simeq T(\nu_M)$. One can check that α induces an isomorphism $H_{n+k}(\mathbb{S}^{n+k},\mathbb{Z}) \to H_{n+k}(T(\nu_M),\mathbb{Z})$ and sends the canonical generator of $H_{n+k}(\mathbb{S}^{n+k},\mathbb{Z})$ to the generator of $H_{n+k}(T(\nu_M),\mathbb{Z})$, which comes from the fundamental class $[M]$ via the Thom isomorphism. In this sense this map is of degree 1.

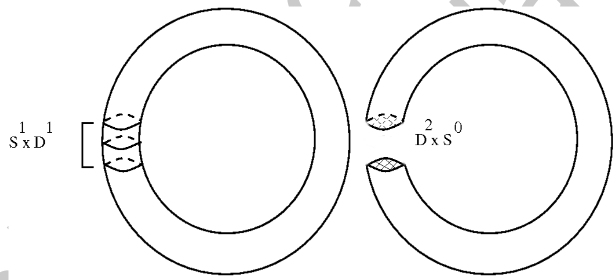
Therefore, for a Poincaré complex X to be homotopy equivalent to a closed oriented smooth manifold it is necessary that there should be a real vector bundle ξ on X and a degree 1 map $\alpha : \mathbb{S}^{n+k} \to T(\xi)$. The Thom space $T(\xi)$ in this generality is defined as follows. Consider the fiber bundle ξ^* over X obtained by taking the one point compactification of the fibers of the bundle ξ. The bundle ξ^* admits a section $s : X \to \xi^*$, sending a point of X to the point at infinity of the fiber over the point. Then $T(\xi)$ is defined as the quotient of the total space of ξ^* by $s(X)$.

Once this information is given we can apply Thom Transversality Theorem to homotope α to β so that $\beta^{-1}(X)$ $(:= K)$ is a (oriented) submanifold of $\mathbb{S}^{n+k}$. Furthermore, the map β restricted to the normal bundle ν_K of K gives a linear bundle map onto ξ and $\beta|_K$ is of degree 1. A data of this type as in the following commutative diagram is called a *normal map* and is denoted by (f, b),

$$\begin{tikzcd}
\nu_M \arrow[r, "f"] \arrow[d] & \xi \arrow[d] \\
M \arrow[r, "b"] & X
\end{tikzcd}$$

where, M is a closed connected smooth oriented manifold, X is a Poincaré complex, ν_M is the normal bundle in some embedding of M in $\mathbb{S}^{n+k}$, and b is a degree 1 map. Here we remark that not all Poincaré complexes admit a normal map. See the example on p.32-33 in [11]. We saw that this map b is obtained from the Transversality Theorem but it is nowhere close to being a homotopy equivalence. In the simply connected and odd high dimension case, in fact, b can be homotoped to a homotopy equivalence ([2]). In the general case, the next general step is to apply Surgery theory to b to get another normal map, which is normally cobordant to the previous one, and to try to get closer to a homotopy equivalence. To achieve this we need that b induces isomorphisms on the homotopy groups level and then apply Whitehead theorem (Theorem 1.1.1).

We digress here a bit to show how surgery works to get rid of some homotopy group element from the kernel of b_*^q. Let $M = \mathbb{S}^1 \times \mathbb{S}^1$ as in the picture. And, suppose we want to get rid of the fundamental group element generated by the first circle, which lies in the kernel of $b_*^1 : \pi_1(M, m) \to \pi_1(X, b(m))$. In the picture we show a tubular neighborhood of the circle, this is an embedded $\mathbb{S}^1 \times \mathbb{D}^1$ (also called an 1-handle). Note that $\partial(\mathbb{S}^1 \times \mathbb{D}^1) = \partial(\mathbb{D}^2 \times \mathbb{S}^0)$. Now remove the interior of this handle and replace it by $\mathbb{D}^2 \times \mathbb{S}^0$. This is called



Surgery on an 1−handle

the *surgery on the* 1-*handle*. The resulting manifold is a 2-sphere and b can be extended to this new manifold as the restrictions of b to the circles $\mathbb{S}^1 \times \mathbb{S}^0$ are homotopic to the constant maps. This new b_*^1 does not have any kernel.

More generally, suppose an element α in the kernel of b_*^q is represented by the embedding of $\mathbb{S}^q$ in M. Since both these manifolds are orientable, the normal bundle of the image of $\mathbb{S}^q$ in M is trivial and hence this image gives rise to an embedding of $\mathbb{S}^q \times \mathbb{D}^{n-q}$ (take the disc bundle of the normal bundle), called a q-*handle*, in M, where M is n-dimensional. A *surgery* along this q-handle is removing the interior of $\mathbb{S}^q \times \mathbb{D}^{n-q}$ and attaching $\mathbb{D}^{q+1} \times \mathbb{S}^{n-q-1}$ to the resulting manifold. This operation kills the kernel element α.

These surgery operations can be done up to dimension $< [\frac{n}{2}]$ and, therefore, by Poincaré duality the main problem lies in dimension $[\frac{n}{2}]$. The complication in this middle dimension gives rise to the Wall's *Surgery obstruction group* $L_n^h(\mathbb{Z}[\pi_1(X)])$.

That is, given a normal map (f, b), there is an obstruction $\sigma(f, b)$, which lies in the group $L_n^h(\mathbb{Z}[\pi_1(X)])$, whose vanishing will ensure that the normal map can be normally cobordant to another normal map (f', b'), where b' is a homotopy equivalence. See [24].

Remark 3.2.1. These surgery groups depend only on the fundamental group and on its orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$. Here, as we are dealing with the oriented case, this homomorphism is trivial. In the general situation we need to incorporate ω in the the surgery groups. But in this article we avoid it for simplicity.

Remark 3.2.2. The upper script 'h' to the notation of the surgery groups is due to 'homotopy equivalence'. There are problems, when one asks for 'simple homotopy equivalence' in the normal map. Then a different surgery problem appears and gives rise to the surgery groups $L_*^s(\mathbb{Z}[-])$. There are many other decorated surgery groups for different surgery problems, like $L_n^{\langle -\infty \rangle}(\mathbb{Z}[-])$. But all of them coincide, once we have the lower K-theory vanishing result of the group. This is checked using Rothenberg's exact sequence. For example, if the Whitehead group of a group G vanishes, then, $L_n^h(\mathbb{Z}[G]) = L_n^s(\mathbb{Z}[G])$ for all n.

Remark 3.2.3. One further remark is that the surgery groups are 4-periodic. That is, $L_n^h(\mathbb{Z}[-]) = L_{n+4}^h(\mathbb{Z}[-])$ for all n and, in fact, is true for all decorations. This is obtained by showing that the surgery obstruction of a normal map on a complex X of dimension n is same as the surgery obstruction of the corresponding normal map on $X \times \mathbb{CP}^2$.

To end this subsection we give a nice application of surgery theory.

Theorem 3.2.1. *There are infinitely many distinct closed manifolds homotopy equivalent to the projective space $\mathbb{CP}^k$ for $k \geq 3$. And, this is true in any of the categories; topological, smooth or piecewise linear (PL).*

3.3. **Whitehead group $Wh(-)$.** Once we have established that a complex is homotopy equivalent to a manifold, the next question is about the uniqueness of the manifold. In surgery theory one gets two such manifolds M_1 and M_2 (when they exist) as h-cobordant, which we define below.

Definition 3.3.1. Let M_1 and M_2 be two connected closed n-dimensional manifolds. A compact manifold W of dimension $n+1$ with two boundary components M_1 and M_2 is called a *cobordism* between M_1 and M_2. In such a situation M_1 and M_2 are called *cobordant.* If the inclusions $M_i \subset W$ are homotopy equivalences then W is called an *h-cobordism* between M_1 and M_2. *And M_1 and M_2 are called h-cobordant.*

Using Morse Theory, one can show that two closed manifolds are cobordant if and only if one of the manifolds can be obtained from the other by finitely many surgery operations.

Given such an h-cobordism W there is an obstruction $\tau(W, M_1)$, which lies in a quotient (called the *Whitehead group* and is denoted by $Wh(\pi_1(M_1))$) of the K_1 of the integral group ring of the fundamental group of M_1. When $\dim W \geq 6$ the element $\tau(W, M_1) \in Wh(\pi_1(M_1))$ has the following property: $\tau(W, M_1) = 0$ implies that W is homeomorphic to $M_1 \times I$ where $I = [0, 1]$. This is called the *s-cobordism theorem* stated below as Theorem 3.4.1. See [9] and [10]. One consequence of this theorem is the Poincaré conjecture in high dimensions.

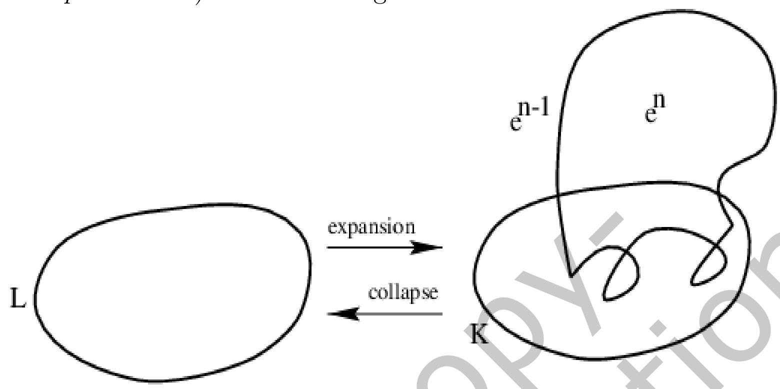
There are similar interpretation of the reduced projective class groups and negative K-groups in terms of some 'special' kind of h-cobordism called *bounded h-cobordism*. See [14] for some more on this matter.

We now recall the original interpretation of the Whitehead group, which says that given a homotopy equivalence $f : K \to L$ between two connected finite complexes K and L there is an element $\tau(f) \in Wh(\pi_1(K))$ whose vanishing ensures that the map f is homotopic to a *simple homotopy equivalence*. See [4]. We already gave a hint to this in Section 2. We describe this important subject below to a certain extent.

The simple homotopy equivalences lie in between homotopy equivalences and homeomorphisms. Even in such a classical vast area of topology the most basic question is not yet answered; namely, if any homotopy equivalence between two finite aspherical complexes is homotopic to a simple homotopy equivalence, which is known as *Whitehead's conjecture* in K-theory. There is an even stronger conjecture which asks; if any homotopy equivalence between two aspherical manifolds is homotopic to a homeomorphism. This is known as *Borel's conjecture.* Recall that, a connected complex X is called *aspherical* if $\pi_i(X) = 0$ for all $i \geq 2$, or equivalently, the universal cover of X is contractible. A result of Chapman says that any homeomorphism of complexes is a simple homotopy equivalence. So, the first step to prove the Borel's conjecture is to verify Whitehead's conjecture in K-theory.

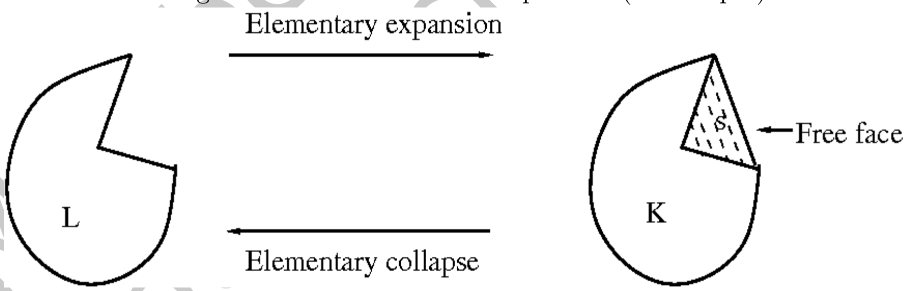
As far as the Borel's conjecture is concerned, dimension 2 is understood completely in the topological, piecewise linear or smooth category. Perelman's proof of the Thurston's Geometrization conjecture completes the picture in dimension 3. Recall that, in dimension ≤ 3 any manifold supports a unique topological, piecewise linear or smooth structure. In dimension greater or equal to 5 an enormous literature exists; where there are enough machinery, language to attack a problem. The critical dimension is 4. In this dimension even the s-cobordism theorem is not yet known. So far this is proved for 4-manifolds with some restriction on the fundamental group; namely, groups with subexponential growth. Also another interesting fact is that a 4-manifold can support infinitely many smooth structures. Even our familiar Euclidean 4-space $\mathbb{R}^4$ has infinitely many smooth structures.

3.4. **Simple homotopy equivalence and Whitehead group.** The reference for this subsection is [4]. Let K and K' be two finite complexes. A homotopy equivalence $f : K \to K'$ is called a *simple homotopy equivalence* provided f is homotopic to a composition of maps of the following kind: $K = K_0 \to K_1 \to \cdots \to K_s = K'$ where the arrows are either an elementary expansion or collapse. A pair of complexes (K, L) is called an *elementary collapse* (and we say K *collapses to* L or L *expands to* K) if the followings are satisfied:



- $K = L \cup e^{n-1} \cup e^n$ where e^{n-1} and e^n are not in L (e^i denotes an i-cell),
- there exist a ball pair $(\mathbb{D}^n, \mathbb{D}^{n-1})$ and a map $\phi : \mathbb{D}^n \to K$ such that
  - (a) $\phi|_{\partial \mathbb{D}^n}$ is a characteristic map for e^n
  - (b) $\phi|_{\partial \mathbb{D}^{n-1}}$ is a characteristic map for e^{n-1}
  - (c) $\phi(P^{n-1}) \subset L^{n-1}$, where $P^{n-1} = \overline{\mathbb{D}^n - \mathbb{D}^{n-1}}$.

From the above figure the definition of an expansion (or collapse) will be clear.

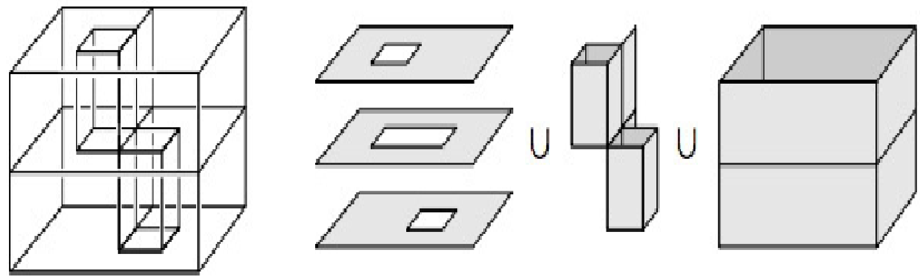


Collapsing a simplex

Above is yet another example in a more concrete situation of a polyhedron. In the example, L is a polyhedron to which we introduce the new simplex s, but the new simplex has a face which is not the face of any other simplex, such a face is called a *free face*. A free face gives the freedom to collapse without changing the homotopy type of the complex.

We now come to a popular example of a contractible topological space which has a 2-dimensional polyhedron structure but has no free face. This is called

Bing's house with two rooms. In this example one has to first expand and then collapse. The picture explains the space. There are two rooms with two different entrances. It is easy to see if we start filling the rooms with square blocks (which are elementary expansions) then at the end we get a three dimensional cube, which can be collapsed to a point as it has free faces.



Bing's house with two rooms

There is an algebraic picture of the above topological construction, which we describe now.

Given any homotopy equivalence $f : K \to K'$, there is an obstruction $\tau(f)$ which lies in an abelian group $Wh(\pi_1(K'))$ (defined below) detecting if f is a simple homotopy equivalence.

Let R be a ring with unity. Let $GL_n(R)$ be the multiplicative group of invertible $n \times n$ matrices with entries in R and $E_n(R)$ be the subgroup of elementary matrices. By definition an *elementary matrix*, denoted by $E_{ij}(a)$, for $i \neq j$, has 1 on the diagonal entries, $a \in R - \{0\}$ at the (i, j)-th position and the remaining entries are 0. Define $GL(R) = \lim_{n\to\infty} GL_n(R)$, $E(R) = \lim_{n\to\infty} E_n(R)$. Here the limit is taken over the following maps:

$$GL_n(R) \to GL_{n+1}(R)$$

$$\begin{pmatrix} A \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The following lemma shows that $E(R)$ is also the commutator subgroup of $GL(R)$.

Lemma 3.4.1 (Whitehead's lemma)**.** *Let R be a ring with unity. Then the commutator subgroup of $GL(R)$ and of $E(R)$ is $E(R)$.*

Proof. At first note the following easy to prove identities.

$E_{ij}(a)E_{ij}(b) = E_{ij}(a + b)$; (1)

$E_{ij}(a)E_{kl}(b) = E_{kl}(b)E_{ij}(a)$, $j \neq k$ and $i \neq l$; (2)

$E_{ij}(a)E_{jk}(b)E_{ij}(a)^{-1}E_{jk}(b)^{-1} = E_{ik}(ab)$, i, j, k distinct; (3)

$E_{ij}(a)E_{ki}(b)E_{ij}(a)^{-1}E_{ki}(b)^{-1} = E_{kj}(-ba)$, i, j, k distinct. (4)

Also note that any upper triangular or lower triangular matrix with 1 on the diagonal belongs to $E(R)$.

Since $E(R) \subset GL(R)$ we have $[E(R), E(R)] \subset [GL(R), GL(R)]$. Using (3) we find that $E_{ij}(a) = [E_{ik}(a), E_{kj}(1)]$ provided i, j and k are distinct. Hence any generator of $E(R)$ is a commutator of two other generators of $E(R)$. Hence $E(R) = [E(R), E(R)]$. We only need to check that $[GL(R), GL(R)] \subset E(R)$. So let $A, B \in GL_n(R)$. Then note the following identity.

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

Now we check that all the factors on the right hand side belong to $E_{2n}(R)$. This follows from the following. Let $A \in GL_n(R)$. Then the following equality is easy to check.

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Again note the following.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now recall our earlier remark that any upper triangular or lower triangular matrix with 1 on the diagonal belong to $E(R)$. This completes the proof of the Whitehead Lemma. □

Definition 3.4.1. Define $K_1(\mathbb{Z}[\pi]) = GL(\mathbb{Z}[\pi])/E(\mathbb{Z}[\pi])$. The *Whitehead group* $Wh(\pi)$ of π is by definition $K_1(\mathbb{Z}[\pi])/N$. Here N is the subgroup of $K_1(\mathbb{Z}[\pi])$ generated by the 1×1 matrices (g) and $(-g)$, for $g \in \pi$.

Note that multiplying a matrix A by an elementary matrix from left (or right) makes an elementary row (or column) operation in A. Now by applying elementary row and column operations one can transform an invertible integral matrix to I or $-I$. This shows that $Wh((1)) = 0$. (Though the matrix multiplication induces the group operation in $Wh(-)$ we write it additively, since $Wh(-)$ is abelian.)

We recall below a transparent topological definition of Whitehead group which is naturally isomorphic to the above one. The proof of this isomorphism is very long and can be found in [4]. But we will give the map from this topological definition to the algebraic definition above. For computational purposes the algebraic definition is more useful as we will see later.

Let K be a fixed finite complex. Let $\mathcal{W}(K)$ be the collection of all pairs of finite complexes (L, K) so that K is a strong deformation retract of L. For any two objects $(L_1, K), (L_2, K) \in \mathcal{W}$ define $(L_1, K) \equiv (L_2, K)$ if and only if L_1 and L_2 are simple homotopically equivalent relative to the subcomplex K. Here by a relative simple homotopy equivalence we mean that in the definition of simple homotopy equivalence we do not collapse (or expand) any cells contained in K. Let $Wh(K) = \mathcal{W}/\equiv$. Let $[L_1, K]$ and $[L_2, K]$ be two classes in $Wh(K)$. Define

$[L_1, K] \oplus [L_2, K] = [L_1 \cup_K L_2, K]$. Here $L_1 \cup_K L_2$ is the disjoint union of L_1 and L_2 identified along the common subcomplex K. This defines an abelian group structure on $Wh(K)$ called the *topological Whitehead group* of K.

Now let (L, K) be an element in $\mathcal{W}$. Consider the universal cover $(\tilde{L}, \tilde{K})$. It can be checked that the inclusion $\tilde{K} \subset \tilde{L}$ is a homotopy equivalence. Equip $(\tilde{L}, \tilde{K})$ with the complex structure lifted from the complex structure of (L, K). Let $S_*(\tilde{L}, \tilde{K})$ be the cellular chain complex of $(\tilde{L}, \tilde{K})$. The covering action of $\pi_1(L)$ on $(\tilde{L}, \tilde{K})$ induces an action on $S_*(\tilde{L}, \tilde{K})$ and makes it a chain complex of $\mathbb{Z}[\pi_1(L)]$-modules. In fact, it is a finitely generated free acyclic chain complex of $\mathbb{Z}[\pi_1(L)]$-modules. Let d be the boundary map. One can find a contraction map δ of degree $+1$ of $S_*(\tilde{L}, \tilde{K})$ so that $d\delta + \delta d = id$ and $\delta^2 = 0$. Consider the module homomorphism $d + \delta : \bigoplus_{i=0}^{\infty} S_{2i+1}(\tilde{L}, \tilde{K}) \to \bigoplus_{i=0}^{\infty} S_{2i}(\tilde{L}, \tilde{K})$. It turns out that this homomorphism is an isomorphism of $\mathbb{Z}[\pi_1(L)]$- modules. The image and range of this homomorphism are finitely generated free modules with a preferred basis coming from the complex structure on $(\tilde{L}, \tilde{K})$. We consider the matrix of this homomorphism $d + \delta$ which is an invertible matrix with entries in $\mathbb{Z}[\pi_1(L)]$ and hence lies in $GL_n(\mathbb{Z}[\pi_1(L)])$ for some n. We take the image of this matrix in $Wh(\pi_1(L))$. The proof that this map (say τ) sending (L, K) to this image in $Wh(\pi_1(L))$ is an isomorphism is given in [4].

Now, consider a homotopy equivalence $f : K \to K'$ between two finite complexes. Let M_f be the mapping cylinder of the map f. Recall that, the *mapping cylinder* M_f of f is by definition, the quotient space of the disjoint union $(K \times I) \cup K'$, under the identifications $(k, 1) = f(k)$, for all $k \in K$.

Consider the pair (M_f, K). Here K is identified with $K \times \{0\}$ in M_f. As f is a homotopy equivalence it is easy to check that $(M_f, K) \in \mathcal{W}$. Now, we recall that f is a simple homotopy equivalence if and only if $\tau([M_f, K])$ is the trivial element in $Wh(\pi_1(K))$.

Finally, we state the s-cobordism theorem. Smale proved the theorem in the simply connected case and it is known as the h-cobordism theorem. He received the Fields medal for this proof, which also implies the high dimensional Poincaré conjecture.

Theorem 3.4.1 (**S-cobordism theorem**). *(Barden, Mazur, Stallings) Let M_1 and M_2 be two compact connected manifolds of dimension ≥ 5 if they have empty boundary, and of dimension ≥ 6 otherwise. Let W be an h-cobordism between M_1 and M_2. If $\tau([W, M_1]) = 0$ in $Wh(\pi_1(M_1))$ then $W \simeq M_1 \times I$. In particular $M_1 \simeq M_2$. Furthermore, any element of $Wh(\pi_1(M_1))$ is realized by an h-cobordism.*

Remark 3.4.1. An h-cobordism between M_1 and M_2 with $\tau([W, M_1]) = 0$, is called an s-cobordism between M_1 and M_2.

Here $\simeq$ denotes a homeomorphism, a piecewise linear homeomorphism or a diffeomorphism according as the manifolds are topological, piecewise linear or smooth respectively.

4. Known results on $\tilde{K}_0(-)$, $Wh(-)$ and $L_*(-)$

Now we come to results about computing the invariants we encountered so far, namely the reduced projective class group, the Whitehead group and the surgery obstruction groups.

The Whitehead conjecture in K-theory asks if $Wh(\pi) = 0$, for any finitely presented torsion free group. This has been checked for several classes of groups: for free abelian groups by Bass-Heller-Swan ([1]) ; for free nonabelian groups by Stallings ([17]); for the fundamental group of any complete nonpositively curved Riemannian manifold by Farrell and Jones ([5]); for the fundamental group of finite $CAT(0)$ complexes by B. Hu. ([8]). Waldhausen proved that the Whitehead group of the fundamental group of any Haken 3-manifold vanishes ([21]). Some results like Whitehead group of finite groups are also known: $Wh(F) = 0$, when F is a finite cyclic group of order $1, 2, 3, 4$ and 6 and $Wh(F)$ is infinite when F is any other finite cyclic group. Also $Wh(S_n) = 0$, here S_n is the symmetric group on n letters. Below, we show by a little calculation that there is an element of infinite order in $Wh(\mathbb{Z}_5)$. In fact this group is infinite cyclic, but that needs a difficult proof and we do not talk about it here.

Lemma 4.0.1. *There exists an element of infinite order in $Wh(\mathbb{Z}_5)$.*

Proof. Let t be the generator of $\mathbb{Z}_5$. Let $a = 1 - t - t^{-1}$. Then

$$(1 - t - t^{-1})(1 - t^2 - t^3) = 1 - t - t^{-1} - t^2 + t^3 + t - t^3 + t^{-1} + t^2 = 1.$$

Hence a is a unit in $\mathbb{Z}[\mathbb{Z}_5]$. Define $\alpha : \mathbb{Z}[\mathbb{Z}_5] \to \mathbb{C}$ by sending t to $e^{2\pi i/5}$. Then α sends $\{\pm g \mid g \in \mathbb{Z}_5\}$ to the roots of unity in $\mathbb{C}$. Hence $x \mapsto |\alpha(x)|$ defines a homomorphism from $Wh(\mathbb{Z}_5)$ into R^*_+, the nonzero positive real numbers. Next note the following.

$$|\alpha(a)| = |1 - e^{2\pi i/5} - e^{-2\pi i/5}| = |1 - 2cos\frac{2\pi}{5}| \approx 0.4.$$

This proves that α defines an element of infinite order in $Wh(\mathbb{Z}_5)$. □

For Whitehead groups of general finite groups see [13]. We have already seen that the Whitehead group of the trivial group is trivial. Next simplest question is what is $Wh(C)$, where C is the infinite cyclic group? Well, this has already been computed by G. Higman in 1940 even before Whitehead group was defined.

Theorem 4.0.1 ([7])**.** *The Whitehead group of the infinite cyclic group is trivial.*

The proof is also given in [4]. One can ask for generalization of this result in two possible directions; namely, what is $Wh(F_r)$ and $Wh(C^n)$? Here F_r is the non-abelian free group on r generators and C^n is the abelian free group on n generators. This question has been answered completely.

Theorem 4.0.2 ([17]). *Let G_1 and G_2 be two groups and let $G_1 * G_2$ denotes their free product. Then $Wh(G_1 * G_2)$ is isomorphic to $Wh(G_1) \oplus Wh(G_2)$ and $\tilde{K}_0(G_1 * G_2)$ is isomorphic to $\tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)$.*

Using Higman's result we get the following important corollary.

Corollary 4.0.1. $Wh(F_r) = 0$.

Theorem 4.0.3 ([1]). $Wh(C^n) = 0$.

We have already mentioned before the following result.

Theorem 4.0.4 ([21], [5], [8]). *Let M be one of the following spaces: a compact Haken 3-manifold, a complete nonpositively curved Riemannian manifold or a finite $CAT(0)$-complex. Then $Wh(\pi_1(M)) = \tilde{K}_0(\mathbb{Z}[\pi_1(M)]) = 0$.*

An important result relating Whitehead group and the reduced projective class group is the Bass-Heller-Swan formula.

Theorem 4.0.5 ([1]). *Let G be a finitely generated group. Then*

$$Wh(G \times C) \simeq Wh(G) \oplus \tilde{K}_0(\mathbb{Z}[G]) \oplus N \oplus N,$$

where N is some Nil-group.

Theorem 4.0.3 is, in fact, a corollary of the above formula.

As we are mostly interested in showing the vanishing of the Whitehead group we do not recall the definition of the group N. For details see [15].

The following two corollaries are now easily deduced.

Corollary 4.0.2. $\tilde{K}_0(\mathbb{Z}[C^n]) = 0$.

Corollary 4.0.3. $\tilde{K}_0(\mathbb{Z}[F_r]) = 0$.

It is also known that $\tilde{K}_0(\mathbb{Z}[G])$ of any finite group G is finite. See [18].

The surgery L-groups involves extremely complicated algebra. We recall here only the computation for the non-abelian free groups, which also gives the computations for the trivial group and the infinite cyclic group.

Theorem 4.0.6 ([3]). *Let F_m be a free group on m generators. Then $L_n(F_m) = \mathbb{Z}, \mathbb{Z}^m, \mathbb{Z}_2, \mathbb{Z}_2^m$ for $n = 4k, 4k+1, 4k+2, 4k+3$ respectively.*

5. Open problems

Finally, to end this article we state the well known conjectures we already mentioned in this article. These conjectures are still open.

Conjecture 1. (W. C. Hsiang) *$\tilde{K}_0(\mathbb{Z}[G])$ of any torsion free group G vanishes.*

Conjecture 2. (J. H. C. Whitehead) *The Whitehead group of any torsion free group vanishes.*

Note that, using Bass-Heller-Swan formula it follows that Conjecture 2 implies Conjecture 1.

Conjecture 3. (A. Borel) *Two aspherical closed manifolds are homeomorphic if their fundamental groups are isomorphic.*

We should mention here that the above conjectures are already known for a large class of groups. We will discuss this in a future article. To explain these groups we need to describe many other concepts which have not been covered here.

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EULER'S PARTITION IDENTITY-FINITE VERSION

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Abstract. Euler proved that the number of partition of n into odd parts equals the number of partitions of n into distinct parts. There have been several refinements of Euler's Theorem which have limited the size of the parts allowed. Each is surprising and difficult to prove. This paper provides a finite version of Glaisher's exquisitely elementary proof of Euler's Theorem.

1. Introduction

Euler is truly the father of the theory of the partitions of integers. He discovered the following prototype of all subsequent partition identities.

Theorem 1. ***Euler's Theorem*** [3]**.** *The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.*

For example, if $n = 10$, then the ten odd partitions of n into distinct parts are

$$\begin{array}{ccccc} 10, & 9+1, & 8+2, & 7+3, & 7+2+1, \\ 6+4, & 6+3+1, & 5+4+1, & 5+3+2, & 4+3+2+1, \end{array}$$

and the ten partitions of n into odd parts are

$$\begin{array}{ccc} 9+1, & 7+3, & 7+1+1+1, \\ 5+5, & 5+3+1+1, & 5+1+1+1+1+1, \\ 3+3+3+1, & 3+3+1+1+1+1, & 3+1+1+1+1+1+1+1, \\ 1+1+1+1+1+1+1+1+1+1. & & \end{array}$$

Euler's proof was an elegant use of generating functions. If $\mathcal{D}(n)$ denotes the number of partitions of n into distinct parts and $\mathcal{O}(n)$ denotes the number of partitions of n into odd parts, then it is immediate (just by multiplying out the products and collecting terms) that

$$\sum_{n\geq 0} \mathcal{D}(n)q^n = \prod_{n=1}^{\infty}(1+q^n) \tag{1.1}$$

and

$$\sum_{n\geq 0} \mathcal{O}(n)q^n = \prod_{n=1}^{\infty}(1+q^{2n-1}+q^{2(2n-1)}+\cdots) \tag{1.2}$$
$$= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}.$$

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Euler's proof is then an algebraic exercise:

$$\sum_{n\geq 0}\mathcal{D}(n)q^n=\prod_{n=1}^{\infty}(1+q^n)=\prod_{n-1}^{\infty}\frac{1-q^{2n}}{1-q^n}$$
$$=\prod_{n-1}^{\infty}\frac{1}{1-q^{2n-1}}=\sum_{n\geq 0}\mathcal{O}(n)q^n. \tag{1.3}$$

It was J. W. L. Glaisher [5] who in 1883 found a purely bijective proof of Euler's Theorem. Glaisher's mapping goes as follows: start with a partition of n into odd parts (here f_i is the number of times $(2m_i-1)$ appears as a part):

$$f_1(2m_1-1)+f_2(2m_2-1)+\cdots+f_r(2m_r-1). \tag{1.4}$$

Now write each f_i in its unique binary representation as a sum of distinct powers of 2, i.e., now we have (with $a_1(i)<a_2(i)<\cdots$)

$$\sum_{i=1}^{r}f_i(2m_i-1)=\sum_{i=1}^{r}(2^{a_1(i)}+2^{a_2(i)}+\cdots+2^{a_j(i)})(2m_i-1)$$
$$=\sum_{i=1}^{r}\left(2^{a_1(i)}(2m_i-1)+\cdots+2^{a_j(i)}(2m_i-1)\right) \tag{1.5}$$

and this last expression is the image partition into distinct parts. To make Glaisher's maps concrete, let us return to the case $n=10$.

$9+1 \quad\to\quad 1\cdot 9+1\cdot 1 \quad\to\quad 9+1$

$7+3 \quad\to\quad 1\cdot 7+1\cdot 3 \quad\to\quad 7+3$

$7+1+1+1 \quad\to\quad 1\cdot 7+3\cdot 1 \quad\to\quad 1\cdot 7+(2{+}1)\cdot 1 \quad\to\quad 7+2+1$

$5+5 \quad\to\quad 2\cdot 5 \quad\to\quad 10$

$5+3+1+1 \quad\to\quad 1\cdot 5+1\cdot 3+1\cdot 2 \quad\to\quad 5+3+2$

$5+1+1+1+1+1 \quad\to\quad 1\cdot 5+5\cdot 1$

$\quad\to\quad 1\cdot 5+(4{+}1)\cdot 1 \quad\to\quad 5+4+1$

$3+3+3+1 \quad\to\quad 3\cdot 3+1\cdot 1 \quad\to\quad (2{+}1)\cdot 3+1\cdot 1 \quad\to\quad 6+3+1$

$3+3+1+1+1+1 \quad\to\quad 2\cdot 3+4\cdot 1 \quad\to\quad 6+4$

$3+1+1+1+1+1+1+1 \quad\to\quad 1\cdot 3+7\cdot 1$

$\quad\to\quad 1\cdot 3+(1{+}2{+}4)\cdot 1 \quad\to\quad 4+3+2+1$

$1+1+1+1+1+1+1+1+1+1 \quad\to\quad 10\cdot 1 \quad\to\quad (8{+}2)\cdot 1 \quad\to\quad 8+2$

It is clear that this map is reversible; just collect together in groups those parts with common largest odd factor.

Now there have been a number of refinements of Euler's Theorem which have, in one way or another, placed restrictions on the size of the parts used. Bousquet-Mélou and Eriksson [1] have a version in which their "lecture hall partitions" occur. Nathan Fine [4] has a version involving the Dyson rank. I. Pak [6] devotes Section 3 of his exhaustive study of partition identities to a variety of refinements of Euler's Theorem.

The point of this short note is to provide a simple Glaisher style proof of the following finite version of Euler's Theorem due to Bradford, Harris, Jones, Komarinski, Matson, and O'Shea that was first stated in [2].

Theorem [2; Sec. 3]. *The number of partitions of n into odd parts each $\leq 2N$ equals the number of partitions of n into parts each $\leq 2N$ in which the parts $\leq N$ are distinct.*

It should be noted that the bijective proofs in [2] as well as those by Bousquet-Melou and Erickson in [1] and Yee in [7] prove much more than the above theorem and are thus much more complicated than our Glaisher-like bijection.

2. First proof of the theorem

This result has an Eulerian proof that has exactly the simplicity of Euler's original proof.

Let $\mathcal{O}_N(n)$ denotes the number of partitions of n in which each part is odd and $\leq 2N$, and $\mathcal{D}_N(n)$ denotes the number of partitions of n in which each part is $\leq 2N$ and all parts $\leq N$ are distinct. Thus

$$\sum_{n\geq 0} \mathcal{O}_N(n)q^n = \prod_{n=1}^{N} \frac{1}{1-q^{2n-1}}$$

and

$$\sum_{n\geq 0} \mathcal{D}_N(n)q^n = \prod_{n=1}^{N} \frac{1+q^n}{1-q^{N+n}}.$$

Finally

$$\begin{aligned}\sum_{n\geq 0} \mathcal{D}_N(n)q^n = \prod_{n=1}^{N} \frac{1-q^{2n}}{(1-q^n)(1-q^{N+n})} &= \frac{\prod_{n=1}^{N}(1-q^{2n})}{\prod_{n=1}^{2N}(1-q^n)} \\ &= \prod_{n=1}^{N} \frac{1}{1-q^{2n-1}} = \sum_{n\geq 0} \mathcal{O}_N(n)q^n.\end{aligned}$$

3. A Glaisher-type proof

Now we return to Glaisher's proof, with the following alteration, namely, for each odd part $(2m_i-1)$ (all being $\leq 2N$) there is a unique $j_i \geq 0$ such that

$$N < (2m_i-1)2^{j_i} \leq 2N.$$

Now instead of rewriting each f_i completely in binary, we write f_i (with $a_1(i) < a_2(i) < \cdots < a_m(i) < f_i$) as $2^{a_1(i)} + 2^{a_2(i)} + \cdots + 2^{a_m(i)} + g_i 2^{j_i}$, where, of course, g_i might be 0. Thus, instead of (1.5) we now have

$$\sum_{i=1}^{r} f_i(2m_i-1) = \sum_{i=1}^{r}(2^{a_1(i)}+2^{a_2(i)}+\cdots+2^{a_m(i)}+g_i2^{j_i})(2m_i-1)$$
$$= \sum_{i=1}^{r}\left(2^{a_1(i)}(2m_i-1)+2^{a_2(i)}(2m_i-1)+\cdots\right)+\sum_{i=1}^{r} g_i2^{j_i}(2m_i-1)$$

and the latter expression is a partition wherein the parts $\leq N$ are distinct and each is $\leq 2N$.

As an example, let us consider a partition of 78 into odd parts each $\leq 2N = 2\cdot 6$:

$$11+11+7+7+5+5+5+3+3+3+3+3+3+3+3+3.$$

Now 11 and 7 lie in $(6,12]$, $2\cdot 5 \in (6,12]$, and $4\cdot 3 \in (6,12]$. Hence this partition is

$$\begin{aligned} &2\cdot 11+2\cdot 7+3\cdot 5+9\cdot 3\\ &\quad = 2\cdot 11+2\cdot 7+(1+2)\cdot 5+(1+2\cdot 4)\cdot 3\\ &\quad = 11+11+7+7+5+10+3+2\cdot(4\cdot 3)\\ &\quad = 11+11+7+7+5+10+3+12+12 \end{aligned}$$

and this last expression has all parts ≤ 12 and no repeated parts ≤ 6.

My thanks to Drew Sills and Carla Savage for alerting me to references [2] and [7].

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A SIMPLE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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Abstract. In this short note we give a proof of the Fundamental Theorem of Algebra using elementary complex analysis.

1. Fundamental Theorem of Algebra

For $x \geq 0$, we define ([1])

$$\log^+ x = \max\{\log x, 0\} = \begin{cases} \log x, & x \geq 1, \\ 0, & 0 \leq x < 1. \end{cases}$$

Then

$$\log x = \log^+ x - \log^+ \frac{1}{x} \text{ for } x > 0. \tag{1.1}$$

Now we recall a well known result from Complex Analysis ([2]):

For every holomorphic function g defined in a simply connected domain $D \subseteq \mathbb{C}$ which never vanishes in $D \subseteq \mathbb{C}$, there exists a holomorphic function $h : D \subseteq \mathbb{C} \to \mathbb{C}$ such that $g(z) = e^{h(z)}$ for all $z \in D \subseteq \mathbb{C}$.

In view of this, if $f(z)$ is a non constant analytic function having no zero in $|z| \leq r$ $(0 < r < \infty)$, then $\log f(z)$ is also analytic in $|z| \leq r$.

Then by Cauchy's Theorem of Residues we have

$$\log f(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{\log f(z)}{z} dz = \frac{1}{2\pi}\int_0^{2\pi} \log f(re^{i\theta}) d\theta. \tag{1.2}$$

Actually equation (1.2) implies that $\log f(0)$ is the average value of $\log f(z)$ taken on $|z| = r$. Now by taking the real parts from both the sides of equation (1.2) and then applying equation (1.1), we get

$$\log |f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})| d\theta \tag{1.3}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})|} d\theta. \tag{1.4}$$

Let $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ $(a_n \neq 0)$ be a non-constant polynomial over $\mathbb{C}$. Assume on the contrary that $P(z)$ has no zero in $\mathbb{C}$. It is clear that

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(1) $\frac{1}{|P|} < 1$ for sufficiently large r because $|P| \to \infty$ as $r \to \infty$ (where $z := re^{i\theta}$), and

(2) $\mid \frac{P(z)}{a_n z^n} \mid \to 1$ for sufficiently large r.

Then equation (1.4) yields that when $r \to \infty$

$$\begin{aligned} \log \mid P(0) \mid &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \mid P(re^{i\theta}) \mid d\theta \\ &= n \log r + O(1). \end{aligned}$$

But $\log r$ is unbounded as $r \to \infty$. Hence our assumption is wrong. It follows that P has atleast one, and hence exactly n, zeros in $\mathbb{C}$.

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HOPF'S UMLAUFSATZ OR THE THEOREM OF TURNING TANGENTS

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Abstract. In this short article we give a brief description of plane curves with a focus on regular curves. We talk about one of the fundamental theorems in the theory of plane curves, famously known as Hopf's Umlaufsatz and try to present a (more or less) self contained proof of the same.

In mathematics a given object is studied from various aspects (such as analytic, geometric, topological, algebraic, etc.) and relating different descriptions or properties of the same object often leads to a deeper understanding. More specifically, in the theory of curves and surfaces (or *smooth manifolds* in general) one tries to relate their *local* (geometric) and *global* (topological) properties and if possible, understand to what extent the local properties of these spaces control the global ones. For example, the famous Gauss-Bonnet theorem (see [1]) says that the sign of the *total* Gaussian curvature (a geometric data which also hints at the shape of the surface) of a closed oriented surface is essentially determined by its *topology*. In the context of *nice* plane curves one would like to ask a similar but simpler question. Can one interpret the total curvature of a plane-curve (i.e., the integral of curvature of the curve) in terms of its topological properties? In this short discussion we talk about a celebrated result due to H. Hopf famously known as *Hopf's Umlaufsatz* which addresses this issue. *Umlaufsatz* is a German word translating in to *circulation rate* in English. It is one of the most fundamental theorems in the study of plane curves that relates a local (geometric) characteristic of a plane curve (provided by its curvature) to a global (topological) characteristic, namely, the number of times the curve revolves around a point. The theorem in particular, says that a closed plane curve without self- intersections can revolve only once around a point.

1. Describing a curve in the Plane

For the purpose of this note we will restrict to plane curves i.e., curves sitting in the Euclidean plane. A *plane curve* is commonly visualized as a *bent* line drawn in the Euclidean plane. In other words, a *straight line* in the plane is a line that is

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not *curved*. A few well known plane curves are *circle*, *parabola*, *ellipse*, *hyperbola*,

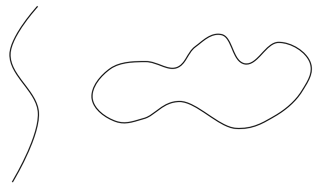


FIGURE 1. Plane curves.

etc. There are several ways for describing a plane curve mathematically. Perhaps the simplest description of a plane curve (including a straight line) would be as a set of points in the Euclidean plane satisfying some mathematical constraints. More precisely, a plane curve C is a set of points in $\mathbb{R}^2$ defined by

$$C = \{(x, y) \in \mathbb{R}^2 : f(x, y) = c\}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a (preferably smooth) function and c is a constant. This describes a plane curve as a *level set* of f and such a curve is often called a *level curve*. Previously mentioned plane curves can all be described as level curves as follows.

(1) The straight line L of *slope* m passing through the origin $(0, 0)$ is the level curve $L = \{(x, y) \in \mathbb{R}^2 : y - mx = 0\}$.

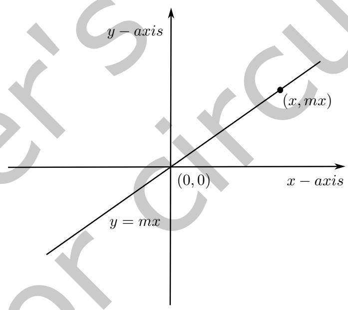


FIGURE 2. The straight line through origin with slope m

(2) The circle C of *radius* r *centered* at the origin is the level curve

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r\}.$$

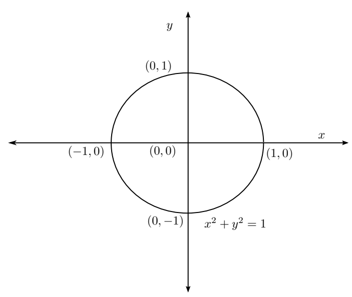


FIGURE 3. The circle of unit radius centered at origin

(3) The parabola P with its *vertex* at the origin and *focus* at the point $(1,0)$ is the level curve $P = \{(x,y) \in \mathbb{R}^2 : y^2 - 4x = 0\}$.

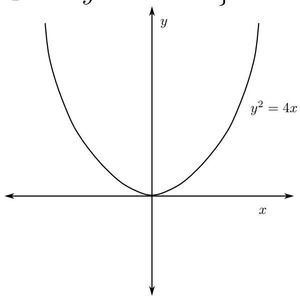


FIGURE 4. Parabola

(4) An ellipse E with x and y-axes as *major* and *minor* axes respectively is represented as the level curve $E = \{(x,y) \in \mathbb{R}^2 : (x^2/a^2) + (y^2/b^2) = 1\}$.

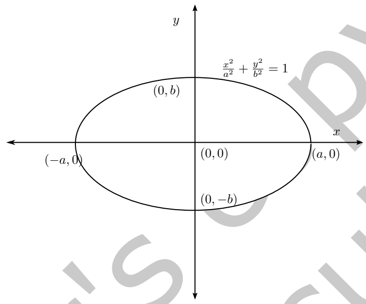


FIGURE 5. An ellipse

(5) A hyperbola H with *center* at the origin and the *transverse* axis aligned with the x-axis has the describing set $\{(x,y) \in \mathbb{R}^2 : (x^2/a^2) - (y^2/b^2) = 1\}$.

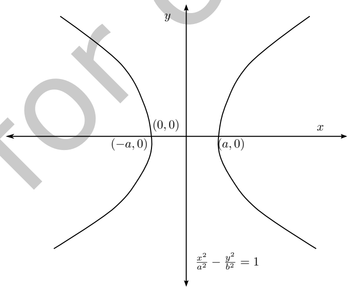


FIGURE 6. A hyperbola

The above description of a plane curve is not useful in understanding the infinitesimal behaviour or the analytic properties of a curve. For this purpose a given plane curve is often viewed as a path traversed by a point moving in the plane. For instance, if $\gamma(t)$ denotes the position vector of a moving point at time t, one obtains a plane curve described by the function γ of a scalar parameter t taking vector values in the Euclidean plane. Then one can talk about the velocity γ' and acceleration γ'' of the moving point which provide significant information about the corresponding curve. The following formal definition makes things precise.

Definition 1.1. A **parametrized plane curve** is a map $\gamma : I \to \mathbb{R}^2$ where $I \subset \mathbb{R}$ is an interval. A parametrized curve γ is **smooth** if it is a smooth (infinitely differentiable) map. Moreover, a smooth parametrized curve γ is

(1) **regular** if the velocity vector γ' is no-where vanishing i.e., $\gamma'(t) = \frac{d\gamma}{dt} \neq 0$ for all $t \in I$ and
(2) **unit speed** or **parametrized by arclength** if γ' has unit length i.e., $\|\gamma'(t)\| = 1$ for all $t \in I$ where $\|.\|$ denotes the Euclidean norm (i.e., for $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, the length of the vector $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))^T$ ($(\gamma_1'(t), \gamma_2'(t))^T$ is the column vector i.e., the transpose of $(\gamma_1'(t), \gamma_2'(t))$) is given by $\|\gamma'(t)\|^2 = \gamma_1'(t)^2 + \gamma_2'(t)^2$).

The interval I in this definition can be bounded, unbounded, open, closed or half-open. A parametrized curve γ is called a *parametrization* of the plane curve described by the image ($Im(\gamma) \subset \mathbb{R}^2$) of the map γ.

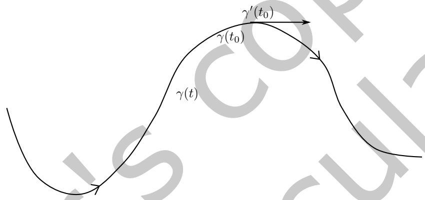


FIGURE 7. A regular parametrized plane curve

We will focus on regular parametrized curves. This excludes the trivial parametrized curve given by the constant map $\gamma : t \mapsto p_0$, $\forall t \in I$ (p_0 is a fixed point in $\mathbb{R}^2$). For computational purposes it sometimes helps to work with unit speed curves. In the following examples well known plane curves discussed so far are realized as regular parametrized curves.

Example 1.2. (1) A *straight line* can be viewed as the locus of a point moving in the plane with a constant (non-zero) velocity, described as the regular parametri-

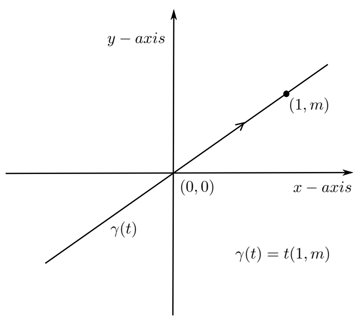


FIGURE 8. Parametrized straight line

zed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = p + tv$ where $p \in \mathbb{R}^2$ denotes the initial position and $v \in \mathbb{R}^2 - \{0\}$ the initial velocity of the moving point. In particular, the straight line passing through the origin with slope $m \neq 0$ is the locus of a moving point starting from the origin and moving with constant velocity $v = (1, m)$ (cf. figure 8).
(2) The *circle* of radius r at any point $(x_0, y_0) \in \mathbb{R}^2$ can be viewed as the regular parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (x_0 + r\cos t, y_0 + r\sin t)$. In fact, it suffices to define γ on the closed interval $[0, 2\pi]$.
(3) The *parabola* with vertex at the origin and focus at $(1, 0)$, can be described as a regular parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (t^2, 2t)$.
(4) The *ellipse* described before can be viewed as the regular parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (a\cos t, b\sin t)$ where a, b are as before.
(5) The *hyperbola* as before can be viewed as the regular parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (a\cosh t, b\sinh t)$ where a, b are as before.

Let us see a few more examples of parametrized curves.

Example 1.3. (1) The *logarithmic spiral* (cf. figure 9) is the regular parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (e^t \cos t, e^t \sin t)$.

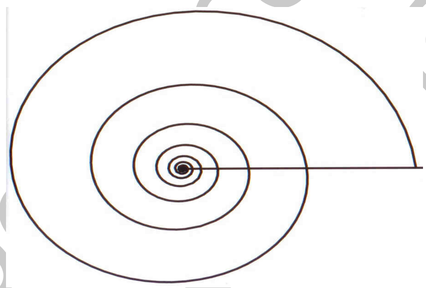


FIGURE 9. Logarithmic Spiral

(2) The *astroid* is the parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos^3 t, \sin^3 t)$.
(3) The *tractrix* is the regular parametrized curve $\gamma : (0, \frac{\pi}{2}) \to \mathbb{R}^2$ defined by $\gamma(t) = (\sin t, \cos t + \ln\tan(t/2))$. The curve γ is not regular when defined on the open interval $(0, \pi)$ as $\gamma'(\frac{\pi}{2}) = 0$.

A parametrized curve γ is much more than $Im(\gamma)$ (i.e., the set of points in the plane lying on the curve) as it also indicates with how much speed and in which direction the point is moving to trace the curve. This information is provided by the velocity vector. We will only consider regular parametrized curves which have well-defined, non-vanishing velocity vectors.

1.1. **Reparametrization of plane curves.** One can show that a given plane curve can be described as different parametrized curves. This is natural as the same path can be travelled with different speed and velocity. In fact, both the smooth parametrized curves $\gamma, \sigma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (t^2, 2t)$, and $\sigma(t) = (4t^2, 4t)$ describe the parabola with vertex $(0,0)$ and focus $(1,0)$. Thus the parametrization of a given plane curve is not unique. One would like to understand the relation between two different parametrizations of the same curve.

Definition 1.4. Let $\gamma : I \to \mathbb{R}^2$ be a parametrised curve as before. A parametrised curve $\sigma : \tilde{I} \to \mathbb{R}^2$ ($\tilde{I} \subset \mathbb{R}$ an interval) is said to be a **reparametrization** of γ if there is a smooth bijective map $\phi : \tilde{I} \to I$ (called a **change of parameter**) such that the inverse map $\phi^{-1} : I \to \tilde{I}$ is also smooth and $\sigma = \gamma \circ \phi$ i.e., $\sigma(t) = \gamma(\phi(t))$ for all $t \in \tilde{I}$.

As ϕ and ϕ^{-1} are both smooth maps, interchanging their role one can view γ as a reparametrization of σ. Also, any parametrized curve $\gamma : I \to \mathbb{R}^2$ as above can be thought as a reparametrization of itself via the trivial change of parameter, namely, the identity map of I. One further observes that if $\sigma : \tilde{I} \to \mathbb{R}^2$ is a reparametrization of $\gamma : I \to \mathbb{R}^2$ and γ a reparametrization of $\eta : \hat{I} \to \mathbb{R}^2$ with respective changes of parameter $\phi : \tilde{I} \to I$ and $\psi : I \to \hat{I}$, then σ is a reparametrization of η with change of parameter $\psi \circ \phi : \tilde{I} \to \hat{I}$. It thus follows that on the space of parametrized plane curves *reparametrization* defines an *equivalence relation*.

Example 1.5.
(1) Cosider regular parametrized curves $\gamma, \sigma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (r\cos t, r\sin t)$ and $\sigma(t) = (r\sin t, r\cos t)$. Both describing the circle of radius r centred at the origin. It is easy to see that σ is a reparametrization of γ, with change of parameter $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = \frac{\pi}{2} - t$.
(2) Recall the smooth parametrized curves $\gamma_i : \mathbb{R} \to \mathbb{R}^2$ (i=1,2) defined by $\gamma_1(t) = (t^2, 2t)$ and $\gamma_2(t) = (4t^2, 4t)$ describing the parabola with vertex at the origin and focus at $(1,0)$. It follows that γ_2 is a reparametrization of γ_1 with the change of parameter $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = 2t$.

One wonders what kind of properties of a parametrized curve remain unchanged under a reparametrization. We observe the following.

Proposition 1.6. *A reparametrisation of a regular parametrized curve is regular.*

Proof. This is a direct consequence of the chain rule of differentiation. Let $\gamma : I \to \mathbb{R}^2$ be a regular parametrised curve and $\sigma : \tilde{I} \to \mathbb{R}^2$ be a reparametrization of γ with a change of parameter $\phi : \tilde{I} \to I$ i.e., $\sigma(t) = \gamma(\phi(t))$ for all $t \in \tilde{I}$. Then $\phi'(t)$ is never zero as according to the chain rule one has $1 = (\phi^{-1} \circ \phi)'(t) = (\phi^{-1})'(\phi(t)) \cdot \phi'(t)$. Again applying the chain rule it follows that for all $t \in \tilde{I}$, $\sigma'(t) = (\gamma \circ \phi)'(t) = \gamma'(\phi(t)) \cdot \phi'(t) \neq 0$. □

An immediate consequence is that a regular curve can not have a reparametrization that is *not* regular. The smooth parametrized curves $\gamma : \mathbb{R} \to \mathbb{R}^2$ and $\sigma : \mathbb{R} \to \mathbb{R}^2$ defined respectively by $\gamma(t) = (t^2, 2t)$ and $\sigma(t) = (t^6, 2t^3)$ both describe the parabola with vertex $(0,0)$ and focus $(1,0)$, but while γ is regular, σ

is not (as $\sigma'(0) = (0,0)$). Thus none of them can be realized as a reparametrization of the other. This also shows that a plane curve can have both regular and non-regular parametrizations.

Another property of a smooth parametrized curve that does not change under a reparametrization is the *length* as described below. One would expect so as the length of a path does not depend on the speed at which it is travelled and a paramerization is one particular description of a plane curve as a path traced by a moving point.

Definition 1.7. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a smooth parametrised curve where $[a,b]$ is a closed interval. The **length** of γ denoted by $l(\gamma)$, is defined by

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

More generally, given any smooth curve $\gamma : I \to \mathbb{R}^2$ defined on any interval $I \subset \mathbb{R}$ (not necessarily closed) and any fixed $t_0 \in I$, the **arc-length** of γ starting at the point $\gamma(to)$ is the function $s : I \to \mathbb{R}$ defined by

$$s(t) = l(\gamma|_[t_0, t]) = \int_{t_0}^t \|\gamma'(\tau)\| d\tau.$$

This definition matches our intuition about the length and arclength of a curve. In fact, If s is the arc-length of a curve γ as above, then $\frac{ds}{dt} = \|\gamma'(t)\|$ i.e., if $\gamma(t)$ stands for the position of a moving point at time t, $\frac{ds}{dt}$ gives the rate of change of position along the curve. according to this definition the circle of radius r (centered at $(0,0)$) as the regular parametrized curve $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (r\cos t, r\sin t)$ has length $2\pi r$. The following result unravels a crucial property of the length of a curve.

Proposition 1.8. *The length of a smooth parametrized curve is unchanged by reparametrization.*

Proof. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a smooth parametrised curve and $\sigma : [c,d] \to \mathbb{R}^2$ be a reparametrization of γ with a change of parameter $\phi : [c,d] \to [a,b]$ satisfying $\sigma = \gamma \circ \phi$. Then

$$\begin{aligned} l(\sigma) = \int_c^d \|\sigma'(t)\| dt &= \int_c^d \|\gamma'(\phi(t))\| |\phi'(t)| dt \text{ (chain rule)} \\ &= \int_a^b \|\gamma'(s)\| ds \text{ (putting } s = \phi(t)) \quad = \quad l(\gamma). \end{aligned}$$

Here the fact that ϕ is either strictly increasing or decreasing (i.e., $\phi'(t)$ is either positive or negative for all $t \in [c,d]$) is used. $\square$

Not all properties are preserved under reparametrizations. Recall that a regular parametrized curve being the locus of a moving point in the plane also indicates a specific direction in which it moves. This direction is given by the velocity (vector) of the curve. In the proof of Proposition 1.6 the key fact is that for a change of parameter $\phi : \tilde{I} \to I$, $\phi'(t)$ is either positive or negative. Thus, as a consequence of chain rule of differentiation again, the direction of a regular parametrized curve (i.e., the direction of its velocity) is either preserved or reversed by a reparametrization depending on the sign of the derivative of corresponding change of parameter.

Definition 1.9. Let $\gamma : I \to \mathbb{R}^2$ be a parametrised curve, $\sigma : \tilde{I} \to \mathbb{R}^2$ a reparametrization of γ with a change of parameter $\phi : \tilde{I} \to I$ i.e., $\sigma(t) = \gamma(\phi(t))$ for all $t \in \tilde{I}$. Then ϕ is called **orientation preserving** (or **reversing**) if $\phi'(t) > 0$ (or < 0) for all $t \in \tilde{I}$.

Clearly, a change of parameter is either orientation preserving or reversing. We consider a couple of simple examples.

Example 1.10.
(1) The change of parameter $\phi(t) = t$ ($\phi(t) = -t$) is orientation preserving (reversing).
(2) The change of parameters discussed in Example 1.5 (1) and (2) are respectively orientation reversing and preserving.

1.2. **Unit speed curves.** It was mentioned in the previous section that for purely technical reasons working with unit speed curves is preferred at times. But to begin with one needs to have a few examples of such curves at one's disposal and if possible, identify curves that can be realized as unit speed curves. As unit speed curves are regular and regularity is preserved by reparametrization, it follows that almost immediately that regularity is a necessary condition for a parametrized curve in order to admit a unit speed reparametrization. It turns out to be also a sufficient condition.

Proposition 1.11. *A regular parametrized curve can be reparametrized as a unit speed curve.*

Proof. Consider a regular parametrized curve $\gamma : I \to \mathbb{R}^2$. Fixing some $t_0 \in I$, consider the arc-length function $s : I \to \mathbb{R}$ of γ starting at the point $\gamma(to)$ given by

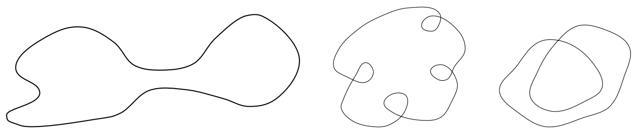
$$s(t) = l(\gamma|_[t_0, t]) = \int_{t_0}^{t} \|\gamma'(\tau)\| d\tau.$$

Then $\frac{ds}{dt} = \|\gamma'(t)\| \neq 0$ as γ is regular. Thus s is strictly monotonically increasing and smooth. Consider the smooth bijection $s : I \to s(I) = J$ ($J \subset \mathbb{R}$ is an interval) and let $\phi = s^{-1} : J \to I$. Then it follows from Inverse Function Theorem (see [**?**] for details) that ϕ is also smooth. Thus ϕ can be treated as an orientation preserving change of parameter as using chain rule one obtains $\phi'(t) = 1/\|\gamma'(\phi(t))\|$. This implies

$$\|(\gamma \circ \phi)'(t)\| = \|\gamma'(\phi(t))\| \cdot |\phi'(t)| = 1,$$

which completes the proof. (This proof justifies the name 'curve parametrized by arc length' for a unit speed curve.) $\square$

Two simple examples are discussed here.



(a) Simple closed curve (b) Closed curves

FIGURE 10

Example 1.12. (1) Recall from Example 1.2(1) that the straight line is the regular parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = p + tv$ ($p \in \mathbb{R}^2$ and $v \in \mathbb{R}^2 - \{0\}$ as before). Consider the change of parameter $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = \frac{t}{\|v\|}$ (obtained by inverting the arc-length function of γ). Then the reparametrization $\tilde{\gamma}$ given by $\tilde{\gamma}(t) = (\gamma \circ \phi)(t) = p + t\frac{v}{\|v\|}$ is a unit speed curve.

(2) The circle of radius r centered at any point $(x_0, y_0) \in \mathbb{R}^2$, viewed as the regular parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (x_0 + r\cos t, y_0 + r\sin t)$, can be reparametrized by arc-length as the unit speed curve $\tilde{\gamma}$: $\mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (x_0 + r\cos\frac{t}{r}, y_0 + r\sin\frac{t}{r})$.

One might ask if unit speed reparametrizations are unique and if not, then one would like to understand the relation between any two unit speed reparametrizations of a regular parametrized curve. Since the reparametrization is an equivalence relation, it follows that any two different unit speed reparametrizations will be reparametrizations of each other as well. One further observes the following.

Lemma 1.13. *Let $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^2$ and $\sigma : J \to \mathbb{R}^2$ be two unit speed reparametrizations of a regular parametrized curve $\gamma : I \to \mathbb{R}^2$. Then the change of parameter function $\phi : \tilde{I} \to J$ satisfying $\tilde{\gamma} = \sigma \circ \phi$ is defined by $\phi(t) = \pm t + c$ for a constant $c \in \mathbb{R}$.*

Proof. Using chain rule of differentiation one can see that

$$1 = \|\tilde{\gamma}'(t)\| = \|\sigma'(\phi(t)) \cdot \phi'(t)\| = \|\sigma'(\phi(t))\| \cdot |\phi'(t)| = |\phi'(t)|.$$

On integration this yields that $\phi(t) = \pm t + c$ where c is the integrating constant. $\square$

1.3. **Simple closed curves.**

Definition 1.14. A parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ is said to be **periodic with period** L if for each $t \in \mathbb{R}$, $\gamma(t) = \gamma(t + L)$ holds for a number $L > 0$ and for any number L' with $0 < L' < L$, $\gamma(t) = \gamma(t + L')$ does not hold for all $t \in \mathbb{R}$. We would call a plane curve **closed** if it admits a (regular) periodic parametrization. Moreover, a closed curve is called **simple** if it has a (regular) peroidic parametrization $\gamma : \mathbb{R} \to \mathbb{R}^2$ with period L such that for each $t \in \mathbb{R}$, the restricted map $\gamma|_{[t,t+L)}$ is injective.

From the definition it follows almost immediately that a closed curve encloses itself around a point and a simple closed curve is a closed curve without self-intersections other than the point where it closes itself (cf. figures 10(a) and 10(b)). For example, the circle of unit radius centered at the origin is a closed curve together with a unit speed periodic parametrization $\gamma : \mathbb{R} \to \mathbb{R}^2$ with period $L = 2\pi$ defined by $\gamma(t) = (\cos t, \sin t)$. In fact, using this parametrization one further observes that the unit circle centered at the origin is a simple closed curve. From now on we will focus on unit speed parametrizations of simple closed curves.

2. Hopf's Umlaufsatz

2.1. **Curvature of plane curves.** How does one describe the *curvature* of a (plane) curve? An intuitive description would be 'something' that decides how much *curved* the curve is i.e., how far it is from being *straight*. A curve does not necessarily bend uniformly at all points on itself and the curvature might vary as a point moves along the curve, but overall it should depend only on the shape of the curve and not on a particular description (or parametrization) of it. Also, the straight line should have zero curvature and more bent a curve is at a point, more should be the curvature. In particular, circles with bigger radii should have smaller curvatures. One can formalize these intuitions mathematically as follows.

Henceforth we will stick to unit speed plane curves. Let $\gamma : I \to \mathbb{R}^2$ be such a curve. One can talk about its normal vector at each point namely, a unit vector perpendicular to the tangent (or velocity) vector $\gamma'(t)$. Then the normal vector is also no where vanishing at each point on the curve just like the tangent vector. In fact, one can see that at each point $\gamma(t)$ there are exactly two choices for the unit normal vector namely, the vectors obtained by rotating the tangent vector $\gamma'(t)$ by 90 degrees in the anti-clockwise and clockwise directions. We choose the **normal vector** at $\gamma(t)$ as the unit vector $n(t)$ obtained by rotating $\gamma'(t)$ by 90 degrees anti-clockwise direction i.e,

$$n(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \gamma'(t).$$

Then for a unit speed plane curve γ as above its normal vector $n(t)$ and acceleration vector $\gamma''(t)$ are scalar multiple of each other where the scalar depends on the time parameter t. To see this first note that γ has unit speed i.e.,

$$\|\gamma'(t)\|^2 = \langle \gamma'(t), \gamma'(t) \rangle = 1, \ \forall \ t \in I$$

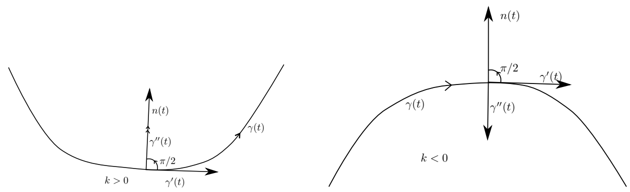
where $\langle ., . \rangle$ is the usual dot product of vectors in $\mathbb{R}^2$. Differentiating this from both sides one obtains

$$2\langle \gamma'(t), \gamma''(t) \rangle = \langle \gamma''(t), \gamma'(t) \rangle + \langle \gamma'(t), \gamma''(t) \rangle = 0,$$

which in particular shows that $\gamma''(t)$ (just like $n(t)$,) is perpendicular to $\gamma'(t)$ for each $t \in I$ and one can write $\gamma''(t)$ as a scalar multiple of $n(t)$ (with the scalar depending on t) as follows : $\gamma''(t) = \kappa(t).n(t)$.

This observation leads to the following definition.

Definition 2.1. Let $\gamma : I \to \mathbb{R}^2$ be a unit speed plane curve with normal vector $n(t)$ and $\gamma''(t) = \kappa(t).n(t)$. Then the scalar function $\kappa : I \to \mathbb{R}$ is called the **signed curvature** and its absolute value the **curvature** of γ.



(a) Positive Curvature (b) Negative Curvature

FIGURE 11

This only defines the curvature of a unit speed curve. As a regular parametrized plane curve can be reparametrized as unit speed curves, one can attempt to define the curvature of such a curve γ (of non-unit speed) to be that of an orientation-preserving unit-speed reparametrization of γ provided the curvature of any such unit speed reparametrization of a given regular curve is the same. This is asserted by the following lemma.

Lemma 2.2. *Let $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^2$ and $\sigma : J \to \mathbb{R}^2$ be two unit speed reparametrizations of a regular parametrized curve $\gamma : I \to \mathbb{R}^2$. Then $\tilde{\gamma}$ and σ have the same curvature. But the signed curvature of $\tilde{\gamma}$ is the opposite to that of σ if the change of parameter $\phi : \tilde{I} \to J$ with $\tilde{\gamma} = \sigma \circ \phi$ is orientation reversing.*

This follows using the definition of curvature and Lemma 1.13. One needs to further check if the definition of curvature matches our intuitive understanding as discussed in the beginning of this section. Firstly, recall that a unit speed straight line $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = p + tv$ where $p, v \in \mathbb{R}^2$ and $\|v\| = 1$, has zero acceleration ($\gamma''(t) = 0$) and hence zero curvature (i.e., $k(t) = 0$ for all $t \in \mathbb{R}$). The curvature of a circle of radius r is computed in the following example.

Example 2.3. Recall from Example 1.12(2) that the circle of radius r centered at a point $(x_0, y_0) \in \mathbb{R}^2$, is the unit speed curve $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (x_0 + r\cos\frac{t}{r}, y_0 + r\sin\frac{t}{r})$. The tangent vector is $\gamma'(t) = (-\cos\frac{t}{r}, \sin\frac{t}{r})^T$ and acceleration vector is $\gamma''(t) = -\frac{1}{r}(\cos\frac{t}{r}, \sin\frac{t}{r})^T$. Also the normal vector is

$$n(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \gamma'(t) = -(\cos\frac{t}{r}, \sin\frac{t}{r})^T$$

and hence the curvature for all $t \in R$, is the constant $k(t) = \frac{1}{r}$ which matches to our intuition.

It follows immediately that the curvature of a parametrized curve is positive when the curve turns in the direction of its normal vector (cf. figure) and negative otherwise.

2.2. **Winding number.** Consider a unit speed curve $\gamma : I \to \mathbb{R}^2$ ($I = [a, b]$ is a closed interval). Then its velocity vector $\gamma'(t)$ is a unit vector in the Euclidean plane and one can write $\gamma'(t) = (\cos\theta(t), \sin\theta(t))^T$ where $\theta(t)$ represents the angle the vector makes with x-axis and is unique upto addition of an integer multiple of

2π. Moreover as a consequence of the following lemma it follows that the angle also changes smoothly with time and is uniquely determined by the number $\theta(b)-\theta(a)$.

Lemma 2.4. *Given a unit speed curve $\gamma : I \to \mathbb{R}^2$ (I is a closed interval) there exists a smooth function $\theta : I \to \mathbb{R}$ such that one has*

$$\gamma'(t) = (\cos\theta(t), \sin\theta(t))^T.$$

Further if $\theta_1, \theta_2 : I \to \mathbb{R}$ are two such smooth functions satisfying the above equation, then there exists $k \in \mathbb{Z}$ such that $\theta_1 = \theta_2 + 2k\pi$.

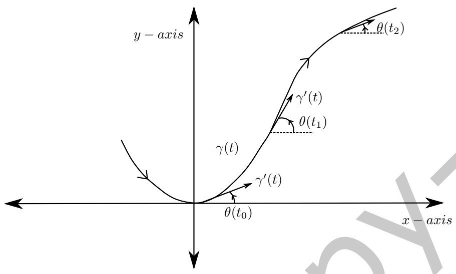


FIGURE 12. A trigonometric function θ

Proof. Consider the four semicircles S_R, S_L, S_T, S_B all contained in the unit circle S^1 described by

$$\begin{aligned} S_R &= \{(x,y)^T \in S^1 : x > 0\}, \\ S_L &= \{(x,y)^T \in S^1 : x < 0\}, \\ S_T &= \{(x,y)^T \in S^1 : y > 0\}, \\ S_B &= \{(x,y)^T \in S^1 : y < 0\}. \end{aligned}$$

First assume that the entire image $\gamma'(I)$ is contained in one of the semi-circles above, say for definiteness, $\gamma'([a,b]) \subset S_R$ i.e., $\gamma_1'(t) > 0$ for all $t \in I$ where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ (and thus $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))^T$). Then one can write

$$\frac{\gamma_2'(t)}{\gamma_1'(t)} = \frac{\sin\theta(t)}{\cos\theta(t)} = \tan\theta(t).$$

The smooth solution to this trigonometric equation is of the form

$$\theta(t) = \arctan\big(\frac{\gamma_2'(t)}{\gamma_1'(t)}\big) + 2k\pi,\ k \in \mathbb{Z}$$

which is uniquely determined for a fixed value of $\theta(a)$. If $\gamma'(I)$ is contained in one of the other three semi-circles, a similar argument shows the existence of a smooth trigonometric function θ with desired properties.

Now consider the general situation when $\gamma'(I)$ is not entirely contained in one of the semi-circles above. In that case, one can divide the closed interval $I = [a,b]$ into $n+1$ subintervals $a = t_0 < t_1 < \cdots < t_n = b$ such that for each $i = 0, \cdots, n-1$, $\gamma'[t_i, t_{i+1}]$ is contained in one of the four semi-circles. Specifying a particular

$\theta(a)$ one can obtain a unique smooth function $\theta : [a, t_1] \to \mathbb{R}$ satisfying $\gamma'(t) = (\cos\theta(t), \sin\theta(t))^T$. This determines $\theta(t_1)$ and proceeding as before one obtains a unique smooth continuation $\theta : [a, t_2] \to \mathbb{R}$ satisfying $\gamma'(t) = (\cos\theta(t), \sin\theta(t))^T$. Repeating this finitely many times one finally obtains the desired smooth function $\theta : [a, b] \to \mathbb{R}$. □

The lemma leads to the following definition.

Definition 2.5. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a unit speed periodic curve with period L and $\theta : \mathbb{R} \to \mathbb{R}$ be a trigonometric scalar function as in Lemma 2.4. Then the **winding number** of γ denoted by n_γ is defined to be

$$n_\gamma = \frac{1}{2\pi}(\theta(L) - \theta(0)).$$

This definition makes sense as the difference $(\theta(L) - \theta(0))$ is well defined irrespective of the choice of θ. As $\gamma'(L) = \gamma'(0)$ it follows from Lemma 2.4 that the n_γ is actually a *whole* number: in fact, $\cos\theta(L) = \cos\theta(0)$ and $\sin\theta(L) = \sin\theta(0)$ together imply that $e^{i\theta(L)} = e^{i\theta(0)}$ i.e., $\theta(L) - \theta(0) = 2k\pi$ for some $k \in \mathbb{Z}$.

Example 2.6. Recall that the circle of radius r centered at a point $(x_0, y_0) \in \mathbb{R}^2$, is the unit speed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (x_0 + r\cos\frac{t}{r}, y_0 + r\sin\frac{t}{r})$ with tangent vector $\gamma'(t) = (-\sin\frac{t}{r}, \cos\frac{t}{r})^T = (\cos(\frac{t}{r} + \frac{\pi}{2}), \sin(\frac{t}{r} + \frac{\pi}{2}))^T$, is a periodic curve with period $L = 2\pi r$. The trigonometric function $\theta : \mathbb{R} \to \mathbb{R}$ can be chosen to be $\theta(t) = \frac{t}{r} + \frac{\pi}{2}$ and hence the winding number is

$$n_\gamma = \frac{1}{2\pi}(\theta(2\pi r) - \theta(0)) = 1.$$

How the winding number changes under reparametrization is answered by the following result.

Lemma 2.7. *Let $\gamma, \sigma : \mathbb{R} \to \mathbb{R}^2$ be two unit speed periodic curves. If σ is a reparametrization of γ with a change of parameter $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\gamma = \sigma \circ \phi$, then $n_\sigma = \pm n_\gamma$ in accordance with the fact that ϕ is orientation preserving or reversing.*

Proof. First, recall from Lemma 1.13 that $\phi(t) = \pm t + c$ depending on whether ϕ is orientation preserving or reversing. Then it follows that γ and σ both have the same period say, L. Let θ be a trigonometric function for σ as in Lemma 2.4. If ϕ is orientation preserving, then one can check that the function $\tilde{\theta}$ defined by $\tilde{\theta} = \theta \circ \phi$ serves as a trigonometric function for γ as in that case $\phi(t) = t + c$ implies

$$\gamma'(t) = \sigma'(t + c) = (\cos(\theta(t + c)), \sin(\theta(t + c)))^T = (\cos(\tilde{\theta}(t)), \sin(\tilde{\theta}(t)))^T.$$

Also, since $\gamma'(t) = \gamma'(t + L)$ for all $t \in \mathbb{R}$, the trigonometric function $\hat{\theta}$ given by $\hat{\theta}(t) = \tilde{\theta}(t + L)$ satisfies

$$\gamma'(t) = (\cos\hat{\theta}(t), \sin\hat{\theta}(t))^T$$

i.e., $\hat{\theta}$ also serves as atrigonometric function for γ as in Lemma 2.4. Combining all these facts one obtains

$$\begin{aligned}
2\pi(n_\gamma - n_\sigma) &= (\tilde{\theta}(L) - \tilde{\theta}(0)) - (\theta(L) - \theta(0)) \\
&= (\tilde{\theta}(L) - \tilde{\theta}(0)) - (\tilde{\theta}(L-c) - \tilde{\theta}(-c)) \\
&= (\tilde{\theta}(0) - \tilde{\theta}(-c))(\hat{\theta}(0) - \hat{\theta}(-c)) = 0.
\end{aligned}$$

In the orientation reversing case i.e., when $\phi(t) = -t + c$ and θ is a trigonometric function for σ as in Lemma 2.4, then one can see that $\tilde{\theta} = \theta \circ \phi + \pi$ is a trigonometric function for γ. Then arguing similarly as in the orientation preserving case the result follows. □

2.3. **Relating curvature to winding number.** There is a natural relation between the curvature and winding number of a unit speed plane curve. Intuitively one would think that if a curve has positive (signed) curvature at a point then it has to curve to the left and has to close itself at some point, thus completing a winding and increasing the winding number. This is summed up mathematically in the following result using a geometric description of signed curvature. The following can be thought as the one dimensional analogue of the Gauss-Bonnet Theorem for closed orientable surfaces.

Theorem 2.8. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a unit speed periodic curve with period L and signed curvature $\kappa : \mathbb{R} \to \mathbb{R}$. Then the winding number n_γ is given by*

$$n_\gamma = \frac{1}{2\pi} \int_0^L \kappa(t)dt.$$

Proof. Using Lemma 2.4 one writes $\gamma'(t) = (\cos\theta(t), \sin\theta(t))^T$ for a trigonometric function $\theta : \mathbb{R} \to \mathbb{R}$. Differentiating with respect to t one obtains

$$\gamma''(t) = (-\sin\theta(t) \cdot \theta'(t), \cos\theta(t) \cdot \theta'(t))^T.$$

On the other hand one has

$$\gamma''(t) = \kappa(t) \cdot n(t) = \kappa(t)(-\sin\theta(t), \cos\theta(t))^T$$

using

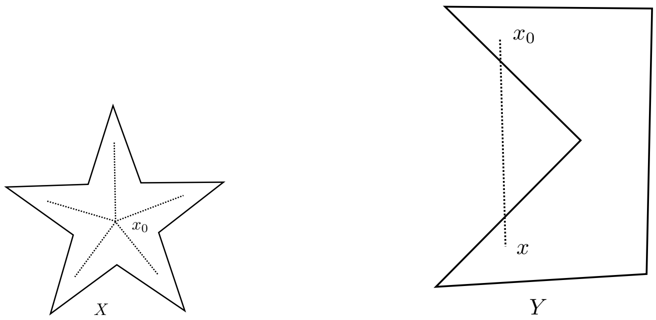
$$n(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \gamma'(t).$$

Then it follows that $\kappa(t) = \theta'(t) = \frac{d\theta}{dt}$ i.e., the signed curvature is the rate of change of the angle the velocity vector makes with the x-axis. One can further use Fundamental Theorem of Calculus to conclude

$$\frac{1}{2\pi}\int_0^L \kappa(t)dt = \frac{1}{2\pi}\int_0^L \theta'(t)dt = \frac{1}{2\pi}(\theta(L) - \theta(0)) = n_\gamma.$$

□

One would intuitively imagine that a unit speed periodic plane curve with winding number of modulus > 2 must wind twice around a point i.e., it must have some self-intersections while one also knows that the winding number of a unit speed circle (which is a simple closed curve) has modulus 1. Putting these threads together it is only natural to ask if the absolute value of winding number



(a) Star Convex Region (b) Not a star Convex Region

FIGURE 13

of a simple closed curve is 1. The following theorem answers this question in the affirmative.

Theorem 2.9 (Hopf's Umlaufsatz)**.** *The winding number of a simple closed curve parametrized by arc-length is* ± 1.

In order to prove the above theorem a small lemma is needed.

Definition 2.10. A set $X \subset \mathbb{R}^2$ is said to be **star-convex** with respect to a point $x_0 \in X$ if for each point $x \in X$ the straight line segment joining x and x_0 is contained in X (i.e., $tx + (1-t)x_0 \in X$ for each t, $0 \leq t \leq 1$).

Lemma 2.11. *Let* $X \subset \mathbb{R}^2$ *be a star-convex set with respect to some* $x_0 \in X$. *For any continuous map* $f : X \to S^1$ *there exists a continuous map* $\theta : X \to \mathbb{R}$ *satisfying*

$$f(x) = (\cos\theta(x), \sin\theta(x))^T, \ \forall\, x \in X$$

where θ *is uniquely determined by the value of* $\theta(x_0)$.

Note that putting $X = [0,1]$, $x_0 = 0$ and $f = \gamma'$ Lemma 2.11 reduces to Lemma 2.4. The only difference is that for a general continuous function f the trigonometric function θ is only continuous, not smooth.

Proof. Let $x \in X$ be an arbitrary but fixed point. As the line segment joining x and x_0 is contained in X, the map $f_x : [0,1] \to S^1$ given by $f_x(t) = f(tx+(1-t)x_0)$ is well defined and continuous. Now arguing in a similar manner as in Lemma 2.4 it follows that there is a unique continuous map $\theta_x : [0,1] \to \mathbb{R}$ satisfying $\theta_x(0) = \theta_0$ (some chosen constant) and $f_x(t) = (\cos(\theta_x(t)), \sin(\theta_x(t)))^T$.

Note that, using uniqueness of θ_x for each $x \in X$ as above one can see that any θ with desired properties, if exists, has to satisfy $\theta_x(t) = \theta(tx + (1-t)x_0)$ which in particular says that $\theta(x) = \theta_x(1)$. This uniquely determines θ provided it exists. To see the existance, define $\theta : X \to \mathbb{R}$ by $\theta_(x) = \theta_x(1)$ where θ_x for each $x \in X$ is as above. Then, by construction,

$$f(x) = f_x(1) = (\cos(\theta_x(1)), \sin(\theta_x(1)))^T = (\cos(\theta(x)), \sin(\theta(x)))^T$$

and $\theta(x_0) = \theta_{x_0}(1) = \theta_0$. It only remains to prove the continuity of θ. The continuity argument is quite similar to the argument used in Lemma 2.4.

Choose $x \in X$ and $\epsilon > 0$. Let $0 = t_0 < t_1 < \cdots < t_N = 1$ be a partition of $[0,1]$ such that for each $i = 0, \cdots, N$, $f_x([t_i, t_{i+1}])$ is entirely contained in one of the four semi circles described in Lemma 2.4. For any $y \in X$ sufficiently close to x one has $\|f_x(t) - f_y(t)\| < \epsilon$ for any $t \in [0,1]$ using continuity of f. As a result choosing ϵ sufficiently small and N sufficiently large one can show that for each $i = 0, \cdots, N$, $f_x([t_i, t_{i+1}])$ and $f_y([t_i, t_{i+1}])$ both lie in the same semi-circle as mentioned. Using a inductive argument similar to Lemma 2.4 on N (the number of subintervals in the partition) one obtains the following trigonometric expressions

$$\theta_x(t) = \arctan\left(\frac{f_x^1(t)}{f_x^2(t)}\right) + 2k\pi, \quad \theta_y(t) = \arctan\left(\frac{f_y^1(t)}{f_y^2(t)}\right) + 2k\pi,$$

(if the images are contained in the left or right semicircle) and

$$\theta_x(t) = \operatorname{arccot}\left(\frac{f_x^2(t)}{f_x^1(t)}\right) + 2k\pi, \quad \theta_y(t) = \operatorname{arccot}\left(\frac{f_y^2(t)}{f_y^1(t)}\right) + 2k\pi,$$

(if the images are contained in the upper or lower semicircle) where $k \in \mathbb{Z}$, $f_x(t) = (f_x^1(t), f_x^2(t))^T$ and $f_y(t) = (f_y^1(t), f_y^2(t))^T$. Thus the continuity of θ is obtained as a consequence of the same of functions f, arctan and arccot. □

With the aid of Lemma 2.11, one can prove Theorem 2.9 as follows.

Proof. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a unit speed simple closed curve with period L. Then $Im(\gamma) = \{\gamma(t) = (\gamma_1(t), \gamma_2(t)) : t \in \mathbb{R}\} = \gamma([0, L])$ is compact and the value $\sup\{\gamma_1(t) : t \in \mathbb{R}\}$ is attained at some $t = t_0$ (say). Without loss of generality assume that $\sup\{\gamma_1(t) : t \in \mathbb{R}\} = \gamma_1(0)$ (if not, consider the unit speed reparametrization of γ with the change of parameter $t \mapsto t + t_0$ having the same winding number as γ). Also, $\gamma'(0) = (\gamma_1'(0), \gamma_2'(0))^T = (0, \gamma_2'(0))^T$ (as $\gamma_1(0)$ is the maximum value for the smooth function $\gamma_1 : \mathbb{R} \to \mathbb{R}$). Since γ has unit speed, $\gamma_2'(0) = \pm 1$. Assume without loss of generality $\gamma'(0) = (0, 1)^T$ (if not, consider the unit speed reparametrization of γ with the change of parameter $t \mapsto -t$ resulting at most a change of sign for the winding number).

Consider the triangular region $X = \{(t, s) \in \mathbb{R}^2 : 0 \le t \le s \le L\}$ (cf. figure) which is star-convex with respect to the origin. Define $f : X \to S^1$ by

$$f(t,s) = \begin{cases} \frac{\gamma(s)-\gamma(t)}{\|\gamma(s)-\gamma(t)\|}, & t < s, (t,s) \neq (0, L); \\ \gamma'(t), & s = t; \\ -\gamma'(0) & (t,s) = (0, L). \end{cases}$$

Then f is a continuous map as $\gamma|_{[0,L]}$ is smooth and injective. By Lemma 2.11 there is a continuous map $\theta : X \to \mathbb{R}$ uniquely determined by $\theta(0,0)$, satisfying $f(t,s) = (\cos(\theta(t,s)), \sin(\theta(t,s)))^T$, $\forall\ (t,s) \in X$. Also $\gamma'(t) = f(t,t) = (\cos(\theta(t,t)), \sin(\theta(t,t)))^T$ means that $t \mapsto \theta(t,t)$ is a trigonometric function for γ

as in Lemma 2.4. Then the winding number n_γ is given by

$$n_\gamma = \frac{1}{2\pi}(\theta(L,L) - \theta(0,0)).$$

To see that n_γ has unit modulus, we first claim that the map $t \mapsto f(0,t)$ is not surjective and its image does not contain the unit vector $(1,0)^T \in S^1$. Assume the contrary. If $f(0,t) = (1,0)^T$ for some $t \in (0,L)$, then it follows that $\gamma_1(t) > \gamma_1(0)$, a contradiction to the fact that $\gamma_1(0)$ is the maximum value of all $\gamma_1(t)$. Also $f(0,L) = -\gamma'(0) = (0,-1)^T$ and $f(0,0) = \gamma'(0) = (0,1)^T$ are both perpendicular to $(1,0)^T$ establishing the claim. Now for each $k \in \mathbb{Z}$, there is a homeomorphism $\psi_k : (2\pi k, 2\pi(k+1)) \to S^1 - (1,0)^T$ defined by $\psi_k(t) = (\cos t, \sin t)^T$ such that the composite map $\psi_k^{-1} \circ f : X \to (2\pi k, 2\pi(k+1))$ is contunuous satisfying

$$f(t,s) = (\cos((\psi_k^{-1} \circ f)(t,s)), \sin((\psi_k^{-1} \circ f)(t,s)))^T, \ \forall \ (t,s) \in X.$$

Then using Lemma 2.11, one can assume $\theta = \psi_k^{-1} \circ f$ for some $k \in \mathbb{Z}$. Then from $f(0,0) = \gamma'(0) = (0,1)^T$ and $f(0,L) = -\gamma'(0) = (0,-1)^T$ follows that $\theta(0,0) = \frac{\pi}{2} + 2\pi k$ and $\theta(0,L) = \frac{3\pi}{2} + 2\pi k$. This yields $\theta(0,L) - \theta(0,0) = \pi$ Similarly showing that the image of the map $t \mapsto f(t,L)$ does not contain the unit vector $(-1,0)^T$, one can further show that $\theta(L,L) - \theta(0,L) = \pi$ and thus

$$n_\gamma = \frac{1}{2\pi}(\theta(L,L) - \theta(0,0)) = \theta(L,L) - \theta(0,L) + \theta(0,L) - \theta(0,0) = 1.$$

□

Remark 2.12. Combining Theorems 2.9 and 2.8 one obtains a nice geometric description of the *average* signed curvature of a simple closed plane curve in terms of the infinitesimal change of angle between the velocity vector and a fixed direction.

2.3.1. *Significance of Hopf's Umlaufsatz.* Hopf's Umlaufsatz together with the Four Vertex theorem and the Isopereimetric Inequality (see [1] for details) are three key results in the theory of plane curves. These results collectively characterize plane curves; in particular, simple closed curves and *convex* curves (see [1] for definition). As a detailed discussion on these topics is beyond the scope of this article, an interested reader is referred to [1] and [2] for further study of plane curves. It turns out that Hopf's Umlaufsatz is an instrumental step used in the characterization of convex plane curves and in the proof of both the Four Vertex theorem and the Isopereimetric Inequality, too. Theorem 2.9 is also used in the proof of the Gauss-Bonnet theorem.

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BOOK-REVIEW

TITLE : **THE FIBONACCI RESONANCE - AND OTHER GOLDEN RATIO DISCOVERIES**
Author : **Clive N. Menhinick**
Publisher: OnPerson International Limited, Poynton, Cheshire, England
First Edition, 2015; Hard cover, xiv + 618 pages
ISBN No. 978-0-9932166-0-2
British Library cataloguing-in-publication Data.
A catalogue record for this book is available from the British Library.
It includes bibliographic references (pp.487-576) and index.
Reviewer : **Amritanshu Prasad**

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

This sequence owes its name to the Italian mathematician *Leonardo of Pisa* (c. 1170 – c. 1250), who, owing to his being a scion of the Bonacci family, is known as *Fibonacci*.

Fibonacci earned a lasting place in the development of European mathematics through his book *Liber Abaci*. This book brought to Europe the decimal system of the Indians, which Fibonacci learned from scholars in Bugia (in present-day Algeria). In this book, we encounter the Fibonacci numbers through a whimsical problem of highly improbable reproducing rabbits. I will not repeat the details of this story here, since we will soon see a more sensible way to interpret this sequence.

Stigler's law of eponymy (discovered by Robert K. Merton) says that no scientific discovery is named after its original discoverer. And so it is with the Fibonacci numbers, which can be traced back to the Indian prosodist *Virahanka* (6th to 8th century), building on the *chanda sutras* of *Pingala* (4th century BCE)[1].

Sanskrit (like many Indian languages) has distinct long and short syllables. Pingala and Virahanka were interested in generating and counting all the different patterns of syllables satisfying certain constraints. Pingala, for example, asked for all the patterns with a short and b long syllables, for fixed numbers a and b.

For example, with 2 short and 2 long syllables there are six possible patterns:

$$1122, 1213, 2112, 2121, 2211.$$

Here 1 denotes a short syllable and 2 denotes a long one. Writing $C(a, b)$ for the number of patterns with a short and b long syllables, Pingala discovered the recurrence relation

$$C(a, b) = C(a - 1, b) + C(a, b - 1),$$

which he presented in the form of a triangle of numbers which he called the *Meru Prastaara*. In accordance with Stigler's law, the Meru Prastaara is called *Pascal's triangle*.

[1]See *A history of Pingala's combinatorics*, by Jayant Shah; available from `www.northeastern.edu/shah/papers/Pingala.pdf`

Along with the initial condition that $C(0,0) = 1$, this rule determines all the numbers $C(a,b)$. Indeed, the number $C(a,b)$ is the binomial coefficient $\binom{a+b}{a}$.

A natural notion of the length of a line of verse is the amount of time it measures out. If a short syllable is taken to measure out one time unit, and a long syllable 2 time units, then the total number of time units in a pattern is obtained by simply adding up the digits that represent it; for example, 122212 is $1+2+2+2+1+2 = 10$ beats long. Let F_n denote the number of patters with n beats. For such a pattern there are two possibilities. If it ends with a 2, then it is obtained from a unique pattern of length $n-2$ by appending a 2. If it ends with a 1, then it is obtained from a unique pattern of length $n-1$ by appending a 1. Thus

$$F_n = F_{n-1} + F_{n-2},$$

a relation which, along with initial conditions $F_0 = 1$, $F_1 = 1$, determines the Fibonacci sequence. This identity can be used to define Fibonacci numbers for positive as well as negative indices. Indeed, we may rewrite it as $F_n = F_{n+1} - F_{n+1}$ to define F_{-1}, F_{-2} etc. The ratios of successive terms, namely the numbers F_n/F_{n-1} converge to a definite limit, which is $(1+\sqrt{5})/2$, and is known as the golden ratio, and denoted by ϕ.

The book titled "The Fibonacci Resonance (and other new golden ratio discoveries)" by Clive N. Menhinick is devoted to the Fiboanacci sequence and the golden ratio. It falls into the genre of recreational mathematics, and uses only high-school level mathematics.

A six-hundred-page book on a simple sequence and an irrational number? But the Fibonacci sequence and the golden ratio have a cult following. They even play a role in Dan Brown's bestseller *The Da Vinci Code*. They apparently appear in contexts as diverse as post-impressionist art, pineapples and fractals. A somewhat disconnected, sometimes unconvincing, account of these appearances forms the first chapter of Menhinick's book.

The second chapter introduces something called the Ori32 geometry. It involves dividing the circle into 32 equal angles. There are inexplicable references to digital SLR cameras. But the significance of this construction was lost on me.

The next chapter introduces the great discovery of this book, namely the "Fibonacci Resonance" mentioned in the title. It refers to the identity:

$$F_n = F_s\phi^{n-s} + F_{n-s}(-\phi)^{-s} \text{ for all integers } n \text{ and } s.$$

This identity is not difficult to prove, but in it, Menhinick searches for, and claims to find, deep visual, accoustic and analytic meaning. He then goes on to generalize it to other sequences which are defined by recurrence relations similar to the ones defining the Fibonacci sequence (he calls these Lucas sequences). Once again, the discussion is at once grandiose, disconnected, rambling, and ultimately unconvincing.

The last Chapter is a collection of articles on how the golden ratio appears in Science. This includes a nice discussion of Penrose's aperiodic tilings and quasicrystals. It draws heavily from Martin Gardner [*Penrose Tiles and Trapdoor Ciphers*, Mathematical Association of America, Wahsington DC, 1989].

The mathematical content of this book is negligible. The *new discoveries* promised in the title are simple exercises. However, the author is a good story-teller with a repertoire of good stories. Many passages are interesting, and the book covers a range of topics. This book demonstrates that one does not need a great deal of mathematical background to ask questions, and enjoy the process of mathematical discovery (even if these discoveries are not going to win the acclaim of the mathematical community). Take at bedtime, with a pinch of salt[2].

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[2]This prescription was spotted on Rahul Siddharthan's website `rsidd.online.fr`, accessed: 15th September 2016.

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PROBLEM SECTION

In the last issue of the Math. Student Vol. **85**, Nos. 1-2, January-June (2016), we had invited solutions from the floor to the remaining problems *12, 13, 14, 15, 16* and *17* of the MS, 84, 3-4, 2015 as well as to the eight new problems *1, 2, 3, 4, 5, 6, 7* and *8* presented therein till October 31, 2016.

We received **one** correct solution from the floor to each of the Problems *12* & *15* of MS, 84, 3-4, 2015 and we publish here the solution received from the floor to the Problem *12* and Proposer's crisp solution to the Problem *15*. No solution was received from the floor to the remaining problems *13, 14, 16* and *17* of the MS, 84, 3-4, 2015 and hence we provide in this issue the Proposer's solution to these problems also.

We also received **six** correct solutions from the floor to the Problem *3*, **one** correct solution from the floor to the Problem *7* and **two** correct solutions from the floor to the Problem *8* of the MS, 85, 1-2, 2016. We publish Proposer's crisp solution to these problems in this issue. Readers can try their hand on the remaining problems *1, 2, 4, 5* and *6* till April 30, 2017.

In this issue we first present **five new problems**. Solutions to these problems as also to the remaining *1, 2, 4, 5* and *6* of MS, 85, 1-2, 2016, received from the floor till April 30, 2017, if approved by the Editorial Board, will be published in the MS, 86, 1-2, 2017.

MS-2016, Nos. 1-2: Problem-9: Proposed by S. K. Tomar .

> Equation of small motion in a uniform elastic solid medium is given as
> $$(\lambda+\mu)\bigtriangledown(\vec{\bigtriangledown}\cdot\vec{u})+\mu\bigtriangledown^2\vec{u}=\rho\ddot{\vec{u}}, \tag{0.1}$$
> where λ and μ are Lame's parameters; ρ and $\vec{u}$ represent respectively the density and displacement vector. Superposed dot represents the temporal derivative and other symbols have their usual meanings.
>
> Using Lie symmetry method, show that (0.1) leads to two waves propagating with distinct speeds.

MS-2016, Nos. 1-2: Problem-10: Proposed by Purusottam Rath, CMI, Chennai; submitted through Clare D'Cruz.

> Let $n \geq 5$ be a natural number. How many distinct topologies, not necessarily Hausdorff, can be given to the alternating group A_n so that it becomes a topological group?

MS-2016, Nos. 3-4: Problem-11: Proposed by B. Sury.

If a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(2t - f(t)) = t = 2f(t) - f(f(t))\ \ \forall t \in \mathbb{R}$ then prove that $\exists\ c \in \mathbb{R}\ \ \ni f(t) = t + c$ for all t.

MS-2016, Nos. 1-2: Problem-12: Proposed by George E. Andrews and Emeric Deutsch..

Show that the number of parts having odd multiplicities in all partitions of n is equal to the difference between the number of odd parts in all partitions of n and the number of even parts in all partitions of n.
Example: $n = 5$. The partitions $5'$, $4'1'$, $3'2'$, $3'11$, $221'$, $2'1'11$, $1'1111$ have 10 parts with odd multiplicities (marked with $'$). On the other hand, the number of odd parts is 15

$$(5, 1, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

while the number of even parts is 5

$$(4,\ 2,\ 2,\ 2,\ 2).$$

MS-2016, Nos. 1-2: Problem-13: Proposed by Aritro Pathak, Dept. of Mathematics, Brandeis University, Waltham, Massachusetts, 02453 USA; currently: 12 East Road, Jadavpur, Kolkata-700032, West bengal, India. E-mail: ap323@brandeis.edu ; submitted through B. Sury.

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying the conditions $|f'(x)| \leq |f(x)|$ and $f(0) = 0$ then show that $f(x) = 0$ for all $x \in \mathbb{R}$. Give also a proof that uses only the Mean Value Theorem.

Solution from the floor: MS-2015, Nos. 3-4: Problem 12: Let ABC be a triangle with medians AD, BE, and FC. Construct semicircles with diameters BD, DC, and BC outwardly. Let T_A be the circle tangent to the three semicircles with diameters BD, DC, and BC, and let A' be the center of T_A. Define B' and C' cyclically. Then give a synthetic (pure geometric) proof of the fact that the segments AA', BB', and CC' are concurrent. (Solution submitted on 10-10-2016 by **Diyath Nelaka Pannipitiya**; No 286, High-Level Road, Maharagama, 10280, Sri Lanka; *diyathnp@yahoo.com*).

Solution. Let $AB = 4t$, $BC = 4r$ and $CA = 4s$. Then $BD = DC = 2r$. Let A'' be the point where BC and AA' meet each other. Define B'' and C'' cyclically.
Let us find the radius of T_A first. Let x be the radius of T_A. Let O be the center of the semicircle with diameter BD. Notice that the line segment OA' passes through the tangent point of the semicircle with center O and T_A. And $\triangle ODA'$ is a right triangle with $\angle A'DO = 90^o$. Then by the Pythagoras theorem we have, $(r + x)^2 = r^2 + (2r - x)^2$. Which implies $x = (2/3)r$.

Without loss of generality consider $\angle BCA$.

Case 1: $\angle BCA < 90^o$. (Refer Figure-1)

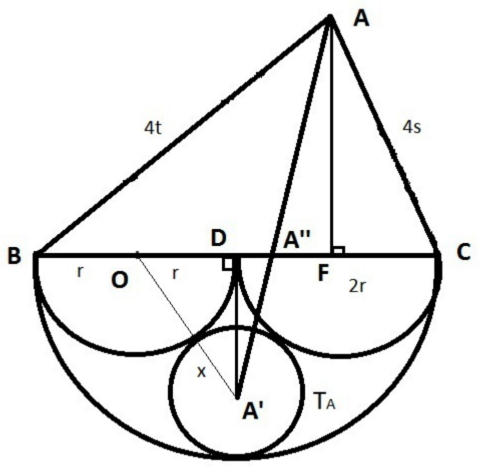

Figure 1

Because $\triangle BAF$ and $\triangle AFC$ are right triangles, by Pythagoras theorem, we have

$$AF^2 = AB^2 - BF^2 = AC^2 - FC^2,$$

that is,

$$16t^2 - BF^2 = 16s^2 - (4r - BF)^2 = 16s^2 - 16r^2 + 8rBF - BF^2.$$

Hence

$$BF = (2/r)(t^2 + r^2 - s^2). \tag{0.2}$$

Therefore

$$AF^2 = 16t^2 - BF^2 = 16t^2 - (4/r^2)(t^2 + r^2 - s^2)^2$$
$$= (4/r^2)(2r^2t^2 + 2t^2s^2 + 2s^2r^2 - t^4 - s^4 - r^4) \ = \ (M^2/r^2),$$

where $M = 2(2r^2t^2 + 2t^2s^2 + 2s^2r^2 - t^4 - s^4 - r^4)^{1/2}$. Hence AF=(M/r). Observe that $\triangle DA''A'$ and $\triangle AA''F$ are similar triangles. Therefore

$$\frac{A''F}{DA''} = \frac{AF}{DA'} = \frac{(M/r)}{2r - (2r/3)} = \frac{3M}{4r^2},$$

and hence

$$A''F = (3M/4r^2)DA''. \tag{0.3}$$

In view of Figure-1, from (0.2) and (0.3), we get

$$(2/r)(t^2 + r^2 - s^2) = BF = BD + DA'' + A''F$$
$$= 2r + DA'' + \frac{3M}{4r^2}DA'' = 2r + \frac{3M + 4r^2}{4r^2}DA''$$

which gives

$$DA'' = 8r(t^2 - s^2)/(4r^2 + 3M). \tag{0.4}$$

Therefore

$$\frac{BA''}{A''C} = \frac{2r + DA''}{2r - DA''} = \frac{2r + \dfrac{8r(t^2 - s^2)}{4r^2 + 3M}}{2r - \dfrac{8r(t^2 - s^2)}{4r^2 + 3M}} = \frac{3M + 4(r^2 + t^2 - s^2)}{3M + 4(r^2 + s^2 - t^2)}. \tag{0.5}$$

Similarly we can show that

$$\frac{CB''}{B''A} = \frac{3M + 4(s^2 + r^2 - t^2)}{3M + 4(s^2 + t^2 - r^2)} \tag{0.6}$$

and

$$\frac{AC''}{C''B} = \frac{3M + 4(t^2 + s^2 - r^2)}{3M + 4(t^2 + r^2 - s^2)}. \tag{0.7}$$

Case 2: $\angle BCA > 90^o$. (Refer Figure-2)

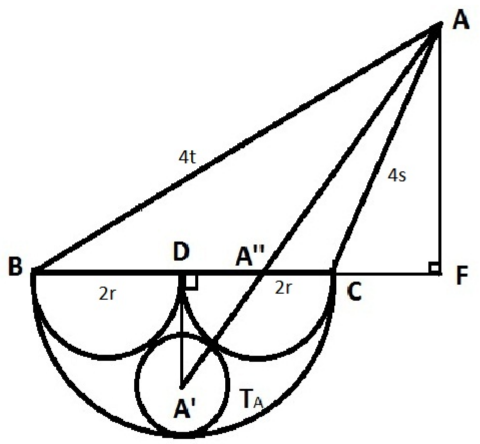

Figure 2

In an exactly the similar way as in the *Case* 1 we can show that

$$\frac{BA''}{A''C} = \frac{3M + 4(r^2 + t^2 - s^2)}{3M + 4(r^2 + s^2 - t^2)}, \tag{0.8}$$

$$\frac{CB''}{B''A} = \frac{3M + 4(s^2 + r^2 - t^2)}{3M + 4(s^2 + t^2 - r^2)} \tag{0.9}$$

and

$$\frac{AC''}{C''B} = \frac{3M + 4(t^2 + s^2 - r^2)}{3M + 4(t^2 + r^2 - s^2)}. \tag{0.10}$$

Case 3: $\angle BCA = 90^o$. (Refer Figure 3)

We have an immediate result

$$t^2 = r^2 + s^2. \tag{0.11}$$

Using this we then get

$$\begin{aligned}(M^2/4) &= 2r^2t^2 + 2t^2s^2 + 2s^2r^2 - t^4 - s^4 - r^4 \\ &= 2(r^2 + s^2)t^2 + 2r^2s^2 - r^4 - s^4 - t^4 \\ &= 2(r^2 + s^2)^2 + 2r^2s^2 - r^4 - s^4 - (r^2 + s^2)^2 \ = \ 4r^2s^2,\end{aligned}$$

and hence

$$M = 4rs. \tag{0.12}$$

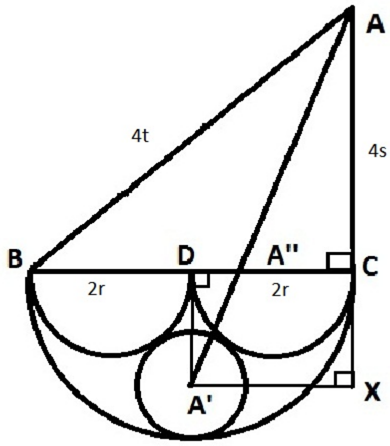

Figure 3

Let X be the point on the extended line segment AC such that $\angle A'XA = 90^o$. Observe that AX is a tangent line segment to the semicircle whose diameter is BC. Thus $A'X = 2r$ and $\triangle AA''C$ and $\triangle AA'X$ are similar triangles. Therefore $(A''C/2r) = (AC/AX) = (4s/(4s + (4r/3))$, which in turn implies

$$A''C = (6sr/(3s + r)). \tag{0.13}$$

Therefore $BA'' = 4r - A''C = 4r - (6sr/(3s + r))$, and hence

$$BA'' = \frac{2r(3s + 2r)}{3s + r}. \tag{0.14}$$

It follows that

$$\frac{BA''}{A''C} = \frac{3s + 2r}{3s}. \tag{0.15}$$

Similarly we can show that

$$\frac{CB''}{B''A} = \frac{3r}{3r + 2s} \tag{0.16}$$

and

$$\frac{AC''}{C''B} = \frac{s(3r + 2s)}{r(3s + 2r)}. \tag{0.17}$$

Notice that (0.15), (0.16) and (0.17) can be obtained from (0.5), (0.6) and (0.7) (or from (0.8), (0.9) and (0.10)) respectively, by using the relations (0.11) and (0.12). Therefore in general

$$\frac{BA''}{A''C} = \frac{3M + 4(r^2 + t^2 - s^2)}{3M + 4(r^2 + s^2 - t^2)} \tag{0.18}$$

$$\frac{CB''}{B''A} = \frac{3M + 4(s^2 + r^2 - t^2)}{3M + 4(s^2 + t^2 - r^2)} \tag{0.19}$$

$$\frac{AC''}{C''B} = \frac{3M + 4(t^2 + s^2 - r^2)}{3M + 4(t^2 + r^2 - s^2)} \tag{0.20}$$

Thus for any triangle we have

$$\frac{BA''}{A''C} \times \frac{CB''}{B''A} \times \frac{AC''}{C''B} = \frac{3M + 4(r^2+t^2-s^2)}{3M+4(r^2+s^2-t^2)} \times \frac{3M+4(s^2+r^2-t^2)}{3M+4(s^2+t^2-r^2)} \times \frac{3M+4(t^2+s^2-r^2)}{3M+4(t^2+r^2-s^2)} \; = \; 1.$$

It follows by Ceva's theorem that AA'', BB'' and CC'' are concurrent, and therefore AA', BB' and CC' are also concurrent. This proves the claim.

Remark 0.1. This result was mentioned (without proof) by the proposer, first on December 22, 2000 in the 'Hyacinthos' group devoted to triangle geometry. More properties of the point of concurrence of AA', BB' and CC' (and other analogous points) can be found at

http://faculty.evansville.edu/ ck6/encyclopedia/ETCPart3.html

where it is listed as triangle center X(3590). In particular, it lies on Kiepert hyperbola corresponding to the angle arctan(2/3).

Solution by the Proposer B. Sury: MS-2015, Nos. 3-4: Problem 13: Let $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Determine the *semigroup* generated by these two matrices (note that we are not allowed to take negative powers of the matrices).

Solution. We shall show that the semigroup generated by S and T consists of all 2×2 integer matrices with non-negative entries and determinant ± 1. It is clear that every matrix in the semigroup generated by S and T must have non-negative integer entries and determinant ± 1 as this is true of S and T. The main point is to show that any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \geq 0$ integers and $ad - bc = \pm 1$ is in the semigroup generated by S, T. We argue by induction on $min(a, b, c, d)$. We may assume $min(a, b, c, d) = a$ because the columns or rows of M may be permuted simply by multiplying by S on the right or left respectively.

Firstly, if $a = 0$, then the determinant condition det $(M) = \pm 1$ forces $b = c = 1$ and det $(M) = -1$. Thus,

$$M = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} = ST^d.$$

Assume now that $min(a, b, c, d) = a > 0$. Therefore, either $d > b$ or $d > c$.

In case $d > b$ we write

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c-a & d-b \end{pmatrix} = STS \begin{pmatrix} a & b \\ c-a & d-b \end{pmatrix}.$$

Similarly, in case $d > c$ we write

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b-a & a \\ d-c & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 = \end{pmatrix} \begin{pmatrix} b-a & a \\ d-c & c \end{pmatrix} ST.$$

By repeatedly applying the above step and, by induction on the minimum of the entries, it follows that M is in the semigroup generated by S and T.

Solution by the Proposer B. Sury: MS-2015, Nos. 3-4: Problem 14: Let $F \in \mathbb{Q}[X]$ be a polynomial of degree at least 2. If F gives a bijection on a subset S of $\mathbb{Q}$, then prove that S must be finite.

Solution. Let $F = \sum_{i=0}^{n} c_i X^i \in \mathbf{Q}[X]$ have degree $n > 1$. Suppose S is an infinite set of rational numbers permuted by F. The polynomial

$$F(X - \frac{c_{n-1}}{nc_n}) + \frac{c_{n-1}}{nc_n}$$

permutes the infinite set $S + (c_{n-1}/(nc_n))$ and its coefficient of X^{n-1} is zero. By renaming, we may assume that $F = \sum_{i=0}^{n} c_i X^i \in \mathbf{Q}[X]$ with $c_{n-1} = 0$. Write

$$F = \frac{1}{d}\sum_{i=0}^{n} a_i X^i$$

with $a_i, d \in \mathbf{Z}$. Consider an arbitrary, fixed element (p/q) of S. Write $p_0 = p, q_0 = q$ and $f(p_i/q_i) = p_{i+1}/q_{i+1}$ in irreducible fractions with positive denominators for all $i \in \mathbf{Z}$. We have

$$q_{i+1}(a_n p_i^n + a_{n-2} p_i^{n-2} q_i^2 + \cdots + a_0 q_i^n) = d p_{i+1} q_i^n.$$

Since $n \geq 2$, q_i^2 divides $q_{i+1} a_n$. In particular, if $|a_n| < q_{i+1}$, then $q_i < q_{i+1}$.
We deduce: *there are infinitely many $k < 0$ such that $q_k \leq |a_n|$;* otherwise, we would have a strictly decreasing sequence of natural numbers.
Now, there exists $M > 0$ such that $|F(x) > |x|$ for $|x| > M$. Fix such an M and let $|F(x)| \leq N$ for all $|x| \leq M$. We will show that the sequence $\{p_r/q_r : r < 0\}$ is periodic. Suppose $|p_k/q_k| > max(M, N)$ for some k. Then, since $|p_k/q_k| = |F(p_{k-1}/q_{k-1})| > N$, we must have $|p_{k-1}/q_{k-1}| > M$ and, therefore

$$|p_k/q_k| = |F(p_{k-1}/q_{k-1})| > |p_{k-1}/q_{k-1}|.$$

We claim that $|p_{k-2}/q_{k-2}| < |p_k/q_k|$ also. Indeed, if not, then

$$|p_{k-2}/q_{k-2}| \geq |p_k/q_k| > M.$$

Looking at the F-values $|p_{k-1}/q_{k-1}| = |F(p_{k-2}/q_{k-2})| > |p_{k-2}/q_{k-2}|$ so that

$$|p_k/q_k| > |p_{k-1}/q_{k-1}| > |p_{k-2}/q_{k-2}| \geq |p_k/q_k|$$

a contradiction. In this manner, we inductively get that *For each $r > 0$, $|p_{k-r}/q_{k-r}| < |p_k/q_k|$.* As we observed, there are infinitely many $s > 0$ such that $q_{-s} \leq |a_n|$, a fixed number. Hence, there must exist arbitrarily large s such that $p_{-s}/q_{-s} = p_{-t}/q_{-t}$ for some $t \neq s$. This clearly implies that the sequence $\{p_r/q_r : r < 0\}$ is periodic and hence, by taking F-values, the whole sequence $\{p_r/q_r\}$ is periodic.
We considered an arbitrary p_0/q_0 element of S and showed that it is a part of a periodic F-orbit. So, S is a union of finite F-invariant sets. As each such

finite F-invariant set contains a rational number with denominator bounded by $|a_n|$ (observed in the beginning), these subsets are only finitely many. Therefore, S is finite.

Solution by the Proposer B. Sury: MS-2015, Nos. 3-4: Problem 15: If N is a positive integer which is a multiple of a number of the form $99\cdots 9$ where 9 is repeated n times, then show that N has at least n non-zero digits. More generally, prove that if $b > 1$ and $b^n - 1$ divides a, then the base-b expression of a has at least n non-zero digits.

Solution. We prove the general statement of which the first one is a special case. Let m be the minimal number of non-zero digits of any non-zero multiple of $b^n - 1$. Among all multiples with m non-zero digits, suppose $A = a_1 b^{k_1} + \cdots + a_m b^{k_m}$ has the smallest digit-sum. Here $0 \leq a_i < b$ and $k_1 > k_2 > \cdots > k_m$. The key claim is that the powers k_i are all distinct mod n. If this is proved, then it would follow that the number $C = a_1 b^{r_1} + \cdots + a_m b^{r_m}$ where $r_i < n$ and $r_i \equiv k_i$ mod n, is a multiple of $b^n - 1$ but is less than or equal to $(b-1)(1+b+\cdots+b^{n-1}) < b^n$. Thus, $C = b^n - 1$ and so $m = n$. Let us prove now that k_i's are distinct mod n. Suppose $i < j$ and $k_i \equiv k_j$ mod n. Choosing d large enough such that $k_j + dn > k_1$, consider the number $B = A - a_i b^{k_i} - a_j b^{k_j} + (a_i + a_j) b^{k_j + dn}$. This is a multiple of $b^n - 1$ as $B - A = a_i b^{k_i}(b^{k_j - k_i + dn} - 1) + a_j b^{k_j}(b^{dn} - 1)$.

Note that by minimality of the number m of non-zero digits, the number $a_i + a_j$ must be $\geq b$. But, then the digit-sum of B is clearly (digit-sum for A) $- a_i - a_j + 1 + (a_i + a_j - b)$ which is less than the digit-sum for A. This contradicts the choice of A. Therefore, the claim is proved.

Correct solution was also received from the floor from:

Diyath Nelaka Pannipitiya; (No 286, High-Level Road, Maharagama, 10280, Sri Lanka; *diyathnp@yahoo.com* ; received on 10-10-2016)

Solution by the Proposer Mathew Francis : MS-2015, Nos. 3-4: Problem 16: Let $S_1, S_2, \ldots, S_k$ be subsets of a universe $U = \bigcup S_i$ such that the union of no t of them covers U. Then prove that there exist $t+1$ sets among them that are pairwise incomparable.

Solution. We may use Dilworth's theorem. The minimum number of chains that cover the partial order equals the cardinality of the maximum antichain. Consider the partial order formed by the given subsets. We need to show here that the maximum antichain cardinality is at least $t + 1$. That is, the minimum number of chains to cover the partial order is at least $t + 1$.

Suppose t chains can cover. Then take the maximal subset from each chain and this collection of t subsets would cover all the elements in U. Else, if $x \in U$ is not covered, there exists some S_i containing x and this S_i is not in any of the chains, for else the maximal subset from that chain would contain x also.

Solution by the Proposer Mahender Singh: MS-2015, Nos. 3-4: Problem 17: Let G be a finite group of order n and let ϕ be the Euler's totient function. It is easy to see that if G is cyclic then $|Aut(G)| = \phi(n)$. Is the converse true?

Solution. For solution to this problem, refer the theorem in the paper: 'J. C. Howarth, On the power of a prime dividing the order of the automorphism group of a finite group, Proc. Glasgow Math. Assoc. 4 (1960), 163-170'

The solution in the aforementioned reference for this simple looking problem involves quite intricate analysis of finite group theory. The proposer, and possibly some of the readers as well, would be interested to know if there are simpler solutions.

Solution by the Proposer B. Sury: MS-2016, Nos. 1-2: Problem 03: Show that the number $11\cdots122\cdots25$ where 1 is repeated 2015 times and 2 is repeated 2016 times, is a perfect square.

Solution. If 1 occurs n times and 2 occurs $n+1$ times, then the number is $(11\cdots1)\times 10^{n+2} + (22\cdots2)\times 10^2 + 25$ which simplifies to $(10^{2n+2} + 10^{n+2} - 200)/9 + 25 = ((10^{n+1}+5)/3)^2$ which is a perfect square as $10^{n+1}+5$ is a multiple of 3.

Correct solutions were also received from the floor from:

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Diyath Nelaka Pannipitiya; (No 286, High-Level Road, Maharagama, 10280, Sri Lanka; *diyathnp@yahoo.com* ; received on 10-10-2016)

Solution by the Proposer Purusottam Rath: MS-2016, Nos. 1-2: Problem 07: Let $M = (a_{ij})$ be an $n \times n$ matrix with $a_{ij} \in \mathbb{R}$ and $|a_{ij}| \leq 1$. Show that the absolute value of determinant of M is at most $n^{n/2}$. When does equality hold?

Solution. $|det(M)|$ is the Euclidean volume of a parallelopiped with sides at most $\sqrt{n}$. Thus the volume is at most the product of the sides which is maximal when the sides are orthogonal and are of length $\sqrt{n}$, that is, when $|a_{ij}| = 1$ and $MM^t = nI$

Correct solution was also received from the floor from:

Manjil P. Saikia; (Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, Universität Wien, 1090 Vienna, Austria; *manjil.saikia@univie.ac.at* ; received on 08-06-2016)

Solution by the Proposer Purusottam Rath: MS-2016, Nos. 1-2: Problem 08: let A and B be finite subsets of integers. Consider the set $A + B$ given by

$$A + B := \{a + b : a \in A, b \in B\}.$$

Show that $|A + B| \geq |A| + |B| - 1$.

Solution. Translation is allowed and hence translate A and B so that $Max(A) = Min(B) = 0$. Observe now that $A \cup B \subset A + B$.

Correct solutions were also received from the floor from:

Manjil P. Saikia; (Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, Universität Wien, 1090 Vienna, Austria; *manjil.saikia@univie.ac.at* ; received on 08-06-2016)

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