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THE MATHEMATICS STUDENT

Edited by M. M. SHIKARE

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3. general survey articles, popular articles, expository papers and Book-Reviews.
4. problems and solutions of the problems,
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6. articles that arouse curiosity and interest for learning mathematics among readers and motivate them for doing mathematics.

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VIEW OF MATHEMATICS BY OUR SOCIETY AND WHAT OUR ROLE COULD BE*

B SURY

It is a great honor to have been elected to the office of the President of the Indian Mathematical Society. The genesis of the Indian Mathematical Society has the following interesting history. On April 4, 1907, V. Ramaswamy Aiyar and 20 founding members founded the 'Analytic club' on the lines of the Edinburgh Mathematical society. The idea was put forth in a letter in December 1906; note that Einstein's theory of relativity and Ramanujan's works came to the world's notice in 1906 - a golden year indeed. The IMS is the oldest mathematical society of our country, and has had such luminaries as Vijayaraghavan, Vaidyanathaswamy, Hansraj Gupta and P L Bhatnagar among its Past Presidents. I am also grateful for the opportunity to address this august gathering. Even though the pandemic has not allowed us to meet in person, it could be a blessing in disguise as perhaps more people can virtually participate.

I would like to share my thought about the following three aspects of Mathematics:
(i) How Indian society views mathematics, and what we should or could do to effectively create awareness among the public to the importance of this discipline;
(ii) Ways and means of motivating students who are not particularly interested in the subject due to the way it is being taught; and
(iii) (This has to do with the mathematical community) What might the future scenario of research in mathematics look like.

∗ This article is based on the text of the Presidential Address (General) given by Prof. B. Sury at the 86th Annual Conference of the IMS-An International meet held at Vellore Inst. of Tech., Vellore (T. N.) using online mode during December 17-20, 2020.

Before talking about these three points, let us recognize that mathematics is a social enterprise that is expected to further human goals and aspirations. A mathematical conference is not only about discussions that are mathematical, but the purpose is also to socialize, reinvigorate friendships, and engage in informal conversations. The point is to remind ourselves that it is about being part of a community that is united by its love for, and belief in, the importance of mathematics. Irrespective of whether our interest is in teaching, or research or reaching out to society, all of us are essential to the continued health of our discipline. In that aspect, an online meeting falls woefully short. The belief is that a good education in mathematics benefits the individual as well as society, but the benefits should include empowerment of the individual, development of analytical/logical thinking and also gains that are practical such as the support of scientific and business aspirations.

Aspect I : Our society's view of mathematics

The view of the mathematical layman towards practising mathematicians like us is based mainly on our holding positions in institutions rather than actual appreciation - this is understandable. It is akin to what many mathematicians think of Ramanujan. Their appreciation is mostly based on reading encomiums about Ramanujan by some top mathematicians, coupled with personal aspects as to how he faced hardships, poverty etc. An appreciation of his original ideas and an understanding of what Ramanujan actually did and how fertile his mind was, is only by a small minority.

Many people think of mathematicians as people who can do fast calculations and always deal with numbers. My favorite reply to this often is that the only numbers I may come across in my research are page numbers of manuscripts! I am sure that most of us face the following experience. In any general get-together, when people hear about your profession, they say, "Oh, I was so bad at mathematics! I just managed to scrape through my exams." We are led to think that they perceive us as higher entities of some sort. But, what is subconsciously conveyed by the person is "see how successful I am in my career and I did not really need your mathematics." This could mean that the manner in which mathematics is taught in our educational system is not very relevant. On the other hand, if we can communicate that a mathematical way of thinking can be helpful in developing our analytical abilities, that would be much more effective to society.

For instance, if I were to ask, "can we estimate (at least roughly) how many people must be sleeping at this moment ¿', a mathematically minded person would know how to analyze and solve such a problem. But, this example may not really impress a common man how this kind of thinking can be all that useful. If we give an example of how to play probabilities to maximize a win in a casino, that would probably impress them more! The British mathematician John Venn wrote in 1866, in reference to probability theory, "To many persons the mention of Probability suggests little else than the notion of a set of rules, very ingenious and profound rules no doubt, with which mathematicians amuse themselves by setting and solving puzzles." Maybe, many in our society would think Venn's comments apply to the whole of mathematics.

So it is important to convince society what is special about learning mathematics.

Convincing the general public that understanding mathematics is significant for all of society is difficult. In all this, the irony is that nowadays, abstract/pure mathematics is used in our daily life hundreds of times more intensively than 20 years back. Concepts involved in practical applications are deeper and more abstract and difficult than before. This is one of the paradoxes of modern times - that deep mathematics may be carefully hidden behind a user friendly smart-phone interface.
Amongst the realm of diverse disciplines learnt by students all over the world, why mathematics may have a special place, is beautifully explained by a mathematical friend Alexander Borovik; I can't do better than quote him on this. He says:
"In many walks of life, to have a happy and satisfying professional career, one has to be future-proof by being able to re-learn the craft, to change his/her way of thinking. How can this skill of changing one's way of thinking be acquired and nurtured? At school level mostly by learning mathematics. Regular and unavoidable changes of mathematical language reflect changes

of mathematical thinking. This makes mathematics different from the majority of other disciplines. The crystallization of a mathematical concept (say, of a fraction), in a child's mind could be like a phase transition in a crystal growing in a rich, saturated and undisturbed|solution of salt. An "Aha!" moment is a sudden jump to another level of abstraction. Such changes in one's mode of thinking are like a metamorphosis of a caterpillar into a butterfly. As a rule, the difficulties of learning mathematics are difficulties of adjusting to change. Pupils who have gained experience of overcoming these difficulties are more likely to grow up future-proof. I lived through sufficiently many changes in technology to become convinced that mathematically educated people are stem cells of a technologically advanced society, they are re-educable, they have a capacity for metamorphosis."

Although it is only to a certain extent that we can communicate mathematics with an outsider to mathematics, it may be harmful if we divorce ourselves from their interests and immerse ourselves entirely into our world. For one thing, we may not get funding!

So, one needs to create general awareness by popular writings:

An aspect regarding mathematical communication (or the lack of it) with our society seems rather specific to our discipline. Unlike science subjects like astronomy or molecular biology, there are very few competent popular communications in mathematics in our country. There is a dearth of "mathematics journalists" - those who are trained enough to understand what is going on in research and possess the talent to communicate effectively. I do not mean experts in mathematics education but trained mathematicians who can comprehend the state of the art sufficiently enough and have the talent to write about them in a simple and interesting manner to reach the reader who is not an active mathematician. This is a need of the hour in my opinion - one reason why mathematicians lag behind in popularity as compared to, say, Astronomy. Astrophysics, as practiced by researchers is often as abstract as mathematics but the public gets a feeling for it due to some talented science expositors.

Aspect II : Communicating mathematics to students who are not already very motivated:

The young student's interest in mathematics is more likely to be kindled more through some fun and games. With them, posing appealing puzzles may be a better way to communicate.

Some years back, I was going by train to Hyderabad from Bangalore and there were some 7 or 8 kids who were co-passengers along with their parents. The youngsters were aged between 7 and 17. When a parent asked me what I did, and I mentioned that I am a mathematician, the parents immediately switched off excepting for a mention of a cousin who became an engineer because she was very good at mathematics! The general perception in the public is that if someone is good at mathematics, she must become an engineer. I decided instead to try and pass on some mathematics under the guise of fun and games. That experience exceeded all my expectations and all the kids, irrespective of their ages, were thoroughly interested and engaged throughout the train journey.

In this respect, I wish to point out a societal drawback - which I call as "Talking down" to students:
The main difficulty lies in making relevant mathematics interesting for the non-motivated student by a lighter-toned talk. There is a fundamental problem here due to our ingrained culture, I feel. We stress on the students being 'well-mannered' and 'respectful' and often this turns them into being 'obedient and even seemingly docile' to the extent that they never question anything we say. This is the worst possible effect. We need to dissuade them from needless practices of offering respect. Instead, we have to encourage a spirit of enquiry and criticism in them. This is very difficult in our society as older people are usually authoritative and seldom act in a friendly or equal manner with the young students. But, this - perhaps desirable quality among teachers - is conspicuous by its absence. All this is compounded by the social environment where any word of dissent or criticism is seen to be a crime!

The manner in which our system functions makes it a laborious process to talk comfortably to students. To give an example, if a college organizes a special invited talk, say, then it is often found that the organizers are more interested in fanfare such as lighting lamps and arranging flowers and singing prayers than the actual programme which often begins very late when participants are tired and uninterested. Nothing really wrong with that but they would be better off displaying instead, mathematical models etc. that some students may perhaps have made.

Even if it goes against our instincts, it is important to - what I call as - Bend rigor a bit; create simple examples even if not precise or only as somewhat far-fetched analogies:

The difficulty is, of course, to come up with inspiring, simple examples of what we are trying to teach. The example should deal with the issue as well as be capable of having a 'worldly' or 'common sense' point of view even if it is not very accurate. We may succeed only rarely. It is possible to give some mathematical examples. For instance, I am sure some of you may know and may have used the following example when we teach a basic concept like the intermediate value theorem. A wobbly, unstable table in a Restaurant can be made stable - not by placing pieces of newspaper under the legs - but simply by rotating the table suitably and we may prove that this works mainly due to the IMVT. A somewhat far-fetched example which turned out surprisingly effective is the following one. While teaching the Urysohn lemma in topology, I drew an increasing net of open sets which look like waves in water, and told my class that in one of those waves, Urysohn may have lost his life. We know Urysohn died in a swimming accident, and some students told me later that somehow this made them look at the proof and grasp it with more interest. Even though this did not provide a mathematical perspective, it produced a human perspective and succeeded in interesting and engaging the students to learn it themselves.

Aspect III : Finally, I say a bit about my perception on the Future scenario of mathematical research:

It is becoming increasingly clear that the future modus operandi of mathematical research - vis-a-vis the big outstanding problems - is through joint ventures like the Polymath project. Perhaps, not all individuals involved in the project may get the desired recognition but there is no doubt that dents can be made on largely unsurmountable problems.

Computers, of course, provide new tools for organizing as well as carrying out parts of our research. But, apart from this and apart from getting information from the web, it is in the computer's help with experimentation that is playing an increasingly powerful role. I have experienced this in some small ways as follows. While writing a paper on class groups of certain number fields, we obtained a general result which was not good enough to answer specific questions. Some experimentation based on it revealed so many patterns that we could conjecture and prove concrete results. This was exciting to me personally. In fact, as a part of a committee that prepares question papers for mathematical olympiads, I have seen this again. Experimentation with some program like Geogebra seems to produce configurations which were hitherto unnoticed, and some of them led to problems that were posed in these olympiads.

Timothy Gowers holds the view that computers will eventually take over from human mathematicians, even if they are unable to intuitively understand answers. I end with his quote:

"What I would hope is that there will be two activities: thinking very hard about the research process from the perspective of explaining to humans how to do it, and the bottom-up process of getting a computer to be able to do most sophisticated things. At some point I would like those two to meet in the middle, so the computer can do the easy stuff and then they can get this sort of advice about how to do harder stuff which they will then be able to act on. "

Acknowledgments. I would like to repeat what I said in the beginning of this talk - it is indeed a great pleasure and privilege to have been been elected to the Presidentship of this August society which had been headed earlier by so many stalwarts. It has given me a platform to express my thoughts on the way mathematics is viewed by our society at large and what we can/must do to improve/correct things.

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LINEAR ALGEBRAIC GROUPS OVER GLOBAL AND LOCAL FIELDS - SOME THEMES*

B. SURY

We discuss roughly four types of questions to which we have been able to contribute something. These are:

- Theme One: Concerns arithmetic groups and we particularly concentrate on ideas surrounding the so-called 'Congruence Subgroup Problem'.
- Theme Two: p-adic groups and their central extensions; though these are of independent interest, they appear naturally in the set-up of the CSP.
- Theme Three: Some of the problems arising under the above two areas lead to natural questions on division algebras which are also of independent interest and the techniques from algebraic group theory do not work. This is the third type of problem we discuss.
- Theme Four: Abstract group theoretic questions on arithmetic groups that can be studied using techniques like class field theory.

Another useful theme concerns the study of matrix groups over special commutative rings satisfying stability conditions. As this topic was discussed earlier (see [2]) in a talk given in honor of the celebrated mathematician Hansraj Gupta, we avoid discussing it here.

The bibliography covering all these topics is too vast to recall; hence, we have given references only to our work mentioned in the article.

$*$ This article is based on the text of the Presidential Address (Technical) given by Prof. B. Sury at the 86th Annual Conference of the IMS-An International meet held at Vellore Inst. of Tech., Vellore (T. N.) using online mode during December 17-20, 2020.

Theme 1: Arithmetic and congruence groups in algebraic groups

The prototype of arithmetic and congruence groups is $SL(n, \mathbb{Z})$ and its subgroups which contain the kernels $\Gamma(k)$ of homomorphisms $SL(n, \mathbb{Z}) \to SL(n, \mathbb{Z}/k\mathbb{Z})$ for various k.
It was proved in the 1960's by Bass-Lazard-Serre and, independently by Mennicke, that every subgroup of finite index in $SL(n, \mathbb{Z})$ contains $\Gamma(k)$ for some k, when $n \geq 3$. Already, in the 19-th century, the analogous statement was known to be false for $n = 2$. Informally, one says that the congruence subgroup property holds for $SL(n, \mathbb{Z})$ for $n \geq 3$ and fails for $n = 2$.

To point out that the congruence subgroup property for $SL(n, \mathbb{Z})$ $(n \geq 3)$ could have surprising, purely group-theoretic consequences, we mention in passing, the following interesting property deduced in collaboration with T N Venkataramana (see [3]):

For any fixed $n \geq 3$, there is a number $N(n)$ depending only on n so that every group of the form

$$Ker(SL_n(\mathbf{Z}) \to SL_n(\mathbf{Z}/k\mathbf{Z}))$$

can be generated by $N(n)$ elements for every $k > 1$. One may also give a description of generators for each k. The analogous assertion is false for $n = 2$.

The CS property was investigated for SL_n and Sp_n over rings of integers of number fields by Bass, Milnor and Serre. The methods essentially use only abelian class field theory. The analogue of the congruence subgroup property can be formulated for algebraic groups over global fields and the problem becomes much more difficult; we will recall this now.

Briefly recall that an algebraic group defined over a field k (considered as a subfield of $\mathbf{C}$) is a subgroup G of $GL(N, \mathbf{C})$ which is also the set of common zeroes of a finite set of polynomial functions $P(g_{ij}, det(g)^{-1})$ in $N^2 + 1$ variables with coefficients from k. Thus, there is a k-embedding $G \to GL(N)$. The definition has to be slightly modified if k is a field of positive characteristic. The group $G(k) = G \cap GL(N, k)$ turns out to be defined independent of the choice of k-embedding $G \leq GL(N)$.

Standard examples of algebraic groups defined over a subfield k of $\mathbf{C}$ are:

$$G = GL(n, \mathbb{C}) \ , \ SL(n, \mathbb{C}).$$

For any symmetric invertible matrix $M \in GL(n, k)$, the orthogonal group

$$O(M) = \{g \in GL(n, \mathbf{C}) :^t gMg = M\}.$$

For any skew-symmetric matrix $\Omega \in GL(2n, k)$, the symplectic group

$$Sp(\Omega) = \{g \in GL(2n, \mathbf{C}) :^t g\Omega g = \Omega\}.$$

Let D be a division algebra with center k - its dimension as a k-vector space must be n^2 for some n. Let $v_i; 1 \leq i \leq n^2$ be a k-basis of D (then it is also a $\mathbf{C}$-basis (as a vector space) of the algebra $D \otimes_k \mathbf{C}$). The right multiplication by v_i gives a linear transformation R_{v_i} from $D \otimes_k \mathbf{C}$ to itself, and thus, one has elements $R_{v_i} \in GL(n^2, \mathbf{C})$ for $i = 1, 2, \cdots, n^2$. The group

$$G = \{g \in GL(n^2, \mathbf{C}) : gR_{v_i} = R_{v_i}g \ \forall \ i = 1, 2, \cdots, n^2\}.$$

$G(k) =$ the nonzero elements of D.

If k denotes an algebraic number field and O_k its ring of integers, if $G \subset SL_n$ is a k-embedding of a linear algebraic group, define $G(O_k) := G \cap SL_n(O_k)$. For any non-zero ideal I of O_k, one has the normal subgroup

$$G(I) := \mathrm{Ker}(G(O_k) \to SL_n(O_k/I))$$

of finite index in $G(O_k)$; it is of finite index as O_k/I is finite.
Unlike $G(k)$, the definitions of $G(O_k)$ and $G(I)$ etc. depend on the k-embedding we started with, but it turns out that for a new k-embedding, the *new* $G(O_k)$ contains an *old* $G(I)$ for some $I \neq 0$.

The group $G(O_k)$ can be realized as a discrete subgroup of a product $G(\mathbf{R})^{r_1} \times G(\mathbf{C})^{r_2}$ of Lie groups, where k has r_1 real completions and r_2 non-conjugate complex completions.

More generally, let S be any finite set of inequivalent valuations containing all the archimedean valuations; the non-archimedean valuations in S correspond to nonzero prime ideals of O_k. The ring O_S of S-units of k consists of the elements of k which admit denominators only from primes in S.
Define $G(O_S) = G \cap SL_n(O_S)$; it is a discrete subgroup of the product $\prod_{v \in S} G(k_v)$ of real, complex and p-adic Lie groups, where k_v denotes the

completion of k with respect to the valuation v.

A subgroup $\Gamma \subset G(k)$ is called an S-congruence subgroup if, for some (and, therefore, any) k-embedding of G, the group Γ contains $G(I)$ as a subgroup of finite index for some nonzero ideal I of O_S.

One also defines a subgroup $\Gamma \subset G(k)$ to be an S-arithmetic subgroup if, for some (and, therefore, any) k-embedding, Γ and $G(O_S)$ are commensurable (i.e., $\Gamma \cap G(O_S)$ has finite index in both groups).
As S-congruence subgroups are clearly S-arithmetic, the converse question is natural, and is the simplest form of the congruence subgroup problem.

The congruence kernel
The two families (the S-arithmetic groups and the S-congruence groups) define topologies T_a, T_c on $G(k)$. The topology T_c is called the S-congruence topology on $G(k)$, and the topology T_a is called the S-arithmetic topology. As S-congruence subgroups are S-arithmetic subgroups, T_a is finer; note that T_a and T_c are equivalent if all subgroups of finite index in $G(O_S)$ are S-congruence subgroups. If the resulting completions of (uniform structures on) $G(k)$ are denoted by $\hat{G}_a$ and $\hat{G}_c$, then there is a continuous surjective homomorphism from $\hat{G}_a$ onto $\hat{G}_c$.
The kernel $C(S,G)$ of the above map, called the *S-congruence kernel*, measures the deviation.

A basic important property is that $C(S,G)$ is a profinite group. In fact, the closures $\widehat{\Gamma_a}, \widehat{\Gamma_c}$ of $G(O_S)$ in $\hat{G}_a$ and $\hat{G}_c$ respectively, are profinite groups and $C(S,G) \subset \widehat{\Gamma_a}$. Note that when $C(S,G)$ is trivial, all S-arithmetic subgroups are S-congruence subgroups.
The congruence subgroup problem (CSP) is the problem of determining the group $C(S,G)$ for any S,G.

Margulis-Platonov conjecture
A conjecture due to Serre predicts precisely for which S, G, the group $C(S,G)$ is finite; the finiteness of $C(S,G)$ has many interesting consequences like super-rigidity. For any S, the association $G \mapsto C(S,G)$ is a functor from the category of k-algebraic groups to the category of profinite groups. Given G, a general philosophy (which can be made very precise) is that the larger

S is, the 'easier' it is to compute $C(S,G)$.

In fact, when S consists of all the places of k excepting the (finitely many possible) non-archimedean places T for which $G(k_v)$ is compact (for the topology induced from k_v), the computation of $C(S,G)$ amounts to a conjecture of Margulis & Platonov. In fact, in most cases T is empty which means that the triviality of $C(S,G)$ (for the S mentioned last) is equivalent to the simplicity of the abstract group $G(k)/center$.

More generally, when S is co-finite, contains the archimedean places and all the nonarchimedean places where $G(k_v)$ is compact, (the more general version of) Margulis-Platonov conjecture has been proved for most cases (in my PhD thesis, this was done for groups of type A_1, B_n, C_n, D_n - see [4]). One important case where the Margulis-Platonov conjecture has still not been proved is that of the special unitary group of a division algebra with center k and with an involution of the second kind.

Necessary conditions for finiteness

The CSP for a general group reduces to CSP for connected reductive groups using essentially only the Chinese remainder theorem.

A reductive k-group G contains a central k-torus T (a connected, abelian k-subgroup $\mathbf{C}$-isomorphic to products of k^*) such that G/T is a semisimple group.

For tori T, the congruence kernel $C(S,T)$ is trivial - this is a theorem due to Chevalley and is essentially a consequence of Chebotarev's density theorem in global class field theory.

Hence, the problem reduces to that for semisimple groups.

For semisimple G, the group $G(O_S)$ (for any embedding) can be identified with a lattice in the group $G_S := \prod_{v \in S} G(k_v)$ under the diagonal embedding of $G(k)$ in G_S; that is, the quotient space $G_S/G(O_S)$ has a finite, G_S-invariant measure.

Finally, it is necessary (as observed by Serre) for the finiteness of $C(S,G)$ that G be simply-connected as an algebraic group - that is, there is no k-algebraic group $\tilde{G}$ admitting a surjective k-map $\pi : \tilde{G} \to G$ with finite nontrivial kernel.

Further, if one can compute $C(S,\tilde{G})$ for a simply-connected 'cover' $\tilde{G}$ of G, then one can compute $C(S,G)$. Thus, the CSP reduces to the problem for semisimple, simply-connected groups.

The reason that G must be simply-connected in order that $C(S,G)$ be finite, is as follows.

Let there exist a k-map : $\tilde{G} \to G$ with kernel μ and $\tilde{G}$ simply-connected. Now, the S-congruence completion $\tilde{G}_c$ can be identified with the S-adelic group $\tilde{G}(A_S)$. If π is the homomorphism from the S-arithmetic completion $\tilde{G}_a$ to $\tilde{G}_c = \tilde{G}(A_S)$, then it is easy to see that $C(S,G)$ contains the infinite group $\pi^{-1}(\mu(A_S))/\mu(k)$.

If G is simply-connected and the group $\prod_{v \in S} G(k_v)$ is noncompact (equivalently, $G(O_S)$ is not finite), one has the strong approximation property.

This means that the closure $\widehat{\Gamma_c}$ of $G(O_S)$ with respect to T_c can be identified with $\prod_{v \notin S} G(O_v)$. Here O_v is the local ring of integers in the p-adic field k_v. Thus, if $C(S,G)$ is trivial, the profinite completion of $G(O_S)$ is also equal to $\prod_{v \notin S} G(O_v)$.

Thus, roughly speaking, when the congruence kernel is trivial the topology given by subgroups of finite index is built out of the p-adic topologies.

For $G(k)$ itself, strong approximation means that G_c can be identified with the 'S-adelic group' $G(A_S)$, the restricted direct product of all $G(k_v), v \notin S$ with respect to the open compact subgroups $G(O_v)$.

As mentioned in the beginning, subgroups of finite index in $SL(n,\mathbf{Z})$ are congruence subgroups when $n \geq 3$ while this is not so when $n = 2$; what distinguishes these groups qualitatively is their rank.

If $G \subset SL_n$ is a k-embedding, one calls a k-torus T in G to be k-split, if there is some $g \in G(k)$ such that gTg^{-1} is a subgroup of the diagonals in SL_n. The maximum of the dimensions of the various k-split tori (if they exist) is called the k-rank of G. For example, k-rank $SL_n = n-1$ and k-rank $Sp_{2n} = n$.

If F is a quadratic form over k, it is an orthogonal sum of an anisotropic form over k and a certain number r of hyperbolic planes (r is called the Witt index of F), by Witt's classical theorem. Then, the group $SO(F)$ is a k-group whose k-rank is this same r.

Serre's conjecture

Let us consider a general semisimple, simply-connected k-group G. Assume that G is absolutely almost simple (that is, G has no connected normal algebraic subgroups).

First, it is easy to see that a necessary condition for finiteness of $C(S,G)$ is that for any nonarchimedean place v in S, the group G has k_v-rank > 0 (equivalently, $G(k_v)$ is non-compact by a theorem of Bruhat-Tits-Rousseau). Indeed, otherwise the whole of $G(k_v)$ is a quotient of $C(S,G)$.

For a finite S containing all the archimedean places, Serre formulated the characterization of the congruence subgroup property (that is, the finiteness of $C(S,G)$) conjecturally as follows.

Conjecture of Serre:

$C(S,G)$ is finite if, and only if, S-rank$(G) := \sum_{v \in S} k_v - \text{rank}(G) \geq 2$ and $G(k_v)$ is noncompact for each nonarchimedean $v \in S$.

When $C(S,G)$ is finite, one says that the CSP is solved affirmatively or that the congruence subgroup property holds for the pair (G,S).

For $k = \mathbf{Q}, S = \{\infty, p\}$ for some prime p and $G = SL_2$, the CSP was solved affirmatively by Mennicke and this fact can be used to show (as was done by Serre and Thompson) that the classical theory of Hecke operators 'lives' only on congruence subgroups of $SL_2(\mathbf{Z})$.

It should be noted that the finitness of $C(S,G)$ as against its being actually trivial, already has strong consequences like super-rigidity, as pointed by Serre.

This is one reason for considering not just the triviality but even the finiteness of $C(S,G)$ as a positive answer of the CSP.

In other words, if $C(S,G)$ is finite, then any abstract homomorphism from $G(O_S)$ to $GL_n(\mathbf{C})$ is essentially algebraic; there is a k-algebraic group homomorphism from G to GL_n which agrees with the homomorphism we started with, at least on a subgroup of finite index in $G(O_S)$.

In particular, $\Gamma/[\Gamma,\Gamma]$ is finite for every subgroup of finite index in $G(O_S)$.

This last fact has already been used to prove in some cases that $C(S,G)$ is NOT finite, by producing a Γ of finite index which surjects onto $\mathbf{Z}$.

Relation with group cohomology

As the S-congruence completion G_c of $G(k)$ can be identified with the 'S-adelic group' $G(A_S)$, we have the exact sequence

$$1 \to C(S,G) \to \hat{G_a} \to G(A_S) \to 1$$

which defines $C(S,G)$.
By looking at continuous group cohomology H^i with the universal coefficients $\mathbf{R}/\mathbf{Z}$, one has the corresponding Hochschild-Serre spectral sequence which gives the exact sequence

$$H^1(G(A_S)) \to H^1(\hat{G_a}) \to H^1(C(S,G))^{G(A_S)} \to H^2(G(A_S)).$$

As the congruence sequence above splits over $G(k)$, the last map actually goes into the kernel of the restriction map from $H^2(G(A_S))$ to $H^2(G(k))$. So, if α denotes the first map, then we have an exact sequence

$$1 \to \text{Coker}\alpha \to H^1(C(S,G))^{G(A_S)} \to Ker(H^2(G(A_S) \to H^2(G(k))).$$

Here $G(k)$ is considered with the discrete topology. The last kernel is called the S-metaplectic kernel and it is a finite group, and has been computed now in all cases. The cokernel of α is a finite group since $[G(k), G(k)]$ has finite index in $G(k)$. Therefore, the middle term $H^1(C(S,G))^{G(A_S)}$ is finite, and the quotient $C(S,G)/[C(S,G), \hat{G_a}]$ is finite.

In other words, we have: $C(S,G)$ *is finite if, and only if, it is contained in the center of* $\hat{G_a}$.

The centrality of $C(S,G)$ has been proved for most cases of S-rank at least 2; the important case of groups of type A_n which have k-rank 0 is wide open.
Split and quasi-split groups were treated by Matsumoto and Deodhar. Raghunathan's path-breaking works in 1976, 1986 solved the problem for all k-isotropic groups.

One general method for proving centrality already appeared in Raghunathan's second paper, and works for anisotropic groups as well in the following way (I had proved a version of this for special cases but Rapinchuk proved it in general).

Let G be an absolutely simple, simply connected algebraic group over a global field k satisfying the M-P conjecture. Assume that for every $v \notin S$, there is a subgroup G_v of $\hat{G}_a$ so that the following conditions are satisfied:
(i) *$\pi(G_v) = G(k_v)$ for all $v \notin S$, where π is the map from $\hat{G}_a$ to $G(\mathbb{A}_S)$ as in the exact sequence defining $C(S, G)$;*
(ii) *G_v and G_w commute element-wise for all $v \neq w$ outside S;*
(iii) *the G_v's for $v \notin S$ generate a dense subgroup of $\hat{G}_a$.*
Then $C(S, G)$ is central in $\hat{G}_a$.

The computation of the metaplectic kernel uses some wonderful results of Calvin Moore on uniqueness of reciprocity laws - these show that the Artin Reciprocity Law is the only general reciprocity law. Robert Steinberg studied central extensions of the group $G(k)$ for a simply connected Chevalley group G over an arbitrary field. Moore used Steinberg's results to connect topological central extensions of $G(k_v)$'s with norm residue symbols of local class field theory. Steinberg had shown that a topological central extension of $G(k_v)$ by $\mathbb{R}/\mathbb{Z}$ can be characterized by a certain 2-cocycle (called the Steinberg cocycle) $k_v^* \times k_v^* \to \mathbb{R}/\mathbb{Z}$.

Moore showed that the corresponding Steinberg cocycle c_v is the composition of the norm residue symbol $(.,.)_v : k_v^* \times k_v^* \to \mu(k_v)$ with a character χ_{c_v} of the group $\mu(k_v)$ of roots of unity.
Given an element of the metaplectic kernel $M(S, G)$, one has a corresponding topological central extension of $G(A_S)$. As it splits over $G(k)$, we have

$$\prod_{v \notin S} \chi_{c_v}((r, s)_v) = 1 \ \forall \ r, s \in k^*;$$

this connects the norm residue symbols of different places of k and can be thought of as an example of a global reciprocity law.

Artin's reciprocity law asserts that

$$\prod_{v \notin S} (r, s)_v^{|\mu(k_v)/\mu(k)|} = 1 \ \forall \ r, s \in k^*.$$

Moore proved the remarkable result that any relation between all the norm residue symbols which holds on $k^* \times k^*$ is a power of the Artin reciprocity law, and used this to compute $M(S, G)$.

For k-rank 0 groups, an analogue of Moore's uniqueness reciprocity laws was proposed by Gopal Prasad (a detailed proof appears in [5]).

Various profinite-group-theoretic formulations equivalent to the CSP have been studied by many authors. For instance, the finiteness of $C(S,G)$ has been characterized in terms of *polynomial index growth* for $\widehat{\Gamma_a}$.

Another characterization was conjectured by Lubotzky and proved in collaboration with Vladimir Platonov ([6]); it shows that an S-arithmetic group has the CS property if, and only if, it can be embedded as a closed subgroup of $\prod_p SL(n, \mathbf{Z}_p)$ for some n.

In that paper, we conjectured the following more general result which was proved by Liebeck, and Pyber:
A topologically finitely generated profinite group Δ which can be continuously embedded in the S-adelic group $\prod_p SL(n, \mathbf{Z}_p)$ for some n, has bounded generation.

The centrality/finiteness of $C(S,G)$ (for the cases where it is conjectured to be finite) is still open for anisotropic groups of type A_n including the two very important cases where new ideas may be required - a solution in these cases may have a lot of relevance to the theory of automorphic forms.
The two cases alluded to are :
(i) $G = SL(1,D)$ for a division algebra with center k, and
(ii) $G = SU(1,D)$ where D is a division algebra with a center K which is a quadratic extension of k and D has a K/k-involution.

Structure of $C(S,G)$ when it is infinite
Melnikov used results on profinite groups to prove that, for $SL_2(\mathbf{Z})$, one has $C(\{\infty\}, SL_2) \cong \hat{F_\omega}$, the free profinite group of countably infinite rank.

With Mason, Premet and Zalesskii, we computed (in [7]) the structure of $C(S,G)$ when G,S are so that $G(O_S)$ can be identified with a lattice in $G(k_v)$ for some non-archimedean completion k_v and k is a global field of any characteristic.

We use group actions on trees and profinite analogues as well as a detailed analysis of unipotent radicals in every rank 1 group over a local field of positive characteristic, to deduce the following structure theorem.

Theorem ([7]).
(i) *If $G(O_S)$ is cocompact in $G(k_v)$ (in particular, if char. $k = 0$), then $C(S,G) \cong \hat{F}_\omega$.*
(ii) *If $G(O_S)$ is nonuniform (therefore, necessarily char. $k = p > 0$), then $C(S,G) \cong \hat{F}_\omega \sqcup T$, a free profinite product of a free profinite group of countably infinite rank and of the torsion factor T which is a free profinite product of groups, each of which is isomorphic to the direct product of $2^{\aleph_0}$ copies of $\mathbf{Z}/p\mathbf{Z}$.*

A surprising consequence of the theorem is that $C(S,G)$ *depends only on the characteristic of* k when $G(\mathcal{O}_S)$ is a lattice in a rank 1 group over a non-archimedean local field.

As a by-product of the above result, we also obtain the following result which is of independent interest:

Let U be the unipotent radical of a minimal k_v-parabolic subgroup of G where G is as in (ii) above, then either U is abelian or is automatically defined over k.

Theme 2: Central extensions
Let G be an abstract group and A an abelian group; central extensions of G by A arise naturally in the context of projective representations.
But, we principally look at the case $G \leq \mathcal{G}(k)$ where $\mathcal{G}$ is a semisimple algebraic group a p-adic field k and A is finite - this arises from the CSP, where the central extensions which arise are topological central extensions.

Let G be an abstract group and A an abelian group. A central extension of G by A is a pair (E, ϕ) where E is a group containing A in its center, $\phi : E \to G$ a surjective homomorphism whose kernel equals A:

$$1 \to A \to E \xrightarrow{\phi} G \to 1.$$

The theory of central extensions generalizes that of covering spaces for topological groups as it generalizes to disconnected groups also. The 'Baer multiplication' gives a group structure on the set of isomorphism classes of parametrized central extensions of G by A.
One may use Eilenberg-Maclane's group cohomology to study central extension. The second cohomology group $H^2(G, A)$ where G acts on A trivially, is isomorphic to the above group of parametrized central extensions of G by A. The associativity of the Baer multiplication reflects as the 2-cocycle condition.

When the groups are (Hausdorf, 2nd countable, locally compact) topological groups, then one can talk about a topological central extension.

When G is a topological group, and A is a closed, central subgroup of a topological group E and $\phi : E \to G$ a continuous surjection such that $Ker\phi = A$, the map $E/A \to G$ induced by ϕ, is a topological isomorphism. The group of topological central extensions is isomorphic to $H^2_{cont}(G, A)$ (sometimes written $H^2_{top}(G, A)$) where G acts trivially and the cohomology is defined by continuous cocycles.
Moreover, in the case when G is a totally disconnected group (like the group of rational points of an algebraic group over a p-adic field), then $H^2_{cont}(G, \mathbf{R}/\mathbf{Z})$ is isomorphic to $H^2_{cont}(G, \mathbf{Q}/\mathbf{Z})$ where $\mathbf{Q}/\mathbf{Z}$ is considered as a discrete group and, it turns out from results of Mackey-Moore-Wigner, the latter groups can be computed as the cohomology $H^2_{meas}(G, \mathbf{Q}/\mathbf{Z})$ based on measurable cochains.

Using the behaviour of power maps on compact, p-adic Lie groups, one can deduce (see '[8]) in certain cases that every abstract central extension of G by A is automatically topological. This was used to deduce another proof of a theorem of Tate on K_2.

Using the Tits building
We can use the Tits building to obtain an application to abstract central extensions of the group of rational points of p-adic algebraic groups by finite p-groups (see [9]).

For a connected, semisimple, simply-connected algebraic group G defined and isotropic over a nonarchimedean local field k, and A is a finite, abelian p-group where p is the characteristic of the residue field of k, then when G of k-rank at least 2, we can show that the group $H^2(G(k), A)$ of abstract central extensions injects into a finite direct sum of $H^2(H(k), A)$ for certain semisimple k-subgroups H of smaller k-ranks.

On the way, we can derive some results interesting results of independent interest, which are valid over a general field k; for instance, one can prove that the analogue of the Steinberg module for $G(k)$ has no non-zero $G(k)$-invariants.

Let k be a nonarchimedean local field and A a finite, abelian p-group, where p is the characteristic of the residue field of k. Let G be an absolutely almost simple, simply-connected algebraic group defined over k with k-rank$(G) = r \geq 2$. Then there exist semisimple k-subgroups $G_1, \cdots, G_r$ without k-anisotropic factors and, each of k-rank equal to k-rank$(G) - 1$ and semisimple k-subgroups G_{ij} of $G_i \cap G_j$ such that the 'restriction' map

$$H^2(G(k), A) \to \bigoplus_{i=1}^{r} H^2(G_i(k), A)$$

of abstract central extensions is injective, and injects into

$$Ker(\bigoplus_{i \leq r} H^2(G_i(k), A) \to \bigoplus_{i<j} H^2(G_{ij}(k), A)).$$

In particular, if the abstract central extensions are automatically topological for all k-subgroups of k-rank $r-1$, then the same holds for G.

The technique cannot address k-anisotropic groups as the Tits building is empty; it is an open problem to determine the central extensions of $SL(1, D)$ by $\mathbf{F}_p$ where D is a division algebra over a p-adic field - we mention partial results later below.

Modus operandi

To compute $H^n(G(k), A)$, the abstract group cohomology $Ext^n(G(k), A)$, we use a resolution of the $G(k)$-module provided by the simplicial Tits building - an idea due to Raghunathan.

The Tits building of G over k is a simplicial complex of dimension $r-1$ (where $r = k$-rank(G)) whose vertices are maximal parabolic k-subgroups;

in fact, by a theorem of Solomon and Tits, it has the homotopy type of a bouquet of spheres, each of dimension $r-1$.
A set $\{P_1, \cdots, P_d\}$ of vertices forms a simplex if and only if the intersection $\cap_{i=1}^{d} P_i$ is a parabolic k-subgroup - this parabolic is precisely the stabilizer of the simplex.
The group $G(k)$ acts on the set of parabolic k-subgroups by conjugation.
The set of s-dimensional simplices in the Tits building is $G(k)$-equivariantly parametrized by $\bigcup_{|\Theta|=s+1} G(k)/P_\Theta(k)$ where Θ's are subsets of a fixed set of simple k-roots, and P_Θ's are corresponding k-parabolics.
Since $G(k)$ acts simplicially on the Tits building, we have a complex of $G(k)$-modules

$$0 \to A \to C^0(A) \to C^1(A) \to \cdots\cdots \to C^{r-1}(A) \to 0$$

where $C^i(A)$ is the group of simplicial i-cochains of the Tits building, with coefficients in A.
Therefore, $C^i(A) = \bigoplus_{|\Theta|=i+1} Ind_{P_\Theta(k)}^{G(k)}(A)$ as a $G(k)$-module.
Here, $Ind_{P_\Theta(k)}^{G(k)}(A)$ stands for the $G(k)$-module induced by the trivial action of $P_\Theta(k)$ on A.
So, the simplicial cohomology groups of the Tits building of G over k with coefficients in A, are all zero excepting the 0-th and the $(r-1)$-th one.
This top cohomology, denoted by $St(A)$, is called the Steinberg module of G over k with corfficients in A.
Therefore, the $G(k)$-complex

$$0 \to A \to C^0(A) \cdots \to C^{r-1}(A) \to St(A) \to 0$$

is exact.
The associated spectral sequence which computes $H^*(G(k), A)$ has its $E_2^{i,j}$-term to be the i-th cohomology of the complex

$$0 \to H^j\left(G(k), C^0(A)\right) \to H^j\left(G(k), C^1(A)\right)$$
$$\to \cdots \to H^j(G(k), St(A)) \to 0.$$

Using Shapiro's lemma, this is just the complex

$$0 \to \bigoplus_{|\Theta|=1} H^j(P_\Theta(k), A) \cdots \to H^j(P_\Delta(k), A) \to H^j(G(k), St(A)) \to 0.$$

The proof of the restriction theorem proceeds in steps - each step reducing the computation of the relevant cohomology groups to a computation for

subgroups of a particular kind like parabolic subgroups, then their Levi subgroups and finally to groups of smaller k-ranks.
For the proof, we will require a result of independent interest which we prove for a general field k.
For an arbitrary field k, let $St(A)$ be defined as the top-dimensional cohomology with coefficients in A of the Tits building of G over k. Assume that if A has even order, then k is infinite. Then, we have $St(A)^{G(k)} = 0$.
During the course of proof of the above lemma on Steinberg invariants, the following result of independent interest is proved.
Let k be any infinite field and let G be a semisimple, algebraic k-group which is k-isotropic, and let S be a maximal k-split torus and let P be a minimal parabolic k-subgroup of G containing S. Let W denote the k-Weyl group and U^- denote the unipotent radical of the parabolic k-subgroup which is opposite to P. Then, $\bigcap_{w \in W} U^-(k)P(k)w \neq \emptyset$.

Theme 3: Division algebras over global and local fields
As mentioned earlier, the M-P (Margulis-Platonov) conjecture is known for all simply connected groups over global fields other than the groups of outer type A. These arise as follows:

Let K/k be a quadratic extension of a global field and let D be a central division algebra over K with a K/k-involution σ. One has the unitary group

$$U(1, D) = \{d \in D^* : \sigma(d) = d^{-1}\}$$

and the special unitary group

$$SU(1, D) := \{d \in U(1, D) : N_{red}(d) = 1\}.$$

This corresponds to the k-points of anisotropic groups of outer type A^n.

In the case of global fields, the M-P conjecture for $SL(1, D)$ has been proved due to the efforts of several mathematicians working on different aspects using diverse methods (Margulis, Raghunathan, Rapinchuk, Segev, Seitz).

The equality $SL(1, D) = [D^*, D^*]$ plays a crucial preliminary role in resolving the M-P conjecture for $SL(1, D)$, but even the analogue of this as to whether $SU(1, D) = [U(1, D), U(1, D)]$ is unsolved yet, let alone M-P for

$SU(1, D)$.

If one considers more general fields like $K(X)$, there are infinitely many examples (as shown in [10]) that the quotient $SU(1, D)/[U(1, D), U(1.D)]$ can be infinite. We proved:

Let $n \geq 3$, and let ζ be a primitive n-th root of unity. Then, there exists a division algebra D of index n with center $\mathbb{Q}(\zeta)(x)$ which has an involution of the second kind such that the corresponding group $SU(1, D)/[U(1, D), U(1.D)]$ is infinite.

In another work ([11]), when D is a division algebra over a global field, we have outlined some partial results towards the equality of $SU(1, D) = [U(1, D), U(1, D)]$ when the degree is odd. In that paper, we have dealt also with quaternion division algebras, and used results of Margulis, to explicitly determine the quotient

$$\left|SU(1, D)/[U(1, D), U(1, D)]\right| = \prod_{v \in T} n_v.$$

Here, n_v is defined as follows.
Denote by q_v the order of the residue field of k_v for any nonarchimedean place v of k and, let:

$n_v = q_v + 1$ if $K \otimes_k k_v$ is an unramified field extension of k_v;

$n_v = 2$ if $K \otimes_k k_v$ is a ramified quadratic extension and the residue characteristic of k_v is not 2;

$n_v = 1$ if $K \otimes_k k_v$ is a ramified quadratic extension and the residue characteristic of k_v is 2;

$n_v = 1$ if $K \otimes_k k_v$ is not a field.

This shows quaternion division algebras over global fields do not always satisfy $SU(1, D) = [U(1, D), U(1, D)]$.
Later, Yanchevskii and his student have given another interesting proof.

Topological central extensions of $SL(1, D)$

We could use the Tits building to study central extensions of $G(k)$ for k-isotropic groups G. For anisotropic groups over a local field k, totally different techniques are needed.

When k is a p-adic field of characteristic zero, the k-anisotropic, absolutely simple, simply connected groups G arise as follows.
D is a finite-dimensional central division algebra over k, the group $G(k) = SL_1(D)$ consisting of elements of reduced norm 1 in D.
It is expected that

$$H^2(G, \mathbf{R}/\mathbf{Z}) \cong \mu(k)_p,$$

where $\mu(k)_p$ is the finite cyclic group of p-th power roots of unity in k.
Gopal Prasad & M.S.Raghunathan proved that $H^2(G, \mathbf{R}/\mathbf{Z})$ is a finite cyclic group containing an isomorphic copy of $\mu(k)_p$ and is trivial if $\mu(k)_p$ is trivial.
As $G(k)$ is totally disconnected, the cohomology group $H^2(G(k), \mathbf{R}/\mathbf{Z})$ (based on continuous cochains) is isomorphic to $H^2(G(k), \mathbf{Q}/\mathbf{Z})$. Calvin Moore showed that the latter group can be computed using measurable rather than continuous cochains.

Results of Gopal Prasad & Raghunathan show that $H^2(G, \mathbf{R}/\mathbf{Z})$ is a finite p-group, and also that if k contains a primitive p^r-th root of unity, then this H^2 also has an element of order p^r. The converse is also expected to hold and they showed this for $r = 1$.

The first non-trivial case left open is $r = 2$. I have been able to prove ([12]) in a special case $p = 3$, $d = 2$ and $e = 6$ that if H^2 has an element of order p^2, then k contains a primitive p^2-th root of unity.

M. Ershov has recently obtained very nice results on $H^2(SL_1(D), \mathbf{R}/\mathbf{Z})$.

Theme 4: Bounded generation and finite width
A remarkable refinement of finite generation came to the fore in the work of A.S.Rapinchuk; this is known as bounded generation (by cyclic subgroups). An abstract group G is said to be boundedly generated of degree $\leq n$ if there exists a sequence of (not necessarily distinct) elements $g_1, \cdots, g_n$ such that

$$G = < g_1 >< g_2 > \cdots < g_n > .$$

A free, non-abelian group (and therefore, $SL_2(\mathbf{Z})$ also) is not boundedly generated.

On the other hand, a group like $SL_n(\mathbf{Z})$ for $n \geq 3$, is boundedly generated by elementary matrices (an elementary proof of this can be given using Dirichlet's theorem on primes in arithmetic progressions).
It turns out that this difference is an attribute of the congruence subgroup property as revealed in the work of V P Platonov & A S Rapinchuk and, independently, that of A Lubotzky.

Finitely generated matrix groups are residually finite, and hence embed in their profinite completions.
A profinite group G is said to be boundedly generated as a profinite group if there exists a sequence of (not necessarily distinct) elements $g_1, \cdots, g_n$ such that

$$G = \overline{< g_1 >< g_2 >} \cdots \overline{< g_n >}$$

where the 'bar' denotes closure. It follows from Lazard's profound work on p-adic Lie groups and the solution to the restricted Burnside problem that a pro-p group has bounded generation (as a profinite group) if and only if it is a p-adic compact Lie group; this can be thought of as an analogue of Hilbert's 5th problem for the p-adic case.

If an abstract group has bounded generation, then so do its pro-p completions for each prime p (as does the full profinite completion). Therefore, we have a nice sufficient criterion for an abstract group to have a faithful linear representation - viz., if it has bounded generation and is virtually residually-p. We have used this idea ([13]) to show that the automorphism group of a free group does not have bounded generation.

The question of existence of bounded generation for matrix groups over number-theoretic rings has rather deep connections with other properties.
The profinite completion of an arithmetic group is boundedly generated if, and only if, it has the congruence subgroup property - this amazing result was proved independently by V.P.Platonov & A.S.Rapinchuk and by A.Lubotzky.

Lubotzky also conjectured that the congruence subgroup property holds for an S-arithmetic group if, and only if, it can be embedded as a closed subgroup of $SL_n(\mathbf{A})$ - a so-called adelic group. As mentioned earlier, this

was proved in collaboration with Platonov, where we also conjectured that finitely generated closed subgroups of adelic groups have bounded generation; since then, it has been proved by Martin Liebeck & Laszlo Pyber.
It is still an intriguing open question as to whether the property of bounded generation for an S-arithmetic group Γ is equivalent to bounded generation for its profinite completion (which is, as mentioned earlier, equivalent to the congruence subgroup property holding good for Γ). The answer is perhaps in the negative and certain arithmetic subgroups of $Sp(n,1)$ could provide counter-examples. Moreover, this is a subtle question specific to arithmetic groups and not for more general profinite groups because:

The group $\prod_{r\geq 1} PSL_n(\mathbf{F}_{2^r})$ is boundedly generated group as a profinite group but none of its discrete subgroups is boundedly generated.

O.Tavgen proved bounded generation of arithmetic groups in rank > 1 groups. However, bounded generation for co-compact arithmetic lattices is still an open question in general excepting the case of quadratic forms; note here there are no unipotent elements. Very recently, Corvaja, Rapinchuk, Ren and Zannier have shown that S-arithmetic groups in anisotropic groups cannot be boundedly generated.

We mention in passing that some matrix groups are finitely generated but are not boundedly generated:

Consider, for example, the group of 2×2 matrices

$$\begin{pmatrix} t^m & t^n f(t) \\ 0 & 1 \end{pmatrix}$$

where f is any polynomial with integer coefficients and m, n are any integers, is an infinite group (it can be identified with the wreath product of $\mathbb{Z}$ with itself; it has an infinitely generated abelian subgroup, but is itself generated by just two matrices:
$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
One can prove by combinatorial methods that the above matrix group does not have bounded generation.

With Nikolay Nikolov, we characterized the wreath products $A \wr B$ which have bounded generation ([14]).

$SL(2, O)$ - work with Morgan and Rapinchuk.
Let $\mathbb{O}$ be the ring of S-integers in a number field k whose group of units $\mathbb{O}^\times$ is infinite. We show that every matrix in $\Gamma = \mathrm{SL}_2(\mathbb{O})$ is a product of at most 9 elementary matrices. As a consequence, we obtain that Γ is boundedly generated as an abstract group by $9[k:\mathbb{Q}]+10$ cyclic subgroups. This work finishes a long line of work by several mathematicians over five decades; it appears in [15].

As a consequence of our results, we can deduce:

Let $\mathbb{O}^\times$ be infinite. Then, for $n \geq 3$, the abstract group $SL_n(\mathbb{O})$ can be boundedly generated by $4 + \frac{n(3n-1)}{2}$ elementary generators.
If $S = V_\infty^k$, our theorems yield a factorization of $\mathrm{SL}_2(\mathscr{O})$ as a finite product $\langle \gamma_1 \rangle \cdots \langle \gamma_d \rangle$ of cyclic subgroups where all generators γ_i are elementary matrices, hence *unipotent*.
On the contrary, when $S \neq V_\infty^k$, the factorization we produce involves some diagonal (*semisimple*) matrices; there exists no factorization with all γ_i unipotent.
With a little bit of work, we can then deduce that $SL_2(\mathbb{O})$ has bounded generation by $9[k:\mathbb{Q}] + 10$ cyclic groups.
If $O = \mathbb{Z}[1/p]$, where $p > 7$ is a prime, it turns out that the assertion that 5 cyclic subgroups suffice. But, not every matrix in $\mathrm{SL}_2(O)$ is a product of four elementary matrices.

The key notion used is that of $\mathbb{Q}$-split prime: we say that a prime $\mathfrak{p}$ of a number field k is $\mathbb{Q}$-split if it is non-dyadic and its local degree over the corresponding rational prime is 1.

Some simple properties of $\mathbb{Q}$-split primes are listed as:

Let $\mathfrak{p}$ be a $\mathbb{Q}$-split prime in $\mathbb{O}$, and for $n \geq 1$ let $\rho_n \colon \mathbb{O} \to \mathbb{O}/\mathfrak{p}^n$ be the corresponding quotient map. Then:
(a) *the group of invertible elements $(\mathbb{O}/\mathfrak{p}^n)^\times$ is cyclic for any n;*

(b) *if* $c \in \mathbb{O}$ *is such that* $\rho_2(c)$ *generates* $(\mathbb{O}/\mathfrak{p}^2)^\times$ *then* $\rho_n(c)$ *generates* $(\mathbb{O}/\mathfrak{p}^n)^\times$ *for any* $n \geq 2$.

Among other things, we prove the following refinement of Dirichlet's theorem:

Let $\mathbb{O}$ *be the ring of* S*-integers in a number field* k *for some finite* $S \subset V^k$ *containing* V^k_∞. *If nonzero* $a, b \in \mathbb{O}$ *are relatively prime (i.e.,* $a\mathbb{O}+b\mathbb{O} = \mathbb{O}$*), then there exist infinitely many principal* $\mathbb{Q}$*-split prime ideals* $\mathfrak{p}$ *of* $\mathbb{O}$ *with a generator* π *such that* $\pi \equiv a$ *(modulo* $b\mathbb{O}$*) and* $\pi > 0$ *in all real completions of* k.

We need number-theoretic results that we prove using class field theory; in particular applying Chebotarev's Density Theorem to a specific automorphism of an appropriate finite Galois extension, we prove the following key result.

To describe the results, we use the following notations:
let $\mu = |\mu(k)|$ be the number of roots of unity in k, let K be the Hilbert S-class field of k, and let $\tilde{K}$ be the Galois closure of K over $\mathbb{Q}$. Suppose we are given two finite sets P and Q of rational primes. Let

$$\mu' = \mu \cdot \prod_{p \in P} p,$$

pick an integer $\lambda \geq 1$ which is divisible by μ and for which $\tilde{K} \cap \mathbb{Q}^{\mathrm{ab}} \subseteq \mathbb{Q}(\zeta_\lambda)$, and set

$$\lambda' = \lambda \cdot \prod_{q \in Q} q.$$

Then, we prove:

Let $u \in \mathbb{O}^\times$ *be a unit of infinite order such that* $u \notin \mu(k)_p (k^\times)^p$ *for every prime* $p \in P$, *and let* $\mathfrak{q}$ *be a* $\mathbb{Q}$*-split prime of* $\mathbb{O}$ *which is relatively prime to* λ'. *Then there exist infinitely many principal* $\mathbb{Q}$*-split primes* $\mathfrak{p} = \pi\mathbb{O}$ *of* $\mathbb{O}$ *with a generator* π *such that:*
(1) for each $p \in P$, *the* p*-primary component of* $\phi(\mathfrak{p})/\mu$ *divides the* p*-primary component of the order of* $u \pmod{\mathfrak{p}}$*;*
(2) π *(mod* $\mathfrak{q}^2$*) generates* $(\mathbb{O}/\mathfrak{q}^2)^\times$*;*
(3) gcd $(\phi(\mathfrak{p}), \lambda') = \lambda$.

As mentioned above, if $S = V_\infty^k$, then our theorems yield a factorization of $\mathrm{SL}_2(\mathscr{O})$ as a finite product $\langle\gamma_1\rangle\cdots\langle\gamma_d\rangle$ of cyclic subgroups where all generators γ_i are elementary matrices, hence unipotent but when $S \neq V_\infty^k$, the factorization must involve some semisimple matrices.

It is worth pointing out in the latter case why there is no factorization with all γ_i unipotent. Indeed, let $v \in S \setminus V_\infty^k$ and let $\gamma \in \mathrm{SL}_2(\mathscr{O})$ be unipotent.

Then there exists $N = N(\gamma)$ such that for any $a = (a_{ij}) \in \langle\gamma\rangle$ we have $v(a_{ij}) \leq N(\gamma)$ for all $i, j \in \{1, 2\}$.

It follows that if $\mathrm{SL}_2(\mathscr{O}) = \langle\gamma_1\rangle\cdots\langle\gamma_d\rangle$ where all γ_i are unipotent, then there exists N_0 such that for any $a = (a_{ij}) \in \mathrm{SL}_2(\mathscr{O})$ we have $v(a_{ij}) \leq N_0$ for $i, j \in \{1, 2\}$, which is absurd.

$SL(n, F[X])$ for F finite field and $n \geq 3$

Here is a result (proved by Bogdan Nica) on bounded elementary generation of $SL(n)$ over the polynomial ring over a finite field F:

If $n \geq 3$, the group $SL_n(F[X])$ is boundedly generated by $29 + \frac{n(3n-1)}{2}$ elementary matrices.

Compare it with the result of van der Kallen that $SL_n(\mathbb{C}[X])$ is not boundedly generated by elementary matrices. The proof over $F[X]$ is based on the following analogue of Dirichlet's theorem due to Kornblum and Artin:

If $f, g \in F[X]$ are relatively prime and $g \neq 0$, then there are infinitely many primes congruent to g mod f. Further, such a prime can have arbitrary degree provided the degree is sufficiently large.

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SOME GLIMPSES INTO NONCOMMUTATIVE RING THEORY*

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Dedicated to Late Professor Hansraj Gupta

Abstract. We will give some glimpses into noncommutative ring theory by discussing some typical results. One of the aim is to demonstrate some basic ways in which noncommutative ring theory is different from its commutative version

1. Introduction

Since the discovery of Division rings by Hamilton in 1843, study of noncommutative rings and modules over them, i.e., noncommutative algebra, has become a major area of research which has attracted several leading mathematicians like Joseph Wedderburn, Emil Artin, Richard Brauer, Nathan Jacobson, P. M. Cohn, Emmy Noether, S. A. Amitsur, I. N. Herstein and more recently K. R. Goodearl, Donald Passman, George Bergman, T. Y. Lam, Agata Smoktunowicz. We refer the reader to [3], [4], [5], [6], [7], [15], [17], [20], [21], [24], [26] for excellent expositions of noncommutative algebra.

This expository article is intended to provide a sneak peek into noncommutative algebra. This is done by discussing some basic themes and some typical results of the subject. In section 2 we discuss some important commutativity results. In section 3 behaviour of nilpotent elements and nil one sided ideals in noncommutative rings is discussed. In section 4 we discuss polynomials over noncommutative rings. In the last section 5 we discuss invertibility of square matrices over noncommutative rings.

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Throughout all rings will contain identity unless specifically stated otherwise.

2. Commutativity Results

Since Wedderburn in 1905 proved the wonderful result that a finite division ring is a field, a major theme in noncommutative ring theory has been the study of commutativity results which involves imposing conditions on rings which imply some sort of commutativity in rings. Stone [31, Theorem 1] in 1936 proved that if $a^2 = a$ for every $a \in R$, then R is commutative. This led to the following natural question.

Question 2.1. *If there exists $n \in \mathbb{N}$ such that $a^n = a$ for every $a \in R$, then is R commutative?*

Note that an affirmative answer to this question would generalize the Wedderburn's Little Theorem. Nothing happened on this question till 1945, when three mathematicians Neal McCoy, Nathan Jacobson and Irving Kaplansky, independently, took a crack at it. Kaplansky [18] narrates the further story as follows.

McCoy and I made some progress but were outdistanced by Jacobson [16], who went the whole way.…There is a little more to tell and I shall tell it for a reason. I wrote up what I had and submitted it to the Duke Journal, it was accepted. Then I learned about Jacobson's work, and he sent me a reprint of [16]. I tossed sleeplessly for a night thinking about what to do. The next morning I decided to withdraw my paper. I have never regretted the decision. My reason for relating this story is that perhaps some young mathematician who faces a similar decision may read it. I hope that he or she will similarly decide to withdraw a superseded paper.

In 1945 Nathan Jacobson [16, Theorem 11] proved much more than what Question 2.1 had asked for.

Theorem 2.2. *If for every $a \in R$ there exists a positive integer $n(a) > 1$, depending on a, such that $a^{n(a)} = a$, then R is commutative.*

Note that the condition

$$a^{n(a)} = a \text{ for every } a \in R$$

is not necessary for R to be commutative. After that I.N. Herstein came to picture and proved a flurry of wonderful results and set the stage for hundreds of similar results. In 1951 Herstein [11, Theorem 18] proved the following result.

Theorem 2.3. *Let R be a ring with center $Z(R)$ and $n > 1$ be a fixed integer. Then R is commutative if and only if $a^n - a \in Z(R)$ for every $a \in R$.*

In 1953 Herstein [12, Theorem 21] and [13, Theorem 19] proved the following two generalizations of Theorem 2.3.

Theorem 2.4. *A ring R is commutative if and only if for every $a \in R$ there exists a positive integer $n(a) > 1$, depending on a, such that $a^{n(a)} - a \in Z(R)$.*

Theorem 2.5. *If for every $a \in R$ there exists a polynomial $p_a(x) \in \mathbb{Z}[x]$ such that*

$$a^2 p_a(a) - a \in Z(R),$$

then R is commutative.

In 1957 Herstein [14, Theorem 6] proved a yet another commutativity result as follows.

Theorem 2.6. *A ring R is commutative if and only if for any $a, b \in R$ there exists a positive integer $n(a, b) > 1$ such that*

$$(ab - ba)^{n(a,b)} = ab - ba.$$

In attempts to prove these commutativity results Herstein discovered a very powerful method of first proving a result in division rings, then extending it to left primitive rings using Jacobson's Density Theorem, then to semiprimitive rings using the fact that every semiprimitive ring is a subdirect product of left primitive rings, and lastly to the arbitrary rings (see [20, Page 208]). Kaplansky [18] in 1995 writes the following about the method discovered by Herstein to prove commutativity results.

To this day I know no way of proving commutativity other than invoking all the Herstein's devices, including a reduction to subdirectly irreducible case. (He once told me that this reduction was an act of desperation.) On this matter of use of subdirect irreducibility, I am reminded of Wigner's paper on the unreasonable effectiveness of mathematics in physical sciences. I feel that subdirect irreducibility is unreasonably effective in proving commutativity theorems.

These results by Herstein motivated an enormous amount of commutativity results which involved myriad of conditions forcing rings to be commutative. This theme is no longer fashionable as, apparently, it has almost reached a saturation point. We refer the reader to [28], where commutativity results from 1950-2005 have been reviewed, and [18] for a very interesting exposition.

3. Prime Ideals, Nilpotency and Köthe's Conjecture

Let $R = \mathbb{M}_2(\mathbb{S})$, for any ring S, and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Then $0 = a^2 \in I$ for every ideal I of R but $a \notin I$ for any ideal $I \neq R$. In particular, the complement of no ideal in R is multiplicative! So the localization, which is an important tool in commutative algebra, is no longer available in noncommutative rings. Also it follows that the elementwise definition of prime ideals is too restrictive for noncommutative rings. So prime ideals in noncommutative rings are defined by replacing elements with ideals. We know that the intersection of all prime ideals in a commutative ring coincides with the set of all nilpotent elements of the ring. But the intersection of all prime ideals in a noncommutative ring may not contain any nonzero nilpotent element of the ring. For instance, take the ring $\mathbb{M}_n(F)$ for any field F and any positive integer $n > 1$.

Unlike commutative rings where nilpotent elements form an ideal, sum of two nilpotent elements may not be nilpotent in noncommutative rings. For instance

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is invertible in $\mathbb{M}_2(R)$ although $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

A subset S of a ring R is said to be nil if every element of S is nilpotent.

Proposition 3.1. *If I is a nil two sided ideal and J is a nil right ideal of any ring R, then $I + J$ is nil.*

Proof. If $x = a + b$, where $a \in I$ and $b \in J$, then $x + I = b + I$ is nilpotent in R/I. So $x^m \in I$ for some $m \in \mathbb{N}$ and as I is nil $x^{mn} = 0$ for some $n \in \mathbb{N}$. □

It follows from Proposition 3.1 that the sum of all nil two sided ideals of a ring R is the largest nil two sided ideal of R which is denoted by $\mathrm{Nil}^*(\mathrm{R})$ and is called the upper nil-radical of R. If R is commutative, then $\mathrm{Nil}^*(\mathrm{R})$ coincides with the prime radical of R. Please see [20, page 178] for an example of a noncommutative ring whose upper nil-radical properly contains the prime radical.

Proposition 3.1 leads to the following natural question.

Question 3.2. *Is a sum of two nil right ideals of R a nil right ideal?*

The simplicity of the proof of Proposition 3.1 may deceive one into believing that Question 3.2 should be easy to answer. But quite surprisingly the question turns out to be very hard and is open since 1930 when it was posed by an Austrian Mathematician Gottfried Köthe and is called *Köthe's conjecture*. In last about ninety years Köthe's conjecture has resisted all attempts of mathematicians. But in the bargain several equivalent questions have come up.

Theorem 3.3. *The following conditions are equivalent.*

1. *Sum of two nil left (respectively right) ideals is nil.*
2. *Sum of two nil one sided ideals is nil.*
3. *Every nil one-sided ideal is contained in a nil two-sided ideal.*
4. *If* $\mathrm{Nil}^*(\mathrm{R}) = 0$, *then R does not a contain a nonzero nil one-sided ideal.*
5. *Every nil one-sided ideal of R is contained in* $\mathrm{Nil}^*(\mathrm{R})$.

6. If S is a nil ring, then so is $\mathbb{M}_n(S)$ for every positive integer n.

7. If S is a nil ring, then so is $\mathbb{M}_2(S)$.

8. For any nil ideal I of a ring R, $I[x] \subseteq J(R[x])$.

9. $\mathrm{Nil}^*(\mathbb{M}_\mathrm{n}(\mathrm{R})) = \mathbb{M}_\mathrm{n}(\mathrm{Nil}^*(\mathrm{R}))$ *for every ring R.*

10. For every ring R, $J(R[x]) = \mathrm{Nil}^*(\mathrm{R})[\mathrm{x}]$.

For proof of some of these equivalent conditions, we refer to [22, Exercise 10.25, Page 160]. The following related result was proved by Amitsur [1, Theorem 1].

Theorem 3.4. *For every ring R, $J(R[x]) = I[x]$ for some nil ideal I of R.*

Amitsur [2] had conjectured that $S[x]$ is nil for every nil ring S. Note that a positive answer to the Amitsur's conjecture would lead to a positive answer to the Köthe's conjecture. But in 2000 Agata Smoktunowicz [29] came up with a wonderful construction showing that Amitsur's conjecture was false. For more information the reader is referred to an excellent expository article [30] by Agata Smoktunowicz.

4. Polynomials over noncommutative rings

A polynomial $f(x) = a_0 + a_1x + \ldots + a_nx^n$ over a commutative ring R is invertible if and only if a_0 is invertible and a_i is nilpotent for all $i \geq 1$. If $R = \mathbb{M}_2(\mathbb{Z})$, then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \in R[x]$ is not invertible although $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a unit and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent. On the other hand $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \in R[x]$ is invertible although $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not nilpotent. This is because, unlike in commutative rings, a sum of a unit and a nilpotent may not be a unit in a noncommutative ring and also a noncommutative ring modulo a prime ideal may not be a domain.

Neal H. McCoy [25, Theorem 2] in 1942 proved the following interesting result.

Theorem 4.1. *Let R be commutative and $f(x) \in R[x]$. If $f(x)g(x) = 0$*

for some nonzero $g(x) \in R[x]$, then there exists a nonzero $r \in R$ such that $f(x)r = 0$.

So $f(x)$ is a zero divisor in $R[x]$, R commutative, if and only if its every coefficient is a zero divisor with a common nonzero annihilator. This is not true for polynomials over noncommutative rings. As shown in [19] if $R = \mathbb{M}_2(S)$,

$$f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x^2,$$

$$g(x) = -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x,$$

then $f(x)g(x) = 0$ despite the fact that one of the coefficients of $f(x)$ is the identity element of R.

The following 2018 result of Khurana et al. [19, Theorem 2.2] generalizes McCoy's Theorem 4.1.

Theorem 4.2. *Let R be any ring and $f(x) \in R[x]$. Then there exists a nonzero $r \in R$ such that $f(x)r = 0$ if and only if there exists a nonzero $g(x)$ such that $f(x)cg(x) = 0$ for every c in the multiplicative monoid in R generated by the coefficients of $f(x)$.*

The above result does not hold if we restrict c to the coefficients of $f(x)$ as the following example [19, Example 2.3] shows. For instance, let $R = \mathbb{M}_5(S)$, $f(x) = a + bx$ and $g(x) = E_{11} - E_{31}x + E_{51}x^2$, where

$$a = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then $f(x)g(x) = f(x)ag(x) = f(x)bg(x) = 0$, but it is easy to see that no nonzero element of R kills $f(x)$ on right.

The evaluation map $f(x) \to f(a)$ from $R[x] \to R$ is not a homomorphism if $a \notin \mathrm{Z(R)}$. In particular a root of the polynomial $f(x)$ may not be a root of $f(x)g(x)$! Also even over a domain, a root of $f(x)g(x)$ may not be a root of $f(x)$ or $g(x)$. So a polynomial of degree n may have more than n roots. In $\mathbb{H}[x]$, k is a root of $(x-i)(x+i) = x^2+1$. Also i is not a root of the polynomial $(x-i)(x-j) = x^2-(i+j)x+k \in \mathbb{H}[x]$. On the other hand note that j is a root of $(x-i)(x-j)$. This is no coincidence. It is true that a root of $g(x)$ is always a root of $f(x)g(x)$ for polynomials over any ring. This fact has interesting consequences.

Let $A \in \mathbb{M}_n(R)$ and I denote the identity matrix, where R is a commutative ring. Cayley-Hamilton Theorem says that A satisfies the polynomial $\mathrm{det(A-xI)I} \in \mathrm{\mathbb{M}_n(R)[x]}$. As $\mathrm{adj(A-xI)(A-xI) = det(A-xI)I}$, $\mathrm{adj(A-xI)}$, $A-xI$ and $det(A-xI)I$ are polynomials in $\mathbb{M}_n(R)[x]$ and A satisfies $A-xI$, so A satisfies $det(A-xI)I$. The same proof yields that if $A \in \mathbb{M}_n(R)$, R commutative, satisfies a polynomial $f(x) \in \mathbb{M}_n(R)[x]$, then A also satisfies $\mathrm{det(f(x)I}$. Mathematicians knew this fact a long back. In fact, in some of the earlier texts Cayley-Hamilton Theorem was derived as a Corollary to this more general fact (see for instance [23, Theorem 14.2 and Corollary 14.21]). In view of this the following is a valid proof of the Cayley-Hamilton Theorem.

As A satisfies $A-xI$, A satisfies $\mathrm{det(A-xI)I}$.

Recently Grover et al. [8, Theorem 2.1] have proved the following result.

Theorem 4.3. *Let $f,\ g,\ h \in R[x_1, x_2, ..., x_n]$ and $h = fg$, then the element $h(a_1,\ a_2,\ ...,\ a_n)$ is contained in the ideal of R generated by $g(a_1, a_2, ..., a_n)$ for every pairwise commuting elements $a_1,\ a_2,\ ..., a_n$ of R. In particular $g(a_1, a_2, ..., a_n) = 0$ implies $h(a_1, a_2, ..., a_n) = 0$.*

As a corollary we have the following generalization of the Cayley-Hamilton Theorem that was proved by H. B. Phillips [27, Theorem I] in 1919.

Theorem 4.4. *Let S be a commutative ring and $f \in \mathbb{M}_n(S)[x_1, x_2, ..., x_n]$. If $A_1, A_2, ..., A_n$ in $\mathbb{M}_n(S)$ are pairwise commuting matrices such that $f(A_1, A_2, ..., A_n) = 0$ and $g = \mathrm{det(f)I}$, then $g(A_1, A_2, ..., A_n) = 0$.*

In view of the Cayley-Hamilton Theorem every matrix $A \in \mathbb{M}_n(R)$, R commutative, satisfies a monic polynomial of degree n in $R[x]$. The following result of Grover et al. [8, Theorem 3.1] shows that this is not true for any non-commutative ring R.

Theorem 4.5. *Let R be a ring and $n > 1$ be a positive integer. If every matrix in $\mathbb{T}_n(R)$ satisfies a monic polynomial of degree n over R, then R is commutative.*

A ring is called left duo if its every left ideal is a two-sided ideal. The following result of Grover et al. [8, Theorem 3.2] characterizes rings R over which every diagonal matrix in $\mathbb{M}_n(R)$ satisfies a monic polynomial of degree n in $R[x]$.

Theorem 4.6. *The following conditions are equivalent for a ring R.*

(1) For any $n > 1$ every diagonal matrix in $\mathbb{M}_n(R)$ satisfies a monic polynomial of degree n over R.

(2) R is left duo.

(3) Every diagonal matrix in $\mathbb{M}_2(R)$ satisfies a monic polynomial of degree two over R.

The following result [8, Theorem 3.7] gives a necessary condition for every square matrix over R to be R-integral.

Theorem 4.7. *If every diagonal matrix of $\mathbb{M}_n(R)$, $n > 1$, satisfies a monic polynomial over R, then R is Dedekind finite.*

There is an example [8, Example 3.8] showing that Dedekind-finiteness is not a sufficient condition for every diagonal matrix in $\mathbb{M}_n(R)$, $n > 1$, to be R-integral. The following problem apparently is quite hard.

Problem 4.8. *Characterize rings R such that every square matrix over R to be R-integral.*

5. Invertible matrices over noncommutative rings

For a commutative ring R, $A \in \mathbb{M}_n(R)$ is invertible if and only if $\det(A)$ is invertible in R. Also in this case $A^{-1} = \text{adj(A)}/\text{det(A)}$. As analogous determinants of matrices over non-commutative rings cannot be defined, it is hard to see when a square matrix over a noncommutative ring is invertible. Even if one knows that a matrix is invertible, still it is very hard to find its inverse over a noncommutative ring. In $\mathbb{M}_2(\mathbb{H})$, $\begin{pmatrix} i & j \\ j & i \end{pmatrix}$ is invertible although $i^2 - j^2 = 0$. On the other hand in $\mathbb{M}_2(\mathbb{H})$, $\begin{pmatrix} i & -j \\ j & i \end{pmatrix}$ is not invertible although $i^2 + j^2$ is a unit in $\mathbb{H}$. It is also interesting to note that a matrix $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ over a noncommutative ring can be invertible even when none of the a and c is invertible in R.

Suppose R is commutative and $A \in \mathbb{M}_n(R)$ is invertible. Then as $\det(\mathrm{A}) = \det(\mathrm{A}^{\mathrm{t}})$, it follows that A^t is also invertible. This also follows from the fact that $(XY)^t = Y^tX^t$ for every $X, Y \in \mathbb{M}_n(R)$ if R is commutative. But the transpose of an invertible matrix over a noncommutative ring may not be invertible. In 1970 R. N. Gupta [9] gave an example of a nilpotent matrix with invertible transpose. It is classically known that for a division ring D if transpose of every invertible matrix in $\mathbb{M}_n(D)$, $n \geq 2$, is invertible, then D is commutative. This makes one ask the following question.

Question 5.1. *What are rings over which transpose of every invertible matrix is invertible?*

This question was answered in 2009 by Gupta et al. [10, Theorem 2.3]. For $a, x, b \in R$, $[a, b] = ab - ba$ and $[a, x, b] = axb - bxa$.

Theorem 5.2. *For a ring R the following are equivalent:*

1. *Transpose of every invertible matrix is invertible.*
2. *$R/J(R)$ is commutative.*
3. *If $\begin{pmatrix} 1 & a \\ b & c \end{pmatrix} \in \mathbb{M}_2(R)$ is invertible, then so is $\begin{pmatrix} 1 & b \\ a & c \end{pmatrix}$.*
4. *$a + bc \in \mathrm{U(R)}$ implies $a + cb \in \mathrm{U(R)}$ for every $a,\ b,\ c \in R$.*
5. *$u + [a, b] \in U(R)$ for every $u \in U(R)$ and $a,\ b \in R$.*
6. *$1 + [a, x, b] \in U(R)$ for every $a,\ x,\ b \in R$.*
7. *$x + abc \in \mathrm{U(R)}$ implies $x + cba \in \mathrm{U(R)}$ for every $a,\ b,\ c,\ x\ \in R$.*

This leads to some commutativity results for semiprimitive rings (i.e., rings with zero Jacobson radical).

Corollary 5.3. *For a semiprimitive ring R the following are equivalent:*

1. R is commutative.

2. $a + bc \in \mathrm{U(R)}$ implies $a + cb \in \mathrm{U(R)}$ for every $a,\ b,\ c \in R$.

3. $u + [a, b] \in\ U(R)$ for every $u \in\ U(R)$ and $a,\ b \in R$.

4. $1 + [a, x, b] \in\ U(R)$ for every $a,\ x,\ b \in R$.

5. $x + abc \in \mathrm{U(R)}$ implies $x + cba \in \mathrm{U(R)}$ for every $a,\ b,\ c,\ x\ \in R$.

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LARGE FOURIER COEFFICIENTS OF MODULAR FORMS*

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Abstract. This expository article is an extended version of an invited talk (V Ramaswami Aiyer Memorial Award Lecture) delivered at the 86th Annual conference of the Indian Mathematical Society (IMS) in December 2020. In this note, we give a brief overview of the theme of size of Fourier coefficients of integral as well as half-integral weight cusp forms. We also give an outline of a recent work with W. Kohnen and K. Soundararajan.

1. Introduction

Perhaps it is apt to start with the following quote attributed to Eichler.

"There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and modular forms."

Modular forms are ubiquitous in Mathematics. These are interwoven with myriad themes in Mathematics: Complex Analysis, Algebraic Geometry, Representation theory, Combinatorics, Number Theory and so on and so forth. We shall however not be able to cover the basics of modular forms for the uninitiated, but there are plethora of sources, depending on one's interest as well as perspective, to study these wondrous objects.

In this note, we give an overview of the theme of size of Fourier coefficients of integral as well as half-integral weight cusp forms. We hasten to add that our exploration is by no means exhaustive, but rather a geodesic tour through some of the tracks that are linked to our interest and taste. Finally, we give an outline of a recent work with W. Kohnen and K. Soundararajan.

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Let us begin by setting up the notations relevant for us. Let $\mathfrak{H}$ denote the Ponicareé upper half plane defined as

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$$

and for any $z \in \mathfrak{H}$, we set $q = e^{2\pi i z}$. Let Γ denote the full modular group given by

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}.$$

This is a group under multiplication and is generated by

$$T \ = \ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S \ = \ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This group acts on the upper half plane $\mathfrak{H}$ where the action is given by

$$\gamma.z := \frac{az+b}{cz+d}$$

for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ and $z \in \mathfrak{H}$.

The quotient space $\mathfrak{H}/\Gamma$ is the set of isomorphism classes of elliptic curves over $\mathbb{C}$ and is one of the many reasons that modular forms have such rich arithmetic content.

Let $k \in \mathbb{Z}$ be a positive integer and $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function. Suppose that for any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, the function f satisfies the following transformation property:

$$f(\gamma.z) = (cz+d)^k f(z).$$

Now for any function f as above,

$$f(T.z) = f(z+1) = f(z).$$

Thus f is a periodic holomorphic function with period 1 which ensues that f has a Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_f(n) q^n, \quad z \in \mathfrak{H}.$$

Definition. Such an f is called a modular form of weight k for Γ if f is holomorphic at ∞ which is the only cusp up to $\mathrm{SL}_2(\mathbb{Z})$ equivalence. Equivalently, $a_f(n) = 0$ for all $n < 0$.

Definition. A modular form f for Γ is called a cusp form of weight k if in addition to above, it vanishes at infinity. Equivalently, $a_f(n) = 0$ for all $n \leq 0$.

The set of modular forms of weight k forms a finite dimensional complex vector space denoted by M_k. The set of cusp forms, denoted by S_k, forms a codimension one subspace of M_k.

Throughout, let $k > 1$ (unless otherwise stated) be an integer and f be a cusp form of weight k for Γ. For such an f to be non-zero, k must be even and at least 12. It turns out S_{12} is one dimensional and is generated by

$$\Delta(z) \;=\; q\prod_{n=1}^{\infty}(1-q^n)^{24} \;=\; \sum_{n\geq 1}\tau(n)q^n.$$

In 1916, Ramanujan made three conjectures (see [22]) about $\tau(n), n \geq 1$ which helped shape the theory of modular forms. The first two of these conjectures are

$$\begin{aligned} \tau(mn) &= \tau(m)\tau(n) \quad \text{for} \quad (m,n)=1 \\ \text{and} \quad \tau(p^{r+1}) &= \tau(p^r)\tau(p) - p^{11}\tau(p^{r-1}), \end{aligned} \tag{1.1}$$

for any prime number p and integer $r \geq 1$. We will discuss about the third conjecture in a short while.

One can define an inner product, the *Petersson inner product*, on the space of cusp forms S_k. This is given by

$$< f, g >= \int_{\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2}.$$

In fact, existence of the integral is ensured if at least one of f and g, say f is a cusp form as $f(z) = O(e^{-2\pi y})$ as $y \to \infty$. Further, since the integrand is $\mathrm{SL}_2(\mathbb{Z})$ invariant, choice of $\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})$ does not matter.

For integers $n \geq 1$, the n-th Hecke operator T_n on the space of cusp forms (one can define it for all modular forms, but it preserves the subspace of cusp forms and we will restrict our attention to cusp forms) is defined as follows: For $f \in S_k$, $T_n(f) \in S_k$ is given by

$$T_n(f)(z) = n^{k-1}\sum_{\substack{a\geq 1, ad=n\\ 0\leq b<d}} d^{-k}f\left(\frac{az+b}{d}\right).$$

The family of Hecke operators $T_n, n \geq 1$ are Hermitian with respect to the Petersson inner product. This implies that each T_n is diagonalisable and their eigenvalues are real. Further these Hecke operators $T_n, n \geq 1$ are commuting and hence the space S_k has a basis consisting of cusp forms which are simultaneous eigen vectors for all these Hecke operators.

For each such cusp form

$$f = \sum_{n=1}^{\infty} a_f(n) q^n,$$

the first Fourier coefficient $a_f(1)$ is necessarily non-zero. We say that f is normalised if $a_f(1) = 1$. For such a normalised eigen form, the n-th Fourier coefficient $a_f(n)$ is an eigen value of T_n. This along with the functional identities

$$\begin{aligned} T_{mn} &= T_m T_n \quad \text{for} \quad (m,n) = 1 \\ \text{and} \quad T_{p^{n+1}} &= T_p T_{p^n} - p^{k-1} T_{p^{n-1}} \quad \text{for primes } p \text{ and integer } n > 1 \end{aligned}$$

allowed Mordell to prove the first two conjectures (1.1) of Ramanujan for the Fourier coefficients of such normalised Hecke eigen cusp forms (see [17, 9, 10] for further details).

2. Growth of Fourier-coefficients of integer weight cusp forms

Let f be a normalised Hecke eigen cusp form of weight k for Γ with Fourier expansion at infinity given by

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n.$$

We now ask the following question:

Question. What can we say about the growth as well as bounds for the sequence $\{a_f(n)\}_{n \geq 1}$ *?*

Here we have the celebrated third conjecture of Ramanujan [22] which was proved by Deligne [4, 5] by appealing to deep tools in algebraic geometry. For a positive integer n, let $d(n)$ denote the number of positive divisors of n. Here is the theorem of Deligne.

Theorem 2.1. *(Deligne) The sequence* $\{a_f(n)\}_{n \geq 1}$ *satisfies*

$$|a_f(n)| \leq d(n) n^{\frac{k-1}{2}}$$

for all $n \geq 1$.

Thus in particular for a prime number p, one has

$$|a_f(p)| \leq 2p^{\frac{k-1}{2}}.$$

On the other hand, the divisor function $d(n)$ is rather enigmatic. While $\liminf_{n\to\infty} d(n) = 2$, one also knows that there exists a constant $c > 0$ such that the following inequality

$$d(n) > \exp\left(c\, \frac{\log n}{\log\log n}\right)$$

holds for infinitely many n. Here $\exp(\alpha) = e^{\alpha}$. These observations naturally beg the following questions :

Questions: Can the quantity $\frac{|a_f(n)|}{n^{\frac{k-1}{2}}}$ *be bounded by an absolute constant as opposed to* $d(n)$*? Can the quantity* $\frac{|a_f(n)|}{n^{\frac{k-1}{2}}}$ *be bounded by a function with much slower order of growth than* $d(n)$*?*

Of course, an affirmative answer to the first question decidedly resolves the second one. The first question was answered in the negative by Rankin [23] in 1973 who proved the following;

Theorem 2.2. *(Rankin) The sequence* $\{a_f(n)\}_{n\geq 1}$ *satisfies*

$$\limsup_{n\to\infty} \frac{|a_f(n)|}{n^{\frac{k-1}{2}}} = \infty.$$

With respect to the second question, Ram Murty [18] in 1983 proved the following result:

Theorem 2.3. *(Ram Murty) There exists an absolute constant* $c > 0$ *such that the inequality*

$$\frac{|a_f(n)|}{n^{\frac{k-1}{2}}} > \exp\left(c\, \frac{\log n}{\log\log n}\right)$$

holds for infinitely many n*.*

The above result is rather satisfactory as it evinced that the bound conjectured by Ramanujan and proved by Deligne is optimal expect for the constant c appearing above. Furthermore it was shown by Ram Murty, using an elegant idea of Rankin, that a similar result holds for any arbitrary non-zero cusp form.

3. Growth of Fourier-coefficients of half-integer weight cusp forms

With this brief preamble about integral weight cusp forms, let us now study analogous questions for half integral weight cusp forms. Inevitably, we shall need to introduce some more notions and notations.

To start with, we have to abandon the comforting full modular group $\mathrm{SL}_2(\mathbb{Z})$ and introduce congruence subgroups. But we shall refrain from generalities and introduce objects on a need-to-know basis to keep the presentation as less technical as possible.

With this, let us define the congruence subgroup $\Gamma_0(N)$ for a positive integer N given by

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c \right\}.$$

Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4)$. Then if $c \neq 0$, let χ_c be the primitive real character associated to the algebraic number field $\mathbb{Q}(\sqrt{c})$. Let ϵ_d be equal to 1 or i according as d is congruent to 1 mod 4 or 3 mod 4. For $z \in \mathfrak{H}$, let

$$j(\gamma, z) = \epsilon_d^{-1} \chi_c(d)(cz + d)^{1/2}$$

where the square root is chosen such that the real part is positive. If $c = 0$, then $j(\gamma, z) = 1$.

Let $4|N$ and χ be a Dirichlet character mod N. A holomorphic function g on $\mathfrak{H}$ is called a modular form of weight $k + \frac{1}{2}$ for $\Gamma_0(N)$ with character χ if

- for any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, we have
$$g(\gamma.z) = \chi(d) j(\gamma, z)^{2k+1} g(z).$$
This is unambiguous since $4|N$.
- the function g is holomorphic at cusps.

As before, it is a cusp form if it vanishes at all the cusps. One has a theory of Hecke operators and Hecke eigen forms in this set up also. But the matter is more delicate and involved than the integral weight case. We refer the interested reader to the foundational paper of Shimura [26] for more details and clarifications.

Let g be non-zero half integral weight cusp form of weight $k+\frac{1}{2}, k>1$ for $\Gamma_0(4)$ and let

$$g(z) = \sum_{n=1}^{\infty} c_g(n) q^n$$

be its Fourier expansion at ∞. The growth of the sequence of Fourier coefficients $\{c_g(n)\}_{n\geq 1}$ is hardly an open book. One has the following folklore conjecture/expectation.

Conjecture. Let g be a non-zero half integral weight cusp form of weight $k+\frac{1}{2}, k>1$ for $\Gamma_0(4)$ and $\epsilon>0$. Then there exists a positive real number $A = A(g,\epsilon)$ such that for any $n \geq 1$,

$$|c_g(n)| \leq A n^{k/2-1/4+\epsilon}.$$

In semblance with the integral weight case, this conjecture is often referred to as the *Ramanujan-Petersson Conjecture* for half-integral weight cusp forms. Historically, the generalisation of the conjecture of Ramanujan on Delta function to arbitrary Hecke eigenforms was proposed by Petersson.

In contrast to the world of integral weight cusp forms, the above conjecture is widely open. One does not know a single example for which this conjecture is true. However the remarkable works of Shimura [26], Waldspurger [29] and Kohnen-Zagier [15] suggest evidence towards the validity of the above conjecture (and not just some wishful thinking on our part).

Till the late 20th century, it was not quite clear how to study the Fourier coefficients of half integral weight modular forms and hence how to make reasonable models. The path breaking work of Shimura in 1973 [26] provided a tremendous impetus to this theme by giving an explicit dictionary between spaces of half integral and integral weight modular forms. The following is a consequence of Shimura's seminal work.

Theorem 3.1. *Let g be a Hecke eigen cusp form of weight $k+\frac{1}{2}$ for $\Gamma_0(4)$ and t be a positive square free integer. Then there exists a Hecke eigen cusp form f of weight $2k$ for Γ such that their Fourier coefficients are related by*

$$c_g(tn^2) = c_g(t) \sum_{d|n} \mu(d) \left(\frac{t}{d}\right) d^{k-1} a_f(n/d).$$

Regrettably, Shimura passed away in 2019.

We note that the right hand side of the above, other than $c_g(t)$ involves objects whose growth is well understood. So the Ramanujan-Petersson conjecture can be relegated to the growth of terms of the type $c_g(t)$.

The sequence of such Fourier coefficients naturally lead to L-functions. For instance, for a cusp form of weight k for Γ with Fourier coefficients $\{a_f(n)\}_{n\geq 1}$, one associates an L-function given by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

which is absolutely convergent for all complex numbers s with $\Re(s) > \frac{k+1}{2}$.

Another very important result in this theme is due to Waldspurger [29] whose remarkable work relates Fourier coefficients of half integral weight Hecke eigen cusp forms with certain special values of L-functions associated to "twists" of integral weight modular forms. This result of Waldspurger along with convexity bounds from analytic number theory shows that

$$|c_g(n)| \leq A(g) n^{\frac{k}{2}}\, d(n)$$

where as before the quantity $A(g)$ depends only on g. The above bound can be proved without using convexity bound (see Iwaniec [12] for details).

Remark. In general, the above upper bound cannot be improved for forms belonging to the subspace of theta functions. An interesting result of Serre and Stark [25] states that all cusp forms of weight $\frac{1}{2}$ are spanned by theta functions.

Coming back to our set up, one of the first non-trivial results in this direction is due to Iwaniec [12] who in 1987 proved the following.

Theorem 3.2. *(Iwaniec) For any cusp form g of weight $k+\frac{1}{2}, k>1$ with Petersson norm one, one has*

$$|c_g(t)| \leq A(k) t^{k/2-1/28} d(t) (\log 2t)^2$$

for all square free integer $t > 0$. Here $A(k)$ depends only on k.

This result was improved by Petrow and Young [20] in 2019.

Theorem 3.3. *(Petrow-Young) Let g be Hecke eigen cusp form of weight $k+\frac{1}{2}, k>1$ for $\Gamma_0(4)$. Then for any square free positive integer t and $\epsilon > 0$, one has*

$$|c_g(t)| \leq A(g,\epsilon) t^{\frac{k}{2}-\frac{1}{12}+\epsilon}.$$

Here $A(g,\epsilon)$ depends only on g and ϵ.

Let $S_{k+\frac{1}{2}}$ be the complex vector space of cusp forms of weight $k+\frac{1}{2}$ for $\Gamma_0(4)$ while as before S_{2k} be the complex vector space of cusp forms of weight $2k$ for Γ. Consider the following distinguished subspace of $S_{k+\frac{1}{2}}$ defined as

$$S^+_{k+\frac{1}{2}} = \left\{ g \in S_{k+\frac{1}{2}} : c_g(n) = 0 \text{ if } (-1)^k n \not\equiv 0, 1 \bmod (4) \right\}.$$

This is called the *Kohnen's plus space.* One knows, courtesy the work of Kohnen [14] (see also Shimura [26]), that

$$S^+_{k+\frac{1}{2}} \cong S_{2k}.$$

In a joint work with Kohnen [7], we prove the following theorem which can be regarded as the half-integral weight analogue of Rankin's Theorem alluded to earlier.

Theorem 3.4. *(Gun-Kohnen) Let $g \in S^+_{k+\frac{1}{2}}, k \geq 1$ be arbitrary non-zero cusp form. Then the sequence $\{c_g(n)\}_{n\geq 1}$ satisfies*

$$\limsup_{n\to\infty} \frac{|c_g(n)|}{n^{\frac{k}{2}-\frac{1}{4}}} = \infty.$$

Furthermore, appealing to Sato-Tate conjecture [1] about equidistribution of normalised Hecke eigen values of T_p (p prime) in $[-2,2]$, we could show the following.

Theorem 3.5. *(Gun-Kohnen) Let $g \in S^+_{k+\frac{1}{2}}, k \geq 1$ be arbitrary non-zero cusp form. Then the sequence $\{c_g(n)\}_{n\geq 1}$ satisfies*

$$|c_g(n)| > n^{\frac{k}{2}-\frac{1}{4}} \exp(c(\log n)^{\delta})$$

for infinitely many n. Here $c > 0$ and $0 < \delta < \frac{1}{2}$ are absolute constants.

We note that if $g \in S^+_{k+\frac{1}{2}}$ is a Hecke eigen form, then the above theorems would follow from the works of Hoffstein and Lockhart [11]. On the other hand, if $g \in S^+_{k+\frac{1}{2}}$ is a Hecke eigenform where $4|k$ and n is square free, then the lower bound in Theorem 3.5 follows from a work of Soundararajan and Young [28].

4. Brief description of some recent work

We now briefly describe some recent joint work with Kohnen and Soundararajan. We shall need some more notations!

For any $g \in S^+_{k+\frac{1}{2}}$ which is a Hecke eigen form, there is a Hecke eigenform in S_{2k} under the Shimura-Kohnen dictionary. We shall denote this by f. Our purpose is to study the growth of $c_g(|D|)$, where D is a fundamental discriminant with $|D| = (-1)^k D > 0$. For such a D, let $\chi_D = \left(\frac{D}{\cdot}\right)$ denote the primitive quadratic character modulo $|D|$ corresponding to the fundamental discriminant D.

The L-series attached to the D-th quadratic twist of f is defined as

$$L(f, \chi_D, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\chi_D(n)}{n^s}.$$

This Dirichlet series converges absolutely for $\Re(s) > k + \frac{1}{2}$ and extends analytically to the whole complex plane. Furthermore, it satisfies the following functional equation

$$\Lambda(f, \chi_D, s) = \left(\frac{|D|}{2\pi}\right)^s \Gamma(s)\, L(f, \chi_D, s) = (-1)^k\, \chi_D(-1)\, \Lambda(f, \chi_D, 2k - s).$$

Consequently when $(-1)^k D < 0$, $L(f, \chi_D, k) = 0$. However, when $(-1)^k D > 0$, a remarkable link was discovered by Waldspurger [29] between $c_g(|D|)$ and $L(f, \chi_D, k)$. An explicit version of such a link was deduced by Kohnen and Zagier [15] which is given by

$$|c_g(|D|)|^2 = \frac{(k-1)!}{\pi^k} \frac{||g||^2}{||f||^2} |D|^{k-\frac{1}{2}} L(f, \chi_D, k)$$

where $||g||$ and $||f||$ are the Petersson norms of g and f respectively. Recall these Petersson norms are defined as

$$\begin{aligned} ||g||^2 &= \frac{1}{6} \int_{\mathfrak{H}/\Gamma_0(4)} |g(z)|^2 y^{k-\frac{1}{2}} dxdy, \\ \text{and} \quad ||f||^2 &= \int_{\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})} |f(z)|^2 y^{2k-2} dxdy \end{aligned}$$

This remarkable formula links the growth of $c_g(|D|)$ to that of $L(f, \chi_D, k)$, albeit when g is a Hecke eigen form. Note that this in particular shows that $L(f, \chi_D, k) \geq 0$ for all D with $(-1)^k D > 0$ as $|c_g(|D|)| \in \mathbb{R}$.

However, when $g \in S^+_{k+\frac{1}{2}}$ is an arbitrary non-zero cusp form, then it is not clear that $c_g(|D|) \neq 0$ for infinitely many D with $(-1)^k D > 0$. Here one has the following 1997 result of Luo and Ramakrishnan [16].

Theorem 4.1. *(Luo-Ramakrishnan) When $g \in S^+_{k+\frac{1}{2}}$ is a non-zero linear combination of two Hecke eigenforms, then*

$$c_g(|D|) \neq 0$$

for infinitely many D with $(-1)^k D > 0$.

For an arbitrary g, this was deduced by Saha [24].

Theorem 4.2. *(Saha) When $g \in S^+_{k+\frac{1}{2}}$ is non-zero, then*

$$c_g(|D|) \neq 0$$

for infinitely many D with $(-1)^k D > 0$.

In our recent work with Kohnen and Soundararajan [8], we give a soft proof of the above result without appealing to Waldspurger's theorem.

Here is a brief summary of our proof. Let $f_1, \ldots, f_r$ be a basis of normalized Hecke eigenforms in S_{2k} and g_ν for $1 \leq \nu \leq r$ be Hecke eigenforms in $S^+_{k+\frac{1}{2}}$ corresponding to f_ν under the Kohnen-Shimura map. Denote the Fourier expansions of g_ν for $1 \leq \nu \leq r$ by

$$g_\nu(z) = \sum_{n=1}^{\infty} c_\nu(n) q^n.$$

For a non-zero cusp form $g \in S^+_{k+\frac{1}{2}}$, let us write

$$g = \sum_{\nu=1}^{r} \lambda_\nu g_\nu$$

for some constants $\lambda_\nu \in \mathbb{C}$, not all zero. For for $s \in \mathbb{C}$ with $\Re(s) = \sigma > \frac{k}{2} + \frac{5}{4}$, we consider the absolutely convergent series

$$D_g(s) \;=\; \sum_{n \geq 1} \frac{\alpha(n)}{n^s}$$

where $\alpha(n) = c_g(|D|)\mu(m)\chi_d(m)m^{k-1}$ and n is written uniquely as $n = |D|m^2$ with D a fundamental discriminant as above. We show that in the region $\sigma > \frac{k}{2} + \frac{5}{4}$, the L-function D_g can be written as

$$D_g(s) \;=\; \sum_{\substack{D \\ (-1)^k D > 0}} \frac{c_g(|D|)}{|D|^s L(2s - k + 1, \chi_D)} \;=\; \sum_{\nu=1}^{r} \lambda_\nu \, \frac{L(g_\nu, s)}{L(f_\nu, s)}.$$

Here $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}, \Re(s) > 1$ is the Dirichlet L-function. The above relations show that $D_g(s)$ has meromorphic continuation to $\mathbb{C}$ and is holomorphic for $\Re(s) = \sigma > \frac{k}{2} + \frac{1}{4}$.

We then show that D_g is identically zero (i.e. $c(|D|) = 0$ for all fundamental discriminants D) if there are only finitely many fundamental discriminant D with $c(|D|) \neq 0$. This is deduced by using the fact that $D_g(s)$ inherits a functional equation from Dirichlet L-function which is inconsistent with the functional equation arising from the modular L-functions.

Using functional equation again, we show that

$$\sum_{\nu=1}^{r} \lambda_\nu c_\nu(|D|) a_\nu(p) \;=\; 0$$

for all odd primes p and for all fundamental discriminants D with $4|D$ and $(-1)^k D > 0$. Here $a_\nu(p)$ denotes the p-th Fourier-coefficient of f_ν.

Finally using Rankin-Selberg theory together with the fact that for each g_ν, there exists a fundamental discriminant D with $4|D$ such that $c_\nu(|D|) \neq 0$ (see [14]), we conclude that $g = 0$, a contradiction. This completes the proof of the theorem.

Furthermore, we show in [8] that many of these $c_g(|D|)$ take large values. More precisely, we prove:

Theorem 4.3. *(Gun - Kohnen - Soundararajan) Let $g \in S^+_{k+\frac{1}{2}}$ be non-zero and $\epsilon > 0$. Then for sufficiently large X, there are at least $X^{1-\epsilon}$ fundamental discriminants D with $X < (-1)^k D < 2X$ such that*

$$|c_g(|D|)| \geq |D|^{\frac{k}{2}-\frac{1}{4}} \exp\left(\frac{1}{82}\frac{\sqrt{\log|D|}}{\sqrt{\log\log|D|}}\right).$$

Remarks.

- Apart from the constant $\frac{1}{82}$, the lower bounds furnished in Theorem 4.3 are the best known, even when g is a Hecke eigenform.
- Recently Theorem 4.3 has been extended to arbitrary level by Jääsaari, Lester and Saha [13].

In another direction, it follows from the conjecture of Farmer, Gonek and Hughes [6] that

Conjecture. For all fundamental discriminants D with $(-1)^k D > 0$

$$|c_g(|D|)| \ll_g |D|^{\frac{k}{2}-\frac{1}{4}} \exp(c\sqrt{\log|D| \log\log|D|})$$

for some absolute constant $c > 0$.

Best known result in this direction is by Radziwiłł and Soundararajan [21].

Theorem 4.4. *(Radziwiłł - Soundararajan) Let $g \in S^+_{k+\frac{1}{2}}$ be non-zero cusp form and $\epsilon > 0$ be a real number. Then for all but $o(X)$ fundamental discriminants D with $X \le (-1)^k D \le 2X$, one has*

$$|c_g(|D|)| \ll_{g,\epsilon} |D|^{\frac{k}{2}-\frac{1}{4}} (\log D)^{-\frac{1}{4}+\epsilon},$$

We deduce Theorem 4.3 as a consequence of the following theorem.

Theorem 4.5. *(Gun - Kohnen - Soundararajan) Let $A > 0$ be a constant, and let X be large. For any $\epsilon > 0$, there are $\gg X^{1-\epsilon}$ fundamental discriminants D with $X < (-1)^k D \le 2X$ such that*

$$L(f_1, \chi_D, k) > A \sum_{\nu=2}^{r} L(f_\nu, \chi_D, k) + \exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big).$$

Remarks:

- We use resonance method [27] developed by Soundararajan to prove Theorem 4.5. For Hecke eigenforms f of integer weight $2k$ for Γ, it follows from the method developed in [27] that there are infinitely many fundamental discriminants D such that

  $$L(f, \chi_D, k) \gg \exp(c\sqrt{\log|D|/\log\log|D|})$$

  for a positive constant c. A weaker result can be found in [11].

- Work of Bondarenko and Seip [2] gives an improvement of the resonance method [27] while producing larger values of Riemann zeta function on the line $\frac{1}{2}$ and a similar improvement for Dirichlet L-function has been obtained in [3]. However, this method exploits positivity of coefficients, and of orthogonality relations in crucial ways, and does not seem to extend to quadratic twists of modular L-functions attached to Hecke eigenforms.

We end with a sketch of the proof of Theorem 4.5. For this, let us introduce a "resonator" relevant for us. Let D be fundamental discriminant with $X < (-1)^k D \le 2X$. We consider the following special value of a Dirichlet polynomial at $k-\frac{1}{2}$:

$$R(D) = \sum_{n \le N} r(n) \frac{a_1(n)}{n^{k-\frac{1}{2}}} \chi_D(n),$$

where $N = X^{\frac{1}{24}}$ and $r(n)$ is a multiplicative function defined as follows. Set $r(n) = 0$ unless n is square-free, and for primes p define, with $L = \frac{1}{8}\sqrt{\log N \log\log N}$

$$r(p) = \begin{cases} \frac{L}{\sqrt{p}\log p} & \text{if } L^2 \le p \le L^4 \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Theorem 4.5 rests on the following intermediate results.

Proposition 4.6. *With notations as above, we have*

$$\sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} |R(D)|^2 \le \frac{2X}{\pi^2}\mathcal{R} + O(X), \tag{4.1}$$

where

$$\mathcal{R} = \prod_{L^2 \le p \le L^4} \Big(1 + r(p)^2 \frac{a_1(p)^2}{p^{2k-1}}\Big).$$

Further

$$\sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} |R(D)|^6 \ll X \exp\Big(O\Big(\frac{\log X}{\log\log X}\Big)\Big). \tag{4.2}$$

Proposition 4.7. *With notations as above, we have*

$$\sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} L(f_1, \chi_D, k)|R(D)|^2 \gg X\mathcal{R} \exp\Big((\frac{1}{2} + o(1))\frac{L}{\log L}\Big),$$

while for all $2 \le \nu \le r$

$$\sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} L(f_\nu, \chi_D, k)|R(D)|^2 \ll X\mathcal{R} \exp\Big(o\Big(\frac{L}{\log L}\Big)\Big). \tag{4.3}$$

We now complete the proof of Theorem 4.5. Let $\mathcal{S}$ be the set of fundamental discriminants D with $X < (-1)^k D \le 2X$ and $D \equiv 1 \bmod 4$ for which

$$L(f_1, \chi_D, k) > A \sum_{\nu=2}^{r} L(f_\nu, \chi_D, k) + \exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big).$$

Note that

$$\begin{aligned}
&\sum_{\substack{X<(-1)^k D\leq 2X\\ D\equiv 1 \mod 4}} L(f_1,\chi_D,k)|R(D)|^2\\
\leq &\sum_{\substack{X<(-1)^k D\leq 2X\\ D\equiv 1 \mod 4}} \Big(A\sum_{\nu=2}^{r} L(f_\nu,\chi_D,k)+\exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big)\Big)|R(D)|^2\\
&\quad + \sum_{D\in\mathcal{S}} L(f_1,\chi_D,k)|R(D)|^2. \qquad (4.4)
\end{aligned}$$

Using (4.1), (4.3) and (4.4), we conclude that

$$\sum_{D\in\mathcal{S}} L(f_1,\chi_D,k)|R(D)|^2 \gg X\mathcal{R}\exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big).$$

Using the Perelli-Pomykała bound [19]

$$L(f_1,\chi_D,k)^2 \ \ll\ X^{1+\epsilon} \quad \text{for any } \epsilon>0$$

Cauchy-Schwarz inequality, Hölder inequality and inequality (4.2), we get

$$\begin{aligned}
&\sum_{D\in\mathcal{S}} L(f_1,\chi_D,k)|R(D)|^2\\
\leq\ & \Big(\sum_{\substack{X<(-1)^k D\leq 2X\\ D\equiv 1 \bmod 4}} L(f_1,\chi_D,k)^2\Big)^{\frac{1}{2}}\Big(\sum_{D\in\mathcal{S}}|R(D)|^4\Big)^{\frac{1}{2}}\\
\ll\ & (X^{1+\epsilon})^{\frac{1}{2}}\Big(|\mathcal{S}|\Big)^{\frac{1}{6}}\Big(\sum_{\substack{X<(-1)^k D\leq 2X\\ D\equiv 1 \mod 4}}|R(D)|^6\Big)^{\frac{1}{3}} \ \ll\ X^{\frac{5}{6}+\epsilon}|\mathcal{S}|^{\frac{1}{6}}.
\end{aligned}$$

This completes the proof of Theorem 4.5.

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AN INVITATION TO DIFFERENTIAL EQUATIONS*

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1. Introduction

Many physical phenomena like the melting of ice, radioactive decay, spring motion, heat conduction and many others have been modeled using differential equations to describe the process. Main motivations for such modeling are to understand the phenomenon, to make predictions and also to regulate the process by using suitable controls.

The use of some basic principles to relate the rate of change of the object of study with various other quantities involved in the process, leads to a mathematical model involving differential equations. Prime example is Newton's equations of motion from 17th century. These are ordinary differential equations (ODE). Then followed partial differential equations(PDE) models. Hamilton's system from Mechanics, vibrating string model and its solution by D'Alembert around 1752, gravitational potential field model by Laplace around 1780, analytic theory of heat conduction by Fourier in 1815, fluid flow models including Navier - Stokes System are some of the notable ones, while modern applications involve biological and financial models.

Once the model is formulated, it can be studied thoroughly using various mathematical tools to analyze the equation theoretically and to solve it numerically if possible. We recall here that Newton introduced derivatives and integrals, to formulate and solve his differential equations to understand the motion of bodies and many other physical phenomena.

1.1. **Main Directions of research.** The early research in differential equations was focussed on finding the model and then solving it explicitly. The initial models were linear and most of them were amenable to various solution techniques. This is what we study in a first course in differential

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equations. Later on, the theory of general partial differential equations started shaping up - classification of the types of PDE and existence and uniqueness of solutions for linear PDE. Hadamard in 1920, proposed that a PDE is well posed if it admits a unique solution and the solution is "continuous" with respect to the data, in the sense that small changes in the data result in correspondingly small changes in the solution. This definition is the one followed even now. One has to set up a suitable functional framework for existence and uniqueness theory and continuous dependence of the solution on the data. This is a major area of research.

Another direction of research is the search for qualitative properties of the solution : Apriori estimates, regularity, boundedness, decay to zero, stability, to name a few. These are studied using the PDE, without even solving them explicitly.

Motivated by practical applications in the 1950s, control theory of differential equations started in the Soviet Union and in the United States. It is currently an active field of research for both ordinary and partial differential equations. A few of the questions studied are:

(i) Can the solution trajectory be controlled optimally to optimize certain costs (optimal control) ?

(ii) Can the system be controlled to reach a target (Controllability)?

(iii) Is the solution stable or not and if not, how to add a control to stabilize it (stabilization) ?

A brief introduction to these topics is given here, starting from finite dimensional optimization, followed by optimal control of ODE and then for PDE, with some simple examples. The reader is invited to browse through the introductory texts like [2], [5], [4], [6], [7].

2. Optimal Control

2.1. **Optimization.** Let us start from one variable calculus : For a function f from $\mathbb{R}^N$ to $\mathbb{R}$, is there a maximum or minimum ? How to find the maxima, minima ?

A theorem of Weirstrass says that if x_0 is a local extremum of a function f which is differentiable at that point, then its total derivative vanishes,

$$\nabla f(x_0) = (\frac{\partial f}{\partial x_1}, \cdots , \frac{\partial f}{\partial x_N}) = 0.$$

Let us now move to optimization with constraints. Suppose that f and g are real valued functions of N variables, and consider a typical problem of minimizing f under the constraint $g(x) = 0$. We recall that if all the functions are continuous and the set of admissible points is bounded then the minimization problem is solvable. How do we find the minimum ? For that, one useful tool is Lagrange's rule. Form the Lagrange augmented function, for λ in $\mathbb{R}$:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

Now minimize in all the variables (x, λ) without any constraints! If the admissible point $(\hat{x}, \hat{\lambda})$ is a local extremum, then

$$\frac{\partial L(\hat{x}, \hat{\lambda})}{\partial x_j} = 0, \ j = 1, \cdots, N; \quad \frac{\partial L(\hat{x}, \hat{\lambda})}{\partial \lambda} = 0.$$

Then it follows that at the point $(\hat{x}, \hat{\lambda})$, we have $\nabla f(\hat{x}) + \hat{\lambda} \nabla g(\hat{x}) = 0$.

In fact, the solution set of the constraint equation $g(x) = 0$ near the minimum point $\hat{x}$, forms a curved surface with the normal direction along $\nabla g(\hat{x})$. Minimizing f along this surface will only force the tangential component of the derivative of f to vanish. Thus $\nabla f(x)$ is in the direction of the normal $\nabla g(x)$ at $x = \hat{x}$. That is to say :

$$\nabla f(\hat{x}) \ = \ \lambda \nabla g(\hat{x}).$$

The number λ is known as the Lagrange multiplier. This idea of passing from constrained minimization to unconstrained minimization via the augmented function, is due to Lagrange. However, a rigorous proof came much later. Although the number of unknown variables have increased, the number of equations also have increased equally.

This is only a necessary condition. However, it helps considerably in reducing our search for minima. See [2] for a neat introduction and a lot of real life examples as applications. See also [6], chapter 1.

2.2. **Optimal Control for ODE.** In many situations, one is confronted with the minimization of a real valued function J from an infinite dimensional function space X. Some famous examples are Dido's isoperimetric problem and Brachistochrone problem, one learns in Calculus of variations. This J is called a functional on X, a linear vector space. J takes a real value for each $x(t)$, an element in the function space X. This $x(t)$ is itself now a function of a real variable t.

We now need new tools to handle the infinite dimensional situation. These function spaces have a linear vector space structure and a notion of distance, "norm" and hence a notion of "derivative" for the functional J, can be defined. Then the necessary optimality conditions can be written down. Refer [6], section 1.3, for more information.

Historically, Euler solved such a problem, using a method of approximation for the curve $x(t)$ in 1740. Lagrange came up with an analytic method in 1755. Euler was impressed with Lagrange's method and coined the term "Calculus of Variations" for that method. In these initial examples, the end points of the curves were fixed. The first order optimality conditions are now known as Euler-Lagrange equations.

In some situations, there may be additional constraints and the constraint may also involve the derivative $\dot{x}(t)$ of the function, $x(t)$. Then we may have a differential equation as a constraint evolving in time. In general, there may be less number of constraints than the dimension of $\dot{x}(t)$. Adding more free variables, if needed, we may end up with $\dot{x}(t) = f(t, x(t), u)$, with u denoting control variables. Can we extend the idea of Lagrange multiplier rule to work in this dynamic optimization problems also? These are the optimal control problems, we will see next.

Let us start with a simple example. A gardener wishes to grow a number of plants to a height 2, by the time $t = 1$. The natural rate of growth is accelerated by artificial lighting $u(t)$, which is now the control variable.

To write down the mathematical model, assume that $x(t)$ is the height of the plant at time t. Then

$$\frac{dx}{dt} = 1 + u,$$

$x(0) = 0$ and $x(1) = 2$. The control u is due to artificial lighting. The gardener has to optimize his cost while achieving the target growth for his plant. Let us form the cost functional :

$$J(u) \ = \ \int_0^1 \frac{1}{2} u^2 dt.$$

How to find a control variable u that produces a solution $x(t)$, subject to the constraints and the boundary conditions, while at the same time allowing for a minimum cost?

We can solve the linear differential equation explicitly :

$$x(t) = \int_0^t (1 + u(\tau))\ d\tau$$

From the boundary condition at $t\ =\ 1$:

$$\int_0^1 u(\tau)\ d\tau = 1.$$

To find the optimal cost, rewrite J suitably :

$$\begin{aligned} J(u) &= \int_0^1 \frac{1}{2}u^2 dt = \int_0^1 \left(\frac{1}{2}(u-1)^2 + u - \frac{1}{2}\right) dt \\ &= \int_0^1 \frac{1}{2}(u-1)^2 + \int_0^1 u - \frac{1}{2} = \int_0^1 \frac{1}{2}(u-1)^2 + 1 - \frac{1}{2} \\ &= \int_0^1 \frac{1}{2}(u-1)^2 + \frac{1}{2}, \end{aligned}$$

using the boundary condition at 1. Thus the minimum value is $J(u) = \frac{1}{2}$ and is attained when $u = 1$. This indeed is the optimal control. The optimal trajectory x is then given by the solution of the linear ODE

$$\frac{dx}{dt}\ =\ 2, \quad x(0) = 0,\ x(1) = 2.$$

This is the so-called "Fixed final time-Fixed final point " type Optimal Control problem.

In practical applications, most equations are nonlinear and cannot be solved explicitly! Let us see one such example, the so-called moon landing problem. How to bring the spacecraft to land softly on the lunar surface using minimum amount of fuel?

Let $h(t)$ be the height of the spacecraft from the surface, $v(t)$ be the velocity of the spacecraft and $m(t)$ be its mass including the fuel. Suppose that g is the acceleration due to gravity and $\alpha(t)$ is the thrust to be applied. From Newton's second law,

$$m(t)\,\frac{d^2}{dt^2}(h(t))\ =\ -gm(t) + \alpha(t).$$

This can be modeled by an ODE system for (v, h, m),

$$\begin{aligned}\dot{v}(t) &= -g + \frac{\alpha(t)}{m(t)} \\ \dot{h}(t) &= v(t) \\ \dot{m}(t) &= -k\alpha(t),\end{aligned}$$

for some constant k. Here the problem is to minimize the fuel or to maximize the remaining amount when the spacecraft lands. From our equations,

$$m_0 - m(\tau) \;=\; \int_0^\tau k\alpha(t)\; dt$$

Form the cost functional

$$J(\alpha(.)) = m(T),$$

where T is the first time when $v(T) = 0$, or when it lands on the surface softly. Then the problem is to maximize $J(\alpha(.))$. Other constraints are

$$h(t) \geq 0, \quad m(t) \geq 0, \quad 0 \leq \alpha(t) \leq 1$$

This is an example of "Free final time - Fixed final point" type Optimal Control problem. See [5] for more such examples.

Let us consider a model problem in an abstract set up. We define the cost function

$$J\;(x(\cdot), u(\cdot)) \;=\; g(x(T)) \;+\; \int_0^T \ell\,(x(t), u(t))\, dt,$$

with g as the terminal cost and ℓ as the running cost. Here the function $x(t)$ evolves in time, according to the differential equation in $\mathbb{R}^N$,

$$\dot{x}(t) = f(x(t), u(t))$$

for $t \in [0, T]$ with initial condition $x(0) \;=\; x_0$. Here T is the time when $x(T) \;=\; x_1$, for a given target x_1. We have to minimize the cost functional under the constraint of the ODE, using the control function $u(t)$, possibly satisfying some additional constraint $u(t) \in U$ in $[0, T]$. In most cases, explicit solutions may not be available. But an extension of the idea of Lagrange multiplier rule to this situation, can give us optimality conditions. It may then be possible to solve numerically the ODE and these optimality conditions together to arrive at the optimal trajectory and an optimal control. Pontryagin Maximum Principle helps us to write down the optimality conditions. The following theorem is "Free final time - Fixed final point" type Optimal Control problem. See [4], section 4.3.

Theorem 2.1. *Let* $u^*(\cdot)$ *be optimal for the problem and let* $x^*(\cdot)$ *be the corresponding trajectory. Then there exists* $p^*(\cdot)$ *from* $[0,T]$ *to* $\mathbb{R}^N$ *such that with control Hamiltonian* $H(x,p,u) = \langle f(x,u), p\rangle + \ell(x,u)$,

$$\begin{aligned} \dot{x}^*(t) &= \nabla_p H(x^*(t), p^*(t), u^*(t)) \\ \dot{p}^*(t) &= -\nabla_x H(x^*(t), p^*(t), u^*(t)) \end{aligned}$$

and for $t \in [0,T]$,

$$\max_{u\in U} H(x^*(t), p^*(t), u) = H(x^*(t), p^*(t), u^*(t))$$

T *being the first time* $x^*(t)$ *hits the target* x_1. *Further*

$$\dot{H}(x^*(t), p^*(t), u^*(t)) = 0, \ t \in [0,T]; \quad p^*(T) = \nabla g(x^*(T)).$$

Here $x(t)$ is called the state variable and $p(t)$, the co-state variable, which in fact is the Lagrange multiplier.

2.3. **Optimal Control for PDE.** Unlike the earlier situation, suppose that the function space X consists of functions x, depending on more than one variable. Then the constraint may involve partial derivatives of x. Thus in this case, both the constraint and the optimality condition may involve PDE, stationary or evolutionary. Many physical processes, like heat conduction, vibration of plates, electro magnetic waves, fluid flows, are described via PDE. For these processes, optimal control problems are quite relevant. But for each PDE, the well-posedness theory has to be done according to the type of the PDE. Only then, the optimal control problem can be tackled.

Let us start with an example of an optimal control problem in stationary heating. The model is an elliptic equation, posed in a domain Ω in $\mathbb{R}^3$ with boundary Γ. The temperature y, reaches a steady state and a heating device is on the boundary of the domain, described by this PDE.

$$-\Delta y(x) + y(x) = f(x) \quad \text{for} \quad x \in \Omega, \quad \frac{\partial y}{\partial n}(x) = u(x) \quad \text{for} \quad x \in \Gamma.$$

Our aim is to minimize the distance between the steady state solution y and the desired distribution y_d and the consumed energy:

$$J(y,u) = \frac{1}{2}\int_\Omega |y - y_d|^2 dx + \frac{\beta}{2}\int_\Gamma |u|^2 ds(x).$$

Here β is some parameter chosen suitably.

In this case also, optimality conditions can be deduced. For simplicity, we avoid mentioning the function spaces and present here a diluted version of a theorem. Refer [7], Chapter 2 for more details. Here again, the co-state vector or adjoint vector p can be viewed as the Lagrange variable.

Theorem 2.2. *If* (y^*, u^*) *is the solution to the optimal control problem, then*

$$u^* \ = \ -\frac{p^*}{\beta}$$

where the adjoint vector p satisfies

$$-\Delta p + p = y - y_d \quad \text{in} \quad \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on} \quad \Gamma.$$

Conversely, if a pair (y^*, p^*) *obeys the original elliptic equation and the above adjoint equation, then the pair* $(y^*, \frac{-p^*}{\beta})$ *is the optimal solution to the optimal control problem.*

Next example is regarding the cooling of molten steel in an industrial steel casting machine. The cooling is carried out by controlled water sprays on the boundary. This complicated process can be modeled by a nonlinear equation

$$\begin{aligned} \rho(T)\ c(T)\ \frac{\partial T}{\partial t}(x,t) &= \text{div}(k(T)\ \nabla T(x,t)) \quad \text{in} \quad \Omega \times (0, t_f), \\ k(T)\frac{\partial T}{\partial n}(x,t) &= R(T, u(x,t)) \quad \text{on} \quad \Gamma \times (0, t_f) \\ T(x,0) &= T_0(x) \quad \text{in} \quad \Omega. \end{aligned}$$

Here T is the temperature of steel in Ω, ρ its density, c is specific heat capacity, k the conductivity of steel at temperature T and $u(x,t)$ is the control for water jets. The cost functional is

$$J(y,u) = \frac{1}{2}\int_\Omega |T(x,t_f) - T_d(x)|^2 dx + \frac{\beta}{2}\int_0^{t_f}\int_\Gamma |u(x,t)|^2\ ds(x)\ dt,$$

subject to some control constraints. Here also one can anticipate for a simplified linear problem, the optimality conditions using adjoint variable. It is not given here as it is not straightforward to state. The reader can refer [7], Chapter 3 and 5.

3. Controllability

Before calculating an optimal control, it is essential that we determine if the system is controllable or not.

3.1. **Controllability of ODE.**

A system of differential equations in $\mathbb{R}^N$

$$\dot{y}(t) = Ay(t) + Bu(t),\ t > 0, \quad y(0) = y^0 \in \mathbb{R}^N$$

with $A,\ N \times N$ matrix and $B,\ N \times m$ matrix, is controllable in time $T > 0$ if there exists an admissible control u which steers the system from any y^0 to a target y^1, that is for any y^0, y^1, the solution y satisfies $y(T) = y^1$.

Let us see some examples.

Example 1 : Consider $\dot{y} = Ay + Bu \in \mathbb{R}^2, \quad y(0) = y^0 \in \mathbb{R}^2$. Take the matrices to be

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the equations are

$$\dot{y_1} = y_1 + u, \quad \dot{y_2} = y_2.$$

Here the solution of the second equation $y_2 = (y^0)_2\ e^t$, is not affected by the control u. Thus we expect the system not to be controllable.

Example 2 : Let us take the matrices to be

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The equations are

$$\dot{y_1} = y_2, \quad \dot{y_2} = u - y_1,$$

or equivalently, $\ddot{y_1} + y_1 = u$. Using this, one can check that choosing u suitably, we can reach any vector in $\mathbb{R}^2$ for any initial condition. Thus we expect the system to be controllable.

It is interesting to have a test to check when a system is controllable. There are ways to decide if a given system is controllable or not.

Kalman rank condition characterizes controllable finite dimensional linear systems. It also shows that if the system is controllable for some T, then it is controllable for any time.

Theorem 3.1. *The linear system is controllable in time $T > 0$ if and only if*

$$\textit{rank}\ [B, AB, \cdots, A^{N-1}B] = N$$

For Example 1, the controllability matrix $[B, AB]$ is

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

It is clear that rank $[B, AB] = 1$ and Kalman rank condition does not hold.

For Example 2, the controllability matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We see that Kalman rank condition holds here.

To find other equivalent conditions for controllability, we use the formula for the solution to the system

$$z' = Az + Bu, \quad z(0) = z_0,$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times m}$:

$$z_{z_0,u}(t) = z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}Bu(s)\,ds.$$

This motivates the study of The operator $L_T : L^2(0, T; U) \longmapsto Z$, from the space of square integrable functions with values in U ($\mathbb{R}^m$ here) into Z ($\mathbb{R}^N$ here),

$$L_T u = \int_0^T e^{(T-s)A}Bu(s)\,ds.$$

Note that the system (A, B) is controllable at time $T < \infty$ if and only if Image $(L_T) = Z$. Further, L_T is surjective if and only if the adjoint operator $L_T^* \in L(Z, L^2(0, T; U))$,

$$(L_T^*\phi)(\cdot) = B^* e^{(T-\cdot)A^*}\phi,$$

is injective. This will be true if there is a lower bound for the norm of $L_T^*\phi$ in terms of the norm of ϕ.

This leads us to the study of the adjoint system :

$$-\phi'(t) = A^*\phi(t),\ t > 0, \quad \phi(T) = \phi_T$$

and its observability : There exists $c > 0$ such that

$$\int_0^T |B^*\phi|^2 \ \geq\ c|\phi_T|^2.$$

The following theorem gives a few controllability criteria. For more details, see [8], Theorem 1.6.

Theorem 3.2. *The system is controllable at time $T < \infty$ if and only if*

- *the adjoint system is observable in time $T > 0$.*
- *the matrix controllability Gramian*

$$L_T L_T^* \;=\; W_{A,B}^T = \int_0^T e^{tA} BB^* e^{tA^*} dt$$

is invertible.
- *Kalman rank condition holds :*

$$\textit{rank}\ [B, AB, \cdots, A^{N-1}B] = N$$

Once we know that a system is controllable, it is also important to compute the control. Using optimal control theory, a control with minimum norm can be found.

3.2. **Controllability of PDE.** Viewing a PDE as an ODE evolving in an infinite dimensional function space Z, a Hilbert space for example, some of the results of the earlier section can be extended to PDE, with additional technical details. Unlike the case of ODE, there are different concepts of controllability of PDE - approximate controllability, exact controllability and null controllability. The controllability will depend on the type of the PDE. Further, the PDE may be controllable for large time only, as in the case of the transport equation, which is a PDE of hyperbolic type. The reader may get a feel for these questions from [3], Chapters 1 and 2.

4. Stabilization

From ODE theory, we recall that the solution of a system of differential equations in $\mathbb{R}^N$

$$y'(t) = Ay(t),\ t > 0, \quad y(0) = y_0 \in \mathbb{R}^N,$$

with a given $N \times N$ matrix A, is $y(t) = e^{tA}(y_0)$. From this expression, one can see that the solution starting near the origin, will tend to 0 exponentially as t tends to infinity, if Real $(\sigma(A)) < 0$, (the spectrum of A is denoted by $\sigma(A)$) i.e if the eigenvalues of A have negative real part. Otherwise, the solution may become unbounded.

It is interesting to ask if one can prevent such blow up of solutions, when some of the eigenvalues have positive real part. Can we add a control term to bring the trajectories to 0? If yes, how to compute this control and implement it numerically to stabilize the system? This is the process of stabilization, well understood for the ODEs, currently being extended to various types of PDEs.

For a system of controlled differential equations in $\mathbb{R}^N$

$$y'(t) = Ay(t) + Bu(t),\ t > 0, \quad y(0) = y^0 \in \mathbb{R}^N,$$

with A, $N \times N$ matrix and B, $N \times m$ matrix, the question is to find a control $u \in \mathbb{R}^m$, steering the solution to 0 in $\mathbb{R}^N$. Further, we would to like to find it in feedback form or the control $u(t)$ depending on the state $y(t)$, more precisely, $u(t) = Ky(t)$ for a linear operator $K : \mathbb{R}^N \longmapsto \mathbb{R}^m$. This is likely to be more robust under perturbations.

If such a feedback control can be found, then the system

$$y'(t) = (A + BK)y(t),\ t > 0$$

will be stable if Real $\sigma(A + BK) < 0$. A matrix test for stabilizability is given by the following theorem. See [1] Chapter 1, Corollary 2.2.

Theorem 4.1. *The pair (A, B) is stabilizable if and only if*

$$\textit{rank}\ [(\lambda I - A), B] = n, \quad \forall\, \lambda \in \mathbb{C},\ \textit{Real}(\lambda) \geq 0.$$

If the matrix pair (A, B) is controllable, then it can be shown to be stabilizable also. Refer [1], Chapter 1, Corollary 2.1.

In the PDE situation, the above criteria for stabilizability extends to certain PDE of parabolic type.

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RECURSIVE SUBSEQUENCE OF VARIOUS FIBONACCI-TYPE SEQUENCES

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Abstract. In this article, we describe special types of recursive subsequence of various well-known sequences like the sequences of Lucas numbers, generalized Fibonacci numbers, Pell numbers, Pell-Lucas numbers and Half-companion Pell numbers and we obtain their recurrence relation in each case.

1. INTRODUCTION

The *Fibonacci sequence* $\{F_n\}$ is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. Also the sequence of *Lucas numbers* $\{L_n\}$ is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$, for all $n \geq 2$ with initial conditions $L_0 = 2$ and $L_1 = 1$. It was observed by Koshy [10] that the Binet formula for Fibonacci sequence and Lucas sequence are respectively given by $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ and $L_n = \alpha^n + \beta^n = \left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$, where $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ is famously referred as *'golden ratio'*. Many papers concerning a variety of generalizations of Fibonacci sequence have appeared in recent years. For further details, one can refer Arvadia, Shah [1, 2], Das, Patel, Shah [4], Diwan, Shah [5, 6], Harne, Singh, Pal [7], Patel, Shah [12].

In this section, we introduce various linear recursive sequences similar to that of Fibonacci/Lucas sequence and subsequently obtain their special types of subsequence in each case along with their recurrence relation.

Badshah, Teeth and Dar [3] considered an interesting combination of Fibonacci and Lucas sequences in the form $M_n = M_{n-1} + M_{n-2}; n \geq 2$ with $M_0 = 2m, M_1 = 1 + m$; where m is fixed positive integer. Here the initial conditions are the sum of the initial conditions of traditional Fibonacci sequence and 'm'' times the initial conditions of traditional Lucas sequence. Patel, Shah [13] further generalized it and considered the same sequence $\{M_n\}$ with the initial conditions as the sum of 'l'' times the initial conditions of traditional Fibonacci sequence and 'm'' times the initial conditions

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of traditional Lucas sequence. Thus $\{M_n\}$ is defined by the recurrence relation $M_n = M_{n-1} + M_{n-2}$, for all $n \geq 2$ with $M_0 = 2m, M_1 = l + m$; where m, l are integers. They obtain several of the fundamental identities related with $\{M_n\}$. They also derive Binet-type formula of M_n given by $M_n = l\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) + m\left(\alpha^n + \beta^n\right)$, where α, β are the roots of $x^2 - x - 1 = 0$.

Next we consider the sequence $\{g_n\}$ defined by the recurrence relation $g_n = g_{n-1} + g_{n-2}; n \geq 2$ with $g_0 = a, g_1 = b$; where a, b are integers. The corresponding Binet-type formula for g_n was derived by Horadam [8, 9] as $g_n = \frac{c\alpha^n - d\beta^n}{\alpha-\beta}$, where $c = \alpha a + (a - b)$ and $d = \beta a + (a - b)$.

We also consider Pell sequence $\{P_n\}$ defined by the relation $P_n = 2P_{n-1} + P_{n-2}; n \geq 2$ with $P_0 = 0, P_1 = 1$. The analogous Binet-type formula for Pell number was given by Koshy [11] as $P_n = \frac{\gamma^n-\delta^n}{\gamma-\delta}$, where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$.

We next consider Pell-Lucas sequence $\{Q_n\}$ defined by the relation $Q_n = 2Q_{n-1} + Q_{n-2}; n \geq 2$ with $Q_0 = 2, Q_1 = 2$. The equivalent Binet-type formula for Pell-Lucas numbers was also given by Koshy [11] as $Q_n = \gamma^n + \delta^n$, where γ and δ are as defined above.

Finally we consider the Half-companion Pell's sequence $\{Q_n\}$ defined by the relation

$$H_n = \begin{cases} 1 & n = 0 \\ H_{n-1} + 2P_{n-1}, & n \geq 1 \end{cases} \quad and \quad P_n = \begin{cases} 0 & n = 0 \\ H_{n-1} + P_{n-1}, & n \geq 1 \end{cases}$$

The corresponding Binet-type formula for Half-companion Pell number was obtained by Koshy [11] as $H_n = \frac{\gamma^n+\delta^n}{2}$; where γ and δ are as defined above.

Özvatan, Pashaev [14] considered the concept of subsequence for the Fibonacci numbers. Here we follow his technique and obtain such subsequence for all the sequences defined above.

2. LUCAS SUBSEQUENCE

To develop the understanding of Lucas subsequence, we first consider the subsequence $\{L_{3n}\}$ of $\{L_n\}$. For brevity, we write $G_n = L_{3n}$, and derive the corresponding recursion relation for G_n with suitable initial conditions for this sequence. Suppose that the recursion relation for G_n is of the type

$$G_{n+1} = AG_n + BG_{n-1} \tag{2.1}$$

where the coefficients A and B are to be determined. We write (2.1) as $L_{3n+3} = AL_{3n} + BL_{3n-3}$. We next express L_{3n+3} in terms of L_{3n} and L_{3n-3}. Using the definition of Lucas numbers, it is easy to show that $L_{3n+3} = 4L_{3n} + L_{3n-3}$. This gives the recurrence relation for the subsequence $\{G_n = L_{3n}\}$ of $\{L_n\}$ as $G_{n+1} = 4G_n + G_{n-1}$.

We use this idea to find the recurrence relation for the equi-Lucas subsequence $\{L_{kn+\tau}\}$ of L_n, where $\tau = 0, 1, 2, ..., k - 1$ and $k = 1, 2, ...$ are xed integers. Here the suffix of each of these numbers are congruent to $\tau(modk)$.

Here we write $G_n^{(k,\tau)} = L_{kn+\tau}$ and obtain the corresponding recursion formula for $\left\{G_n^{(k,\tau)}\right\}$.

Theorem 2.1. *The sequence* $\left\{G_n^{(k,\tau)} = L_{kn+\tau}\right\}$ *satisfies the recurrence relation*

$$G_{n+1}^{(k,\tau)} = L_k G_n^{(k,\tau)} + (-1)^{(k-1)} G_{n-1}^{(k,\tau)};$$

where L_k is k^{th} Lucas number.

Proof. First we rewrite the recursion formula to be proved as

$$L_{kn+\tau+k} = L_k L_{kn+\tau} + (-1)^{k-1} L_{kn+\tau-k}.$$

We prove this result using Binet's formula for Lucas numbers and the fact that $\alpha\beta = -1$. Now

$L_k L_{kn+\tau} + (-1)^{k-1} L_{kn+\tau-k}$
$= (\alpha^k + \beta^k)(\alpha^{kn+\tau} + \beta^{kn+\tau}) + (-1)^{k-1}(\alpha^{kn+\tau-k} + \beta^{kn+\tau-k}$
$= \alpha^{kn+\tau+k} + \beta^{kn+\tau+k} + \alpha^k\beta^{kn+\tau} + \beta^k\alpha^{kn+\tau} + (-1)^{k-1}\alpha^{kn+\tau}\alpha^{-k}$
$\quad +(-1)^{k-1}\beta^{kn+\tau}\beta^{-k}$
$= \alpha^{kn+\tau+k} + \beta^{kn+\tau+k} + \alpha^k\beta^{kn+\tau} + \beta^k\alpha^{kn+\tau} - \alpha^{kn+\tau}\beta^k - \beta^{kn+\tau}\alpha^k$
$= \alpha^{kn+\tau+k} + \beta^{kn+\tau+k}$
$= L_{kn+\tau+k}$, as required □

If we consider $\tau = 0, 1, 2$ successively, we get

$G_n^{(3;0)} = L_{3n} = L_{0(mod3)} = 2, 4, 18, ...$
$G_n^{(3;1)} = L_{3n+1} = L_{1(mod3)} = 1, 7, 29, ...$
$G_n^{(3;2)} = L_{3n+2} = L_{2(mod3)} = 3, 11, 47, ...$

These three sequences starts with different initial values while covering the whole Lucas sequence. Thus, we describe the sequences $L_{kn}, L_{kn+1}, ...$, $L_{kn+(k+1)}$ using the above recursion with the different initial conditions and covering the whole Lucas sequence.

In the following table we show the recursive relations of the subsequence $\left\{G_n^{(k,\tau)} = L_{kn+\tau}\right\}$ for the various values of τ:

SEQUENCES	DIFFERENCE	VALID RECURSION RELATION
$G_n^{(1;0)} = \{L_n\}$	0	$G_{n+1}^{(1;0)} = G_n^{(1;0)} + G_{n-1}^{(1;0)}, k = 1, \tau = 0$
$G_n^{(2;\tau)} = \{L_{2n}, L_{2n+1}\}$	1	$G_{n+1}^{(2;\tau)} = 3G_n^{(2;\tau)} - G_{n-1}^{(2;\tau)}, k = 2, \tau = 0, 1$
$G_n^{(3;\tau)} = \{L_{3n}, L_{3n+1}, L_{3n+2}\}$	2	$G_{n+1}^{(3;\tau)} = 4G_n^{(3;\tau)} + G_{n-1}^{(3;\tau)}, k = 3, \tau = 0, 1, 2$
⋮	⋮	⋮
$G_n^{(k;\tau)} = \left\{L_{kn}, \cdots, L_{kn+(k-1)}\right\}$	$k-1$	$G_{n+1}^{(k;\tau)} = L_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$, $\tau = 0, \cdot, k-1$

TABLE 1

We next derive subsequence of the sequence $\{M_n\}$.

3. SEQUENCE FOR $\{M_n\}$

In this section, we consider the subsequence of the sequence $\{M_n\}$. We define $G_n^{k;\tau} = M_{kn+\tau}$.

Theorem 3.1. *The sequence* $\left\{G_n^{(k;\tau)} = M_{kn+\tau}\right\}$ *satisfies the recurrence relation*

$$G_{n+1}^{(k;\tau)} = L_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$$

where L_k is the k^{th} Lucas number.

Proof. We rewrite this recursion formula as

$$M_{kn+\tau+k} = L_k M_{kn+\tau} + (-1)^{k-1} M_{kn+\tau-k}.$$

We prove this result using Binet-type formula for M_k. Now
$L_k M_{kn+\tau} + (-1)^{k-1} M_{kn+\tau-k}$
$= \left(\alpha^k + \beta^k\right)\left\{l\left(\frac{\alpha^{kn+\tau}-\beta^{kn+\tau}}{\alpha-\beta}\right) + m\left(\alpha^{kn+\tau} + \beta^{kn+\tau}\right)\right\}$
$+ (-1)^{k-1}\left\{l\left(\frac{\alpha^{kn+\tau-k}-\beta^{kn+\tau-k}}{\alpha-\beta}\right) + m\left(\alpha^{kn+\tau-k} + \beta^{kn+\tau-k}\right)\right\}$
$= \left\{l\left(\frac{\alpha^{kn+\tau+k}-\beta^{kn+\tau+k}}{\alpha-\beta}\right) + m(\alpha^{kn+\tau+k} + \beta^{kn+\tau+k})\right\} - \frac{l\alpha^k\beta^{kn+\tau}}{\alpha-\beta} + \frac{l\beta^k\alpha^{kn+\tau}}{\alpha-\beta} +$
$m\alpha^k\beta^{kn+\tau} + m\beta^k\alpha^{kn+\tau} + (-1)^{k-1} l\left(\frac{\alpha^{kn+\tau}\alpha^{-k}-\beta^{kn+\tau}\beta^{-k}}{\alpha-\beta}\right) + (-1)^{k-1} m(\alpha^{kn+\tau}\alpha^{-k} +$
$\beta^{kn+\tau}\beta^{-k})$.
Using the fact that $\alpha\beta = -1$, we get
$L_k + M_{kn+\tau} + (-1)^{k-1} M_{kn+\tau-k} = l\left(\frac{\alpha^{kn+\tau+k}-\beta^{kn+\tau+k}}{\alpha-\beta}\right) + m(\alpha^{kn+\tau+k} +$
$\beta^{kn+\tau+k}) = M_{kn+\tau+k}$.
Thus $G_{n+1}^{k;\tau} = L_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$, as required.

□

The following table shows recursive relations of the subsequence $\left\{G_n^{(k;\tau)} = M_{kn+\tau}\right\}$ for the various values of τ:

SEQUENCES	DIFFERENCE	VALID RECURSION RELATION
$G_n^{(1;0)} = \{M_n\}$	0	$G_{n+1}^{(1;0)} = G_n^{(1;0)} + G_{n-1}^{(1;0)}, k = 1, \tau = 0$
$G_n^{(2;\tau)} = \{M_{2n}, M_{2n+1}\}$	1	$G_{n+1}^{(2;\tau)} = 3G_n^{(2;\tau)} - G_{n-1}^{(2;\tau)}, k = 2, \tau = 0, 1$
$G_n^{(3;\tau)} = \{M_{3n}, M_{3n+1}, M_{3n+2}\}$	2	$G_{n+1}^{(3;\tau)} = 4G_n^{(3;\tau)} + G_{n-1}^{(3;\tau)}, k = 3, \tau = 0, 1, 2$
$\vdots$	$\vdots$	$\vdots$
$G_n^{(k;\tau)} = \left\{M_{kn}, \cdots, M_{kn+(k-1)}\right\}$	$k-1$	$G_{n+1}^{(k;\tau)} = L_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$, $\tau = 0, \cdot, k-1$

TABLE 2

4. SUBSEQUENCE FOR $\{g_n\}$

In this section, we consider the subsequence of the sequence $\{g_n\}$. We define $G_n^{k;\tau} = g_{kn+\tau}$.

Theorem 4.1. *The sequence* $\left\{G_n^{(k;\tau)} = g_{kn+\tau}\right\}$ *satisfies the recurrence relation*

$$G_{n+1}^{(k;\tau)} = L_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$$

where L_k *is the* k^{th} *Lucas number.*

Proof. We rewrite this recursion formula as

$$g_{kn+\tau+k} = L_k g_{kn+\tau} + (-1)^{k-1} g_{kn+\tau-k}.$$

We prove this result using Binet-type formula for g_k. Now
$L_k g_{kn+\tau} + (-1)^{k-1} g_{kn+\tau-k}$
$= (\alpha^k + \beta^k)\left(\frac{c\alpha^{kn+\tau} - d\beta^{kn+\tau}}{\alpha-\beta}\right) + (-1)^{k-1}\left(\frac{c\alpha^{kn+\tau-k} - d\beta^{kn+\tau-k}}{\alpha-\beta}\right)$
$= \left(\frac{c\alpha^{kn+\tau+k} - d\beta^{kn+\tau+k}}{\alpha-\beta}\right) - \frac{d\alpha^k\beta^{kn+\tau}}{\alpha-\beta} + \frac{c\beta^k\alpha^{kn+\tau}}{\alpha-\beta} + (-1)^{k-1}\left(\frac{c\alpha^{kn+\tau}\alpha^{-k} - d\beta^{kn+\tau}\beta^{-k}}{\alpha-\beta}\right)$
Using the fact that $\alpha\beta = -1$, we get
$L_k + g_{kn+\tau} + (-1)^{k-1} g_{kn+\tau-k} = \left(\frac{c\alpha^{kn+\tau+k} - d\beta^{kn+\tau+k}}{\alpha-\beta}\right) = g_{kn+\tau+k}.$
Thus $G_{n+1}^{k;\tau} = L_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$, as required. □

The following table shows recursive relations of the subsequence $\left\{G_n^{(k;\tau)} = g_{kn+\tau}\right\}$ for the various values of τ:

SEQUENCES	DIFFERENCE	VALID RECURSION RELATION
$G_n^{(1;0)} = \{g_n\}$	0	$G_{n+1}^{(1;0)} = G_n^{(1;0)} + G_{n-1}^{(1;0)}, k = 1, \tau = 0$
$G_n^{(2;\tau)} = \{g_{2n}, g_{2n+1}\}$	1	$G_{n+1}^{(2;\tau)} = 3G_n^{(2;\tau)} - G_{n-1}^{(2;\tau)}, k = 2, \tau = 0, 1$
$G_n^{(3;\tau)} = \{g_{3n}, g_{3n+1}, g_{3n+2}\}$	2	$G_{n+1}^{(3;\tau)} = 4G_n^{(3;\tau)} + G_{n-1}^{(3;\tau)}, k = 3, \tau = 0, 1, 2$
$\vdots$	$\vdots$	$\vdots$
$G_n^{(k;\tau)} = \{g_{kn}, \cdots, g_{kn+(k-1)}\}$	$k-1$	$G_{n+1}^{(k;\tau)} = L_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$, $\tau = 0, \cdot, k-1$

TABLE 3

5. SUBSEQUENCE FOR PELL SEQUENCE

In this section, we consider the subsequence of the Pell sequence $\{P_n\}$. We define $G_n^{k;\tau} = P_{kn+\tau}$.

Theorem 5.1. *The sequence* $\left\{G_n^{(k;\tau)} = P_{kn+\tau}\right\}$ *satisfies the recurrence relation*

$$G_{n+1}^{(k;\tau)} = 2H_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$$

where H_k *is the* k^{th} *Half-Companion Pell number.*

Proof. First we rewrite this recursion formula as

$$P_{kn+\tau+k} = 2H_k P_{kn+\tau} + (-1)^{k-1} P_{kn+\tau-k}.$$

We prove this result using Binet-type formula for P_k and H_k. Now
$2H_kP_{kn+\tau} + (-1)^{k-1}P_{kn+\tau-k}$
$= 2\left(\frac{\gamma^k+\delta^k}{2}\right)\left(\frac{\gamma^{kn+\tau}-\delta^{kn+\tau}}{\gamma-\delta}\right) + (-1)^{k-1}\left(\frac{\gamma^{kn+\tau-k}-\delta^{kn+\tau-k}}{\gamma-\delta}\right)$
$= \frac{\gamma^{kn+\tau+k}-\delta^{kn+\tau+k}}{\gamma-\delta} + \frac{\delta^k\gamma^{kn+\tau}}{\gamma-\delta} - \frac{\gamma^k\delta^{kn+\tau}}{\gamma-\delta} + (-1)^{k-1}\left(\frac{\gamma^{kn+\tau-k}-\delta^{kn+\tau-k}}{\gamma-\delta}\right)$
Using the fact that $\gamma\delta = -1$, we get
$2H_kP_{kn+\tau} + (-1)^{k-1}P_{kn+\tau-k} = \frac{\gamma^{kn+\tau+k}-\delta^{kn+\tau+k}}{\gamma-\delta} = P_{kn+\tau+k}.$, as required.

□

In the following table we present recursive relations of the subsequence $\left\{G_n^{(k;\tau)} = P_{kn+\tau}\right\}$ for the various values of τ:

SEQUENCES	DIFFERENCE	VALID RECURSION RELATION
$G_n^{(1;0)} = \{P_n\}$	0	$G_{n+1}^{(1;0)} = 2G_n^{(1;0)} + G_{n-1}^{(1;0)}, k = 1, \tau = 0$
$G_n^{(2;\tau)} = \{P_{2n}, P_{2n+1}\}$	1	$G_{n+1}^{(2;\tau)} = 6G_n^{(2;\tau)} - G_{n-1}^{(2;\tau)}, k = 2, \tau = 0, 1$
$G_n^{(3;\tau)} = \{P_{3n}, P_{3n+1}, P_{3n+2}\}$	2	$G_{n+1}^{(3;\tau)} = 14G_n^{(3;\tau)} + G_{n-1}^{(3;\tau)}, k = 3, \tau = 0, 1, 2$
$\vdots$	$\vdots$	$\vdots$
$G_n^{(k;\tau)} = \{P_{kn}, \cdots, P_{kn+(k-1)}\}$	$k-1$	$G_{n+1}^{(k;\tau)} = 2H_kG_n^{(k;\tau)} + (-1)^{k-1}G_{n-1}^{(k;\tau)}$, $\tau = 0, \cdot, k-1$

TABLE 4

6. SUBSEQUENCE FOR PELL-LUCAS SEQUENCE

Next we consider the subsequence of the Pell-Lucas sequence $\{Q_n\}$. We define $G_n^{k;\tau} = Q_{kn+\tau}$.

Theorem 6.1. *The sequence* $\left\{G_n^{(k;\tau)} = Q_{kn+\tau}\right\}$ *satisfies the recurrence relation*

$$G_{n+1}^{(k;\tau)} = 2H_kG_n^{(k;\tau)} + (-1)^{k-1}G_{n-1}^{(k;\tau)}$$

where H_k *is the* k^{th} *Half-Companion Pell number.*

Proof. First we rewrite this recursion formula as

$$Q_{kn+\tau+k} = 2H_kQ_{kn+\tau} + (-1)^{k-1}Q_{kn+\tau-k}.$$

We prove this result using Binet-type formula for Q_k and H_k. Now
$2H_kQ_{kn+\tau} + (-1)^{k-1}Q_{kn+\tau-k}$
$= 2\left(\frac{\gamma^k+\delta^k}{2}\right)\left(\gamma^{kn+\tau} + \delta^{kn+\tau}\right) + (-1)^{k-1}\left(\gamma^{kn+\tau-k} + \delta^{kn+\tau-k}\right)$
$= \gamma^{kn+\tau+k} + \delta^{kn+\tau+k} + \delta^k\gamma^{kn+\tau} + \gamma^k\delta^{kn+\tau} + (-1)^{k-1}\left(\gamma^{kn+\tau-k} + \delta^{kn+\tau-k}\right)$
Using the fact that $\gamma\delta = -1$, we get
$2H_kQ_{kn+\tau} + (-1)^{k-1}Q_{kn+\tau-k} = \gamma^{kn+\tau+k} + \delta^{kn+\tau+k} = Q_{kn+\tau+k}.$, as required.

□

In the following table we present recursive relations of the subsequence $\left\{G_n^{(k;\tau)} = Q_{kn+\tau}\right\}$ for the various values of τ:

SEQUENCES	DIFFERENCE	VALID RECURSION RELATION
$G_n^{(1;0)} = \{Q_n\}$	0	$G_{n+1}^{(1;0)} = 2G_n^{(1;0)} + G_{n-1}^{(1;0)}, k = 1, \tau = 0$
$G_n^{(2;\tau)} = \{Q_{2n}, Q_{2n+1}\}$	1	$G_{n+1}^{(2;\tau)} = 6G_n^{(2;\tau)} - G_{n-1}^{(2;\tau)}, k = 2, \tau = 0, 1$
$G_n^{(3;\tau)} = \{Q_{3n}, Q_{3n+1}, Q_{3n+2}\}$	2	$G_{n+1}^{(3;\tau)} = 14G_n^{(3;\tau)} + G_{n-1}^{(3;\tau)}, k = 3, \tau = 0, 1, 2$
$\vdots$	$\vdots$	$\vdots$
$G_n^{(k;\tau)} = \{Q_{kn}, \cdots, Q_{kn+(k-1)}\}$	$k-1$	$G_{n+1}^{(k;\tau)} = 2H_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$, $\tau = 0, \cdot, k-1$

TABLE 5

7. SUBSEQUENCE FOR HALF-COMPANION PELL SEQUENCE

Finally, we consider the subsequence of the Half-Companion Pell sequence $\{H_n\}$. We define $G_n^{k;\tau} = H_{kn+\tau}$.

Theorem 7.1. *The sequence* $\left\{G_n^{(k;\tau)} = H_{kn+\tau}\right\}$ *satisfies the recurrence relation*

$$G_{n+1}^{(k;\tau)} = 2H_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}.$$

Proof. First we rewrite this recursion formula as

$$H_{kn+\tau+k} = 2H_k H_{kn+\tau} + (-1)^{k-1} H_{kn+\tau-k}.$$

We prove this result using Binet-type formula for H_k. Now
$2H_k H_{kn+\tau} + (-1)^{k-1} H_{kn+\tau-k}$
$= 2\left(\frac{\gamma^k+\delta^k}{2}\right)\left(\frac{\gamma^{kn+\tau}+\delta^{kn+\tau}}{2}\right) + (-1)^{k-1}\left(\frac{\gamma^{kn+\tau-k}+\delta^{kn+\tau-k}}{2}\right)$
$= \frac{\gamma^{kn+\tau+k}+\delta^{kn+\tau+k}}{2} + \frac{\delta^k\gamma^{kn+\tau}}{2} + \frac{\gamma^k\delta^{kn+\tau}}{2} + (-1)^{k-1}\left(\frac{\gamma^{kn+\tau-k}+\delta^{kn+\tau-k}}{2}\right)$
Using the fact that $\gamma\delta = -1$, we get
$2H_k H_{kn+\tau} + (-1)^{k-1} H_{kn+\tau-k} = \frac{\gamma^{kn+\tau+k}+\delta^{kn+\tau+k}}{2} = H_{kn+\tau+k}$, as required. □

In the following table we present recursive relations of the subsequence $\left\{G_n^{(k;\tau)} = H_{kn+\tau}\right\}$ for the various values of τ:

SEQUENCES	DIFFERENCE	VALID RECURSION RELATION
$G_n^{(1;0)} = \{H_n\}$	0	$G_{n+1}^{(1;0)} = 2G_n^{(1;0)} + G_{n-1}^{(1;0)}, k = 1, \tau = 0$
$G_n^{(2;\tau)} = \{H_{2n}, H_{2n+1}\}$	1	$G_{n+1}^{(2;\tau)} = 6G_n^{(2;\tau)} - G_{n-1}^{(2;\tau)}, k = 2, \tau = 0, 1$
$G_n^{(3;\tau)} = \{H_{3n}, H_{3n+1}, H_{3n+2}\}$	2	$G_{n+1}^{(3;\tau)} = 14G_n^{(3;\tau)} + G_{n-1}^{(3;\tau)}, k = 3, \tau = 0, 1, 2$
$\vdots$	$\vdots$	$\vdots$
$G_n^{(k;\tau)} = \{H_{kn}, \cdots, H_{kn+(k-1)}\}$	$k-1$	$G_{n+1}^{(k;\tau)} = 2H_k G_n^{(k;\tau)} + (-1)^{k-1} G_{n-1}^{(k;\tau)}$, $\tau = 0, \cdot, k-1$

TABLE 6

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ON DECOMPOSITION OF A RATIONAL PRIME IN A CUBIC FIELD

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Abstract. This article gives an expanded proof of the theorem of Llorente and Nart, [10], on the decomposition of a rational prime in a cubic field. We use this theorem to determine the decomposition of a prime p in a cyclic cubic field as well as in a pure cubic field.

1. Introduction

How a rational prime p decomposes in a number field is one of the fundamental problems in the field of algebraic number theory. It has many applications. One such is the determination of the class number of a number field. In general, it is a difficult task to determine how a prime decomposes in a number field of fixed degree. For a few number fields, such decomposition is known. For example, it is known for quadratic, cyclotomic, cubic and some special number fields of fixed degree. Almost every book on algebraic number theory describes, how a prime splits in a quadratic field and in a cyclotomic field. But for a cubic field, it is partly given in [7] and [9]. Theorem 2.13 of [9], states the theorem from [10] which gives simple criteria to find how a prime decomposes in a cubic field in terms of the coefficients of a generating polynomial of the cubic field.

In 2006, S. Alaca, B. K. Spearman and K. S. Willams [2] also gave a proof of such decomposition in a cubic field. They used the theory of p-integral bases, whereas Llorente and Nart's proof used the results from classical number theory and p-adic numbers. However, it should be pointed out that Alaca, Spearman and Williams [2] gave the explicit form of the prime ideals occurring in the decomposition of the prime p not just the

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type of the decomposition. One still finds the proof of Llorente and Nart interesting as it involves results from various number theoretic topics

In this expository article, we elaborate the proof of Llorente and Nart of decomposition of a rational prime in a cubic field. This article has four sections. The first section is an introduction where we develop the background material for a general number field. In the second section, we develop the theory exclusive to cubic fields and state the important results going to be used in the third section. To keep the article brief, we will state all those results whose proofs are easily available in the literature otherwise we will give the proof. The third section is the main part of the article where we will see Llorente and Nart's proof on the decomposition of a rational prime in a cubic field. In the fourth section, as an application, we determine when and how a prime splits in a cyclic and in a pure cubic field.

Let K be an algebraic number field of degree n and let $\mathcal{O}_K$ be its the ring of integers. It is a well-known result that $\mathcal{O}_K$ is a Dedekind domain, so every ideal in $\mathcal{O}_K$ is written uniquely as the product of prime ideals of K. For a prime p let,

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_r^{e_r}, \tag{1.1}$$

where the $\mathfrak{p}_i$'s are the distinct prime ideal of $\mathcal{O}_K$ and the e_i's are the positive integers. The number e_i is called the ramification index of the ideal $\mathfrak{p}_i$ in $p\mathcal{O}_K$ and is denoted by $e(\mathfrak{p}_i/p)$. Being a Dedekind domain every prime ideal in $\mathcal{O}_K$ is a maximal ideal. Further $\mathcal{O}_K$ has the finite norm property, that is for an ideal I of $\mathcal{O}_K$, the order of $\mathcal{O}_K/I$ is finite. Hence, for any prime ideal $\mathfrak{p}_i$ in Equation 1.1, $\mathcal{O}_K/\mathfrak{p}_i$ is a finite field and the index $[\mathcal{O}_K/\mathfrak{p}_i : \mathbb{Z}/\langle p\rangle]$ is finite. We call this index the residual degree of $\mathfrak{p}_i$ and denote it by $f(\mathfrak{p}_i/p)$. Thus we have two numbers related to the factorization of ideal $p\mathcal{O}_K$. Dedekind's theorem on the factorization of ideals relates these numbers in the following way:

Theorem 1.1 ([11], Theorem 21). *Let K be an algebraic number field of degree n and $\mathcal{O}_K$ its ring of integers. Let p be a prime. If $p\mathcal{O}_K = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_r^{e_r}$ and f_i is the residual degree of $\mathfrak{p}_i$ for all i then*

$$\sum_{i=1}^{r} e_i f_i = n.$$

Moreover, if $K/\mathbb{Q}$ is a normal extension, then $e_1 = e_2 = \cdots = e_r = e$, $f_1 = f_2 = \cdots = f_r = f$ and $ref = n$.

Depending on the values of e_i and r, the splitting of $p\mathcal{O}_K$ is described as follows:

(1) When $r = n$ (i.e., e_i and f_i equal 1 for all i), we say $p\mathcal{O}_K$ splits completely in K.
(2) When $e_i = 1$ for all i, we say $p\mathcal{O}_K$ is unramified in K, otherwise it is said to be ramified in K.
(3) When r=1, e_1=n, we say $p\mathcal{O}_K$ is totally (completely) ramified in K.
(4) When $r = 1$, $e_1 = 1$, we say $p\mathcal{O}_K$ is inert (or remains prime) in K.
(5) When $e_i > 1$ but such e_i do not divide p, we say $p\mathcal{O}_K$ is tamely ramified in K.
(6) When $e_i > 1$ and at least one such e_i divides p, we say $p\mathcal{O}_K$ is wildly ramified in K.

Again due to Dedekind it is well-known that the ramified primes in K are finite and they are precisely those rational primes that divide the discriminant D of K. Hence once the discriminant is known, it is not difficult to find the ramified primes in K. But finding the discriminant itself is a nontrivial task.

Next, we quote the theorems regarding the decomposition of primes in quadratic and cyclotomic fields.

Theorem 1.2 (Prime Decomposition in a quadratic field, [11], Theorem 25)**.** *Let p be a prime and $K = \mathbb{Q}(\sqrt{m})$, where m is a square free integer. Then,*

$$p\mathcal{O}_K =< p, \sqrt{m} >^2 \quad \textit{if } p \mid m.$$

If $p = 2$ and m is odd, then

$$2\mathcal{O}_K = \begin{cases} \langle 2, 1+\sqrt{m}\rangle^2 & \textit{if } m \equiv 3 \pmod 4, \\ \left\langle 2, \dfrac{1+\sqrt{m}}{2}\right\rangle \left\langle 2, \dfrac{1-\sqrt{m}}{2}\right\rangle & \textit{if } m \equiv 1 \pmod 8, \\ \textit{remains prime} & \textit{if } m \equiv 5 \pmod 8. \end{cases}$$

If p is odd and $p \nmid m$, then

$$p\mathcal{O}_K = \begin{cases} \langle p, n+\sqrt{m}\rangle \langle p, n-\sqrt{m}\rangle & \textit{if } \left(\dfrac{m}{p}\right) = 1, \ \textit{where } n^2 \equiv m \pmod p, \\ \textit{remains prime} & \textit{if } \left(\dfrac{m}{p}\right) = -1. \end{cases}$$

Theorem 1.3 (Prime Decomposition in a cyclotomic field, [12], Theorem 4.40)**.** *Let $K = \mathbb{Q}(\zeta_m)$, where ζ_m is a primitive m^{th} root of unity, $m \neq 1$, $m \not\equiv 2 \pmod 4$ and let p be a prime.*

If $p \nmid m$ and f is the order of p mod m, i.e., the least positive integer with $p^f \equiv 1 \pmod m$, then $p\mathcal{O}_K$ is the product of $\phi(m)/f$ distinct prime ideal of K having degree f, where $\phi(m)$ is the number of positive integers less then m and relative prime to m.

If $p \mid m$, let $m = p^a m_1$, with $p \nmid m_1$, and f_1 is the order of p mod m_1, then

$$p\mathcal{O}_K = (\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_r)^e,$$

with $e = \phi(p^a), r = \phi(m_1)/f_1$ and distinct prime ideals $\mathfrak{p}_i$'s of degree f_1.

2. Important Results and Definitions

Let $K = \mathbb{Q}(\theta_2)$ be a cubic field. Let $f_2(x) = x^3 + a_1x^2 + a_2x + a_3$ be the minimal polynomial of θ_2. The substitution $x = y - a_1/3$ reduces the polynomial $f_2(x)$ to $f_1(y) = y^3 - a_4y + a_5$, where $a_4 = a_1^2/3 - a_2$ and $a_5 = (1/27)(2a_1^3 - 9a_1a_2 + 27a_3)$ (for details refer to [3], section 2). The coefficients a_4, a_5 need not be integers. But multiplying by a suitable integer (27 in this case) they can be made integers. Hence we can assume that $f_1(x) = x^3 - a_4x + a_5$, with $a_4, a_5 \in \mathbb{Z}$. Let θ_1 be its root. Further, if p is a prime such that $p^2 \mid a_4$ and $p^3 \mid a_5$ then we can reduce the polynomial $f_1(x)$ to $f(x) = (x/p)^3 - (a_4/p)(x/p) + (a_5/p^3)$ with the root $\theta = \theta_1/p$. Note that the roots of $f_2(x)$ and $f_1(x)$ differ by a constant, whereas the roots of $f_1(x)$ are constant multiples of that of $f(x)$, hence these roots generate the same cubic field.

Hence without loss of generality, we can assume that if $K = \mathbb{Q}(\theta)$ is a cubic field then θ is a root of a polynomial of the form $f(x) = x^3 - ax + b$, where $a, b \in \mathbb{Z}$. Further, there is no prime p such that $p^2 \mid a$ and $p^3 \mid b$. The discriminant of $f(x)$ is[1] $\triangle = 4a^3 - 27b^2$. In K, the ring $\mathbb{Z}[\theta]$ is a finitely generated submodule of rank 3. Hence, the index $[\mathcal{O}_K : \mathbb{Z}[\theta]]$ is finite. Denote this index by $i(\theta)$. It is a well-known result that $\triangle = i(\theta)^2 D$, where D is the discriminant of the field K (for proof see [12], Proposition 2.13 and corollary thereafter).

The discriminant of a number field tells us that, which primes $p \in \mathbb{Z}$ ramify in K. One may ask:

[1]For details refer [3], section 2

(1) Which prime ideals of K, ramify in K?

(2) What is the largest possible value of an integer m such that $p^m \mid D$?

The answer to these questions lies in the notion of the different ideal of K which we will deal within the next subsection. But before this, let us fix some notation.

For a prime p and an integer k denote by $v_p(k)$, the largest integer m such that $p^m \mid k$. For sake of convenience, we denote $v_p(\triangle)$ by s_p and $\triangle_p = \triangle/p^{s_p}$.

2.1. **Different Ideal.** Define $\mathcal{O}_K^\vee = \{x \in K \,:\, \mathrm{Tr}(x\mathcal{O}_K) \subseteq \mathbb{Z}\}$, where Tr x denote the usual trace of the number x. That is $\mathcal{O}_K^\vee$ consists of all $x \in K$ such that $\mathrm{Tr}(x\alpha) \in \mathbb{Z}$ for all $\alpha \in \mathcal{O}_K$. Then $\mathcal{O}_K^\vee$ is a fractional ideal of K. The inverse of $\mathcal{O}_K^\vee$ is defined as, $(\mathcal{O}_K^\vee)^{-1} = \{x \in K \,:\, x\,\mathcal{O}_K^\vee \subset \mathcal{O}_K\}$ is called the different ideal[2] and is denoted by $\mathfrak{D}$. If $\mathcal{O}_K = \mathbb{Z}[\theta]$, $\mathfrak{D} = \langle f'(\theta)\rangle$, where $f(x)$ is the minimal polynomial of θ. For the quadratic field $K = \mathbb{Q}(\sqrt{m})$, where m is a squarefree integer,

$$\mathfrak{D} = \begin{cases} \langle\sqrt{m}\rangle & \text{if } d \equiv 1 \pmod 4, \\ \langle 2\sqrt{m}\rangle & \text{if } d \not\equiv 1 \pmod 4. \end{cases}$$

The norm of $\mathfrak{D}$ is the absolute value of the discriminant D. (For a proof see [6], Theorem 4.6). The following theorem due to Dedekind gives the answers to the questions asked in the last paragraph, before this subsection.

Theorem 2.1 ([6], Theorem 4.8). *The prime ideal factors of $\mathfrak{D}$ are the primes in K that ramify over $\mathbb{Q}$. More precisely, for any prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ lying over a prime number p, with the ramification index $e = e\,(\mathfrak{p}/p)$, the exact power of $\mathfrak{p}$ in $\mathfrak{D}$ is $e-1$ if $p \nmid e$, and $\mathfrak{p}^e \mid \mathfrak{D}$ if $p \mid e$.*

As a consequence of Theorem 2.1 we have the following corollary.

Corollary 2.2 ([6], Corollary 4.10). Let $p\mathcal{O}_K = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_r^{e_r}$ and $f_i = f(\mathfrak{p}_i/p)$ for all i. If no e_i is a multiple of p then the multiplicity of p in the discriminant of K is

$$(e_1-1)f_1 + (e_2-1)f_2 + \cdots + (e_r-1)f_r = n - \sum_{i=1}^{r} f_i.$$

If some e_i is a multiple of p then the multiplicity of p in the discriminant of K is larger than this number.

[2] For details refer to [6].

It was proved that, if $v_{\mathfrak{p}}(\mathfrak{D})$ denotes the multiplicity of $\mathfrak{p}$ in $\mathfrak{D}$ then

$$e - 1 \le v_{\mathfrak{p}}(\mathfrak{D}) \le e - 1 + e v_p(e), \tag{2.1}$$

where $\mathfrak{p} \mid p$ and $e = e\,(\mathfrak{p}/p)$.

Using these results we derive the following lemma.

Lemma 2.3. *Let K be a cubic field and let p be a prime number. Then,*

(1) For $p \neq 2$, $p = \mathfrak{p}_1\mathfrak{p}_2^2$ in K if and only if $v_p(D) = 1$.
(2) For $p \neq 3$, $p = \mathfrak{p}_1^3$ in K if and only if $v_p(D) = 2$.
(3) If $2 = \mathfrak{p}_1\mathfrak{p}_2^2$ in K then $v_2(D) = 2$ or 3.
(4) If $3 = \mathfrak{p}_1^3$ in K then $v_3(D) = 3, 4,$ or 5.

Proof. We begin by proving (1). Let us assume that the prime $p \neq 2$ and $p = \mathfrak{p}_1\mathfrak{p}_2^2$. Then $e_1 = 1, e_2 = 2, f_1 = 1$ and $f_2 = 1$. As p does not divide any e_i, by Theorem 2.1, the exact multiplicity of p in D is $3 - (1 + 1) = 1$. Therefore $v_p(D) = 1$. Conversely let $v_p(D) = 1$. Hence the prime p is ramified in K. So $p = \mathfrak{p}_1\mathfrak{p}_2^2$ or $p = \mathfrak{p}_1^3$. But if $p = \mathfrak{p}_1^3$ then, $v_p(D)$ is at least 2. Hence $p = \mathfrak{p}_1\mathfrak{p}_2^2$.

The proof of (2) is similar to that of (1).

We now prove (3). Let $2 = \mathfrak{p}_1\mathfrak{p}_2^2$. Then $e_1 = 1$, $e_2 = 2$, $f_1 = 1$ and $f_2 = 1$. Let $v_{\mathfrak{p}_2}(\mathfrak{D}) = l$. By Equation 2.1, we have,

$$e_2 - 1 \le l \le e_2 - 1 + e_2 v_2(e_2).$$

That is $1 \le l \le 3$. But $2 \mid e_2$, hence l is at least 2. Further, $\mathfrak{p}_2^l \mid \mathfrak{D}$. Hence $N(\mathfrak{p}_2)^l \mid N(\mathfrak{D})$. But $N(\mathfrak{p}_2) = p^{f_2} = p$. Hence $p^l \mid D$ and the result follows.

The proof of (4) follows similarly. □

We need one of the applications of Hensel's lemma for finding the square root of a number in a p-adic number field $\mathbb{Q}_p$.

2.2. **Square Root of a p-adic number.** A p-adic integer is a formal series $\sum_{i\ge 0} a_i p^i$ with integral coefficients a_i satisfying $0 \le a_i \le p - 1$ for all i. The set of all p-adic integers forms an integral domain and is denoted by $\mathbb{Z}_p$. The field of fractions of $\mathbb{Z}_p$ is called the field of p-adic numbers and is denoted by $\mathbb{Q}_p$.

Let $a = \sum_{i\ge 0} a_i p^i$ be a p-adic integer. If $a \neq 0$ then, there is a first index $v \ge 0$ such that $a_v \neq 0$. This index is the p-adic order of a and we denote it by $v_p(a)$. An element a of $\mathbb{Z}_p$ is invertible if its order $v_p(a) = 0$. We denote by $\mathbb{Z}_p^*$, the set of all invertible elements of $\mathbb{Z}_p$. It can be proved that $\mathbb{Z}_p$ has a unique maximal ideal which is $p\mathbb{Z}_p$ and $\mathbb{Z}_p^* = \mathbb{Z}_p - p\mathbb{Z}_p$. Hence every p-adic integer a can be written uniquely as $a = p^v a'$ where $v = v_p(a)$ is the order of a and $a' \in \mathbb{Z}_p^*$. The same is true for every p-adic number, with the only difference being that the order v can be a negative integer. Hence $\mathbb{Q}_p = p^{\mathbb{Z}}\mathbb{Z}_p^*$. For more details see [13]. Surprisingly there are very few quadratic extensions of $\mathbb{Q}_p$ that can be determined by Hensel's lemma.

Theorem 2.4 (Hensel's Lemma, [13], Page 48)**.** *Assume $f(x) \in \mathbb{Z}_p[x]$ and $a \in \mathbb{Z}_p$ satisfies $f(a) \equiv 0 \pmod{p^n}$. If $k = v_p(f'(a)) < n/2$, then there exists a unique root α of $f(x)$ in $\mathbb{Z}_p$ such that $\alpha \equiv a \pmod{p^{n-k}}$ and $v_p(f'(\alpha)) = k$.*

We will use this theorem to characterize the squares in $\mathbb{Q}_p$.

Corollary 2.5. An integer $a \in \mathbb{Z}$ is a square in $\mathbb{Z}_p$ if and only if $a = p^{2r}c$, where $r \in \mathbb{Z}$, $c \in \mathbb{Z}_p^*$ and $\begin{cases} \left(\dfrac{c}{p}\right) = 1 & \text{if } p \neq 2, \\ c \equiv 1 \pmod 8 & \text{if } p = 2. \end{cases}$

Proof. Let us assume that $a = \beta^2$ in $\mathbb{Z}_p$. Let $a = p^s c$ where $c \in \mathbb{Z}_p^*$. Then $s = v_p(a) = v_p(\beta^2) = 2v_p(\beta)$. Therefore s is even say $2r$. As $a = p^{2r}c$ is a square in $\mathbb{Z}_p$, hence c is a square in $\mathbb{Q}_p$. Let $c = \gamma^2$. But $0 = v_p(c) = 2v_p(\gamma)$. Hence $v_p(\gamma) = 0$, so $\gamma \in \mathbb{Z}_p^*$.

First assume $p \neq 2$. Let (a_n) be the Cauchy sequence of integers converging to γ (here we used the fact that $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$). Then (a_n^2) converges to c. Hence there exists $n_0 \in \mathbb{N}$ such that $|a_n^2 - c|_p < 1/p$ for all $n \geq n_0$. Hence $a_{n_0}^2 \equiv c \pmod p$ implies $\left(\dfrac{c}{p}\right) = 1$.

For $p = 2$, first observe that $\mathbb{Z}_2^* = 1 + 2\mathbb{Z}_2 = \{\pm 1\}(1 + 4\mathbb{Z}_2)$. The first equality is from the fact that the only non zero constant term in $\mathbb{Z}_2$ is 1. For the second equality note that elements of $1 + 2\mathbb{Z}_2$ are either of the form $1 + 4a_1 + 8a_2 + \cdots$ or $3 + 4a_1 + 8a_2 + \cdots$. Hence $(\mathbb{Z}_2^*)^2 = 1 + 8\mathbb{Z}_2$. Therefore if $c \in \mathbb{Z}_2^*$ is a square, then $c \in 1 + 8\mathbb{Z}_2$, equivalently $c \equiv 1 \pmod 8$.

Conversely let $a = p^{2r}c$, where c satisfies the hypothesis of the corollary. It is enough to prove c is a square. This is trivially an application of Hensel's lemma. When $p \neq 2$ then, $\left(\dfrac{c}{p}\right) = 1$, hence there exists an integer b such that $b^2 \equiv c \pmod p$. Take $f(x) = x^2 - c$, then $f(b) = 0 \pmod p$, $v_p(f'(b)) = 0 < 1/2$. Hence Hensel's lemma ensure that c is a square in $\mathbb{Z}_p$.

For $p = 2$, $c \equiv 1 \pmod 8$. Define $f(x) = x^2 - c$. Then $f(1) = 0 \pmod{2^3}$ and $v_p(f'(1)) = 1 < 3/2$. Again by Hensel's lemma, c is a square. $\square$

2.2.1. *Quadratic Extension of* $\mathbb{Q}_p$. First Assume that $p \neq 2$. Then $\mathbb{Q}_p = p^{\mathbb{Z}}\mathbb{Z}_p^*$. Therefore $\mathbb{Q}_p/\mathbb{Q}_p^2 = p^{\mathbb{Z}}/p^{2\mathbb{Z}} \times \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus $\mathbb{Q}_p/\mathbb{Q}_p^2$ is a group of type $(2,2)$. Note that p is not a square in $\mathbb{Z}_p^*$ as the power of p is odd. Choose $u \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $\left(\frac{u}{p}\right) = -1$. Then $\{1, p, u, pu\}$ generates $\mathbb{Q}_p/\mathbb{Q}_p^2$. Hence any quadratic extension of $\mathbb{Q}_p$ is isomorphic to $\mathbb{Q}_p(\sqrt{p})$, $\mathbb{Q}_p(\sqrt{u})$ or $\mathbb{Q}_p(\sqrt{pu})$.

Let $p = 2$. We saw that $\mathbb{Z}_2^* = \{\pm 1\}(1 + 4\mathbb{Z}_2)$. and $(\mathbb{Z}_2^*)^2 = 1 + 8\mathbb{Z}_2$. Then

$$\mathbb{Q}_2/\mathbb{Q}_2^2 = 2^{\mathbb{Z}}/2^{2\mathbb{Z}} \times \{\pm 1\}(1 + 4\mathbb{Z}_2)/(1 + 8\mathbb{Z}_2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Thus $\mathbb{Q}_2/\mathbb{Q}_2^2$ is of the type $(2,2,2)$. Hence in particular we get $\mathbb{Q}_2/\mathbb{Q}_2^2 = \{\pm 1, \pm 2, \pm(1+4), \pm 10\}$. Therefore any quadratic extension of $\mathbb{Q}_2$ is isomorphic to $\mathbb{Q}_p(\sqrt{-1})$, $\mathbb{Q}_p(\sqrt{\pm 2})$, $\mathbb{Q}_p(\sqrt{\pm 5})$ or $\mathbb{Q}_p(\sqrt{\pm 10})$.

Remark 2.6. Among all the quadratic extensions of $\mathbb{Q}_2$ only $\mathbb{Q}_2(\sqrt{5})$ is the unramified extension of $\mathbb{Q}_2$ (see [12], Corollary 10 of Theorem 5.15).

The following are some results which we need subsequently.

Lemma 2.7. *Let $K = \mathbb{Q}(\theta)$ be a cubic field. Let $f(x) = x^3 - ax + b$ be the minimal polynomial of θ and $\triangle$ its discriminant. Then for a prime p,*

(i) If $p = \mathfrak{p}$ or $p = \mathfrak{pqr}$ then $\triangle \in \mathbb{Q}_p^2$.
(ii) If $p = \mathfrak{pq}$ or $p = \mathfrak{pq}^2$ then, $\triangle \notin \mathbb{Q}_p^2$.

Lemma 2.8 ([7], $18). *Let $K = \mathbb{Q}(\theta)$ be a number field of degree n and $f(x)$ the minimal polynomial of θ. Let $p \in \mathbb{Z}$ be a prime and let*

$$f(x) = \phi_1(x)^{e_1}\phi_2(x)^{e_2}\cdots\phi_r(x)^{e_r} \pmod p$$

be the factorization of $f(x)$ into irreducible factors $\pmod p$. *If p does not divide $i(\theta)$, the index of θ, then*

$$p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_r^{e_r}$$

where $\mathfrak{p}_i = (p, \phi_i(\theta))$ and $f(\mathfrak{p}_i/p) = deg\ \phi_i(x)$.

We will frequently use a result from the theory of the Newton polygon which we will see in the following subsection.

2.3. **Newton Polygon.** Let F be a field and let $f(x) = a_nx^n + \cdots + a_1x + a_0 \in F[x]$ be such that $a_0a_n \neq 0$. To find the Newton polygon of $f(x)$ with respect to the prime p we do the following:

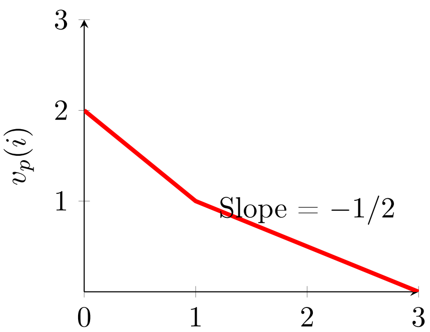
(1) For each a_i of $f(x)$, plot the points $(i, v_p(a_i))$ on the xy-plane, ignoring all those points for which $a_i = 0$.

(2) Start with the point $(0, v_p(a_0))$. Rotate a ray passing through this point in the anticlockwise direction, till it reaches a point $(k, v_p(a_k))$ (Note that this point may not be $(1, v_p(a_1))$). Break the ray here and from $(k, v_p(a_k))$ rotate another ray in the anticlockwise direction till it reaches to another point. We continue in this way till we reach to the last point $(n, v_p(a_n))$.

(3) Thus we get convex hull[3] of the points $(k, v_p(a_k))$, $0 \leq k \leq n$ that is the Newton polygon of $f(x)$ with respect to p.

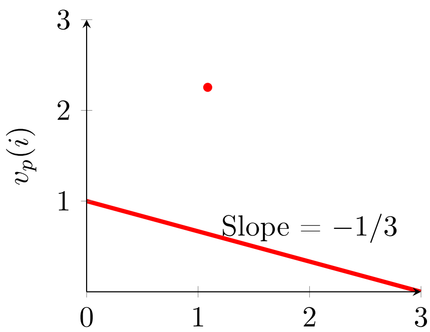
The following are two examples of the Newton polygon.

$f(x) = x^3+10x+25$ and $p=5$



Graph 1

$f(x) = x^2-25x+5$ and $p=5$



Graph 2

Let $K = \mathbb{Q}(\theta)$ be a number field and $f(x)$ the minimal polynomial of θ. The following lemma gives a connection between the prime ideals of K occurring in the factorization of the prime p and the slopes of the sides of the Newton polygon of $f(x)$ with respect to p.

Lemma 2.9 (Bauer, [4])**.** *Let $K = \mathbb{Q}(\theta)$ be a number field of degree n. Let $f(x)$ be the minimal polynomial of θ. Let $p \in \mathbb{Z}$ be a prime. Then for every prime ideal $\mathfrak{p}$ of K lying over p, the quotient $v_{\mathfrak{p}}(\theta)/e(\mathfrak{p}/p)$ is equal to the slope of one of the sides of the Newton polygon of $f(x)$ with respect to*

[3]For a given set of points, the convex hull is the smallest convex polygon that has all the points of it.

p. Conversely if $\lambda \in \mathbb{Q}$ is the slope of one of the sides of that polygon, then there exists a prime $\mathfrak{p}$ of K lying over p such that $v_{\mathfrak{p}}(\theta)/e(\mathfrak{p}/p) = \lambda$.

3. Decomposition of a prime in a cubic field

We use all the tools developed in the previous sections to determine the splitting of a rational prime in a cubic field. Let us revisit all the notation that we are going to use.

Let $K = \mathbb{Q}(\theta)$ be a cubic field such that the minimal polynomial of θ is $f(x) = x^3 - ax + b$. The discriminants of $f(x)$ and K are denoted by $\triangle$ and D respectively with $\triangle = 4a^3 - 27b^2$. If $i(\theta) = [\mathcal{O}_K : \mathbb{Z}[\theta]]$ is the index of θ then $\triangle = i(\theta)^2 D$. For $m \in \mathbb{Z}$, $v_p(m)$ is the largest power of p such that $p^{v_p(m)} \mid m$. Let $v_p(\triangle) = s_p$ and $\triangle_p = \triangle / p^{s_p}$.

Further, we have chosen the coefficients a and b of $f(x)$ in such a way that for any prime p, we have $v_p(a) < 2$ or $v_p(b) < 3$.

For the sake of convenience, we have divided the theorem of Llorente and Nart on the splitting of primes into three parts, (i) $p > 3$, (ii) $p = 2$ and (iii) $p = 3$. In order to reduce the space in all tables, (mod m) is abbreviated as (m) only.

Theorem 3.1 (Llorente, Nart, [10], Theorem 1)**.** *The prime $p > 3$ decomposes in K as follows*

<table>
<tr><th>Case</th><th colspan="3">Sub case</th><th>Factorization</th></tr>
<tr><td rowspan="2">$p \mid a, p \mid b$</td><td colspan="3">$1 = v_p(a) < v_p(b)$</td><td>$p = \mathfrak{p}\mathfrak{q}^2$</td></tr>
<tr><td colspan="3">$1 \leq v_p(b) \leq v_p(a)$</td><td>$p = \mathfrak{p}^3$</td></tr>
<tr><td rowspan="3">$p \mid a, p \nmid b$</td><td colspan="3">$p \equiv -1(3)$</td><td>$p = \mathfrak{p}\mathfrak{q}$</td></tr>
<tr><td rowspan="2">$p \equiv 1(3)$</td><td colspan="2">$\left(\frac{b}{p}\right)_3 = 1$</td><td>$p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td colspan="2">$\left(\frac{b}{p}\right)_3 \neq 1$</td><td>$p = \mathfrak{p}$</td></tr>
<tr><td rowspan="2">$p \nmid a, p \mid b$</td><td colspan="3">$\left(\frac{a}{p}\right) = 1$</td><td>$p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td colspan="3">$\left(\frac{a}{p}\right) = -1$</td><td>$p = \mathfrak{p}\mathfrak{q}$</td></tr>
<tr><td rowspan="4">$p \nmid ab$</td><td colspan="3">s_p is odd</td><td>$p = \mathfrak{p}\mathfrak{q}^2$</td></tr>
<tr><td rowspan="3">s_p is even</td><td rowspan="2">$\left(\frac{\Delta_p}{p}\right) = 1$</td><td>$f(x)$ has some root (p)</td><td>$p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td>$f(x)$ has no roots (p)</td><td>$p = \mathfrak{p}$</td></tr>
<tr><td colspan="2">$\left(\frac{\Delta_p}{p}\right) = -1$</td><td>$p = \mathfrak{p}\mathfrak{q}$</td></tr>
</table>

Table 1: Decomposition of Prime $p > 3$

Proof. Case (i) : $p \mid a,\ p \mid b$

By hypothesis $v_p(a) < 2$ or $v_p(b) < 3$. Therefore $3v_p(a) \neq 2v_p(b)$. Hence we have the following two subcases. $3v_p(a) < 2v_p(b)$ and $3v_p(a) > 2v_p(b)$. In the first subcase we have $1 = v_p(a) < v_p(b)$ whereas in the second, we have $1 \leq v_p(b) \leq v_p(a)$. In the first subcase the coefficients of the Newton polygon for $f(x)$ with respect to p are $(0, v_p(b)), (1, 1), (2, \infty)$ and $(3, 0)$ with $v_p(b) > 1$ whereas in the second subcase the corresponding coefficients are $(0, v_p(b)), (1, v_p(a)), (2, \infty)$ and $(3, 0)$ with $v_p(b) = 1$ or 2. In the first subcase the Newton polygon consists of two sides with the slope of one side equal to $-1/2$ (see Graph 1), hence by Lemma 2.9, $p = \mathfrak{p}\mathfrak{q}^2$. While in the second subcase the Newton polygon consists of only one side with the slope either $-1/3$ or $-2/3$ (see Graph 2). Thus in this case we obtain $p = \mathfrak{p}^3$.

Case (ii) : $p \mid a,\ p \nmid b$.

Since $p > 3$, we have $s_p = 0$. Hence by Lemma 2.8 the decomposition of p depends on how $f(x) = x^3 + b$ splits in $\mathbb{Z}/p\mathbb{Z}$. First observe that $(\mathbb{Z}/p\mathbb{Z})^*$ is a cyclic group of order $p-1$. Hence the polynomial $x^3 - 1$ splits completely in $(\mathbb{Z}/p\mathbb{Z})^*$ if and only if $(\mathbb{Z}/p\mathbb{Z})^*$ contains a cyclic subgroup of order 3, that is if and only if $3 \mid (p-1)$, equivalently $p \equiv 1 \pmod 3$. Hence for $p \equiv 1 \pmod 3$, the polynomial $x^3 + b$ splits completely in $\mathbb{Z}/p\mathbb{Z}$ if and only if the polynomial has one root in $\mathbb{Z}/p\mathbb{Z}$. That is if and only if the cubic reciprocity symbol $\left(\dfrac{b}{p}\right)_3 = 1$. Hence $p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$ or $p = \mathfrak{p}^3$ according as $\left(\dfrac{b}{p}\right)_3 = 1$ or $\left(\dfrac{b}{p}\right)_3 \neq 1$ respectively.

When $p \equiv -1 \pmod 3$, then $(\mathbb{Z}/p\mathbb{Z})^* \cong (\mathbb{Z}/p\mathbb{Z})^{*3}$ (since the map $x \to x^3$ has the trivial kernel). Hence in this case $x^3 + b$ has only one root in $\mathbb{Z}/p\mathbb{Z}$. So $p = \mathfrak{p}\mathfrak{q}$.

Case (iii) : $p \nmid a, p \mid b$

As $p \neq 2$, $s_p = 0$. Hence according to Lemma 2.8 the splitting of p is governed by the splitting of $f(x) = x^3 - ax$ in $\mathbb{Z}/p\mathbb{Z}$. Now $x^3 - ax$ splits completely in $\mathbb{Z}/p\mathbb{Z}$ if and only if a is a quadratic residue modulo p, that is $\left(\dfrac{a}{p}\right) = 1$ and into a linear and a quadratic factor if and only if $\left(\dfrac{a}{p}\right) = -1$. Hence the result follows.

Case (iv) : $p \nmid ab$.

By Lemma 2.3, $p = \mathfrak{p}\mathfrak{q}^2$ if and only if $v_p(D) = 1$, that is if s_p is odd. So assume s_p is even. Since $p \nmid ab$, hence $f(x) = (x + c)^3 \pmod p$ is not possible for any $c \in \mathbb{Z}$. Therefore we know that $p \neq \mathfrak{p}^3$. Combining Lemma 2.3 and Lemma 2.7 we get, if $\triangle \notin \mathbb{Q}_p^2$ then $p \neq \mathfrak{pq}$ and $p \neq \mathfrak{pqr}$. So the only case left out is $p = \mathfrak{pq}$, which occurs when s_p is even and $\triangle \notin \mathbb{Q}_p^2$ or $\left(\frac{\triangle_p}{p}\right) = -1$. When s_p is even and $\left(\frac{\triangle_p}{p}\right) = 1$, the only cases left to consider us are $p = \mathfrak{p}$ and $p = \mathfrak{pqr}$ which can be distinguished according as $f(x)$ is irreducible or splits completely in $\mathbb{Z}/p\mathbb{Z}$.

With this the case $p > 3$ is concluded. □

Theorem 3.2 (Llorente, Nart, [10], Theorem 1)**.** *The rational prime $p = 2$ decomposes in K as follows*

Case	*Sub case*		*Factorization*
a, b even	$1 = v_2(a) < v_2(b)$		$2 = \mathfrak{pq}^2$
	$1 \leq v_2(b) \leq v_2(a)$		$2 = \mathfrak{p}^3$
a even, b odd	—		$2 = \mathfrak{pq}$
a odd, b even	s_2 *odd*		$2 = \mathfrak{pq}^2$
	s_2 *even*	$\triangle_2 \equiv 3\ (4)$	$2 = \mathfrak{pq}^2$
		$\triangle_2 \equiv 5\ (8)$	$2 = \mathfrak{pq}$
		$\triangle_2 \equiv 1\ (8)$	$2 = \mathfrak{pqr}$
a, b odd	—		$2 = \mathfrak{p}$

Table 2: Decomposition of Prime $p = 2$

Proof. Case (i) : a and b are both even.

The proof is similar to case 1 of $p > 3$.

Case (ii) : a even, b odd.

In this case $2 \nmid \triangle$. Therefore by Lemma 2.8, 2 splits in K according as to how $f(x) \equiv x^3 + 1 \pmod 2$ splits in $\mathbb{Z}/2\mathbb{Z}$. Since $f(x)$ has only one root in $\mathbb{Z}/2\mathbb{Z}$ (namely $x = 1$), therefore $f(x)$ is the product of a linear and a quadratic factor over $\mathbb{Z}/2\mathbb{Z}$, this implies $2 = \mathfrak{pq}$.

Case (iii) : a is odd and b is even.

In this case $f(x)$ reduces to $x^3 - x$. Thus $f(x)$ is not irreducible (hence $2 \neq \mathfrak{p}$). Similarly $f(x) = (x + 1)^3 \pmod 2$ is also not possible (hence $2 \neq \mathfrak{p}^3$). Therefore $2 = \mathfrak{pq}$ or $\mathfrak{pq}^2$ or $\mathfrak{pqr}$. If s_2 is odd then $2 \mid D$. Therefore $2 = \mathfrak{pq}^2$. Assume now s_2 is even. By Lemma 2.7, $2 = \mathfrak{pqr}$ if and only if $\triangle_2 \in \mathbb{Q}_p^2$ that is $\triangle_2 \equiv 1 \pmod 8$ (see Corollary 2.5).

We assume now $\triangle_2 \not\equiv 1 \pmod 8$. As $\triangle_2$ is odd, we have the two possibilities $\triangle_2 \equiv 5 \pmod 8$ or $\triangle_2 \equiv 3 \pmod 4$. As far as the decomposition of 2 is concerned we have the two possibilities, $2 = \mathfrak{pq}$ and $2 = \mathfrak{pq}^2$. If

$2 = \mathfrak{pq}$ then 2 is unramified in the quadratic extension $\mathbb{Q}_2(\sqrt{\triangle_2})$. This means $\triangle_2 \equiv 5 \pmod 8$, as the only unramified extension of $\mathbb{Q}_2$ is $\mathbb{Q}_2(\sqrt{5})$ (see the Remark 2.6). The last case left out occurs when $\triangle_2 \equiv 3 \pmod 4$.

Case (iv) a, b are both odd

In this case $s_2 = 0$ and $f(x) = x^3 - x + 1 \pmod 2$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$. Hence $2 = \mathfrak{p}$. This gives the complete factorization of 2. □

Theorem 3.3 ([Llorente, Nart , [10], Part III). *The prime $p = 3$ decomposes in K as follows*

<table>
<tr><th>Case</th><th colspan="5">Sub case</th><th>Factors</th></tr>
<tr><td rowspan="2">$3 \mid a$, $3 \mid b$</td><td colspan="5">$1 = v_3(a) < v_3(b)$</td><td>$3 = \mathfrak{pq}^2$</td></tr>
<tr><td colspan="5">$1 \leq v_3(b) \leq v_3(a)$</td><td>$3 = \mathfrak{p}^3$</td></tr>
<tr><td rowspan="3">$3 \nmid a$</td><td colspan="5">$a \equiv -1\ (3)$</td><td>$3 = \mathfrak{pq}$</td></tr>
<tr><td rowspan="2">$a \equiv 1\ (3)$</td><td colspan="4">$3 \nmid b$</td><td>$3 = \mathfrak{p}$</td></tr>
<tr><td colspan="4">$3 \mid b$</td><td>$3 = \mathfrak{pqr}$</td></tr>
<tr><td rowspan="7">$3 \mid a$, $3 \nmid b$</td><td rowspan="2">$a \not\equiv 3\ (9)$</td><td colspan="4">$b^2 \not\equiv a + 1\ (9)$</td><td>$3 = \mathfrak{p}^3$</td></tr>
<tr><td colspan="4">$b^2 \equiv a + 1\ (9)$</td><td>$3 = \mathfrak{pq}^2$</td></tr>
<tr><td rowspan="5">$a \equiv 3\ (9)$</td><td rowspan="4">$b^2 \equiv a + 1\ (27)$</td><td colspan="3">s_3 is odd</td><td>$3 = \mathfrak{pq}^2$</td></tr>
<tr><td rowspan="3">s_3 even</td><td colspan="2">$\triangle_3 \equiv -1(3)$</td><td>$3 = \mathfrak{pq}$</td></tr>
<tr><td rowspan="2">$\triangle_3 \equiv 1(3)$</td><td>$s_3 = 6$</td><td>$3 = \mathfrak{p}$</td></tr>
<tr><td>$s_3 > 6$</td><td>$3 = \mathfrak{pqr}$</td></tr>
<tr><td colspan="4">$b^2 \not\equiv a + 1(27)$</td><td>$3 = \mathfrak{p}^3$</td></tr>
</table>

Table 3: Decomposition of Prime $p = 3$

Proof. Case (i) : $3 \mid a,\ 3 \mid b$.

The proof is similar to the case (i) of $p > 3$.

Case (ii) : $3 \nmid a$.

Observe that in this case $3 \nmid \triangle$. So 3 is an unramified prime. Hence the splitting of 3 in K depends on how $\overline{f}(x) = x^3 - \overline{a}x + \overline{b}$ factorizes in $\mathbb{Z}/3\mathbb{Z}$. Here $\overline{a} = \pm 1, \overline{b} = 0, \pm 1$. Note that when $a \equiv -1 \pmod 3$, then for any value of $\overline{b}$, $\overline{f}(x)$ has exactly one root. Hence in this case $f(x)$ is a product of a linear and a quadratic factor $\pmod 3$. Therefore $3 = \mathfrak{pq}$.

Now assume that $a \equiv 1 \pmod 3$. When $\overline{b} = 0$, that is when $3 \mid b$, then $\overline{f}(x) = x(x-1)(x+1)$, hence $3 = \mathfrak{pqr}$. On other hand, when 3 does not divide b, then $\overline{f}(x)$ has no root $\pmod 3$. Hence we deduce that $3 = \mathfrak{p}$.

Case (iii) : $3 \mid a,\ 3 \nmid b$.

In this case $\overline{f}(x) = x^3 \pm 1 = (x \pm 1)^3 \pmod 3$. We change the polynomial by substituting $\theta_1 = \theta + b$, to get a new polynomial $f_1(x) = x^3 - 3bx^2 +$

$(3b^2-a)x - b(b^2-a-1)$. As $3 \mid a$, therefore $a \equiv \pm 3 \pmod 9$. First we assume $a \not\equiv 3 \pmod 9$. Then $3b^2 \equiv 3 \not\equiv a \pmod 9$. Hence $9 \nmid (3b^2-a)$. Note that $3 \mid (b^2-a-1)$, so there are two cases $b^2 \not\equiv a+1 \pmod 9$ and $b^2 \equiv a+1 \pmod 9$ to consider. How 3 splits in K is determined by the Newton polygon of $f_1(x)$ with respect to 3. The coefficients of the Newton polygon of $f_1(x)$ with respect to 3 are $(0,1)$, $(1,1)$, $(2,1)$, $(3,0)$, when $b^2 \not\equiv a+1 \pmod 9$ and $(0,r)$, $r>1$, $(1,1)$, $(2,1)$, $(3,0)$ when $b^2 \equiv a+1 \pmod 9$. Therefore in the first case the Newton polygon consists of the line with slope $-1/3$. Hence $3 = \mathfrak{p}^3$. In the other case it consist of two lines with the slope of one line $-1/2$, so $3 = \mathfrak{p}\mathfrak{q}^2$.

Now assume that $a \equiv 3 \pmod 9$. Then $9 \mid (3b^2-a)$. Here again we consider two cases $b^2 \not\equiv a+1 \pmod{27}$ and $b^2 \equiv a+1 \pmod{27}$. When $b^2 \not\equiv a+1 \pmod{27}$, then the coefficients of the Newton polygon of $f_1(x)$ are $(0,r)$, $r=1$ or $2, (1,s), s \geq 2, (2,1)$ and $(3,0)$. The Newton polygon in this case consists of one line with the slope $-1/3$, hence $3 = \mathfrak{p}^3$.

When $b^2 \equiv a+1 \pmod{27}$, then the coefficients of x^2, x and the constant term of $f_1(x)$ are divisible by 3, 3^2 and 3^3 respectively. So we replace the polynomial by substituting $\theta_2 = \theta_1/3$ to obtain the new polynomial

$$f_2(x) = x^3 - bx^2 + \frac{3b^2-a}{9}x - \frac{b^3-ab-b}{27}.$$

As $3 \nmid b$, hence $f_2(x) \not\equiv (x+c)^3 \pmod 3$ for any $c \in \mathbb{Z}$. Thus $3 \neq \mathfrak{p}^3$. By Lemma 2.3, $3 = \mathfrak{p}\mathfrak{q}^2$ if $v_3(D) = 1$, that is if s_3 is odd.

Now assume s_3 is even. Here once again we invoke Lemma 2.7 in order to distinguish between the various cases regarding the decomposition of 3. First assume that $\triangle_3 \notin \mathbb{Q}_3^2$ (or $\left(\frac{\triangle_3}{3}\right) = -1$), which is equivalent to $\triangle_3 \equiv -1 \pmod 3$. Hence by Lemma 2.7 we have $3 \neq \mathfrak{p}$ and $3 \neq \mathfrak{p}\mathfrak{q}\mathfrak{r}$. Thus $3 = \mathfrak{p}\mathfrak{q}$ or $3 = \mathfrak{p}\mathfrak{q}^2$. But $3 = \mathfrak{p}\mathfrak{q}^2$, when s_3 is odd. Hence we left with the case $3 = \mathfrak{p}\mathfrak{q}$ and this happens when $\triangle_3 \equiv -1 \pmod 3$. Now assume $\triangle_3 \equiv 1 \pmod 3$. The remaining possibilities for the decomposition of 3 are $3 = \mathfrak{p}$ or $3 = \mathfrak{p}\mathfrak{q}\mathfrak{r}$ which depend on whether $f_2(x)$ is irreducible or splits into linear factors $\pmod 3$. Also note that by assumption it is not possible for $f_2(x)$ to have one linear and one quadratic factor. Denote by D_2 the discriminant of $f_2(x)$. Then $D_2 = \triangle/3^6$.

Now, $s_3 = 6 \Longleftrightarrow 3 \nmid D_2 \Longleftrightarrow f_2(x)$ is irreducible $\pmod 3 \Longleftrightarrow 3 = \mathfrak{p}$. The last case is possible if $s_3 > 6$. With this the determination of the decomposition of 3 in K is complete. □

4. Application

As an application, we will use the theorem of Llorente and Nart to determine the splitting of a prime in a cyclic cubic field and in a pure cubic field.

4.1. **Cyclic Cubic Field.** A number field with a cyclic Galois group is called a cyclic number field. Let $K = \mathbb{Q}(\theta)$ be a cubic field and let $f(x)$ be the minimal polynomial of θ. Then the Galois group of $f(x)$ is either $\mathbb{Z}/3\mathbb{Z}$ or S_3, depending on whether the discriminant $\triangle$ of $f(x)$ is a square or not and conversely (see [3], Theorem 2.1). Thus the cubic field K is cyclic if and only if the discriminant $\triangle$ of $f(x)$ is a square. We will see how and when a prime splits in such a cubic field. Throughout this subsection, $K = \mathbb{Q}(\theta)$ is a cyclic cubic field and $f(x)$ is the minimal polynomial of θ. We start with the following two remarks.

Remark 4.1. Since a Galois extension is a normal extension, the cyclic cubic field K is a normal extension. Therefore by Theorem 1.1, the value of r, the number of prime ideals in the decomposition of a prime p, is 1 or 3. When $r = 1$, $p = \mathfrak{p}$ with $f(\mathfrak{p}/p) = 3$ or $p = \mathfrak{p}^3$ with $f(\mathfrak{p}/p) = 1$. When $r = 3$, then $p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ with each $f(\mathfrak{p}_i/p) = 1$. These are the only three ways in which the prime p can split in K.

Remark 4.2. Since the discriminant $\triangle$ of $f(x)$ is a square, therefore for any prime p, the value of s_p is always even and $\triangle_p$ is a square. Therefore for any prime p, the quadratic residue symbol $\left(\dfrac{\triangle_p}{p}\right) = 1$.

The generating polynomial $f(x)$ of K comes in a particular form which can be seen from the following theorem.

Theorem 4.3 ([5], Lemma 6.4.5)**.** *For any cyclic cubic field K, there exists a unique pair of integers e and u such that e is a product of distinct primes congruent of 1 modulo 3 and $u \equiv 2 \pmod 3$, such that $K = \mathbb{Q}(\theta_1)$ where θ_1 is a root of the polynomial $g(x) = x^3 - (e/3)x - eu/27$. Equivalently $K = \mathbb{Q}(\theta)$, where θ is a root of $f(x) = 27g(x/3) = x^3 - 3ex - eu$.*

Remark 4.4. If $f(x) = x^3 - ax + b$ is a generating polynomial of the cyclic cubic field K, then from the above theorem, $a = 3e$ and $b = -eu$, where e and u are as in the above theorem. Therefore in the case of a cyclic cubic field, the only prime divisors of a are 3 and primes congruent to $1 \pmod 3$. Moreover because $e \equiv 1 \pmod 3$, we always have $a \equiv 3 \pmod 9$.

Remark 4.5. Let $p \neq 2$ be a prime such that $1 = v_p(a) < v_p(b)$. Let $a = p\alpha$, $p \nmid \alpha$. Let $v_p(b) = r > 1$. Then $b = p^r\beta$ such that $p \nmid \beta$. The discriminant of $f(x)$,

$$\triangle = 4a^3 - 27b^2 = p^3(4\alpha^3 - 27p^{2r-3}\beta^2).$$

As $\triangle$ is a square, therefore p must divide $4\alpha^3 - 27p^{2r-3}\beta^2$. Since $r > 1$, therefore $2r - 3$ is positive integer, hence p must divide $4\alpha^3$. This means $p \mid \alpha$, a contradiction.

Hence the case $1 = v_p(a) \leq v_p(b)$ does not arise for a cyclic cubic field.

Using these remarks and Theorem 4.3, we describe the decomposition of a prime p in a cyclic cubic field K in the following theorem.

Theorem 4.6. *The prime p in a cyclic cubic field decomposes as follows*

<table>
<tr><th>Prime</th><th colspan="3">Case</th><th>Factorization</th></tr>
<tr><td rowspan="6">$p > 3$</td><td colspan="3">$p \mid a, p \mid b$</td><td>$p = \mathfrak{p}^3$</td></tr>
<tr><td rowspan="2">$p \mid a$ and $p \nmid b$</td><td colspan="2">$\left(\frac{b}{p}\right)_3 = 1$</td><td>$p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td colspan="2">$\left(\frac{b}{p}\right)_3 \neq 1$</td><td>$p = \mathfrak{p}$</td></tr>
<tr><td colspan="3">$p \nmid a$ and $p \mid b$</td><td>$p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td rowspan="2">$p \nmid ab$</td><td colspan="2">$f(x)$ has some roots $\pmod p$</td><td>$p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td colspan="2">$f(x)$ has no root $\pmod p$</td><td>$p = \mathfrak{p}$</td></tr>
<tr><td rowspan="2">$p = 2$</td><td colspan="3">a odd, b even</td><td>$2 = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td colspan="3">a, b odd</td><td>$2 = \mathfrak{p}$</td></tr>
<tr><td rowspan="4">$p = 3$</td><td colspan="3">$3 \mid a, 3 \mid b$</td><td>$3 = \mathfrak{p}^3$</td></tr>
<tr><td rowspan="3">$3 \mid a, 3 \nmid b$</td><td colspan="2">$b^2 \not\equiv a + 1 \pmod{27}$</td><td>$3 = \mathfrak{p}^3$</td></tr>
<tr><td rowspan="2">$b^2 \equiv a + 1 \pmod{27}$</td><td>$s_3 = 6$</td><td>$3 = \mathfrak{p}$</td></tr>
<tr><td>$s_3 > 6$</td><td>$3 = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
</table>

Table 5: Decomposition of a prime in cyclic cubic field

Remark 4.7. The generating polynomial of a cyclic cubic field is called a cyclic cubic polynomial. Coefficients of such polynomials can be represented by a family involving two parameters, say α and β.

In fact, for any rational values of α and β, the polynomial $x^3 - 3(\alpha^2 + \alpha\beta + \beta^2)x - (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$ has a discriminant which is a perfect square of rational numbers. Hence if this polynomial is irreducible, then it will generate a cyclic cubic field.

Conversely, if $x^3 - ax + b$ is an irreducible polynomial with square discriminant, then the values of the parameter α and β can be found by solving the equations $\alpha - \beta = 3a/b$ and $\alpha + \beta = -\sqrt{\triangle}/3a$ (refer [7], Chapter 2, Section 24). Hence one can compute the values of e and u used in Theorem 4.3.

4.2. **Pure Cubic Field.** Another simple class of cubic fields are those of the form $K = \mathbb{Q}(\sqrt[3]{b})$, fields generated by a cubic polynomial $x^3 - b$, where b is a cubefree integer. The discriminant $\triangle$ of such a polynomial is $-27b^2$. The splitting of any prime p in K is determined by only two conditions, $p \mid b$ and $p \nmid b$. The following theorem describes the decomposition of any prime p.

Theorem 4.8. *The prime p decomposes in a pure cubic field as follows;*

<table>
<tr><th>Prime</th><th colspan="3">Case</th><th>Factorization</th></tr>
<tr><td rowspan="4">$p > 3$</td><td colspan="3">$p \mid b$</td><td>$p = \mathfrak{p}^3$</td></tr>
<tr><td rowspan="3">$p \nmid b$</td><td colspan="2">$p \equiv -1 \pmod 3$</td><td>$p = \mathfrak{p}\mathfrak{q}$</td></tr>
<tr><td rowspan="2">$p \equiv 1 \pmod 3$</td><td>$\left(\frac{b}{p}\right)_3 = 1$</td><td>$p = \mathfrak{p}\mathfrak{q}\mathfrak{r}$</td></tr>
<tr><td>$\left(\frac{b}{p}\right)_3 \neq 1$</td><td>$p = \mathfrak{p}$</td></tr>
<tr><td rowspan="2">$p = 2$</td><td colspan="3">$2 \mid b$</td><td>$2 = \mathfrak{p}^3$</td></tr>
<tr><td colspan="3">$2 \nmid b$</td><td>$2 = \mathfrak{p}\mathfrak{q}$</td></tr>
<tr><td rowspan="3">$p = 3$</td><td colspan="3">$3 \mid b$</td><td>$3 = \mathfrak{p}^3$</td></tr>
<tr><td rowspan="2">$3 \nmid b$</td><td colspan="2">$b \equiv \pm 1 \pmod 9$</td><td>$3 = \mathfrak{p}\mathfrak{q}^2$</td></tr>
<tr><td colspan="2">$b \not\equiv \pm 1 \pmod 9$</td><td>$3 = \mathfrak{p}^3$</td></tr>
</table>

Table 6: Decomposition of a prime in a pure cubic field

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HYPERBOLIC K-FIBONACCI AND K-LUCAS QUATERNIONS

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Abstract. The hyperbolic quaternions form a 4-dimensional non-associative and non-commutative algebra over the set of real numbers. In this paper, we introduce the hyperbolic k-Fibonacci and k-Lucas quaternions. We present generating functions and Binet formula for the k-Fibonacci and k-Lucas hyperbolic quaternions, and establish binomial and congruence sums of hyperbolic k-Fibonacci and k-Lucas quaternions.

1. Introduction

The well-known integer sequence, Fibonacci sequence is defined by the numbers which satisfy the second-order recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers have many interesting properties and applications in various research areas such as Architecture, Engineering, Nature and Art. The Lucas sequence is a companion sequence of Fibonacci sequence defined by the Lucas numbers which are defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. Binet's formula for the Fibonacci and Lucas numbers are

$$F_n = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$L_n = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2-x-1 = 0$. The positive root r_1 is known as the golden ratio. The Fibonacci and Lucas sequences are generalized by changing the initial conditions or changing the recurrence relation. One of the generalizations of the Fibonacci sequence is k-Fibonacci sequence first introduced by Falcon and Plaza [5]. The k-Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$

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with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. Falcon [6] defined the k-Lucas sequence that is companion sequence of k- Fibonacci sequence defined with the k- Lucas numbers which are defined with the recurrence relation $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$ with the initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$. Binet's formulas for the k- Fibonacci and k-Lucas numbers are

$$F_{k,n} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$L_{k,n} = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{k + \sqrt{k^2 + 4}}{2}$ and $r_2 = \frac{k - \sqrt{k^2 + 4}}{2}$ are the roots of the characteristic equation $x^2 - kx - 1 = 0$. The characteristic roots r_1 and r_2 satisfy the properties

$$r_1 - r_2 = \sqrt{k^2 + 4} = \sqrt{\delta}, \quad r_1 + r_2 = k, \quad r_1 r_2 = -1.$$

The reader can refer to [4, 7, 8, 9, 10, 11, 12] for properties and applications of k-Fibonacci and k-Lucas numbers.

The quaternions are generalized numbers. The quaternions first introduced by Irish mathematician William Rowan Hamilton in 1843. Hamilton [27] introduced the set of quaternions form a 4-dimensional real vector space with a multiplicative operation. The quaternions are used in applied sciences such as physics, computer science and Clifford algebras in mathematics. In particular, they are important in mechanics [14], chemistry [15], kinematics [16], quantum mechanics [17], differential geometry, pure algebra. A quaternion a, with real components a_0, a_1, a_2, a_3 and basis $1, i, j, k$, is an element of the form

$$a = a_0 + a_1 i + a_2 j + a_3 k = \big(a_0, a_1, a_2, a_3\big),$$

where

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = k = -ji, jk = i = -kj, ki = j == ik.$$

Horadam[18] defined n^{th} Fibonacci and n^{th} Lucas quaternions as

$$\bar{F_n} = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k = \big(F_n, F_{n+1}, F_{n+2}, F_{n+3}\big)$$

and

$$\bar{L_n} = L_n + L_{n+1} i + L_{n+2} j + L_{n+3} k = \big(L_n, L_{n+1}, L_{n+2}, L_{n+3}\big)$$

respectively.

Ramirez [19] has defined and studied the k-Fibonacci and k-Lucas quaternions as

$$\bar{F_{k,n}} = F_{k,n} + F_{k,n+1} i + F_{k,n+2} j + F_{k,n+3} k = \big(F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}\big)$$

and

$$\bar{L_{k,n}} = L_{k,n} + L_{k,n+1} i + L_{k,n+2} j + L_{k,n+3} k = \big(L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}\big)$$

respectively. Where $F_{k,n}$ is the n^{th} k-Fibonacci sequence and $L_{k,n}$ is the n^{th} k-Lucas sequence.

Different quaternions of sequences have been studied by different researchers. For example, Iyer [20, 21] obtained various relations containing the Fibonacci and Lucas quaternions. Halici [22] studied some combinatorial properties of Fibonacci quaternions. Akyigit et al. [23, 24] established and investigated the Fibonacci generalized quaternions and split Fibonacci quaternions. Catarino [25] obtained different properties of the $h(x)$-Fibonacci quaternion polynomials. Polatli and Kesim [26] have introduced quaternions with generalized Fibonacci and Lucas number components.
A hyperbolic quaternion h is an expression of the form

$$h = h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 i_4 = \big(h_1, h_2, h_3, h_4\big),$$

with real components h_1, h_2, h_3, h_4 and i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the non-commutative multiplication rules

$$i_2{}^2 = i_3{}^2 = i_4{}^2 = i_2 i_3 i_4 = +1, i_1 = 1$$

$$i_2 i_3 = i_4 = -i_3 i_2, i_3 i_4 = i_2 = -i_4 i_3, i_4 i_2 = i_3 = -i_2 i_4. \tag{1.1}$$

The scalar and the vector part of a hyperbolic quaternion h are denoted by $S_h = h_1$ and $\overrightarrow{V}_h = h_2 i_2 + h_3 i_3 + h_4 i_4$, respectively. Thus, a hyperbolic quaternion h is given by $h = S_h + \overrightarrow{V}_h$. For any two hyperbolic quaternion $h^{(1)} = h_1^{(1)} i_1 + h_2^{(1)} i_2 + h_3^{(1)} i_3 + h_4^{(1)} i_4$ and $h^{(2)} = h_1^{(2)} i_1 + h_2^{(2)} i_2 + h_3^{(2)} i_3 + h_4^{(2)} i_4$.

A. Cariow and G. Cariow [28] state low multiplicative complexity algorithm for multiplying two hyperbolic octonions. The conjugate of hyperbolic quaternion h is denoted by $\widehat{h}$ and it is

$$\widehat{h} = h_1 i_1 - h_2 i_2 - h_3 i_3 - h_4 i_4 = \big(h_1, -h_2, -h_3, -h_4\big).$$

The norm of h is defined as

$$N_h = h \cdot \widehat{h} = h_1{}^2 - h_2{}^2 - h_3{}^2 - h_4{}^2.$$

In the present paper, our main aim is to define hyperbolic k-Fibonacci quaternion $\bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n}$ and hyperbolic k-Lucas quaternion $\bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n}$ and derive the relations connecting the hyperbolic k-Fibonacci and k-Lucas quaternions. We have adapted the methods of Carlitz [2] and Zhizheng Zhang [3] to the hyperbolic k-Fibonacci and k-Lucas quaternions and derived some fundamental and congruence identities for these quaternions.

2. Some Fundamental Properties of Hyperbolic k-Fibonacci and k-Lucas Quaternions

In this section, we establish certain elementary properties of the hyperbolic k-Fibonacci and k-Lucas quaternions.

Definition 2.1. For $n \geq 0$, the hyperbolic k-Fibonacci and k-Lucas quaternions $\bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ and $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$ are defined by

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} = F_{k,n}i_1 + F_{k,n+1}i_2 + F_{k,n+2}i_3 + F_{k,n+3}i_4 \tag{2.1}$$
$$= \big(F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}\big)$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = L_{k,n}i_1 + L_{k,n+1}i_2 + L_{k,n+2}i_3 + L_{k,n+3}i_4 \tag{2.2}$$
$$= \big(L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}\big)$$

respectively, where $F_{k,n}$ is n^{th} k-Fibonacci sequence and $L_{k,n}$ is n^{th} k-Lucas sequence. Here i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the multiplication rule as in table (1.1).

Theorem 2.2. *For all* $n \geq 0$,

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+2} = k\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}, \tag{2.3}$$

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+2} = k\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{L}}}_{k,n}, \tag{2.4}$$

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{L}}}_{k,n-1}. \tag{2.5}$$

Proof. i. From equations (2.1) and (2.2),

$$\begin{aligned}
k\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} + \bar{\mathcal{H}^{\mathcal{F}}}_{k,n} &= k\big[F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}i_4\big] \\
&\quad + \big[F_{k,n}i_1 + F_{k,n+1}i_2 + F_{k,n+2}i_3 + F_{k,n+3}i_4\big] \\
&= \big[kF_{k,n+1} + F_{k,n}\big]i_1 + \big[kF_{k,n+2} + F_{k,n+1}\big]i_2 \\
&\quad + \big[kF_{k,n+3} + F_{k,n+2}\big]i_3 + \big[kF_{k,n+4} + F_{k,n+3}\big]i_4 \\
&= F_{k,n+2}i_1 + F_{k,n+3}i_2 + F_{k,n+4}i_3 + F_{k,n+5}i_4 \\
&= \bar{\mathcal{H}^{\mathcal{F}}}_{k,n+2}.
\end{aligned}$$

The proofs of (ii) and (iii) are similar to (i), using equations (2.1) and (2.2). □

Theorem 2.3. ***(Binet Formulas)***. *For all* $n \geq 0$,

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} = \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2} \tag{2.6}$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n \tag{2.7}$$

where, $\bar{r_1} = i_1 + r_1i_2 + {r_1}^2i_3 + {r_1}^3i_4 = \big(1, r_1, {r_1}^2, {r_1}^3\big)$, $\bar{r_2} = i_1 + r_2i_2 + {r_2}^2i_3 + {r_2}^3i_4 = \big(1, r_2, {r_2}^2, {r_2}^3\big)$ and i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the multiplication rule as in table (1.1).

Proof. Using the definition of $\bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ and the Binet formulas of k-Fibonacci and k-Lucas sequences, we have

$$\begin{aligned}
\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} &= F_{k,n}i_1 + F_{k,n+1}i_2 + F_{k,n+2}i_3 + F_{k,n+3}i_4 \\
&= \Big[\frac{{r_1}^n - {r_2}^n}{r_1 - r_2}\Big]i_1 + \Big[\frac{{r_1}^{n+1} - {r_2}^{n+1}}{r_1 - r_2}\Big]i_2 + \Big[\frac{{r_1}^{n+2} - {r_2}^{n+2}}{r_1 - r_2}\Big]i_3 \\
&+ \Big[\frac{{r_1}^{n+3} - {r_2}^{n+3}}{r_1 - r_2}\Big]i_4 \\
&= \frac{{r_1}^n}{r_1 - r_2}\big(i_1 + r_1 i_2 + {r_1}^2 i_3 + {r_1}^3 i_4\big) \\
&- \frac{{r_2}^n}{r_1 - r_2}\big(i_1 + r_2 i_2 + {r_2}^2 i_3 + {r_2}^3 i_4\big) \\
&= \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} &= L_{k,n}i_1 + L_{k,n+1}i_2 + L_{k,n+2}i_3 + L_{k,n+3}i_4 \\
&= \big[{r_1}^n + {r_2}^n\big]i_1 + \big[{r_1}^{n+1} + {r_2}^{n+1}\big]i_2 + \big[{r_1}^{n+2} + {r_2}^{n+2}\big]i_3 + \\
&\big[{r_1}^{n+3}{r_2}^{n+3}\big]i_4 \\
&= {r_1}^n\big(i_1 + r_1 i_2 + {r_1}^2 i_3 + {r_1}^3 i_4\big) + {r_2}^n\big(i_1 + r_2 i_2 + {r_2}^2 i_3 + {r_2}^3 i_4\big) \\
&= \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n.
\end{aligned}$$

□

Lemma 2.4. *For $\bar{r_1}$ and $\bar{r_2}$,*

(i) $\bar{r_1} - \bar{r_2} = \sqrt{\delta}\bar{\mathcal{H}^{\mathcal{F}}}_{k,0}$,

(ii) $\bar{r_1} + \bar{r_2} = \bar{\mathcal{H}^{\mathcal{L}}}_{k,0}$,

(iii) $\bar{r_1}\bar{r_2} = \big(0, 2r_2, 2{r_2}^2, {r_1}^3 + {r_2}^3 + r_1 - r_2\big)$,

(iv) $\bar{r_2}\bar{r_1} = \big(0, 2r_1, 2{r_1}^2, {r_1}^3 + {r_2}^3 - r_1 + r_2\big)$,

(v) $\bar{r_1}^2 = \big(-1 + {r_1}^2 + {r_1}^4 + {r_1}^6\big) + 2\bar{r_1}$,

(vi) $\bar{r_2}^2 = \big(-1 + {r_2}^2 + {r_2}^4 + {r_2}^6\big) + 2\bar{r_2}$,

(vii) $\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1} = 2\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - 2\big)$,

(viiii) $\bar{r_1}\bar{r_2} - \bar{r_2}\bar{r_1} = 2\sqrt{\delta}\big(0, -1, -k, 1\big)$,

(ix) $\bar{r_1}^2 - \bar{r_2}^2 = \sqrt{\delta}\big(F_{k,2} + F_{k,4} + F_{k,6} + 2\bar{\mathcal{H}^{\mathcal{F}}}_{k,0}\big)$,

(x) $\bar{r_1}^2 + \bar{r_2}^2 = \big(-L_{k,0} + L_{k,2} + L_{k,4} + L_{k,6}2\bar{\mathcal{H}^{\mathcal{L}}}_{k,0}\big)$.

Theorem 2.5. *For all $s, t \in Z^+$, $s \geq t$ and $n \in N$, the generating functions for the hyperbolic k-Fibonacci and k-Lucas quaternions $\bar{\mathcal{H}^{\mathcal{F}}}_{k,tn}$ and $\bar{\mathcal{H}^{\mathcal{L}}}_{k,tn}$*

are

$$(i) \quad \sum_{n=0}^{\infty} \bar{\mathcal{H}^{\mathcal{F}}}_{k,tn} x^n = \frac{\bar{\mathcal{H}^{\mathcal{F}}}_{k,0} + \big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} F_{k,t} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\big) x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(ii) \quad \sum_{n=0}^{\infty} \bar{\mathcal{H}^{\mathcal{L}}}_{k,tn} x^n = \frac{\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - \big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} L_{k,t} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\big) x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(iii) \quad \sum_{n=0}^{\infty} \bar{\mathcal{H}^{\mathcal{F}}}_{k,tn+s} x^n = \frac{\bar{\mathcal{H}^{\mathcal{F}}}_{k,s} + (-1)^t x \bar{\mathcal{H}^{\mathcal{F}}}_{s,s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(iv) \quad \sum_{n=0}^{\infty} \bar{\mathcal{H}^{\mathcal{L}}}_{k,tn+s} x^n = \frac{\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} + (-1)^t x \bar{\mathcal{H}^{\mathcal{L}}}_{s-t}}{1 - xL_{k,t} + x^2(-1)^t}.$$

Proof. (1). Using theorem (2.3), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \bar{\mathcal{H}^{\mathcal{F}}}_{k,tn} x^n &= \sum_{n=0}^{\infty} \frac{\bar{r_1} {r_1}^{tn} - \bar{r_2} {r_2}^{tn}}{r_1 - r_2} x^n \\
&= \frac{\bar{r_1}}{r_1 - r_2} \sum_{n=0}^{\infty} \big({r_1}^t\big)^n x^n - \frac{\bar{r_2}}{r_1 - r_2} \sum_{n=0}^{\infty} \big({r_2}^t\big)^n x^n \\
&= \frac{\big(\frac{\bar{r_1} - \bar{r_2}}{r_1 - r_2}\big) + \big[\big(\bar{r_1} + \bar{r_2}\big)\big(\frac{{r_1}^t - {r_2}^t}{r_1 - r_2}\big) - \big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\big)\big] x}{1 - \big({r_1}^t + {r_2}^t\big) x + x^2 (r_1 r_2)^t} \\
&= \frac{\bar{\mathcal{H}^{\mathcal{F}}}_{k,0} + \big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} F_{k,t} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\big) x}{1 - xL_{k,t} + x^2(-1)^t}
\end{aligned}$$

The proofs of (ii), (iii) and (iv) are similar to (i), using theorem (2.3). □

Theorem 2.6. *For all $t \in Z^+$ and $n \in N$, the exponential generating functions for the hyperbolic k-Fibonacci and k-Lucas quaternions $\bar{\mathcal{H}^{\mathcal{F}}}_{k,tn}$ and $\bar{\mathcal{H}^{\mathcal{L}}}_{k,tn}$ are*

$$\sum_{n=0}^{\infty} \frac{\bar{\mathcal{H}^{\mathcal{F}}}_{k,tn}}{n!} x^n = \frac{\bar{r_1} e^{{r_1}^t x} - \bar{r_2} e^{{r_2}^t x}}{r_1 - r_2} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty} \frac{\bar{\mathcal{H}^{\mathcal{L}}}_{k,tn}}{n!} x^n = \bar{r_1} e^{{r_1}^t x} + \bar{r_2} e^{{r_2}^t x}. \tag{2.9}$$

Theorem 2.7. *For all $n \in N$,*

$$(i) \qquad \sum_{i=0}^{n} \binom{n}{i} k^i \bar{\mathcal{H}^{\mathcal{F}}}_{k,i} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,2n},$$

$$(ii) \qquad \sum_{i=0}^{n} \binom{n}{i} k^i \bar{\mathcal{H}^{\mathcal{L}}}_{k,i} = \bar{\mathcal{H}^{\mathcal{L}}}_{k,2n}.$$

Lemma 2.8. *For all $t \geqslant 0$ and $m \geqslant n$, we have*

$$(i)\quad \frac{\bar{r_1}\bar{r_2}{r_2}^t - \bar{r_2}\bar{r_1}{r_1}^t}{r_1 - r_2} = \big(0, -2F_{k,t+1}, -2F_{k,t+2}, - F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big),$$

$$(ii)\quad \frac{{r_1}^{m-n}\bar{r_1}\bar{r_2} - {r_2}^{m-n}\bar{r_2}\bar{r_1}}{r_1 - r_2} = \big(0, -2F_{k,m-n-1}, 2F_{k,m-n-2}, F_{k,m-n+3} - F_{k,m-n-3} + F_{k,m-n+1} + F_{k,m-n-1}\big).$$

Theorem 2.9. ***(Catalan's Identity)****. For any integer t and s,*

$$(i)\quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,n-t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} - \bar{\mathcal{H}^{\mathcal{F}}}^2_{k,n} = (-1)^{n-t}F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, - 2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big),$$

$$(ii)\quad \bar{\mathcal{H}^{\mathcal{L}}}_{k,n-t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+t} - \bar{\mathcal{H}^{\mathcal{L}}}^2_{k,n} = \delta(-1)^{n-t+1}F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, - 2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big).$$

Theorem 2.10. ***(Cassini's Identity)****. For all $n \geq 1$,*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n-1}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{F}}}^2_{k,n} = (-1)^n\big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big) \tag{2.10}$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,n-1}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{L}}}^2_{k,n} = \delta(-1)^{n-1}\big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big). \tag{2.11}$$

Theorem 2.11. ***(Vajda's Identity)****. For any two natural numbers i, j, we have*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+i}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+j} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+i+j} = (-1)^{n+1}F_{k,i} \tag{2.12}$$

$$\big(0, 2F_{k,j+1}, -2F_{k,j+2}, -F_{k,j+3} + F_{k,j-3} + F_{k,j+1} + F_{k,j-1}\big). \tag{2.13}$$

Theorem 2.12. ***(d'Ocagne's Identity)****. Let n be any non-negative integer and t a natural number. If $t \geq n+1$, then we have*

$$(i)\quad \bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,t+1}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n} = (-1)^n\big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2}, F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big),$$

$$(ii)\quad \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n+1} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t+1}\bar{\mathcal{H}^{\mathcal{L}}}_{k,n} = (-1)^{n+1}\delta\big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2}, F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big).$$

Theorem 2.13. *For any integer t,*

$$(i)\quad \bar{\mathcal{H}^{\mathcal{F}}}^2_{k,t} + \bar{\mathcal{H}^{\mathcal{L}}}^2_{k,t} = \frac{2(k^2+5)}{k}\bar{\mathcal{H}^{\mathcal{L}}}_{k,2t} + \delta(k^2+5)L_{k,2t+3} + 2(-1)^t\frac{(k^2+3)}{\delta}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - 2\big),$$

$$(ii)\quad \bar{\mathcal{H}^{\mathcal{F}}}^2_{k,t} - \bar{\mathcal{H}^{\mathcal{L}}}^2_{k,t} = \frac{2(k^2+3)}{\delta}\bar{\mathcal{H}^{\mathcal{L}}}_{k,2t} + (k^2+3)(k^2+2)L_{k,2t+3} + 2(-1)^{t+1}\frac{(k^2+5)}{\delta}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - 2\big).$$

Theorem 2.14. *For any integer r, s ≥ t,*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,r+s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,r+t} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,r+t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,r+s} = 2(-1)^{r+t}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - 2\big)F_{k,s-t}.$$

Theorem 2.15. *For any integer s, and t,*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,s+t} + (-1)^t\bar{\mathcal{H}^{\mathcal{F}}}_{k,s-t} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,s}L_{k,t} \tag{2.14}$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,s+t} + (-1)^t\bar{\mathcal{H}^{\mathcal{L}}}_{k,s-t} = \bar{\mathcal{H}^{\mathcal{L}}}_{k,s}L_{k,t}. \tag{2.15}$$

Theorem 2.16. *For any integer s ≤ t,*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,s}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s} = 2(-1)^s F_{k,t-s}\big(0,-1,-k,1\big) \tag{2.16}$$

and

$$\bar{\mathcal{H}^{\mathcal{L}}}_{k,s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,t} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} = 2(-1)^{s+1} F_{k,t-s}\delta\big(0,-1,-k,1\big). \tag{2.17}$$

Theorem 2.17. *For any integer s ≤ t,*

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,s}\bar{\mathcal{H}^{\mathcal{L}}}_{k,t} = 2(-1)^s F_{k,t-s}\big(\bar{\mathcal{H}^{\mathcal{L}}}_{k,0} - 2\big) \tag{2.18}$$

and

$$\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}\bar{\mathcal{H}^{\mathcal{L}}}_{k,s} - \bar{\mathcal{H}^{\mathcal{L}}}_{k,t}\bar{\mathcal{H}^{\mathcal{F}}}_{k,s} = 2(-1)^s \bar{r_2}\big[\bar{\mathcal{H}^{\mathcal{F}}}_{k,t-s} - {r_2}^{t-s}\big(0,1,k,k^2+1\big)\big]. \tag{2.19}$$

The proofs of theorem 2.6–2.17 are similar to theorem 2.5, using theorem 2.3 and lemma 2.4.

3. Some Binomial and Congruence Properties of Hyperbolic k-Fibonacci and k- Lucas Quaternions

In this section, we explore some binomial and congruence properties of the hyperbolic k-Fibonacci and k-Lucas quaternions.

Lemma 3.1. *Let* $u = r_1$ *or* r_2. *Then*

$$(a) \quad u^n = uF_{k,n} + F_{k,n-1},$$
$$(b) \quad u^{2n} = u^n L_{k,n} - (-1)^n,$$
$$(c) \quad u^{tn} = u^n \frac{F_{k,tn}}{F_{k,n}} - (-1)^n - \frac{F_{k,(t-1)n}}{F_{k,n}},$$
$$(d) \quad u^{sn}F_{k,rn} - u^{rn}F_{k,sn} = (-1)^{sn}F_{k,(r-s)n}.$$

Proof. We prove only (a) and (c) since the proofs of (b) and (d) are similar.
(a) Since r_1 and r_2 are roots of $r^2 - kr - 1 = 0$, then we have $r_1^2 = kr_1 + 1$ and $r_2^2 = kr_2 + 1$. Therefore, we have

$$\begin{aligned} u^{2n} &= F_{k,n}u^{n+1} + u^n F_{k,n-1} \\ &= F_{k,n}(uF_{k,n+1} + F_{k,n}) + u^n F_{k,n-1} \\ &= uF_{k,n}F_{k,n+1} + F_{k,n-1}u^n + F_{k,n}^2 \end{aligned}$$

$$= (u^n - F_{k,n-1})F_{k,n+1} + F_{k,n-1}u^n + F^2_{k,n}$$
$$= u^n(F_{k,n+1} + F_{k,n-1}) + F^2_{k,n} - F_{k,n}F_{k,n-1}.$$

Using $F_{k,n-1}F_{k,n+1} - F^2_{k,n} = (-1)^n$ and $F_{k,n+1} + F_{k,n-1} = L_{k,n}$, we obtain $u^{2n} = L_{k,n}u^n - (-1)^n$.

This completes the proof of (a).

(c) If $u = r_1$, then

$$\begin{aligned} F_{k,tn}r_1^n - (-1)^n F_{k,(t-1)n} &= (\frac{r_1^{tn} - r_2^{tn}}{r_1 - r_2})r_1^n - (r_1r_2)^n(\frac{r_1^{(t-1)n} - r_2^{(t-1)n}}{r_1 - r_2}) \\ &= (\frac{r_1^n - r_2^n}{r_1 - r_2})r_1^{tn} \\ &= F_{k,n}r_1^{tn}. \end{aligned}$$

This completes the proof of (c). □

Theorem 3.2. *For all $n, r, s, t \geq 1$, we have*

(i) $\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} = F_{k,n}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t+1} + F_{k,n-1}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}$,

(ii) $\bar{\mathcal{H}^{\mathcal{F}}}_{k,2n+t} = L_{k,n}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} - (-1)^n\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}$,

(iii) $\bar{\mathcal{H}^{\mathcal{F}}}_{k,sn+t} = \frac{F_{k,sn}}{F_{k,n}}\bar{\mathcal{H}^{\mathcal{F}}}_{k,n+t} - (-1)^n\frac{F_{k,(s-1)n}}{F_{k,n}}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}$,

(iv) $\bar{\mathcal{H}^{\mathcal{F}}}_{k,sn+t}F_{k,rn} - \bar{\mathcal{H}^{\mathcal{F}}}_{k,rn+t}F_{k,sn} = (-1)^{sn}\bar{\mathcal{H}^{\mathcal{F}}}_{k,t}F_{k,(r-s)n}$.

Theorem 3.3. *For all $n, r, s, t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$,*

(i) $\mathcal{G}_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}F^i_{k,r}F^{n-i}_{k,r-1}\mathcal{G}_{k,i+t}$,

(ii) $\mathcal{G}_{k,2rn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}L^i_{k,r}\mathcal{G}_{k,ri+t}$,

(iii) $\mathcal{G}_{k,trn+l} = \frac{1}{F^n_{k,r}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}F^{n-i}_{k,(t-1)r}F^i_{k,tr}\mathcal{G}_{k,ri+l}$,

(iv) $\sum_{i=0}^{n}\binom{n}{i}(-1)^i\mathcal{G}_{k,r(n-i)+i+t}F^i_{k,r} = \mathcal{G}_{k,t}F^n_{k,r-1}$,

(v) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\mathcal{G}_{k,ri+t}F^{(n-i)}_{k,r-1} = \mathcal{G}_{k,n+t}F^n_{k,r}$,

(vi) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}F^{(n-i)}_{k,sm}F^{(i)}_{k,rm}\mathcal{G}_{k,m[rn+i(s-r)]+t} = (-1)^{smn}\mathcal{G}_{k,t}F^n_{k,(r-s)m}$.

Lemma 3.4. *If $L_{k,n}$ is n^{th} k-Lucas sequence and $u = r_1$ or r_2, then*

$$1 + ku + u^{2(2^{n+1}+1)} = L_{k,2^{n+1}} u^{2(2^n+1)}.$$

Theorem 3.5. *For all $t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$,*

$$(i) \quad \mathcal{G}_{k,t+2^{n+1}+2} = \frac{\mathcal{G}_{k,t} + k\mathcal{G}_{k,t+1} + \mathcal{G}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}},$$

$$(ii) \quad \mathcal{G}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n} (-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$(iii) \quad \mathcal{G}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j} (-1)^{j+s} L_{k,2^{r+1}}{}^{i} \mathcal{G}_{k,(2^{r+1}+2)i+j+t},$$

$$(iv) \quad \mathcal{G}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j} L_{k,2^{r+1}}{}^{-n} \mathcal{G}_{k,(2^{r+1}+2)i+j+t}.$$

Lemma 3.6. *If $u = r_1$ or r_2, then for $l_n = \sum_{i=1}^{n} L_{k,2^i}$ and for every $n, t \geq 1$,*

$$1 + u^{2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}} u^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}} u^{2^{n-t}} - l_{n-1} \sum_{i=2}^{t} \dfrac{1}{l_{n-i}}, \quad \textit{If } t = 2, 3, 4, \ldots, n-2 \ ; \\ l_{n-1} u^2 - l_{n-1} \sum_{i=2}^{n-1} \dfrac{1}{l_{n-i}}. \end{cases}$$

Theorem 3.7. *For $l_n = \sum_{i=1}^{n} \mathcal{L}_{k,2^i}$, for every $n, t \geq 1$ and $\mathcal{G}_{k,n} = \mathcal{F}_{k,n}$ or $\mathcal{L}_{k,n}$, we have*

$$(i) \quad \mathcal{G}_{k,t+2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}} \mathcal{G}_{k,t+2^{n-1}} - \mathcal{G}_{k,t}; \\ \dfrac{l_{n-1}}{l_{n-t-1}} \mathcal{G}_{k,t+2^{n-s}} - l_{n-1} \sum_{i=2}^{s} (1 + \dfrac{1}{l_{n-i}}) \mathcal{G}_{k,t}, \\ \qquad \textit{If } s = 2, 3, 4, \ldots, n-2 \ ; \\ l_{n-1} \mathcal{G}_{k,t+2} - l_{n-1} \sum_{i=2}^{n-1} (\dfrac{1}{l_{n-i}} + 1) \mathcal{G}_{k,t}. \end{cases}$$

$$(ii) \quad \mathcal{G}_{k,2^r n+t} = \begin{cases} \sum_{i+j=n} \binom{n}{i} (\dfrac{l_{r-1}}{l_{r-2}})^i (-1)^j \mathcal{G}_{k,2^{r-1}i+t}; \\ \sum_{i+j=n} \binom{n}{i} (\dfrac{l_{r-1}}{l_{r-s-1}})^i (-1)^j (\sum_{h=2}^{s} (1 + \dfrac{l_{r-1}}{l_{r-h}})^j \mathcal{G}_{k,2^{n-s}i+t}, \\ \qquad \textit{If } s = 2, 3, 4, \ldots, n-2 \ ; \\ \sum_{i+j=n} \binom{n}{i} (l_{r-1})^i (-1)^j (\sum_{h=2}^{s} (1 + \dfrac{l_{r-1}}{l_{r-h}})^j \mathcal{G}_{k,2i+t}. \end{cases}$$

Lemma 3.8. *For all* $t \geq 1$,

$$(i) \quad r_1^{2t} = \frac{F_{k,2t}}{k} r_1 \sqrt{\delta} - \frac{L_{k,2t-1}}{k}, r_2^{2t} = -\frac{F_{k,2t}}{k} r_2 \sqrt{\delta} - \frac{L_{k,2t-1}}{k}.$$

$$(ii) \quad r_1^{2t+1} = \frac{L_{k,2t+1}}{k} r_1 - \frac{F_{k,2t}}{k} \sqrt{\delta}, r_2^{2t+1} = \frac{L_{k,2t+1}}{k} r_2 + \frac{F_{k,2t}}{k} \sqrt{\delta}.$$

Theorem 3.9. *For* $s, t \geq 1$,

$$(i) \quad \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+2t} = \frac{F_{k,2t}}{k} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+1} - \frac{L_{k,2t-1}}{k} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s},$$

$$(ii) \quad \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+2t} = \frac{F_{k,2t}}{k} \delta \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+1} - \frac{L_{k,2t-1}}{k} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s},$$

$$(iii) \quad \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+2t} - \frac{F_{k,2t}}{k} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+2} + \frac{F_{k,2t-2}}{k} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s} = 0,$$

$$(iv) \quad \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+2t} - \frac{F_{k,2t}}{k} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+2} + \frac{F_{k,2t-2}}{k} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s} = 0.$$

Theorem 3.10. *For all* $s, t \geq 1$,

$$(i) \quad \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+1} - \frac{F_{k,2t}}{k} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s},$$

$$(ii) \quad \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+1} - \delta \frac{F_{k,2t}}{k} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s},$$

$$(iii) \quad \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s} = 0,$$

$$(iv) \quad \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)} \delta \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,s} = 0.$$

Theorem 3.11. *For* $n, s, t \geq 1$,

$$(i) \quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} (L_{k,2t-1})^{(n-i)} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,2ti+s}$$
$$= \begin{cases} k^{-n} (\mathcal{F}_{k,2t})^n \delta^{\frac{n}{2}} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n+s}, & \textit{if } n \textit{ is even;} \\ k^{-n} (\mathcal{F}_{k,2t})^n \delta^{\frac{n-1}{2}} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n+s}, & \textit{if } n \textit{ is odd,} \end{cases}$$

$$(ii) \quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} (L_{k,2t-1})^{(n-i)} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,2ti+s}$$
$$= \begin{cases} k^{-n} (F_{k,2t})^n \delta^{\frac{n}{2}} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,n+s}, & \textit{if } n \textit{ is even;} \\ k^{-n} (F_{k,2t})^n \delta^{\frac{n+1}{2}} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,n+s}, & \textit{if } n \textit{ is odd.} \end{cases}$$

$$(iii) \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} k^{-i} (L_{k,2t+1})^i \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,2t(n-i)+n}$$
$$= \begin{cases} \delta^{\frac{n}{2}} \bar{\mathcal{H}}^{\mathcal{F}}{}_{k,0}, & \textit{if } n \textit{ is even;} \\ \delta^{\frac{n-1}{2}} \bar{\mathcal{H}}^{\mathcal{L}}{}_{k,0}, & \textit{if } n \textit{ is odd,} \end{cases}$$

$$(iv)\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^{i}\bar{\mathcal{H}^{\mathcal{L}}}_{k,2t(n-i)+n} = \begin{cases}\delta^{\frac{n}{2}}\bar{\mathcal{H}^{\mathcal{L}}}_{k,0}, & \textit{if } n \textit{ is even;}\\ \delta^{\frac{n+1}{2}}\bar{\mathcal{H}^{\mathcal{F}}}_{k,0}, & \textit{if } n \textit{ is odd.}\end{cases}$$

The following theorem deals with congruence properties of the hyperbolic k-Fibonacci and k-Lucas quaternions.

Theorem 3.12. *For $n, t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$, we have*

$$(i)\qquad \mathcal{G}_{k,n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2^{r+1}}},$$

$$(ii)\qquad \mathcal{G}_{k,(2^{r+2}+2)n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{j}(-1)^{n}\mathcal{G}_{k,j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

Proof. From theorem (3.5; (ii)), for all $n, t \geq 1$ and $\mathcal{G}_{k,n} = \bar{\mathcal{H}^{\mathcal{F}}}_{k,n}$ or $\bar{\mathcal{H}^{\mathcal{L}}}_{k,n}$, we have

$$\begin{aligned}\mathcal{G}_{k,n+t} &= \sum_{i+j+s=n;i\neq 0}\binom{n}{i,j}k^{-n}(-1)^{j+s}L_{k,2^{r+1}}{}^{i}\mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t}\\ &\quad + \sum_{i+j+s=n;i=0}\binom{n}{i,j}k^{-n}(-1)^{j+s}L_{k,2^{r+1}}{}^{i}\mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t},\\ &= \sum_{i+j+s=n;i\neq 0}\binom{n}{i,j}k^{-n}(-1)^{j+s}L_{k,2^{r+1}}{}^{i}\mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t}\\ &\quad + \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t}.\end{aligned}$$

$$\begin{aligned}&\mathcal{G}_{k,n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t}\\ &= \sum_{i+j+s=n;i\neq 0}\binom{n}{i,j}k^{-n}(-1)^{j+s}L_{k,2^{r+1}}{}^{i}\mathcal{G}_{k,2^{r+1}(i+2j)+2(i+j)+t},\end{aligned}$$

$$\therefore L_{k,2} \quad \text{divides} \quad (\mathcal{G}_{k,n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t}),$$

$$\therefore \mathcal{G}_{k,n+t} - \sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^{n}\mathcal{G}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2}}.$$

This completes the proof of (i).
The proof of (ii) is similar to (i), using theorem (3.5; (iii)). □

4. Conclusion

In this paper, we defined the hyperbolic k-Fibonacci and k-Lucas quaternions and presented some well-known identities such as generating functions, Binet formula, Catalan's identity, Cassini's identity, Vajda's identity and d'Ocagne's identity for these quaternions. In the future, we can extend this approach to k-Fibonacci and k-Lucas hyperbolic sedenions.

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UNIQUE COMMON FIXED POINT RESULTS OF INTEGRAL TYPE CONTRACTION CONDITION IN 2-BANACH SPACE

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Abstract. In this paper we have proved some fixed point theorems for integral type contraction condition on 2-Banach space.

1. Introduction

In a series of papers, Gähler ([4]-[6]) initiated the concept of 2-norm and 2-Banach spaces. Gähler studied the topological property of 2-metric as well as 2-normed spaces. Interested research workers can see the properties in the papers ([4]-[6]). It was Branciari [2] who established fixed point theorem for contractive mapping of integral type on metric spaces. Thereafter, many authors have used this result in 2-metric spaces. Liu et. al. [10], Okeke et. al. [14], Moradi [11], Sarwar [17], Badehian [1], Liu et. al.[9] have worked on integral type contractive condition in metric space. Also many authors have investigated fixed point theorems using various contractive conditions on 2-Banach space. Gangopadhyay et. al.[7], Okeke et. al.[12], Das et. al.[3], Saluja [16], Okeke and Olaleru [13] have proved fixed point theorems on 2-Banach spaces. Gupta et. al. [8], Prajapati et. al. [15] have worked on 2-Banach space for contractive condition of integral type mappings. In this paper we have proved some unique common fixed point theorems for integral type contractive condition on 2-Banach space. We also have used F-contraction to obtain the results and have given some corollaries of these results.

2. Definitions

We have collected the following definitions from Gähler [4].

Definition 2.1. (*2-norm*) Let X be a linear space and $\|.,.\|$ be a real valued function defined on X, where

i) $\|a, b\| = 0$ if and only if a and b are linearly dependent;

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ii) $\|a, b\| = \|b, a\|$;
iii)$\|a, xb\| = |x|\|a, b\|$;
iv)$\|a, b + c\| \leq \|a, b\| + \|a, c\|$
for all $a, b, c \in X$ and $x \in \mathbb{R}$. Then $\|., .\|$ is called a 2-norm and the pair $(X, \|., .\|)$ is called a 2-normed space.

Definition 2.2. (*Convergent*) A sequence $\{x_n\}$ in a 2-normed space X is said to be convergent if there is a point $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$ for all $a \in X$.

Definition 2.3. (*Cauchy Sequence*) A sequence $\{x_n\}$ in a 2-normed space X is called a Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m, a\| = 0$ for all $a \in X$.

Definition 2.4. (*2-Banach Space*) A linear 2-normed space is said to be complete if every Cauchy sequence in X is convergent in X. Then we say X is a 2-Banach Space.

Wardowski[18] has defined F-contraction in metric space which we can defined it in 2-normed space as follows:

Definition 2.5. (*F-contraction*) Let $\mathbf{F}$ be the collection of all functions $\mathbf{F} = \{F : \mathbb{R}_+ \to \mathbb{R}\}$ satisfying the following conditions:
i) F is strictly increasing;
ii) for all sequence $\{\alpha_n\} \in \mathbb{R}, \lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
iii) there exists $0 < k < 1$ such that $\lim_{\alpha\to 0_+} \alpha^n F(\alpha) = 0$.
Then a function $T : X \to X$ is said to be F-contraction if there exists a function $F \in \mathbf{F}$ such that for all $x, y, a \in X$,

$$\tau \in \mathbb{R}_+ \;\Rightarrow \tau + F(\|Tx - Ty, a\|) \leq F(\|x - y, a\|).$$

3. Preliminaries

Throughout the paper we denote the followings:
i) We write X as a 2-Banach space.
ii) $\Phi = \{\phi$ where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}_+$ satisfying the conditions:
a) $\int_0^\epsilon \phi(t)dt > 0$ for each ϵ and
b)$\int_0^{a+b} \phi(t)dt \leq \int_0^a \phi(t)dt + \int_0^b \phi(t)dt\}$.
iii) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be strictly increasing such that

$$\psi(0) = 0. \tag{3.1}$$

iv) $\eta : {\mathbb{R}_+}^3 \to \mathbb{R}_+$ defined by

$$\eta(s, t, u) = \alpha\frac{\min\{s, t, u\}}{1 + \max\{s, t, u\}} + \beta\frac{s + t}{1 + u} \tag{3.2}$$

$\forall s, t, u \in \mathbb{R}_+$ and α, β are arbitrary constants.

v) $F : F \in \mathbf{F}$.
We have used the following lemmas to prove the main results.

Lemma 3.1. [9] *Let $\phi \in \Phi$ and $\{s_n\}$ be a of non-negative sequence with $\lim_{n\to\infty} s_n = s$. Then*

$$\lim_{n\to\infty} \int_0^{s_n} \phi(t)dt = \int_0^{s} \phi(t)dt.$$

Lemma 3.2. [9] *Let $\phi \in \Phi$ and $\{s_n\}$ be a of non-negative sequence. Then* $\lim_{n\to\infty} \int_0^{s_n} \phi(t)dt = 0$ *if and only if* $\lim_{n\to\infty} s_n = 0$.

4. Main Results

In this section we have proved some results as follows:

Theorem 4.1. *Let $\{T_i\}_{i=1}^{\infty}$ be the sequence of self-maps on X satisfying the relation*

$$\int_0^{\|T_ix-T_jy,a\|} \phi(t)dt \leq \int_0^{M(x,y,a)} \phi(t)dt,$$

where

$M(x,y,a) = \alpha\|x-y,a\| + \beta\max\{\|x-T_ix,a\|, \|y-T_jy,a\|\} + \gamma\max\{\|x-y,a\|, \|x-T_jy,a\|\}$, $\alpha+\beta+\gamma < 1$. *Then $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point.*

Proof. For $x_0 \in X$, we get a sequence $\{x_n\}$ in X for a fixed $i \in \mathbb{N}$ by setting $x_{n+1} = T_ix_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Since $x_n = T_ix_n \Rightarrow x_{n+1} = x_n$, then x_n is a common fixed point of $\{T_i\}_{i=1}^{\infty}$ and the proof is over. So we assume that $x_{n+1} \neq x_n$.
Now,

$\int_0^{\|x_{n+1}-x_n,a\|\}} \phi(t)dt = \int_0^{\|T_ix_n-T_jx_{n-1},a\|} \phi(t)dt$
$\leq \int_0^{M(x_n,x_{n-1},a)} \phi(t)dt$
$= \int_0^{\alpha\|x_n-x_{n-1},a\|+\beta\|x_n-x_{n+1},a\|+\gamma\|x_n-x_{n-1},a\|} \phi(t)dt$ [by simplifying]
$= \int_0^{(\alpha+\beta+\gamma)\|x_n-x_{n-1},a\|} \phi(t)dt$
$\leq \int_0^{(\alpha+\beta+\gamma)^2\|x_{n-1}-x_{n-2},a\|} \phi(t)dt$
$\vdots$
$\leq \int_0^{(\alpha+\beta+\gamma)^n\|x_1-x_0,a\|} \phi(t)dt.$

Therefore,

$\lim_{n\to\infty} \int_0^{\|x_{n+1}-x_n,a\|} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{(\alpha+\beta+\gamma)^n\|x_1-x_0,a\|} \phi(t)dt = 0$ [by **Lemma 3.1**].

By **Lemma 3.2** we get, $\lim_{n\to\infty} \|x_{n+1}-x_n,a\| = 0$.
Again let, $p,n,m \in \mathbb{N}; n = p+m$.
Then,

$\lim_{n,m\to\infty} \int_0^{\|x_{n+1}-x_{m+1},a\|} \phi(t)dt = \lim_{n,m\to\infty} \int_0^{\|T_ix_n-T_jx_m,a\|} \phi(t)dt$
$\leq \lim_{n,m\to\infty} \int_0^{M(x_n,x_m,a)} \phi(t)dt$
$= \lim_{n,m\to\infty} \int_0^{\alpha\|x_n-x_m,a\|+\gamma\|x_n-x_{m+1},a\|} \phi(t)dt$ [by simplifying $M(x_n,x_m,a)$ and since $\lim_{n,m\to\infty} \|x_n-x_m,a\| \leq \lim_{n,m\to\infty} \|x_n-x_{m+1},a\|$

$+ \lim_{n,m\to\infty} \|x_{m+1} - x_m, a\| = \lim_{n,m\to\infty} \|x_n - x_{m+1}, a\|]$
$\leq \lim_{n,m\to\infty} \int_0^{(\alpha+\gamma)\|x_n - x_{m+1}, a\|} \phi(t)dt$
$\leq \lim_{n,m\to\infty} \int_0^{(\alpha+\gamma)^2\|x_{n-1} - x_{m+1}, a\|} \phi(t)dt$
$\vdots$
$\leq \lim_{n,m\to\infty} \int_0^{(\alpha+\gamma)^{p+1}\|x_{n-p} - x_{m+1}, a\|} \phi(t)dt$
$= \lim_{n,m\to\infty} \int_0^{(\alpha+\gamma)^{p+1}\|x_m - x_{m+1}, a\|} \phi(t)dt$
$= 0.$
which implies, $\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, a\| = 0$ [By **Lemma 3.2**].
Therefore $\{x_n\}$ is a Cauchy sequence in X. Thus there exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$.
Since

$\lim_{n\to\infty} \|T_i x - x, a\| \leq \lim_{n\to\infty} \|T_i x - x_n, a\| + \lim_{n\to\infty} \|x_n - x, a\|,$

we get

$\lim_{n\to\infty} \int_0^{\|T_i x - x, a\|} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{\|T_i x - x_n, a\|} \phi(t)dt$
$= \lim_{n\to\infty} \int_0^{\|T_i x - T_j x_{n-1}, a\|} \phi(t)dt$
$\leq \lim_{n\to\infty} \int_0^{M(x, x_{n-1}, a)} \phi(t)dt$
$= \lim_{n\to\infty} \int_0^{\alpha.0 + \beta\|x - T_i x, a\| + \gamma.0} \phi(t)dt$ [by simplifying $M(x, x_{n-1}, a)$]
implies, $\lim_{n\to\infty} \|T_i x - x, a\| \leq \lim_{n\to\infty} \beta\|x - T_i x, a\|$
implies, $\|x - T_i x, a\| = 0$ implies, $T_i x = x$.
To show the uniqueness of x, let y be another common fixed point.
Now

$\int_0^{\|x-y,a\|} \phi(t)dt = \int_0^{\|T_i x - T_j y, a\|} \phi(t)dt$
$\leq \int_0^{\alpha\|x-y,a\| + \beta\max\{\|x - T_i x, a\|, \|y - T_j y, a\|\} + \gamma\max\{\|x-y,a\|, \|x - T_j y, a\|\}} \phi(t)dt$
$= \int_0^{\alpha\|x-y,a\| + \beta\max\{\|x-x,a\|, \|y-y,a\|\} + \gamma\max\{\|x-y,a\|, \|x-y,a\|\}} \phi(t)dt$
$= \int_0^{(\alpha+\gamma)\|x-y,a\|} \phi(t)dt$
implies, $\|x - y, a\| \leq (\alpha+\gamma)\|x - y, a\|$
implies, $\|x - y, a\| = 0$ implies, $x = y$.
Thus $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point. □

Corollary 4.2. *Let T_1 and T_2 be two self-maps on X satisfying the relation*

$$\int_0^{\|T_1 x - T_2 y, a\|} \phi(t)dt \leq \int_0^{M(x,y,a)} \phi(t)dt,$$

where

$M(x,y,a) = \alpha\|x-y,a\| + \beta\max\{\|x - T_1 x, a\|, \|y - T_2 y, a\|\} + \gamma\max\{\|x - y, a\|, \|x - T_2 y, a\|\}$, $\alpha+\beta+\gamma < 1$. *Then T_1 and T_2 have a unique common fixed point.*

Proof. Putting $T_i = T_1$ and $T_j = T_2$ in the above **Theorem 4.1** we get the result. □

Corollary 4.3. *Let T be a self-map on X satisfying the relation*

$$\int_0^{\|Tx - Ty, a\|} \phi(t)dt \leq \int_0^{M(x,y,a)} \phi(t)dt,$$

where

$M(x,y,a) = \alpha\|x-y,a\| + \beta\max\{\|x-Tx,a\|, \|y-Ty,a\|\} + \gamma\max\{\|x-y,a\|, \|x-Ty,a\|\}$, $\alpha+\beta+\gamma<1$. *Then T have a unique common fixed point.*

Proof. Putting $T_i = T_j = T$ in the above **Theorem 4.1** we get the required result. □

Theorem 4.4. *Let $\{T_i\}_{i=1}^{\infty}$ be the sequence of self-maps on X satisfying the relation*

$$\int_0^{\|T_ix-T_jy,a\|}\phi(t)dt \le \int_0^{m(x,y,a)}\phi(t)dt,$$

where

$m(x,y,a) = \alpha\frac{\|x-T_ix,a\|+\|x-T_jy,a\|}{1+\|y-T_ix,a\|} + \beta\max\{\|x-y,a\|, \|y-T_jy,a\|, \|x-T_ix,a\|\} + \gamma\min\{\|x-T_jy,a\|, \|y-T_ix,a\|\}, \alpha+\beta+\gamma<1$. *Then $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point.*

Proof. Since each of T_i is a self-map, we construct a sequence $\{x_n\}$ such that $x_{n+1} = T_ix_n$ for all $n \in \mathbb{N}\cup\{0\}$ for a fixed $i \in \mathbb{N}$ where $x_0 \in X$ is an initial approximation.

If $x_{n+1} = T_ix_n$ i.e., $T_ix_n = x_n$, then x_n is common fixed point of $\{T_i\}_{i=1}^{\infty}$ and the proof is completed. So we assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}\cup\{0\}$.

Now,

$$\int_0^{\|x_{n+1}-x_n,a\|}\phi(t)dt = \int_0^{\|T_ix_n-T_jx_{n-1},a\|}\phi(t)dt = \int_0^{m(x_n,x_{n-1},a)}\phi(t)dt, \quad (4.1)$$

where

$m(x_n,x_{n-1},a)$
$= \alpha\frac{\|x_n-T_ix_n,a\|+\|x_n-T_jx_{n-1},a\|}{1+\|x_{n-1}-T_ix_n,a\|} + \beta\max\{\|x_n-x_{n-1},a\|, \|x_{n-1}-T_jx_{n-1},a\|, \|x_n-T_ix_n,a\|\} + \gamma\min\{\|x_n-T_jx_{n-1},a\|, \|x_{n-1}-T_ix_n,a\|\}$
$= \alpha\frac{\|x_n-x_{n+1},a\|+\|x_n-x_n,a\|}{1+\|x_{n-1}-x_{n+1},a\|} + \beta\max\{\|x_n-x_{n-1},a\|, \|x_{n-1}-x_n,a\|, \|x_n-x_{n+1},a\|\} + \gamma\min\{\|x_n-x_n,a\|, \|x_{n-1}-x_{n+1},a\|\}$
$\le \alpha\|x_n-x_{n+1},a\| + \beta\max\{\|x_n-x_{n-1},a\|, \|x_n-x_{n+1},a\|\} + \gamma.0$

If $\|x_n-x_{n-1},a\| \le \|x_{n+1}-x_n,a\|$, then $m(x_n,x_{n-1},a) \le (\alpha+\beta)\|x_{n+1}-x_n,a\|$.

Therefore from (4.1),we get

$$\int_0^{\|x_{n+1}-x_n,a\|}\phi(t)dt \le \int_0^{(\alpha+\beta)\|x_{n+1}-x_n,a\|}\phi(t)dt$$

implies, $\|x_{n+1}-x_n,a\| \le (\alpha+\beta)\|x_{n+1}-x_n,a\|$
implies, $1 \le (\alpha+\beta)$,
which is a contradiction.

Therefore $\|x_{n+1}-x_n,a\| \le \|x_n-x_{n-1},a\|$. Thus $\{\|x_{n+1}-x_n,a\|\}$ is a monotone decreasing sequence of real numbers and bounded below. Therefore it is convergent. Suppose $\lim_{n\to\infty}\|x_{n+1}-x_n,a\| = r$.

Then

$\int_0^r \phi(t)dt = \lim_{n\to\infty} \int_0^{\|x_{n+1}-x_n,a\|} \phi(t)dt \leq \int_0^{(\alpha+\beta)\|x_n-x_{n-1},a\|} \phi(t)dt = \int_0^{(\alpha+\beta)r} \phi(t)dt$
implies, $r \leq (\alpha+\beta)r$ i.e., $r = 0$.
Thus $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = 0$.
Now we show that $\{x_n\}$ is a Cauchy sequence.

Let $n, m \in \mathbb{N}$. Then

$$\int_0^{\|x_{n+1}-x_{m+1},a\|} \phi(t)dt = \int_0^{\|T_i x_n - T_j x_m, a\|} \phi(t)dt \leq \int_0^{m(x_n,x_m,a)} \phi(t)dt \tag{4.2}$$

where
$m(x_n, x_m, a) = \alpha \frac{\|x_n - T_i x_n, a\| + \|x_n - T_j x_m, a\|}{1+\|x_m - T_i x_n, a\|} + \beta \max\{\|x_n - x_m, a\|, \|x_m - T_j x_m, a\|, \|x_n - T_i x_n, a\|\} + \gamma \min\{\|x_n - T_j x_m, a\|, \|x_m - T_i x_n, a\|\}$
$\leq \alpha\{\|x_n - x_{n+1}, a\| + \|x_n - x_{m+1}, a\|\} + \beta \max\{\|x_n - x_m, a\|, \|x_m - x_{m+1}, a\|, \|x_n - x_{n+1}, a\|\} + \gamma \min\{\|x_n - x_{m+1}, a\|, \|x_m - x_{n+1}, a\|\}$
$\leq \alpha\{\|x_n - x_{n+1}, a\| + \|x_n - x_m, a\| + \|x_m - x_{m+1}, a\|\} + \beta \max\{\|x_n - x_m, a\|, \|x_m - x_{m+1}, a\|, \|x_n - x_{n+1}, a\|\} + \gamma \min\{\|x_n - x_m, a\| + \|x_m - x_{m+1}, a\|, \|x_m - x_n, a\| + \|x_n - x_{n+1}, a\|\}$.
Taking limit as $n, m \to \infty$, we get

$\lim_{n,m\to\infty} m(x_n, x_m, a)$
$\leq \alpha \lim_{n,m\to\infty} \|x_n - x_m, a\| + \beta \lim_{n,m\to\infty} \|x_n - x_m, a\| + \gamma \lim_{n,m\to\infty} \|x_n - x_m, a\|$
$= (\alpha + \beta + \gamma) \lim_{n,m\to\infty} \|x_n - x_m, a\|$.
So from (4.2), we get,
$\lim_{n,m\to\infty} \int_0^{\|x_{n+1}-x_{m+1},a\|} \phi(t)dt \leq \lim_{n,m\to\infty} \int_0^{(\alpha+\beta+\gamma)\|x_n - x_m,a\|} \phi(t)dt$
implies, $\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, a\| \leq (\alpha+\beta+\gamma) \lim_{n,m\to\infty} \|x_n - x_m, a\|$
implies, $\lim_{n,m\to\infty} \|x_n - x_m, a\| = 0$.
Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$.
Again

$\int_0^{\|T_i x - x, a\|} \phi(t)dt = \lim_{n\to\infty} \int_0^{\|T_i x - x_n, a\|} \phi(t)dt$

$$= \lim_{n\to\infty} \int_0^{\|T_i x - T_j x_{n-1}, a\|} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{m(x,x_{n-1},a)} \phi(t)dt, \tag{4.3}$$

where,
$m(x, x_{n-1}, a) = \alpha \frac{\|x - T_i x, a\| + \|x - T_j x_{n-1}, a\|}{1+\|x_{n-1} - T_i x, a\|} + \beta \max\{\|x - x_{n-1}, a\|, \|x_{n-1} - T_j x_{n-1}, a\|, \|x - T_i x, a\|\} + \gamma \min\{\|x - T_j x_{n-1}, a\|, \|x_{n-1} - T_i x, a\|\}$
$\leq \alpha\|x - T_i x, a\| + \|x - x_n, a\| + \beta \max\{\|x - x_{n-1}, a\|, \|x_{n-1} - x_n, a\|, \|x - T_i x, a\|\} + \gamma \min\{\|x - x_n, a\|, \|x_{n-1} - T_i x, a\|\}$.
Therefore,

$\lim_{n\to\infty} m(x, x_{n-1}, a) \leq \alpha\|x - T_i x, a\| + \beta\|x - T_i x, a\| + \gamma.0 = (\alpha + \beta)\|T_i x - x, a\|$.
Therefore from equation (4.3) we get

$\int_0^{\|T_ix-x,a\|}\phi(t)dt \leq \int_0^{(\alpha+\beta)\|T_ix-x,a\|}\phi(t)dt$
implies, $\|T_ix-x,a\| \leq (\alpha+\beta)\|T_ix-x,a\|$
implies, $\|T_ix-x,a\| = 0$ implies, $T_ix = x$.
Thus x is a common fixed point of $\{T_i\}_{i=1}^{\infty}$.
To prove the uniqueness , let y be another common fixed point.
Then

$$\int_0^{\|x-y,a\|}\phi(t)dt = \int_0^{\|T_ix-T_jy,a\|}\phi(t)dt \leq \int_0^{m(x,y,a)}\phi(t)dt, \tag{4.4}$$

where
$m(x,y,a) = \alpha\frac{\|x-T_ix,a\|+\|x-T_jy,a\|}{1+\|y-T_ix,a\|} + \beta\max\{\|x-y,a\|,\|y-T_jy,a\|,\|x-T_ix,a\|\} + \gamma\min\{\|x-T_jy,a\|,\|y-T_ix,a\|\}$
$\leq \alpha\{\|x-x,a\|+\|x-y,a\|\} + \beta\max\{\|x-y,a\|,\|y-y,a\|,\|x-x,a\|\} + \gamma\min\{\|x-y,a\|,\|y-x,a\|\}$
$= (\alpha+\beta+\gamma)\|x-y,a\|$.
Then from (4.4) we get
$\int_0^{\|x-y,a\|}\phi(t)dt \leq \int_0^{\|x-y,a\|}\phi(t)dt$
implies, $\|x-y,a\| \leq (\alpha+\beta+\gamma)\|x-y,a\| = 0$
implies, $x = y$.
Hence the theorem. □

Corollary 4.5. *Let T_1 and T_2 be two self-maps on X satisfying the relation*

$$\int_0^{\|T_1x-T_2y,a\|}\phi(t)dt \leq \int_0^{m(x,y,a)}\phi(t)dt,$$

where
$m(x,y,a) = \alpha\frac{\|x-T_1x,a\|+\|x-T_2y,a\|}{1+\|y-T_1x,a\|} + \beta\max\{\|x-y,a\|,\|y-T_2y,a\|,\|x-T_1x,a\|\} + \gamma\min\{\|x-T_2y,a\|,\|y-T_1x,a\|\}, \alpha+\beta+\gamma < 1$. *Then T_1 and T_2 have a unique common fixed point.*

Proof. Replacing T_i by T_1 and T_j by T_2 in **Theorem 4.4** we get the result. □

Corollary 4.6. *Let T be a self-map on X satisfying the relation*

$$\int_0^{\|Tx-Ty,a\|}\phi(t)dt \leq \int_0^{m(x,y,a)}\phi(t)dt,$$

where
$m(x,y,a) = \alpha\frac{\|x-Tx,a\|+\|x-Ty,a\|}{1+\|y-Tx,a\|} + \beta\max\{\|x-y,a\|,\|y-Ty,a\|,\|x-Tx,a\|\} + \gamma\min\{\|x-Ty,a\|,\|y-Tx,a\|\}, \alpha+\beta+\gamma < 1$. *Then T have a unique common fixed point.*

Proof. Replacing T_i and T_j by T in **Theorem 4.4** we get the result. □

Theorem 4.7. *Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of self-maps on X satisfying the relation,*

$$\int_0^{\psi(\|T_ix-T_jy,a\|)}\phi(t)dt \leq \int_0^{\eta(\|x-y,a\|,\|x-T_jy,a\|,\|y-T_ix,a\|)}\phi(t)dt,$$

where $\alpha+2\beta+\gamma < 1$, $\psi(t)$ satisfy (3.1) and $\eta(t)$ satisfy (3.2). Then $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point.

Proof. Let us construct a sequence $\{x_n\}$ in X for a fixed $i \in \mathbb{N}$ such that $x_{n+1} = T_i x_n$ with an initial approximation $x_0 \in X$. If $x_n = T_i x_n$ i.e., $x_{n+1} = x_n$, then x_n is a common fixed point of $\{T_i\}_{i=1}^{\infty}$ and the proof is over. So we assume that $x_{n+1} \neq x_n$.
Since

$$\int_0^{\psi(\|x_{n+1}-x_n,a\|)} \phi(t)dt = \int_0^{\psi(\|T_i x_n - T_j x_{n-1},a\|)} \phi(t)dt$$

$$\leq \int_0^{\eta(\|x_n-x_{n-1},a\|,\|x_n-T_j x_{n-1},a\|,\|x_{n-1}-T_i x_n,a\|)} \phi(t)dt \qquad (4.5)$$

Now,

$$\eta(\|x_n - x_{n-1}, a\|, \|x_n - T_j x_{n-1}, a\|, \|x_{n-1} - T_i x_n, a\|)$$
$$= \alpha\left(\frac{\min\{\|x_n-x_{n-1},a\|,\|x_n-T_j x_{n-1},a\|,\|x_{n-1}-T_i x_n,a\|\}}{1+\max\{\|x_n-x_{n-1},a\|,\|x_n-T_j x_{n-1},a\|,\|x_{n-1}-T_i x_n,a\|\}}\right)+\beta\frac{\|x_n-x_{n-1},a\|+\|x_n-T_j x_{n-1},a\|}{1+\|x_{n-1}-T_i x_n,a\|}$$
$$= \alpha\left(\frac{\min\{\|x_n-x_{n-1},a\|,\|x_n-x_n,a\|,\|x_{n-1}-x_{n+1},a\|\}}{1+\max\{\|x_n-x_{n-1},a\|,\|x_n-x_n,a\|,\|x_{n-1}-x_{n+1},a\|\}}\right) + \beta\frac{\|x_n-x_{n-1},a\|+\|x_n-x_n,a\|}{1+\|x_{n-1}-x_{n+1},a\|}$$
$$\leq \alpha\left(\frac{0}{1+\|x_{n-1}-x_{n+1},a\|}\right) + \beta\frac{\|x_n-x_{n-1},a\|}{1+\|x_{n-1}-x_{n+1},a\|} = \beta\|x_n - x_{n-1}, a\|$$
$$\leq \|x_n - x_{n-1}, a\|. \qquad (4.6)$$

From equation (4.5) we get,

$$\int_0^{\psi(\|x_{n+1}-x_n,a\|)} \phi(t)dt \leq \int_0^{\psi(\|x_n-x_{n-1},a\|)} \phi(t)dt$$

implies, $\psi(\|x_{n+1} - x_n, a\|) \leq \psi(\|x_n - x_{n-1}, a\|)$
implies, $\|x_{n+1} - x_n, a\| \leq \|x_n - x_{n-1}, a\|$.
Thus $\{\|x_{n+1} - x_n, a\|\}$ is a monotone decreasing bounded below sequence of real numbers. Therefore it is convergent.

Suppose, $\lim_{n\to\infty} \|x_{n+1}-x_n, a\| = r$ implies, $\lim_{n\to\infty} \psi(\|x_{n+1}-x_n, a\|) = \psi(r)$.
Now,

$$\int_0^{\psi(r)} \phi(t)dt = \lim_{n\to\infty} \int_0^{\psi(\|x_{n+1}-x_n,a\|)} \phi(t)dt$$
$$\leq \lim_{n\to\infty} \int_0^{\psi(\|x_{n+1}-x_n,a\|)} \phi(t)dt \text{ [by (4.6)]}$$
$$\leq \int_0^{\psi(r)} \phi(t)dt$$

implies, $\psi(r) = 0$ implies, $r = 0$ i.e., $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = 0$.
Let, $n > m \in \mathbb{N}$.
Then

$$\int_0^{\psi(\|x_{n+1}-x_{m+1},a\|)} \phi(t)dt = \int_0^{\psi(\|T_i x_n - T_j x_m,a\|)} \phi(t)dt$$

$$\leq \int_0^{\eta(\|x_n-x_m,a\|,\|x_n-T_j x_m,a\|,\|x_m-T_i x_n,a\|)} \phi(t)dt. \qquad (4.7)$$

Now,

$$\eta(\|x_n - x_m, a\|, \|x_n - T_j x_m, a\|, \|x_m - T_i x_n, a\|)$$
$$= \alpha\frac{\min\{\|x_n-x_m,a\|,\|x_n-T_j x_m,a\|,\|x_m-T_i x_n,a\|\}}{1+\max\{\|x_n-x_m,a\|,\|x_n-T_j x_m,a\|,\|x_m-T_i x_n,a\|\}} + \beta\frac{\|x_n-x_m,a\|+\|x_n-T_j x_m,a\|}{1+\|x_m-T_i x_n,a\|}$$
$$= \alpha\frac{\min\{\|x_n-x_m,a\|,\|x_n-x_{m+1},a\|,\|x_m-x_{n+1},a\|\}}{1+\max\{\|x_n-x_m,a\|,\|x_n-x_{m+1},a\|,\|x_m-x_{n+1},a\|\}} + \beta\frac{\|x_n-x_m,a\|+\|x_n-x_{m+1},a\|}{1+\|x_m-x_{n+1},a\|}$$
$$\leq \alpha\frac{\min\{\|x_n-x_m,a\|,\|x_n-x_m,a\|+\|x_m-x_{m+1},a\|,\|x_m-x_n,a\|+\|x_n-x_{n+1},a\|\}}{1+\max\{\|x_n-x_m,a\|,\|x_n-x_m,a\|+\|x_m-x_{m+1},a\|,\|x_m-x_n,a\|+\|x_n-x_{n+1},a\|\}}+\beta(\|x_n-x_m, a\| + \|x_n - x_{m+1}, a\|)$$

$\leq \alpha\|x_n - x_m, a\| + \beta(\|x_n - x_m, a\| + \|x_n - x_m, a\| + \|x_m - x_{m+1}, a\|)$
$= \alpha\|x_n - x_m, a\| + 2\beta\|x_n - x_m, a\| + \beta\|x_m - x_{m+1}, a\|$
$= (\alpha + 2\beta)\|x_n - x_m, a\| + \beta\|x_m - x_{m+1}, a\|$.
Therefore from (4.7) we get
$\lim_{n,m\to\infty} \int_0^{\psi(\|x_{n+1}-x_{m+1},a\|)} \phi(t)dt$
$\leq \lim_{n,m\to\infty} \int_0^{\psi((\alpha+2\beta)\|x_n-x_m,a\|+\beta\|x_m-x_{m+1},a\|)} \phi(t)dt$
implies, $\lim_{n,m\to\infty} \psi(\|x_{n+1}-x_{m+1}, a\|) \leq \lim_{n,m\to\infty}[\psi((\alpha+2\beta)\|x_n-x_m, a\|+ \beta\|x_m - x_{m+1}, a\|)]$
implies, $\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, a\| \leq \lim_{n,m\to\infty}[(\alpha + 2\beta)\|x_n - x_m, a\| + \beta\|x_m - x_{m+1}, a\|]$
implies, $\lim_{n,m\to\infty} \|x_n - x_m, a\| \leq (\alpha + 2\beta) \lim_{n,m\to\infty} \|x_n - x_m, a\|$
implies, $\lim_{n,m\to\infty} \|x_n - x_m, a\| = 0$.

Therefore $\{x_n\}$ is a Cauchy sequence. Thus there exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$.
Again,
$\lim_{n\to\infty} \int_0^{\psi(\|T_ix-x,a\|)} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{\psi(\|T_ix-x_n,a\|+\|x_n-x,a\|)} \phi(t)dt$
$= \lim_{n\to\infty} \int_0^{\psi(\|T_ix-T_jx_{n-1},a\|)} \phi(t)dt$

$$\leq \lim_{n\to\infty} \int_0^{\psi(\eta(\|x-x_{n-1},a\|,\|x-T_jx_{n-1},a\|,\|x_{n-1}-T_ix\|))} \phi(t)dt. \tag{4.8}$$

Now,
$\lim_{n\to\infty} \eta(\|x - x_{n-1}, a\|, \|x - T_jx_{n-1}, a\|, \|x_{n-1} - T_ix\|)$
$= \lim_{n\to\infty} \alpha \frac{\min\{\|x-x_{n-1},a\|,\|x-T_jx_{n-1},a\|,\|x_{n-1}-T_ix\|\}}{1+\max\{\|x-x_{n-1},a\|,\|x-T_jx_{n-1},a\|,\|x_{n-1}-T_ix\|\}}$
$+ \lim_{n\to\infty} \beta \frac{\|x-x_{n-1},a\|+\|x-T_jx_{n-1},a\|}{1+\|x_{n-1}-T_ix,a\|}$
$= \lim_{n\to\infty} \alpha \frac{0}{1+\|x_{n-1}-T_ix,a\|} + \beta.0$
$= 0$.
Therefore from(4.8) we get,
$\lim_{n\to\infty} \int_0^{\psi(\|T_ix-x,a\|)} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{\psi(0)} \phi(t)dt$
implies, $\psi(\|T_ix - x, a\|) \leq \psi(0) = 0$
implies, $\|T_ix - x, a\| = 0$ i.e., $T_ix = x$.
Thus x is a common fixed point of $\{T_i\}_{i=1}^{\infty}$.
Again let y be another common fixed point. Then
$\int_0^{\psi(\|x-y,a\|)} \phi(t)dt = \int_0^{\psi(\|T_ix-T_jy,a\|)} \phi(t)dt$
$\leq \int_0^{\eta(\|x-y,a\|,\|x-T_jy,a\|,\|y-T_ix,a\|)} \phi(t)dt$

$$= \int_0^{\eta(\|x-y,a\|,\|x-y,a\|,\|y-x,a\|)} \phi(t)dt, \tag{4.9}$$

where,
$\eta(\|x - y, a\|, \|x - y, a\|, \|y - x, a\|)$
$= \alpha(\frac{\min\{\|x-y,a\|,\|x-y,a\|,\|y-x,a\|\}}{1+\max\{\|x-y,a\|,\|x-y,a\|,\|y-x,a\|\}}) + \beta \frac{\|x-y,a\|+\|x-y,a\|}{1+\|y-T_ix,a\|}$

$\leq \alpha\|x-y,a\| + 2\beta\|x-y,a\| = (\alpha+2\beta)\|x-y,a\|$.
Therefore from equation (4.9) we get,
$\int_0^{\psi(\|x-y,a\|)} \phi(t)dt \leq \int_0^{\psi((\alpha+2\beta)\|x-y,a\|)} \phi(t)dt$
implies, $\psi(\|x-y,a\|) \leq \psi((\alpha+2\beta)\|x-y,a\|)$
implies, $\|x-y,a\| \leq (\alpha+2\beta)\|x-y,a\|$
implies, $\|x-y,a\| = 0$ implies, $x = y$.
Thus x is a unique common fixed point of $\{T_i\}_{i=1}^{\infty}$. □

Corollary 4.8. *Let T_1 and T_2 be two self-maps on X satisfying the relation,*
$$\int_0^{\psi(\|T_1x-T_2y,a\|)} \phi(t)dt \leq \int_0^{\eta(\|x-y,a\|,\|x-T_2y,a\|,\|y-T_1x,a\|)} \phi(t)dt,$$
where,
$\alpha+2\beta+\gamma<1$ *,* $\psi(t)$ *satisfy (3.1) and* $\eta(t)$ *satisfy (3.2). Then* T_1 *and* T_2 *have a unique common fixed point.*

Proof. By putting T_1 in place of T_i and T_2 in place of T_j in the above **Theorem 4.7**, we get the result. □

Corollary 4.9. *Let T be a self-map on X satisfying the relation,*
$$\int_0^{\psi(\|Tx-Ty,a\|)} \phi(t)dt \leq \int_0^{\eta(\|x-y,a\|,\|x-Ty,a\|,\|y-Tx,a\|)} \phi(t)dt,$$
where,
$\alpha+2\beta+\gamma<1$ *,* $\psi(t)$ *satisfy (3.1) and* $\eta(t)$ *satisfy (3.2). Then* T *have a unique common fixed point.*

Proof. By putting T in place of T_i in the above **Theorem 4.7**, we get the result. □

Theorem 4.10. *Let$\{T_i\}_{i=1}^{\infty}$ be the sequence of self-maps on X satisfy the relation*
$$\int_0^{\psi(\|T_ix-T_jy,a\|)} \phi(t)dt \leq \int_0^{\psi(\eta(x,y,a))} \phi(t)dt,$$
where,
$$\eta(x,y,a) = \alpha\|x-y,a\| + \beta\max\{\|x-y,a\|, \|x-T_jy,a\|, \|y-T_ix,a\|\}$$
$$+\gamma\frac{\min\{\|x-T_jy,a\|,\|y-T_ix,a\|\}}{1+\max\{\|x-T_jy,a\|,\|y-T_ix,a\|\}} + \delta(\|x-y,a\|+\|y-T_jy,a\|), \alpha+\beta+\gamma+2\delta<1.$$
Then $\{T_i\}_{i=1}^{\infty}$ *have a unique common fixed point.*

Proof. Construct the sequence $\{x_n\}$ as **Theorem 4.7** where $x_{n+1} = T_ix_n$. Since $x_n = T_ix_n$ i.e., $x_{n+1} = x_n$, then x_n is a common fixed point of $\{T_i\}_{i=1}^{\infty}$ and the proof is completed. So we assume that $x_n \neq x_{n+1}$.
Now
$$\int_0^{\psi(\|x_{n+1}-x_n,a\|)} \phi(t)dt = \int_0^{\psi(\|T_ix_n-T_jx_{n-1},a\|)} \phi(t)dt$$
$$\leq \int_0^{\phi(\eta(x_n,x_{n-1},a))} \phi(t)dt, \tag{4.10}$$
where
$\eta(x_n, x_{n-1}, a)$
$= \alpha\|x_n-x_{n-1},a\| + \beta\max\{\|x_n-x_{n-1},a\|, \|x_n-T_ix_n,a\|, \|x_{n-1}-T_jx_{n-1},a\|\}$

$+\gamma\frac{\min\{\|x_n-T_jx_{n-1},a\|,\|x_{n-1}-T_ix_n,a\|\}}{1+\max\{\|x_n-T_jx_{n-1},a\|,\|x_{n-1}-T_ix_n,a\|\}}+\delta(\|x_n-T_jx_{n-1},a\|+\|x_{n-1}-T_jx_{n-1},a\|)$
$=\alpha\|x_n-x_{n-1},a\|+\beta\max\{\|x_n-x_{n-1},a\|,\|x_n-x_{n+1},a\|,\|x_{n-1}-x_n,a\|\}$
$+\gamma\frac{\min\{\|x_n-x_n,a\|,\|x_{n-1}-x_{n+1},a\|\}}{1+\max\{\|x_n-x_n,a\|,\|x_{n-1}-x_{n+1},a\|\}}+\delta(\|x_n-x_n,a\|+\|x_{n-1}-x_n,a\|)$
$=\alpha\|x_n-x_{n-1},a\|+\beta\max\{\|x_n-x_{n-1},a\|,\|x_n-x_{n+1},a\|\}$

$$+\gamma\frac{0}{1+\|x_{n-1}-x_{n+1},a\|}+\delta\|x_n-x_{n-1},a\| \qquad (4.11)$$

If $\|x_n-x_{n-1},a\|\leq\|x_n-x_{n+1},a\|$, then from (4.11)
$\eta(x_n,x_{n-1},a)\leq(\alpha+\beta+\delta)\|x_n-x_{n+1},a\|$.
So from (4.10) we get
$\int_0^{\psi(\|x_{n+1}-x_n,a\|)}\phi(t)dt\leq\int_0^{\psi((\alpha+\beta+\delta)\|x_{n+1}-x_n,a\|)}\phi(t)dt$
implies, $\psi(\|x_{n+1}-x_n,a\|)\leq\psi((\alpha+\beta+\delta)\|x_{n+1}-x_n,a\|)$
implies, $\|x_{n+1}-x_n,a\|\leq(\alpha+\beta+\delta)\|x_{n+1}-x_n,a\|$
implies, $1\leq\alpha+\beta+\delta$, which is a contradiction.
Therefore,
$\|x_n-x_{n+1},a\|\leq\|x_n-x_{n-1},a\|\leq\|x_{n-1}-x_{n-2},a\|\leq\ldots\leq\|x_1-x_0,a\|$.
Thus $\|x_n-x_{n+1},a\|$ is a monotone decreasing bounded below sequence of real numbers. So it is convergent. Suppose $\lim_{n\to\infty}\|x_{n+1}-x_n,a\|=r\neq0$. Then from equation (4.10),
$\int_0^{\psi(r)}\phi(t)dt=\lim_{n\to\infty}\int_0^{\psi(\|x_{n+1}-x_n,a\|)}\phi(t)dt$
$\leq\int_0^{\psi((\alpha+\beta+\delta)\|x_n-x_{n-1},a\|)}\phi(t)dt\leq\int_0^{\psi((\alpha+\beta+\delta)r)}\phi(t)dt$
implies, $\psi(r)\leq\psi((\alpha+\beta+\delta)r)$

implies, $r\leq(\alpha+\beta+\delta)r$
implies, $1\leq(\alpha+\beta+\delta)$, which is again a contradiction.
Therefore $\lim_{n\to\infty}\|x_{n+1}-x_n,a\|=0$.
Let, $n>m\in\mathbb{N}$.
Then
$\int_0^{\psi(\|x_{n+1}-x_{m+1},a\|)}\phi(t)dt=\int_0^{\psi(\|T_ix_n-T_jx_m,a\|)}\phi(t)dt$

$$\leq\int_0^{\psi(\eta(x_n,x_m,a))}\phi(t)dt, \qquad (4.12)$$

where,
$\eta(x_n,x_m,a)$
$=\alpha\|x_n-x_m,a\|+\beta\max\{\|x_n-x_m,a\|,\|x_n-T_ix_n,a\|,\|x_m-T_jx_m,a\|\}$
$+\gamma\frac{\min\{\|x_n-T_jx_m,a\|,\|x_m-T_ix_n,a\|\}}{1+\max\{\|x_n-T_jx_m,a\|,\|x_m-T_ix_n,a\|\}}+\delta(\|x_n-T_jx_m,a\|+\|x_m-T_ix_n,a\|)$
$=\alpha\|x_n-x_m,a\|+\beta\max\{\|x_n-x_m,a\|,\|x_n-x_{n+1},a\|,\|x_m-x_{m+1},a\|\}$
$+\gamma\frac{\min\{\|x_n-x_{m+1},a\|,\|x_m-x_{n+1},a\|\}}{1+\max\{\|x_n-x_{m+1},a\|,\|x_m-x_{n+1},a\|\}}+\delta(\|x_n-x_{m+1},a\|+\|x_m-x_{n+1},a\|)$
$\leq\alpha\|x_n-x_m,a\|+\beta\max\{\|x_n-x_m,a\|,\|x_n-x_{n+1},a\|,\|x_m-x_{m+1},a\|\}$
$+\gamma\frac{\min\{\|x_n-x_m,a\|+\|x_m-x_{m+1},a\|,\|x_m-x_n,a\|+\|x_n-x_{n+1},a\|\}}{1+\max\{\|x_n-x_m,a\|+\|x_m-x_{m+1},a\|,\|x_m-x_n,a\|+\|x_n-x_{n+1},a\|\}}+\delta(\|x_n-x_m,a\|+$
$\|x_m-x_{m+1},a\|+\|x_m-x_n,a\|+\|x_n-x_{n+1},a\|)$.
Taking $\lim_{n,m\to\infty}$ in the above inequality we get

$\lim_{n,m\to\infty} \eta(x_n, x_m, a)$
$\leq \alpha \lim_{n,m\to\infty} \|x_n - x_m, a\| + \beta \lim_{n,m\to\infty} \|x_n - x_m, a\| + \gamma \lim_{n,m\to\infty} \|x_n - x_m, a\| + 2\delta \lim_{n,m\to\infty} \|x_n - x_m, a\|$
$\leq (\alpha + \beta + \gamma + 2\delta) \lim_{n,m\to\infty} \|x_n - x_m, a\|$.
Therefore from (4.12)we get,
$\lim_{n,m\to\infty} \int_0^{\psi(\|x_{n+1}-x_{m+1},a\|)} \phi(t)dt \leq \lim_{n,m\to\infty} \int_0^{\psi((\alpha+\beta+\gamma+2\delta)\|x_n-x_m,a\|)} \phi(t)dt$
implies, $\lim_{n,m\to\infty} \psi(\|x_{n+1}-x_{m+1}, a\|) \leq \lim_{n,m\to\infty} \psi((\alpha+\beta+\gamma+2\delta)\|x_n - x_m, a\|)$
implies, $\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, a\| \leq \lim_{n,m\to\infty}(\alpha + \beta + \gamma + 2\delta)\|x_{n+1} - x_{m+1}, a\|$
implies, $\lim_{n,m\to\infty} \|x_n - x_m, a\| = 0$.
Thus $\{x_n\}$ is a Cauchy sequence. There exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$.
Again since, $\|T_i x - x, a\| \leq \|T_i x - x_n, a\| + \|x_n - x, a\|$,
we have
$\lim_{n\to\infty} \int_0^{\psi(\|T_i x-x,a\|)} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{\psi(\|T_i x-x_n,a\|)} \phi(t)dt$

$$= \lim_{n\to\infty} \int_0^{\psi(\|T_i x-T_j x_{n-1},a\|)} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{\psi(\eta(x,x_{n-1},a))} \phi(t)dt, \quad (4.13)$$

where,
$\eta(x, x_{n-1}, a)$
$= \alpha\|x - x_{n-1}, a\| + \beta \max\{\|x - x_{n-1}, a\|, \|x - T_i x, a\|, \|x_{n-1} - T_j x_{n-1}, a\|\}$
$+ \gamma \frac{\min\{\|x-T_j x_{n-1},a\|,\|x_{n-1}-T_i x,a\|\}}{1+\max\{\|x-T_j x_{n-1},a\|,\|x_{n-1}-T_i x,a\|\}} + \delta(\|x - T_j x_{n-1}, a\| + \|x_{n-1} - T_i x, a\|)$
implies, $\lim_{n\to\infty} \eta(x, x_{n-1}, a)$
$= \alpha.0 + \beta\|x - T_i x, a\| + \gamma.0 + \delta \lim_{n\to\infty} \|x_{n-1} - T_i x, a\|$
$\leq (\beta + \delta) \lim_{n\to\infty} \|x_{n-1} - T_i x, a\|$.
So from (4.13) we get,
$\lim_{n\to\infty} \int_0^{\psi(\|T_i x-x,a\|)} \phi(t)dt \leq \lim_{n\to\infty} \int_0^{\psi((\beta+\delta)\|x_{n-1}-T_i x,a\|)} \phi(t)dt$
implies, $\lim_{n\to\infty} \psi(\|T_i x - x, a\|) \leq \lim_{n\to\infty} \psi((\beta + \delta)\|x_{n-1} - T_i x, a\|)$
implies, $\lim_{n\to\infty} \|T_i x - x, a\| \leq \lim_{n\to\infty}(\beta+\delta)\|x_{n-1} - T_i x, a\| = (\beta+\delta)\|x - T_i x, a\|$
implies, $\|x_{n-1} - T_i x, a\| = 0$[otherwise, it will lead to a contradiction]
implies, $T_i x = x$.
Thus x is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. Now let, y be another common fixed point.
Then

$$\int_0^{\psi(\|x-y,a\|)} \phi(t)dt = \int_0^{\psi(\|T_i x-T_j y,a\|)} \phi(t)dt \leq \int_0^{\psi(\eta(x,y,a))} \phi(t)dt, \quad (4.14)$$

where,
$\eta(x, y, a)$
$= \alpha\|x - y, a\| + \beta \max\{\|x - y, a\|, \|x - T_i x, a\|, \|y - T_j y, a\|\}$
$+ \gamma \frac{\min\{\|x-T_j y,a\|,\|y-T_i x,a\|\}}{1+\max\{\|x-T_j y,a\|,\|y-T_i x,a\|\}} + \delta(\|x - T_j y, a\| + \|y - T_i x, a\|)$

$= \alpha\|x-y,a\| + \beta\max\{\|x-y,a\|, \|x-x,a\|, \|y-y,a\|\}$
$+ \gamma\frac{\min\{\|x-y,a\|,\|y-x,a\|\}}{1+\max\{\|x-y,a\|,\|y-x,a\|\}} + \delta(\|x-y,a\| + \|y-x,a\|)$
$\leq \alpha\|x-y,a\| + \beta\|x-y,a\| + \gamma\|x-y,a\| + 2\delta\|x-y,a\|$
$= (\alpha+\beta+\gamma+2\delta)\|x-y,a\|$.
Therefore from (4.14)

$$\int_0^{\psi(\|x-y,a\|)}\phi(t)dt \leq \int_0^{\psi((\alpha+\beta+\gamma+2\delta)\|x-y,a\|)}\phi(t)dt$$

implies, $\psi(\|x-y,a\|) \leq \psi((\alpha+\beta+\gamma+2\delta)\|x-y,a\|)$
implies, $\|x-y,a\| \leq (\alpha+\beta+\gamma+2\delta)\|x-y,a\|$
implies, $\|x-y,a\| = 0$ implies, $x = y$.
Therefore $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point in X. □

Corollary 4.11. *Let T_1 and T_2 be two self-maps on X satisfy the relation*

$$\int_0^{\psi(\|T_1x-T_2y,a\|)}\phi(t)dt \leq \int_0^{\psi(\eta(x,y,a))}\phi(t)dt,$$

where,

$$\eta(x,y,a) = \alpha\|x-y,a\| + \beta\max\{\|x-y,a\|, \|x-T_2y,a\|, \|y-T_1x,a\|\}$$
$$+\gamma\frac{\min\{\|x-T_2y,a\|,\|y-T_1x,a\|\}}{1+\max\{\|x-T_2y,a\|,\|y-T_1x,a\|\}} + \delta(\|x-y,a\|+\|y-T_2y,a\|), \alpha+\beta+\gamma+2\delta < 1.$$

Then T_1 and T_2 have a unique common fixed point.

Proof. Putting $T_i = T_1$ and $T_j = T_2$ in **Theorem 4.10** we get the required result. □

Corollary 4.12. *Let T be a self-map on X satisfy the relation*

$$\int_0^{\psi(\|Tx-Ty,a\|)}\phi(t)dt \leq \int_0^{\psi(\eta(x,y,a))}\phi(t)dt,$$

where,

$$\eta(x,y,a) = \alpha\|x-y,a\| + \beta\max\{\|x-y,a\|, \|x-Ty,a\|, \|y-Tx,a\|\}$$
$$+\gamma\frac{\min\{\|x-Ty,a\|,\|y-Tx,a\|\}}{1+\max\{\|x-Ty,a\|,\|y-Tx,a\|\}} + \delta(\|x-y,a\|+\|y-Ty,a\|), \alpha+\beta+\gamma+2\delta < 1.$$

Then T have a unique common fixed point.

Proof. Putting $T_i = T_j = T$ in **Theorem 4.10** we get the desired result. □

Theorem 4.13. *Let $\{T_i\}_{i=1}^{\infty}$ be the sequence of self-maps on X such that each of them is F-contraction satisfying the condition,*

$$\int_0^{\tau+F(\|T_ix-T_jy,a\|)}\phi(t)dt \leq \int_0^{F(\|x-y,a\|)}\phi(t)dt,$$

where $F \in \boldsymbol{F}$ and $\tau > 0$. Then $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\{x_n\}$ be a sequence constructed as **Theorem 4.7** where $x_{n+1} = T_ix_n$ for all $n \in \mathbb{N} \cup \{0\}$.

If $x_n = T_ix_n$ i.e., $x_{n+1} = x_n$, then x_n is a common fixed point of $\{T_i\}_{i=1}^{\infty}$ and the proof is over. So we assume that $x_{n+1} \neq x_n$.
Now,

$$\int_0^{F(\|x_{n+1}-x_n,a\|)}\phi(t)dt \leq \int_0^{\tau+F(\|T_ix_n-T_jx_{n-1},a\|)}\phi(t)dt \leq \int_0^{F(\|x_n-x_{n-1},a\|)}\phi(t)dt$$

implies, $F(\|x_{n+1}-x_n,a\|) \leq F(\|x_n-x_{n-1},a\|)$
implies, $\|x_{n+1}-x_n,a\| \leq \|x_n-x_{n-1},a\|$.
Therefore,

$\|x_{n+1}-x_n,a\| \le \|x_n-x_{n-1},a\| \le \|x_{n-1}-x_{n-2},a\| \le \ldots \le \|x_1-x_0,a\|$.
Thus $\{\|x_{n+1}-x_n,a\|\}$ is a monotone decreasing sequence of real numbers and bounded below and so is convergent.
Suppose $\lim_{n\to\infty}\|x_{n+1}-x_n,a\| = r \neq 0$.
Then
$\int_0^{F(r)}\phi(t)dt = \lim_{n\to\infty}\int_0^{F(\|x_{n+1}-x_n,a\|)}\phi(t)dt$
$\le \lim_{n\to\infty}\int_0^{\tau+F(\|T_ix_n-T_jx_{n-1},a\|)}\phi(t)dt \le \lim_{n\to\infty}\int_0^{F(\|x_n-x_{n-1},a\|)}\phi(t)dt$
$= \int_0^{F(r)}\phi(t)dt$,
where is a contradiction.
Therefore,
$\lim_{n\to\infty}\|x_{n+1}-x_n,a\| = 0$.
Now let $n > m \in \mathbb{N}$, then
$\lim_{n,m\to\infty}\int_0^{F(\|x_{n+1}-x_{m+1},a\|)}\phi(t)dt = \lim_{n,m\to\infty}\int_0^{\tau+F(\|T_ix_n-T_jx_m,a\|)-\tau}\phi(t)dt$
$\le \lim_{n,m\to\infty}\int_0^{F(\|x_n-x_m,a\|)-\tau}\phi(t)dt$
$\le \lim_{n,m\to\infty}\int_0^{F(\|x_{n-1}-x_{m-1},a\|)-2\tau}\phi(t)dt$
$\le \lim_{n,m\to\infty}\int_0^{F(\|x_{n-2}-x_{m-2},a\|)-3\tau}\phi(t)dt$
$\vdots$
$\le \lim_{n,m\to\infty}\int_0^{F(\|x_{n-m-1}-x_0,a\|)-(m+1)\tau}\phi(t)dt$
implies, $\lim_{n,m\to\infty}F(\|x_{n+1}-x_{m+1},a\|) \le \lim_{n,m\to\infty}F(\|x_{n-m-1}-x_0,a\|)-(m+1)\tau$
implies, $\lim_{n,m\to\infty}F(\|x_{n+1}-x_{m+1},a\|) = -\infty$
implies, $\lim_{n,m\to\infty}\|x_{n+1}-x_{m+1},a\| = 0$ i.e., $\lim_{n,m\to\infty}\|x_n-x_m,a\| = 0$.
Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists an $x \in X$ such that $\lim_{n\to\infty}\|x_n-x,a\| = 0$.
Again,
$\lim_{n\to\infty}\int_0^{F(\|T_ix-x,a\|)}\phi(t)dt \le \lim_{n\to\infty}\int_0^{F(\|T_ix-x_n,a\|+\|x_n-x,a\|)}\phi(t)dt$
$= \lim_{n\to\infty}\int_0^{\tau+F(\|T_ix-x_n,a\|)}\phi(t)dt = \lim_{n\to\infty}\int_0^{F(\|T_ix-T_jx_{n-1},a\|)}\phi(t)dt$
$\le \lim_{n\to\infty}\int_0^{F(\|x-x_{n-1},a\|)}\phi(t)dt$
implies, $\lim_{n\to\infty}F(\|T_ix-x,a\|) \le \lim_{n\to\infty}F(\|x-x_{n-1},a\|)$
implies, $\lim_{n\to\infty}\|T_ix-x,a\| \le \lim_{n\to\infty}\|x-x_{n-1},a\| = 0$
i.e., $\|T_ix-x,a\| = 0$ implies, $T_ix = x$.
Therefore, x is a common fixed point of $\{T_i\}_{i=1}^{\infty}$.
Now we show that x is unique. Let $y \neq x$ be a common fixed point.
Since,
$\int_0^{F(\|x-y,a\|)}\phi(t)dt \le \int_0^{\tau+F(\|T_ix-T_jy,a\|)}\phi(t)dt \le \int_0^{F(\|x-y,a\|)}\phi(t)dt$,
which is again a contradiction.
Thus the sequence of self-maps $\{T_i\}_{i=1}^{\infty}$ have a common fixed point in X. □

Corollary 4.14. *Let T_1 and T_2 be two self-maps on X such that each of them is F-contraction satisfying the condition*

$$\int_0^{\tau+F(\|T_1x-T_2y,a\|)}\phi(t)dt \le \int_0^{F(\|x-y,a\|)}\phi(t)dt,$$

where $F \in \boldsymbol{F}$ and $\tau > 0$. Then T_1 and T_2 have a unique common fixed point.

Proof. Putting $T_i = T_1$ and $T_j = T_2$ in **Theorem 4.13** the result follows. □

Corollary 4.15. *Let T be a self-map on X such that T is F-contraction satisfying the condition,*

$$\int_0^{\tau+F(\|Tx-Ty,a\|)} \phi(t)dt \leq \int_0^{F(\|x-y,a\|)} \phi(t)dt,$$

where $F \in \boldsymbol{F}$ and $\tau > 0$. Then T have a unique common fixed point.

Proof. Putting $T_i = T_j = T$ in **Theorem 4.13** the result follows. □

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INITIAL VALUE PROBLEM FOR N^{TH} ORDER RANDOM DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we investigate initial value problem of the n^{th}- order random differential inclusions and prove the existence result through multi-valued version of Schaefer's fixed point theorem of Martelli using caratheodory condition.

1. Statement of the Problem

Let R denote the real line and let $J = [0, a]$ be a closed and bounded interval in R.

Consider the initial value problem of n^{th} order random differential inclusions

$$x^{(n)}(t, \omega) \in F(t, x(t, \omega), \omega) \quad a.e. \quad t \in J, \omega \in \Omega \tag{1.1}$$
$$x^{(i)}(0, \omega) = x_i \in R, \omega \in \Omega$$

where $F : J \times R \times \Omega \to 2^R, i \in \{0, 1, \cdots, n-1\}$ and 2^R is the class of all nonempty subsets of R.

By a random solution of problem (1.1) we mean a function $x \in AC^{n-1}(J, R, \Omega)$ whose n^{th} Derivative $x^{(n)}$ exists and is a member of $L^1(J, R, \Omega)$ in $F(t, x, \omega)$, there exists $av \in L^1(J, R, \Omega)$ such that $v(t, \omega) \in F(t, x(t, \omega), \omega)$ a.e $t \in J$, and $x^{(i)}(0, \omega) = x_i \in R, i = 0, 1, \ldots, n-1$, where $AC^{n-1}(J, R, \Omega)$ is the space of all continuous real-valued functions whose $(n-1)$ derivatives exist and are absolutely continuous on J.

The method of upper and lower solutions has been applied to the problem of nonlinear differential inclusions. In this direction, we quote some of the results of Heikkila and Laksmikantham [6], Halidias and Papagerogiou [5], Benchohra [1]. In this paper we apply the multi-valued version of

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Schaefer's fixed point theorem of Martelli [8] and prove the existence of solutions between the given lower and upper solutions, using the caratheodory condition.

2. Auxiliary Results

We quote the following fixed point theorem of Martelli useful to prove the main existence result.

Theorem 2.1. *Let $T : X \to KC(X)$ be a completely continuous multi-valued map. If the set $\varepsilon = \{u \in X : \lambda u \in Tu \quad \text{for some } \lambda > 1\}$ is bounded, then* T *has a fixed point.*

We need the following definitions.

Definition 2.2. A multi-valued map $F : J \to KC(R)$ is said to be measurable if for every $y \in R$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| :\in F(t)\}$ is measurable.

Definition 2.3. A multi-valued map $F : J \times R \times \Omega \to 2^R$ is said to be L^1- random caratheodory if

(i) $(t, \omega) \to F(t, x(t, \omega), \omega)$ is measurable for each $x \in R, \omega \in \Omega$.

(ii) $x \to F(t, x(t, \omega), \omega)$ is upper semi-continuous for almost all $t \in J$ and $\omega \in \Omega$.

(iii) for each real number k > 0, there exists a function $h_k \in L^1(J, R, \Omega)$ such that
$\|F(t, x(t, \omega), \omega)\| = \sup\{|v| : v \in F(t, x(t, \omega), \omega)\} \le h_k(t, \omega)$, a. e. $t \in J, \omega \in \Omega$ for all $x \in R$ with $|x| \le k$.

Denote

$$S_F^1(x, \omega) = \left\{v \in L^1(J, R, \Omega) : v(t, \omega) \in F(t, x(t, \omega), \omega) \text{ a.e. } t \in J, \omega \in \Omega\right\}$$

We have quote the following lemmas due to Lasota and Opial [7].

Lemma 2.4. *If* $\dim(X) < \infty$ *and* $F : J \times X \to KC(X)$ *then* $S_F^1 \ne \varnothing$ *for each* $x \in X$.

Lemma 2.5. *Let X be a Banach space, F an L^1 - Caratheodory multivalued map with $S_F^1 \ne \varnothing$ and $K : L^1(J, X) \to C(J, X)$ be a linear continuous mapping. Then the operator* $\mathrm{Ko}S_F^1 : C(J, X) \to \mathrm{KC}(C(J, X))$ *is a closed graph operator in $C(J, X) \times C(J, X)$.*

We define the partial ordering $\le$ in $W^{n,1}(J, R, \Omega)$ (the sobolev class of functions $x : J \to R$ for which $x^{(n-1)}$ are absolutely continuous and $x^{(n)} \in L^1(J, R, \Omega)\big)$ as follower. Let $x, y \in W^{n,1}(J, R, \Omega)$. Then we define $x \le y \Leftrightarrow x(t, \omega) \le y(t, \omega), \forall t \in J, \omega \in \Omega$. if $a, b \in W^{n,1}(J, R, \Omega)$ and $a \le b$. then we define an order interval $[a, b]$ in $W^{n,1}(J, R, \Omega)$ by

$$[a, b] = \left\{x \in W^{n,1}(J, R, \Omega) : a \le x \le b\right\}.$$

Definition 2.6. A function $\alpha \in W^{n,1} \to (J, R, \Omega)$ is called a lower solution of problem (1.1) if there exists

$$v_1 \in L^1(J, R, \Omega) \quad \text{with } v_1(t, w) \in F(t, \alpha(t, \omega), \omega) \text{ a.e. } t \in J, \omega \in \Omega.$$

We have that $\alpha^{(n)}(t, \omega) \leq v_1(t, \omega) \quad a.e. \quad t \in J, \omega \in \Omega$ and $\alpha^{(i)}(0, \omega) \leq x_i, i = 0, 1, \cdots, n-1$.
Similarly a function $\beta \in W^{n,1}(J, R, \Omega)$ is called an upper solution of problem (1.1) if there exists $v_2 \in L^1(J, R, \Omega)$ with $v_2(t, w) \in F(t, \beta(t, \omega), \omega)$ a. e. $t \in J, \omega \in \Omega$. We have that $\beta^{(n)}(t, \omega) \geq v_2(t, \omega) \quad a.e. \quad t \in J, \omega \in \Omega$ and $\beta^{(i)}(0, \omega) \geq x_i, i = 0, 1, \cdots, n-1$.

3. Existence Results

We consider the following assumptions to prove main result.

- (A_1) The multi-valued $F(t, x(t, \omega), \omega)$ has compact and convex values for each $(t, x, \omega) \in J \times R$.
- (A_2) $F(t, x(t, \omega), \omega)$ is L^1- random caratheodory.
- (A_3) The problem (1.1) has a lower random solution α and an upper random solution β with $\alpha \leq \beta$.

Theorem 3.1. *Assume that $(A_1) - (A_3)$ hold. Then the problem (1.1) has at least one random solution x such that*

$$\alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega), \text{for all } t \in J, \omega \in \Omega.$$

Proof. First, we transform problem (1.1) into a fixed point inclusion in a suitable Banach space. Consider the problem

$$\begin{aligned} x^{(n)}(t, \omega) \in F(t, \tau x(t, \omega), \omega) \quad a.e. t \in J, \omega \in \Omega \\ x^{(i)}(0, \omega) = x_i \in R \end{aligned} \tag{3.1}$$

For all $i \in \{0, 1, \dots\dots, n-1\}$, where $\tau : C(J, R) \to C(J, R)$ is the truncation operator defined by

$$(\tau x)(t, \omega) = \begin{cases} \alpha(t, \omega), & \text{if } x(t, \omega) < \alpha(t, \omega) \\ x(t, \omega), & \text{if } \alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega) \\ \beta(t, \omega), & \text{if } \beta(t, \omega) < x(t, \omega) \end{cases} \tag{3.2}$$

The problem of existence of a solution to problem (1.1) reduces to finding the solution of the integral inclusion

$$x(t, \omega) \in \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s, \tau x(s, \omega), \omega) ds, \quad t \in J, \omega \in \Omega \tag{3.3}$$

We study the integral inclusion (3.3) in the space $C(J, R)$ of all continuous real-valued functions on J with a supremum norm $\| \cdot \|_c$. Define a multi-valued map $T : C(J, R) \to 2^{C(J,R)}$ by

$$Tx = \left\{ u \in C(J, R) : u(t, \omega) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s, \omega) ds, v \in \overline{S_F^1}(\tau x) \right\} \tag{3.4}$$

where
$\overline{S_F^1}(\tau x) = \{v \in \overline{S_F^1}(\tau x) : v(t, \omega) \geq \alpha(t, \omega)$ a. e. $t \in A_1$ and $v(t, \omega) \leq \beta(t, \omega)$, a. e. $t \in A_2, \omega \in \Omega\}$
and

$$A_1 = \{t \in J : x(t, \omega) < \alpha(t, \omega) \leq \beta(t, \omega)\},$$
$$A_2 = \{t \in J : \alpha(t, \omega) \leq \beta(t, \omega) < x(t, \omega)\},$$
$$A_3 = \{t \in J : \alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega)\}$$

By Lemma 2.4, $\overline{S_F^1}(\tau x) \neq \varnothing$ for each $x \in C(J, R)$ which further yields that $\overline{S_F^1}(\tau x) \neq \varnothing$ for each $x \in C(J, R)$. Indeed, if $v \in \overline{S_F^1}(x)$ then the function $w \in L^1(J, R)$ defined by

$$w = \alpha \chi A_1 + \beta \chi A_2 + u \chi A_3$$

is in $\overline{S_F^1}(\tau x)$ by virtue of decomposability of w.

We shall show that the multi valued T satisfies all the conditions of theorem 3.1.

Step I. First ,we prove that T(x) is a convex subset of C(J, R) for each $x \in C(J, R)$ Let $u_1, u_2 \in T(x)$. Then there exists v_1 and v_2 in $\overline{S_F^1}(\tau x)$ such that

$$u_3(t, \omega) = \sum_{i=0}^{n-1} \frac{x_i t^{\to i}}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_j(s, \omega) ds, \quad j = 1, 2$$

since $F(t, x(t, \omega), \omega)$ has convex values, one has for $0 \leq k \leq 1$

$$|kv_1 + (1-k)v_2| (t, \omega) \in S_F^1(\tau x)(t, \omega), \forall t \in J, \omega \in \Omega$$

As a result we have

$$[ku_1 + (1-k)u_2] (t, \omega) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |kv_1(s, \omega) + (1-k)v_2(t, \omega)| \, ds$$

Therefore $[ku_1 + (1-k)u_2] \in Tx$ and consequently T has convex values in C(J, R).

Step II. T maps bounded sets into bounded sets in C(J, R). To see this, let B be a bounded set in C(J, R). Then there exists a real number r > 0 such that $\|x\| \leq r, \forall x \in B$.

Now for each $u \in Tx$, there exists a $v \in \overline{S^1_F}(\tau x)$ such that

$$u(t,\omega) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s,\omega) ds$$

Then for each $t \in J, \omega \in \Omega$.

$$\begin{aligned} | u(t,\omega) | &\leq \sum_{i=0}^{n-1} \frac{|x_i|\, a^i}{i!} + \int_0^t \frac{a^{n-1}}{(n-1)!} \mid v(s,\omega) \mid ds \\ &\leq \sum_{i=0}^{n-1} \frac{|x_i|\, a^i}{i!} + \int_0^t \frac{a^{n-1}}{(n-1)!} h_r(s,\omega) ds \\ &\leq \sum_{i=0}^{n-1} \frac{|x_i|\, a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_r\|\, L^{\rightarrow} \end{aligned}$$

This further implies that

$$\|u\|_C \leq \sum_{i=0}^{n-1} \frac{|x_i|\, a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_r\|\, L^1$$

For all $u \in Tx \subset \cup T(B)$. Hence $\cup T(B)$ is bounded.
Step III. Next, we show that T maps bounded sets into equi-continuous sets. Let B be a bounded set as in step II, and $u \in Tx$ for some $x \in B$. Then there exists $u \in \overline{S^1_F}(\tau x)$ such that

$$u(t,\omega) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s,\omega) ds$$

Then for any $t_1, t_2 \in J$ we have

$$\begin{aligned} &|u(t_1,\omega) - u(t_2,\omega)| \\ &\leq \left| \textstyle\sum_{i=0}^{n-1} \frac{x_i t_1^i}{i!} - \sum_{i=0}^{n-1} \frac{x_i i_2^i}{i!} \right| + \int_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!} v(s,\omega) ds - \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s,\omega) ds \mid \\ &\leq |q(t_1,\omega) - q(t_2,\omega)| + \left| \int_0^{t} \frac{(t_1-s)^{n-1}}{(n-1)!} v(s,\omega) ds - \int_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s,\omega) ds \right| \\ &\quad + \left| \int_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s,\omega) ds - \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s,\omega) ds \right| \\ &\leq |q(t_1,\omega) - q(t_2,\omega)| + \int_0^{t_1} \left| \frac{(t_1-s)^{n-1}}{(n-1)!} - \frac{(t_2-s)^{n-1}}{(n-1)!} \right| |v(s,\omega)| ds \\ &+ \left| \int_{t_1}^{t_2} \right| \frac{(t_2-s)^{n-1}}{(n-1)!} ||v(s,\omega)| ds| \\ &\leq |q(t_1,\omega) - q(t_2,\omega)| + |p(t_1,\omega) - p(t_2,\omega)| \\ &\quad + \frac{1}{(n-1)!} \int_0^{t_1} \left| (t_1-s)^{n-1} - (t_2-s)^{n-1} \right| \|F(s,u(s,\omega),\omega)\| ds \\ &\leq |q(t_1,\omega) - q(t_2,\omega)| + |p(t_1,\omega) - p(t_2,\omega)| \\ &\quad + \frac{1}{(n-1)!} \int_0^{a} \left| (t_1-s)^{n-1} - (t_2-s)^{n-1} \right| h_r(s,\omega) ds \end{aligned}$$

where

$$q(t,\omega)=\sum_{i=0}^{n-1}\frac{x_it^i}{i!} \text{ and } p(t,\omega)=\int_0^t\frac{(a-s)^{n-1}}{(n-1)!}h_r(s,\omega)ds.$$

Now the functions p and q are continuous on the compact interval J, hence they are uniformly continuous on J. Hence we have

$$\mid u(t_1,\omega)-u(t_2,\omega)\mid\to 0\text{as } t_1\to t_2.$$

As a result $\cup T(B)$ is an equicontinuous set in $C(J,R)$. Now an application of Arzela-ascoli theorem gets that the multi T is totally bounded on $C(J,R)$.

Step IV. Next, we prove that T has a closed graph. Let $\{x_n\}\subset C(J,R)$ be a sequence such that $x_n\to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n\in Tx_n$ for each $n\in N$ such that $\underline{y_n}\to y_*$. We just show that $y_*\in Tx_*$. Since $y_n\in Tx_n$, there exists $av_n\in\overline{S_F^1}(\tau x_n)$ such that

$$y_n(t,\omega)=\sum_{i=0}^{n-1}\frac{x_it^i}{i!}+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}v_n(s,\omega)ds$$

Consider the linear and continuous operator $\kappa: L^1(J,R)\to C(J,R)$ defined by

$$\kappa v(t,\omega)=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}v(s,\omega)ds$$

Now

$$\begin{aligned}\left|y_n(t,\omega)-\textstyle\sum_{t=0}^{n-1}\frac{|x_i|t^i}{i!}-y_*(t,\omega)-\sum_{t=0}^{n-1}\frac{|x_i|t^i}{i!}\right|\\ \le|y_n(t,\omega)-y_*(t,\omega)|\\ \le\|y_n\to y_*\|\,c\to 0\text{ as } n\to\infty\end{aligned}$$

From Lemma 2.5 it follows that $\Big(\kappa\circ\overline{S_F^1}$) is a closed graph operator and from the definition of κ one has

$$y_n(t,\omega)-\sum_{i=0}^{n-1}\frac{x_it^i}{i!}\in\Big(\kappa\circ\overline{S_F^1}(\tau x_n)\Big).$$

As $x_n\to x_*$ and $y_n\to y_*$ there is a $v_*\in\overline{S_F^1}(\tau x_n)$ such that

$$y_*=\sum_{i=0}^{n-1}\frac{x_it^i}{i!}+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}v_*(s,\omega)ds.$$

Hence the multi T is an upper semi-continuous operator on C(J, R).

Step V**.** Finally we show that the set

$$\varepsilon=\{x\in C(J,R):\lambda x\in Tx\text{ for some }\lambda>1\}$$

is bounded. Let $u \in \varepsilon$ be any element. Then there exists a $av \in \overline{S_F^1}(\tau x)$ such that

$$u(t,\omega) = \lambda^{-1} \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \lambda^{-1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s,\omega) ds$$

Then

$$|u(t,\omega)| \leq \sum_{i=0}^{n-1} \frac{|x_i|\, a^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |v(s,\omega)| ds$$

Since $\tau x \in |\alpha, \beta|, \forall x \in C(J,R)$, we have

$$\|\tau x\|_c \leq \|\alpha\|_c + \|\beta\|_c := l$$

By (A2) there is a function $h_l \in L^1(J,R)$ such that

$$\|F(t,\tau x,\omega)\| = \sup\{|u| : u \in F(t,\tau x,\omega)\} \leq h_l(t,\omega) \quad \text{a.e. } t \in J, \omega \in \Omega\}$$

for all $x \in C(J,R)$. therefore

$$\|u\|_C \leq \sum_{i=0}^{n-1} \frac{|x_i|\, a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \int_0^a h_l ds = \sum_{i=0}^{n-1} \frac{|x_i|\, a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_l\|\, L^{\to 1}$$

And so, the set ε is bounded in C(J, R). Thus T satisfies all the conditions of theorem 2.1 and so an application of this theorem yields that the multi T has a random fixed point. Consequently (3.2) has a random solution u on J.

Next we show that u is also a solution of (1.1) on J. First we show that $u \in [\alpha, \beta]$ Suppose not. The either $\alpha \not\leq u$ or $u \not\leq \beta$ on some subinterval J^1 of J. if $u \not\geq \alpha$

Then there exist $t_0, t_1 \in J, t_0 < t_1$ such that $u\,(t_0, \omega) = \alpha\,(t_0, \omega)$ and $\alpha(t,\omega) > u(t,\omega)$ for all $t \in (t_0, t_1) \subset J$. From the definition of the operator τ it follows that

$$u^{(n)}(t,\omega) \in F(t, \alpha(t,\omega), \omega) \quad \text{a.e.} \quad t \in J, \omega \in \Omega$$

Then there exists a $v(t,\omega) \in F(t,\alpha(t,\omega),\omega)$ such that $v(t,\omega) \geq v_1(t,\omega), \forall t \in J, \omega \in \Omega$. with

$$u^{(n)}(t,\omega) = v(t,\omega) \text{ a.e. } t \in J, \omega \in \Omega$$

Integrating from t_0 to t, n times yields

$$u(t,\omega) - \sum_{i=0}^{n-1} \frac{u_i(0,\omega)\,(t-t_0)^i}{i!} = \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} v(s,\omega) ds$$

since α is a lower solution of (1.1) we have

$$\begin{aligned} u(t,\omega) &= \sum_{i=0}^{n-1} \frac{u_i(0,\omega)\,(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} v(s,\omega)ds \\ &\geq \sum_{i=0}^{n-1} \frac{\alpha_i(0,\omega)\,(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \alpha(s,\omega)ds \\ &= \alpha(t,\omega) \end{aligned}$$

For all $t \in (t_0, t_1)$. This is a contradiction. Similarly if $u \not\leq \beta$ on some subinterval of J, then also we get a contradiction. Hence $\alpha \leq u \leq \beta$ on J. As a result (3.2) has a random solution u in $[\alpha, \beta]$. Finally since $\tau x = x, \forall x \in [\alpha, \beta], u$ is a required random solution of (1.1) on J

□

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EXTENSIONS OF RAMANUJAN'S THREE SERIES FOR $1/\pi$ AND RELATED SERIES

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Abstract. The paper deals with extensions of Ramanujan's three series for $1/\pi$ and present some families of similar series for $1/\pi$ and $1/\pi^2$. Seven theorems of Wei and Dong are combined into unified formulae. Two new families of series are derived using a ${}_7F_6$ summation formula of Gasper and Rahman hypergeometric series approach.

1. Prehistory of Ramanujan's series

Human beings are endowed with an innate capacity to be fascinated by figures (shape) and numbers (magnitude), the twin topics of mathematics. Pythagoras remarked that of all solids the sphere was the most beautiful; of all plane figures, the circle. The investigation of the relation between the circumference (C) of a circle and its diameter (D) began in antiquity. Numerical estimates of the ratio $\frac{C}{D}$, now denoted by π, found in the records of ancient civilizations point to the early realization of the fact that the ratio was constant irrespective of the size of the circle (proved in Euclid's *Elements,* Bk.XII, Prop.2). The computation of π is probably the only common mathematical activity between the ancient and modern mathematicians. Archimedes initiated a systematic approach to compute π by calculating the perimeters of two regular polygons — one circumscribing and the other inscribed in the circle. He thereby inferred the estimate: $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Archimedean algorithm was employed by mathematicians for centuries using regular polygons with more and more vertices (and sides) for better approximation. But the polygonal method had its limit and any further precision could only be achieved through some novel method. As luck would have it, the discovery of the device of infinite series was made.

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The Madhava-Gregory series for the arctan function came to be developed which made it possible to approximate π more efficiently and with greater precision. Leibniz discovered his celebrated series for $\pi/4$ in 1673[1]. Though simple and beautiful, it is practically useless due to its slow speed. Euler developed a technique for accelerating the convergence of slow alternating series and applied his scheme to the Leibniz series thereby obtaining a faster series with positive terms only.

Ramanujan's remarkably rapid series for $1/\pi$ published in 1914 earned him a permanent place in the history of π. Indian mathematicians in ancient and medieval times neither explained the 'derivation' nor gave 'demonstration' of their formulae. Ramanujan too recorded his series without explicit derivation or any clue to proof. J. M. Borwein and P. B. Borwein were able to prove all 17 formulae of Ramanujan in a 1987 paper (see survey article [1]). Guillera [11] proved some Ramanujan-type formulae for $1/\pi$ using the WZ-method created by H. Wilf and D. Zeilberger.

1.1. **Morphology of Ramanujan's series.** Ramanujan had no notation for hypergeometric series and stated his formulas by writing out the first few terms in each series.[3, p.8] A compact notation often obscures the aesthetic beauty; Ramanujan's style made the elegance of his formulas noticeable. We may group his seventeen series for $1/\pi$ [17] by Pochhammer symbols:

$$R_2 = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!^3} (An+a)\,\alpha^n. \quad (28)-(30) \tag{1.1a}$$

$$R_3 = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n}{n!^3} (Bn+b)\,\beta^n. \quad (31)-(32) \tag{1.1b}$$

$$R_6 = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} (Cn+c)\,\gamma^n. \quad (33)-(34) \tag{1.1c}$$

$$R_4 = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} (Dn+d)\,\delta^n. \quad (35)-(39) \tag{1.1d}$$

$$R_5 = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} (En+f)\,\epsilon^n. \quad (40)-(44) \tag{1.1e}$$

[1]By the end of the year 1673 he [Leibniz] had in all probability discovered the equivalence $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$. A. Rupert Hall, *Philosophers at War: The quarrel between Newton and Leibniz*, Cambridge University Press, 1980, p.53

1.1.1. *Predecessors of Ramanujan.* It may be pointed out that Ramanujan's series are not unprecedented (see [14]). In §15 of [17], Ramanujan noted that the series which he employed to derive his series for $1/\pi$ is very closely connected with the perimeter of an ellipse. Ramanujan cites the following formula (eq. 45) wherein κ is the eccentricity of ellipse:

$$P = 2\pi a \left\{ 1 - \frac{1}{2^2}\kappa^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2}\kappa^4 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2}\kappa^6 - \dots \right\}$$

due to C. Maclaurin (*A Treatise on Fluxions,* vol. 2, 1742).[4, p.146, (3.1)] We can visualize that at $\kappa = 1$ (limiting case), ellipse degenerates into a pair of overlapping straight lines with combined length $P = 4a$ yielding:

$$\frac{2}{\pi} = 1 - \frac{1}{2^2} - \frac{1^2 3}{2^2 4^2} - \frac{1^2 3^2 5}{2^2 4^2 6^2} - \frac{1^2 3^2 5^2 7}{2^2 4^2 6^2 8^2} - \cdots .$$

E. Catalan[5, p.140, eq. L] had discovered in 1858 the following series which was again derived by Forsyth [6] in 1883:

$$\frac{4}{\pi} = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 + \cdots$$

Bauer [2, §4, p.110] obtained in 1859 the following alternating series:

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3 - 13\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^3 + - \cdots$$

It is noted on p.92, Chapter XII of the Manuscript Book 1, on p.118, Chapter X of the Manuscript Book 2 of Ramanujan (see [3, pp.23-4, Ex.14]) and was communicated (among others as his own formula) to Hardy in a letter of January 16, 1913. Hardy found the series "intriguing"[12, p.143(2), 144] though it already appeared in a 1875 book by I. Todhunter (*An Elementary Treatise on Laplace's functions, Lamé's functions and Bessel's functions,* p. 114, §145) who noted (p.116) that it came from *Crelle's Journal,* Vol. 56. Whipple too erroneously called it 'Ramanujan's series'[20, p.140].

Glaisher [10] obtained 18 series for $1/\pi$ in 1905 from the expansions of Complete Elliptic integrals. Levrie [13] used Bauer's method to obtain two general formulas. The author obtained many Forsyth-Glaisher type series by using only an elementary method [15].

Section 2 contains an extension of Ramanujan's series (28) by Wei and Gong. Section 3 employs their method but uses another identity due to Gosper for deriving a family of alternating series. Sections 4 deals with an extension of Ramanujan's series (40) by Wei and Gong using an identity of

Gasper. Section 5 uses an identity of Gasper and Rahman to obtain some new sums. Section 6 gives an extension of Ramanujan's series (35).

2. Extension of Ramanujan's series involving $(6n+1)4^{-n}$

Recall the generalized hypergeometric function defined by:

$${}_pF_q\left[\begin{matrix}a_1,\, a_2,\, \ldots,\, a_p\\ b_1,\, b_2,\, ,\ldots,\, b_q\end{matrix}\;\middle|\; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\cdot\frac{z^n}{n!}$$

where $a_i, b_j \in \mathbb{C}, b_j \neq 0, -1, -2, \ldots$ and $(\alpha)_n$ are rising shifted factorials defined by $(\alpha)_0 = 1,\ (\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, $n \in \mathbb{N}$. We will use these relations: $(a)_{n+m} = (a)_m\,(a+m)_n = (a)_n(a+n)_m$ and $\frac{(a+1)_n}{(a)_n} = \frac{a+n}{a}$. When $p = q+1$, the series ${}_pF_q$ converges for $|z| < 1$. It converges absolutely at $|z| = 1$ provided $\Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0$.

${}_pF_q$ is a useful tool in the evaluation of series. Whenever it reduces to a quotient of the products of the Gamma function, it yields some interesting result. Mathematicians derived numerous formulae involving π by employing some known hypergeometric series summation results.

Gessel and Stanton[9, eq. (1.7)] derived the following identity:

$${}_7F_6\left[\begin{matrix}2a, 2b, 1-2b, 1+(2a/3), a+d+n+(1/2), a-d, -n\\ a-b+1, a+b+(1/2), 2a/3, -2d-2n, 2d+1, 1+2a+2n\end{matrix}\;\middle|\; 1\right]$$
$$= \frac{(2a+1)_{2n}\,(d+b+\frac{1}{2})_n\,(d-b+1)_n}{(2d+1)_{2n}\,(a+b+\frac{1}{2})_n\,(a-b+1)_n}.$$

Let us change $2a$ to a, $2b$ to b and $a-d$ to c. Now we have: $(a)_{2n} = 2^{2n}\left(\dfrac{a}{2}\right)_n\left(\dfrac{a+1}{2}\right)_n$. [4, p.110, eq.(7.10)] Thus the above-noted identity becomes eq.(3) in Wei and Gong [19]:

$${}_7F_6\left[\begin{matrix}a, b, 1-b, 1+(a/3), a-c+n+(1/2), c, -n\\ 1+\frac{a-b}{2}, \frac{a+b+1}{2}, a/3, -a+2c-2n, 1+a-2c, 1+a+2n\end{matrix}\;\middle|\; 1\right]$$
$$= \frac{(\frac{1+a}{2})_n\,(1+\frac{a}{2})_n\,(\frac{1+a+b}{2}-c)_n\,(1+\frac{a-b}{2}-c)_n}{(\frac{1+a}{2}-c)_n\,(1+\frac{a}{2}-c)_n\,(\frac{1+a+b}{2})_n\,(1+\frac{a-b}{2})_n}.$$

Since $\dfrac{(a-c+n+(1/2))_k}{(1+a+2n)_k} \to 1/2$ and $\dfrac{(-n)_k}{(-a+2c-2n)_k} \to 1/2$ as $n \to \infty$, on letting $n \to \infty$ the previous formula transforms into:

$$
{}_5F_4\left[\begin{matrix} a,\ 1+(a/3),\ b,\ 1-b,\ c \\ a/3,\ 1+(a-b)/2,\ (1+a+b)/2,\ 1+a-2c \end{matrix}\,\middle|\, \frac{1}{4}\right]
$$
$$
= \frac{(\frac{1+a}{2})_n\,(1+\frac{a}{2})_n\,(\frac{1+a+b}{2}-c)_n\,(1+\frac{a-b}{2}-c)_n}{(\frac{1+a}{2}-c)_n\,(1+\frac{a}{2}-c)_n\,(\frac{1+a+b}{2})_n\,(1+\frac{a-b}{2})_n}. \tag{2.1}
$$

Choosing $a=\frac{1}{2}+2p, b=\frac{1}{2}+2q, c=\frac{1}{2}+r$ with $p,q,r \in \mathbb{Z}$ such that $\min\{p+q, p-q\} \geq 0$, (2.1) yields:

$$
\sum_{k=0}^{\infty} \frac{(\frac{1}{2}+2p)_k\,(\frac{1}{2}+2q)_k\,(\frac{1}{2}-2q)_k\,(\frac{1}{2}+r)_k}{k!\,(1+p+q)_k\,(1+p-q)_k\,(\frac{1}{2}+2p-2r)_k} \cdot \frac{6k+4p+1}{(4p+1)\,4^k}
$$
$$
= \frac{(p+q)!\,(p-q)!(\frac{1}{4})_{p-r}\,(\frac{3}{4})_{p-r}}{(\frac{1}{2})_{p+q-r}\,(\frac{1}{2})_{p-q-r}\,(\frac{3}{4})_p(\frac{1}{4})_{p+1}} \cdot \frac{1}{\pi}.
$$

The equation (Theorem 1 of Wei and Gong) gives a formula for $\frac{1}{\pi}$ with three free parameters. Obviously p,q cannot be both negative. With $p=q=r=0$, it reduces to Ramanujan's series (28):

$$
\sum_{n=0}^{\infty} \frac{\left((\frac{1}{2})_n\right)^3 (6n+1)}{n!^3\,2^{2n}} = \sum_{n=0}^{\infty} \frac{(6n+1)\binom{2n}{n}^3}{2^{8n}} = \frac{4}{\pi}.
$$

Again, choosing $a=\frac{3}{2}+2p, b=\frac{3}{2}+2q, c=\frac{1}{2}+r$ in (2.1), Wei and Gong similarly deduced their Theorem 2 involving $6k+4p+3$.

We may unify their two theorems by taking $p=q$ and $r=2p$, thereby getting (for $m=0,1,2,\dots$) cubic power of one parameter only:

Theorem 2.1.

$$
\sum_{n=0}^{\infty} \frac{\left((\frac{2m+1}{2})_n\right)^3 (6n+2m+1)}{n!^2\,(n+m)!\,2^{2n}} = \frac{2^{3m+2}}{(2m-1)!!\pi},
$$

where $(2m-1)!! = 1\cdot 3\cdots(2m-1)$.

The choice $q=q$ and $r=0$, yields a formula containing one fixed and two other distinct half parameters with sum 1:

Theorem 2.2.

$$
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n\,(\frac{2m+1}{2})_n\,(\frac{-2m+1}{2})_n\,(6n+2m+1)}{n!^2\,(n+m)!\,2^{2n}}
$$

$$= \frac{2^{m+2}}{(2m-1)!!\pi}, \quad m = 0, 1, 2, \ldots$$
$$= (-1)^{-m} \frac{(-2m-1)!!}{2^{-m-2}\,\pi}, \quad m = -1, -2, -3, \ldots \textit{(sum starts from } n\text{=-}m.)$$

We have these special cases for $m = 1, 2$ and for $m = -1, -2$:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n\,(\frac{3}{2})_n\,(\frac{-1}{2})_n\,(6n+3)}{n!^2\,(n+1)!\,2^{2n}} = \frac{8}{\pi}; \quad \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n\,(\frac{5}{2})_n\,(\frac{-3}{2})_n\,(6n+5)}{n!^2\,(n+2)!\,2^{2n}} = \frac{16}{3\pi}.$$
$$\sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n\,(\frac{-1}{2})_n\,(\frac{3}{2})_n\,(6n-1)}{n!^2\,(n-1)!\,2^{2n}} = -\frac{2}{\pi}; \quad \sum_{n=2}^{\infty} \frac{(\frac{1}{2})_n\,(\frac{-3}{2})_n\,(\frac{5}{2})_n\,(6n-3)}{n!^2\,(n-2)!\,2^{2n}} = \frac{3}{\pi}.$$

3. An alternating series involving $(6n+1)8^{-n}$

Gessel and Stanton [9, eq.(1.2)] mention that in a letter to R. Askey, R. Gosper communicated a list of hypergeometric series evaluations including:

$${}_5F_4\left[\begin{matrix} 2a,\ 2b,\ 1-2b,\ 1+(2a/3), -n \\ a-b+1,\ a+b+\frac{1}{2},\ 2a/3,\ 1+2a+2n \end{matrix} \,\middle|\, \frac{1}{4}\right]$$
$$= \frac{(a+\frac{1}{2})_n\,(a+1)_n}{(a+b+\frac{1}{2})_n\,(a-b+1)_n}.$$

As was done earlier, we change $2a$ to a, $2b$ to b and let $n \to \infty$ getting:

$${}_4F_3\left[\begin{matrix} a,\ b,\ 1-b,\ 1+(a/3) \\ 1+(a-b)/2,\ (a+b+1)/2,\ a/3 \end{matrix} \,\middle|\, \frac{-1}{8}\right]$$
$$= \frac{(\frac{1+a}{2})_n\,(1+\frac{a}{2})_n}{(\frac{1+a+b}{2})_n\,(1+\frac{a-b}{2})_n}. \tag{3.1}$$

Again with the same choice for a, b, namely, $a = \frac{1}{2} + 2p, b = \frac{1}{2} + 2q$ we straightway obtain a sum involving $6k + 4p + 1$

$$\sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{2}+2p)_k\,(\frac{1}{2}+2q)_k\,(\frac{1}{2}-2q)_k}{k!\,(1+p+q)_k\,(1+p-q)_k} \cdot \frac{6k+4p+1}{(4p+1)\,8^k}$$
$$= \frac{(p+\frac{3}{4})_k(p+\frac{5}{4})_k}{(1+p+q)_k\,(1+p-q)_k}.$$

Recall the relation: $\Gamma(3/4) \cdot \Gamma(5/4) = \frac{\pi}{2\sqrt{2}}$ for simplifying RHS.

The second choice, viz. $a = \frac{3}{2} + 2p, b = \frac{3}{2} + 2q$, yields a sum involving $6k + 4p + 3$. These are theorems 3 and 4 of Wei and Gong derived by them from ${}_7F_6$. The two results so derived can be combined together in the following formula for $m = 0, 1, 2, \ldots$

Theorem 3.1.

$$\sum_{n=0}^{\infty}(-1)^n \frac{(\frac{-2m+1}{2})_n \left((\frac{2m+1}{2})_n\right)^2 (6n+2m+1)}{n!^2\,(n+m)!\,2^{3n}} = \frac{2^{3m+1}m!\,\sqrt{2}}{(2m)!\,\pi}.$$

It gives with $m=2$:

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left((\frac{5}{2})_n\right)^2 (\frac{-3}{2})_n\,(6n+5)}{n!^2\,(n+2)!\,8^n} = \frac{32\,\sqrt{2}}{3\,\pi}.$$

The fixed choice $b=1/2$ in the first pair of values yields:

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left((\frac{1}{2})_n\right)^2 (\frac{4m+1}{2})_n(6n+4m+1)}{n!\,(n+m)!^2\,2^{3n}} = \frac{2^{6m+1}(2m)!\sqrt{2}}{(4m)!\,\pi}.$$

For $m=1$, we get:

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left((\frac{1}{2})_n\right)^2 (\frac{5}{2})_n\,(6n+5)}{n!\,(n+1)!^2\,8^n} = \frac{32\,\sqrt{2}}{3\,\pi}$$

Compare it with the previous series with identical sum.

4. Extension of Ramanujan's series involving $(8n+1)9^{-n}$

This is the series (no.40) of Ramanujan:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n\,(\frac{1}{2})_n\,(\frac{3}{4})_n\,(8n+1)}{n!^3\,3^{2n}} = \frac{2\sqrt{3}}{\pi}.$$

Gasper[7, eq(5.23)] obtained this identity:

$$\begin{aligned}
&{}_7F_6\left[{3a,\,1+(3a/4),\,3b,\,(1-3b)/2,\,1-(3b/2),\,c,\,2a+b-c \atop 3a/4,\,1+a-b,\,(1+3a+3b)/2,\,3(a+b)/2,\,1+3a-3c,\,1+3c-3a-3b};1\right]\\
&= \frac{\Gamma(3a+3b)\,\Gamma(1+3a-3c)\,\Gamma(a+2b-c)\,\Gamma(1+a-b)}{\Gamma(3a+1)\,\Gamma(3a+3b-3c)\,\Gamma(a+2b)\,\Gamma(1+a-b-c)}\\
&\times\left\{1+\frac{\sin 3\pi b\,\sin\pi c}{\sin 3\pi(a+b-c)\,\sin\pi(a+2b)}\right\}.
\end{aligned}$$

Replacing c by $-n$ and simplifying RHS, Wei and Gong expressed it as

$$\begin{aligned}
&{}_7F_6\left[{3a,\,1+(3a/4),\,3b,\,(1-3b)/2,\,1-(3b/2),\,-n,\,2a+b+n \atop 3a/4,\,1+a-b,\,(1+3a+3b)/2,\,3(a+b)/2,\,1+3a+3n,\,1-3a-3b-3n};1\right]\\
&= \frac{(a+2b)_n\,(a+\frac{1}{3})_n\,(a+\frac{2}{3})_n\,(a+1)_n}{(1+a-b)_n\,(a+b)_n\,(a+b+\frac{1}{3})_n\,(a+b+\frac{2}{3})_n},
\end{aligned}$$

using $\Gamma(a+n)=(a)_n\Gamma(a)$ and $\Gamma(3x)=\frac{1}{2\pi}3^{2x-1/2},\Gamma(x)\,\Gamma(x+\frac{1}{3})\,\Gamma(x+\frac{2}{3})$.

Letting $n \to \infty$ in this expression, they obtained:

$$
{}_5F_4\left[\begin{matrix} 3a,\ 1+(3a/4),\ 3b,\ (1-3b)/2,\ 1-(3b/2) \\ 3a/4,\ 1+a-b,\ (1+3a+3b)/2,\ 3(a+b)/2 \end{matrix} \,\middle|\, \frac{1}{9}\right]
= \frac{\Gamma(a+b)\,\Gamma(1+a-b)\,\Gamma(a+b+\frac{1}{3})\,\Gamma(a+b+\frac{2}{3})}{\Gamma(a+2b)\,\Gamma(a+\frac{1}{3})\,\Gamma(a+\frac{2}{3})\,\Gamma(a+1)}.
$$

Using this and following the same technique as above, Wei and Gong [19] obtained three more formulae (their theorems 5, 6 and 7) with $(8k+6p+1)$ via $a = \frac{1}{6}+p, b = \frac{1}{6}+q$ followed with $(8k+6p+3)$ via $a = \frac{1}{2}+p, b = \frac{1}{2}+q$ and $(8k+6p+5)$ via $a = \frac{5}{6}+p, b = \frac{5}{6}+q$. Their three formulae can be unified into a general formula for $m = 0, 1, 2, \ldots$:

Theorem 4.1.

$$
\sum_{n=0}^{\infty} \frac{(\frac{-2m+1}{4})_n\,(\frac{2m+1}{2})_n\,(\frac{-2m+3}{4})_n(8n+2m+1)}{n!^2\,(n+m)!\,3^{2n}} = \frac{2^{3m+1}\sqrt{3}}{3^m\,(2m-1)!!\pi}.
$$

5. Two families of series involving $(4n+1)9^{-n}$

We now take up an identity due to Gasper and Rahman [8, eq.(1.6)]:

$$
{}_7F_6\left[\begin{matrix} a,\ a+(1/2),\ (a/2)+1,\ b,\ 1-b,\ c,\ -c+(2a+1)/3 \\ 1/2,\ a/2,\ 1+(2a-b)/3,\ (2+2a+b)/3,\ 3c,\ 1+2a-3c \end{matrix} \,\middle|\, 1\right]
$$

$$
= \frac{2}{\sqrt{3}} \cdot \frac{\Gamma(\frac{3+2a-b}{3})\,\Gamma(\frac{2+2a+b}{3})}{\Gamma(\frac{2+2a}{3})\,\Gamma(\frac{3+2a}{3})} \cdot \frac{\Gamma(c+\frac{1}{3})\,\Gamma(c+\frac{2}{3})}{\Gamma(c+\frac{1+b}{3})\,\Gamma(c+\frac{2-b}{3})}
\times \frac{\Gamma(\frac{3+2a}{3}-c)\,\Gamma(\frac{2+2a}{3}-c)}{\Gamma(\frac{3+2a-b}{3}-c)\,\Gamma(\frac{2+2a+b}{3}-c)} \cdot \sin\frac{\pi(b+1)}{3}
$$

Replacing c by n and letting $n \to \infty$ in this expression, we have:

$$
LHS = {}_5F_4\left[\begin{matrix} a,\ a+(1/2),\ (a/2)+1,\ b,\ 1-b \\ 1/2,\ a/2,\ 1+(2a-b)/3,\ (2+2a+b)/3 \end{matrix} \,\middle|\, \frac{1}{9}\right]
$$

which on choosing $a = \frac{1}{2}+p, b = \frac{1}{2}+q$ becomes:

$$
\sum_{k=0}^{\infty} \frac{(\frac{1+2p}{2})_k\,(1+p)_k\,(\frac{1+2q}{2})_k\,(\frac{1-2q}{2})_k}{k!\,(\frac{1}{2})_k\,(\frac{4p-2q+7}{6})_k\,(\frac{4p+2q+7}{6})_k} \cdot \frac{4k+2p+1}{(2p+1)\,9^k}
$$

After taking the limit $n \to \infty$, we are left with:

$$
RHS = \frac{2}{\sqrt{3}} \cdot \frac{\Gamma(\frac{7+4p-2q}{6})\,\Gamma(\frac{7+4p+2q}{6})}{\Gamma(\frac{3+2p}{3})\,\Gamma(\frac{4+2p}{3})} \cdot \sin\frac{\pi(3+2q)}{6}.
$$

We thus evaluate RHS for $p = q = 0$:

$$RHS = \frac{2}{\sqrt{3}} \cdot \frac{\Gamma(\frac{7}{6})^2}{\Gamma(\frac{4}{3})} = \frac{\sqrt[3]{2}\,\left(\Gamma(\frac{1}{3})\right)^3}{4\sqrt{3}\,\pi}.$$

It results in the following sum:

$$\sum_{n=0}^{\infty} \frac{\left((\frac{1}{2})_n\right)^2 (4n+1)}{\left((\frac{1}{6})_n\right)^2 (6n+1)^2\, 3^{2n}} = \frac{\sqrt{3}\,\sqrt[3]{2}\,\left(\Gamma(\frac{1}{3})\right)^3}{12\,\pi}. \tag{5.1}$$

The choice $p = 1, q = 1$ yields:

$$\sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n\,(\frac{3}{2})_n\,(n+1)\,(4n+3)}{(\frac{1}{2})_n\,(\frac{13}{6})_n\,3^{2n}} = \frac{7\sqrt{3}\,\sqrt[3]{4}\,\left(\Gamma(\frac{1}{3})\right)^3}{64\,\pi}. \tag{5.2}$$

We get another family of series with sum of different type. For example, $p = 1, q = 0$ leads to the sum:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n, (\frac{3}{2})_n\,(n+1)\,(4n+3)}{\left((\frac{11}{6})_n\right)^2\, 3^{2n}} = \frac{25\sqrt[3]{2}\,\pi^2}{6\left(\Gamma(\frac{1}{3})\right)^3}. \tag{5.3}$$

And with $p = 0, q = 1$, we obtain:

$$\sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n\,(4n+1)}{(\frac{5}{6})_n\,3^{2n}} = \frac{\sqrt[3]{2}\,\pi^2}{\left(\Gamma(\frac{1}{3})\right)^3}. \tag{5.4}$$

6. Extension of Ramanujan's series with $(20n+3)(-4)^{-n}$

We found this extension of Ramanujan's (35) series:

Theorem 6.1.

$$\sum_{n=0}^{\infty}(-1)^n \frac{(\frac{-2m+1}{4})_n\,(\frac{1}{2})_n\,(\frac{-2m+3}{4})_n\,(20n+2m+3)}{n!^2\,(n+m)!\,2^{2n}}$$
$$= \frac{2^{m+3}}{(2m-1)!!\pi}, \quad m = 0, 1, 2, \ldots$$
$$= (-1)^{-m}\frac{(-2m-1)!!}{2^{-m-3}\,\pi}, \quad m = -1, -2, -3, \ldots\, (\textit{sum starts from } n\text{=-}m.)$$

Remark 6.2. The theorem has been derived empirically and tested for a large number of cases (m) and found valid. We have no formal proof of the theorem yet.

Ramanujan's series is a special case $m = 0$:

$$\sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{4})_n\,(\frac{1}{2})_n\,(\frac{3}{4})_n\,(20n+3)}{n!^2\,(n+0)!\,2^{2n}} = \frac{8}{\pi}.$$

Two other special cases are:

$$\sum_{n=0}^{\infty}(-1)^n \frac{(\frac{-1}{4})_n\,(\frac{1}{2})_n\,(\frac{1}{4})_n\,(20n+5)}{n!^2\,(n+1)!\,2^{2n}} = \frac{16}{\pi}.$$

$$\sum_{n=2}^{\infty}(-1)^n \frac{(\frac{5}{4})_n\,(\frac{1}{2})_n\,(\frac{7}{4})_n\,(20n-1)}{n!^2\,(n-2)!\,2^{2n}} = \frac{6}{\pi}.$$

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A CONTRACTIBLE GRAPH WITH ELEVEN VERTICES AND NO GLUABLE EDGE

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Abstract. Responding to a challenge posed by Frías-Armenta (2020), who gave an example of a contractible graph with thirteen vertices that does not have any gluable edge, we give a similar example with eleven vertices. These examples illustrate that Axiom 3.4 of Ivashchenko (1994) is violated, and so are several follow-up theorems.

1. Introduction

As an aid to study molecular spaces and digital topology, Ivashchenko (1985) [2] defined a family of contractible graphs. These contractible graphs can be constructed starting from the trivial graph $K(1)$ (that is, a graph with only one vertex and no edge) via contractible transformations, which allow: (1) deletion/gluing of a vertex when the subgraph induced by its neighbors is a contractible graph, and (2) deletion/gluing of an edge when the subgraph induced by the intersection of neighbors of its two end-points is a contractible graph. Such a vertex or an edge is called contractible, as are the subgraphs induced by the neighbors of the vertex and the joint neighbors of the end-points of the edge.

Ivashchenko proved that these transformations do not change the Euler characteristics of a graph (see [3]), or the graph homology (see [4]). He also derived several properties of contractible graphs in [5]. Most of the new results in [5] depend on Axiom 3.4, which was easily verified for contractible graphs with up to four vertices.

Axiom 3.4 of [5]**:** "For a contractible graph G, and a vertex $v \in G$ which has some non-adjacent vertices, there exists a non-adjacent vertex $u \in G$

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such that the subgraph induced by the intersection of their neighbors is a contractible graph, making edge $[uv]$ gluable."

When Axiom 3.4 holds, graph G can be extended to another contractible graph by gluing $[uv]$, the edge with end vertices u and v. If this axiom were always true, then by repeated application of this axiom, any contractible graph would expand to a complete graph by gluing the missing edges in some suitable sequence; conversely, any contractible graph would be obtainable from a complete graph by deleting edges in the reverse sequence. However, recognizing correctly that the claim could not be proved in general, Ivashchenko stated it as an axiom. Thereafter, assuming Axiom 3.4, he went on to prove several useful theorems.

Recently, in [1], Frías-Armenta constructed a contractible graph with thirteen vertices that does not satisfy Axiom 3.4 of [5]. He challenged the reader to construct a similar example with fewer vertices. We construct such an example with eleven vertices. Consequently, results 3.5 through 3.10 in [5], which depend on Axiom 3.4, do not hold for our example.

2. Some preliminaries on contractible transformations

What we wrote in words in Section 1, let us restate in mathematical terms. For a graph $G = (V(G), E(G))$ and a subset S of $V(G)$, let us define the joint neighbors of S as

$$N_G(S) = \bigcap_{v \in S} \{u \in V(G) : [uv] \in E(G)\}.$$

The induced graph of $N_G(S)$, denoted by $L_G(S)$, has vertex set S and edge set consisting of all adjacent pairs of vertices in S. For a singleton $S = \{v\}$, we denote $N_G(\{v\})$ and $L_G(\{v\})$ by $N_G(v)$ and $L_G(v)$ respectively.

As mentioned before, Ivashchenko [2] introduced a family of graphs whose members are called *contractible graphs*, if each member is obtainable from the trivial graph $K(1)$ through a sequence of contractible transformations.

Definition 2.1. A family $\mathcal{F}$ of graphs $G_1, G_2, \cdots, G_n, \cdots$ is called *contractible* if

(1) The trivial graph is in $\mathcal{F}$.

(2) Any graph of $\mathcal{F}$ can be obtained from the trivial graph by finite sequence of contractible transformations $\{T_1, T_2, T_3, T_4\}$ defined as follow:

T_1 (Deleting a vertex v): A vertex v of a graph G can be deleted, if $L_G(v) \in \boldsymbol{\mathcal{F}}$.

T_2 (Gluing a new vertex v): If a subgraph G_1 of a graph G is contractible; that is, if $G_1 \in \boldsymbol{\mathcal{F}}$, then the new vertex v, not in G, can be glued to G in such a way that $L_{G+v}(v) = G_1$, where the vertex-extended graph $G + v = (V(G) \cup \{v\}, E(G) \cup \{[uv] : u \in G_1\})$.

T_3 (Deleting an edge $[v_1v_2]$): An edge $[v_1v_2]$ of G can be deleted if $L_G(\{v_1, v_2\}) \in \boldsymbol{\mathcal{F}}$.

T_4 (Gluing a new edge $[v_1v_2]$): Let two vertices v_1 and v_2 of a graph G be non-adjacent. The edge $[v_1v_2]$ of G can be glued if $L_G(\{v_1, v_2\}) \in \boldsymbol{\mathcal{F}}$, where the edge-extended graph $G + [uv] = (V(G), E(G) \cup [uv])$.

Any graph $G \in \boldsymbol{\mathcal{F}}$ is called a *contractible graph.* A vertex v is deletable (gluable) if contractible transformation T_1 (T_2) can be applied to it. Similarly, an edge $[uv]$ is deletable (gluable) if contractible transformation T_3 (T_4) can be applied to it.

A complete graph is a contractible graph as it can be constructed by starting from $K(1)$ and successively gluing new vertices along with edges joining each new vertex to all existing vertices (that is, by using a finite number of transformations of type T_2). A triangulated graph (also known as a chordal graph) is a contractible graph as it can be constructed by starting from $K(1), K(2)$ and successively adding vertices along with edges joining each new vertex with one or two existing vertices to form each triangular face (that is, by using a finite number of transformations of type T_2). A cycle with four or more edges is non-contractible. Since each contractible transformation results in a connected graph, any contractible graph is connected.

3. A CONTRACTIBLE GRAPH WITH NO GLUABLE EDGE

Figure 1 shows a contractible graph on eleven vertices that violates Axiom 3.4 (in [5]) of Ivashchenko.

Let us show that G is a contractible graph: (1) The black part of the graph G is a triangulated graph, and hence contractible using transformations of type T_2. (2) To the black part, we glue contractible edges sequentially (these are transformations of type T_4): purple $[8, 10]$; green $[2, 9]$,

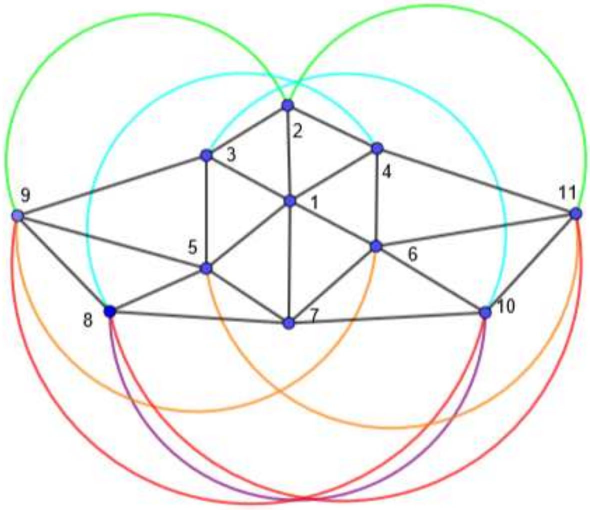

FIGURE 1. This graph G on 11 vertices violates Axiom 3.4 of [5].

$[2, 11]$; red $[9, 10]$, $[8, 11]$; brown $[6, 9]$, $[5, 11]$; and blue $[8, 4]$, $[3, 10]$. Thus, G is a contractible graph.

Now we shall show that G violates Axiom 3.4 in [5]. First, note that Vertex 2 is the only deletable vertex of G: This is because the neighborhood $L_G(2)$, being a path on vertices $\{9, 3, 1, 4, 11\}$, is contractible; but for any other vertex u, the neighborhood $L_G(u)$ is either disconnected or cyclic with five/six edges, hence not contractible. Moreover, even after Vertex 2 is deleted, no other vertex can be deleted. Second, note that Vertex 1, which has several non-adjacent vertices, does not admit any gluable edge through it, whether or not we delete Vertex 2: This is because for any vertex v non-adjacent to Vertex 1, the neighborhood $L_G(\{1, v\})$ is disconnected. Hence, G violates Axiom 3.4 in [5].

It is immediate that Theorem 3.5, Corollary 3.6, Theorem 3.7 and Theorem 3.8 in [5] do not hold for graph G. Theorem 3.9 in [5] does not hold because the induced graph generated by $N_G(2) \cup \{2\}$ does not admit a non-adjacent deletable vertex. Theorem 3.10 in [5] does not hold because subgraph G_1 induced by vertices $\{1, 5, 6, 7\}$ does not extend to G only by gluing contractible vertices.

We invite the reader to construct an example with fewer vertices that violate Axiom 3.4 of [5] or prove that ours is such an example with the fewest vertices.

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A NEW APPROACH TO GOOD MATRICES

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Abstract. In this paper we consider a particular type of partition of $\mathbb{Z}_n$, called skew H-partitions and H-partitions [7] and obtain a necessary and sufficient condition for existence of a set of one circulant and three symmetric back-circulant matrices of order n to obtain a good matrices of order $4n$ in terms of such partitions when $n(\geq 3)$ odd.

1. Introduction:

A $(1,-1)$ matrix H of order n is called a Hadamard matrix, if $HH^{'} = nI$, where $H^{'}$ is the transpose of H. If H is a Hadamard matrix of order n then $n = 2$ or $n \equiv 0(mod\ 4)$. The converse of this seems to be true and is known as the Hadamard conjecture.

Many exciting results have stemmed from the following basic idea put forward by Williamson. Consider the array

$$H = \begin{pmatrix} W & X & Y & Z \\ -X & W & -Z & Y \\ -Y & Z & W & -X \\ -Z & -Y & X & W \end{pmatrix}$$

If W, X, Y and Z are replaced by square matrices A, B, C and D of order n respectively, then H becomes a square matrix of order $4n$. Williamson proved that a sufficient condition for H to be a Hadamard matrix is that A, B, C and D are $(1,-1)$ matrices of order n with

$$AA' + BB' + CC' + DD' = 4nI \tag{1.1}$$

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and for every pair $X, Y \in \{A, B, C, D\}$ are circulant and symmetric such matrices satisfy

$$XY' = YX' \tag{1.2}$$

If A, B, C, D are $(1, -1)$ matrices of order n with the properties

(i) $XY' = YX'$ for $X, Y \in \{A, B, C, D\}$
(ii) $(A - I)' = -(A - I)$, $B' = B$, $C' = C$ and $D' = D$
(iii) $AA' + BB' + CC' + DD' = 4nI$

will be called good matrices[1],[10].
Also it can be observed that the array H becomes a skew-Hadamard matrix i.e. H^* is a skew symmetric matrix, where H^* is obtained from H by replacing principal diagonal entries of H by zero.
The basic difficulty lies in finding the matrices A, B, C and D which satisfy the above three properties of good matrices. In this article we give a necessary and sufficient condition for the existence of such matrices A, B, C and D. Our result also gives a method for finding a set of such matrices.

2. Definitions:

Definition 2.1. For any odd integer $n \geq 3$, let $\mathbb{Z}_n$ be the cyclic group of integers modulo n under addition. Let A_i be a proper subset of $\mathbb{Z}_n$ such that $0 \in A_i$ and $A_i = -A_i$. Then $\{A_i, B_i = \mathbb{Z}_n - A_i\}$ is clearly a partition of $\mathbb{Z}_n$ such that $B_i = -B_i$. We call such a partition of $\mathbb{Z}_n$ to be an **H-partition** of $\mathbb{Z}_n$

Addition in H-partition: For an H-partition $\{A_i, B_i\}$ of $\mathbb{Z}_n$, let $A_i + B_i = \{a + b(mod\ n) \mid a \in A_i,\ b \in B_i\}$. Let C_i denotes the set of distinct elements of $A_i + B_i$. For any $c \in C_i$ the frequency of occurrence of c in $A_i + B_i$ is denoted by n_c^i. Clearly $0 \notin C_i$ for any H-partition $\{A_, B_i\}$ of $\mathbb{Z}_n$.

Example 2.2. For $n = 7$, $\mathbb{Z}_7$ be the cyclic group of order 7.
Let

(1) $A_1 = \{0, 1, 2, 5, 6\}$, $B_1 = \{3, 4\}$.
(2) $A_2 = \{0, 2, 5\}$, $B_2 = \{1, 3, 4, 6\}$.

are H-partitions of $\mathbb{Z}_7$.
Addition of H partitions are given by

(1) $A_1+B_1 = \{\{0,1,2,5,6\}+\{3,4\} \mod 7\} = \{3,4,5,1,2,4,5,6,2,3\}$
$\Rightarrow A_1+B_1 = \{1,2,2,3,3,4,4,5,5,6\}$, $C_1 = \{1,2,3,4,5,6\}$
and n_c^1 are $1,2,2,2,2,1$ for $c = 1,2,3,4,5,6$ respectively.

(2) $A_2+B_2 = \{\{0,2,5\}+\{1,3,4,6\} \mod 7\} = \{1,3,4,6,3,5,6,1,6,1,2,4\}$
$\Rightarrow A_2+B_2 = \{1,1,1,2,3,3,4,4,5,6,6,6\}$, $C_2 = \{1,2,3,4,5,6\}$
and n_c^2 are $3,1,2,2,1,3$ for $c = 1,2,3,4,5,6$ respectively.

Definition 2.3. For any odd integer $n \geq 3$, let $\mathbb{Z}_n$ be the cyclic group of integers modulo n under addition. Let A_i be a proper subset of $\mathbb{Z}_n$ with $\frac{n+1}{2}$ elements in A_i, such that $0 \in A_i$ and $a(\neq 0) \in A_i$ implies $-a \in B_i$ i.e. $B_i = -1(A_i - \{0\})$. Clearly $\{A_i, B_i\}$ is a partition of $\mathbb{Z}_n$ called **skew H-partition** of $\mathbb{Z}_n$.

Subtraction in skew H-partition:

For a skew H-partition $\{A_i, B_i\}$ of $\mathbb{Z}_n$, let $A_i - B_i = \{a-b(\mod n) | a \in A_i,\ b \in B_i\}$. Let C_i denotes the set of distinct elements of $A_i - B_i$. For any $c \in C_i$ the frequency of occurrence of c in $A_i - B_i$ is denoted by n_c^i. Clearly $0 \notin C_i$ for any skew H- partition $\{A_i, B_i\}$ of $\mathbb{Z}_n$.

Example 2.4. For $n = 7$, $\mathbb{Z}_7$ be the cyclic group of order 7.
Let

(1) $A_1 = \{0,1,2,3\}$, $B_1 = \{4,5,6\}$;
(2) $A_2 = \{0,1,3,5\}$, $B_2 = \{2,4,6\}$

are skew H-partitions of $\mathbb{Z}_7$.

Subtractions of skew H-partitions are given by

(1) $A_1-B_1 = \{\{0,1,2,3\}-\{4,5,6\} \mod 7\} = \{3,2,1,4,3,2,5,4,3,6,5,4,5\}$
$\Rightarrow A_1-B_1 = \{1,2,2,3,3,3,4,4,4,5,5,6\}$, $C_1 = \{1,2,3,4,5,6\}$
and n_c^1 are $1,2,3,3,2,1$ for $c = 1,2,3,4,5,6$ respectively.

(2) $A_2-B_2 = \{\{0,1,3,5\}-\{2,4,6\} \mod 7\} = \{5,3,1,6,4,2,1,6,4,3,1,6\}$
$\Rightarrow A_2-B_2 = \{1,1,1,2,3,3,4,4,5,6,6,6\}$,$C_1 = \{1,2,3,4,5,6\}$
and n_c^2 are $3,1,2,2,1,3$ for $c = 1,2,3,4,5,6$ respectively.

Definition 2.5. The **shift matrix** T of order n is a $(0,1)$-square matrix defined as $T = [u_{ij}]$, where

$$u_{ij} = \begin{cases} 1, & \text{if } j-i \equiv 1 \pmod n \\ 0, & \text{otherwise.} \end{cases}$$

We observe that $T^n = I$ (*identity matrix*).

Definition 2.6. For any matrix A, the **match matrix** $A^{(m)}$ of A is defined as $A^{(m)} = [n_{ij}]$, where n_{ij} = number of places in which the i^{th} row and j^{th} row of A have the same non-zero entry at corresponding places.

Definition 2.7. For any matrix A with nonzero entries, the **mis-match matrix** $A^{(mm)}$ of A is defined as $A^{(mm)} = [\acute{n}_{ij}]$, where $\acute{n}_{ij}$ = number of places in which the i^{th} row and j^{th} row of A have different entries at corresponding places.

Definition 2.8. Let $a_0, a_1, \ldots, a_{n-1}$ be a sequence of n elements. Then a matrix $U = [c_{ij}]$ is called a **circulant matrix** with entries $a_0,\ a_1, \ldots, a_{n-1}$ if $c_{ij} = a_{(j-i)(mod\ n)}$, for 1 ≤ i, j ≤ n.

Clearly U is a circulant matrix if and only if $U = \sum_{i=0}^{n-1} a_i T^i$.

Definition 2.9. Let $a_0, a_1,\ \ldots, a_{n-1}$ be a sequence of n elements. Then a matrix $L = [c_{ij}]$ is called a back-circulant matrix with entries $a_0, a_1, ..., a_{n-1}$ if $c_{ij} = a_{(i+j-2)(mod\ n)}$, for $1 \le i, j \le n$.

Clearly L is a **back-circulant matrix** [10] if and only if

$$L = \sum_{i=0}^{n-1} a_{n-(i+1)} T^i R$$

where T is shift matrix and

$$R = \begin{bmatrix} 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \\ . & .. & . & ... & . \\ 0 & 1 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \end{bmatrix}.$$

Note: It can be noted that:-

(1) $R^2 = RR' = I$ $(identity\ matrix)$.

(2) For any a, $0 \le a \le n-1$

$$RT^{n-a} = T^a R. \tag{2.1}$$

3. Necessary and Sufficient Condition

Theorem 3.1. *There exists a set of four good matrices A, B, C, D, where A is skew type circulant and B, C, D are symmetric back-circulant matrices of order n if and only if there exists four partitions $\{A_i, B_i\}$, $i = 1, 2, 3, 4$ where $\{A_1, B_1\}$ is skew H-partition and $\{A_i, B_i\}$ $i = 2, 3, 4$ are H-partitions of $\mathbb{Z}_n$,* *not necessarily distinct such that* $S = \bigcup_{i=1}^{4} C_i = \mathbb{Z}_n - \{0\}$ *and* $\sum_{i=1}^{4} n_c^i = n$ *for each* $c \in \mathbb{Z}_n - \{0\}$*, where* n_c^1 *denotes the occurrence number of* c *in* $A_1 - B_1$*, and* n_c^i *denotes the occurrence number of* c *in* $A_i + B_i$ *for* $i = 2, 3, 4$*.*

Proof: Let T be the shift matrix of order n For any set of four partitions $\{A_i, B_i\}$ $i = 1, 2, 3, 4$ of $\mathbb{Z}_n$, let

$$P_1 = \sum_{a_1 \in A_1} T^{a_1},\ N_1 = \sum_{b_1 \in B_1} T^{b_1}$$

Now

$$N_1' = (\sum_{b_1 \in B_1} T^{b_1})' = \sum_{b_1 \in B_1} (T^{b_1})' = \sum_{b_1 \in B_1} T^{n-b_1} = \sum_{b_1 \in B_1} T^{-b_1}$$

$$\Rightarrow N_1' = \sum_{b_1 \in B_1} T^{-b_1}$$

Then P_1 and N_1 are circulant $(0, 1)$ matrices and

$$P_1 N_1' = \sum_{a_1 \in A_1} T^{a_1} \sum_{b_1 \in B_1} T^{-b_1}$$

$$P_1 N_1' = \sum_{a_1 - b_1 \in A_1 - B_1} T^{a_1 - b_1}$$

$$= \sum_{c \in C_1} n_c^1 T^c \tag{3.1}$$

Also $P_i = \sum_{a_i \in A_i} T^{n-(a_i+1)} R$ and $N_i = \sum_{b_i \in B_i} T^{n-(b_i+1)} R$, for $i = 2, 3, 4$.

Then P_i and N_i are symmetric back-circulant $(0, 1)$ matrices and

$$P_i N_i' = \sum_{a_i \in A_i} T^{n-(a_i+1)} R (\sum_{b_i \in B_i} T^{n-(b_i+1)} R)'$$

$$= \sum_{a_i \in A_i} T^{n-(a_i+1)} RR (\sum_{b_i \in B_i} T^{b_i+1}) \quad [from (2.1)]$$

$$= \sum_{c \in C_i} n_c^i T^c. \tag{3.2}$$

From (3.1) and (3.2) we get,

$$P_i N_i' = \sum_{c \in C_i} n_c^i T^{c_i} \quad i = 1, 2, 3, 4$$

$$\Rightarrow \sum_{i=1}^{4} P_i N_i^{'} = n \sum_{c \in Z_n - \{0\}} T^c = n(J - I). \tag{3.3}$$

Now let $X_i = P_i - N_i$ for $i = 1, 2, 3, 4$ then X_1 is skew-type circulant matrix and X_2, X_3, X_4 are symmetric back circulant matrices with entries 1 and -1. From definition 2.6 it is clear that for a $(0, 1)$ matrix A the match matrix $A^{(m)} = AA^{'}$. Since P_i's and N_i's are $(0, 1)$ matrices, $P_i^m = P_i P_i^{'}$ and $N_i^m = N_i N_i^{'}$ and $X_i^m = P_i^m + N_i^m$ for $i = 1, 2, 3, 4$. Since $\{A_i, B_i\}$ is a partition of $\mathbb{Z}_n$, so

$$X_i^m = P_i P_i^{'} + N_i N_i^{'} \ ; \ i = 1, 2, 3, 4. \tag{3.4}$$

From definition 2.7 it is clear that, for a $(1, -1)$ matrix A of order n, the mis-match matrix $A^{(mm)} = [\acute{n}_{ij}] = [n - n_{ij}]$, where n_{ij} is the $(i,\ j)^{th}$ entry of $A^{(m)}$.

Therefore $A^{(mm)} = nJ - A^{(m)}$, where J is the square matrix with entry 1. Since X_i is a $(1, -1)$ matrix,

$$X_i^{(mm)} = nJ - X_i^{(m)}$$

$$X_i^{(mm)} = nJ - (P_i P_i^{'} + N_i N_i^{'}) \ ; \ i = 1, 2, 3, 4. \tag{3.5}$$

Also, since X_1 is a skew-type $(1, -1)$ matrix and X_2, X_3, X_4 are symmetric $(1, -1)$ matrix so $X_i X_i^{'} = [x_{kl}]$, where x_{kl} = inner product of the k^{th} row and l^{th} row of X_i =(number of places in which the k^{th} row and l^{th} row of X_i have the same entries)$-$ (number of places in which the k^{th} row and l^{th} row of X_i have different entries).

Thus

$$X_i X_i^{'} = X_i^{(m)} - X_i^{(mm)}$$

$$\Rightarrow X_i X_i^{'} = 2(P_i P_i^{'} + N_i N_i^{'}) - nJ \ ; \ i = 1, 2, 3, 4$$

$$\Rightarrow \sum_{i=1}^{4} X_i X_i^{'} = 2(\sum_{i=1}^{4} P_i P_i^{'} + \sum_{i=1}^{4} N_i N_i^{'}) - 4nJ \tag{3.6}$$

Again

$$\sum_{i=1}^{4} X_i X_i^{'} = \sum_{i=1}^{4} (P_i - N_i)(P_i - N_i)^{'}$$

$$= \sum_{i=1}^{4} P_i P_i^{'} + \sum_{i=1}^{4} N_i N_i^{'} - 2 \sum_{i=1}^{4} P_i N_i^{'} \quad [form(2.1)]$$

$$\Rightarrow \sum_{i=1}^{4} P_i P_i^{'} + \sum_{i=1}^{4} N_i N_i^{'} = \sum_{i=1}^{4} X_i X_i^{'} + 2\sum_{i=1}^{4} P_i N_i^{'} \qquad (3.7)$$

From equations (3.6) and (3.7)

$$\sum_{i=1}^{4} X_i X_i^{'} = 2(\sum_{i=1}^{4} X_i X_i^{'} + 2\sum_{i=1}^{4} P_i N_i^{'}) - 4nJ$$

$$\Rightarrow \sum_{i=1}^{4} X_i X_i^{'} = 4nJ - 4\sum_{i=1}^{4} P_i N_i^{'} \qquad (3.8)$$

So equations (3.3) and (3.8) imply

$$\sum_{i=1}^{4} X_i X_i^{'} = 4nJ - 4n(J-I) = 4nI.$$

Thus $X_i,\ i = 1,2,3,4$ form a set of four good matrices for a Hadamard matrix of order $4n$.

Conversely, Let $X_i,\ i = 1,2,3,4$ form a set of four good matrices for a Hadamard matrix of order n, where X_1 is skew-type circulant and X_2, X_3, X_4 are symmetric back circulant matrices of order n. Then

$$\sum_{i=1}^{4} X_i X_i^{'} = 4nI. \qquad (3.9)$$

Since X_i is a $(1,\ -1)$ back circulant matrix it can be written as

$$X_i = \sum_{k=0}^{n-1} a_{n-(k+1)} T^k R\ ;\ a_{n-(k+1)} = \pm 1\ ;\ i = 2,3,4 \qquad (3.10)$$

and X_1 is $(1,-1)$ circulant matrix so it can be written as

$$X_1 = \sum_{k=0}^{n-1} a_k T^k\ ;\ a_k = \pm 1. \qquad (3.11)$$

Let $A_1 = \{k \mid k \in \mathbb{Z}_n, a_k = +1\}$ and $B_1 = \{k \mid k \in \mathbb{Z}_n, a_k = -1\}$

and

$A_i = \{k \mid k \in \mathbb{Z}_n, a_{n-(k+1)} = +1\}$ and $B_i = \{k \mid k \in \mathbb{Z}_n, a_{n-(k+1)} = -1\}$; $i = 2,\ 3,\ 4$ then clearly $\{A_i, B_i\}$, $i = 1,2,3,4$ are four partitions of $\mathbb{Z}_n$ and exactly one of A_i and B_i contains 0. Since equation (3.9) remains valid if X_i is replaced by $-X_i$, replacing X_i by $-X_i$, if necessary, we can assume that A_i contains 0, for $i = 1, 2, 3, 4$. As $\pm X_1$ is a skew type circulant and $\pm X_i$; $i = 2, 3, 4$ are symmetric back-circulant matrix $k \in A_1$, $(n-k) \in B_1$; $k \in A_i$, $(n-k) \in A_i$ for $i = 2, 3, 4$. Therefore $\{A_i, B_i\}$; $i = 1, 2, 3, 4$ are four partitions of $\mathbb{Z}_n$.

Let

$$P_1 = \sum_{k \in A_1} T^k \text{ and } N_1 = \sum_{k \in B_1} T^k$$

Also

$$P_i = \sum_{k \in A_i} T^{n-(k+1)} R \quad \text{and } N_i = \sum_{k \in B_i} T^{n-(k+1)} R \quad \text{where} \quad i = 2, 3, 4.$$

Then $X_i = P_i - N_i$, $i = 1, 2, 3, 4$ where P_1, N_1 are circulant matrices and P_i, N_i $i = 2, 3, 4$ are symmetric back-circulant matrices with entries $(0, 1)$. Thus $P_i^{(m)} = P_i P_i'$ and $N_i^{(m)} = N_i N_i'$ and $X_i^{(m)} = P_i^{(m)} + N_i^{(m)}$ for $i = 1, 2, 3, 4$. So

$$X_i^{(m)} = P_i P_i' + N_i N_i' \; ; \; i = 1, 2, 3, 4. \tag{3.12}$$

Since X_i is a $(1, -1)$ matrix from definition 2.7

$$X_i^{(mm)} = nJ - X_i^{(m)}. \tag{3.13}$$

Using equations (3.9), (3.12) and (3.13) we get

$$\sum_{i=1}^{4} P_i N_i' = n(J - I). \tag{3.14}$$

Now, if possible let us assume that for some element k $\in \mathbb{Z}_n - \{0\}$, $\sum_{i=1}^{4} n_k^i = n_k \neq n$.

As $P_i N_i' = \sum_{c \in C_i} n_c^i T^c$; $i = 1, 2, 3$ and 4, where C_i is the set determined by $A_1 - B_1$ and $A_i + B_i$ $\;i = 2, 3, 4$.

$$\begin{aligned}\sum_{i=1}^{4} P_i N_i^{'} &= \sum_{i=1}^{4}(\sum_{c\in C_i} n_c^i T^c)\\ &= \sum_{c\in S}(\sum_{i=1}^{4} n_c^i)T^c, \quad where \;\; S=\bigcup_{i=1}^{4} C_i\\ &= \sum_{c\in S-\{k\}}(\sum_{i=1}^{4} n_c^i)T^c + \sum_{i=1}^{4} n_k^i T^k\\ &= \sum_{c\in S-\{k\}}(\sum_{i=1}^{4} n_c^i)T^c + n_k T^k\\ &\neq n(J-I), \quad as \; n_k \neq n.\end{aligned}$$

which contradicts

$$\sum_{i=1}^{4} P_i N_i^{'} = n(J-I).$$

So $S=\bigcup_{i=1}^{4} C_i = \mathbb{Z}_n - \{0\}$ and $\sum_{i=1}^{4} n_c^i = n$ for each $c \in \mathbb{Z}_n - \{0\}$.
Hence the theorem. □

4. Examples

Example 4.1. For n=5 ; let $A_1=\{0,1,2\}$, $B_1=\{3,4\}$; $A_2=\{0,2,3\}$, $B_2=\{1,4\}$; $A_3=\{0\}$, $B_3=\{1,2,3,4\}$; $A_4=\{0\}$, $B_4=\{1,2,3,4\}$. Then $A_1-B_1=\{1,2,2,3,3,4\}$; $A_2+B_2=\{1,1,2,3,4,4\}$; $A_3+B_3=\{1,2,3,4\}$; $A_4+B_4=\{1,2,3,4\}$. These four partitions clearly satisfy the condition of the theorem and yield a set of four good matrices whose first rows are given by

+1	+1	+1	-1	-1
+1	-1	+1	+1	-1
+1	-1	-1	-1	-1
+1	-1	-1	-1	-1

Example 4.2. For $n=7$; let $A_1=\{0,1,2,3\}$, $B_1=\{4,5,6\}$; $A_2=\{0,2,5\}$, $B_2=\{1,3,4,6\}$; $A_3=\{0,3,4\}$, $B_3=\{1,2,5,6\}$; $A_4=\{0\}$, $B_4=\{1,2,3,4,5,6\}$. Then $A_1-B_1=\{1,2,2,3,3,3,4,4,4,5,5,6\}$; $A_2+B_2=\{1,1,1,2,3,3,4,4,5,6,6,6\}$; $A_3+B_3=\{1,1,2,2,2,3,4,5,5,5,6,6\}$;

$A_4+B_4 = \{1,2,3,4,5,6\}$. These four partitions clearly satisfy the condition of the theorem and yield a set of four good matrices whose first rows are given by

+1	+1	+1	+1	-1	-1	-1
+1	-1	+1	-1	-1	+1	-1
+1	-1	-1	+1	+1	-1	-1
+1	-1	-1	-1	-1	-1	-1

5. Possible size of partitions for good matrices

Theorem 5.1. *Let* $\{A_i, B_i\}$; $i = 1,2,3,\ 4$ *where* $\{A_1, B_1\}$ *is a skew H-partition and* $\{A_i, B_i\}$; $i = 2,3,4$ *be a set of H-partition of* $\mathbb{Z}_n$*, which gives rise to a set of good matrices. Then* $\sum_{i=2}^{4} k_i(n-k_i) = \dfrac{(n-1)(3n-1)}{4}$ *where* $k_i = |A_i|$; $i = 1,2,3,4$.

Proof: Let $\{A_i, B_i\}$; $i = 1,2,3,4$ are stated type partitions of $\mathbb{Z}_n$, which gives rise to a good matrix of order $4n$. Then $\sum_{i=1}^{4} n_c^i = n$ for all $c \in \mathbb{Z}_n - \{0\}$. Let $k_i = |A_i|$; $i = 1,2,3,4$. Obviously, by definition 2.3, $k_1 = |A_1| = \frac{n+1}{2}$.

Without loss of generality we can assume that $0 \in A_i$; $i = 2,3,4$. As $A_i = -A_i$, for $i = 2,3,4$ k_i is an odd positive integer and consequently $|B_i| = n - k_i$ is an even integer for all $i = 2,3,4$. Since $A_i + B_i$ forms a $k_i \times (n-k_i)$ sub-matrix of the matrix corresponding to the composition table of $\mathbb{Z}_n$, for $i = 2,3,4$ we have

$$\sum_{c\in\mathbb{Z}_n} n_c^i = k_i(n-k_i)\ ;\ i = 1,2,3,4.$$

$$\Rightarrow \sum_{i=2}^{4}\Big(\sum_{c\in Z_n} n_c^i\Big) = \sum_{i=2}^{4} k_i(n-k_i) \tag{5.1}$$

Again

$$\begin{aligned}\sum_{i=2}^{4}\Big(\sum_{c\in\mathbb{Z}_n} n_c^i\Big) &= \sum_{i=2}^{4}\Big(\sum_{c\in\mathbb{Z}_n-\{0\}} n_c^i\Big) \quad as \quad n_0^i = 0\ ;\ i = 2,3,4\\ &= \sum_{c\in\mathbb{Z}_n-\{0\}}\Big(\sum_{i=2}^{4} n_c^i\Big)\end{aligned}$$

$$\Rightarrow \sum_{c \in Z_n} (\sum_{i=2}^{4} n_c^i) = \sum_{c \in Z_n - \{0\}} (\sum_{i=1}^{4} n_c^i - n_c^1)$$

$$= \sum_{c \in Z_n - \{0\}} (\sum_{i=1}^{4} n_c^i) - \sum_{c \in Z_n - \{0\}} n_c^1$$

$$= n(n-1) - \sum_{c \in Z_n - \{0\}} n_c^1$$

$$= n(n-1) - k_1(n-k_1)$$

$$= n(n-1) - \frac{n+1}{2}(n - \frac{n+1}{2}) \;\; as \; k_1 = \frac{n+1}{2}$$

$$\Rightarrow \sum_{i=2}^{4} (\sum_{c \in Z_n} n_c^i) = \frac{(n-1)(3n-1)}{4}. \tag{5.2}$$

From (5.1) and (5.2) we have

$$\sum_{i=2}^{4} k_i(n-k_i) = \frac{(n-1)(3n-1)}{4}. \; \square$$

So the possible size of A_i, $i = 1, 2, 3, 4$ are k_1, k_2, k_3 and k_4 respectively, where $k_1 = \frac{n+1}{2}$ and $\{k_2, k_3, k_4\}$ is a set of odd integers which satisfy the equation

$$x(n-x) + y(n-y) + z(n-z) = \frac{(n-1)(3n-1)}{4}.$$

Theorem 5.2. *The equation*

$$x(n-x) + y(n-y) + z(n-z) = \frac{(n-1)(3n-1)}{4}$$

has an integer solution if and only if there exists an integer solution of the equation

$$X_2 + X_3 + X_4 = n - 1$$

in $\{m(m-1)\}_{m=1}^{\infty}$.

Proof: Let $\{k_2, k_3, k_4\}$ be an integer solution of the equation

$$x(n-x) + y(n-y) + z(n-z) = \frac{(n-1)(3n-1)}{4}. \tag{5.3}$$

Thus

$$\sum_{i=2}^{4} k_i(n-k_i) = \frac{(n-1)(3n-1)}{4}.$$

Let $X_i = (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i),\ i = 2,3,4.$

Since $k_i+(n-k_i) = n$; $i = 2,3,4$, so $(\frac{n-1}{2})(\frac{n+1}{2}) \geq k_i(n-k_i)$; $i = 2,3,4$

$\Rightarrow X_i = (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i) \geq 0$; $i = 2,3,4$

Then

$$\begin{aligned}\sum_{i=2}^{4} X_i &= \sum_{i=2}^{4}\{(\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i)\}\\ &= \frac{3(n-1)(n+1)}{4} - \sum_{i=2}^{4} k_i(n-k_i)\\ &= \frac{3(n-1)(n+1)}{4} - \frac{(n-1)(3n-1)}{4} \quad [from(5.3)]\\ &= n-1\end{aligned}$$

Now we have to show that $X_i \in \{m(m-1)\}_{m=1}^{\infty}$ for all $i = 2,3,4$.
For all $i = 2,3,4$ we have

$$\begin{aligned}X_i &= (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i)\\ &= (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(\frac{n+1}{2}) + k_i(\frac{n+1}{2}) - k_i(n-k_i)\\ &= (\frac{n+1}{2})(\frac{n-1}{2} - k_i) - k_i(\frac{n-1}{2} - k_i)\\ &= (\frac{n+1}{2} - k_i)(\frac{n-1}{2} - k_i)\\ &= m_i(m_i-1) \quad [say \quad m_i = \frac{n+1}{2} - k_i].\end{aligned}$$

If $\frac{n+1}{2} > k_i \Rightarrow m_i > 0 \Rightarrow m_i(m_i-1) \geq 0 \Rightarrow X_i \geq 0.$

If $\frac{n+1}{2} \leq k_i \Rightarrow m_i \leq 0 \Rightarrow m_i(m_i-1) \geq 0 \Rightarrow X_i \geq 0.$

Thus for all $i = 2,3,4$; $X_i \in \{m(m-1)\}_{m=1}^{\infty}$.

Conversely, let $m_i(m_i-1)$; $i = 2,3,4$ be an integer solution of

$$X_2 + X_3 + X_4 = n-1. \tag{5.4}$$

Then $\sum_{i=2}^{4} m_i(m_i - 1) = n - 1$. We claim that for all $i = 2, 3, 4$; $m_i \leq \frac{n-1}{2}$.
If not suppose for some $i = 2, 3, 4$; $m_i > \frac{n-1}{2} \Rightarrow m_i(m_i - 1) > (\frac{n-1}{2})(\frac{n+1}{2})$ for $n \geq 3$. For $n = 1$, $X_2 = X_3 = X_4 = 0$ is a solution of (5.4) and the corresponding solution of (5.3) is $x = y = z = 1$.
Now consider $k_i = \frac{n+1}{2} - m_i$; $i = 2, 3, 4$.
Then

$$\begin{aligned}
\sum_{i=2}^{4} k_i(n - k_i) &= \sum_{i=2}^{4} (\frac{n+1}{2} - m_i)\{n - (\frac{n+1}{2} - m_i)\} \\
&= \sum_{i=2}^{4} (\frac{n+1}{2} - m_i)(\frac{n-1}{2} + m_i) \\
&= \sum_{i=2}^{4} \{(\frac{n+1}{2})(\frac{n-1}{2}) + m_i(\frac{n+1}{2} - \frac{n-1}{2}) - m_i^2\} \\
&= 3(\frac{n+1}{2})(\frac{n-1}{2}) - \sum_{i=2}^{4} m_i(m_i - 1) \\
&= 3(\frac{n+1}{2})(\frac{n-1}{2}) - (n - 1) \\
&= (\frac{(n-1)(3n-1)}{4}).
\end{aligned}$$

So $k_i(n - k_i)$; $i = 2, 3, 4$ is a solution set of equation (5.3).

Example: For n = 13 ; the solutions of the equation

$$X_2 + X_3 + X_4 = n - 1$$

in $\{m(m + 1)\}_{m=0}^{\infty}$ are given by

(i) (0, 0, 12), (ii) (0, 6, 6).

Using theorem (5.2) the corresponding solutions of

$$x(n - x) + y(n - y) + z(n - z) = \frac{(n-1)(3n-1)}{4}.$$

are (a) (7, 7, 3) (b) (7, 9, 9) [taking all odd solutions] respectively. So possible size of part A_i of the H-partitions $\{A_i, B_i\}$; $i = 2, 3, 4$ are given by one of the solutions (a) and (b) only. Using these concepts the exhaustive search becomes quite easy as other sizes of partitions are disposed off.

Let us consider the solution (a) (7, 7, 3), by hit and trial method we obtain

$A_1 = \{0, 1, 3, 7, 8, 9, 11\},$

$A_2 = \{0, 4, 5, 6, 7, 8, 9\},$

$A_3 = \{0, 1, 3, 6, 7, 10, 12\}$

and $A_4 = \{0, 5, 8\},$

such that the frequencies n_j^i ; $j = 1, 2, 3, 4, 5, \ldots, 12$; $i = 1, 2, 3, 4$ are as follows:

$n_j^1 = \{4, 3, 4, 4, 3, 3, 3, 3, 4, 4, 3, 4\};$

$n_j^2 = \{2, 3, 4, 3, 4, 5, 5, 4, 3, 4, 3, 2\};$

$n_j^3 = \{4, 4, 3, 3, 5, 2, 2, 5, 3, 3, 4, 4\};$

$n_j^4 = \{3, 3, 2, 3, 1, 3, 3, 1, 3, 2, 3, 3\};$

Since $\sum_{i=1}^{4} n_j^i = 13$ for $j = 1, 2, 3, 4, ..., 12$, the condition of the theorem (3.1) are satisfied by this set of four partitions and we have a set of four good matrices giving rise to a Hadamard matrix of order 4×13.

Conclusion: It is very tedious job to search a set of good matrices to obtain a skew-type Williamson Hadamard matrix. The paper also includes possible size of partitions for Williamson skew type symmetric matrices, which helps to disposed off some cases of exhaustive search of such matrices. These matrices are very useful because of its applications.

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COMPLEX INVERSION FORMULA FOR MODIFIED-STIELTJES TRANSFORM

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Abstract. This paper deals with Modified- Stieltjes transform. The present paper mainly provides complex inversion formula for the Modified-Stieltjes transform. The aim of the paper is to extend inverse Modified-Stieltjes transform and also prove complex inversion theorem.

1. INTRODUCTION

The basic aim of the transform method is to convert a given problem into one that is simpler to solve. Intrgral transform methods provide effective ways to solve a variety of problems arising in engineering and physical sciences.In this paper we have proved complex inversion theorem for modified Stieltjes transform.The classical Stieltjes transform has been given by Sumner [6]. Its convergence in S' instead of point wise convergence, is for the $\Gamma(r+1)T_{r+1}(f)$, $r > -1$ where f belongs to a subspace of T_{r+1}- transformable tempered distributions.

2. Definitions

Definition 2.1. SPACE $L'(r)$: We extend the definition of the space $I'(r)$ given in [5] and using the same idea we provided the definition of **space** $L'(r)$.
$L'(r), r \in \Re\backslash(-N)$ denotes the space of all distributions $f \in S'_{+}(\Re)$ such that there exists $k \in N_0$ and locally integrable function F, supp $F \subset [0, \infty)$, so that f is of the form

$$f = t^{-r} D^k F \tag{2.1}$$

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and there exists $C = C(F)$ and $\epsilon = \epsilon(F) > 0$ such that

$$|F(x)| \leq C(1+x)^{r+k-\epsilon}, x \geq 0 \tag{2.2}$$

Definition 2.2. Stieltjes Transform: The Stieltjes Transform $S_r(f)(s), r \in \mathbf{R}\backslash(-N)$ is complex valued function, defined by

$$S_r(f(t))(s) = \int_0^\infty \frac{f(t)}{(s+t)^{r+1}} dt, s \in C\backslash(-\infty, 0], 0 < t < \infty, r \in \mathbf{R}\backslash(-N) \tag{2.3}$$

Definition 2.3. Modified Stielejes Transformation: The Modified Stielejes Transformation $T_{r+1}(f), r \in R\backslash(-N)$ is a complex valued function defined by

$\Gamma(r+1)T_{r+1}(f)(s) = (r+1)_k \int_0^\infty \frac{F(t)}{(s+t)^{(r+1+k)}} dt, \quad r \in R\backslash(-N)$, $s \in C\backslash 0 < t < \infty$.

Where $(r+1)_k = (r+1)(r+2) - - - (r+k-1) = \frac{\Gamma(r+k+1)}{\Gamma(r+1)}$

3. MAIN RESULTS

First we provide some preliminary results. Let $\eta > 0, t \in \Re$, we denote $C_{\eta t}$, the contour in $/\!\!c$ which starts at the point, $-t - i\eta$, proceeds along the straight line $Imz = -\eta$ to the point $-i\eta$, then along the semicircle $|z| = \eta, Rez \geq 0$ to the point $i\eta$ and finally along the line $Imz = \eta$ to the point $-t + i\eta$. We noted that these contours were observed in [4] only for $t > 0$. Let $K(u,t) = K_1(t-u), t, u \in \Re, t \neq u$ where
$K_1(x) = x^{-1}\Big((-\eta - ix)^{-r-1} - (\eta + ix)^{-r-1}\Big), x \in \Re\backslash\{0\}, \eta > 0.$
For convenience we took that determination of $(s+t)^{-r-k-1}$, which occur in this section for which $arg(s+t)^{-r-k-1}(argz^{-r-s-1}, s \in N_0)$ has its principal value.
We need the following identities:

$$\int_{C_{\eta t}} \frac{(z+t)^r}{(z+t)^{r+2}} dz = \frac{\eta^{r+1}}{r+1} k(u,t),\ u > 0, t \in \Re, t \neq u, \eta > 0, ([6], p.180) \tag{3.1}$$

$$\frac{\partial^i K(u,t)}{\partial t^i} = (-1)^i \frac{\partial^i K(u,t)}{\partial u^i}, t, u \in \Re, u \neq t, \eta > 0$$

By Leibnitz formula we have

$$\Big|\frac{\partial^i K(u,t)}{\partial t^i}\Big| \leq C_1 \sum_{p=0}^{i} |t - u^{-i-1+p}| \Big(\eta^2 + (t-u)^2\Big)^{-(r+1+p)/2}$$

where $C_1 = 2max\Big\{\binom{i}{p}(i-p)!(r+p)_p; 0 \leq p \leq i\Big\}$.
Thus with $C_0 = (i+1)C_1$, it holds

$$|\frac{\partial^i K(u,t)}{\partial t^i}| \leq C_0 |t-u|^{-i-1}\Big(\eta^2 + (t-u^2)\Big)^{-(r+1)/2} \tag{3.2}$$

We shall suppose that $r > -1$. We denote by $\tilde{L}'(r), (r > -1)$ a subset of $L'(r+1)$ such that $f \in L'(r)$ if $f = t^r D^k F, k \in N_0$, F is continuous and if instead of (1.2.1) it holds

$$|F(x)| \leq C(1+x)^{r+1-\epsilon}, x \geq 0\, for some\, C > 0\, and\, some\, \epsilon > 0 \tag{3.3}$$

Lemma 3.1. *Let F be a continuous function on $\Re$ with $suppF \subset [0,\infty)$ and let* (3.2) *hold. Then for every $k \in N_0$ and $t_0 \in \Re$*

$$.\int_{-\infty}^{\infty} F(u)(\frac{\partial^i K(u,t)}{\partial t^i})|_{t=t_0} du = \frac{d^k}{dt^k}\Bigg[\int_{-\infty}^{\infty} F(u)K(u,t)du\Bigg]\Bigg|_{t=t_0} \tag{3.4}$$

Lemma 3.2. *Let F satisfies the conditions of lemma (3.1)and let*

$$\phi_i(t) = \int_{-\infty}^{\infty} |F(u)\frac{\partial^i K(u,t)}{\partial t^i}|du,\, t \in \Re, i \in N_0 \tag{3.5}$$

(i) *There exists constants $k(i,\eta)$ and polynomial $P_i(t)$ such that*

$$\phi_i(t) \leq k(i,\eta)P_i(t), t \in \Re, i \in N$$

(ii) *There exists constant k_0 (which does not depend on η) and a polynomial $P_0(t)$ such that $\eta^{r+1}\phi_0(t) \leq k_0 p_0(t), t \in \Re$.*

Proof. (i) Let $t > 1$, we have

$$\phi_i(t) \leq \Bigg[\int_0^{t-1}\int_{t-1}^{t+1}\int_{t+1}^{\infty}\Bigg]\Big(|\frac{F(u)\partial^i K(u,t)}{\partial t^i}|\Big)du = J_1 + J_2 + J_3$$

By (3.2) and (3.3) we have

$$J_1 \leq CC_0 \int_0^{t-1} \frac{(1+u)^{r+1-\epsilon}}{|t-u|^{i+1}(\eta^2 + (t-u)^2)^{(r+1)/2}} du \leq CC_0 t(1+t)^{r+1-\epsilon}$$

because for $t > 0$ and $u \in (0, t-1), (t-u)^{i+1}(\eta^2 + (t-u)^2)^{(r+1)/2} > 1$

$$J_3 \leq CC_0 \int_{t+1}^{\infty} \frac{(1+u)^{r+1-\epsilon}du}{|t-u|^{i+1}(\eta^2 + (t-u)^2)^{(r+1)/2}}$$

$$= CC_0 \int_0^\infty \frac{(2+t+v)^{r+1-\epsilon}}{(v+1)^{i+1}(\eta^2+(t-u)^2)^{(r+1)/2}} dv$$

Since $r-\epsilon > -1$, from $(2+t+v)^{r+1-\epsilon} \leq 2^{r+1-\epsilon}\Big((2+t)^{r+1-\epsilon} + v^{r+1-\epsilon}\Big)$, ($\because$ Putting $u = t+v+1 \therefore |t-u| = |v+1|$) $v > 0$, we obtain

$$J_3 \leq 2^{r+1-\epsilon} C_0 C \Bigg[(2+t)^{r+1-\epsilon} \int_0^\infty \frac{dv}{(v+1)^{i+1}(\eta^2+(v+1)^2)^(r+1/2)}$$
$$+ \int_0^\infty \frac{v^{r+1-\epsilon}dv}{(v+1)^{i+1}(\eta^2+(v+1)^2)^(r+1/2)} \Bigg]$$

For J_2 we have

$$J_2 \leq \sup_{t-1 \leq u \leq t+1} \{|F(u)|\} \int_{t-1}^{t+1} |\frac{\partial^i K(u,t)}{\partial t^i}| du \leq C(t+2)^{r+1-\epsilon} \int_{-1}^{1} |\frac{\partial^i K(0,s)}{\partial t^i}| ds$$

Since the function $H(0,s), s \in [-1,1]$,

where $H(u,t) = \begin{cases} K(u,t), u \neq t \\ \frac{2(r+1)^i}{\eta^{r+2}}, u = t, (u,t) \in R^2 \end{cases}$

is smooth one . We obtain that for some constant M_i which depends on η $J_2 \leq M_i(t+2)^{r+1-\epsilon}$.

Estimations for J_1, J_2 and J_3 imply that the assertion holds if $t > 1$.

Let $0 \leq t \leq 1$ then we have $\phi_i(t) \leq \Big[\int_0^2 + \int_2^\infty\Big]\Big(|\frac{F(u)\partial^i K(u,t)}{\partial t^i}|\Big)du$, and by the similar arguments as above we can prove that the assertion holds.

If $t > 1$ there is no need to divide the integral in (3.5) and assertion (i)follows by arguments given by above.

(ii) Let $t > 1$. From the first part of the lemma we conclude that only in the calculation of the integral J_2 the constant $k(i,\eta)$ depends on η. But, on setting $s = \eta tg\phi$ in $\int_{-1}^{1} |\frac{\partial^i K(0,s)}{\partial t^i}| ds$, the same way as in the proof of Lemma 4.b from [4], we prove the assertion.For $t \leq 1$, we have to use arguments given above.

Hence the proof of lemma. □

Lemma 3.3. *Let F be a continuous function on $\Re$ with $suppF \subset [0,\infty)$ and let $|F(x)| \leq C(1+x)^{r+1-\epsilon}$, $x \geq 0$ hold. Then*
$\lim_{\eta \to 0^+} \frac{\eta^{r+1}}{2\pi i} \int_{-\infty}^{\infty} F(u)K(u,t)du = F(t), t \in \Re.$

Proof. For $t \geq 0$ the proof follows from ([6],Lemma 4.C) since for enough large $\Re$, $\lim_{\eta \to 0^+} \int_{IR}^{\infty} F(u)K(u,t)du = 0$

Since $r > -1$ and
$\int_0^\infty |F(u)K(u,t)|dt \leq \int_0^\infty |F(u)|(|t|+u)^{-r-1}du < \infty,\ t < 0.$
We obtain $\lim\limits_{\eta\to 0^+} \frac{\eta^{r+1}}{2\pi i}\int_{-\infty}^{\infty} F(u)K(u,t)du = 0,\ t < 0$
This completes proof. □

If $f \in L'(r)$, then for $t \in \Re$ we have
$(r+1)\int_{C_{\eta t}}(z+t)^r(\Gamma(r+2)T_{r+2}F)(z)dz$
$= (r+1)_{k+1}\int_{C_{\eta t}}(z+t)^r(\Gamma(r+k+2)T_{r+k+2}F)(z)dz$
$= (r+1)_{k+1}\int_{C_{\eta t}}(z+t)^r\left(\int_{-\infty}^{\infty}\frac{F(u)}{(z+u)^{r+k+2}}du\right)dz$
$= (r+1)_{k+1}\int_{-\infty}^{\infty}F(u)\left(\int_{C_{\eta t}}\frac{(z+t)^r}{(z+u)^{r+k+2}}dz\right)du.$
The last equality holds on the basis of the uniform convergence of

$$\int_{-\infty}^{\infty}\frac{F(u)}{(z+u)^{r+k+2}}du,\ for z \in C_{\eta t}$$

. Thus we have by (3.1)
$(r+1)\int_{C_{\eta t}}(z+t)^r(\Gamma(r+2)T_{r+2}f)(z)dz = (-1)^k\eta^{r+1}\int_{-\infty}^{\infty}F(u)(\frac{\partial^k K(u,t)}{\partial t^k})du,$
$t \in \Re$.

Theorem 3.4. Complex Inversion: *Let $f \in L'(r)$. Then for every $\phi \in S$.*

$$\lim_{\eta\to 0^+}\left(\frac{r+1}{2\pi i}\left\langle \int_{C_{\eta t}}(z+t)^r(\Gamma(r+2)T_{r+2}f)(z)dz, \phi(t)\right\rangle\right) = \langle f(t), \phi(t)\rangle.$$

Proof. We have
$\frac{r+1}{2\pi i}\left\langle \int_{C_{\eta t}}(z+t)^r(\Gamma(r+2)T_{r+2}f)(z)dz, \phi(t)\right\rangle$
$= \frac{1}{2\pi i}(r+1)_{k+1}\left\langle \int_{C_{\eta t}}(z+t)^r(\Gamma(r+k+2)T_{r+k+2}F)(z)dz, \phi(t)\right\rangle (\because by (3.5))$
$= \frac{(r+1)_{(k+1)}}{2\pi i}\left\langle \int_{C_{\eta t}}(z+t)^r\left[\int_{-\infty}^{\infty}\frac{F(u)}{(z+u)^{r+k+2}}dz\right]du, \phi(t)\right\rangle$
$= \frac{(r+1)_{(k+1)}}{2\pi i}\left\langle \int_{-\infty}^{\infty}F(u)\left[\int_{C_{\eta t}}\frac{(z+t)^r}{(z+u)^{r+k+2}}dz\right]du, \phi(t)\right\rangle$
$= \frac{(-1)^k}{2\pi i}(r+1)\left\langle \int_{-\infty}^{\infty}F(u)\left[\frac{\partial^k}{\partial u^k}\int_{C_{\eta t}}\frac{(z+t)^r}{(z+u)^{r+2}}dz\right]du, \phi(t)\right\rangle$
$= \frac{(-1)^k}{2\pi i}\eta^{r+1}\left\langle \int_{-\infty}^{\infty}F(u)(\frac{\partial^k K(u,t)}{\partial t^k})du, \phi(t)\right\rangle\ (\because by (3.1))$

$= \frac{\eta^{r+1}}{2\pi i}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}F(u)(\frac{\partial^k K(u,t)}{\partial t^k})du\right]\phi(t)dt.$
$= \frac{\eta^{r+1}}{2\pi i}\int_{-\infty}^{\infty}\frac{d^k}{dt^k}\left[\int_{-\infty}^{\infty}F(u)K(u,t)du\right]\phi(t)dt\ (byLemma 3.1)$
$= \frac{\eta^{r+1}}{2\pi i}(-1)^k\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}F(u)K(u,t)du\right]\phi^{(k)}(t)dt.$ (by partial integration)
Thus by lemma (3.2)and lemma (3.1)(ii)and the Lebesgue's theorem, we obtain
$\lim_{\eta\to 0^+}\frac{r+1}{2\pi i}\left\langle\int_{C_{\eta t}}(z+t)^r(\Gamma(r+2)T_{r+2}f)(z)dz,\phi(t)\right\rangle$
$= (-1)^k\langle F(t),\phi^{(k)}(t)\rangle = \langle f,\phi\rangle.$
∴ The proof is complete. □

4. CONCLUSION

A definitions for Modified-Stieltjes and its complex inverse are introduced in this research work. Complex inversion theorem has been proved.

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PROBLEM SECTION

In Vol. 89 (3-4) 2020 of The Mathematics Student, we had invited solutions from the floor to Problem 3 of MS 88 (3-4) 2019, solutions to Problems 5 and 7 posed in MS 89 (1-2) 2020 as well as solutions to the nine new problems till March 15, 2021.

We did not receive any solution to Problem 3 of MS 88 (1-2) 2019 so we print at the end of this section the solution provided by the proposer of the problem.

As regards to solutions to Problems 5 and 7, mentioned in MS 89(1-2) 2020, we received one solution to Problem 5 but it was not correct. We did not receive any solution to problem 7. So the solutions provided by the proposers of these Problems are being printed later at the end of this section.

As far as solutions to the nine new Problems of MS 89(3-4) 2020 are concerned, we received one correct solution to Problem 4 and one correct solution to Problem 7. These solutions are being printed here.

We pose eight new problems in this volume. We invite Solutions to these problems and solutions to the remaining problems of MS 89 (3-4) 2020 from the researchers till August 20, 2021. Correct solutions received from the floor by August 20, 2021 will be published in Volume 90 (3-4) 2021 of The Mathematics Student. This volume is scheduled to be published in September 2021.

The following five problems have been posed by Prof. B. Sury, Indian Statistical Institute, Bangalore.

MS 90 (1-2) 2021: Problem 1. We know that the diagonals of a rhombus intersect at a point that divides the rhombus into four congruent triangles. Prove that this characterizes a rhombus. That is, if $ABCD$ is a quadrilateral such that for some point P on the plane of the quadrilateral (not necessarily inside the quadrilateral), the four triangles PAB, PBC, PCD, PDA are all congruent, then $ABCD$ must be a rhombus.

MS 90 (1-2) 2021: Problem 2. Let $f : [0,1] \to \mathbb{R}$ be differentiable, and let $f(0) = 0, f(1) = 1$. Prove that there exist $t_1, \cdots, t_{2021} \in [0,1]$ such that $2021 = \sum_{i=1}^{2021} \frac{1}{f'(t_i)}$.

MS 90 (1-2) 2021: Problem 3. Let A be an $n \times n$ matrix with rational entries. If the rank of A is 1, show that $det(I_n + A) - trace(A) = 1$.

MS 90 (1-2) 2021: Problem 4. Let $A \subset \mathbb{R}^2$ be a finite set such that no three points in A are collinear. Assuming that the circle passing through any three points of A contains a fourth point of A, prove that all the points of A are concyclic.

MS 90 (1-2) 2021: Problem 5. Let p be a prime number which has at least ten digits. If $4p+1$ is also a prime, show that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits $0, 1, \cdots, 9$.

The following problem has been proposed by Dr. Siddhi Pathak, Department of Mathematics, Pennsylvania State University, State College, PA 16802, USA.

MS 90 (1-2) 2021: Problem 6. Let $\mathbb{Q}^+$ denote the set of positive rational numbers, and $P : \mathbb{Q}^+ \to \mathbb{N}$ be defined as $P(m/n) = mn$ for $\gcd(m,n) = 1$. Show that

$$\sum_{q \in \mathbb{Q}^+} \frac{1}{P(q)^2} = \frac{5}{2}.$$

Dr. Anup Dixit, Institute of Mathematical Sciences, Chennai has posed the following two problems.

MS 90 (1-2) 2021: Problem 7. Show that the series

$$\sum_{n=1}^{\infty} \frac{-1^{\lfloor \sqrt{n} \rfloor}}{n^{5/9}}$$

is convergent. Here $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

MS 90 (1-2) 2021: Problem 8. Show that

$$\int_0^1 \left\{ \frac{1}{x} \right\}^2 x^2 \, dx = 1 - \frac{1}{3}(\zeta(2) + \zeta(3)),$$

where $\{x\}$ is the fractional part of x and $\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}$ denotes the Riemann zeta-function for $k > 1$.

Solutions to the Old Problems

MS 88 (3-4) 2019: Problem 3 (posed by Prof. B. Sury, ISI, Bangalore) Let u_n denote the number of essentially different ways to tile the $2n \times 2n$ chess board with dominos (1×2 pieces). For instance $u_1 = 2$.

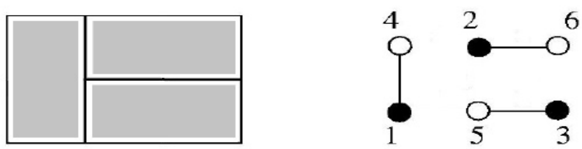
Prove

$$u_n = 2^{2n^2} \prod_{r=1}^{n} \prod_{s=1}^{n} (cos^2 \frac{r\pi}{2n+1} + cos^2 \frac{s\pi}{2n+1}).$$

Solution (by Prof. B. Sury).
This result is due independently to Kasteleyn and Temperley-Fisher. Let us first describe how one attaches a bipartite graph to each tiling and solves the problem through graph theory. More generally, we prove the following amazing formula for the number of domino tilings of an $m \times n$ grid where m is even (our proof is based on the discussion in the book 'Combinatorial Problems and Exercises, by Laszlo Lovasz, AMS Chelsea Publishing, Indian edition 2012):

$$\prod_{r=1}^{m/2} \prod_{s=1}^{n} 2\sqrt{\cos^2\left(\frac{r\pi}{m+1}\right) + \cos^2\left(\frac{s\pi}{n+1}\right)}.$$

To describe the associated graph, look at a 2×3 grid tiled by dominos where the six vertices represent the tiles and the edges represent the dominos.



A 1-matching (or just matching) of a graph is a set of edges so that no two of them share a common vertex. It is called a perfect matching when each vertex is incident to exactly one edge of the matching. We wish to find the number of perfect matchings of the graph corresponding to the grid as this is the number of domino tilings. The proof uses Pfaffians which are defined as follows.

For a $2n \times 2n$ real, skew-symmetric, invertible matrix $A = \{a_{ij} : 1 \leq i, j \leq 2n\}$, consider the set of permutations σ of $\{1, \cdots, 2n\}$ such that

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1);$$

$$\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \cdots, \sigma(2n-1) < \sigma(2n).$$

Define the Pfaffian of A as

$$Pf(A) = \sum_{\sigma \in S_{2n}} sgn(\sigma) a_{\sigma(1),\sigma(2)} a_{\sigma(3),\sigma(4)} \cdots a_{\sigma(2n-1),\sigma(2n)}$$

It is known that $Pf(A) = \sqrt{det(A)}$.
The idea of the proof of the formula for the number of domino tilings of an $m \times n$ grid with m even is to use graphs as mentioned. If we assign an ordered pair (i, j) with $i \leq m, j \leq n$ to each vertex (=tile), then (i, j) can be connected by an edge to (i', j') by a domino if, and only if $|i - i'| = 1, j = j'$ or $i = i'$ and $|j - j'| = 1$. To find a generating function for the number of domino tilings, it is convenient to call $g(h, v)$ to be the number of domino tilings with h horizontal dominos and v vertical dominos.

The generating function is the polynomial

$$Z_{m,n}(x, y) = \sum_{2(h+v)=mn} g(h, v) x^h y^v.$$

Hence the number we seek is $Z_{m,n}(1, 1)$. In order to keep track of the configuration of each domino tiling, it is convenient to represent the tile (i, j) by the number $p = (i-1)n + j$. For instance, for $m = n = 4$, we have:

p=1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

In order to specify a valid configuration of domino tiling, we order the tiles based on their p-numbers:

$$p_1 < p_3 < \cdots < p_{mn-1};$$

$$p_1 < p_2, p_3 < p_4, \cdots, p_{mn-1} < p_{mn}.$$

This ordering allows us to write a configuration uniquely; for instance,

$$C = (p_1, p_2)(p_3, p_4) \cdot \cdots \cdot (p_{mn-1}, p_{mn})$$

is a valid configuration but

$$(p_3, p_4)(p_1, p_2) \cdot \cdots \cdot (.,.)$$

is not. The conditions for a valid configuration resemble those for defining Pfaffians. Indeed, the idea is to define a skew-symmetric matrix D whose Pfaffian equals the generating function $Z_{m,n}(x, y)$. In fact, we will construct D in such a way that each non-zero term in $Pf(D)$ corresponds to a configuration of domino tiling, and vice versa. We shall define a weighted adjacency matrix D of size $mn \times mn$. This requires a delicate study of the configurations to deduce:

For any $\sigma \in S_{mn}$ that satisfies the conditions of a valid configuration C; viz.

$$p_1 < p_3 < \cdots < p_{mn-1};$$

$$p_1 < p_2, p_3 < p_4, \cdots, p_{mn-1} < p_{mn},$$

it can be proved that each polygon in C contributes a factor -1 to $sgn(\sigma)$. One may then define $D_{(i,j),(i+1,j)} = x;\ i \leq m-1, j \leq n$; and $D_{(i,j),(i,j+1)} = (-1)^i y;\ i \leq m, j \leq n-1$.
Define $D_{p,p'} = 0$ in all other cases. Another description of D is then

$$D = x(I_n \otimes Q_m) + y(Q_n \otimes F_m)$$

where:

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \ F = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using this, we may find the eigenvalues and eigenvectors and write a matrix of 2×2 blocks which is similar to D. This gives finally

$$det(D) = \prod_{k=1}^{m/2} \prod_{l=1}^{n} det \begin{pmatrix} 2ix \cos \frac{k\pi}{m+1} & -2iy \cos \frac{l\pi}{n+1} \\ 2iy \cos \frac{l\pi}{n+1} & -2ix \cos \frac{k\pi}{m+1} \end{pmatrix}$$

whose square root evaluates to the Pfaffian giving the asserted formula.

□

MS 89 (1-2) 2020 : Problem 5 (Posed by Dr. Anup Dixit, IMSc, Chennai). Find all natural numbers $n > 1$ such that $n \mid 2^n + 1$.

Solution (by Dr. Anup Dixit). We start by observing that n is odd. Let p be the smallest prime divisor of n. Then, $2^n \equiv -1 \bmod p$ and hence, $2^{2n} \equiv 1 \bmod p$. On the other hand, Fermat's little theorem states that $2^{p-1} \equiv 1 \bmod p$. Therefore,

$$2^{\gcd(2n,(p-1))} \equiv 1 \bmod p.$$

Since p is the smallest prime factor of n, we deduce that $\gcd(2n, p-1) = 2$. Hence, $2^2 \equiv 1 \bmod p$, which implies $p = 3$.

Suppose n has 2 or more prime factors. Let $q > 3$ be the second smallest prime factor of n. A similar argument as above leads to

$$2^{\gcd(2n,(q-1))} \equiv 1 \bmod q.$$

As q is the smallest prime divisor of n after 3, we conclude that $\gcd(2n, q-1)$ is either 2 or 6. If the gcd is 2, then we get $2^2 \equiv 1 \bmod q$, which implies $q = 3$. This contradicts the assumption that $q > 3$. Hence, we deduce that $\gcd(2n, q-1) = 6$. So, $q \mid 2^6 - 1 = 63$ and thus $q = 7$.

If n is divisible by both 3 and 7, then $2^n \equiv 1 \bmod 7$ as $2^3 \equiv 1 \bmod 7$. However, $7 \mid 2^n + 1$ which leads to a contradiction. Therefore, 3 is the only prime factor of n.

Now, we claim that every n of the form $n = 3^k$ satisfies the condition $2^n \equiv -1 \bmod n$. By Euler's theorem, we have $2^{\phi(m)} \equiv 1 \bmod m$ for any odd natural number m, where ϕ denotes the Euler-totient function. As $\phi(3^k) = 2 \cdot 3^{k-1}$, we get that

$$2^{3^k} = 2^{3^{k-1}} 2^{2 \cdot 3^{k-1}} \equiv 2^{3^{k-1}} \bmod 3^k.$$

Therefore, it suffices to show that $2^{3^{k-1}} \equiv -1 \bmod 3^k$. Since $3^{k-1} = \phi(3^k)/2$, and the only solutions for $x^2 \equiv 1 \bmod 3^k$ are 1 and -1, we have $2^{3^{k-1}}$ is either 1 or $-1 \bmod 3^k$. If $2^{3^{k-1}} \equiv 1 \bmod 3^k$, then it is also $1 \bmod 3$. This is a contradiction because 2 raised to an odd number is $-1 \bmod 3$.

Hence, all such n are precisely given by 3^k where $k \geq 1$.

□

The following problem was taken from Graph Theory Prolems / solutions (smograph.pdf). The solution presented there is printed below.

MS 89 (1-2) 2020 : Problem 7. At the end of a birthday party, the hostess wants to give away candies. She has 6 types of cookies. Each child is given a gift packet which contains two types of cookies. Each type of cookie is used is combination with at least three others. Prove there are three children, who between them, have all the six types of cookies.

Solution. Form a graph with each type of candies corresponding to a vertex. Two vertices are joined by an edge if the corresponding types of candies are used together in a gift pack. In this graph every vertex is of degree ≥ 3. To solve the problem, we need to show that the graph contains three edges which are pairwise nonadjacent (such a set of edges are said to be independent). Let a be a vertex and b, c, d be 3 of its neighbours. Let the remaining two vertices be e, f (these may also be neighbours of a). Finally, let $A = \{a, b, c, d\}$ and $B = \{b, c, d\}$. Note that $|N(e) \cap A| \geq 2$ and $|N(f) \cap A| \geq 2$. If $|(N(e) \cup N(f)) \cap B| \geq 2$, then there exists 2 vertices in B, say b and c, such that be and cf are edges. Then be, cf and ad are 3 independent edges. If $|(N(e) \cup N(f)) \cap B| = 1$, say b is the common neighbour of e and f, then e and f are both adjacent to a. Since the degree of d is ≥ 3, and d is not adjacent to e, f, d must be adjacent to c and b. Thus ae, bf and cd are 3 independent edges.

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MS 89 (3-4) 2020 : Problem 4 (Posed by Prof. B. Sury, ISI, Banagalore). In a meeting, n participants sit around a table and are served drinking water in the following manner. First, any one of the participants is selected and served. Then, moving clockwise, one participant is skipped and the next is served. The next two are skipped, and the next one is served; the next 3 are skipped and the next person is served and so on. After a while, everyone has been served water at least once. Prove that if

$n > 1$, it cannot be odd.

A correct solution is given by Rakesh Dwivedi (M. Sc. student), Department of Mathematics, University of Allahabad, Allahabad (U. P.). Prof. Sury proves a stronger result here and also point out on the way what Dwivedi's solution was.

Solution. The hypothesis is equivalent to asserting that the set of triangular numbers $\{t_k = \frac{k(k+1)}{2} | k > 0\}$ contains all the residues modulo n among them. Call such numbers n as ideal. We will prove the stronger result that ideal numbers are just the powers of 2. But, first we describe Dwivedi's proof of the original problem.

Dwivedi notices that if $t_k \equiv a \bmod n$, then $8t_k + 1 = (2k+1)^2 \equiv 8a+1 \bmod n$. This means that the residues of $8a+1$ as a varies (these are distinct if n is odd) are all squares modulo n. In particular, if n is odd, for each prime p dividing n, all the residues are quadratic residues mod p which contradicts the fact that only half of them are squares.

Let us return to our proof. The residues of t_k modulo n repeat after t_n; indeed,

$$t_{n+k} - t_k = \frac{n(2k+n+1)}{2}$$

which is a multiple of n.

From this, it follows immediately that an odd $n > 1$ cannot be ideal. This is because $t_{n-1} = \frac{(n-1)n}{2}$ and $t_n = \frac{n(n+1)}{2}$ are both multiples of n for odd $n > 1$. Hence, some residues modulo n are not values. Thus, n cannot be an odd integer > 1.

To see that the set of all ideal numbers consists precisely of powers of 2, note first that if n is ideal and d divides n, then d is ideal.

Thus, an ideal number $n > 1$ must be a power of 2. Every number n of the form 2^r is ideal because the numbers

$$t_1, t_3, t_5, \cdots, t_{2n-1}$$

give different residues modulo n. Indeed,

$$t_{2k-1} - t_{2l-1} = (2k-1)k - (2l-1)l = (2k+2l-1)(k-l)$$

can be a multiple of $n = 2^r$ only if $k - l$ is a multiple (as the other factor is odd).

□

MS 89 (3-4) 2020: Problem 7 (Posed by Dr. Anup Dixit, IMSc., Chennai).

For a real number x, Let $[x]$ denotes the largest integer $\leq x$. Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{4n+1}]}}{n(n+1)}$$

.

Mr. Rohit Yadav (M. Sc. student), Department of Mathematics, University of Allahabad, Allahabad (U. P.) provided the correct solution to this problem. The solution is given below.

Solution.

Step 1: Since $[\sqrt{4n+1}]$ are same for such $n \in N$, which is satisfying -

$$k^2 \leq 4n+1 < (k+1)^2, k \in N$$

$$\Rightarrow k^2 - 1 \leq 4n < k^2 + 2k............(1)$$

So, $[\frac{2k+1}{4}] + 1$ number of distinct $n \in N$which satisfy (1).

Step 2: Now we grouping of terms of given series as kth group contains $[\frac{2k+1}{4}] + 1$ terms.

By step 1, $[\sqrt{4n+1}]$ are same for each term of k^{th} group and for k^{th} group $[\sqrt{4n+1}]$ = k+1 Also, $(2m)^{th}$ and $(2m+1)^{th}$ groups containing equal number of terms $\forall m \in N$.

Hence by above discussion,

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{4n+1}]}}{n(n+1)} = \frac{(-1)^2}{1.2} + (-1)^3(\frac{1}{2.3} + \frac{1}{3.4}) + (-1)^4(\frac{1}{4.5} + \frac{1}{5.6})$$

$$+(-1)^5(\frac{1}{6.7} + \frac{1}{7.8} + \frac{1}{8.9}) + (-1)^6(\frac{1}{9.10} + \frac{1}{10.11} + \frac{1}{11.12}) + ...$$

$$= (1 - \frac{1}{2}) - (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{6}) - (\frac{1}{6} - \frac{1}{9}) + (\frac{1}{9} - \frac{1}{12}) + ...$$

$$= 1 - 2(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{12} - \frac{1}{16} + ...)$$

$$= 1 - 2[(\frac{1}{2} - \frac{1}{4}) + (\frac{1}{6} - \frac{1}{9}) + (\frac{1}{12} - \frac{1}{16}) + ...]$$

$$= 1 - 2[\frac{2}{2.4} + \frac{3}{6.9} + \frac{4}{12.16} + ...]$$

$$= 1 - 2\sum_{m=1}^{\infty} \frac{1}{m(m+1)^2} = 1 - 2\sum_{m=1}^{\infty}(\frac{1}{m(m+1)} - \frac{1}{(m+1)^2})$$

$$= 1 - 2(1 - \sum_{m=1}^{\infty} \frac{1}{(m+1)^2})$$
$$= 1 - 2(1 - \frac{\pi^2}{6} + 1)$$
$$= \frac{\pi^2}{3} - 3$$

.

We conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{4n+1}]}}{n(n+1)} = \frac{\pi^2}{3} - 3$.

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(See Rule 8)

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