ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 91, Nos. 1-2, January - June (2022)
(Issued: February, 2022)

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
www.indianmathsociety.org.in

THE MATHEMATICS STUDENT

Edited by M. M. SHIKARE

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6. articles that arouse curiosity and interest for learning mathematics among readers and motivate them for doing mathematics.

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Published by Prof. Satya Deo for the Indian Mathematical Society, type set by M. M. Shikare, "Krushnakali", Survey No. 73/6/1, Gulmohar Colony, Jagtap Patil Estate, Pimple Gurav, Pune 411061 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune-411 041 (India).

Printed in India.

The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022)

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Vol. 91, Nos. 1-2, January-June (2022), 1–8

MATHEMATICS IN THE POST-INDEPENDENCE INDIA*

DIPENDRA PRASAD

Abstract. This is the text of the Presidential address given at the Annual meeting of the Indian Mathematical Society on December 04, 2021. It takes a stock of how we, as a nation, are doing in mathematics as India celebrates its 75th anniversary of Independence soon. This also involves thinking of the profession of Mathematics, and the changing scenario in it, which I touch upon in this article.

At the time of Independence, India had very little infrastructure in terms of scientific institutions, and our first Prime Minister, Pt. Nehru, rightly emphasized science and technological institutions to be the modern temples of the country. Thus the IITs, the Central universities, IIMs, AIIMS, BARC, NPL, PRL, ISRO etc. were setup in the first decades of Independence. Some other institutes which had started functioning before Independence such as the Indian Statistical Institute (ISI) and the Indian Institute of Science, Bangalore (IISc), were recognized: ISI as an institute of national importance in 1959, and IISc as a "deemed to be university" in 1958. TIFR too which had just been founded before Independence, acquired its campus after Independence which was inaugurated by Pt. Nehru in 1962.

Many of the aforementioned institutions have done very well and are among the most important institutes the country has for science, engineering, medicine and management, and are the coveted institutions every student of the country aspires to go to.

* This article is based on the text of the Presidential Address (General) given by Prof. Dipendra Prasad at the 87th Annual Conference of the IMS - An International meet held at MGM Unuversity, Aurangabad during December 4-7, 2021 using online mode.

India is a very large country and with the setting up of these institutions, although India made a good beginning, they were by no means adequate to cater to the many millions of students finishing the 12th standard each year. Keeping this in mind, in the first decade of this millennium, Indian government tried to enlarge the existing IITs, IIMs etc. to many more so that each state had one of each of these, and also created more central universities, and created IIT-like institutions for sciences: the Indian Institutes of Science Education and Research (IISER), which are there now in 7 cities: Kolkata, Pune, Mohali, Bhopal, Thiruvananthapuram, Tirupati, Berhampur, and are, in about a decade-and-half of existence, already well-recognized for their quality.

I myself grew up in the post-independence India, and when I was growing up in the 70's, there were very limited opportunities to pursue careers in science and technology in the country, and almost nothing outside these. In my youth, most students looked for jobs in banks, or civil services. Going to IITs was starting to become popular. Of course, the IITs have grown so much in their importance now that they offer the most coveted admissions in the country, and for which every year more than 2 million students appear in what's known to all students (and their parents!) in the country as JEE (joint entrance examination), selecting about 20 thousand students for the IITs, thus about 1 for each 100 student appearing for the examination.

But the India of today has changed considerably from the India of my youth. There are many more opportunities for young students to pursue many career options, including mathematics. Of course, here I talk only about mathematics which can be pursued at many of the institutions mentioned above, other universities, as well as at the Chennai Mathematical Institute (CMI), after finishing the higher secondary grade (12th standard). There are programs to complete undergraduate (BSc degree) in 3 years in most places, but now more and more places are offering 4 year undergraduate programs (so that students can go to USA more easily!), and then there are also many places having 5 year integrated BSc-MSc programs.

Although the number of the *top talents* of the country taking up mathematics in these undergraduate programs compared to the available pool of talented students at the 12th standard, is minuscule, it is not insignificant, let's say a few hundreds to a few thousands each year, depending on where we put the bar, who are quite good to excellent (so we are reduced from a billion plus population to a few thousands!). However, these students in the undergraduate programs in Mathematics are spread over the length and breadth of the country, and except in some places such as ISI and CMI, the number of good students may be only a handful. The best students from these undergraduate programs by and large choose to go abroad for which many excellent opportunities exist, leaving very few of the most talented undergraduates in the country to go for a PhD program in mathematics in the country. Of course, some of the students who go for a PhD abroad do return to the country — and such numbers have been increasing in the recent past, but most do not come back, and thus India is steadily exporting excellent mathematics students abroad which should be visible to all.

India is a country which has had a considerable presence in mathematics, not only because two of the greatest mathematicians of the 20th century, Srinivasa Ramanujan and Harish-Chandra, came from India. But we have had a host of other distinguished mathematicians in the post-independence India, many from the Tata Institute of Fundamental research which has towered over Indian mathematics after independence with all the 3 Fellows of Royal Society the country produced in mathematics coming from there, and another one who became an FRS after going abroad having been at TIFR for many years. Here we are not counting Prof. CR Rao, a colossal figure in Statistics, who became FRS in 1967. Till recently, most of the invited speakers at the International Congress of Mathematicians (ICM), at least in Pure Mathematics, have come from TIFR. (To be sure, there have been speakers from India in Statistics, Probability, Stochastic process, Computer Science... too, who came from outside TIFR.) Last time, in 2018, Ritabrata Munshi of ISI Kolkata, and this time for the 2022 ICM, Neena Gupta of ISI Kolkata and Mahesh Kakde of IISc, Bangalore are the invited speakers who do not come from TIFR. While all the three were trained at the Indian Statistical Institute (ISI) for their undergraduate degrees, two of these three got their PhD abroad and also did some post-doctoral studies before returning to India, whereas one is indigenous. Although one cannot read too much from this sample size of three, it is indicative of many things, including that good mathematics is happening in many places in India now.

I now want to make some comments on the dynamics of mathematics as a profession. Popular perception considers it for the most part a solo activity needing only pen and paper to do mathematics. Nothing could be farther from the truth, unless you are a Ramanujan. For younger students at least till the time they finish PhD, and then also for many years as professional mathematicians, they need to be immersed in an atmosphere of intense intellectual activity with peers as friends and competitors. A young faculty member joining a department in which she/he is the only one in nearby topics, is certainly going to feel lost or lose momentum at times whereas being surrounded by colleagues in related areas can bring her/him back to research. One knows of many examples of distinguished mathematicians who quit normal academic life: teaching, research, students, visitors, seminars, conferences, dinners (yes, that helps a lot to do research!), and the associated social aspects, hoping to devote themselves more fully to research, but never coming back to it.

I know of many departments of small as well as large faculty who have not much common interest, and no one can talk to each other on the subject which has brought them together; it is clearly very undesirable. Rather, it would be much better to have all the faculty in one subject, certainly another extreme. For example, at one point, Harvard especialised in Algebraic geometry and Number theory, *everyone* in the world who was *anybody* in the subject had to pass through Harvard! In fact everyone in Algebraic geometry and Number theory was at Harvard either as a student or as a post-doc or as a visiting faculty or as a permanent faculty, some in all roles such as David Mumford. And they did this with a faculty size of just 15 or 20, running both the undergraduate and the graduate programs!

The world of mathematics is becoming more and more competitive in the last 30 years (since the Chinese started to come in mathematics in a big way). Now even to get invited to a good international conference as a participant, and not even as a speaker, in one's own subject, not to speak of larger all-purpose conferences such as the ICM, is no mean achievement — department heads should pay attention to grant leaves to such people! What used to be middle level journals in the 80's and 90's (say below first 20 but in the top 100 in MathSciNet) are already becoming quite difficult to reach, and now a publication in such a journal calls for a celebration for the individual as well as for the country. Just to look at numbers, every year, there might be about 10 thousand papers being published in these top 100 journals (according to the MathSciNet MCQ), and my rough estimate is that only about 50 of these papers may have an author who is in India. One measure of where we belong is how many of us (based in India) are in the first shortlist (say of a few hundreds) for a Fields medal, and I would not hazard a guess to disappoint you! It will be curious to compare how India fares in mathematics (or, sciences in general for that matter) with how it does in Olympic games, or the Olympiad programs.

Mathematics is also becoming increasingly compartmentalized. In most cases, Algebraists or Algebraic Geometers or Arithmetic Geometers cannot talk to each other. Not only that, even in their own subjects, they can talk to people of only certain interests. Different areas in mathematics have different cultures and these affect departments where they might be together. And for the departments, judging of individuals often gets reduced to the grants and awards/honors the individual brings, and the impact factor of journals where the individual publishes, without any regard to the quality or depth of what the individual does. It is clearly undesirable even if there is no obvious way out. The problem is compounded especially as everyone seems busy writing papers but no one is interested in refereeing papers (why should they!), and many of those who do, do a very perfunctory job. Thus both top journals and those considered not-as-good journals have a spectrum of quality depending mostly on the job the referees and the editors do – not to speak of their biases and preferences, and then if you hire someone based on a top journal publication, how fair are you to one whom you do not hire. There is a further complicating factor. It is becoming common these days for most better journals to take a year or so to accept/reject papers, and I know of many cases in which it has taken two to three years, sometimes only to be told that, "your paper is not up to the level expected for the journal". Now compound this with the possibility that you may have two or three rejections before an acceptance, and that for many jobs in India (even post-docs!), people count how many published papers you have (in high impact journals!).

One of my friends, a director of an IISER, used to periodically ask me, "How many crores are needed to get a paper in the Annals of Mathematics," as he was willing to make any investment to get his institute on the world map of mathematics (and had identified a bright young faculty whom he wanted to encourage). I had no answer (and I did not give him a false hope!), as getting a paper in the Annals is the end result of a long journey in which the country needs to invest in training many talented students hoping that one of them will make it to the top. Perhaps my response should have been a question to this director, himself a distinguished scientist, as to what he thought is the money needed to get a Nobel prize for India in science (which the country has not got since Independence). Back of the envelope calculation would show that India must spend instead of present 0.6% of GDP to at least 1% of GDP in higher education (to match other countries, for example China spends 2.1% and USA 2.7% of their GDP, see[1]) for next 20 years, and then hope that *perhaps* someone in India gets a Nobel prize. This comes (at the present 3 trillion dollars GDP – a figure that every Indian knows these days!) to an extra budget of about 12 billion dollars each year for at least next 20 years to get us a Nobel prize! Of course, we can wait for a Ramanujan or CV Raman to be born again, but even more importantly, hope that they do not go away to the West for better opportunities.

On the other hand, the support needed to run good activities in mathematics is considerably less. For example, the National Center of Mathematics (NCM), a joint venture of IIT Bombay and the TIFR has a budget only of a few crores (rupees!) and supports all kinds of undergraduate and post-graduate training programs in the country, but still has to fight for getting the financial resources. National Board of Higher Mathematics also has only a minimal budget to support undergraduate and postgraduate students in mathematics. Chennai Mathematical Institute is perpetually fighting for its survival. Many other centers of mathematics, such as Kerala School of Mathematics (KSOM) or Bhaskaracharya Pratishthan in Pune seem to survive on a shoe-string budget. The Indian Mathematical Society (IMS) has acquired a piece of land in Pune to house its headquarters, but has no money for constructing one, or to run any activity! I hope the country invests in mathematics — which means training of mathematics students from young students going for the Olympiad programs to those in the post-doctoral years, and those in faculty positions who do not stop to

[1]https://www.brookings.edu/wp-content/uploads/2019/11/Reviving-Higher-Education-in-India-email.pdf

do research for paucity of funds for travel or to attend conferences. Going to conferences is not a luxury trip, it is essential to do sciences, where you not only learn the latest on your subject, you make contacts, so essential in any human endeavor. It is important to get out of our comfort zone at least once in a while and talk to our peers, juniors as well as seniors. For us in the third world when most of the sciences is done elsewhere, it is even more essential so that we do *relevant* research at the *forefronts.*

Most of the Indian students belong to the University system (India had just about a score of universities at the time of Independence, and now about a thousand). It is imperative that the University system be improved, and especially those doing well, are supported better. Students in the country with inclination to mathematics need to be connected better through mathematical activities, perhaps through the existing programs of NCM, NBHM, MTTS, etc. For this to happen, we must strengthen these agencies considerably with more generous funds. The mathematical societies, such as the Indian Mathematical Society and the Ramanujan Mathematical Society, should have more activities both national and regional, and should have a larger national presence.

I hope mathematics can be a serious career option for some of the most talented in the country, and that they try to reach greater and greater heights in mathematics, so that India continues to be a presence in mathematics, which is not easy as Lewis Carroll says in “Alice in Wonderland”:

It takes all the running you can do, to keep in the same place.
If you want to get somewhere else,
you must run at least twice as fast as that!

I want to end by making a case for the country as working and living conditions offered by many of our Institutes are excellent — perhaps even better than most places in the world, and salaries are good (in the Indian context, not to be converted to US dollars!). There seems nothing to prevent us from doing good work. But we are not doing so, except for some small exceptions. Most of us are not trying as hard as our counterparts in the Western or Eastern world. I would exhort students of Indian origin finishing a PhD outside India to consider coming back to India. India needs you and there are plenty of jobs available in mathematics. Your presence will surely make a difference to India!

Acknowledgement: The author thanks U.K. Anandavardhanan, Amartya K. Dutta, S. Ghorpade, C.S. Rajan, B. Sury and J.K. Verma for looking at an earlier version of the article offering corrections and comments.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 9–28

ARITHMETIC IN L-VALUES*

DIPENDRA PRASAD

Abstract. This article consists of three parts. In part one we begin with the Riemann zeta function, and some of its properties, and define the Dedekind zeta functions and L-functions. This part is meant for a very general audience in mathematics without presuming any prior knowledge of number theory. The second part discusses various results and conjectures, due among others to Klingen, Siegel, Harder, Borel, Lichtenbaum, Deligne, Beilinson, Bloch, and Kato. We are necessarily very brief here though I have tried to give some of the assertions precisely. Part 3 is a summary of some of my own works [Pr] which tries to understand denominators in Artin L-values at $s = 0$ when it is known a priori that it is a rational number, and relates them to order of certain classgroups.

1. The Riemann zeta function

Let us recall that the Riemann zeta function is defined as the infinite sum,

$$\zeta(s) = \sum_n \frac{1}{n^s},$$

where $n \geq 1$ is an integer and $s > 1$ is a real number.

Using the indefinite integral,

$$\int \frac{1}{x^s} dx = \frac{x^{-s+1}}{-s+1},$$

it is easy to see that the infinite sum defining the Riemann zeta function converges for real $s > 1$, and even converges for complex s as long as $re(s) > 1$.

* This article is based on the text of the Presidential Address (Technical) given by Prof. Dipendra Prasad at the 87th Annual Conference of the IMS - An International meet held at MGM Unuversity, Aurangabad during December 4-7, 2021 using online mode.

One fundamental reason why the Riemann zeta function is of interest to number theorists is the following product expansion,

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \frac{1}{(1-\frac{1}{p^s})},$$

where the product is taken over all prime numbers $p = 2, 3, 5, 7, 11, \cdots$. The product expansion also is "convergent" for $re(s) > 1$.

The above product expansion of the Riemann zeta function is often called the "Euler product", as it was first noticed by Euler, who also evaluated the Riemann zeta function in particular cases, such as the famous identity:

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90},$$

more generally, we have

$$\zeta(2n) \in \pi^{2n}\mathbb{Q}^\times.$$

Noticing,

$$\frac{1}{(1-\frac{1}{p^s})} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \cdots,$$

we see that the product expansion of the Riemann zeta function is a way of expressing the following fundamental fact about integers.

Unique Factorization Property: Every integer $n \geq 1$ can be writeen uniquely as a (finite) product of powers of primes: $n = p_1^{d_1} \cdot p_2^{d_2} \cdot p_3^{d_3} \cdots$.

As a first consequence of the product expansion of the Riemann zeta function, we give a proof of the infinitude of prime.

Suppose to the contrary, that there are only finitely many primes $2, 3, 5, \cdots p_d$. Then we find that the (finite) product over the primes:

$$\prod_p \frac{1}{(1-\frac{1}{p^s})},$$

must be convergent at $s = 1$, and must be equal to,

$$\sum_n \frac{1}{n},$$

which we all know rather well is a divergent series, proving that primes must be infinitely many.

2. Basic Propertes of the Riemann zeta function

As we just discussed, the infinite series defining the zeta function:

$$\zeta(s) = \sum_n \frac{1}{n^s},$$

is convergent for $re(s) > 1$, and hence defines what is called a complex analytic function or a holomorphic function of the complex variable s in the domain $re(s) > 1$.

A fundamental fact discovered by Riemann is that this function can be extended to an analytic function in the whole of the complex plane with a unique singularity at $s = 1$ (remember $\sum_n \frac{1}{n}$ is divergent!).

Furthermore, if we define

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx,$$

is the famous Gamma function (with $\Gamma(n) = (n-1)!$), then we have the functional equation:

$$\xi(s) = \xi(1-s).$$

Thus, to summarize, the Riemann zeta function has the following basic properties:

(1) It is defined by an infinite series of the form:

$$f(s) = \sum_{n\geq 1} \frac{a_n}{n^s},$$

(in our case, $a_n = 1$ for all n) which is convergent in some right half plane (in our case, $re(s) > 1$), defining a complex analytic function in this domain, which has an analytic continuation to the whole of complex plane with possibly some poles. Such infinite series are called Dirichlet series.

(2) The extended function to the complex plane satifies a simple enough functional equation of the form,

$$f(s) = f(1-s).$$

(3) The infinite series defining $f(s)$ has an Euler product:

$$f(s) = \sum_n \frac{a_n}{n^s} = \prod_p \frac{1}{(1 - \frac{a_p}{p^s})},$$

We will call a Dirichlet series having an Euler product, an L-function, hoping that they also have analytic continuation to the whole of complex plane and a functional equation, though often these take considerable effort to prove.

After the Riemann zeta function, the first Dirichlet series (and which was indeed defined by Dirichlet) is:

$$L(s,\chi) = \sum_n \frac{\chi(n)}{n^s} = \prod_p \frac{1}{(1 - \frac{\chi(p)}{p^s})},$$

where

$$\chi : (\mathbb{Z}/d)^\times \to \mathbb{C}^\times$$

is a finite order character (extended to all integers n by declaring $\chi(n) = 0$ if $(d,n) \neq 1$).

The function $L(s,\chi)$ has all the properties enumerated above of the Riemann zeta function. This L-function was constructed to prove an important theorem due to Dirichlet: any arithmetic progression $an + b$ with $(a,b) = 1$ contains infinitely many primes.

3. Dedekind Zeta functions

Start with a monic polynomial

$$f(X) = X^n + a_1 X^{n-1} + a_2 X^{n-2} + \cdots + a_n \in \mathbb{Z}[X],$$

where $\mathbb{Z}[X]$ refers to the ring of polynomials with integral coefficients. One standard thing to do in number theory is to consider the reduction modulo primes of this polynomial, and write it as a product of irreducible polynomials:

$$\bar{f}(X) = \prod p_i(X)^{n_i} \in \mathbb{Z}/p[X],$$

where $\bar{f}$ denotes reduction of f modulo a prime p. (This is very similar to writing integers as a product of prime numbers.)

It is easy to see that for most primes p, $\bar{f}$, the reduction modulo p of f has no multiple factors, i.e., $n_i = 1$. Thus for most primes p, we have a factorization: $\bar{f}(X) = \prod p_i(X)$ as a product of distinct irreducible monic polynomials in $\mathbb{Z}/p[X]$. The basic question here is: what are the possible degrees of the polynomials $p_i(X)$ as p varies? Can any possible degree occur as long as the sum of degrees = n? Can one describe the set of primes p where $\bar{f}(X)$ is a product of linear factors, or where it remains irreducible?

All this information can be encoded in a zeta function $\zeta_f(s)$ defined as follows:

$$\zeta_f(s) = \prod_p \prod_i \frac{1}{(1 - \frac{1}{p^{d_i s}})},$$

where the product is taken over all primes p in $\mathbb{N}$ where $\bar{f}(X)$ is a product of distinct irreducible factors of degree d_i.

To be able to nicely express $\zeta_f(s)$ as product of known L-functions, such as Dirichlet L-function, Artin was led to a general reciprocity law generalizing the law of quadratic reciprocity which bears his name, *Artin reciprocity*, but valid only for certain kind of polynomials (whose Galois group is abelian).

The zeta function $\zeta_f(s)$ that we have introduced is more commonly associated to number fields, i.e., finite extensions F of $\mathbb{Q}$ inside $\mathbb{C}$, and this is what we too will do in what follows, and use the notation $\zeta_F(s)$ for the corresponding zeta function, called the Dedekind zeta function.

4. Klingen-Siegel theorem, refinement by Harder

If F is a totally real number field, then by a theorem due to Klingen and Siegel, $\zeta_F(-m)$ is a nonzero rational number for m an odd integer ≥ 1. The following theorem due to Harder refines this theorem of Klingen and Siegel, gives an estimate on the denominator of these zeta values, and also interprets the zeta values as an Euler characteristic.

Theorem 4.1. *For each integer $n \geq 1$, F a totally real number field of degree d over $\mathbb{Q}$, we have,*

$$\prod_{m=1}^{n} \zeta_F(1-2m) = 2^{n(d-1)} \chi(\mathrm{Sp}_{2n}(\mathcal{O}_F)).$$

Further, if q is the least common multiple of the orders of finite subgroups of $\mathrm{Sp}_{2n}(\mathcal{O}_F)$ (known to be finitely many up to conjugacy), then,

$$q\zeta_F(1-2m) \in \mathbb{Z}.$$

5. Borel-Lichtenbaum

According to the class number formula, we have,

$$\zeta_F(s) = -\frac{hR}{w} s^{r_1+r_2-1} + \text{ higher order terms},$$

where r_1, r_2, h, R, w are the standard invariants associated to F: r_1, the number of real embeddings; r_2, number of pairs of complex conjugate embeddings which are not real; h, the class number of F; R, the regulator, and w the number of roots of unity in F.

One can re-interpret the class number formula from the K-theoretic point of view. For this, note that,

(1) $K_0(\mathcal{O}_F) = \mathbb{Z} \oplus Cl(F)$,
(2) $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times$,

so,

$$\frac{h}{w} = \frac{K_0(\mathcal{O}_F)_{\text{tor}}}{K_1(\mathcal{O}_F)_{\text{tor}}},$$

where we denote the torsion in an abelian group A by A_{tor}.

By the work of Borel (building on the work of Quillen), we have:

(1) $K_{2n}(\mathcal{O}_F)$ are finite for $n \geq 1$, and
(2) $K_{2n-1}(\mathcal{O}_F) \cong \mathbb{Z}^{d_n} +$ finite group, for $n \geq 1$, where d_n is the order of vanishing of $\zeta_F(s)$ at $s = 1-n$.

Further, Borel constructed a map:

$$K_{2n-1}(\mathcal{O}_F) \to \mathbb{R}^{d_n},$$

with finite kernel and whose image is a lattice in $\mathbb{R}^{d_n}$, the covolume of which is called the Borel regulator, which we will denote by $R_n^{\text{Borel}}(F)$. Further, Borel proved that if $\zeta_F^*(1-n)$ is the leading term in the Taylor expansion of $\zeta_F(s)$ at $s = 1-n$, then $\zeta_F^*(1-n)/R_n^{\text{Borel}}(F)$ is a rational number, which has been made precise in the following conjecture of Lichtenbaum.

Conjecture 5.1. *(Lichtenbaum) For F any number field, and n any integer $n \geq 1$, one has*

$$\zeta_F^*(1-n) = \pm \frac{|K_{2n-2}(\mathcal{O}_F)|}{|K_{2n-1}(\mathcal{O}_F)_{\text{tor}}|} R_n^{\text{Borel}}(F),$$

where $\zeta_F^*(1-n)$ is the leading term in the Taylor expansion of $\zeta_F(s)$ at $s = 1-n$.

In the above conjecture, if we take F totally real, $n = 2$, then there is no Borel regulator to consider (since $d_2 = 0$), and the Lichtenbaum conjecture reduces to an earlier conjecture due to Birch-Tate which we now recall.

If F is a totally real number field, $\zeta_F(-1)$ is a nonzero rational number. The Milnor K-group $K_2(\mathcal{O}_F)$ (defined in terms of the Steinberg extension of the subgroup of $\mathrm{GL}(\mathcal{O}_F)$ generated by elementary matrices) is a finite abelian group, and the conjecture of Birch-Tate asserts that:

$$|K_2(\mathcal{O}_F)| = |w_2(F)| \cdot \zeta_F(-1),$$

where $w_2(F)$ is the largest integer N such that attaching primitive N-th roots of unity to F gives an abelian extension of F with Galois group an elementary abelian 2-group. For example, if $F = \mathbb{Q}$, then $w_2(\mathbb{Q}) = 24$, $\zeta_{\mathbb{Q}}(-1) = -1/12$, and $K_2(\mathbb{Z}) = \mathbb{Z}/2$.

The Birch-Tate conjecture is a consequence of the work of Mazur-Wiles for abelian extensions of $\mathbb{Q}$, and for totally real fields by a subsequent work of Wiles.

To relate the Birch-Tate conjecture to the Lichtenbaum conjectures for totally real number fields, one also needs to identify the order of $K_3(\mathcal{O}_F) \cong K_3(F)$ to $|w_2(F)| = |H^0(F, \mu^2)|$ which in fact is done using the following exact sequence:

$$0 \to K_3^M(F) \cong (\mathbb{Z}/2)^d \to K_3(F) \to H^0(F, \mu^2) \to 0,$$

where $K_3^M(F)$ is the Milnor K-group of F (whereas $K_3(F)$ is the Quillen K-group).

6. Deligne-Beilinson-Bloch-Kato

For $\mathcal{M}$ a motive over $\mathbb{Q}$, or a cohomological automorphic representation on $G(\mathbb{A})$ where G is a reductive group over $\mathbb{Q}$, there is a notion of its L-function, $L(\mathcal{M}, s)$, with all the properties discussed earlier for the Riemann zeta function such as analytic continuation, Euler product and the functional equation.

The general conjectures in this section have the flavor of suggesting that:

$$L^*(\mathcal{M}, k) = (\text{algebraic number}) \times (\text{period}) \times (\text{regulator}), \quad (1)$$

where $L^*(M, k)$ is the leading term in the Taylor expansion of $L(\mathcal{M}, s)$ in the neighborhood of $s = k$; further, the algebraic number belongs to the field of coefficients E of $\mathcal{M}$. Thus, the L-function $L(\mathcal{M}, s)$, which is a

Dirichlet series, has coefficients belonging to E, and hence $L(\mathcal{M}, s)$ is to be considered as a holomorphic function in $s \in \mathbb{C}$ with values in $E \otimes_{\mathbb{Q}} \mathbb{C}$, in particular, $L(M, k)$, $L^*(M, k)$ belong to $E \otimes_{\mathbb{Q}} \mathbb{C}$, and the decomposition in (1) belongs to $E \times \mathbb{C} \subset E \otimes_{\mathbb{Q}} \mathbb{C}$. This, Deligne says in his Corvalis article (for his critical points) is inspired by a comment of B. Gross.

For instance, the conjecture of Birch and Swinnerton-Dyer suggests in a very precise way the conjecture expressed in equation (1) above for abelian varieties, and this is what the classnumber formula proves.

Although the terms, *period* and *regulator* are of very different origin, they are defined using one notion from Linear algebra: If we have two vector spaces V and W over $\mathbb{Q}$ of the same dimension $d < \infty$, and which come equipped with an isomorphism $\phi : V \otimes \mathbb{R} \to W \otimes R$, then $\Lambda^d_{\mathbb{R}}\phi$ gives an isomorphism of two one dimension vector space over $\mathbb{R}$ with $\mathbb{Q}$-structures, defining a real number well defined up to $\mathbb{Q}^{\times}$. For the periods, either for a motive or for an algebraic variety over $\mathbb{Q}$, these come equipped with the de Rham as well as Betti cohomology both of which have a $\mathbb{Q}$-structure, and there is a comparison isomorphism between these $\mathbb{Q}$-structures when tensored with $\mathbb{C}$. Similarly, regulators (defined by Beilinson in some generality) arise from comparing two $\mathbb{Q}$ structures on a finite dimensional vector space, such as in the case of an abelian variety A, there is the Neron-Tate pairing $A(\mathbb{Q}) \times A(\mathbb{Q}) \to \mathbb{R}$ which gives an isomorphism of $A(\mathbb{Q}) \otimes \mathbb{R}$ with its dual, both coming with natural $\mathbb{Z}$-structures, defining the regulator of an abelian variety.

The regulator term in (1) is present if and only if the order of vanishing of $L(\mathcal{M}, s)$ at k is $d > 0$, and is then given by the Linear algebra notion of the previous paragraph, equivalently, by a certain $d \times d$ determinant.

The work of Deligne on $L(\mathcal{M}, k)$ proposes a conjecture of the form given in (1) in the case when one can omit the regulator term in (1), thus if $L(\mathcal{M}, k) = 0$, Deligne's conjecture says nothing, and if $L(\mathcal{M}, k)$ is nonzero, it says as in (1), that $L(\mathcal{M}, k)$ is an algebraic multiple of the period. Deligne's conjecture is only for critical points $k \in \mathbb{Z}$ which according to him are those integers for which gamma factors at infinity attached to $\mathcal{M}$, $L_\infty(\mathcal{M}, s)$ (resp., attached to $\mathcal{M}^\vee$, $L_\infty(\mathcal{M}^\vee, s)$) have no poles at $s = k$ (resp., $1-k$). Thus non-critical points are precisely the integers k for which $L(\mathcal{M}, k)$ or $L(\mathcal{M}^\vee, 1-k)$ acquire a zero because of the gamma factors at infinity – such zeros are also sometimes called the trivial zeros.

In the case of the Riemann zeta function, k, the even negative integers, or odd positive integers are non-critical points, outside the purview of Deligne's conjecture (but to which Borel's theorems and Lichtenbaum's conjectures discussed earlier apply).

In the case of an elliptic curve E over $\mathbb{Q}$, with L-function $L_E(s)$, and completed L-function

$$\Lambda_E(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s),$$

where N is the conductor of E, one has the functional equation,

$$\Lambda_E(s) = \epsilon(E)\Lambda_E(2-s),$$

where $\epsilon(E) = \pm 1$ is the global root number of E. As $\Lambda_E(s)$ is an entire function of s, because of the presence of $\Gamma(s)$ in the expression for $\Lambda_E(s)$, which has poles exactly at integers $\{0, -1, -2, -3, \cdots\}$, we find that $L_E(s)$ must have zeroes at these integers $\{0, -1, -2, -3, \cdots\}$, thus 1 is the only critical point of E.

Beilinson makes a conjecture of the form (1), of course that means making the regulator term precise, for all integers k critical or not, but still with the same ambiguity as in (1) of being up to $\bar{\mathbb{Q}}^\times$ which is remedied by Bloch and Kato up to sign. We now review the conjecture of Beilinson.

Let X be a smooth projective variety over $\mathbb{Q}$ with $\mathcal{X}$ a regular (flat) model over $\mathbb{Z}$. Let $H^j_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(j-n))$, called the motivic cohomology of X, be the subspace of $K_m(\mathcal{X}) \otimes \mathbb{Q}$ on which the Adams operators act with weight $j-n$ where $m = j-2n$. Beilinson defines homomorphisms (as Chern character)

$$H^j_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(j-n)) \to H^j_{\mathcal{D}}(X, \mathbb{R}(j-n)),$$

where $H^j_{\mathcal{D}}(X, \mathbb{R}(j-n))$ is the Deligne cohomology, a finite dimensional $\mathbb{R}$-vector space, on which Beilinson constructs a rational structure and conjectured that as long as $m \geq 2$, the above homomorphism tensored with $\mathbb{R}$ is an isomorphism, allowing Beilinson to define a regulator, a real number, well-defined up to $\mathbb{Q}^\times$, which is now called the Beilinson regulator. One way to define the Deligne cohomology $H^j_{\mathcal{D}}(X, \mathbb{R}(j-n))$ (according to Beilinson) is:

$$\mathrm{Ext}^j(\mathbb{Z}(n), H^*(X)) := \mathrm{Hom}(\mathbb{Z}(n), H^*(X)[j]),$$

where Ext and Hom are computed in the derived category of Hodge structures where $H^*(X)$ lives (by a fundamental work of Deligne), and where $\mathbb{Z}(m)$ denotes the Tate-Hodge structure.

It is easy to see that for any $n \in \mathbb{Z}$, $n < j/2$, the order of zero of the L-function $L(H^{j-1}(X, \mathbb{Q}), n)$ is dictated by the poles of the archimedean component $L_\infty(H^{j-1}(X, \mathbb{Q}), n)$, and thus by the Hodge structure of X. Further, the order of zero of the L-function $L(H^{j-1}(X, \mathbb{Q}), n)$ is the same as the rank of the $\mathbb{Q}$-vector space $H^j_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(j-n))$ as well as the rank of the $\mathbb{R}$-vector space $H^j_{\mathcal{D}}(X, \mathbb{R}(j-n))$. (The analogous assertions for $m = 0, 1$, corresponding to the central point for the L-function if j is even, or the integer nearest to the central point if j is odd, are different; the case of j odd corresponds to Tate cycles, but the case of j even, and the central point is subtler!)

7. Artin L-functions

Rest of the paper is an exposition on my own work in [Pr], clarifying an issue which was missed out in this which is contained in remarks 10.2, and 11.3. At the end of this paper, we list a few papers and books relevant to this as well as other sections, but without giving precise references.

Now on, we consider certain Artin representations $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \mathrm{GL}_n(\mathbb{C})$ for which we know a priori that $L(0, \rho)$ is a nonzero algebraic number (in particular, F will be totally real), and try to understand the nature of the algebraic number $L(0, \rho)$: to know the prime divisors of the numerator and the denominator of $L(0, \rho)$, to know if it is an algebraic integer, but if not, what are its possible denominators. We think of the possible denominators in $L(0, \rho)$, as existence of poles for $L(0, \rho)$, at the corresponding prime ideals of $\bar{\mathbb{Z}}$. It is thus analogous to the conjecture of Artin, both in its aim — and as we will see — in its formulation. Recall that the conjecture of Artin asserts that $L(s, \rho)$ has an analytic continuation to an entire function on $\mathbb{C}$ unless ρ is the trivial representation, in which case it has a unique pole at $s = 1$ which is simple.

Remark 7.1. For an Artin representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \mathrm{GL}_n(\mathbb{C})$, one knows that $L(0, \rho)$ is nonzero if and only if F is a totally real number field, and ρ cuts out a CM extension of F such that $\rho(c) = -1 \in \mathrm{GL}_n(\mathbb{C})$, where c is the complex conjugation on $\mathbb{C}$. In this situation, by theorems of Siegel and Klingen, $L(0, \rho)$ is an algebraic number (belonging to the field generated by the trace of ρ). On the other hand, one expects that if $L(0,\rho)=0$, then the first nonzero derviative of $L(s,\rho)$ at $s=0$ is always transcendental, a certain determinant of logarithm of algebraic numbers.

If E is a CM field with F its totally real subfield, then r_1+r_2 is the same for E as for F, and the regulators of E and F too are the same except for a possible power of 2. Therefore for τ the complex conjugation on $\mathbb{C}$, the classnumber formulae for E, F give rise to:

$$(\zeta_E/\zeta_F)(0) = \prod_{\rho(\tau)=-1} L(0,\rho)^{\dim\rho} = \frac{h_E/h_F}{w_E/w_F},$$

where each of the L-values $L(0,\rho)$ in the above expression are nonzero algebraic numbers by a theorem of Klingen and Siegel. In fact, we can even remove the denominator w_E/w_F by dividing by the same formula for the cyclotomic extension of $\mathbb{Q}$ obtained by attaching w_E-th root of unity to $\mathbb{Q}$.

In this identity, observe that the L-functions are associated to complex representations of $\mathrm{Gal}(E/\mathbb{Q})$, whereas the classgroups of E and F are finite Galois modules. Modulo some details, I formulated a conjecture (Conjecture 1 in [Pr]) which basically asserts that the two sides are not only the same as a product taken over certain representations of $\mathrm{Gal}(E/\mathbb{Q})$, but the individual factors on the two sides are the same. In particular, this demands that as the class groups have integral order, so must the L-functions!

Thus we like to separate out as two conclusions:

(1) $L(0,\rho)$ are integral except for certain "obvious" ρ.
(2) The classgroup as a module for $\mathrm{Gal}(E/\mathbb{Q})$ has a particular (mod p) representation appearing in it with nonzero multiplicity if and only if the corresponding L value is nonzero (mod p).

This is exactly what happens for $E=\mathbb{Q}(\zeta_p)$ by the theorems of Herbrand and Ribet which is one of the main motivating example for all that we do here, and this is what we will review next, but before that we review Dirichlet characters.

8. Dirichlet L-functions and the class number

Let $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. It is known that if χ is an odd character of $(\mathbb{Z}/f)^\times$, $L(0,\chi)$ is an algebraic number which is given in terms of the generalized Bernoulli number $B_{1,\chi}$ as follows:

$$L(0,\chi) = -B_{1,\chi} = -\frac{1}{f}\sum_{a=1}^{a=f} a\chi(a),$$

If, furthermore, $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$ is a quadratic primitive Dirichlet character, defining the quadratic field $K = \mathbb{Q}(\sqrt{\pm f})$, then its class number is given by

$$h_K = L(0,\chi) = -B_{1,\chi} = -\frac{1}{f}\sum_{a=1}^{a=f} a\chi(a).$$

It is a consequence of this result that if $p \equiv 3 \bmod 4$, then for $K = \mathbb{Q}(\sqrt{-p})$,

$$h_K = L(0,\chi) = -B_{1,\chi} = \frac{1}{2-\chi(2)}\sum_{a=1}^{(p-1)/2} \chi(a).$$

In what follows, $\omega_p : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to (\mathbb{Z}/p)^\times \hookrightarrow \mathbb{Z}_p^\times$ is the action of Galois group on the p-th roots of unity.

9. The Herbrand-Ribet theorem

In this section we recall the Herbrand-Ribet theorem from the point of view of this lecture.

Consider the class number formula

$$\zeta_F(s) = -\frac{hR}{w}s^{r_1+r_2-1} + \text{ higher order terms},$$

both for $F = \mathbb{Q}(\zeta_p)$ as well as its maximal real subfield $F^+ = \mathbb{Q}(\zeta_p)^+$. It is known that,

$$R/R^+ = 2^{\frac{p-3}{2}},$$

where R is the regulator for $\mathbb{Q}(\zeta_p)$ and R^+ is the regulator for $\mathbb{Q}(\zeta_p)^+$. We will similarly denote h and h^+ to be the order of the two class groups, with $h^- = h/h^+$, an integer.

Dividing the class number formula of $\mathbb{Q}(\zeta_p)$ by that of $\mathbb{Q}(\zeta_p)^+$, we find,

$$\prod_{\chi \text{ an odd character of } (\mathbb{Z}/p)^\times} L(0,\chi) = \frac{1}{p}\cdot\frac{h}{h^+}\cdot 2^{\frac{p-3}{2}}, \qquad (*)$$

the factor $1/p$ arising because there are $2p$ roots of unity in $\mathbb{Q}(\zeta_p)$ and only 2 in $\mathbb{Q}(\zeta_p)^+$.

It is easy to see that $pB_{1,\omega_p^{p-2}} \equiv (p-1) \bmod p$ since $a\omega_p^{p-2}(a)$ is the trivial character of $(\mathbb{Z}/p)^\times$ whereas for all the other characters of $(\mathbb{Z}/p)^\times$, $L(0,\chi)$ is not only an algebraic number but is p-adic integral (Schur orthogonality!); all this is clear by looking at the expression:

$$L(0,\chi) = -B_{1,\chi} = -\frac{1}{p}\sum_{a=1}^{a=p} a\chi(a).$$

Rewrite the class equation for $\mathbb{Q}(\zeta_p)$ divided by the class equation for $\mathbb{Q}(\zeta_p)^+$ as,

$$\prod_{\substack{\chi \text{ an odd character of } (\mathbb{Z}/p)^\times \\ \chi \neq \omega_p^{p-2} = \omega_p^{-1}}} L(0,\chi) = \frac{h}{h^+},$$

where we note that both sides of the equality are p-adic integral elements as just observed.

This when interpreted — just an interpretation in the spirit of this lecture without any suggestions for proof in either direction! — for each χ component on the two sides of this equality amounts to the theorem of Herbrand and Ribet which asserts that p divides $L(0,\chi) = -B_{1,\chi}$ for χ an odd character of $(\mathbb{Z}/p)^\times$, which is not ω_p^{p-2}, if and only if the corresponding χ^{-1}-eigencomponent of the classgroup of $\mathbb{Q}(\zeta_p)$ is nontrivial. (Note the χ^{-1}, and not χ!) Furthermore, the character ω_p does not appear in the p-classgroup of $\mathbb{Q}(\zeta_p)$. It can happen that $L(0,\chi)$ is divisible by higher powers of p than 1, and one expects — this is not proven yet! — that in such cases, the corresponding χ^{-1}-eigencomponent of the classgroup of $\mathbb{Q}(\zeta_p)$ is $\mathbb{Z}/p^{(\mathrm{val}_p L(0,\chi))}$, in particular, it still has p-rank 1. (By Mazur-Wiles, χ^{-1}-eigencomponent of the classgroup of $\mathbb{Q}(\zeta_p)$ is of order $p^{(\mathrm{val}_p L(0,\chi))}$.)

The work of Ribet was to prove that if $p|B_{1,\chi}$, χ^{-1}-eigencomponent of the classgroup of $\mathbb{Q}(\zeta_p)$ is nontrivial by constructing an unramified extension of $\mathbb{Q}(\zeta_p)$ by using a congruence between a holomorphic cusp form and an Eisenstein series on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

10. Integrality

The following conjecture (Conjecture 1 of [Pr]) about $L(0,\rho)$ extends the integrality properties of

$$L(0,\chi) = -B_{1,\chi} = -\frac{1}{p}\sum_{a=1}^{a=p} a\chi(a),$$

encountered and used earlier.

Conjecture 10.1. *(mod p analogue of the Artin conjecture) Let ρ be an irreducible representation of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *cutting out a CM extension E of $\mathbb{Q}$ with $\rho(-1) = -1$ where -1 is the complex conjugation in* $\mathrm{Gal}(E/\mathbb{Q})$. *Then unless ρ is a one dimensional representation factoring through* $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ *(for some prime p) with $\bar{\rho}$ the reduction of ρ modulo p being $\bar{\rho} = \omega_p^{-1}$, $L(0,\rho) \in \bar{\mathbb{Q}}$ is integral outside 2, i.e., $L(0,\rho) \in \bar{\mathbb{Z}}[\frac{1}{2}]$.*

Remark 10.2. In several places in this work, as well as in [Pr], where we asert: "$L(0,\rho)$ is integral unless ρ is a one dimensional representation factoring through $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ (for some prime p) with $\bar{\rho}$ the reduction of ρ modulo p being $\bar{\rho} = \omega_p^{-1}$, $L(0,\rho) \in \bar{\mathbb{Q}}$" needs care. The problem is with the usage of "the reduction of ρ modulo p" since ρ takes values in an algebraic number field, say $E/\mathbb{Q}$, to reduce it modulo p, we must choose a prime in E over p, and different choices give rise to different "reductions of ρ modulo p". In the particular case when χ is an odd character $\chi : (\mathbb{Z}/p)^\times \to \mathbb{C}^\times$, which is of order $(p-1)$, $L(0,\chi) \in E = \mathbb{Q}(\zeta_{p-1})$, and for the field, $\mathbb{Q}(\zeta_{p-1})$, the prime p is split, so there are as many primes in $\mathbb{Q}(\zeta_{p-1})$ over p as $\phi(p-1)$, same as number of characters $\chi : (\mathbb{Z}/p)^\times \to \mathbb{C}^\times$ of order $(p-1)$. Therefore for any $\chi : (\mathbb{Z}/p)^\times \to \mathbb{C}^\times$ of order $(p-1)$, there is a place $\wp$ of $\mathbb{Q}(\zeta_{p-1})$ over p such that $\bar{\rho} = \omega_p^{-1}$, so for any $\chi : (\mathbb{Z}/p)^\times \to \mathbb{C}^\times$ of order $(p-1)$, $L(0,\chi)$ is nonintegral at the place $\wp$ of $\mathbb{Q}(\zeta_{p-1})$. Thus the above conjecture for odd characters $\chi : (\mathbb{Z}/p)^\times \to \mathbb{C}^\times$ is that $L(0,\chi)$ is integral if and only if χ is not of order $(p-1)$, and the statement of the conjecture should be replaced by, "$L(0,\rho)$ is integral unless ρ is a one dimensional representation factoring through $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ (for some prime p) of order $(p-1)p^{n-1}$." One way to eliminate this problem of "reduction modulo p" is to consider representations not with values in $\mathrm{GL}_n(\mathbb{C})$, but with values in $\mathrm{GL}_n(\bar{\mathbb{Q}}_p)$, and only talk of p-integrality and not integrality, so one prime at a time.

We next recall the following theorem of Deligne-Ribet (cf. [DR]) which could be considered as a weaker version of Conjecture 10.1.

Theorem 10.3. *Let F be a totally real number field, and let $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \bar{\mathbb{Q}}^\times$ be a character of finite order cutting out a CM extension K of F (which is not totally real). Let w be the order of the group of roots of unity in K. Then,*

$$wL(0, \chi) \in \bar{\mathbb{Z}}.$$

In fact Conjecture 10.1 can be used to make precise the above theorem of Deligne-Ribet as follows. This involves a simple argument using the fact that the Artin L-function is invariant under induction from $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Conjecture 10.4. *Let F be a totally real number field, and $\chi : \mathbb{A}_F^\times/F^\times \to \bar{\mathbb{Z}}_p^\times$ a finite order character, cutting out a non-real but CM extension. Then if $L_F(0, \chi) \notin \bar{\mathbb{Z}}_p$,*

1. *χ mod p is ω_p^{-1}.*
2. *χ is a character of $\mathbb{A}_F^\times/F^\times$ associated to a character of the Galois group $\mathrm{Gal}(F(\zeta_q)/F)$ for some q which is a power of p.*

Remark 10.5. In the examples that I know, which are for characters $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \bar{\mathbb{Q}}_p^\times$ with $\chi = \omega_p^{-1} \pmod p$, if $L(0, \chi)$ has a (mod p) pole, the pole is of order 1; more precisely, if $L = \mathbb{Q}_p[\chi(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p))]$ is the subfield of $\bar{\mathbb{Q}}_p$ generated by the image under χ of the decomposition group at p, then $L(0, \chi)$ is the inverse of a uniformizer of this field L. It would be nice to know if this is the case for characters χ of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ for F arbitrary. This would be in the spirit of classical Artin's conjecture where the only possible poles of $L(1, \rho)$, for ρ an irreducible representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$, are simple.

11. Integrality of Abelian L-values for $\mathbb{Q}$

The aim of this section is to recall certain results on integrality of $L(0, \chi)$ for χ an odd Dirichlet character of $\mathbb{Q}$ which go as first examples of all the integrality conjectures made in this lecture. Of course, these are all well-known results.

Lemma 11.1. *For integers $m > 1, n > 1$, with $(m, n) = 1$, let $\chi = \chi_1 \times \chi_2$ be a primitive Dirichlet character on $(\mathbb{Z}/mn\mathbb{Z})^\times = (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/n\mathbb{Z})^\times$*

with $\chi(-1) = -1$. *Then,*

$$L(0,\chi) = -B_{1,\chi} = -\frac{1}{mn}\sum_{a=1}^{mn} a\chi(a),$$

is an algebraic integer, i.e., belongs to $\bar{\mathbb{Z}} \subset \bar{\mathbb{Q}}$.

Lemma 11.2. *For* p *a prime, let* χ *be a primitive Dirichlet character on* $(\mathbb{Z}/p^n\mathbb{Z})^\times$ *with* $\chi(-1) = -1$. *Write* $(\mathbb{Z}/p^n\mathbb{Z})^\times = (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}/1+p^n\mathbb{Z})$, *and the character* χ *as* $\chi_1 \times \chi_2$ *with respect to this decomposition. Then,*

$$L(0,\chi) = -B_{1,\chi} = -\frac{1}{p^n}\sum_{a=1}^{p^n} a\chi(a),$$

is an algebraic integer, i.e., belongs to $\bar{\mathbb{Z}} \subset \bar{\mathbb{Q}}$ *if and only if* $\chi_1 \neq \omega_p^{-1}$.

Remark 11.3. Just as in Remark 10.2, the above lemma from [Pr] needs to be made more carefully to take into account the issue of choosing a prime in $\mathbb{Q}(\chi)$ for "reduction modulo p". Here is the precise statement. For an odd primitive character $\chi : (\mathbb{Z}/p^n)^\times \to \mathbb{C}^\times$, $L(0,\chi)$ is integral if and only if χ is not of order $(p-1)p^{n-1}$.

The following proposition follows by putting the previous two lemmas together.

Proposition 11.4. *Primitive (odd) Dirichlet characters* $\chi : (\mathbb{Z}/n)^\times \to \bar{\mathbb{Z}}_p^\times$ *for which* $L(0,\chi)$ *does not belong to* $\bar{\mathbb{Z}}_p$ *are exactly those for which:*

(1) $n = p^d$.

(2) $\chi = \omega_p^{-1}$ *mod* $\mathfrak{p}$.

The following consequence of the proposition suggests that prudence is to be exercised when discussing congruences of L-values for Artin representations which are congruent.

Corollary 11.5. *Let* p, q *be odd primes with* $p|(q-1)$. *For any character* χ_2 *of* $(\mathbb{Z}/q\mathbb{Z})^\times$ *of order* p, *define the character* $\chi = \omega_p^{-1} \times \chi_2$ *of* $(\mathbb{Z}/pq\mathbb{Z})^\times$. *Then although the characters* ω_p^{-1} *and* χ *have the same reduction modulo* p, $L(0,\omega_p^{-1})$ *is* p*-adically non-integral whereas* $L(0,\chi)$ *is integral.*

Question 11.6. Let $\chi : (\mathbb{Z}/p^d m)^\times \to \bar{\mathbb{Z}}_p^\times$ with $(p,m) = 1$, $m > 1$, be a primitive Dirichlet character for which $\chi = \omega_p^{-1}$ mod $\mathfrak{p}$ so that by Proposition 11.4, $L(0,\chi)$ is $\mathfrak{p}$-integral. Is it possible to have $L(0,\chi) = 0$ modulo $\mathfrak{p}$, the maximal ideal of $\bar{\mathbb{Z}}_p$? My proofs are good to detect integrality, but not good for questions modulo $\mathfrak{p}$. The question is relevant to see if the character ω_p appears in the classgroup H/H^+ for $E = \mathbb{Q}(\zeta_{p^d m})$; such a character is known not to appear in the classgroup of H/H^+ for $E = \mathbb{Q}(\zeta_{p^d})$.

12. Congruences of L-values

Congruence of L-values is one of the dominant themes of contemporary number theory, but as much of mathematics, its role has been felt for a long time, in this case in terms of what is called Kummer congruence. Thus Kummer discovered that irregular primes not in terms of p dividing $L(0,\chi)$, but in terms of p dividing a Bernoulli number.

We just state the precise result here.

(1) $L(-n,\chi) = -\frac{B_{n+1,\chi}}{n+1}$.
(2) $B_{1,\omega_p^n} \equiv \frac{B_{n+1}}{n+1} \bmod p$ if $n \not\equiv -1 \bmod (p-1)$.
(3) $L(0,\omega_p^n) \equiv \frac{B_{n+1}}{n+1} \bmod p$ if $n \not\equiv -1 \bmod (p-1)$.

In particular, $p|L(0,\chi)$ if and only if $p|B_i$ for some $i = 2, 4, ...p-3$ which is Kummer's criterion for the irregularity of a prime.

13. Congruences and their failure for L-values

This lecture considers integrality properties of certain Artin L-functions at 0. It may seem most natural that if two such Artin representations $\rho_1, \rho_2 : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ have the same semi-simplification mod-p and do not contain the character ω_p^{-1}, then $L(0,\rho_1)$ and $L(0,\rho_2)$ which are in $\bar{\mathbb{Z}}_p$ by Conjecture 1, have the same reduction mod-p. This is not true even in the simplest case of Dirichlet characters for $\mathbb{Q}$ as we just saw. It is possible to fix this problem for abelian characters of $\mathbb{Q}$, and more generally for any totally real number field which is what this section strives to do, cf. Proposition 13.1. The recipe given in Proposition 13.1 immediately suggests itself in the non-abelian case, but we have not spelt it out.

The problem that we find dealing with abelian characters χ_1, χ_2 is that they may be congruent for some prime, but may have different conductors in which case it is not the L-values $L(0,\chi_1)$ and $L(0,\chi_2)$ which are congruent, but a modified L-value, say $L_f(0,\chi)$ which gives the right congruence; these L-values are product of $\prod_{\wp}(1-\chi(\wp))$ with $L(0,\chi)$ where $\wp$ are all primes dividing either the conductor of χ_1 or of χ_2. We begin with some elementary lemmas which go into congruences of L-values at 0 of Dirichlet characters, and then we consider totally real number fields.

The following proposition which is a consequence of the work of Deligne-Ribet, discusses integrality as well as congruence of abelian L-functions.

Proposition 13.1. *Let F be a totally real number fields with $G = \mathrm{Gal}(\bar{\mathbb{Q}}/F)$ its Galois group. For $\chi : G \to \bar{\mathbb{Z}}_p^\times$, a character of finite order of G, which is also to be considered as a character of the groups of ideals coprime to a nonzero ideal $\mathfrak{f}$ in K by classfield theory (so $\mathfrak{f}$ is divisible by the conductor of χ, but may not be the conductor of χ). Let $L(s,\chi)$ be the 'true' L-function associated to the character χ, and define $L_\mathfrak{f}(s,\chi)$ by*

$$L_\mathfrak{f}(s,\chi) = \sum_{(\mathfrak{a},\mathfrak{f})=1} \frac{\chi(\mathfrak{a})}{(N\mathfrak{a})^s},$$

where $N\mathfrak{a}$ denotes the norm of an integral ideal $\mathfrak{a}$ in F. Then,

(1) $L_\mathfrak{f}(s,\chi) = \prod_{\wp|\mathfrak{f}}(1 - \frac{\chi(\wp)}{(N\wp)^s}) \cdot L(s,\chi)$.
(2) For any integral ideals $\mathfrak{c}$ in F coprime to $p\mathfrak{f}$, integers $k \geq 1$,

$$\Delta_\mathfrak{c}(1-k,\chi) = (1-\chi(\mathfrak{c})N\mathfrak{c}^k)L_\mathfrak{f}(1-k,\chi),$$

are in $\bar{\mathbb{Z}}_p$.
(3) If χ_1 and χ_2 are two characters of G with values in $\bar{\mathbb{Z}}_p^\times$ with conductors dividing $\mathfrak{f}$, such that neither of the two reductions $\bar{\chi}_1, \bar{\chi}_2 : G \to \bar{\mathbb{F}}_p^\times$ is ω_p^{-1}, then $L_\mathfrak{f}(0,\chi_1)$ and $L_\mathfrak{f}(0,\chi_2)$ are in $\bar{\mathbb{Z}}_p$, and if χ_1 and χ_2 are congruent modulo the maximal ideal in $\bar{\mathbb{Z}}_p$, so is the case for $L_\mathfrak{f}(0,\chi_1)$ and $L_\mathfrak{f}(0,\chi_2)$.

14. A question on Algebraicity

I want to conclude by posing a question. Recall that in this lecture, for $\mathcal{M}$ a motive over $\mathbb{Q}$, or a cohomological automorphic representation on $G(\mathbb{A})$ where G is a reductive group over $\mathbb{Q}$, we considered the L-function, $L(\mathcal{M},s)$, and especially its values at integers. More precisely, we discussed the general conjecture:

$$L^*(\mathcal{M},k) = (\text{algebraic number}) \times (\text{period}) \times (\text{regulator}), \quad (1)$$

where $L^*(M,k)$ is the leading term in the Taylor expansion of $L(\mathcal{M},s)$ in the neighborhood of $s=k$.

It is natural to understand for which motives $\mathcal{M}$ over $\mathbb{Q}$, $L^*(\mathcal{M}, k)$ can be algebraic for some integer k. One expects periods as well as regulators to be transcendental numbers and of independent nature, thus the most optimistic guess would be that such motives are the (direct sum of) Tate twists of Artin motives discussed in this work which arise from representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which cut out a totally real extension or a CM extension of $\mathbb{Q}$. In particular, I would expect that if $L^*(\mathcal{M}, k)$ is algebraic for some integer k, then $L(M, k)$ must be nonzero.

I do not know if this question has been considered or investigated in the literature.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 29–43

EFFECT OF SLOW-FAST TIME SCALE ON SPATIO-TEMPORAL PATTERN FORMATION*

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Abstract. Main objective of this article is to explain the effect of slow-fast time scale on the dynamics of prey-predator interaction. Effect of slow-fast time scale on temporal dynamics and spatio-temporal patterns are described briefly.

1. Introduction

Mathematical modelling of interacting populations is an interesting and challenging research area for researchers and scientists from various disciplines. A wide range of mathematical models is proposed and analyzed with the help of different types of mathematical tools. Here we confine ourselves within a small domain, namely, Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE) based mathematical models of two interacting species. ODE-based two species interaction models are divided into three types: competition, cooperation, and prey-predator, depending upon the interaction between the individuals of two interacting species under consideration [16]. Here we focus on prey-predator type interaction-based ODE and PDE models, this type of interaction also known as resource-consumer dynamics. To be precise, two species prey-predator type interaction models are the building blocks of large food chains, and food webs [6, 10].

Based upon the available food source for the predator species, prey-predator models involve either specialist predators or generalist predators. In the case of the two component prey-predator model with generalist predators, it is assumed that the predator species under consideration has some alternative food source apart from the prey species whose density is considered in the model explicitly. Note that other kinds of classification are also available for example, based upon the functional response, which we will discuss in the next section briefly with the appropriate mathematical formulation. The main objective behind the formulation of ODE and PDE based

* This article is based on the text of 35th P. L. Bhatnagar Memorial Award lecture given at the 87th Annual Conference of the IMS - An International meet held at MGM Unuversity, Aurangabad during December 4-7, 2021 using online mode.

models and their detailed analysis is to understand the dynamic nature of the interaction and their effect on the long-time survival of the constituent species. ODE models assume the homogeneous distribution of the individuals over the domain under consideration, but the heterogeneous population distribution over the habitats leads to formulation of PDE based models. Irrespective of the mathematical description, researchers are mainly interested to understand the nature of coexistence and possible mechanism of destabilization leading to the extinction of one or both species. As a result, researchers are mainly focused on the stability and bifurcation of various steady-states and the path of convergence of the solution trajectories starting from different initial population densities. In order to achieve this goal, we mostly ignore initial transient dynamics, that is the part of the trajectory in between the initial point and final destination (attractor). However, we mainly observe in nature the transient dynamics only due to the multi-time-scale variation of several factors responsible for shaping the dynamics of the interacting species under consideration.

Organization of this report are as follows: In Section 2, we describe the modified Rosenzweig-MacArthur model incorporated with multiplicative weak Allee effect, discuss the equilibrium points and perform linear stability analysis. In Section 3 we modify the model into slow-fast time-scale setting and describe the slow-fast dynamics exhibited by the model. We have used the blow-up technique to derive the singular Hopf bifurcation curve and maximal canard curve in a two parametric domain. The corresponding spatial model is considered in Section 4. Traveling wave solution in one-dimensional space and corresponding pattern in two-dimension space is discussed in the subsequent subsections. The transient dynamics observed in the model is discussed briefly in Section 5. We conclude our work in Section 6.

2. ODE Model

Three types of ODE models are available in the literature, Lotka-Volterra type model, Kolmogorov type model, and Gause type model for prey-predator interaction [6, 16], here we concentrate on the later one. If $u \equiv u(t)$ and $v \equiv v(t)$ denote the population density of the prey and the predator species, respectively, at time t, then their growth equations are two coupled nonlinear ODEs

$$\frac{du}{dt} = ur(u) - p(u,v)v, \ \frac{dv}{dt} = q(u,v)v - m(v)v, \tag{2.1a}$$

subject to non-negative initial conditions. $r(u)$ is per capita growth rate of the prey, $p(u,v)$ is the functional response (describes the grazing pattern of prey by the predator), $q(u,v)$ is numerical response (contribution of consumed prey biomass to the growth of the predator through the birth of new offsprings) and $m(v)$ is the per capita death rate of the predator. For mathematical simplicity, we consider $q(u, v) = ep(u, v)$ where $0 < e < 1$ is known as the conversion efficiency [10]. If $p(u, v)$ is function of u only then the functional response is called prey-dependent otherwise it is known as predator-dependent functional response without any ambiguity. From above formulation, per capita predator growth rate is $q(u, v) - m(v)$. For specialist predator, we must have $q(0, v) - m(v) < 0$ and in case of generalist predator $q(0, v) - m(v) > 0$ for a range of values of v and $-m(v)$ is interpreted as prey independent per capita growth rate of the generalist predator (interested readers can check the model in [18]).

First we consider the dynamics of a modified Rosenzweig-MacArthur type prey-predator model with multiplicative weak Allee effect (see [4] for details) in prey growth

$$\frac{du}{dt} = \gamma u(1-u)(u+\beta) - uv/(1+\alpha u) = f(u, v), \tag{2.2a}$$

$$\frac{dv}{dt} = uv/(1+\alpha u) - \delta v \equiv g(u, v). \tag{2.2b}$$

Comparing this model with (2.1), we find $r(u) = \gamma(1-u)(u+\beta)$, $p(u, v) = u/(1+\alpha u)$ and $m(v) = \delta$. The model is written in terms of dimensionless variables and parameters, all the parameters are positive. Weak Allee effect is characterised by $0 < \beta < 1$, α is a parameter involved with the Holling type II functional response and δ is intrinsic death rate of predators. These interpretations are in terms of dimensionaless parameters. The model admits at most three equilibrium points, (i) extinction equilibrium $E_0(0, 0)$, (ii) prey only equilibrium $E_1(1, 0)$ and (iii) interior equilibrium point $E_*(u_*, v_*)$. Components of E_* are given by

$$u_* = \delta/(1-\alpha\delta), \quad v_* = \gamma(1-u_*)(u_*+\beta)(1+\alpha u_*),$$

which is feasible under the parametric restriction $\delta(\alpha + 1) < 1$.

Stability of various equilibrium points can be studied with the help of linear stability analysis [10]. Here we mention the stability condition of the coexistence equilibrium point E_* only. Evaluating the Jacobian matrix for the model (2.2) at $E_*(u_*, v_*)$ we find

$$J_* = \begin{pmatrix} \gamma(u_*(2-3u-2\beta)+\beta) - \frac{v_*}{(1+\alpha u_*)^2} & -\frac{u_*}{1+\alpha u_*} \\ \frac{v_*}{(1+\alpha u_*)^2} & 0 \end{pmatrix}. \tag{2.3}$$

Using u_*, $v_* > 0$, we can easily verify that $\mathrm{Det}(J_*) > 0$. Hence E_* is stable if $\mathrm{Tr}(J_*) < 0$, loses stability through a super-critical Hopf bifurcation when $\mathrm{Tr}(J_*) = 0$ and is unstable for $\mathrm{Tr}(J_*) > 0$. The Hopf-bifurcation threshold in terms of $\delta \equiv \delta_H$ is given by

$$\delta_H = \frac{1+\alpha^2\beta - \sqrt{1+\alpha+\alpha^2-\alpha\beta+\alpha^2\beta+\alpha^2\beta^2}}{\alpha(-1-\alpha+\alpha\beta+\alpha^2\beta)}. \tag{2.4}$$

The coexistence equilibrium point E_* is stable for $\delta > \delta_H$ and it is surrounded by a stable limit cycle for $\delta < \delta_H$.

3. ODE Model with slow-fast time scale

Depending on the species traits, the growth and death rate of the species may vary on different timescales. Assuming that the prey population often grows much faster than its predator a small time-scale parameter ε, $0 < \varepsilon < 1$ can be introduced in the temporal model (2.2). The parameter ε is interpreted as the ratio between the growth rate of the predator and the growth rate of the prey [8, 17]. The assumption $\varepsilon < 1$ implies that one generation of predator can encounter several generations of prey [9, 12]. Therefore considering the difference in the time scale, the slow-fast version of the dimensionless model (2.2) can be written as

$$\frac{du}{dt} = f(u,v) = \gamma u(1-u)(u+\beta) - \frac{uv}{1+\alpha u}, \tag{3.1a}$$

$$\frac{dv}{dt} = \varepsilon g(u,v) = \varepsilon v\Big(\frac{u}{1+\alpha u} - \delta\Big), \tag{3.1b}$$

with initial conditions $u(0)$, $v(0) \geq 0$. Since the prey population grows faster compared to the predator, u and v are referred to as fast and slow variables, respectively and time t is called fast time. We now describe the dynamics of the slow-fast system (3.1) when $0 < \varepsilon \ll 1$. To understand the dynamics of the system for sufficiently small ε (> 0) we need to consider the behaviour of two subsystems obtained for $\varepsilon = 0$. The system in its singular limit, $\varepsilon = 0$ is given by

$$\frac{du}{dt} = f(u,v) = \gamma u(1-u)(u+\beta) - \frac{uv}{1+\alpha u}, \ \frac{dv}{dt} = 0. \tag{3.2a}$$

The above system is known as fast subsystem or layer system corresponding to the slow-fast system (3.1). The fast flow is given by constant predator density determined by the initial condition $v(0) = c$, and by integrating the differential equation

$$\frac{du}{dt} = \gamma u(1-u)(u+\beta) - \frac{uc}{1+\alpha u}, \tag{3.3}$$

with initial condition $u(0) > 0$. The direction of the fast flow depends on the choice of initial conditions $u(0)$, $v(0)$ and other parameter values. Green horizontal lines are fast solution trajectories with appropriate direction as shown in Fig. 1a.

Now writing system (3.1) in terms of the slow time $\tau := \varepsilon t$, we get the equivalent system in terms of slow time derivatives,

$$\varepsilon\frac{du}{d\tau} = f(u,v),\ \frac{dv}{d\tau} = g(u,v). \tag{3.4a}$$

Substituting $\varepsilon = 0$ in the above system we find the following differential algebraic system of equation,

$$0 = f(u,v),\ \frac{dv}{d\tau} = g(u,v), \tag{3.5a}$$

which is known as the slow subsystem corresponding to the slow-fast system (3.4). The solution of the above system is constrained to the set $\{(u,v) \in \mathbb{R}^2_+ : f(u,v) = 0\}$ known as critical manifold and in general is denoted by C_0. This set has one-one correspondence with the set of equilibrium of the fast subsystem (3.3). The critical manifold consists of two manifolds

$$C_0^0 = \{(u,v) \in \mathbb{R}^2_+ : u = 0, v \geq 0\},$$

$$C_0^1 = \Big\{(u,v) \in \mathbb{R}^2_+ : v = q(u) := \gamma(1-u)(u+\beta)(1+\alpha u),\ 0 < u < 1,\ v > 0\Big\},$$

such that $C_0 = C_0^0 \cup C_0^1$ where C_0^0 is the positive v-axis and C_0^1 is a portion of the cubic curve in the first quadrant shown in Fig. 1a, marked with black colour. The slow flow on the critical manifold is given by

$$\frac{du}{d\tau} = \frac{g(u,q(u))}{\dot{q}(u)}, \tag{3.6}$$

where '.' refers to the differentiation with respect to u. The solution of the system (3.1) for sufficiently small $\varepsilon > 0$ cannot be approximated from its limiting solution at $\varepsilon = 0$. The critical manifold C_0^1 can be divided into two

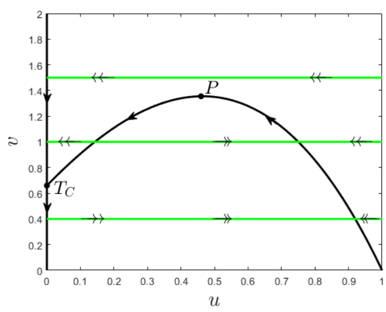


FIGURE 1. (a) Dynamics of the slow-fast system (3.1) where single arrow represnt slow flow and double arrow represent fast flow for $\delta = 0.36$ (magenta) and other parameters are $\alpha = 0.5$, $\beta = 0.22$, $\gamma = 3$, and $\delta = 0.3$.

parts (attracting and repelling sub-manifolds) by a non-degenerate fold point

$P(u_f, v_f)$ obtained from the following conditions [11]

$$\frac{\partial f}{\partial u}(u_f, v_f) = 0,\ \frac{\partial f}{\partial v}(u_f, v_f) \neq 0,\ \frac{\partial^2 f}{\partial u^2}(u_f, v_f) \neq 0,\ \text{and } g(u_f, v_f) \neq 0.$$

The components of the fold point, for the model under consideration, is given by

$$u_f = \frac{(\alpha - \alpha\beta - 1) + (1 + \alpha + \alpha^2 - \alpha\beta + \alpha^2\beta + \alpha^2\beta^2)^{\frac{1}{2}}}{3\alpha},$$
$$v_f = \gamma(1 - u_f)(u_f + \beta)(1 + \alpha u_f).$$

and the attracting ($C_0^{1,a}$) and repelling ($C_0^{1,r}$) sub-manifold is given by

$$C_0^{1,a} = \Big\{(u, v) \in \mathbb{R}_+^2 : v = q(u), u_f < u \leq 1\Big\}$$
$$C_0^{1,r} = \Big\{(u, v) \in \mathbb{R}_+^2 : v = q(u), 0 \leq u < u_f\Big\}.$$

The point of intersection of C_0^1 with the vertical v-axis is $T_C(0, \beta\gamma)$, which is the transcritical bifurcation point for the fast subsystem. C_0^1 can be written explicitly as $v = q(u)$, and it follows from Fenichel's theorem [5, 12], that for ε small enough there exist locally invariant slow sub-manifolds C_ε^1 and C_ε^0 which are diffeomorphic to the respective critical manifolds C_0^1 and C_0^0, except at the non-hyperbolic points P and T. The system is singular at these two non-hyperbolic points. However, when the fold point P coincides with the interior equilibrium of the system (3.1), thus satisfying $g(u_f, v_f) = 0$ along with the conditions of the fold point, there exists a solution passing through the vicinity of the fold point P. With this condition, the fold point is known as canard point, and the corresponding solution trajectory is called canard solution. Therefore, to study the dynamics in the vicinity of the point P, we blow up the non-hyperbolic fold point into $S^3 = \{x \in \mathbb{R}^4 : ||x|| = 1\}$ to remove this singularity. To simplify our calculations we first consider a topologically equivalent form of the system (3.1) by re-scaling the time with the help of a transformation $dt \to (1 + \alpha u)dt$ [13], where $(1 + \alpha u) > 0$ as follows

$$\begin{aligned} \frac{du}{dt} &= \gamma u(1 - u)(u + \beta)(1 + \alpha u) - uv \equiv F(u, v, \delta), \\ \frac{dv}{dt} &= \varepsilon\left(uv - \delta v(1 + \alpha u)\right) \equiv \varepsilon G(u, v, \delta). \end{aligned} \tag{3.7}$$

Using the transformation $U = u - u_*,\ V = v - v_*,\ \lambda = \delta - \delta_*$, the system reduces to the required slow-fast normal form near $(0, 0)$ as follows

$$\begin{aligned} \frac{dU}{dt} &= -Vh_1(U, V) + U^2 h_2(U, V) + \varepsilon h_3(U, V), \\ \frac{dV}{dt} &= \varepsilon\left(Uh_4(U, V) - \lambda h_5(U, V) + Vh_6(U, V)\right), \end{aligned} \tag{3.8}$$

where

$$h_1(U,V) = u_* + U, \quad h_3(U,V) = 0, \quad h_5(U,V) = (v_* + V)(1 + \alpha u_*) + U v_* \alpha,$$

$$h_2(U,V) = -\gamma(-1+6u_*^2\alpha+3u_*(1+\alpha(\beta-1))+\beta-\alpha\beta) - U\gamma(1+\alpha(4u_*+\beta-1)),$$

$$h_4(U,V) = (v_* + V)(1 - \alpha\delta_*), \quad h_6(U,V) = u_* - (1 + u_*\alpha)\delta_*.$$

We now apply a geometric transformation by which the non-hyperbolic fold point is "blown up" to a 3-sphere, $S^3 = \{(\bar{U},\bar{V},\bar{\lambda},\bar{\varepsilon}) \in \mathbb{R}^4 : \bar{U}^2+\bar{V}^2+\bar{\lambda}^2+\bar{\varepsilon}^2 = 1\}$ known as blow-up space [11]. Let $\mathcal{I} := [0,\rho]$ where $\rho > 0$ is a small constant and $\bar{r} \in \mathcal{I}$. We define a manifold $\mathcal{M} := S^3 \times \mathcal{I}$ and the blow-up map, $\Phi : \mathcal{M} \to \mathbb{R}^4$ where

$$\Phi(\bar{U},\bar{V},\bar{\lambda},\bar{\varepsilon},\bar{r}) = (\bar{r}\bar{U}, \bar{r}^2\bar{V}, \bar{r}\bar{\lambda}, \bar{r}^2\bar{\varepsilon}) := (U,V,\lambda,\varepsilon). \tag{3.9}$$

To study the dynamics of the transformed system on and around the hemisphere $S^3_{\varepsilon\geq 0}$ we will introduce the charts with direction blow-up maps [12]. The slow-fast normal form (3.8) can be extended to $\mathbb{R}^4$ and with the help of above blow-up map we transform the system as explained in [3]. Along each direction of the coordinate axis we define the charts K_1, K_2, K_3 and K_4 by setting $\bar{V} = 1$, $\bar{\varepsilon} = 1$, $\bar{U} = 1$ and $\bar{\lambda} = 1$ respectively. The charts K_1 and K_3 describe the dynamics in the neighborhood of the equator of S^3 and K_2 describes the dynamics in a neighborhood of the positive hemisphere. Here, we mainly focus on chart K_2 to prove the existence of a periodic solution for $0 < \bar{\varepsilon} \ll 1$. On chart K_2, that is, for $\bar{\varepsilon} = 1$ the above blow-up transformation reduces to

$$\bar{r} = \sqrt{\varepsilon}, \; U = \sqrt{\varepsilon}\bar{U}, \; V = \varepsilon\bar{V}, \; \lambda = \sqrt{\varepsilon}\bar{\lambda}, \tag{3.10}$$

and the transformed system is given as follows, here we remove the overbars without any loss of generality,

$$\begin{aligned} U_t &= -b_1 V + b_2 U^2 + \sqrt{\varepsilon}\mathcal{G}_1(U,V) + O(\sqrt{\varepsilon}(\lambda + \sqrt{\varepsilon})), \\ V_t &= b_3 U - b_4\lambda + \sqrt{\varepsilon}\mathcal{G}_2(U,V) + O(\sqrt{\varepsilon}(\lambda + \sqrt{\varepsilon})), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} &b_1 = u_*, \quad b_2 = -\gamma(-1 + 6u_*^2\alpha + 3u_*(1 + \alpha(\beta - 1)) + \beta - \alpha\beta), \\ &b_3 = v_*(1 - \alpha\delta_*), \quad b_4 = v_*(1 + \alpha u_*), \end{aligned} \tag{3.12}$$

and

$$\mathcal{G}_1(U,V) = a_1 U - a_2 UV + a_3 U^3, \quad \mathcal{G}_2(u,V) = a_4 U^2 + a_5 V. \tag{3.13}$$

Theorem 3.1 *Let* $(U,V) = (0,0)$ *be the canard point of the transformed system (3.8) at* $\lambda = 0$ *such that* $(0,0)$ *is a folded singularity and* $G(0,0,0) = 0$*. Then for sufficiently small* ε *there exist a singular Hopf bifurcation curve* $\lambda = \lambda_H(\sqrt{\varepsilon})$ *such that the equilibrium point of the system (3.8) is stable for*

$\lambda > \lambda_H(\sqrt{\varepsilon})$ *with*

$$\lambda_H(\sqrt{\varepsilon}) = -\frac{b_3(a_1 + a_5)}{2b_2 b_4}\varepsilon + O(\varepsilon^{3/2}). \tag{3.14}$$

Proof. Let us denote the equilibrium point of the system (3.11) is (U_e, V_e), $U_e = \frac{b_4 \lambda}{b_3} + O(2)$ and $V_e = O(2)$ where $O(2) := O(\lambda^2, \lambda\sqrt{\varepsilon}, \lambda)$. Linearizing the system about this equilibrium point we have the Jacobian matrix as

$$\mathcal{J} := \begin{pmatrix} 2U_e b_2 + a_1\sqrt{\varepsilon} + O(2) & -b_1 + O(2) \\ b_3 + O(2) & a_5\sqrt{\varepsilon} + O(2) \end{pmatrix} \tag{3.15}$$

At the Hopf bifurcation we have Trace $\mathcal{J} = 0$ which implies

$$\frac{2b_2 b_4 \lambda}{b_3} + \sqrt{\varepsilon}(a_1 + a_5) + O(2) = 0. \tag{3.16}$$

and applying the blow-down map $\lambda_{\mathcal{H}} = \lambda\sqrt{\varepsilon}$ we get the singular Hopf bifurcation curve $\lambda_{\mathcal{H}}(\sqrt{\varepsilon})$ for the slow-fast normal form (3.8) as

$$\lambda_{\mathcal{H}}(\sqrt{\varepsilon}) = -\frac{b_3(a_1 + a_5)}{2b_2 b_4}\varepsilon + O(\varepsilon^{3/2}). \tag{3.17}$$

The singular Hopf bifurcation curve for the system (3.7) is thus given by

$$\begin{aligned} \delta_H(\sqrt{\varepsilon}) =& \frac{1 + \alpha^2\beta - \sqrt{1 + \alpha + \alpha^2 - \alpha\beta + \alpha^2\beta + \alpha^2\beta^2}}{\alpha(-1 - \alpha + \alpha\beta + \alpha^2\beta)} - \\ & \frac{b_3(a_1 + a_5)}{2b_2 b_4}\varepsilon + O(\varepsilon^{3/2}). \end{aligned} \tag{3.18}$$

In Fig. 2 the singular Hopf bifurcation curve (red) is plotted in $\delta - \varepsilon$ parametric plane, it clearly explains how the singular Hopf-bifurcation threshold changes with the variation in ε. Once the coexistence equilibrium loses stability through Hopf bifurcation, at the Canard point, we find a closed orbit as attractor surrounding the unstable equilibrium point. From this point small amplitude stable canard cycle originates enclosing the point P and then forms canard cycle with head depending on the parameter values. The following theorem provides an anlaytical expression of the maximal canard curve in $(\lambda - \varepsilon)$ plane.

Theorem 3.2 *Let* $(U, V) = (0, 0)$ *be the canard point of the slow-fast normal form (3.8) at* $\lambda = 0$ *such that* $(0, 0)$ *is a folded singularity and* $G(0, 0, 0) = 0$. *Then for* $\varepsilon > 0$ *sufficiently small there exists maximal canard curve* $\lambda = \lambda_c(\sqrt{\varepsilon})$, *i.e. the parametric curve of maximal canard solution such that the slow flow on the normally hyperbolic invariant submanifolds* $\mathcal{M}_\varepsilon^{1,a}$ *connects with* $\mathcal{M}_\varepsilon^{1,r}$ *in the blow-up space. And* $\lambda_c(\sqrt{\varepsilon})$ *is given by*

$$\lambda_c(\sqrt{\varepsilon}) = -\frac{1}{A_5}\Big(\frac{3A_1}{4A_4^2} + \frac{A_2}{2A_4} + A_3\Big)\varepsilon + O(\varepsilon^{3/2}) \tag{3.19}$$

Proof. Please refer to [3] for the proof.

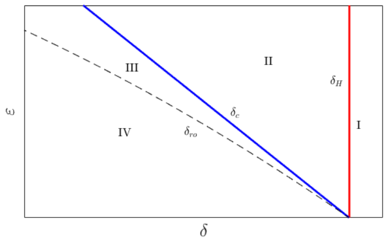


FIGURE 2. Schematic diagram showing singular Hopf bifurcation curve $\delta_{\mathcal{H}}$ (red), maximal canard curve δ_c (blue), and δ_{ro} (dashed black) relaxation oscillation cycle.

4. PDE MODEL: SPATIAL PATTERNS

Considering the heterogeneous distribution of both the species and assuming their random movement within the habitat, we can extend the temporal model (3.1) to a spatio-temporal model by incorporating self-diffusion terms as follows,

$$u_t = \gamma u(1-u)(u+\beta) - \frac{uv}{1+\alpha u} + \nabla^2 u, \tag{4.1a}$$

$$v_t = \varepsilon\left(\frac{uv}{1+\alpha u} - \delta v\right) + d\nabla^2 v, \tag{4.1b}$$

where $d\,(>0)$ is the ratio of diffusivity coefficients of predator over prey and ∇^2 is the Laplacian operator. Here $u \equiv u(t,\mathbf{x})$ and $v \equiv v(t,\mathbf{x})$ denote prey and predator densities, respectively, at time t and at spatial location $\mathbf{x}$. For one dimensional (1D) space $\mathbf{x} \equiv x \in \mathbb{R}$ and $\mathbf{x} \equiv (x,y) \in \mathbb{R}^2$ for two dimensional (2D) space. For simplicity we assume that $\mathbf{x}$ belongs to a bounded domain. The model under consideration fails to produce any stationary pattern due to Turing instability as the Jacobian matrix of the model (3.1) evaluated at E_* fails to satisfy the requisite sign pattern necessary to satisfy the Turing instability condition [1]. In other words, if we denote the entries of the Jacobian matrix (2.3) by j_{rs}, $(r,\, s = 1,\, 2)$ then j_{11} and j_{22} must have opposite signs at the coexistence equilibrium point such that the conditions $j_{11}+j_{22}$ and $dj_{11}+j_{22}$ are satisfied simultaneously for $d>0$. For the model under consideration, $j_{22}=0$ and hence Turing instability conditions can not be satisfied. However, the model (4.1) is capable to produce some dynamic patterns. In the context of PDE model of prey-predator interaction with self-diffusion terms, the dynamic patterns include travelling wave, periodic travelling wave and spatio-temporal chaos [2, 14]. With the help of mathematical tools one can prove the existence of travelling wave, stability of travelling wave and determine the minimum wave speed [15]. However, there is a lack of mathematical machinery which can establish the existence of spatio-temporal chaos rather its detection solely depends upon exhaustive numerical simulation for parameter values chosen from the Hopf domain or Turing-Hopf domain [2].

4.1. **1D travelling wave.** Here we consider the existence of traveling wave joining the predator free steady-state E_1 with the coexistence steady-state E_* whenever the parametric restriction $\delta(\alpha+1) < 1$ holds. Existence of such traveling wave implies the successful invasion of the predators and this requirement leads to the consideration of the model (4.1) subject to the conditions

$$\begin{aligned} u(t,x) = 1, \text{ and } \ v(t,x) = 0, \text{ as } x \to -\infty, \ \forall\, t, \\ u(t,x) = u_*, \text{ and } \ v(t,x) = v_*, \text{ as } x \to \infty, \ \forall\, t. \end{aligned}$$

Let us consider the traveling wave solution of the model (4.1) in the form $u(t,x) = \phi(\xi)$, $v(t,x) = \psi(\xi)$ where $\xi = x - ct$ and c is the speed of travelling wave. Substituting $u = \phi(\xi)$ and $v = \psi(\xi)$ in (4.1), we find

$$\frac{d^2\phi}{d\xi^2} + c\frac{d\phi}{d\xi} + f(\phi,\psi) = 0, \ d\frac{d^2\psi}{d\xi^2} + c\frac{d\psi}{d\xi} + \varepsilon g(\phi,\psi) = 0. \tag{4.2}$$

We introduce two new variables $p(\xi) = -\frac{d\phi}{d\xi}$ and $q(\xi) = -\frac{d\psi}{d\xi}$ to transform the system (4.2) to the first order coupled ODEs as follows

$$\begin{aligned} \frac{d\phi}{d\xi} &= -p, \ \frac{dp}{d\xi} = -cp + f(\phi,\psi), \\ \frac{d\psi}{d\xi} &= -q, \ \frac{dq}{d\xi} = \frac{1}{d}(-cq + \varepsilon g(\phi,\psi)). \end{aligned} \tag{4.3}$$

The homogeneous steady states E_1 and E_* of (4.1) corresponds to the equilibrium points of the system (4.3) given by $Q_1(1,0,0,0)$ and $Q_*(u_*,0,v_*,0)$ respectively. Successful invasion of the predator is reflected through the existence of a heteroclinic connection joining Q_1 with Q_*. The Jacobian matrix of the system (4.3) evaluated at Q_1

$$J_{Q_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\gamma(1+\beta) & -c & -\frac{1}{1+\alpha} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\varepsilon}{d}(\frac{1}{1+\alpha} - \delta) & -\frac{c}{d} \end{pmatrix}, \tag{4.4}$$

has eigenvalues $\lambda_{1,2} = \frac{c}{2} \pm \frac{\sqrt{c^2+4g(1+\beta)}}{2}$ and $\lambda_{3,4} = -\frac{c(1+\alpha)}{2} \pm \frac{\sqrt{\Gamma}}{d(1+\alpha)}$ with $\Gamma = (1+\alpha)^2c^2 - 4\varepsilon d(1+\alpha)(1-\delta-\alpha\delta)$. First two eigenvalues are real, but $\lambda_{3,4}$ are real whenever $c^2 \geq \frac{4\varepsilon d(1-\delta-\alpha\delta)}{1+\alpha}$. Feasible heteroclinic connection exist for four real eigenvalues only. The minimum wave speed of traveling wave emerging from Q_1 is given by

$$c_{min} = \Big[\frac{4\varepsilon d(1-\delta-\alpha\delta)}{1+\alpha}\Big]^{1/2}. \tag{4.5}$$

Clearly c_{min} depends on ε and hence the slow-fast time scale regulates the speed of traveling wave. It is difficult to obtain the explicit expressions for the eigenvalues of the Jacobian matrix evaluated at Q_* and hence we provide below numerical example. For $c \geq c_{min}$ and depending upon the nature of eigenvalues of J_{Q_*} (real or complex), we find monotone traveling wave, non-monotone as well as periodic traveling waves.

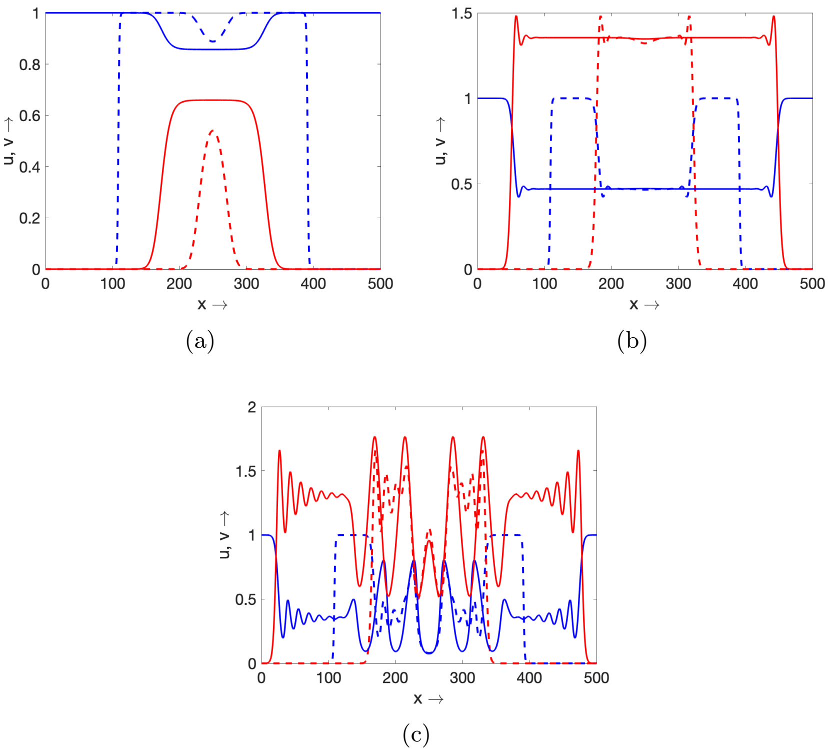
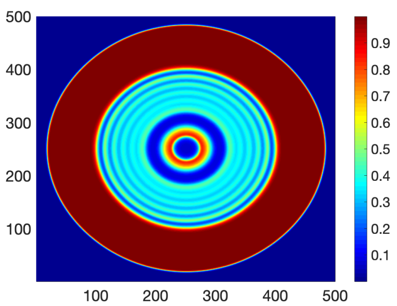


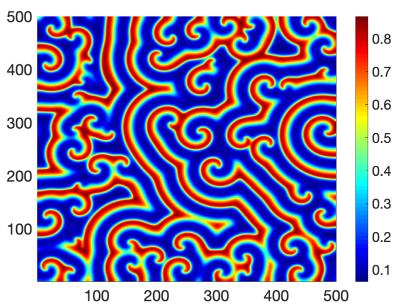
FIGURE 3. (a) $\delta = 0.6$, monotone traveling wave, (a) $\delta = 0.38$, non-monotone traveling wave, (a) $\delta = 0.3$, periodic traveling wave. Prey (blue) and predator (red) distributions are shown at $t = 100$ (dashed) and $t = 200$ (continuous).

For numerical illustration, we choose fixed parameter values $\alpha = 0.5$, $\beta = 0.22$, $\gamma = 3$, $\varepsilon = 1$, $d = 1$ and vary δ. Using (4.5) one can verify that traveling wave exists for $\delta < 2/3$. All the four eigenvalues of J_{Q_*} are real for $0.52 < \delta < 0.667$ and we find two complex conjugate eigenvalues for $\delta \leq 0.52$. Complex conjugate eigenvalues with negative real parts imply the existence of non-monotone traveling wave joining E_1 and E_* whereas positive real part correspond to periodic traveling wave. Monotone, non-monotone and periodic traveling waves are shown in Fig. 3 for three different values of δ as mentioned at the caption of the figure.

4.2. **Pattern formation in 2D.** The model (4.1) produces dynamic patterns which are partly analogous to the patterns which we have explained above. In this subsection we just highlight the two-dimensional interacting spiral pattern produced by the model for same set of parameters as above but a specific choice of initial condition. We have used Euler scheme for the simulation of temporal part and five point finite difference scheme for the Laplacian with no-flux boundary condition along the boundary. As initial condition, we assume that prey and predator population is zero everywhere except a small circular domain at the center. Within a circular domain at the center, the prey is at its carrying capacity and some non-zero predator population in introduced within a smaller circular patch at the center of the domain. Then it evolves like a periodic traveling wave at the core and large circular plateau with prey population at the carrying capacity (=1) as initial transient pattern (see Fig. 4a). Once the periodic travelling wave hits the boundary, the periodic wave structure is destroyed and we find interacting spiral pattern which is chaotic in space as well as time. In order to break the spatial symmetry, we have used a specific type of initial condition as described in the book [14]. This interacting spiral pattern (see Fig. 4b) persists as a dynamic pattern, neither they goes to complete extinction nor settle down to any stationary pattern.



(a)

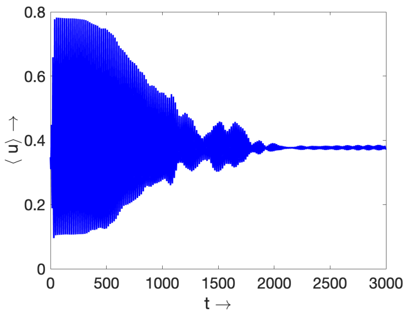


(b)

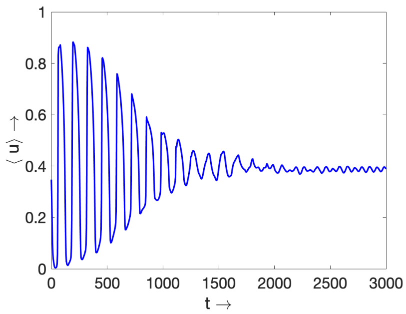
FIGURE 4. Two dimensional spatio-temporal pattern at two different time (a) $t = 1400$, (b) $t = 30000$, for prey population density. Parameter values are same as in Fig. 3c.

5. Transient dynamics

As observed in the previous sections, the slow-fast systems shows variety of interesting dynamics that are believed to be more ecologically justifiable. In particular, one of the crucial and recently observed phenomenon is the presence of long transient in slow-fast systems. There are many factors that can influence the duration and nature of the transients which includes system properties, the background bifurcation parameter, the initial state of the system, etc. Consideration of spatio-temporal model with slow-fast time scale can significantly alter the transient time and its duration increases with dimension and size of the domain considered [7]. In the ODE model, we can observe long transient along the critical manifold where the dynamics of the system is slow and there exists long period of stasis and rapid oscillations. However, interestingly, in the corresponding spatio-temporal system, due to the presence of multiple timescale, the nature of the transient gets more regularized. The number of maxima and minima obtained for the model (4.1) for $\varepsilon = 1$ is significantly more than that obtained for $\varepsilon = 0.1$ (cf. Fig. 5). For instance, the number of complete prey-predator cycles from time $t = 0$ to $t = 2000$ decreases with the decrease in magnitude of ε. The time evolution of spatial average of the prey population corresponding to the pattern presented in Fig. 4 is depicted in Fig. 5a, and the time evolution of prey population for $\varepsilon = 0.1$ is presented in Fig. 5b. As discussed in the previous section, in case of 1D spatial component, the speed of the travelling wave decreases for $0 < \varepsilon \ll 1$, thus altering the transient behaviour of the waves. For further details, interested readers can refer to [3].



(a)



(b)

FIGURE 5. Time evolution of spatial average of prey population density against time for two different values of ε, (a) $\varepsilon = 1$, (b) $\varepsilon = 0.1$.

6. DISCUSSIONS

Systematic study of transient dynamics have received some attention from the researchers but the mathematical condition ensuring the onset of transient dynamics is yet to be investigated. So far as our knowledge goes, our recent work [3] is the only available literature which discussed in detail the effect of slow-fast time scale on spatio-temporal pattern formation for interacting population model. We have shown that the presence of slow-fast time scale can influence the transient dynamics as well as spatio-temporal pattern and also capable to stabilize the spatio-temporal chaotic oscillation. However, its effect on stationary pattern remains an open area of further investigation. Our future goal is to explore canard explosion and relaxation oscillation in the context of self-organizing spatial pattern.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 45–54

RAMANUJAN GRAPHS*

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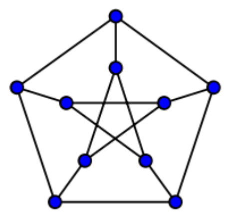
Abstract. We give a gentle introduction to the construction of infinite families of bipartite Ramanujan graphs of all regularities following the work of Marcus, Spielman and Srivastava.

The goal of this expository note is an introduction to the beautiful ideas of Marcus, Spielman and Srivastava in [MrcSplSrv2015] on interlacing polynomials and their application to the construction of infinite families of bipartite Ramanujan graphs of all regularities. We will begin with absolute preliminaries and build up to their main result.

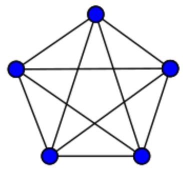
For our purposes, a graph G consists of finite sets V of vertices and E of edges. Each element of E is an unordered pair of vertices. Thus all graphs will be finite, unoriented and simple - no multiple edges and no loops at vertices. An example to keep in mind is the famous Petersen graph with 10 vertices and 15 edges.



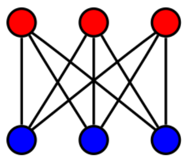
One pleasant feature of the Petersen graph is that it is 3-regular, i.e., each of its vertices has degree 3. We will mainly be interested in d-regular graphs for arbitrary d. A simple example of a d-regular graph, for a general

* This article is based on the text of the 32nd V. Ramaswami Aiyer Memorial Award Lecture delivered by the author at the 87th Annual Conference of the IMS - An International meet held at MGM University, Aurangabad during December 4 - 7, 2021 using online mode

d, is the complete graph on $d+1$-vertices with $\binom{d+1}{2}$ edges - illustrated in the next figure for $d = 4$.



The graphs of interest to us will further be bipartite graphs. These are graphs where the vertex set comes with a partition into two subsets so that there are no edges between vertices in the same subset. Equivalently, there are no odd cycles in the graph. Here is the complete bipartite graph on $(3, 3)$ vertices.



The notion of a Ramanujan graph is a spectral one, i.e., it depends on the spectrum of the graph. Recall that to any graph G is associated its adjacency matrix A, which is a square matrix with rows and columns indexed by vertices of G and (v, w)-entry being 1 if $\{v, w\}$ is an edge of G and 0 otherwise. This is a certain real symmetric matrix. The spectrum of G is defined to be the spectrum of this matrix, namely, the set of eigenvalues of A. From linear algebra the eigenvalues of A are real and if G has n-vertices, these can be arranged in non-increasing order as $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}$. An equivalent way of saying that the eigenvalues of A are real is that the characteristic polynomial of A is real-rooted. It is important to note this since real-rooted polynomials and their generalisations play an important role in this story.

The spectrum of the Petersen graph, for instance, can be calculated to be the set $\{3, 1, 1, 1, 1, 1, -2, -2, -2, -2\}$, or equivalently, the characteristic polynomial of its adjacency matrix is $(x-3)(x-1)^5(x+2)^4$. The adjacency matrix of the complete graph on 5 vertices is $(J_5 - I_5)$ where J_5 is the all-entries-1 matrix. The matrix J_5 clearly has minimal polynomial $x^2 - 5x$ and hence eigenvalues 0 and 5 with 0 having multiplicity 4 (since J is of rank 1) and so the spectrum of K_5 is $\{4, -1, -1, -1, -1\}$. Equivalently the characteristic polynomial of its adjacency matrix is $(x-4)(x+1)^4$. The complete bipartite graph on $(3, 3)$ vertices has adjacency matrix

$$A = \begin{bmatrix} 0 & J_3 \\ J_3 & 0 \end{bmatrix}$$

This has minimal polynomial $x^3 - 9x$ and hence eigenvalues ± 3 and 0 with 0 having multiplicity 4 (since A is of rank 2). Therefore the characteristic polynomial of the adjacency matrix in this case is $x^4(x-3)(x+3)$.

These examples illustrate an easily proved result in the spectral theory of graphs - if a graph G is d-regular, then $\mu_0 = d$ and has multiplicity 1 iff it is connected and in this case, $\mu_{n-1} \geq -d$ with equality iff it is bipartite, in which case, the spectrum is symmetric about the origin. The eigenvalues d and $-d$ (when it occurs) are called the trivial eigenvalues of G and the difference $d - \mu_1$ is called the spectral gap of G. It can be shown that the spectral gap is a good measure of how well-connected G is and so, from the point of view of applications, it turns out to be important to be able to construct graphs with large spectral gap.

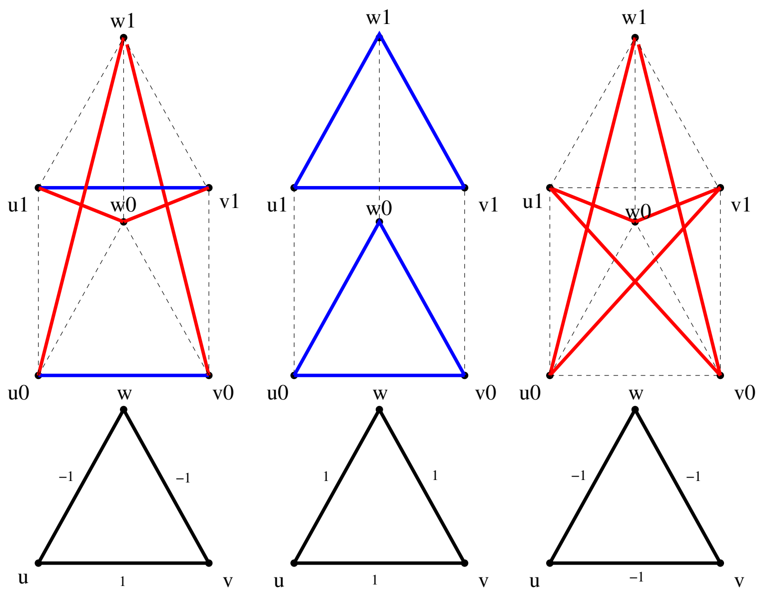
For a fixed $d \geq 3$, a family of connected d-regular graphs $G_1, G_2, G_3, \cdots$ is said to be family of expanders if $|V_n| \to \infty$ and $d - \mu_1(G_n) \geq \epsilon$ for some positive ϵ, i.e., the spectral gap of these graphs is bounded away from 0. We would like ϵ to be as large as possible for good expanders. But there are limits on how large ϵ can be for such a family to exist. A non-trivial result of Alon and Boppana - see [Nli1991] - asserts that for a sequence of connected d-regular graphs $G_1, G_2, G_3, \cdots$ with $|V_n| \to \infty$, the inequality $\underline{lim}\ \mu_1(G_n) \geq 2\sqrt{d-1}$ holds. Thus, for any number $\alpha < 2\sqrt{d-1}$, all but finitely many $\mu_1(G_n) \geq \alpha$. A little thought then shows that the number of connected d-regular graphs G with $\mu_1(G) < \alpha$ is finite.

This motivates the definition of a Ramanujan graph. A connected d-regular graph is said to be a Ramanujan graph if every non-trivial eigenvalue is at most $2\sqrt{d-1}$ in modulus. The Petersen graph, the complete graph and the complete bipartite graph are all examples of Ramanujan graphs. Observe that for a bipartite graph G, being a Ramanujan graph is equivalent to satisfying $\mu_1(G) \leq 2\sqrt{d-1}$ by symmetry of the spectrum about the origin. A family $G_1, G_2, G_3, \cdots$ of d-regular Ramanujan graphs with $|V_n| \to \infty$ thus gives an optimal family of expanders.

Until the work of Marcus, Spielman and Srivastava, all known constructions of infinite families of Ramanujan graphs of fixed degree involved deep number theory - hence, in fact, the name Ramanujan graphs due to Lubotzky, Phillips and Sarnak in [LbtPhlSrn1988]. Another construction was due to Margulis - see [Mrg1988]. Besides they could only produce Ramanujan graphs of regularity $p^n + 1$ with p being a prime. The appearance of $p^n + 1$ might lead one to guess that these constructions involve finite fields and indeed, this guess is correct.

To proceed further, we need to understand a process of doubling graphs called 2-lifting. Suppose that $G = (V, E)$ is a graph. A two lift of G is a graph $\hat{G} = (\hat{V}, \hat{E})$ with the following properties. For every vertex $v \in V$ there are two vertices $v_0, v_1 \in \hat{V}$. For every edge $\{v, w\} \in E$ there are two edges in $\hat{E}$. These are either the pairs $\{v_0, w_0\}, \{v_1, w_1\}$ (which we will refer to as uncrossed edges) or the pairs $\{v_0, w_1\}, \{v_1, w_0\}$ (which we will refer to as crossed edges).

Here are three examples of lifts of the complete graph on 3-vertices - the triangle graph.



If only uncrossed edges are used the resulting graph is two copies of the original graph while if only crossed edges are used the resulting graph is called the double cover of the original graph. Using a little more terminology, a 2-lift is a 2-fold covering space of the graph.

Note that if $\hat{G}$ is a 2-lift of G, then the degree of any vertex v of G is the same as those of both v_0 and v_1. In particular, if G is d-regular, then so is $\hat{G}$. Also, it can be seen that if G is bipartite, then so is $\hat{G}$.

How many 2-lifts are there of a given graph $G = (V, E)$? It is clear that for every edge of G we need to make a choice of what edges to include in $\hat{G}$ - either the uncrossed ones or the crossed ones. Thus the number of 2-lifts of G is 2^m, where $m = |E|$. We will index the 2-lifts of G by signings of G where a signing is a map $s : E \to \{\pm 1\}$. These correspond to 2-lifts in the following way. If $s(e) = 1$ we take the uncrossed edges above e while if $s(e) = -1$, we take the crossed edges.

Here is a very simple minded idea to construct large Ramanujan graphs with fixed regularity - if it works! Begin with one Ramanujan graph. Suppose we can show that one of its 2-lifts is also a Ramanujan graph. Then we iterate this procedure and we are done. To actually implement this procedure we need to understand the spectrum of a 2-lift of a graph.

This was done by Bilu and Linial in [BluLnl2006]. To a signing s of G, associate the signed adjacency matrix A_s whose (v, w)-entry is $s(v, w)$ (instead of 1 as in the adjacency matrix) if $\{v, w\}$ is an edge of G and 0 otherwise. They show (and it is not too difficult) that for the corresponding 2-lift $\hat{G}_s$, the spectrum is the union of the spectra of A and of A_s. They also conjectured that for a d-regular graph, some signing has A_s with spectrum contained in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. If true, this would immediately imply that the corresponding 2-lift is a Ramanujan graph provided the initial one is. Marcus, Speilman and Srivastava prove this conjecture for bipartite graphs using an ingenious argument related to the probabilistic method.

This brings us to the next idea in the development of this proof. A matching in a graph $G = (V, E)$ is a subset of E no two elements of which share a common vertex. Let m_i be the number of matchings with i elements - set $m_0 = 1$. The matching polynomial of G is defined by the equation:

$$\mu_G(x) = \sum_{i \geq 0} (-1)^i m_i x^{n-2i},$$

where n is the number of vertices of G. This is a monic polynomial of degree n, just as the characteristic polynomial of the adjacency matrix of G is.

This was defined by Heilmann and Lieb in [HlmLeb1972] who showed that $\mu_G(x)$ is real-rooted and further, that if the maximal degree of a vertex of G is d, then all roots of $\mu_G(x)$ are at most $2\sqrt{d-1}$ in modulus. Later, Godsil and Gutman in [GdsGtm1981] showed that the matching polynomial is the average of the characteristic polynomials of all A_s over all signings s of G. Thus, if $f_s(x) = det(xI - A_s)$ denotes the characteristic polynomial of A_s, then

$$\mu_G(x) = \frac{1}{2^m} \sum_{s \in \{\pm\}^m} f_s(x).$$

The Bilu-Linial conjecture is the assertion that some $f_s(x)$ has all roots at most $2\sqrt{d-1}$ in modulus. The Heilmann-Lieb and Godsil-Gutman results imply that their average, which is $\mu_G(x)$, has this property. But what information can be obtained about the roots of individual polynomials from a knowledge of the roots of their average? In general, not very much, unless something special happens, and that is exactly the phenomenon of interlacing which is the next topic we will consider.

We will be interested, in the rest of this note, in real rooted polynomials with positive leading coefficients. Suppose that f is a real rooted polynomial of degree n with roots $\beta_1, \cdots, \beta_n$ and g is a real rooted polynomial of degree $n-1$ with roots $\alpha_1, \cdots, \alpha_{n-1}$. We say that g interlaces f if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n.$$

This can be shown to be equivalent to several other conditions on f and g. One is that in the partial fraction expansion

$$\frac{g(x)}{f(x)} = \sum_i \frac{c_i}{x - \gamma_i},$$

all the c_i are positive. Another, related to real-rootedness, is that $f(x) + \lambda g(x)$ is real-rooted for all real λ.

The study of interlacing has a long history going back to Cauchy and Hermite. One of the beautiful results in this area is the Cauchy interlacing theorem which states that for a real symmetric matrix A, if B is the submatrix of A obtained by deleting its first row and column, then, the characteristic polynomial of B interlaces that of A. Given the equivalent characterisations of interlacing of the previous paragraph, Cauchy's interlacing theorem has a very brief proof discovered by Fisk - see [Fsk2005] - that reads as follows. Write A as

$$\begin{bmatrix} \rho & u^T \\ u & B \end{bmatrix},$$

for some real ρ and vector u. Linearity of the determinant implies that for arbitrary real λ and indeterminate x,

$$\begin{vmatrix} x+\lambda-\rho & -u^T \\ -u & xI-B \end{vmatrix} = \begin{vmatrix} x-\rho & -u^T \\ -u & xI-B \end{vmatrix} + \begin{vmatrix} \lambda & 0 \\ -u & xI-B \end{vmatrix}.$$

Thus with C denoting the matrix

$$\begin{bmatrix} \rho-\lambda & u^T \\ u & B \end{bmatrix},$$

we have that $\chi_C = \chi_A + \lambda\chi_B$. Since χ_C, χ_B, χ_A are all real-rooted, the equivalent characterisations of interlacing show that $\chi_B(x)$ interlaces $\chi_A(x)$, as was to be seen.

We say that real-rooted polynomials $f_1, \cdots, f_k$ - all of the same degree - have a common interlacing, if there is a single g that interlaces each of them. Again, this is equivalent to a real-rootedness condition - namely, that every convex combination of the $f_1, \cdots, f_k$ is real-rooted.

Here is a very simple but extraordinarily useful observation. If $f_1, \cdots, f_k$ have a common interlacing and f_ϕ is their sum, then there is an i for which the largest root of f_i is at most the largest root of f_ϕ. To prove this, take a common interlacing polynomial g and say its largest root is α_{n-1}. Now consider any f_i. It can't be positive at α_{n-1} since it has only one root that is greater than or equal to α_{n-1} and is eventually positive. Hence $f_\phi(\alpha_{n-1}) \leq 0$. Since f_ϕ is also eventually positive, its largest root, say β_n is greater than or equal to α_{n-1}. Since $f_\phi(\beta_n) = 0$, some $f_i(\beta_n) \geq 0$. But $f_i(\alpha_{n-1}) \leq 0$ and f_i has exactly one root (its largest root) greater than or equal to α_{n-1} this largest root necessarily lies in $[\alpha_{n-1}, \beta_n]$.

Our polynomials f_s are indexed by $s \in \{\pm 1\}^n$ so we need an inductive generalisation of the observation above. It is a little messy to state but easy to prove. First we need a definition. Let $S_1, \cdots, S_m$ be finite sets and for each $(s_1, \cdots, s_m) \in S_1 \times \cdots \times S_m$, let $f_{(s_1, \cdots, s_m)}$ be a real-rooted polynomial of degree n with positive leading coefficient. For a partial assignment $(s_1, \cdots, s_k) \in S_1 \times \cdots \times S_k$ with $k < m$ set

$$f_{(s_1,\cdots,s_k)} = \sum_{(s_{k+1},\cdots,s_m)\in S_{k+1}\times\cdots\times S_m} f_{(s_1,\cdots,s_m)}.$$

Naturally, set

$$f_\phi = \sum_{(s_1,\cdots,s_m)\in S_1\times\cdots\times S_m} f_{(s_1,\cdots,s_m)}.$$

The polynomials $f_{(s_1,\cdots,s_m)}$ are said to form an interlacing family if for every $k = 0, 1, 2, \cdots, m-1$ and $(s_1, \cdots, s_k) \in S_1 \times \cdots \times S_k$ the polynomials $f_{(s_1,\cdots,s_m,t)}$ for $t \in S_{k+1}$ have a common interlacing.

The inductive generalisation of the above observation that we need is that if $f_{(s_1,\cdots,s_m)}$ form an interlacing family then, for some $(s_1, \cdots, s_m) \in S_1 \times \cdots \times S_m$, the largest root of $f_{(s_1,\cdots,s_m)}$ is at most the largest root of f_ϕ.

The Bilu-Linial conjecture, at least for bipartite graphs, would follow if the polynomials f_s for $s \in \{\pm 1\}^m$ form an interlacing family of polynomials. The restriction to bipartite graphs is because only the largest root is bounded above, but for bipartite graphs the symmetry about the origin implies the corresponding lower bound on the smallest root. And this, indeed is the main technical result. By the real-rootedness equivalent condition for having a common interlacing, it suffices to show that every convex combination of $f_{(s_1,\cdots,s_k,1)}$ and $f_{(s_1,\cdots,s_k,-1)}$ is real-rooted. This is proved by establishing the following far-reaching generalisation of the results of Heilmann-Lieb and Godsil-Gutman.

Suppose that $p_1, p_2, \cdots, p_m \in [0, 1]$, thought of as probabilities for choosing the signings 1 or -1 on the edges. Then, the polynomial

$$\sum_{s \in \{\pm\}^m} \left(\prod_{i:s_i=1} p_i \right) \left(\prod_{i:s_i=-1} (1-p_i) \right) f_s(x)$$

which is the weighted average of the $f_s(x)$ is real-rooted. This immediately implies that $\lambda f_{(s_1,\cdots,s_k,1)} + (1-\lambda) f_{(s_1,\cdots,s_k,-1)}$ is real-rooted by choosing appropriate values of $p_1, \cdots, p_m$. Specifially let $p_{k+1} = \lambda$, $p_{k+2} = p_{k+3} = \cdots = p_m = \frac{1}{2}$, and $p_i = \frac{1}{2}(1+s_i)$ for $i \leq k$.

Several ingredients go into the proof of this step. The first is to consider a multivariate generalisation of real-rootedness called real-stability. A polynomial $f \in \mathbb{R}[z_1, \cdots, z_n]$ is said to be real-stable if it is either the 0 polynomial or never vanishes when the imaginary part of each z_i is strictly positive. A large supply of real stable polynomials comes from a result of Borcea and Brändén in [BrcBrn2008] that for positive semidefinite matrices $A_1, A_2, \cdots, A_m$, the polynomial $det(z_1 A_1 + \cdots + z_m A_m)$ is real stable. Further, real-stable polynomials are closed under specialisation of one of their variables to a real value.

Secondly, they use the matrix determinant lemma sometimes referred to as a rank 1 update result for the determinant, which states that for an invertible matrix A, and vectors u, v,

$$det(A + uv^T) = det(A)(1 + v^T A^{-1} u),$$

which is not too hard to see. Although not needed, observe that the invertibility hypothesis on A can be removed by writing the right hand side of the above equation as $det(A) + v^T adj(A)u$.

Lastly they use more results of Borcea and Brändén in [BrcBrn2010] on differential operators that preserve real-stability - in particular, that for real $a, b \geq 0$ and variables x, y, the operator $1 + a\partial_x + b\partial_y$ preserves real-stability.

The weighted average of the $f_s(x)$ is then shown to be real-stable by exhibiting it explicitly as the image of a real-stable polynomial in $2m + 1$ variables under a composition of operators preserving real-stability. Since a univariate real-stable polynomial is real-rooted, this concludes the proof.

In a striking application of the same techniques - using interlacing families - they show in the following paper in the same issue of the Annals of Mathematics - see [MrcSplSrv2015(2)] - that the Kadison-Singer question from operator theory that had been open for several decades, has an affirmative answer. One version of this question asks whether for all $\epsilon > 0$, there is a $k \in \mathbb{N}$ such that for any linear operator T on $\mathbb{C}^n$ with 0 diagonal, there is a partition of $\{1, \cdots, n\}$ into k sets $A_1, \cdots, A_k$ such that $||P_j T P_j|| \leq \epsilon ||T||$ for $j = 1, \cdots, k$.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 55–78

BIRKHOFF-JAMES ORTHOGONALITY : ITS ROLE IN THE GEOMETRY OF BANACH SPACE*

KALLOL PAUL AND ARPITA MAL

Abstract. Birkhoff-James orthogonality generalizes the concept of usual orthogonality in an Euclidean space, that we are familiar with from schooldays. In this article, we discuss the notion of Birkhoff-James orthogonality ($\perp_B$) in a Banach space and explore its role in the geometry of Banach space, with a special emphasis to strict convexity, smoothness and reflexivity. We also explore the role of Birkhoff-James orthogonality in the study of smoothness in the space of bounded linear operators.

1. Introduction

One of the most important and useful theorem in Euclidean geometry is the Baudhayana Sulba Sutra (popularly known as Pythagorus theorem) that includes the notion of perpendicularity. It is well-known from our school days that two vectors $\vec{a}$ and $\vec{b}$ in the real plane $\mathbb{R}^2$ are perpendicular or orthogonal if $\vec{a} \cdot \vec{b} = 0$. This notion of perpendicularity of vectors can be extended to orthogonality of two elements x and y in a Hilbert space as $\langle x, y \rangle = 0$. This orthogonality notion plays a very important role in understanding the geometry of Hilbert space. It is easy to see that orthogonality of two elements x and y in a real Hilbert space is equivalent to each of the following:

(i) $\|x + \lambda y\| \geq \|x\|$ for all scalars λ.

(ii) $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

* This article is based on the text of the 15th Ganesh Prasad Memorial Award Lecture given by the first author at the 87th Annual Conference of the IMS - An International meet held at MGM Unuversity, Aurangabad during December 4-7, 2021 using online mode.

(iii) $\|x+y\| = \|x-y\|$.

Each of the above equivalent conditions is described in terms of the norm induced by the inner product. In an arbitrary Banach space $\mathbb{X}$, there is no natural notion of orthogonality of two elements like $\langle x, y\rangle = 0$, as the norm is not necessarily induced by an inner product. Motivated by the above equivalent conditions, the following notions of orthogonality are introduced in a Banach space.

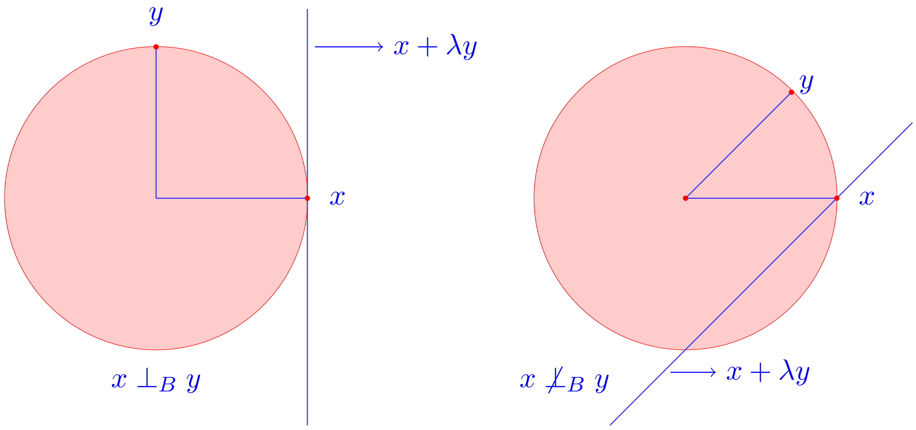
Definition 1.1. [3, 11] (**Birkhoff-James orthogonality**) Let $\mathbb{X}$ be a Banach space and $x, y \in \mathbb{X}$. Then x is said to be Birkhoff-James orthogonal to y, written as $x \perp_B y$, if for any scalar λ, $\|x + \lambda y\| \geq \|x\|$.

Definition 1.2. [12] (**Isosceles orthogonality**) Let $\mathbb{X}$ be a Banach space and $x, y \in \mathbb{X}$. Then x is said to be Isosceles orthogonal to y, written as $x \perp_I y$, if $\|x+y\| = \|x-y\|$.

Definition 1.3. [12] (**Pythagorean orthogonality**) Let $\mathbb{X}$ be a Banach space and $x, y \in \mathbb{X}$. Then x is said to be orthogonal to y in the sense of Pythagorean or Pythagorean orthogonal to y, written as $x \perp_P y$, if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$.

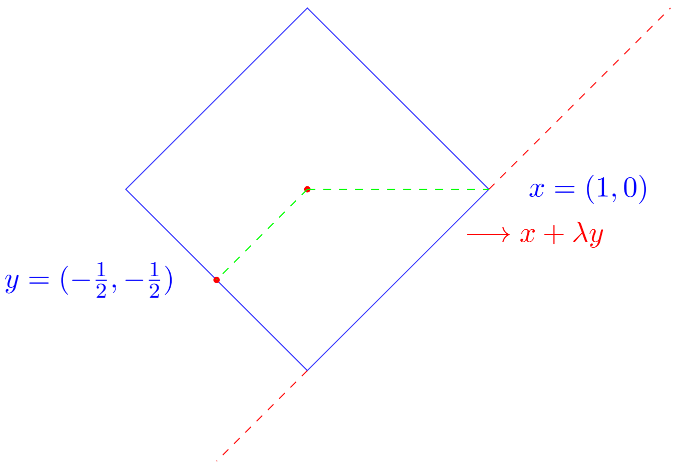
As the article is presented with a geometrical motivation the underlying space here is always considered over the real field unless mentioned otherwise. Geometrical interpretation of Birkhoff-James orthogonality is as follows: Given two elements x and y in a Banach space $\mathbb{X}$ with $\|x\| = 1$, we see that $x \perp_B y$ if and only if the line passing through x parallel to y is outside the open unit ball.

We note that in a Hilbert space, Birkhoff-James orthogonality is equivalent to the usual inner product orthogonality even if the space is complex but the case is not so for Isosceles orthogonality and Pythagorean orthogonality. Although each of the above three notions of orthogonality are equivalent in a real Hilbert space, they are not so in a Banach space. In the following examples we demonstrate the independence of relation between Birkhoff-James orthogonality and Pythagorean orthogonality as well as Birkhoff-James orthogonality and Isosceles orthogonality. The same can be done for Isosceles orthogonality and Pythagorean orthogonality.



Example 1.4. (i) (Birkhoff-James orthogonality does not imply Pythagorean orthogonality)

Consider the real Banach space ℓ_1^2. Let $x = (1,0), y = (-\frac{1}{2}, -\frac{1}{2}) \in \ell_1^2$. Then $\|x + \lambda y\| = \|(1 - \frac{1}{2}\lambda, -\frac{1}{2}\lambda)\| = |1 - \frac{1}{2}\lambda| + |\frac{1}{2}\lambda| \geq 1$ and so $x \perp_B y$. But $\|x - y\|^2 = \|(1,0) - (-\frac{1}{2}, -\frac{1}{2})\|^2 = 4 \neq \|(1,0)\|^2 + \|(-\frac{1}{2}, -\frac{1}{2})\|^2 = \|x\|^2 + \|y\|^2$. Thus, $x \not\perp_P y$.

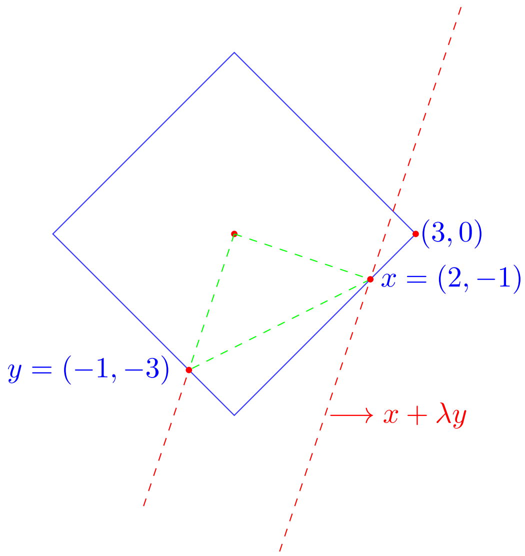


Unit sphere of $\ell_1^2 : x \perp_B y$ but $x \not\perp_P y$

(ii) (Pythagorean orthogonality does not imply Birkhoff-James orthogonality)

Consider the real Banach space ℓ_1^2. Let $x = (2,-1), y = (-1,-3) \in \ell_1^2$. Then $\|x - y\|^2 = \|(2,-1) - (-1,-3)\|^2 = 25 = \|(2,-1)\|^2 + \|(-1,-3)\|^2 = \|x\|^2 + \|y\|^2$. So $x \perp_P y$. But $\|x - \frac{1}{3}y\| = \|(2,-1) - \frac{1}{3}(-1,-3)\| = \|(2 +$

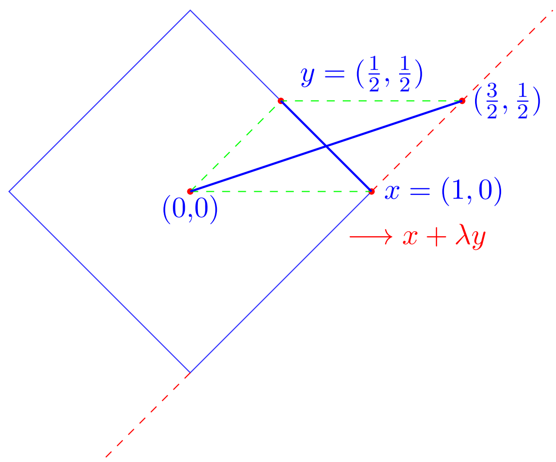
$\frac{1}{3}, 0)\| = 2 + \frac{1}{3} < 3 = \|(2,-1)\|$. So $x \not\perp_B y$.



Unit sphere of $\ell_1^2 : x \perp_P y$ but $x \not\perp_B y$

(iii) (Birkhoff-James orthogonality does not imply Isosceles orthogonality)

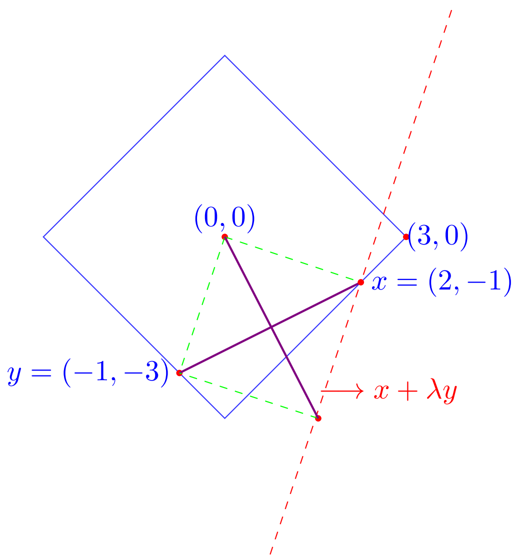
Consider the real Banach space ℓ_1^2. Let $x = (1,0), y = (\frac{1}{2}, \frac{1}{2}) \in \ell_1^2$. Then $\|x + \lambda y\| = \|(1 + \frac{1}{2}\lambda, \frac{1}{2}\lambda)\| = |1 + \frac{1}{2}\lambda| + |\frac{1}{2}\lambda| \geq 1$ and so $x \perp_B y$. But $\|x - y\| = \|(1,0) - (\frac{1}{2}, \frac{1}{2})\| = \|(\frac{1}{2}, \frac{1}{2})\| = 1 \neq 2 = \|(1 + \frac{1}{2}, \frac{1}{2})\| = \|(1,0) + (\frac{1}{2}, \frac{1}{2})\| = \|x + y\|$. Thus, $x \not\perp_I y$.



Unit sphere of $\ell_1^2 : x \perp_B y$ but $x \not\perp_I y$

(iv) (Isosceles orthogonality does not imply Birkhoff-James orthogonality)

Consider the real Banach space ℓ_1^2. Let $x = (2, -1), y = (-1, -3) \in \ell_1^2$. Then $\|x - y\| = \|(2, -1) - (-1, -3)\| = \|(3, 2)\| = 5 = \|(1, -4)\| = \|(2, -1) + (-1, -3)\| = \|x + y\|$. So $x \perp_I y$. But $\|x - \frac{1}{3}y\| = \|(2, -1) - \frac{1}{3}(-1, -3)\| = \|(2 + \frac{1}{3}, 0)\| = 2 + \frac{1}{3} < 3 = \|(2, -1)\|$. So $x \not\perp_B y$.



Unit sphere of $\ell_1^2 : x \perp_I y$ but $x \not\perp_B y$

Observe that the Hahn-Banach theorem guarantees the existence of orthogonal elements in the sense of Birkhoff-James in a Banach space $\mathbb{X}$. Given $(0 \neq) x \in \mathbb{X}$, consider $M = span\{x\}$. Define $f : M \to \mathbb{R}$ by $f(ax) = a\|x\|$ for all scalars a. Then f is a bounded linear functional on M and $\|f\| = 1$. By the Hahn-Banach theorem f can be extended to a bounded linear functional $\widetilde{f}$ on the space $\mathbb{X}$ preserving the norm and so $\|\widetilde{f}\| = 1, \widetilde{f}(x) = \|x\|$. Let $H = \ker(\widetilde{f})$ and $y \in H$. Then for any scalar λ, $\|x + \lambda y\| \geq |\widetilde{f}(x + \lambda y)| = |\widetilde{f}(x)| = \|x\|$, so that $x \perp_B y$. Moreover, for any $z \in \mathbb{X}$, if $a = -\frac{\widetilde{f}(z)}{\widetilde{f}(x)}$, then $\widetilde{f}(ax + z) = 0$ and so $x \perp_B (ax + z)$. Thus we have the following theorem.

Theorem 1.5. [11, Th. 2.2, Cor. 2.2] *Let $\mathbb{X}$ be a Banach space. Then for every non-zero $x \in \mathbb{X}$, there exists a closed hyperspace H of $\mathbb{X}$ such that $x \perp_B H$.*
Moreover, for any $z \in \mathbb{X}$, there exists $a \in \mathbb{R}$ such that $x \perp_B (ax + z)$.

On the other hand, given $x, y \in \mathbb{X}$, consider the set $S = \{\|\lambda x + y\| : \lambda \text{ is a scalar}\}$. Then S is a non-empty subset of non-negative real numbers and so the infimum of S exists. By continuity of the norm function it is easy to see that there exists a scalar a such that $\|ax+y\| = \inf_\lambda \|\lambda x + y\|$. Then for any scalar λ, $\|\lambda x + y\| \geq \|ax + y\|$ which implies that $\|(ax+y) + \lambda x\| \geq \|ax + y\|$ so that $ax + y \perp_B x$. Thus we have the following theorem.

Theorem 1.6. [11, Th. 2.3] *Let $\mathbb{X}$ be a Banach space. Given $x, y \in \mathbb{X}$, there exists a scalar a such that $ax + y \perp_B x$.*

Remark 1.7. Following Theorem 1.5 we raise two natural questions : (i) Given a closed hyperspace H of $\mathbb{X}$. Does there exist $(0 \neq)x \in \mathbb{X}$ such that $x \perp_B H$? (ii) Given $(0 \neq)y \in \mathbb{X}$. Does there exist a hyperspace H such that $H \perp_B y$? The reader can think about it for the time being and we will address the same in due course of time.

In the following theorem, we provide a wonderful characterization of Birkhoff-James orthogonality in terms of linear functionals.

Theorem 1.8. [11, Th. 2.1] *Let $\mathbb{X}$ be a Banach space. Then for $(0 \neq)x \in \mathbb{X}$ and a closed hyperspace H of $\mathbb{X}$, $x \perp_B H$ if and only if there exists a bounded linear functional f such that $f(x) = \|x\|, \|f\| = 1$ and $f(h) = 0$ for all $h \in H$.*

Proof. **Necessary part** : Each element $z \in \mathbb{X}$ can be written as $z = \alpha x + h$, where α is a scalar and $h \in H$. Define a functional f on $\mathbb{X}$ as follows: $f(z) = \alpha\|x\|$. If $\|z\| = 1$ then $1 = \|\alpha x + h\| \geq |\alpha|\|x\|$ and so $|f(z)| = |\alpha|\|x\| \leq \|z\|$. Moreover, $f(x) = \|x\|$ implies that $\|f\| = 1$. Thus f is a bounded linear functional on $\mathbb{X}$ such that $f(x) = \|x\|, \|f\| = 1$ and $f(h) = 0$ for all $h \in H$.
Sufficient part : Consider $H = \ker(f)$. Then H is a closed hyperspace of $\mathbb{X}$. Observe that for any scalar λ and for each $h \in H$, we have

$$\|x\| = \|f\|\|x\| = |f(x)| = |f(x + \lambda h)| \leq \|f\|\|x + \lambda h\| = \|x + \lambda h\|.$$

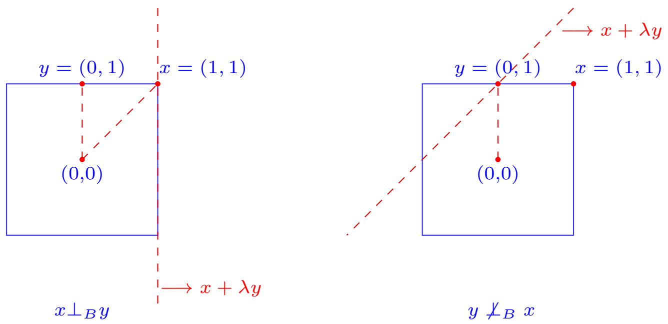
This shows that $x \perp_B H$.

□

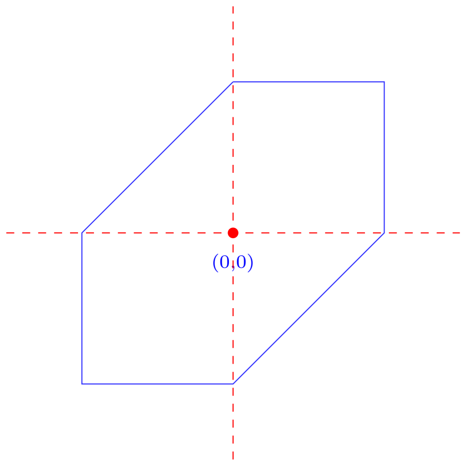
There are many basic properties of the inner product orthogonality like homogeneity, additivity, symmetricity which may or may not be true for Birkhoff-James orthogonality. In a Banach space $\mathbb{X}$, we note that

(1) Birkhoff-James orthogonality is **homogeneous** : If $x, y \in \mathbb{X}$ be such that $x \perp_B y$ then $\alpha x \perp_B \beta y$ for all scalars α, β.
(2) Birkhoff-James orthogonality is **not left additive** : If $x, y, z \in \mathbb{X}$ be such that $x \perp_B z$ and $y \perp_B z$ then $(x+y)$ may or may not be Birkhoff-James orthogonal to z.
(3) Birkhoff-James orthogonality is **not right additive** : If $x, y, z \in \mathbb{X}$ be such that $x \perp_B y$ and $x \perp_B z$ then x may or may not be Birkhoff-James orthogonal to $(y+z)$.
(4) Birkhoff-James orthogonality is **not symmetric** : For $x, y \in \mathbb{X}$, if $x \perp_B y$ then it is not necessarily true that $y \perp_B x$. Similarly, if $y \perp_B x$ then it is not necessarily true that $x \perp_B y$.

In due course of time, we will further discuss about left additivity and right additivity of Birkhoff-James orthogonality. For the time being, we note few facts about symmetry of Birkhoff-James orthogonality. Observe that in the real Banach space ℓ_∞^2, for $x = (1,1)$ and $y = (0,1)$, $x \perp_B y$. However, $y \not\perp_B x$.



The symmetricity of Birkhoff-James orthogonality is a very strong property and it induces an inner product on the space if the dimension of the space is greater or equal to 3. However, there are two-dimensional Banach spaces, where Birkhoff-James orthogonality is symmetric but the norm is not induced by an inner product. Such two-dimensional Banach spaces are known as Radon planes. See [1, 4, 13, 18] and the references therein for construction and characterizations of Radon planes. In [10] James provided a way to construct Radon planes. The following figure gives an example of the unit sphere of a Radon plane $\mathbb{X}$, where for $(a,b) \in \mathbb{X}$, $\|(a,b)\| = \max\{|a|,|b|\}$ if $ab \geq 0$ and $\|(a,b)\| = |a| + |b|$ if $ab < 0$.



Unit sphere of a Radon plane constructed from $\ell_\infty(\mathbb{R}^2)$ and its dual

The geometry of a Banach space is much more involved and it demands further research to have a better understanding. In due course of time we will see that Birkhoff-James orthogonality plays a very effective role in understanding the geometry of Banach space. The notion of strict convexity, uniform convexity, smoothness, reflexivity in a Banach space can be studied using Birkhoff-James orthogonality, which was wonderfully done in the remarkable article by James [11]. In recent times, many researchers are actively engaged in studying the geometry of spaces of bounded linear operators defined between Hilbert as well as Banach spaces using the notion of Birkhoff-James orthogonality (see [2, 14, 15, 16, 22, 23, 25]). We here try to demonstrate the role of Birkhoff-James orthogonality both in the geometry of ground space as well as in the space of operators.

2. Birkhoff-James orthogonality and strict convexity

Suppose $(\mathbb{X}, \|.\|)$ is a Banach space and $\mathbb{X}^*$ denotes the dual space of $\mathbb{X}$. $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ be the unit ball and the unit sphere of the space $\mathbb{X}$, respectively. Let S be a non-empty convex subset of the Banach space $\mathbb{X}$. An element $x \in S$ is said to be an extreme point of S if $x = (1-t)y + tz$ for some $t \in (0, 1)$ and $y, z \in S$ then $x = y = z$. A Banach space $\mathbb{X}$ is said to be strictly convex if and only if every element of the unit sphere is an extreme point of the unit ball. It is easy to observe that every Hilbert space is strictly convex. On the other hand, ℓ_p is strictly convex and $\ell_1, \ell_\infty, C[a,b]$ are not strictly convex. It follows from Theorem 1.6 that for $x, y \in \mathbb{X}$ there always exists a scalar a such that $ax + y \perp_B x$. The Birkhoff-James orthogonality is said to be left unique (see [11]) if and only if for no elements $x (\neq 0)$ and y there exists more than one scalar a such that $ax + y \perp_B x$. The following theorem gives a characterization of strictly convex Banach space in terms of left uniqueness of Birkhoff-James orthogonality.

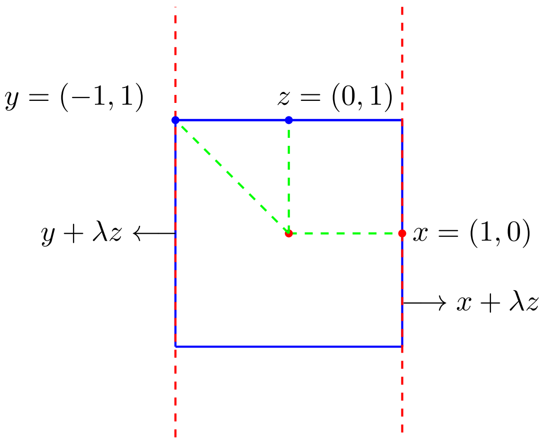
Theorem 2.1. [11, Th. 4.3] *A Banach space $\mathbb{X}$ is strictly convex if and only if Birkhoff-James orthogonality is left unique.*

Proof. Let $\mathbb{X}$ be strictly convex. If possible, let Birkhoff-James orthogonality be not left unique. Then there exist $x, y \in \mathbb{X}$ and $a, b \in \mathbb{R}$ such that $ax + y \perp_B x$ and $bx + y \perp_B x$, where $a \neq b$. Let $a < b$. Now, $\|ax + y + \lambda x\| \geq \|ax + y\|$, for all scalars λ. Putting $\lambda = b - a$, we have, $\|bx + y\| \geq \|ax + y\|$. Again, $\|bx + y + \lambda x\| \geq \|bx + y\|$, for all scalars λ. Putting $\lambda = a - b$, we have, $\|ax + y\| \geq \|bx + y\|$. Thus, $\|ax + y\| = \|bx + y\|$. So, $\|kx + y\| = \|ax + y\|$ for all $a \leq k \leq b$. This contradicts that $\mathbb{X}$ is strictly convex. Therefore, $\mathbb{X}$ is strictly convex implies that Birkhoff-James orthogonality is left unique.
Conversely, let Birkhoff-James orthogonality be left unique. Let $\mathbb{X}$ be not strictly convex. Then the unit sphere of $\mathbb{X}$ contains a line segment. So, there exist $u, v \in S_{\mathbb{X}}$ such that $u \neq v$ and $(1-t)u + tv \in S_{\mathbb{X}}$ for all $0 \leq t \leq 1$. Thus, $\|u + \lambda(v - u)\| = \|(1-\lambda)u + \lambda v\| = 1$, for all $0 \leq \lambda \leq 1$. Using the convexity of norm, it can be shown that $\|u + \lambda(v - u)\| \geq 1$ for all $\lambda \geq 0$. Now, let $\lambda < 0$. If possible, let $\|u + \lambda(v - u)\| < 1$. Then $u = (1 - \frac{\lambda}{\lambda - 1})(u + \lambda(v - u)) + \frac{\lambda}{\lambda - 1} v$ and $0 \leq \frac{\lambda}{\lambda - 1} \leq 1$. So,

$$\|u\| \leq (1 - \frac{\lambda}{\lambda - 1})\|(u + \lambda(v - u))\| + \frac{\lambda}{\lambda - 1}\|v\| < (1 - \frac{\lambda}{\lambda - 1}) + \frac{\lambda}{\lambda - 1} = 1,$$

a contradiction. Therefore, $\|u + \lambda(v - u)\| \geq 1 = \|u\|$ for all $\lambda < 0$. Hence, $u \perp_B (v - u)$. Similarly, it can be shown that $v \perp_B (v - u)$, i.e., $u + (v - u) \perp_B (v - u)$. This yields that Birkhoff-James orthogonality is not left unique, a contradiction. Therefore, $\mathbb{X}$ is strictly convex. $\square$

We have already noted that Birkhoff-James orthogonality is not necessarily left additive. As for example, consider the real Banach space ℓ_∞^2. Let $x = (1, 0), y = (-1, 1), z = (0, 1)$. Then $x \perp_B z, y \perp_B z$ but $x + y \not\perp_B z$.



$x \perp_B z, y \perp_B z$ but $x + y \not\perp_B z$

Unit sphere of ℓ_∞^2

Next, we give a characterization of two-dimensional strictly convex Banach space in terms of left additivity of Birkhoff-James orthogonality.

Theorem 2.2. *A two-dimensional Banach space $\mathbb{X}$ is strictly convex if and only if Birkhoff-James orthogonality is left additive.*

Proof. Let $\mathbb{X}$ be strictly convex. Then Birkhoff-James orthogonality is left unique. Let $(0 \neq) x, y, z \in \mathbb{X}$ be such that $y \perp_B x$ and $z \perp_B x$. Clearly, the sets $\{x, y\}$ and $\{z, x\}$ are linearly independent. Therefore, $z = ax + by$ for some scalars a, b, where $b \neq 0$. Now, from $ax + by \perp_B x$, it follows that $\frac{a}{b}x + y \perp_B x$. Since, Birkhoff-James orthogonality is left unique and $y \perp_B x$, we get $\frac{a}{b} = 0$. Thus $a = 0$, which gives $z = by$. Now, $y \perp_B x$ implies that $(1 + b)y \perp_B x$, i.e., $y + z \perp_B x$. Thus Birkhoff-James orthogonality is left additive.
Conversely, let Birkhoff-James orthogonality be left additive. If possible, let $\mathbb{X}$ be not strictly convex. Then the unit sphere of $\mathbb{X}$ contains a line segment. So, there exist $u, v \in S_{\mathbb{X}}$ such that $u \neq v$ and $(1-t)u + tv \in S_{\mathbb{X}}$ for all $0 \leq t \leq 1$. Hence, $\|u + \lambda(v - u)\| = \|(1 - \lambda)u + \lambda v\| = 1$, for all $0 \leq \lambda \leq 1$. Now, let $\lambda < 0$. If possible, let $\|u + \lambda(v - u)\| < \|u\|$. Then $u = (1 - \frac{\lambda}{\lambda-1})(u + \lambda(v - u)) + \frac{\lambda}{\lambda-1}v$ and $0 \leq \frac{\lambda}{\lambda-1} \leq 1$. So, $\|u\| \leq (1 - \frac{\lambda}{\lambda-1})\|(u + \lambda(v - u))\| + \frac{\lambda}{\lambda-1}\|v\| < (1 - \frac{\lambda}{\lambda-1})\|u\| + \frac{\lambda}{\lambda-1}\|u\| = \|u\|$, a contradiction. Therefore, $\|u + \lambda(v - u)\| \geq \|u\|$ for all $\lambda < 0$. So, $u \perp_B (v - u)$. Similarly, it can be shown that $v \perp_B (v - u)$. Since Birkhoff-James

orthogonality is left additive, $(v-u) \perp_B (v-u)$. Therefore, $v-u=0$, i.e., $v=u$, a contradiction. Thus $\mathbb{X}$ is strictly convex. □

The following theorem gives a characterization of Hilbert space when the dimension of the space is greater than or equal to 3.

Theorem 2.3. [10, Th. 2] *Let $\mathbb{X}$ be a Banach space such that* $\dim(\mathbb{X}) \geq 3$. *Then $\mathbb{X}$ is a Hilbert space if and only if Birkhoff-James orthogonality is left additive.*

Proof. Let us first prove the sufficient part. Since Birkhoff-James orthogonality coincides with usual inner product orthogonality in a Hilbert space, it is easy to see that Birkhoff-James orthogonality is left additive in a Hilbert space.
Let us now prove the necessary part. Suppose Birkhoff-James orthogonality is left additive. Let $x, y \in \mathbb{X}$ be such that $\{x, y\}$ is linearly independent. Then there exist hyperspaces H_x and H_y in $\mathbb{X}$ such that $x \perp_B H_x$ and $y \perp_B H_y$. Consider, $H = H_x \bigcap H_y$. Then $x, y \notin H$ and $codimH = 2$. Since Birkhoff-James orthogonality is left additive, we have, $ax + by \perp_B H$, for all $a, b \in \mathbb{R}$. It is easy to see that any $z \in \mathbb{X}$ can be written as $z = p(z) + w$, where $p(z) = ax + by$ and $w \in H$. Observe that, $\|p(z)\| = \|ax + by\| \leq \|p(z) + w\| = \|z\|$, since $p(z) \perp_B w$. Now, from $\|p(x)\| = \|x\|$ it follows that $\|p\| = 1$. So, there exists a projection of norm 1 on any two-dimensional subspace of $\mathbb{X}$. Since there is a projection of norm 1 on any given two-dimensional subspace of $\mathbb{X}$, we have, $\mathbb{X}$ is a Hilbert space. □

We next address the question raised in Remark 1.7. Given a closed hyperspace H there is no guarantee on the existence of non-zero x such that $x \perp_B H$. However, it is true if the space is reflexive. In fact, this gives a characterization of reflexive Banach spaces.

Theorem 2.4. *A Banach space $\mathbb{X}$ is reflexive if and only if for any closed hyperspace $H \subseteq \mathbb{X}$, there exists $x(\neq 0) \in \mathbb{X}$ such that $x \perp_B H$.*

Proof. We first prove the sufficient part of the theorem. We note from [9, Th. 2] that a Banach space $\mathbb{X}$ is reflexive if and only if every bounded linear functional on $\mathbb{X}$ attains its norm. Let f be a non-zero bounded linear functional on $\mathbb{X}$ and $H = \ker(f)$. Clearly, H is a closed hyperspace of $\mathbb{X}$. By hypothesis, there exists $x(\neq 0) \in \mathbb{X}$ such that $x \perp_B H$. Clearly, $f(x) \neq 0$, since $f \neq 0$. Let $f(x) = r \neq 0$. Now, every element z of $\mathbb{X}$

can be uniquely written as $z = ax + h$, where $a \in \mathbb{R}$ and $h \in H$. So, $\|z\| = \|ax + h\| \geq |a|\|x\|$. Now, $|f(z)| = |af(x)| = |r||a| \leq |r|\frac{\|z\|}{\|x\|}$. Thus, $\|f\| \leq \frac{|r|}{\|x\|}$. Again, $|f(\frac{x}{\|x\|})| = \frac{|r|}{\|x\|}$. Therefore, $\|f\| = \frac{|r|}{\|x\|}$ and f attains its supremum on $\frac{x}{\|x\|}$. Therefore, every bounded linear functional attains its supremum on $\mathbb{X}$. Hence, $\mathbb{X}$ is reflexive.
For the necessary part, suppose that $\mathbb{X}$ is reflexive. Let H be a closed hyperspace of $\mathbb{X}$. Let f be a bounded linear functional on $\mathbb{X}$ such that $H = \ker(f)$. Since $\mathbb{X}$ is reflexive, f attains its supremum at $x \in S_{\mathbb{X}}$, say. Let $y \in H$. Then for any $\lambda \in \mathbb{R}$, $\|f\|\|x + \lambda y\| \geq |f(x + \lambda y)| = |f(x)| = \|f\|\|x\|$. So, $\|x + \lambda y\| \geq \|x\|$ for all scalars λ. Thus, $x \perp_B y$. Hence, $x \perp_B H$. This completes the proof of the theorem. □

The next theorem by James answers the other question raised in Remark 1.7.

Theorem 2.5. [10, Th. 5] *Let* $\mathbb{X}$ *be a Banach space such that* $\dim(\mathbb{X}) \geq 3$. *Then* $\mathbb{X}$ *is a Hilbert space if and only if for any element* $x \in \mathbb{X}$, *there exists a hyperspace* H *such that* $H \perp_B x$.

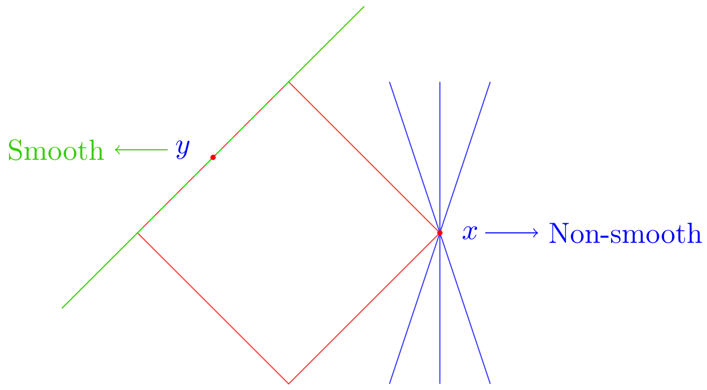
3. Birkhoff-James orthogonality and Smoothness

One of the most important geometric notion in a Banach space is smoothness.

Definition 3.1. An element $x \in S_{\mathbb{X}}$ is said to be a smooth point if there is a unique linear functional $f \in \mathbb{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\| = 1$.

Equivalently, a geometric definition of a smooth point is as follows:

Definition 3.2. An element $x \in S_{\mathbb{X}}$ is said to be a smooth point if there is a unique hyperplane H supporting $B_{\mathbb{X}}$ at x.



Unit sphere of ℓ_1^2

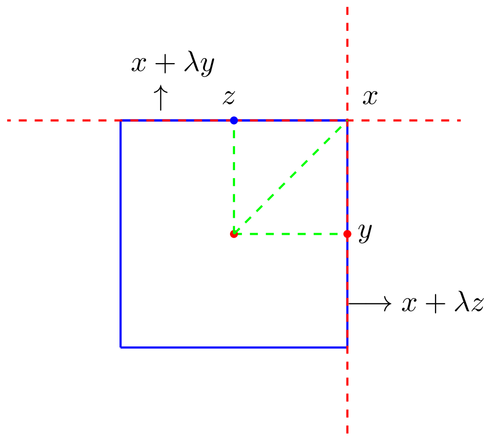
Let us recall from Theorem 1.5 that given $(0 \neq) x \in \mathbb{X}$, there exists a hyperspace H such that $x \perp_B H$. The following theorem shows that x is smooth if and only if $x \perp_B y$ implies that $y \in H$.

Theorem 3.3. *A Banach space $\mathbb{X}$ is smooth if and only if for any non-zero $x \in \mathbb{X}$, $H_x = \{y \in \mathbb{X} : x \perp_B y\}$ is a subspace.*

Proof. Let $\mathbb{X}$ be smooth and x be any non-zero element in $\mathbb{X}$. Let g be the unique norming linear functional of x, i.e., $g(x) = \|x\|$ and $\|g\| = 1$. We show that $\ker(g) = H_x$, where $H_x = \{y \in \mathbb{X} : x \perp_B y\}$. Let $y \in \ker(g)$. Then for any scalar λ, $\|x+\lambda y\| = \|g\|\|x+\lambda y\| \geq |g(x+\lambda y)| = |g(x)| = \|x\|$. So $x \perp_B y$. Thus, $y \in H_x$. Therefore, $\ker(g) \subseteq H_x$. Now, let y be a non-zero element in H_x. Then $x \perp_B y$. Let $Z = span\{x, y\}$. Define $f : Z \to \mathbb{R}$ by $f(ax + by) = a\|x\|$. Then clearly, $\|f\| = 1, f(x) = \|x\|$ and $f(y) = 0$. By the Hahn Banach theorem, there exists $h \in \mathbb{X}^*$ such that $\|h\| = \|f\| = 1,\ h(x) = f(x) = \|x\|$ and $h(y) = f(y) = 0$. Thus h is a norming linear functional of x. Since, g is the unique norming linear functional of x, we get $g = h$. Thus $y \in \ker(h) = \ker(g)$. Hence, $H_x \subseteq \ker(g)$. Therefore, $H_x = \ker(g)$. Since, $\ker(g)$ is a subspace, H_x is a subspace.
Conversely, let H_x be a subspace for each non-zero $x \in \mathbb{X}$. If possible, let $\mathbb{X}$ be not smooth. Then there exists a non-zero $x \in \mathbb{X}$ such that x has at least two norming linear functionals f, g. Since $f \neq g$, $\ker(f) \neq \ker(g)$. Since f, g are norming linear functionals of x, proceeding as above it can be shown that $\ker(f) \subseteq H_x$ and $\ker(g) \subseteq H_x$. Since $x \notin H_x$, H_x is a proper subspace of $\mathbb{X}$. Again, $\ker(f)$ and $\ker(g)$ are maximal proper subspaces of $\mathbb{X}$. Therefore, $\ker(f) = H_x = \ker(g)$. This gives that $f = \lambda g$ for some scalar

λ. Again, $f(x) = \|x\| = g(x)$ gives that $\lambda = 1$. Thus $f = g$, a contradiction. Therefore, $\mathbb{X}$ is smooth. $\square$

We have already noted that Birkhoff-James orthogonality is not necessarily right additive. As for example, consider the real Banach space ℓ_∞^2. Let $x = (1,1), y = (1,0), z = (0,1)$. Then $x \perp_B y, x \perp_B z$ but $x \not\perp_B y + z (= x)$.



$x \perp_B y, x \perp_B z$ but $x \not\perp_B y + z (= x)$

Unit sphere of ℓ_∞^2

We now explore the relation between right additivity of Birkhoff-James orthogonality and smoothness in a Banach space (see [11]) .

Theorem 3.4. *A Banach space $\mathbb{X}$ is smooth if and only if Birkhoff-James orthogonality is right additive.*

Proof. Let $\mathbb{X}$ be smooth. Let $x, y, z \in \mathbb{X}$ be such that $x \perp_B y$ and $x \perp_B z$. Then $y, z \in H_x$, where $H_x = \{w \in \mathbb{X} : x \perp_B w\}$. Since $\mathbb{X}$ is smooth, H_x is a subspace. So, $y + z \in H_x$. Hence, $x \perp_B (y + z)$. Thus, Birkhoff-James orthogonality is right additive.
Conversely, let Birkhoff-James orthogonality be right additive. Let x be a non-zero element in $\mathbb{X}$. Let $H_x = \{w \in \mathbb{X} : x \perp_B w\}$. We show that H_x is a subspace. Let $y, z \in H_x$. Then $x \perp_B y$ and $x \perp_B z$. Since, Birkhoff-James orthogonality is right additive, $x \perp_B (y+z)$. Thus, $y+z \in H_x$. Again, since Birkhoff-James orthogonality is homogeneous, $\alpha y \in H_x$, for each scalar α. Thus, H_x is a subspace. Therefore, $\mathbb{X}$ is smooth. $\square$

Next, we explore the relation between smoothness and right uniqueness of Birkhoff-James orthogonality. Given $x, y \in \mathbb{X}$ there always exists a scalar

a such that $x \perp_B (ax + y)$. The Birkhoff-James orthogonality is said to be right unique (see [11]) if and only if for no elements $x(\neq 0)$ and y there exists more than one scalar a such that $x \perp_B (ax + y)$.

Theorem 3.5. [11] *A Banach space $\mathbb{X}$ is smooth if and only if Birkhoff-James orthogonality is right unique.*

Proof. Let $\mathbb{X}$ be smooth. Then Birkhoff-James orthogonality is right additive. Let $(0 \neq)x, y \in \mathbb{X}$ be such that $x \perp_B (ax + y)$ and $x \perp_B (bx + y)$, where $a, b \in \mathbb{R}$. Then $x \perp_B (ax + y - bx - y)$, i.e., $x \perp_B (ax - bx)$. Hence, $a - b = 0 \Rightarrow a = b$. So, Birkhoff-James orthogonality is right unique.
Conversely, let Birkhoff-James orthogonality be right unique. Let x be a non-zero element in $\mathbb{X}$. Let g, h be two norming linear functionals of x. We show that $g = h$. Let $y \in \ker(g)$. Then for any scalar λ, $\|x + \lambda y\| = \|g\|\|x + \lambda y\| \geq |g(x + \lambda y)| = \|x\|$. Thus, $x \perp_B y$. If possible, let $y \notin \ker(h)$. Then $h(y) = r \neq 0$. Let $z = x - \frac{\|x\|}{r}y$. Then $h(z) = h(x) - \frac{\|x\|}{r}h(y) = \|x\| - \frac{\|x\|}{r}r = 0$. Now, similarly it can be shown that $x \perp_B z \Rightarrow x \perp_B (x - \frac{\|x\|}{r}y) \Rightarrow x \perp_B (-\frac{r}{\|x\|}x + y)$, contradicting that Birkhoff-James orthogonality is right unique. Therefore, $y \in \ker(h)$. So, $\ker(g) \subseteq \ker(h)$. Again, $\ker(h) \neq \mathbb{X}$, since $x \notin \ker(h)$. Hence, $\ker(g) = \ker(h)$. Moreover, $g(x) = h(x) = \|x\|$. So, $g = h$. Thus, x has unique norming linear functional. Hence, $\mathbb{X}$ is smooth. □

Remark 3.6. We note that Theorem 3.4 is also locally true. More precisely, an element $x \in \mathbb{X}$ is smooth if and only if Birkhoff-James orthogonality is right additive at x, i.e., x is smooth if and only if for $y, z \in \mathbb{X}$, $x \perp_B y, x \perp_B z$ implies that $x \perp_B (y + z)$. The same holds for Theorem 3.5.

4. Smoothness in operator space through Birkhoff-James orthogonality

Suppose $\mathbb{X}, \mathbb{Y}$ stand for Banach spaces and $\mathbb{H}$ stands for Hilbert space. Let $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ $(\mathcal{L}(\mathbb{H}))$ denote the space of all bounded linear operators defined from $\mathbb{X}$ to $\mathbb{Y}$ (on $\mathbb{H}$). The aim of this section is to discuss smoothness of an element $T \in \mathcal{L}(\mathbb{H})$ or $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ using the notion of Birkhoff-James orthogonality. To verify smoothness of T, we check whether $T \perp_B A_1, T \perp_B A_2$ implies $T \perp_B (A_1 + A_2)$ for arbitrary bounded linear operators A_1 and A_2. The most important point of study now is : Given a bounded linear operator T, what are the conditions so that $T \perp_B A$, where A is a bounded linear operator? We observe that if there exists an element $x \in \mathbb{X}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$ then for any scalar λ, $\|T + \lambda A\| \geq \|Tx + \lambda Ax\| \geq \|Tx\| = \|T\|$ so that $T \perp_B A$. The converse of this is not always true, as is clear from the following example.

Example 4.1. [20, Ex.2.3] Consider the bounded linear operators T and A on the Banach space $(\mathbb{R}^2, \|\cdot\|_\infty)$, given as, $T = \begin{pmatrix} 1 & -4 \\ 2 & 3 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$. Then $\|T\|_\infty = 5$ and T attains its norm at the points $\pm(1,1)$ and $\pm(1,-1)$. It is easy to check that $\|T\|_\infty \leq \|T + \lambda A\|_\infty$ for all $\lambda \in \mathbb{R}$ but there exists no $x = (x_1, x_2) \in \mathbb{R}^2$ with $\|x\|_\infty = 1$ such that $\|T\|_\infty = \|Tx\|_\infty \leq \|(T + \lambda A)x\|_\infty$ for all $\lambda \in \mathbb{R}$.

However, if the operators are defined on a finite-dimensional Hilbert space $\mathbb{H}$, then from [2, 19] it follows that $T \perp_B A$ if and only if there exists $x \in \mathbb{H}, \|x\| = 1$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$. The relevant result can be stated in the form of following theorem:

Theorem 4.2. *Let $\mathbb{H}$ be a finite-dimensional Hilbert space and $T, A \in \mathcal{L}(\mathbb{H})$. Then $T \perp_B A$ if and only if there exists $x \in \mathbb{H}$, $\|x\| = 1$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$.*

It is now appropriate time to introduce the notion of norm attainment set studied in [26].

Definition 4.3 (Norm attainment set)**.** Let $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$. Then the norm attainment set of T, denoted as M_T, is defined as the collection of all unit vectors x such that T attains its norm at x, i.e.,

$$M_T = \{x \in \mathbb{X} : \|x\| = 1, \|Tx\| = \|T\|\}.$$

It follows from the above discussion that the norm attainment set plays a crucial role in connecting the orthogonality of elements in the ground space $\mathbb{Y}$ with that in the operator space $\mathcal{L}(\mathbb{X}, \mathbb{Y})$. In case of a Hilbert space, using parallelogram law it is easy to observe that the norm attainment set of an operator is either empty or the unit sphere of some subspace. However, the structure of the norm attainment set is complicated in case of operators acting on Banach spaces (see [17]) and it needs to be explored further. At this point we do not want to digress too much in this direction. We are now in a position to discuss smoothness of a bounded linear operator defined on a Hilbert space.

Theorem 4.4. *Let $\mathbb{H}$ be a finite-dimensional Hilbert space and $T \in \mathcal{L}(\mathbb{H})$. Then T is smooth if span M_T is one-dimensional.*

Proof. Suppose $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{H}}$. We claim that Birkhoff-James orthogonality is right additive at T. Let $T \perp_B A_1$ and $T \perp_B A_2$. Then by Theorem 4.2, we get $Tx \perp_B A_1x$ and $Tx \perp_B A_2x$. Note that a Hilbert space is smooth. Therefore, the smoothness of Tx implies that $Tx \perp_B (A_1+A_2)x$. This along with the fact that $x \in M_T$ implies that $T \perp_B (A_1 + A_2)$. Thus, Birkhoff-James orthogonality is right additive at T and so T is smooth. □

Observe that in the last result we have used the fact that Tx, being an element of a Hilbert space, is smooth. Continuing in the same spirit, if we want to study smoothness of operators defined between Banach spaces, then we need a result analogous to Theorem 4.2. This follows from the remarkable result obtained in [27, Th.2.1].

Theorem 4.5. [27, Th.2.1] *Let $T, A \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, where $\mathbb{X}$ is a finite-dimensional Banach space. Assume that $M_T = D \cup (-D)$, where D is a non-empty connected subset of the unit sphere $S_{\mathbb{X}}$ of $\mathbb{X}$. Then $T \perp_B A \Leftrightarrow Tx \perp_B Ax$ for some $x \in M_T$.*

The following theorem on smoothness of an operator defined on a finite-dimensional Banach space now follows easily.

Theorem 4.6. *Let $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, where $\mathbb{X}$ is a finite-dimensional Banach space. Then T is smooth if span M_T is one-dimensional, i.e., $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{X}}$ and Tx is smooth.*

The converse of the above result is also true.

Theorem 4.7. *Let $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, where $\mathbb{X}$ is finite-dimensional. If T is smooth then span M_T is one-dimensional, i.e., $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{X}}$ and Tx is smooth.*

Proof. Since the space $\mathbb{X}$ is finite-dimensional, there exists $x \in S_{\mathbb{X}}$ such that $\|Tx\| = \|T\|$. We first show that *span* M_T is one-dimensional. If possible, suppose $y \in M_T$ and $\{x, y\}$ is linearly independent. Then there exists $f \in \mathbb{X}^*$ such that $f(x) = 1, f(y) = 0$. Define $A : \mathbb{X} \to \mathbb{Y}$ such that $A(u) = f(u)Tx$. Note that $Ay = 0$ and $(T - A)x = 0$, this along with the fact that $x, y \in M_T$ imply that $T \perp_B A, T \perp_B (T - A)$. But $T \not\perp_B T = (A + T - A)$, a contradiction to that fact that Birkhoff-James orthogonality is right additive at T. We next show that Tx is a smooth point in $\mathbb{Y}$. If possible, let Tx be not smooth. Then there exist $y, z \in \mathbb{Y}$ such that $Tx \perp_B y, Tx \perp_B z$ but $Tx \not\perp_B y + z$. There exists a hyperspace H such that $x \perp_B H$. Define two operators $A_1, A_2 : \mathbb{X} \longrightarrow \mathbb{Y}$ as follows :

$$A_1(ax + h) = ay, \;\; A_2(ax + h) = az, \;\text{ where } h \in H, \;\; a \in \mathbb{R}.$$

Then clearly $T \perp_B A_1$, $T \perp_B A_2$. But $T \not\perp_B A_1 + A_2$, otherwise since $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{X}}$, we have by Theorem 4.5, $Tx \perp_B (y+z)$, which is not possible. This contradiction shows that Tx is a smooth point. $\square$

Thus, on a finite-dimensional Banach space $\mathbb{X}$, a bounded linear operator T is smooth if and only if *span* M_T is one-dimensional, i.e., $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{X}}$ and Tx is a smooth point. If the underlying space is a Hilbert space, then T is smooth if and only if *span* M_T is one-dimensional. However, the situation is not so in case of infinite-dimensional space. Let us look at the following example:

Example 4.8. Consider $T \colon \ell_2 \to \ell_2$ defined by $Te_1 = -e_1$, and $Te_n = (1 - 1/n)e_n$ for $n \geq 2$, where $\{e_n \colon n \in \mathbb{N}\}$ is the usual orthonormal basis for the Hilbert space ℓ_2. Then T attains its norm only at $\pm e_1$ so that *span* M_T is one-dimensional. Consider $A_1, A_2 \colon \ell_2 \to \ell_2$ defined by $A_1(e_1) = 0, A_1(e_n) = (1 - 1/n)e_n$, for $n \geq 2$ and $A_2(e_1) = -e_1, A_2(e_n) = 0$, for $n \geq 2$. Then it is easy to check that $T \perp_B A_1, T \perp_B A_2$ but $T \not\perp_B (A_1 + A_2) = T$ and so T is not smooth.

The main tool that is used to characterize smoothness of bounded linear operators on finite-dimensional Hilbert space is Theorem 4.2. A further look at the Example 4.8 shows that this is not true for infinite-dimensional spaces, where $T \perp_B I$ and *span* M_T is one-dimensional but $Te_1 \not\perp_B Ie_1$. However, in case of infinite-dimensional Hilbert spaces we do have the following characterization from [21].

Theorem 4.9. [21, Th. 3.1] *Let $\mathbb{H}$ be an arbitrary Hilbert space and $T \in \mathcal{L}(\mathbb{H})$. Then for any $A \in \mathcal{L}(\mathbb{H})$, $T \perp_B A \Leftrightarrow Tx_0 \perp Ax_0$ for some $x_0 \in M_T$ if and only if $M_T = S_{H_0}$, where H_0 is a finite-dimensional subspace of $\mathbb{H}$ and $\|T\|_{{H_0}^{\perp}} < \|T\|$.*

We are now in a position to characterize the smoothness of a bounded linear operator defined on an arbitrary Hilbert space.

Theorem 4.10. [21, Th. 4.6] *Let $\mathbb{H}$ be a Hilbert space. Then $T \in \mathcal{L}(\mathbb{H})$ is a smooth point if and only if span M_T is one-dimensional, i.e., $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{H}}$ and* $\sup\{\|Ty\| : x \perp_B y, y \in S_{\mathbb{H}}\} < \|T\|$.

Proof. The sufficient part of the proof follows in the same spirit as that of Theorem 4.4. We prove the necessary part in the following three steps.
(i) T attains its norm at some point of $S_{\mathbb{H}}$.
(ii) *span* M_T is one-dimensional, i.e., $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{H}}$.
(iii) $\sup\{\|Ty\| : x \perp y, y \in S_{\mathbb{H}}\} < \|T\|$.

Step (i) : If T does not attain its norm then there exists a sequence $\{e_n\}$ of orthonormal vectors such that $\|Te_n\| \to \|T\|$. Then there exists an orthonormal basis $\mathcal{B}$ containing $\{e_n : n = 1, 2, \ldots\}$. Let $H_0 = \overline{span\ \langle\{e_{2n} : n \in \mathbb{N}\}\rangle}$. Then $\mathbb{H} = H_0 \oplus H_0^{\perp}$. Every element z in $\mathbb{H}$ can be written as $z = x + y$ where $x \in H_0, y \in H_0^{\perp}$. Define a linear operator A_1 on $\mathbb{H}$ as : $A_1(z) = Tx$. We show that A_1 is bounded. Now $\|A_1 z\| = \|Tx\| \leq \|T\|\|x\| \leq \|T\|$ for every z with $\|z\| = 1$. Thus A_1 is bounded. Consider another bounded operator $A_2 = T - A_1$. Next we claim that $T \perp_B A_1, T \perp_B A_2$. Now, $\|Te_{2n+1}\| \to \|T\|$ and $\langle Te_{2n+1}, A_1 e_{2n+1}\rangle \to 0$ and so by [19, Lemma 2], we get $T \perp_B A_1$. Similarly, we can show that $T \perp_B A_2$. But $T \not\perp_B (A_1 + A_2)$. This contradicts the fact that T is smooth. Hence, (i) holds.

Step (ii) : Suppose $x, y \in M_T$ where $\{x, y\}$ is linearly independent. By [27, Th. 2.2], the norm attaining set M_T is the unit sphere of some subspace of $\mathbb{H}$. So without loss of generality, we assume that $x \perp y$. Let $H_0 = \langle\{x, y\}\rangle$. Then $\mathbb{H} = H_0 \oplus H_0^{\perp}$. Clearly, each $z \in \mathbb{H}$ can be written as $z = \alpha x + \beta y + h$ where $h \in H_0^{\perp}$ and α, β are scalars. Define $A_1, A_2 : \mathbb{H} \longrightarrow \mathbb{H}$ as : $A_1(\alpha x + \beta y + h) = \alpha Tx$, $A_2(\alpha x + \beta y + h) = \beta Ty + Th$. Then as before it is easy to check that both A_1, A_2 are bounded linear operators and $T \perp_B A_1, T \perp_B A_2$. But $T = A_1 + A_2$ and so $T \not\perp_B (A_1 + A_2)$, which contradicts the fact that T is smooth. Thus (ii) holds.

Step (iii) : Assume that $M_T = span\langle\{x\}\rangle \cap S_{\mathbb{X}}$ and $\sup\{\|Ty\| : x \perp y, y \in S_{\mathbb{H}}\} = \|T\|$. Let $H_0 = span\ \langle\{x\}\rangle$. Then $\mathbb{H} = H_0 \oplus H_0^{\perp}$. Each element $z \in \mathbb{H}$ can be written as $z = x + y$, where $x \in H_0$ and $y \in H_0^{\perp}$. Define $A_1, A_2 : \mathbb{H} \longrightarrow \mathbb{H}$ as : $A_1 z = Tx, A_2 z = Ty$. It is easy to check that both

A_1, A_2 are bounded linear operators and $T \perp_B A_1, T \perp_B A_2$. But $T = A_1 + A_2$ and so $T \not\perp_B (A_1 + A_2)$, which contradicts the fact that T is smooth. Thus (iii) holds.

□

Remark 4.11. Theorem 4.10 completes the study of smoothness of bounded linear operators defined on arbitrary Hilbert spaces through Birkhoff-James orthogonality. However, the study is much more involved when the operators are defined between arbitrary infinite-dimensional Banach spaces. Interested readers may follow [6, 7, 8, 21, 24, 29, 30] for further study on smoothness of operators.

5. Birkhoff-James orthogonality and strict convexity in the space of operators

We begin this section with the observation that in a strictly convex space Birkhoff-James orthogonality takes a nice form as follows:

Theorem 5.1. [28, Th.2.2] *Suppose $\mathbb{X}$ is a Banach space. Then $\mathbb{X}$ is strictly convex if and only for $x, y \in S_{\mathbb{X}}$, $x \perp_B y$ implies $\|x + \lambda y\| > 1$, for each non-zero scalar λ.*

Proof. **Necessary Part** : Assume that there exist $x, y \in S_{\mathbb{X}}$ such that $x \perp_B y$ but $\|x + \alpha y\| = 1$ for some $\alpha \neq 0$. Then $\|x\| = \|x + \alpha y\| = 1$. Now, by convexity of the norm function, we get $\|(1-t)x + t(x + \alpha y)\| = 1$ for each $t \in [0, 1]$, which contradicts the fact that $\mathbb{X}$ is strictly convex.
Sufficient part : If possible, suppose the unit sphere $S_{\mathbb{X}}$ contains a straight line segment, i.e., there exist $u, v \in S_{\mathbb{X}}$ with $\|tu + (1-t)v\| = 1$ for all $t \in [0, 1]$. Considering $x = u, y = \frac{1}{\|v-u\|}(v - u)$, it is easy to see that x, y violates the hypothesis. Thus $\mathbb{X}$ is strictly convex. □

Our main aim in this section is to show that this result is carried to some extent to the space of operators.

Theorem 5.2. [5, Th.2.1] *Let $\mathbb{X}$ be a finite-dimensional strictly convex Banach space. Let $T, A \in \mathcal{L}(\mathbb{X}, \mathbb{X})$ and $T \perp_B A$. Then either $\|T + \lambda A\| > \|T\|$ for each non-zero scalar λ or $Ax = 0$ for some $x \in M_T$.*

Proof. If $Ax = 0$ for some $x \in M_T$ then we are done. Assuming $Ax \neq 0$ for all $x \in M_T$ we show that $\|T + \lambda A\| > \|T\|$ for each non-zero scalar λ. We consider the following two cases :

Case 1. There exists $x \in M_T$ such that $Tx \perp_B Ax$.
Since $\mathbb{X}$ is strictly convex, by Theorem 5.1, we get $\|Tx + \lambda Ax\| > \|Tx\|$ for each non-zero scalar λ. Then for each non-zero scalar λ, we have $\|T+\lambda A\| \geq \|Tx + \lambda Ax\| > \|Tx\| = \|T\|$.
Case 2. There exists no $x \in M_T$ such that $Tx \perp_B Ax$.
For any $x \in M_T$, either $\|Tx + \lambda Ax\| \geq \|T\| \ \forall \ \lambda \geq 0$ or $\|Tx + \lambda Ax\| \geq \|T\| \ \forall \ \lambda \leq 0$. We show that there exist $x_1, x_2 \in M_T$ such that $\|Tx_1 + \lambda Ax_1\| \geq \|T\| \ \forall \ \lambda \geq 0$ and $\|Tx_2 + \lambda Ax_2\| \geq \|T\| \ \forall \ \lambda \leq 0$. If not, without loss of generality, we may assume that

$$\forall x \in M_T, \ \|Tx + \lambda Ax\| \geq \|T\| \ \forall \ \lambda \leq 0. \tag{5.1}$$

For each $n \in \mathbb{N}$, the operator $(T + \frac{1}{n}A)$ attains its norm. So there exists $x_n \in S_{\mathbb{X}}$ such that $\|T+\frac{1}{n}A\| = \|(T+\frac{1}{n}A)x_n\|$. Since $\mathbb{X}$ is finite-dimensional, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$ (say) in $B_{\mathbb{X}}$. Without loss of generality, we assume that $x_n \to x_0$. Then $Tx_n \to Tx_0$, $Ax_n \to Ax_0$. As $T \perp_B A$ we have $\|T + \frac{1}{n}A\| \geq \|T\| \ \forall n \in \mathbb{N}$ and so $\|Tx_n + \frac{1}{n}Ax_n\| \geq \|T\| \geq \|Tx_n\| \ \forall n \in \mathbb{N}$. Letting $n \to \infty$ we get $\|Tx_0\| \geq \|T\| \geq \|Tx_0\|$. So $x_0 \in M_T$. For any $\lambda > \frac{1}{n}$, we claim that $\|Tx_n + \lambda Ax_n\| \geq \|Tx_n\|$. Otherwise

$$\begin{aligned}
Tx_n + \frac{1}{n}Ax_n &= (1 - \frac{1}{n\lambda})Tx_n + \frac{1}{n\lambda}(Tx_n + \lambda Ax_n) \\
\Rightarrow \|Tx_n + \frac{1}{n}Ax_n\| &\leq (1 - \frac{1}{n\lambda})\|Tx_n\| + \frac{1}{n\lambda}\|Tx_n + \lambda Ax_n\| \\
\Rightarrow \|Tx_n + \frac{1}{n}Ax_n\| &< \|Tx_n\| \text{ , a contradiction.}
\end{aligned}$$

Choose $\lambda > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\lambda > \frac{1}{n_0}$ and so for all $n \geq n_0$ we get, $\|Tx_n + \lambda Ax_n\| \geq \|Tx_n\|$. Letting $n \to \infty$ we get $\|Tx_0 + \lambda Ax_0\| \geq \|Tx_0\|$ - which holds for any $\lambda \geq 0$. This along with (5.1) shows that $Tx_0 \perp_B Ax_0$, which violates hypothesis of Case 2. Hence, there exist $x_1, x_2 \in M_T$ such that $\|Tx_1 + \lambda Ax_1\| \geq \|T\| \ \forall \ \lambda \geq 0$ and $\|Tx_2 + \lambda Ax_2\| \geq \|T\| \ \forall \ \lambda \leq 0$. As $\mathbb{X}$ is strictly convex, we have $\|Tx_1 + \lambda Ax_1\| > \|T\| \ \forall \ \lambda > 0$ and $\|Tx_2 + \lambda Ax_2\| > \|T\| \ \forall \ \lambda < 0$. For $\lambda > 0$, $\|T+\lambda A\| \geq \|(T+\lambda A)x_1\| > \|T\|$ and for $\lambda < 0$, $\|T + \lambda A\| \geq \|(T + \lambda A)x_2\| > \|T\|$. This completes the proof. □

If in addition to the previous theorem, we assume that A is injective then we get the following theorem:

Theorem 5.3. [5, Th.2.2] *Let $\mathbb{X}$ be a finite-dimensional strictly convex Banach space and $A \in \mathcal{L}(\mathbb{X}, \mathbb{X})$ be injective. Then for any $T \in \mathcal{L}(\mathbb{X}, \mathbb{X})$, $T \perp_B A \Rightarrow \|T + \lambda A\| > \|T\|$ for each non-zero scalar λ.*

Remark 5.4. The following example shows that the above theorem can not be improved on to bounded linear operators:

Consider $T\colon \ell_2 \to \ell_2$ defined by $Te_1 = -e_1$, and $Te_n = (1 - 1/n)e_n$ for $n \geq 2$, where $\{e_n : n \in \mathbb{N}\}$ is the usual orthonormal basis for the Hilbert space ℓ_2. Then T attains norm only at $\pm e_1$. Let $A = I$, the identity operator on ℓ_2. Then both T, A are bounded linear operators. It is easy to check that $T \perp_B A$ but there exists a non-zero scalar λ such that $\|T + \lambda A\| = \|T\|$.

In our next theorem, we give a sufficient condition for strict convexity of any finite-dimensional Banach space $\mathbb{X}$.

Theorem 5.5. [5, Th.2.4] *Let $\mathbb{X}$ be a finite-dimensional Banach space. Suppose that for any $T, A \in \mathcal{L}(\mathbb{X}, \mathbb{X})$ with $T \perp_B A$ either $\|T + \lambda A\| > \|T\|$ for each non-zero scalar λ or $Ax = 0$ for some $x \in M_T$. Then $\mathbb{X}$ is strictly convex.*

Proof. If possible, suppose that $\mathbb{X}$ is not strictly convex. Then there exist $x, y \in S_{\mathbb{X}}$ such that $x \neq y$ and $\|tx + (1 - t)y\| = 1$ for all $t \in (0, 1)$. Then $x \perp_B y - x$. By Theorem 1.5, there exists a hyperspace H such that $x \perp_B H$. Each element $z \in \mathbb{X}$ can be written as $z = \alpha x + h$ where α is a scalar and $h \in H$. Define two operators $T, A \colon \mathbb{X} \longrightarrow \mathbb{X}$ as follows : $Tz = \alpha x$ and $Az = \alpha(x - y)$. It is easy to check that both T and A are linear operators. We have, $x \in M_T$, $Tx \perp_B Ax$ and so $T \perp_B A$. From the construction of T, A it follows that $\|T - A\| = 1 = \|T\|$.

We next show that $Aw \neq 0$ for any $w \in M_T$. Let $w = \beta x + h \in M_T$. As $\|Tw\| = 1$, we have $|\beta| = 1$. So $Aw = \beta(x - y) \neq 0$ for any $w \in M_T$. This contradiction proves that $\mathbb{X}$ is strictly convex. □

Combining the last two results, we get the following characterization for strictly convex Banach spaces.

Theorem 5.6. *Let $\mathbb{X}$ be a finite-dimensional Banach space. Then $\mathbb{X}$ is strictly convex if and only if for any $T, A \in \mathcal{L}(\mathbb{X}, \mathbb{X})$, $T \perp_B A \Rightarrow$ either $\|T + \lambda A\| > \|T\|$ for each non-zero scalar λ or $Ax = 0$ for some $x \in M_T$.*

Remark 5.7. We would like to end this article with the remark that although all the theorems in this article are discussed for real Banach (Hilbert) spaces, most of the theorems hold for complex spaces with suitable modification.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 79–94

LIDSTONE INTERPOLATION I : ONE VARIABLE*

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Abstract. According to Lidstone interpolation theory, an entire function of exponential type $< \pi$ is determined by it derivatives of even order at 0 and 1. This theory can be generalized to several variables. Here we survey the theory for a single variable. Complete proofs are given. This first paper of a trilogy is devoted to Univariate Lidstone interpolation; Bivariate and Multivariate Lidstone interpolation will be the topic of two forthcoming papers.

1. Introduction

In 1930, in a seminal paper [3], G. J. Lidstone introduced a basis $\Lambda_k(z)$ $(k \geq 0)$ of the space $\mathbb{C}[z]$ of polynomials in a single variable, which has the property that any polynomial $f \in \mathbb{C}[z]$ has a finite expansion

$$f(z) = \sum_{k\geq 0} f^{(2k)}(0)\Lambda_k(1-z) + \sum_{k\geq 0} f^{(2k)}(1)\Lambda_k(z),$$

where

$$f^{(2k)} = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} f.$$

Two years later, H. Poritsky [4] J. M. Whittaker [8] extended these expansions to entire functions of exponential type $< \pi$. In 1936, I. J. Schoenberg [5] proved that the only entire functions of finite exponential type which vanish at the two points 0 and 1 together with all their derivatives of even order are the linear combinations with constant coefficients of the functions $\sin(k\pi z)$, with $k \in \mathbb{N}$. This result follows from an expansion formula for such functions which was obtained by R. C. Buck in 1955 [2].

* This article is based on the text of the plenary talk given by the author at the 87th Annual Conference of the IMS - An International meet held at MGM University, Aurangabad, India during December 4 - 7, 2021 using online mode.

We recall the basic facts concerning Lidstone expansion in a single variable. In this section z and ζ are in $\mathbb{C}$. It will be convenient to use notations which can be generalized to several variables. The classical Lidstone polynomials [3, §6 p. 18] $\Lambda_k(z)$ $(k \geq 0)$ are denoted here $\Lambda_{2k,1}(z)$, while the polynomials $\Lambda_k(1-z)$ are written here $\Lambda_{2k,0}(z)$. The successive derivatives of a function f of a single variable are denoted f', f'', ..., $f^{(k)}$.

The Lidstone polynomials are introduced in Theorem 1. They are the solution of a system of differential equations (Lemma 2). The unicity of the expansion for an entire function of exponential type $< \pi$ (Theorem 2) is easy to prove, the existence (Theorem 3) needs more work - both results are due to H. Poritsky and J. M. Whittaker. Next we prove integral formulae for the Lidstone polynomials (Propositions 1 and 2), and we give proofs of the results of Buck (Proposition 3) and Schoenberg (Corollary 2).

This paper is self contained, full proofs are given. It is an introduction to two forthcoming papers, [6] where we extend the theory to two variables and [7] where we extend the theory to an arbitrary number of variables.

2. Definition of the univariate Lidstone polynomials

Let us recall the definitions of the *order* of an entire function f:

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log|f|_r}{\log r} \quad \text{where} \quad |f|_r = \sup_{|z|=r} |f(z)|$$

and of the *exponential type* of f:

$$\tau(f) = \limsup_{r\to\infty} \frac{\log|f|_r}{r}\cdot$$

If the exponential type of f is finite, then f has order ≤ 1. If f has order < 1, then the exponential type is 0. Using Cauchy's estimate for the coefficients of the Taylor series together with Stirling's formula for $n!$, one deduces [8, Lemma 1] that if f has exponential type $\tau(f)$, then for all $z_0 \in \mathbb{C}$,

$$\limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} = \tau(f).$$

For $\zeta \in \mathbb{C} \setminus \{0\}$, the function $\mathrm{e}^{\zeta z}$ has order 1 and exponential type $|\zeta|$.

We denote by $2\mathbb{N}$ the set of even nonnegative integers. The starting point of the theory of Lidstone interpolation is the following.

Lemma 1. *Let f be a polynomial satisfying*

$$f^{(t)}(0) = f^{(t)}(1) = 0 \text{ for all } t \in 2\mathbb{N}. \tag{1}$$

Then $f = 0$.

We give three proofs of this lemma, the arguments are slightly different and will be used again.

First proof. By induction on the total degree of the polynomial f.

If f has degree ≤ 1, say $f(z) = a_0 z + a_1$, the conditions $f(0) = f(1) = 0$ imply $a_0 = a_1 = 0$, hence $f = 0$.

If f has degree $\leq d$ with $d \geq 2$ and satisfies the hypotheses, then f'' also satisfies the hypotheses and has degree $< d$, hence by induction $f'' = 0$ and therefore f has degree ≤ 1.

Lemma 1 follows. □

Second proof. Let f be a polynomial satisfying (1). The assumption $f^{(t)}(0) = 0$ for all $t \in 2\mathbb{N}$ means that f is an odd function: $f(-z) = -f(z)$. The assumption $f^{(t)}(1) = 0$ for all $t \in 2\mathbb{N}$ means that $f(1-z)$ is an odd function: $f(1-z) = -f(1+z)$. We deduce

$$f(z+2) = f(1+z+1) = -f(1-z-1) = -f(-z) = f(z),$$

hence the polynomial f is periodic, and therefore is a constant.

Since $f(0) = 0$, we conclude $f = 0$. □

Third proof. Assume (1). Write

$$f(z) = a_1 z + a_3 z^3 + a_5 z^5 + a_7 z^7 + + \cdots + a_{2m+1} z^{2m+1} + \cdots$$

(finite sum). We have $f(1) = f''(1) = f^{(\mathrm{iv})}(1) = \cdots = 0$:

$$\begin{array}{cccccccc}
a_1 & +a_3 & +a_5 & +a_7 & +\cdots & +a_{2n+1} & +\cdots = 0 \\
 & 6a_3 & +20a_5 & +42a_7 & +\cdots & +2m(2m+1)a_{2m+1} & +\cdots = 0 \\
 & & 120a_5 & +840a_7 & +\cdots & +\frac{(2m+1)!}{(2m-3)!}a_{2m+1} & +\cdots = 0 \\
 & & & & & \ddots & \vdots
\end{array}$$

The matrix of this system is triangular with maximal rank. We conclude $a_1 = a_3 = a_5 = \cdots = 0$. □

The fact that this matrix has maximal rank means that a polynomial f is uniquely determined by the numbers

$$f^{(t)}(0) \quad \text{and} \quad f^{(t)}(1) \quad \text{for} \quad t \in 2\mathbb{N}.$$

Let $T \geq 0$ be even. The space $\mathbb{C}[z]_{\leq T+1}$ of polynomials of degree $\leq T+1$ has dimension $T+2$. All elements $f \in \mathbb{C}[z]_{\leq T+1}$ satisfy $f^{(k)} = 0$ for $k \geq T+2$. Lemma 1 shows that the linear map

$$\begin{array}{ccc} \mathbb{C}[z]_{\leq T+1} & \longrightarrow & \mathbb{C}^{T+2} \\ f & \longmapsto & \bigl(f^{(t)}(0),\ f^{(t)}(1)\bigr)_{0\leq t\leq T,\, t\in 2\mathbb{N}} \end{array}$$

is injective. Hence it is an isomorphism.

Given numbers a_t and b_t, ($t \in 2\mathbb{N}$), where all but finitely many of them are 0, there is a unique polynomial f such that

$$f^{(t)}(0) = a_t \quad\text{and}\quad f^{(t)}(1) = b_t \text{ for all } t \in 2\mathbb{N}.$$

In particular, for each $t \in 2\mathbb{N}$, there is a unique polynomial $\Lambda_{t,0}$ which satisfies

$$\Lambda_{t,0}^{(\tau)}(0) = \delta_{t,\tau} \quad\text{and}\quad \Lambda_{t,0}^{(\tau)}(1) = 0 \quad\text{for}\quad \tau \in 2\mathbb{N}$$

(Kronecker symbol), and there is a unique polynomial $\Lambda_{t,1}$ which satisfies

$$\Lambda_{t,1}^{(\tau)}(0) = 0 \quad\text{and}\quad \Lambda_{t,1}^{(\tau)}(1) = \delta_{t,\tau} \quad\text{for}\quad \tau \in 2\mathbb{N}.$$

Therefore:

Theorem 1 (G. J. Lidstone (1930)). *There exist two sequences of polynomials,* $\bigl(\Lambda_{t,0}(z)\bigr)_{t\in 2\mathbb{N}}$, $\bigl(\Lambda_{t,1}(z)\bigr)_{t\in 2\mathbb{N}}$, *such that any polynomial* f *can be written as a finite sum*

$$f(z) = \sum_{t\in 2\mathbb{N}} f^{(t)}(0)\Lambda_{t,0}(z) + \sum_{t\in 2\mathbb{N}} f^{(t)}(1)\Lambda_{t,1}(z). \tag{2}$$

The involution $z \mapsto 1-z$:

$$0 \mapsto 1, \quad 1 \mapsto 0, \quad 1-z \mapsto z$$

shows that $\Lambda_{t,0}(z) = \Lambda_{t,1}(1-z)$.

At this point, we can make an analogy with Taylor series, where the polynomials $z^m/m!$ satisfy

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z^m}{m!}\right)_{z=0} = \delta_{mk} \quad\text{for}\quad m \geq 0 \quad\text{and}\quad k \geq 0.$$

Given a sequence $(a_m)_{m\geq 0}$ of complex numbers, the unique analytic solution (if it exists) f of the interpolation problem

$$f^{(m)}(0) = a_m \text{ for all } m \geq 0$$

is given by the Taylor expansion

$$f(z) = \sum_{m\geq 0} a_m \frac{z^m}{m!}.$$

Lidstone expansion replaces the single point 0 and the sequence of all derivatives with two points 0 and 1 and only the derivatives of even order at these two points.

The first Lidstone polynomial is $\Lambda_{0,1}(z) = z$:

$$\Lambda_{0,1}(0) = 0, \quad \Lambda_{0,1}(1) = 1, \quad \Lambda_{0,1}^{(t)}(0) = \Lambda_{0,1}^{(t)}(1) = 0 \text{ for } t \in 2\mathbb{N},\ t \geq 2.$$

The next lemma provides an inductive way for finding all of them.

3. Differential equation

Lemma 2. *The sequence of Lidstone polynomials $\big(\Lambda_{t,1}\big)_{t\in 2\mathbb{N}}$ is determined by $\Lambda_{0,1}(z) = z$ and*

$$\Lambda''_{t,1} = \Lambda_{t-2,1} \text{ for } t \geq 2 \text{ even},$$

with the initial conditions $\Lambda_{t,1}(0) = \Lambda_{t,1}(1) = 0$ for $t \in 2\mathbb{N}$, $t \geq 2$. More precisely, let $\big(L_t\big)_{t\in 2\mathbb{N}}$ be a sequence of polynomials satisfying $L_0(z) = z$ and

$$L''_t = L_{t-2} \text{ for } t \in 2\mathbb{N},\ t \geq 2,$$

with the initial conditions $L_t(0) = L_t(1) = 0$ for $t \in 2\mathbb{N}$, $t \geq 2$; then $L_t = \Lambda_{t,1}$ for all $t \in 2\mathbb{N}$.

Notice that the assumption $L_0(z) = z$ cannot be omitted: given any polynomial A, there is a unique sequence $\big(L_t\big)_{t\in 2\mathbb{N}}$ satisfying all other assumptions but with $L_0 = A$.

Proof. That the sequence $\big(\Lambda_{t,1}\big)_{t\in 2\mathbb{N}}$ satisfies these conditions is plain. We now prove the unicity. Let $\big(L_t\big)_{t\in 2\mathbb{N}}$, be a sequence of polynomials satisfying the conditions of Lemma 2. By assumption $L_0(z) = z$. By induction, assume that for some $t \geq 2$ we know that $L_{t-2} = \Lambda_{t-2,1}$. Then the difference $g = L_t - \Lambda_{t,1}$ satisfies $g'' = 0$, hence g has degree ≤ 1. The assumptions $L_t(0) = L_t(1) = 0$ for $t \in 2\mathbb{N}$, $t \geq 2$ imply $g = 0$. □

For $t \in 2\mathbb{N}$, the polynomial $\Lambda_{t,1}$ is odd, it has degree $t + 1$ and leading term $\frac{1}{(t+1)!} z^{t+1}$. For instance

$$\Lambda_{2,1}(z) = \frac{1}{6}(z^3 - z) = \frac{1}{6} z(z-1)(z+1),$$

$$\Lambda_{2,0}(z) = \Lambda_{2,1}(1-z) = -\frac{z^3}{6} + \frac{z^2}{2} - \frac{z}{3} = -\frac{1}{6}z(z-1)(z-2),$$

$$\Lambda_{4,1}(z) = \frac{1}{120}z^5 - \frac{1}{36}z^3 + \frac{7}{360}z = \frac{1}{360}z(z^2-1)(3z^2-7),$$

and

$$\begin{aligned}\Lambda_{4,0}(z) = \Lambda_{4,1}(1-z) &= -\frac{1}{120}z^5 + \frac{1}{24}z^4 - \frac{1}{18}z^3 + \frac{1}{45}z \\ &= -\frac{1}{360}z(z-1)(z-2)(3z^2-6z-4).\end{aligned} \tag{3}$$

4. Recurrence formula

For $t \in 2\mathbb{N}$, the polynomial $f_t(z) = z^{t+1}$ satisfies

$$f_t^{(\tau)}(0) = 0 \quad \text{for} \quad \tau \in 2\mathbb{N}, \qquad f_t^{(\tau)}(1) = \begin{cases} \frac{(t+1)!}{(t-\tau+1)!} & \text{for } 0 \le \tau \le t,\ \tau \in 2\mathbb{N} \\ 0 & \text{for } \tau \ge t+2,\ \tau \in 2\mathbb{N}. \end{cases}$$

From Theorem 1 one deduces, for $t \in 2\mathbb{N}$,

$$z^{t+1} = \sum_{\substack{0 \le \tau \le t \\ \tau \in 2\mathbb{N}}} \frac{(t+1)!}{(t-\tau+1)!} \Lambda_{\tau,1}(z),$$

which yields the recurrence formula

$$\Lambda_{t,1}(z) = \frac{1}{(t+1)!} z^{t+1} - \sum_{\substack{0 \le \tau \le t-2 \\ \tau \in 2\mathbb{N}}} \frac{1}{(t-\tau+1)!} \Lambda_{\tau,1}(z). \tag{4}$$

Another consequence of Theorem 1 is

$$\frac{z^t}{t!} = \Lambda_{t,0}(z) + \sum_{\substack{0 \le \tau \le t \\ \tau \in 2\mathbb{N}}} \frac{1}{(t-\tau)!} \Lambda_{\tau,1}(z) \tag{5}$$

for $t \in 2\mathbb{N}$.

5. Unicity for entire functions

According to Theorem 1, a polynomial is determined by the values of its derivatives of even order at the two points 0 and 1. H. Poritsky [4] and J. M. Whittaker [8] proved that he same is true more generally for an entire function of exponential type $< \pi$:

Theorem 2 (H. Poritsky, J. M. Whittaker 1932)**.** *Let f be an entire function of exponential type $< \pi$ satisfying $f^{(t)}(0) = f^{(t)}(1) = 0$ for all sufficiently large $t \in 2\mathbb{N}$. Then f is a polynomial.*

Proof. We combine some arguments that we used for polynomials in the three proofs of Lemma 1. Let $\tilde{f} = f - P$, where P is the polynomial satisfying

$$P^{(t)}(0) = f^{(t)}(0) \quad \text{and} \quad P^{(t)}(1) = f^{(t)}(1) \quad \text{for} \quad t \in 2\mathbb{N}.$$

We have $\tilde{f}^{(t)}(0) = \tilde{f}^{(t)}(1) = 0$ for all $t \in 2\mathbb{N}$.

The functions $\tilde{f}(z)$ and $\tilde{f}(1-z)$ are odd, hence $\tilde{f}(z)$ is periodic of period 2. Therefore there exists a function g, analytic in $\mathbb{C}^\times$, such that $\tilde{f}(z) = g(\mathrm{e}^{\pi \mathrm{i} z})$. Since $\tilde{f}(z)$ has exponential type $< \pi$, using Cauchy's inequalities for the coefficients of the Laurent expansion of g at the origin, we deduce $\tilde{f} = 0$ and $f = P$. □

Theorem 2 is best possible in the following two directions:

- The entire function $\sin(\pi z)$ has exponential type π and satisfies $f^{(t)}(0) = f^{(t)}(1) = 0$ for all $t \in 2\mathbb{N}$.
- If we assume only $f^{(t)}(0) = f^{(t)}(1) = 0$ for all even t outside a finite set, then the conclusion is still valid – this follows from Theorems 1 and 2. However, if we remove an infinite subset of conditions in the assumptions of Theorem 2, then the conclusion is no more valid:

Lemma 3. *Let $\mathcal{E}$ be an infinite subset of the set of $(t,i) \in 2\mathbb{N} \times \{0,1\}$. Then there exists a non countable set of transcendental entire functions f of order 0, with rational Taylor coefficients at the origin, such that $f^{(t)}(i) = 0$ for all $(t,i) \in (2\mathbb{N} \times \{0,1\}) \setminus \mathcal{E}$.*

Proof. Let $(P_m)_{m \geq 0}$ be an infinite sequence of polynomials belonging to the set $\{\Lambda_{t,i} \mid (t,i) \in \mathcal{E}\}$. Let d_m be the degree of P_m. We assume the sequence $(d_m)_{m \geq 0}$ to be increasing. Let $(c_m)_{m \geq 0}$ be a sequence of rational numbers such that

$$|P_m|_r \leq |c_m| r^{d_m}$$

for all $r \geq 1$ and $m \geq 0$. For $m \geq 0$, set

$$u_m = \frac{1}{c_m (d_m!)^2}.$$

The series

$$\sum_{m \geq 0} u_m P_m(z)$$

is uniformly convergent on any compact subset of $\mathbb{C}$, its sum $f(z)$ is an entire function of order 0. From the uniform convergence of the series, we deduce, for all $t \in 2\mathbb{N}$ and $i \in \{0,1\}$,

$$f^{(t)}(i) = \begin{cases} u_m & \text{if } P_m = \Lambda_{t,i}, \\ 0 & \text{if } P_m \neq \Lambda_{t,i}. \end{cases}$$

The conclusion of Lemma 3 follows. □

6. Expansion of entire functions and generating series

Lidstone finite expansion for polynomials (2) has been extended in [4, 8] to an infinite expansion for entire functions of exponential type $< \pi$ as follows:

Theorem 3 (H. Poritsky, J. M. Whittaker 1932)**.** *The expansion (2) holds for any entire function f of exponential type $< \pi$, where, for each $z \in \mathbb{C}$, the series*

$$\sum_{t \in 2\mathbb{N}} f^{(t)}(0)\Lambda_{t,0}(z) \quad \text{and} \quad \sum_{t \in 2\mathbb{N}} f^{(t)}(1)\Lambda_{t,1}(z)$$

are absolutely convergent.

Notice that Theorem 2 is a consequence of Theorem 3.

We will deduce from Theorem 3 explicit formulae for the following two generating series:

$$M_1(\zeta, z) := \sum_{t \in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t \quad \text{and} \quad M_0(\zeta, z) := \sum_{t \in 2\mathbb{N}} \Lambda_{t,0}(z)\zeta^t.$$

Corollary 1. *For $|\zeta| < \pi$, we have*

$$M_1(\zeta, z) = \frac{\sinh(\zeta z)}{\sinh(\zeta)} \tag{6}$$

and

$$M_0(\zeta, z) = \cosh(\zeta z) - \sinh(\zeta z)\coth(\zeta). \tag{7}$$

Since $\Lambda_{t,0}(z) = \Lambda_{t,1}(1-z)$, we have $M_0(\zeta, z) = M_1(\zeta, 1-z)$ and the trigonometric relation

$$\sinh(z_1 - z_2) = \sinh(z_1)\cosh(z_2) - \cosh(z_1)\sinh(z_2)$$

shows that the two formulae (6) and (7) are equivalent.

Proof of (6) *as a consequence of Theorem 3.* Let $\zeta \in \mathbb{C}$ satisfy $|\zeta| < \pi$. We use Theorem 3 and formula (2) for the function $f_\zeta(z) = \mathrm{e}^{\zeta z}$. Since $f_\zeta^{(t)}(0) = \zeta^t$ and $f_\zeta^{(t)}(1) = \mathrm{e}^\zeta \zeta^t$, we deduce

$$\mathrm{e}^{\zeta z} = \sum_{t\in 2\mathbb{N}} \Lambda_{t,0}(z)\zeta^t + \mathrm{e}^\zeta \sum_{t\in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t. \tag{8}$$

Replacing ζ with $-\zeta$ yields

$$\mathrm{e}^{-\zeta z} = \sum_{t\in 2\mathbb{N}} \Lambda_{t,0}(z)\zeta^t + \mathrm{e}^{-\zeta} \sum_{t\in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t.$$

Hence

$$\mathrm{e}^{\zeta z} - \mathrm{e}^{-\zeta z} = (\mathrm{e}^\zeta - \mathrm{e}^{-\zeta}) \sum_{t\in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t.$$

This proves (6). □

From (6) one readily deduces the following relation [8, Equation(3.5)] for $t \in 2\mathbb{N}$,

$$\Lambda_{t,1}(z) = \frac{2^{t+1}}{(t+1)!} B_{t+1}\left(\frac{1+z}{2}\right),$$

between the Lidstone polynomials and the Bernoulli polynomials; the latter are defined by

$$t\frac{\mathrm{e}^{tz}-1}{\mathrm{e}^t-1} = \sum_{n=1}^{\infty} B_n(z)\frac{t^n}{n!}\cdot$$

From Corollary 1 we deduce $M_1(\zeta, z+1) - M_1(\zeta, z-1) = 2\cosh(\zeta z)$, which means

$$\Lambda_{t,1}(z+1) - \Lambda_{t,1}(z-1) = 2\frac{z^t}{t!}\cdot$$

This relation also follows from the functional equation

$$B_n(z+1) - B_n(z) = nz^{n-1}$$

of the Bernoulli polynomials.

Our proof of Theorem 3 below will rest on (6), hence we need to give a direct proof of it.

Direct proof of (6). We start with the formula

$$\mathrm{e}^{\zeta z} = \frac{\sinh(\zeta(1-z))}{\sinh(\zeta)} + \mathrm{e}^\zeta \frac{\sinh(\zeta z)}{\sinh(\zeta)}, \tag{9}$$

which holds for $\zeta \in \mathbb{C}$, $\zeta \notin \pi\mathrm{i}\mathbb{Z}$ and $z \in \mathbb{C}$. For $\zeta \in \mathbb{C}$, $\zeta \notin \pi\mathrm{i}\mathbb{Z}$, the entire function

$$f(z) = \frac{\sinh(\zeta z)}{\sinh(\zeta)} = \frac{\mathrm{e}^{\zeta z} - \mathrm{e}^{-\zeta z}}{\mathrm{e}^\zeta - \mathrm{e}^{-\zeta}}$$

satisfies
$$f'' = \zeta^2 f, \quad f(0) = 0, \quad f(1) = 1,$$
hence $f^{(t)}(0) = 0$ and $f^{(t)}(1) = \zeta^t$ for all $t \in 2\mathbb{N}$.

For $z \in \mathbb{C}$ and $|\zeta| < \pi$, let
$$F(\zeta, z) = \frac{\sinh(\zeta z)}{\sinh(\zeta)}$$
with $F(0, z) = z$. Fix $z \in \mathbb{C}$. The map $\zeta \mapsto F(\zeta, z)$ is analytic in the disc $|\zeta| < \pi$ and is even: $F(-\zeta, z) = F(\zeta, z)$. Consider its Taylor expansion at the origin:
$$F(\zeta, z) = \sum_{t \in 2\mathbb{N}} c_t(z) \zeta^t$$
with $c_0(z) = z$. For fixed $z \in \mathbb{C}$, this Taylor series is absolutely and uniformly convergent on any compact subset of the disc $|\zeta| < \pi$. We have $F(\zeta, 0) = 0$, $F(\zeta, 1) = 1$, and
$$F(\zeta, z) = \frac{\mathrm{e}^{\zeta z} - \mathrm{e}^{-\zeta z}}{\mathrm{e}^{\zeta} - \mathrm{e}^{-\zeta}} \cdot$$
From
$$c_t(z) = \frac{1}{t!} \left(\frac{\partial}{\partial \zeta} \right)^t F(0, z)$$
it follows that $c_t(z)$ is a polynomial. From
$$\left(\frac{\partial}{\partial z} \right)^2 F(\zeta, z) = \zeta^2 F(\zeta, z)$$
we deduce
$$c_t'' = c_{t-2} \ \text{ for } \ t \in 2\mathbb{N},\ t \geq 2.$$
Since $c_t(0) = c_t(1) = 0$ for $t \in 2\mathbb{N}$, $t \geq 2$, we deduce from Lemma 2 that $c_t(z) = \Lambda_{t,1}(z)$.

This completes the proof of (6), hence the proof of (8) for all $\zeta \in \mathbb{C}$ with $|\zeta| < \pi$. □

We are going to prove Theorem 3 by means of the Laplace transform, which is a special case of the method of kernel expansion of R.C. Buck [2]; see also [1, Chap.I §3]. Let
$$f(z) = \sum_{k \geq 0} \frac{a_k}{k!} z^k$$

be an entire function of exponential type $\tau(f)$. The *Laplace transform* of f, viz.

$$F(\zeta)=\sum_{k\geq 0} a_k\zeta^{-k-1}, \tag{10}$$

is analytic in the domain $\{\zeta\in\mathbb{C} \mid |\zeta|>\tau(f)\}$. From Cauchy's residue Theorem we deduce, for $r>0$,

$$\frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=r} \mathrm{e}^{\zeta z}\zeta^{-k-1}\mathrm{d}\zeta=\frac{z^k}{k!}\cdot \tag{11}$$

From the absolute and uniform convergence of the series in the right hand side of (10) on $|\zeta|=r$, it follows that for $r>\tau(f)$ we have

$$f(z)=\frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=r} \mathrm{e}^{\zeta z}F(\zeta)\mathrm{d}\zeta$$

and

$$f^{(t)}(z)=\frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=r} \zeta^t\mathrm{e}^{\zeta z}F(\zeta)\mathrm{d}\zeta.$$

Proof of Theorem 3. Let f be an entire function of exponential type $\tau(f)$ satisfying $\tau(f)<\pi$. Let r satisfy $\tau(f)<r<\pi$. From the uniform convergence of the series (8) on the compact set $\{\zeta\in\mathbb{C} \mid |\zeta|=r\}$, we deduce

$$f(z)=\sum_{t\in 2\mathbb{N}}\left(\frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=r}\zeta^t F(\zeta)\mathrm{d}\zeta\right)\Lambda_{t,0}(z)+ \\ \sum_{t\in 2\mathbb{N}}\left(\frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=r}\zeta^t \mathrm{e}^{\zeta} F(\zeta)\mathrm{d}\zeta\right)\Lambda_{t,1}(z),$$

and therefore (formula of Poritsky and Whittaker (2) for entire functions of exponential type $<\pi$)

$$f(z)=\sum_{t\in 2\mathbb{N}} f^{(t)}(0)\Lambda_{t,0}(z)+\sum_{t\in 2\mathbb{N}} f^{(t)}(1)\Lambda_{t,1}(z),$$

where the two series are absolutely convergent.

This completes the proof of Theorem 3. □

7. Integral formulae for Lidstone polynomials

Using Cauchy's residue Theorem, we deduce from (6) the following integral formula [8, (4.1)]:

Proposition 1. *For $z \in \mathbb{C}$, $t \in 2\mathbb{N}$ and $K \geq 0$, we have*

$$\Lambda_{t,1}(z) = (-1)^{t/2} \frac{2}{\pi^{t+1}} \sum_{k=1}^{K} \frac{(-1)^{k+1}}{k^{t+1}} \sin\bigl(k\pi z\bigr) + \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=(2K+1)\pi/2} \zeta^{-t-1} \frac{\sinh(\zeta z)}{\sinh(\zeta)} \mathrm{d}\zeta.$$

Proof. Let $z \in \mathbb{C}$. Inside the disc $\{\zeta \in \mathbb{C} \mid |\zeta| \leq (2K+1)\pi/2\}$, the function $\zeta \mapsto \zeta^{-t-1} \frac{\sinh(\zeta z)}{\sinh(\zeta)}$ has a pole of order $t+1$ at $\zeta = 0$ and only simple poles at $\zeta = k\pi\mathrm{i}$ with $k \in \mathbb{Z}$, $0 < |k| \leq K$. The residue at 0 is $\Lambda_{t,1}(z)$, while for $k \in \mathbb{Z} \setminus \{0\}$, the residue at $k\pi\mathrm{i}$ is

$$(-1)^k \mathrm{i}^{-t} (k\pi)^{-t-1} \sin\bigl(k\pi z\bigr). \tag{12}$$

Since t is even, the function is odd and the residues at $k\pi\mathrm{i}$ and at $-k\pi\mathrm{i}$ are the same. □

In particular, with $K = 1$ we have [8, (4.3)]

$$\Lambda_{t,1}(z) = (-1)^{t/2} \frac{2}{\pi^{t+1}} \sin(\pi z) + \frac{1}{2\pi\mathrm{i}} \int_{|\zeta|=3\pi/2} \zeta^{-t-1} \frac{\sinh(\zeta z)}{\sinh(\zeta)} \mathrm{d}\zeta.$$

Since $|\sinh(\zeta)| \geq 1$ for $|\zeta| = 3\pi/2$, one deduces, for $t \geq 0$ and $r > 0$,

$$\begin{cases} \left| \Lambda_{t,1}(z) - (-1)^{t/2} \dfrac{2}{\pi^{t+1}} \sin(\pi z) \right| \leq \left(\dfrac{2}{3\pi} \right)^t \mathrm{e}^{3\pi r/2}, \\ \left| \Lambda_{t,0}(z) - (-1)^{t/2} \dfrac{2}{\pi^{t+1}} \sin(\pi z) \right| \leq \mathrm{e}^{3\pi/2} \left(\dfrac{2}{3\pi} \right)^t \mathrm{e}^{3\pi r/2}. \end{cases} \tag{13}$$

These estimates enable Whittaker [8, Theorem 1] to solve the Lidstone interpolation problem as follows. Let $(a_t)_{t\in 2\mathbb{N}}$ and $(b_t)_{t \in 2\mathbb{N}}$ be two sequences of complex numbers. If the series

$$\sum_{t \in 2\mathbb{N}} (-1)^{t/2} \frac{a_t}{\pi^t} \quad \text{and} \quad \sum_{t \in 2\mathbb{N}} (-1)^{t/2} \frac{b_t}{\pi^t}$$

are convergent, then

$$\sum_{t \in 2\mathbb{N}} a_t \Lambda_{t,0}(z) + \sum_{t \in 2\mathbb{N}} b_t \Lambda_{t,1}(z) \tag{14}$$

is uniformly convergent on any compact of $\mathbb{C}$ and its sum $f(z)$ is an entire function satisfying

$$f^{(t)}(0) = a_t \quad \text{and} \quad f^{(t)}(1) = b_t \text{ for all } t \in 2\mathbb{N}.$$

If one of the series

$$\sum_{t\in 2\mathbb{N}} (-1)^{t/2}\frac{a_t}{\pi^t}, \quad \sum_{t\in 2\mathbb{N}} (-1)^{t/2}\frac{b_t}{\pi^t}$$

is not convergent, then (14) cannot converge for any non integral value of z.

Another consequence of (13) is, for $t \in 2\mathbb{N}$ and $r \geq 0$,

$$|\Lambda_{t,1}|_r \leq 2\pi^{-t}\mathrm{e}^{3\pi r/2} \quad \text{and} \quad |\Lambda_{t,0}|_r \leq 2\mathrm{e}^{3\pi/2}\pi^{-t}\mathrm{e}^{3\pi r/2}. \tag{15}$$

We now prove another integral formula for the polynomials $\Lambda_{t,0}$.

Proposition 2. *For $t \in 2\mathbb{N}$ and for $K \geq 0$, we have*

$$\Lambda_{t,0}(z) = \frac{z^t}{t!} + (-1)^{t/2}\frac{2}{\pi^{t+1}}\sum_{k=1}^{K}\frac{1}{k^{t+1}}\sin\bigl(k\pi z\bigr) - \frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=(2K+1)\pi/2}\zeta^{-t-1}\sinh(\zeta z)\coth(\zeta)\mathrm{d}\zeta.$$

Proof. Proposition 2 is equivalent to Proposition 1 by changing the variable z to $1-z$. We give another proof by repeating the same arguments as for the proof of Proposition 1. Inside the disc $\{\zeta \in \mathbb{C} \mid |\zeta| \leq (2K+1)\pi/2\}$, the function $\zeta \mapsto \zeta^{-t-1}\sinh(\zeta z)\coth(\zeta)$ has only simple poles at $k\pi\mathrm{i}$ with $k \in \mathbb{Z}$, $|k| \leq K$. From (7), it follows that the residue at $\zeta = 0$ is

$$\frac{z^t}{t!} - \Lambda_{t,0}(z),$$

while for $k \in \mathbb{Z}\setminus\{0\}$, the residue at $k\pi\mathrm{i}$ is

$$\mathrm{i}^{-t}(k\pi)^{-t-1}\sin\bigl(k\pi z\bigr). \tag{16}$$

Since t is even, the function we integrate is odd and the residues at $k\pi\mathrm{i}$ and at $-k\pi\mathrm{i}$ are the same. □

8. Functions of finite exponential type

We follow [2]. Let $K \geq 1$. The function $\zeta \mapsto \frac{\sinh(\zeta z)}{\sinh(\zeta)}$ is even and has only simple poles at $k\pi\mathrm{i}$ with $k \in \mathbb{Z}\setminus\{0\}$, with residue given by (12) with $t=-1$, namely $(-1)^k\mathrm{i}\sin(k\pi z)$. The sum of the residues at k and $-k$ is 0 and we have[1]

$$\frac{(-1)^k\mathrm{i}\sin(k\pi z)}{\zeta - k\pi\mathrm{i}} - \frac{(-1)^k\mathrm{i}\sin(k\pi z)}{\zeta + k\pi\mathrm{i}} = 2\pi(-1)^{k+1}\frac{k\sin(k\pi z)}{\zeta^2+k^2\pi^2}.$$

[1] A factor 2 is missing in [2, p.795] and [1, Chap.I §4 p.15].

Hence the function $G_K(\zeta, z)$ defined by

$$\frac{\sinh(\zeta z)}{\sinh(\zeta)} = 2\pi \sum_{k=1}^{K} \frac{(-1)^{k+1} k \sin(k\pi z)}{\zeta^2 + k^2\pi^2} + G_K(\zeta, z) \tag{17}$$

is analytic in the domain $\{(\zeta, z) \in \mathbb{C}^2 \mid |\zeta| < (K+1)\pi\}$. Notice that for $|\zeta| < k\pi$ and $k \geq 1$ we have

$$\frac{1}{\zeta^2 + k^2\pi^2} = \sum_{t \in 2\mathbb{N}} \frac{(\mathrm{i}\zeta)^t}{(k\pi)^{t+2}}.$$

The function $z \mapsto G_K(\zeta, z)$ is odd. Since the function $\zeta \mapsto G_K(\zeta, z)$ is even, its Taylor expansion at the origin can be written

$$G_K(\zeta, z) = \sum_{t \in 2\mathbb{N}} g_t(z)\zeta^t$$

where the functions $g_t(z)$ are odd entire functions. This Taylor series is absolutely and uniformly convergent for ζ in any compact subset of the disc $\{\zeta \in \mathbb{C} \mid |\zeta| < (K+1)\pi\}$. The Taylor coefficient $g_t(z)$ is the sum of $\Lambda_{t,1}(z)$ and a finite trigonometric sum of exponential type $\leq K\pi$, namely

$$g_t(z) = \Lambda_{t,1}(z) + 2(-1)^{t/2} \sum_{k=1}^{K} (-1)^k (k\pi)^{-t-1} \sin(k\pi z).$$

Using (9) we deduce, for $|\zeta| < (K+1)\pi$,

$$\mathrm{e}^{\zeta z} = \sum_{t \in 2\mathbb{N}} g_t(1-z)\zeta^t + \mathrm{e}^{\zeta} \sum_{t \in 2\mathbb{N}} g_t(z)\zeta^t + 2\pi \sum_{k=1}^{K} \frac{k \sin(k\pi z)}{\zeta^2 + k^2\pi^2} \left(1 + (-1)^{k+1}\mathrm{e}^{\zeta}\right). \tag{18}$$

Proposition 3 (R.C. Buck, 1955)**.** *Let K be a positive integer. Let f be an entire function of finite exponential type $\tau(f) < (K+1)\pi$ and let $F(\zeta)$ be the Laplace transform of f. Then for $z \in \mathbb{C}$ we have*

$$f(z) = \sum_{t \in 2\mathbb{N}} f^{(t)}(0) g_t(1-z) + \sum_{t \in 2\mathbb{N}} f^{(t)}(1) g_t(z) + \sum_{k=1}^{K} C_k \sin(k\pi z),$$

where the series are absolutely convergent and

$$C_k = -k\mathrm{i} \int_{|\zeta|=r} \frac{1 + (-1)^{k+1}\mathrm{e}^{\zeta}}{\zeta^2 + k^2\pi^2} F(\zeta)\mathrm{d}\zeta \quad (1 \leq k \leq K) \tag{19}$$

for any r in the range $\tau(f) < r < (K+1)\pi$.

Proof. Let r satisfy $\tau(f) < r < (K+1)\pi$. From the absolute and uniform convergence on $|\zeta| = r$ of the series in the right hand side of (18), we deduce

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \mathrm{e}^{\zeta z} F(\zeta) \mathrm{d}\zeta \\
&= \sum_{t \in 2\mathbb{N}} g_t(1-z) \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \zeta^t F(\zeta) \mathrm{d}\zeta + \sum_{t \in 2\mathbb{N}} g_t(z) \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \zeta^t \mathrm{e}^{\zeta} F(\zeta) \mathrm{d}\zeta \\
&\qquad + \sum_{k=1}^{K} C_k \sin(k\pi z),
\end{aligned}
$$

with

$$
\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \zeta^t F(\zeta) \mathrm{d}\zeta = f^{(t)}(0) \quad \text{and} \quad \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \zeta^t \mathrm{e}^{\zeta} F(\zeta) \mathrm{d}\zeta = f^{(t)}(1).
$$

□

Example. The Laplace transform of $f(z) = \sin(\pi z)$ is $F(\zeta) = \frac{\pi}{\zeta^2 + \pi^2}$ and for $\pi < r < 2\pi$ we have

$$
\int_{|\zeta|=r} \frac{1 + \mathrm{e}^{\zeta}}{(\zeta^2 + \pi^2)^2} \mathrm{d}\zeta = \frac{\mathrm{i}}{\pi},
$$

hence for this function f, we have

$$
C_1 = -\mathrm{i} \int_{|\zeta|=r} \frac{1 + \mathrm{e}^{\zeta}}{\zeta^2 + \pi^2} F(\zeta) \mathrm{d}\zeta = -\mathrm{i}\pi \int_{|\zeta|=r} \frac{1 + \mathrm{e}^{\zeta}}{(\zeta^2 + \pi^2)^2} \mathrm{d}\zeta = 1,
$$

as expected.

In [6] and [7], we will need the following variant of (17). The function $H_K(\zeta, z)$ defined by

$$
\sinh(\zeta z) \coth(\zeta) = -2\pi \sum_{k=1}^{K} \frac{k \sin(k\pi z)}{\zeta^2 + k^2 \pi^2} + H_K(\zeta, z), \tag{20}
$$

is analytic in the domain $\big\{(\zeta, z) \in \mathbb{C}^2 \;\; | \;\; |\zeta| < (K+1)\pi\big\}$. The map $\zeta \mapsto H_K(\zeta, z)$ is even and the map $z \mapsto H_K(\zeta, z)$ is odd. Replacing z with $1 - z$ in (17) yields

$$
H_K(\zeta, z) = \cosh(\zeta z) - G_K(\zeta, 1 - z).
$$

Corollary 2 (I.J. Schoenberg, 1936)**.** *Let f be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(t)}(0) = f^{(t)}(1) = 0$ for all $t \in 2\mathbb{N}$. Then*

$$
f(z) = \sum_{k=1}^{K} C_k \sin(k\pi z).
$$

with $K \le \tau(f)/\pi$ *and with the constants* $C_1, \dots, C_K$ *given by* (19).

A side result is that the exponential type $\tau(f)$ of a function f satisfying the assumption of Corollary 2 is an integer multiple of π.

Acknowledgments. This is an expanded version of the first part of a plenary talk given at the 87th Annual Conference of the Indian Mathematical Society in December 2021. The second part of the talk was devoted to several variables. This paper and the two forthcoming ones on two and several variables were completed at The Institute of Mathematical Sciences (IMSc) Chennai, India, where the author gave a course in December 2021 under the Indo-French Program for Mathematics. The author is thankful to IMSc for its hospitality and the Laboratoire International Franco–Indien for its support.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 95–146

THE SOJOURN OF A MATRIX AND ITS QUINTESSENTIAL GEOMETRY

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(Received : 20 - 12 - 2020 ; Revised : 12 - 03 - 2021)

Abstract. The set of all 2×2 real matrices does not form a group under matrix multiplication, as the singular matrices among them are not 'invertible' matrix theoretically. However, if some specific subsets from those singular matrices be chosen, they do form a group under matrix multiplication, where the group theoretic inverse of the matrix theoretically non-invertible matrices sometimes turn out to be their corresponding Moore-Penrose inverses. In the first part of this paper we investigate the whole family of 2×2 real singular matrices and linear algebraically pin down those specific subclasses. Now, 2×2 real matrices are nice geometric objects, as their realization in the guise of linear transformation from $\mathbb{R}^2$ to itself can really be *visualized* on the Cartesian plane $\mathbb{R}^2$ through pictorial presentations via the fundamental subspaces of those matrices. This natural interplay between linear algebra and its pictorial geometric presentation available in the present context then motivates us to see these different subclasses of matrices geometrically on $\mathbb{R}^2$, an odyssey through the beautifully synchronized and harmonious geometric presentation of their locations, finally culminating in a well-cataloged library of all these 2×2 real singular matrices with respect to the above mentioned property.

1. Introduction : A Test Case

In introductory Group Theory, it is indeed a common part of rudimentary knowledge, that the set of all non-singular real matrices of some specific order, say 2×2, usually denoted by $GL(2, \mathbb{R})$, makes a non-Abelian group under usual matrix multiplication, where the non-singularity is crucial for the invertibility of a matrix, and that inverse matrix in turn plays the role of

2020 Mathematics Subject Classification: 15A09, 15B99
Key words and phrases: Moore-Penrose inverse, simultaneously orthogonally (resp. non-orthogonally) diagonalizable matrices, eigenvalue, Jordan matrix, affine space, family of circles, family of right circular double cylinders

group theoretic inverse as well, while the identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ plays the role of identity element. In contrast, one somewhat counter-intuitive example for the beginners happens to be the set

$$\mathfrak{A} = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} : x \in \mathbb{R}, x \neq 0 \right\}.$$

In spite of being made of some singular (*and hence non-invertible*) matrices, this set still forms a group under matrix multiplication, and that too an Abelian one, with identity element $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \bar{I}$ (say), as can be routinely verified. In this example the group theoretic unique inverse of the singular matrix $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ with respect to its identity element $\bar{I}$ turns out to be $X = \begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$. Since the singular matrix A does not admit any *matrix theoretic inverse*, it becomes worthy of investigation whether its *group theoretic inverse* X with reference to the matrix $\bar{I}$ has any specific linear algebraic significance as well or not. Now, as it is well known in linear algebra that every $m \times n$ matrix, irrespective of its singularity or non-singularity has a corresponding uniquely determined generalized inverse, called the Moore-Penrose inverse, it seems appropriate to check that particular inverse of the matrix A.

1.1. **The Moore-Penrose inverse of A.** In the theory of various (not necessarily unique) generalized inverses of a given $m \times n$ matrix A (here we have chosen to stick to real entries only, though the general theory is formed over the complex matrices), the Moore-Penrose inverse enjoys the distinction of being uniquely definable corresponding to every given matrix whatsoever. It is well known [3, 4] in linear algebra that, for every given matrix $A_{m\times n}$ over $\mathbb{R}$, there exists exactly one matrix $A^+ \in \mathbb{R}^{n\times m}$, that meets all the following four conditions:

$$AA^+A = A, \qquad A^+AA^+ = A^+,$$
$$AA^+ = (AA^+)^T, \qquad A^+A = (A^+A)^T.$$

A^+ is called the **Moore-Penrose inverse** (*henceforth referred to as the MP inverse*) of A. There are more than one ways to compute the MP inverse of a given matrix, the most common being through singular value decomposition. However, we shall use the method via the rank factorization of the given matrix, following Bapat [1]. Here $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} = x \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = xA_1$ (say), where $x \neq 0$. First we find a rank factorization of A_1 and then use it to compute the MP inverse of A, denoted by A^+ in due course. Now, $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, whence $\rho(A_1) = 1$ shows that $A_1 = BC$ where $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is of full column rank and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ is of full row rank, whence

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = BC$$

is a rank factorization. Now,

$$B^+ = (B^T B)^{-1} B^T = \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and

$$C^+ = C^T (CC^T)^{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

so that the MP inverse of A is given by

$$A^+ = (xA_1)^+ = x^{-1} A^+ = \frac{1}{x} C^+ B^+ = \frac{1}{x} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix} = X.$$

So we see that, the group theoretic inverses of the elements of the group $\mathfrak{A}$ from the standpoint of abstract algebra, turned out to be, from the standpoint of linear algebra, the corresponding MP inverses of those elements matrix theoretically. However, this amazing feature does not pervade through all the possible groups that can be made out of various specific 2×2 singular matrices.

1.2. **A Counter Example.** Choose any arbitrary $n \in \mathbb{R} \setminus \{1\}$ and fix it. Now consider

$$\mathfrak{B} = \left\{ \begin{bmatrix} x & nx \\ x & nx \end{bmatrix} : x \in \mathbb{R}^* \right\}.$$

It is routine to check that $\mathfrak{B}$ forms an Abelian group under matrix multiplication, where the matrix $\begin{bmatrix} \frac{1}{n+1} & \frac{n}{n+1} \\ \frac{1}{n+1} & \frac{n}{n+1} \end{bmatrix}$ plays the role of identity element and the matrix $\begin{bmatrix} \frac{1}{(n+1)^2 x} & \frac{n}{(n+1)^2 x} \\ \frac{1}{(n+1)^2 x} & \frac{n}{(n+1)^2 x} \end{bmatrix}$ turns out to be the group theoretic inverse element for the matrix $\begin{bmatrix} x & nx \\ x & nx \end{bmatrix}$. To make the calculation simple for the sake of our counter example, let us take $n = 2$. Straightforward calculations then show that, the group theoretic inverse of the matrix $B_1 = x \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ in $\mathfrak{B}_1$ (with $n = 2$) is $B_1^{-1} = \frac{1}{x} \begin{bmatrix} \frac{1}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}$.

We now compute the MP inverse of $B_1 = xB$, say, where $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Following the method in Section 1.1 it can be shown that $B_1^+ = (xB)^+ = x^{-1}B^+ = \frac{1}{x} \begin{bmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$.

Clearly, the matrix theoretic unique MP inverse of B_1, not only is *distinct from the group theoretic inverse* of B_1, but also it goes out of the set $\mathfrak{B}_1$.

This counter example motivates us to frame the fundamental question that will guide us through the rest of our investigation here. **What are the specific subclasses of 2×2 real singular matrices that form Abelian groups under usual matrix multiplication, where the group theoretic inverse of each of the matrices is the same as their respective unique MP inverse?**

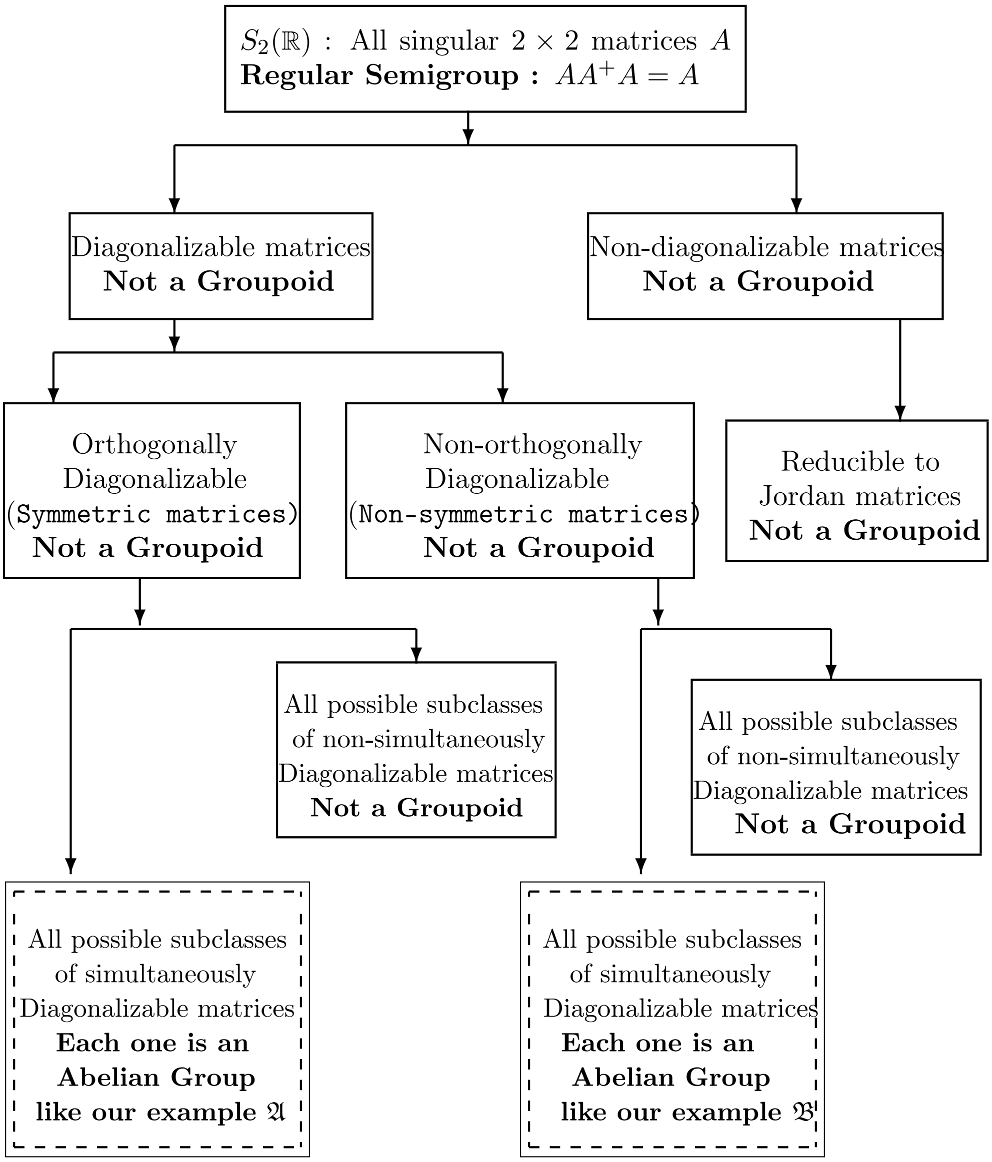
1.3. **Comparing $\mathfrak{A}$ and $\mathfrak{B}$ Linear Algebraically.** In our endeavor to pin down the specific subclasses of 2×2 real singular matrices with the desired property as mentioned above, let us look into the structures of the matrices in $\mathfrak{A}$ vis-a-vis with those in $\mathfrak{B}$ from the angle of linear algebra, and try to assess their structural dissimilarity, that may throw some light towards a judicious guess about the possible subclass to begin the investigation with.

The matrices in the group $\mathfrak{A}$ are nonzero scalar multiples of the symmetric matrix $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, whence, *their row space is the same as their* *column space, which is geometrically the line* $y = x$ *in* $\mathbb{R}^2$ looked upon as a standard Cartesian plane. The matrix A_1, when acts (as a transformation) on any vector in $\mathbb{R}^2$, crushes it by first orthogonally projecting that vector onto the row space $\mathcal{R}(A_1)$, i.e., the line $y = x$, and then to produce the outcome, transforms it to the column space $\mathcal{C}(A_1)$, i.e., in this case, keeps it right on that same line $y = x$, but merely stretches the projection by a factor of 2, *the only nonzero eigen value* of A_1 (the other being 0, as A_1 is singular). Again, the matrices in $\mathfrak{A}$ being symmetric, they are orthogonally diagonalizable and their eigen values 0 and $2x$ ensure that each of them is similar to the diagonal matrix $D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2x \end{bmatrix}$. Furthermore, the commutativity present in the group structure indicates that these matrices are simultaneously diagonalizable, i.e., they must share a common orthogonal matrix towards diagonalization under similarity. On the other hand, it is quite evident that the set of *all symmetric* 2×2 *real matrices cannot be our desired subclass*, as it is not even closed under matrix multiplication.

Turning the matter around, if we look at the group $\mathfrak{B}$ of some other specific singular matrices, that fails to attain the desired property, the first structural contrast we see is that, these groups are not symmetric, whence their respective *row spaces are different from their corresponding column spaces.* However, having two distinct eigen values 0 and $(1+n)x$, these are still diagonalizable, but not orthogonally this time, and all the matrices of one such group (for a pre-chosen $n \in \mathbb{R} \setminus \{1\}$) are similar to the diagonal matrix $D_2 = \begin{bmatrix} 0 & 0 \\ 0 & (1+n)x \end{bmatrix}$. Further, the commutative nature of the group, as before, indicates the simultaneous diagonalizability of the matrices among themselves, within every such group.

In any case, one thing is quite clear. If we try to classify all the 2×2 singular real matrices from various linear algebraic standpoint, our targeted subclasses must first become a group under usual matrix multiplication, which is not automatic, and this condition along with what we have seen so far, allows us to narrow down our focus for the search considerably, as can be seen in the following tabular presentation.

1.4. **Classifying All** 2×2 **Real Singular Matrices.** We shall now narrow our search towards pinning down the exact possible forms of the desired matrices. In the light of the following tabular classification, it will be good enough to look for them only in the relevant subclasses of 2×2 singular real matrices endowed with the property that they make an Abelian group. In some of these groups, as we shall see, inverse element is the same as the corresponding MP inverse and in some others, those are different.



We shall now investigate into the structural difference of the two classes reached at the end of the above chart from the standpoint of linear algebra, to begin with.

2. Specifying the Exact Subclasses of Matrices

2.1. **Orthogonally (Simultaneously) Diagonalizable Matrices.** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a singular matrix in one such subclass. Because orthogonal diagonalizability induces symmetry, we have $c = b$. Since A must be singular, so $ad = c^2$. Hence the matrix must be of the form

$$A = \begin{bmatrix} a & c \\ c & \frac{c^2}{a} \end{bmatrix} = a \begin{bmatrix} 1 & \frac{c}{a} \\ \frac{c}{a} & \frac{c^2}{a^2} \end{bmatrix},$$

provided[1], $\boldsymbol{a \neq 0}$. Thus $A = a \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$ for some $m = \frac{c}{a} \in \mathbb{R}\,(a \neq 0)$. Let us consider the set

$$\boxed{\boldsymbol{G(m) = \left\{ x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} : x \in \mathbb{R}^* \right\}}.}$$

Proposition 2.1.1. *$G(m)$ is precisely the set of those orthogonally diagonalizable (i.e., symmetric) matrices that are simultaneously diagonalizable with the matrix $M = \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$.*

Proof. In fact, suppose there is an orthogonal matrix P such that

$$P^{-1}MP = \begin{bmatrix} 0 & 0 \\ 0 & 1+m^2 \end{bmatrix} \text{ with } M = \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}.$$

Then, a matrix B is orthogonally simultaneously diagonalizable with M $\iff P^{-1}BP = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}, \exists r \in \mathbb{R}^* \iff P^{-1}BP = \frac{r}{1+m^2} \begin{bmatrix} 0 & 0 \\ 0 & 1+m^2 \end{bmatrix} = \frac{r}{1+m^2}(P^{-1}MP) \iff P^{-1}BP = P^{-1}(xM)P$ for $x = \frac{r}{1+m^2} \iff B = x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$. □

Now, the multiplication among elements of $G(m)$ is commutative (due to the simultaneous diagonalizability), whence the set $G(m)$ can be seen to be closed under matrix multiplication (however, though the product of symmetric matrices remain symmetric if and only if the product commutes,

[1]The **case when $\boldsymbol{a = 0}$** shall be dealt with separately.

still mere orthogonal diagonalizability i.e., symmetry alone was not good enough for the 'closure' in this case). Clearly $G(m)$ is a semigroup, as associativity is duly inherited from the set $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices under matrix multiplication. Again, as every matrix must admit its unique MP inverse, so for $P = x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$, we should have unique P^+ satisfying the four required conditions, as stated before.

Proposition 2.1.2. For every $P \in G(m)$, we have $P^+ \in G(m)$.

Proof. For $P = x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$, we have $P^+ = x^+ \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}^+$, where $x^+ = x^{-1}$ (as $x \neq 0$). Suppose $M = \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$. Then by the method of rank factorization, it can be shown that $M^+ = \frac{1}{(1+m^2)^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$, whence

$$P^+ = \frac{1}{x(1+m^2)^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} \in G(m).$$

□

Theorem 2.1.3. *$G(m)$ is a commutative group with the desired property.*

Proof. From the Proposition 2.1.2, we see that $G(m)$ is a regular semigroup, whence the commutativity available in its structure ensures that $PP^+ = P^+P$, so that $G(m)$ becomes completely regular. Since

$$x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} = x^2(1+m^2) \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix},$$

we see that the requirement for idempotence is $x(1 + m^2) = 1$, whence $\frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$ is the unique idempotent in $G(m)$, whence $G(m)$ is a group under matrix multiplication, with $\frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$ as its identity. Again, as both P^+P and PP^+ are candidates for idempotent elements in this structure, so $P^+P = PP^+ = e$ (say, the identity element of $G(m)$), which indicates, $P^+ = P^{-1}$ in this group, i.e., the group theoretic inverse of the matrix P is the same as its Moore Penrose inverse. □

Remark 2.1.4. It is now time to settle the **case for $\boldsymbol{a = 0}$** in the singular matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belonging to the class under consideration. We see that, such matrices must have the general form $x_1 \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ i.e., equivalently $x \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for some $x \in \mathbb{R}^*$, or, *as an extreme case*, it can be the null matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as well. However, *keeping the extreme case aside for the time being*, it is interesting to see that though the matrices in

$$\boxed{\boxed{\boldsymbol{G_0 = \left\{ x \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \ : x \in \mathbb{R}^* \right\}}}}$$

may appear to be different in look compared to those in $G(m)$, actually each of them belongs to the same similarity class with some element of $G(m)$, in the sense that for every matrix in $G(m)$ (for a fixed m), there exists some orthogonal (hence invertible) matrix P, such that under similarity, every matrix A of $G(m)$ will take the diagonal form $P^{-1}AP = P^TAP \in G_0$.

Note 2.1.5. Indeed, if the diagonal form mentioned above, turns out to be $diag(x, 0)$ rather than $diag(0, x)$, as is necessary for the membership of G_0, the columns of corresponding P may be permuted (i.e., the ordered orthonormal basis of $\mathbb{R}^2$ may be suitably reordered) to have the desired result, with the new matrix Q (made of orthonormal eigen vectors) being still orthogonal and playing the role of P for everyone. We now show the following result.

Proposition 2.1.6. *The set G_0 as defined above, is a commutative group with the desired property.*

Proof. In view of the Remark 2.1.4, it is now sufficient to show that the matrices of the form $P^{-1}AP$, where $A \in G(m)$ and P is a suitably chosen diagonalizing matrix, form a commutative group with the desired property about MP inverse. It is routine to check that these matrices do form a group with $P^{-1}\bar{I}P$ as the identity (where $\bar{I}$ is the identity of $G(m)$) and $P^{-1}A^{-1}P$ is the inverse of $P^{-1}AP$, and this group is commutative due to commutativity of $G(m)$. It now remains to show that, these new form of

matrices $B = P^{-1}AP\,[\in G_0]$ do adhere to the fact that $B^{-1} = B^{+}$. Clearly, $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P$, whence it will be sufficient to show that, $(P^{-1}AP)^{+} = P^{-1}A^{-1}P$. Now, from the standard properties of MP inverse [3], it is known that, in case of real matrices
(i) if $|P| \neq 0$ then $P^{-1} = P^{+}$
(ii) if U, V are orthogonal, then $(UAV)^{+} = V^{T}A^{+}U^{T}$.
Using these results we see that,

$$\begin{aligned}
(P^{-1}AP)^{+} &= (P^{+}AP)^{+} && (\text{since } |P| \neq 0)\\
&= P^{T}A^{+}(P^{+})^{T} && (\text{since } P, P^{-1} \text{ are orthogonal})\\
&= P^{-1}A^{-1}(P^{-1})^{T} && (\text{since we have } A^{+} = A^{-1} \text{ in } G(m))\\
&= P^{-1}A^{-1}P && (\text{since } P^{-1} = P^{T}).
\end{aligned}$$

□

Note 2.1.7. Finally, we take up the *extreme case* of the null matrix $\theta_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, as mentioned in Remark 2.1.4. Now, as a singleton set $\{\theta_{2\times 2}\}$ satisfies all the axioms of a commutative group under matrix multiplication trivially. Further, $\theta^{-1} = \theta^{+}(= \theta)$ as well, justifying its membership in this desired class.

Remark 2.1.8. So, to sum up from abstract algebraic standpoint (with underlying linear algebraic link), we have a family $\mathcal{G}$ of mutually disjoint classes $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ where

$$\mathcal{G}_1 = \{G(m) \,:\, m \in \mathbb{R}\},\ \mathcal{G}_2 = \{G_0\},\ \mathcal{G}_3 = \{\theta_{2\times 2}\}.$$

Here $\mathcal{G}_1$ is an infinite family of commutative groups like $G(m)$, one group for every value of $m \in \mathbb{R}$, while $\mathcal{G}_2$ and $\mathcal{G}_3$ comprise of one commutative group each, as described. Note that,

$$G = \bigcup \mathcal{G} = \left(\bigcup_{m\in\mathbb{R}} G(m)\right) \cup G_0 \cup \{\theta_{2\times 2}\}$$

is not a semigroup, as the set product $G(m)G(n) \notin \mathcal{G}$ (for $m \neq n$ with $mn \neq -1$), though it is visibly a union of groups. Hence $G = \cup\,\mathcal{G}$ is a class of matrices, where every matrix M (say), is an element of some commutative group with the *desired property*.

Next, we shall investigate whether any matrix that lies outside the class $G = \cup\,\mathcal{G}$, as mentioned above, may still have the desired property. In search of a conclusive answer to this question, we now turn towards a disjoint family of matrices, the family of non-orthogonally simultaneously diagonalizable 2×2 singular matrices.

2.2. **Non-orthogonally (Simultaneously) Diagonalizable Matrices.** Consider a diagonalizable matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which lies outside the set $G = \cup \mathcal{G}$. Then A is a non-orthogonally diagonalizable 2×2 real singular matrix. Hence it is non-symmetric, i.e., $b \neq c$. From singularity of A we get that $ad = bc$. Thus, the matrices A can be written as

$$A = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix} = a \begin{bmatrix} 1 & \frac{b}{a} \\ \frac{c}{a} & \frac{c}{a}\frac{b}{a} \end{bmatrix}$$

provided[2] $\boldsymbol{a \neq 0}$. So, $A = a \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}$ for some $m = \frac{c}{a} \in \mathbb{R}$ and $n = \frac{b}{a} \in \mathbb{R}$, $(a \neq 0)$. Clearly, $m \neq n$. Since A is nonzero singular matrix and also diagonalizable, so $1 + mn \neq 0$. Let us consider the set

$$\boxed{\boxed{\boldsymbol{G(m,n) = \left\{ x \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix} : x \in \mathbb{R}^* \right\}}}}.$$

Proposition 2.2.1. *The set $G(m,n)$ consists precisely of those non-orthogonally diagonalizable matrices that are simultaneously diagonalizable with the matrix $M = \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}$.*

Proof. Note that the matrix M in $G(m,n)$ is (non-orthogonally) similar to the diagonal matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1+mn \end{bmatrix}$. The rest of the proof is essentially similar to that of Proposition 2.1.1, and hence is omitted. $\square$

Now, simultaneous diagonalizability of the matrices in $G(m,n)$ with the matrix M, as mentioned above, ensures that the set $G(m,n)$ is closed under matrix multiplication. Indeed, if P be the common (non-orthogonal) invertible matrix diagonalizing the matrices in $G(m,n)$, and if $C, D \in G(m,n)$, then we must have $P^{-1}CP = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$ and $P^{-1}DP = \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}$, for some

[2]The **case for $\boldsymbol{a = 0}$** shall be treated later.

$r, s \in \mathbb{R}^*$, whence

$$P^{-1}(CD)P = (P^{-1}CP)(P^{-1}DP) = \begin{bmatrix} 0 & 0 \\ 0 & rs \end{bmatrix} \qquad \ldots (i)$$

indicates that CD is also (non-orthogonally) simultaneously diagonalizable with M, whence $CD \in G(m,n)$. Again, from (i), due to the commutativity of multiplication of reals, we have $P^{-1}(CD)P = P^{-1}(DC)P$, whence $CD = DC$. Thus, $G(m,n)$ is a commutative semigroup (associativity being routinely inherited as before).

Notice that, for any $x, y \in \mathbb{R}^*$,

$$x \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix} y \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix} = xy(1+mn) \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}. \qquad \ldots (ii)$$

Thus for any $x \in \mathbb{R}^*$, $x \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}$ is an idempotent element of the semigroup $G(m,n)$ iff $x^2(1+mn) = x$, i.e., $x = \frac{1}{1+mn}$. Hence $\frac{1}{1+mn} \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}$ is the unique idempotent element of the commutative semigroup $G(m,n)$.

Now, from (ii), it can be observed that, for any $C \in G(m,n)$, there exists $D \in G(m,n)$ such that $CDC = C$ (and $DCD = D$). Indeed, if $C = x \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}$ for $x \in \mathbb{R}^*$, then $D = \frac{1}{x(1+mn)^2} \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}$ serves the purpose. Thus, $G(m,n)$ is a regular semigroup with unique idempotent, whence it is a (commutative) group, in which the above mentioned matrices C and D play the role of mutual inverses of each other.

The above discussion leads us to the following result.

Proposition 2.2.2. *The set $G(m,n)$ of matrices, as defined above with $m \neq n, 1+mn \neq 0$, forms a commutative group under matrix multiplication.*

Note 2.2.3. It is of a definite significance for our future discourse that the *column space* and *row space* of each of the matrices P in the above mentioned group $G(m,n)$ are $\mathcal{C}(P) = \{x(1,m) : x \in \mathbb{R}\}$ and $\mathcal{R}(P) = \{x(1,n) : x \in \mathbb{R}\}$ respectively.

Remark 2.2.4. Let us consider $C = x \begin{bmatrix} 1 & n \\ m & mn \end{bmatrix} \in G(m,n)$. If we calculate its unique MP inverse, by the same technique as shown in Proposition 2.1.2, (detailed calculation is essentially similar and hence omitted), we see that

$$\boxed{\boldsymbol{C^+} = \tfrac{1}{x(1+m^2)(1+n^2)}\begin{bmatrix} 1 & \mathbf{m} \\ \mathbf{n} & mn \end{bmatrix} \qquad \text{while} \qquad \boldsymbol{C^{-1}} = \tfrac{1}{x(1+mn)^2}\begin{bmatrix} 1 & \mathbf{n} \\ \mathbf{m} & mn \end{bmatrix}}$$

which are clearly distinct. Moreover, we would like to point out the fact that, while the group theoretic inverse of the matrix C is given by $C^{-1} \in G(m,n)$, its unique MP inverse $C^+ \in G(n,m)$, where $G(n,m) = \{C^T : C \in G(m,n)\}$. This leads to the following fundamental result, in contrast with the Theorem 2.1.3.

Theorem 2.2.5. *The commutative group $G(m,n)$ as described earlier,* **does not** *enjoy the desired property.*

Though the commutative group $G(m,n)$ does not enjoy the desired property (unlike the other commutative group $G(m)$), it is worth mentioning some interesting features of the MP inverses of the matrices belonging to a group of the form $G(m,n)$ for some $m,n \in \mathbb{R}$.

Lemma 2.2.6. *For any $m,n,p,q \in \mathbb{R}$ with $m \neq n, p \neq q$ such that none of mn, pq, np and mq is equal to -1, we have $G(m,n)G(p,q) = G(m,q)$.*

Proof. It is a routine argument involving set theoretic product and hence is omitted. □

Remark 2.2.7. If we extend our notation naturally, so as to rename the group $G(m)$ (for a particular m) as $G(m,m)$, then from the Lemma 2.2.6 we have, for any $m,n \in \mathbb{R}$,

$$G(m,n)G(n,m) = G(m,m) = G(m).$$

Note 2.2.8. The condition $1 + np \neq 0$ i.e., $np \neq -1$ can never be dropped from the Lemma 2.2.6, as then the row space of every matrix in $G(m,n)$ will be orthogonal to each matrix in the column space of $G(p,q)$, which will result in $G(m,n)G(p,q) = \{\theta_{2\times 2}\}$.

For $P \in G(m,n)$ with $m,n \in \mathbb{R}$, we have already seen that $P^+ \in G(n,m)$. However, it is known from linear algebra that, both the matrices PP^+ (orthogonal projection onto $\mathcal{C}(P)$) and P^+P (orthogonal projection onto $\mathcal{R}(P)$) are idempotent symmetric matrices, whence they are idempotent elements in the multiplicative semigroup $M_{2\times 2}(\mathbb{R})$. Next Lemma puts them at their respective rightful places commensurately.

Lemma 2.2.9. *For any $P \in G(m,n)$,* (i) $PP^+ = \mathcal{I}_{G(m)}$ *and* (ii) $P^+P = \mathcal{I}_{G(n)}$, where $\mathcal{I}_A$ stands for the identity element of the group A.

Proof. Follows from Lemma 2.2.6 directly. □

The next result establishes a rather beautiful relation among the matrices $P^{-1} \in G(m,n)$ and $P^+ \in G(n,m)$, that have already been proved to be distinct for any $P \in G(m,n)$.

Lemma 2.2.10. *For any $P \in G(m,n)$ we have, $P^{-1} = \mathcal{I}_{G(m,n)}\, P^+\, \mathcal{I}_{G(m,n)}$.*

Proof. Here $PP^+P = P \implies P^{-1}(PP^+P)P^{-1} = P^{-1}PP^{-1} \implies (P^{-1}P)P^+(PP^{-1}) = P^{-1} \implies P^{-1} = \mathcal{I}_{G(m,n)}\, P^+\, \mathcal{I}_{G(m,n)}$. □

Following Corollary is easy to see from Lemma 2.2.9 and Lemma 2.2.10.

Corollary 2.2.11. $\mathcal{I}_{G(m)}\mathcal{I}_{G(m,n)} = \mathcal{I}_{G(m,n)} = \mathcal{I}_{G(m,n)}\mathcal{I}_{G(n)}$.

Remark 2.2.12. It is well known from linear algebra that for any $P, Q \in M_{2\times 2}(\mathbb{R})$, it is **not in general true** that $(PQ)^+ = Q^+P^+$. However, if we prefer to choose these matrices from the group $G(m,n)$, we can reach really close to the equality, as is shown below.

Proposition 2.2.13. *Let $m, n, p, q \in \mathbb{R}$ such that none of the products mq, np, mn, pq are equal to -1. Then for any $P \in G(m,n)$ and $Q \in G(p,q)$, there exists $x \in \mathbb{R}^*$ such that $(PQ)^+ = x(Q^+P^+)$.*

Proof. Suppose $P \in G(m,n)$ and $Q \in G(p,q)$. Then,

$$
\begin{aligned}
& PQ \in G(m,n)G(p,q) = G(m,q) && \text{[by Lemma 2.2.6]}\\
\implies\quad & (PQ)^+ \in G(q,m) && \\
\implies\quad & (PQ)^+ = a\begin{bmatrix} 1 & m \\ q & qm \end{bmatrix} && \text{(for some } a \in \mathbb{R}^*)
\end{aligned}
$$

Again

$$
\begin{aligned}
& P^+ \in G(n,m), Q^+ \in G(q,p) && \\
\implies\quad & Q^+P^+ \in G(q,p)G(n,m) = G(q,m) && \text{[by Lemma 2.2.6]}\\
\implies\quad & Q^+P^+ = b\begin{bmatrix} 1 & m \\ q & qm \end{bmatrix} && \text{(for some } b \in \mathbb{R}^*)
\end{aligned}
$$

Hence, $(PQ)^+ = x(Q^+P^+)$, where $x = \frac{b}{a} \in \mathbb{R}^*$ $(a \neq 0)$. □

Remark 2.2.14. At the beginning of this section we have considered the case of those non-orthogonally diagonalizable matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \neq 0$. It is now time **to investigate the corresponding case of the matrices, with $\boldsymbol{a = 0}$**. Since A is singular, here we must have $bc = 0$ as well. Again A being non-orthogonally diagonalizable, (i.e., non-symmetric), we must have $b \neq c$, whence exactly one of b and c is zero.

Let us first take up the **subcase, $\boldsymbol{b \neq 0}$ and $\boldsymbol{c = 0}$**. Note that, here diagonalizability of A indicates that $d \neq 0$. Hence the matrix A must be of the form, $A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ 0 & \frac{d}{b} \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ 0 & m \end{bmatrix}$, for some $m = \frac{d}{b} \in \mathbb{R}^*$(as $d \neq 0, b \neq 0$). Now, for each particular choice of $m \in \mathbb{R}^*$, let us consider the set

$$\boxed{\boxed{\boldsymbol{G_0(m,0)} = \left\{ \boldsymbol{x} \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \boldsymbol{m} \end{bmatrix} \ \boldsymbol{:} \ \boldsymbol{x} \in \mathbb{R}^* \right\}}}.$$

Now, in an essentially similar manner as in Proposition 2.2.1, it can be shown that all matrices in $G_0(m,0)$ are simultaneously diagonalizable with the matrix $\begin{bmatrix} 0 & 1 \\ 0 & m \end{bmatrix}$, whence, arguments parallel to the case of $G(m,n)$ lead us to the similar conclusion that $G_0(m,n)$ is a commutative semigroup. Some further routine computations then establish that $G_0(m,0)$ is a group with identity $\begin{bmatrix} 0 & \frac{1}{m} \\ 0 & 1 \end{bmatrix}$. It is now easy to check that the group theoretic inverse of $C = x \begin{bmatrix} 0 & 1 \\ 0 & m \end{bmatrix}$ is clearly distinct from its MP inverse. Indeed, it can be shown that

$$\boxed{\frac{1}{xm^2} \begin{bmatrix} 0 & 1 \\ 0 & m \end{bmatrix} = \boldsymbol{C^{-1} \neq C^{+}} = \frac{1}{x(1+m^2)} \begin{bmatrix} 0 & 0 \\ 1 & m \end{bmatrix}}.$$

Hence, we conclude that, like the groups $G(m,n)$, the present class $G_0(m,0)$ *as well, does not have the desired property.*

Turning to the other possible **subcase, $\boldsymbol{b = 0}$ and $\boldsymbol{c \neq 0}$**, we now see that, here the matrix A must be of the form, $A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} = c \begin{bmatrix} 0 & 0 \\ 1 & \frac{d}{c} \end{bmatrix} = c \begin{bmatrix} 0 & 0 \\ 1 & m \end{bmatrix}$, for some $m = \frac{d}{c} \in \mathbb{R}^*$(as $d \neq 0, c \neq 0$). As before, for each particular choice of $m \in \mathbb{R}^*$, we consider the set

$$\boxed{\boxed{\boldsymbol{G_0(0, m) = \left\{ x \begin{bmatrix} 0 & 0 \\ 1 & m \end{bmatrix} \ : \ x \in \mathbb{R}^* \right\}}}}.$$

Clearly, $G_0(0, m) = \{C^T : C \in G_0(m, 0)\}$. Now, for a particular $m \in \mathbb{R}^*$, since $G_0(m, 0)$, is a class of all non-orthogonally simultaneously diagonalizable matrices, so there exists a (non-orthogonal) invertible matrix, say P, which diagonalizes every matrix in $G_0(m, 0)$. As matrices in $G_0(0, m)$ are nothing but transposes of the matrices in $G_0(m, 0)$, we can say that the invertible matrix $Q = (P^{-1})^T$ diagonalizes every matrix in $G_0(0, m)$. Thus $G_0(0, m)$ is also a class of non-orthogonally simultaneously diagonalizable matrices. Arguing identically as above, we may then conclude that $G_0(0, m)$ is also a commutative group under matrix multiplication with identity element $\begin{bmatrix} 0 & 0 \\ \frac{1}{m} & 1 \end{bmatrix}$. Here also, for every $D = x \begin{bmatrix} 0 & 0 \\ 1 & m \end{bmatrix}$, the group theoretic inverse D^{-1} is distinct from the corresponding MP inverse, as a routine computation may establish. Indeed, it can be shown that

$$\boxed{\tfrac{1}{xm^2} \begin{bmatrix} 0 & 0 \\ 1 & m \end{bmatrix} = \boldsymbol{D^{-1} \neq D^{+}} = \tfrac{1}{x(1+m^2)} \begin{bmatrix} 0 & 1 \\ 0 & m \end{bmatrix}}.$$

Hence, we conclude that, for any $m \in \mathbb{R}^*$, the class $G_0(0, m)$ *also does not have the desired property.*

Note 2.2.15. It is interesting to observe that, for any $C \in G_0(m, 0), C^+ \in G_0(0, m)$ while for any $D \in G_0(0, m), D^+ \in G_0(m, 0)$. It is now easy to see that, for any, $m, n \in \mathbb{R}^*$ with $mn \neq -1$,

$$G_0(m, 0)G_0(0, n) = G(m, n) \ \text{ while } \ G_0(0, m)G_0(n, 0) = G_0.$$

Note that, whenever $mn = -1$, we have $G_0(0, m)G_0(n, 0) = \{\theta_{2\times 2}\}$. So, similarly as in the case of the groups $G(m, n)$, we may arrive at the following identical results here.

1. For any $m \in \mathbb{R}^*$, if $P \in G_0(m,0)$, then $PP^+ \in G_0(m,0)G_0(0,m) = G(m,m) = G(m)$ and $P^+P \in G_0(0,m)G_0(m,0) = G_0$, whence $PP^+ = \mathcal{I}_{G(m)}$ and $P^+P = \mathcal{I}_{G_0}$.

2. For any $m \in \mathbb{R}^*$ and for any $P \in G_0(m,0)$, $P^{-1} = \mathcal{I}_{G_0(m,0)}P^+\mathcal{I}_{G_0(m,0)}$, which along with the result above, leads to the fact that

$$\mathcal{I}_{G(m)}\mathcal{I}_{G_0(m,0)} = \mathcal{I}_{G_0(m,0)} = \mathcal{I}_{G_0(m,0)}\mathcal{I}_{G_0}.$$

2.3. **Non-diagonalizable Matrices.** Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a singular non-null 2×2 non-diagonalizable matrix. Let us first consider the **case** $\boldsymbol{a \neq 0}$. Since A is singular, we have $ad = bc$ and also one of its eigenvalue must be 0. Since A is non-diagonalizable, with $tr(A) = a+d$, we see that $a+d = 0$ (as otherwise distinct eigenvalues would have made A diagonalizable). Hence, $d = -a$ indicates that, $ad = bc = -a^2$ i.e., $\frac{b}{a} = -\frac{a}{c}$ (clearly, $c \neq 0$ as $a \neq 0$). Thus the matrix A is seen to have the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & \frac{b}{a} \\ \frac{c}{a} & \frac{d}{a} \end{bmatrix} = a\begin{bmatrix} 1 & -\frac{1}{m} \\ m & -1 \end{bmatrix}$$

for some $m = \frac{c}{a} \in \mathbb{R}^*$. Now, for each $m \in \mathbb{R}^*$, let us consider the set

$$\boxed{\boldsymbol{J(m) = \left\{ x \begin{bmatrix} 1 & -\frac{1}{m} \\ m & -1 \end{bmatrix} : x \in \mathbb{R}^* \right\}}}.$$

The matrices in $J(m)$, being singular 2×2 non-diagonalizable matrices of trace zero, it is easy to see that they are similar to the Jordan matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is an interesting observation that for such matrices A, the column space $\mathcal{C}(A) = \{x(1,m) : x \in \mathbb{R}^*\}$ and the row space $\mathcal{R}(A) = \{x(1,-\frac{1}{m}) : x \in \mathbb{R}^*\}$ are mutually orthogonal. Later, in section 3, this feature will be instrumental in locating this matrices geometrically on $\mathbb{R}^2$. Right now, however, looking at them algebraically, we see that this property compels the product of any two matrices in $J(m)$ to become null matrix $\theta_{2\times 2} \notin J(m)$, whence the set $J(m)$ is not even a groupoid under matrix multiplication. Clear enough, this class of matrices remains excluded from our search of the commutative groups with the desired property.

Note 2.3.1. As a passing remark, it may be noted that, the class $J(m) \cup \{\theta_{2\times 2}\}$ can go up to a commutative semigroup (which is not a group) under matrix multiplication.

Remark 2.3.2. Under the **case** $\boldsymbol{a = 0}$, the above mentioned matrix A looks like $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ to begin with, and it can be dealt with, in a spirit similar as above. Indeed, while singularity indicates here that one eigenvalue must be zero, non-diagonalizability implies that the other eigenvalue must also be zero as well, whence $tr(A) = d$ enforces $d = 0$. Furthermore, $bc = 0$ from singularity, along with the fact that A is non-null, tells us that exactly one of b and c must be zero in this case. So, under the **subcase when** $\boldsymbol{b = 0}$, we have $A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} = c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, which is itself in Jordan form and can be made similar to the Jordan matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ as well. Thus we have another set of matrices given by

$$\boxed{\boldsymbol{J^0 = \left\{ x \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : x \in \mathbb{R}^* \right\}}}$$

where the matrices are of different form than before, (the column space is $\{x(0,1) : x \in \mathbb{R}^*\}$) but the set is similar in algebraic structural properties with $J(m)$.

Finally, if we take up the **subcase when** $\boldsymbol{c = 0}$, we have $A = b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which is similar to the Jordan matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This leads to the last possible set of matrices in our present discussion, given by

$$\boxed{\boldsymbol{J_0 = \left\{ x \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : x \in \mathbb{R}^* \right\}}}$$

where the matrices are of yet another form, (column space is $\{x(1,0) : x \in \mathbb{R}^*\}$) but the set is similar in algebraic structural properties with $J(m)$ and J^0.

So, the whole class J of non-diagonalizable (singular) 2×2 real matrices (i.e., matrices similar to Jordan matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$) may be seen as an union of three families

$$J = \Big(\bigcup_{m \in \mathbb{R}^*} J(m) \Big) \cup J_0 \cup J^0$$

and clearly no subclass under consideration succeeds to form a group, (indeed not even a groupoid), whence *these matrices remain outside the realm of our desired property.*

Remark 2.3.3. Since the MP inverse of every matrix exists uniquely, we put down a list of features of MP inverses of these non-diagonalizable matrices.

1. For a fixed $m \in \mathbb{R}^*, P \in J(m) \implies P^+ \in J(-\frac{1}{m})$;
2. $P \in J_0 \implies P^+ \in J^0$ and $Q \in J^0 \implies Q^+ \in J_0$;
3. For any $m, n \in \mathbb{R}^*$, with $m \neq n$, $J(m)J(n) = G(m, -\frac{1}{n})$;
4. For any $m \in \mathbb{R}^*$, $J(m)J(-\frac{1}{m}) = G(m)$ [follows from 3 above];
5. $J_0 J^0 = G(1, 0), J^0 J_0 = G_0$;
6. For any $P \in J(m), PP^+ = \mathcal{I}_{G(m)}$ and $P^+P = \mathcal{I}_{G(-\frac{1}{m})}$ [follows from 4 above];
7. For any $P \in J_0, PP^+ = \mathcal{I}_{G(1,0)}$ and $P^+P = \mathcal{I}_{G_0}$;
8. For any $P \in J^0, PP^+ = \mathcal{I}_{G_0}$ and $P^+P = \mathcal{I}_{G(1,0)}$.

3. A Geometric Odyssey via Algebraic Roads

Now that we have thoroughly classified all the members of the family of 2×2 singular matrices, with respect to the validity of the desired property, relating their group theoretic inverse (whenever exists), with the corresponding MP inverse, we embark upon a rather interesting geometric journey towards *locating these various classes of matrices as well as their member matrices on the Euclidean plane* $\mathbb{R}^2$. The journey culminates into some beautifully synchronized geometric patterns evolving out of their positions on $\mathbb{R}^2$, harmonious with the corresponding rich algebraic structures. In a sense it justifies the old saying that, '*Algebra is Geometry in writing and Geometry is Algebra en-figured*'.

3.1. **Examining the Groups in $\mathcal{G}$ with the 'Desired Property'.** Exploiting the size 2×2 along with the singularity of the matrices under consideration, let us try to fix them on $\mathbb{R}^2$ through their fundamental subspaces. It can be shown that the matrices in $\mathcal{G}_1$ and $\mathcal{G}_2$ correspond to 1-dimensional subspaces in $\mathbb{R}^2$, while that in $\mathcal{G}_3$ corresponds to the zero dimensional subspace. Indeed, matrices in $G(m)$ [for a specific $m \in \mathbb{R}$] of $\mathcal{G}_1$ are those, whose both the row space and column space are the same line $y = mx$ through origin (including the case for $m = 0$, which gives the x-axis), whereas, for the matrices in $\mathcal{G}_2 = \{G_0\}$, it is the y-axis, i.e., the line $x = 0$ (the only line through origin in $\mathbb{R}^2$ that cannot be expressed in the form $y = mx$), while the only matrix in $\mathcal{G}_3 = \{\theta_{2\times 2}\}$ is linked with $\{(0,0)\}$. So, *for every possible proper subspace of* $\mathbb{R}^2$ *there is a group with the 'desired property'.*

3.1.1. *Connecting $G(m)$ with G_0 via Similarity.* When m runs over $\mathbb{R}$, the matrices in $\mathcal{G}_1$ belonging to different groups of type $G(m)$ can be thought of to be arranged in a doubly infinite array; for fixed m, think of the matrices in the infinite group $G(m)$ in a row, while such rows are made of different values of $m \in \mathbb{R}$. Now, for every matrix in the group G_0, there exists exactly one matrix belonging to every group $G(m)$ (where m runs over $\mathbb{R}$), such that all these matrices are similar to the pre-chosen matrix from G_0. This happens due to the orthogonal diagonalizability prevalent in this class of matrices. In this way, the matrices of the form $\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$ from G_0, where $r \neq 0$, originating out of different vectors $(0, r)$ on y-axis, play a dominant role in understanding the linear algebraic significance of the various group elements, coming one each from every $G(m)$, via similarity.

Suppose we take $\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix} = B \in G_0$ (with $r \neq 0$) and try to see which matrices belong to its similarity class as proposed above. If $A = k\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} \in G(m)$ has to be similar with B, they must have the same eigenvalues, whence we have $r = k(1 + m^2)$, so that $k = \frac{r}{1+m^2}$ tells us that $A = r\begin{bmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{bmatrix} \in G(m)$ is orthogonally similar to B. Allowing m to run over $\mathbb{R}$, we get the whole family of similar matrices representing the same singular linear operator on $\mathbb{R}^2$, given by $T(x, y) = (0, ry)$ under different choices of orthonormal bases.

3.1.2. *Dissecting the Matrices in $G(m)$ linear algebraically.* Note that, the orthogonal projection matrix onto the line $y = mx$ is given by

$$\frac{\begin{bmatrix}1\\m\end{bmatrix}\begin{bmatrix}1 & m\end{bmatrix}}{\begin{bmatrix}1 & m\end{bmatrix}\begin{bmatrix}1\\m\end{bmatrix}} = \frac{\begin{bmatrix}1 & m\\m & m^2\end{bmatrix}}{1+m^2} = \begin{bmatrix}\frac{1}{1+m^2} & \frac{m}{1+m^2}\\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}\end{bmatrix}$$

which indicates to the *spectral decomposition* of the symmetric matrix $A = k\begin{bmatrix}1 & m\\m & m^2\end{bmatrix}$ where $k = \frac{r}{1+m^2}$, which is orthogonally similar to $\begin{bmatrix}0 & 0\\0 & r\end{bmatrix}$, $(r \neq 0)$ in the form $rP_1 + 0P_2$ where $P_1 = \begin{bmatrix}\frac{1}{1+m^2} & \frac{m}{1+m^2}\\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}\end{bmatrix}$ happens to be *the identity element of the group* $G(m)$ as well. Note that, here the one-dimensional eigen space belonging to the eigenvalue r turns out to be the line $y = mx$, (which is the row space and the column space of A also), while the line through the origin, perpendicular to $y = mx$ i.e., the null space of A will then become the eigen space belonging to the eigen value 0. This explains the spectral decomposition $rP_1 + 0P_2$ of A (where P_2 is of no consequence to us due to the eigen value 0), which shows that the matrix A actually projects every vector in $\mathbb{R}^2$ orthogonally onto the line $y = mx$ and then multiplies the projection by r, its nonzero eigenvalue, to give the outcome.

Again, from another perspective, we see that the matrix A, when acts on any vector $\alpha \in \mathbb{R}^2$, it actually breaks α uniquely as $\alpha_{\mathcal{R}(A)} + \alpha_{\mathcal{N}(A)}$ (as $\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathbb{R}^2$) and then $A\alpha = A\alpha_{\mathcal{R}(A)}$ (as $A\alpha_{\mathcal{N}(A)} = 0$) is enacted. Note that, here $\alpha_{\mathcal{R}(A)}$ is the orthogonal projection of α onto $\mathcal{R}(A)$ (i.e., $y = mx$) i.e., $\alpha_{\mathcal{R}(A)} = P_1\alpha$. Now if we begin with the matrix P_1 instead of A and do the same routine to see how it works, we have $P_1\alpha = P_1\alpha_{\mathcal{R}(P_1)} = P_1\alpha_{\mathcal{R}(A)} = P_1(P_1\alpha) = P_1^2\alpha \ \forall \alpha \in \mathbb{R}^2$, whence $P_1 = P_1^2$ shows its idempotence, which justifies its 'identity' (i.e., do nothing) role in the group $G(m)$.

3.1.3. *Locating the Matrices from $\mathcal{G}$ on $\mathbb{R}^2$ Geometrically.* We first pin down the matrix $A = r\begin{bmatrix}\frac{1}{1+m^2} & \frac{m}{1+m^2}\\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}\end{bmatrix} \in G(m)$ (*corresponding to* $y = mx$), which is similar to $B = \begin{bmatrix}0 & 0\\0 & \mathbf{r}\end{bmatrix} \in G_0$ (with $r \neq 0$), on the line $y = mx$ geometrically, by associating a unique point $\boldsymbol{M_r(m)}$ at a distance, say s, from the origin on the line $y = mx$.

Suppose $x = \frac{r}{1+m^2}$, so that $mx = \frac{mr}{1+m^2}$, whence (x, mx) is a point on $y = mx$. This is the point M_r (or, $M_r(m)$). If the distance of M_r from origin is s, then

$$s^2 = x^2 + m^2x^2 = \frac{r^2}{(1+m^2)^2} + \frac{m^2r^2}{(1+m^2)^2} = \frac{r^2}{1+m^2}.$$

So, $s = \frac{|r|}{\sqrt{1+m^2}}$.

Thus, for the non-null vector $(0, \mathbf{r})$ on y-axis that corresponds to the matrix $B \in G_0$, the *family of orthogonally similar matrices*, one from each group $G(\mathbf{m})$, (as m runs over $\mathbb{R}$), can be made to *correspond to the points lying on the* **line** $\boldsymbol{y = mx}$, respectively at a distance $\boldsymbol{s = \frac{|r|}{\sqrt{1+m^2}}}$ from the origin, *each matrix being r times the 'identity' of the corresponding group* $G(m)$.

These points, corresponding to a fixed $r \in \mathbb{R}^*$, trace out a curve $\boldsymbol{C_r}$ on $\mathbb{R}^2$ whose *parametric equation* is given by

$$x = \frac{r}{1+m^2}, y = \frac{rm}{1+m^2}.$$

The curve C_r is indeed a *circle punctured at a single point* $(0, 0)$ (as $r \neq 0$). The Cartesian equation of the circle corresponding to C_r $(r \neq 0)$ is $x^2 + y^2 = rx$. Thus for a fixed $r \in \mathbb{R}^*$, the curve C_r can be referred to as a **circle**

$$\boldsymbol{x^2 + y^2 = rx} \text{ with } \ x \neq 0.$$

Geometrically, it is immediate to see that these circles have their respective centers at $(\frac{r}{2}, 0)$ on x-axis, with radius $\frac{|r|}{2}$, and are symmetric about the positive (resp. negative) side of x-axis when $r > 0$ (resp. $r < 0$), while the y-axis is tangential to them (not to C_r in true sense) at $(0, 0)$. So we have a one-parameter family of curves $\{C_r : r \in \mathbb{R}^*\}$, having two distinct subfamilies governed by the conditions $r > 0$ or $r < 0$. Clearly, the *differential equation* of this one-parameter family is

$$2xy\frac{dy}{dx} + x^2 = y^2$$

where y is a function of x, continuous everywhere on $\mathbb{R}$, except having a removable discontinuity at the point $x = 0$.

3.1.4. *Visual Illustrations.* The family of curves $\{C_r : r \in \mathbb{R}^*\}$ can be visualized diagrammatically as follows :

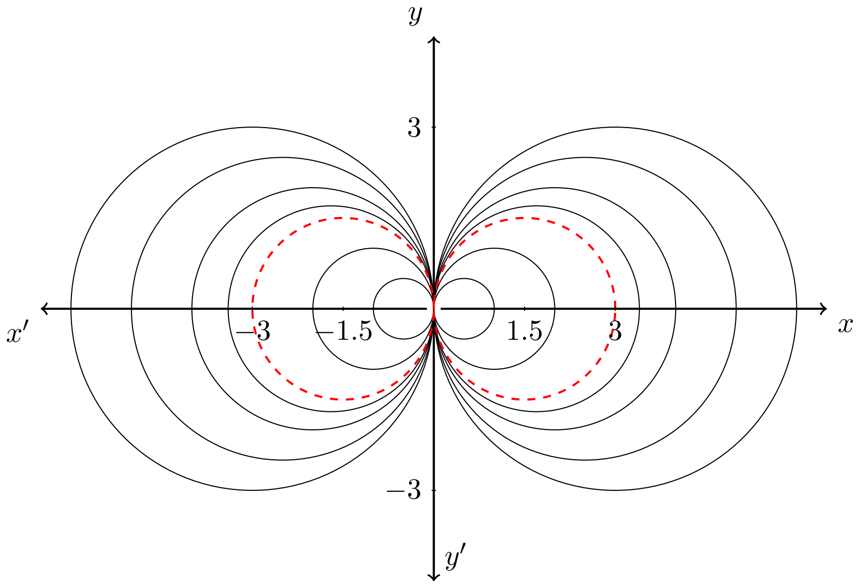
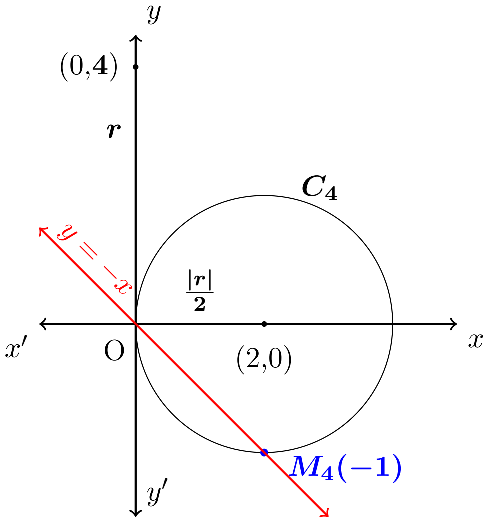


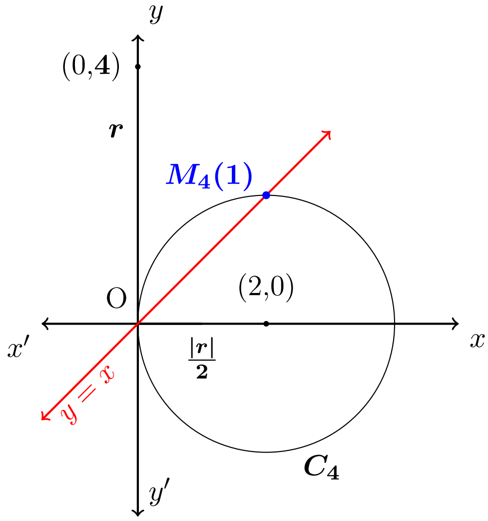
Fig. 1

This description of the family $\{C_r : r \in \mathbb{R}^*\}$ enables us to locate the point on $\mathbb{R}^2$ that may represent a matrix in the group $G(m)$ for every $m \in \mathbb{R}$. More precisely, this description establishes a one-to-one correspondence between the set $G_0 \cup \left(\bigcup_{m\in\mathbb{R}} G(m)\right)$ and $\mathbb{R}^2$. Note that, the axis of y, represents the whole group G_0 i.e., every matrix $r\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is uniquely represented by the point $(0, r)$ on y-axis. Note further that, here y-axis is the column space (also the row space) of each of the matrices in G_0. For any $A = x\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} \in G(m)$ (for $m \in \mathbb{R}$), the point on $\mathbb{R}^2$ that represents A is found as follows : Let $r = x(1+m^2)$. Draw the circle with radius $\frac{|r|}{2}$, fixing the centre at $(\frac{r}{2}, 0)$. Draw the line $y = mx$. The point $M_r(m)$ on $\mathbb{R}^2$ that represents the matrix $A \in G(m)$, is the point of intersection of the circle and the line (as shown in the following Figures 2(a)-(d)).



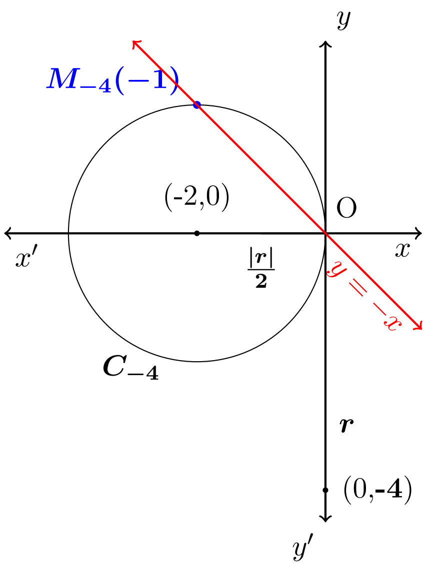
$m < 0, r > 0$ (when $x > 0$)
Here $m = -1, r = 4$

Fig. 2(a)



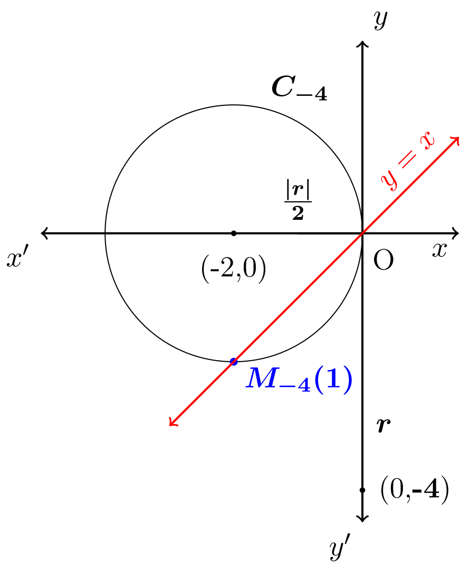
$m > 0, r > 0$ (when $x > 0$)
Here $m = 1, r = 4$

Fig. 2(b)



$m < 0, r < 0$ (when $x < 0$)
Here $m = -1, r = -4$

Fig. 2(c)



$m > 0, r < 0$ (when $x < 0$)
Here $m = 1, r = -4$

Fig. 2(d)

In view of the above pictorial description of one-to-one correspondence, it is clearly understood that the group $G(m)$, for each $m \in \mathbb{R}$, is represented by the line $y = mx$, which in turn, is the row space (and the column space as well) of each matrix in $G(m)$. This also suggests the *way of finding the matrix, say A, in the set $\bigcup_{m\in\mathbb{R}} G(m)$ which is represented on $\mathbb{R}^2$ by a point $P(h,k)$* say, taken under consideration. In fact, take the line through origin and the point $P(h,k)$ considered. Determine the angle θ made by this line with the positive direction of x-axis. Let $m = \tan\theta$. Then draw a perpendicular on this line at P to meet the x-axis at Q. Take the measure of the line segment OQ. Suppose $OQ = r$ and $x = r(1+m^2)$. Then the point P is the representative of the matrix $A = x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} \in G(m)$ [Fig. 3].

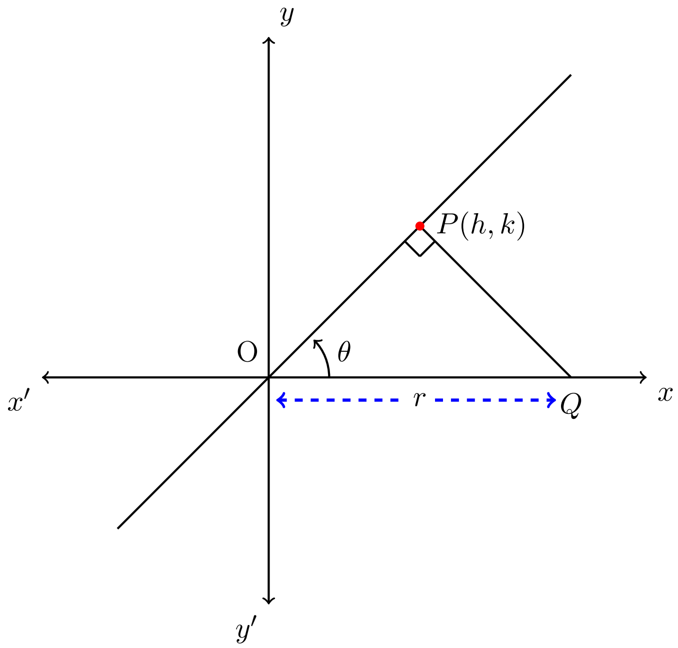


Fig. 3

3.1.5. *A Group Theoretic Interlude : Harmony of Curves and Groups.* It is interesting to note that the family of curves $\mathfrak{C} = \{C_r : r \in \mathbb{R}^*\}$ forms a commutative group under the operation $C_rC_s = C_{rs}$ (for any $r, s \in \mathbb{R}^*$) with identity element C_1 in which for any $C_r \in \mathfrak{C}, C_r^{-1} = C_{\frac{1}{r}}$. This group $\mathfrak{C}$ is eventually isomorphic to the multiplicative group $\mathbb{R}^*$ under the isomorphism $C_r \mapsto r$. Every point on the identity element C_1 (i.e., on the curve C_1)

represents the identity element of exactly one group of matrices of the form $G(m)$. If a point P on C_r represents a matrix A in the group $G(m)$, then the MP inverse A^+ of A (i.e., A^{-1} in the group $G(m)$) is represented by the point Q on the group inverse $C_r^{-1} = C_{\frac{1}{r}}$ of C_r in the group $\mathfrak{C}$, which is the point of intersection of the curve $C_{\frac{1}{r}}$ and the line $y = mx$. Following diagram [Fig. 4] depicts this beautifully synchronized harmony.

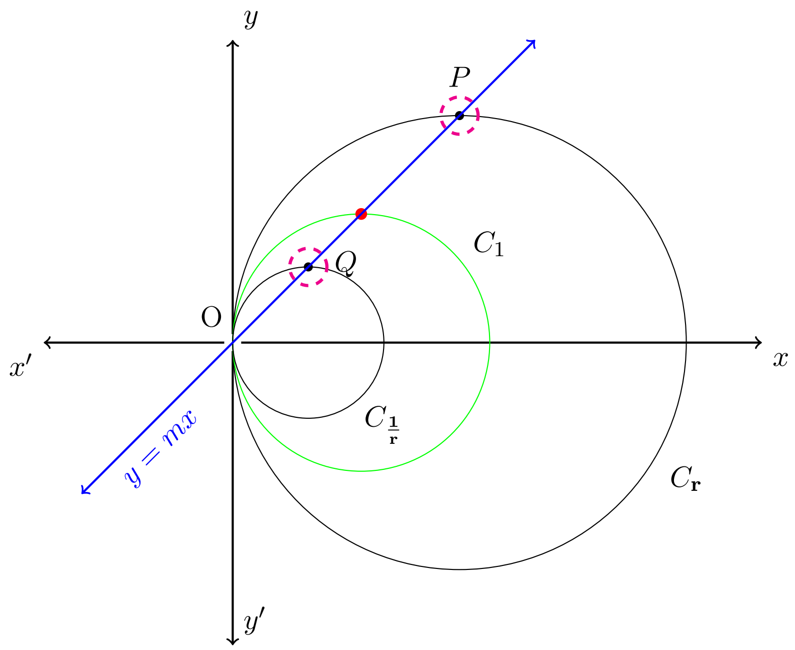


Fig. 4

Let us now define a mapping

$$f : G(m) \to \mathfrak{C} \text{ by } f\left(x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\right) = C_r, \text{ where } r = x(1+m^2).$$

Now, for any $x, y \in \mathbb{R}^*$, if $r = x(1+m^2)$ and $s = y(1+m^2)$, then $f\left(x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} y \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\right) = f\left(xy(1+m^2) \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\right) = C_{xy(1+m^2)^2} = C_{x(1+m^2)}C_{y(1+m^2)} = C_rC_s = f\left(x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\right) f\left(y \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\right)$, whence f is a group homomorphism.

For any $r \in \mathbb{R}^*$, $x = \frac{r}{1+m^2} \in \mathbb{R}^*$ and so $f\left(x \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\right) = C_r$. Again,

$f\left(x\begin{bmatrix}1 & m\\ m & m^2\end{bmatrix}\right) = C_1 \implies C_{x(1+m^2)} = C_1 \implies x = \frac{1}{1+m^2} \implies$ $x\begin{bmatrix}1 & m\\ m & m^2\end{bmatrix}$ is the identity element of the group $G(m)$.

Thus the group of matrices $G(m)$ is isomorphic to the group of curves $\mathfrak{C}$ for each $m \in \mathbb{R}$. Again the *group of curves $\mathfrak{C}$ is isomorphic to the multiplicative group* $\mathbb{R}^*$ by isomorphism $C_r \mapsto r$. Hence for each $m \in \mathbb{R}$, $G(m) \cong \mathbb{R}^*$ by the isomorphism

$$A = x\begin{bmatrix}1 & m\\ m & m^2\end{bmatrix} \mapsto C_r \mapsto r$$

where $r = x(1+m^2)$, which is the nonzero eigenvalue of A. Thus from the linear algebraic standpoint, we can say that, since for every $A \in G(m)$, there exists unique $r \in \mathbb{R}^*$, such that A is orthogonally similar to the matrix $B = x\begin{bmatrix}0 & 0\\ 0 & r\end{bmatrix} \in G_0$ and thereby r is the nonzero eigenvalue of A, so the group isomorphism for which $G(m) \cong \mathbb{R}^*$, maps A into its nonzero eigenvalue r. Geometrically, if a point P on $\mathbb{R}^2$ (not on y-axis) on the line $y = mx$ represents a matrix A in the group $G(m)$, then the group isomorphism for $G(m) \cong \mathbb{R}^*$ maps A into the point Q, at which the perpendicular to the line $y = mx$ at P, intersects the x-axis. It follows immediately from the following diagram [Fig. 5] along with the diagram in Fig. 3.

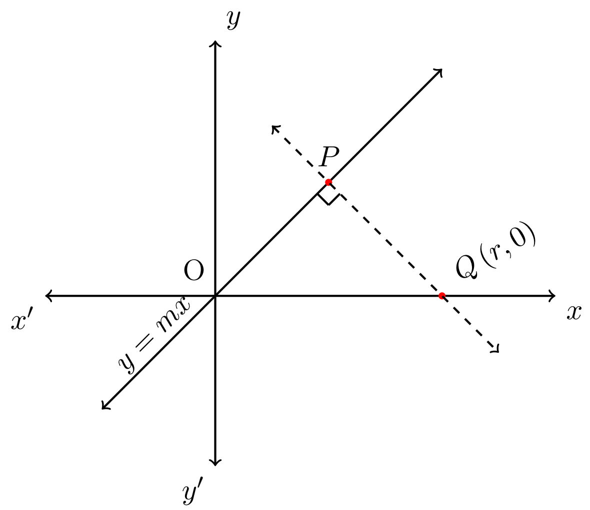


Fig. 5

Note that, for any $a, b \in \mathbb{R}^*$, if the points $M_a(m)$ and $M_b(m)$ on $y = mx$ represent the matrices A and B in $G(m)$, then $M_{ab}(m)$ represents the matrix AB in $G(m)$, on $y = mx$. This indicates that the set $M = \{M_a(m) : a \in \mathbb{R}^*\}$ turns out to be a commutative group under the operation defined by $M_a(m)M_b(m) = M_{ab}(m)$. The identity element of the group M is $M_1(m)$ which represents the identity element $\mathcal{I}_{G(m)}$ of the group $G(m)$. The group theoretic inverse $(M_a(m))^{-1}$ of $M_a(m)$ in the group M is $M_{\frac{1}{a}}(m)$. In other words, if $M_a(m)$ represents a matrix A in $G(m)$, then the inverse of $M_a(m)$ in the group M represents the inverse of A in the group $G(m)$.

3.2. **Examining the Groups without the 'Desired Property'.**

3.2.1. *Connecting $G(m,n)$ with $G_0 \in \mathcal{G}$ via Similarity.* In order to organize the matrices in $G(m,n)(m, n \in \mathbb{R}$ with $m \neq n, 1 + mn \neq 0)$ in a systematic manner, we first consider the class of all matrices with column space $y = mx$ (**for a fixed $\boldsymbol{m} \in \mathbb{R}$**), which are *non-orthogonally similar to the diagonal matrix* $\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$ **for a fixed $\boldsymbol{r} \in \mathbb{R}^*$**. This class is indeed formed by taking exactly one matrix from each group $G(m,n)$ when $m \in \mathbb{R}$ is fixed and n (satisfying $n \neq m$ and $1 + mn \neq 0$) runs over $\mathbb{R}$. One such matrix from $G(m,n)$ is $A = r \begin{bmatrix} \frac{1}{1+mn} & \frac{n}{1+mn} \\ \frac{m}{1+mn} & \frac{mn}{1+mn} \end{bmatrix}$. We pin down the matrix A *on the line $y = nx$ which is the row space of* A, by associating a unique point, denoted by $\boldsymbol{M_r(m,n)}$ (for a fixed m) at a distance say s, from the origin, **on the line $\boldsymbol{y = nx}$**.

Suppose $x = \frac{r}{1+mn}$, so that $nx = \frac{nr}{1+mn}$, whence (x, nx) is a point on $y = nx$. Let us take $M_r(m,n) = (x, nx)$ to locate the matrix $A \in G(m,n)$ on $\mathbb{R}^2$. Now, we note that the distance s of this point $M_r(m,n)$ is given by

$$s^2 = x^2 + n^2x^2 = \frac{r^2(1+n^2)}{(1+mn)^2}.$$

So $s = \frac{|r|\sqrt{1+n^2}}{|1+mn|}$.

3.2.2. *Locating the matrices in $G(m,n)$ on $\mathbb{R}^2$ Geometrically.* For the non-null vector $(0, r)$ on the y-axis that corresponds to the matrix $B \in G_0$, the family of non-orthogonally similar matrices, one from each group $G(m,n)$ (as n runs over $\mathbb{R}$, keeping m fixed, with the restrictions already stated) can be made to correspond to the points lying on the line $y = nx$, respectively at a distance $s = \frac{|r|\sqrt{1+n^2}}{|1+mn|}$ from the origin, each matrix being r times the 'identity' of the corresponding group $G(m,n)$.

These points, corresponding to a fixed $r \in \mathbb{R}^*$, trace out a line on $\mathbb{R}^2$, whose parametric equation is given by

$$x = \frac{r}{1+mn}, \quad y = \frac{rn}{1+mn} \text{ (with parameter } n\text{)}.$$

This is indeed the line $\boldsymbol{x + my = r}$ on $\mathbb{R}^2$, not passing through the origin. Note that, for a fixed $r \in \mathbb{R}^*$, geometrically this line is perpendicular to $y = mx$ and intersects it at the typical point $M_r(m)$, which is, in fact, the point representation of the matrices in $G(m)(\in \mathcal{G})$, orthogonally similar to the matrix $\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$. Thus, towards exactly locating the desired class of points in the present case, *we must keep the point of intersection of the lines $x + my = r$ and $y = mx$ (for each fixed $r \in \mathbb{R}^*$) out of our consideration.* Thus, if r runs over $\mathbb{R}^*$, we get that, for each fixed $m \in \mathbb{R}$, *a family of parallel lines* (each being perpendicular to $y = mx$) given by $\mathcal{L} = \{x + my = r : r \in \mathbb{R}^*\}$, *which are broken at the points $M_r(m)$ $(r \in \mathbb{R}^*)$, represents the classes of matrices those are non-orthogonally similar to* $\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$, $(r \in \mathbb{R}^*)$ [Fig. 6].

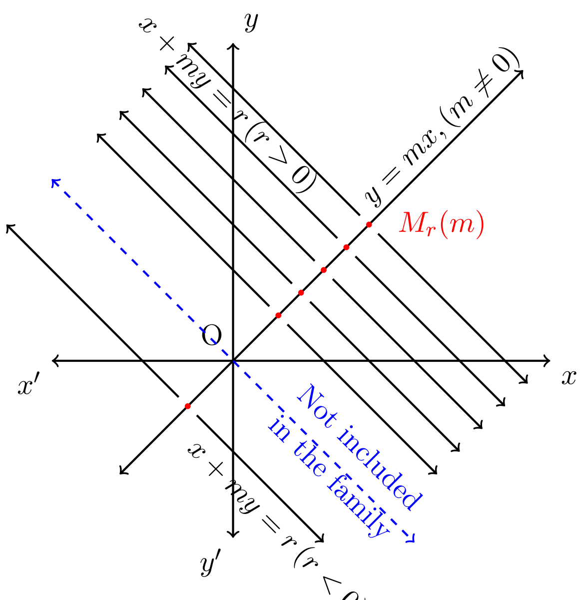
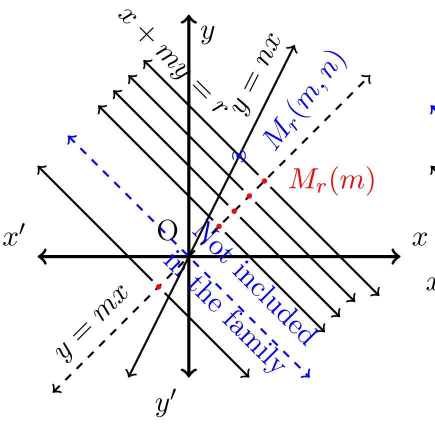


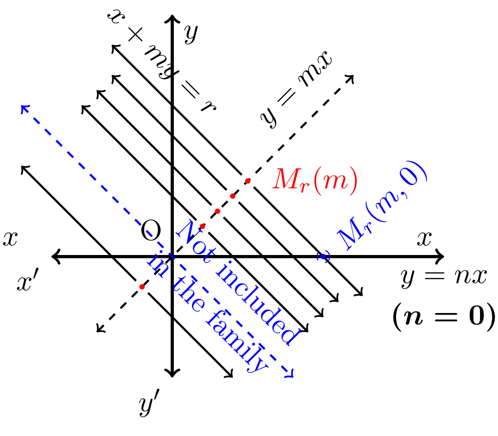
Fig. 6

Note that, the whole line like $x + my = r$ here happens to be a representation of the class of all similar singular (symmetric or non-symmetric) matrices (with column space $y = mx$), diagonalizable to the matrix $\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$. As a passing remark, here we point out that, every line in the aforesaid family is indeed an *affine subspace* in $\mathbb{R}^2$; in fact, every such line is the solution set (in $\mathbb{R}^2$) of the matrix equation $A\underline{x} = \underline{b}$, where $A = \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} r \\ 0 \end{bmatrix}$ for some $r \in \mathbb{R}^*$.

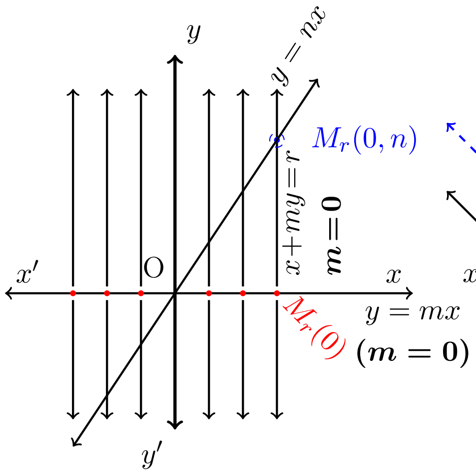
3.2.3. *Visual Locations of Various Subclasses on* $\mathbb{R}^2$. So we have seen that, for a fixed $m \in \mathbb{R}$ and a fixed $r \in \mathbb{R}^*$, as n runs over $\mathbb{R}$ with the usual restrictions, a matrix $P \in G(m, n)$ with nonzero eigenvalue r, is located on $\mathbb{R}^2$ by a point on the line $x + my = r$. Geometrically, it means that, due to the rotation of the line $y = mx$ through an angle α with $0 < \alpha < \frac{\pi}{2}, \frac{\pi}{2} < \alpha < \pi$ (anticlockwise), when it arrives at the line $y = nx$, then its point of intersection with the line $x + my = r$ becomes the point representation (P.R.) $M_r(m, n)$ of P on $\mathbb{R}^2$. [Fig. 7, 8, 9]. The line $y = mx$ may rotate through some particular α to arrive at the line $x = 0$. The point of intersection $(0, \frac{r}{m})$ in this case represents the matrix $P \in G_0(m, 0)$ with nonzero eigenvalue r. [Fig.10]



P.R. of $\boldsymbol{A} \in \boldsymbol{G(m,n)}$
with eigenvalue $r \in \mathbb{R}^*$
$\boldsymbol{M_r(m,n) = (\frac{r}{1+mn}, \frac{rn}{1+mn})}$
Fig. 7

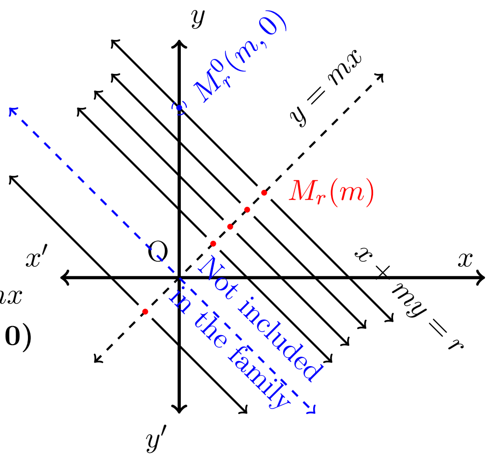


P.R. of $\boldsymbol{A} \in \boldsymbol{G(m,0)}$
with eigenvalue $r \in \mathbb{R}^*$
$\boldsymbol{M_r(m,0) = (r,0)}$
Fig. 8



P.R. of $\boldsymbol{A \in G(0, n)}$
with eigenvalue $r \in \mathbb{R}^*$
$\boldsymbol{M_r(0, n) = (r, nr)}$

Fig. 9

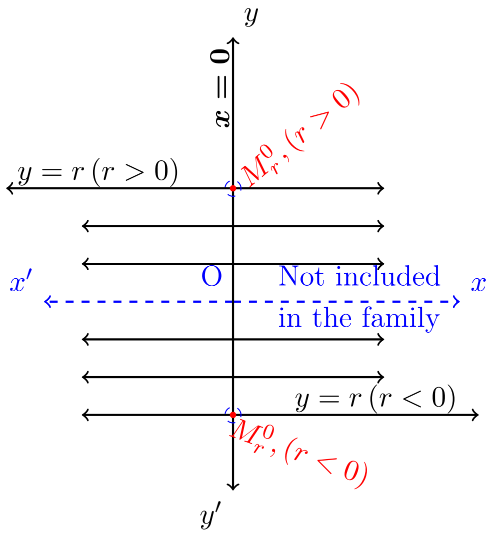


P.R. of $\boldsymbol{A \in G_0(m, 0)}$
with eigenvalue $r \in \mathbb{R}^*$
$\boldsymbol{M_r^0(m, 0) = (0, \frac{r}{m})}$

Fig. 10

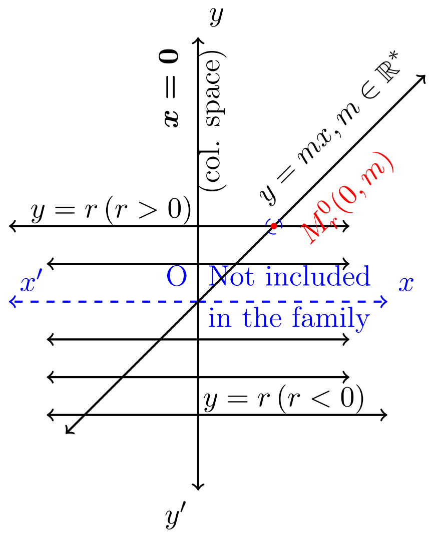
It is interesting to note that, while the family of curves $\mathfrak{C} = \{C_r : r \in \mathbb{R}^*\}$ (introduced in 3.1.5) is useless to represent the matrices in $G(m, n)$ on $\mathbb{R}^2$, the family of lines $\mathcal{L} = \{x + my = r : r \in \mathbb{R}^*\}$ may be used for locating the matrices in $G(m)\,(m \in \mathbb{R})$. In fact, if $n \neq m$ is dropped and if $G(m)$ be written as $G(m, m)$, then the point-representation $M_r(m)$ of a matrix $A \in G(m)$ with nonzero eigenvalue r, is nothing but the point of intersection of $y = mx$ and the line $x + my = r$ taken from the family $\mathcal{L}$. This not only suggests an easier way to trap the points on $\mathbb{R}^2$ that represents the symmetric matrices in $G(m)$, but also provides us with a way out for representing the symmetric matrices in G_0 as well. Recall that, both the row space and column space of every matrix in G_0 are the y-axis ($x = 0$), which has no point in common with any of the curves of the family $\mathfrak{C}$; so in no way the family $\mathfrak{C}$ may help to locate the matrices in G_0. However, we may form a family corresponding to G_0, essentially analogous to the family $\mathcal{L}$, that may serve our purpose. Recall that the family of parallel lines $\mathcal{L}$, for $G(m)$ and $G(m, n)$ both, are the lines perpendicular to the column space $y = mx$ for every matrix in $G(m)$ (and $G(m, n)$). Guided by this feature of forming the family of lines, we consider for G_0, the family of lines $\{y = r : r \in \mathbb{R}^*\}$, which are perpendicular to the column space ($x = 0$)

of every matrix in G_0. Then similar to the case of $G(m)$, we can say that the point-representation of a matrix $A \in G_0$, (with nonzero eigenvalue r), is merely the point of intersection of $x = 0$ and $y = r$, i.e., the point $(0, r)$ [Fig. 11].



P.R. of $\boldsymbol{A \in G_0}$
with eigenvalue $r \in \mathbb{R}^*$
$\boldsymbol{M_r^0 = (0, r)}$

Fig. 11



P.R. of $\boldsymbol{A \in G_0(0, m)}$
with eigenvalue $r \in \mathbb{R}^*$
$\boldsymbol{M_r^0(0, m) = (\frac{r}{m}, r)}$

Fig. 12

Focusing on the form of the matrices in $G_0(0, m)$, we see that, $x = 0$ is the column space of all the matrices here, which happens to be the same as those in the class G_0; so we may go by the method applied for the matrices in G_0, to locate the matrices in $G_0(0, m)$ on $\mathbb{R}^2$. Hence the point representations of $A \in G_0(0, m)$ with eigenvalue $r \neq 0$, is the solution of the equations $y = r$ and $y = mx$ (*row space*), which is clearly the point $(\frac{r}{m}, r)$ and then $A = \frac{r}{m}\begin{bmatrix} 0 & 0 \\ 1 & m \end{bmatrix} \in G_0(0, m)$ [Fig. 12].

3.2.4. *Non-diagonalizable Matrices and Jordan Line.* Note that, for any $m \in \mathbb{R}^*$, a matrix $J = r\begin{bmatrix} 1 & -\frac{1}{m} \\ m & -1 \end{bmatrix} \in J(m)\,(r \in \mathbb{R}^*)$ is represented by $(r, -\frac{r}{m})$ lying on the line $x + my = 0$. We call it the **Jordan line for**

$\boldsymbol{m} \in \mathbb{R}^*$. Further, observe that, a matrix $J_1 = r\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in J_0\,(r \in \mathbb{R}^*)$ is represented by $(0, r)$ (which lies on the *row space* of every matrix in J_0) on $\mathbb{R}^2$. So, the row space $x = 0$ is termed as the **Jordan line for $\boldsymbol{m} = \mathbf{0}$**. Again, each matrix $J_2 = r\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in J^0\,(r \in \mathbb{R}^*)$ is represented on $\mathbb{R}^2$ by the unique point $(r, 0)$ lying on $y = 0$. We wish to call this line $y = 0$ as a *Jordan like line* following the two other cases, but one must appreciate that this particular line cannot be represented as $x + my = 0$ or, equivalently as $y = -\frac{1}{m}x\,(m \in \mathbb{R}^*)$ for any $m \in \mathbb{R}$. However, note that, $y = -\frac{1}{m}x$ is a Jordan line for every $m \in \mathbb{R}^*$, and as $m \in \mathbb{R}^*$ grows larger and larger, these lines approach (in limit) to this typical line $y = 0$. So, the line $y = 0$, representing the matrices in J^0 on $\mathbb{R}^2$ can be idealized as **Jordan line on $\mathbb{R}^2$ at infinity**.

3.2.5. *Putting all the Results in a Nutshell.* We can now say that all the singular 2×2 real matrices have been classified from our context. In fact, we have categorized all such singular matrices into some classes, each of which, with respect to the operation of usual matrix multiplication, either possesses a rich algebraic structure of an *Abelian group*, or else fails to have even the simplest structure of a groupoid. Also we have considered the member matrices of these classes structurally from various perspectives of linear algebra towards analyzing and understanding their abstract algebraic behaviors that have been discussed so far. Let us now present, in a compact tabular form, all of these different classes of 2×2 singular real matrices with their main aspects and features, as have so far been revealed through our investigation, in the following table.

For the sake of typographical space management, we shall use the following abbreviations in the table.
LAA : Linear Algebraic Aspects,
AAA: Abstract Algebraic Aspects,
MM : matrix multiplication,
LR: Line-representation on $\mathbb{R}^2$,
PR: Point-representation on $\mathbb{R}^2$.

All Singular 2×2 Matrices over $\mathbb{R}$				
Class	**LAA**	**AAA** (under MM)	**Geometric Aspects**	
			LR of the Class	**PR** of a Matrix A of the Class
$G(m)$; $m \in \mathbb{R}$	Class of orthogonally simultaneously diagonalizable (i.e. Symmetric) matrices **Column space** : $y = mx$ **Row space** : $y = mx$	An Abelian Group $\boxed{A^{-1} = A^{+}}$	$y = mx$	$(\frac{r}{1+m^2}, \frac{rm}{1+m^2})$: $r \neq 0$ is eigenvalue of A Point of intersection of $y = mx$ and $x^2 + y^2 = rx$ **OR**, $y = mx$ and $x + my = r$
G_0	Class of orthogonally simultaneously diagonalizable (i.e. Symmetric) matrices **Column space** : $x = 0$ **Row space** : $x = 0$	An Abelian Group $\boxed{A^{-1} = A^{+}}$	$x = 0$	$(0, r)$: $r \neq 0$ is eigenvalue of A
$\{\theta_{2\times 2}\}$	Trivially diagonalizable (i.e. Symmetric) matrix	A trivial Group $\boxed{A^{-1} = A^{+}}$	—	$(0, 0)$
$G(m, n)$; $m, n \in \mathbb{R}$ $n \neq m$, $mn \neq -1$	Class of non-orthogonally simultaneously diagonalizable (i.e.non-Symmetric) matrices **Column space** : $y = mx$ **Row space** : $y = nx$	An Abelian Group $\boxed{A^{-1} \neq A^{+}}$	$y = nx$	$(\frac{r}{1+mn}, \frac{rn}{1+mn})$: $r \neq 0$ is eigenvalue of A Point of intersection of $y = nx$ and $x + my = r$
$G_0(m, 0)$; $m \in \mathbb{R}^*$	Class of non-orthogonally simultaneously diagonalizable (i.e.non-Symmetric) matrices **Column space** : $y = mx$ **Row space**: $x = 0$	An Abelian Group $\boxed{A^{-1} \neq A^{+}}$	$x = 0$	$(0, \frac{r}{m})$: $r \neq 0$ is eigenvalue of A Point of intersection of $x = 0$ and $x + my = r$
$G_0(0, m)$; $m \in \mathbb{R}^*$	Class of non-orthogonally simultaneously diagonalizable (i.e.non-Symmetric) matrices **Column space** : $x = 0$ **Row space**: $y = mx$	An Abelian Group $\boxed{A^{-1} \neq A^{+}}$	$y = mx$	$(\frac{r}{m}, r)$: $r \neq 0$ is eigenvalue of A

[*Continued*]

Classification of all Singular 2×2 Matrices over $\mathbb{R}$				
Class	**LAA**	**AAA** (under MM)	**Geometric Aspects** **LR** of the Class	**PR** of a Matrix A of the Class
$J(m)$; $m \in \mathbb{R}^*$	Class of non-diagonalizable (i.e. similar to Jordan) matrices **Column space** : $y = mx$ **Row space** : $x + my = 0$	Not even a Groupoid	$x + my = 0$ **Jordan line** for $m \in \mathbb{R}^*$	$(r, -\frac{r}{m})$: when $A = r\begin{pmatrix} 1 & -\frac{1}{m} \\ m & -1 \end{pmatrix}$ with $r \in \mathbb{R}^*$
J_0	Class of non-diagonalizable (i.e.Jordan) matrices **Column space** : $y = 0$ **Row space** : $x = 0$	Not even a Groupoid	$x = 0$ **Jordan line** for 0	$(0, r)$: when $A = r\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with $r \in \mathbb{R}^*$
J^0	Class of non-diagonalizable (i.e.Jordan) matrices **Column space**: $x = 0$ **Row space** : $y = 0$	Not even a Groupoid	$y = 0$ **Jordan line** at *infinity*	$(r, 0)$: when $A = r\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $r \in \mathbb{R}^*$

It is apparent from the table above that there are different classes which are represented by the same line on $\mathbb{R}^2$. As for example, the classes $G(n)$ and $G(m,n)$ (for $m, n \in \mathbb{R}$ with $m \neq n$) are represented by the same line $y = nx$, though one of them is a class of symmetric matrices and the other is of non-symmetric matrices. Moreover, matrices belonging to different classes are sometimes being represented by the same point on $\mathbb{R}^2$. In fact, the orthogonally diagonalizable (symmetric) matrix $r\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in G(0)$, (a group of matrices '*with the desired property*'), and the non-diagonalizable Jordan matrix $r\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in J^0$, (a set '*without the desired property*') both are represented by the same point $(r, 0)$ on $\mathbb{R}^2$. This shows that the bijectivity between various classes and their line-representation, and also that between the member-matrices and their point-representations seem to be missing. However, to achieve this bijection, it is necessary to take parallel copies of $\mathbb{R}^2$ on which the line and the point representations of the classes and matrices may be spread out systematically in such a way that no line (point) on the same copy of $\mathbb{R}^2$ may represent two or more distinct classes (respectively, matrices). In the next subsection, we undertake such a geometric presentation of the state of affairs.

3.3. **The Visual Library of Real 2×2 Singular Matrices in $\mathbb{R}^3$.** Let us consider the 3-dimensional space $\mathbb{R}^3$ and fix a three dimensional Cartesian reference frame. For each fixed $m \in \mathbb{R}$ we take two fixed 2-dimensional planes $y = mx$ (i.e. $mx - y + 0 \cdot z = 0$) and $z = m$ which are orthogonal to each other. The line of intersection of these planes is $\frac{x}{1} = \frac{y}{m} = \frac{z-m}{0}$, which is a fixed line on the plane $z = m$. We shall refer to this plane $z = m$, which is parallel to the xy-plane at a height (or, depth) of m from origin O as the **m-plane, for each m** $\in \mathbb{R}$ [Fig. 13].

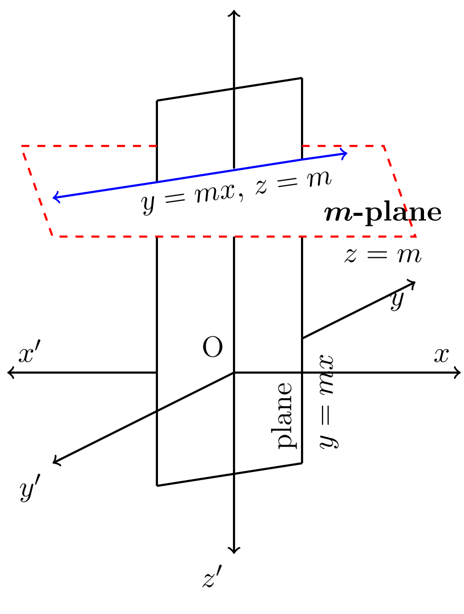


Fig.13

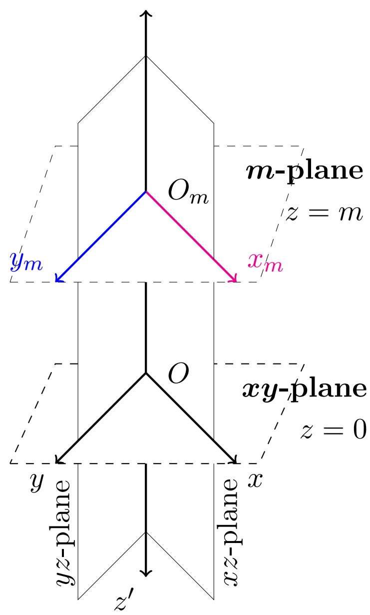


Fig.14

Note that, such an m-plane (for each $m \in \mathbb{R}$) is not a subspace of $\mathbb{R}^3$. However, an m-plane is an affine space in $\mathbb{R}^3$. Indeed, we see that $AX = b$, where $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 & m \end{bmatrix}^T$ has the solution set $\{(x, y, m : x, y \in \mathbb{R}\}$ which is virtually the m-plane $z = m$. Take the xz-plane and also yz-plane. Suppose they meet the m-plane $z = m$ in the line $x_m O_m x'_m$ and $y_m O_m y'_m$ [Fig. 14]. The point O_m is naturally the point at which the z-axis intersects the m-plane $z = m$. Thus, for every choice of $m \in \mathbb{R}$, the three dimensional Cartesian Coordinate system (x, y, z) is translated into another system (x_m, y_m, z_m) through $x_m = x, y_m = y, z_m = z - m$, where every point on m-plane is $(x_m, y_m, 0)$. This in turn yields a two-dimensional frame on m-plane, (for each $m \in \mathbb{R}$) on which the fixed line $\frac{x}{1} = \frac{y}{m} = \frac{z-m}{0}$ gets its equation as $y_m = mx_m$, or, for all practical purpose, as $y = mx$ (by writing O_m, x_m, y_m as O, x, y respectively, if no confusion occurs.) Recall that, with the points on this affine space $z = m$, *looked upon as a copy of* $\mathbb{R}^2$, we already have a one-to-one correspondence with those 2×2 real singular matrices, that have their column space as the fixed line $y = mx$.

3.3.1. *Organizing the Catalogue of the Library.* The effects of the aforesaid translation (for different $m \in \mathbb{R}$) of the pre-fixed 3-dimensional reference frame indulges our imagination of the old-fashioned manual '*card-cataloguing*' system of books racked in a library. Imagine that all our groups of singular 2×2 real matrices (of the form $G(m), G(m, n), m, n \in \mathbb{R}$) and the sets of all possible Jordan 2×2 matrices are, as if, being *catalogued* on the *catalogue cards* of affine planes (m-plane, $m \in \mathbb{R}$), which are stacked in a vertically placed huge *catalogue drawer* of $\mathbb{R}^3$, all being fixed with a *catalogue rod* of z-axis pierced through all the cards of affine planes, endowing each plane with an '*origin*' for a 2-dimensional reference plane at the piercing point, (*a copy of* $\mathbb{R}^2$ *indeed*), corresponding to the origin of the pre-fixed 3-dimensional reference frame in $\mathbb{R}^3$. [Fig. 15] The *catalogue cards* of m-planes ($m \in \mathbb{R}$) are arranged in the *catalogue drawer* of $\mathbb{R}^3$ vertically, according to the natural 'total order' of the reals $m \in \mathbb{R}$ on the real line. That is the key to handle this *catalogue drawer* in order to locate our groups and Jordan matrix sets.

3.3.2. *Locating Different Subclasses of Matrices on Copies of* $\mathbb{R}^2$. For a fixed $m \in \mathbb{R}$, to locate the position of the group $G(m)$, one must start from the 0-plane (i.e., xy-plane) and then has to move upward (if $m > 0$) or downward (if $m < 0$) along the catalogue rod of z-axis to find the 'm-th card' (i.e., m-plane) of the drawer and then **read this card by its fixed line $y = mx$** [refer to Fig. 15]. Our group $G(m)$ is described on this card by the *fixed line* $y = mx$(excluding the origin). For $n \in \mathbb{R}$ with $n \neq m$, the group $G(n)$ cannot have any description on the mth card; to determine it one has to find the 'nth card' (i.e., n-plane) of the drawer, on which $y = nx$ is the fixed line, which (excluding the 'origin') describes the group $G(n)$. The group $G(m, n)$ is located on the mth card (i.e., m-plane with fixed line $y = mx$), but it is described by the line $y = nx$ (excluding the origin).

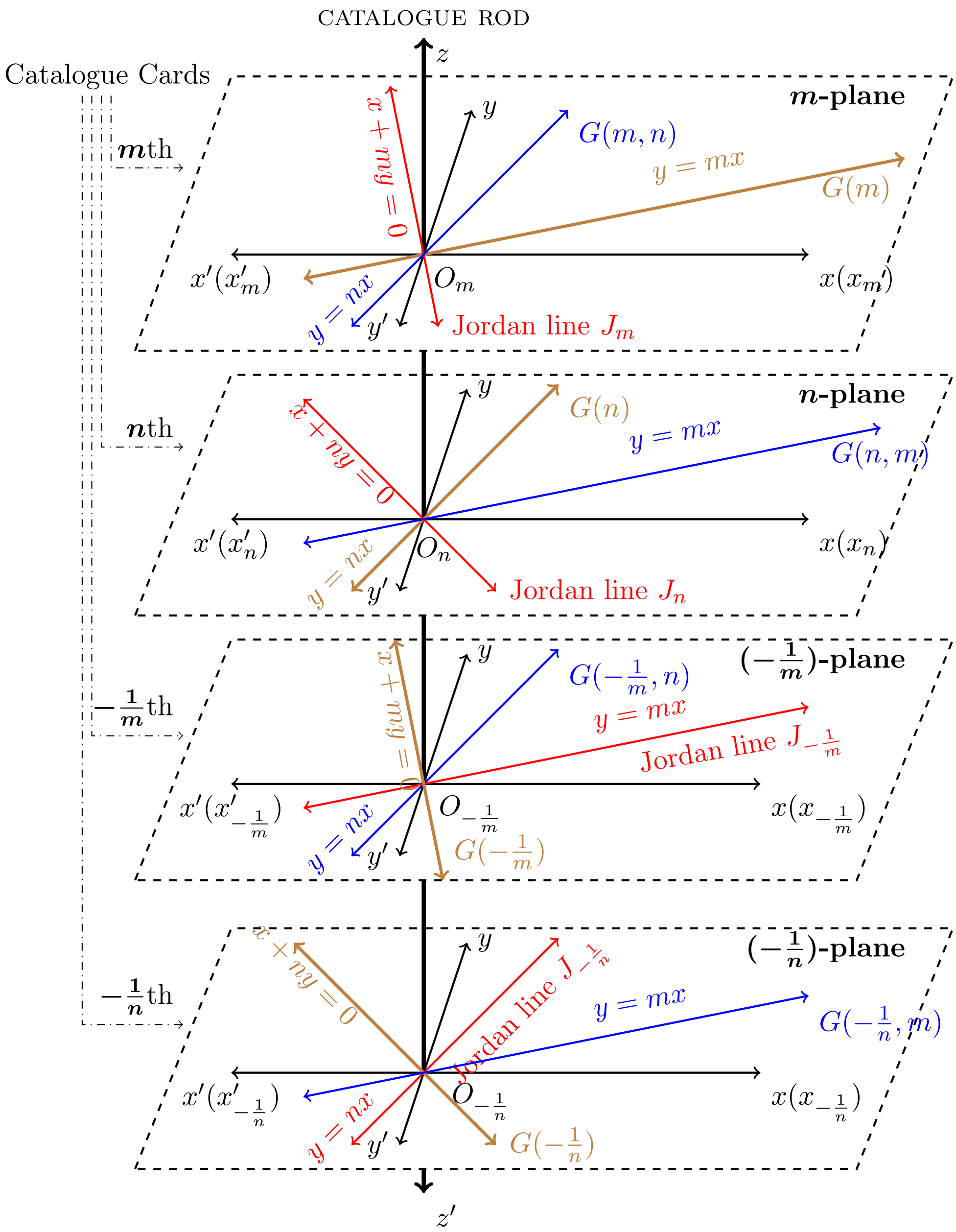
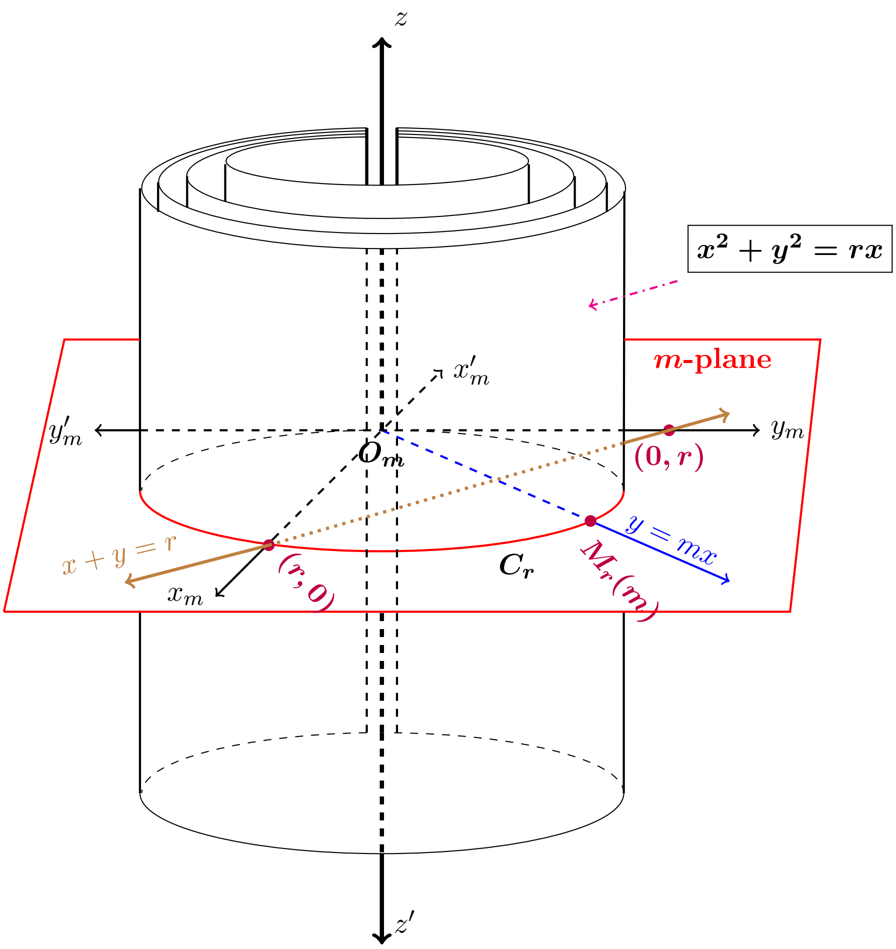


Fig. 15

On the other hand, the group $G(n,m)$ is located on the nth card (i.e.,n-plane, with fixed line $y = nx$) but it is described by the line $y = mx$ (excluding the origin). Again, on the 'mth card' (i.e., m-plane), the set J_m of Jordan matrices of the form $r \begin{bmatrix} 1 & -\frac{1}{m} \\ m & -1 \end{bmatrix}$ are described by the line $x + my = 0$ (excluding origin) whence we call it as the *Jordan line on the m-plane*, while the line $x+ny = 0$ on the 'nth card' (i.e., n-plane) describes the set J_n of Jordan matrices whence it is the *Jordan line on n-plane*. [Fig. 15]

3.3.3. *Locating Individual Matrices on Different Copies of* $\mathbb{R}^2$. Note that, reading the mth catalogue card (i.e., m-plane) for each $m \in \mathbb{R}$ does not help us **to locate an individual matrix on that plane**; rather it helps us to locate some specific classes (like, $G(m), G(m,n), J_m$) in which individual matrices of identical characteristics may be contained. It thus, in a sense, provides a '*call number*' of an individual matrix. What we need further, is something like an '*accession number*' for individual matrices, in order to locate them on various m-planes. Towards this, we have to draw some curves (and/or lines) on a particular m-plane, in such a way that, the individual matrices may be 'accessed' (i.e., located) at the respective points of intersection of these curves (and/or lines) with suitably chosen lines through origin on m-planes, a line that represents the classes of the individual matrices. What are these families of curves and lines on a m-plane for a fixed $m \in \mathbb{R}$ have already been derived in Sections 3.1.3 and 3.2.3.

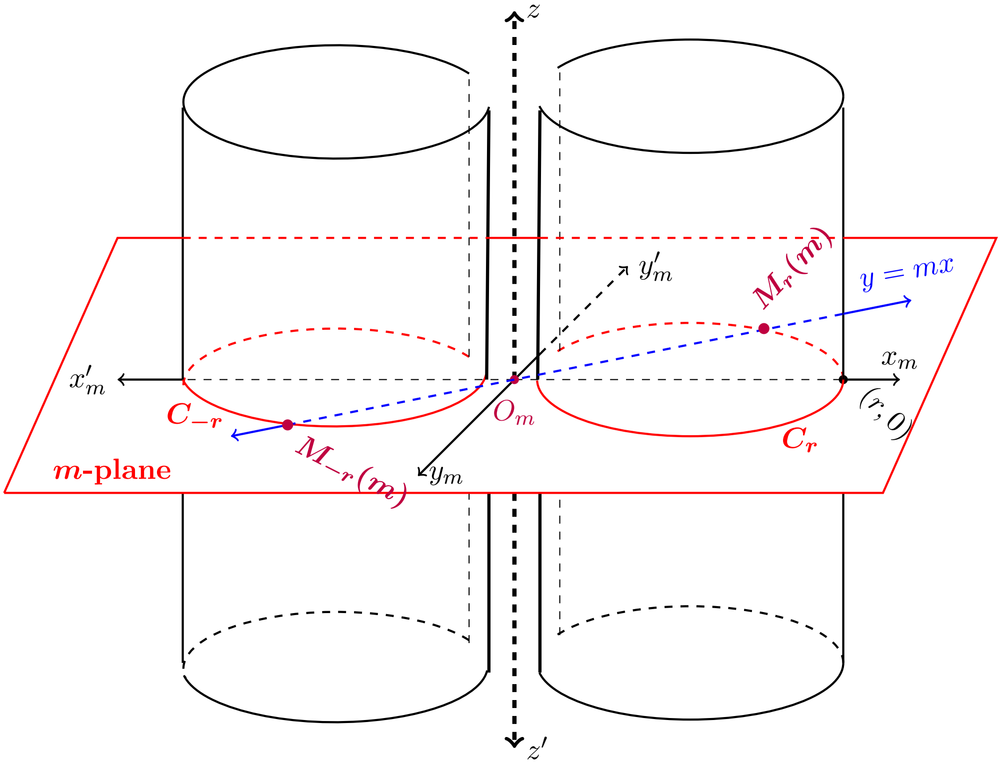
Recall that, for any particular $m \in \mathbb{R}$, the family of curves on the corresponding m-plane (viewed as a copy of $\mathbb{R}^2$) is $\mathfrak{C} = \{C_r :: x^2 + y^2 = rx : x, y \in \mathbb{R}, (x,y) \neq (0,0), r \in \mathbb{R}^*\}$, where each C_r may be looked upon as a 2-dimensional space curve on the m-plane. However, in actuality, they should be appreciated as the cross section in $\mathbb{R}^3$ made by the affine space $z = m$ (i.e.,m-plane) with the surface $x^2 + y^2 = rx$ for $r \in \mathbb{R}^*$, whence they are 3-dimensional space curves having equations $x^2 + y^2 = rx, z = m$ with the point $(0,0,m)$ missing. Now, since all the m-planes ($m \in \mathbb{R}$) as affine spaces in $\mathbb{R}^3$, are parallel to the xy-plane, (i.e, 0-plane), and since in that plane the curves of the aforesaid family (for each $r \in \mathbb{R}^*$) are indeed circles, it is easy to see that the surface $x^2 + y^2 = rx$ in $\mathbb{R}^3$ is a **right circular cylinder** having the *guiding curve* $x^2+y^2 = rx, z = 0$ and a *generating line* $\frac{x-r}{0} = \frac{y}{0} = \frac{z}{1}$ (perpendicular to xy-plane). Note that the z-axis lies on the cylinder $x^2 + y^2 = rx$ for each $r \in \mathbb{R}^*$. If, by a so called one-point fine cut, just the z-axis is slashed out of the cylinder, then we have a one-parameter family of surfaces $S_r (r \in \mathbb{R}^*)$ in $\mathbb{R}^3$, where $S_r = x^2+y^2 = rx, (x, y) \neq (0, 0)$, which when cross-sectioned by the affine spaces like $z = m(m \in \mathbb{R})$, give rise to our known family $\mathfrak{C}$ on m-planes. [Fig.16] It is interesting to note that, for every $r \in \mathbb{R}^*$, the equation $x^2 + y^2 = |r|x; (x, y) \neq (0, 0)$ gives a surface in $\mathbb{R}^3$ which is everything of a double cylinder, except a one point fine cut along its z-axis. [fig. 17]



The point $M_r(m) = (\frac{r}{1+m^2}, \frac{rm}{1+m^2})$ on C_r representing $A = \frac{r}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix} \in G(m)$

Fig. 16

Thus in sooth, the whole scenario is like the following: For two $m, n \in \mathbb{R}$ with $m \neq n$,
(i) the groups $G(m)$ and $G(n)$ are represented on the two different planes $z = m$ (m-plane) and $z = n$ (n-plane) respectively.
(ii) In m-plane, the group $G(m)$ is represented by the line $y = mx, z = m$; whereas, $G(n)$ is represented on n-plane by the line $y = nx, z = n$.
(iii) A matrix A belonging to $G(m)$ is represented on the m-plane by the point of intersection of the line $y = mx, z = m$ and the cylinder $x^2 + y^2 = rx$ whenever r is the nonzero eigenvalue of A. To locate a matrix $B \in G(n)$ with nonzero eigenvalue r, on the n-plane, we have to specify the point at which the line $y = nx, z = n$ intersects the cylinder $x^2 + y^2 = rx$, as discussed in the previous section.



Double Cylinder $x^2 + y^2 = |r|x$
with one point fine cut along z-axis

Fig.17

Apart from the 2-dimensional cylindrical surface sitting in $\mathbb{R}^3$ and producing 1-dimensional curves in 2-dimensional m-planes when sliced by them, we get another family of surfaces, this time 3-dimensional and sitting in $\mathbb{R}^4$, given by $\mathfrak{S} = \{x + yz = r \,:\, r \in \mathbb{R}^*\}$. For a fixed $r \in \mathbb{R}^*$, the cross section of the surfaces of this family by an m-plane gives a family of lines $\mathfrak{L} = \{x+my = r \,:\, m \in \mathbb{R}\}$. The role of this family in determining the point representation of the matrices in $G(m,n)$ (with $n \neq m, mn \neq -1), G_0(m,0)$ and some of the matrices in $G(m)$ has already been discussed in Section 3.2.3.

It is interesting to observe that, by virtue of this '*Card Cataloguing*' of singular 2×2 matrices, each line of R^3 that lies on an unique m-plane ($m \in \mathbb{R}$) and each points on those lines respectively represent a unique class and a unique matrix belonging to those classes. Recall that, the idea of this typical '*Card Cataloguing*' rests upon the act of taking the lines $y = mx (m \in \mathbb{R})$ as the **fixed lines** on the $\boldsymbol{m}$**-plane** and then to represent (by its points) all those singular 2×2 matrices which have **the line $\boldsymbol{y = mx}$ as their column space**.

This idea may use up each of the points in $\mathbb{R}^3$ (being considered as a point on some 2-dimensional affine space in $\mathbb{R}^3$) towards representing a unique singular 2×2 real matrix; but interestingly, *still there will remain some classes (and their matrices) which will not get any line (respectively, point) representation on any affine space (m-plane). These matrices are precisely those which have the* $\boldsymbol{y}$**-axis (i.e., line $\boldsymbol{x = 0}$) as their column space.** Note that, as this is the only line that cannot be expressed in the form $y = mx$, we do not get an affine space parallel to xy-plane in $\mathbb{R}^3$ that may contain this line as a '*fixed line*.' Hence no '*catalogue card*' (m-plane) in the '*catalogue drawer*' of $\mathbb{R}^3$ records the '*call number*' (representing lines) and the '*accession numbers*' (representing points) for the matrices which have $x = 0$ as their column spaces. So, apart from all the copies of $\mathbb{R}^2$, that have already been tagged by reals $m \in \mathbb{R}$, we do need exactly one more copy of $\mathbb{R}^2$ to represent the classes $G_0, G_0(0,m), J^0$ containing those singular matrices that have $x = 0$ as their column spaces. This copy of $\mathbb{R}^2$ cannot be placed inside $\mathbb{R}^3$. However, if we take the extended real line $\mathbb{R} \cup \{\infty\}$, then this copy of $\mathbb{R}^2$ can be thought of as a *plane at infinity* ('∞-plane'), containing $x = 0$ as its *fixed line*. Then all those particular classes as stated above and all of their matrices can be represented on that '∞-plane' by its lines and its points respectively and that may be done in the manner similar to what has been adopted for m-planes ($m \in \mathbb{R}$). This brings an end to our pictorial geometric odyssey with the 2×2 singular real matrices and we digress once again to algebraic perspectives.

4. Epilogue : An Abstract Algebraic Detour

We started our inquisition from a purely abstract algebraic set up, indeed, that of a commutative group, where the elements were matrices and hence were linear algebraic objects. This motivated us initially to look into the possible interplay at their structural level towards resolving one specific question about the group theoretic inverse. Later, thanks to the favourable size of the matrices under consideration, we indulged our visual imagination, to try and locate all these matrices geometrically via pictorial presentation in $\mathbb{R}^3$, successfully to a great extent. All this said and done, we now go back to algebra again, and use abstract algebraic tools to dissect those particular classes, that have already been pinned down, one in favour of what we wanted, while the other failing to satisfy that property. It is now time to see what makes them 'tick'.

4.1. Why the Groups in $G(m)$ are the way they are?

4.1.1. *An Action of $G(m)$ and A Group of Permutations.* We have seen that for each $m \in \mathbb{R}$, the set $M(m) = \{M_a(m) \in \mathbb{R}^2 \;:\; a \in \mathbb{R}^2\}$ forms a commutative group under the operation

$$M_a(m)M_b(m) = M_{ab}(m).$$

Eventually, this group $M(m)$ is seen to be isomorphic to $G(m)$, under the isomorphism given by $M_a(m) \mapsto A \in G(m)$, whenever $\boldsymbol{a}$ **is the nonzero eigenvalue of** $\boldsymbol{A}$. Thus, group theoretically, for each $m \in \mathbb{R}$, our group $G(m)$ is isomorphic to a subgroup of the group $S_{M(m)}$ of permutations of $M(m)$. In fact, we see that, the group $G(m)$ (for $m \in \mathbb{R}$) acts on the set $M(m)$ from left by the action defined by

$$A \cdot M_s(m) = M_{as}(m),$$

whenever a is the nonzero eigenvalue of $A \in G(m)$. Thus the mapping ϕ : $G(m) \to S_{M(m)}$, given for any $A \in G(m)$ with a as its nonzero eigenvalue, by $\phi(A) = \sigma_A$ is a homomorphism, where $\sigma_A : M(m) \to M(m)$ is defined by

$$\sigma_A(M_s(m)) = A \cdot M_s(m) = M_{as}(m).$$

Now for any $A \in G(m)$ with nonzero eigenvalue a, $\phi(A) = \sigma_{\mathcal{I}_{G(m)}}$ (identity permutation of $M(m)$) $\implies \sigma_A = \sigma_{\mathcal{I}_{G(m)}} \implies \sigma_A(M_s(m)) = \sigma_{\mathcal{I}_{G(m)}}(M_s(m)) \implies M_{as}(m) = M_s(m) \implies as = s \implies a = 1 \implies A = \mathcal{I}_{G(m)}$. Thus, $\phi : G(m) \to S_{M(m)}$ is a monomorphism, whence $G(m)$ is embedded in the group $S_{M(m)}$ of permutations of $M(m)$.

4.1.2. *Permutations as a Linear Operator.* The set $M(m)$, with one point $(0, 0)$ adjoined, is indeed the subspace $y = mx$ in $\mathbb{R}^2$. Thus, each permutation σ_A ($A \in G(m)$) of $M(m)$ can be shown to be a linear operator on the subspace $y = mx$, when the image of $(0, 0)$ under σ_A is defined to be $(0, 0)$ (for any $A \in G(m)$). Clearly this linear operator is invertible. It is interesting to note that, for each $A \in G(m)$, the inverse of the linear operator σ_A on $y = mx$ is $\sigma_{A^{-1}}$ where A^{-1} is the group inverse of A in the group $G(m)$.

Note that whenever we talk about a linear operator T on $\mathbb{R}^2$, we usually associate a 2×2 matrix A (say) with T in such a way that $T = L_A$ (left multiplication transformation by A) and $[T]_\epsilon = A$, where ϵ is the standard ordered basis for $\mathbb{R}^2$ and vice-versa. Thus, a 2×2 matrix over $\mathbb{R}$ may be perceived as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, but cannot actually be thought of as a linear transformation from its row space (as its '*domain*') to its column space (as '*codomain*'), although it does map the row space to the column space (as subspaces of $\mathbb{R}^2$). Moreover, the matrix representation of $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to any ordered basis for $\mathbb{R}^2$ is a 2×2 matrix, whereas the matrix representation from the row space to column space of A may be an 1×1 matrix, in some cases. These are precisely the distinction between A (i.e., $T = L_A$) and σ_A for every $A \in G(m)$, despite the fact that

$$\sigma_A\Big(M_s(m)\Big) = A \cdot M_s(m) \text{ for any } s \in \mathbb{R}^*.$$

Also note that, for every $A \in G(m)$ (for a fixed $m \in \mathbb{R}$), $T = L_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a non-invertible linear transformation, while the transformation $\sigma_A : \mathcal{R}(A) \to \mathcal{C}(A)$, as seen above, is invertible, where both $\mathcal{R}(A)$ and $\mathcal{C}(A)$ are same as the space $y = mx$. The invertibility of σ_A (for $A \in G(m)$) is the key factor in establishing a kind of generalized invertibility (MP-invertibility) of the non-invertible linear operator $T(= L_A)$ on $\mathbb{R}^2$; and on the way towards that, we understand as to how does the group structure of $G(m)$ play there its role, thereby resolving our inquisition of *why the MP inverse of $A \in G(m)$ coincides with its group inverse in $G(m)$.*

4.1.3. *Why* $\boldsymbol{A^{-1} = A^{+}}$ *for all* $\boldsymbol{A \in G(m)}$*?* Let $T \in \mathfrak{L}(\mathbb{R}^2)$ be singular and self-adjoint and $A = [T]_\epsilon$, where ϵ is the standard orthonormal ordered basis for $\mathbb{R}^2$. Clearly, $A \in G(m)$ for some $m \in \mathbb{R}$, and also $N(T)^\perp = R(T) = M(m) \cup \{(0,0)\}$. In order to find the MP inverse T^+ of T, we first take the restriction $T\Big|_{N(T)^\perp} = f$. It is well-known from linear algebra that this restriction $f : N(T)^\perp \to R(T)$ is an invertible linear transformation, admitting the inverse $f^{-1} : R(T) \to N(T)^\perp$. Then the linear transformation $T' : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T'(v) = f^{-1}(v_1)$ for all $v \in \mathbb{R}^2$, where $v_1 \in R(T)$ is the unique vector of $\mathbb{R}^2$ obtained from the direct sum $v = v_1 + v_2 \in R(T) \oplus R(T)^\perp (= \mathbb{R}^2)$, turns out, by definition, to be the MP inverse T^+ of T.

Here, in the present context, since $T = L_A$, it is seen at once, that the restriction of T on $N(T)^\perp$ is $f = \sigma_A$ (the permutation of $M(m)$ by $A \in G(m)$) and also $f^{-1} = \sigma_{A^{-1}}$ (the permutation of $M(m)$ by $A^{-1} \in G(m)$). Here lies the subtle role of the group structure of $G(m)$, that is indeed, worth pointing out. One can now compute the unique $v_1 \in R(T) = M(m) \cup \{(0,0)\}$ for any $v = (x, y) \in \mathbb{R}^2$ as

$$v_1 = \frac{x + my}{1 + m^2}(1, m) = M_{x+my}(m),$$

whence

$$T'(v) = \sigma_{A^{-1}}\Big(M_{x+my}(m)\Big) = A^{-1}v = L_{A^{-1}}(v).$$

Thus we find $T^+ = L_{A^{-1}}$, when $T = L_A$ i.e., $L_A^+ = L_{A^{-1}}$. Observe the pattern : The MP inverse of the left-multiplication transformation by the non-invertible matrix $A \in G(m)$ is nothing but the left-multiplication transformation by the group theoretic inverse A^{-1} of A in the group $G(m)$. This eventually resolves our inquisition; in fact, from linear algebra, we have

$$[L_A]_\epsilon^+ = [L_A^+]_\epsilon = [L_{A^{-1}}]_\epsilon \implies A^+ = A^{-1}.$$

4.2. **Why the Groups in $G(m, n)$ are the way they are?**

4.2.1. *Reaching the Group $G(m, n)$ through the same tool.* Now, for a fixed $m \in \mathbb{R}$, we search the nonzero singular matrices B, *not belonging to* $G(m)$, such that the restriction of the aforesaid linear operator over $y = mx$ is an element σ_A of the permutation group $S_{M(m)}$, for some $A \in G(m)$. Since σ_A is a permutation of $M(m)$, we have $\sigma_A(M(m)) = M(m)$, whence $L_B(M(m)) = M(m)$. Hence $y = mx$ is an L_B-invariant subspace of $\mathbb{R}^2$ (as $L_B(0,0) = (0,0) = \sigma_A(0,0)$), and so it is the *column space of* B. Thus, $B = \begin{bmatrix} p & q \\ pm & qm \end{bmatrix}$. Now, $(p,q) \neq (0,0)$ (as $B \neq \theta_{2\times 2}$) and so the one dimensional subspace $\{r(p,q) : r \in \mathbb{R}\}$ is the *row space* of B. Again, $q \neq pm$ (since, $B \notin G(m)$, its row space is different from its column space $y = mx$). Also, since $(0,0) \notin M(m)$ and σ_A is a permutation of $M(m)$, so $BM_s(m) = L_B(M_s(m)) = \sigma_A(M_s(m)) \neq (0,0)$ for any $s \in \mathbb{R}^*$. Hence the row space $\{r(p,q) : r \in \mathbb{R}\}$ is not orthogonal to its column space $y = mx$; i.e., $(p,q) \cdot (1,m) \neq 0$, i.e., $p + qm \neq 0$. Thus, the class of all the matrices B, which satisfy the condition that the restriction of L_B on the subspace $y = mx$ is the permutation σ_A of $M(m)$, for some $A \in G(m)$ is

$$S = \left\{ \begin{bmatrix} p & q \\ pm & qm \end{bmatrix} : p, q \in \mathbb{R}, (p,q) \neq (0,0), q \neq pm, p + qm \neq 0 \right\}.$$

Now we fix $p, q \in \mathbb{R}$ such that $B = \begin{bmatrix} p & q \\ pm & qm \end{bmatrix} \in S$. Since for each $x \in \mathbb{R}^*, (xp, xq) \neq (0,0), xq \neq xpm$, and $xp + xqm \neq 0$, so $xB \in S$ for each $x \in \mathbb{R}^*$. Consider the subset $S(B) = \{xB : x \in \mathbb{R}^*\}$. It can be readily checked that $S(B)$ is a commutative semigroup under multiplication, since for any $x, y \in \mathbb{R}^*, (xB)(yB) = tB$ for some $t = xy(p + qm) \in \mathbb{R}^*$. Also it can be shown that, for any $C \in S(B), A \in G(m)$ and $s \in \mathbb{R}^*$, $L_C(M_s(m)) = \sigma_A(M_s(m))$ if and only if the matrices A and C have the same eigenvalues. Thus the mapping $\phi : G(m) \to S(B)$, defined by $\phi(A) = B_A \in S(B)$, whenever B_A has its nonzero eigenvalues same as that of A, is well defined. Note that, if $A, A' \in G(m)$ have nonzero eigen values a and a' respectively, then the nonzero eigen value of $AA' \in G(m)$ is aa'. Let $B_A = rB$ and $B_{A'} = r'B$ for some $r, r' \in \mathbb{R}^*$. Then $r(p + qm) = a$ and $r'(p + qm) = a'$. Again, $B_A B_{A'} = (rB)(r'B) = rr'(p + qm)B$ which has the nonzero eigen value $rr'(p + qm)^2 = aa'$. Thus, $\phi(AA') = B_A B_{A'} = \phi(A)\phi(A')$, which indicates that $\phi : G(m) \to S(B)$ is a homomorphism. Again for any $A, A' \in G(m)$, $B_A = B_{A'} \implies \sigma_A(M_s(m)) = \sigma_{A'}(M_s(m))$ for all $s \in \mathbb{R}^* \implies M_{as}(m) = M_{a's}(m) \implies as = as' \implies a = a' \implies A = A'$. So,

ϕ is injective. Now, for $B \in S(B)$, $\exists P \in G(m)$ such that $\phi(P) = B$. Then clearly $xP \in G(m)$ $(\forall\, x \in \mathbb{R}^*)$ and for any $xB \in S(B)$, $\phi(xP) = xB$, whence ϕ is surjective. Thus $\phi : G(m) \to S(B)$ is a semigroup isomorphism. However, $G(m)$ is known to be a commutative group, whence $S(B)$ must also be so.

Clearly, then $I = \phi(\mathcal{I}_{G(m)}) \in S(B)$ is the identity element of the group $S(B)$, where $\mathcal{I}_{G(m)}$ is the identity element of the group $G(m)$, having nonzero eigenvalue 1. Then, I has nonzero eigenvalue 1 and since $I \in S(B)$, there exists $s \in \mathbb{R}^*$, such that $I = sB$. Then, $s(p + mq) = 1$ implying that $I = \frac{1}{p+mq}\begin{bmatrix} p & q \\ pm & qm \end{bmatrix}$. Now, if any $C = sB \in S(B)$ has eigenvalue $r \in \mathbb{R}^*$, then $s(p+qm) = r$ i.e., $s = \frac{r}{p+mq}$. So, $C = \frac{r}{p+mq}B = rI$. So we may write $S(B) = \{rI \,:\, r \in \mathbb{R}^*\}$.

Now, for any $C \in S(B)$ if $C = \phi(A)$ for some $A \in G(m)$ with eigenvalue $r \in \mathbb{R}^*$, then the group theoretic inverse of C in the group $S(B)$ is $C^{-1} = \phi(A^{-1})$ which has nonzero eigenvalue $\frac{1}{r}$ (as $A^{-1} \in G(m)$ has $\frac{1}{r}$ as its eigenvalue). Thus, the group theoretic inverse of $C = rI$ in $S(B)$ is $C^{-1} = \frac{1}{r}I \in S(B)$.

Observe that, for any $B \in S, S(B) \subseteq S$. Again, it is immediate to observe that, for $B, B' \in S$, $S(B) = S(B')$ *if and only if* B *and* B' *have the same row space*. So, the class of all distinct disjoint groups of the form $S(B)$ is $\{S(B) \,:\, B \in \zeta\}$, where ζ is the calss of all possible matrices of different row spaces chosen randomly from S i.e.,

$$\bigcup_{B\in\zeta} S(B) = S.$$

Thus $\{S(B) \,:\, B \in \zeta\}$ is a set theoretic partition of S.

Since every $B \in S$ is singular, so the row space of every matrix $B \in \zeta$ is a line through the origin (*other than the line* $y = mx$). Thus, the row space $\{r(p, q) \,:\, r \in \mathbb{R}\}$ of every matrix in the group $S(B)$ is either the y-axis $(x = 0)$ or a line $y = nx$ for some $n \in \mathbb{R}$, with $n \neq m$ and $mn \neq -1$. The whole scenario thus leads to the fact that $S(B)$ turns either into the group $G_0(m, 0)$ or into the group $G(m, n)$ $(m \neq n)$, as described in Section 2.2.

4.2.2. *Why* $\boldsymbol{P^{-1} \neq P^{+}}$ *for all* $\boldsymbol{P \in G(m,n)}$ *?* Let r be the nonzero eigen value of the matrix $P = t\begin{bmatrix} 1 & n \\ m & mn \end{bmatrix} \in G(m,n)$, $(t \in \mathbb{R}^*)$. Then, $P = r\mathcal{I}_{G(m,n)}$, where $\mathcal{I}_{G(m,n)} = \frac{1}{1+mn}\begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}$ is the identity of the group $G(m,n)$ and $\boldsymbol{r = t(1+mn)}$. So, the linear operator $T = L_P$ on $\mathbb{R}^2$ can explicitly be written as,

$$T(x,y) = \frac{r}{1+mn}\begin{bmatrix} 1 & n \\ m & mn \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{r(x+ny)}{(1+mn)}(1,m)\ldots\ldots(i)$$

Evidently, the range space $R(T)$ of T is the column space $\mathcal{C}(P) = \{x(1,m) : x \in \mathbb{R}\}$, (i.e., the line $y = mx$). The row space of P, i.e., $\mathcal{R}(P) = \{x(1,n) : x \in \mathbb{R}\}$, (i.e., the line $y = nx$) is eventually the orthogonal complement $N(T)^{\perp}$ of the null space $N(T)$ of T. So the restriction f of T over $\mathcal{R}(P)$ is the invertible linear transformation $f : N(T)^{\perp} \to R(T)$. Clearly, then, for any $x \in \mathbb{R}$, we get from (i) above that,

$$f(x(1,n)) = \frac{rx(1+n^2)}{1+mn}(1,m)\ldots\ldots(ii)$$

Then, the inverse of f is the transformation $f^{-1} : R(T) \to N(T)^{\perp}$, given by

$$f^{-1}(x(1,m)) = \frac{x(1+mn)}{r(1+n^2)}(1,n)\ldots\ldots(iii)$$

The restriction $T = L_P$ over the column space (i.e. $R(T)$), is the permutation σ_A (of $M(m)$) for some $A \in G(m)$, whence r is the nonzero eigenvalue of A as well. Thus, $A = r\mathcal{I}_{G(m)}$ where $\mathcal{I}_{G(m)} = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$. Note that, $r(1,m) = r(1+m^2)(\frac{1}{1+m^2}, \frac{m}{1+m^2}) = M_{r(1+m^2)}(m) \in M(m)$ for any $r \in \mathbb{R}^*$ (as described in Section 3.1.3). So, we have, from (iii) above,

$$(f^{-1}\sigma_A)(1,m) = f^{-1}[\sigma_A(M_{1+m^2}(m))] = f^{-1}[M_{r(1+m^2)}(m)] =$$
$$f^{-1}(r(1,m)) = \frac{1+mn}{1+n^2}(1,n)$$

(since r is the eigenvalue of A).

Let us now consider the real number

$$s = \frac{(1+mn)^2}{r(1+m^2)(1+n^2)} \in \mathbb{R}^*.$$

Let $B \in G(n)$ be the matrix whose nonzero eigenvalue is s. Now, since the group $G(n)$ is embedded in the group $S_{M(n)}$ of permutations of the set $M(n)$, so for $B \in G(n)$, σ_B stands for a permutation of $M(n)$. So, from (ii) above,

$$(f\sigma_B)(1,n) = f[\sigma_B(M_{1+n^2}(n))] = f[M_{s(1+n^2)}(n)] = f(s(1,n)) =$$

$$\frac{rs(1+n^2)}{1+mn}(1,m) = \frac{1+mn}{1+m^2}(1,m)$$

(since s is the eigenvalue of B).

Now, the matrix $Q \in G(n,m)$ which is similar to $B = s\mathcal{I}_{G(n)}$, must have s as its nonzero eigenvalue. So,

$$Q = s\mathcal{I}_{G(n,m)} = \frac{s}{1+mn}\begin{bmatrix} 1 & m \\ n & mn \end{bmatrix} = \frac{1+mn}{r(1+m^2)(1+n^2)}\begin{bmatrix} 1 & m \\ n & mn \end{bmatrix} =$$

$$\frac{1}{t(1+m^2)(1+n^2)}\begin{bmatrix} 1 & m \\ n & mn \end{bmatrix}$$

which is indeed P^+, **the MP inverse of P**, as we have already pointed out in Remark 2.2.4.

Furthermore, as $P \in G(m,n)$, we must have $P^{-1} \in G(m,n)$ as well, while $P^+ \in G(n,m)$. Now, from what we have seen so far, it can be concluded that

$$P^{-1} = \frac{1}{r}\mathcal{I}_{G(m,n)} \quad \text{whence} \quad P^+ = s\,\mathcal{I}_{G(n,m)}$$

where $r, s \in \mathbb{R}^*$ must be the (nonzero) eigenvalues of P and P^+ respectively. Also recall that

$$\mathcal{I}_{G(m,n)} = \frac{1}{1+mn}\begin{bmatrix} 1 & n \\ m & mn \end{bmatrix} \quad \text{and} \quad \mathcal{I}^+_{G(m,n)} = \frac{1+mn}{(1+m^2)(1+n^2)}\begin{bmatrix} 1 & m \\ n & mn \end{bmatrix}.$$

Then,

$$\begin{aligned} P^{-1} &= \tfrac{1}{r}\,\mathcal{I}_{G(m,n)} = \tfrac{1}{r}\,\mathcal{I}_{G(m,n)}\,\mathcal{I}^+_{G(m,n)}\,\mathcal{I}_{G(m,n)} \\ &= \tfrac{1}{r}\,\mathcal{I}_{G(m,n)}\,\tfrac{(1+mn)^2}{(1+m^2)(1+n^2)}\mathcal{I}_{G(n,m)}\,\mathcal{I}_{G(m,n)}, \quad \text{as } \mathcal{I}_{G(n,m)} = \tfrac{1}{1+mn}\begin{bmatrix} 1 & m \\ n & mn \end{bmatrix} \\ &= \mathcal{I}_{G(m,n)}\,\tfrac{(1+mn)^2}{t(1+mn)(1+m^2)(1+n^2)}\,\mathcal{I}_{G(n,m)}\,\mathcal{I}_{G(m,n)}, \quad \text{as } r = t(1+mn) \end{aligned}$$

$$= \mathcal{I}_{G(m,n)} \tfrac{(1+mn)}{t(1+m^2)(1+n^2)} \mathcal{I}_{G(n,m)} \mathcal{I}_{G(m,n)}$$

$$= \mathcal{I}_{G(m,n)} P^+ \mathcal{I}_{G(m,n)},$$

which is the result stated in Lemma 2.2.10 earlier.

So, in short, we have achieved the following :

Let $P \in G(m,n)$ and $T = L_P$. If the respective restrictions of T over $\mathcal{R}(P)$ and $\mathcal{C}(P)$ are given by f and the permutation σ_A for some $A \in G(B)$, then

$$(f^{-1}\sigma_A)(1,m) = \tfrac{1+mn}{1+n^2}(1,n) \qquad \ldots (I)$$

$$(f\sigma_B)(1,n) = \tfrac{1+mn}{1+m^2}(1,m) \text{ for some } B \in G(n). \qquad \ldots (II)$$

Also, the matrix $Q \in G(n,m)$ which is similar to $B \in G(n)$ is the MP inverse of P, which is clearly distinct from the corresponding group theoretic inverse $P^{-1} \in G(m,n)$.

It is interesting to note that the above result, in effect, leads us to *another method for calculating MP inverse of the matrices under consideration.*

4.2.3. *Another Method for Calculating MP Inverse in $G(m,n)$ and $G(m)$.* Observe that, for any $s \in \mathbb{R}^*$ and $n \in \mathbb{R}$, $s(1,n) = M_{s(1+n^2)}(n) \in M(n)$. Thus for the permutation σ_B of the set $M(n)$, we see that $\sigma_B(1,n) = \sigma_B[M_{1+n^2}(n)] = M_{s(1+n^2)}(n) = s(1,n)$, where **$s$ is the nonzero eigenvalue of $B \in G(n)$**. This fact about σ_B, playing from behind the curtain, considerably shortens up the assertion made in (II) above and we get,

$$(f\sigma_B)(1,n) = f(s(1,n)) = \frac{1+mn}{1+m^2}(1,m) \ldots\ldots (*)$$

Now since $f = T\Big|_{\mathcal{R}(P)}$, so we can find $f(s(1,n))$ by writing T explicitly, thereby getting a simple linear equation in s. Solving this, we first find the nonzero eigenvalue s of P^+, and finally P^+ itself in $G(n,m)$ for the $P \in G(m,n)$.

To cross-check this technique, let us take the matrix $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ from Section 1.2 again. Clearly, $B \in G(1,2)$. Let $T = L_B$. Then for any

$(x, y) \in \mathbb{R}^2$,

$$T(x,y) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x+2y)(1,1).$$

So, $f(s(1,2)) = 5s(1,1) \implies \frac{1+1\times 2}{1+1^2}(1,1) = 5s(1,1)$ [by (*) above], whence $5s = \frac{3}{2}$ i.e., $s = \frac{3}{10}$. Since $B^+ \in G(2,1)$, we have

$$B^+ = s\mathcal{I}_{G(2,1)} = \frac{3}{10}\begin{bmatrix} \frac{1}{1+2\times 1} & \frac{1}{1+2\times 1} \\ \frac{2}{1+2\times 1} & \frac{2}{1+2\times 1} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

which corroborates with what we have found earlier by the method of rank factorization in Section 1.2.

Concluding comments

Since we started this journey with the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in Section 1.1, it only seems befitting to end with a verification of the unique MP inverse of that same matrix by the above method as well.

Clearly, $A \in G(1) = G(1,1)$ as we have already explained. Let $T = L_A$. Then for any $(x, y) \in \mathbb{R}^2$,

$$T(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x+y)(1,1).$$

So, $f(s(1,1)) = (1+1)s(1,1) = 2s(1,1)$ and on the other hand $f(s(1,1)) = \frac{1+1\times 1}{1+1^2}$ as well; whence, $2s = 1$ i.e., $s = \frac{1}{2}$. So, in this case $A^+ \in G(1,1)$ is given by

$$A^+ = s\mathcal{I}_{G(1,1)} = \frac{1}{2}\begin{bmatrix} \frac{1}{1+1\times 1} & \frac{1}{1+1\times 1} \\ \frac{1}{1+1\times 1} & \frac{1}{1+1\times 1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

which turns out to be the expected matrix as calculated before.

On the one hand, while this observation somewhat satisfactorily completes the circle of our specific inquisition about the 2×2 singular real matrices, on the other, it leaves us to ponder upon the possible generalization of this method of finding MP inverse of a matrix, at least for the square matrices of order n, to begin with.

Acknowledgment: We gratefully acknowledge the constant encouragement of our teacher Dr. M. K. Sen, who has all along been a fountainhead of inspiration. We also thank the communicating editor Prof. C. S. Aravinda and the learned referee for their support.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 147–158

A FAMILY OF CONGRUENCES FOR $(2,\beta)-$REGULAR BIPARTITIONS

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Abstract. The congruence of certain restricted partition functions known as regular bipartition is discussed in this paper. We particularly investigate the $(2,\beta)$-regular bipartitions of n, denoted by $B_{2,\beta}(n)$, and establish certain congruences for $B_{2,\beta}(n)$ when $\beta \geq 3$. We derive infinite families of congruences modulo 4 for the $(2,3)$-regular bipartition. We also obtain a generalisation of the regular bipartition for modulo p and p^2.

1. Introduction

Partition of an integer $n \geq 0$ is the method of expressing n as a sum of positive integers $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k$; that is, as $n = \alpha_1+\alpha_2+\alpha_3+...+\alpha_k$, where the ordering of the integers is irrelevant and $p(n)$ represents the number of partitions of $n \geq 0$. Trivially, $p(0)$ has only one partition called the empty partition, whereas for $n = 7$, we have $p(7) = 15$ and these partitions are:

$$7, \quad 6+1, \quad 5+2, \quad 5+1+1, \quad 4+3, \quad 4+2+1, \quad 4+1+1+1,$$
$$3+3+1, \quad 3+2+2, \quad 3+2+1+1, \quad 3+1+1+1+1, \quad 2+2+2+1,$$
$$2+2+1+1+1, \quad 2+1+1+1+1+1, \quad 1+1+1+1+1+1+1$$

In [12], Ramanujan obtained some properties of the partition function for any non-negative integer n, modulo 5, 7 and 11

$$p(5n+4) \equiv 0 \pmod{5} \tag{1.1}$$
$$p(7n+5) \equiv 0 \pmod{7} \tag{1.2}$$
$$p(11n+6) \equiv 0 \pmod{11} \tag{1.3}$$

2010 Mathematics Subject Classification: 05A17, 11P83
Key words and phrases: partitions, generating functions, m-regular partition, (α,β)-regular bipartition, 3-dissections

For any positive integer a, q-series notation is given by

$$f_a = (q^a; q^a)_\infty$$

and for any complex number β and $|z| < 1$,

$$(\beta, z)_\infty = \prod_{n=1}^{\infty} (1 - \beta z^{n-1}) \tag{1.4}$$

Euler introduced the generating functions for partition function $p(n)$ as

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q; q)_\infty} = \frac{1}{f_1} \tag{1.5}$$

Partition is said to be an m-regular partition if none of its summands are congruent to 0 (mod m). An ordered pair (η_1, η_2) is said to be a bipartition of n if the sum of all the parts of η_1 and η_2 is n. For integers $\alpha, \beta \geq 0$, an (α, β)-regular bipartition of n is a bipartition (η_1, η_2) of n such that η_1 is α-regular and η_2 is β-regular. If $B_{\alpha,\beta}(n)$ denotes the number of $(\alpha, \beta)-$regular bipartition of n then its generating function is given by

$$\sum_{n=0}^{\infty} B_{\alpha,\beta}(n) q^n = \frac{f_\alpha f_\beta}{f_1^2} \tag{1.6}$$

In [8], Furcy and Penniston obtained many results on congruences for various values of ℓ-regular partition functions modulo 3. Dandurand and Penniston [6], obtained various congruences modulo ℓ for divisibility of ℓ-regular partition functions $b_\ell(n)$, where $\ell \in \{5, 7, 11\}$. Cui and Gu [5], studied p-dissection identities for Ramanujan's theta functions for a given prime p and derived infinite families of congruences modulo 2 for certain l-regular partition functions where $l \in \{2, 4, 5, 8, 13, 16\}$.

In [10], Lin studied the 13-regular bipartitions of n, and proved infinite family of congruences for non-negative integers α and n

$$B_{13}\left(3^\alpha.n + 2.3^{\alpha-1} - 1\right) \equiv 0 \pmod{3}. \tag{1.7}$$

And in [9], Lin studied l-regular bipartitions of n and proved infinite family of congruences for non-negative integers α and n

$$B_7\left(3^\alpha.n + \frac{5.3^{\alpha-1} - 1}{2}\right) \equiv 0 \pmod{3}. \tag{1.8}$$

For all non-negative integers α and n, Dou [7] proved the following:

$$B_{3,11}\left(3^{\alpha}+\frac{5.3^{\alpha-1}-1}{2}\right)\equiv 0 \pmod{11} \tag{1.9}$$

In the same paper, he also mentioned two conjectures for $B_{5,7}(n)$ and $B_{3,7}(n)$. The Conjecture for $B_{3,7}(n)$ was later proved by Xia and Yao in [14] using q-series identities. Liu and Du [11] discussed the congruence properties for broken 3-diamond partitions, introduced by Andrews et al. in [2], based on the MacMohan's partition analysis. Adiga and Dasappa [1] gave a simpler proof for the conjecture stated by Wang [13], and also studied the arithmetic properties of $B_{3,t}$ and $B_{5,t}$ for any integer $t > 0$.

2. Preliminaries

In this section some of the 3-dissection formulae are recalled as lemmas, and will be used later for proving theorems.

Lemma 2.1. *The following* 3*-dissections hold*

$$\frac{f_1^2}{f_2}=\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9} \tag{2.1}$$

$$\frac{f_2}{f_1^2}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^2}{f_3^6} \tag{2.2}$$

Proof. See (Lemma 2.2 in [3]) □

Lemma 2.2. *The following* 3*-dissection holds true*

$$\frac{f_2^2}{f_1}=\frac{f_6f_9^2}{f_3f_{18}}+q\frac{f_{18}^2}{f_9} \tag{2.3}$$

Proof. See([4] corollary (ii) p.49) □

The following congruences are derivable from the binomial theorem. These congruences are used at appropriate places, as and when necessary, in proving the theorems.

$$f_k^2\equiv f_{2k} \pmod{2} \tag{2.4}$$

$$f_k^4\equiv f_{2k}^2 \pmod{4} \tag{2.5}$$

$$f_k^8\equiv f_{2k}^4 \pmod{8} \tag{2.6}$$

$$f_k^{16}\equiv f_{2k}^8 \pmod{16} \tag{2.7}$$

3. Congruences for $(2,3)$-Regular bipartitions

Theorem 3.1. *For any $n \geq 1$, we have*

$$B_{2,3}(9n+7) \equiv 0 \ (mod\ 16),$$
$$B_{2,3}(27n+19) \equiv 0 \ (mod\ 16),$$
$$B_{2,3}(81n+64) \equiv 0 \ (mod\ 16)$$

Proof. From equations (1.6) and (2.2), we see that

$$\sum_{n=0}^{\infty} B_{2,3}(n)q^n = \frac{f_2 f_3}{f_1^2}$$
$$= 2q\frac{f_6^3 f_9^3}{f_3^6} + \frac{f_6^4 f_9^6}{f_3^7 f_{18}^3} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^5} \tag{3.1}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} on both sides of equation (3.1), we obtain

$$\sum_{n=0}^{\infty} B_{2,3}(3n)q^n = \frac{f_2^4 f_3^6}{f_1^7 f_6^3} \tag{3.2}$$
$$\sum_{n=0}^{\infty} B_{2,3}(3n+1)q^n = 2\frac{f_2^3 f_3^3}{f_1^6} \tag{3.3}$$
$$\sum_{n=0}^{\infty} B_{2,3}(3n+2)q^n = 4\frac{f_2^2 f_6^3}{f_1^5} \tag{3.4}$$

By (2.2) and (3.3),

$$\sum_{n=0}^{\infty} B_{2,3}(3n+1)q^n = 2f_3^3\left(\frac{f_2}{f_1^2}\right)^3$$
$$\equiv \left(2\frac{f_6^{12} f_9^{18}}{f_3^{21} f_{18}^9} + 12q\frac{f_6^{11} f_9^{15}}{f_3^{20} f_{18}^6}\right) \ (\text{mod } 16) \tag{3.5}$$

By extracting q^{3n+2} from the above congruence (3.5), we see that

$$B_{2,3}(9n+7) \equiv 0 \ (\text{mod } 16). \tag{3.6}$$

Extracting the terms involving q^{3n} from equation (3.5),

$$\begin{aligned}\sum_{n=0}^{\infty} B_{2,3}(9n+1)q^n &\equiv 2\frac{f_2^4 f_3^2}{f_1^5 f_6} \pmod{16}\\ &\equiv 2\frac{f_3^2}{f_6}\left(\frac{f_2^2}{f_1}\right)\left(\frac{f_2}{f_1^2}\right)^2 \pmod{16}\\ &\equiv \left(2\frac{f_6^8 f_9^{14}}{f_3^{15} f_{18}^7} + 10q\frac{f_6^7 f_9^{11}}{f_3^{14} f_{18}^4} + 8q^3\frac{f_6^5 f_9^5 f_{18}^2}{f_3^{12}}\right) \pmod{16}\end{aligned} \tag{3.7}$$

Since there are no terms containing q^{3n+2} in right side of (3.7), it can be concluded that

$$B_{2,3}(27n+19) \equiv 0 \pmod{16} \tag{3.8}$$

If we extract those terms involving q^{3n+1} in equation (3.7), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{2,3}(27n+10)q^n &\equiv 10\frac{f_2^7 f_3^{11}}{f_1^{14} f_6^4} \pmod{16}\\ &\equiv 10\left(\frac{f_3^{11}}{f_6^4}\right)\left(\frac{f_1^2}{f_2}\right) \pmod{16}\\ &\equiv \left(10\frac{f_3^{11} f_9^2}{f_6^4 f_{18}} + 12q\frac{f_3^{12} f_{18}^2}{f_6^5 f_9}\right) \pmod{16}\end{aligned} \tag{3.9}$$

Extracting the terms involving q^{3n+2} on both sides of (3.9) we get,

$$B_{2,3}(81n+64) \equiv 0 \pmod{16} \tag{3.10}$$

□

Theorem 3.2. *For any $\delta \geq 1$ and $n \geq 1$, we have*

$$\begin{aligned}&B_{2,3}(27n+1) \equiv 2B_{2,3}(3n) \pmod{4},\\ &B_{2,3}\left(9^\delta n + \frac{9^\delta - 1}{8}\right) \equiv B_{2,3}(9n+1) \pmod{4}\end{aligned}$$

Proof. From (3.2) we have,

$$\begin{aligned}\sum_{n=0}^{\infty} B_{2,3}(3n)q^n &= \frac{f_2^4 f_3^6}{f_1^7 f_6^3}\\ &\equiv \frac{f_3^2 f_1}{f_6} \pmod{4}\end{aligned} \tag{3.11}$$

From congruence (3.3), extracting those terms which involve q^{3n} on both the sides, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} B_{2,3}(9n+1)q^n &\equiv 2\frac{f_2^2 f_3^2}{f_1 f_6} \pmod{4} \\ &\equiv 2\left(\frac{f_3^2}{f_6}\right)\left(\frac{f_2^2}{f_1}\right)\ (mod\ 4) \\ &\equiv \left(2\frac{f_3 f_9^2}{f_{18}} + 2q\frac{f_3^2 f_{18}^2}{f_6 f_9}\right) \pmod{4} \qquad (3.12)\end{aligned}$$

From equation (3.12), extracting the terms involving q^{3n} we see that

$$\sum_{n=0}^{\infty} B_{2,3}(27n+1)q^n \equiv 2\frac{f_1 f_3^2}{f_6} \pmod{4} \qquad (3.13)$$

By comparing congruences (3.11) and (3.13), the following result can be achieved

$$B_{2,3}(27n+1) \equiv 2B_{2,3}(3n) \pmod{4} \qquad (3.14)$$

Extracting those terms involving q^{3n+1} in (3.12), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{2,3}(27n+10)q^n &\equiv 2\frac{f_1^2 f_6^2}{f_2 f_3} \pmod{4} \\ &\equiv 2\left(\frac{f_6^2}{f_3}\right)\left(\frac{f_1^2}{f_2}\right) \pmod{4} \\ &\equiv 2\frac{f_6^2 f_9^2}{f_3 f_{18}} \pmod{4} \qquad (3.15)\end{aligned}$$

Extracting q^{3n} on both the sides of congruence (3.15), we have

$$\sum_{n=0}^{\infty} B_{2,3}(81n+10)q^n \equiv 2\frac{f_2^2 f_3^2}{f_1 f_6} \pmod{4} \qquad (3.16)$$

By (3.12) and (3.16), it can be seen that

$$B_{2,3}(81n+10) \equiv B_{2,3}(9n+1) \pmod{4} \qquad (3.17)$$

Iterating n by $9n+1$ in $B_{2,3}(81n+10)$, we obtain

$$B_{2,3}\left(9^{\delta}n + 9^{\delta-1} + 9^{\delta-2} + 9^{\delta-3} + \cdots + 1\right) \equiv B_{2,3}(9n+1) \pmod{4}$$

That is, $B_{2,3}\left(9^{\delta}n + \dfrac{9^{\delta}-1}{8}\right) \equiv B_{2,3}(9n+1) \pmod{4}$

□

Corollary 3.3. *For any $\delta \geq 1$ and $n \geq 1$, we have*

$$B_{2,3}\left(9^{\delta}n + \frac{9^{\delta}-1}{8}\right) \equiv 0 \pmod{2}$$

Proof. From (3.3) we have,

$$B_{2,3}(9n+1) \equiv 0 \pmod{2} \tag{3.18}$$

From (3.18) and the second congruence of the above theorem, we obtain the required congruence. □

4. Congruences for $(2,\beta)$-regular bipartitions modulo p and p^2

Theorem 4.1. *For $\beta \geq 3$ and $d \in \mathbb{N}$ such that $(12d+1)a^2$, for all $a \in \mathbb{N}$, is a quadratic non-residue modulo p. If there exists some prime $p(\geq 5)|\beta$ then*

$$B_{2,\beta}(pn+d) \equiv 0 \pmod{p} \text{ for all } n \geq 0. \tag{4.1}$$

In order to prove the above theorem, we first discuss the following lemma.

Lemma 4.2. *For any non negative integer b, and any prime p,*

$$(q^p;q^p)_{\infty}^{b} \equiv (q^{bp};q^{bp})_{\infty} \pmod{p} \tag{4.2}$$

Proof. To prove the congruence (4.2), it is sufficient to show that

$$(1-q^b)^p \equiv (1-q^{bp}) \pmod{p}$$

By Binomial theorem, we have,

$$\begin{aligned}(1-q^p)^b &= \sum_{i=0}^{b}\binom{b}{i}q^{pi}\\ &= \binom{b}{0} - \binom{b}{1}q^p + \binom{b}{2}q^{2p} - \ldots - \binom{b}{b}q^{bp}\\ &= (1-q^{bp}) \pmod{p}\end{aligned}$$

Hence, the Lemma. □

We now present the proof of Theorem 4.1, based on the above lemma.

Proof. From equation (1.6), we have,

$$\sum_{n=1}^{\infty} B_{2,\beta}(n)q^n = \frac{(q^2,q^2)_\infty(q^\beta;q^\beta)_\infty}{(q;q)_\infty^2} \tag{4.3}$$

Since $p|\beta$, $\beta = pr$ for some $r \in \mathbb{N}$. By Lemma 4.2

$$\sum_{n=1}^{\infty} B_{2,\beta}(n)q^n = \frac{(q^2,q^2)_\infty(q^p,q^p)_\infty^r}{(q,q)_\infty^2} \tag{4.4}$$

Euler pentagonal number theorem gives

$$(q,q)_\infty = \sum_{m=1}^{\infty}(-1)^m q^{\left(\frac{3}{2}m^2-\frac{1}{2}m\right)} \tag{4.5}$$

$$\Rightarrow (q^p,q^p)_\infty = \sum_{m=1}^{\infty}(-1)^m q^{\left(\frac{3}{2}m^2-\frac{1}{2}m\right)p} \tag{4.6}$$

Substituting equation (4.6) in (4.4) we get,

$$\begin{aligned}\sum_{n=1}^{\infty} B_{2,\beta}(n)q^n = \left(\sum_{s_1=1}^{\infty}(-1)^{s_1}q^{\left(3s_1^2-s_1\right)}\right)\left(\sum_{s_2=1}^{\infty}(-1)^{s_2}q^{\left(\frac{3}{2}s_2^2-\frac{1}{2}s_2\right)p}\right)^r \\ \left(\sum_{j=1}^{\infty}p(j)q^j\right)^2 \pmod{p}\end{aligned} \tag{4.7}$$

Suppose there are $s_1,\ s_2$ and j, such that the sum of the powers of q equals $pn+d$, then

$$\begin{aligned} d &\equiv 3s_1^2 - s_1 \pmod{p} \\ 12d+1 &\equiv 36s_1^2 - 12s_1 + 1 \pmod{p} \\ (12d+1)a^2 &\equiv (36s_1^2 - 12s_1 + 1)a^2 \pmod{p} \\ (12d+1)a^2 &\equiv [(6s_1+1)a]^2 \pmod{p} \end{aligned} \tag{4.8}$$

The congruence (4.8) contradicts the fact that $(12d+1)a^2$ is a quadratic non residue modulo p. This proves the theorem. $\square$

The Lemma 4.2 can further be extended for the integral multiple of p^r for every $r \in \mathbb{N}$, which helps in proving congruences for regular bipartition modulo p^2.

Lemma 4.3. *For any prime $p \geq 3$ and integer $r \geq 2$,*

$$(q;q)_\infty^{np^r} \equiv (q^p;q^p)_\infty^{np^{r-1}} \pmod{p^2} \tag{4.9}$$

where $n \in \mathbb{N}$.

Proof. We first prove the following congruence

$$(q;q)_\infty^{p^2} \equiv (q^p;q^p)_\infty^{p} \pmod{p^2} \tag{4.10}$$

and hence deduce the congruence (4.9). In order to prove the above congruence, it is sufficient to prove the following:

$$(1-q)^{p^2} \equiv (1-q^p)^p \pmod{p^2} \tag{4.11}$$

By Binomial theorem, we have,

$$\begin{aligned}
(1-q)^{p^2} &= \sum_{i=0}^{p^2} \binom{p^2}{i} q^i \\
&= \binom{p^2}{0} - \binom{p^2}{1} q + \binom{p^2}{2} q^2 - \ldots - \binom{p^2}{p^2} q^{p^2} \\
&\equiv (1-q^p)^p \pmod{p^2}
\end{aligned}$$

Now by the successive multiplication of congruence (4.11) upto np^{r-2} times for all $n \in \mathbb{N}$ and $r \geq 2$, we arrive at the required congruence (4.9). □

Using the above lemma, we now prove the following theorem:

Theorem 4.4. *If $p^r|\beta$ for $p \geq 5$ and $r \geq 2$ then for all $n \in \mathbb{N}$*

$$B_{2,\beta}(pn+d) \equiv 0 \pmod{p^2} \tag{4.12}$$

where $d \in \mathbb{N}$ such that $(12d+1)a^2$ is a quadratic non-residue modulo p^2.

Proof. Using (1.6), we have,

$$\sum_{n=1}^{\infty} B_{2,\beta}(n) q^n = \frac{(q^2,q^2)_\infty (q^\beta;q^\beta)_\infty}{(q;q)_\infty^2} \tag{4.13}$$

Since $p^r|\beta$, $\beta = \alpha p^r$ for some $\alpha \in \mathbb{N}$, we get

$$\sum_{n=1}^{\infty} B_{2,\beta}(n) q^n = \frac{(q^2,q^2)_\infty (q^{\alpha p^r},q^{\alpha p^r})_\infty}{(q,q)_\infty^2} \tag{4.14}$$

By Lemma 4.3 we have,

$$\sum_{n=1}^{\infty} B_{2,\beta}(n)q^n = \frac{(q^2,q^2)_\infty (q^p,q^p)_\infty^{\alpha p^{r-1}}}{(q,q)_\infty^2} \pmod{p^2} \tag{4.15}$$

Replacing q by q^2 in equation (4.5) we have,

$$(q^2,q^2)_\infty = \sum_{m=1}^{\infty} (-1)^m q^{3m^2-m} \tag{4.16}$$

Substituting equation (4.6), (4.16) and (1.5) in (4.15) we have,

$$\sum_{n=1}^{\infty} B_{2,\beta}(n)q^n = \left(\sum_{m_1=1}^{\infty} (-1)^{m_1} q^{3m_1^2-m}\right)\left(\sum_{m_2=1}^{\infty} (-1)^{m_2} q^{\left(\frac{3}{2}m_2^2-\frac{1}{2}m_2\right)p}\right)^{\alpha p^{r-1}} \left(\sum_{j=1}^{\infty} p(j)q^j\right)^2 \pmod{p^2} \tag{4.17}$$

Suppose there are m_1, m_2 and j, such that the sum of the powers of q equals $pn+d$, then

$$\begin{aligned} d &\equiv 3m_1^2 - m_1 \pmod{p^2} \\ 12d+1 &\equiv 36m_1^2 - 12m_1 + 1 \pmod{p^2} \\ (12d+1)a^2 &\equiv (36m_1^2 - 12m_1 + 1)a^2 \pmod{p^2} \\ (12d+1)a^2 &\equiv [(6m_1-1)a]^2 \pmod{p^2} \end{aligned} \tag{4.18}$$

Since $(12d+1)a^2$ is a quadratic non residue modulo p^2, the congruence (4.18) contradicts this fact. Hence, the theorem. $\square$

Concluding comments

In this article, first it is shown that the number of $(2,3)$-regular bipartitions of $9n+7$, $27n+19$ and $81n+64$ are divisible by 16 for all $n \geq 1$. Moreover, infinite families of congruences modulo 4 have been derived $(2,3)$ regular bipartitions. Then $(2,\beta)$-regular bipartitions is investigated and certain congruences are established for $B_{2,\beta}(pn+d)$ for all $\beta \geq 3$. Furthermore, a generalisation of the regular bipartitions for modulo p and modulo p^2 is obtained.

Acknowledgement: We express our heartfelt gratitude to Dr. C S Aravinda for his unconditional support and we are grateful to the referees for the comments which helped us improve the quality of the manuscript.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 159–170

MAGIC SQUARES FROM SIMPLE SQUARES

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(Received : 06 - 01 - 2021 ; Revised : 06 - 06 - 2021)

Abstract. This article studies the relation between the magic square and the simple square arrangement of numbers. It is shown that the cycles and reverse cycles of the simple square forms the rows and columns of odd magic square, constructed using the Siamese method. For singly even magic square, corresponding arrangement arising from cycles and reverse cycles is presented. For singly even magic square, a Conways LUX-like construction method is presented which forms an almost dual. The duality between simple square and doubly even magic square constructed using reflecting certain entries is discussed. The article concludes with few observational remarks on this duality in general. It is an open question to find the dual singly even magic square.

1. Introduction

A normal magic square is an arrangement of natural numbers from $1, 2, \cdots, n^2$ into a square, such that every row, every column, and diagonals have the same sum. The sum is called magic sum and it is equal to $n\frac{n^2+1}{2}$. There are several methods for constructing the magic squares which involve following an elementary pattern, adjoining two or more elementary squares, and so on. One such method is well known for constructing the odd ordered magic squares which follow diagonal filling of the square, this was proposed by Simon de la Loubere [2] also known as Siamese method [3]. Let us first look at the the relationship between the simple square arrangement of numbers from one to n^2 and the odd magic square obtained by the Siamese method. Also, the analysis shows why the method is not applicable for constructing even ordered magic squares. Further, we present a construction method for singly even magic square.

2010 Mathematics Subject Classification: 00A08, 05A19, 06D50
Key words and phrases: Magic squares Duality Algorithms

Let I_n be the identity matrix of dimension $n \times n$. Let C be the adjacency matrix of a directed cycle, denoted by $C = \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}$. This matrix can also be described by the relation $i - j = 1 \mod n$ having one for the i, j entry satisfying the relation and zero otherwise. Let J be the left to right column flip of identity matrix, denoted by the relation $i = -j + 1 \mod n$. A be the square arrangement of the integers from 1 to n^2, having entry $A(i,j) = (i-1)n + j$ where $1 \leq i, j \leq n$. The matrix CJ is denoted by the relation $i + j = 2 \mod n$. Let $A.B$ denote the hadamard product of two matrices A and B. The non zero entris of $A.C^j$, are called entries of j^{th} cycle of A. Similarly the non zero entries of $A.(C^iJ)$ are called the entries of i^{th} reverse cycle of A.

2. Results

Lemma 2.1. *The sum of entries of square arrangement A along any cycles and along any reverse cycles is equal to the magic sum.*

Proof. Any cycle is defined by the relation $i - j = l \mod n$ for $1 \leq l \leq n$. So we get,

$$\sum_{i-j=l \mod n} A(i,j) = \sum_{i-j=l \mod n} (i-1)n + j. \tag{2.1}$$

For every i we have a corresponding j in the relation, so when i varies among 1 to n, we have j also varying fron 1 to n. So the summation can be split,

$$\sum_{i-j=l \mod n} A(i,j) = \sum_{i=1}^{n} (i-1)n + \sum_{j=1}^{n} j, \tag{2.2}$$

$$= \frac{n^2(n+1)}{2} - n^2 + \frac{n(n+1)}{2}, \tag{2.3}$$

$$= \frac{n(n^2+1)}{2}. \tag{2.4}$$

Similarly, we can prove for reverse cycles. □

Also, note that the sum of the central row and central column entries of odd-dimensional square arrangement is equal to the magic sum. The central column sum of the matrix A is given by,

$$\sum_{k=0}^{n-1} \frac{n+1}{2} + kn, \tag{2.5}$$

$$= n\frac{n+1+(n-1)(n)}{2}, \tag{2.6}$$

$$= \frac{n(n^2+1)}{2}. \tag{2.7}$$

Similarly the central row sum in the matrix A is given by,

$$\sum_{k=1}^{n} \frac{n-1}{2}n + k, \tag{2.8}$$

$$= n\frac{n(n-1)+n+1}{2}, \tag{2.9}$$

$$= \frac{n(n^2+1)}{2}. \tag{2.10}$$

2.1. **Row-Column systems.** Here we axiomatize rows and columns of a square arrangement,

(1) There are n rows in the square arrangement each having n elements, no two rows intersect each other (i.e. have a common element).
(2) There are n non intersecting columns in the square arrangement each having n elements and every column intersects every row exactly once (i.e. only one common element).

From a set of n^2 elements if we can identify such n rows and n columns, then it is possible to put them in a square arrangement. Specifying the order of rows and order of columns uniquely specifies the square arrangement hence defining the diagonals.

Now we verify that the properties 1 and 2 are satisfied by n cycles and n reverse cycles of the square arrangent A, when n is odd.
Property 1 : It is easy to see that C^k and C^l do not intersect, as $i - j = l \mod n$ uniquely determines i given j. Each C^i has n nonzero elements.
Property 2 : There are n reverse cycles. The relation $i + j = l \mod n$ uniquely determines j given i. To see the intersection of the rows and

column, we need i, j satisfying

$$i - j = l \mod n, \tag{2.11}$$

$$i + j = m \mod n. \tag{2.12}$$

For solving this, we get

$$2i = l + m \mod n, \tag{2.13}$$

$$2j = m - l \mod n \tag{2.14}$$

This system has unique solution when modulo inverse of 2 is defined with respect to n. This is possible for odd n. So every row (cycle) intersect every column (reverse cycle) exactly once, when n is odd.
However when n is even, we have two cases,
l and m both even or both odd : in this case we have two solutions for i and j. Which are $\left(\frac{l+m}{2}, \frac{l-m}{2}\right)$ and $\left(\frac{l+m\pm n}{2}, \frac{l-m\pm n}{2}\right)$.
l is even m is odd or m is even l is odd : Then the system is not solvable.

Hence, for odd n, the square arrangement can be rearranged to obtain the magic square. This can be done by looking at diagonals in the square lattice arrangement by A. By looking at the diagonal arrangement, the linear rows and columns of the matrix A will be reshaped into cycles and reverse cycles whereas the cycles and reverse cycles will be reshaped into rows and columns.

2.2. **Duality.** Let (R, L) and (S, T) be two row column systems. When a square A is reshaped into a square B by reordering the elements in R_i (and L_j) of A into S_i (and T_j) of B. If this reordering also maps S_i (and T_j) of A into R_i(and L_j) of B for $1 \leq i, j \leq n$, then we say A and B have dual row column systems.

Let $vec(A)$ be the $n^2 \times 1$ vector obtained by appending all the rows of A in a single column. Let P be the permutation matrix which maps $vec(A) \rightarrow vec(M)$.

Theorem 2.2. *The two row column systems are dual of each other if and only if $P^2 = q \otimes s$ for order n permutation matrices q and s.*

Proof. Since P maps simple square to the magic square, in order for the row column system to be dual, it should map magic square to the permuted simple square. Thus we have,

$$Pvec(A) = vec(M), \tag{2.15}$$

$$Pvec(M) = vec(\tilde{A}), \tag{2.16}$$

$$P^2 vec(A) = vec(\tilde{A}) \tag{2.17}$$

Here the vector $vec\tilde{A}$ has permutation of row elemnts due to column rearrangements and permutation of row positions. All the row permutations must be same because otherwise result into different column elements than the simple square. Thus if two row column systems are dual, then $P^2 = q \otimes s$. $\square$

As an example the 3×3 simple square $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$ is mapped to magic square $\begin{array}{|c|c|c|}\hline 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \\ \hline \end{array}$ via a matrix P by the relation,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 6 \\ 3 \\ 5 \\ 7 \\ 4 \\ 9 \\ 2 \end{bmatrix}.$$

The matrix P satisfies $P^2 = J \otimes J$ with J being left to right flip of identity matrix.

So we pose the question : *Is there a dual magic squares for a simple square of any order?* We can see from the previous section that, the answer is true for odd order magic square. It is also easy to see this holds for doubly even order magic square from the discussion in section 3. The question remains open for singly even magic square.

2.3. **Lattice embedding.** Consider the 5 by 5 arrangement,

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

.

The corresponding lattice arrangement is given by,

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
11	12	13	14	15	11	12	13	14	15	11	12	13	14	15
16	17	18	19	20	16	17	18	19	20	16	17	18	19	20
21	22	23	24	25	21	22	23	24	25	21	22	23	24	25
1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
11	12	13	14	15	11	12	13	14	15	11	12	13	14	15
16	17	18	19	20	16	17	18	19	20	16	17	18	19	20
21	22	23	24	25	21	22	23	24	25	21	22	23	24	25
1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
11	12	13	14	15	11	12	13	14	15	11	12	13	14	15
16	17	18	19	20	16	17	18	19	20	16	17	18	19	20
21	22	23	24	25	21	22	23	24	25	21	22	23	24	25

.

By looking at the diagonals in the green cells, the corresponding magic square is given by

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

.

Note that cycles and reverse cycles of this magic square are rows and columns of 5 by 5 simple square arrangement. Magic squares formed by green cells and semi magic square of pink cells together cover the lattice.

One can also look at the lattice formed by A^T, for example,

1	4	7
2	5	8
3	6	9

.

By arranging this into a lattice we get,

1	4	7	1	4	7	1	4	7
2	5	8	2	5	8	2	5	8
3	6	9	3	6	9	3	6	9
1	4	7	1	4	7	1	4	7
2	5	8	2	5	8	2	5	8
3	6	9	3	6	9	3	6	9
1	4	7	1	4	7	1	4	7
2	5	8	2	5	8	2	5	8
3	6	9	3	6	9	3	6	9

.

The entries colored in the green are given by

8	1	6
3	5	7
4	9	2

, which is an order 3 magic square.

In the case of even n, cycles and reverse cycles fail to form rows and columns. but we obtain the following type of arrangement by listing the even and odd cycles seperately. Consider the arrangement from order 4 square,

4	10	5	15
13	7	2	12
3	9	8	14
6	16	11	1

.

Here every row sum is a magic sum and every colored square sum is the magic sum, but every column sum is not the magic sum.

Similarly from the order 6 simple square arrangement, by listing even cycles and odd cycles separately, we get

6	21	14	35	28	7
11	26	19	4	18	33
16	31	9	30	23	2
5	20	12	27	13	34
10	25	17	32	3	24
15	36	22	1	8	29

.

2.4. **A LUX-like method for construct singly even magic square.** The method of constructing an odd magic square can be used for constructing a magic square when n is a singly even number (i.e. it is twice an odd integer). Conway's LUX method is one such method [3]. We dedicate this section to John H Conway.

Let A be a matrix denoting the simple square. Then A can be looked as the odd square of 2×2 blocks denoted by $A_2(i)$ for $1 \leq i \leq \left(\frac{n}{2}\right)^2$. Let B be the matrix obtained by rearranging the blocks $A_2(i)$ according to the position of i in the Siamese magic square of order $\left(\frac{n}{2}\right)$. Then we can see that sum of pair of columns, pair of rows, and pair of diagonals (corresponding to the blocks) in B will be twice the magic sum. We call the matrix B as the almost magic square. For example, consider the simple square of order six,

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

,

by re-arranging the 2×2 blocks of this simple square according to 3×3 magic square

8	1	6
3	5	7
4	9	2

we get,

$$\begin{array}{|c|c|c|c|c|c|}\hline 27 & 28 & 1 & 2 & 17 & 18 \\ \hline 33 & 34 & 7 & 8 & 23 & 24 \\ \hline 5 & 6 & 15 & 16 & 25 & 26 \\ \hline 11 & 12 & 21 & 22 & 31 & 32 \\ \hline 13 & 14 & 29 & 30 & 3 & 4 \\ \hline 19 & 20 & 35 & 36 & 9 & 10 \\ \hline \end{array}. \tag{2.18}$$

Every 2×2 block is of the form $\begin{bmatrix} k & k+1 \\ n+k & n+k+1 \end{bmatrix}$. The permutations of entries in this block can be uniquely represented by three corodinates (a, b, c) where $a = \sum$(column2 -column1) , $b = \sum$(row2 -row1) and $c = \sum$ diagonal $-$ anti diagonal. Thus a, b and c taking possible values from $0, \pm 2, \pm 2n$. The allowed co ordinates are,

$$\begin{aligned} \mathcal{S} =& \{(\pm 2, \pm 2n, 0), (\pm 2n, \pm 2, 0), (\pm 2, 0, \pm 2n), (\pm 2n, 0, \pm 2) \\ & (0, \pm 2, \pm 2n), (0, \pm 2n, \pm 2)\}. \end{aligned}$$

Now the problem of constructing the magic square reduces to the problem of constructing odd ordered square of size $\frac{n}{2}$, with entries from $\mathcal{S}$, such that:

- Sum of first co-ordinate along every column is zero.
- Sum of second co-ordinate along every row is zero.
- Sum of third co-ordinate along diagonal and anti diagonal are respectively zero.

One such possibility is :

- Starting with $v = [2; -2; 2; -2; 2; -2; \cdots; 0]_{k\times 1}$, the first co ordinates are $L = [v, v, v, \cdots Cv]_{k\times k}$. Where $C = \begin{bmatrix} 0 & 1 \\ I_{k-1} & 0 \end{bmatrix}$.
- Starting with $w = [0, 2n, -2n, 2n, -2n, \cdots]_{1\times k}$, the second co ordinates are $M = [w; w; w; \cdots wC]_{k\times k}$.
- Starting with $x = [2n; 0; 0; 0; 0; \cdots; 2]_{k\times 1}$, the third co ordinates are $N = [x, 0, 0, 0, \cdots, -Jx]_{k\times k}$. Where J is the left to right flip of identity matrix.

The entries in the blocks of the matrix A_2 permuted according to $P_{k\times k}$ such that $P(i,j) = (L(i,j), M(i,j), N(i,j))$ will give the magic square.

For $n = 6$ we have $k = 3$ and we get the matrix

$$P = \begin{bmatrix} (2,0,2n) & (2,2n,0) & (0,-2n,-2) \\ (-2,0,2n) & (-2,2n,0) & (2,-2n,0) \\ (0,2n,2) & (0,-2n,2) & (-2,0,-2n) \end{bmatrix}.$$

By permuting the square (2.19) according to P, we get

33	28	1	2	23	24
27	34	7	8	18	17
12	5	16	15	31	32
6	11	22	21	25	26
14	13	36	35	4	9
19	20	29	30	10	3

. (2.19)

3. Observations

The duality between the simple square arrangement and the magic square also holds in the construction of the doubly even magic square which involves reflecting elements about the center of the simple square in x patterns (criss-cross patterns) [1]. Some other construction methods for doubly even magic squares are given in [4]. In reflecting method, there are an even number of flips. Thus the matrix P which maps $vec(A) \to vec(M)$ satisfies $P^2 = I$. This is because combining two rows (or columns) equidistant from the center in the simple square has all the elements corresponding two rows (or columns) of the magic square.

As an example we have order 4 magic square and simple square,

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

,

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

.

With rows in the original simple square being,

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Corresponding columns L_i are obtained by the transpose of the matrices R_i,

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, L_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Sum of entries of A along every row R_i and sum along every column L_j is equal to the magic sum 34 (eg: in the green cells). In the magic square rows of A are reshaped as R_i columns reshaped as L_j.
In general the magic square of order $4n$ constructed by flipping entries about center in simple square S can be represented by,

$$M = A + JBJ$$

Here A is $P \cdot S$, with P representing the elementary pattern of block size 4,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

repeated to form order $4n$ matrix. And $(X \cdot Y)$ representing hadamard product of X and Y. The matrix B is $S \cdot (U - P)$, here U being matrix with all one entries.
So entries corresponding to k^{th} row/column of S in M will be the entries in the k^{th} row/column of M.
Now we consider the example of order 6 magic square and see that there need not exist such a duality between the simple square and magic square always.

For the order six magic square and simple square,

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

,

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

.

The sum of the entries corresponding to the inverse map of the first row elements is 84, however, the magic sum is 111.

Conclusion

Using the duality between the simple square arrangement and magic squares, odd ordered magic squares are obtained from the lattice arrangement of the simple squares. The duality is also observed for the doubly even magic square. A Conway's LUX-like method is presented for constructing singly even magic square. However the existence of duality for singly even magic square remains an open question. It is also noted by an example that such duality may not exist for all magic squares.

4. Summary

This article encourages the reader to look at the magic squares from simple square perspective as in the lattice embedding of the odd magic squares. It poses the question on the existence of dual singly even magic square.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 171–176

ON ABSOLUTE CONVERGENCE OF DOUBLE FOURIER-HAAR COEFFICIENTS SERIES

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(Received : 10 - 07 - 2020 ; Revised : 02 - 05 - 2021)

Abstract. In this paper we obtain a sufficient condition for the absolute convergence of the double Fourier-Haar coefficients series of a function f whose continuous partial derivatives f_x and f_y are of bounded partial $p-$variation on the unit square.

1. Introduction

The complete orthonormal system of functions $\{\chi_n\}_{n=1}^{\infty}$ was constructed by A. Haar in 1909 (see [5]). Let us recall the definition of this system. Let $\chi_1(x) = 1$ on $\mathbb{I} = [0,1]$. Introduce the denotation $I_i^k = \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right)$, $i = 1, 2, \cdots, 2^k$, $k = 0, 1, \cdots$ We represent a positive integer $n \geq 2$ in the form $n = 2^k + i$, $i = 1, 2, \cdots, 2^k$, $k = 0, 1, \cdots$, and put $\chi_n(x) = \chi_k^i(x) = \sqrt{2^k}$ for $x \in I_{2i-1}^{k+1}$, $\chi_n(x) = -\sqrt{2^k}$ for $x \in I_{2i}^{k+1}$, and $\chi_n(x) = 0$ for $x \in \mathbb{I} \backslash \overline{I}_i^k$, where $\overline{I}_i^k$ is the closure of the interval I_i^k. If $x \in (0,1)$ is a point of discontinuity of χ_n then we set $\chi_n(x) = (\chi_n(x-0)+\chi_n(x+0))/2$. At the endpoints of the interval $\mathbb{I}$ we set $\chi_n(0) = \chi_n(0+0)$ and $\chi_n(1) = (1-0)$. The Haar functions $\chi_n(x)$ are step functions.

A double Haar system $\chi_{mn}^{(ks)}(x,y) = \chi_m^{(k)}(x)\chi_n^{(s)}(y)$ is defined on $\mathbb{I}^2$. The Fourier-Haar coefficients are defined in the form

$$c_{rq}(f) = c_{mn}^{(ks)}(f) = \int_0^1 \int_0^1 f(x,y)\ \chi_m^{(k)}(x)\ \chi_n^{(s)}(y)\ dx\ dy,$$

where $k = 1, 2, \cdots, 2^m$, $s = 1, 2, \cdots, 2^n$, $r = 2^m + k$ and $q = 2^n + s$.

Let $p \geq 1$, $\tau_1 = \{0 = x_0 < x_1 < \cdots < x_r = 1\}$ and $\tau_2 = \{0 = y_0 <$

2010 Mathematics Subject Classification: 42A20, 42A16, 26B30

Key words and phrases: absolute convergence, Fourier-Haar coefficients, functions of bounded partial $p-$variation

$y_1 < \cdots < y_l = 1\}$. A function f defined on $\mathbb{I}^2$ is said to be of bounded partial $p-$variation (that is, $f \in PBV_1^{(p)}(\mathbb{I}^2)$) if

$$V_1(f)_p = \sup_{0\le y\le 1} \sup_{\tau_1} \left\{ \left(\sum_{i=1}^{r} |f(x_i, y) - f(x_{i-1}, y)|^p \right)^{\frac{1}{p}} \right\} < \infty.$$

Analogously, we say $f \in PBV_2^{(p)}(\mathbb{I}^2)$ if

$$V_2(f)_p = \sup_{0\le x\le 1} \sup_{\tau_2} \left\{ \left(\sum_{j=1}^{l} |f(x, y_j) - f(x, y_{j-1})|^p \right)^{\frac{1}{p}} \right\} < \infty.$$

It is well known that if a function $f(x)$ has a finite variation or a continuous derivative, then its Fourier-Haar coefficients series converges absolutely (see [1, 4, 8]). On the other hand, if a function f of two variables has continuous partial derivatives f_x and f_y, then its Fourier coefficients series does not necessarily absolutely converges with respect to a multiple Haar system (see [7]). So, in order to state the question about the absolute convergence of the double Fourier-Haar coefficients series, it is necessary to impose additional conditions on the partial derivatives f_x and f_y. In 2008, L. D. Gogoladze and V. Tsagareishvili [2] obtained a sufficient condition for the absolute convergence of the double Fourier-Haar coefficients series of a function f whose continuous partial derivatives f_x and f_y are of bounded partial variation. Generalizing these result, in this paper we obtain a sufficient condition for the absolute convergence of the double Fourier-Haar coefficients series of a function f whose continuous partial derivatives f_x and f_y are of bounded partial $p-$variation on the unit square.
Set

$$A(\beta, \varphi) = \left\{ f : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}(f)|^{\beta} < \infty \right\},$$

where

$$c_{mn}(f) = \int_0^1 \int_0^1 f(x, y)\ \varphi_{mn}(x, y)\ dx\ dy$$

in which $\varphi_{mn}(x, y)$ is an orthonormal system on $\mathbb{I}^2$.

2. Main results

We prove the following results.

Theorem 2.1. *If $f_x \in C(\mathbb{I}^2)\cap PBV_2^{(p)}(\mathbb{I}^2)$ and $f_y \in C(\mathbb{I}^2)\cap PBV_1^{(p)}(\mathbb{I}^2)$ ($p \geq 1$), then $f \in A(\beta,\chi)$, where $\frac{2p}{2p+1} < \beta < p$ and χ is the Haar system.*

Proof. Let $\Delta_{mk} = \left(\frac{k-1}{2^m}, \frac{k}{2^m}\right)$, $\Delta^+_{mk} = \left(\frac{2k-1}{2^{m+1}}, \frac{2k-2}{2^{m+1}}\right)$, $\Psi_m^{(k)}(x) = \int_0^x \chi_m^{(k)}(t)\ dt$ and

$$F_m^{(k)}(y) = -\int_0^1 f_x(x,y)\ \Psi_m^{(k)}(x)\ dx.$$

Since $\Psi_m^{(k)}(0) = \Psi_m^{(k)}(1) = 0$ and $\Psi_m^{(k)}(x) = 0$ for $x \in [0,1]\backslash\Delta_{mk}$, integration by parts, we obtain

$$\int_0^1 f(x,y)\ \chi_m^{(k)}(x)\ dx = f(x,y)\ \Psi_m^{(k)}(x)|_0^1 - \int_0^1 f_x(x,y)\ \Psi_m^{(k)}(x)\ dx$$

$$= -\int_0^1 f_x(x,y)\ \Psi_m^{(k)}(x)\ dx = -\int_{\Delta_{mk}} f_x(x,y)\ \Psi_m^{(k)}(x)\ dx = F_m^{(k)}(y). \quad (2.1)$$

Then the definition of the Haar function and equality (2.1) imply the equality

$$c_{mn}^{(ks)}(f) = \int_0^1 \left(\int_0^1 f(x,y)\ \chi_m^{(k)}(x)\ dx\right)\ \chi_n^{(s)}(y)\ dy = \int_0^1 F_m^{(k)}(y)\ \chi_n^{(s)}(y)\ dy$$

$$= 2^{n/2}\int_0^{1/2^{n+1}} \left[F_m\left(y+\frac{2s-2}{2^{n+1}}\right) - F_m\left(y+\frac{2s-1}{2^{n+1}}\right)\right]\ dy$$

$$= -2^{n/2}\int_0^{1/2^{n+1}}\int_{\Delta_{mk}} \Delta f_x^s(x,y)\ \Psi_m^{(k)}(x)\ dx\ dy,$$

where

$$\Delta f_x^s(x,y) = f_x\left(x, y+\frac{2s-2}{2^{n+1}}\right) - f_x\left(x, y+\frac{2s-1}{2^{n+1}}\right).$$

Since $|\Psi_m^{(k)}(x)| \leq 2^{-m/2}$, we have

$$|c_{mn}^{(ks)}(f)| \leq 2^{n/2}\ 2^{-m/2}\int_0^{1/2^{n+1}}\int_{\Delta_{mk}} |\Delta f_x^s(x,y)|\ dx\ dy.$$

Applying Hölder's inequality on right side of the above inequality, we have

$$|c_{mn}^{(ks)}(f)| \leq 2^{n/2}\ 2^{-m/2}\left(\int_0^{1/2^{n+1}}\int_{\Delta_{mk}} |\Delta f_x^s(x,y)|^p\ dx\ dy\right)^{1/p}$$

$$\times\left(\int_0^{1/2^{n+1}}\int_{\Delta_{mk}} dx\ dy\right)^{1-1/p}.$$

Thus,

$$|c_{mn}^{(ks)}(f)|^p \leq 2^{1-p}\ 2^{n(1-p/2)}\ 2^{m(1-3p/2)} \int_0^{1/2^{n+1}} \int_{\Delta_{mk}} |\Delta f_x^s(x,y)|^p\ dx\ dy$$

and therefore

$$\begin{aligned}
&\sum_{k=1}^{2^m}\sum_{s=1}^{2^n} |c_{mn}^{(ks)}(f)|^p \\
&\leq 2^{1-p}\ 2^{n(1-p/2)}\ 2^{m(1-3p/2)} \sum_{k=1}^{2^m}\sum_{s=1}^{2^n} \int_0^{1/2^{n+1}} \int_{\Delta_{mk}} |\Delta f_x^s(x,y)|^p\, dx\, dy \\
&= 2^{1-p}\ 2^{n(1-p/2)}\ 2^{m(1-3p/2)} \int_0^{1/2^{n+1}} \int_0^1 \sum_{s=1}^{2^n} |\Delta f_x^s(x,y)|^p\, dx\, dy \\
&\leq 2^{-p}\ 2^{-np/2}\ 2^{m(1-3p/2)} (V_2(f_x)_p)^p.
\end{aligned} \tag{2.2}$$

Using Hölder's inequality with $0 < \beta < p$, from (2.2) we obtain

$$\begin{aligned}
\sum_{k=1}^{2^m}\sum_{s=1}^{2^n} |c_{mn}^{(ks)}(f)|^\beta &\leq (2^m 2^n)^{1-\beta/p} \left(\sum_{k=1}^{2^m}\sum_{s=1}^{2^n} |c_{mn}^{(ks)}(f)|^p\right)^{\beta/p} \\
&\leq K_1\ 2^{m(1-\frac{3\beta}{2})}\ 2^{n\left(1-\frac{\beta}{2}-\alpha\beta\right)},
\end{aligned}$$

where $\alpha = 1/p$ and $K_1 = 2^{-\beta}(V_2(f_x)_p)^\beta$.

Similarly, we can prove the inequality

$$\sum_{k=1}^{2^m}\sum_{s=1}^{2^n} |c_{mn}^{(ks)}(f)|^\beta \leq K_2\ 2^{n(1-\frac{3\beta}{2})}\ 2^{m\left(1-\frac{\beta}{2}-\alpha\beta\right)},$$

where $K_2 = 2^{-\beta}(V_1(f_y)_p)^\beta$.

Using these inequalities and taking into account the fact that

$$1 - \frac{3}{2}\beta < 1 - \frac{3}{2}\frac{2}{2+\alpha} \leq 0,$$

we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{2^m}\sum_{s=1}^{2^n} |c_{mn}^{(ks)}(f)|^\beta \leq 2K_1 \sum_{m=1}^{\infty} 2^{m\left(1-\frac{\beta}{2}-\alpha\beta\right)} \sum_{n=m}^{\infty} 2^{n(1-\frac{3\beta}{2})}$$

$$+K_2 \sum_{m=1}^{\infty} 2^{m(1-\frac{3\beta}{2})} \sum_{n=1}^{m-1} 2^{n\left(1-\frac{\beta}{2}-\alpha\beta\right)}$$

$$< 2(K_1+K_2) \sum_{m=1}^{\infty} 2^{m(2-(2+\alpha)\beta)}.$$

This complete the proof of theorem. □

In particular case when $p = 1$ in Theorem 2.1, we have the following result proved by L. D. Gogoladze and V. Tsagareishvili [2, Theorem 6, p. 15].

Corollary 2.2. *If* $f_x \in C(\mathbb{I}^2) \cap PBV_2(\mathbb{I}^2)$ *or* $f_y \in C(\mathbb{I}^2) \cap PBV_1(\mathbb{I}^2)$*, then* $f \in A(\beta, \chi)$*, where* $\frac{2}{3} < \beta < 1$.

If $f \in C(\mathbb{I}^2)$, then the partial modules of continuity are defined as follows:

$$\omega_1(\delta, f) = \sup_{|h|<\delta} \|f(x+h, y) - f(x, y)\|_C,$$

$$\omega_2(\delta, f) = \sup_{|k|<\delta} \|f(x, y+k) - f(x, y)\|_C,$$

where $x + h, y + k \in \mathbb{I}$.

For $\alpha \in (0, 1]$, if $\omega_1(\delta, f) = O(\delta^\alpha)$, then we say that $f \in Lip^\dagger\alpha$, and if $\omega_2(\delta, f) = O(\delta^\alpha)$, then we say that $f \in Lip^\sharp\alpha$.

For $\alpha \in (0, 1]$, we say that $f \in \Lambda_\alpha$ if $f_x, f_y \in C(\mathbb{I}^2)$, $\omega_2(\delta, f_x) = O(\delta^\alpha)$ and $\omega_1(\delta, f_y) = O(\delta^\alpha)$.

Note that if $f \in \Lambda_\alpha$, then $f_x \in C(\mathbb{I}^2) \cap Lip^\sharp\alpha$ and $f_y \in C(\mathbb{I}^2) \cap Lip^\dagger\alpha$.

For a complete orthonormal system (φ_{mn}) and $\alpha \in (0, 1]$, L. D. Gogoladze and V. Tsagareishvili [2, Theorem 7, p. 16] find a function $f_\alpha(x, y)$ such that $f_\alpha \in \Lambda_\alpha$, but $f_\alpha \notin A(\beta, \varphi), i.e.,$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}(f_\alpha)|^\beta = +\infty, \tag{2.3}$$

where $\beta = \frac{2}{2+\alpha}$.

Putting $\alpha = 1/p$ and taking into account the embedding $Lip^\sharp\frac{1}{p} \subset PBV_2^{(p)}(\mathbb{I}^2)$ and $Lip^\dagger\frac{1}{p} \subset PBV_1^{(p)}(\mathbb{I}^2)$, the sharpness of Theorem 2.1 follows from equation (2.3).

Acknowledgement: Author is thankful to the referee for the valuable suggestions.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January- June (2022), 177–186

INCLUSION RELATIONS AND INTEGRAL PRESERVING PROPERTY FOR THE Q-ANALOGUE OF RUSCHEWEYH-TYPE DIFFERENCE OPERATOR

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(Dedicated to Prof. Shashi Kant on his 56'th birthday.)

(Received : 23 - 07 - 2020 ; Revised : 20 - 08 - 2021)

Abstract. In this paper we define q- starlike class $\mathcal{ST}_q(\phi)$ and q- close to convex class $\mathcal{KT}_q(\psi)$. By making use of these classes we obtained various inclusions relations for the q- analogue of Ruscheweyh-type difference operator. Moreover integral preserving property for this operator was also obtained.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{H}(\mathbb{U})$ which are normalized by $f(0) = 0, f'(0) = 1$ and have the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

Given two functions $f \in \mathcal{H}(\mathbb{U})$ and $g \in \mathcal{H}(\mathbb{U})$, we say that f is subordinated to g in $\mathbb{U}$ and write $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in $\mathbb{U}$, with $w(0) = 0, |w(z)| < |z|$, $z \in \mathbb{U}$, such that $f(z) = g(w(z))$ in $\mathbb{U}$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, we have the following equivalence (see [10]):

$$f(z) \prec g(z), \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad [f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U})].$$

We denote by $\mathcal{P}$ a class of analytic function in $\mathbb{U}$ with $p(0) = 1$ and $\Re(p(z)) > 0$ is of the form $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$, analytic in $\mathbb{U}$

2010 Mathematics Subject Classification: 30C45, 30C50
Key words and phrases: Analytic function, Univalent function, q-Starlike function class, q-Convex class, Ruscheweyh-type q- difference operator

such that

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+\omega(z)}{1-\omega(z)}$$

for some functions $\omega \in \Omega$ and for all $z \in \mathbb{U}$. It is well known that a function $f \in \mathcal{H}$ is called starlike $f \in \mathcal{S}^*$, convex $f \in \mathcal{C}$ and close-to-convex $f \in \mathcal{K}$, quasi-convex $f \in \mathcal{C}^*$ if there exist a function $p(z)$ in $\mathcal{P}$ such that $p(z)$ must be expressed respectively, by the following relations:

$$\frac{zf'(z)}{f(z)} = p(z), \quad 1 + \frac{zf''(z)}{f'(z)} = p(z)$$

and

$$\frac{zf'(z)}{g(z)} = p(z) \ , \quad \frac{(zf'(z))'}{g'(z)} = p(z), \text{ where } g(z) \in \mathcal{S}^*$$

for all $z \in \mathbb{U}$. For definitions and properties of these classes see [2] and [10]. Ma and Minda [17] unified various subclasses of starlike and convex functions for which either one of the quantities $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. We express $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ functions as $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$, where function ϕ with positive real part in $\mathbb{U}$, $\phi(0) = 1$, $\phi'(0) > 0$. Here we assume that $\phi \in \mathcal{P}$ satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Also ϕ has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ..., (B_1 > 0). \tag{1.2}$$

In 1909 and 1910, Jackson [12, 13, 14] initiated a study of q difference operator $(0 < q < 1)$ for$f(z) \in \mathcal{H}$;

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & (z \neq 0), \\ \\ f'(0), & (z = 0). \end{cases}$$

For a function $f(z) = z^n$, we observe that

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1}.$$

Therefore we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} a_n [n]_q z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$. Clearly, as $q \to 1^-$, $[n]_q \to n$. The q- Gamma function is given by

$$\Gamma_q(n+1) = [n]_q \Gamma_q(n)$$

and q-factorial is given by

$$[n]_q! = [1]_q[2]_q...[n-1]_q[n]_q, \quad (n > 0) \text{ and } [0]_q! = 1.$$

Also, the q-shifted factorial is given by

$$([n]_q)_m = [n]_q[n+1]_q...[n+m-1]_q, \quad (m \geq 1) \text{ and } ([n]_q)_0 = 1.$$

The well-known q-extension of calculus and q-analysis have been studied by many eminent mathematicians for a long time. It has a great importance in the applied science, dynamical systems, combinatorics, orthogonal polynomials, basic hypergeometric functions and quantum physics. A detailed and systematic development can be seen in the papers of Jackson [13], Gasper and Rahman [9], Ismail [11], Srivastava [23], Raghvendra and Swaminathan [20], Andrews[4], Fine[8], Kac [15]. Recently research in this field includes the work of Agarwal and Sahoo [1], Aldweby and darus [3], Arif et al. [5], Cetinkaya [6, 7], Mahmood and Sokol [18], Purohit and Raina [19] .

Definition 1.1. In [16], Kanas and Raducanu introduced the following Ruscheweyh-type q-differential operator for $f \in \mathcal{H}$ by

$$R_q^\lambda f(z) = f(z) * F_{q,\lambda+1}(z) \quad (z \in \mathbb{U},\ \lambda > -1) \tag{1.3}$$

where

$$F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda+n)}{[n-1]_q!\Gamma_q(1+\lambda)} z^n = z + \sum_{n=2}^{\infty} \frac{([\lambda+1]_q)_{n-1}}{[n-1]_q!} z^n. \tag{1.4}$$

The symbol $*$ stands for Hadamard product (or convolution).
From (1.3) we obtain that

$$R_q^0 f(z) = f(z), \qquad R_q^1 f(z) = zD_q f(z)$$

and

$$R_q^m f(z) = \frac{zD_q^m(z^{m-1}f(z))}{[m]_q!} \qquad (m \in \mathbb{N}).$$

Making use of (1.3) and (1.4), the power series of $R_q^\lambda f(z)$ for f of the form (1.1) is given by

$$R_q^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{([\lambda+1]_q)_{n-1}}{[n-1]_q!} a_n z^n. \tag{1.5}$$

Note that
$$\lim_{q\to 1^-} F_{q,\lambda+1}(z) = \frac{z}{(1-z)^{\lambda+1}}$$
and
$$\lim_{q\to 1^-} R_q^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}.$$
Thus, we can say that Ruscheweyh q-differential operator reduces to the differential operator defined by Ruscheweyh [21] in the case when $q \to 1^-$. It is easy to check that
$$zD_q(F_{q,\lambda+1}(z)) = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right) F_{q,\lambda+2}(z) - \frac{[\lambda]_q}{q^\lambda} F_{q,\lambda+1}(z) \quad (z \in \mathbb{U}). \quad (1.6)$$
Making use of (1.3), (1.6) and the properties of Hadamard product, we obtain the following equality
$$zD_q(R_q^\lambda f(z)) = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right) R_q^{\lambda+1} f(z) - \frac{[\lambda]_q}{q^\lambda} R_q^\lambda f(z) \quad (z \in \mathbb{U}). \quad (1.7)$$
If $q \to 1^-$, the equality (1.7) implies
$$z(R^\lambda f(z))' = (\lambda+1) R^{\lambda+1} f(z) - \lambda R^\lambda f(z) \quad (z \in \mathbb{U}),$$
which is the well known formula for Ruscheweyh differential operator.

Definition 1.2. A function $f \in \mathcal{H}$ is said to be in the class $\mathcal{ST}_q(\phi)$ if it satisfies the following condition
$$\frac{z(D_q f(z))}{f(z)} \prec \phi(z), \quad (\phi \in \mathcal{P})$$
where D_q is the q difference operator.
Also a function $f \in \mathcal{H}$ is said to be in the class $\mathcal{C}V_q(\phi)$, if and only if
$$z(D_q f(z)) \in \mathcal{ST}_q(\phi). \quad (1.8)$$

When $q \to 1^-$ in the limiting sense, the classes $\mathcal{ST}_q(\phi)$ and $\mathcal{C}V_q(\phi)$ reduce to the traditional classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$. Using the operator defined above, we define the following subclasses for starlike and convex functions .

Definition 1.3. Let $f \in \mathcal{H}$, $\lambda > -1$. Then
$$f \in \mathcal{ST}_q^\lambda(\phi) \text{ if and only if } R_q^\lambda f(z) \in \mathcal{ST}_q(\phi) \quad (1.9)$$
and
$$f \in \mathcal{C}V_q^\lambda(\phi) \text{ if and only if } R_q^\lambda f(z) \in \mathcal{C}V_q(\phi), \quad (1.10)$$

it is clear that

$$f \in \mathcal{C}V_q^{\lambda}(\phi) \text{ if and only if } z(D_q f(z)) \in \mathcal{ST}_q^{\lambda}(\phi). \tag{1.11}$$

Definition 1.4. A function $f \in \mathcal{H}$ is said to be in the class $\mathcal{KT}_q(\psi)$ if it satisfies the following condition

$$\frac{z(D_q f(z))}{g(z)} \prec \psi(z), \quad \textit{where } g(z) \in \mathcal{ST}_q(\phi) \textit{ and } \psi(z) \in \mathcal{P}.$$

Also a function $f \in \mathcal{H}$ is said to be in the class $\mathcal{C}^*V_q(\psi)$, if and only if

$$z(D_q f(z)) \in \mathcal{KT}_q(\psi). \tag{1.12}$$

Similarly when $q \to 1^-$ in the limiting sense, the classes $\mathcal{KT}_q(\psi)$ and $\mathcal{C}^*V_q(\psi)$ reduce to the traditional classes of close-to-convex and quasi convex $\mathcal{K}(\psi)$ and $\mathcal{C}^*(\psi)$ respectively. Using the operators defined above, we define the following subclasses for the close-to-convex and quasi convex functions.

Definition 1.5. Let $f \in \mathcal{H}$, $\lambda > -1$. Then

$$f \in \mathcal{KT}_q^{\lambda}(\psi) \text{ if and only if } \frac{zD_q(R_q^{\lambda} f(z))}{R_q^{\lambda} g(z)} \prec \psi(z), \tag{1.13}$$

where $R_q^{\lambda} g(z) \in \mathcal{ST}_q(\phi)$.

$$f \in \mathcal{C}^*V_q^{\lambda}(\psi) \text{ if and only if } \frac{zD_q(zD_q R_q^{\lambda} f(z))}{D_q R_q^{\lambda} g(z)} \prec \psi(z), \tag{1.14}$$

where $R_q^{\lambda} g(z) \in \mathcal{ST}_q(\phi)$. It is clear that

$$f \in \mathcal{C}^*V_q^{\lambda}(\psi) \text{ if and only if } z(D_q f) \in \mathcal{KT}_q^{\lambda}(\psi). \tag{1.15}$$

In order to prove our results we require the following lemma.

Lemma 1.6. [22] *Let β and γ be complex numbers with $\beta \neq 0$ and let $h(z)$ be analytic in $\mathbb{U}$ with $h(0) = 1$ and $\Re\{\beta h(z) + \gamma\} > 0$. If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in $\mathbb{U}$, then*

$$p(z) + \frac{zD_q p(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that $p(z) \prec h(z)$.

2. Inclusion Results

Theorem 2.1. *Let $q \in (0,1)$ and $\lambda > -1$, $\phi(z)$ be analytic and convex univalent function with $\phi(0) = 1$ and $\Re(\phi(z)) > 0$, then for $z \in \mathbb{U}$*

$$\mathcal{ST}_q^{\lambda+1}(\phi) \subset \mathcal{ST}_q^{\lambda}(\phi).$$

Proof. Let $f \in \mathcal{ST}_q^{\lambda+1}(\phi)$ and we set

$$\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} = p(z), \tag{2.1}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Using the identity (1.7) and (2.1), we get

$$\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right) \frac{R_q^{\lambda+1} f(z)}{R_q^\lambda f(z)} - \frac{(\lambda)_q}{q^\lambda}.$$

Equivalently

$$\left(1 + \frac{[\lambda]_q}{q^\lambda}\right) \frac{R_q^{\lambda+1} f(z)}{R_q^\lambda f(z)} = p(z) + \gamma_q, \quad \text{where } \gamma_q = \frac{[\lambda]_q}{q^\lambda}.$$

q-Logarithmic differential yields,

$$\frac{zD_q(R_q^{\lambda+1} f(z))}{R_q^{\lambda+1} f(z)} = p(z) + \frac{zD_q(p(z))}{p(z) + \gamma_q}. \tag{2.2}$$

Since $f \in \mathcal{ST}_q^{\lambda+1}(\phi)$, so from (2.2), we have

$$p(z) + \frac{zD_q(p(z))}{p(z) + \gamma_q} \prec \phi(z).$$

Now by applying Lemma 1.6, we conclude that $p(z) \prec \phi(z)$, consequently $\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} \prec \phi(z)$ this implies $f \in \mathcal{ST}_q^{\lambda}(\phi)$. □

Theorem 2.2. *Let $q \in (0,1)$, $\lambda > -1$ and $\phi(z)$ be analytic and convex univalent function with $\phi(0) = 1$ and $\Re(\phi(z)) > 0$ then,*

$$\mathcal{C}V_q^{\lambda+1}(\phi) \subset \mathcal{C}V_q^{\lambda}(\phi).$$

Proof. Let $f \in \mathcal{C}V_q^{\lambda+1}(\phi)$. Then by Alexander's type relation (1.11), we can write $z(D_q f) \in \mathcal{ST}_q^{\lambda+1}(\phi)$. From theorem (2.1), we know that $\mathcal{ST}_q^{\lambda+1}(\phi) \subset \mathcal{ST}_q^{\lambda}(\phi)$, so we have $z(D_q f) \in \mathcal{ST}_q^{\lambda}(\phi)$. We get required result by again using Alexander's type relation (1.11). □

Corollary 2.3. *Let $q \in (0,1)$ and $\lambda > -1$, then for $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, we have*

$$\mathcal{ST}_q^{\lambda+1}\left(\frac{1+Az}{1+Bz}\right) \subset \mathcal{ST}_q^{\lambda}\left(\frac{1+Az}{1+Bz}\right).$$

Moreover, For $A = 0$ and $B = -q$ and for $A = 1$ and $B = -q$,

$$\mathcal{ST}_q^{\lambda+1}\left(\frac{1}{1-qz}\right) \subset \mathcal{ST}_q^{\lambda}\left(\frac{1}{1-qz}\right) \text{ and } \mathcal{ST}_q^{\lambda+1}\left(\frac{1+z}{1-qz}\right) \subset \mathcal{ST}_q^{\lambda}\left(\frac{1+z}{1-qz}\right)$$

respectively.

Theorem 2.4. *Let $\psi(z)$ be analytic and convex function with $\psi(0) = 1$ and $\Re(\psi(z)) > 0$, then for $z \in \mathbb{U}$ and $\lambda > -1$*

$$\mathcal{KT}_q^{\lambda+1}(\psi) \subset \mathcal{KT}_q^{\lambda}(\psi).$$

Proof. Let $f \in \mathcal{KT}_q^{\lambda+1}(\psi)$, then there exist $R_q^{\lambda+1}g(z) \in \mathcal{ST}_q(\phi)$ such that

$$\frac{zD_q(R_q^{\lambda+1}f(z))}{R_q^{\lambda+1}g(z)} \prec \psi(z). \tag{2.3}$$

Since $\mathcal{ST}_q^{\lambda+1}(\phi) \subset \mathcal{ST}_q^{\lambda}(\phi)$, therefore we have

$$\frac{zD_q(R_q^{\lambda}g(z))}{R_q^{\lambda}g(z)} = h(z), \tag{2.4}$$

where $h(z)$ is analytic in $\mathcal{H}$, with $h(0) = 1$. Now we set

$$\frac{zD_q(R_q^{\lambda}f(z))}{R_q^{\lambda}g(z)} = p(z), \tag{2.5}$$

where $p(z)$ is of the form $p(z) = 1 + p_1 z + p_2 z^2 + ...$ To obtain the required result, we shall prove that $p(z) \prec \phi(z)$. On the differentiation of the identity (1.7), we get

$$zD_q(D_q(R_q^{\lambda}f(z))) = \left(1 + \frac{[\lambda]_q}{q^{\lambda}}\right) D_q(R_q^{\lambda+1}f(z)) - \left(1 + \frac{[\lambda]_q}{q^{\lambda}}\right) D_q(R_q^{\lambda}f(z)) \quad (z \in \mathbb{U}). \tag{2.6}$$

In view of the identity (1.7) for the function $g(z)$ and identity (2.6), we have

$$\frac{\frac{zD_q\left(R_q^{\lambda}f(z)\right)}{R_q^{\lambda}g(z)}\left(\frac{z\left(D_q(D_q(R_q^{\lambda}f(z)))\right)}{D_q(R_q^{\lambda}f(z))}\right) + \left(1 + \frac{[\lambda_q]}{q^{\lambda}}\right)}{\frac{zD_q(R_q^{\lambda}g(z))}{R_q^{\lambda}g(z)} + \frac{[\lambda]_q}{q^{\lambda}}} = \frac{zD_q(R_q^{\lambda+1}f(z))}{R_q^{\lambda+1}g(z))}. \tag{2.7}$$

Differentiating logarithmically (2.5), we get

$$\frac{zD_q\left(D_q(R_q^\lambda f(z))\right)}{D_q(R_q^\lambda f(z))} = \frac{zD_q(R_q^\lambda g(z))}{R_q^\lambda g(z))} + \frac{zD_q(p(z))}{p(z)} - 1. \tag{2.8}$$

Equation (2.4) and (2.8) gives us

$$\frac{zD_q\left(D_q(R_q^\lambda f(z))\right)}{D_q(R_q^\lambda f(z))} = h(z) + \frac{zD_q(p(z))}{p(z)} - 1. \tag{2.9}$$

Now substitute the values from (2.4), (2.5) and (2.9) in (2.8), we get

$$p(z) + \frac{zD_q(p(z))}{h(z) + \frac{[\lambda]_q}{q^\lambda}} = \frac{zD_q\left(R_q^{\lambda+1} f(z)\right)}{R_q^{\lambda+1} g(z)}. \tag{2.10}$$

In view of (2.3), we have

$$p(z) + \frac{zD_q p(z)}{h(z) + \frac{[\lambda]_q}{q^\lambda}} \prec \psi(z). \tag{2.11}$$

Now using Lemma 1.6, we conclude that $p(z) \prec \psi(z)$ or equivalently $\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda g(z)} \prec \psi(z)$, that is $f \in \mathcal{KT}_q^\lambda(\psi)$. That completes the proof of the Theorem 2.4. □

Theorem 2.5. *Let $\psi(z)$ be analytic and convex function with $\psi(0) = 1$ and $\Re(\psi(z)) > 0$, then for $z \in \mathbb{U}$ and $\lambda > -1$, we have*

$$\mathcal{C}^* V_q^{\lambda+1}(\psi) \subset \mathcal{C}^* V_q^\lambda(\psi).$$

Proof. Let $f \in \mathcal{C}^* V_q^{\lambda+1}(\psi)$. Then by Alexander's type relation (1.15), we can write $z(D_q f) \in \mathcal{KT}_q^{\lambda+1}(\psi)$. From Theorem 2.4, we know that $\mathcal{KT}_q^{\lambda+1}(\psi) \subset \mathcal{KT}_q^\lambda(\psi)$, so we have $z(D_q f) \in \mathcal{KT}_q^\lambda(\psi)$. We get required result by again using Alexander's type relation (1.15). □

3. Integral Preserving Property

Theorem 3.1. *Let $q \in (0, 1)$, $\lambda > -1$ and $\phi(z)$ be analytic and convex univalent function with $\phi(0) = 1$ and $\Re(\phi(z)) > 0$ then, we have $J_{q,\lambda}^1 f \in \mathcal{ST}_q^\lambda(\phi)$, where*

$$J_{q,\lambda}^1 f(z = \frac{[1+\lambda]_q}{z^\lambda} \int_0^z t^{\lambda-1} f(t) d_q t \ . \tag{3.1}$$

Proof. Let $f \in \mathcal{ST}_q^\lambda(\phi)$ and if we set for $F(z) = J_{q,\lambda}^1 f(z)$, then we get

$$\frac{zD_q(R_q^\lambda F(z))}{R_q^\lambda F(z)} = Q(z), \tag{3.2}$$

where $Q(z)$ is analytic in $\mathbb{U}$ with $Q(0) = 1$.
From (3.1), we can write

$$\frac{D_q(z^\lambda F(z))}{(1+\lambda)_q} = z^{\lambda-1}F(z). \tag{3.3}$$

Knowing the fact that $R_q^{\lambda-1}f(z) = R_q^\lambda(J_{q,\lambda}^1 f(z))$, using product rule of q difference operator, we get

$$zD_q(F(z)) = \left(1+\frac{[\lambda]_q}{q^b}\right)f(z) - \frac{(\lambda)_q}{q^\lambda}F(z). \tag{3.4}$$

From (3.2), (3.4) and (1.7), we have

$$Q(z) = \left(1+\frac{[\lambda]_q}{q^b}\right)\frac{R_q^\lambda f(z)}{R_q^\lambda F(z)} - \frac{[\lambda-1]_q}{q^{\lambda-1}}. \tag{3.5}$$

On q-logarithmic differentiation of (3.5) , we get

$$\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} = Q(z) + \frac{zD_q(Q(z))}{Q(z)+\frac{[\lambda-1]_q}{q^{\lambda-1}}} \quad . \tag{3.6}$$

Since $f \in \mathcal{ST}_q^\lambda(\phi)$, so (3.6) implies $Q(z) + \frac{zD_q(Q(z))}{Q(z)+\frac{(\lambda-1)_q}{q^{\lambda-1}}} \prec \phi(z)$. Now by applying Lemma 1.6, we conclude $Q(z) \prec \phi(z)$. Consequently $\frac{zD_q(R_q^\lambda F(z))}{R_q^\lambda F(z)} \prec \phi(z)$, hence $F = J_{q,\lambda}^1 f \in \mathcal{ST}_q^\lambda(\phi)$. □

Acknowledgment: We are grateful to the referee for valuable comments which improved the merit of the paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 187–199

ON CERTAIN TRIGONOMETRIC IDENTITIES AS A CONSEQUENCE OF BAILEY'S SUMMATION FORMULA

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(Received : 23 - 06 - 2020 ; Revised : 17 - 03 - 2021)

Abstract. In this article, we give an elementary proof of nine non trivial trigonometric identities from Bailey's summation formula. We also generalize two of those identities.

1. Introduction

For any complex numbers a and q, with $|q| < 1$ we define

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ if } n \text{ is a positive integer,}$$

$$(a)_0 := (a;q)_0 = 1,$$

and

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

Ramanujan's general theta function is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \tag{1.1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1.$$

B. C. Berndt and L. C. Zhang [6] discovered the following two interesting identities for trigonometric sums by using certain identities satisfied by Ramanujan's general theta function $f(a,b)$:

$$\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{3\pi}{7}\right)} - \frac{\sin\left(\frac{\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)} + \frac{\sin\left(\frac{3\pi}{7}\right)}{\sin^2\left(\frac{\pi}{7}\right)} = 2\sqrt{7} \tag{1.2}$$

2010 Mathematics Subject Classification: 11L03, 11F27, 11M36
Key words and phrases: Ramanujan's theta-function, Trigonometric sum, Eisenstein series

and

$$\frac{\sin^2\left(\frac{3\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)} - \frac{\sin^2\left(\frac{2\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)} + \frac{\sin^2\left(\frac{\pi}{7}\right)}{\sin\left(\frac{3\pi}{7}\right)} = 0. \tag{1.3}$$

Motivated by the above works, Z. G. Liu [8] deduced and proved eight new non-trivial identities from complex variable theory for elliptic functions. In [5], Berndt and A. Zahareseu have generalized certain identities proved by Liu [8]. All the eight specific identities proved by Liu, involve multiples of $\frac{\pi}{7}$. Motivated by these, in this paper we deduce certain trigonometric identities involving the angles which are multiples of $\frac{\pi}{14}$ and $\frac{\pi}{18}$ by making use of following beautiful identity due to W. N. Bailey [1]:

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right] = \frac{af_1^6\, f\left(-ab, \frac{-q}{ab}\right) f\left(\frac{-b}{a}, \frac{-aq}{b}\right)}{f^2\left(-a, \frac{-q}{a}\right) f^2\left(-b, \frac{-q}{b}\right)}, \tag{1.4}$$

where $f_1 = f\left(-q, -q^2\right)$. Bailey proved the above identity by making use of elliptic function theory. It can also be deduced from Bailey ${}_6\psi_6$ well-poised summation formula, relations between modular equations and theta functions recorded by Ramanujan.

We close this section by recalling certain definitions and elementary results required to prove the trigonometric identities. Ramanujan defines the following special cases of (1.1):

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

For convenience, we set $f_n := f(-q^n) = (q^n;q^n)_\infty$ for any positive integer n

and

$$\chi(q) := (-q;q^2)_\infty.$$

The Eisenstein series $P(q)$ is defined as

$$P(q) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \quad |q|<1.$$

For convenience, we set

$$P_n = P(q^n) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^{nk}}{1-q^{nk}}.$$

We make use of the following Eisenstein series identities:

$$-P_1 + P_2 + 7P_7 - 7P_{14} = 72\left[q^2\frac{f_2^4 f_{14}^4}{f_1^2 f_7^2} + qf_1f_2f_7f_{14}\right], \tag{1.5}$$

$$P_1 - 2P_2 = -16q\psi^4(q^2) - \varphi^4(q), \tag{1.6}$$

$$\begin{aligned}-P_1 + 7P_7 = {} & 24q^2\psi^2(q)\psi^2(q^7) + 6\varphi^2(-q)\varphi^2(-q^7)\\ & +24q\varphi(-q)\varphi(-q^7)\psi(q)\psi(q^7),\end{aligned} \tag{1.7}$$

$$-P_2 + 9P_{18} = \left[\frac{\varphi^4(q^3) + 3\varphi^2(q)\varphi^2(q^9)}{4}\right]^2 \frac{\varphi^2(q^3)}{\varphi^3(q)\varphi^3(q^9)} \tag{1.8}$$

and

$$-P_1 + P_2 + 3P_3 - 3P_6 = 24q\psi^2(q) - \psi^2(q^3). \tag{1.9}$$

Equations (1.5) and (1.7) were proved by S. Cooper and Dongxi Ye in [7] by using theory of modular forms and also K. R. Vasuki and Veeresh R. G. have given an elementary proof of it in [10]. Proofs of (1.6) and (1.8) can be found in [2], (1.9) is given in [11].

In the unorganized pages of his second notebook [9, p. 309], Ramanujan recorded the following identities which relates a ratio of theta functions to a ratio of trigonometric functions

If $m+n = p+q = k$, then $\dfrac{f(-x^m,-x^n)}{f(-x^p,-x^q)} = \dfrac{\sin\left(\frac{m\pi}{k}\right)}{\sin\left(\frac{p\pi}{k}\right)}$, when $x=1$.

Berndt expressed the above identity in Chapter 25 of his edited notebook of Ramanujan Part IV [3, p. 140], which is as follows:

Let m, n, p, r and k be positive integers such that $m+n = p+r = k$, then as q tends to 1^- we have

$$\frac{f(-q^m,-q^n)}{f(-q^p,-q^r)} \sim \frac{\sin\left(\frac{m\pi}{k}\right)}{\sin\left(\frac{p\pi}{k}\right)}.$$

Since the right hand side is a constant (free of q), applying the definition as 'asymptotic to', we can write

$$\lim_{q\to1^-}\frac{f(-q^m,-q^n)}{f(-q^p,-q^r)}=\frac{\sin\left(\frac{m\pi}{k}\right)}{\sin\left(\frac{p\pi}{k}\right)}. \tag{1.10}$$

The Gaussian hypergeometric series ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,\ \ |z|<1,$$

where a,b and c are any complex numbers. The elliptic integral of first kind is defined by

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\sin^2\varphi}}d\varphi,\ \ |k|<1.$$

Here, k is called the modulus and $k'=\sqrt{1-k^2}$ is called the complementary modulus.

Following is the relation between Gaussian hypergeometric series and elliptic integral of first kind:

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right)=\frac{\pi}{2}K(k).$$

Theorem 1.1. [2] *Suppose* $0<\alpha<1,\ \ y=\pi\,\dfrac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}$, *and* $q=e^{-y}$ *then*

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)=\varphi^2(q).$$

If $0<\alpha,\beta<1$, and the equality

$$n\,\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}$$

holds, then any relationship induced between α and β by the above equations is called a modular equation of degree n. We say that β is of degree n over α. The multiplier connecting α and β is defined by

$$m=\frac{z_1}{z_n}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}.$$

We now prove some elementary theorems which are necessary to prove main identities.

Theorem 1.2. *For every natural number n and for any $\alpha \in (0,1)$ there exists $\beta \in (0,1)$ such that β is of degree n over α.*

Proof. The function F as defined above is a bijection from [0, 1] onto [0, 1]. Hence if $q \in (0,1)$, then there exists $\beta \in (0,1)$ such that $q^n = F(\beta)$ (since $q^n \in (0,1)$). In other words, β is of degree n over α. □

Theorem 1.3. *If $0 < \alpha < 1$ then as $q \to 1^-, \alpha \to 1^-$. Consequently if β is of degree n over α then as $q \to 1^-,\ \beta \to 1^-$.*

Proof. Defined $F(\alpha) = e^{-y}$, where y is as defined in Theorem 1.1 and $0 < \alpha < 1$. F can be extended to [0, 1] by defining $F(0) = 0$ and $F(1) = 1$. It can be easily seen that F is continuous on [0, 1]. Moreover, F is monotonically increasing [0, 1] and hence has a continuous inverse. In other words, we can treat α as a function of q. Since as $\alpha \to 1^-, q \to 1^-$ and F is monotonically increasing, as $q \to 1^-, \alpha$ should tend to 1^-.

The same argument can be made for β too, since $q^n = F(\beta)$. □

Theorem 1.4. *If β is of degree n over α and if m is the multiplier connecting α and β, then $\lim\limits_{q\to 1^-} m = n$.*

Proof. By definition $m = \dfrac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}$. Since β is of degree n over α, we have

$$\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)} = n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}.$$

Thus $m = \dfrac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}$. As $q \to 1^-, (1-\alpha) \to 0$ and $(1-\beta) \to 0$. Hence, we have as $q \to 1^-, m \to n$. □

2. Main Results

In this section we prove the following theorem:

Theorem 2.1. *We have*

$$\frac{\sin^2\left(\frac{5\pi}{14}\right)}{\sin\left(\frac{3\pi}{7}\right)} - \frac{\sin^2\left(\frac{3\pi}{14}\right)}{\sin\left(\frac{\pi}{7}\right)} + \frac{\sin^2\left(\frac{\pi}{14}\right)}{\sin\left(\frac{2\pi}{7}\right)} = 0, \tag{2.1}$$

$$\frac{\sin^2\left(\frac{\pi}{7}\right)}{\sin^2\left(\frac{5\pi}{14}\right)} + \frac{\sin^2\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{3\pi}{14}\right)} + \frac{\sin^2\left(\frac{3\pi}{7}\right)}{\sin^2\left(\frac{\pi}{14}\right)} = 21, \tag{2.2}$$

$$\frac{\sin^2\left(\frac{5\pi}{14}\right)}{\sin^2\left(\frac{\pi}{7}\right)} + \frac{\sin^2\left(\frac{3\pi}{14}\right)}{\sin^2\left(\frac{2\pi}{7}\right)} + \frac{\sin^2\left(\frac{\pi}{14}\right)}{\sin^2\left(\frac{3\pi}{7}\right)} = 5, \tag{2.3}$$

$$\frac{\sin^2\left(\frac{2\pi}{9}\right)}{\sin^2\left(\frac{\pi}{18}\right)}+\frac{\sin^2\left(\frac{3\pi}{9}\right)}{\sin^2\left(\frac{3\pi}{18}\right)}+\frac{\sin^2\left(\frac{2\pi}{9}\right)}{\sin^2\left(\frac{5\pi}{12}\right)}+\frac{\sin^2\left(\frac{\pi}{9}\right)}{\sin^2\left(\frac{7\pi}{18}\right)}=36, \tag{2.4}$$

$$\frac{\sin^2\left(\frac{\pi}{18}\right)}{\sin^2\left(\frac{2\pi}{9}\right)}+\frac{\sin^2\left(\frac{3\pi}{18}\right)}{\sin^2\left(\frac{3\pi}{9}\right)}+\frac{\sin^2\left(\frac{5\pi}{12}\right)}{\sin^2\left(\frac{2\pi}{9}\right)}+\frac{\sin^2\left(\frac{7\pi}{18}\right)}{\sin^2\left(\frac{\pi}{9}\right)}=\frac{28}{3}, \tag{2.5}$$

$$\frac{\sin^4\left(\frac{\pi}{7}\right)}{\sin^4\left(\frac{5\pi}{14}\right)}+\frac{\sin^4\left(\frac{2\pi}{7}\right)}{\sin^4\left(\frac{3\pi}{14}\right)}+\frac{\sin^4\left(\frac{3\pi}{7}\right)}{\sin^4\left(\frac{\pi}{14}\right)}=371, \tag{2.6}$$

$$\frac{\sin^4\left(\frac{5\pi}{14}\right)}{\sin^4\left(\frac{\pi}{7}\right)}+\frac{\sin^4\left(\frac{3\pi}{14}\right)}{\sin^4\left(\frac{2\pi}{7}\right)}+\frac{\sin^4\left(\frac{\pi}{14}\right)}{\sin^4\left(\frac{3\pi}{7}\right)}=19, \tag{2.7}$$

$$\frac{\sin^6\left(\frac{\pi}{7}\right)}{\sin^6\left(\frac{5\pi}{14}\right)}+\frac{\sin^6\left(\frac{2\pi}{7}\right)}{\sin^6\left(\frac{3\pi}{14}\right)}+\frac{\sin^6\left(\frac{3\pi}{7}\right)}{\sin^6\left(\frac{\pi}{14}\right)}=7077, \tag{2.8}$$

and

$$\frac{\sin^6\left(\frac{5\pi}{14}\right)}{\sin^6\left(\frac{\pi}{7}\right)}+\frac{\sin^6\left(\frac{3\pi}{14}\right)}{\sin^6\left(\frac{2\pi}{7}\right)}+\frac{\sin^6\left(\frac{\pi}{14}\right)}{\sin^6\left(\frac{3\pi}{7}\right)}=\frac{563}{7}. \tag{2.9}$$

Proof of (2.1). Replacing q by q^{14} in (1.4), we find that

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^{14n}}{(1-aq^{14n})^2}-\frac{bq^{14n}}{(1-bq^{14n})^2}\right]=\frac{af_{14}^6\, f\left(-ab,\frac{-q^{14}}{ab}\right)f\left(\frac{-b}{a},\frac{-aq^{14}}{b}\right)}{f^2\left(-a,\frac{-q^{14}}{a}\right)f^2\left(-b,\frac{-q^{14}}{b}\right)}. \tag{2.10}$$

Setting $a=q$ and $b=q^3$ in (2.10), we find that

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{14n+1}}{(1-q^{14n+1})^2}-\frac{q^{14n+3}}{(1-q^{14n+3})^2}\right]=q\frac{f_2^3f_7^2}{f_1^2}\;\frac{f^2(-q^5,-q^9)}{f(-q^6,-q^8)}. \tag{2.11}$$

Setting $a=q$ and $b=q^5$ in (2.10), we find that

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{14n+5}}{(1-q^{14n+5})^2}-\frac{q^{14n+1}}{(1-q^{14n+1})^2}\right]=-q\frac{f_2^3f_7^2}{f_1^2}\;\frac{f^2(-q^3,-q^{11})}{f(-q^2,-q^{12})}. \tag{2.12}$$

Setting $a=q^3$ and $b=q^5$ in (2.10), we find that

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{14n+3}}{(1-q^{14n+3})^2}-\frac{q^{14n+5}}{(1-q^{14n+5})^2}\right]=q^3\frac{f_2^3f_7^2}{f_1^2}\;\frac{f^2(-q,-q^{13})}{f(-q^4,-q^{10})}. \tag{2.13}$$

Adding (2.11), (2.12) and (2.13), we obtain

$$\frac{f^2(-q^5,-q^9)}{f(-q^6,-q^8)}-\frac{f^2(-q^3,-q^{11})}{f(-q^2,-q^{12})}+q^2\frac{f^2(-q,-q^{13})}{f(-q^4,-q^{10})}=0.$$

Using equation (1.10), tending q to 1^-, we get the required result (2.1). $\square$

Proof of (2.2). Setting $b = q^7$ in (2.10), we find that

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^{14n}}{(1-aq^{14n})^2}-\frac{q^{14n+7}}{(1-q^{14n+7})^2}\right]=a\psi^4(q^7)\frac{f^2\left(-aq^7,\frac{-q^7}{a}\right)}{f^2\left(-a,\frac{-q^{14}}{a}\right)}. \quad (2.14)$$

By setting $a = q, q^3$ and q^5 respectively in the above equation (2.14), we find that

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{14n+1}}{(1-q^{14n+1})^2}-\frac{q^{14n+7}}{(1-q^{14n+7})^2}\right]=q\psi^4(q^7)\frac{f^2\left(-q^6,-q^8\right)}{f^2\left(-q,-q^{13}\right)}, \quad (2.15)$$

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{14n+3}}{(1-q^{14n+3})^2}-\frac{q^{14n+7}}{(1-q^{14n+7})^2}\right]=q^3\psi^4(q^7)\frac{f^2\left(-q^4,-q^{10}\right)}{f^2\left(-q^3,-q^{11}\right)} \quad (2.16)$$

and

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{14n+5}}{(1-q^{14n+5})^2}-\frac{q^{14n+7}}{(1-q^{14n+7})^2}\right]=q^5\psi^4(q^7)\frac{f^2\left(-q^2,-q^{12}\right)}{f^2\left(-q^5,-q^9\right)}. \quad (2.17)$$

Adding (2.15), (2.16) and (2.17) and then using (1.5), we find that

$$\begin{aligned}&q^5\frac{f^2(-q^2,-q^{12})}{f^2(-q^5,-q^9)}+q^3\frac{f^2(-q^4,-q^{10})}{f^2(-q^3,-q^{11})}+q\frac{f^2(-q^6,-q^8)}{f^2(-q,-q^{13})}\\&=\frac{1}{24\psi^4(q^7)}[-P_1+P_2+7P_7-7P_{14}]=3q^2\frac{f_2^4f_7^2}{f_1^2f_{14}^4}+q\frac{f_1f_2f_7^5}{f_{14}^7}.\end{aligned}$$

We express the last part of the above equation in terms of α, β and m, where β has degree 7 over α and m is a multiplier connecting α and β [2, Entry 12, p. 124], we find that

$$\begin{aligned}&q^5\frac{f^2(-q^2,-q^{12})}{f^2(-q^5,-q^9)}+q^3\frac{f^2(-q^4,-q^{10})}{f^2(-q^3,-q^{11})}+q\frac{f^2(-q^6,-q^8)}{f^2(-q,-q^{13})}\\&=m\left[3q\left(\frac{\alpha}{\beta}\right)^{1/4}+2\left(\frac{\alpha}{\beta^3}\right)^{1/8}[(1-\alpha)(1-\beta)]^{1/4}\right].\end{aligned}$$

As q tends to 1^-, we know that α and β tend to 1 and $m \to 7$, we get the required result (2.2). □

Proof of (2.3). Setting $a = q^2, q^4$ and q^6 respectively in the equation (2.14) and adding the resulting equations, we find that

$$\begin{aligned}&q^2\frac{f^2(-q^5,-q^9)}{f^2(-q^2,-q^{12})}+q^4\frac{f^2(-q^3,-q^{11})}{f^2(-q^4,-q^{10})}+q^6\frac{f^2(-q,-q^{13})}{f^2(-q^6,-q^8)}\\&=\frac{1}{24\psi^4(q^7)}[-P_2-5P_{14}+6P_7]\end{aligned}$$

$$= \frac{1}{24\psi^4(q^7)}[-P_2 + 7P_{14} + 6(P_7 - 2P_{14})].$$

Using (1.5) and (1.6), we can write the RHS of the above equation as

$$= \frac{1}{24}\left\{24q^2\frac{\psi^2(q)}{\psi^2(q^2)} + 6\frac{\varphi^2(-q)\varphi^2(-q^2)}{\psi^4(q^2)} + 24q\frac{\varphi(-q)\varphi(-q^2)\psi(q)}{\psi^3(q^7)}\right.$$
$$\left. -6\left[16q^2\frac{\psi^4(q^{14})}{\psi^4(q^2)} + \frac{\varphi^4(q^2)}{\psi^4(q^2)}\right]\right\}.$$

When expressing this in terms of α, β and m [2, Entry 10, 11, pp. 122, 123], we get

$$= m\left\{q^{3/2}\left(\frac{\alpha}{\beta}\right)^{1/4} + q^{7/2}\left[\frac{(1-\alpha)(1-\beta)}{\beta}\right]^{1/2} + q^3\frac{[(1-\alpha)(1-\beta)\alpha]^{1/4}}{(\beta)^{3/8}}\right\}$$
$$-q^{7/2}(\beta)^{1/2} - q^{-7/2}(\beta)^{1/2}.$$

As q tends to 1^-, we know that α and β tend to 1 and $m \to 7$, we get the required result (2.3). □

Proof of (2.4). Replacing q by q^{18} and $b = q^9$ in (1.2), we find that

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^{18n}}{(1-aq^{18n})^2} - \frac{q^{18n+9}}{(1-q^{18n+9})^2}\right] = a\psi^4(q^9)\frac{f^2\left(-aq^9, \frac{-q^9}{a}\right)}{f^2\left(-a, \frac{-q^{18}}{a}\right)}. \quad (2.18)$$

Setting $a = q, q^3, q^5$ and q^7 respectively in the equation (2.18) and adding the resulting equations, we find that

$$q\frac{f^2(-q^8,-q^{10})}{f^2(-q,-q^{17})} + q^3\frac{f^2(-q^6,-q^{12})}{f^2(-q^3,-q^{15})} + q^4\frac{f^2(-q^4,-q^{14})}{f^2(-q^5,-q^{13})} + q^6\frac{f^2(-q^2,-q^{16})}{f^2(-q^7,-q^{11})}$$
$$= \frac{1}{24\psi^4(q^9)}[-P_1 + P_2 + 9P_2 - 9P_{18}].$$

Replacing q by q^3 in the equation (1.9) and then adding three times of the resultant equation with (1.9), we find that

$$\frac{1}{24q\psi^4(q^9)}[-P_1 + P_2 + 9P_9 - 9P_{18}] = \frac{\psi^2(q)\psi^2(q^3)}{\psi^4(q^9)} + 3q^2\frac{\psi^2(q^3)}{\psi^2(q^9)}.$$

Expressing the RHS of the above equation in terms of α, β, γ, m and m', where β and γ are of degree 3 and 9 respectively with respect to α and $m = \dfrac{z_1}{z_3}$ and $m' = \dfrac{z_3}{z_9}$, we find that

$$\frac{1}{24q\psi^4(q^9)}[-P_1 + P_2 + 9P_9 - 9P_{18}] = q^{7/2}m'\left[mm'\frac{(\alpha\beta)^{1/4}}{(\gamma)^{1/2}} + 3\left(\frac{\beta}{\gamma}\right)^{1/8}\right].$$

Hence by tending $q \to 1^-$, we have $\alpha \to 1, \beta \to 1$ and $m' \to 3$ and $mm' \to 9$ to get (2.4). $\square$

Proof of (2.5). Setting $a = q^2, q^4, q^6$ and q^8 respectively in the equation (2.18) and adding the resultant equations, we find that

$$q^8 \frac{f^2(-q,-q^{17})}{f^2(-q^8,-q^{10})} + q^6 \frac{f^2(-q^3,-q^{15})}{f^2(-q^6,-q^{12})} + q^4 \frac{f^2(-q^5,-q^{13})}{f^2(-q^4,-q^{14})} + q^2 \frac{f^2(-q^7,-q^{11})}{f^2(-q^2,-q^{16})}$$

$$= \frac{1}{24\psi^4(q^9)}[-P_1 - 7P_{18} + 8P_9] = \frac{1}{24\psi^4(q^9)}[-P_2 + 9P_{18} + 8(P_9 - 2P_{18})].$$

Using (1.6) and (1.8), we get

$$= \frac{1}{48}\left[\frac{\varphi^{10}(q^3)}{\varphi^3(q)\varphi^3(q^9)\psi^4(q^9)} + \frac{9\varphi(q)\varphi(q^9)\varphi^2(q^3)}{\psi^4(q^9)} + \frac{6\varphi^6(q^3)}{\varphi(q)\varphi(q^9)\psi^4(q^9)}\right]$$
$$-\frac{16}{3}q^9\psi^4(q^{18}) - \frac{1}{3}\varphi^4(q^9).$$

Let α, β, γ, m and m' be as in the proof of (2.4). Letting $q \to 1^-$, we get the required equation (2.5). $\square$

Proof of (2.6), (2.7), (2.8) and (2.9). Define $A = 3q\dfrac{f_2^4 f_7^2}{f_1^2 f_{14}^4} + \dfrac{f_1 f_2 f_7^5}{f_{14}^7}$,

$$B = q^{-4}\frac{\psi^2(q)}{\psi^2(q^7)} + \frac{q^{-6}}{4}\frac{[\varphi^2(-q)\varphi^2(-q^7) - \varphi^4(q^7)]}{\psi^4(q^7)}$$
$$+ q^{-5}\frac{\varphi(-q)\varphi(-q^7)\psi(q)}{\psi^3(q^7)} - 4q\frac{\psi^4(q^{14})}{\psi^4(q^7)}$$

and $C = q^6\dfrac{\psi^2(q)}{\psi^2(q^7)}$.

Using the formulas $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xyz\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$ and

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3xyz(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 3xyz,$$

we find the following equations:

$$q^8\frac{f^4(-q^2,-q^{12})}{f^4(-q^5,-q^9)} + q^4\frac{f^4(-q^4,-q^{10})}{f^4(-q^3,-q^{11})} + \frac{f^4(-q^6,-q^8)}{f^4(-q,-q^{13})} = A^2 - 2BC,$$

$$q^{-8}\frac{f^4(-q^5,-q^9)}{f^4(-q^2,-q^{12})} + q^{-4}\frac{f^4(-q^3,-q^{11})}{f^4(-q^4,-q^{10})} + \frac{f^4(-q,-q^{13})}{f^4(-q^6,-q^8)} = B^2 - \frac{2A}{C},$$

$$q^{12}\frac{f^6(-q^2,-q^{12})}{f^6(-q^5,-q^9)} + q^6\frac{f^6(-q^4,-q^{10})}{f^6(-q^3,-q^{11})} + \frac{f^6(-q^6,-q^8)}{f^6(-q,-q^{13})} = A^3 + 3C - 3ABC,$$

and

$$q^{-12}\frac{f^6(-q^5,-q^9)}{f^6(-q^2,-q^{12})}+q^{-6}\frac{f^6(-q^3,-q^{11})}{f^6(-q^4,-q^{10})}+\frac{f^6(-q,-q^{13})}{f^6(-q^6,-q^8)}=B^3+\frac{3}{C}-\frac{3AB}{C}.$$

Letting $q\to 1^-$ in the above equations, we get the required equations (2.6), (2.7), (2.8) and (2.9). □

We now find a recurrence relation for the sum of the kind

$$a(n)=tan^{-2n}\left(\frac{\pi}{14}\right)+tan^{-2n}\left(\frac{3\pi}{14}\right)+tan^{-2n}\left(\frac{5\pi}{14}\right),$$

where $n\in\mathbb{Z}$ and in general, we prove that $a(n)$ is a positive integer for all $n\geq 0$ and $a(n)$ is a positive integer divided by some power of 7 for all $n<0$.

We first prove the following theorem:

Theorem 2.2. *If* $x=tan^{-2}\left(\frac{\pi}{14}\right), y=tan^{-2}\left(\frac{3\pi}{14}\right)$ *and* $z=tan^{-2}\left(\frac{5\pi}{14}\right)$ *then* $xyz=7$.

Proof. Consider

$$\frac{f^2(-q^6,-q^8)}{f^2(-q,-q^{13})}\times\frac{f^2(-q^4,-q^{10})}{f^2(-q^3,-q^{11})}\times\frac{f^2(-q^2,-q^{12})}{f^2(-q^5,-q^9)}=\frac{\psi^2(q)}{\psi^2(q^7)}=m\,q^{3/2}\left(\frac{\alpha}{\beta}\right)^{1/4} \tag{2.19}$$

As q tends to 1^-, we know that α and β tend to 1 and $m\to 7$. Taking q tends to 1^- on both sides of the equation (2.19), and using equation (1.10), we obtain the required result. □

We now move into the recurrence relation for $a(n)$.

Theorem 2.3. *Let*

$$a(n):=tan^{-2n}\left(\frac{\pi}{14}\right)+tan^{-2n}\left(\frac{3\pi}{14}\right)+tan^{-2n}\left(\frac{5\pi}{14}\right),$$

and

$$c(n):=7\left[a(n-1)\,a(-1)-a(n-2)\right],\ \forall\, n\geq 1.$$

Then

$$(a(1))^n=a(n)+\sum_{k=1}^{n-1}c(k)\,[a(1)]^{n-k-1}. \tag{2.20}$$

Proof. Let $x=tan^{-2}\left(\frac{\pi}{14}\right), y=tan^{-2}\left(\frac{3\pi}{14}\right)$ and $z=tan^{-2}\left(\frac{5\pi}{14}\right)$.
By the elementary formula,

$$(x+y+z)^2=x^2+y^2+z^2+2xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

and the fact that $xyz = 7$, it easily follows that (2.20) holds for $n = 2$. Assume that (2.20) holds for all $3 \leq k \leq (n-1)$.

Consider $(x+y+z)^n = (x+y+z)^{n-1}(x+y+z)$

$$= \left[a(n-1) + \sum_{k=1}^{n-2} c(k)\,[a(1)]^{n-k-2}\right](x+y+z)$$

$$= a(n-1)\ (x+y+z) + \sum_{k=1}^{n-2} c(k)\,[a(1)]^{n-k-1} \tag{2.21}$$

Since $a(n-1)\ (x+y+z) = \left(x^{n-1}+y^{n-1}+z^{n-1}\right)(x+y+z)$

$$= (x^n+y^n+z^n) + \left(x^{n-1}+y^{n-1}+z^{n-1}\right) xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) - \left(x^{n-2}+y^{n-2}+z^{n-2}\right) xyz$$

$$= a(n) + c(n-1),$$

equation (2.21) becomes equation (2.20) and this completes the proof. □

Theorem 2.4. *Let*

$$d(n) := \frac{1}{7}\left[a(1-n)\,a(1) - a(2-n)\right], \forall\, n \geq 1.$$

Then

$$(a(-1))^n = a(-n) + \sum_{k=1}^{n-1} d(k)\,[a(-1)]^{n-k-1}. \tag{2.22}$$

Proof of this is similar to the proof of the above Theorem 2.3.

Corollary 2.5. *We have*

(1) For $n \geq 0$, $a(n)$ are all positive integers.
(2) For $n > 0$, $a(-n)$ are positive integers divided by some power of 7.

Proof. Clearly $a(-1), a(0)$ and $a(1)$ are integers. Now by mathematical induction, and the above two recurrence relations (2.20) and (2.22), the corollary follows. □

Remark

Berndt and B. P. Yeap [4] have generalized the sum $a(-n)\,(n>0)$ for any base other than 7. Their generalized formula, which is obtained through contour integration technique, involves Bernoulli numbers.

Acknowledgement: The authors would like to thank the anonymous referee for his/her valuable suggestions and motivating us to prove Theorem 2.3, 2.4 and Corollary 2.5.

The authors would like to thank Prof. Z. -G. Liu and Prof. K. R. Vasuki for their valuable suggestions during the preparation of this article.

Harshitha K. N. is supported by grant UGC-Ref. No.:982/(CSIR-UGC NET DEC.2017) by the funding agency UGC, INDIA, under CSIR-UGC JRF.

Yathirajsharma M. V. is supported by grant No.09/119(0221)/2019-EMR-1 by the funding agency CSIR, INDIA, under CSIR-JRF.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 201–208

SOME PROPERTIES OF A SEMICONFORMAL CURVATURE TENSOR ON A RIEMANNIAN MANIFOLD

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(Received : 10 - 07 - 2020 ; Revised : 12 - 05 - 2021)

Abstract. The purpose of the present paper is to study a semiconformally flat on the Riemannian manifolds with a perfect fluid spacetime that satisfies Einstein's field equation without cosmological constant. Many geometric features related to weakly semiconformally symmetric manifolds have been studied with the weakly symmetric and the weakly Ricci symmetric manifolds.

1. Introduction

Kim [8] introduced a curvature-like tensor called the semiconformal curvature tensor such that its $(1,3)$ components remain invariant under conharmonic transformation and the melodic of the harmonic semiconformal curvature tensor has yielded several results then the year was 2017. The semiconformal curvature tensor $\mathcal{P}$ of type $(1,3)$ on a Riemannian manifold (M^n, g) is defined by the tensor

$$\mathcal{P}(X,Y)Z = -(n-2)b\,\mathcal{C}(X,Y)Z + [a+(n-2)b]\mathcal{H}(X,Y)Z, \tag{1.1}$$

where a, b are constant not simultaneously zero, $\mathcal{C}$ and $\mathcal{H}$ are the conformal curvature tensor of type $(1,3)$ and the conharmonic curvature tensor $\mathcal{H}$ of type $(1,3)$ respectively are defined by the tensors

$$\begin{aligned}\mathcal{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ -g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{1.2}$$

2010 Mathematics Subject Classification: 53C15, 53C25
Key words and phrases: Semiconformal curvature tensor, Weakly symmetric manifold, Pseudo symmetric manifold, Einstein manifold

and

$$\mathcal{H}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \quad (1.3)$$

where r is the scalar curvature and Q denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is $g(QX,Y) = S(X,Y)$.

From (1.1), (1.2) and (1.3), we can conclude it

$$\mathcal{P}(X,Y)Z = a\ \mathcal{H}(X,Y)Z - \frac{br}{n-1}[g(Y,Z)X - g(X,Z)Y]. \quad (1.4)$$

Also (1.4), we can write that

$$\tilde{\mathcal{P}}(Y,Z,U,V) = a\ \tilde{R}(Y,Z,U,V) - \frac{a}{n-2}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) + S(Y,V)g(Z,U) - S(Z,V)g(Y,U)] - \frac{br}{n-1}[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)], \quad (1.5)$$

where $\tilde{\mathcal{P}}(Y,Z,U,V) = g(\mathcal{P}(Y,Z)U,V)$ and $\tilde{R}(Y,Z,U,V) = g(R(Y,Z)U,V)$ are the semiconformal curvature tensor of type $(0,4)$ and the curvature tensor of the manifold of type $(0,4)$ respectively.

Chaki [5] studied that a non flat Riemannian manifold (M^n, g), $n \geq 2$ is considered a *pseudo symmetric manifold* [5] if its curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the tensor type in 1987

$$(\nabla_X\tilde{R})(Y,Z,U,V) = 2A(X)\tilde{R}(Y,Z,U,V) + A(Y)\tilde{R}(X,Z,U,V) + A(Z)\tilde{R}(Y,X,U,V) + A(U)\tilde{R}(Y,Z,X,V) + A(V)\tilde{R}(Y,Z,U,X),$$

where the 1-form A and a vector field ρ_1 is given by $g(X,\rho_1) = A(X)$. The pseudo-Ricci symmetric manifold can be identified by the following equation [5]

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y).$$

In 1989, Tamassy and Binh [12] investigated that a non flat Riemannian manifold (M^n, g), $n > 3$ is considered a *weakly symmetric manifold* [12] if there exist 1-forms A, B, D, E and F present that met the condition

$$\begin{aligned}(\nabla_X \tilde{R})(Y,Z,U,V) = A(X)\tilde{R}(Y,Z,U,V) + B(Y)\tilde{R}(X,Z,U,V)\\ +D(Z)\tilde{R}(Y,X,U,V) + E(U)\tilde{R}(Y,Z,X,V)\\ +F(V)\tilde{R}(Y,Z,U,X).\end{aligned} \tag{1.6}$$

And the weakly-Ricci symmetric manifold [12] can be identified by the following condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(X,Y). \tag{1.7}$$

In a recent research paper, De and Suh [7] has done research, "on *weakly semiconformally symmetric manifold*". A manifold is a the weakly semiconformally symmetric manifolds (M^n, g) if the semiconformal curvature tensor $\tilde{\mathcal{P}}$ of type $(0,4)$ the synthesis is as follows [7]

$$\begin{aligned}(\nabla_X \tilde{\mathcal{P}})(Y,Z,U,V) = A(X)\tilde{\mathcal{P}}(Y,Z,U,V) + B(Y)\tilde{\mathcal{P}}(X,Z,U,V)\\ +D(Z)\tilde{\mathcal{P}}(Y,X,U,V) + E(U)\tilde{\mathcal{P}}(Y,Z,X,V)\\ +F(V)\tilde{\mathcal{P}}(Y,Z,U,X).\end{aligned} \tag{1.8}$$

The semiconformal curvature tensor $\tilde{\mathcal{P}}$ of type $(0,4)$ satisfies the condition [7]

$$\begin{aligned}(\nabla_X \tilde{\mathcal{P}})(Y,Z,U,V) = 2A(X)\tilde{\mathcal{P}}(Y,Z,U,V) + A(Y)\tilde{\mathcal{P}}(X,Z,U,V)\\ +A(Z)\tilde{\mathcal{P}}(Y,X,U,V) + A(U)\tilde{\mathcal{P}}(Y,Z,X,V)\\ +A(V)\tilde{\mathcal{P}}(Y,Z,U,X),\end{aligned} \tag{1.9}$$

then the Riemannian manifolds (M^n, g), $n \geq 4$ is the *pseudo semiconformally symmetric manifolds.* The semiconformal curvature tensor also studied by Ali and Pundeer [1], Ali, Pundeer and Suh [2] and many others.

A reminiscence carries many Weakly symmetric structures on a Riemannian manifold (M^n, g) have been studied by Barman ([3], [4]), Ozen and Altay [10], De and Mallik [6] and many others.

Inspired by the above research in the present research paper, we are featured some geometric properties of the semiconformal curvature tensor on a Riemannian manifold.

The present paper is organized as follows : After introduction in section 2, the semiconformal curvature tensor on a Riemannian manifold vanishes and a perfect fluid spacetime satisfies Einstein's field equation without cosmological constant have been studied. Finally, we deals with some geometric properties of weakly semiconformally symmetric manifolds with the weakly symmetric and the weakly Ricci symmetric manifolds.

2. Semiconformally flatness and without cosmological constant spacetime on Riemannian manifold

Definition 2.1. A Riemannian manifold (M^n, g) is said to have Einstein manifold if its Ricci tensor S [9] satisfies the condition

$$S(X, Y) = \lambda_1 g(X, Y), \tag{2.1}$$

where λ_1 is the scalar function.

Theorem 2.2. *If the semiconformal curvature tensor vanishes in a Riemannian manifold, then the manifold is the Einstein manifold.*

Proof. Putting $\tilde{\mathcal{P}}(Y, Z, U, V) = 0$ in (1.5), we implies that

$$\begin{aligned} a\ \tilde{R}(Y, Z, U, V) = {} & \frac{a}{n-2}[S(Z, U)g(Y, V) - S(Y, U)g(Z, V) + S(Y, V)g(Z, U) \\ & -S(Z, V)g(Y, U)] + \frac{br}{n-1}[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)]. \end{aligned} \tag{2.2}$$

Taking the contrations of (2.2), it follows that

$$S(Z, U) = -\frac{(a + nb - 2b)r}{2a} g(Z, U). \tag{2.3}$$

Comparing (2.1) and (2.3), we have

$$\lambda_1 = -\frac{(a + nb - 2b)r}{2a},$$

that means, the manifold of semiconformal curvature tensor is the Einstein manifold. The proof is completed. □

Theorem 2.3. *If the semiconformally flat in a Riemannian manifold and a perfect fluid spacetime satisfies Einstein's field equation without cosmological constant, then the scalar curvature of the manifold is equal to* $-\frac{2[n\kappa p + \kappa(\sigma + p)\pi(\rho)]}{a(2n - nb + 2b)}$.

Proof. The General relativity given by Einstein's equation [9] is the lower form of flow

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \lambda g(X,Y) = \kappa T(X,Y), \tag{2.4}$$

where $S(X,Y)$ is the Ricci tensor of type $(0,2)$ of the spacetime, r is the scalar curvature, $T(X,Y)$ is the energy-momentum tensor of type $(0,2)$, λ is the cosmological constant and κ is the gravitational constant. Einstein's gave the equation without the cosmological constant as follows

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa T(X,Y). \tag{2.5}$$

From Einstein's equations (2.4) and (2.5), we conclude that "matter determines the geometry of spacetimes and conversely that the motion of matter is determined by the metric tensor of the space which is not fiat [11]".

The energy-momentum tensor describes a Perfect fluid [9] if

$$T(X,Y) = (\sigma + p)\pi(X)\pi(Y) + \rho g(X,Y), \tag{2.6}$$

where σ is the energy density and p is the isotropic pressure of the fluid.

Ali and Pundeer [1] proved that for Einstein's field equation with cosmological constant are as follows:

1. A semiconformally-flat spacetime is an Einstein-like space [1].

2. A semiconformally-flat spacetime is an Einstein space for $\sigma = 0$ and λ non-zero constant [1].

3. A semiconformally-flat spacetime is of constant curvature [1].

Combining (2.5) and (2.6), it implies that

$$S(X,Y) - (\frac{1}{2}r + \kappa p)g(X,Y) = \kappa(\sigma + p)\pi(X)\pi(Y). \tag{2.7}$$

Adding (2.3) and (2.7), it yields that
$r = -\frac{2[n\kappa p + \kappa(\sigma+p)\pi(\rho)]}{a(2n-nb+2b)}$. So, the Theorem 2.3 is proved. □

3. SOME PROPERTIES OF WEAKLY SEMICONFORMALLY SYMMETRIC MANIFOLDS

Theorem 3.1. *If the Ricci tensor vanishes on a Riemannian manifold, then the weakly semiconformally symmetric manifold is a weakly symmetric manifold.*

Proof. Putting (1.5) in (1.8), we obtain

$$\begin{aligned}
a(\nabla_X\tilde{R})(Y,Z,U,V) - \frac{a}{n-2}[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)\\
+(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)] = A(X)[a\tilde{R}(Y,Z,U,V)\\
-\frac{a}{n-2}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) + S(Y,V)g(Z,U)\\
-S(Z,V)g(Y,U)] - \frac{br}{n-1}[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)]]\\
+B(Y)[a\tilde{R}(X,Z,U,V) - \frac{a}{n-2}[S(Z,U)g(X,V) - S(X,U)g(Z,V)\\
+S(X,V)g(Z,U) - S(Z,V)g(X,U)] - \frac{br}{n-1}[g(Z,U)g(X,V)\\
-g(X,U)g(Z,V)]] + D(Z)[a\tilde{R}(Y,X,U,V) - \frac{a}{n-2}[S(X,U)g(Y,V)\\
-S(Y,U)g(X,V) + S(Y,V)g(X,U) - S(X,V)g(Y,U)]\\
-\frac{br}{n-1}[g(X,U)g(Y,V) - g(Y,U)g(X,V)]] + E(U)[a\tilde{R}(Y,Z,X,V)\\
-\frac{a}{n-2}[S(Z,X)g(Y,V) - S(Y,X)g(Z,V) + S(Y,V)g(Z,X)\\
-S(Z,V)g(Y,X)] - \frac{br}{n-1}[g(Z,X)g(Y,V) - g(Y,X)g(Z,V)]]\\
+F(V)[a\tilde{R}(Y,Z,U,X) - \frac{a}{n-2}[S(Z,U)g(Y,X) - S(Y,U)g(Z,X)\\
+S(Y,X)g(Z,U) - S(Z,X)g(Y,U)] - \frac{br}{n-1}[g(Z,U)g(Y,X)\\
-g(Y,U)g(Z,X)]].
\end{aligned} \tag{3.1}$$

If the Ricci tensor (S) vanishes, that means, the scalar curvature also vanishes of the manifold. Then (3.1) we can to write

$$\begin{aligned}
(\nabla_X\tilde{R})(Y,Z,U,V) = A(X)\tilde{R}(Y,Z,U,V) + B(Y)\tilde{R}(X,Z,U,V)\\
+D(Z)\tilde{R}(Y,X,U,V) + E(U)\tilde{R}(Y,Z,X,V)\\
+F(V)\tilde{R}(Y,Z,U,X),
\end{aligned}$$

Which is the weakly symmetric manifolds on the Riemannian manifolds. The Theorem 3.1 is proved. □

Theorem 3.2. *If the weakly semiconformally symmetric manifold satisfies the weakly symmetric and the weakly Ricci-symmetric conditions, then the scalar curvature of the manifold is equal to* $-\frac{a[(n-1)Q\rho_3+Q\rho_5+2Q\rho_4]}{b(n-2)[\rho_1-n\rho_2-\rho_3+\rho_5]-a[(n+1)\rho_2+\rho_5]}$.

Proof. Using (1.6) and (1.7), we have from (3.1)

$$\begin{aligned}\frac{a}{n-2}&[B(Y)S(Z,U)g(X,V)+B(Y)S(Z,V)g(X,U)+D(Z)S(Y,U)g(X,V)\\&+D(Z)S(Y,V)g(X,U)+E(U)S(Y,V)g(Z,X)+E(U)S(Z,V)g(X,Y)\\&+F(V)S(Z,U)g(X,Y)+B(V)S(Y,U)g(X,Z)]+\frac{br}{n-1}[A(X)g(Z,U)g(Y,V)\\&-A(X)g(Y,U)g(Z,V)+B(Y)g(Z,U)g(X,V)-B(Y)g(X,U)g(Z,V)\\&+D(Z)g(X,U)g(Y,V)-D(Z)g(Y,U)g(X,V)+E(U)g(Z,X)g(Y,V)\\&-E(U)g(X,Y)g(Z,V)+F(V)g(Z,U)g(X,Y)\\&-F(V)g(Y,U)g(Z,X)]=0\end{aligned}\qquad(3.2)$$

Contracting (3.2) over X and V, we obtain

$$\begin{aligned}\frac{a}{n-2}&[\{(n+1)B(Y)+F(Y)\}S(Z,U)+\{(n+1)D(Z)+F(Z)\}S(Y,U)\\&+2E(U)S(Y,Z)]+\frac{br}{n-1}[\{A(Y)+(n-1)B(Y)+F(Y)\}g(Z,U)\\&+\{(n-1)D(Z)-A(Z)\}g(Y,U)-F(Z)g(Y,U)]=0\end{aligned}\qquad(3.3)$$

Again contracting (3.3) over Z and U, we conclude that

$$\begin{aligned}\frac{a}{n-2}&[r\{(n+1)B(Y)+F(Y)\}+(n+1)S(Y,\rho_3)+S(Y,\rho_5)+2S(Y,\rho_4)]\\&+\frac{br}{n-1}[n\{A(Y)+(n-1)B(Y)+F(Y)\}+(1-n)D(Y)-A(Y)\\&-F(Y)]=0\end{aligned}\qquad(3.4)$$

where $D(Z)=g(Y,\rho_3)$, $E(U)=g(Y,\rho_4)$ and $F(V)=g(Y,\rho_5)$.
Further contracting Y, we decide that

$$r=-\frac{a[(n-1)Q\rho_3+Q\rho_5+2Q\rho_4]}{b(n-2)[\rho_1-n\rho_2-\rho_3+\rho_5]-a[(n+1)\rho_2+\rho_5]},\qquad(3.5)$$

where $S(X,Y) = g(QX,Y)$, $A(X) = g(X,\rho_1)$ and $B(Y) = g(Y,\rho_2)$. The proof is completed. □

Theorem 3.3. *If the weakly semiconformally symmetric manifold satisfies the weakly symmetric and the weakly Ricci-symmetric conditions, then the scalar curvature of the manifold vanishes provided one constantof the semiconformal curvature tensor $a = 0$.*

Proof. If $a = 0$ in (3.5), then $r = 0$. Proved the Theorem 3.3. □

Acknowledgement: The author is grateful to the referee for the comments which improved the quality of the paper.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 209–217

TRANSCENDENTAL NUMBERS

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(Received : 1 - 08 - 2021 ; Revised : 26 - 08 - 2021)

Abstract: These notes are based on a talk I gave at a webinar organized by Vimala College, Trissur, Kerala, in December 2020. This is an exposition of some basic results in the theory of transcendental numbers. No originality is claimed and no effort was taken to make this up to date. The aim is to arouse curiosity in the reader about the rich theory of transcendental numbers. She or he should consult textbooks and research articles to explore and gain deeper understanding of the subject.

1. Introduction

A complex number α is called an *algebraic number* if it is the root of a non-constant polynomial over the field of rational numbers $\mathbb{Q}$ (i.e., with coefficients in $\mathbb{Q}$). We say that $\alpha \in \mathbb{C}$ is *transcendental* if it is not an algebraic number. Equivalently, $\alpha \in \mathbb{C}$ is transcendental if it is not a root of any non-constant polynomial over $\mathbb{Q}$.

If α, β are algebraic, then so are $\alpha + \beta, \alpha\beta$ and, if $\beta \neq 0$, then α/β is also algebraic. These statements follow easily using basic facts from field theory, as we shall now explain.

First, recall that if $\alpha \in \mathbb{C}$, and if $F \subset \mathbb{C}$ is a subfield, then $F[\alpha]$ is the subring of $\mathbb{C}$ that consists of all complex numbers of the form $\sum_{0\leq k\leq r} a_j\alpha^j$ where the a_j vary over F and r varies over non-negative integers.

If X denotes an indeterminate, we have a unique homomorphism of rings $\phi : \mathbb{Q}[X] \to \mathbb{Q}[\alpha]$ that sends X to α. If α is not *algebraic over* F, that is, α is not the root of any non-zero polynomial over F, then ϕ is an isomorphism of rings. On the other hand, suppose that α is algebraic over F. Then ϕ is not one-to-one. For, if α is a root of a polynomial $f(X) \in \mathbb{Q}[X]$, then $\phi(f(X)) = f(\alpha) = 0$. Since $F[X]/\ker\phi \cong F[\alpha]$ is an integral domain, $\ker(\phi)$ is a prime ideal. Since $F[X]$ is a PID, $\ker(\phi)$ is a maximal ideal.

2010 Mathematics Subject Classification: 11-01, 11J 99
Key words and phrases: transcendental numbers

Therefore $F[\alpha]$ is a field and equals $F(\alpha)$, the smallest subfield of $\mathbb{C}$ that contains F as well as α. (The elements of $F(\alpha)$ are of the form $a(\alpha)/b(\alpha)$ where $a(X), b(X) \in F[X]$ with $b(\alpha) \neq 0$.)

Let $h(X)$ be a generator of $\ker(\phi)$. (The generators of $\ker(\phi)$ are unique up to scalar multiples of each other. So there is a unique monic polynomial that generates $\ker(\phi)$.) Then $h(X)$ is irreducible and its degree is the smallest among all polynomials over F of which α is a root. It is readily verified that the $\dim_F F[\alpha]$ equals the degree of $h(X)$ and is called the *degree of* $F[\alpha]$ *over* F. When $F = \mathbb{Q}$, $\deg_{\mathbb{Q}}[\alpha]$ is called the degree of the algebraic number α.

If E is a field that contains F then we may regard E as a vector space over F and the dimension $\dim_F E$ of E over F is called the *degree* of E over F, denoted $\deg_F E$. It is readily seen that if $F \subset E \subset K$ where K is also a field, then $\deg_F K = \deg_E K . \deg_F E$. In particular, if $\deg_F E$ and $\deg_E K$ are finite, then so is $\deg_F K$. Conversely, if $\deg_F K < \infty$, then so are $\deg_F E$ and $\deg_E K$.

Suppose that $\alpha, \beta \in \mathbb{C}$ are algebraic. Then, by the above discussion, $\deg_{\mathbb{Q}} \mathbb{Q}(\alpha), \deg_{\mathbb{Q}} \mathbb{Q}(\beta)$ are finite. Also, since β is algebraic (over $\mathbb{Q}$), it is algebraic over $\mathbb{Q}(\alpha)$. Hence, writing $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha)(\beta)$, we have $\deg_{\mathbb{Q}(\alpha)} \mathbb{Q}(\alpha, \beta)$ is finite. It follows that $\deg_{\mathbb{Q}} \mathbb{Q}(\alpha, \beta)$ is finite by the observation in the previous paragraph. Hence any $\gamma \in \mathbb{Q}(\alpha, \beta)$ is algebraic in view of the fact that $\mathbb{Q} \subset \mathbb{Q}(\gamma) \subset \mathbb{Q}(\alpha, \beta)$. It follows that $\alpha + \beta, \alpha\beta, \alpha/\beta$ (when $\beta \neq 0$) are algebraic.

From the above discussion we see that set $\overline{\mathbb{Q}} \subset \mathbb{C}$ of all algebraic number is a subfield of $\mathbb{C}$.

Denote by $\mathcal{T} \subset \mathbb{C}$ the set of all transcendental numbers and by $\mathcal{T}_0$ the set $\mathcal{T} \cap \mathbb{R}$. Note that there are only countably many polynomials in X with coefficients in $\mathbb{Q}$. Since each polynomial of degree n has at most n roots (ignoring multiplicity!) in $\mathbb{C}$, we can write $\overline{\mathbb{Q}}$ as countable union of finite sets. Hence $\overline{\mathbb{Q}}$ is countable. Since $\mathbb{C}, \mathbb{R}$ are uncountable, it follows that $\mathcal{T}$ and $\mathcal{T}_0$ are uncountable. Also, $\mathcal{T}$ and $\mathcal{T}_0$ have full Lebesgue measures and everywhere dense residual sets. Thus these sets are *large*. It is surprising that, in spite of the largeness of the sets $\mathcal{T}, \mathcal{T}_0$, it is in general a difficult problem to write down families of (infinitely many) transcendental numbers. It is even more difficult to decide whether a given number is transcendental.

Unlike $\overline{\mathbb{Q}}$, the set of transcendental numbers is not closed under addition or multiplication: For example, if α is transcendental, so is $1-\alpha$ and their sum is 1 and $\alpha.\alpha^{-1}=1$. However, $\mathcal{T}$ is closed under multiplication by non-zero algebraic numbers and translation by algebraic numbers. That is, if $\lambda,\mu\in\overline{\mathbb{Q}},\lambda\neq 0$, and if $\alpha\in\mathcal{T}$, then $\lambda\alpha+\mu\in\mathcal{T}$. Also, performing algebraic operation such as taking square roots, cube roots, etc (repeatedly) on a transcendental number result in transcendental numbers. More precisely, if $\alpha\in\mathbb{C}, P(X)\in\overline{\mathbb{Q}}[X]$ such that $P(\alpha)\in\mathcal{T}$, then $\alpha\in\mathcal{T}$. (If $\alpha\in\overline{\mathbb{Q}}$, then $P(\alpha)\in\overline{\mathbb{Q}}$, a contradiction to our assumption that $P(\alpha)\in\mathcal{T}$.) If $\sigma:\mathbb{C}\to\mathbb{C}$ is any field automorphism, then $\sigma(\mathcal{T})=\mathcal{T}$. (Note that $\sigma(\mathbb{Q})=\mathbb{Q}$ and so $\sigma(\overline{\mathbb{Q}})=\overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}},\mathcal{T}$ form a partition of $\mathbb{C}$, the assertion follows.)

Well-known examples of transcendental numbers are e (the base of the natural logarithm) and π. The transcendence of e was established by Hermite in 1873 and that of π by Lindemann who actually proved that if $\alpha\neq 0$ is algebraic, then e^{α} is transcendental. Since $e^{i\pi}=-1$ is algebraic, it follows that $i\pi$ is transcendental. Since i is algebraic, it follows that π is transcendental. Lindemann's theorem also readily implies that $\log m$ is transcendental for all $m\in\mathbb{N}, m>1$.

It is a lesser known fact that $2^{\sqrt{2}}$ and the Liouville number $\sum_{n\geq 1}10^{-n!}$ are transcendental. These are special instances of more general results on transcendental numbers, which we discuss next.

2. Liouville numbers

J. Liouville proved, in 1844, an important result that led him to exhibit plenty of transcendental numbers. The proof of the result involves nothing more than mean value theorem.

If $\alpha\in\overline{\mathbb{Q}}$, the degree of α, written $\deg\alpha$, equals $\deg h(X)$ where $h(\alpha)=0$ and $h(X)\in\mathbb{Q}[X]$ irreducible.

Theorem 2.1. *Suppose that α is a real algebraic number of degree $n\geq 2$. Then there exists a constant $C=C(\alpha)>0$ such that $|\alpha-p/q|>C/q^n$ for any rational number p/q where $p,q\in\mathbb{Z}$, $q\geq 1$.*

Before we prove the theorem, we note that if $\alpha=a/b\in\mathbb{Q},(a,b\in\mathbb{Z},b>0)$, then we may take p/q to be α and there cannot be a $C>0$ as in the theorem. However, for any other rational number p/q, we see that $|\alpha-p/q|=|aq-bp|/bq>C/q$ where $C:=1/2b$ since $|aq-bp|\geq 1$. When $\deg(\alpha)>1$, then α is irrational so the possibility $\alpha=p/q$ is ruled out.

Proof. Let $h(X) = \sum_{0\leq j\leq n} a_jX^j$ be an irreducible polynomial such that $h(\alpha) = 0$ and $\alpha \notin \mathbb{Q}$. We may (and do) assume that the coefficients a_j of $h(X)$ are all *integers*. Since $h(X)$ is irreducible, $a_0 \neq 0$ and since $\deg \alpha = n > 1$, α is not a rational number. By clearing the denominators, we may assume that $h(X)$ has integer coefficients. We have $h(\alpha) = \sum a_j\alpha^j = 0$.

We may (and do) choose p/q to be in the interval $J := [\alpha - 1, \alpha + 1]$.

We know that $h(p/q) \neq 0$ since $h(X)$, being irreducible, has no rational roots. (Otherwise $h(X)$ would have to be a constant multiple of the linear polynomial $X - p/q$ which is a contradiction since $\deg h(X) = n > 1$ and $h(X)$ is irreducible.) So, by mean value theorem, $0 \neq h(p/q) = h(p/q) - h(\alpha) = (p/q - \alpha).h'(\beta)$ for a suitable β in the open interval with end points $p/q, \alpha$. Therefore we have

$$0 < \frac{|h(p/q)|}{|h'(\beta)|} = |\alpha - p/q|. \tag{1}$$

Let $M = \max_{t\in J} |h'(t)|$. Then $0 < |h'(\beta)| \leq M$ since $\beta \in J$ as both $p/q, \alpha \in J$. Note that the value of M depends only on α (having fixed $h(X)$).

We now obtain a lower bound for $|h(p/q)|$. We have

$$|h(p/q)| = |\sum a_jp^j/q^j| = q^{-n}|\sum a_jp^jq^{n-j}| \geq q^{-n}$$

since the a_j are integers and $\sum a_jp^jq^{n-j}$ is a non-zero *integer*. So, from (1) we obtain that

$$|\alpha - p/q| = |h(p/q)|/|h'(\beta) \geq q^{-n}/M. \tag{2}$$

Let $C := M + 1$. Then for *any* $p/q \in \mathbb{Q}$ we have $|\alpha - p/q| > C/q^n$. $\square$

Definition 2.2. A real number α is called a *Liouville number*, if there exists an infinite sequence of rational numbers $(p_n/q_n)_{n\geq1}$, $p_n, q_n \in \mathbb{Z}$ with $q_n \geq 2$, such that $0 < |\alpha - p_n/q_n| < 1/q_n^n$.

Denote by $\mathcal{L}$ the set of all Liouville numbers. As an application of the above theorem, we have

Theorem 2.3. *Any Liouville number is transcendental.*

Proof. Let $\alpha \in \mathcal{L}$. First we show that α is irrational. Suppose, on the contrary, that α is rational, say $\alpha = p_0/q_0$ in least form, where $p_0, q_0 \in \mathbb{Z}, q_0 \geq 1$. Let $p, q \in \mathbb{Z}$ and $\alpha \neq p/q$. Suppose that $q \geq q_0 + 1$. Then we have $|\alpha - p/q| = |p_0q - pq_0|/q_0q > 1/q^2$.

Let p_n/q_n is an infinite sequence of rational number with $p_n, q_n \in \mathbb{Z}, q \geq 1$, such that $0 < |\alpha - p_n/q_n| < 1/q_n^n$. Then $\alpha \neq p_n/q_n$ for any n. We choose $n \geq 2$ and so that $q_n > q_0$. As has already been observed we have $|\alpha - p_n/q_n| > 1/q_n^2$. Hence we have $1/q_n^2 < |\alpha - p_n/q_n| < 1/q_n^n$, i.e., $1/q_n^2 < 1/q_n^n$, which is a contradiction. So α is not a rational number.

Now if α is algebraic, then $N := \deg(\alpha) \geq 2$ since α is not rational. By Theorem 2.1, there exists a $C > 0$ such that $|\alpha - p/q| > C/q^N$ for any rational number $p/q, p, q \in \mathbb{Z}, q \geq 2$. Since there are at most finitely many rational numbers with denominator $1 \leq q \leq 1/C$, we can find an $n \geq N+1$ large so that $q_n > \max\{2, 1/C\}$. Now $C/q_n^N < |\alpha - p_n/q_n| < 1/q_n^n$ implies that $C/q_n^N < 1/q_n^n \leq 1/q_n^{N+1}$ which implies that $q_n < 1/C$, a contradiction. $\square$

Let $b > 1$ be any integer. Let $(a_k)_{k\geq 1}$ be any sequence of integers where $0 \leq a_k < b$ for all k with infinitely many a_k being non-zero. Consider the series $\sum_{k\geq 1} a_k/b^{k!}$. A comparision with the geometric series $\sum(b-1)/b^k$ shows that that the series converges to a real number, say α. We claim that α is transcendental. We will show that α is a Liouville number. Let s_n be the partial sum $\sum_{1\leq k\leq n} a_k/b^{k!} = p_n/b^{n!} \in \mathbb{Q}$ for some positive integer p_n. We note that p_n is not divisible by b if a_n is non-zero. So the fraction $p_n/b^{n!}$ is in least form. Then $\alpha - s_n = \sum_{k>n} a_k/b^{k!} < (b-1)b^{-(n+1)!}(\sum_{r\geq 0} b^{-r}) = (b-1)b^{-(n+1)!}\frac{b}{b-1} = b^{1-(n+1)!}$. Note that $(n+1)! - 1 > n.n!$ if $n \geq 2$. So setting $q_n = b^{n!}$ we have $0 < |\alpha - p_n/q_n| < q_n^{-n}$ whenever $a_n \neq 0$. This proves that α is a Liouville number. .

Observe that if $b \geq 2$, there are uncountably many sequences (a_n) such that $a_n \in \{0, 1, \ldots, b-1\}$ with the property that infinitely many of the a_n are non-zero. Therefore $\mathcal{L}$ is uncountable.

Although $\mathcal{L}$ is uncountable, it can be shown that the Lebesgue measure of $\mathcal{L} \subset \mathbb{R}$ is zero. So most transcendental numbers are not Liouville numbers.

3. Gelfond-Schneider theorem

A. O. Gelfond and T. Schneider independently proved around 1934 the following result, thereby settling the Hilbert's seventh problem.

Before we state the result, recall that if $\alpha, \beta \in \mathbb{C}$ and if $\alpha \neq 0$ and β is not an integer, then α^β is defined as $e^{\beta \log \alpha} \in \mathbb{C}$. [1] Since log is multivalued,

[1]The base of log is understood to be e unless indicated otherwise.

we see that α^β is multivalued. For example, $1^{1/3}$ has three values, namely, $1, (1 \pm i\sqrt{3})/2$. As another example, $i = e^{i\pi/4+2ki\pi}, k \in \mathbb{Z}$. Therefore $i^i = \{e^{-\pi/4-2k\pi} \mid k \in \mathbb{Z}\}$. However, when $\alpha, \beta > 0$, then there exists a unique positive number $\alpha^\beta \in \mathbb{R}$ although α^β is multivalued as a complex number (except when β is an integer).

Theorem 3.1. (A. O. Gelfond and T. Schneider) *Let $\alpha \neq 0, 1$ be algebraic and let β be an algebraic number which is not rational. Then any value of α^β is transcendental.*

The proof of this theorem is beyond the scope of these notes. We merely illustrate the result and make some remarks.

Example 3.2. *1. Consider the real number e^π. Since $(e^\pi)^i = e^{i\pi} = -1$ and since i is algebraic, it follows that e^π is transcendental. The number e^π is called the Gelfond constant. Also, setting $\alpha = 2^{\sqrt{2}}$ we have $\alpha^{\sqrt{2}} = 2^2 = 4$. Since $\sqrt{2}$ is irrational and algebraic, it follows that $\alpha = 2^{\sqrt{2}}$ is transcendental. The number $2^{\sqrt{2}}$ is called the Gelfond-Schneider constant.*

2. Suppose that $m, n > 1$ are multiplicatively independent, i.e., there does not exist a positive integer p, q such that $m^q = n^p$. Then $\lambda = \log m / \log n = \log_n m$ is irrational. For, otherwise, writing $\log_n m = p/q$ we see that $m = n^\lambda = n^{p/q}$, that is, $m^q = n^p$ contradicting our hypothesis on m, n. Now, if $\lambda = \log m/ \log n$ were algebraic, Gelfond-Scheider theorem would imply that $m = n^\lambda$ is transcedental. Therefore $\lambda := \log m/ \log n = \log_n m$ is transcendental.

3. Consider three numbers a, b, e^{ab} where a is non-zero and b is not rational. Then at least one of the numbers e^a, b, e^{ab} is transcendental. We will use the Gelfond-Schneider theorem repeatedly. We may assume that $a \neq 1$ (since e is transcendental). Note that if e^a and b are algebraic, then (as $e^a \neq 0, 1, b \notin \mathbb{Q}$) e^{ab} is transcendental. If e^a and e^{ab} are algebraic, then b has to be transcedental since $b \notin \mathbb{Q}$. Again if e^{ab} and b are algebraic, then $(e^{ab})^{1/b} = e^a$ is transcendental since $b \notin \mathbb{Q}$.

As remarked in the introduction, Lindemann proved that if α is a non-zero algebraic number, then e^α is transcendental. This result was generalized by Weierstrass. Before we state the Lindemann-Weierstrass theorem, we recall the definition of algebraic independence. We say that $a_1, \ldots, a_n \in \mathbb{C}$ are *algebraically independent* if there is no non-zero polynomial $P(X_1, \ldots, X_n)$ over $\mathbb{Q}$ such that $P(a_1, \ldots, a_n) = 0$. Note that if

$a_1, \ldots, a_n$ are algebraically independent, then necessarily they are transcendental. For a non-example, $a_1 = \pi, a_2 = \sqrt{\pi + \sqrt{\pi + 2}}$ are transcendental, but not algebraically independent. We see that, setting $P(X_1, X_2) = (X_2^2 - X_1)^2 - X_1 - 2$, we have $P(a_1, a_2) = 0$.

Theorem 3.3. (Lindemann-Weierstrass Theorem) *If $\alpha_1, \ldots, \alpha_k$ are algebraic numbers that are $\mathbb{Q}$-linearly independent, then $e^{\alpha_1}, \ldots, e^{\alpha_k}$ are algebraically independent.*

Next we state 'the qualitative form' of a result of A. Baker. For this work, and its applications to some Diophantine problems, he was awarded the Fields Medal in 1970.

Theorem 3.4. *(A. Baker) Let $a_1, \ldots, a_n \in \mathbb{C}$ be non-zero numbers such that $e^{a_1}, \ldots, e^{a_n}$ are algebraic numbers. If $a_1, \ldots, a_n$ are $\mathbb{Q}$-linearly independent, then, $1, a_1, \ldots, a_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

Baker's theorem yields the Gelfond-Schneider theorem as a special case. Let α be algebraic, $\alpha \notin \{0, 1\}$ and let β be algebraic but not a rational number.

Suppose that $\alpha^\beta = e^{\beta \log \alpha}$ is algebraic. Also $e^{\log \alpha} = \alpha$ is algebraic. Taking $a_1 = \log \alpha, a_2 = \beta \log \alpha$, the hypotheses of Baker's theorem are satisfied since $\beta \notin \mathbb{Q}$ implies that a_1, a_2 are linearly independent over $\mathbb{Q}$. Baker's theorem says that $1, a_1, a_2$ are linearly independent over $\overline{\mathbb{Q}}$. But evidently, $\beta.a_1 - a_2 = 0$, a contradiction since $\beta \in \overline{\mathbb{Q}}$. Therefore α^β has to be transcendental.

Also, Baker's theorem implies the following result that generalizes the Gelfond-Schneider theorem:

Theorem 3.5. (Baker) *Suppose $\alpha_1, \ldots, \alpha_n$ are algebraic and none of them belongs to $\{0, 1\}$. Suppose that $\beta_1, \ldots, \beta_n$ are algebraic and $1, \beta_1, \ldots, \beta_n$ are $\mathbb{Q}$-linearly independent. Then $\alpha_1^{\beta_1} \ldots . \alpha_n^{\beta_n}$ is transcendental.*

We state a conjecture due to S. Schanuel. We say that a field $E \subset \mathbb{C}$ has *transcendence degree* at least n over a subfield $F \subset E$ if there exists elements $\lambda_1, \ldots, \lambda_n \in E$ that are algebraically independent over F.

Schanuel's conjecture: *If $a_1, \ldots, a_n$ are $\mathbb{Q}$-linearly independent then the field*
$\mathbb{Q}(a_1, \ldots, a_n, e^{a_1}, \ldots e^{a_n}) \subset \mathbb{C}$ has transcendence degree at least n over $\mathbb{Q}$.

The special case when $a_1, \ldots, a_n$ are $\mathbb{Q}$-linearly independent algebraic numbers, then the above conjecture is valid as it reduces to the statement of Lindemann-Weierstrass theorem. On the other hand, taking $a_1 = \log 2, a_2 = \log 3$, it is readily seen that a_1, a_2 are $\mathbb{Q}$-linearly independent. Schanuel's conjecture says that $\log 2, \log 3$ are algebraically independent. Although $\log 3/\log 2 = \log_2 3$ is transcendental (by Gelfond-Schneider since $2^{\log_2 3} = 3$), it appears to be unknown whether $\log 2. \log 3$ is transcendental. As another example, taking $a_1 = 1, a_2 = 2i\pi$, Schanuel's conjecture says that $e, 2i\pi$—and hence e, π—are algebraically independent. This would imply that $e + \pi$ is transcendental, but this is an open problem. What we do know is that at least one of $e + \pi, e\pi$ is transcendental. Indeed if both are algebraic, then e, π would also algebraic since they are roots of the quadratic $X^2 - (e + \pi)X + e\pi$.

The following *six exponentials theorem* may be viewed as a generalization of Gelfond-Schneider theorem. It was proved by S. Lang and K. Ramachandra independently in the 1960s.

Theorem 3.6. (Lang , Ramachandra) *Suppose that $z_1, z_2, z_3 \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$ and $\beta_1, \beta_2 \in \mathbb{C}$ are also linearly independent over $\mathbb{Q}$. Then at least one of the six numbers $e^{z_j\beta_k}, 1 \leq j \leq 3, 1 \leq k \leq 2$ is transcendental.*

Example 3.7. *Let $z_1 = \log p_1, z_2 = \log p_2, z_3 = \log p_3; \beta_1 = \pi, \beta_2 = 1$ where p_1, p_2, p_3 are distinct primes. Note that $\log p_1, \log p_2, \log p_3$ are $\mathbb{Q}$-linearly independent since $p_1^{a_1} p_2^{a_2} p_3^{a_3} = 1$ with $a_1, a_2, a_3 \in \mathbb{Z}$ implies that $a_j = 0, j = 1, 2, 3$. The six exponentials $e^{z_j\beta_k}$ are $p_1^\pi, p_2^\pi, p_3^\pi, p_1, p_2, p_3$. So at least one of the three numbers $p_1^\pi, p_2^\pi, p_3^\pi$ has to be transcendental. Thus, at most two members of the set $\{p^\pi \mid p \in \mathbb{N} \text{ prime}\}$ are algebraic.*

There are also *five exponentials theorem* and *four exponentials conjecture* which have very interesting consequences. We discuss them very briefly and refer the reader to [2] for details.

Theorem 3.8. *Suppose that each of the two pairs of complex numbers α_1, α_2 and β_1, β_2 is $\mathbb{Q}$-linearly independent. If γ is a non-zero algebraic number, then at least one of the following numbers is transcendental*

$$e^{\gamma\alpha_2/\alpha_1}, e^{\alpha_1\beta_1}, e^{\alpha_1\beta_2}, e^{\alpha_2\beta_1}, e^{\alpha_2\beta_2}.$$

Four exponentials conjecture: *Suppose that α_1, α_2 and β_1, β_2 are complex numbers such that each pair is $\mathbb{Q}$-linearly independent. Then at least one of the four numbers $e^{\alpha_i \beta_j}, 1 \leq i, j \leq 2$, is transcendental.*

Taking p, q to be distinct primes and $\alpha_1 = \log p, \alpha_2 = \log q, \beta_1 = 1$, and $\beta_2 = b$ any irrational number, the above conjecture implies that at least one of p^b, q^b has to be transcendental, which seems to be unsettled.

I have included a few references which the reader can consult for detailed proofs and for an introduction to the fascinating subject of transcendental number theory.

Acknowledgments: I thank Anjaly Kishore, Vimala College, Trissur, Kerala, for inviting organizing my talk, which formed the basis for this article. I thank Kotyada Srinivas for his valuable comments, for pointing out to me the references [6], [7], and for suggesting that I publish this expository note. I thank Sanoli Gun for her valuable comments.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 219–223

A CONCEPTUAL APPROACH TOWARDS UNDERSTANDING MATRIX COMMUTATORS

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Abstract. It is well known that a square matrix over a field is a commutator if and only if it has trace zero. We give a conceptual proof of this result for square matrices over algebraically closed fields.

1. Introduction

Let $\mathbb{K}$ be a field. For a natural number n, let $M_n(\mathbb{K})$ denote the $\mathbb{K}$-algebra of $n \times n$ matrices with entries from $\mathbb{K}$. The *commutator* of two matrices $P, Q \in M_n(\mathbb{K})$ is defined as $[P, Q] := PQ - QP$. A *trace* τ on $M_n(\mathbb{K})$ is a $\mathbb{K}$-valued linear map on $M_n(\mathbb{K})$ such that $\tau(PQ) = \tau(QP)$ for all $P, Q \in M_n(\mathbb{K})$. In other words, a trace on $M_n(\mathbb{K})$ is a $\mathbb{K}$-valued linear map on $M_n(\mathbb{K})$ that vanishes on the set of commutators in $M_n(\mathbb{K})$. The function which maps a matrix in $M_n(\mathbb{K})$ to the sum of its diagonal entries, is a $\mathbb{K}$-valued trace on $M_n(\mathbb{K})$. In fact, it is the unique $\mathbb{K}$-valued trace on $M_n(\mathbb{K})$ upto scalar multiplication.

In [2], Shoda showed that an element of $M_n(\mathbb{K})$ with trace zero is necessarily a commutator when $\mathbb{K}$ has characteristic zero. The result for arbitrary fields was shown by Albert and Muckenhoupt in [1]. The present author is of the opinion that the proofs available in the literature lean more towards the technical side and do not throw as much light on why we should expect such a result to hold. In this note, we provide a conceptual proof of the fact that trace zero matrices over an algebraically closed field are commutators.

2. Trace zero matrices and commutators

In this section, $\mathbb{K}$ denotes an algebraically closed field. For $X, Y \in M_n(\mathbb{K})$, we say that X is *similar* to Y if there is an invertible matrix $S \in M_n(\mathbb{K})$ such that $X = SYS^{-1}$. It is easy to see that the relation, *similarity*,

2010 Mathematics Subject Classification: 15A18, 15A21
Key words and phrases: commutators, trace, similarity orbit

is an equivalence relation on $M_n(\mathbb{K})$. The equivalence class of an element $X \in M_n(\mathbb{K})$ is said to be the *similarity orbit* of X.

Let the matrix $X \in M_n(\mathbb{K})$ be a commutator, so that there are matrices $P, Q \in M_n(\mathbb{K})$ such that $X = [P, Q]$. Note that $[P, Q] = [P - \alpha I, Q]$ for every $\alpha \in \mathbb{K}$. Since $\mathbb{K}$ has infinitely many elements (by virtue of being algebraically closed), there exists an $\alpha \in \mathbb{K}$ such that $P - \alpha I$ is invertible and hence without loss of generality, we may assume that P is invertible. Let us rewrite X as $P(QP)P^{-1} - QP$ and define $T := QP$. Thus $PTP^{-1} = T + X$ or in other words, T and $T + X$ are similar. Conversely, it is easy to see by retracing our steps that if there is a matrix $T \in M_n(\mathbb{K})$ such that T and $T + X$ are similar, then X is a commutator. Also note that if $X = [P, Q]$, then $SXS^{-1} = [SPS^{-1}, SQS^{-1}]$ for every invertible matrix $S \in GL_n(\mathbb{K})$. We summarize the preceding discussion in the following lemma.

Lemma 2.1. *For a matrix $X \in M_n(\mathbb{K})$, the following are equivalent:*

(i) *X is a commutator;*

(ii) *There is a matrix in the similarity orbit of X which is a commutator;*

(iii) *There is a matrix $T \in M_n(\mathbb{K})$ such that T and $T + X$ are similar.*

In view of Lemma 2.1 and recalling that $\mathbb{K}$ is assumed to be algebraically closed, we will take the liberty of viewing X in its Jordan canonical form. In this note, our main goal is to show the following result, which combined with Lemma 2.1, shows that trace zero matrices over an algebraically closed field are commutators.

Theorem 2.2. *Let X be a matrix in $M_n(\mathbb{K})$. Then there is a matrix $T \in M_n(\mathbb{K})$ such that T and $T + X$ are similar if and only if X has trace zero.*

If T and $T + X$ are similar, they must have the same trace and hence it easily follows that X must have trace zero. A key element of our proof of the other direction is the following basic result from linear algebra.

Proposition 2.3. *Let $\mathbb{K}$ be an algebraically closed field. If every eigenvalue of a matrix $A \in M_n(\mathbb{K})$ has multiplicity 1, then A is diagonalizable. Consequently, if two matrices $A, B \in M_n(\mathbb{K})$ have the same set of eigenvalues with each eigenvalue having multiplicity 1, then A and B are similar.*

Note that nilpotent matrices in $M_n(\mathbb{K})$ have trace zero. We discuss Theorem 2.2 for the special case of nilpotent matrices before proceeding to the general case. Let $X \in M_n(\mathbb{K})$ be a nilpotent matrix in Jordan canonical form so that it is an upper triangular matrix with zeroes on the principal diagonal. Since the field $\mathbb{K}$ has infinitely many elements, we may choose distinct elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ from $\mathbb{K}$. Let $D := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Note that D and $D + X$ have the same set of eigenvalues. By Proposition 2.3, D and $D + X$ are similar. By Lemma 2.1, X is a commutator. We conclude that every nilpotent matrix in $M_n(\mathbb{K})$ is a commutator.

The proof of the main theorem (Theorem 2.2) involves use of the following combinatorial lemma. For the sake of brevity, we use the notation $[n] := \{1, 2, \ldots, n\}$ in the lemma. We denote the group of permutations of $[n]$ by Σ_n.

Lemma 2.4. *Let $(G; +)$ be an abelian group with infinitely many elements and n be a positive integer. Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-tuple of elements from G satisfying*

$$\sum_{i=1}^{n} \lambda_i = 0.$$

Then there is an n-tuple $(\mu_1, \mu_2, \ldots, \mu_n)$ of elements from G and a permutation $\sigma \in \Sigma_n$ such that $\mu_i \neq \mu_j$ for $i \neq j$, and $\lambda_i = \mu_i - \mu_{\sigma(i)}$ for $1 \leq i \leq n$.

Proof. We proceed inductively. For $n = 1$, the lemma is obvious. Consider a positive integer $m \geq 2$, and assume that the assertion is true for $n = 1, 2, \ldots, m - 1$.

Let $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of elements from G satisfying $\sum_{i=1}^{m} \lambda_i = 0$. We say that a subset S of $[m]$ has property $\mathfrak{P}$ if $\sum_{i \in S} \lambda_i = 0$ and $\sum_{i \in S'} \lambda_i \neq 0$ for any non-empty proper subset, S', of S. Let R be a non-empty subset of $[m]$ with smallest cardinality such that $\sum_{i \in R} \lambda_i = 0$. Clearly R has property $\mathfrak{P}$.

<u>Case I</u>: $R = [m]$.

For $1 \leq k \leq m$, we define μ_k to be the cumulative sum $\sum_{i=1}^{k} \lambda_i$. Let σ be the n-cycle defined by

$$\sigma(i) = \begin{cases} i - 1 & \text{if } 2 \leq i \leq m \\ m & \text{if } i = 1. \end{cases}$$

From the hypothesis of the lemma, $\mu_m = \sum_{i=1}^{m} \lambda_i = 0$. Furthermore, for $1 \leq k < \ell \leq m$, we have $\mu_\ell - \mu_k = \sum_{i=k+1}^{\ell} \lambda_k \neq 0$ (that is, $\mu_k \neq \mu_\ell$). For

$k = 1$, we have $\mu_k - \mu_{\sigma(k)} = \mu_1 - \mu_{\sigma(1)} = \mu_1 - \mu_m = \lambda_1$. For $2 \le k \le m$, we observe that $\mu_k - \mu_{\sigma(k)} = \mu_k - \mu_{k-1} = \sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k-1} \lambda_i = \lambda_k$. Thus the m-tuple $(\mu_1, \mu_2, \dots, \mu_m)$ satisfies the conditions described in the lemma.

Case II: R is a non-empty proper subset of $[m]$.

By suitable permutation of the entries of $(\lambda_1, \lambda_2, \dots, \lambda_m)$, we may assume that $R = [\ell]$ for some $1 \le \ell \le m - 1$. Note that $(\lambda_1, \dots, \lambda_\ell)$ is an ℓ-tuple satisfying the hypothesis of the lemma. Using the induction hypothesis for $n = \ell$, we obtain an ℓ-tuple $(\mu_1, \mu_2, \dots, \mu_\ell)$ and a permutation of $[\ell]$, σ_1, such that $\lambda_i = \mu_i - \mu_{\sigma_1(i)}$ for $1 \le i \le \ell$ and $\mu_i \ne \mu_j$ for distinct $i, j \in [\ell]$.

Note that
$$\sum_{i=\ell+1}^{m} \lambda_i = \sum_{i=1}^{m} \lambda_i - \sum_{i=1}^{\ell} \lambda_i = 0 - 0 = 0.$$
Thus $(\lambda_{\ell+1}, \dots, \lambda_m)$ is an $(m - \ell)$-tuple satisfying the hypothesis of the lemma. Using the induction hypothesis for $n = m - \ell$, we obtain an $(m-\ell)$-tuple $(\mu_{\ell+1}, \dots, \mu_m)$ and a permutation of $[m]\backslash[\ell]$, σ_2, such that $\lambda_i = \mu_i - \mu_{\sigma_2(i)}$ for $\ell + 1 \le i \le m$ and $\mu_i \ne \mu_j$ for distinct $i, j \in [m]\backslash[\ell]$.

Let σ be the permutation of $[m]$ which acts as σ_1 on $[\ell]$ and as σ_2 on $[m]\backslash[\ell]$. Clearly $\lambda_i = \mu_i - \mu_{\sigma(i)}$ for $1 \le i \le m$. The set $\Lambda := \{\mu_i - \mu_j | i \in [\ell], j \in [m]\backslash[\ell]\}$ is a finite subset of G. Since G is infinite, we may choose $\alpha \in G\backslash\Lambda$. We define an m-tuple $(\nu_1, \nu_2, \dots, \nu_m)$ as follows:
$$\nu_i := \begin{cases} \mu_i - \alpha & \text{if } i \in [\ell] \\ \mu_i & \text{if } \ell + 1 \le i \le m \end{cases}$$
We observe that $\lambda_i = \nu_i - \nu_{\sigma(i)}$ for $1 \le i \le m$, and for distinct $i, j \in [m]$, we have $\nu_i \ne \nu_j$. Thus the m-tuple $(\nu_1, \nu_2, \dots, \nu_m)$ satisfies the conditions described in the lemma. □

Proof of Theorem 2.2. As mentioned before, it is straightforward to see that X has trace zero if there is a matrix T such that T and $T + X$ are similar.

For the converse, without loss of generality, we may assume that X is in Jordan canonical form. Let $(\lambda_1, \lambda_2, \dots, \lambda_n)$ denote the n-tuple of eigenvalues of X such that
$$X = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) + N$$

where N is an upper triangular nilpotent matrix. Since $\mathrm{tr}(X) = \sum_{i=1}^{n} \lambda_i = 0$, from Lemma 2.4, we have an n-tuple $(\mu_1, \mu_2, \dots, \mu_n)$ of elements of $\mathbb{K}$ such that $\mu_i \neq \mu_j$ for distinct $i, j \in [n]$ and a permutation $\sigma \in \Sigma_n$ such that $\lambda_i = \mu_i - \mu_{\sigma(i)}$. Let

$$D_\sigma := \mathrm{diag}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \dots, \mu_{\sigma(n)}), \text{ and } D := \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_n).$$

By Proposition 2.3, D_σ and $D + N$ are similar as they have the same set of eigenvalues and each of their eigenvalues has multiplicity 1. Since $D + N = D_\sigma + X$, we conclude that D_σ and $D_\sigma + X$ are similar. Thus $T := D_\sigma$ satisfies the assertion in the theorem. □

From Lemma 2.1 and Theorem 2.2, we conclude that a square matrix over an algebraically closed field is a commutator if and only if it has trace zero.

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The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022), 225–232

PROBLEM SECTION

Through Volume 90 (3-4) 2021 of The Mathematics Student, we had invited solutions from readers to the Problems 1, 2, 3 and 5 mentioned in MS 89 (3-4) 2020, solutions to the problems 1, 2, 4 and 5 mentioned in MS 90 (1-2) 2021, as well as solutions to the ten new problems till January 10, 2022.

As regards to solutions to the four Problems mentioned in MS 89(3-4) 2020, we did not receive any solution to any of the four problems. Therefore solutions provided by Prof. B. Sury, the proposer of the problems, are being published in this section.

As far as solutions to the four Problems mentioned in MS 90 (1-2) 2021 are concerned, we have not received received any solution to any of the problems from the readers of the Mathematics Student. However, we feel that several readers can provide solutions to these problems. Hence we give one more opportunity to the readers to provide their solutions to these problems until April 20, 2022.

As far as solutions to the ten new problems mentioned in MS 90 (3-4) 2021 are concerned, we received one correct solution to Problem 1 which will be printed in this section. Two correct solutions have been received to Problem 7 and we publish the solution which is more precise and elegant.

We pose **Twelve new problems** in this section. We invite Solutions from the readers to the Problems 1, 2 4 and 5 of MS 90 (1-2) 2021, solutions to the remaining eight problems of MS 90 (3-4) 2021 and solutions to the Twelve new problems till April 20, 2022. Correct solutions received from the readers by this date will be published in Volume 91 (3-4) 2022 of The Mathematics Student. This volume is scheduled to be published in May 2022.

New Problems.

Ilir Demiri and **Prof. Shpetim Rexhepi**, Mother Teresa University, Skopje, North Macedonia proposed the following three problems.

MS 91 (1-2) 2022: Problem 1. Prove that

$$\int_0^1 \left(t^{1/n} \;-\; t^{1-1/n}\right)^{n-1} dt = \frac{n}{(1+1/n)\,(1+1/2n)\ldots(1+1/n(n+1))}$$

,

for $n \in \mathbb{N}$.

MS 91 (1-2) 2022 : Problem 2. For the beta function, prove with usual meaning that

$$\sum_{n=0}^{\infty} (B(2,n) - B(3,n)) = \frac{1}{2} \quad \text{where} \quad n \in \mathbb{N}.$$

MS 91 (1-2) 2022 : Problem 3. For the beta function, prove with usual meaning that

$$B(k,n) = \frac{(k-1)B(k-1,n)}{n+k-1} \quad \text{where } k,n \in \mathbb{N}.$$

The following five problems have been proposed by **Prof. B. Sury**, Indian Statistical Institute, Bangalore.

MS 91 (1-2) 2022 : Problem 4. Let $f(x) = cot(x)$. Note that $f'(x) = -f(x)^2 - 1$. More generally, for each $n \geq 0$, write

$$n!f(x)^{n+1} = a_n + b_{n0}f(x) + b_{n1}f'(x) + \cdots b_{nn}f^{(n)}(x)$$

for some a_n's and b_{nm}'s. Find a recursion for the b_{ij}'s and determine all the a_n's.

MS 91 (1-2) 2022 : Problem 5. Let a_n denote the number of different ways of putting brackets on a sequence of n objects. For example,

$$(p_1p_2p_3), (p_1)(p_2p_3), (p_1p_2)(p_3)$$

shows that $a_3 = 3$. Check for instance that $a_4 = 11$. Consider also the number b_n of paths from $(0,0)$ to (n,n) which never go above the line $y = x$ and where each step is a unit north, east or north-east (that is, cross). For instance, if we write h, v, c respectively for an eastward horizontal step, a northward vertical step and a southwest-northeast cross step, then

$$hvc, hcv, hhvv, hvhv, cc, chv$$

are the six paths from $(0,0)$ to $(2,2)$ which shows $b_2 = 6$. Prove that $b_n = 2a_{n+1}$ for all n.

MS 91 (1-2) 2022 : Problem 6. Let $f : [0,1] \to \mathbb{R}$ be differentiable and satisfy $f(0) = f'(0) = f'(1) = 0$. Then, show that there exists $t \in (0,1)$ such that $f(t) = tf'(t)$.

MS 91 (1-2) 2022 : Problem 7. Let $C(x)$ be the Cantor singular function; this is the unique non-decreasing function on $[0,1]$ such that if $x = \sum_{n\geq 1} a_n/3^n$ where a_n's are 0 or 2, then $C(x) = \sum_{n\geq 1} \frac{a_n/2}{2^n}$. Find the values of $\int_0^1 C(x)^n dx$ for $1 \leq n \leq 5$.

MS 91 (1-2) 2022 : Problem 8. For a positive real number x, let $AT(x)$ denote the value of $arctan(x)$ (that is, $tan^{-1}(x)$) in $[0, \pi/2]$. Observe $AT(3) = 3AT(1) - AT(2)$, $AT(8) = 2AT(1) + AT(3) - AT(5)$. Prove that a positive integer n has the property that $AT(n) = \sum_{i=1}^{n-1} a_i AT(i)$ for some integers a_i if, and only if, $n^2 + 1$ has a prime factor $p \geq 2n$.

Dr. Anup Dixit, Institute of Mathematical Sciences, Chennai suggested the following two problems.

MS 91 (1-2) 2022 : Problem 9. Let $f : \mathbb{N} \to \mathbb{N}$ be a function on positive integers satisfying the property

$$f(f(n)) + f(n+1) = n + 2.$$

Show that $f(n) = \lfloor n\alpha \rfloor + 1$, where $\alpha = (\sqrt{5} - 1)/2$.

MS 91 (1-2) 2022 : Problem 10. For a complex number $s = \sigma + it$ with $\sigma > 1$, let $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ denote the Riemann zeta function. Show that for a fixed $\sigma > 1$,

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |\zeta(\sigma + it)| \leq \zeta(\sigma).$$

Dr. Siddhi Pathak, Chennai Mathematical Institute, Chennai proposed the following two problems.

MS 91 (1-2) 2022 : Problem 11. Fix an integer $k \geq 2$. A positive integer n is said to be k-free if it is not divisible by p^k for any prime p.

Evaluate the summation

$$\sum_{\substack{n\geq 1,\\ n \text{ is 4-free}}} \frac{1}{n^2}.$$

MS 91 (1-2) 2022 : Problem 12. Let

$$I := \int_0^\infty \frac{(\log x)^4}{1+x^2}\,dx.$$

Prove that I is an algebraic number times π^5.

Solutions to the Old Problems.

Problems 1, 2, 3 and 5 in MS 89 (1-2) 2020 were proposed by Prof. B. Sury. The problems and their solutions provided by him are given below.

MS 89 (1-2) 2020 Problem 1. Note that $(x, y) \mapsto (y, -x)$ is a bijection of the plane. In contrast, show that there is no bijection f from $\mathbb{R}^3$ to itself such that for any two points $P \neq Q$, the line joining P and Q is perpendicular to the line joining $f(P)$ and $f(Q)$.

Solution.
Suppose such a bijection f exists. Without loss of generality, we may assume f maps 0 to 0 (else, we may consider the new bijection $x \mapsto f(x) - f(0)$). Viewing the points of $\mathbb{R}^3$ as vectors, we have for all $a, b \in \mathbb{R}^3$ that

$$(a-b).(f(a)-f(b)=0.$$

Hence $a.f(a) = 0$ (taking $b = 0$). Thus, we have

$$a.f(b) + b.f(a) = 0 \ \forall\ a, b.$$

From this, we can see easily that for all $a, b, c \in \mathbb{R}^3$ and any scalars $u, v \in \mathbb{R}$ that

$$a.(uf(b) + vf(c) - f(ub + vc)) = 0.$$

As a is arbitrary, this implies $uf(b) + vf(c) = f(ub + vc)$; that is, f is linear. The conditions $a.f(b) = -b.f(a)$ for all a, b then implies that the corresponding matrix F of f with respect to the canonical basis is a real skew-symmetric 3×3 matrix; that is, $F = -F^t$. But then $det(F) = -det(F^t) = -det(F)$ gives $det(F) = 0$. So, f cannot be a bijection. This proves no such bijection can exist.

MS 89 (1-2) 2020 Problem 2. Find all positive integral solutions of $a^3 = b^2 + p^3$ where p is a prime and 3 and p do not divide b.

Solution.
For any solution, we have

$$(a-p)(a^2+ap+p^2) = b^2.$$

The factors on the left are relatively prime; else, any common prime factor would divide both b^2 and $3p^2$. Therefore, $a - p = u^2, a^2 + ap + p^2 = v^2$ for some integers u, v.
Now, putting $a = u^2 + p$, we have

$$a^2+ap+p^2 = u^4+2u^2p+p^2+(u^2+p)p+p^2 = u^4+3u^2p+3p^2 = (u^2+3p/2)^2+3p^2/4.$$

So, $4v^2 = 4(a^2 + ap + p^2) = (2u^2 + 3p)^2 + 3p^2$.
Thus, $3p^2 = (2v + 2u^2 + 3p)(2v - 2u^2 - 3p)$.
The only three possibilities for the right hand side factors are $(3p^2, 1), (p^2, 3), (3p, p)$.
The first possibility cannot occur as we see by viewing modulo 3. The third case is also not possible as $v = u = 0$ is not possible. In the second case $2v + 2u^2 + 3p = p^2, 2v - 2u^2 - 3p = 3$.
So $4u^2 = p^2 - 6p - 3 < (p-3)^2$. On the other hand, $p^2 - 6p - 3 > (p-4)^2$ when $p > 10$. Thus, the only possibilities are $p = 2, 3, 5, 7$. But p^2-6p-3 is negative when $p = 2, 3, 5$. So, the unique solution is when $p = 7$ which gives $u = 1$ and hence $a = 8$ and $b = 13$. The only solution is $(a, b, p) = (8, 13, 7)$.

MS 89 (1-2) 2020 Problem 3. A positive integer is written in each square of an 100 by 100 chess board. The difference between the numbers in any two adjacent squares that share an edge is at the most 10. Prove that at least six squares must contain the same number. Generalize?

Solution.
We prove this replacing 100 by any n^2 in which case 'six' in the problem is replaced by $[n/2] + 1$.
Consider the smallest and largest numbers m and M that are written on this $n^2 \times n^2$ chess board. They are separated by at most $n^2 - 1$ squares horizontally as well as vertically. This means that there is a path from one to the other whose length is at the most $2(n^2 - 1)$. By hypothesis, any two successive squares differ by at most n. Hence, we have $M - m \leq 2n(n^2 - 1)$.
Since all the numbers written on the board are integers are in $[m, M]$, there can be at the most $2n(n^2 - 1) + 1$ distinct numbers on the board. But

$n^4 > (2n(n^2-1)+1)(n/2)$, we must have more than $n/2$ squares where the same number occurs.

MS 89 (1-2) Problem 5. For any continuous function f on $[-1/2, 3/2]$, prove that

$$\int_{-1/2}^{3/2} f(3x^2-2x^3)dx = 2\int_0^1 f(3x^2-2x^3)dx;$$

$$\int_{-1/2}^{3/2} xf(3x^2-2x^3)dx = 2\int_0^1 xf(3x^2-2x^3)dx;$$

$$\int_{-1/2}^{3/2} x^2 f(3x^2-2x^3)dx = \int_0^1 (3x-x^2)f(3x^2-2x^3)dx.$$

Solution.
Break the interval $[-1/2, 3/2]$ into the 3 parts $[-1/2, 0], [0, 1], [1, 3/2]$. Note that the relation $3x^2 - 2x^3 = u$ has a unique solution in each interval. Complete the solution.

MS 90 (3-4) 2021 : Problem 1 (Posed by Prof. B. Sury, Indian Statistical Institute, Bangalore).
Find all 4-tuples (a_1, a_2, a_3, a_4) of positive integers such that $a_i s$ are distinct and each a_i divides the sum of the other three.

Mr. Ritesh Dwivedi, Rera, Jasra, Prayagraj, Uttar Pradesh provided the solution to the problem. The solution is presented below.

Solution.
In order to find out all required 4-tuples, it is sufficient to determine all 4-tuples $(a_1,\ a_2,\ a_3,\ a_4)$ s.t. each a_i divides the sum of the other three alongwith the conditions that $a_1 < a_2 < a_3 < a_4$ and $\gcd(a_1,\ a_2,\ a_3,\ a_4)$ $= 1$. Now we note that for such a 4-tuple, a_3 must be an integral multiple of $a_1 + a_2$. So let $a_3 = n.(a_1 + a_2)$. Now applying the condition that a_4 divides $a_1 + a_2 + a_3$, we get that $a_4 = (n+1).(a_1 + a_2)$. Further applying the condition that $a_3 (= n.(a_1 + a_2))$ divides $a_1 + a_2 + a_4$, we get that $n =$ 1 or 2. Now we proceed for these two cases separately:

Case I. In this case, we suppose that $n = 1$. Then we have the 4-tuple $(a_1, a_2, a_1 + a_2, 2a_1 + 2a_2)$. Now since a_2 divides the sum of the other three, we have $r.a_2 = 4a_1 + 3a_2$, for some positive integer r. This implies that $(r-3).a_2 = 4a_1$, i.e. a_2 divides $4a_1$. Now since $a_1 < a_2$, we must have $4a_1 = a_2$, $2a_2$ or $3a_2$. This gives the 4-tuples as (1, 4, 5, 10), (1, 2, 3, 6) and (3, 4, 7, 14). However, the last one does not fullfill our criterion, so we shall discard it. Therefore in the first case when $n = 1$, we get the desired 4-tuples as (1, 4, 5,10) and (1, 2, 3, 6).

Case II. In this case, we suppose that $n = 2$. Then we have the 4-tuple $(a_1, a_2, 2a_1 + 2a_2, 3a_1 + 3a_2)$. Now since a_2 divides the sum of the other three, we have $r.a_2 = 6a_1 + 5a_2$, for some positive integer r. This implies that $(r - 5).a_2 = 6a_1$, i.e. a_2 divides $6a_1$. Now since $a_1 < a_2$, we must have $6a_1 = a_2$, $2a_2$, $3a_2$, $4a_2$ or $5a_2$. This gives the 4-tuples as (1,6,14,21), (1,3,8,12), (1,2,6,9), (2,3,10,15) and (5,6,22,33). However, the last one does not fulfill our criterion, so we shall discard it. Therefore in the second case when $n = 2$, we get the desired 4-tuples as (1,6,14,21), (1,3,8,12), (1,2,6,9) and (2,3,10,15).

After the discussion of above two cases, we are ready to give the complete list of all 4-tuples (a_1, a_2, a_3, a_4) of positive integers s.t. $a_i's$ are distinct and each a_i divides the sum of the other three. The required set of all 4-tuples consists of all positive integral multiple of the permutations of all 4-tuples, namely (1,2,3,6), (1,2,6,9), (1,3,8,12), (1,4,5,10), (1,6,14,21) and (2,3,10,15), obtained in the above two cases. This completes the solution.

MS 90 (3-4)2021: Problem 7 (Proposed by Dr. Anup Dixit, IMSc., Chennai).

Show that $\zeta(3)<5/4$, where $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$.

Mr. Hemant Dubey, Central University of South Bihar, Gaya, Bihar and **Miss Emil Inochkin**, ADA University, Azerbaijan provided different solutions to the problem. The solution given by Miss Inochkin is short and elegant and it is produced below.

Solution (by Miss Emil Inochkin).

We can represent the sum as the sum of areas of rectangles with sides $\frac{1}{n^3}$ and 1. All these rectangles lie below the function f(n) $= \frac{1}{n^3}$. Then the sum of areas is less than the area below the curve from 1 to infinity plus the area of the first rectangle, which is equal to

$$\int_1^{\infty} \frac{1}{n^3} \text{ dn } + 1 = 1.5 = 6/4$$

.

The number we got is a bit bigger than that we are trying to find (5/4). Then we can subtract error (the area between curve and rectangles, which is equal to $\int_a^b \frac{1}{n^3}\mathrm{dn} - Area\ (rectangle)$). To prove that the sum is less than 5/4, it is enough to subtract the error from second rectangle, which is equal to 1/4.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^3} &< 1 + \int_1^{\infty} \frac{1}{n^3}\mathrm{dn} - \left(\int_1^2 \frac{1}{n^3}\mathrm{dn} - \frac{1}{8}\right) \\ &= 1 + \left(-\frac{1}{\infty} - \left(-\frac{1}{2}\right)\right) - \left(-\frac{1}{8} - \left(-\frac{1}{2}\right) - \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{4} \\ &= \frac{5}{4}.\end{aligned}$$

Thus $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^3}} = \zeta\,(\mathbf{3})\ < \frac{\mathbf{5}}{\mathbf{4}}$ as desired.

We would also like to present below the solution provided by **Dr. Anup Dixit** since the solution points to the fact that the identity observed in the solution was the first step in Apéry's proof of $\zeta(3)$ being irrational.

Solution (By Anup Dixit).
We start with the following observation.

$$\frac{1}{n^3} = \frac{1}{(n-1)n(n+1)} - \frac{1}{(n-1)n^3(n+1)}.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^3} &< 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{2(n+1)} \left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \\ &= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{4} \\ &= \frac{5}{4}.\end{aligned}$$

FORM IV
(See Rule 8)

1.	Place of Publication:	PUNE
2.	Periodicity of publication:	QUARTERLY
3.	Printer's Name: Nationality: Address:	DINESH BARVE INDIAN PARASURAM PROCESS 38/8, ERANDWANE PUNE-411 004, INDIA
4.	Publisher's Name: Nationality: Address:	SATYA DEO INDIAN GENERAL SECRETARY THE INDIAN MATHEMATICAL SOCIETY HARISH CHANDRA RESEARCH INSTITUTE CHHATNAG ROAD, JHUNSI ALLAHABAD-211 019, UP, INDIA
5.	Editor's Name: Nationality: Address:	M. M. SHIKARE INDIAN CENTER FOR ADVANCED STUDY IN MATHEMATICS, S. P. PUNE UNIVERSITY PUNE-411 007, MAHARASHTRA, INDIA
6.	Names and addresses of individuals who own the newspaper and partners or sharehold-ers holding more than 1% of the total capital:	THE INDIAN MATHEMATICAL SOCIETY

I, Satya Deo, the General Secretary of the IMS, hereby declare that the particulars given above are true to the best of my knowledge and belief.

SATYA DEO

Dated: February 28, 2022 Signature of the Publisher

Published by Prof. Satya Deo for the Indian Mathematical Society, type set by M. M. Shikare, "Krushnakali", Servey No. 73/6/1, Gulmohar Colony, Jagtap Patil Estate, Pimple Gurav, Pune 411061 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune 411 041 (India). Printed in India

The Mathematics Student ISSN: 0025-5742
Vol. 91, Nos. 1-2, January-June (2022)

THE INDIAN MATHEMATICAL SOCIETY
Founded in 1907
Registered Office: Center for Advanced Study in Mathematics
Savitribai Phule Pune University, Pune - 411 007

Back volumes of our periodicals, except for a few numbers out of stock, are available.

Edited by M. M. Shikare and published by Satya Deo, the General Secretary of the IMS, for the Indian Mathematical Society.

Typeset by M. M. Shikare, “Krushnakali”, Survey No. 73/6/1, Gulmohar Colony, Jagtap Patil Estate, Pimple Gurav, Pune 411061 and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No.129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune - 411 041, Maharashtra, India. Printed in India