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(Dedicated to Prof. N. K. Thakare in honour of his long and dedicated services to the Indian Mathematical Society)

THE MATHEMATICS STUDENT

Edited by M. M. SHIKARE

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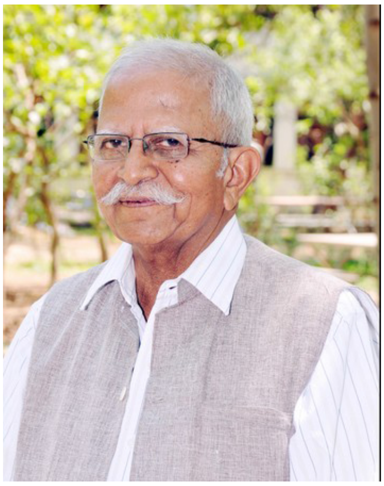
PROFESSOR N. K. THAKARE: A BRIEF MATHEMATICAL PROFILE*

SHRIRAM K. NIMBHORKAR, MEENAKSHI P. WASADIKAR AND MADHUKAR M. PAWAR



Professor Nimba Krishna Thakare was born on April 22, 1938 at the village Morane, Dist. Dhule in Maharashtra State, India. After completing his school education in Dhule, he completed graduation from Fergusson College, Pune and M. Sc. (Mathematics) from the Royal Institute of Science (Govt. Institute of Science), Bombay in 1962.

After M. Sc. he served in some colleges under Bombay university, including, its P. G. Centre at Panaji, Goa. He completed Ph.D. (Mathematics) in 1972 under the guidance of Late Prof. B. R. Bhonsle, in Special Functions. He was the first researcher to receive the Ph. D. degree in Mathematics from Shivaji University, Kolhapur.

Days at Shivaji University (1969–1977):

Prof. Thakare joined Shivaji University in 1969. During his tenure at Shivaji University he introduced several new courses for M. Sc. students such as Banach Algebra, Distribution Theory, Lattice Theory, Topological Vector Spaces and Lie theoretic approach to Special Function Theory. In fact, he started researches in several areas such as orthogonal polynomials, bi-orthogonal polynomials, Geometry of Banach spaces, Operator theory and Lattice theory. His general approach for research was to tackle the open problems posed by eminent mathematicians in that particular area.

∗ This article is written on the occasion of retirement of Prof. N. K. Thakare as the General Secretary of the Indian Mathematical Society.

R. Askey, [*J. Math. Anal. Appli., 24(1968), 677–685*] queried:

“ *Whether dual series equations involving Jacobi polynomials of different indices could be solved.* ”

N. K. Thakare [*Zentl. Aung. Math. Mech. 54(1974), 283–284*] explicitly solved the dual series equations involving Jacobi polynomials of different indices by modifying Noble's multiplying factor technique. K. R. Patil and N. K. Thakare [*J. Math. Phy., 18(1977), 1724–1726*] gave solutions to the dual series equations involving Laguerre - Konhauser polynomials of the first kind and of different indices. N. K. Thakare and B. K. Karande obtained several q-identities and gave solutions to many problems posed by L. Carlitz.

Prof. Thakare's substantial contribution is the unification of all the classical orthogonal polynomials; namely the Jacobi polynomials, the Laguerre polynomials and the Hermite polynomials. It is being called Szegö–Fujiwara–Thakare approach of unification by few authors. N. K. Thakare, B. R. Bhonsle and R. D. Prasad [*Indian J. Pure Appl. Math. 6 (1975), 637–647*] unified the study of Jacobi Transform, Laguerre Transform and Hermite Transform by this approach. In a rather different direction B. K. Karande and N. K. Thakare [*Indian J. Pure Appl. Math. (1975), 98–107*] unified the Bernoulli and Euler polynomials. K. R. Patil and N. K. Thakare [*SIAM Jour. Math. Anal., 9(1978), 921–923*] obtained a multilateral generating function involving both the Laguerre–Konhauser biorthogonal polynomial sets, namely $Y_n^\alpha(x;k)$ and $Z_n^\alpha(x;k)$.

In Geometry of Banach spaces the Dunford–Pettis property and the Schur property play important roles. The question of characterizing the Schur property in Banach spaces is still open. In that direction P. V. Pethe and N. K. Thakare [*Indiana Univ. Math. J., 27 (1978), 91–92*] proved the following result.

(1) *If X has the Dunford–Pettis Property and is isomorphic to a subspace of Y^*, where $Y \not\supseteq l_1$, then X has the Schur property.*
(2) *The dual X^* of a real Banach space X has the Schur property if and only if X has the Dunford–Pettis property and $X \not\supseteq l_1$.*

P. V. Pethe and N. K. Thakare [*J. Indian Math. Soc., 41(1977), 371–373*] provided a partial answer to R. R. Phelp's conjecture by proving the result that in an Asplund space X, the following statements are equivalent :

(1) *X has the property (I),*

(2) *The strongly exposed points of the unit disc U^* of X^* are dense in the unit sphere S^* of X^*.*

P. V. Pethe and N. K. Thakare [*Kyungpook Math. J., 17(1977), 109–115*] discussed interrelationship between the Dunford-Pettis property, the Schur property and the famous Radon-Nikodym property in real Banach spaces along with semi-separability, almost reflexivity, smoothness and related concepts and settled a few open problems. N. K. Thakare [*Proc. Int. Symposium Modern Anal. Appl. Edited by H. L. Manocha, Prentice Hall (1986), 80–114*] has adequately surveyed the results obtained by them in Geometry of Banach Spaces and posed several new problems.

S. V. Phadke and N. K. Thakare [*Kyungpook Math. J., 18 (1978), 201–205*] studied different types of operators on Hilbert spaces such as ρ-convexoid operators, ρ-oid operators and spectroloid operators and related topics such as spectrum, numerical ranges and their generalizations. S. V. Phadke and N. K. Thakare [*Math. Japonica, 23 (1978), 494–500; ibid, 9(1978), 1263–1270*] obtained many results involving ρ-oid operators, spectroloid operators, M-hyponormal operators and Spectral localization and supplied several counter examples to falsify some conjectures. S. K. Khasbardar and N. K. Thakare [*Boll. Union. Math. Italiana, 15 (1978), 581–584*] obtained interesting results concerning commutativity and numerical ranges for elements of a Banach algebra. S. V. Phadke and N. K. Thakare [*Linear Algebra and Applications, 23(1979), 191–199*] established conditions under which the operator equations $AX = C, AXA^* = C$ have Hermitian and non negative definite solutions, it is being assumed that A is a relatively invertible operator on a Hilbert space. Two open problems posed by them in the said paper received overwhelming response.

Prof. N. K. Thakare tried to solve open problems posed by G. Birkhoff [*Lattice Theory, Amer. Math. Soc. Colloq. Publ., 25, 1979*], F. Maeda and S. Maeda [*Theory of Symmetric Lattices, Springer Verlag, 1970*], G. Grätzer [*General Lattice Theory, Academic Press, 1978*, *First Concepts and Distributive Lattices, Freeman, 1971*]. Birkhoff gave a list of 166 open problems, F. Maeda and S. Maeda listed around 13 unsolved problems and Grätzer has listed 193 open problems. Grätzer [*op. cit. 1971, 1978*] posed the question:

"*How to characterize $\mathcal{P}(L)$, the poset of all prime ideals of a distributive lattice L?*"

Y. S. Pawar and N. K. Thakare [*Algebra Universalis, 7 (1977), 259–263*] introduced the concept of a pm-lattice and succeeded in characterizing $\mathcal{P}(L)$ as a normal space in terms of pm-ness. Y. S. Pawar and N. K. Thakare [*Canadian Math. Bull., 21 (1978), 469–475; ibid, 23(1980), 291–298*] investigated 0-distributive semilattices. They obtained several interesting characterizations for prime semilattices. Y. S. Pawar and N. K. Thakare [*Periodica Math. Hungarica, 13(1982), 237–246; ibid 13(1982), 309–319*] investigated extensively the minimal prime ideals in a 0-distributive lattice. They characterized the minimal spectrum of a 0-distributive semilattice and succeeded in obtaining topological and algebraic properties of the minimal spectrum of a 0-distributive semilattice by using lattice theoretic techniques, a very pleasant situation indeed.

Prof. Thakare along with his two Ph.D. students D. A. Patil and S. R. Kulkarni has studied the long standing open problem in the Geometric Function Theory, namely Bieberbach's conjecture of 1916. In that context they introduced in separate papers, several new classes of univalent and multivalent functions. The most conspicuous part of their investigations is the use of Krein–Milman theorem of Functional Analysis to determine convex hulls of extreme points of a few classes of univalent functions; see N. K. Thakare and D. A. Patil [*Mathematica(cluj), 24(47) (1982), 112-116*] and [*Bull. Math. Soc. Sci. Math., R. S. Roumanie (N.S.), 27(75) (1983), 145-160*].

M. Ward and R. P. Dilworth [*Trans. American Math. S0c., 45 (1939), 335–354*] developed an abstract formulation of the Noether decomposition theorems for commutative rings via the vehicle of multiplicative lattices. The spectral theory of various algebraic systems such as that of rings, distributive lattices was being pursued as separate and unrelated identities. N. K. Thakare and C. S. Manjarekar with the active assistance of Prof. S. Maeda of Japan built up an abstract spectral theory through a series of papers; see N. K. Thakare and C. S. Manjarekar [*Algebra and its Applications; Editors - H. L. Manocha and J. B. Shrivastava, Marcel Dekker (1984), 265–276*]. They liberally used the character theory to develop smoothly the abstract spectral theory encompassing in it both the spectral theory of commutative rings as well as of distributive lattices; see Thakare, Manjarekar and Maeda [*Acta Sci. Math. (Szeged)*, 52(1–2) (1980), 53–67]. Here note that a character is nothing but a homomorphism from a multiplicative lattice onto the two element chain $\{0, 1\}$; see also N. K. Thakare and C. S. Manjarekar [*Studia Sci. Math. Hungarica, 18 (1982), 13–19*] wherein they have studied radicals and uniqueness theorems in multiplicative lattices with chain conditions.

Neeta Bharambe of Thakare's group weakened further the concept of a character and generalized abstract character theory; see Neeta Bharambe [*A study of some generalized continuous and multiplicative lattices, Ph.D. Thesis (1994), University of Poona*].

Days at Dr. Babasaheb Ambedkar Marathwada University, Aurangabad (1977–1985):

Prof. Thakare joined the then Marathwada University, Aurangabad in 1977. There he continued his research in orthogonal polynomials with his former student H. C. Madhekar.

The concept of orthogonality of a sequence $\{P_n(x)\}$ in the Hilbert space $L_2[a, b]$ is classical. It gets readily extended to bi-orthogonality wherein two sets $\{R_n(x)\}$ and $\{S_n(x)\}$ are involved. The realization of the orthogonality concept via Jacobi polynomials, Laguerre polynomials and Hermite polynomials is several decades old. However, that is not the case in the case of bi-orthogonality.

Konhauser [*J. Math. Anal. Appl. 11(1965), 242-260*] established general properties of bi-orthogonal polynomials in $L_2[a, b]$. Later the concrete realization of a pair of bi-orthogonal polynomials with respect to the Laguerre weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$ was realized by Konhauser [*Pacific J. Math. 21(1967), 303-314*] who constructed a pair of bi-orthogonal polynomials $Z_n^\alpha(x, k)$ and $Y_n^\alpha(x, k), (\alpha > -1, k \textit{ a positive integer})$. It is the most general pair of bi-orthogonal polynomials with respect to Laguerre weight function $x^\alpha e^{-x}(\alpha > -1)$. These two sets $Z_n^\alpha(x, k)$ and $Y_n^\alpha(x, k)$ are now called Laguerre-Konhauser bi-orthogonal polynomials of the first kind and second kind, respectively. These investigations naturally led to the following queries.

Question–(A): *Can a bi-orthogonal pair of polynomials be constructed with respect to the Jacobi weight function* $(1-x)^\alpha(1+x)^\beta, (\alpha, \beta > -1)$ *over the interval (-1, 1)?*

Question–(B): *Can a bi-orthogonal pair of polynomials be constructed with respect to the Hermite weight function* e^{-x^2} *over the interval* $(-\infty, \infty)$*?*

In a surprising breakthrough Question-(A) has been completely settled by N. K. Thakare and H. C. Madhekar [*Pacific J. Math. 100(1982), 417–424*] by constructing the pair of bi-orthogonal polynomials $J_n(\alpha, \beta, k; x)$ and $K_n(\alpha, \beta, k; x)$ which are now called Jacobi–Thakare–Madhekar polynomials of the first kind and the second kind, respectively.

The Question–(B) is also completely settled by Thakare and Madhekar [*Indian J. Pure Appl. Math. 17(1986) 1031-1041*] wherein they constructed a pair of polynomials $S_n(x, k)$ and $T_n(x, k)$ (k, an odd integer) that are bi-orthogonal with respect to the Hermite weight function e^{-x^2} over the interval $(-\infty, \infty)$. These are now called Hermite -Thakare- Madhekar bi-orthogonal polynomials of the first kind and the second kind, respectively. Thakare and Madhekar [*International J. Math. and Math. Sci. 4(1988), 763–768*] also succeeded in constructing a pair of bi-orthogonal polynomials with respect to the Szegö–Hermite weight function $|x|^{2\mu} exp{-}x^2, (\mu > -\frac{1}{2})$ over the interval $(-\infty, \infty)$.

Interestingly, Alpana Kulkarni [*M. Phil. Dissertation (1990)*, University of Poona] succeeded in unifying all these different pairs of bi-orthogonal polynomials with respect to classical weight functions (i.e. Jacobi, Laguerre and Hermite) by Szegö–Fujiwara–Thakare approach.

There is vast scope for research in the general theory of bi-orthogonal polynomials on the lines of rich theory available for orthogonal polynomials. In this context see the survey article of N. K. Thakare [*Proc. Soc. Special Functions and Its Appl. 3 (2002) 13–40.*] wherein difficult open problems are listed for the first time.

The concept of a modular pair was available for many years for lattices. Much of this theory is well documented in F. Maeda and S. Maeda [*op. cit.*]. Birkhoff [*op. cit. p. 109*] posed the following problem.

Birkhoff's Open Problem - I : How should one define a modular pair in a general poset?

N. K. Thakare, M. P. Wasadikar and S. Maeda, [*Algebra Universalis, 19(1984) 235–265*], and S. Maeda, N. K.Thakare and M. P. Wasadikar, [*ibid, 20(1985), 229–242*] have satisfactorily built up the theory of modular pairs in semilattices with 0; see also N. K. Thakare, [*Math. Student, 57(1989), 225–235*] wherein he has adequately surveyed the concepts of symmetricity such as modular pairs, covering property, exchange property etc. in semilattices.

The $*-$algebra of operators on a Hilbert space containing the identity operator that are closed in the weak operator topology are called von Neumann algebras (also called W^*-algebras). In an attempt to propose an algebraic setting for the theory of von Neumann algebras, Kaplansky [*Ann. Math. 53(1957), 235–249*], [*Rings of operators, Benjamin, 1968*] introduced the study of Baer $*$-rings and AW^* algebras. There are weakened versions of the concept of Baer $*$-rings such as Rickart $*$-rings, weakly Rickart $*$-rings. Earlier the abstract spectral theory for lattices and commutative rings were discussed in the context of multiplicative lattices. Surprisingly N. K. Thakare and S. K. Nimbhorkar [*J. Pure and Appl. Algebra, 27(1983), 75–85*] succeeded in extending this to noncommutative rings as well as to Rickart $*$-rings and Baer $*$-rings. They proved a crucial separation theorem for Rickart $*$-rings and studied the prime spectrum of such rings that resulted in acquiring a sheaf representation; see also N. K. Thakare and S. K. Nimbhorkar [*Indian J. Pure Appl. Math. 22(1991), 63–72*]. Moreover, N. K. Thakare and S. K. Nimbhorkar [*J. Indian Math. Soc. 59(1993), 179–190*] extended the study of symmetricity such as modular pairs, compactible pairs and comparability axioms to Rickart $*$-rings. We shall allude to these topics at a later stage.

Days at Savitribai Phule Pune University, Pune and North Maharashtra University, Jalgaon (1985-1998):

The fruits of sustained and continuous research of Prof. Thakare and his students became conspicuously visible when he joined the then University of Poona in 1985 as a Professor of Mathematics (Lokmanya Tilak chair). Most of his students interested in research continued converging in the Mathematics Department of Pune University and tried to concentrate on problems that were not fully solved earlier. During 1990 he was appointed as the first Vice-Chancellor of the North Maharashtra University, Jalgaon. This university was established for Dhule and Jalgaon districts, which were under the jurisdiction of University of Poona at that time. He showed his ability as the best administrator. Inspite of his busy schedule as the vice-chancellor, he guided his research students to tackle some of the open problems posed by Kaplanky, Berberian and Birkhoff. It gives us a great pleasure to state that, Prof. Thakare was not only a guide but a father figure for his students. During his stay at Jalgoan, most of his research students coming for their work, used to stay at his rented and modest residence and after completing his hectic work, he used to guide the students till late night. As a result Prof. Thakare and his group could continue their tradition to provide solutions (in totality or partially) to some of the open problems.

It is well known that Baer $*$-rings are an algebraic compromise between operator algebras and lattice theory. I. Kaplansky's student, S. K. Berberian lists more than 30 open problems in his book [*Baer $*$-rings, Springer Verlag, 1972*]. It was stated earlier that this theory has its roots in von Neumann's theory of rings of operators and in the book of Kaplansky [*op. cit.*]. Let us list some of these problems for which Thakare's group could find either complete solution or a partial solution. For unexplained terms refer to S. K. Berberian [*op. cit.*].

KB Problem-I (Kaplansky-Berberian's Problem - I) : *Can a unitary of a properly infinite AW^*-algebra be expressed as a product of four symmetries?*

KB Problem–II: *Can any weakly Rickart $*$-ring be embedded in a Rickart $*$-ring with preservation of right projections (RPs)?*

KB Problem–III: *Does every Rickart C^*-algebra have a polar decomposition (PD)?*

KB Problem–IV: *Let R be a Baer $*$-ring and R_n, the $*$-ring of $n \times n$ matrices over R. Is R_n a Baer $*$-ring?*

KB Problem-V : *Let R be a Baer $*$-ring with general comparability (GC) and let e, f be projections in R such that $e \sim f$. Do there exist orthogonal decompositions $e = e_1 + e_2, f = f_1 + f_2$ with $e_i, f_i (i = 1, 2)$ unitarily equivalent?*

Let us begin with KB Problem–I. The roots of this problem lie in the important result of P. R. Halmos and S. Kakutani [*Bull. Amer. Math Soc. 52(1958), 77–78*] that says that a unitary operator on an infinite dimensional Hilbert space is a product of at most four symmetries (i.e. self adjoint unitaries). N. K. Thakare and Asha Baliga [*Proc. Amer. Math. Soc. 123(1995). 1005–1008*] answered KB Problem-I in the affirmative using purely algebraic techniques.

B. N. Waphare [*M. Phil. Dissertation (1988), University of Poona*] succeeded in answering the KB Problem–II in two different directions and closed the gap substantially in solving the said problem; see also N. K. Thakare and B. N. Waphare [*Indian J. Pure Appl. Math. 28(1997), 189–195*]. A. Baliga [*Ph.D. Thesis (1992), University of Poona*] parially answered KB Problem–III and KB Problem–V. B. N. Waphare [*Symmetricity in Posets, Comparability Axioms in orthomodular Lattices with Applications, Ph.D. Thesis, University of Poona, (1992)*], gave a ready reckoner that classifies the matrix $*$-rings over the ring $\mathbb{Z}_m$ of integers modulo m into the Baer $*$-rings and non Baer $*$-rings giving a partial solution to KB Problem–IV. It may be mentioned here that, there are still many unsolved problems in Baer $*$-rings; see N. K. Thakare and B. N. Waphare [*Some open problems in Baer $*$-rings: A Survey, Proc. International Symp. Theory of Rings and Modules, Edited by S. Tariq Rizvi and S. M. Zaidi (2005), 145–159, Anamaya Publishers, New Delhi*].

Alongwith Birkhoff's Open Problem–1 mentioned earlier, Prof. Thakare's group also concentrated on the following open problems posed by Birkhoff in his book Lattice Theory (1940) and are to be found in the revised editions also.

Birkhoff's Open Problem - II: Compute for small 'n'all non-isomorphic lattices on a set of 'n'elements. Also find asymptotic estimates and bounds for this rate of growth of the number of non-isomorphic lattices with 'n' elements.

Birkhoff's Open Problem - III: Enumerate all finite lattices up to isomorphism by their diagrams.

Birkhoff's Open Problem - IV: Define natural extensions of the notions of neutral elements and standard elements to posets.

In the context of Birkhoff's Open Problem - II, D. N. Kolhe [*M. Phil. Dissertation, Univerisity of Poona, (1989)*] obtained the diagrams of all non-isomorphic semilattices upto $n = 8$ elements and all non-isomorphic semilattices with $n = 7, 8$ and 9 elements by constructing an algorithm that permits one to pass from a poset with n elements to a semilattice with $n + 1$ elements and later to a lattice and so on.

M. M. Pawar [*M. Phil. Dissertation, University of Poona (1991)*] covers substantial ground by completely determining the number of lattices upto isomorphism with n elements and n edges by introducing an ingenious and novel notion of 2-sum of a lattice and a chain; see also M. M. Pawar and B. N. Waphare [*Indian J. Math. 45(2003), 315–323*]. But much

ground was covered by N. K. Thakare, M. M. Pawar and B. N. Waphare [*Periodica Math. Hungarica 45(2002)* 147–160] wherein the concept of, 'adjunct operation' of two lattices with respect to a pair of elements was introduced that gave a nice Structure Theorem for Dismantlable Lattices. The investigations therein led them to find:
(i) The number of lattices with exactly two reducible elements.
(ii) The number of lattices with n elements and up to $n+1$ edges.
(iii) The number of distributive lattices and number of modular lattices with n elements and up to $n+1$ edges.

We are tempted to list the number for the case–(ii) above.

For every integer $n \geq 5$ the number of lattices with n elements and up to $n+1$ edges is

$$(n-2) + \sum_{k=0}^{[\frac{n-5}{2}]}(n-2k-4)\left\langle \frac{14k^4+54k^3+68k^2+48k+21}{12} \right\rangle + \sum_{k=0}^{[\frac{n-6}{2}]}(n-2k-5)\left[\frac{(k+2)(7k^3+27k^2+31k+19)}{6} \right].$$

Here $[a]$ denotes the largest integer less than or equal to the real number a and $< a >$ denotes the integer nearest to a: see also N. K. Thakare and M. M. Pawar [*South East Asian J. Math. Sci., 10 (2012), 97-113*]. Their findings is a big breakthrough in the context of Birkhoff's Open Problems II and III.

The research work carried out in the context of Birkhoff's Open problems I and IV by Professor Thakare's group is discussed in the papers N. K. Thakare, M. M. Pawar and B. N. Waphare [*J. Indian Math. Soc., 71(2004), 13–53*] and N. K. Thakare, S. Maeda and B. N. Waphare [*ibid, 70(2013), 229–253*]. They defined natural extensions of the concepts of Neutral elements, Standard elements, Distributive elements and Modular elements in a general poset and obtained several characterizations and deep results envisaged by G. Birkhoff.

Earlier we mentioned the attempts of N. K. Thakare, M. P. Wasadikar and S. Maeda, in making breakthrough in defining a modular pair in a join-semilattice with 0. They obtained many characterizations and deep results involving their concept of a modular pair. But the restriction of the presence of 0 was too severe. Later the group of Thakare consisting of B. N. Waphare [*op. cit.*], Meenakshi Dhanke [*Covering Property, exchange property and related results in meet-semilattices, M. Phil. dissertation, University of Poona (1989)*] and M. M. Pawar [*Symmetricity and Enumeration* *of Posets, Ph.D. Thesis, North Maharashtra University, Jalgaon (1999)*] removed that restriction as well as extended the concept of a modular pair to posets. B. N. Waphare [*op. cit.*] gave two independent definitions and later M. M. Pawar [*op. cit.*] gave additional three definitions of modular pairs. This opened up a broad window for further research; see N. K. Thakare, S. Maeda and B. N. Waphare [*ibid., 70(1–4), (2003),229–253*] and N. K. Thakare, M. M. Pawar and B. N. Waphare [*J. Indian Math. Soc. 71(1–4),(2004) 13–53*].

Now, we highlight the research he did with his last Ph.D. student V. S. Kharat [*Reducibility, Heyting operation and Related Aspects in Finite Posets, Ph. D. Thesis, University of Poona, (2001)*]. The question of extending the concept of reducibility from finite lattices to finite posets was wide open. V. S. Kharat, B. N. Waphare and N. K. Thakare [*European J. Combinatorics, 22(2001), 197–205*] studied in depth the concept of reducibility for posets. They gave many counter examples to show that the classes reducible for lattices are not reducible for posets. They have shown that the class of pseudocomplemented n-posets are reducible. Interestingly, V. S. Kharat, B. N. Waphare and N. K. Thakare [*Discrete Appl. Math. 155 (2007) 2069–2076*] introduced and studied the reducibility number of the lattice of the power set of an n element set with respect to the class of distributive and modular lattices. They gave an interesting application of this new concept to group theory namely:

"*If the reducibility number of $L(G)$, the lattice of all subgroups of a finite group G with respect to the class of distributive lattices is a prime number, then the order of G is divisible by exactly three distinct prime numbers.*"

The problem of characterizing posets in terms of forbidden structures was open. By introducing the novel ideas of Strongness, Balancedness and Consistantness in posets, V. S. Kharat, B. N. Waphare and N. K. Thakare [*Algebra Universalis, 51(2004), 114-124*] characterized upper semimodular posets as strong posets in number of ways involving forbidden structures.

Visits abroad

Prof. Thakare visited many leading Universities and institutions throughout the world. He visited Hungary as a visiting professor at the University of Budapest (March–September 1979) under the Indo-Hungarian cultural exchange program. He was invited to deliver the Key Note Address at the 4^{th} International Symposium on Operator Theory, held at Timisioara, Rumania (1979). Professor Izuru Fujiwara invited him to the University of Osaka Prefeeture, Japan under the programme of Japanese Society for Promotion of Science. He was invited as a visiting professor at Department of Mathematics and Statistics, McGill university, Montreal, Canada (January–March 1989). U. G. C. nominated him as a visiting professor at Steklov Institute of Mathematics, Moscow, U. S. S. R. (May-July 1989). He was invited to deliver the Key Note Address at the International conference on Geometry of Banach Spaces, held at Varna, Bulgaria in 1989.

Other activities:

Prof. Thakare has organized several International conferences/ Symposia and delivered more than hundred talks in various conferences held in India and abroad.

During the Birth centenary year of Shrinivas Ramanujan, in 1987, he organized an "International Symposium on Analysis"at Pune. Many renowned foreign mathematicians including R. Asky, G. E. Andrews, A. M. Mathai, Mourad Ismail, M. Rahman, M. A. Al–Bassam and several others actively participated in the symposium. The proceedings of this symposium was edited by Professor Thakare with the assistance of Prof. T. T. Raghunathan and Prof. K. C. Sharma and published by McMillan.

In 1988, the 75^{th} Session of the Indian Science Congress was held at the University of Poona. Prof. Thakare played an important part in its organization. At the 79^{th} Session of the Indian Science Congress Association, held at Baroda in 1992, Professor Thakare was the *President for the Section of Mathematics.*

Professor Thakare was elected as a *Fellow of the National Academy of Sciences* and also as a *Fellow of the Maharashtra Academy of Sciences.* He is a member of the Indian Science Congress Association, the Indian Mathematical Society, American Mathematical Society, Society for Special Functions and Its Applications, etc.

Prof. Thakare has guided 17 students for Ph.D. and 10 students for M.Phil. He has published about 200 research articles in reputed journals.

The Indian Mathematical Society and Professor Thakare (1988–2019):

In December 1988, Professor Thakare successfully organized 54^{th} Annual Conference of the IMS at Pune University. Prof. Thakare is associated with the IMS as a council member since 1989. He was the *President* of the

IMS for 1995–96, and presided over the deliberations of the 61^{st} Annual Conference of the IMS held at Aligarh Musllim University, Aligarh in December 1995.

He was the *Academic Secretary* of the IMS from 1998 to 2007. Later he shouldered the responsibility of the *Editor of the Journal of the Indian Mathematical Society* during 2007-2013. Under his initiative the 'Special Centenary Volume' of the Journal of the Indian Mathematical Society was published in 2007. He used modern technology such as computers and internet. All the editorial work of the journal such as submission, refereeing etc. was started through e-mail. This reduced the time period between submission of a paper and decision on it. The authors of the accepted papers were asked to submit the final copy in Latex format. This reduced the time in typesetting of the final copy of the Journal. As a result the publication was on time. The Journal of the Indian Mathematical Society is supposed to be a quarterly periodical. But for many years all the four issues were published as a single volume. Prof. Thakare started publishing JIMS in two parts by combining issues 1 and 2 in Part 1 and issues 3 - 4 in Part 2. The issues were sent to the subscribers in January and July, so many new subscribers were added. After the retirement of Late Prof. V. M. Shah as the General Secretary from IMS, Prof. Thakare became the *General Secretary* (2013 - 2019). The IMS started publishing the JIMS on the online platform of **Informatics India**. The old issues of JIMS have been digitized. The Indian Mathematical Society progressed a lot under his leadership.

Other publications:

Apart from mathematical publications, Prof. Thakare also authored a collection of short stories entitled *"Nimbol"(1984)* and a collection of poems entitled *"Shunya aani Infinity"(1996)* in vernacular Marathi language. He was recipient of prestigious Reoo pratishthan award for his story *"Jyula"* in 1984. He was appointed as a member of Marathi Shabdkosh Mandal. He is also a member of several literary and cultural associations.

A Mathematician as an Academic Administrator :

It is not only our pleasure but also a privilege to be Ph.D. students of Prof. Thakare during the time when he was acting as an administrator of academic institutions. S. K. Nimbhorkar and M. P. Wasadikar worked with Principal N. K. Thakare at Jaihind College, Dhule during 1982-1985.

M. M. Pawar was a teacher fellow when Prof. N. K. Thakare was the Vice-Chancellor (1990-1996) at North Maharashtra University, Jalgaon. We note in brief his administrative work. His administrative contributions are publicly more admired than his mathematical contributions. We present here some main qualitative aspects in his administration.

As the founder principal of Jai Hind College, Dhule. He showed novel creativity in each action. He started new subjects which were not taught outside Pune city at that time. He arranged visits of some foreigners to the college.

As the first Vice-Chancellor of North Maharashtra University, he started innovative courses at the university such as M.Sc. in Pesticides and Agro-chemicals, M.C.A., M.Sc. Computer Science, M.Sc. Computational Mathematics, B.Tech. in Paints, Oil and Vaxes, M.Sc. Microbiology and Biotechnology etc. in Science faculty. B.B.S.(Bachelor in Business Studies), B.B.A., B.B.M. and M.B.A. in Commerce and Management faculty and M.A. in Comparative literature and dialects in Arts faculty. He adopted school concept for the first time in a state university. In fact he himself used to teach some courses to the students of M.Sc. (Mathematics). With the help of enthusiastic and sincere faculty members in the university and affiliated colleges, low cost innovative study material including text books were made available for students. He used his mathematical ability and personal contacts in the constructions of quality buildings with minimum cost. This fact was widely appreciated.

Observing the loopholes in the examination system he made many examination reforms which included abnormal size of the answer sheets, regionwise flying squads visiting the examination centers to stop the malpractices. He introduced centralized admission process for professional courses. The same was appreciated by the State Government and was implemented in other universities in Maharashtra.

It is a matter of great pride that even after relinquishing the office of the Vice–Chancellor some twenty three years ago, the staff in the said university respectfully recall the golden six years they spent with Hon'ble Thakare Sir.

He not only worked wholeheartedly but inspired his colleagues to work hard and with devotion. As a result, his work as the principal for three years at Jai Hind college and Vice–Chancellor for six years at North Maharashtra

University, Jalgaon, is and will be remembered forever in the university and affiliated colleges of North Maharashtra University.

In 2003 Professor Thakare established a trust in the name of his parents. He concentrated on improving primary and secondary education. He and his wife sold their houses at Pune and Dhule and diverted their retirement benefits for the development of the school. Prof. N. K. Thakare and Mrs Pushpal Thakare work hard for the school from morning to evening.

As gratitude to Prof. Thakare and his old friend / colleague Prof. T. T. Raghunathan, their students in University of Poona had organized a National Conference in 2013 on the occasion of completion of 75 years of age of both Prof. Thakare and Prof. T. T. Raghunthan. Many good mathematicians from various parts of India graced the occasion. Even at the age of 80, he helps his students in academic activities. Prof. Thakare was instrumental in organizing an "International Conference on Algebra, Discrete Mathematics and Applications" during December 2017 at Dr. Babasaheb Ambedkar Marathwada University, Aurangabad. This conference was organized by Meenakshi Wasadikar. Many world known mathematicians, including Fields Medalist Prof. Efim Zelmanov participated in the conference.

We wish and pray for long, prosperous, peaceful and healthy life for both Prof. Thakare and Madam Thakare.

S. K. Nimbhorkar and M. P. Wasadikar
Formerly of Dept. of Mathematics,
Dr. B. A. M. University, Aurangabad-431004, India
E-mail: *sknimbhorkar@gmail.com* ; *wasadikar@yahoo.com*

M. M. Pawar
Formerly of Dept. of Mathematics,
Dr. P. R. Ghogrey Science College, Dhule-424002, India.
E-mail: *m2pawar@yahoo.com*

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PROFESSOR N. K. THAKARE - ON HIS RETIREMENT FROM THE INDIAN MATHEMATICAL SOCIETY

SATYA DEO

Professor N.K.Thakare, who has been heading the Indian Mathematical Society for the past six years as the General Secretary, has announced his retirement from the IMS w.e.f April 1, 2019. We, the remaining office bearers of the IMS, made strong appeals that he should continue at least for one more term of three years, but he did not accept our request. He is a man of principles and normally no amount of persuasion can change his decisions. I gladly mention that there is no health issue with Prof Thakare; he is fit and fine at the age of 80. He often tells me that "I have no BP, no sugar, nothing of that sort!" However, perhaps he now wants to spend his time in things important to his heart like writing books, essays and poetry etc. He is an excellent writer in English as well as in Marathi and has already published a number of very popular books. This, of course, is the best option of an experienced academician after his retirement who can deliver to the society through his writings. I know that he also gives a lot of his time in running a very good school in his hometown Dhulia.

Prof Thakare served the Indian Mathematical Society with distinction and total dedication for a long time. He was elected President of IMS for the year 95-96, Academic Secretary for nine years (1998-2007), Editor of the JIMS for six years (2007-2013) and finally, the General Secretary for six years (2013-2019). He took over as the General Secretary from Prof V. M. Shah and continued to serve the IMS with same spirit and enthusiasm as he had served other institutions such as the Pune University as Head of the Mathematics Department, North Maharashtra University, Jalgaon as Vice Chancellor. Prof Thakare has been a very powerful and tough administrator as the founding Vice Chancellor of the North Maharastra University, Jalgaon. I met him for the first time only in that university. He is an eminent mathematician and a very sound academician. He has been very supportive of the noble cause of promoting mathematics at the national level through the activities of the Indian Mathematical Society. He realizes and is proud of the prestige of the IMS as one of the best mathematical societies of the world. Very often I heard him saying "IMS is my life!"

There is an interesting lighter side of his last name "Thakare". Prof Thakare was the President of the IMS in 1995-96 and the annual session of the IMS was being held at Aligarh Muslim University, Aligarh that year. On the very day when he delivered his presidential address, Mrs Thakare went missing while she had gone for shopping in the Aligarh city. Everyone, in the group of people accompanying her, was highly worried as to what happened! Local organizers were more worried since the place was the city of Aligarh and the last name of the missing person had the word Thakare of the famous Bal Thakare of Maharashtra. While I was with Prof Thakare, I noticed that this thought also bothered him but only for a while. Fortunately, very soon our Pune friends located her and she was found safe and sound!

When he took over as the Academic Secretary of IMS, there was a problem during the last year of his term. Late Prof. M. K. Singal, the Administrative Secretary, the de facto manager of the IMS, had just died of heart attack and it was not certain who will lead the IMS after him. Prof V. M. Shah, who had not asserted himself by that time, was the General Secretary. Then suddenly, the group of IMS office bearers showed an exemplary unity and some major decisions were taken. The headquarters of the IMS were moved from Delhi to Pune, the place where the IMS was born, with very little cooperation from the outgoing team of the IMS. There were difficulties for the new office bearers of the IMS, especially the treasurer, to put everything in order. But this was done by Prof Shah, Prof Thakare and other office bearers, and the IMS was now more united than ever before. Prof Thakare's initiative and hard work at Pune made everything easy and straightforward. Dr Shashi Prabha Arya, the former Treasurer of IMS, Prof H. P. Dikshit, former Vice Chancellor, IGNOU, both told me on several occasions that Prof Thakare is, of course, a great organizer!

When Prof Thakare took over as the editor of the JIMS, the Journal had a lot of backlog. I had joined the IMS administration as the Academic

Secretary. There were problems for the JIMS of finding a new type-setter, a new printing house etc. all in Pune. He worked hard along with his students at Pune and solved all problems. Very quickly it was brought to a level where the Journal had no backlog. Many other things were also streamlined. An additional special issue of the JIMS was published during the 125^{th} birth anniversary of Srinivas Ramanujan. Apart from this, the special issues of the JIMS and the Mathematics Student were also published during the centenary year celebrations of the IMS in 2007. As the General Secretary of IMS, I found him omnipresent during any annual conference with his usual dynamism. His sense of responsibility and alertness to make sure that the conference is held with perfection was simply amazing. He also faced many challenges in keeping the identity of the IMS distinct, and also in retaining the glorious past of the IMS than any time before. There was always a healthy competition with other mathematical societies of India and abroad. Once an idea of having a joint meeting with the American Mathematical Society was mooted and the IMS was ready to be a part of the same provided "the president of the IMS rubs shoulder with the president of AMS," declared Prof Thakare in the inaugural function of the IMS at Dhanbad. There was a spontaneous clapping approval by one and all present on that occasion. Somehow, the joint meeting never materialized. The annual conferences of the Indian Mathematical Society are very important. They are organized and hosted by various educational institutions on voluntary basis. Good amount of funds are needed and a lot of hard work is required on the part of the organizers. Consequently, there are not many volunteers, and the General Secretary including other office bearers of IMS, have to find an institution who can accept the responsibility and do a good justice. Whenever we were to take a decision and there were no good invitations available, Prof Thakare would say that Pune people are always ready to host the conference if no one is available. In other words, the IMS had an open invitation from Pune to host an annual conference anytime, thanks to Prof N. K. Thakare and his group of students. They have already organized several such conferences in the past with record attendance of participants. This was, however, never needed by the IMS since we always had a nice invitation from some other institution from various parts of the country. The centenary year conference of 2007 was also organized at Pune with Prof Thakare as the leader of organizing team.

Prof Thakare is a great and convincing orator. He is highly original in his ideas, in his writings, in his speeches and in taking decisions. He is always conscious of his responsibilities as a leader. No one, however great or powerful he may be, can change his views and his beliefs. At the same time, he is very respectful towards his seniors as well as juniors. I wish Prof. Thakare and Mrs. Thakare to a have very good health and cheerful long life.

Satya Deo
Harish-Chnadra Research Institute
Chhatnag Road, Jhusi, Allahabad-211 019, (UP), India
E-mail: sdeo@hri.res.in

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PROFESSOR N. K. THAKARE: SOME REMINISCENCES

A. K. AGARWAL

It was in late nineteen seventies when I was doing my Ph.D. at the Indian Institute of Technology (IIT), Delhi with Prof. H. L. Manocha that I came to know about Prof. N. K. Thakare (NKT) for the first time. I was working on a research problem related to Konhauser biorthogonal polynomials, I came across two papers related to my problem co-authored by NKT, namely: (1) B. K. Karande and N. K. Thakare, Some results for Konhauser biorthogonal polynomials and dual series equations, Indian J. Pure Appl. Math. 7, (1976), 635–646 and (2) K. R. Patil and N. K. Thakare, Multilinear generating function for the Konhauser biorthogonal polynomials sets, SIAM J. Math. Anal. 9(5), (1978), 921–923. Joseph D. E. Konhauser in his paper [Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math. Vol. 21, (1967), 303–314)] introduced a pair of biorthogonal polynomials with respect to the weight function $x^a exp(-x)$ over the interval zero to infinity. In (1), the authors obtained generating functions, bilinear generating functions, recurrence relations and related results for Konhauser biorthogonal polynomials and in (2) a multilinear generating function was derived. The work of NKT and his collaborators on Konhauser biorthogonal polynomials was helpful to us in our research on this topic. We further studied these biorthogonal sets. Our results were published in the paper: A. K. Agarwal and H. L. Manocha, A note on Konhauser sets of biorthogonal polynomials, Nederl. Akad. Wetensch. Indag. Math. 42(2) (1980), 113–118. In collaboration with H. C. Madhekar, NKT has also constructed pairs of biorthogonal polynomials with respect to Jacobi weight function, Hermite weight function and Szego-Hermite weight function. Their study in this area is helpful in the calculation of gamma rays through matter.

My active interaction with NKT began with my participation in the annual conferences of the Indian Mathematical Society (IMS) after my return from the USA, where I had gone to pursue postdoctoral research with Prof. George E. Andrews at the Pennsylvania State University. I was on a National Scholarship for Higher Study Abroad, Ministry of Education & Culture, Government of India, 1984–86. In my opinion, in the recent times, NKT has been the strongest pillar of IMS after late Professors R. P. Agarwal, M. K. Singal and V. M. Shah. He has served IMS in various capacities - as President in 1995-96, as an Academic Secretary for nine years (1998-2007), as an Editor of the Journal of Indian Mathematical Society (2007-2011). Presently, he is the General Secretary of IMS . He was impressed by my collaborative work with Prof. Andrews who is an internationally renowned expert in q-series and partition theory and is a leading exponent in Ramanujan Mathematics. NKT himself is very much interested in Ramanujan Mathematics. To mark the Ramanujan Birth Centenary Year in 1987, he organized an International Symposium on Analysis at Pune University and edited the Proceedings Volume which was published by McMillan.

His quality which I like most is: his ability to take other people along with and encourage them to participate in the academic activities of IMS. I remember that in 1997 when I was participating in the 63^{rd} Annual Conference of IMS at Ahmednagar, he asked me to chair an academic session and in the following year, that is, in 1998, when he was the academic secretary of IMS, I was invited to deliver the Ramanujan Memorial Award Lecture at the 64^{th} Annual Conference of IMS at Haridwar. I got the opportunity to interact with him on several other occasions also. For examples, during 2008-2009 when I was the President of IMS and during 2009-2010 as Immediate Past President, I attended several meetings of IMS Executive Committee along with him. He was the editor of the Journal of Indian Mathematical Society at that time. I always found him very sincere, inspiring and having exceptional leadership qualities. On my invitation, he has visited our department - the Centre for Advanced Study in Mathematics, Panjab University, Chandigarh and gave a series of lectures. His lectures were well taken by the audience. I have also visited the Mathematics Department at Pune University on several occasions. I was deeply impressed by seeing the excellent academic atmosphere that existed there.

I was told that it was mainly due to NKT. The 75^{th} (Platinum Jubilee) Annual Conference of IMS was held at the Kalaslingam University, Tamil Nadu, in December, 2009. After the Conference was over, NKT, Prof. B. N. Waphare and myself went on an excursion tour to nearby temples by taxi. All of us enjoyed a lot. I have fond memories of that trip. Just after that trip, we went to Thiruvananthapuram, where the 97^{th} Session of the Indian Science Congress Association was being held at the University of Kerala. I was the President of the Mathematics (including Statistics) Section. As a part of the academic program, I had organized a special symposium on Discrete Mathematics. NKT had helped me in the organization of this symposium right from the beginning till the end. He had suggested some names for possible speakers and had also chaired this special session. I really appreciate his helpful nature. My association with NKT is also through the Society for Special Functions and their Applications (SSFA). I was the President of SSFA for eleven years (from 2007-08 to 2017-18). Late Prof. R. P. Agarwal was the founder patron of SSFA. After his demise in 2008, NKT was elected unanimously as the patron of SSFA by the Executive Committee in its meeting during the 9^{th} Annual Conference held at Jiwaji University, Gwalior in 2010. He is continuing in this position till date. He chaired my last Presidential Address in the 16^{th} Annual Conference of SSFA held at the Government College of Engineering & Technology, Bikaner during November 02-04, 2017. In this conference he was also felicitated for his commendable achievements and service to mathematics community.

In conclusion, I would like to remark that NKTs contributions are not limited to mathematics research only and his services are not confined to mathematics societies only. His other social activities are also remarkable. Presently, he along with his wife Mrs. Pushpa Thakare is engaged in a very noble mission. From the proceeds of the sale of their properties in Pune and Dhule and with their life long savings, they have established primary and secondary schools at their native village and are spreading education in rural and tribal belt of North Maharashtra. This is an exemplary act. I wish him a long, happy and healthy life ahead and hope that he will continue motivating others to strengthen Indian mathematics and to serve our great nation wholeheartedly.

A. K. Agarwal
Professor Emeritus
Centre for Advanced Study in Mathematics
Panjab University, Sector 14,
Chandigarh-160 014, (Punjab), India
E-mail: aka@pu.ac.in

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PROF. N. K. THAKARE: THROUGH MY EYES

J. R. PATADIA

It was during the 54th Annual Conference of the Indian Mathematical Society (IMS) in December 1988 at the then University of Poona that I first met Prof. Thakare as a receipient of the A. Narasinga Rao Gold Medal awarded by the Society. I again got an opportunity to meet him in my department when he was the Sectional President of the Section of Mathematical Sciences during the 79th Session of the Indian Science Congress Association held at the M. S. University of Baroda, Vadodara during 1991-92. I understand it is mainly because of him and Late Prof. V. M. Shah that I was inducted as a council member of IMS since April 01, 2003; and thanks to the then office brearers and the entire council of the Society that subsequently I got an opportunity to serve the Society as Editor, the Math. Student with effect from April 01, 2004 in view of the sad and sudden demise of Prof. H. C. Khare, the then Editor.

Once I was an office bearer of the Society, I came in much close contact with Prof. Thakare and his group of Pune students. It is indeed heartening to see that he is an elderly and respected *family friend* of all his research students. Along with their academic growth, whenever needed, I see him taking interest in their personal matters as well providing all the support. He and the entire team of his Pune students is sincere and hard working.

I observed that he recognizes good researchers and has a soft corner and appreciation for researchers from universities in general and for those with rural background in particular, may be because they have to do researches along with huge teaching load, meager resources and taking care of large number of *average* students. He is bold, assertive and frank in expressing his views. Being a person of integrity, working with him is a pleasure.

Over the years I have found him an honest and true custodian of the Society overseeing that not a single paisa of IMS funds is wasted. As a General Secretary he gave full freedom to the colleagues once the task is given to them after judicious decision, no interference, no manipulation, full support and appreciation for hard work and honesty.

By nature he puts his heart into the tasks on hand and remains committed. I recall couple of my visits to the North Maharashtra University, Jalgaon, as a resource person of a refresher course, during Prof. Thakare's tenure there as the Vice-Chancellor. I still remeber *the subtle and enthusiastic way, with a sense of pride and satisfaction,* he showed and explained to me the development he carried out there as the founding Vice-Chancellor!!

Along with mathematics, he is interested in literature as well. In fact, he writes poetry in marathi and is respected in a certain literary group of marathi poerty as well.

Prof. Thakare has many contacts in different fields. Late Prof. V. M. Shah used to tell me that Prof. Thakare is very resourceful - always ready to hold IMS Conference at Pune whenever required. I could see that once the IMS council under Prof. Shah decided to have its own office premises at Pune, Prof. Thakare not only managed to find a suitable office premices at a prime location in Pune but all the necessary relevant formalities were also promptly completed by him and his team - Prof. Shah had to just sign the documents along with witnesses on a certain day.

Perhaps many may not be aware of the fact that Prof. Thakare and Madam Thakare are dedicated social and educational workers with a zeal to work for rural and tribal children of the region around his native place Dhule in Maharashtra. In fact, without any assistance from any corner, completely through their own funds and on their own land, they have not only established a school in Dhule, primary till higher secondary and serving mainly tribal children, but both of them are personally devoting, at this age of around eighty, their complete time looking after the school. During office hours, if one wishes to contact him, one has to call him in the school !! Personally, I am very much impressed - though could never visit the school.

I take this opportunity to wish Mr. and Mrs. Thakare all the best in this noble activity and pray for a very healthy, happy and long life.

J. R. Patadia
(Former Head, Dept. of Mathematics, The M. S. University of Baroda)
5, Arjun Park, Near Patel Colony, Behind Dinesh Mill
Shivanand Marg, Vadodara-390 007, (Gujarat), India.
E-mail: jamanadaspat@gmail.com

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THE ROLE OF HISTORY IN LEARNING AND TEACHING MATHEMATICS: A PERSONAL PERSPECTIVE*

SUDHIR R. GHORPADE

At the outset, I would like to express my gratitude to the Indian Mathematical Society for electing me as its President for the year 2018-19. I feel humbled by this honour and I consider it a privilege to address this august gathering of fellow mathematicians and invited guests. The Indian Mathematical Society, or in short, IMS, has a distinguished history of well over a century and many stalwarts have held the high office of the President of the IMS. Just to name a few, I might mention R. Vaidyanathaswamy, T. Vijayaraghavan, Hansraj Gupta, R. P. Bambah, R. B. Bapat, R. Sridharan, and S. G. Dani. This conference is rather unique because it is perhaps for the first time that it is scheduled in the month of November and also the first time it is held in the state of Jammu and Kashmir.

The topic I have chosen for this general talk is the role of history in learning and teaching mathematics. It would be beyond my competence to give here a comprehensive account of the history of mathematics or to give a scholarly narration befitting a professional historian. Instead, I will attempt to provide a personal perspective by relating some incidents in my own journey as a student and teacher of mathematics, especially those related with the history and development of the magnificent edifice that we call mathematics. Along the way, I will also mention some of my favourite books and articles. It may be hoped that this could be of some value and interest to younger students and researchers. I am reminded here of the following lines from a highly readable, instructive, and M A A Chauvenet

* This is a slightly revised version of the text of the Presidential Address (general) delivered at the 84^{th} Annual Conference of the Indian Mathematical Society-An International Meet, held at Sri Mata Vaishno Devi University, Katra - 182 301, Jammu and Kashmir, India during November 27-30, 2018. The author gratefully acknowledges comments and suggestions received from several colleagues, especially U. K. Anandavardhanan and Amartya K. Dutta.

prize winning article [1] by my teacher Prof. Shreeram Abhyankar:

> Now it is true that the history of mathematics should primarily consist of an account of the achievements of great men; but it may be of some interest to also see their impact on an average student like me. With this in mind I may be allowed some personal reminiscences.

I think Abhyankar was exceptionally humble to call himself *an average student* here. But if I express a similar sentiment, then it would not be a matter of humility, but simply being truthful. Nonetheless, I hope you will permit some indulgence at this age and stage. I would also like to use the opportunity to pay a tribute to some of my teachers who played an important role in shaping my career.

Coming back to the theme of this address, the importance of history can hardly be overstated. In the preface to his book on differential equations, Simmons [25] mentions an old Armenian saying "He who lacks a sense of the past is condemned to live in the narrow darkness of his own generation", and goes on to write:

> Mathematics without history is mathematics stripped of its greatness: for, like the other arts–and mathematics is one of the supreme arts of civilization–it derives its grandeur from the fact of being a human creation.

At any rate, this book of Simmons, and especially the historical notes in it, which I read during the second year of my undergraduate study had a considerable influence on me then. Moreover, it may have sown the seeds of eventually opting to make a career in mathematics. I was also blessed to have had very good college teachers like Prof. (Mrs.) S. B. Kulkarni and Prof. V. R. Kulkarni. In the third year of my B.Sc., they encouraged me to become in-charge of college's Mathematics Association and edit a wallpaper. This wallpaper was to contain snippets from the history of mathematics, problems, quotable quotes, etc. I remember writing several pieces under different names due to a lack of contributions from fellow students. An old book of Cajori [6] that I found in the college library was of great help to me then. This book has on its title page the following nice quote of J. W. L. Glaisher:

> I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history.

For my Master's degree in Mathematics, I joined IIT Bombay in 1982 and this opened the doors to its well stacked library. It was quite exciting to be able to walk among books and browse through anything that seemed interesting. It was also thrilling to read the following lines by Sarvapalli Radhakrishnan that are engraved on a stone plaque in the foyer of the Convocation Hall of the Institute.

> Knowledge is not something to be packed away in some corner of our brain, but what enters into our being, colours the emotion, haunts our soul, and is as close to us as life itself. It is the overmastering power which through the intellect moulds the whole personality, trains the emotion and disciplines the will.

At IIT Bombay, I had the good fortune to be taught by some wonderful teachers such as Professors M. V. Deshpande, K. D. Joshi, B. V. Limaye, and D. V. Pai. In the summer vacation between the two years of my M.Sc., I had an opportunity to attend a summer school on Analysis and Probability conducted by the Indian Statistical Institute at the University of Mysore. Prior to going there, my friend Subathra Ramanathan gifted me with a copy of E. T. Bell's *Men of Mathematics.* This book has 29 chapters, each giving an account of the life and work of a great mathematician in an absorbing and lucid manner. While I was in Mysore for a month (around the same time when India won the Cricket World Cup for the first time!), there would be a stern knock on the door very early in the morning to serve coffee, and after that one had to wait a considerable time for breakfast. During this period I would read one chapter each of Bell's book everyday, and could thus finish reading the book by the end of the month long school. The Mysore summer school was also memorable for me because of excellent teachers such as Professors B. V. Rao, S. C. Bagchi, and V. S. Sunder, and brilliant fellow students like C. S. Rajan and V. Balaji. I was to learn later that although the historical account in Bell's book is interesting and readable, it wasn't always the most accurate. However, the book does give a broad perspective of the subject and some of its heroes in a manner that can captivate an impressionable mind, and can still be recommended to young readers. But as indicated above, Bell's treatment, especially of ancient Indian mathematics and mathematicians, could be tempered with other sources. See, for example, Amartya Dutta's article [9] on and in

Bhāvanā, and the references therein, as well as the article [23] of Seidenberg (who is related to me by mathematical geneaology [1]), which is also a good reference for real "Vedic Mathematics". Furthermore, today's reader should complement Bell [3] with a book such as [5] on women of mathematics, and read something (for instance, [2]), about the first female Fields medalist Maryam Mirzakhani.

It was also in the year 1983 that I first met Prof. Shreeram Abhyankar at Bombay University (rather accidentally, as I have mentioned in [12]) and later went on to do Ph.D. under his supervision at Purdue University. In the course of my long association with Prof. Abhyankar, I learned a great deal not only about mathematics and many mathematicians he knew personally, but also about his view of the historical development of mathematics, especially, algebra and algebraic geometry. Abhyankar spoke highly of Felix Klein's *Entwicklung*, and while I couldn't read the German original, I did read parts of the English translation [17] and could see what a valuable resource it was. Somewhere around this time, probably much earlier, I came across G. H. Hardy's *Apology* [14]. What a remarkable book this is! It gives a brilliant insight into the mind of a mathematician. Even if some of what Hardy writes is controversial and perhaps untrue, I would heartily recommend this book to anyone interested in mathematics, or for that matter, any layman curious about mathematics and mathematicians. I certainly find myself going back to this book from time to time, even if just to admire Hardy's turn of phrase. For a more in-depth look into the working of a mathematician's mind, Hadamard's essay [13] (which I have only skimmed through) could be consulted. Perhaps a more riveting account is given by Poincaré in [19] where, among other things, he narrates how the crucial idea about automorphic functions came to him just as he was about to put his foot on a bus. The book also has a scholarly preface by Bertrand Russell, which makes for an interesting read. In his ICM address [27], André Weil highlights the importance of history for creative and would-be creative scientists, and includes (an English translation of) the following words of Leibniz:

> Its use is not just that History may give everyone his due and that others may look forward to similar praise, but

[1] One can trace mathematical geneaology at: https://www.genealogy.math.ndsu.nodak.edu

> also that the art of discovery be promoted and its methods known through illustrious examples.

Biographies of great mathematicians can also teach us a lot and help gain a perspective on their work. There are several that are available, but I will mention just one that I remember reading with pleasure. Namely, Constance Reid's biography of Hilbert [21], which Freeman Dyson called "a poem in praise of mathematics." Another inspiring book of a more recent vintage is Sreenivasan's compilation [24] of accounts by about 100 distinguished scientists, including several mathematicians, that outline their motivations for taking up science and a brief description of their contributions to the subject. I would add to this collection a fascinating and instructive interview [7] with one of the greatest mathematicians of our time, J.-P. Serre, whom I have had the pleasure of meeting at Purdue, Paris and Luminy.

While biographical details and anecdotes about the life and times of mathematicians can be interesting, a mathematician finds true expression in his or her mathematical work. Thus for a serious student of mathematics, there is often no better alternative to learn the subject than to go through original sources. Isaac Newton famously spoke of being able to see farther because he was standing on the shoulders of giants. Abel puts this point across more directly when he says:

> ... if one wants to make progress in mathematics one should study the masters not the pupils.

This particular quote appears at the top of the translators' preface to Hecke's *Vorlesungen* [15], which is a masterly treatment of the theory of algebraic numbers. Speaking of algebraic number theory, Hilbert's *Zahlbericht* [16] is now available in English and it is still an excellent introduction to the subject. As for elementary number theory, it is still profitable to read Gauss's *Disquisitiones* [11], which was first published in Latin in 1801 and is available in English. Many good libraries contain collected works of eminent mathematicians, and they make it easier for us to understand the evolution of ideas of these mathematicians and the paths taken in their mathematical journeys. Today, thanks to the Internet, it has become simpler to access many original sources and it may suffice to cite the Digital Mathematics Library (https://www.math.uni-bielefeld.de/~rehmann/DML/) and the AMS Digital Mathematics Registry (https://mathscinet.ams.org/dmr/).

Let me turn to the role of history in teaching mathematics. Here too, rather than pontificating about generalities, I would like to draw from personal experiences. By the end of 1989, I returned to IIT Bombay to join as a faculty member, and taught for the first time a course on algebra for M.Sc. students in Spring 1990. (Incidentally, another thing I did in 1990 was to become a life member of the Indian Mathematical Society, and to my surprise, I was invited to give a talk at the annual conference of the IMS at Surat later that year.) Partly because of the course I was teaching and partly due to my own interest, I consulted the book of van der Waerden on history of algebra [26]. It was then that I understood why abelian groups were named after Abel. This book also helped me to understand better the evolution of theory of equations and the advents made by Scipione del Ferro, Tartaglia, Cardano, Ferrari, and how they culminated in the monumental work of Évariste Galois (1811-1832) which brought to fore the basic notion of a group. Later I could use this (I think with good effect) in lectures for younger audiences that explained the development of algebra. Over the years, my hobby of picking up bits and pieces of history of mathematics has, I believe, stood me in good stead, and perhaps enhanced the quality of my teaching. Students are often amused by snippets of history such as for instance, the fact that the basic notion of an ideal in a ring came about in Kummer's attempt to prove Fermat's Last Theorem, or that many of the fundamental results of commutative algebra and algebraic geometry (such as Hilbert's basis theorem, Nullstellensatz, and syzygy theorem) were, in fact, lemmas in Hilbert's breakthrough work on invariant theory. Even in a very basic course such as Calculus, students are sometimes surprised to learn that the development of calculus did not happen in the order it is given in their textbooks. The notion of integration came much before the notion of differentiation was discovered! Realizing this, one can perhaps have a much better appreciation of the fundamental theorem of calculus, which relates seemingly disparate geometric problems of finding area under a curve in the plane and determining the (slope of) tangents to plane curves. Once again, thanks to the Internet, it is no longer necessary for an interested student to leaf through dusty volumes in the library, but simply access the requisite information on a digital media of his or her choice in order to get a fairly good sense of the history of a topic in mathematics or a mathematician. For them, I would recommend using the MacTutor History of Mathematics archive (http://www-history.mcs.st-and.ac.uk/). Many a time, the Wikipedia is also a good source of information. The historical notes appearing at the end of Bourbaki's *Elements* have long been recognized as an authoritative source. A compilation of most of these historical notes is available in book form [4]. A very nice collection of articles about American mathematics can be found in the first 3 volumes of the AMS series on History of Mathematics [8]. I should also mention a few good sources about mathematics in India and Indian mathematicians. Here. I warmly recommend Volumes 3 and 5 of the *Culture and History of Mathematics* series of HBA [10, 22], and the scholarly articles of Raghavan Narasimhan [18] and M. S. Raghunathan [20].

Let me end with a somewhat philosophical remark. Why do we do mathematics? And why should we learn something about its history? The first question is relatively easy to answer: we do mathematics because it brings us joy (not to mention a job and a salary ...). I have dwelt on this question in greater details elsewhere[2], quoting Hardy [14] rather extensively, and so I will not discuss it further, except to say that we mathematicians yearn for those rare moments of exhilaration when suddenly we understand some seemingly complex topic very clearly, with everything falling into place, or, if we are luckier still, then we discover something new and nice. In a way, this brings us closer to divinity, albeit for a brief fleeting moment. Needless to say, for first rate mathematicians, such moments of exhilaration would be more frequent and of a much higher order. By dwelling upon the history of mathematics and the achievements of distinguished mathematicians, we not only see glimpses of greatness, but an easier way to be a step closer to divinity, even if vicariously. We also feel a greater connect with the mathematical milieu. At any rate, whatever be your reason, I wish you many pleasant moments with mathematics and its history, and an enjoyable conference!

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Sudhir R. Ghorpade
Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai-400 076, India.
E-mail: srg@math.iitb.ac.in
URL: http://www.math.iitb.ac.in/∼srg/

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RESOLUTION OF SINGULARITIES AND LOCAL UNIFORMIZATION*

STEVEN DALE CUTKOSKY**

Abstract. An overview is given of Resolution of Singularities of Algebraic Varieties. We begin with a quick survey of the foundations of Algebraic Geometry, then discuss some of the major results in the field of resolution of singularities and present some open problems. Some basic techniques in resolution are explained, and some of the obstructions to proving resolution of singularities in positive characteristic are presented.

1. Algebraic Geometry

We begin by giving a very quick introduction to algebraic geometry. We refer to [8] for a thorough and extensive treatment. Let k be an algebraically closed field. An algebraic variety X over k is a space which is locally given by the vanishing of polynomial equations in $\mathbb{A}^n = k^n$. The most basic nontrivial example is the n dimensional projective space

$$\mathbb{P}^n = \mathbb{A}^{n+1}/\sim$$

where $\sim$ is the equivalence relation $(a_0, a_1, \ldots, a_n) \sim (b_0, b_1, \ldots .b_n)$ if there exists $0 \neq \lambda \in k$ such that $(a_0, a_1, \ldots, a_n) = \lambda(b_0, b_1, \ldots .b_n)$. We denote an equivalence class by $(a_0 : a_1 : \ldots : a_n)$. Projective space has the homogeneous coordinates $x_0, \ldots, x_{n+1}$. Although the value of x_i at a point of $\mathbb{P}^n$ is not well defined, the vanishing of a homogeneous polynomial at a point is; that is, if $F(x_0, \ldots, x_n)$ is a homogeneous polynomial, and $(a_0 : a_1 : \ldots : a_n) = (b_0 : b_1 : \ldots, b_n)$, then $F(a_0, a_1, \ldots, a_n)$ and $F(b_0, b_1, \ldots, b_n)$

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are simultaneously zero or nonzero. The projective space $\mathbb{P}^n$ naturally has an open covering by $n+1$ copies of $\mathbb{A}^n$. Let $U_i = \mathbb{P}^n \setminus \{p \mid x_i(p) = 0\} \cong \mathbb{A}^n$ by the (well defined) algebraic map

$$(a_0 : a_1 : \ldots : a_n) \mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{a_n}{a_i}\right).$$

The inverse map is

$$(b_1, \ldots, b_n) \mapsto (b_1 : \ldots : b_{i-1} : 1 : b_i : \ldots : b_n).$$

We have that $\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}$ are coordinates on U_i. If $k = \mathbb{C}$, this gives a natural isomorphism of $\mathbb{P}^1$ with the Riemann sphere.

Projective varieties are irreducible closed subsets of projective space. The projective varieties are the sets

$$X = \{(a_0 : \ldots : a_n) \mid F(a_0, \ldots, a_n) = 0 \text{ for all } F \in \mathfrak{P}\}$$

where $\mathfrak{P}$ is a homogeneous prime ideal in the polynomial ring $k[x_0, \ldots, x_n]$. Such an X has the affine cover $X = \cup_{i=0}^{n+1} V_i$ where $V_i = U_i \cap X$. We have that

$$V_i = \{(b_1, \ldots, b_n) \in U_i \cong \mathbb{A}^n \mid f(b_1, \ldots, b_n) = 0 \text{ for all } f \in \mathfrak{P}_i\}$$

where $\mathfrak{P}_i$ is the prime ideal in the polynomial ring $k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$ defined by

$$\mathfrak{P}_i = \left\{F(\frac{x_0}{x_i}, \ldots, 1, \ldots, \frac{x_n}{x_i}) \mid F \in \mathfrak{P}_i \text{ is homogeneous}\right\}.$$

Projective varieties have the important property that they are proper (if k is the complex numbers $\mathbb{C}$, then they are compact in the Euclidean topology).

An important theorem in commutative algebra is the Hilbert basis theorem ([3, Theorem 7.5], [8, Theorem 1.26]). This theorem says that every ideal I in a polynomial ring $k[x_1, \ldots, x_n]$ has a finite basis; that is, there exist $f_1, \ldots, f_r \in I$ such that

$$I = (f_1, \ldots, f_r) = \{a_1 f_1 + \cdots + a_r f_r \mid a_1, \ldots, a_r \in k[x_1, \ldots, x_n]\}.$$

Suppose that X is a variety and $q \in X$. We can assume that $X \subset \mathbb{A}^n$ and $q = (0, \ldots, 0) \in \mathbb{A}^n$ (by replacing X with an affine neighborhood of q in X and performing a translation). Suppose that the ideal of polynomials vanishing on X is

$$I = (f_1, \ldots, f_t).$$

For $1 \le i \le t$, write

$$f_i = L_i + \text{ higher order terms}$$

where L_i is a linear form.

The tangent space to X at q is the linear space

$$T_q(X) = \{p \in \mathbb{A}^n \mid L_1(p) = \ldots = L_t(p) = 0\}.$$

This space does not depend on a choice of generators of I. We always have ([8, Theorem 10.10]) that

$$\dim T_q(X) \geq \dim(X).$$

The point q is said to be a nonsingular point of X if $\dim T_q(X) = \dim(X)$.

Singularity can be detected from the Jacobian matrix of $f_1, \ldots, f_t$ (Theorem [8, Proposition 10.14]). Specifically, the singular points of X are the points $p \in X$ such that the rank s of the $t \times n$ matrix

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right)$$

is $s = n - r$ where r is the dimension of X. It follows that the nonsingular points are an open subset of X. Further, the nonsingular points are dense in X.

2. Resolution of singularities

Suppose that X is a variety. A resolution of singularities of X is a proper morphism (proper algebraic map) $\pi : X_1 \to X$ such that π is birational (an isomorphism over a dense open subset of X) and X_1 is nonsingular. If $k = \mathbb{C}$, then proper means that the inverse image of a compact set is compact (in the Euclidean topology).

Suppose that X is a closed subvariety of a nonsingular variety Y. Then an embedded resolution of singularities of X is a proper birational morphism $\pi : Y_1 \to Y$ such that Y_1 is nonsingular, π is an isomorphism in a neighborhood of some point of X, and there exists a subvariety X_1 of Y_1 such that the induced map $\pi : X_1 \to X$ is a resolution of singularities.

The basic example of a proper birational map is the blow up $\Phi : Y_1 \to Y$ of a nonsingular subvariety Z of a nonsingular variety Y. Geometrically, the blowup is obtained by replacing Z with a hypersurface (a subvariety of one less dimension than Y). So a point Z on a surface Y is replaced with a curve, a point or curve Z on a three dimensional variety is replaced with a surface. The strict transform of a subvariety X of Y under the blow up of Z is defined to be the closure X_1 of $\Phi^{-1}(X \setminus Z)$. The map Φ can be constructed locally over affine open subsets of Y, so we may assume that Y is a suitable affine neighborhood of the origin q. There are coordinates (regular parameters) $x_1, \ldots, x_n$ at q such that $Z = \{x_1 = \cdots = x_s = 0\}$ for some $s \leq n = \dim X$. We have that Y_1 is the closure of the set

$$\{(p, (x_1(p) : \ldots : x_s(p)) \mid p \in Y \mid \text{ and } x_i(p) \neq 0 \text{ for some } i \text{ with } 1 \leq i \leq s\}$$

in $Y \times \mathbb{P}^{s-1}$. The map Φ is the projection onto the first factor. The affine open subsets $W_i = Y_1 \cap (Y \times U_i)$, where $U_i = \mathbb{P}^s \setminus \{p \mid x_i(p) = 0\}$, are an open cover of Y_1. In W_i, we have that

$$(1)\qquad x_1' = \frac{x_1}{x_i}, \ldots . x_{i-1}' = \frac{x_{i-1}}{x_i}, x_i' = x_i, x_{i+1}' = \frac{x_{i+1}}{x_i}, \ldots, x_s' = \frac{x_s}{x_i}, \\ x_{s+1}' = x_{s+1}, \ldots, x_n' = x_n$$

are coordinates on W_i. The map Φ is defined on W_i as

$$(x_1', \ldots, x_n') \mapsto (x_1'x_i', \ldots, x_{i-1}'x_i', x_i', x_{i+1}'x_i', \ldots, x_i'x_s', x_{s+1}', \ldots, x_n').$$

The preimage $W_i \cap \Phi^{-1}(Z)$ is defined by the vanishing of the single equation $x_i' = 0$, showing that $\Phi^{-1}(Z)$ is a hypersurface.

Using the blowup, we now contruct an example of a resolution of singularities. Let $Y = \mathbb{A}^2$, $f = x_2^2 - x_1^3 \in k[x_1, x_2]$ and X be the curve

$$X = \{q \in Y \mid f(q) = 0\} \subset Y.$$

The only singular point on the curve X is $q = (0, 0)$, as we detect from the Jacobian criterion: $(0, 0)$ is the only solution in Y to the equations

$$f = x_2^2 - x_1^3 = 0, \frac{\partial f}{\partial x_1} = -3x_1^2 = 0, \frac{\partial f}{\partial x_2} = 2x_2 = 0$$

(even when k has characteristic 2 or 3).

Let $\pi : Y_1 \to Y$ be the blow up of q. The curve $E = \pi^{-1}(q)$ is a one dimensional projective space $\mathbb{P}^1$ (a Riemann sphere if $k = \mathbb{C}$). There is an induced map $\pi : X_1 \to X$ where X_1 is the strict transform of X. We will show that X_1 is nonsingular, so that $\pi : X_1 \to X$ is a resolution of singularities.

The blow up $Y_1 = B_1 \cup B_2$, where $B_1 \cong \mathbb{A}^2$, has coordinates $x_1, x_2/x_1$ and $B_2 \cong \mathbb{A}^2$ has coordinates $x_2, \frac{x_1}{x_2}$. Further, $f = x_1^2\left((x_2/x_1)^2 - x_1\right)$ on B_1. We have that $x_1 = 0$ is an equation of E in B_1 and $f_1 = (x_2/x_1)^2 - x_1$ is an equation of the strict transform X_1 in B_1, which is nonsingular since $\frac{\partial f_1}{\partial x_1} = 1$ is never 0. Similarly, $1 - (x_1/x_2)^3 x_2 = 0$ is an equation of X_1 in B_2, which is also nonsingular.

The proof of the existence of resolution of singularities for all varieties of characteristic zero was established by Hironaka. Since then, there have been significant simplifications in the proof by Hironaka and others. Many of these references are discussed in [6].

Theorem 2.1 (Hironaka, 1964 [10])**.** *A resolution of singularities exists for all varieties over a field of characteristic 0.*

Some examples of fields of characteristic zero are the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$.

Beginning with the first proof of resolution of singularities of positive characteristic surfaces in [1], Abhyankar proved resolution of singularities of 3-dimensional varieties of characteristic p with $p > 5$.

Theorem 2.2 (Abhyankar, 1966 [2]). *A resolution of singularities exists for all varieties of dimension* 3 *over an algebraically closed field of positive characteristic* $p > 5$.

A short and self contained proof of this theorem is given in [7].

The existence of a resolution of singularities was established in dimension 3 and the remaining characteristics $p = 2, 3, 5$ by Cossart and Piltant ([4], [5]) in 2009.

This leads to the following important question.

Question 2.3. *Does every variety over a field of positive characteristic p and of arbitrary dimension have a resolution of singularities?*

Suppose that Y is nonsingular and X is a (singular) subvariety of Y. The multiplicity ([17, Section 10, Chapter VIII]) of a point of X is the fundamental invariant for measuring the singularity. The multiplicity at a point q is always a positive integer, and is equal to 1 if and only if q is a nonsingular point of X. The multiplicity is just the order of a polynomial at a point if X is a hypersurface.

Let r be the maximum of multiplicities of points on X. The locus of points in X of maximum multiplicity r is a closed subset of X. Suppose that $q \in X$ has maximal multiplicity r. A fundamental property is that the maximal multiplicity remains $\leq r$ under blowing up. The difficulty is to find a sequence of blowups which leads to a decrease of multiplicity everywhere when $r > 1$. To achieve this, we make a fundamental definition ([8, page 177]).

Definition 2.4. *A nonsingular hypersurface H in a neighborhood U of q in Y is called a hypersurface of maximal contact for X at q if*

1) *All points of multiplicity r in U are contained in H.*
2) *If $Y_n \to \cdots \to Y_1 \to Y$ is a sequence of blowups of nonsingular subvarieties contained in the multiplicity r locus of the strict transform X_i of X in Y_i for all i then all points of multiplicity r on X_n above*

U are contained in the strict transform H_n of H on Y_n, which is a nonsingular hypersurface.

If a hypersurface of maximal contact exists, then resolution of singularities of X is reduced to a resolution problem on the hypersurface of maximal contact, which has smaller dimension than X, allowing for an inductive proof of resolution of singularities. The existence of hypersurfaces of maximal contact in characteristic zero was first realized by Abhyankar.

Theorem 2.5. *(Abhyankar) In characteristic zero, hypersurfaces of maximal contact always exist.*

Suppose that $f = x_n^r + a_1 x_n^{r-1} + \cdots + a_r$ with $a_i \in k[x_1, \ldots, x_{n-1}]$ and

$$X = \{p \in \mathbb{A}^n \mid f(p) = 0\}$$

where the origin q is a point of maximal multiplicity $r > 1$. Let

$$x_n' = x_n - \frac{a_1}{r}. \tag{2}$$

Then we have an expression $f = (x_n')^r + a_1'(x_n')^{r-2} + \cdots + a_r'$, where we have removed the $(x_n')^{n-1}$ term. The hypersurface H defined by $x_n' = 0$ is then a hypersurface of maximal contact for X in a neighborhood of q.

Notice that this is the type of transformation used to "complete the square" to solve quadratic equations (in one variable) by radicals. This transformation is also used to solve cubic and quartic equations, but it is not always possible to solve polynomials of degree greater than or equal to 5 by radicals.

We will show that if we blow up the point q by $\pi : Y_1 \to Y = \mathbb{A}^n$, X_1 is the strict transform of X by π and $q_1 \in \pi^{-1}(q) \cap X_1$ has maximal multiplicity r, then $q \in H_1$ where H_1 is the nonsingular hypersurface which is the strict transform of H. To simplify notation, we may assume that f already has no x^{r-1} term, so that H is the hypersurface $x_r = 0$.

Now using our local description of the blowup of q (equation (1)) we see that at q_1 there are regular parameters $y_1, \ldots, y_n$, where

$$y_j = \frac{x_j}{x_i} - \alpha_j \text{ if } j \neq i \text{ and } y_i = x_i \tag{3}$$

for some i and $\alpha_j \in k$. Regular parameters at q_1 are coordinates which vanish at q_1. Now the strict transform X_1 of X has a local equation $f_1 = f/x_i^r = 0$ at q_1, and substituting from (3) into f_1, we compute that since

q_1 has multiplicity r on X_1, we have that $i \neq n$ and $\alpha_n = 0$ so that q_1 is on the strict transform $\frac{x_n}{x_i} = 0$ of H.

Observe that if X is defined over a field k of positive characteristic p, then the hypersurface $x'_n = 0$ of (2) is not defined if p divides r (as then $r = 0$ in k). In fact, hypersurfaces of maximal contact do not in exist in general in positive characteristic. This is the fundamental reason that resolution of singularities is so difficult in positive characteristic p.

Theorem 2.6 (Narasimhan, 1983 [13])**.** *Hypersurfaces of maximal contact do not always exist in positive chracteristic $p > 0$.*

3. Local Uniformization

Resolution of singularities can be formulated locally as local uniformization, the problem of resolving singularities at the center of a specified valuation.

A valuation ν of a field K is a surjective map

$$\nu : K \setminus \{0\} \to \Gamma_\nu = \text{totally ordered Abelian group}$$

such that

$$\nu(fg) = \nu(f) + \nu(g) \qquad \text{and} \qquad \nu(f+g) \geq \min\{\nu(f), \nu(g)\}.$$

The fundamentals of valuation theory are developed in [17, Chapter 9].

Let X be a projective variety and let K be the function field of X. The local ring $\mathcal{O}_{X,q}$ of germs of functions at a point q of X is a subring of K with K as its quotient field. Associated to a valuation ν of K is the valuation ring

$$V_\nu = \{f \in K \mid \nu(f) \geq 0\}.$$

A projective variety has the property that for every valuation ν of its function field K (which is zero on $k \setminus \{0\}$), there exists a unique point $q \in X$ such that $\mathcal{O}_{X,q} \subset V_\nu$ and the maximal ideal of V_ν contracts to the maximal ideal of $\mathcal{O}_{X,q}$. The point q is called the center of ν on X.

The problem of local uniformization was introduced by Zariski in his proof of resolution of (characteristic zero) surface singularities. Zariski used the following algorithm to resolve the singularities of a surface S. Let $S_1 \to S$ be normalization followed by the blowup of the finitely many remaining singular points. Normalization removes singularities in dimension one less than the dimension of the variety. Repeat to construct

$$\cdots \to S_n \to S_{n-1} \to \cdots \to S_1 \to S.$$

Theorem 3.1 (Zariski, 1939 [14]). *If the characteristic is zero, then the sequence terminates in finitely many steps with a nonsingular S_n.*

This theorem was shown to be true in all characteristics by Lipman in 1978, [12].

We now indicate the idea of the proof of Theorem 3.1. If the alogorithm does not terminate, then there exists an infinite sequence of points $q_n \in S_n$ which are singular on S_n, and $q_{n+1} \mapsto q_n$ for all n.

We have that $\cup_n \mathcal{O}_{S_n,q_n}$ is a valuation ring V_ν of the function field of S, and q_n is the center of ν on S_n for all n. The value group of ν need not be finitely generated (it could be $\mathbb{Q}$ for instance). Zariski proves that the singularity must be resolved eventually along the valuation (local uniformization) giving a contradiction, showing that the sequence must terminate after a finite number of steps with a nonsingular S_n.

Zariski later showed that local uniformization is true for varieties of characteristic zero and any dimension.

Theorem 3.2 (Local uniformization: Zariski, 1940 [15]). *Suppose that ν is a valuation of the function field of a characteristic zero variety. Then there exists a variety X_1 with a proper birational morphism $X_1 \to X$ such that X_1 is nonsingular at the center of ν.*

Zariski [16] used this result to prove the existence of a resolution of singularities of a 3-dimensional variety of characteristic zero, using a patching argument. As the essential part of his proof of resolution of singularities of 3-dimensional varieties in positive characteristic > 5, Abhyankar proved local uniformization. Cossart and Piltant proved local uniformization in dimension three and all characteristics.

Theorem 3.3 (Abhyankar, [2]). *Local uniformization is true in characteristic $p > 0$ for surfaces and in characteristic $p > 5$ in $\dim X = 3$ (Cossart and Piltant in characteristic 2,3,5 [4], [5]).*

This leads us to the following fundamental question.

Question 3.4. *Is local uniformization true in characteristic $p > 0$ and for varieties of arbitrary dimension?*

4. Local uniformization and defect

We show in Theorem 4.3 of this section that the only obstruction to local uniformization in positive characteristic is the existence of defect in a general linear projection of a hypersurface singularity.

Suppose that $L \to K$ is a finite field extension, ν is a valuation on K and $\omega = \nu|L$. An important invariant of this extension of valued fields is the defect, $\delta(\nu/\omega)$. If the defect $\delta(\nu/\omega)$ is 1, then the extension can be understood by knowledge of the quotient group Γ_ν/Γ_ω and the field extension V_ν/m_ν of V_ω/m_ω. The part of the field extension which comes from nontrivial defect is extremely complicated and not well understood. In characteristic zero, the defect is always 1, which is one explanation for why local uniformization is much simpler in characteristic zero. The defect $\delta(\nu/\omega)$ is always a power of the residue characteristic p of V_ν ($\delta = 1$ if $p = 0$).

The defect can be computed from Ostrowski's lemma: If ν is the unique extension of ω, then

$$[K : L] = e(\nu/\omega)f(\nu/\omega)\delta(\nu/\omega),$$

where $e(\nu/\omega) = [\Gamma_\nu : \Gamma_\omega]$ is the index of value groups and $f(\nu/\omega)$ is the degree of the residue field extension of valuation rings of ν and ω. The defect for a general finite extension can be calculated from this formula using Galois theory.

The fact that defect is an obstruction to local uniformization in positive characteristic was observed by Kuhlmann in [11]. The role of defect as an obstruction to local uniformization in characteristic $p > 0$ is not readily visible in the proofs of local uniformization or resolution in characteristic $p > 0$ in dimension ≤ 3.

A valuation ν of an algebraic function field K/k is said to be zero dimensional if the transcendence degree of the residue field of the valuation ring of ν over k is zero. This is equivalent to the statement that for every projective variety X with algebraic function field K, if p is the center of ν on X, then $\dim \mathcal{O}_{X,p} = \dim X$ (p is a closed point of X). The essential case of local uniformization is for 0-dimensional valuations ([15]). For simplicity, we will assume that this condition holds. Further, since the ground field k is algebraically closed, we may assume (by the theorem of the primitive element) that we are in the essential case of a hypersurface singularity in a polynomial ring. We now state some definitions, within the context which we have just established.

An extension of domains $R \to S$ is said to be birational if R and S have the same quotient field.

Definition 4.1. *Embedded local uniformization holds in dimension m (ELU holds in dimension m) if the following is true: Suppose that $A = k[x_1,\ldots,x_m]$ is a polynomial ring in m variables over an algebraically closed field k and ν is a zero dimensional valuation of the quotient field K of A which dominates $A_{\mathfrak{m}}$ where $\mathfrak{m} = (x_1,\ldots,x_m)$. Then if $0 \neq f \in A$, there exists a birational extension $A \rightarrow A_1$ where A_1 is a polynomial ring $A_1 = k[x_1(1),\ldots,x_m(1)]$ such that ν dominates $(A_1)_{\mathfrak{m}_1}$ where $\mathfrak{m}_1 = (x_1(1),\ldots,x_m(1))$, there exists $n \le m$ such that $\{\nu(x_1(1)),\ldots,\nu(x_n(1))\}$ is a rational basis of $\Gamma_\nu \otimes \mathbb{Q}$ and $f = x_1(1)^{b_1}\cdots x_n(1)^{b_n}\overline{f}$ with $b_1,\ldots,b_n \in \mathbb{N}$ and $\overline{f} \in A_1 \setminus \mathfrak{m}_1$.*

Definition 4.2. *Local reduction of multiplicity holds in dimension m (LRM holds in dimension m) if the following is true: Suppose that $A = k[x_1,\ldots,x_m,x_{m+1}]$ is a polynomial ring in $m+1$ variables over an algebraically closed field k, $f \in A$ is irreducible and K is the quotient field of $A/(f)$, with $f \in \mathfrak{m} = (x_1,\ldots,x_{m+1})$ and $r = ord(f(0,\ldots,0,x_{m+1}))$ satisfies $1 < r < \infty$. Suppose that ν is a zero dimensional valuation of the quotient field of $A/(f)$ which dominates $(A/(f))_{\mathfrak{m}}$. Then there exists a birational extension $A \rightarrow A_1$ where A_1 is a polynomial ring $A_1 = k[x_1(1),\ldots,x_{m+1}(1)]$ such that ν dominates $(A_1/f_1)_{\mathfrak{m}_1}$ where f_1 is a strict transform of f in A_1, $\mathfrak{m}_1 = (x_1(1),\ldots,x_{m+1}(1))$, and $1 \le r_1 = ord(f_1(0,\ldots,0,x_{m+1}(1)) < r$.*

Now ELU in dimension m immediately implies LU (local uniformization) in dimension $m-1$ for hypersurfaces, which implies LU (local uniformization) in dimension $m-1$ by the primitive element theorem.

Consider the following statements:

(4) LRM in dimension m implies ELU in dimension $m+1$.

and

(5) ELU in dimension m implies LRM in dimension m.

If statements (4) and (5) are true, then we could immediately deduce that ELU in dimension m implies ELU in dimension $m+1$, and we would then know that LU holds in all dimensions.

The proofs of local uniformization and resolution of singularities in characteristic zero cited above involve proving the two statements (4) and (5). The proofs of resolution in dimension three and positive characteristic cited above involve tricks to obtain a proof that ELU in dimension 3 and

LRM in dimension 3 in the special case $r = p = \text{char k}$ implies LU in dimension 3. The problem is that we do not know ELU in dimension 4, so we are unable to proceed to LU in dimension 4.

Now (4) is not so difficult. In fact, induction on r in LRM in dimension m almost gives ELU in dimension $m+1$. So the really hard thing that needs to be proven (to obtain LU) is (5). Now there are methods, for instance in [14], to reduce LRM to the case of rank 1 valuations, so we see that the really essential problem is to prove (5) for rank 1 valuations (the value group Γ_ν of ν is order isomorphic to a subgroup of $\mathbb{R}$).

The key statement in Zariski's original characteristic zero proof in [14] of (5) which fails in positive characteristic is that we have that in characteristic $p > 0$,

$$\binom{r}{r-1} = 0 \text{ if } p \text{ divides } r.$$

In other words, the problem is the failure of the binomial theorem in positive characteristic, or put more positively, the fact that the Frobenius map is a homomorphism in positive characteristic.

It follows from Theorem 4.3 below, that if embedded local uniformization is true within nonsingular varieties of the dimension of K, and a suitable linear projection of a hypersurface singularity with function field K which is dominated by a given valuation ν can always be found such that if L is the function field of the linear projection and $\omega = \nu|L$ then the defect $\delta(\nu/\omega) = 1$, then local uniformization holds in K.

Theorem 4.3 (C. Mourtada [9]). *Let assumptions be as above and suppose that embedded local uniformization (Definition 4.1) is true in dimension* $m-1$,

$$r = ord\ f(0, \ldots, x_m) > 1$$

and the defect $\delta(\nu^*/\nu) = 1$. *Then there exists a birational transform along* ν^*, $T \to T' = k[x'_1, \ldots, x'_m]$, *such that* $(x'_1, \ldots, x'_m)$ *is the center of* ν^* *on* T', *and if* f' *is the strict transform of* f *in* T', *so that* $S' = T'/(f')$ *is a birational extension* $S \to S'$ *along* ν^*, *we have that ord* $f'(0, \ldots, 0, x'_m) < r$.

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Steven Dale Cutkosky
Department of Mathematics
University of Missouri, Columbia, MO 65211, USA
E-mail: cutkoskys@missouri.edu

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HYDRODYNAMIC INSTABILITIES IN POROUS MEDIA FLOWS

MANORANJAN MISHRA

Abstract. The flows through porous media encounter various changes in the physical properties of the fluids, resulting in hydrodynamic instabilities. A rigorous mathematical analysis is needed to understand these flow instabilities. The miscible fluids pose mathematical challenges due to the presence of a non-autonomous system. Tackling these challenges, lead to various developments in the literature. Mathematically, the miscible displacement process is described by coupling the continuity and Darcy equations with a convection-diffusion equation for solute concentration that determines the viscosity. A detailed investigation about various control strategies ranging from an optimum flow rate to the use of reactive fluids (with convection-reaction-diffusion system) are discussed, in particular miscible viscous fingering, using both linear stability analysis and robust direct numerical simulations of coupled partial differential equations.

1. Introduction

Infinitesimal disturbances are invariably present in nature, which may amplify triggering certain undesirable results, often termed as instability. The presence and growth of infinitesimal perturbations in a fluid system may result in a flow in a stationary system or in a transition to some other flow. The subject of hydrodynamic instabilities has been of keen interest to a generation of applied mathematicians. In fact, the experimental and theoretical foundation of hydrodynamic stability was laid by Lord Kelvin, Hermann von Helmholtz, Lord Rayleigh and Osborne Reynolds during the

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nineteenth century [9]. Ever since many researchers have been attracted to understanding various hydrodynamic instabilities, where perturbations grow due to a change in the fluid properties. For instance Rayleigh-Taylor instability is a consequence of a change in the density of the fluids; a variation in the viscosity results in Saffman-Taylor instability; Kelvin-Helmholtz instability arises due to a velocity gradient in the flow; while a difference in the temperature leads to Rayleigh-Bérnard instability.

In this work, we focus on the instability arising due to the mobility gradient of the flowing fluids. S. Hill [10] first observed this phenomenon during the sugar purification experiments where more mobile water channels through less mobile sugar liquor. He named this phenomenon as *channeling through packed columns.* However, now this instability is called the Saffman-Taylor instability, named after Philip Geoffrey Saffman and Geoffrey Ingram Taylor for their seminal work on displacement of viscous liquids by injecting air [30]. The less viscous fluid displacing a more viscous one results in finger like patterns at the interface, thus this interfacial instability is commonly known as viscous fingering (VF).

Various porous media flows *viz.,* oil recovery, CO_2 sequestration, contaminant transport in aquifers, to name a few, are prone to various hydrodynamic instabilities. CO_2 sequestration is a way to control the growing levels of green house gas in the atmosphere by storing the carbon dioxide coming from industries into the porous reservoirs. At a depth of 800 meters, CO_2 becomes less dense and less viscous than ambient brine [16], resulting in the Rayleigh-Taylor and the Saffman-Taylor instability. Thus, hydrodynamic instabilities play a key role in determining the feasibility and risk involved in sequestration. The extraction of oil through porous wells also witnesses VF. It is detrimental to oil recovery as less viscous water being used to extract oil during enhanced oil recovery, fingers out and reaches the extraction well before oil. Sometimes a fluid miscible with oil is used for enhanced oil recovery and this process is called solvent drive [1]. So, VF can be broadly classified as miscible VF and immiscible VF, depending upon the nature of the fluid involved. Immiscible VF has been extensively studied but the miscible counterpart poses various mathematical challenges which we discuss in this work.

VF is also observed in the chemistry labs during a separation process called chromatography separation. The silica gel used in the chromatography column acts as a porous medium and the viscosity gradient between the consequent components results in VF and hinders the separation process. The contaminant transport in aquifers is amplified due to VF which enhances mixing of the contaminants. Further VF is used as a paradigm of pattern formation. The radial viscous fingering patterns resemble snowflake formation and diffusion limited aggregation [19]. Recently, a similarity of the growth of a particular type of biological cell fragment and viscous fingering is reported [3]. Also the VF experiments feature proportionate growth observable in the human tissues [2]. Hence, it is of profound interest to understand VF due to its relation with a variety of fields.

In order to have a mathematical insight into miscible VF, a system of coupled non-linear partial differential equations (PDEs) is dealt with. In an attempt to obtain critical parameters, an analysis of the equations linearised around the base state is done. We focus on the difficulties introduced due to the miscible nature of fluids and the various solutions proposed. We also discuss the numerical methods used to solve the non linear PDEs governing miscible VF.

2. Mathematical Model

The mathematical modelling for a flow phenomena involves an equation for the conservation of momentum along with an equation describing the conservation of mass of species, if present. The conservation of momentum for porous media flows is governed by Darcy's law which is an empirical relation given by Henri Darcy [7] and states that velocity of a fluid in a porous medium is directly proportional to the applied pressure gradient. The homogeneity of the porous medium is decided by its permeability, κ, which is a constant for a homogeneous porous medium. We assume the fluids to be incompressible, Newtonian and miscible in nature, which may or may not react upon contact. The governing equations for miscible viscous fingering constitute a system of coupled non-linear partial differential equations:

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{2.1}$$

$$\vec{\nabla} p = -\frac{\mu(c_i)}{\kappa} \vec{u}, \tag{2.2}$$

$$\frac{\partial c_i}{\partial t} + \vec{u} \cdot \vec{\nabla} c_i = \nabla \cdot \nabla (D_i c_i) + f(c_i), \tag{2.3}$$

where $\mu(c_i) = g(c_i)$, a function of the concentration of the species involved. The viscosity may be a monotonic or a non-monotonic function of concentration. In either case, the dynamics are really interesting and have been explored since many years [11, 14]. Equation (2.1) is the conservation of mass for Newtonian, incompressible fluids and $\vec{u}$ is the Darcy velocity. The pressure is p and equation (2.2) is the Darcy's law for conservation of momentum. The fluids being miscible in nature, diffuse into each other resulting a change in the concentration. As such, equation (2.3) is the equation for the conservation of the mass of the i^{th} species $i = 1, 2, ..., n$; $f(c_i)$ is the source term which is non-zero in case the fluids are reactive. D_i is the diffusion coefficient of i^{th} species in some solvent. The value of i depends upon the nature of the fluids. For the non-reactive miscible fluids, $i = 1$ and $f(c_i) = 0$, while for a reaction involving two reactants and a product, $i = 3$.

In order to carry out the VF experiments, a transparent glass cell called Hele-Shaw cell is used. Due to the glass plates, a Hele-Shaw cell provides an ease in the visibility during the experiments. A Hele-Shaw cell contains two glass plates separated by a gap much less than the other dimensions of the plates, so that the flow is treated to be quasi-two dimensional and the quantities are gap averaged. Consequently the flow through it is a mathematical analogue to Darcy's law [11] with the permeability of the porous medium given in terms of the gap between the two plates as $\kappa = \frac{b^2}{12}$. Various different kinds of Hele-Shaw cells, *viz.* rectilinear, radial, quarter five spot, etc (see Figure 1 on the next page) [11] are used for the experimental, numerical and theoretical analysis of VF. The difference in the different Hele-Shaw cells is in the way the fluid is injected. This results in different initial and boundary conditions to be analysed which significantly affect the dynamics. In a rectilinear Hele-Shaw cell, a fluid is injected rectilinearly from one end to displace the other more or less viscous fluid, while in a radial Hele-Shaw cell, the fluid is injected from a point source. A quarter five spot is a square having a source at the centre and a sink at each corner. A radial cell is the most suitable for the experiments [8], while numerical and theoretical work is preferably done with a rectilinear geometry [11] and the quarter five spot is studied in context with oil recovery [4, 22].

Since any flow phenomenon is independent of the units of dimensions, proper non-dimensionalisation must be done before carrying out the mathematical analysis. This non-dimensionalisation provides various non-dimens-

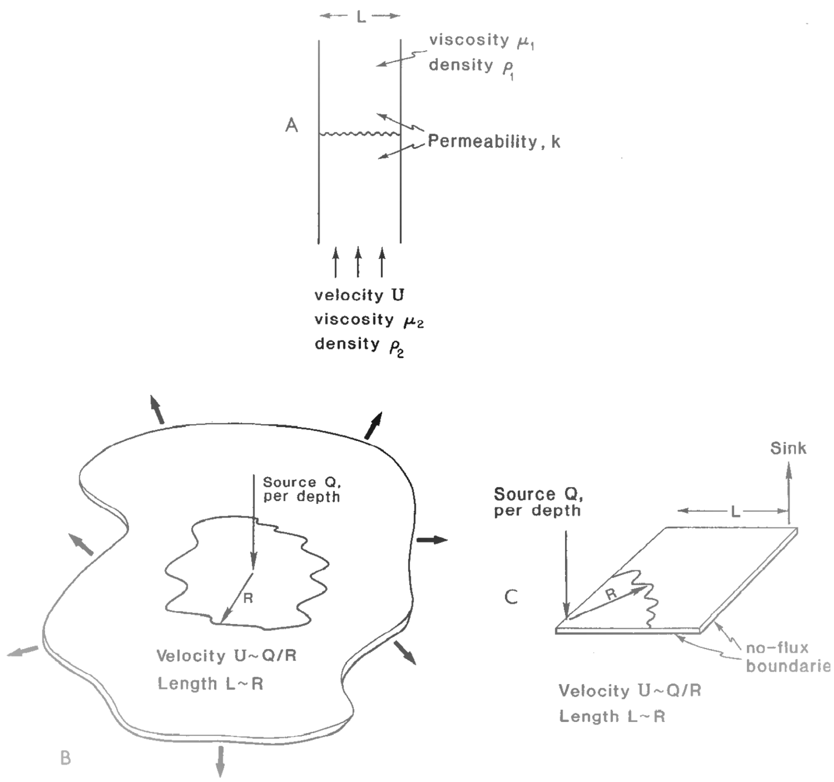

FIGURE 1. Different flow geometries used to understand miscible VF [11].

ional parameters which help in the classification of the flow. Various non dimensionalisations have been used in miscible VF. The uniform velocity with which the fluid is injected is used as characteristic velocity for rectilinear geometry, while the characteristic length scale is chosen in a variety of ways in the literature. For instance, Tan and Homsy [34] use diffusive length scale and got a dimensionless parameter Péclet number (Pe) appearing in the boundary conditions. Many authors use permeability [21, 35], while others [23, 24] use the finite width of the cell as the length scale. This results in Pe appearing within the equation. For the radial geometry, there is no uniform velocity, so a proper characteristic velocity needs to be chosen. This is achieved by fixing the flow rate and/or the time [5, 33].

Another non-dimensional parameter appearing in the miscible VF is the log-mobility ratio ($R = \ln(\mu_2/\mu_1)$), which comes from the viscosity

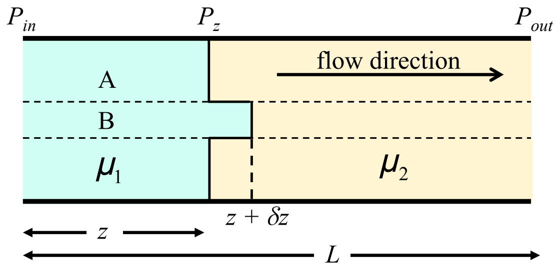

FIGURE 2. Schematic of the flow problem. *Fluid 1* of viscosity μ_1 displaces *fluid 2* of viscosity μ_2 Rousseaux et al.[29]

concentration relation. It is a measure of the viscosity contrast between the two fluids.

The viscous fingering occurs only when a less viscous fluid displaces a more viscous one in a porous medium. Due to the contrast in the mobility of the two fluids owing to their different viscosity, the natural disturbances which are pertinent to any system grow only when a more viscous fluid gets displaced by a less viscous fluid. Rousseaux *et al.*[29] gave a physical mechanism of viscous fingering in context with chromatographic column. On similar lines, we present here a mathematical prove for the growth of perturbations when a less viscous fluid displaces a more viscous one in a Hele-Shaw cell. The detail of the explanations are briefly written here. The fluids are considered to be neutrally buoyant. Figure 2 shows the schematic of the pressure driven flow where the fluid with viscosity μ_1, *fluid 1*, displaces the fluid with viscosity μ_2, *fluid 2*. A constant pressure gradient $\Delta p = P_{in} - P_{out}$ is maintained. A perturbation at the interface $z = z$ displaces the *fluid 1* ahead (or behind) in the region B. The velocity of the fluids differ in the region A and B, being u_A and u_B respectively. There is no momentum exchange between the two parts so that the velocities in the two parts are independent of each other [29]. From the Darcy's law (2.2), we have

$$u = -\frac{\kappa}{\mu}\frac{dp}{dz}. \tag{2.4}$$

Applying this to part A, we have

$$u_A = \frac{\kappa}{\mu_1}\left(\frac{P_{in} - P_z}{z}\right) = \frac{\kappa}{\mu_2}\left(\frac{P_z - P_{out}}{L - z}\right) \tag{2.5}$$

which implies

$$\frac{\mu_1 z}{\kappa L} u_A = \frac{P_{in} - P_z}{L} \quad \text{and} \tag{2.6}$$

$$\frac{\mu_2 (L - z)}{\kappa L} u_A = \frac{P_z - P_{out}}{L}. \tag{2.7}$$

Adding Equation (2.6) and (2.7), we get

$$\frac{\Delta p}{L} = \frac{u_A}{\kappa}\left(\frac{\mu_1 z}{L} + \mu_2\left(1 - \frac{z}{L}\right)\right). \tag{2.8}$$

For the part B, changing u_A to u_B and z to $z + \delta z$, we get

$$\frac{\Delta p}{L} = \frac{u_B}{\kappa}\left(\frac{\mu_1 z}{L} + \mu_2\left(1 - \frac{z}{L}\right)\right) + \frac{\delta z}{L}\frac{u_B(\mu_1 - \mu_2)}{\kappa}.$$

Subtracting Equation (2) from (2.8), we have

$$u_A - u_B = \left(\frac{\delta z}{L}\frac{u_B}{\kappa}(\mu_1 - \mu_2)\right)\frac{\kappa}{\frac{\mu_1 z}{L} + \mu_2\left(1 - \frac{z}{L}\right)} \tag{2.9}$$

Clearly $\mu_1 - \mu_2 > 0, \implies u_A > u_B$, hence the fluid in part A moves faster and prevents the perturbation to grow, resulting in a stable displacement. On the other hand, for $\mu_1 < \mu_2$, the perturbations move faster resulting in the fingering out of the less viscous fluid through the more viscous one, that is, viscous fingering.

3. Discussion and review of results

In this section we discuss various mathematical advancements in the field of miscible VF. The hydrodynamic instabilities are analysed with two aims. The first being finding some critical parameters to gain insight into the transition to some other regime or the onset of the instability, while the second one emphasises on the dynamics once the instability has set in. The miscibility of the fluids poses various challenges which lead to many different methods for achieving the above said aims. The first aim is accomplished by doing the linear stability analysis (LSA), in which the governing non-linear PDEs are linearised by introducing infinitesimal perturbations to the base state

$$f(x, y, t) = f_b(x, y, t) + \epsilon f'(x, y, t), \tag{3.1}$$

where $f \in \{c, u, v, p, \mu\}$ for miscible viscous fingering and $\epsilon << 1$ is the amplitude of the perturbations f'. Usually wave-like perturbations are studied and the base state may be a function of only one space coordinate, may or may not be time-dependent.

First of all, we discuss the LSA of miscible VF with non-reactive fluids in rectilinear geometry, attempted by Tan and Homsy [34]. A fluid with viscosity μ_1, concentration $c = 0$ is injected into a two dimensional porous medium with uniform velocity U to displace another fluid of viscosity μ_2, concentration $c = 1$. The base state solution of the system of equations (1) - (3) is:

$$\vec{u}_b(x,t) = 0, \tag{3.2}$$

$$p_b(x,t) = -\int_{-\infty}^{x} \mu(c_b)ds, \tag{3.3}$$

$$c_b(x,t) = \frac{1}{2}\left[1 + erf\left(\frac{x}{2\sqrt{t}}\right)\right]. \tag{3.4}$$

Here, $erf(z) = (2/\sqrt{\pi})\int_0^z e^{-w^2}dw$ is the error function and $\mu(c) = e^{Rc}, R = ln\left(\frac{\mu_2}{\mu_1}\right)$. The above base state is in a reference frame moving with the uniform velocity U, consequently, $c_b(x,t)$ is the solution of the diffusion equation with a step up initial condition and Neumann boundary condition $\frac{\partial c}{\partial x} \to 0$ as $x \to \pm\infty$. In the moving reference frame, the linearised equations thus obtained by neglecting product and higher order terms of the perturbation quantities are [34]:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \tag{3.5}$$

$$\frac{\partial p'}{\partial x} = -\mu_b u' - \mu', \tag{3.6}$$

$$\frac{\partial p'}{\partial y} = -\mu_b v', \tag{3.7}$$

$$\mu' = \left.\frac{d\mu}{dc}\right|_{c_b} c', \tag{3.8}$$

$$\frac{\partial c'}{\partial t} + \frac{\partial c_b}{\partial x}u' = \frac{\partial^2 c'}{\partial x^2} + \frac{\partial^2 c'}{\partial y^2}. \tag{3.9}$$

The time-dependence of the base state poses limitations in analysing the growth or decay of the perturbations which is the main aim of LSA. In order to get rid of the time-dependence, Tan and Homsy [34] assumed the base state to be evolving much slower than the perturbations so that the base state can be assumed to be quasi steady and hence the name Quasi Steady State Approximation (QSSA). In QSSA, the base state is frozen at some time t_0 and the following coupled linear partial differential equations with variable coefficients are obtained by eliminating p', v' from equation

(3.5) - (3.9):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\mu_b(x,t_0)}\frac{d\mu_b(x,t_0)}{dx}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}\right)u'(x,y,t) = -\frac{1}{\mu_b}\frac{d\mu}{dc}\bigg|_{c_b}\frac{\partial^2 c'}{\partial y^2}, \quad (3.10)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)c'(x,y,t) = -\frac{dc_b(x,t_0)}{dx}u'(x,y,t). \quad (3.11)$$

Clearly, the above two equations are homogeneous in y and t, consequently, in the transverse direction, the perturbations are decomposed in the Fourier modes with amplitude depending upon x which grows exponentially with time as:

$$(u', c')(x,y,t) = (\hat{u}, \hat{c})(x,t_0)e^{iky}e^{\sigma(t_0)t}, \quad (3.12)$$

where k is the wavenumber of the modes and $\sigma(k,t_0)$ is the growth rate of the respective mode. Substituting above Fourier mode decomposition into equation (3.10), (3.11), following coupled ordinary differential equations are obtained:

$$\left(\frac{d^2}{dx^2} + \frac{1}{\mu_b(x,t_0)}\frac{d\mu_b(x,t_0)}{dx}\frac{d}{\partial x} - k^2\right)\hat{u}(x,t_0) = \frac{k^2}{\mu_b}\frac{d\mu}{dc}\bigg|_{c_b}\hat{c}(x,t_0), \quad (3.13)$$

$$\left(\sigma(t_0) - \frac{d^2}{\partial x^2} + k^2\right)\hat{c}(x,t_0) = -\frac{dc_b(x,t_0)}{dx}\hat{u}(x,t_0). \quad (3.14)$$

The above equations form an Eigen value problem (EVP) with eigen value σ. The EVP is solved subject to far field boundary conditions, $\hat{u}, \hat{c} \to 0$ as $x \to \pm\infty$. The evp is solved using shooting method or finite difference method.

The QSSA method is extensively used to explore the stability of a rectilinear displacement with anisotropic dispersion [35] and the influence of non-monotonic viscosity profile on the onset of instability [20]. The onset of instability in chromatographic column is predicted using the LSA of Tan and Homsy [34] by Rousseaux *et al.* [29], while Rogerson and Meiburg [28] used QSSA to analyse the interfacial instability between two fluids of different viscosities and densities in a porous medium when the gravity is aligned in various angles to the interface. Despite wide applicability of QSSA, it fails to capture initial diffusion, paving way to finding some other technique for the LSA.

For this, Ben *et al.* presented a spectral theory based on similarity transformation. Pramanik *et al.,* [23] used a self similarity transformation in order to get rid of the time dependent base state without freezing the time,

$$\xi = \frac{x}{2\sqrt{t}}, \quad (3.15)$$

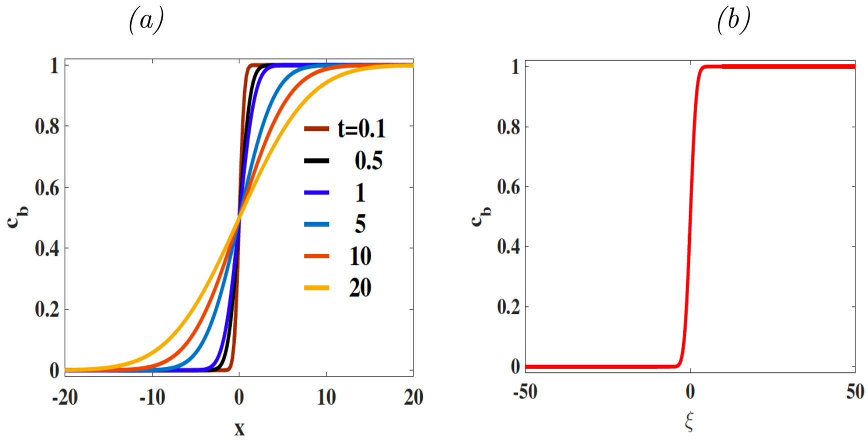

FIGURE 3. The base state concentration profile in *(a)* (x, t) coordinate, *(b)* (ξ, t) domain. [23]

so that the base state becomes time independent in the self similarity (ξ, t) domain, that is,

$$c_b(\xi, t) = \frac{1}{2}[1 + erf(\xi)]. \tag{3.16}$$

Figure 3 shows the base state profiles in (x, t) as well as (ξ, t) domain. Along with capturing the early time behaviour of the perturbations, this technique guaranteed the localisation of the perturbations around the interface in the (ξ, t) domain (Figure 4 on the next page). This was absent in Tan and Homsy's QSSA [34]. But the linearised equations similar to equations (3.10),(3.11) in the self-similarity domain contain time t explicitly. As such, Pramanik *et al.* [23] had to freeze the base state at time t_0 to analyse the growth of the perturbations, thus calling their technique as Self Similar Quasi Steady State Approximation (SS-QSSA). Hence for the LSA through the evp formulation, time needs to be frozen. This is not the case for immiscible VF where analytical dispersion relations are also possible [30]. Thus, the miscibility of the fluids bring in various mathematical challenges which have to be tackled keeping in mind the physics of the problem.

If we do Fourier mode decomposition only in the transverse direction as

$$(u', c')(x, y, t) = (\hat{u}, \hat{c})(x, t)e^{iky}, \tag{3.17}$$

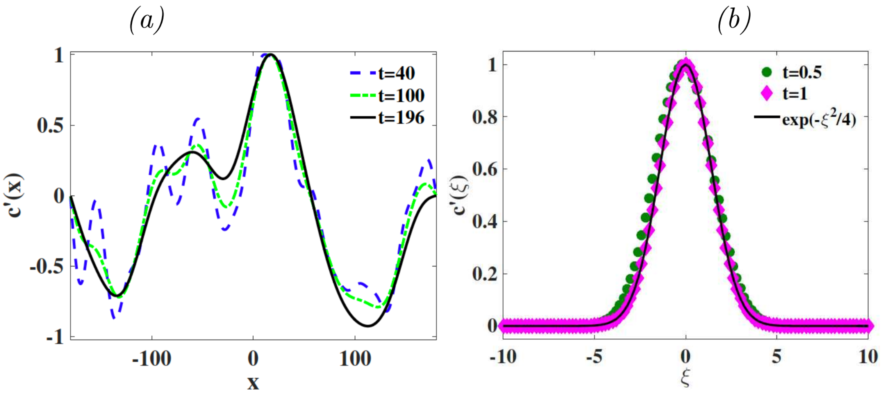

FIGURE 4. Perturbations in the (x, t) as well as (ξ, t) domain. The perturbations are localised around the interface in the self similarity domain *(b)*[13].

we get an initial boundary value problem

$$\left(\frac{d^2}{dx^2} + \frac{1}{\mu_b(x,t)}\frac{d\mu_b(x,t)}{dx}\frac{d}{\partial x} - k^2\right)\hat{u}(x,t) = \frac{k^2}{\mu_b}\frac{d\mu}{dc}\bigg|_{c_b}\hat{c}(x,t), \tag{3.18}$$

$$\left(\frac{\partial}{\partial t} - \frac{d^2}{\partial x^2} + k^2\right)\hat{c}(x,t) = -\frac{dc_b(x,t)}{dx}\hat{u}(x,t), \tag{3.19}$$

which can be reduced into an initial value problem(IVP) [34, 12]

$$\frac{d\hat{c}}{dt} = A(t)\hat{c}, \tag{3.20}$$

with some random initial condition. Tan and Homsy [34] solved the IVP and found the perturbations become unstable after some initial time, consistent with what physics demands that initially due to diffusion dominance, the perturbations should decrease. But the random initial condition being used is a drawback of the IVP analysis. Further the time-dependence of the operator $A(t)$ is overlooked in [34]. Since the non-autonomous systems are non-normal, the perturbation modes may amplify at short times then eventually decreasing at later time, a phenomenon called transient growth. The transient growth [31] is susceptible in miscible VF, which was captured by Hota *et al.* [13] for the very first time by using the energy approach. Using the non-modal analysis (NMA) via propagator matrix approach, Hota *et al.* [12, 13] discuss the initial condition which is amplified the most in the linear regime. The NMA also captured the diffusion dominating regime which was not captured by any of the previous LSA techniques. Figure 5 on the next page shows the growth rate as predicted using QSSA,

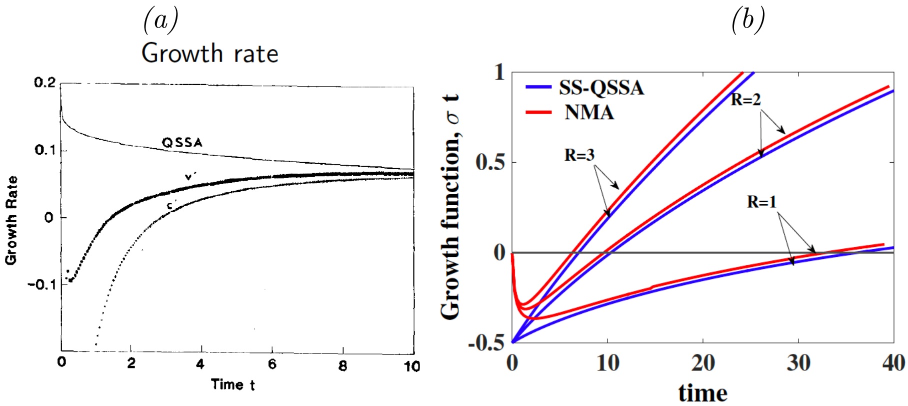

FIGURE 5. *(a)* The growth rate calculated using QSSA and IVP. The c', v' curves are obtained using IVP analysis. This figure is taken from [34]. *(b)* The growth function calculated using SS-QSSA and the NMA [13].

IVP analysis, SS-QSSA and the NMA. It is evident from Figure 5(a), that QSSA predicts the system to be unstable right from the beginning which violates the physics. IVP and SS-QSSA shows the growth rate to be negative for some initial time, indicating an initial stable zone. But the growth rate calculated from NMA is non-monotonic, signifying the dominance of dispersion at early times. Thus, the non-autonomous system introduced due to the miscible nature of the fluids, can be best described through non-modal analysis.

Tan and Homsy [35] did the LSA of the radial displacement of non-reactive miscible fluids. Due to the non constant velocity, radial displacement is different from the rectilinear counter part. As such, QSSA is not applicable in radial displacement and the perturbations grow algebraically in comparison to the exponentially growing perturbations in rectilinear displacement. Further, the instability is found to be dependent on the Peclét number Pe in radial displacement. The effect of initial condition in three dimensional flow is analysed by Riaz *et al.* [26] by considering the effect of axial and helical perturbations while performing the LSA of the radial displacement. Riaz *et al.* [27] investigated the influence of velocity-induced dispersion and concentration-dependent diffusion on the linearly unstable modes in a radial displacement flow by carrying LSA in similarity transformation domain. The concentration-dependent diffusion coefficients are

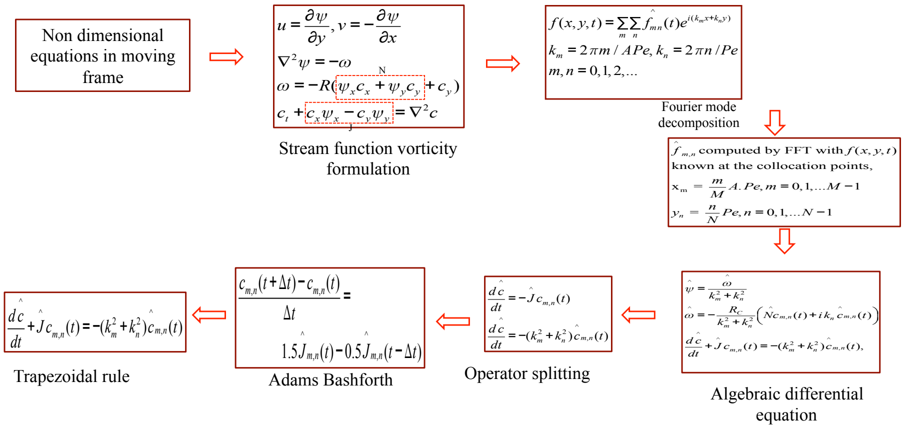

FIGURE 6. A flow chart of the Fourier pseudo-spectral method used for the non-linear simulations.

found to affect the stability in competitive ways with an overall destabilising effect for all Pe and log-mobility ratios. The velocity-dependent dispersion is found to dominate over diffusion for large values of Pe, and as the velocity reduces radially, the growth rate asymptotically approach those corresponding to only molecular diffusion.

The LSA gives information about the growth or decay of the perturbations but in order to have an insight into the dynamics, complete non linear equations must be considered. Many numerical techniques have been used to solve equations (2.1)-(2.3). The most common approach is by using stream function vorticity formulation which is obtained by eliminating pressure from the governing equations. The stream function ψ, is the solution of the continuity equation (2.1) and is easily obtainable in two dimensions as $u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$. Figure 6 gives a flow chart of the Fourier pseudo spectral method [36] for the non-reactive fluids to solve the stream function vorticity equations. The viscous fingering patterns obtained when a less viscous fluid (red) displaces a more viscous one (blue) in a moving reference frame are shown in Figure 7*(a)*. The Fourier pseudo-spectral method is also used to see the fingering dynamics when a fluid is sandwiched between another fluid of less/more viscosity. In Figure 7*(b)*, the more viscous fluid is sandwiched between the layers of a less viscous fluid. As such, the front interface is prone to fingering and the rear interface where more viscous fluid displaces less viscous one acts as a barrier and prevents the fingers from growing further. Consequently, the fingers grow in the upward direction. The less

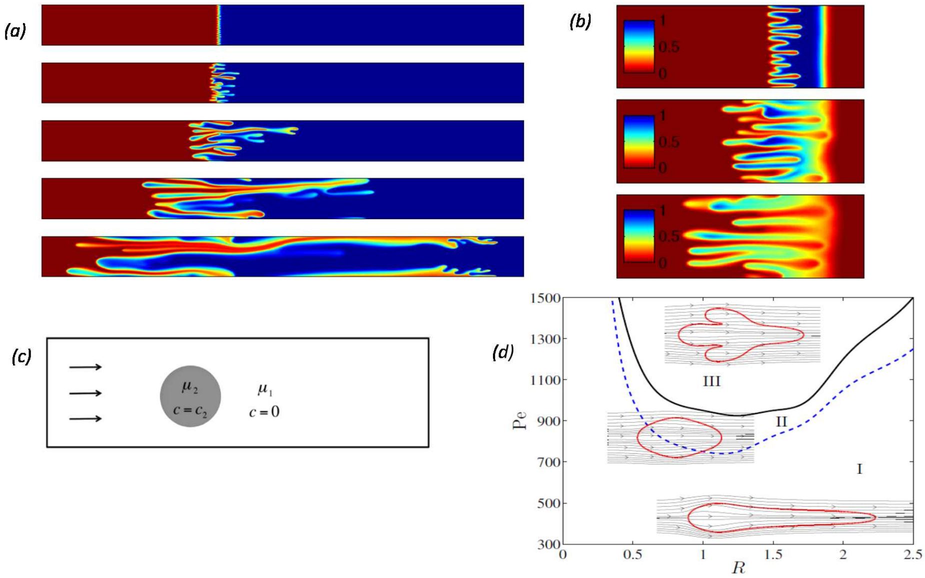

FIGURE 7. The density plots of concentration at different times, *(a)* less viscous fluid diapces a more viscous one. *(b)* The more viscous fluid sandwiched between the less viscous fluid. *(c)* Schematic of more viscous blob in the viscinity of the less viscous fluid[25]. *(d)* The $R - Pe$ plane obtained in [25]

viscous fluid may not be sandwiched as a slice but many be any arbitrary shape. In context with contaminant transport, Pramanik *et al.* [25] considered a more viscous blob displaced by a less viscous ambient fluid as shown in Figure 7*(c)*. The dynamics are found to be Pe dependent, dividing the $R - Pe$ plane into three zones as in Figure 7*(d)*. In zone I and II, no VF is observed despite a favourable viscosity contrast and the zone III undergoes fingering at the front zone. This is attributed to the effect of the curvature [25].

But the Fourier pseudo spectral method has convergence issues due to the stiff nature of the differential equations. As such many other numerical techniques have been used to study miscible viscous fingering. Jha *et al.* [18] used finite volume method and obtained convergence upto $R = 5$. In this regard, Sharma *et al.* [32] used finite element based COMSOL [6] simulations to obtain convergence upto $R = 10$ and discussed the behaviour of a highly viscous circular blob. Many hybrid methods *viz* compact finite

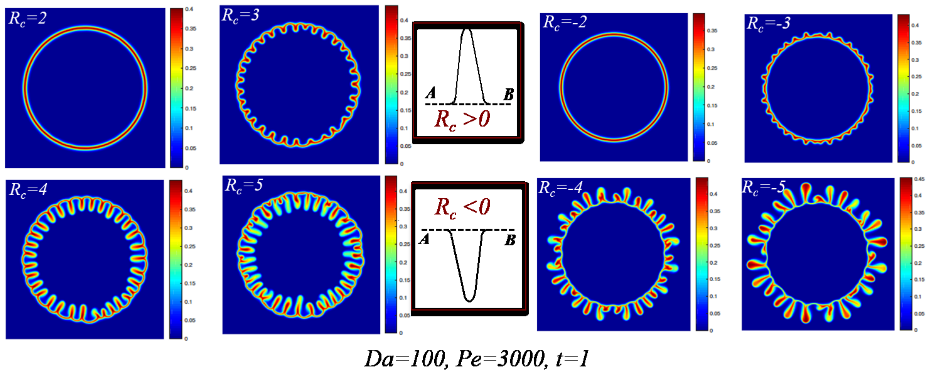

FIGURE 8. Temporal evolution of the product concentration for various parameters. Viscosity profiles for $R_c \neq 0$ are also shown.

difference and pseudo spectral method [5]; alternating direction implicit and pseudo spectral method [17] have been utilised by many researchers to understand the dynamics when a less viscous fluid displaces a more viscous one. Still developing a more efficient numerical tool to handle the coupled partial differential equations is a topic of current research.

We discuss the VF with reactive miscible fluids. The LSA of the reactive miscible fluids is poorly explored while the interest has been focussed on understanding the dynamics numerically and experimentally [8]. Hejazi *et al.* [15] carried the LSA of the reactive miscible fluids undergoing a second order chemical reaction to form a product having same or different viscosity than the two reactants, using QSSA on similar lines as Tan and Homsy [34]. The NMA of reactive miscible fluids has still never been attempted. The non-linear simulations of miscible viscous fingering usually employ the Fourier pseudo spectral method and is carried out in rectilinear geometry. But the experiments are performed in a radial Hele-Shaw cell. Recently, Sharma *et al.*[33] carried out the non linear simulations in a radial geometry using a hybrid compact finite difference and the pseudo-spectral method. This scheme was used to solve the convection-diffusion-reaction system for the first time. Their work paved a way towards a direct comparison of experiments and the numerical work. Figure 8 shows the temporal evolution of the product concentration for various parameters when reaction produces a more viscous product ($R_c > 0$) as well as $R_c < 0$ when product is less viscous than the two reactants.

4. Conclusion

Instabilities in porous media require a rigorous mathematical treatment for a better understanding and prediction of various critical parameters. A review of various advancements in the LSA and the non linear simulations of miscible VF is presented. The aim of LSA is to predict the growth or decay of perturbations with respect to time. A modal analysis in which the perturbations are decomposed into the Fourier modes growing exponentially in time is used for the LSA. The LSA of miscible VF is mathematically challenging mainly due to the time-dependent base state. QSSA forms the first attempt to tackle this, by freezing the base state making it quasi steady. In QSSA, it is assumed that the base state evolves slowly with respect to the perturbations. LSA of many miscible VF problems used QSSA but QSSA could not predict the stabilising effect of diffusion. As such, a new technique called SS-QSSA making base state time independent using self similarity transform is formulated and used for LSA. But SS-QSSA also require freezing the time, as such solving the initial value problem with random initial condition is sought as a technique to see the growth of the perturbations. The non-normality of the coefficient matrix in the IVP indicate the transient growth which is susceptible in many stability problems. Recent advances made to capture transient growth through NMA in miscible VF are presented. We also discuss briefly various numerical techniques available in literature to gain an insight into the VF dynamics. VF is not just an instability observed in porous media flows but has relation to many other fields, consequently, understanding VF is of interest to researcher from multi-disciplinary fields.

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Manoranjan Mishra
Dept. of Mathematics and Dept. of Chemical Engineering
Indian Institute of Technology Ropar, Ropar-140001,
Rupnagar, Punjab, India.
E-mail: manoranjan@iitrpr.ac.in

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TRENDS IN BORDISM, G-BORDISM THEORY AND FIXED POINT SET: A SURVEY*

S. S. KHARE

Abstract. The objective of this survey is to give a brief account of (1) bordism classifications of manifolds and G-bordism classifications of G-manifolds through characteristic numbers (2) G-bordism classifications of G-manifolds through fixed point set with emphasis on involutions and $\mathbb{Z}_2^k$-actions. (3) generators of unoriented bordism algebra MO_* (4) bounding and independence of Dold, Milnor and Wall manifolds and their span.

1. Introduction

The article needs some prerequisits like homotopy, tangent space, tangent bundle, normal bundle, vector bundle, universal vector bundle, singular homology and cohomology, cup and cap product, Stiefel-Whitney classes, Euler characteristic, fundamental class, etc. Although we will try to give brief account of some of them, whenever they appear first, the reader is advised to consult [34] for ready reference as and when needed.

Classification of smooth manifolds has been a basic problem in differential topology. Since it is quite difficult to classify smooth manifolds up to diffeomorphism, mathematicians focussed their attention to classify them up to homotopy or up to bordism. In this article we will confine our attention to classification up to bordism or G-bordism (for a G-manifold, G a group). First such effort was made by Pontrjagin [38] in 1947 by studying bordism classification through algebraic invariant, namely, Stiefel-Whitney numbers [34]. This was further investigated by R. Thom [54] in 1954 and Wall [56] for oriented smooth manifolds in 1960. Lee and Wasserman [60], in 1972, studied

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2010 Mathematics Subject Classification: 55M20, 55N22, 57R20, 57R25, 57R85.
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G-bordism classification for smooth G-manifolds by developing new algebraic invariant, namely, G-characteristic numbers. In 1976, we studied classification of singular G-manifolds in a G-space X by developing new algebraic invariants [15, 16].

Conner and Floyd [6, 7], in 1964-65, started studying classification of G-manifolds through fixed point set of G. Later Stong [46, 47, 48, 49, 50, 51], Khare [17, 18, 19], Capobianco [4, 5], Royster [39], Torrence [52, 53], Boardman [2], Kosniowski [31], Pergher [36, 37], Shaker [40, 41], Wang [57, 58, 59], Wu [62, 63, 64, 65], Ma [64, 65], Feng [9], Geng [10] and He J. Y. [11] contributed significantly towards G-bordism classification through its fixed point set.

Throughout the article, by a closed manifold, we will mean a compact and smooth manifold without boundary. An unoriented closed manifold M_1^n is said to be *bordic* or *bordant* to another such manifold M_2^n, if there exists an unoriented manifold W^{n+1} with the boundary $\partial W^{n+1} \approx M_1^n \sqcup M_2^n$. Here by $\approx$, we mean diffeomorphic and by $\sqcup$, we mean disjoint union. An unoriented closed manifold M^n is said to *bound* if there exists an unoriented manifold W^{n+1} with $\partial W^{n+1} \approx M^n$. Bordism is an equivalence relation on the set of all closed manifolds of dimension n and $[M^n]$ denotes the *bordism class* of M^n. The set MO_n of all bordism classes of closed manifolds of dimension n forms an abelian group under the operation of addition given by

$$[M_1^n] + [M_2^n] = [M_1^n \sqcup M_2^n].$$

MO_n is called the *n-dimensional unoriented bordism group*. Further, $MO_* = \sum_{n=1}^{\infty} MO_n$ is a graded algebra over $\mathbb{Z}_2$ with multiplication

$$[M_1^{n_1}] \cdot [M_2^{n_2}] = [M_1^{n_1} \times M_2^{n_2}].$$

MO_* is called the *unoriented bordism algebra* over $\mathbb{Z}_2$.

Similarly, in oriented case, an oriented closed manifold M_1^n is said to be *bordic* to another such manifold M_2^n, if there exists an oriented manifold W^{n+1} with $\partial W^{n+1} \approx M_1^n \sqcup (-M_2^n)$ under orientation preserving diffeomorphism, $-M_2^n$ being M_2^n with reverse orientation. The set MSO_n of all bordism classes of oriented closed manifolds of dimension n forms an abelian group with respect to addition operation and is called the *n-dimensional oriented bordism group*. $MSO_* = \sum_{n=1}^{\infty} MSO_n$ is a graded algebra over $\mathbb{Z}_2$ with multiplication defined as above.

2. Dold, Milnor and Wall manifolds as generators of the bordism algebra MO_*, their bounding and independence

Dold investigated generators of the bordism algebra MO_* and constructed a new closed manifold $P(m, n)$ for given nonnegative integers m and n as fol lows: $P(m,n) = (S^m \times \mathbb{C}P^n)/\sim$, $\sim$ being a relation given by $(x,\ [z]) \sim (-x,\ [\overline{z}])$. Here $\mathbb{C}P^n$ is n-dimensional complex projective space. Clearly $\sim$ is an equivalence relation. $P(m,n)$ is $(m+2n)$-dimensional closed manifold, called *Dold manifold* [8]. Dold [8] proved in 1956 that MO_* is a polynomial algebra over $\mathbb{Z}_2$ with generators $[\mathbb{R}P^{2i}]$ and $[P(2^r-1, s2^r)]$, $i, r, s \geq 1$, where $\mathbb{R}P^{2i}$ is $2i$-dimensional real projective space.

While investigating other set of generators of MO_*, Milnor [33] constructed another set of closed manifolds $H(m,n)$ in 1965 for $m, n \geq 1$. $H(m,n)$ consists of all pairs $([x_0, \cdots, x_m], [y_0, \cdots, y_n])$ in $\mathbb{R}P^m \times \mathbb{R}P^n$ such that $x_1y_1 + \cdots + x_ty_t = 0$, t being the minimum of m and n. $H(m,n)$ is called *Milnor manifold.* Milnor [33] proved in 1965 that the set $\{[\mathbb{R}P^{2i}], [H(2^k,\ 2t2^k)]\}$, $i, k, t \geq 1$, generates MO_*.

In 1960, Wall [56] gave another set of generators of MO_*. He constructed a new closed manifold $Q(m,n)$ of dimension $m+2n+1$. He defined $Q(m,n)$ as the quotient manifold $(P(m,n) \times I)/\sim$, where $(p,0) \sim (A(p),1)$ and $A : P(m,n) \to P(m,n)$ is an automorphism given by $A([(x,[z])]) = [(T(x),[z])]$. Here, $T : S^m \to S^m$ is given by $T(x_0, \cdots, x_m) = (x_0, \cdots, x_{m-1}, -x_m)$. $Q(m,n)$ is called *Wall manifold.* Wall proved in 1960 that the set $\{[\mathbb{R}P^{2i}], [Q(2^r-2, s2^r)];\ i, r, s \geq 1\}$ generates the unoriented bordism graded algebra MO_*.

In an effort to study the bounding and independence problems of Dold, Milnor and Wall manifolds, we proved the following results.

Theorem 2.1 (Khare [20]). *A Dold manifold $P(m,n)$ bounds if and only if (i) n is odd or (ii) n is even, m is odd and $m > n$ with $2^{\nu(m-n-1)} > n$. Here $2^{\nu(m-n-1)}$ is the highest power of 2 dividing $m-n-1$.*

Theorem 2.2 (Khare [20]). *For a given positive integer d, the set $\{[P(m,n)] \neq 0 : m+2n = d\}$ is linearly independent in the vector space MO_d over $\mathbb{Z}_2$.*

Theorem 2.3 (Khare-Das [21]). *A Milnor manifold $H(m,n)$, $m \leq n$ bounds if and only if at least one of the following holds:*
(i) $m = n$ (ii) $m = 1$ (iii) Both m and n are odd
(iv) $n \equiv 2 (mod\ 4)$ and $m+1 < 2^{\nu(n+2)}$.

Theorem 2.4 (Khare and Dutta [23]). *For a given even integer d or for an odd integer d with $d+1 = 2^\nu \cdot odd$, $\nu > 1$, the set $\{[H(m,n)] \neq 0 : m+n-1 = d\}$ is linearly independent in the vector space MO_d over $\mathbb{Z}_2$.*

Theorem 2.5 (Khare [22]). *(1) A Wall manifold $Q(m,n)$ bounds if and only if n is odd or $n = 0$ with m odd. (2) For a given positive integer d, the set*

$\{[Q(m,n)] \neq 0 : m+2n+1 = d\}$ is linearly independent in the vector space MO_d over $\mathbb{Z}_2$.

3. Vector fields and span of specific manifolds

A *vector field* on a manifold M^n is a section $s : M^n \to T(M^n)$ of the tangent bundle $\tau : T(M^n) \to M^n$, i.e., $\tau \circ s = 1_{M^n}$. A vector field s on a manifold M^n is said to be *nowhere vanishing*, if $s(x) \neq 0$, $\forall x \in M^n$. A set $\{s_1, \cdots, s_r\}$ of nowhere vanishing vector fields is said to be *everywhere linearly independent* if $\{s_1(x), \cdots, s_r(x)\}$ is linearly independent for all $x \in M^n$. The maximum number of linearly independent vector fields on M^n is called the *span of* M^n and is denoted by *span(M^n)*. By *stable span of* M^n, we mean $\text{span}(M^n \times S^1) - 1$, and is denoted by *stabspan(M^n)*.

Condition for the existence of nowhere vanishing vector field on M^n, condition for the existence of k-linearly independent vector fields in terms of some algebraic invariants and computation of span of specific manifolds like $S^n, \mathbb{R}P^n$, lense spaces, Dold manifolds, Milnor manifolds, Wall manifolds, flag manifolds, Stiefel manifolds, etc. have been of long standing interest. Hopf [12], Hurwitz [13], Khare [24, 25, 26, 27], Korbas [29, 30], Novotny [35], Shankaran [42, 43], Steenrod and Whitehead [44] and Zvengrowski [66, 67, 68] contributed significantly towards these problems. Some results in this direction are as follows.

Theorem 3.1 (Hopf [12]). *M^n admits a nowhere vanishing vector field if and only if the Euler characteristic $\chi(M^n)$ mod 2 is zero. Here $\chi(M^n) = \sum_{i=0}^{n}(-1)^i \beta_i$, where β_i (=dimension of $H_i(M^n;Q)$) is the i^{th} Betti number of M^n.*

Theorem 3.2 ([68]). *For a manifold M^n, if $\omega_{n-k}(M^n) \neq 0$, then $span(M^n) \leq k$. Here $\omega_{n-k}(M^n)$ is $(n-k)^{th}$ Stiefel-Whitny class of M^n.*

Theorem 3.3 (Hurwitz [13]). *For odd n with $n+1 = 2^{c+4d}$·odd, $span(S^n) \leq \rho(n) - 1$, where $\rho(n) = 2^c + 8d$.*

Theorem 3.4 (Adams [1]). *For odd n with $n+1 = 2^{c+4d}$·odd, $span(S^n) = \rho(n) - 1$.*

Theorem 3.5. *$span(\mathbb{R}P^n) = span(S^n)$.*

Theorem 3.6 (Novotny [35]). *Given a pair (m,n) of non-negative integers, let $n = 2^{\nu(n+1)} \cdot k + \ell$, $\quad 0 \leq \ell < 2^{\nu(n+1)}$. Then*
(1) $\rho(m) - 1 \leq (stab)span(P(m,n)) \leq 2^{\nu(n+1)}[2^{\nu(k+1)} + 1] - 2$.
(2) If $m, n \equiv 1(mod\ 4)$, then $span(P(m,n)) = 2$.
(3) If $m \equiv 1(mod\ 4)$ and $n \equiv 3(mod\ 4)$, then $span(P(m,n)) \geq 3$.

Theorem 3.7 (Khare [24]). *Following holds true.*
(1) If $\alpha_m < \cdots < \alpha_1 \leq \beta_n < \cdots < \beta_1$, *then* $\rho(2^{\alpha_m} - 1) - 1 \leq span(H(m-1, n-1)) \leq 2^{\alpha_m} - 1$, *where* $m = 2^{\alpha_1} + \cdots + 2^{\alpha_m}$ *and* $n = 2^{\beta_1} + \cdots + 2^{\beta_n}$.
(2) If $\alpha_m = 1, 2$ *or* 3, *then* $span(H(m-1, n-1)) = 2^{\alpha_m} - 1$.
(3) $1 \leq span(H(m, n)) \leq 2^{\alpha_m} - 1$, *if* $m = 2^{\alpha_1} + \cdots + 2^{\alpha_m}$ *and* $n = 2^{\beta_1} + \cdots + 2^{\beta_n}$.
(4) $\rho(2^{\alpha_m} - 1) - 1 \leq span(H(m-1, n)) \leq 2^{\alpha_m} - 1$.
(5) If $\alpha_m = 1, 2$ *or* 3, *then span* $(H(m-1, n)) = 2^{\alpha_m} - 1$.
(6) If $\beta_n < \cdots < \beta_1 < \alpha_m < \cdots < \alpha_1$, *then* $span(H(2^{\alpha_1} + \cdots + 2^{\alpha_m} - 1, 2^{\beta_1} + \cdots + 2^{\beta_n})) = 0$.

Theorem 3.8 (Khare [25]). *(i)* $Span(Q(m, n)) \geq 1$, *for all* m *and* n.
(ii) $Span(Q(m, n)) \geq 3$ *if* $4|m$ *and* $n = 2^s \cdot odd - 1$, $s \geq 2$.
(iii) $Span(Q(m, n)) \leq 2^{\nu(n+1)}[2^{\nu(k+1)} + 1] - 1$, *where* $m = 2^{\nu(n+1)} \cdot k + \ell$, $0 \leq \ell < 2^{\nu(n+1)}$.
(iv) $Span(Q(m, 0)) = 1$ *for all* m.
(v) If $4|m$ *and* $n \equiv 1 (mod\ 4)$, *then* $2 \leq span(Q(m, n)) \leq 3$.

Theorem 3.9 (Khare [25]). *Total Stiefel-Whitney class of* $Q(m, n)$ *is*
$$W(Q(m, n)) = (1 + c + x)(1 + c)^{m-1}(1 + c + d)^{n+1}$$
with $x^2 = 0 = d^{n+1}$ *and* $c^{m+1} = x\, c^m$.

Very recently, we investigated about parallelizability of $Q(m, n)$ and tried to find some sharper bounds of $span(Q(m, n))$.

Theorem 3.10 (Khare [26]). *No Wall manifold is parallelizable, i.e., for no Wall manifold* $span(Q(m, n)) = m + 2n + 1$.

Proof. Korbas [30] proved in 2013 that Dold manifolds $P(m, n)$ are not stably parallelizable for $(m, n) \notin \{(1, 0), (3, 0), (7, 0), (0, 1), (2, 1), (6, 1)\} = T$, i.e., $stabspan(P(m, n)) < m + 2n$, for all $(m, n) \notin T$. This implies that
$$span(Q(m, n)) \leq stabspan(Q(m, n)) \leq stabspan(P(m, n)) + dim S^1,$$
by [25]. Therefore
$$span(Q(m, n)) < (m + 2n + 1), \quad \text{for all } (m, n) \notin T.$$
Further, by Theorem (3.8 (iv)), $span(Q(m, 0)) = 1$, for all m, so that $Q(1, 0)$, $Q(3, 0)$ and $Q(7, 0)$ are not parallelizable. Also, by Theorem (3.9), total Stiefel-Whitney class of $Q(0, 1)$ is given by
$$\begin{aligned} W(Q(0,1)) &= (1 + c + x)(1 + c)^{-1}(1 + c + d)^2 \\ &= (1 + c + x)(1 + c)^{-1}(1 + c)^2, \text{ as } d^2 = 0 \\ &= (1 + c + x)(1 + c) = 1 + c \text{ as } c^2 = cx = 0 \text{ and } c = x. \end{aligned}$$

Therefore, the highest nonzero Stiefel-Whitney class is $\omega_1(Q(0,1)) = c \neq 0$, so that $span(Q(0,1)) \leq 2+1-1 = 2$. Hence, $Q(0,1)$ is not parallelizable. Next,

$$W(Q(2,1)) = (1+c+x)(1+c)(1+c+d)^2 = (1+c+x)(1+c)^3.$$

Thus, the highest nonzero Stiefel-Whitney class

$$\omega_3(Q(2,1)) = c^3 + c^3 + c^2x = c^2x \neq 0.$$

Therefore, $span(Q(2,1)) \leq 2+2+1-3 = 2$, which implies that $Q(2,1)$ is not parallelizable. Similarly,

$$W(Q(6,1)) = (1+c+x)(1+c+c^2+\cdots+c^7),$$

so that the highest nonzero Stiefel-Whitney class $\omega_7(Q(6,1)) = c^6x \neq 0$. Thus $span(Q(6,1)) \leq 2$, which shows that $Q(6,1)$ is not parallelizable.

Thus, $Q(m,n)$ is never parallelizable. Observe that in the process, we have also shown that for every $m,n \in T$, $1 \leq span(Q(m,n)) \leq 2$. □

Theorem 3.11 ([26]). *$Span(Q(m,n)) = 1$ for all m, if n is even.*

Proof. By Theorem (3.8 (i)), $span(Q(m,n)) \geq 1$ for all m and n. Also, for $n = 2r$, by Theorem (3.9)

$$\begin{aligned} W(Q(m,n)) &= [(1+c)^m + x(1+c)^{m-1}](1+c+d)^{2r+1} \\ &= [(1+c)^m + x(1+c)^{m-1}][(1+c)d^{2r} + \text{lower degree terms in } d]. \end{aligned}$$

Therefore, highest non-zero Stiefel-Whitney class is

$$\omega_{m+4r} = \binom{m+1}{1} c^m d^{2r} + \binom{m}{1} xc^{m-1}d^{2r} = \begin{cases} c^m d^{2r}, & \text{if } m \text{ is even;} \\ xc^{m-1}d^{2r}, & \text{if } m \text{ is odd.} \end{cases}$$

Hence $span(Q(m,n)) \leq m+4r+1-(m+4r) = 1$, if $n = 2r$, for all m. This completes the proof. □

Theorem 3.12 ([26]). *(i) If $n+1 = 2(2q+1)$ then $1 \leq span(Q(m,n)) \leq 2$.*
(ii) If $n+1 = 2^r(2q+1)$, $r \geq 2$ and $2^r | m$ then $3 \leq span(Q(m,n)) \leq 2^{r+1}-2$.

Proof. (i) By Theorem (3.8 (i)), $span(Q(m,n)) \geq 1$. Also, $n+1 = 2(2q+1)$ implies

$$\begin{aligned} W(Q(m,n)) &= [(1+c)^m + x(1+c)^{m-1}](1+c+d)^{4q+2} \\ &= [(1+c)^m + x(1+c)^{m-1}][(1+c)^2 d^{4q} + \text{lower degree terms in } d]. \end{aligned}$$

Therefore, highest non-zero Stiefel-Whitney class is

$$\begin{aligned} \omega_{m+2n-1}(Q(m,n)) &= [\binom{m+2}{1} c^{m+1} + \binom{m+1}{1} xc^m] d^{4q} \\ &= \begin{cases} c^{m+1}d^{n-1}, & \text{if } m \text{ is odd;} \\ xc^m d^{n-1}, & \text{if } m \text{ is even,} \end{cases} \neq 0, \end{aligned}$$

which implies that

$$span(Q(m,n)) \leq (m+2n+1)-(m+2n-1) = 2.$$

This completes the proof of (i).

(ii) By Theorem (3.8 (ii)), $span(Q(m,n)) \geq 3$ because $r \geq 2$ implies

$$D = dim(Q(m,n)) = m+2n+1 = 2^r\lambda + 2^{r+1}(2q+1) - 1 \equiv 3(mod\ 4).$$

Also,

$$\begin{aligned} W(Q(m,n)) &= [(1+c)^m + x(1+c)^{m-1}](1+c+d)^{2^r(2q+1)} \\ &= [(1+c)^m + x(1+c)^{m-1}][(1+c)^{2^r} d^{(n+1)-2^r} + \text{lower degree terms in } d]. \end{aligned}$$

Therefore, the highest non-zero Stiefel-Whitney class is

$$\begin{aligned} \omega_{m+2n+1-(2^{r+1}-2)} &= [\binom{m+2^r}{2^r-1} c^{m+1} + \binom{m+2^r-1}{2^r-1} xc^m] d^{(n+1)-2^r} \\ &= xc^m d^{n+1-2^r}, \quad \text{since } \ m = 2^r\lambda \\ &\neq 0, \end{aligned}$$

which implies that

$$1 \leq span(Q(m,n)) \leq (m+2n+1) - (m+2n+1-2^{r+1}+2) = 2^{r+1}-2.$$

This completes the proof of (ii). □

Theorem 3.13 ([26])**.** *(i) For every m, $1 \leq span(Q(m,1)) \leq 2$.*

(ii) $span(Q(4\lambda,1)) = 2$ for all $\lambda \geq 1$.

Proof. (i) By Theorem (3.8 (i)), $span(Q(m,1)) \geq 1$. We consider two cases.

case(a). m is even. In this case

$$\begin{aligned} W(Q(m,1)) &= (1+c+x)(1+c)^{m-1}(1+c+d)^2 \\ &= (1+c+x)(1+c)^{m-1}(1+c)^2, \ \text{ as } \ d^2 = 0 \\ &= (1+c+x)(1+c)^{m+1}. \end{aligned}$$

Therefore, the highest non-zero Stiefel-Whitney class is

$$\omega_{m+1}(Q(m,1)) = c^{m+1} + \binom{m+1}{1} c^m(c+x) = c^m x \neq 0.$$

It follows that $span((Q(m,1)) \leq 2$, which completes the proof in this case.

case(b). m is odd with $m+1 = 2^r(2q+1)$. In this case

$$W(Q(m,1)) = (1+c+x)(1+c)^{m+1}$$

so that the highest non-zero Stiefel-Whitney class is

$$\omega_{2^r(2q+1)}(Q(m,1)) = \omega_{m+1}(Q(m,1)) = c^{2^r(2q+1)} = c^{m+1} \neq 0,$$

which implies that $span(Q(m,1)) \leq (m+3)-(m+1) = 2$, and hence the proof in this case.

(ii) For $m = 4\lambda$, $W(Q(4\lambda,1)) = (1+c+x)(1+c)^{4\lambda+1}$. Clearly,

$$\omega_1(Q(4\lambda,1)) = x \neq 0 \text{ and } \omega_1^2(Q(4\lambda,1)) = x^2 = 0.$$

Also, $D = dim(Q(4\lambda, 1)) = 4\lambda + 2 + 1 \equiv 3(mod\ 4)$. Further,
$\omega_{D-1}(Q(4\lambda, 1)) = c^{4\lambda+1}c + c^{4\lambda+1}x = xc^{4\lambda}c + c^{4\lambda+1}x = c^{4\lambda+1}x + c^{4\lambda+1}x = 0.$
Therefore, by Result 3.3 of [27], $span(Q(4\lambda, 1)) \geq 2$. Thus, $span(Q(4\lambda, 1)) = 2$ in view of (i). □

4. Bordism and G-bordism through characteristic numbers

Given a closed manifold M^n, Stiefel [45] and Whitney [61] independently defined specific classes $\omega_i(M^n)$ lying in the i^{th} cohomology group $H^i(M^n; \mathbb{Z}_2)$ of M^n specifying some specific properties. These classes significantly helped in bordism classification of closed manifolds and $\omega_i(M^n)$ is called the i^{th} *Stiefel-Whitney class of manifold* M^n [34]. For a given n, $0 \leq i_1 \leq \cdots \leq i_t \leq n$ is called a *partition of* n if $i_1 + \cdots + i_t = n$. Given such a partition of n, the Krönecker product [34]

$$\langle \omega_{i_1} \cup \cdots \cup \omega_{i_t}, \overline{[M^n]} \rangle \in \mathbb{Z}_2$$

is called *Stiefel-Whitney number of* M^n [34] *for the given partition of* n. Here, $\overline{[M^n]} \in H_n(M^n; \mathbb{Z}_2)$ is the *fundamental class of* M^n and $\cup$ is the *cup product* [34]. In 1947, Pontrjagin [38] proved that if a closed manifold M^n bounds then Stiefel-Whitney numbers of M^n are zero for all partitions of n. René Thom [54] proved in 1954 the converse by showing that if the Stiefel-Whitney numbers of M^n for all partitions of n are zero then M^n bounds. In 1960, Wall [56] proved analogous result for oriented [34] closed manifolds using both Stiefel-Whitney numbers and Pontrjagin numbers [34].

Given a closed manifold M^n and a topological space X, by a pair (M^n, f), we mean a continuous map $f : M^n \to X$. A pair (M_1^n, f_1) is defined to be bordic to (M_2^n, f_2) if there exists a manifold W^{n+1} and a continuous map $F : W^{n+1} \to X$ such that $\partial W^{n+1} = M_1^n \sqcup M_2^n$ and $F|M_i^n = f_i$. The set $MO_n(X)$ of all bordism classes $[M^n, f]$ forms an abelian group with operation

$$[M_1^n, f_1] + [M_2^n, f_2] = [M_1^n \sqcup M_2^n, f_1 \sqcup f_2].$$

The group $MO_n(X)$ is called the n-dimensional unoriented singular bordism group of X.

Given a map $f : M^n \to X$, M^n a closed manifold, to every cohomology class $h \in H^m(X; \mathbb{Z}_2)$ and to every partition $\omega \equiv 0 \leq i_1 \leq i_2 \leq \cdots \leq i_t = n-m$, the number

$$\langle \omega_{i_1} \cup \cdots \cup \omega_{i_t} \cup f^*(h), \overline{[M^n]} \rangle \in \mathbb{Z}_2$$

is called the *Stiefel-Whitney number of* (M^n, f) *associated to* h *and the partition* ω. It can be proved that $[M^n, f]$ is zero in $MO_n(X)$ if and only if all Stiefel-Whitney numbers of (M^n, f) defined as above are zero.

A *compact Lie group* G is a comact manifold having group structure such that the multiplication operation $\circ : G \times G \to G$ is a smooth map and the inverse map $x \to x^{-1}$ is a diffeomorphism from G to G. Clearly a finite group G is a compact Lie group of dimension zero. Given a compact Lie group G, a *G-manifold* (M^n, θ) is a smooth manifold M^n along with a smooth group action $\theta : G \times M^n \to M^n$ on M^n (i.e., θ is a smooth map). If $G = \mathbb{Z}_2$, (M^n, θ) is called an *involution* and is usually denoted as (M^n, T). A closed G-manifold (M_1^n, θ_1) is defined to be G-bordic to another G-manifold (M_2^n, θ_2) if there exists a compact G-manifold (W^{n+1}, ϕ) such that $\partial W^{n+1} \sim M_1^n \sqcup M_2^n$ and $\phi | M_i^n = \theta_i,\ i = 1, 2$. The set $MO_n(G)$ of all G-bordism classes $[M^n, \theta]$ of n-dimensional unoriented closed G-manifolds is an abelian group with respect to addition given by

$$[M_1^n, \theta_1] + [M_2^n, \theta_2] = [M_1^n \sqcup M_2^n, \theta_1 \sqcup \theta_2].$$

$MO_n(G)$ is called *n-dimensional unoriented G-bordism group.* In the oriented case, one gets $MSO_n(G)$. G-bordism classification can be represented by the following diagram.

G-bordism classification
↓
- ↓ Study through characteristic classes
- ↓ Study through group action
  ↓
  - ↓ Fixed point set $F = \phi$ *or* S^1 or $\mathbb{R}P^m$ or their product and union
  - ↓ Fixed point set is nonempty and is of general nature

In 1972, Wasserman and Lee [60] introduced the notion of G-characteristic numbers of a G-manifold (M^n, θ) and proved that G-bordism class $[M^n, \theta]$ is zero if and only if all G-characteristic numbers of (M^n, θ) are zero.

Given a G-space X, a *singular G-manifold* in X is a triple (M^n, f, θ) with G-manifold (M^n, θ) and an equivariant continuous map $f : M^n \to X$, i.e., $f(g(x)) = g(f(x))$, for all $g \in G$ and $x \in M^n$. A singular G manifold (M_1^n, f_1, θ_1) in X is said to be G-bordic to another such manifold (M_2^n, f_2, θ_2) if there exists a compact G-manifold (W^{n+1}, ϕ) with a continuous map $F : W^{n+1} \to X$ such that $F(g(x)) = g(F(x))$ for all $g \in G$ and $x \in W^{n+1}$, $\partial W^{n+1} \approx M_1^n \sqcup M_2^n$, $F | M_i^n = f_i$ and $\phi | M_i^n = \theta_i, i = 1, 2$. The set $MO_n(G; X)$ of G-bordism classes $[M^n, f, \theta]$ forms an abelian group with respect to operation

$$[M_1^n, f_1, \theta_1] + [M_2^n, f_2, \theta_2] = [M_1^n \sqcup M_2^n, f_1 \sqcup f_2, \theta_1 \sqcup \theta_2].$$

$MO_n(G;X)$ is called *n-dimensional unoriented singular G-bordism group in* X. Similarly, in oriented case, one gets group $MSO_n(G;X)$. In 1976, we [15] gave the notion of G-characteristic numbers for a singular unoriented G-manifold (M^n, f, θ) in X and proved that $[M^n, f, \theta] = 0$ in $MO_n(G;X)$ if and only if all G-characteristic numbers of (M^n, f, θ) are zero. The oriented case was settled in [16] by us.

5. G-BORDISM CLASSIFICATION, WHEN FIXED POINT SET IS ϕ OR S^1 OR $\mathbb{R}P^m$ OR THEIR PRODUCT/UNION

Given a G-manifold (M^n, θ), the set

$$F = \{x \in M^n : gx = x, \forall g \in G\}$$

is called the *fixed point set.* In general, $F = \sqcup_{i=0}^{n} F^i$, where F^i is the i-dimensional component of F. In 1964, Conner and Floyd [6] started the study of G-bordism classification of G-manifolds using its fixed point set F. They [6] proved that if $\mathbb{Z}_2^k$ acts on a closed manifold M^n with fixed point set $F = \phi$, then M^n bounds. In 1970, Stong [47] strengthened the result of Conner and Floyd by proving that under the same hypothesis, M^n is also $\mathbb{Z}_2^k$-boundary. We tried to study this problem for a general finite group. Let G be a finite group with $G_2(c)$, the subgroup of G consisting of all elements of order 2 lying in the center of G, being nonempty. In 1984, we proved [17] that if (M^n, θ) is a G-manifold with $G_2(c)$ not fixing any point of M^n, then M^n is a G-boundary. The interesting point worth noting here is that as far as G-bounding is concerned, fixed point free action of $G_2(c)$ is enough. In 1985, we [18] extended the result for a compact Lie group partially.

Coming to G-bordism classification of G-manifolds for actions with fixed point set $F \neq \phi$, Conner and Floyd [6], in 1964, proved that any closed involution (M^n, T) with $F = pt. \sqcup S^1$ must be $\mathbb{Z}_2$-bordic to $(\mathbb{R}P^2, T_0)$ with $T_0([x_0, x_1, x_2]) = [-x_0, x_1, x_2]$. Later, Stong [46] proved in 1966 that if (M^n, T) is a closed involution with $F = \mathbb{R}P^{2r}$ then $n = 4r$ and (M^n, T) is $\mathbb{Z}_2$-bordic to $(\mathbb{R}P^{2r} \times \mathbb{R}P^{2r}, Twist)$, Twist: $\mathbb{R}P^{2r} \times \mathbb{R}P^{2r} \to \mathbb{R}P^{2r} \times \mathbb{R}P^{2r}$ being twist involution given by

$$Twist([x_0, x_1, \cdots, x_{2r}], [y_0, y_1, \cdots, y_{2r}]) = ([y_0, y_1, \cdots, y_{2r}], [x_0, x_1, \cdots, x_{2r}]).$$

Capobianco [4] proved in 1976 that if (M^n, T) is a closed involution with $F = \mathbb{R}P^{2m+1}$, then (M^n, T) is a $\mathbb{Z}_2$-boundary. Later Royster [39] proved in 1980 the same result when $F = \mathbb{R}P^m \sqcup \mathbb{R}P^p$, m and p being odd. He proved some partial results for the case $F = \mathbb{R}P^{2k} \sqcup \mathbb{R}P^{2j+1}$ and conjectured that if $F = \mathbb{R}P^{2k} \sqcup \mathbb{R}P^{2j}$, then (M^n, T) is $\mathbb{Z}_2$-boundary. Torrence and Hou [53] proved this conjecture in 1996. The case $F = \sqcup\, \mathbb{R}P^{2\ell+1}$, for fix ℓ, has been settled by Torrence [52]. The case F=union of products $(\mathbb{R}P^1)^k,\ k = 0, \cdots, m$ has been proved by Stong [51] in 1993 by showing that in this case n must be $2m$ and $[M^n, T] = [(\mathbb{R}P^2)^m, T_0^m]$, where $T_0([x_0, x_1, x_2]) = ([-x_0, x_1, x_2])$. Later, Das and Khare [19] proved the following.

Theorem 5.1. *Let (M^n, T) be an involution with $F = \mathbb{R}P^1 \sqcup \mathbb{R}P^{2r}$, $r \geq 1$. Then one of the following must be true.*

(i) $n = 2r$ *and* (M^n, T) *is* $\mathbb{Z}_2$*-bordic to* $(\mathbb{R}P^{2r}, Id)$.

(ii) $n = 4r$ *and* $[M^n, T] = [(\mathbb{R}P^{2r} \times \mathbb{R}P^{2r}, Twist)]$.

(iii) $n = 2r + 2$ *and* $[M^n, T] = [\mathbb{R}P^{2r+2}, t]$,
$t[x_0, x_1, x_2, \cdots, x_{2r+2}] = [-x_0, -x_1, x_2, \cdots, x_{2r+2}]$.

In order to make the proof of this theorem better understandable, we would like to briefly describe the notion of k-dimensional real vector bundle, k-dimensional universal vector bundle and bundle bordism. An *n-dimensional real vector bundle* $\xi : E(\xi) \xrightarrow{\pi} B$ is a triple $(E(\xi), B, \pi)$ of a base topological space B, a top topological space $E(\xi)$ and a continuous map $\pi : E(\xi) \to B$, called projection map, such that for every $b \in B,\ \pi^{-1}(b)$ is a real vector space of dimension n and for all $b \in B$, there exists an open neighbourhood U of b in B and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ with $x \to h(b, x)$ giving a vector space isomorphism from $\mathbb{R}^n$ to $\pi^{-1}(b)$. Consider $V_n(\mathbb{R}^{n+k}) = \{(x_1, \cdots, x_n) \in (\mathbb{R}^{n+k})^n : x_1, x_2, \cdots, x_n$ lying in $\mathbb{R}^{n+k}$ are liearly independent$\}$ with subspace topology on $V_n(\mathbb{R}^{n+k}) \subset (\mathbb{R}^{n+k})^n$. Consider $G_n(\mathbb{R}^{n+k})$ as the set of all n-dimensional subspaces of $\mathbb{R}^{n+k}$. Let $q : V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$ be defined as $q(x_1, \cdots, x_n)$ = the subspace generated by $x_1, \cdots, x_n$. Consider the quotient topology on $G_n(\mathbb{R}^{n+k})$. Define BO_n as the direct limit of the sequence

$$G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \cdots \subset G_n(\mathbb{R}^{n+k}) \subset \cdots \subset G_n(\mathbb{R}^\infty) = BO_n.$$

Give a topology on BO_n by defining a subset X of BO_n open if $X \cap G_n(\mathbb{R}^{n+k})$ is open for all k. Define EO_n as a subset of $BO_n \times \mathbb{R}^\infty$ defined as

$$EO_n = \{(P, x) : P \in BO_n \text{ and } x \in P\}.$$

EO_n has subspace topology from $BO_n \times \mathbb{R}^\infty$. Define $\gamma^n : EO_n \xrightarrow{\pi} BO_n$ as $\pi(P, x) = P$. γ^n is an n-dimensional real vector bundle called *n-dimensional universal vector bundle* in the sense that given an arbitrary n-dimensional real vector bundle $\xi : E(\xi) \xrightarrow{p} B$, B a compact space, there exists a bundle map $(\hat{f}, f)$ from the bundle ξ to γ^n such that the bundle $\xi : E(\xi) \to B$ is isomorphic

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\hat{f}} & EO_n \\ \downarrow p & & \downarrow \pi \\ B & \xrightarrow{f} & BO_n \end{array}$$

to the pull back bundle $f^{-1}(\gamma^n)$, where the top space of the pull back bundle $f^{-1}(\gamma^n)$ is subspace of $B \times EO_n$ consisting of elements (b, y) such that $f(b) = \pi(y)$ and the base space is B with projection map taking (b, y) to b. Also, the map f is unique up to bundle homotopy, i.e., if there exists another bundle map $(\tilde{g}, g)$ from ξ to γ^n with $g^{-1}(\gamma^n)$ isomorphic to ξ then there exists a pair $(\tilde{h}, h)$ of the bundle map from the bundle $E(\xi) \times I \to B \times I$ to $EO_n \to BO_n$ such that

$$\begin{array}{ccc} E(\xi) \times I & \xrightarrow{\hat{h}} & EO_n \\ \downarrow p \times 1 & & \downarrow \pi \\ B \times I & \xrightarrow{h} & BO_n \end{array}$$

$h(b, 0) = f(b), h(b, 1) = g(b),\ \forall\ b \in B$ and $\hat{h}(x, 0) = \hat{f}(x)$ and $\hat{h}(x, 1) = \hat{g}(x),\ \forall\ x \in E(\xi)$. The unique map f (up to homotopy) is called the *classifying map* for bundle $\xi : E(\xi) \to B$.

Now consider $MO_n(BO_k)$. An element of it is represented by a map $f : V^n \to BO_k$. If two such maps are homotopic then they define the same bordism class in $MO_n(BO_k)$. Thus, we may think a bordism class in $MO_n(BO_k)$ as given by a closed manifold V^n together with a homotopy class of maps into BO_k. But the homotopy class $[f]$ of maps into BO_k are in $1-1$ correspondence with the orthogonal k-plane bundle over V^n namely $f^{-1}(\gamma^k)$. Thus given any element $[V^n \xrightarrow{f} BO_k]$, we have a unique orthogonal k-plane vector bundle $f^{-1}(\gamma^k)$. Two orthogonal k-plane bundles $\xi : E(\xi) \xrightarrow{p_1} B_1^n$ and $\eta : E(\eta) \xrightarrow{p_2} B_2^n$ are said to be bundle bordic if there exists a bundle $\zeta : E(\zeta) \xrightarrow{p} \tilde{B}^{n+1}$ such that $\partial\tilde{B}^{n+1} \sim B_1^n \sqcup B_2^n$ and $\zeta|B_1^n : p^{-1}(B_1^n) \xrightarrow{p} B_1^n$ is isomorphic to $\xi : E(\xi) \xrightarrow{p_1} B_1^n$ and $\zeta|B_2^n : p^{-1}(B_2^n) \xrightarrow{p} B_2^n$ is bundle isomorphic to $\eta : E(\eta) \xrightarrow{p_2} B_2^n$. The bordism class of a k-plane vector bundle $\xi : E(\xi) \to V^n$ is denoted by $[\xi : E(\xi) \to V^n]$. Also, a k-plane vector bundle $\xi_1 : E(\xi_1) \to V^n$ is bordic to another such bundle $\xi_2 : E(\xi_2) \to V^n$ if and only if their Stiefel-Whitney numbers are same. Now we come to the proof of Theorem 5.1.

Proof of Theorem 5.1. The proof is based on the fact that the bordism class of an involution is uniquely determined by the bordism class of the normal bundle of the fixed point set F.

Since $F = \mathbb{R}P^1 \sqcup \mathbb{R}P^{2r},\ r \geq 1,\ F$ has odd Euler characteristic (as Euler characteristic $\mathcal{X}$ of $\mathbb{R}P^m$ is odd if m is even and is even if m is odd). Thus Euler characteristic $\mathcal{X}$ of M^n is odd (as $\mathcal{X}(M^n) \equiv \mathcal{X}(F)$ mod 2). Also Euler characteristic $\mathcal{X}$ mod 2 of a closed manifold M^{2r+1} is zero. Therefore M^n must be even dimensional.

The normal bundle of F in M^n is disjoint union of the normal bundle $\eta^{n-1} : E(\eta^{n-1}) \to \mathbb{R}P^1$ and $\eta^{n-2r} : E(\eta^{n-2r}) \to \mathbb{R}P^{2r}$. Also the bundle η^{n-1} is uniquely determined (up to bordism) by the classifying map $h_1 : \mathbb{R}P^1 \to BO_{n-1}$ and the bundle η^{n-2r} is uniquely determined (up to bordism) by the classifying map $h_2 : \mathbb{R}P^{2r} \to BO_{n-2r}$. With this identification $[E(\eta^{n-1}) \to \mathbb{R}P^1] + [E(\eta^{n-2r}) \to \mathbb{R}P^{2r}]$ lies in $\sum_{k=0}^{n} MO_k(BO_{n-k})$.

Case I. $[E(\eta^{n-1}) \to \mathbb{R}P^1] = 0$ in $MO_1(BO_{n-1})$. In this case, the fixed data of the involution (M^n, T) becomes $[E(\eta^{n-2r}) \to \mathbb{R}P^{2r}]$, showing that (M^n, T) is bordant to an involution fixing $\mathbb{R}P^{2r}$. Therefore by [6, 29.1], either $n = 2r$ with $[M^{2r}, T] = [\mathbb{R}P^{2r}, 1]$ or $n = 4r$ and $[M^{4r}, T] = [\mathbb{R}P^{2r} \times \mathbb{R}P^{2r}, t]$, where t is the twist involution.

Case II. $[E(\eta^{n-1}) \to \mathbb{R}P^1] \neq 0$ in $MO_1(BO_{n-1})$.

In this case, the the Stiefel-Whitney class $\omega_1 \neq 0$ for the vector bundle η^{n-1} and so by Torrence [52]

$$[E(\eta^{n-1}) \to \mathbb{R}P^1] = [E(\gamma_1 \oplus (n-2)\mathbb{R}) \to \mathbb{R}P^1]$$

in $MO_1(BO_{n-1})$, where γ_1 is the canonical line bundle over $\mathbb{R}P^1$. Also, in this case the fixed point set $\mathbb{R}P^1 \sqcup \mathbb{R}P^{2r}$ of (M^n, T) lies entirely in one component of M^n. For if N^n denotes the connected component of M^n containing only $\mathbb{R}P^1$ then (N^n, T) is an involution with fixed point data $[E(\eta^{n-1}) \to \mathbb{R}P^1]$ so that by Conner and Floyd [6, 22.1 and 23.2] it follows that $[E(\eta^{n-1}) \to \mathbb{R}P^1] = 0$ in $MO_1(BO_{n-1})$, which is not the case. Further, the involution (M^n, T) is fixed point free on the complement of the component containing F. Therefore, without any loss, we can assume that M^n is connected. Hence $F^n = \phi$ which implies that $n > 2r$. Here F^n is the union of n-dimensional components of F.

In order to apply 28.3 of [6], the only candidate is $\mathbb{R}P^{2r}$ and so $n/2 \leq 2r < n$ and the Stiefel-Whitney class $\omega_{n-2r} \neq 0$ for the bundle η^{n-2r}, since $[E(\eta^{n-2r}) \to \mathbb{R}P^{2r}]$ is not in the image of

$$I_* : MO_{2r}(BO_{n-2r-1}) \to MO_{2r}(BO_{n-2r}).$$

Here I_* is the homeomorphism induced by the inclusion map $I : BO_{n-2r-1} \to BO_{n-2r}$. Hence $n = 2r + B$ with B even and $B \leq 2r$. Also, the fixed data of the involution (M^n, T) is now given by

$$[E(\gamma_1 \oplus (2r + B - 2)\mathbb{R}) \to \mathbb{R}P^1] + [E(\eta^B) \to \mathbb{R}P^{2r}]$$

in $\sum_{k=0}^{2r+B} MO_k(BO_{2r+B-k})$ with $\omega_B \neq 0$ for the bundle η^B on $\mathbb{R}P^{2r}$. Therefore by Conner-Floyd [6, 22.1] we have

$$[S(\gamma_1 \oplus (2r + B - 2)\mathbb{R}), T] = [S(\eta^B), T] \tag{5.1}$$

in $MO_{2r+B-1}(\mathbb{Z}_2)$. Here $S(\eta^B)$ denotes the sphere bundle of the bundle η^B. The total Stiefel-Whitney class of the bundle $\gamma_1 \oplus (2r+B-2)\mathbb{R}$ on $\mathbb{R}P^1$ is given by $(1+a)$, a being the generator of $H^1(\mathbb{R}P^1;\mathbb{Z}_2)$ and by Conner-Floyd ([6, 21.4]), i^{th}Stiefel-Whitney class of the tangent bundle to $\mathbb{R}P(\gamma_1\oplus(2r+B-2)\mathbb{R})$ is given by

$$\omega_i(\mathbb{R}P(\gamma_1 \oplus (2r+B-2)\mathbb{R})) = \sum_{x+y=i} \binom{1}{x}\binom{2r+B-1-x}{y} p^*(a^x)c^y, \quad (5.2)$$

where c is the charcteristic class of the involution $(S(\gamma_1 \oplus (2r+B-2)\mathbb{R}), T)$. Here c is defined as follows. Since $(S(\gamma_1 \oplus (2r+B-2)\mathbb{R}), T)$ is a fixed point free involution, it induces unique (up to homotopy) map $f : S(\gamma_1 \oplus (2r+B-2)\mathbb{R})/T \to \mathbb{R}P^\infty$. Let $\tilde{c}$ be the nonzero element in $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$ and $c = f^*(\tilde{c}) \in H^1(S(\gamma_1\oplus(2r+B-2)\mathbb{R})/T;\mathbb{Z}_2)$. Then c is called the characteristic class of the involution $(S(\gamma_1 \oplus (2r+B-2)\mathbb{R}), T)$. Also, c satisfies the identity

$$c^{2r+B-1} + p^*(a)c^{2r+B-2} = 0. \quad (5.3)$$

Here $p^* : H^*(\mathbb{R}P^1;\mathbb{Z}_2) \to H^*(\mathbb{R}P(\gamma_1\oplus(2r+B-2)\mathbb{R});\mathbb{Z}_2)$ is the monomorphism induced by the map $p : \mathbb{R}P(\gamma_1 \oplus (2r+B-2)\mathbb{R}) \to \mathbb{R}P^1$.
$H^*(\mathbb{R}P(\gamma_1 \oplus (2r+B-2)\mathbb{R});\mathbb{Z}_2)$ is a free module over $H^*(\mathbb{R}P^1;\mathbb{Z}_2)$ with basis $1, c, c^2, \cdots c^{2r+B-2}$ under the monomorphism p^*.

On the other hand, from the structure of the Gröthendieck ring $KO(\mathbb{R}P^{2r})$, we know that the total Stiefel-Whitney class of the bundle η^B over $\mathbb{R}P^{2r}$ is $(1+b)^m$ for some integer $m \geq 0$, b being the generator of $H^1(\mathbb{R}P^{2r};\mathbb{Z}_2)$ and by Conner and Floyd [6, 21.4], the tangential Stiefel-Whitney class of $\mathbb{R}P(\eta^\beta)$ is given by

$$\omega_i(\mathbb{R}P(\eta^\beta)) = \sum_{x+y+z=i} \binom{2r+1}{z}\binom{m}{x}\binom{B-x}{y} q^*(b^{x+z})e^y, \quad (5.4)$$

where e is the characteristc class of the involution $(S(\eta^\beta), T)$ and it satisfies the identity

$$e^\beta + \binom{m}{1}q^*(b)e^{B-1} + \binom{m}{2}q^*(b^2)e^{B-2} + \cdots + \binom{m}{B}q^*(b^B) = 0 \quad (5.5)$$

and $q^* : H^*(\mathbb{R}P^{2r};\mathbb{Z}_2) \to H^*(\mathbb{R}P(\eta^B);\mathbb{Z}_2)$ is the monomorphism under which $H^*(\mathbb{R}P(\eta^B);\mathbb{Z}_2)$ is a free $H^*(\mathbb{R}P^{2r};\mathbb{Z}_2)$-module with basis $1, e, e^2, \cdots, e^{B-1}$. It may be noted that since the Stiefel-Whitney class $\omega_B \neq 0$ for the bundle η^B over $\mathbb{R}P^{2r}$, we have $\binom{m}{B} = 1$ and so $m \geq B$. Hence, by (5.2) and (5.3), the involution number for the involution $(S(\gamma_1 \oplus (2r+B-2)\mathbb{R}), T)$ is $\omega_1^{2r}c^{B-1} = (c+p^*(a))^{2r}c^{B-1} = c^{2r+B-1} \neq 0$, since $a^2 = 0$. Thus in view of 5.1, the class $\omega_1(\mathbb{R}P(\eta^B)) \neq 0$. Therefore, by (5.4), we have $\binom{m}{1}q^*(b) + \binom{B}{1}e + \binom{2r+1}{1}q^*(b) \neq 0$, so that $\binom{m}{1} = 0$, as B is even. Thus m must be even. Again by (5.2) and (5.3), the involution number $\omega_{2r+B-2} \cdot c = c^{2r+B-2} \cdot c$, as B is even $= c^{2r+B-1} \neq 0$ for the involution $(S(\gamma_1 \oplus (2r+B-2)\mathbb{R}), T)$. Therefore, in view of (5.1), $\omega_{2r+B-2}(\mathbb{R}P(\eta^\beta)) \neq 0$. Thus from (5.4), using the fact that for nonzero terms, we must have x even with $x+z \leq 2r$ and $x+y \leq B$, we get

$$\binom{2r+1}{2r-2} q^*(b^{2r-2})e^B + \binom{2r+1}{2r}\binom{B}{B-2} q^*(b^{2r})e^{B-2} + \binom{2r+1}{2r-2}\binom{B-2}{B-2} q^*(b^{2r})e^{B-2} \neq 0, \qquad \text{that is,}$$

$$\binom{2r+1}{2r-2}\left(q^*(b^{2r-2})e^B + q^*(b^{2r})e^{B-2}\right) + \binom{B}{B-2} q^*(b^{2r})e^{B-2} \neq 0,$$

that is, $\binom{B}{B-2} \neq 0$, using (5.5) together with the fact that m is even and $b^j = 0$ for $j > 2r$. Thus, $B = 2\cdot$ odd and hence $m = 2\cdot$ odd, as $\binom{m}{B} = 1$. Thus we can write $m = 2 + 2^t\cdot$ odd, where $t > 1$.

We claim that $2^t > 2r$. Since $\binom{m}{2^t} = 1$ and $E(\eta^\beta) \to \mathbb{R}P^{2r}$ is a B-plane bundle, we have either $B \geq 2^t$ or $2^t > 2r$. Suppose $B \geq 2^t$. Then B must be greater than 2^t as $B = 2\cdot$odd and $t > 1$ so that $2r > 2^t$, since $2r \geq B$. Now, from (5.5), we have

$$e^B = q^*(b^2)e^{B-2} + q^*(b^{2^t})e^{B-2^t} + (\text{terms involving higher powers of } b). \quad (5.6)$$

Therefore, using induction and the fact that $b^j = 0$, if $j > 2r$, we have

$$q^*(b^{2r-k-1})e^{B+k} = q^*(b^{2r})e^{B-1} \neq 0, \quad (5.7)$$

for every k with k odd and $1 \leq k \leq 2^t - 3$. Again, from (5.6), we have

$$q^*(b^{2r-2^t})e^{B+2^t-1} = q^*(b^{2r-2^t+2})e^{B+2^t-3} + q^*(b^{2r})e^{B-1} = 0,$$

by (5.7). Also, from (5.4), we have $\omega_1(\mathbb{R}P(\eta^B)) = q^*(b)$. Hence it follows that the involution number $\omega_1^{2r-2^t}e^{B+2^t-1} = 0$ for the involution $(S(\eta^B), T)$. But for the involution $(S(\gamma_1 \oplus (2r+B-2)\mathbb{R}), T)$, the number

$$\omega_1^{2r-2^t}e^{B+2^t-1} = (p^*(a)+c)^{2r-2^t}c^{B+2^t-1} = c^{2r+B-1} \ (\text{since } a^2 = 0) \ \neq 0,$$

using (5.2) and (5.3), and so we have a contradiction to (5.1). Hence 2^t must be greater than $2r$. Therefore $B = 2$, as $2 \leq B \leq 2r$, $\binom{m}{B} = 1$ and

$$(1+b)^m = (1+b)^2(1+b)^{2^t\cdot odd} = (1+b)^2(1+b^{2^t})^{odd} = (1+b)^2.$$

Thus $n = 2r+2$ and the bundle $\eta^2 \to \mathbb{R}P^{2r}$ has the same total Stiefel-Whitney class as the bundle $\gamma_{2r} \oplus \gamma_{2r} \to \mathbb{R}P^{2r}$, γ_{2r} being the canonical line bundle over $\mathbb{R}P^{2r}$, so that the two bundles are bordic to each other. Thus the fixed data of the involution (M^n, T) is

$$[\gamma_1 \oplus 2r\mathbb{R} \to \mathbb{R}P^1] + [\gamma_{2r} \oplus \gamma_{2r} \to \mathbb{R}P^{2r}] \quad (5.8)$$

in $\sum_0^{2r+2} MO_k(BO_{2r+2-k})$. Also the fixed data of the involution $(\mathbb{R}P^{2r+2}, t)$ is same as (5.8), where $t([x_0, x_1, x_2, \cdots, x_{2r+2}]) = [-x_0, -x_1, x_2, \cdots, x_{2r+2}]$. Therefore $[M^n, T] = [\mathbb{R}P^{2r+2}, t]$ in $MO_{2r+2}(\mathbb{Z}_2)$. □

Theorem 5.2. *Let (M^n, T) be an involution with $F = pt. \sqcup \mathbb{R}P^{2^i-1}, i \geq 2$. Then $n = 2^i$ and $[M^n, T] = [\mathbb{R}P^{2^i}, t]$, where*

$$t([x_0, x_1, \cdots, x_{2^i}]) = [-x_0, x_1, \cdots, x_{2^i}].$$

Proof. In this case also $\chi(F)$ is even, so that n is even and the fixed data of (M^n, T) is $[E(n\mathbb{R}) \to pt.] + [E(\eta^{n-2^i+1}) \to \mathbb{R}P^{2^i-1}]$ in $\sum_{k=0}^n MO_k(BO_{n-k})$. Using the same argument as in Theorem 5.1, we can assume without any loss that M^n is connected and so $F^n = \phi$, which means that $n > 2^i - 1$. Therefore, by Conner and Floyd [6, 28.3], we have $(n/2) \leq 2^i - 1 < n$ and the class $\omega_{n-2^i+1} \neq 0$ for the bundle $E(\eta^{n-2^i+1}) \to \mathbb{R}P^{2^i-1}$. Thus we have $n = 2^i + B$ with $0 \leq B \leq 2^i - 2$ and B even. By Conner-Floyd [6, 22.1], $[S^{2^i+B-1}, A] = [S(\eta^{B+1}), T] \in MO_{2^i+B-1}(\mathbb{Z}_2)$, A being the antipodal map. However, if $B > 0$ then using the result of [6, 21.3], one can easily see that the involution number for (S^{2^i+B-1}, A) is $\omega_{2^i} \cdot c^{B-1} \neq 0$, where as the corresponding involution number for $(S(\eta^{B+1}), T)$ is zero. Hence $B = 0$. Thus it follows that $n = 2^i$ and that the fixed data of (M^n, T) is $[n\mathbb{R} \to pt.] + [\gamma_{2^i-1} \to \mathbb{R}P^{2^i-1}] \in \sum_{k=0}^{2^i} MO_k(BO_{2^i-k})$, γ_{2^i-1} being the canonical line bundle over $\mathbb{R}P^{2^i-1}$. Note that this is same as the fixed data of the involution $(\mathbb{R}P^{2^i}, t)$, where $t([x_0, x_1, \cdots, x_{2^i}]) = [-x_0, x_1, \cdots, x_{2^i}]$. Hence $[M^n, T] = [\mathbb{R}P^{2^i}, t]$ in $MO_{2^i}(\mathbb{Z}_2)$. □

6. G-bordism when $F \neq \phi$ and F is of general nature

Let (M^n, T) be a closed involution with fixed point set F. In 1964, Conner and Floyd[6] proved the following

(a) If $F = \sqcup_{m=0}^n F^m$ and all the normal Whitney classes of the normal bundle $E(\eta^{n-m}) \to F^m$ are zero, for all $m = 0, 1, \cdots, n-1$, then F^m bounds for all $m = 0, 1, \cdots, n-1$ and $[M^n] = [F^n]$.
(b) If $F = F^m$ and all the Stiefel-Whitney classes of F^m are zero then $[M^n] = 0$.
(c) (M^n, T) can not have only odd number of fixed points.
(d) If the normal bundle of F in M^n is trivial, then (M^n, T) is $\mathbb{Z}_2$-boundary.

In 1967, Boardman [2] proved his celebrated Five halves theorem according to which if $n > (5/2)$ dim$F$, then M^n bounds and if $F = F^m$ with $n > 2m$ then M^n bounds. Here dim$F$ is the maximum of dimensions of the components of F.

Kosniowski and Stong [31], in 1978, generalized the result of Boardman by proving that (M^n, T) is a $\mathbb{Z}_2$-boundary if either $n > (5/2)\text{dim}F$, or $F = F^m$ with $n > 2m$. Note that the bound of m (while say, $n > 2m$) is best, as $(F^m \times F^m, T)$ with $T(x, y) = (y, x)$ has fixed point set F homeomorphic to F^m and $F^m \times F^m$ can be taken as non-bounding by choosing F^m as non-bounding.

In 1981, Stong [49] proved that a closed manifold M^{2n-1} admits a $\mathbb{Z}_2$ action with $F = F^n$ if and only if all the Stiefel-Whitney numbers of M^{2n-1} involving a product of two odd dimensional Stiefel-Whitney classes are zero. Later Stong [50], in 1986, studied the following problem.

Problem. Which groups can admit an action on a closed oriented manifold with $F = \{pt.\}$?

He settled the problem by proving that every non-abelian group admits such action. Further, a finite abelian group G admits such an action if and only if $G = \mathbb{Z}_2^k$ or G is an abelian p-group for odd p. Moreover, if G is abelian with dim $G > 0$ and $G_0 = (S^1)^r$ is the component of G having identity, then G admits such an action if and only if G/G_0 is of even order.

In 1973, Stong [48] showed that a bordism class $\alpha \in MO_n$ is represented by a manifold M^n with an involution having fixed point set F of dimension $n-2$ if and only if $\omega_1^n(\alpha) = 0$, where $\omega_1(\alpha)$ is the first Stiefel-Whitney class of α. In 1992, Pergher [37] extended Stong's result for $\mathbb{Z}_2^k$-action. He tried to investigate those bordism classes $\alpha \in MO_n$ such that the bordism class α has a representative M^n which admits a $\mathbb{Z}_2^k$-action having fixed point set F of fixed codimension r (codimension of F is defined as n - dim$F$). For this he considered

$$J^r_{n,k} = \{\alpha \in MO_n : \alpha \text{ has a representative } M^n \text{ admitting } \mathbb{Z}_2^k\text{-action with fixed point set } F \text{ having codimension } r\}.$$

Clearly $J^r_{n,k}$ is a subgroup of MO_n. Consider $J^r_{*,k} = \sum_{n \geq r} J^r_{n,k}$. Then $J^r_{*,k}$ is a graded ring with operation given as follows.

$$\text{For } \alpha = [M_1^{n_1}], \quad \beta = [M_2^{n_2}], \quad \alpha \cdot \beta = [M_1^{n_1} \times M_2^{n_2}].$$

In 1965, Conner and Floyd [7] showed that $J^1_{*,1} = 0$. Stong computed $J^2_{*,1}$ in [48]. Capobianco computed $J^3_{*,1}$ in [5] and $J^4_{*,1}$ in [4]. Iwata [14], Wada [55], Wu [62] and Kikuchi [28] computed $J^5_{*,1}$, $J^6_{*,1}$, $J^7_{*,1}$ and $J^8_{*,1}$. In 1992, Pergher [37] computed $J^1_{*,k}$ and $J^2_{*,k}$, for $k \geq 2$. He proved the following.

Theorem 6.1. *(i)* $J^1_{*,k} = 0$ *(ii)* $J^2_{*,k} = \oplus_{n=2}^{\infty} MO_n,\ k \geq 2$.

In 1994, Shaker [40] gave most important contribution in the computation of

$J^r_{*,k}$. The crucial point was the discovery of a method for constructing models of $\mathbb{Z}_2^k$-actions on certain manifolds so that indecomposability of the manifold and the codimension of the fixed point set can be controlled simultaneously. In this way, he provided a method to construct generators for unoriented bordism graded ring with desired codimension of the fixed point set under $\mathbb{Z}_2^k$-action. Using this method he proved the following.

Theorem 6.2. *For* $0 < r < 2^k$

$$J^r_{*,k} = \begin{cases} \{0\}, & r = 1, \\ \sum_r^\infty MO_n, & r \text{ even}, \\ \sum_r^\infty MO_n \cap ker\chi, & r \text{ odd} \geq 3, \end{cases}$$

where $\chi : MO_n \to \mathbb{Z}_2$ *denotes mod 2 Euler characteristic class map.*

Later, in 1995, Shaker[41] settled the case $r = 2^k$. He used Dold manifolds to calculate $J^r_{*,k}$ for $r = 2^k$. His master stroke was to give suitable $\mathbb{Z}_2^k$-action to some indecomposable Dold manifolds $P(m, n)$ and to ensure that the fixed point set has codimension 2^k. He proved the following.

Theorem 6.3. *For* $k > 1$, $J^{2^k}_{*,k}$ *consists of all bordism classes in dimension greater than* 2^k *and decomposables in dimension* 2^k.

Sketch of the Proof. For $2^k + 1 \leq n \leq 2^k + 2^{k-1}$, $\mathbb{C}P^{2^{k-1}}$ is given suitable $\mathbb{Z}_2^k$-action θ such that the fixed point set is a set of $(2^{k-1} + 1)$ points. Define $\mathbb{Z}_2^k$-action on $P(m, 2^{k-1}) = (S^m \times \mathbb{C}P^{2^{k-1}}) \ / \sim,\ 1 \leq m \leq 2^{k-1}$ given by

$$\phi[(x, [z])] = [(x, \theta[z])].$$

It has been shown that $P(m, 2^{k-1})$ is indecomposable and the fixed point set is $(2^{k-1} + 1)$ copies of S^m which has codimension $(m + 2^k) - m = 2^k$.

For other dimensions $\neq 2^k$, he took help of real projective spaces. For dimension 2^k, he showed that there exists $\alpha \in MO_m$, $\beta \in MO_n$ with $2 \leq m, n \leq 2^k$, $m + n = 2^k$ and $\alpha \in J^r_{*,k}$, $\beta \in J^{2^k-r}_{*,k}$ so that $\alpha \cdot \beta \in J^r_{*,k} \cdot J^{2^k-r}_{*,k} \subset J^{2^k}_{*,k}$. Thus $\alpha \cdot \beta$ is the required decomposable member of $J^{2^k}_{*,k}$ in dimension 2^k. □

These two papers of Shaker gave a neat scheme of attacking the next classes. Using this scheme of Shaker and complicated formula of Kosniowski and Stong given in [31], Wu, Wang and Ma solved the case $r = 2^k + 1$ in [65].

Another significant contribution in computation of $J^r_{n,k}$ for $r > 2^k$ has been made by Wang, Wu and Ding [59] by showing that for a given $r > 2^k$, there exists number $g(r)$, normally much larger than r, so that for $n \geq g(r)$, $J^r_{n,k}$ is explicitly computable. Thus they were successful in reducing the problem of computation of $J^r_{*,k}$, $r > 2^k$ to only finitely many values of n, i.e., $r \leq n < g(r)$. Using this they computed $J^r_{n,k}$ for $r = 2^k + 3$ for $k \geq 4$ and $n \geq 2^k + 4$. In 2005, Wu and Liu [63] studied the case $r = 2^k + 2$. Wu, Li and Ma [64] settled the case $r = 2^k + 4$ in 2005. Feng [9], in 2006, computed $J^r_{n,k}$ for $r = 2^k + 5$. Geng [10] studied $J^{2^k+6}_{n,k}$ in 2006 and J. Y. He [11] computed $J^{2^k+7}_{n,k}$ in 2007. Li and Wang settled the case $r = 2^k + 8$ in 2010. They proved the following.

Theorem 6.4. *For $k \geq 4$, the ideal $J^{2^k+8}_{*,k}$ of MO_* consists of all classes in dimensions greater than $2^k + 8$ and decomposable classes in dimension $2^k + 8$ which contain only factors with dimension less than 2^k.*

Sketch of the proof. To prove this they used generalized Dold manifolds instead of Dold manifolds as used by Shaker. For a given closed manifold M^n and a positive integer m, the *generalized Dold manifold* $P(m, M^n)$ is the quotient manifold $(S^m \times M^n \times M^n)\ /\ \sim$, where equivalence relation $\sim$ identifies (u, x, y) and $(-u, y, x)$. $P(m, M^n)$ is a closed manifold of dimension $(m + 2n)$ and is indecomposable if and only if M^n is so and $\binom{m+n-1}{m-1}$ is odd.

They proved the theorem by dividing into different cases for n. We illustrate the idea of proof for $n > 2^k + 8$, n odd and not of the form $2^i - 1$. In this case $(n-1)/2$ also will not be of the form $2^i - 1$. Also, $(n-1)/2 \geq (2^k + 9 - 1)/2 = 2^{k-1} + 4$. Therefore, by Shaker's theorem, there exists a bordism class $[x]$ in $J^{2^{k-1}+4}_{(n-1)/2,k}$ (since $2^{k-1} + 4 < 2^k$), so that $[x]$ has a representative M of dimension $(n-1)/2$ admitting $\mathbb{Z}_2^k$-action with fixed point set F' of dimension $(n-1)/2 - (2^{k-1} + 4) = (n - 2^k - 9)/2$. Let involution $T_i^{'}$ denote the $\mathbb{Z}_2^k$-action on M, $i = 1, 2, \cdots, k$. This gives a $\mathbb{Z}_2^k$-action on $S^1 \times M \times M$ given by

$$T_i(u, x, y) = (u, T_i^{'}(x), T_i^{'}(y)), \qquad 1 \leq i \leq k.$$

Involutions T_i induce involutions $\overline{T_i}$ on $(S^1 \times M \times M)/\sim\ = P(1, M)$, given by $\overline{T_i}[u, x, y] = [u, T_i^{'}(x), T_i^{'}(y)]$.

This gives a $\mathbb{Z}_2^k$-action on $P(1, M)$ with fixed point set $F = (S^1 \times M \times M)\ /\sim\ = P(1, F^{'})$ of codimension $n - (1 + (n - 2^k - 9)/2 + (n - 2^k - 9)/2) = 2^k + 8$. Taking $x_n = [P(1, M)]$, we have $x_n \in J^{2^k+8}_{*,k}$. Further, $[M]$ is indecomposable in MO_* and $\binom{(n-1)/2+1-1}{1-1} = \binom{(n-1)/2}{0} = 1$=odd. This shows that $x_n = [P(1, M)]$ is indecomposable.

Similarly, for even $n > 2^k + 8$ also, such indecomposable classes x_n are constructed which have fixed point set of codimension $2^k + 8$.

Finally, they proved that for $n = 2^k + 8$, there exists classes $x_{i_1}, \cdots, x_{i_m}$

with $m \geq 2$ and $2 \leq i_1 \leq \cdots \leq i_m \leq 2^k$ such that $i_1 + \cdots + i_m = 2^k + 8$ and $x_{i_t} \in J^{i_t}_{*,k}$ so that $x_{i_1} \cdots x_{i_m} \in J^{i_1}_{*,k} \cdots J^{i_m}_{*,k} \subset J^{2^k+8}_{*,k}$. □

Wang and Meng [58], in 2012, determined $J^r_{*,k}$ for $k \geq 3,\ r = 2^k + t,\ 1 \leq t \leq 2^{k-2} - 1$ and t odd, by using mathematical induction and by constructing indecomposable classes having desired codimension of fixed point set under $\mathbb{Z}_2^k$-action. The interesting part is that as k increases, the value of t, for which computation is possible, also increases. For example, if $k = 10$, t can go up to 225. Their result is as follows.

Theorem 6.5. *(A). For $k \geq 3$, let $2^k + 2\nu + 1 = 2^k + 2^{r_s} + \cdots + 2^{r_0}$ with $k - 3 \geq r_s > \cdots > r_0 = 0$ and $\{x_i\}$ be a system of generators of MO_*, $i \neq 2^u - 1$. Then*

$$J^{2^k+2\nu+1}_{n,k} = \begin{cases} MO_n \cap Ker\chi, & \text{if } n \geq 2^k + 2\nu + 2, \\ \left\{\sum x_{i_1} \cdots x_{i_m} : 2 \leq i_1 \leq \cdots \leq i_m < 2^k,\ m \geq 2\right\}, & \text{if } n = 2^k + 2\nu + 1. \end{cases}$$

(B). For $k \geq 2$, let $2^k+2\nu+1 = 2^k+2^{r_s}+\cdots+2^{r_0}$ with $k-2 = r_s > \cdots > r_0 = 0$. Then $J^{2^k+2\nu+1}_{n,k} = MO_n \cap Ker\chi$, for $n \geq 2^{k+1} + 2^{k-1} - 2$.
(C) For $k \geq 2$, let $2^k+2\nu+1 = 2^k+2^{r_s}+\cdots+2^{r_0}$ with $k-1 = r_s > \cdots > r_0 = 0$. Then $J^{2^k+2\nu+1}_{n,k} = MO_n \cap Ker\chi$, for $n \geq 2^{k+2} + 2^k - 2$.
Further, they conjectured the following.
Conjecture 6.6. For a general $r > 2^k$ and sufficiently large k,

$$J^r_{n,k} = \begin{cases} MO_n \cap Ker\chi, & \text{if } n > r \text{ odd, } r > 1, \\ MO_n, & \text{if } n > r \text{ even,} \\ \left\{\sum x_{i_1} \cdots x_{i_m} : 2 \leq i_1 \leq \cdots \leq i_m < 2^k, m \geq 2\right\}, & \text{if } n = r. \end{cases}$$

Here $\{x_i\}$ is a system of generators of MO_*, $i \neq 2^u - 1$.
Note 6.7. Wang and Meng [58] could compute $J^{2^k+2t+1}_{*,k}$ for some $(2t + 1)$ only. For rest of the values of $(2t + 1)$, the problem is stll open. For $J^{2^k+2u}_{n,k}$, some partial results are available, but still by and large the computation of $J^{2^k+2u}_{n,k}$ is wide open.
Further, for $0 \leq r_1 < \cdots < r_t \leq n$, let

$$J^{r_1,\cdots,r_t}_{n,k} = \{\alpha \in MO_n \colon \alpha \text{ has a representative } M^n \text{ admitting } \mathbb{Z}_2^k\text{-action with the fixed point set } F = \sqcup_{i=1}^t F^{n-r_i}\},$$

Lu, Zhi and Liu Xibo [32], in 1996, computed $J^{r_1,\cdots,r_t}_{n,1}$ partially. The study is wide open for $J^{r_1,\cdots,r_t}_{n,k}$ for $k \geq 2$.

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S. S. Khare
(Former Pro Vice Chancellor,

North Eastern Hill University
Shillong-793 022, Meghalaya, India).
521, Meerapur, Allahabad-211 003, U.P., India.
E-mail: kharess1947@gmail.com

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THE SPACE $\mathcal{C}[0,1]$ CAN NOT BE LOCALLY COMPACT: A MEASURE THEORETIC PROOF

R. P. PAKSHIRAJAN
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Abstract. That a locally compact Banach space is necessarily finite dimensional is well known. Consequently, the Banach space $\mathcal{C}$ of all continuous functions defined on [0, 1] endowed with the uniform norm can not be locally compact since it is infinite dimensional. *In the present paper we give a measure theoretic proof of this result.* The proof rests heavily on a property of compact sets in $\mathcal{C}$.

1. Introduction

Let $\mathcal{C}$ denote the space of all real continuous functions defined on $[0,\ 1]$. Let $C \subset \mathcal{C}$ denote the set of all members of $\mathcal{C}$ vanishing at 0. Endow $\mathcal{C}$ with the sup norm, denoted by $\|.\|$ and note that $(\mathcal{C},\ \|.\|)$ is a Banach space. C is clearly a closed subset of $\mathcal{C}$ and is infinite dimensional since it contains the infinite set $(f_n),\ n = 1,\ 2,\ ...$ of linearly independent functions, where $f_n(t) = t^n,\ 0 \le t \le 1$.

The object of this paper is to give a measure theoretic proof of the familiar result that the infinite dimensional Banach space $\mathcal{C}$ can not be locally compact.

Let μ denote the standard Wiener measure on $\mathscr{C}$, the resulting Borel σ-field of C [1, Remark 5.6.2, p. 338].

We record [1, p. 364], for future application, that

$$\mu(x\ :\ \|x\| \le 1) \ge \frac{4}{\pi}\{e^{-\frac{\pi^2}{8}} - e^{-\frac{9\pi^2}{8}}\} > 0. \tag{1}$$

Let $\mathcal{D}_n = \{\frac{j}{2^n},\ j = 0,\ 1,\ 2,\ ..,\ 2^n\}$ and $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$. Clearly $\mathcal{D}_n \uparrow \mathcal{D}$. Note that $\mathcal{D}$ is a countable dense subset of $[0,\ 1]$.

For $x \in C$, define

$$\wp_n x = \big(x(\frac{1}{2^n}),\ x(\frac{2}{2^n}) - x(\frac{1}{2^n}),\ x(\frac{3}{2^n}) - x(\frac{2}{2^n}), \cdots,\ x(\frac{2^n}{2^n}) - x(\frac{2^n-1}{2^n})\big). \tag{2}$$

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If $A \subset C$, then $\wp_n A = \{\wp_n x,\ x \in A\}$. That the mapping $\wp_n$ from C into R^{2^n} is continuous is easily seen.

For an arbitrary subset K of $\mathcal{C}$, define $E_n = \wp_n K$ and $Q_n = \wp_n^{-1} E_n$ and note that $Q_n \supset Q_{n+1} \supset K,\ n \geq 1$. Let $\lim_{n\to\infty} Q_n = \bigcap_{n=1}^{\infty} Q_n = A$. Trivially, $K \subset A$.

The main theorem rests on the result, describing an essential property of compact sets in $\mathscr{C}$, stated and proved in the following lemma.

Lemma 1. *If $K \subset C$ is a compact set, then*

$$K = \lim_{n\to\infty} \wp_n^{-1} \wp_n K = A. \tag{3}$$

Proof. That $K \subset A$ has already been noted. It remains to show that $A \subset K$. Let x be an arbitrary member of A. Hence for every $n,\ x \in Q_n$. There exists therefore $y_n \in K$ such that

$$\wp_n x = \wp_n y_n \tag{4}$$

Since K is a compact set, sequence $\{y_n\}$ admits a convergent subsequence, say, $\{y_m\}$ converging to, say, $y_0 \in K$. For $x \in C$, define $\mathbf{P}_n x = \left(x(\frac{1}{2^n}),\ x(\frac{2}{2^n}), \cdots\ x(\frac{2^n}{2^n})\right)$ and note that (4) is equivalent to

$$\mathbf{P}_n x = \mathbf{P}_n y_n \quad \text{for all } n. \tag{5}$$

Fix r. Let $m > r$ be arbitrary. Define map $\pi_r : R^{2^m} \to R^{2^r}$ such that

$$\pi_r(u_1,\ u_2, \cdots, u_{2^m}) = (u_{2^{m-r}},\ u_{2\times 2^{m-r}}, \cdots, u_{j2^{m-r}}, \cdots, u_{2^r 2^{m-r}}).$$

We then have for all $m > r$

$$\begin{aligned}
\mathbf{P}_r x &= \pi_r \mathbf{P}_m x && \text{(by direct verification)}\\
&= \pi_r \mathbf{P}_m y_m && \text{(in view of (5))}\\
&= \mathbf{P}_r y_m && \text{(by direct verification)}\\
&\to \mathbf{P}_r y_0 && \text{(since } y_n \to y_0 \text{ in } \mathcal{C}).
\end{aligned}$$

Since the relation $\mathbf{P}_r x = \mathbf{P}_r y_0$ holds for all $r \geq 1$, it follows that $x(t) = y_0(t)$ for every $t \in \mathcal{D}$. Since x and y_0 are continuous functions and since $\mathcal{D}$ is a dense subset of $[0,\ 1]$, it follows that $x(t) = y_0(t)$ for all $t \in [0,\ 1]$. Thus $x \in K$. The proof that $A \subset K$ is now complete. □

Theorem 2. *The space $(\mathcal{C},\ \|.\|)$ is not locally compact.*

Proof. If possible, let $(\mathcal{C},\ \|.\|)$ be locally compact. That would imply that C is locally compact, since a closed subset of a locally compact space is locally compact. That, in turn, would imply that the closed sphere $K = \{x\ :\ x \in C,\ \|x\| \leq 1\}$ is a compact set in C. We have, by (3): $\mu(K) = \lim_{n\to\infty} \mu(Q_n)$. Now, $\mu(Q_n) = \mu\big(\wp_n^{-1} E_n\big) = \nu_n(E_n)$ where ν_n is the measure generated on $\mathscr{R}^{2^n}$ by the continuous mapping $\wp_n$. We note the measure ν_n is the measure corresponding to a multivariate normal vector variable with independent and identically distributed components, each component normally distributed with zero mean and variance $1/2^n$.

$$E_n = \wp_n K = \Big\{(x_1,\ x_2 - x_1,\ ...,\ x_{2^n} - x_{2^n-1}) :\ |x_i| \le 1,\ 1 \le i \le 2^n\Big\}$$
$$\subset \Big\{|x_j - x_{j-1}| \le 2,\ 1 \le j \le 2^n,\ x_0 = 0\Big\}$$

Hence, using (1),

$$0 < \mu(K) = \lim_{n\to\infty} \mu(Q_n) \le \lim_{n\to\infty} \Big\{ \int\limits_{|y|\le 2} \tfrac{1}{\sqrt{2\pi 2^n}} e^{-\frac{1}{2}\frac{y^2}{2^n}}\, \mathrm{d}y \Big\}^{2^n}$$
$$\le \lim_{m\to\infty} \Big\{ \int\limits_{|y|\le\frac{2}{\sqrt{m}}} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, \mathrm{d}y \Big\}^m \le \lim_{m\to\infty} \{\tfrac{2}{\sqrt{m}}\}^m = 0$$

This contradiction establishes the claim in the theorem. □

Acknowledgment. A comment by the referee (thanks to him) led to the additional explanatory steps in lines 10 to 16 from the bottom on page 2 making the arguments easier to follow.

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R. P. Pakshirajan
227, 18th Main, 6th Block, Koramangala
Bangalore-560 095, Karnataka, India
E-mail: vainatheyarajan@yahoo.in

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THE SENDOV CONJECTURE FOR POLYNOMIALS*

PRASANNA KUMAR
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Abstract. The Sendov conjecture states that for a polynomial with every zero in $|z| \leq 1$ in the complex plane, every closed disk of unit radius centered at a zero contains a critical point of the polynomial. The conjecture has so far been verified for general polynomials of degree n having less than or equal to eight distinct zeros. The problem, in its full generality, remains open despite intense ongoing efforts. While a large number of papers addressing this problem mainly discuss some special cases, certain strategies for attacking the problem are being refined and significantly updated over the years.

This purely expository article intends to present some of the developments related to this conjecture, and its stronger and weaker variants.

1. Introduction

The geometric and analytic theory of complex polynomials is an interesting branch of complex analysis, which evolved and started getting an attractive shape around the first quarter of the last century. But from the perspective of complex analysts, this approach constitutes the study of properties of polynomials from purely a non-algebraic plank. When viewed through the prism of geometric function theory, one discerns a rich geometric flavor, as a result of which this field is often called the 'Geometry of Zeros of Polynomials' in the complex plane.

The delightful monograph *Geometry of Polynomials* [30] by the celebrated mathematician Morris Marden, who is the originator of the above nomenclature for the topic, provides an excellent introduction to the subject. Two other equally informative monographs on the subject are *Analytic Theory of Polynomials* by Rahman and Schmeisser [44], and *Topics in Polynomials: extremal problems, inequalities, zeros* by Milovanović et al.

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[41]. These books discuss classical propositions and problems on analytic, as well as geometric theory of polynomials including associated results on growth of polynomials in a region.

The problems in this field largely circle around the study of zeros of polynomials as functions of relevant parameters. These parameters are mostly, the coefficients of given polynomials or zeros or the coefficients of polynomials derived out of the given polynomial. The variation in the location of the zeros of the given polynomial or of the associated polynomials as functions of certain parameters related to the given polynomial is a very significant component in the study of behavior of complex polynomials.

A good number of results in classical real analysis are direct or distant consequences of the Rolle's Theorem and the mean value theorems. More generally, in the case of mapping from a subset of one Banach space into another [16], the mean value theorem is an inequality, which may be sufficiently strong enough in many applications, but fails in establishing a Rolle's Theorem in the sharpest form of an equality, as this theorem does not exist for multi-variable case.

For instance, one can easily see that Rolle's Theorem is not true for analytic functions of a complex variable in general sense. This can be seen by the example of an entire function $f(z) = e^z - 1$. Observe that $f(z) = 0$ whenever $z = 2k\pi i$ for any integral value of k, but $f'(z) = e^z$ vanishes nowhere in the complex plane.

In an attempt towards finding the complex analogue of it, Dieudonne, [16] in 1930 published a necessary and sufficient condition for the existence of a zero of $f'(z)$ in the interior of a circle with diameter AB when $f(z)$ is analytic and $f(A) = f(B) = 0$. Marden [31], [32], [33], furnished results about the relative locations of the zeros of a complex polynomial and the zeros of its derivative. Schoenberg [50] also conjectured an analogue of the Rolle's Theorem for polynomials with complex coefficients.

One consequence of Rolle's Theorem is that, if a real polynomial $P(x)$ of a real variable x has only real zeros, all lying in an interval I of the real axis, then all the zeros of $P'(x)$ also lie on I. This geometric version of Rolle's Theorem leads its generalization to polynomials in the complex domain. The fundamental result which is closer in spirit to Rolle's Theorem is the famous Gauss-Lucas Theorem, which may be stated as follows.

Theorem 1.1. *Any convex polygon, which contains all the zeros of a polynomial $P(z)$, also contains all the zeros of the derivative $P'(z)$.*

A simple consequence of the Gauss-Lucas Theorem, is a result for zeros lying on and inside a circle.

Corollary 1.1. *Any circle C, which encloses all the zeros of a polynomial $P(z)$, also encloses all the zeros of the derivative $P'(z)$.*

The first known proof of the Gauss-Lucas Theorem was due to Lucas, but Gauss's work already contained this result, and hence this is known as Gauss-Lucas Theorem. This result is deservedly considered as one of the earliest theorems in the area of Geometry of Polynomials.

During the second half of last century, a new invigoration arose, galvanizing an unabated momentum for research in the Geometry of Polynomials, due to a conjecture formulated by Bl. Sendov in 1958 [51]. In the last sixty years, more than a hundred papers have been appeared on the conjecture and a good number of partial results have been recorded in the literature. Attempts to solve the conjecture have resulted in the discovery of some new properties, open problems and solution techniques in the area of 'Geometry of Polynomials'. This conjecture might be a source of thoughts for many more new approaches in solving specific problems of Geometric Function Theory.

One may now introduce the Sendov conjecture by first stating a specific case of the Gauss-Lucas Theorem presented in Corollary 1.1.

Remark 1.2. *If all the zeros of a polynomial $P(z)$ lie in the unit disk $D = \{z|\ |z| \leq 1\}$, then all the zeros of $P'(z)$ lie in D.*

It should be noted at this juncture that, the disc D cannot be squeezed further in the sense that, no proper subset of D can be guaranteed to contain even one zero of $P'(z)$, since given any proper subset of D, one can easily construct a polynomial whose zeros and critical points lie outside it; for example, $P(z) = (z-1)^n$.

This leads to the question of a more precise location of critical points, when additional information about the zeros of $P(z)$ is given. This is a general problem of the behavior of a subset of the set of zeros and critical points of a polynomial, when the rest of the zeros and critical points are treated as parameters.

In the wake of an arising need of the situation, in the Remark 1.2, one may assume that, one of the zeros of $P(z)$ lies in D and obtain more precise information about the location of critical points lying around it. This constraint will make our task easier to determine the location of a critical point in the vicinity of the particular zero. Thus, our question now is: what is the smallest region depending on a zero a and the degree n of the polynomial $P(z)$, where a critical point of $P(z)$ is assumed to lie?

A general problem of this kind is too difficult to answer. The subject of this article, the Sendov conjecture pushes us into a most challenging situation of this type!

On the other hand, Corollary 1.1 tells us that, if all the zeros of a complex polynomial $P(z)$, of degree $n \geq 2$, lie in a disk with radius r and a a zero of $P(z)$, then the disk with center a and radius $2r$ contains all zeros of the derivative $P'(z)$.

This, is the point of genesis of this famous problem of the last century. On the driving motivation for formulating this problem, Senodv [51] writes

> *In 1958, I was intrigued by the $2r$ in this result and started thinking of what would happen if we have just r there. My thoughts led me to a statement which I decided must be true, and I formulated it as a conjecture.*

We are now in a position to state this conjecture of Sendov which, as already mentioned, is about the location of nearest critical point to each individual zero unlike Gauss-Lucas Theorem which deals with the location of all critical points vis-a-vis the location of all zeros of the polynomial.

Sendov Conjecture: *Let $P(z) = (z - z_1)(z - z_2)...(z - z_n), n \geq 2$, having all its zeros $z_1, z_2..., z_n$ in the closed unit disk $|z| \leq 1$. Then each of the n zeros of $P(z)$ is at a distance less than or equal to 1 from at least one critical point.*

The disk of 'unit' radius is best possible, and it can be well understood through the extremal polynomial $P(z) = z^n - 1$. One can observe over here that, this example restricts us replacing the unit disc by any smaller disc!

This conjecture became first known as Ilieff's conjecture because L. Ilieff and W. K. Hayman attended the International Conference on Theory of Analytic Functions in Erevan, during September 1965. At this conference, Ilieff presented this conjecture, mentioning Sendov's name as its author. Hayman remembered the conjecture as coming from Ilieff and included it in his book entitled Research Problems in Function Theory [20, Problem 4.5.] as Ilieff's conjecture. But, as a matter of fact, this conjecture was already formulated by the Bulgarian mathematician Bl. Sendov in 1958

[32, page 267], [51], [54].

2. Physical and Geometric viewpoint

The next natural question that comes to our mind is, what is the physical and geometric significance of the statement in the Sendov conjecture? Marden [32] answered this elegantly by presenting the following two fascinating interpretations:

Physical interpretation: *Recall Gauss's theorem that the critical points of a polynomial which are not multiple zeros of the polynomial, are the equilibrium points in a certain force field. This field is due to particles placed at the zeros of the polynomials, the particles having masses equal to the multiplicity of the zeros and attracting with a force inversely proportional to the distance from the particle. These critical points therefore can not be too close to any one zero, since the force due to the particle at the zero would be relatively large. On the other hand, the conjecture would imply that, if all the particles are of unit mass and situated on the disk* $|z| \leq 1$, *then at least one equilibrium point will lie within unit distance from each particle.*

Geometric interpretation: *It is known that the critical points* ξ_1 *and* ξ_2 *different from* z_1, z_2, z_3 *of the polynomial*

$$P(z) = (z - z_1)^{m_1}(z - z_2)^{m_2}(z - z_3)^{m_3}$$

lie at the foci of the ellipse, which touches the line segments $(z_1, z_2), (z_2, z_3)$ *and* (z_3, z_1) *in the points that divides these segments in the ratio* m_1/m_2, m_2/m_3 *and* m_3/m_1 *respectively. The conjecture implies that, if the vertices of this triangle* z_1, z_2, z_3 *all lie in the disk* $|z| \leq 1$, *each vertex is within unit distance from one of the foci of the inscribed ellipse.*

A similar interpretation can be given for the critical points of the polynomial,

$$P(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$$

as foci of the curve Γ *of class* n *which is tangent to the sides of the polygon with vertices at the points* z_j, $j = 1, 2, 3, \ldots, n$. *If true, the conjecture would imply that within unit distance of each vertex lies at least one focus of* Γ.

3. Stronger and weaker statements

When an open problem does not itself point towards a proof naturally, then the mathematicians normally approach it either with additional hypotheses or with some weaker outcomes. In the case of the Sendov conjecture too, we do see such a thing happening more often than not. Both these approaches have been tried and tested.

The possible sub-classes of the set of polynomials satisfying the hypotheses of the conjecture have been studied for establishing the proofs. Sometimes this has led to obtaining stronger and sharper results. We will present a set of statements which will help us understand how different thoughts at different levels yielded different stronger and weaker statements in connection to the Sendov conjecture. The hierarchy of these statements in comprehension is interesting and intriguing, which every reader would not want to miss.

Proposition 3.1. *Let* $P(z) = \prod_{k=1}^{n}(z - z_k)$ *be a polynomial of degree* $n \geq 2$ *with all its zeros in* $|z| \leq 1$*. Let* $H(P)$ *be the convex hull containing all the zeros of* $P(z)$*. Then every statement in (b)-(f) is included in the preceding ones.*

(a) Ratti-Schmeisser conjecture [18], [48]*: For each* $k \in \{1, 2, \dots, n\}$*. the open disc centered at* $z_k/2$ *with radius* $1 - (|z_k|/2)$ *contains a critical point of* $P(z)$ *unless all the critical points of* $P(z)$ *lie on its boundary.*

(b) For each vertex v *of* $H(P)$*, the open disc centered at* $v/2$ *with radius* $1 - (|v|/2)$ *contains a critical point of* $P(z)$ *unless all the critical points of* $P(z)$ *lie on its boundary.*

(c) For each w *in* $H(P)$*, the open disc centered at* w *with unit radius contains a critical point of* $P(z)$ *unless* $P(z) = z^n - e^{i\theta}$ *where* θ *is any real.*

(d) Phelps-Rodrigues conjecture [43]*: For each* $k = 1, 2, \dots, n$ *the open disk* $|z - z_k| < 1$ *contains a critical point of* $P(z)$ *unless* $P(z) = z^n - e^{i\theta}$ *where* θ *is any real.*

(e) The Sendov conjecture for complex polynomials as stated already.

(f) There exists an $r \in [1, 2)$ *such that each of the closed discs centered at* z_k *and radius* r*, with* $1 \leq k \leq n$*, contains a critical point of* $P(z)$*.*

The validity of the above statements with or without any extra hypotheses is yet to be ascertained. It is known that *(a)* is true for polynomials of degree less than or equal to 5, but polynomials of degree greater than 5 do not obey *(b)* and related examples have been provided.

Statement *(a)* was proved for $n = 3, 4$ by Schmeisser [48], and for $n = 5$ by Gacs [17]. Jarnicka [21] and Vernon [62] showed independently that, it always holds for those zeros which are sufficiently close to the origin. Expectation of it holding in general without any additional assumptions was disproved by Miller [37]. Using computational techniques, he constructed polynomials of degree $6, 8, 10, 12$ for which *(a)* and *(b)* are both false.

Kumar and Shenoy [27] added counter examples to statement *(a)* for the degrees $7, 9, 11$. On the other hand, Kumar and Shenoy [29] verified that *(a)* holds for polynomials of arbitrary degree n provided that the number k of distinct zeros is subject to restriction. In particular, it holds (i) for $k \leq 5$ and arbitrary $n \geq k$, (ii) for $k = 6$ and $n \geq 7$, (iii) for $k = 7$ and $n \geq 9$, and (iv) for $k = 8$ and $n \geq 11$.

A slightly weaker form of *(a)* was proven by Goodman et al. [18], and independently, as an auxiliary result for establishing similar one, by Schmeisser [48].

In the similar way, it is still a mystery whether *(c)*, *(d)* and *(e)* hold in general or not. It has been pointed out in [49] that, many of the results in the literature connected to the Sendov conjecture extend to the statement *(c)*.

As given in *(d)*, Phelps and Rodrigues [43] conjectured that the extremal polynomials in the Sendov conjecture are always of the form $z^n - e^{i\theta}$ where θ is any real, which is a stronger form of the Sendov conjecture. This idea has been exploited by Katsoprinakis [25] to some extent while proving *(e)* for polynomials of degree n with six distinct zeros.

The statement *(f)* takes us to a comfortable zone, since it follows directly as a consequence of the Gauss-Lucas Theorem. But then the aim is to determine a smallest possible r which will serve the purpose. As we know, the Sendov conjecture premises that the smallest possible value of r must be equal to 1.

The following useful auxiliary result due to Rubinstein [46] is also equally significant in the study of the extremal cases related to *(e)*.

Theorem 3.1. *Let $P(z) = (z - a)\prod_{k=1}^{n-1}(z - z_k)$, is a polynomial of degree $n \geq 2$, where $|a| = 1$, and all other zeros of $P(z)$ lie in $|z| \leq 1$. If $P(z)$ has no critical point in the open disc $|z - a| < 1$ then $P(z) = z^n - a^n$.*

At this point, let us recall the famous Biernacki Theorem [30], which asserts that, if in the situation of the Sendov conjecture, one of the closed unit discs centered at any one of the n zeros contains all the critical points of $P(z)$, then each of the remaining such $n - 1$ closed unit discs contains at least one critical point of $P(z)$. This substantiates the fact that the Sendov conjecture is true whenever polynomial vanishes at the origin. However, it was first shown in [48] that the Sendov conjecture is true whenever the polynomial has a zero at the origin. The proof was based on the Schur-Szegõ Theorem [58].

Subsequently Gacs [17] gave an alternative proof as well. Bojanov et al. [3] proved an intersting result on verifying asymptotic behavior of degree and zeros of a polynomial in connection to Sendov conjecture by proving,

Theorem 3.2. *If $P(z) = \prod_{k=1}^{n}(z - z_k)$ has all its zeros in the disc $|z| \leq 1$, then each of the discs centered at z_k with radius $(1 + |z_1.z_2 \cdots z_n|)^{1/n}$, $k = 1, 2, \ldots, n$, contains at least one zero of $P'(z)$.*

We already discussed about the validity of the Sendov conjecture when $P(0) = 0$ via Biernacki's Theorem, but the same fact once again can be seen as a direct consequence of Theorem 3.2. It is almost unusual and unbelievable that, the Sendov conjecture has not revealed anything on real polynomials in general, except for a subclass of Cauchy polynomials. In fact, it even holds in a much stronger form, which is presented below.

Theorem 3.3. *Let $P(z) = \prod_{k=1}^{n}(z - z_k) = z^n - \sum_{k=0}^{n-1} a_k z^k$ be a polynomial of degree $n \geq 2$ having all its zeros in the closed unit disc $|z| \leq 1$. Suppose that $a_k \geq 0, 0 \leq k \leq n-1$. Then each of the open unit discs centered at $z_1, \ldots, z_n$ contains a critical point of $P(z)$ unless $P(z) = z^n - 1$.*

Proof. By the hypothesis, it follows that $P(z)$ cannot have any real positive zero greater than 1. The fact that, $P(x) > 0$ for large positive values of x implies that $P(1) \geq 0$, or in otherwords,

$$a_0 + a_1 + \cdots + a_{n-1} \leq 1. \tag{3.1}$$

Now let $a \in \{z_1, z_2, \ldots, z_n\}$, and let $\theta := \arg a$. Then

$$P'(a) = P'(a) - (n/2)e^{-i\theta}P(a)$$

$$= (n/2)e^{-i\theta}a_0 - \sum_{k=1}^{n-1} a_k a^{k-1}\left(k - (n/2)|a|\right) + a^{n-1}\left(n - (n/2)|a|\right). \tag{3.2}$$

Define the following functions

$$\phi_k(x) := x^{k-1}\Big|k - (n/2)x\Big|, \; k = 1, 2, \ldots, n$$

on the interval $[0, 1]$. Clearly $\phi_1(x)$ attains its greatest value at one of the endpoints of $[0, 1]$. It can be easily verified that, $\phi_k(x)$ attains its only local maximum at $2(k-1)/n$ if $1 < k < n/2 + 1$, and is strictly increasing on $[0, 1]$ if $n/2 + 1 \leq k \leq n$. This implies that $\phi_k(x) \leq n/2$ on $[0, 1]$ with equality if and only if either $k = n$ and $x = 1$, or $n = 2, k = 1$, and $x = 0$. Together with (3.1), we obtain from (3.2) that,

$$|P'(a)| \leq (n/2)(a_0 + a_1 + \cdots + a_{n-1} + 1) \leq n,$$

by which it follows that, $|P'(a)| \le n$ unless $a_1 = a_2 = \cdots = a_{n-1} = 0$ and equality is attained in (3.1). Let $\xi_1, \xi_2 \ldots, \xi_{n-1}$ be the critical points of $P(z)$, and then by applying the last formula of Viéte to $P'(z+a)$ we get

$$\prod_{k=1}^{n} |\xi_k - a| \le \frac{|P'(a)|}{n}.$$

Thus there exists a critical point ξ_k for some k, $1 \le k \le n-1$, such that $|\xi_k - a| < 1$ unless $P(z) = z^n - 1$. Since a is an arbitrary zero, the proof of the theorem completes. □

It may be remarked over here that a paper due to Brown [11] discusses the Sendov conjecture for the case of polynomials with real critical points. Tomohiro [61] derived some quantitative results on the conjecture near a zero lying near the unit circle.

In the course of continual perseverance in cracking this problem, it was proved by Kumar and Shenoy [28] that the Sendov conjecture holds true at a zero, if another zero lies within a specific distance from it. This result reduced the subsequent proofs to only the remaining part of the complete proof required. It is presented below with a simple proof.

Theorem 3.4. *Let $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$ be a polynomial having all its zeros in $|z| \le 1$, with $0 \le a \le 1$. If $|a - z_k| \le 2\sin(\pi/n)$ for at least one k, $1 \le k \le n-1$, then $P'(z)$ has at least one zero in the disc $|z-a| \le 1$.*

Proof. Assume that $a \ne z_k$ for any $1 \le k \le n-1$, otherwise the result is trivial. Observe that for any k, $1 \le k \le n-1$, the polynomial $P'(z+a)$ is apolar [30] (see Definition 6.1 later) to the polynomial

$$Q(z) := \frac{(z + a - z_k)^n - z^n}{(a - z_k)},$$

which is basically a polynomial of degree $n-1$. The zeros of $Q(z)$ are $\dfrac{a - z_k}{\omega^s - 1}$ for $s = 1, 2, \ldots, n-1$, where ω is a primitive n^{th} root of unity and all the roots of $Q(z)$ lie in the disc $|z| \le \dfrac{|a - z_k|}{2\sin(\pi/n)}$ for each k, $1 \le k \le n-1$. Therefore by Grace's Theorem [19] (see Theorem 6.2 later), it follows that, $P'(z+a)$ has at least one root in the disc $|z| \le \dfrac{|a - z_k|}{2\sin(\pi/n)}$ for any k, $1 \le k \le n-1$, which is equivalent to saying $P'(z)$ has at least one zero in $|z - a| \le \dfrac{|a - z_k|}{2\sin(\pi/n)}$. This, together with the hypothesis proves the theorem. □

4. Validity for polynomials of small degree

During the early days of the Sendov conjecture, initial attempts tried proving it for polynomials of small degrees. There is nothing really to prove for the case $n = 2$, as its very easily seen. The Sendov conjecture for $n = 3$ was proved by Brannan [7], Cohen and Smith [14] for $3 \leq n \leq 4$ by Rubinstein [46], Cohen and Smith [13], and in stronger forms by Joyal [22], and Schmeisser [48], and for $n = 5$ by Meir and Sharma [35] and in a little stronger form by Gacs [17].

The case $n = 6$ appeared in the literature almost twenty years later and Brown [9] is the first one in this list. Katsoprinakis [24] claimed a complete proof for the conjecture due to Phelps and Rodriguez, but his result later proved to be wrong because of employing a lemma which was incorrectly stated in a book. The first correct proofs for the conjecture of Phelps and Rodriguez for $n = 6$ are due to Borcea [4] and Katsoprinakis [25].

For $n = 7$, the Sendov conjecture was verified by Borcea [5] and Brown [10], and for $n = 8$ by Brown and Xiang [12]. Recently in second half of 2017, Meng [36] announced the proof of polynomials of degree 9 on arXiv platform, which is still under review.

The results for small degree n have been generalized in various ways. Denote by Q_n, the set of all polynomials of arbitrary degree satisfying the hypotheses of the Sendov conjecture, but with atmost n distinct zeros. The Sendov conjecture was proved for Q_3 by Saff and Twomey [47], for Q_4 by Cohen and Smith [13], and independently by Brown [8], for Q_5 by Kumar and Shenoy [28], for Q_6 by Borcea [4], and for Q_8 by Brown and Xiang [12]. Partial results for Q_7 were also obtained by Borcea [5]. Schmessier [49] showed that statement *(c)* of Proposition 3.1 holds for the set of all polynomials for which the convex hull of zeros has at most n vertices.

Another way of generalizing the results for polynomials of degree atmost n with fixed n, is to consider polynomials of arbitrary degree, but with only $n+1$ non-zero terms. Results related to this, with certain constraints, may be found in [2], [3], [49], etc.

The statement of the Sendov conjecture is trivial for those zeros, which are multiple ones. However, Tarique [59] tried to obtain the smallest positive number ρ_m such that, for each zero of multiplicity m, the disc of radius ρ_m around such a zero contains at least m critical points. Partial results obtained by Tarique [60], and Rahman and Tarique [45] suggest that

$\rho_m = 2m/(m+1)$. This is exactly the same as what the Sendov conjecture asserts for a simple zero.

5. Deviations from standard methods

During all these years of research on the Sendov conjecture, it seems that the standard methods from the theory of polynomials have been more or less exhausted and there was always an attempt to look for new avenues. Two promising approaches which do not seem to have been effectively exploited yet, could be to study the conjecture as an extremal problem by characterizing it adequately, and the second one is via a certain apolarity condition (see Definition 6.1 later), which might be the shortest and most elegant one if found, and a sample of what is explained towards the end.

In this context, Phelps and Rodriguez [43] were the first to think of the Sendov conjecture as an extremal problem and tried to characterize it; what they conjectured is precisely *(d)* of Proposition 3.1.

Miller [38], and also McCoy [34] proved that the conjecture of Phelps and Rodrigues is true for all polynomials which are sufficiently close to these presumed extremal polynomials. The paper due to Miller [37] sought the polynomials which satisfy the hypothesis of the Sendov conjecture, have a fixed zero at $a \in (0, 1)$, and are such that a disc centered at a of largest radius r is devoid of critical points. Based on extensive computer searches, he conjectured that, such a polynomial must have all its critical points on the boundary of this disc and as many zeros as possible on the unit circle.

Miller [39] further showed that, for a real zero a, an associated extremal polynomial need not have real coefficients. Hence it would not be enough to verify the Sendov conjecture for the real zeros of a polynomial with real coefficients.

The second innovative deviation is effective use of apolarity relation between two polynomials [30]. Using Grace's apolarity condition, Pflug and Schmieder [42] established some inequalities which are sufficient conditions for the validity of the Sendov conjecture. This idea was continued by Borcea [6], who succeeded in giving inequalities which are sufficient and necessary as well.

Kasten [23] interpreted the Sendov conjecture as a maximum principle for a certain real valued non-negative function $d(z_1, z_2, \ldots, z_n)$ in C^n, and obtained some partial results. Another possible approach via variational method was discussed in a paper due to Borcea [6].

Recently Degót [15] proved a surprising statement that the the Sendov conjecture is not true for polynomials of large degree:

Let $P(z)$ be a polynomial of degree $n \geq 2$ satisfying the hypothesis of the Sendov conjecture and a be one of its zeros. Fix $0 < a < 1$. In the paper due to Degót [15], it was shown that there exists an integer N such that the Sendov conjecture is true for all polynomial of degree bigger than N. Assuming that $P(z)$ contradicts the Sendov conjecture, the author estimates below and above the positive real number$|P(c)|$ for some c satisfying $0 < c < a$. This leads to a contradiction for high values of the degree of $P(z)$ which would eventually contradict the conjecture for large values of degree n.

A natural converse problem of the Sendov conjecture was formulated and solved by Aziz [1], a simple geometric proof of which follows from the Gauss-Lucas Theorem.

Theorem 5.1. *If all the zeros of the polynomial $P(z)$ of degree $n \geq 2$ lie in the unit disc $|z| \leq 1$ and ξ is any zero of $P'(z)$, then $P(z)$ has at least one zero in the disc $|z - \xi| \leq 1$ and also in the disc $|z - 2\xi| \leq 1$.*

The Sendov conjecture estimates the Hausdorff deviation of the set of zeros from the set of critical points whereas Theorem 5.1 estimates the Hausdorff deviation of the set of the critical points from the set of zeros. This observation motivated the idea to consider the Hausdorff distance between the two sets. This side of the conjecture has been very elaborately structured and narrated in two significant papers, [51] and [52], due to Sendov.

Miller [40] found a set of locally extremal polynomials in his recent paper which may be briefed as follows: Let $S(n)$ be the set of all polynomials of degree n with all roots in the unit disk, and define $d(P)$ to be the maximum of the distances from each of the roots of a polynomial P to that root's nearest critical point. Then the Sendov conjecture asserts that $d(P)$ is at most 1 for every P in $S(n)$. Define P in $S(n)$ to be locally extremal if $d(P)$ is at least $d(Q)$ for all nearby Q in $S(n)$. He determined sufficient conditions for real polynomials of degree n with a root strictly between 0 and 1 and a real critical point of order $n-3$ to be locally extremal, and he used these conditions to find locally extremal polynomials of this form of degrees $8, 9, 12, 13, 14, 15, 19, 20$, and 26.

The latest two significant papers in this direction are due to Bl. Sendov and H. S. Sendov [56], [57] who studied the extremal problems using the notion of *locus of a complex polynomial* based on an Grace's Theorem, a theorem which is a central point in the next section.

6. Approach through Grace's Theorem on apolarity

In this section, we [26] give a proof for the Sendov conjecture for polynomials of degree 3 using Theorem of Grace [19]. This might be an appropriate companion to the similar note of Brannan [7], Rubinstein [46], and Cohen and Smith [14], but perhaps with the shortest possible proof until now! As it requires apolarity conditions on polynomial pairs, we first define the notion of apolarity on polynomials.

Definition 6.1. *Let* $R(z)=\sum_{j=0}^{n}\binom{n}{j}a_j z^j$ *and* $S(z)=\sum_{j=0}^{n}\binom{n}{j}b_j z^j$ *then* $R(z)$ *and* $S(z)$ *are said to be apolar if* $\sum_{j=0}^{n}(-1)^j\binom{n}{j}a_{n-j}b_j = 0$.

Let us now state the well-known theorem due to Grace [19].

Theorem 6.2. *If the polynomials* $R(z)$ *and* $S(z)$ *are apolar, then every circular domain containing all the zeros of one polynomial also contains at least one zero of the other.*

Let $P(z)$ be a polynomial of degree n having all its zeros in the closed unit disc. Let a be any one of the zeros of $P(z)$. If we are able to find a polynomial $R(z)$ apolar to the polynomial $P'(z)$ having all its zeros at unit distance from a, then by Theorem 6.2, it follows that the Sendov Conjecture is true. Let us demonstrate this approach for the case $n = 3$.

Consider a polynomial of degree 3 as $P(z) = (z-a)(z-z_1)(z-z_2)$, where $|a| \le 1,\ |z_i| \le 1,\ 1 \le i \le 2$. Also let $Q(z) = (z-\xi_1)(z-\xi_2)$. Then using the definition of apolarity, one can easily arrive that $Q(z)$ and $P'(z)$ are apolar to each other if and only if

$$3\xi_1\xi_2 = (\xi_1+\xi_2)(a+z_1+z_2) - az_1 - az_2 - z_1z_2. \tag{6.1}$$

If we take

$$\xi_1 = a - \frac{(z_1-a)\omega + (z_2-a)\omega^2}{3}, \quad \xi_2 = a - \frac{(z_1-a)\omega^2 + (z_2-a)\omega}{3}, \tag{6.2}$$

where ω is the primitive cube root of unity, then we can see that $\xi_1,\ \xi_2$ satisfy the equation (6.1).

Further, it is very straightforward that $|\xi_i - a| \le 1,\ 1 \le i \le 2$. Hence the polynomial $Q(z) = (z-\xi_1)(z-\xi_2)$ has both the zeros lying in the disc $|z-a| \le 1$ and is apolar to $P'(z)$. Therefore $P'(z)$ has at least one zero in the disc $|z-a| \le 1$. Since a is an arbitrary zero of $P(z)$, we have proved the conjecture for $n = 3$.

Looking at the pattern of apolar polynomial $Q(z)$ in the above case, the present author, as suggested by Sendov, was trying to construct such a $Q(z)$ apolar to the given polynomial $P(z)$ of degree n by extending the expression given in (6.2) in the form as follows:

For any $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$,

$$\xi_k = a - \frac{1}{n}\sum_{l=1}^{n-1}(z_l - a)\omega^{<k+l-1>}, \tag{6.3}$$

where $\omega^{<k>} = \omega^k$ for all $k = 1, 2, \ldots, n-1$, and $\omega^{<k>} = \omega^{k-n+1}$ for all $k = n, n+1, \ldots, 2(n-1)$.

But unfortunately, for a polynomial of degree 4, one can easily see that when $a = z_1 = 0,\ z_2 = 1, z_3 = i$, the above technique of constructing the required apolar polynomial having zeros as given in (6.3) does not work.

Therefore the construction of such a polynomial $Q(z)$ with restricted zeros and apolar to $P'(z)$ for a higher degree polynomial $P(z)$ could definitely be a herculean task. And as per our immediate guess, the corresponding ξ_ks might exist as a non-linear combination of n^{th} roots of unity with the reference zero a, but proving it is anybody's guess!

7. Generalizations to higher derivatives

In the previous sections, we have considered only the relations between the zeros of a polynomial and the zeros of its first derivative. In this section, the extension of the Sendov conjecture for higher derivatives is discussed. First, it was Meir and Sharma [35] who considered the deviation of zeros of a special polynomial from the zeros of its s^{th} derivative.

Theorem 7.1. *Let $P(z)$ be a polynomial of degree n having a 1 as a zero of multiplicity m, and all remaining $n-1$ zeros lie in the disc $|z| \leq 1$. and 1. Then at least one zeros of $P^{(s)}(z)$, $1 \leq s \leq n-1$, lies in the disc $|z - \frac{m}{s+1}| \leq 1 - \frac{m}{s+1}$.*

The Sendov conjecture has been extended to higher derivatives by Sendov [51] as follows:

The Sendov conjecture for higher derivatives. *If $P(z)$ is a polynomial of degree n having all its zeros in the unit disc $|z| \leq 1$ and $s = 1, 2, \ldots, n-1$, then there exist a zero of $P^{(s)}(z)$ in the disc $|z-a| \leq 2s/(s+1)$. For $s = 1$ this conjecture is nothing but the Sendov conjecture.*

Interestingly, the above coinjecture is true for arbitrary n and $s = n-1$ as well. And the proof is simple.

Let $P(z) = \prod_{k=1}^{n}(z - z_k)$, then the only zero of $P^{(n-1)}(z)$ is $\xi = (z_1 + z_2 + \cdots + z_n)/n$ and hence

$$|z_1 - \xi| = \frac{|(n-1)z_1 - (z_2 + z_3 + \cdots + z_n)|}{n} \leq \frac{2(n-1)}{n}.$$

The above conjecture is true whenever $n = 3, 4$. The conjecture is sharp for arbitrary n and $s = n - 1$. The conjecture is sharp also for $n = 3$, but for the case $n = 4,\ s = 2$, it will not remain sharp! This can be seen in the example

$$P(z) = (z-1)\left(z^2 + (2/3)z + 1\right).$$

Observe that at a distance $2/\sqrt{3}$ from any zero of $P(z)$, there lies a zero of $P''(z)$ which is much less than the corresponding bound $\frac{4}{3}$, given by the conjecture.

The conjecture has been verified for special class of polynomials whose zeros satisfy $|z_k| \leq (2n/(s+1))^{1/(n-s)}$ for $k = 1, 2 \ldots, m$ and $|z_k| = 1$ for $k = m, m+1, \ldots, n$.

We hope that the exposition in this article gives an idea of the Sendov conjecture, the finer nuances involved in a fuller understanding of the problem, available information in the literature and some directions presently in vogue towards further verification of the conjecture. We recommend the interested readers to refer the paper about the open problems emerged out of the Sendov conjecture by Sendov [55] and also [53]. We close this article with the remark that although a considerable amount of research has been done in the direction of the conjectures discussed over here, their comprehensive validity remains unanswered yet.

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Prasanna Kumar
Department of Mathematics, Birla Institute of Technology and Science,
Pilani, K. K. Birla Goa Campus, Goa-403 726, India
E-mail: prasannak@goa.bits-pilani.ac.in

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A SIMPLE PROOF THAT $\zeta(2) = \pi^2/6$

M. RAM MURTY
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Abstract. We give a simple proof that $\zeta(2) = \pi^2/6$ using only first-year calculus. An elementary recursion allows us to deduce that generally, $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, a celebrated theorem of Euler.

1. Introduction

In 1650, Pietro Mengoli posed the problem of explicitly evaluating

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and this had come to be known as the Basel problem. The solution is $\pi^2/6$ and it was discovered by Euler almost a century later in 1734, when Euler was 28 years old. Euler's proof, though correct, was far from rigorous and had to await further developments in complex analysis to put it on a sure footing. We will give a slick proof that uses only ideas from first year calculus. In an equally simple way, we will show that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, where $\zeta(s)$ is the Riemann zeta function which for $s > 1$ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

2. The Proof

We first observe that

$$\int_0^\infty \left(\frac{2y}{1+y^2} - \frac{2y}{x^2+y^2} \right) dy = \lim_{N\to\infty} \int_0^N \left(\frac{2y}{1+y^2} - \frac{2y}{x^2+y^2} \right) dy$$
$$= \left[\log \frac{1+y^2}{x^2+y^2} \right]_{y=0}^{y=\infty} = 2\log x, \qquad (2.1)$$

since, for fixed x,

$$\lim_{y\to\infty} \log \frac{1+y^2}{x^2+y^2} = 0.$$

Simplifying the integrand in (2.1), we deduce

$$-\frac{\log x}{1-x^2} = \int_0^\infty \frac{y\,dy}{(1+y^2)(x^2+y^2)}. \qquad (2.2)$$

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Now consider the integral

$$I = \int_0^1 \frac{\log x}{1-x^2} dx.$$

Expanding $1/(1-x^2)$ as a power series and integrating by parts term by term, we get

$$I = \sum_{n=0}^{\infty} \int_0^1 x^{2n} (\log x) dx = -\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Thus

$$I = -\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = -\left(1 - \frac{1}{2^2}\right)\zeta(2). \tag{2.3}$$

Changing x to $1/x$ in the integral leads to

$$I = \int_{\infty}^1 \frac{\log 1/x}{1-(1/x^2)} \left(-\frac{dx}{x^2}\right) = \int_1^{\infty} \frac{\log x}{1-x^2} dx$$

so that

$$2I = \int_0^{\infty} \frac{\log x}{1-x^2} dx. \tag{2.4}$$

Inserting (2.2) into (2.4), we deduce that

$$-2I = \int_0^{\infty} \int_0^{\infty} \frac{y dx dy}{(1+y^2)(x^2+y^2)}.$$

In the inner integral, changing x to yx leads to

$$-2I = \int_0^{\infty} \int_0^{\infty} \frac{dx dy}{(1+x^2)(1+y^2)} = \pi^2/4 \tag{2.5}$$

since

$$\int_0^{\infty} \frac{dx}{1+x^2} = [\arctan x]_0^{\infty} = \pi/2.$$

Combining (2.5) with (2.3) gives the desired result.

3. The formula for $\zeta(2k)$

It is possible to show $\zeta(2k) \in \pi^{2k}\mathbb{Q}$ using the following simple recursion which certainly goes back to Euler.

Theorem 1. *For $k \geq 2$,*

$$(k + (1/2))\zeta(2k) = \sum_{i=1}^{k-1} \zeta(2i)\zeta(2k-2i). \tag{3.1}$$

This gives a recursion from which $\zeta(2k) \in \pi^{2k}\mathbb{Q}$ is easily deduced. Our proof of this relies on the following lemma.

Lemma 2. *For any natural number n,*

$$\sum_{m=1}^{\prime\infty} \frac{1}{m^2 - n^2} = 3/4n^2,$$

where the dash on the summation means that we exclude $m = n$.

Proof. The sum is

$$\lim_{N\to\infty} \frac{1}{2n} \sum\nolimits'^{N}_{m=1} \left(\frac{1}{m-n} - \frac{1}{m+n}\right)$$

$$= \lim_{N\to\infty} \frac{1}{2n} \left(-\sum\nolimits_{j=1}^{n-1} \frac{1}{j} + \sum\nolimits_{j=1}^{N-n} \frac{1}{j} - \sum\nolimits_{j=n+1}^{N+n} \frac{1}{j} + \frac{1}{2n}\right).$$

Using the elementary fact that as $N \to \infty$,

$$\sum\nolimits_{j=1}^{N} \frac{1}{j} = \log N + \gamma + o(1),$$

where γ is Euler's constant, the sum in the brackets is

$$\log \frac{N-n}{N+n} + \frac{3}{2n} + o(1)$$

so that our limit is $3/4n^2$, as claimed. □

We can now prove Theorem 1. Inserting the series for the zeta function, the right hand side of (3.1) is

$$\sum\nolimits_{n=1}^{\infty} \sum\nolimits_{m=1}^{\infty} \sum\nolimits_{i=1}^{k-1} \frac{1}{n^{2i}} \frac{1}{m^{2k-2i}}.$$

When $n \neq m$, the innermost sum being a geometric sum, is easily seen to be

$$(n^{2-2k} - m^{2-2k})/(m^2 - n^2).$$

Separating the contribution from $m = n$, we deduce that our sum is

$$(k-1)\zeta(2k) + \sum\nolimits_{n=1}^{\infty} \sum\nolimits'^{\infty}_{m=1} \left(\frac{1}{n^{2k-2}} - \frac{1}{m^{2k-2}}\right) \frac{1}{m^2 - n^2}.$$

Using the lemma, we see that the double sum contributes $(3/2)\zeta(2k)$, which gives the result as claimed.

M. Ram Murty
Dept. of Mathematics and Statistics
Queen's University, Kingston, Ontario, Canada, K7L 3N6
E-mail: murty@queensu.ca

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SOME SERIES INVOLVING THE CENTRAL BINOMIAL COEFFICIENT WITH SUMS CONTAINING $\pi^2/5$

AMRIK SINGH NIMBRAN AND PAUL LEVRIE
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Abstract. We evaluate here the sums of a class of series involving the central binomial coefficient and whose sums contain multiples of $\pi^2/5$.

An old issue of the *American Mathematical Monthly* [2] contains a nice problem concerning the sum of an infinite series:

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{2^{4n}(2n+1)^2}.$$

We intend to obtain here an alternative solution and seek general formulas to obtain more such sums.

1. The basic series

The following binomial expansion can be easily derived by means of the Binomial Theorem:

$$(1 + x^2/4)^{-1/2} = \sum_{n=0}^{\infty} \frac{\binom{-1/2}{n}}{4^n} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{2^{4n}} x^{2n}.$$

We now take the integral from 0 to x thereby getting

$$\int_0^x \frac{2\,dt}{\sqrt{t^2+4}} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{2^{4n}} \frac{x^{2n+1}}{2n+1}.$$

Recall the formula $\int \frac{dx}{\sqrt{x^2+a^2}} = \log(x + \sqrt{x^2+a^2}) + C$ or $\sinh^{-1}(x/a) + C$, assuming both x and a are positive. Using this formula and then dividing the two sides by x we obtain

$$2\sinh^{-1}(x/2)\; x^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{2^{4n}} \frac{x^{2n}}{2n+1} \tag{1.1}$$

valid for $x \in [-2, 2]$. Now taking the definite integral from 0 to 1 (which is allowed as a consequence of uniform convergence of this power series in the interval $[-1, 1]$), we get

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$$\int_0^1 \frac{\sinh^{-1}(x/2)}{x/2}\,dx = \sum\nolimits_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}}{2^{4n}(2n+1)^2}.$$

Using integration by parts, we can rewrite the left hand side as:

$$\int_0^1 \frac{\sinh^{-1}(x/2)}{x/2}\,dx = -2\int_0^1 \frac{\log x}{\sqrt{4+x^2}}\,dx.$$

Note that $\sinh^{-1}(x/2) = \log(\ (x/2) + \sqrt{1+x^2/4}\ )$. We now use the substitution $x = (1/t) - t$ in the integral:

$$\begin{aligned} -\int_0^1 \frac{\log x}{\sqrt{4+x^2}}\,dx &= \int_1^{\varphi} \frac{\log(\frac{1}{t}-t)}{t}\,dt \\ &= \int_1^{\varphi} \frac{\log(1-t)}{t}\,dt + \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt - \int_1^{\varphi} \frac{\log t}{t}\,dt \end{aligned}$$

with $\varphi = (-1+\sqrt{5})/2$, the golden ratio, which satisfies $\varphi^2 + \varphi = 1$.

The last term is equal to $-(\log^2(\varphi)/2)$. By combining the first two terms in a smart way (see [2]), it is possible to find the exact value of the sum of these two integrals. Indeed:

$$\begin{aligned} &\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt + \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt \\ &= \frac{6}{5}\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt + \frac{4}{5}\int_1^{\varphi} \frac{\log(1+t)}{t}\,dt \qquad (1.2) \\ &- \frac{1}{5}\left(\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt - \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt\right). \qquad (1.3) \end{aligned}$$

We now use two different methods to rewrite these integrals:

$$\begin{aligned} \int_1^{\varphi} \frac{\log(1-t)}{t}\,dt &= \int_1^{\varphi} \frac{\log(1-t^2)}{t}\,dt - \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt \\ &= -\frac{1}{2}\int_0^{\varphi} \frac{\log(1-u)}{u}\,du + \log^2\varphi - \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt. \end{aligned}$$

Here we have used the substitution $u = 1 - t^2$ followed by the partial integration. Note that $1 - \varphi^2 = \varphi$. Hence we find:

$$\frac{3}{2}\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt + \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt = -\frac{1}{2}\int_0^1 \frac{\log(1-t)}{t}\,dt + \log^2\varphi.$$

Note that

$$\int_0^1 \frac{\log(1-t)}{t}\,dt = -\int_0^1 \left(\sum\nolimits_{k=1}^{\infty} \frac{t^{k-1}}{k}\right)dt = -\pi^2/6.$$

As a consequence we find for the sum of the two integrals (1.2) :

$$\frac{6}{5}\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt + \frac{4}{5}\int_1^{\varphi} \frac{\log(1+t)}{t}\,dt = (\pi^2/15) + (4/5)\log^2\varphi.$$

For (1.3) we use partial integration and the substitution $v = 1/t$:

$$\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt = \log(1-\varphi)\log\varphi + \int_1^{\varphi} \frac{\log t}{1-t}\,dt$$
$$= 2\log^2\varphi + \int_1^{1+\varphi} \log v \cdot \left(\frac{1}{v-1} - \frac{1}{v}\right) dv.$$

With the substitution $z = v - 1$, this leads to:

$$\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt = 2\log^2\varphi + \int_0^{\varphi} \frac{\log(1+z)}{z}\,dz - (1/2)\log^2\varphi.$$

This is equivalent to:

$$\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt - \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt = (3/2)\log^2\varphi + \int_0^1 \frac{\log(1+z)}{z}\,dz$$

where

$$\int_0^1 \frac{\log(1+z)}{z}\,dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \pi^2/12.$$

Hence we find for (1.3):

$$-\frac{1}{5}\left(\int_1^{\varphi} \frac{\log(1-t)}{t}\,dt - \int_1^{\varphi} \frac{\log(1+t)}{t}\,dt\right) = -(3/10)\log^2\varphi - \pi^2/60.$$

Combining the results for (1.2) and (1.3), we finally get:

$$-\int_0^1 \frac{\log x}{\sqrt{4+x^2}}\,dx = \pi^2/20.$$

1.1. **Using the dilogarithm.** Alternatively, we could directly evaluate the integrals by means of the dilogarithm function (first discussed by Euler [1, Caput IV, §196–§200]) defined as

$$\mathrm{Li}_2(x) = \sum\nolimits_{m=1}^{\infty} \frac{x^m}{m^2} = -\int_0^x \frac{\log(1-t)}{t}\,dt = -\int_0^x \frac{dt}{t}\int_0^t \frac{du}{1-u} \quad (|x| \le 1).$$

Leaving the trivial $x = 0$, there are only seven known values of x for which $\mathrm{Li}_2(x)$ assumes closed form [3], the first three given by Euler:

$$\mathrm{Li}_2(1) = \pi^2/6, \quad \mathrm{Li}_2(-1) = -\pi^2/12, \quad \mathrm{Li}_2(1/2) = \pi^2/12 - (\ln^2 2)/2.$$

$$\mathrm{Li}_2(\varphi) = \frac{\pi^2}{10} - \ln^2\left(\frac{1}{\varphi}\right), \quad \mathrm{Li}_2\left(-\frac{1}{\varphi}\right) = -\frac{\pi^2}{10} - \ln^2\left(\frac{1}{\varphi}\right) \quad \text{(incorrect in [3])}.$$

$$\mathrm{Li}_2(-\varphi) = -\pi^2/15 + (1/2)\ln^2(1/\varphi), \quad \mathrm{Li}_2(1-\varphi) = \pi^2/15 - \ln^2(1/\varphi).$$

Since

$$-\int_0^x \frac{\log(1-t)}{t}\,dt = \mathrm{Li}_2(x), \qquad -\int_0^x \frac{\log(1+t)}{t}\,dt = \mathrm{Li}_2(-x), \quad \text{and}$$
$$\int_1^x \frac{\log t}{t}\,dt = \log^2(x)/2$$

we get:

$$-\int_0^1 \frac{\log x}{\sqrt{4+x^2}}\,dx = \mathrm{Li}_2(1) - \mathrm{Li}_2(\varphi) + \mathrm{Li}_2(-1) - \mathrm{Li}_2(-\varphi) - \log^2(\varphi)/2.$$

Using the required values noted above, we obtain: $-\int_0^1 (\log x/\sqrt{4+x^2})\,dx = \pi^2/20$. We thus have:

$$\sum\nolimits_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}}{(2n+1)^2\,2^{4n}} = 1 - \frac{1}{2}\frac{1}{4^1 3^2} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{4^2 5^2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{4^3 7^2} + \ldots = \frac{\pi^2}{10}. \quad (1.4)$$

In fact, we have this general formula for integer $k \geq 4$ and $r = \sqrt{k/4}$:

$$\sum\nolimits_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}}{(2n+1)^2 k^n} = r\,\mathrm{Li}_2((-1/r) - (\sqrt{r^2+1}/r)) -$$
$$r\,\mathrm{Li}_2((r-1)/r - (\sqrt{r^2+1}/r)) - (r/2)\ln^2((1/r) + (\sqrt{r^2+1}/r))$$
$$+\, r\ln((1/r) + (\sqrt{r^2+1}/r))\ln((r+1)/r + (\sqrt{r^2+1}/r)) + (r\,\pi^2/12).$$

For comparison, we give here the indefinite integral evaluated by Zagier [3, p.8]:

$$\int \frac{\sinh^{-1}(x/2)}{x}\,dx = \int \frac{\log(x/2 + \sqrt{1+(x^2/4)})}{x}\,dx$$
$$= -\frac{1}{2}\mathrm{Li}_2\left[\left(-(x/2) + \sqrt{1+(x^2/4)}\right)^2\right] - \frac{1}{2}\log^2\left((x/2) + \sqrt{1+(x^2/4)}\right)$$
$$+ \log(x)\,\log\left((x/2) + \sqrt{1+(x^2/4)}\right) + C.$$

In fact, this general formula is returned by *Mathematica*

$$\int \frac{\sinh^{-1}(x/a)}{x}dx = -\mathrm{Li}_2\left(e^{-2\sinh^{-1}(x/a)}\right) + (1/2)\left(\sinh^{-1}(x/a)\right)^2$$
$$+ \sinh^{-1}(x/a)\log\left(1 - e^{-2\sinh^{-1}(x/2)}\right) + C$$

and this is the series expansion of the integral:

$$\int \frac{\sinh^{-1}(x/a)}{x}dx = \sum\nolimits_{n=0}^{\infty}(-1)^n\binom{2n}{n}2^{-2n}\,(2n+1)^{-2}\left(\frac{x}{a}\right)^{2n+1} \quad (|x| < a)$$
$$= \frac{\ln^2(-2x/a)}{2} - \sum\nolimits_{n=1}^{\infty}(-1)^n\frac{\binom{2n}{n}}{2^{2n}\,(2n)^2}\left(\frac{a}{x}\right)^{2n} \quad (x < -a)$$
$$= \frac{\ln^2(2x/a)}{2} + \sum\nolimits_{n=1}^{\infty}(-1)^n\frac{\binom{2n}{n}}{2^{2n}\,(2n)^2}\left(\frac{a}{x}\right)^{2n} \quad (x > a).$$

2. Some related series

Consider the series (1.1) derived in the previous section. Setting $x = 1$ in it yields $\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}(2n+1)^{-1}2^{-4n} = 2\ln(1+\sqrt{5})/2$. We also have the following related series:

$$\sum\nolimits_{n=0}^{\infty}(-1)^n\binom{2n}{n}(n+1)^{-1}\,2^{-4n} = 4(\sqrt{5}-2). \quad (2.1)$$

We use the binomial series ($|x|<1/4$): $1/\sqrt{1-4x}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n$ to prove this. Changing x into $-x$ in the preceding expansion gives $\sum_{n=0}^{\infty}\binom{2n}{n}(-x)^n$ $=1/\sqrt{1+4x}$. Since $\int(ax+b)^{-1/2}\,dx=(2/a)\sqrt{ax+b}+C$, the result follows by taking the definite integral from 0 to 1/16.

These two series can be used to prove the following result

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n-1)^3 2^{4n}}=-\frac{\sqrt{5}}{2}-\frac{\pi^2}{40}+\frac{1}{2}\ln\frac{1+\sqrt{5}}{2} \tag{2.2}$$

by shifting the index in the series. In fact, the left hand side of (2.2) is

$$=-1+\sum_{n=1}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n-1)^3\,2^{4n}}=-1-\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n+2}{n+1}}{(2n+1)^3\,2^{4n+4}}$$

$$=-1-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}(2n+1)^{-2}(2n+2)^{-1}\;2^{-4n}$$

$$=-1-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n+1)^2\,2^{4n}}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n+1)\,2^{4n}}-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n+2)\,2^{4n}}$$

$$=-1-\frac{\pi^2}{40}+\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)-\frac{1}{2}(\sqrt{5}-2)=-\frac{\sqrt{5}}{2}-\frac{\pi^2}{40}+\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

which may be written as

$$1-\frac{1}{2}\frac{1}{4^1 1^2}+\frac{1}{2\cdot 4}\frac{1}{4^2 3^2}-\frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{1}{4^3 5^2}+\ldots=-\frac{\sqrt{5}}{2}-\frac{\pi^2}{40}+\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right). \tag{2.3}$$

Again, this is a special case of the general formula for integer $k\geq 4$ and $r=\sqrt{k/4}$:

$$\sum_{n=0}^{\infty}(-1)^n\frac{\binom{2n}{n}}{(2n-1)^3\,k^n}=\frac{1}{r}\,\mathrm{Li}_2\left(\frac{r-1}{r}-\frac{\sqrt{r^2+1}}{r}\right)-\frac{1}{r}\mathrm{Li}_2\left(-\frac{1}{r}-\frac{\sqrt{r^2+1}}{r}\right)$$

$$+\frac{1}{2r}\ln^2\left(\frac{1}{r}+\frac{\sqrt{r^2+1}}{r}\right)+\frac{1}{r}\ln\left(\frac{1}{r}+\frac{\sqrt{r^2+1}}{r}\right)$$

$$-\frac{1}{r}\ln\left(\frac{1}{r}+\frac{\sqrt{r^2+1}}{r}\right)\ln\left(\frac{r+1}{r}+\frac{\sqrt{r^2+1}}{r}\right)-\frac{\sqrt{r^2+1}}{r}-\pi^2/12r.$$

More generally, using a shift of the index we obtain:

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n+2k-1)^2 2^{4n}}$$

$$=\frac{1}{(2k-1)^2}-\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)\binom{2n}{n}}{(n+1)(2n+2k+1)^2\,2^{4n}}=\frac{1}{(2k-1)^2}$$

$$-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n+2k+1)^2\,2^{4n}}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{(2n+2)(2n+2k+1)^2\,2^{4n}}.$$

Using a partial fraction expansion we can rewrite the last term:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2)(2n+2k+1)^2 2^{4n}} = \frac{1}{(2k-1)^2} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2)\, 2^{4n}}$$

$$- \frac{1}{(2k-1)^2} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2k+1) 2^{4n}} - \frac{1}{(2k-1)} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2k+1)^2\, 2^{4n}}$$

We thus get a recurrence relation valid for $k = 1, 2, 3, \ldots$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2k+1)^2 2^{4n}} = -\frac{2(2k-1)}{k} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2k-1)^2 2^{4n}}$$

$$- \frac{1}{2k(2k-1)} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2k+1)\, 2^{4n}} + \frac{1}{4k(2k-1)} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(n+1)\, 2^{4n}} + \frac{2}{k(2k-1)}.$$

If we introduce the notation:

$$y_k = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2k-1)^2\, 2^{4n}} \quad , \quad z_k = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+2k-1)\, 2^{4n}},$$

we can rewrite this recurrence succinctly using (2.1):

$$y_{k+1} = -2(2k-1)k^{-1} y_k - (2k(2k-1))^{-1} z_{k+1} + \sqrt{5}(k(2k-1))^{-1}. \quad (2.4)$$

Similarly, we easily derive this recurrence relation for z_k valid for $k = 1, 2, 3 \ldots$

$$z_{k+1} = 2(2k-1)k^{-1} z_k + \sqrt{5}\, k^{-1}. \quad (2.5)$$

It may be pointed out that one can also write for $k = 0, 1, 2, \ldots$

$$z_{k+1} = a_k \ln(\ (1+\sqrt{5})/2\ ) + b_k \sqrt{5},$$

where a_k is directly given: $a_k = 2(-1)^k \binom{2k}{k}$ but b_k has to be computed thus: $b_k = -2(2k-1)k^{-1} b_{k-1} + (1/k)$ beginning with $b_0 = 0$. A closed form for b_k is given by:

$$b_k = (-1)^k \binom{2k}{k} \sum_{j=1}^{k} \frac{(-1)^j}{j \binom{2j}{j}}$$

leading to:

$$z_{k+1} = (-1)^k \binom{2k}{k} \left(2 \ln \frac{1+\sqrt{5}}{2} + \sqrt{5} \sum_{j=1}^{k} \frac{(-1)^j}{j \binom{2j}{j}} \right).$$

We can obtain an infinity of series by means of these recurrence relations. E.g.,

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+3) 2^{4n}} = \sqrt{5} - 4 \ln \frac{1+\sqrt{5}}{2}$$

which, in conjunction with (1.4), yields an infinite sum involving $\pi^2/5$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+3)^2\, 2^{4n}} = \frac{\sqrt{5}}{2} - \frac{\pi^2}{5} + 2 \ln \frac{1+\sqrt{5}}{2} \quad (2.6)$$

which in turn gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+5)^2\, 2^{4n}} = -\frac{9\sqrt{5}}{8} + \frac{3\pi^2}{5} - 7 \ln \frac{1+\sqrt{5}}{2}. \quad (2.7)$$

Note that the terms in (2.4) can be reordered and lead to a first-order recurrence for a combination of y_k and z_k: if for $k = 1, 2, 3, \ldots$

$$x_k = y_k + h'_{2k-2}\, z_k \quad \text{with} \quad h'_n = \sum\nolimits_{j=1}^{n} (-1)^{j-1}\, j^{-1}$$

we have that

$$x_{k+1} = -2(2k-1)\ k^{-1} x_k + (\sqrt{5}/k) h'_{2k-1}.$$

The solution of this recurrence is given by:

$$x_{k+1} = (-1)^k \binom{2k}{k} \left(\pi^2/10 + \sqrt{5} \sum\nolimits_{j=1}^{k} (-1)^j\, h'_{2j-1}\, j^{-1} \binom{2j}{j}^{-1} \right).$$

By combining previous results we can get a closed form expression for y_k.

Note that the same techniques can be used to find the sum of related series. For instance, by a shift of the index we get:

$$\sum\nolimits_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n-1)^3\,(2n+1)\,2^{4n}} = -1 + \sum\nolimits_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n-1)^3\,(2n+1)\,2^{4n}}$$

$$= -1 - (1/8) \sum\nolimits_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+1)^2\,(n+1)(2n+3)\,2^{4n}}$$

and using a partial fraction expansion, we have:

$$-\frac{1}{8}\frac{1}{(2n+1)^2\ (n+1)(2n+3)} = -\frac{1}{8}\frac{1}{(2n+1)^2} - \frac{1}{8}\frac{1}{n+1} + \frac{3}{16}\frac{1}{2n+1} + \frac{1}{16}\frac{1}{2n+3}.$$

So using various values computed earlier we deduce another sum involving $\pi^2/5$:

$$\sum\nolimits_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n-1)^3\ (2n+1)\ 2^{4n}} = -\frac{7\sqrt{5}}{16} - \frac{\pi^2}{80} + \frac{1}{8} \ln \frac{1+\sqrt{5}}{2}. \tag{2.8}$$

This way one can bring $(2n+3), (2n+5), (2n+7)$, etc. in the denominator. Similarly

$$\sum\nolimits_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n-1)^3 (2n+1)^2 2^{4n}} = -1 + \frac{1}{8} \sum\nolimits_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n-1)^3 (2n+1)^2 2^{4n}}$$

$$= -1 - \frac{1}{8} \sum\nolimits_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n+1)^2\,(n+1)(2n+3)^2\,2^{4n}}$$

and using a partial fraction expansion, we have:

$$-\frac{1}{8}\frac{1}{(2n+1)^2\,(n+1)(2n+3)^2}$$

$$= -\frac{1}{16}\frac{1}{(2n+1)^2} + \frac{1}{16}\frac{1}{(2n+3)^2} - \frac{1}{8}\frac{1}{n+1} + \frac{1}{8}\frac{1}{2n+1} + \frac{1}{8}\frac{1}{2n+3}$$

again yielding another sum involving $\pi^2/5$:

$$\sum\nolimits_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{(2n-1)^3 (2n+1)^2 2^{4n}} = -\frac{11\sqrt{5}}{32} - \frac{3\pi^2}{160} - \frac{1}{8} \ln \frac{1+\sqrt{5}}{2}. \tag{2.9}$$

Thus our method enables us to bring more and more linear and square factors in the denominator leading to infinite sums involving $\pi^2/5$.

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Amrik Singh Nimbran
B3-304, Palm Grove Heights
Ardee City, Gurgaon, Haryana, India
E-mail: amrikn622@gmail.com

Paul Levrie
Faculty of Applied Engineering, University of Antwerp
Groenenborgerlaan 171, 2020 Antwerpen;
Department of Computer Science
KU Leuven, P.O. Box 2402, 3001 Heverlee, Belgium
E-mail: paul.levrie@cs.kuleuven.be

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SUMS OF SERIES INVOLVING CENTRAL BINOMIAL COEFFICIENTS AND HARMONIC NUMBERS

AMRIK SINGH NIMBRAN
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Abstract. This paper contains a number of series whose coefficients are products of central binomial coefficients & harmonic numbers. An elegant sum involving $\zeta(2)$ and two other nice sums appear in the last section.

1. Introduction and Preliminary resullts

1.1. **Beginnings.** Euler investigated the partial sums of the harmonic series, finding connection between them and $\log(n)$. These sums: $H_n = \sum_{k=1}^{n}(1/k) = \int_0^1((1-x^n)/(1-x))\,dx$ are generally known as *harmonic numbers.* He then discovered the formula for $\zeta(2m) = \sum_{n=1}^{\infty}(1/n^{2m})$. He also introduced the dilogarithm function:

$$\mathrm{Li}_2(x) = \sum_{m=1}^{\infty}\frac{x^m}{m^2} = -\int_0^x \frac{\log(1-t)}{t}dt = -\int_0^x \frac{dt}{t}\int_0^t \frac{du}{1-u} \qquad (|x| \le 1)$$

and computed these values: $\mathrm{Li}_2(1) = (\pi^2/6)$, $\mathrm{Li}_2(-1) = -(\pi^2/12)$, $\mathrm{Li}_2(1/2) = (\pi^2/12) - (\log^2 2/2)$.

Further, he investigated the double sums $\sum_{n=1}^{\infty}(H_n/n^m)$ for some fixed $m \ge 2$ which have been a popular topic of research in recent years.

1.2. **Generating functions for central binomial coefficients and Catalan numbers.** Recall that the *generating function* for the sequence $a_0, a_1, a_2, \cdots$ is defined to be the function represented by power series: $G(x) := \sum_{n=0}^{\infty} a_n\, x^n$. It is always permissible to integrate (and differentiate) a power series term by term over any closed interval lying entirely within its interval of convergence.

The binomial coefficient $\binom{2n}{n}$, the largest coefficient of the polynomial $(1+x)^{2n}$, forms the central column of Pascal's triangle and so is always an

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integer. The sequence of these numbers is generated by

$$(\ 1/(\sqrt{1-4x})\ ) = \sum\nolimits_{n=0}^{\infty} \binom{2n}{n} x^n. \tag{1.1}$$

The expansion is a consequence of the binomial theorem as for $n \in \mathbb{N}$:

$$\frac{1}{(1-x)^{1/n}} = 1 + \frac{1}{n}x + \frac{1\cdot(1+n)}{n\cdot 2n}x^2 + \frac{1\cdot(1+n)\cdot(1+2n)}{n\cdot 2n\cdot 3n}x^3 + \dots.$$

The series on the R.H.S. of (1.1) converges if $|x| < (1/4)$. Lehmer[11] obtained some 'interesting series' by repeated integrations of (1.1)

If we integrate (1.1) from 0 to x and then divide the result by x we get generating function for $\frac{1}{n+1}\binom{2n}{n}$, known as the *Catalan numbers* and denoted by C_n.

$$(1-\sqrt{1-4x})/2x = \sum\nolimits_{n=0}^{\infty} C_n x^n. \tag{1.2}$$

Transposing the first term of the right side of (1.1) to the left, dividing both sides by x and then integrating, one gets [11, (6)]:

$$\sum\nolimits_{n=1}^{\infty} \frac{1}{n}\binom{2n}{n} x^n = 2\log\ (\ (1-\sqrt{1-4x})/(2x)\ ). \tag{1.3}$$

1.3. **Generating function for harmonic numbers and few series.** Using Euler's integral representation of harmonic numbers, we have:

$$\begin{aligned}\sum\nolimits_{n=1}^{\infty} H_n\, x^n\, dx &= \sum\nolimits_{n=1}^{\infty} \left(\int_0^1 (\,(1-u^n)/(1-u)\,)\, du \right) x^n \\ &= \int_0^1 (\,1/(1-u)\,) \left(\sum\nolimits_{n=1}^{\infty} (x^n - (ux)^n) \right) du \\ &= \int_0^1 \frac{1}{1-u} \left(\frac{1}{1-x} - \frac{1}{1-ux} \right) du \ = \ x \int_0^1 \frac{du}{1-ux}.\end{aligned}$$

Since

$$(\,x/(1-x)\,) \int_0^1 \frac{du}{1-ux} = -(\,\ln(1-x)/(1-x)\,),$$

we get the generating function for harmonic numbers given in [5, p.54, 1.514.6]:

$$-\ (\ \ln(1-x)/(1-x)\ ) = \sum\nolimits_{n=1}^{\infty} H_n x^n \qquad (x^2 < 1). \tag{1.4}$$

Considering the partial sums of the alternating harmonic series (having sum $\log 2$), with notation

$$H'_n = \sum\nolimits_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \log 2 + \frac{(-1)^n}{2}\left[\psi\left(\frac{n+1}{2}\right) - \psi\left(\frac{n+2}{2}\right) \right],$$

where $\psi(x)$ is the digamma function, we have this generating function:

$$\ln(1+x)/(1-x) = \sum\nolimits_{n=1}^{\infty} H'_n\, x^n \qquad (x \neq 1). \tag{1.5}$$

Now

$$\int \frac{\log(1+x)}{1-x}\,dx = -\mathrm{Li}_2(\,(1+x)/2\,) - \log(\,(1-x)/2\,)\log(1+x) + C.$$

This integral, evaluated between -1 and 0, gives the sum:

$$\sum_{n=1}^{\infty}(-1)^{n+1}(\,H'_n/(n+1)\,) = (\pi^2/12) - (1/2)\log^2 2. \tag{1.6}$$

Further,

$$\int \frac{\log(1+x)}{x(1-x)}\,dx = \int \frac{\log(1+x)}{1-x}\,dx + \int \frac{\log(1+x)}{x}\,dx$$
$$= -\mathrm{Li}_2(x) - \mathrm{Li}_2\left(\frac{1+x}{2}\right) - \log\left(\frac{1-x}{2}\right)\log(1+x) + C$$

which yields the following sum valid for $-1 \le x < 1$:

$$\sum_{n=1}^{\infty}\frac{H'_n}{n}\,x^n = \mathrm{Li}_2\left(\frac{1}{2}\right) - \mathrm{Li}_2(-x) - \mathrm{Li}_2\left(\frac{1+x}{2}\right) - \log\left(\frac{1-x}{2}\right)\log(1+x). \tag{1.7}$$

Its alternative form occurs in [12, p.302, A.2.8 (1)] [3, (5)]

$$\sum_{n=1}^{\infty}\frac{H'_n}{n}\,x^n = \mathrm{Li}_2(\,(1-x)/2\,) - \mathrm{Li}_2(\,1/2\,) - \mathrm{Li}_2(-x) - \log(1-x)\log 2$$

and Boyadzhiev [3, (10)] computes the sum differently:

$$\sum_{n=1}^{\infty}(-1)^{n+1}(H'_n/n) = (\ \pi^2/12\ ) + (\ 1/2\ )\log^2 2. \tag{1.8}$$

So the sum and difference of (1.6) and (1.8) result in

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H'_n}{n(n+1)} = \log^2 2; \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n+1)\,H'_n}{n(n+1)} = \pi^2/6.$$

The two integrals preceding equations (1.6) and (1.7) evaluated between 0 and 1/2 yield sums:

$$\sum_{n=1}^{\infty}\frac{H'_n}{n\,2^n} = \frac{1}{2}\mathrm{Li}_2\left(\frac{1}{4}\right) + \log^2 2 = -\frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) + \frac{\pi^2}{36} - \frac{\log^2 3}{3} + \log 2\log 3, \tag{1.9}$$

and

$$\sum_{n=1}^{\infty}\frac{H'_n}{(n+1)\,2^{n+1}} = -\frac{1}{3}\mathrm{Li}_2\left(\frac{1}{9}\right) - \frac{\pi^2}{36} - \frac{1}{2}\log^2 2 - \frac{2}{3}\log^2 3 + 2\log 2\log 3. \tag{1.10}$$

Multiplying (1.10) by 2 and by subtracting the result from (1.9), we get

$$\sum_{n=1}^{\infty}\frac{(3n+4)H'_n}{n(n+1)\,2^n} = 2\pi^2/3 + (\log 2)^2 = 8\sum_{n=1}^{\infty}(-1)^{n+1}(H'_n/n). \tag{1.11}$$

1.4. **Relation between binomial coefficients and harmonic numbers.** Harmonic numbers can be expressed in term of binomial coefficients:

$$H_n = \sum\nolimits_{k=0}^{n-1} \frac{1}{k+1} = \sum\nolimits_{k=0}^{n-1} \left(\int_0^1 x^k\,dx \right) = \int_0^1 \left(\sum\nolimits_{k=0}^{n-1} x^k \right) dx$$

$$= \int_0^1 (\,(1-x^n)/(1-x)\,)\,dx \qquad \text{(partial sum of the G.P.)}$$

$$= \int_0^1 (\,(1-(1-y)^n)/y\,)dy \qquad \text{(on setting } (1-x)=y\text{)}$$

$$= \int_0^1 \frac{\sum_{k=1}^n \binom{n}{k} (-1)^{k-1} y^k}{y}\,dy \qquad \text{(removing 1 from numerator)}$$

$$= \sum\nolimits_{k=1}^n \binom{n}{k} (-1)^{k-1} \int_0^1 y^{k-1}\,dy \;=\; \sum\nolimits_{k=1}^n \binom{n}{k} (-1)^{k-1}(1/k).$$

No H_n is an integer for $n > 1$. We find in [6, p.192, (5.48)] this inversion formula:

$$g(n) = \sum\nolimits_k \binom{n}{k} (-1)^k f(k) \Longleftrightarrow f(n) = \sum\nolimits_k \binom{n}{k} (-1)^k g(k).$$

Since $H_n = \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{1}{k}$, we have: $\frac{1}{n} = \sum_{k=1}^n (-1)^k \frac{1}{k} \binom{n}{k} H_k$. This relation is used by Boyadzhiev [1] [2] for derivation of certain series whose coefficients are products of the central binomial coefficients and harmonic numbers, and whose sums are expressible in term of logarithms.

We intend to obtain here series whose sums involve π, $\zeta(2)$ and Catalan's constant and thereby supplement Boyadzhiev's work.

2. Generating function for $\binom{2n}{n} H_n$ and known series

2.1. **Using Euler's transformation of series.** We find in [10, p.469] this version of Euler's transformation of series:

$$\sum\nolimits_{k=0}^{\infty} a_k x^{k+1} = \sum\nolimits_{k=0}^{\infty} a_k \left(\frac{y}{1-y} \right)^{k+1} = \sum\nolimits_{n=0}^{\infty} \left\{ \sum\nolimits_{m=0}^{n} \binom{n}{m} a_m \right\} y^{n+1}.$$

Boyadzhiev comes up with this formula for $\alpha \in \mathbb{C}$ in [1, (2.4)] and [2, (10)]:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n a_n\, z^n = (z+1)^{\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} \left(\frac{z}{z+1} \right)^n$$

Setting $z = 4x$, $a_k = (-1)^{k-1} H_k$, $\alpha = -1/2$ and using the relation we derived above, Boyadzhiev obtained for $|x| < \frac{1}{4}$ these generating functions for the product of the harmonic numbers and the central binomial coefficients:

$$\sum\nolimits_{n=0}^{\infty} H_n \binom{2n}{n} (-1)^{n+1} x^n = \frac{2}{\sqrt{1+4x}} \log \left(\frac{2\sqrt{1+4x}}{1+\sqrt{1+4x}} \right). \tag{2.1}$$

$$\sum\nolimits_{n=0}^{\infty} H_n \binom{2n}{n} x^n = \frac{2}{\sqrt{1-4x}} \log \left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}} \right). \tag{2.2}$$

2.2. **Alternative derivation.** We can also derive (2.2) in a different way as follows:

$$\sum_{n=1}^{\infty} \binom{2n}{n} H_n x^n = \sum_{n=1}^{\infty} \binom{2n}{n} x^n \int_0^1 \frac{1-t^n}{1-t}\,dt$$
$$= \int_0^1 \frac{1}{1-t} \left(\sum_{n=1}^{\infty} \binom{2n}{n} (x^n - (xt)^n) \right) dt$$
$$= \int_0^1 \frac{1}{1-t} \left(\sum_{n=1}^{\infty} \binom{2n}{n} x^n - \sum_{n=1}^{\infty} \binom{2n}{n} (xt)^n \right) dt$$
$$= \int_0^1 \frac{1}{1-t} \left(\frac{1}{\sqrt{1-4x}} - \frac{1}{\sqrt{1-4xt}} \right) dt \text{ [by using (1.1)]}.$$

We assumed that swapping of summation and integration is permissible here. Now $(1/\sqrt{1-4x}) \int (1/(1-t))dt = -(\log(1-t)/\sqrt{1-4x})$. To evaluate the second integral we make the substitution: $u = \sqrt{1-4xt}$ so that $du = -(2x/\sqrt{1-4xt})dt$ and $(1-t) = ((4x-1+u^2)/4x)$. Then,

$$\int \frac{dt}{(1-t)\sqrt{1-4xt}} = 2x \int \frac{du}{(1-4x)-u^2} = \frac{1}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x}+\sqrt{1-4xt}}{\sqrt{1-4x}-\sqrt{1-4xt}}.$$

Thus the integral becomes:

$$-\frac{1}{\sqrt{1-4x}} \log \frac{(\sqrt{1-4x}+\sqrt{1-4xt})(1-t)}{(\sqrt{1-4x}-\sqrt{1-4xt})}$$
$$= -\frac{2}{\sqrt{1-4x}} \log \frac{(\sqrt{1-4x}+\sqrt{1-4xt})(1-t)}{-4x(1-t)}$$
$$= -\frac{2}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x}+\sqrt{1-4xt}}{-4x}.$$

which, at $t=1$, has the value $-\frac{2}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x}}{-2x}$ and at $t=0$ it becomes: $\frac{2}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{-4x}$. Thus the definite integral becomes:

$$\frac{2}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{-4x} \frac{-2x}{\sqrt{1-4x}} = \frac{2}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}$$

which is the formula (2.2). □

Integrating the power series (2.2), using the substitution $(1-4x) = y^2$ for the RHS, one obtains for every $|x| \leq (1/4)$,

$$\sum_{n=0}^{\infty} H_n \binom{2n}{n} \frac{x^{n+1}}{n+1} = (\sqrt{1-4x}) \log(2\sqrt{1-4x})$$
$$- (1+\sqrt{1-4x}) \log(1+\sqrt{1-4x}) + \log 2. \quad (2.3)$$

Putting $x = 1/4$ in (2.3) yields:

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(n+1)\,2^{2n}} = 4\log 2. \quad (2.4)$$

By shifting the index we get:

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(2n-1)\,2^{2n}} = \sum_{n=0}^{\infty} \frac{H_{n+1} \binom{2n+2}{n+1}}{(2n+1)\,2^{2n+2}} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(H_n + \frac{1}{n+1}) \binom{2n}{n}}{(n+1)\,2^{2n}}\right]$$
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{H_n \binom{2n}{n}}{(n+1)\,2^{2n}} + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^2\,2^{2n}}\right]$$

and since (see [4, pp.251-252, §1081])

$$\sum\nolimits_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^2\,2^{2n}} = 4 - 4\log 2 \tag{2.5}$$

we obtain by using (2.4) and (2.5):

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(2n-1)\,2^{2n}} = 2. \tag{2.6}$$

Edwards deduced (2.5) via the integral $\int_0^1 \frac{\log(1/x)}{\sqrt{1-x}}\,dx$ and putting $x = \sin^2\theta$. He also gave: $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2\,2^{2n}} = (\pi \ln 2)/2$ (pp.252–253). It may be interesting to mention here that $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2\,2^{3n}} = \sqrt{2}\left[\frac{\pi \ln 2}{8} + \frac{G}{2}\right]$, where G is the Catalan's constant defined by $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159655941\ldots$ We can similarly derive with $x = -(1/16)$ in (2.3) and by shifting the index:

$$\sum\nolimits_{n=1}^{\infty} (-1)^{n+1} \frac{H_n \binom{2n}{n}}{(n+1)\,2^{4n}} = 16\log(4\sqrt{5}-8) + 8\sqrt{5}\log(10-4\sqrt{5}), \tag{2.7}$$

and
$$\sum\nolimits_{n=1}^{\infty} (-1)^{n+1} \frac{H_n \binom{2n}{n}}{(2n-1)\,2^{4n}} = (\sqrt{5}-2) + \sqrt{5}\log\left(\frac{1}{\sqrt{5}} + \frac{1}{2}\right). \tag{2.8}$$

By a variation on the method, we will now derive some interesting series whose sums involve π, $\zeta(2)$ and Catalan's constant.

3. New sums involving $\pi, \zeta(2)$ *and* G

This is the series that we aim to obtain:

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{n\,2^{2n}} = \pi^2/3. \tag{3.1}$$

Dividing both sides of (2.2) by x and integrating between limits $x = 0$ to $x = 1/4$, after using the substitution $1-4x = y^2$, $x = (1-y^2)/4$, $dx = -(y/2)dy$ we get on the R.H.S.:

$$I = \int_1^0 \frac{2\cdot 4}{(1-y^2)y} \log\left(\frac{1+y}{2y}\right) \cdot \frac{-y}{2}\,dy = \int_0^1 \frac{4}{(1-y^2)} \log\left(\frac{1+y}{2y}\right) dy.$$

The software *WolframAlpha* at https://www.wolframalpha.com gives:

$$\int 4\log\left(\frac{1+y}{2y}\right)\frac{dy}{1-y^2} = -2\mathrm{Li}_2(1-y) - 2\mathrm{Li}_2(-y) - 2\mathrm{Li}_2\left(\frac{y+1}{2}\right) - \log^2(y+1) +$$

$$2\log\left(1+\frac{1}{y}\right)\log(1-y)+2\log(2)\log(1-y)-2\log(1-y)\log(y)+C,$$

and $$\int_0^1 4\log\left(\frac{1+y}{2y}\right)\frac{dy}{1-y^2}=\pi^2/3.$$

Consequently, $\sum_{n=1}^{\infty}\frac{H_n\binom{2n}{n}}{n\,2^{2n}}=\frac{\pi^2}{3}$. Let us try to evaluate the integral I by partial fractions decomposition:

$$\begin{aligned}I&=\int_0^1\frac{2}{(1-y)}\log\left(\frac{1+y}{2y}\right)dy+\int_0^1\frac{2}{(1+y)}\log\left(\frac{1+y}{2y}\right)dy\\&=2\int_0^1\frac{\log(1+y)}{1-y}dy-2\int_0^1\frac{\log 2}{1-y}dy-2\int_0^1\frac{\log y}{1-y}dy\\&+2\int_0^1\frac{\log(1+y)}{1+y}dy-2\int_0^1\frac{\log 2}{1+y}dy-2\int_0^1\frac{\log y}{1+y}dy.\end{aligned}$$

These are the relevant indefinite integrals:

$$\begin{aligned}I_1&=\int\frac{\log(1+y)}{1-y}dy=-\mathrm{Li}_2\left(\frac{1+y}{2}\right)-\log\left(\frac{1-y}{2}\right)\log(1+y)+C\\&=-\mathrm{Li}_2\left(\frac{1+y}{2}\right)-\log(1-y)\log(1+y)+\log 2\log(1+y)+C,\end{aligned}$$

$$I_2=\int\frac{\log(1+y)}{1+y}dy=\frac{\log^2(1+y)}{2}+C,$$

$$I_3=\int\frac{\log 2}{1-y}dy=-\log 2\ \log(1-y)+C,$$

$$I_4=\int\frac{\log 2}{1+y}dy=\log 2\ \log(1+y)+C,$$

$$I_5=\int\frac{\log y}{1-y}dy=\mathrm{Li}_2(1-y)+C,\qquad\text{and}$$

$$I_6=\int\frac{\log y}{1+y}dy=\mathrm{Li}_2(-y)+\log(y)\log(1+y)+C.$$

Remark. *There arises a problem when one puts $y=1$ in the two integrals (with opposite signs) namely I_1 and I_3. Though we get two indeterminate terms $\log(0)\times\log(2)$ with opposite signs, we cannot simply cancel them to get 0. Again, when we put $y=0$ in I_6, we get an indeterminate term $\log(0)\log(1)=\infty\times 0$ which cannot be straightway taken to be 0; for this, we will take limit as $y\to 0$.*

For I_1-I_3, we notice as in [3, (6)]

$$\frac{d}{dx}\mathrm{Li}_2\left(\frac{1-\mathrm{x}}{2}\right)=\frac{1}{1-x}\log\frac{1+x}{2}=\frac{\log(1+x)}{1-x}-\frac{\log(2)}{1-x}.$$

Therefore,

$$\int_0^1 \left(\frac{\log(1+x)}{1-x} - \frac{\log 2}{1-x}\right) = \mathrm{Li}_2\left(\frac{1-\mathrm{x}}{2}\right)\Big|_0^1 = (\,(\log 2)^2/2\,) - (\pi^2/12).$$

For I_6 we have: $\lim_{t\to 0} \log t \log(1+t)$, and an application of l'Hôpital's rule gives:

$$\lim_{t\to 0} \frac{\log t}{1/\log(1-t)} = \lim_{t\to 0} \frac{(1-t)\log^2(1-t)}{t}$$
$$= \lim_{t\to 0} \left[-\log^2(1-t) - 2\log(1-t)\right] = 0.$$

Thus the indeterminate expression has value 0. Consequently, we get (on using the values of the logarithm given by Euler) the same value of the integral $I = \frac{\pi^2}{3}$ as returned by the *Wolfram* software. □

Combining (2.4) and (3.1), we get:

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{n(n+1)\, 2^{2n}} = (\pi^2/3) - 4\log 2. \tag{3.2}$$

By a shift of the index, we have:

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{n\, 2^{2n}} = \sum_{n=0}^{\infty} \frac{H_{n+1} \binom{2n+2}{n+1}}{(n+1)\, 2^{2n+2}} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(H_n + \frac{1}{n+1})\,(2n+1)\binom{2n}{n}}{(n+1)^2\, 2^{2n}}\right]$$

and the expression within brackets on the extreme right can be expanded as

$$2\sum_{n=0}^{\infty} \frac{H_n \binom{2n}{n}}{(n+1)\, 2^{2n}} - \sum_{n=0}^{\infty} \frac{H_n \binom{2n}{n}}{(n+1)^2\, 2^{2n}} + 2\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^2\, 2^{2n}} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^3\, 2^{2n}}.$$

Dividing both sides of (1.3) by $2x$, and then integrating it yields:

$$\sum\nolimits_{n=1}^{\infty} \frac{\binom{2n}{n} x^n}{2n^2\, 2^{2n}} = \int \frac{\log((1-\sqrt{1-4x})/2x)}{x} dx = -Li_2\left(\frac{1+\sqrt{1-4x}}{2}\right)$$
$$- \frac{1}{2}\log^2(1+\sqrt{1-4x}) - \log\left(\frac{1-\sqrt{1-4x}}{2}\right)\log(1+\sqrt{1-4x})$$
$$- \log(-4x)\log\left(\frac{1+\sqrt{1-4x}}{2}\right) + \log(-4x)\log(1+\sqrt{1-4x}) + C.$$

This in turn gives us

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{1}{n^2} = \zeta(2) - 2(\log 2)^2. \tag{3.3}$$

Prof. Paul Levrie drew my attention to this formula in [14, p.354, (22)]

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n\, n!} H_{n-1} = \zeta(2) + 2(\log 2)^2 \tag{3.4}$$

that involves the Pochhammer symbol $(a)_n = \prod_{k=0}^{n-1}(a+k)$ so that $\left(\frac{1}{2}\right)_n = \binom{2n}{n}/(n!\,2^{2n})$. The formula can be written as $\sum_{n=1}^{\infty}\left(\binom{2n}{n} H_n\right)/(n\,4^n) - \sum_{n=1}^{\infty}\binom{2n}{n}/(n^2\,4^n) = \zeta(2) + 2(\log 2)^2$ which follows immediately from (3.1) and (3.3). Now

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{2^{2n}}\frac{1}{n^2}=\sum_{n=0}^{\infty}\frac{\binom{2n+2}{n+1}}{(n+1)^2\,2^{2n+2}}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(n+1)^2\,2^{2n}}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(n+1)^3\,2^{2n+2}}$$

so using (2.5) and (3.3) we deduce:

$$\sum\nolimits_{n=0}^{\infty}\frac{\binom{2n}{n}}{(n+1)^3\,2^{2n}} = 8 - 8\log 2 - (\pi^2/3) + 4(\log 2)^2. \tag{3.5}$$

And by using the sums (2.4), (2.5), (3.1) and (3.5), we obtain:

$$\sum\nolimits_{n=1}^{\infty}\frac{H_n\binom{2n}{n}}{(n+1)^2\,2^{2n}} = -(\pi^2/3) - 4(\log 2)^2 + 8\log 2. \tag{3.6}$$

Further, combining (3.5) and (3.6), we get:

$$\sum_{n=1}^{\infty}\frac{H_n\binom{2n}{n}}{n(n+1)^2\,2^{2n}} = \frac{2\pi^2}{3} + 4(\log 2)^2 - 12\log 2. \tag{3.7}$$

Let us replace x by x^2 in (2.2) and set $\sqrt{1-4x^2} = y$. We then take the integral from $y = 1$ to $y = 0$, which means taking x from 0 to $1/2$. Assuming the value of the integral to be [7, p.173, Table 120, (1)]

$$2\int_0^1 \log\left(\frac{1+y}{2y}\right)\frac{dy}{\sqrt{1-y^2}} = 2\int_0^1 \frac{\log(1+u)}{\sqrt{1-u^2}}\,du = 4G - \pi\log 2,$$

we obtain :

$$\sum_{n=1}^{\infty}\frac{H_n\binom{2n}{n}}{(2n+1)\,2^{2n}} = 4G - \pi\log 2, \tag{3.8}$$

where G is Catalan's constant. Using a shift of the index, we have:

$$\sum_{n=1}^{\infty}\frac{H_n\binom{2n}{n}}{(2n-1)^2\,2^{2n}} = \sum_{n=0}^{\infty}\frac{H_{n+1}\binom{2n+2}{n+1}}{(2n+1)\,2^{2n+2}}$$

$$= \sum_{n=0}^{\infty}\frac{H_n\binom{2n}{n}}{(2n+1)\,2^{2n}} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{H_n\binom{2n}{n}}{(n+1)\,2^{2n}} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)(n+1)^2\,2^{2n}}.$$

By putting $x = 1$ in the expansion:

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

we obtain $\sum_{n=0}^{\infty}\left(\binom{2n}{n}/(2n+1)2^{2n}\right) = \frac{\pi}{2}$. We already have two sums $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(n+1)2^{2n}}$ and $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(n+1)^2 2^{2n}}$ so that by combining the three sums we get:

$$\sum\nolimits_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)(n+1)^2\,2^{2n}} = 2\pi + 4\log 2 - 8. \tag{3.9}$$

And using (3.8) and (3.9) we derive this result:

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(2n-1)^2\, 2^{2n}} = \pi(1-\log 2) - 4(1-G). \tag{3.10}$$

where G is Catalan's constant. Our integral also gives:

$$4\int_{1/3}^{1} \log\left(\frac{1+y}{2y}\right) \frac{dy}{1-y^2} = 2\,\mathrm{Li}_2(-1/3) + 4\,\mathrm{Li}_2(2/3) - (\pi^2/6) - (\log 2)^2 + 3(\log 3)^2 - 4\,\log 2 \log 3,$$

which yields another beautiful formula:

$$\sum_{n=1}^{\infty} \frac{2^n\, H_n \binom{2n}{n}}{n\, 3^{2n}} = (\pi^2/6) - (\log 2)^2. \tag{3.11}$$

Morris[13, p.781] notes that $6\mathrm{Li}_2(3) - 3\mathrm{Li}_2(-3) = 2\pi^2$ using which and various relations from [12, p.283], we found:

$$2\,\mathrm{Li}_2(1/3) - \mathrm{Li}_2(-1/3) = (\pi^2/6)(-1/2)\,(\log 3)^2 \tag{3.12}$$

which we used in the derivation of (3.11). Also refer to [8, p.155, (2.3)] [9, p.89, 6(i)].

Further, our integral, in conjunction with the relations [12, p.283, (7) & (13)]: $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = (\pi^2/6) - \log(x)\log(1-x)$ and $\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = (1/2)\mathrm{Li}_2(x)$, yields:

$$4\int_{1/2}^{1} \log\left(\frac{1+y}{2y}\right) \frac{dy}{1-y^2} = \sum_{n=1}^{\infty} \frac{3^n\, H_n \binom{2n}{n}}{n\, 2^{4n}} = \mathrm{Li}_2(3/4) + 2(\log 2)^2 - (\log 3)^2$$

while the relation [12, p.283, (12)] $\mathrm{Li}_2(\,1/(1+x)\,) - \mathrm{Li}_2(-x) = (\pi^2/6) - (1/2)\log(1+x)\log(\,(1+x)/x^2\,)$ $x > 0$ transforms it into:

$$\sum_{n=1}^{\infty} \frac{3^n\, H_n \binom{2n}{n}}{n\, 2^{4n}} = (\pi^2/6) + \mathrm{Li}_2(-1/3) - (1/2)\,(\log 3)^2$$

and using (3.12) we obtained this lovely result:

$$\sum_{n=1}^{\infty} \frac{3^n\, H_n \binom{2n}{n}}{n\, 2^{4n}} = 2\,\mathrm{Li}_2\ (1/3). \tag{3.13}$$

The R.H.S. can also be written as: $(1/3)\mathrm{Li}_2(1/9) + (\pi^2/9) - (1/3)\,(\log 3)^2$.

Concluding remarks: We have given a host of interesting sums here. The reader may try to compute the sums: $\sum_{n=1}^{\infty}(\,H_n \binom{2n}{n}/n^2\, 2^{2n}\,)$ and $\sum_{n=1}^{\infty}(\,H_n \binom{2n}{n}/(2n+1)^2\ 2^{2n})$ which we couldn't.

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Amrik Singh Nimbran
B3-304, Palm Grove Heights
Ardee City, Gurgaon- , Haryana, India.
E-mail: amrikn622@gmail.com

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ON $N(\kappa)$-PARA CONTACT 3-MANIFOLDS WITH RICCI SOLITONS

AVIJIT SARKAR AND RAJESH MONDAL
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Abstract. The present paper contains the study of $N(\kappa)$-para contact 3-manifolds with Ricci solitons. It is established that such solitons are steady. In addition, if a soliton is gradient Ricci soliton, then the manifold is flat. An example is provided to support the obtained results.

1. Introduction

It is well known that the Poincare Conjecture, posed by Henry Poincare in 1904, was a long standing unsolved problem of geometry and topology. The century long problem was solved by G. Perelman in 2002[13]. In order to solve the problem Perelman used the techniques of Ricci flow developed by R. S. Hamilton [8]. Around the same time Friedan [5] introduced the theory of Ricci flow in physics with a different motivation. In geometric topology Hamilton's Ricci flow is a heat type parabolic partial differential equation of the following form:

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij}, \quad g_{ij}(0) = g_0. \tag{1.1}$$

A Ricci soliton is a constant solution of the Ricci flow equation upto diffeomorphism and scaling. On a Riemannian manifold M with metric g, a Ricci soliton is expressed by

$$(\text{Ł}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{1.2}$$

Here Ł_V denotes the Lie derivative operator along a complete vector field V which is known as a potential vector field. It is assumed that V is complete. Here λ is a constant, known as soliton constant and S is the Ricci curvature of the manifold M. X, Y are arbitrary vector fields of M. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero

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or positive respectively. If the vector field V is the gradient of potential function f, then g is called a gradient Ricci soliton. For details we refer [2]. Ramesh Sharma [14], first introduced the study of Ricci solitons in the field of contact manifolds. Later several authors have studied Ricci solitons on contact and almost contact manifolds [6], [15], [16]. In the papers [10], [11] T. Ivey gave new examples of Ricci solitons. A Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3 [7], [9].

Again a Ricci soliton on a compact manifold is a gradient Ricci soliton [2].The development of the theory of para contact geometry was pioneered by Keneyuki and Williams [12]. Zamkovoy [19] studied canonical connections on para contact manifolds. The first author with De [3] studied para Sasakian manifolds . Ricci solitons on para contact manifolds have been studied in the papers [1], [4]. The first author of the present paper has also studied Ricci soliton on α-para Kenmotsu 3-manifolds. In this paper we would like to give some more characterizations of $N(\kappa)$-para contact 3-manifolds with Ricci solitons [4]. The present paper is organized is as follows:
After the introduction, we give a brief account of $N(\kappa)$-para contact metric manifolds and required formulas. Section 3 consists of the study of $N(\kappa)$-para contact 3-manifolds equipped with Ricci solitons. In this section it is shown that such solitons are steady. Gradient Ricci solitons on $N(\kappa)$-para contact 3-manifolds have been considered in Section 4 and it is established that such solitons make the manifolds flat. The last section contains an illustrative examples.

2. $N(\kappa)$-para contact metric manifolds

Para contact structures are analogous to contact structure. However there are some basic differences that prompts differential geometers to investigate para contact geometry. Before going to the brief introduction of para contact and $N(\kappa)$-para contact geometry let us point out the differences.

A tangent space of an almost contact metric manifold of dimension $2n+1$ admits a pseudo orthonormal ϕ-basis of vector fields of the form $\{\xi, e_1, e_2, ..., e_n, \phi e_1, \phi e_2, ..., \phi e_n\}$. Here $\xi, e_1, e_2, ..., e_n$ are space like vector fields where as $\phi e_1, \phi e_2, ..., \phi e_n$ are time like. But in contact structure all are space like. Due to the existence of time like vector fields, para contact structures have become important tools to the researchers of relativity and cosmology. Another difference is that if a contact metric manifold

with Riemannian curvature R satisfies $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$, then the structure is Sasakian. But mere satisfaction of the analogous relation $R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y)$, for para contact structure does not ensure that the structure is para Sasakian [4] (p.1574).
In the following we give a brief account of para contact and $N(\kappa)$-para contact geometry.

A differentiable manifold of dimension $2n+1$ is said to have para contact structure if it possess a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$form η such that [12]

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

As a consequence of (2.1), we obtain

$$\phi(\xi) = 0, \quad \eta(\phi X) = 0. \tag{2.2}$$

The tensor field ϕ generates an almost para complex structure on each fibre of $D = Ker(\eta)$, i.e., the eigen distributions D^+_ϕ and D^-_ϕ of ϕ corresponding to eigen values 1 and -1, respectively has same dimension n.
If there exists a pseudo Riemannian metric g satisfying

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M, then the manifold is called almost para contact metric manifold and (ϕ, ξ, η, g) is said to be an almost para contact metric structure. Like almost contact structure, an almost para contact structure is said to be normal [19] if and only if the torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \otimes \xi = 0$, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. When $g(X, \phi Y) = d\eta(X,Y)$, the almost para contact structure is called para contact structure. In case of a three dimensional almost para contact structure any local pseudo orthonormal basis of $Ker(\eta)$ determines a $\phi-$basis, upto sign. If $\{e_2, e_3\}$ is a local pseudo orthonormal basis of $Ker(\eta)$ and e_3 is time like, then $\phi e_2 \in Ker\eta$ is also time like and orthogonal to e_2, so $\phi e_2 = e_1$ and $\{\xi, e_2, e_3\}$ is a $\phi-$basis [1].
In a para contact metric manifold, there exists a symmetric, trace free $(1,1)$ tensor field $h = \frac{1}{2}\mathrm{L}_\xi \phi$ satisfying [1]

$$\phi h + h\phi = 0, \quad h\xi = 0 \tag{2.4}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.5}$$

for any vector fields on M.
The tensor field h vanishes if and only if ξ is a killing vector field and in such case, the structure is known as $K-$para contact structure. An almost

para contact metric manifold is called para Sasakian manifold if and only if [19]

$$(\nabla_X \phi)Y = -g(X,Y) + \eta(Y)X, \tag{2.6}$$

for any vector fields X, Y of M. A para contact metric manifold becomes para Sasakian if it is normal and

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.7}$$

A para contact metric manifold is called $N(\kappa)-$para contact metric manifold [4] if there exists a real number κ such that

$$R(X,Y)\xi = -\kappa(\eta(Y)X - \eta(X)Y), \tag{2.8}$$

for any vector fields X, Y of M. For a three dimensional $N(\kappa)$ contact metric manifold [4], we have

$$QX = (r/2 - \kappa)X + (3\kappa - r/2)\eta(X)\xi, \tag{2.9}$$

where Q is the Ricci operator of M and r is the scalar curvature of the manifold.

$$S(X,Y) = (r/2 - \kappa)g(X,Y) + (3\kappa - r/2)\eta(X)\eta(Y), \tag{2.10}$$

where S is the Ricci curvature of M.

$$\begin{aligned} R(X,Y)Z &= (r/2 - 2\kappa)(g(Y,Z)X - g(X,Z)Y) + (3\kappa - r/2)(g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y). \end{aligned} \tag{2.11}$$

Here R is the Riemann curvature. Some consequences of the above mentioned formulas are

$$S(X,\xi) = 2\kappa\eta(X). \tag{2.12}$$

$$R(\xi,X)Y = \kappa(g(X,Y)\xi - \eta(Y)X). \tag{2.13}$$

Again a consequence of (2.5) is

$$(\nabla_X \eta) = g(X,\phi Y) - g(hX,\phi Y), \tag{2.14}$$

3. Three dimensional $N(\kappa)$-para contact metric manifolds with Ricci solitons

In this section, we would like to study three dimensional $N(\kappa)-$ para contact metric manifolds with Ricci solitons.

We know that [18] $(Ł_V\nabla)$ is a symmetric tensor of type $(1,2)$ and it is given by

$$(Ł_V\nabla)(X,Y) = Ł_V(\nabla_X Y) - \nabla_X(Ł_V Y) - \nabla_{[V,X]}Y. \tag{3.1}$$

Now

$$(\nabla_X Ł_V g)(Y,Z) = \nabla_X(Ł_V g)(Y,Z) - (Ł_V g)(\nabla_X Y, Z) - (Ł_V g)(Y, \nabla_X Z). \tag{3.2}$$

Again

$$(Ł_V g)(Y,Z) = \nabla_V g(Y,Z) - g(Ł_V Y, Z) - g(Y, Ł_V Z). \tag{3.3}$$

By virtue of the above three equations, we get

$$(\nabla_X Ł_V g)(Y,Z) = g((Ł_V\nabla)(X,Y),Z) + g((Ł_V\nabla)(X,Z),Y). \tag{3.4}$$

Using (2.10) in (1.2), we have

$$(Ł_V g)(Y,Z) = (2\kappa - r - 2\lambda)g(Y,Z)(r-6\kappa)\eta(Y)\eta(Z). \tag{3.5}$$

Differentiating both sides of (3.5) covariantly and using (2.14) we see that

$$\begin{aligned}(\nabla_X Ł_V g)(Y,Z) &= (r-6\kappa)(g(X,\phi Y)\eta(Z) + g(X,\phi Z)\eta(Y) \\ &- g(hX,\phi Y)\eta(Z) - g(hX,\phi Z))\eta(Y)).\end{aligned} \tag{3.6}$$

Using (3.6) in (3.4)

$$\begin{aligned}&g((Ł_V\nabla)(X,Y),Z) + g((Ł_V\nabla)(X,Z),Y) = (r-6\kappa)(g(X,\phi Y)\eta(Z) + \\ &\quad g(X,\phi Z)\eta(Y) - g(hX,\phi Y)\eta(Z) - g(hX,\phi Z))\eta(Y)).\end{aligned} \tag{3.7}$$

Interchanging X, Y, Z cyclically in (3.7), we have

$$\begin{aligned}&g((Ł_V\nabla)(Y,Z),X) + g((Ł_V\nabla)(Y,X)Z) = (r-6\kappa)(g(Y,\phi Z)\eta(X) + \\ &\quad g(Y,\phi X)\eta(Z) - g(hY,\phi Z)\eta(X) - g(hY,\phi X))\eta(Z)).\end{aligned} \tag{3.8}$$

Similarly from (3.8)

$$\begin{aligned}&g((Ł_V\nabla)(Z,X),Y) + g((Ł_V\nabla)(Z,Y)X) = (r-6\kappa)(g(Z,\phi X)\eta(Y) \\ &\quad + g(Z,\phi Y)\eta(X) - g(hZ,\phi X)\eta(Y) - g(hZ,\phi Y))\eta(X)).\end{aligned} \tag{3.9}$$

Subtracting (3.9) from (3.8), and using (2.4) and symmetry of h, we get

$$\begin{aligned}&g((Ł_V\nabla)(Y,X),Z) - g((Ł_V\nabla)(Z,X),Y) \\ &= (r-6\kappa)(2g(Y,\phi Z)\eta(X) + g(Y,\phi X)\eta(Z) - g(Z,\phi X)\eta(Y) \\ &\quad - g(hY,\phi X))\eta(Z) + g(hZ,\phi X)\eta(Y)).\end{aligned} \tag{3.10}$$

Adding (3.7) and (3.10), we have

$$\begin{aligned}g((Ł_V\nabla)(Y,X),Z) &= (r-6\kappa)(g(X,\phi Z)\eta(Y) + \\ &\quad g(Y,\phi Z)\eta(X) - g(hX,\phi Z)\eta(Y)).\end{aligned} \tag{3.11}$$

The above equation can also be written as

$$\begin{aligned}g((Ł_V\nabla)(Y,X),Z) &= (r-6K)(-g(\eta(Y)\phi X,Z) \\ &\quad - g(\eta(X)\phi Y,Z) + g(\eta(Y)\phi(X,Z)).\end{aligned} \tag{3.12}$$

Comparing both sides of the above equation, we see that

$$(Ł_V\nabla)(Y,X) = (r-6\kappa)(\eta(Y)\phi hX - \eta(Y)\phi X - \eta(X)\phi Y). \tag{3.13}$$

Putting $X=\xi$, we get from above

$$(Ł_V\nabla)(Y,\xi) = -(r-6\kappa)\phi Y. \tag{3.14}$$

We know that [18] , $Ł_V\nabla$ is a symmetric tensor of type $(1,2)$. Its covariant derivative is [18]

$$\begin{aligned}(\nabla_X Ł_V\nabla)(Y,Z) &= \nabla_X((Ł_V\nabla)(Y,Z)) \\ &- (Ł_V\nabla)(\nabla_X Y,Z) - (Ł_V\nabla)(Y,\nabla_X Z).\end{aligned} \tag{3.15}$$

In (3.15) putting $Z=\xi$ and using (3.15) and (2.5), we get

$$\begin{aligned}(\nabla_X \text{Ł}_V\nabla)(Y,\xi) &= -(r-6\kappa)\nabla_X - (\text{Ł}_V\nabla)(\nabla_X Y,\xi)\\ &+ (\text{Ł}_V\nabla)(Y,\phi X) - (\text{Ł}_V\nabla)(Y,\phi hX) \quad (3.16)\end{aligned}$$

From [18], we know

$$(\text{Ł}_V R)(X,Y)\xi = (\nabla_X \text{Ł}_V\nabla)(Y,\xi) - (\nabla_Y \text{Ł}_V\nabla)(X,\xi). \quad (3.17)$$

Applying (3.16) in (3.17), we have

$$\begin{aligned}(\text{Ł}_V R)(X,Y)\xi &= (r-6\kappa)\nabla_Y(\phi X) - (r-6\kappa)\nabla_X(\phi Y)\\ &- (\text{Ł}_V\nabla)([X,Y],\xi) + 2(\text{Ł}_V\nabla)(Y,\phi X) \quad (3.18)\end{aligned}$$

But, Lie derivative of the Riemann curvature gives

$$(\text{Ł}_V R)(X,Y)Z = \text{Ł}_V R(X,Y)Z - R(\text{Ł}_V X,Y)Z - R(X,\text{Ł}_V Y)Z - R(X,Y)\text{Ł}_V Z.$$

Putting $Z=\xi$ in the above equation and using (2.8), we get

$$\begin{aligned}(\text{Ł}_V R)(X,Y)\xi &= -R(X,Y)\text{Ł}_V\xi + \text{Ł}_V(K(\eta(Y)X-\eta(X)Y))\\ &- \kappa(\eta(Y)\text{Ł}_V X - \eta(\text{Ł}_V X)Y) - \kappa(\eta(\text{Ł}_V Y)X - \eta(X)\text{Ł}_V Y). \quad (3.19)\end{aligned}$$

By virtue of (3.18) and (3.19), it follows that

$$\begin{aligned}&(r-6\kappa)\nabla_Y(\phi X) - (r-6\kappa)\nabla_X(\phi Y) - (\text{Ł}_V\nabla)([X,Y],\xi)\\ &+2(\text{Ł}_V\nabla)(Y,\phi X) = -R(X,Y)\text{Ł}_V\xi + \kappa\text{Ł}_V(\eta(Y)X-\eta(X)Y)\\ &-\kappa(\eta(Y)\text{Ł}_V X - \eta(\text{Ł}_V X)Y) - \kappa(\eta(\text{Ł}_V Y)X - \eta(X)\text{Ł}_V Y). \quad (3.20)\end{aligned}$$

Using (3.13) in the above equation, we have

$$\begin{aligned}&(r-6\kappa)\nabla_Y(\phi X) - (r-6\kappa)\nabla_X(\phi Y) - \phi[X,Y]\\ &+2(r-6\kappa)(\eta(Y)\phi h\phi X - \eta(Y)\phi^2 X)\\ &= -R(X,Y)\text{Ł}_V\xi + \kappa\text{Ł}_V(\eta(Y)X-\eta(X)Y) - \kappa(\eta(Y)\text{Ł}_V X\\ &- \eta(\text{Ł}_V X)Y) - \kappa(\eta(\text{Ł}_V Y)X - \eta(X)\text{Ł}_V Y). \quad (3.21)\end{aligned}$$

Putting $X=\xi$, we get from above

$$\begin{aligned}&- (r-6\kappa)\nabla_\xi(\phi Y) + \phi[\xi,Y] = R(Y,\xi)\text{Ł}_V\xi + K\text{Ł}_V(\eta(Y)\xi - Y)\\ &- K(\eta(Y)\text{Ł}_V\xi - \eta(\text{Ł}_V\xi)Y) - K(\eta(\text{Ł}_V Y)\xi - \text{Ł}_V Y). \quad (3.22)\end{aligned}$$

Using (2.5) and (2.6) and simplifying the left hand side of the above equation, we get

$$\begin{aligned}\phi^2 Y - hY &= R(Y,\xi)\text{Ł}_V\xi + \kappa\text{Ł}_V(\eta(Y)\xi - Y)\\ &- \kappa(\eta(Y)\text{Ł}_V\xi - \eta(\text{Ł}_V\xi)Y) - \kappa(\eta(\text{Ł}_V Y)\xi - \text{Ł}_V Y). \quad (3.23)\end{aligned}$$

Putting $V=\xi$, we get from above

$$\phi^2 Y - hY = \kappa\text{Ł}_\xi(\eta(Y)\xi - Y) - \kappa(\eta(\text{Ł}_\xi Y)\xi - \text{Ł}_\xi Y). \quad (3.24)$$

Putting $Y=e_2$ in the above equation

$$\phi^2 e_2 - he_2 = -\kappa\text{Ł}_\xi e_2 - \kappa(\eta(\text{Ł}_\xi e_2)\xi - \text{Ł}_\xi e_2). \quad (3.25)$$

Taking inner product in both sides of the above equation with respect to ξ, we get

$$\kappa = 0. \tag{3.26}$$

In (1.2) putting $V = \xi$ and using (2.5), we get

$$g(\phi hX, Y) + S(X, Y) + \lambda g(X, Y) = 0. \tag{3.27}$$

Putting $X = \xi$, and using (2.12), we get from above

$$(2\kappa + \lambda)\eta(Y) = 0. \tag{3.28}$$

The above equation is true for all Y. So,

$$\lambda = -2\kappa. \tag{3.29}$$

Thus,by (3.26) and (3.29), we have obtained $\lambda = \kappa = 0$, So, we conclude the following

Theorem 3.1: A three-dimensional $N(\kappa)$-para contact Ricci Soliton is steady.

4. Gradient Ricci Soliton

A Ricci soliton is called gradient Ricci soliton if the vector field V is the gradient of a potential function -f. If the Ricci soliton is the gradient Ricci soliton, then (1.2) reduces to [2]

$$\nabla_Y Df = QY + \lambda Y, \tag{4.1}$$

where D is the gradient operator of g and Q is the Ricci operator. Using (4.1), we have

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{4.2}$$

In view of (2.5), (2.9) and (2.14) we get

$$\begin{aligned}(\nabla_X Q)Y \quad &= (3\kappa - r/2)(g(X, \phi Y)\xi - g(hX, \phi Y)\xi \\ &\quad -\eta(Y)\phi X + \eta(Y)\phi hX).\end{aligned} \tag{4.3}$$

Using (4.3) in (4.2), we get

$$\begin{aligned}R(X, Y)Df \quad &= \quad (3\kappa - r/2)(2g(X, \phi Y)\xi - \eta(Y)\phi X + \eta(Y)\phi hX \\ &\quad \eta(X)\phi Y - \eta(X)\phi hY).\end{aligned} \tag{4.4}$$

In the above equation putting $X = \xi$, we get

$$R(\xi, Y)Df = (3K - r/2)(\phi Y - \phi hY). \tag{4.5}$$

In the above equation taking inner product in both sides with W, we get

$$g(R(\xi, Y)Df, W) = (3\kappa - r/2)g(\phi Y - \phi hY, W).$$

Putting $Y = W = e_i$ and taking summation over i, we get v

$$S(\xi, Df) = (r/2 - 3\kappa)\sum_{i=1}^{3} g(\phi he_i, e_i). \tag{4.6}$$

Using (2.10) in the above equation, we get

$$2\kappa\eta(Df) = (r/2 - 3\kappa)\sum_{i=1}^{3} g(\phi he_i, e_i). \tag{4.7}$$

But from previous section $\kappa = 0$. So, $r = 0$.
Hence in view of (2.11) $R(X,Y)Z = 0$. Hence, we conclude the following
Theorem 4.1: A three-dimensional $N(\kappa)$-para contact metric manifold admitting gradient Ricci soliton is flat.

5. Example

In this section we intend to provide an example of $N(\kappa)$-para contact Ricci Soliton of dimension three which is compatible with the obtained results.
Let $M = \{(x,y,z) \in R^3\}$. Here (x,y,z) are standard coordinates in R^3. Let us define a metric on M as follows:
$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1; \quad g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0,$
where e_1, e_2 and e_3 are three linearly independent tangent vectors of $M = R^3$ satisfying

$$[e_1,e_2] = 2e_3, \quad [e_2,e_3] = 2e_1, \quad [e_3,e_1] = 0.$$

If we take $e_1 = \xi$ and $\phi e_2 = e_2, \quad \phi e_3 = -e_3$, then we can easily verify the properties of para contact structure.
Using Koszul formula, we have

$$\begin{array}{lll} \nabla_{e_1} e_3 = 0 & \nabla_{e_1} e_2 = 0 & \nabla_{e_1} e_1 = 0 \\ \nabla_{e_2} e_3 = 2e_1 & \nabla_{e_2} e_2 = 0 & \nabla_{e_2} e_1 = -2e_3 \\ \nabla_{e_3} e_3 = 0 & \nabla_{e_3} e_2 = 0 & \nabla_{e_3} e_1 = 0. \end{array}$$

We can verify that all the components of Riemann curvature are zero and it is seen that the manifold is $N(0)$ para contact metric manifold. Since all the components of Riemann curvature are zero, all the components of Ricci curvature are so. Thus it easily follows that

$$(Ł_V g)(e_i,e_i) + 2S(e_i,e_i) + 2\lambda g(e_i,e_i) = 0$$

for any complete vector field V of M and $\lambda = 0$. Hence the manifold is a steady Ricci soliton and it is also flat.

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Avijit Sarkar
Department of Mathematics, University of Kalyani,
Kalyani, Pin-741 235, Nadia, West Bengal, India.
E-mail: avjaj@yahoo.co.in
and
Rajesh Mondal
Department of Mathematics, University of Kalyani,
Kalyani, Pin-741 235, Nadia, West Bengal, India.
E-mail: rajeshmail.das@gmail.com

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NUMBER OF NON-PRIMES IN THE SET OF UNITS MODULO n

ABHIJIT A. J. AND A. SATYANARAYANA REDDY
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Abstract. In this work, we studied various properties of arithmetic function $\tilde{\varphi}$, where $\tilde{\varphi}(n) = |\{m \in \mathbb{N} | 1 \le m \le n, (m,n) = 1, m \text{ is not a prime}\}|$.

1. Introduction

For a fixed positive integer n, let

$$U_n = \{k : 1 \le k \le n,\ \gcd(k,n) = 1\}. \tag{1.1}$$

If $|S|$ denotes the cardinality of the set S, then $|U_n| = \varphi(n)$ - the well known *Euler-totient function.* Let

$$E_n = \{m \in \mathbb{N}\ :\ 1 \le m \le n,\ (m,n) = 1,\ m \text{ is not a prime}\}, \tag{1.2}$$

that is, $E_n = \{m \in U_n \mid m \text{ is not a prime}\}$. It is known that $\varphi(n) = n - 1$ if and only if n is prime, hence

$$E_n = \{m \in U_n\ :\ \varphi(m) \ne m - 1\}. \tag{1.3}$$

Let $\tilde{\varphi}(n) = |E_n|$. It is clear that $\tilde{\varphi}$ is an arithmetic function. First twenty values of $\tilde{\varphi}$ are given in the following tables.

n	E_n	$\tilde{\varphi}(n)$
1	$\{1\}$	1
2	$\{1\}$	1
3	$\{1\}$	1
4	$\{1\}$	1
5	$\{1,4\}$	2
6	$\{1\}$	1
7	$\{1,4,6\}$	3
8	$\{1\}$	1
9	$\{1,4,8\}$	3
10	$\{1,9\}$	2

n	E_n	$\tilde{\varphi}(n)$
11	$\{1,4,6,8,9,10\}$	6
12	$\{1\}$	1
13	$\{1,4,6,8,9,10,12\}$	7
14	$\{1,9\}$	2
15	$\{1,4,8,14\}$	4
16	$\{1,9,15\}$	3
17	$\{1,4,6,8,9,10,12,14,15,16\}$	10
18	$\{1\}$	1
19	$\{1,4,6,8,9,10,12,14,15,16,18\}$	11
20	$\{1,9\}$	2

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The following results provide few immediate properties of $\tilde{\varphi}$ function. Recall that for $n \in \mathbb{N}$, $\pi(n)$ denotes the number of primes less than or equal to n and $\omega(n)$ denotes the number of distinct prime divisors of n. We denote $N_j = p_1 p_2 \cdots p_j$, the product of first j primes, for example $N_3 = 30$. It would be interesting to note that $\sqrt{n}/2 < \varphi(n) < n - \sqrt{n}$ and by prime number theorem, for a large enough n, $\pi(n) \approx n/log(n)$. But, $\omega(n)$ does not have any ordinary behaviour. In fact for any natural number k, one can find a subsequence $\{n_i\}$ of natural numbers such that $\omega(n_i) = k$ for all i. To know more properties of $\pi(n), \omega(n), \varphi(n)$ and U_n, refer any one of [1, 2, 4, 5].

Proposition 0. *If p_k denotes the k^{th} prime, then $\tilde{\varphi}(p_k) = p_k - k$. In general $\tilde{\varphi}(n) = \varphi(n) - \pi(n) + \omega(n)$.*

Proof. First part follows from the observation that

$$E_{p_k} = U_{p_k} \setminus \{p_1, p_2, \ldots, p_{k-1}\}.$$

Let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$. We claim

$$U_n \cup \{q_1, q_2, \ldots, q_k\} = E_n \cup \{p_1, p_2, \ldots, p_{\pi(n)}\}. \tag{1.4}$$

Let $x \in U_n \cup \{q_1, q_2, \ldots, q_k\}$. If x is prime, then $x \in \{p_1, p_2, \ldots, p_{\pi(n)}\}$ and if x is not a prime then $x \in E_n$. Thus $x \in E_n \cup \{p_1, p_2, \ldots, p_{\pi(n)}\}$.

Now let $y \in E_n \cup \{p_1, p_2, \ldots, p_{\pi(n)}\}$. If $y \in E_n$ then $y \in U_n$ and if $y \in \{p_1, p_2, \ldots, p_{\pi(n)}\}$ then $y \in U_n$ or $y \in \{q_1, q_2, \ldots, q_k\}$ according as $y \nmid n$ or $y|n$ respectively. This proves the claim (1.4).

Next, by definition, $E_n \cap \{p_1, p_2, \ldots, p_{\pi(n)}\} = \emptyset = U_n \cap \{q_1, q_2, \ldots, q_k\}$. Hence $\tilde{\varphi}(n) + \pi(n) = |E_n| + |\{p_1, p_2, \ldots, p_{\pi(n)}\}| = |U_n| + |\{q_1, q_2, \ldots, q_k\}| = \varphi(n) + \omega(n)$. This completes the proof. □

Proposition 1. *We have the following.*

(1) If n_0 is the square free part of n then $\tilde{\varphi}(n) \geq \tilde{\varphi}(n_0)$.
(2) Let $n, k \in \mathbb{N}$ and $n \geq 3$. Then $\tilde{\varphi}(n^k) < \tilde{\varphi}(n^{k+1})$.
(3) Let $n, p, q \in \mathbb{N}$, where p, q are primes with $p < q$ and $(p, n) = 1$. Then $\tilde{\varphi}(np) \leq \tilde{\varphi}(nq)$.
(4) If $a \in \mathbb{N}$ and $\omega(a) = i$, then $\tilde{\varphi}(N_i) \leq \tilde{\varphi}(a)$.

Proof. Since n_0 and n have same set of prime factors, $E_{n_0} \subseteq E_n$, and hence *(1)*.

It is easy to see that $E_{n^k} \subseteq E_{n^{k+1}}$. Hence, *(2)* will be proved if we show that $E_{n^{k+1}} \setminus E_{n^k} \neq \emptyset$. Since $n \geq 3$, there exists $\ell \in \{2, 3, \ldots\}$ such that $(n-1)^{\ell-1} \leq n^k < (n-1)^{\ell}$. Consequently $n^k < (n-1)^{\ell} < (n-1)^{\ell-1} \cdot n \leq n^{k+1}$. Clearly, the element $(n-1)^{\ell}$ belongs to $E_{n^{k+1}} \setminus E_{n^k}$.

To prove *(3)*, let $x \in E_{np}$. Note that $q^m || x$ for some $m \in \mathbb{N} \cup \{0\}$ and hence $x = q^m y$ for some unique $y \in \mathbb{N}$ with $q \nmid y$. Observe that (i) $p < q \Rightarrow p^m y < q^m y = x \leq np < nq$; (ii) $p^m y$ is obviously not a prime and (iii) $(p^m y, nq) = 1$, because $(y, q) = 1$ and since by hypothesis p, q are prime with $p < q$, $(p, n) = 1$, we also have $(p^m, n) = 1$. It follows that $p^m y \in E_{nq}$. If we now define a map $f : E_{np} \to E_{nq}$ as $f(x) = p^m y$ then it is easy to see that f is one-one. Hence *(3)* follows.
Finally, let $q_1, q_2, \ldots, q_i$ be all distinct prime divisors of a and $s = q_1 q_2 \cdots q_i$. Then from *(1)*, we have $\tilde{\varphi}(s) \leq \tilde{\varphi}(a)$. In view of the facts that $N_i \leq s$ and $E_{N_i} \setminus \{q_1, q_2, \ldots, q_i\} \subseteq E_s \setminus \{p_1, p_2, \ldots, p_i\}$, *(4)* now follows. □

2. Main results

It is easy to see that $\tilde{\varphi}(n)=1$ if and only if $n \in \{1, 2, 3, 4, 6, 8, 12, 18, 24, 30\}$. That is, $\tilde{\varphi}(\ell) > 1$ whenever $\ell \geq 31$. In general, let us prove

Theorem 2. *If $n \in \mathbb{N}$ then there exists $N \in \mathbb{N}$ such that $\tilde{\varphi}(m) > n$ for all $m > N$.*

Lemma 3. *Let $n \in \mathbb{N}$. There exists an $M_n \in \mathbb{N}$ such that $\tilde{\varphi}(N_k) > n$ for all $k \in \mathbb{N}$, $k \geq M_n$.*

Proof. Let p_i denote the i^{th} prime. For $i \geq 4$, we define $P_i = \{m \in \mathbb{N} : \varphi(m) = m - 1, m \in U_{N_i}\}$, $Q_i = \{r \in P_i : \ rp_{i+1} < N_i\}$. From Bertrand's Postulate, $p_{i+1} \in Q_i$. Suppose $Q_i = \{p_{i+1}, p_{i+2}, \ldots, p_h\}$ then $|Q_i| = h - i$. Further $p_{i+2} < 2p_{i+1}$ and $p_{h+2} < 2p_{h+1} < 4p_h$. Thus $p_{i+2} p_{h+2} < 8 p_{i+1} p_h < 8N_i < N_{i+1}$ as $p_{i+1} > 8$. Hence $\{p_{i+2}, \ldots, p_{h+2}\} \subseteq Q_{i+1}$ consequently $|Q_i| < |Q_{i+1}|$. Now $\{rp_{i+1} | r \in Q_i\} \subset E_{N_i}$. Thus $\tilde{\varphi}(N_i) > |Q_i|$. Hence the result follows from the fact that $|Q_i|$ increases with i. □

Since $|Q_4| = 4$, therefore for all $n \geq 4$, $|Q_n| \geq n$ and $\tilde{\varphi}(N_k) > k$ when $k \geq 4$. Hence $M_n \leq \max\{4, n\}$. Further, it is easy to see that $\tilde{\varphi}$ is an increasing function on $\{N_3, N_4, N_5, \ldots\}$.

Lemma 4. *Let $a, b \in \mathbb{N}$ and $A(a, b) = \{n \in \mathbb{N} : \ \tilde{\varphi}(n) = a, \ \omega(n) = b\}$. Then $A(a, b)$ is finite.*

Proof. Let p be the $(b + \ell)^{th}$ prime, where $\ell(\ell + 1)/2 > a$. Let $n \in \mathbb{N}$ with $\omega(n) = b$ and $n > p^2$. Since there are at least ℓ primes less than p which are co-prime to n, we can choose two such primes q_1, q_2 less than p. Then $q_1 q_2 < p^2 < n$. Thus $q_1 q_2 \in E_n$ as a consequence $\tilde{\varphi}(n) > \ell(\ell + 1)/2 > a$. Hence the result follows. □

We now prove the Theorem 2.

Proof. Let $n \in \mathbb{N}$. There exists $k \in \mathbb{N}$ such that if $\omega(t) > k$ then $\tilde{\varphi}(t) \geq \tilde{\varphi}(N_{\omega(t)}) > n$. By Lemma 4, for each $i \in \mathbb{N}$, $i \leq k$, there exists $m_i \in \mathbb{N}$ such that $\tilde{\varphi}(t) > n$ for all $t \in \mathbb{N}$ with $\omega(t) = i$ and $t > m_i$. Thus if $N = \max\{m_1, m_2, \ldots, m_n\}$ and $\ell > N$, then $\tilde{\varphi}(\ell) > n$. □

3. Future work

The following question is natural.

Let $k \in \mathbb{N}$ be fixed. Find all n such that $\tilde{\varphi}(n) = k$.

One can see the following result by calculating the appropriate bounds and checking accordingly.

Lemma 5. *Let $s(k) = \{n \in \mathbb{N} : \tilde{\varphi}(n) = k\}$. Then the set of values $k \in \{1, 2, \ldots, 100\}$ such that $s(k)$ is empty set is $\{13, 31, 70\}$.*

Conjecture: If $M = \{\tilde{\varphi}(n) : n \in \mathbb{N}\}$, then $\mathbb{N} \setminus M$ contains infinite number of elements.

Also recall the Carmichael conjecture [3]: if $N(m) = |\{n \in \mathbb{N} : \varphi(n) = m\}|$ then $N(m) \neq 1$. This conjecture is no longer true for $\tilde{\varphi}$ as $\tilde{\varphi}(n) = 16$ only for $n = 144$. If $s(k)$ is as defined in the Lemma 5, then the values of $k \in \{1, \ldots, 100\}$ such that $s(k)$ contains a single element is $\{16, 39, 47, 49, 53, 57, 58, 65, 66, 76, 85, 91, 94\}$.

The following tables and observations may be useful in answering the above questions.

Given a positive integer k, the following tables provide the smallest value of n such that $\tilde{\varphi}(n) = k$. For example $\tilde{\varphi}(1) = \tilde{\varphi}(2) = \tilde{\varphi}(3) = \tilde{\varphi}(4) = 1$, hence the smallest value n such that $\tilde{\varphi}(n) = 2$ is 5.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n	1	5	7	15	26	11	13	38	102	17	19	25	??	23	35

k	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
n	144	74	198	29	31	75	57	104	94	37	55	69	41	43	118

k	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
n	??	47	81	128	87	134	53	93	480	146	77	59	61	117	111

k	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
n	166	172	67	250	91	71	73	350	194	129	202	79	206	212	83

The following observations are useful for the question posed in the beginning of the section for the case $k \leq 20$.

(1) $\tilde{\varphi}(n) = 1 \Leftrightarrow n \in \{2, 3, 4, 6, 8, 12, 18, 24, 30\}$.

(2) $\tilde{\varphi}(n) = 2 \Leftrightarrow n \in \{5, 10, 14, 20, 42, 60\}$.
(3) $\tilde{\varphi}(n) = 3 \Leftrightarrow n \in \{7, 9, 16, 36, 48, 90\}$.
(4) $\tilde{\varphi}(n) = 4 \Leftrightarrow n \in \{15, 22, 54, 84\}$.
(5) $\tilde{\varphi}(n) = 5 \Leftrightarrow n \in \{26, 28, 66, 120\}$.
(6) $\tilde{\varphi}(n) = 6 \Leftrightarrow n \in \{11, 21, 32, 40, 72, 78, 210\}$.
(7) $\tilde{\varphi}(n) = 7 \Leftrightarrow n \in \{13, 34, 50\}$.
(8) $\tilde{\varphi}(n) = 8 \Leftrightarrow n \in \{38, 44, 70, 150\}$.
(9) $\tilde{\varphi}(n) = 9 \Leftrightarrow n \in \{102, 114, 126\}$.
(10) $\tilde{\varphi}(n) = 10 \Leftrightarrow n \in \{17, 27, 46, 56, 96, 108, 180\}$.
(11) $\tilde{\varphi}(n) = 11 \Leftrightarrow n \in \{19, 33, 52, 132\}$.
(12) $\tilde{\varphi}(n) = 12 \Leftrightarrow n \in \{25, 45, 80, 168\}$.
(13) $\tilde{\varphi}(n) = 14 \Leftrightarrow n \in \{23, 39, 58, 62, 110, 138\}$.
(14) $\tilde{\varphi}(n) = 15 \Leftrightarrow n \in \{35, 64, 68, 156, 240\}$.
(15) $\tilde{\varphi}(n) = 16 \Leftrightarrow n = 144$.
(16) $\tilde{\varphi}(n) = 17 \Leftrightarrow n \in \{74, 76, 100, 140\}$.
(17) $\tilde{\varphi}(n) = 18 \Leftrightarrow n \in \{198, 270, 330\}$.
(18) $\tilde{\varphi}(n) = 19 \Leftrightarrow n \in \{29, 51, 88, 98, 162, 174, 420\}$.
(19) $\tilde{\varphi}(n) = 20 \Leftrightarrow n \in \{31, 63, 82, 130\}$.

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Abhijit A. J.
Department of Mathematics
Shiv Nadar University, NH 91, Tehsil Dadri
Gautam Buddha Nagar - 201 314, Uttar Pradesh, India
E-mail: aj448@snu.edu.in
and

A. Satyanarayana Reddy
Department of Mathematics
Shiv Nadar University, NH 91, Tehsil Dadri
Gautam Buddha Nagar - 201 314, Uttar Pradesh, India
E-mail: satyanarayana.reddy@snu.edu.in

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INSTABILITY OF LAGRANGE INTERPOLATION ON $H^1(\Omega) \cap C(\overline{\Omega})$

MOHAMMAD YASIR FEROZ KHAN, RAHUL BISWAS AND RAMESH CHANDRA SAU

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Abstract. It is known that the Lagrange interpolation is not defined for $H^1(\Omega)$ functions. But, for functions in $H^1(\Omega) \cap C(\overline{\Omega})$ point values are well defined and hence the Lagrange interpolation can be introduced. An important feature of an interpolation that many analysis require is its stability under some norm. In this article, we show that the Lagrange interpolation on the space $H^1(\Omega) \cap C(\overline{\Omega})$ is not stable under the $H^1(\Omega)$ norm by giving a counter example.

1. Notations, Preliminaries and Introduction

Let Ω denote a domain in $\mathbb{R}^n$, $\partial\Omega$ denote its boundary, $L^2(\Omega)$ denote the space of square integrable functions on Ω, $\mathcal{D}(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω and $\|.\|_\Omega$ denote the L^2-norm on Ω. For appropriate functions f and g, $f * g$ represents the *convolution* of the two functions defined by $f * g(x) = \int_{\mathbb{R}^2} f(y)g(x-y)dy$ for all $x \in \mathbb{R}^2$.

We say [1, page no. 33] that the boundary of a domain Ω in $\mathbb{R}^n$ is *Lipschitz boundary* if there exists a collection of open sets O_i, a positive parameter ϵ, an integer N and a finite number M, such that for all $x \in \partial\Omega$ a ball of radius ϵ centered at x is contained in some O_i, no more than N of the sets O_i intersect nontrivially and each domain $O_i \cap \Omega = O_i \cap \Omega_i$, where Ω_i is a domain whose boundary is a graph of a Lipschitz function ϕ_i (i.e., $\Omega_i = \{(x,y) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1},\ y < \phi_i(x)\}$) satisfying $\|\phi_i\|_{Lip(\mathbb{R}^{n-1})} \leq M$ in which the norm $\|\cdot\|_{Lip(\Omega)}$ is defined by

$$\|f\|_{Lip(\Omega)} = \|f\|_{L^\infty(\Omega)} + \sup\{|f(x) - f(y)|/|x-y| \ : \ x, y \in \Omega;\ x \neq y\}.$$

For Ω with Lipschitz boundary, *the first order Sobolev space* $H^1(\Omega)$ is defined as follows [3]:

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$$H^1(\Omega) = \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega)\ \forall i = 1, 2, ..., n\},$$

wherein $\frac{\partial v}{\partial x_i}$ belongs to $L^2(\Omega)$ in the sense of distribution, that is, there exists a function $g \in L^2(\Omega)$ such that for all $\phi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega v \frac{\partial \phi}{\partial x_i} = -\int_\Omega g\phi.$$

The $H^1(\Omega)$ norm for $u \in H^1(\Omega) \cap C(\overline{\Omega})$ is defined as

$$\|u\|^2_{1,\Omega} = \|u\|^2_\Omega + \|\nabla u\|^2_\Omega,$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n})$.

Definition 1.1. [3] *Mollifiers on $\mathbb{R}^2$ are defined as follows*

$$\rho_\epsilon(x) = \begin{cases} k\epsilon^{-2}\exp(-\epsilon^2/(\epsilon^2 - |x|^2)), & |x| < \epsilon, \\ 0, & |x| \geq \epsilon, \end{cases}$$

where

$$k^{-1} = \int_{|x|\leq 1} \exp(-1/(1 - |x|^2))dx.$$

Definition 1.2. [1, page no. 69] *Let*

(i) *$K \subseteq \mathbb{R}^n$ be a closed bounded set with nonempty interior and piecewise smooth boundary (the element domain),*

(ii) *$\mathcal{P}$ be a finite-dimensional space of functions on K (the space of shape functions) and*

(iii) *$\mathcal{N} = \{N_1, N_2, ..., N_k\}$ be a basis for $\mathcal{P}'$ (the set of nodal variables).*

Then $(K, \mathcal{P}, \mathcal{N})$ is called a finite element.

Definition 1.3. [1, page no. 77] *Given a finite element $(K, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i : 1 \leq i \leq k\} \subset \mathcal{P}$ be the basis dual to $\mathcal{N}$. If v is a function for which all $N_i \in \mathcal{N}, i = 1, 2, ..., k$, are defined, then we define the local interpolant by*

$$I_h^K v := \sum_{i=1}^k N_i(v)\phi_i.$$

In the above definition 1.3, if the function v is continuous upto boundary of K and $N_i(v) = v(x_i)$ where x_i are nodes of K, then I_h^K is called the *Lagrange interpolant*. The Lagrange interpolant I_h^K is said to be *stable* (bounded) if there exists a positive constant C such that

$$\left\|I_h^K u\right\| \leq C\|u\| \quad \forall u \in H^1(K) \cap C(\overline{K}).$$

It may be noted that point values for a function in $H^1(T)$ are not defined for $T \subset \mathbb{R}^2$. For any function in $H^1(T) \cap C(\overline{T})$, a dense subset of $H^1(T)$ [4], point values are well defined and we can define its Lagrange interpola-

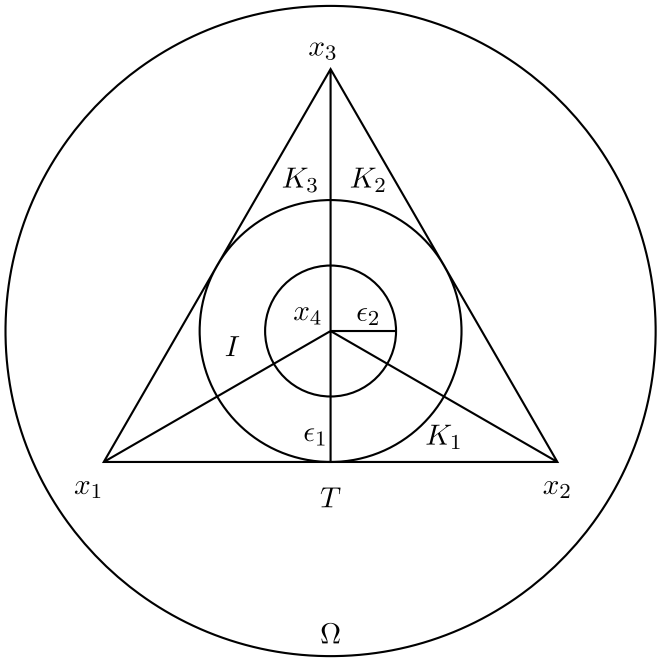

FIGURE 1.1

tion. We prove in this article that for a triangular domain T the Lagrange interpolation is not stable under the $H^1(T)$ norm for $H^1(T) \cap C(\overline{T})$ functions which means that there exists no constant C such that

$$\|I_h u\|_{1,T} \leq C \|u\|_{1,T} \quad \forall u \in H^1(T) \cap C(\overline{T}).$$

We need one more definition.

Definition 1.4. [1, page no. 79] *Let Ω be a domain with a subdivision K_i. Assume that each part of the subdivision forms a finite element. For $u \in C(\overline{\Omega})$ the global interpolant is defined by*

$$I_h u\big|_{K_i} = I_h^{K_i} u.$$

From here on we consider the domain Ω to be the ball $B(0; \frac{1}{2}) \subset \mathbb{R}^2$. We choose an equilateral triangle T with edge length h in such a way that $T + I \subset \Omega$, where I is the incircle to the triangle T. Let the vertices of T be x_1, x_2, x_3 with centroid $x_4 = (0, 0)$. Let K_1, K_2, K_3 be the division of T as shown in the FIGURE 1.1 above.

2. MAIN RESULT

Theorem 2.1. *There exists no constant $C > 0$ such that*

$$\|I_h v\|_{1,T} \leq C \|v\|_{1,T} \quad \forall v \in H^1(T) \cap C(\overline{T}).$$

Proof. We begin with the function $u(x) = \log\log\frac{2}{|x|} \in H^1(\Omega)$ [3, page no. 78]. Since $u \in H^1(\Omega)$ and Ω is a C^∞ domain, we can define an extension operator P such that Pu extends as a H^1 function to whole of $\mathbb{R}^2$.

We define a sequence of mollifiers ρ_{ϵ_m}, where $\epsilon_m = \frac{h}{2^m\sqrt{3}}$. We know that the sequence $u_m = \rho_{\epsilon_m} * Pu\big|_\Omega \to u$ in $H^1(\Omega)$, as $m \to \infty$ by the approximation property of $H^1(\Omega)$ functions [3, page no. 58]. Clearly the functions u_m are $H^1(\Omega) \cap C(\overline{\Omega})$ in its own right.

Now we compute the nodal value of u_m at $x_i,\ i = 1, 2, 3, 4$.

$$\begin{aligned} u_m(x_i) &= \rho_{\epsilon_m} * Pu\big|_\Omega(x_i) \\ &= \int_{|y|<\epsilon_m} \rho_{\epsilon_m}(y)\, u(x_i - y) dy \quad (\text{since } Pu = u \text{ on } \Omega) \\ &= \int_{|y|<\epsilon_m} k\epsilon_m^{-2} \exp\frac{-{\epsilon_m}^2}{{\epsilon_m}^2 - |y|^2} \log\log\frac{2}{|x_i - y|} dy. \end{aligned}$$

For $i = 1, 2, 3,$ we have $\log\log\frac{2}{|x_i-y|} \geq \log\log\frac{2}{z_m}$, where $z_m = \sup_{y\in B(0;\epsilon_m)} |x_i - y|$. Hence,

$$u_m(x_i) \geq \log\log\frac{2}{z_m} \int_{|y|<\epsilon_m} k\epsilon_m^{-2} \exp\frac{-{\epsilon_m}^2}{{\epsilon_m}^2 - |y|^2} dy = \log\log\frac{2}{z_m}. \qquad (2.1)$$

For i=4 we have

$$u_m(x_4) = \int_{|y|<\epsilon_m} k\epsilon_m^{-2} \exp\frac{-{\epsilon_m}^2}{{\epsilon_m}^2 - |y|^2} \log\log\frac{2}{|y|} dy \geq \log\log\frac{2}{\epsilon_m}. \qquad (2.2)$$

Here we have used the facts that $\log\log\frac{2}{|y|} \geq \log\log\frac{2}{\epsilon_m}$ for $|y| < \epsilon_m$ and $\int_{|y|<\epsilon_m} k\epsilon_m^{-2} \exp\frac{-{\epsilon_m}^2}{{\epsilon_m}^2-|y|^2} = \int \rho_{\epsilon_m} = 1$.

Now we compute the Lagrange interpolation of u_m over the triangle T. The triangle T is divided into three sub triangles K_1, K_2, K_3 as shown in figure 2.1. Hence $I_h u_m = \sum_{j=1}^3 I_h^{K_j} u_m$, where $I_h^{K_j}$ denotes the Lagrange interpolation on the triangle K_j. We also define the local basis elements $\phi_{i,j}$ which are local basis functions on the triangle K_j corresponding to the nodes $x_i(\ i = 1, 2, 3, 4)$

$$I_h^{K_1} u_m = u_m(x_1).\phi_{1,1} + u_m(x_2).\phi_{2,1} + u_m(x_4).\phi_{4,1}.$$

From (2.1) and (2.2), we have $I_h^{K_1} u_m \geq \log\log\frac{2}{z_m}.(\phi_{1,1}+\phi_{2,1}) + \log\log\frac{2}{\epsilon_m}.\phi_{4,1}$.
The construction of basis functions associated with the finite element K_j is such that $\phi_{i,j}(x_k) = \delta_{ik}$.
Hence for each finite element K_j, $\sum_{i=1}^3 \phi_{i,j} = 1$. Using this we get

$$I_h^{K_1} u_m \geq \log\log\frac{2}{z_m}.(1 - \phi_{4,1}) + \log\log\frac{2}{\epsilon_m} \cdot \phi_{4,1} = C_m.\phi_{4,1} + A_m,$$

where $C_m = \log\Big(\ \log\frac{2}{\epsilon_m}/\log\frac{2}{z_m}\ \Big)$ and $A_m = \log\log(2/z_m)$. Now,

$$\left\|I_h^{K_1}u_m\right\|^2_{L^2(K_1)} \geq \Big(C_m^2\int_{K_1}\phi_{4,1}^2 + \int_{K_1}A_m^2 + 2A_mC_m\int_{K_1}\phi_{4,1}\Big).$$

Using quadrature formula [1, page 92], we have $\int_{K_1}\phi_{4,1}^2{=}\frac{|K_1|}{6}$ and $\int_{K_1}\phi_{4,1}{=}\frac{|K_1|}{3}$, where $|K_1|$ is the area of the triangle K_1. Using these, we get

$$\left\|I_h^{K_1}u_m\right\|^2_{L^2(K_1)} \geq \alpha_0\,C_m^2, \quad \text{where } \alpha_0 = |K_1|/6.$$

For the triangles K_2 and K_3 we have similar results. Hence,

$$\|I_hu_m\|^2_{1,T} = \sum\nolimits_{i=1}^{3}\left\|I_h^{K_i}u_m\right\|^2_{L^2(K_i)} + \|\nabla_h(I_hu_m)\|^2_{L^2(T)} \geq \alpha C_m^2, \tag{2.3}$$

where $\alpha = |T|/6$. Here ∇_h denotes the piece-wise gradient over each triangle K_i. Since $u_m \to u$ in $H^1(T)$, we can have some $\gamma_0 > 0$ such that

$$\|u_m\|_{1,T} \leq \gamma_0\,\|u\|_{1,T} = \gamma. \tag{2.4}$$

Now using (2.3) and (2.4), we have

$$\Big(\ \|I_hu_m\|_{1,T}\,/\,\|u_m\|_{1,T}\ \Big) \geq (\sqrt{\alpha}/\gamma)C_m.$$

Now as $m\to\infty$, $C_m\to\infty$ which says that there is no C such that $\|I_hv\|_1 \leq C\,\|v\|_1 \quad \forall v \in H^1(T)\cap C(\overline{T})$. Hence Lagrange interpolation is not stable on the space $H^1(T)\cap C(\overline{T})$. □

3. Conclusions

The result above is proved on a triangle. This can be extended to any arbitrary polygonal domain $P\subset\Omega$. To do so one just needs to have the triangle T(in the proof above) as a part of the triangulation of the polygonal domain. Then if we compute the $H^1(P)$ norm of the function I_hu_m(as defined in the proof) we get

$$\|I_hu_m\|^2_{1,P} = \|I_hu_m\|^2_{1,P\setminus T} + \|I_hu_m\|^2_{1,T}\,.$$

The second part on the right hand side is already greater than some constant multiple of C_m^2. Hence

$$\Big(\|I_hu_m\|_{1,P}\,/\,\|u_m\|_{1,P}\Big) \geq C\ C_m.$$

This proves the result for polygonal domains.

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Md Yasir Feroz Khan
Department of Mathematics, Hansraj College
University of Delhi, Delhi-110 007, India
E-mail: ykhan9501@gmail.com

Rahul Biswas
Department of Mathematics, Indian Institute of Science
Bangalore-560012, Karnataka, India
E-mail: rahulbiswas@iisc.ac.in

and
Ramesh Chandra Sau
Department of Mathematics, Indian Institute of Science
Bangalore-560012, Karnataka, India
E-mail: rameshsau@iisc.ac.in

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ON A CONJECTURE OF GEORGE BECK. II

SHANE CHERN
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Abstract. In this paper, we give a combinatorial proof of a generating function identity concerning the sum of the smallest parts in the distinct partitions of n.

1. Introduction

As usual, a *partition* of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n.

In my previous paper [5], I proved the following conjecture due to George Beck (cf. A034296 in [7]).

Theorem 1.1. *The number of gap-free partitions (i.e. partitions with the difference between each pair of consecutive parts being at most* 1*) of n is also the sum of the smallest parts in the distinct partitions (i.e. partitions with distinct parts) of n with an odd number of parts.*

Let $\mathscr{DO}$ denote the set of distinct partitions with an odd number of parts. The main idea in [5] is to use the differentiation technique to study the generating function of $\mathrm{sspt}_{\mathscr{DO}}(n)$, the sum of the smallest parts in the partitions of n in $\mathscr{DO}$. More precisely, I showed that

$$\sum\nolimits_{n\geq 1} \mathrm{sspt}_{\mathscr{DO}}(n)q^n = \sum\nolimits_{t\geq 1}(q^t/(1-q^t))(-q)_{t-1}, \tag{1.1}$$

where and in what follows, we use the standard q-series notation

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \qquad \text{and} \qquad (a)_\infty = (a;q)_\infty := \prod_{k\geq 0}(1-aq^k).$$

In a subsequent paper [8], Yang also provided a combinatorial proof of Beck's conjecture.

Let $\mathscr{D}$ be the set of distinct partitions. Here we will not include the empty partition unless otherwise specified.

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For any partition π, we denote by $|\pi|$ the sum of the parts of π, by $\Lambda(\pi)$ the largest part of π, by $\sigma(\pi)$ the smallest part of π, and by $\sharp(\pi)$ the number of parts of π.

The following general result is almost shown in [5] (in which the cases $z = \pm 1$ are proved).

Theorem 1.2. *It holds that*

$$\sum\nolimits_{\pi\in\mathscr{D}} \sigma(\pi) z^{\sharp(\pi)} q^{|\pi|} = \sum\nolimits_{t\geq 1} (q^t/(1-q^t))\big((-z)_t - 1\big). \tag{1.2}$$

The proof of this generating function identity can still be deduced by the differentiation technique. In fact, it is essentially the same as the proof of (2.3) in [5]:

$$(-q)_\infty \sum\nolimits_{m\geq 1} (mq^m/(-q)_m) = \sum\nolimits_{t\geq 1} (q^t/(1-q^t))\big((-1)_t - 1)\big)$$

together with the observation on p. 648 of [5]:

$$\sum\nolimits_{\pi\in\mathscr{D}} \sigma(\pi) q^{|\pi|} = \sum\nolimits_{m\geq 1} mq^m (-q^{m+1})_\infty = (-q)_\infty \sum\nolimits_{m\geq 1} (mq^m/(-q)_m).$$

In this paper, we will focus on the combinatorial viewpoint.

2. A combinatorial approach

Our starting point is the following double counting argument, which appears to be able to be adapted to many types of partition sets.

For a nonnegative integer t, we define

$$\mathscr{D}_t := \big\{\pi \in \mathscr{D} \,:\, \Lambda(\pi) \geq t+1 \text{ and } \Lambda(\pi) - \sigma(\pi) \leq t\big\}.$$

Now given any $\pi \in \mathscr{D}$, if $\pi \in \mathscr{D}_t$, then $\Lambda(\pi) - \sigma(\pi) \leq t \leq \Lambda(\pi) - 1$ by the definition. Hence, π is exactly contained in the following $\sigma(\pi)$ partition sets: $\mathscr{D}_{\Lambda(\pi)-1}$, $\mathscr{D}_{\Lambda(\pi)-2}$, $\ldots$, $\mathscr{D}_{\Lambda(\pi)-\sigma(\pi)}$. We therefore have

Theorem 2.1. *It holds that*

$$\sum\nolimits_{\pi\in\mathscr{D}} \sigma(\pi) z^{\sharp(\pi)} q^{|\pi|} = \sum\nolimits_{t\geq 0} \sum\nolimits_{\pi\in\mathscr{D}_t} z^{\sharp(\pi)} q^{|\pi|}. \tag{2.1}$$

One immediately sees that the remaining task is to study the generating function for $\mathscr{D}_t$ with $t \geq 0$.

We remark that, for certain partition sets, if we only require the difference between the largest and smallest parts to be bounded by t, then the generating functions are studied in a series of papers [1–4, 6]. In particular, in my joint work with Yee [6], we provided a combinatorial approach that can be easily adapted to obtain the generating function for $\mathscr{D}_t$.

For convenience, we now consider the generating function for $\mathscr{D}_{t-1}$ with $t \geq 1$.

Let $\mathscr{B}_t$ be the set of partition pairs (μ, ν) where μ is nonempty and its parts all have size t, and ν is a nonempty distinct partition with 0 being allowed as a part and the largest part being at most $t-1$. For example,

$$\big((5,5,5,5,5),\ (4,2,1,0)\big) \in \mathscr{B}_5.$$

Furthermore, we use $|(\mu,\nu)|$ to denote $|\mu| + |\nu|$.

For $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ in $\mathscr{D}_{t-1}$, we put $s = \lfloor \pi_\ell / t \rfloor$ with $\lfloor x \rfloor$ being the conventional floor function. We also let k be the positive integer such that $\pi_k \geq (s+1)t$ and $\pi_{k+1} < (s+1)t$. If there is no such k, then we let $k = 0$.

Now we construct a map $\phi_t : \mathscr{D}_{t-1} \to \mathscr{B}_t$ defined by

$$\phi_t \colon (\pi_1, \pi_2, \ldots, \pi_\ell)$$
$$\mapsto \big((\underbrace{t, t, t, \ldots, t}_{\substack{s(\ell-k)+(s+1)k \\ \text{times}}}), (\pi_{k+1} - st, \ldots, \pi_\ell - st, \pi_1 - (s+1)t, \ldots, \pi_k - (s+1)t)\big).$$

Note that the second subpartition can be treated as $(\pi_1, \pi_2, \ldots, \pi_\ell)$ reduced modulo t, cyclically permuted such that they are weakly decreasing.

Similar to Theorem 2.1 of [6], we have

Lemma 2.2. *ϕ_t is a weight preserving map from $\mathscr{D}_{t-1}$ to $\mathscr{B}_t$. Furthermore, the number of parts is preserved by the second subpartition of the image.*

Proof. Let $(\mu, \nu) = \phi_t(\pi)$. We first show that μ is nonempty. Since $\pi \in \mathscr{D}_{t-1}$, we have $\pi_1 \geq (t-1) + 1 = t$. Hence we take out at least one t from π_1 to form μ, which implies that μ is not empty.

On the other hand, we know that π is a distinct partition. Since $\pi_1 - \pi_\ell \leq t - 1 < t$, $s = \lfloor \pi_\ell / t \rfloor$, and $\pi_k \geq (s+1)t > \pi_{k+1}$, we have

$$t > \pi_{k+1} - st > \cdots > \pi_\ell - st > \pi_1 - (s+1)t > \cdots > \pi_k - (s+1)t.$$

Note that $\pi_k - (s+1)t$ could be 0 since π_k could be $(s+1)t$. Hence ν satisfies the conditions. It follows that $(\mu, \nu) \in \mathscr{B}_t$.

At last, it is obvious from the definition of ϕ_t that $|\phi_t(\pi)| = |\pi|$ and $\sharp(\nu) = \sharp(\pi)$. □

The rest of the argument is different to that in [6]. We shall show

Lemma 2.3. *ϕ_t is invertible.*

Proof. Let $(\mu, \nu) \in \mathscr{B}_t$. Let the number of t in μ be $r \geq 1$ and let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$. Now we write $r = m\ell + r^*$ with $m \geq 0$ and $0 \leq r^* \leq \ell - 1$ being integers. We construct the inverse $\phi_t^{-1} : \mathscr{B}_t \to \mathscr{D}_{t-1}$ as follows.

$$\phi_t^{-1} \colon (\mu, \nu) \mapsto (\nu_{\ell - r^* + 1} + (m+1)t, \ldots, \nu_\ell + (m+1)t, \nu_1 + mt, \ldots, \nu_{\ell - r^*} + mt).$$

We now show that the image is in $\mathscr{D}_{t-1}$. Recall that $0 \le \nu_\ell < \cdots < \nu_1 \le t-1$. If $r^* \neq 0$, since $\nu_\ell + t > \nu_1$, we have

$$\nu_{\ell-r^*+1} + (m+1)t > \cdots > \nu_\ell + (m+1)t > \nu_1 + mt > \cdots > \nu_{\ell-r^*} + mt.$$

Notice that $\nu_{\ell-r^*+1} + (m+1)t \ge t$. We further notice that $\nu_{\ell-r^*}$ is not the smallest part of ν, and hence $\nu_{\ell-r^*} > 0$. At last, we have $(\nu_{\ell-r^*+1} + (m+1)t) - (\nu_{\ell-r^*} + mt) = t - (\nu_{\ell-r^*} - \nu_{\ell-r^*+1}) \le t-1$. Hence in this case the image is in $\mathscr{D}_{t-1}$.

If $r^* = 0$, then $m \ge 1$ since $r \ge 1$. We have $\nu_1 + mt > \cdots > \nu_\ell + mt > 0$ and $\nu_1 + mt \ge t$. We also have $(\nu_1 + mt) - (\nu_\ell + mt) = \nu_1 - \nu_\ell \le t-1$. Hence the image is also in $\mathscr{D}_{t-1}$.

From the definition of ϕ_t and ϕ_t^{-1}, it is apparent that $\phi_t^{-1}(\phi_t(\pi)) = \pi$. Hence ϕ_t is invertible. □

Example 2.1. For the partition sets $\mathscr{D}_4$ and $\mathscr{B}_5$, we have

$$(9,7,6,5) \underset{\phi_5^{-1}}{\overset{\phi_5}{\rightleftharpoons}} \big((5,5,5,5),\ (4,2,1,0)\big) \qquad \text{and}$$

$$(10,9,7,6) \underset{\phi_5^{-1}}{\overset{\phi_5}{\rightleftharpoons}} \big((5,5,5,5,5),\ (4,2,1,0)\big).$$

It follows from Lemmas 2.2 and 2.3 that ϕ_t is a bijection from $\mathscr{D}_{t-1}$ to $\mathscr{B}_t$. Hence, for $t \ge 1$,

$$\sum\nolimits_{\pi\in\mathscr{D}_{t-1}} z^{\sharp(\pi)} q^{|\pi|} = \sum\nolimits_{(\mu,\nu)\in\mathscr{B}_t} z^{\sharp(\nu)} q^{|\mu|+|\nu|}. \tag{2.2}$$

The generating function for $\mathscr{B}_t$ is easy to get:

$$\sum\nolimits_{(\mu,\nu)\in\mathscr{B}_t} z^{\sharp(\nu)} q^{|\mu|+|\nu|} = (q^t/(1-q^t))\big((-z)_t - 1\big), \tag{2.3}$$

where $q^t/(1-q^t)$ comes from the first subpartition whereas $(-z)_t - 1$ comes from the second subpartition. Consequently, we have

Theorem 2.4. *For $t \ge 1$, it holds that*

$$\sum\nolimits_{\pi\in\mathscr{D}_{t-1}} z^{\sharp(\pi)} q^{|\pi|} = (q^t/(1-q^t))\big((-z)_t - 1\big). \tag{2.4}$$

Together with (2.1), we have

$$\sum\nolimits_{\pi\in\mathscr{D}} \sigma(\pi) z^{\sharp(\pi)} q^{|\pi|} = \sum\nolimits_{t\ge1} (q^t/(1-q^t))\big((-z)_t - 1\big),$$

which completes the proof of Theorem 1.2.

3. Closing remarks

As I showed in [5], (1.1) follows since

$$\sum\nolimits_{n\geq 1} \mathrm{sspt}_{\mathscr{DO}}(n)q^n = \frac{1}{2}\sum\nolimits_{\pi\in\mathscr{D}} \sigma(\pi)\Big(1^{\sharp(\pi)} - (-1)^{\sharp(\pi)}\Big)q^{|\pi|}.$$

Note that the $z = 1$ and $z = -1$ cases of (1.2) respectively correspond to (2.3) and (2.2) in [5].

I was also pointed out by Dazhao Tang that another conjecture of Beck, which is proposed in [7, A237665], can be proved in the same way.

Conjecture 3.1 (Beck)**.** *The number of gap-free partitions of n with at least two different parts is also the sum of the smallest parts in the distinct partitions of n with an even number of parts.*

Theorem 3.2. *Conjecture 3.1 is true.*

Proof. Let $\mathrm{sspt}_{\mathscr{DE}}(n)$ denote the sum of the smallest parts in the distinct partitions of n with an even number of parts. We have

$$\begin{aligned}\sum_{n\geq 1} \mathrm{sspt}_{\mathscr{DE}}(n)q^n &= \frac{1}{2}\sum_{\pi\in\mathscr{D}} \sigma(\pi)\Big(1^{\sharp(\pi)} + (-1)^{\sharp(\pi)}\Big)q^{|\pi|}\\ &= \sum_{t\geq 1}(q^t/(1-q^t))\big((-q)_{t-1} - 1\big).\end{aligned}$$

We next observe that the conjugates of gap-free partitions with at least two different parts are partitions with at least two different parts where only the largest part may repeat.

Let $\mathrm{gf}_2(n)$ count the number of gap-free partitions of n with at least two different parts. We have

$$\begin{aligned}\sum_{n\geq 1} \mathrm{gf}_2(n)q^n &= \sum_{t\geq 1}(q^t/(1-q^t))(-q)_{t-1} - \sum_{t\geq 1}(q^t/(1-q^t))\\ &= \sum_{t\geq 1}(q^t/(1-q^t))\big((-q)_{t-1} - 1\big).\end{aligned}$$

Here $\sum_{t\geq 1}(q^t/(1-q^t))(-q)_{t-1}$ is the generating function of partitions where only the largest part may repeat, and $\sum_{t\geq 1}(q^t/(1-q^t))$ is the generating function of partitions with only one different part. We conclude that $\mathrm{gf}_2(n) = \mathrm{sspt}_{\mathscr{DE}}(n)$. □

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Shane Chern
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802, USA
E-mail: shanechern@psu.edu

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A SUFFICIENT CONDITION FOR A 2-DIMENSIONAL ORBIFOLD TO BE GOOD

S. K. ROUSHON

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Abstract. We prove that a connected 2-dimensional orbifold with finitely generated and infinite orbifold fundamental group is good. We also describe all the good 2-dimensional orbifolds with finite orbifold fundamental groups.

1. Introduction

We start with a short introduction to orbifolds.

The concept of orbifold was first introduced in [2], and was called 'V-manifold'. Later, it was revived in [4], with the new name *orbifold*, and *orbifold fundamental group* of a connected orbifold was defined.

Definition 1.1. An *orbifold* is a second countable and Hausdorff topological space M, which at every point looks like the quotient space of $\mathbb{R}^n$, for some n, by some finite group action. More precisely, there is an open covering $\{U_i\}_{i\in I}$ of M together with a collection $\mathcal{O}_M = \{(\tilde{U}_i, p_i, G_i)\}_{i\in I}$ (called *orbifold charts*), where for each $i \in I$, $\tilde{U}_i$ is an open set in $\mathbb{R}^n$, for some n, G_i is a finite group acting on $\tilde{U}_i$ and $p_i : \tilde{U}_i \to U_i$ is the quotient map, via an identification of $\tilde{U}_i/G_i$ with U_i by some homeomorphism.

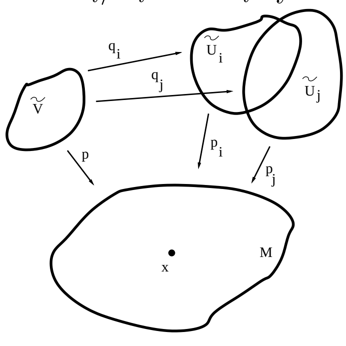

Figure 1: Orbifold charts

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Furthermore, the following compatibility condition is satisfied. Given $x \in U_i \cap U_j$, for $i, j \in I$ there is a neighborhood $V \subset U_i \cap U_j$ of x and a chart $(\tilde{V}, p, G) \in \mathcal{O}_M$ together with embeddings $q_i : \tilde{V} \to \tilde{U}_i$ and $q_j : \tilde{V} \to \tilde{U}_j$ such that, $p_i \circ q_i = p$ and $p_j \circ q_j = p$ are satisfied.

The pair $(M, \mathcal{O}_M)$ is called an *orbifold* and M is called its *underlying space*. The finite groups $\{G_i\}_{i \in I}$ are called the *local groups*. The union of the images (under p_i) of the fixed point sets of the action of G_i on $\tilde{U}_i$, when G_i acts non-trivially, is called the *singular set*. Points outside the singular set are called *regular points*. In case the local group is cyclic (of order k) acting by rotation about the origin on the Euclidean space, the image of the origin is called a *cone point* of *order* k. If the local group at some point acts trivially then it is a regular point. A regular point is also called a *manifold point*.

Clearly, if all the local groups are either trivial or acts trivially then the orbifold is a manifold. The easiest example of an orbifold is the quotient of a manifold by a finite group. Also see Example 1.1 below.

In the rest of the paper we will not use the full notation of an orbifold as defined above, unless it is explicitly needed. In general, we will use the same letter M both for the orbifold and its underlying space, which will be clear from the context.

As in the case of a manifold, *dimension* of a connected orbifold is defined. One can also define an *orbifold with boundary* in the same way we define a manifold with boundary. In the definition we have to replace $\mathbb{R}^n$ by the upper half space $\mathbb{R}^n_+$.

In this paper we consider orbifolds with boundary (may be empty).

A boundary component of the underlying space of an orbifold has two types, one which we call *manifold boundary* and the other *orbifold boundary*. These are respectively defined as points on the boundary (of the underlying space) with local group acting trivially or non-trivially.

The notion of *orbifold covering space* was defined in [4]. We refer the reader to this source for the basic materials and examples. But, we recall below the definition and properties we need.

Definition 1.2. A connected orbifold $(\hat{M}, \mathcal{O}_{\hat{M}})$ is called an *orbifold covering space* of a connected orbifold $(M, \mathcal{O}_M)$ if there is a surjective map $f : \hat{M} \to M$, called an *orbifold covering map*, such that given $x \in M$ there is an orbifold chart $(\tilde{U}, p, G)$ with $x \in U$ and the following is satisfied.

- $f^{-1}(U) = \cup_{i \in I} V_i$, where for each $i \in I$, $(\tilde{V}_i, q_i, H_i) \in \mathcal{O}_{\hat{M}}$, V_i is a component of $f^{-1}(U)$, there is an injective homomorphism $\rho_i : H_i \to G$ and V_i is homeomorphic to $\tilde{U}/(\rho_i(H_i))$ making the two squares in the following diagram commutative.

$$\begin{array}{ccccc} \tilde{U}/(\rho_i(H_i)) & \longrightarrow & V_i & \xleftarrow[q_i]{} & \tilde{V}_i \\ \downarrow & & \downarrow{\scriptstyle f|_{V_i}} & & \downarrow{\scriptstyle \tilde{f}} \\ \tilde{U}/G & \longrightarrow & U & \xleftarrow[p]{} & \tilde{U}, \end{array}$$

where $\tilde{f}$ is ρ_i-equivariant.

Here note that the map on the underlying spaces of an orbifold covering map need not be a covering map in the ordinary sense.

Example 1.1. Given a group Γ and a properly discontinuous action of Γ on a manifold M, the quotient space M/Γ has an orbifold structure and the quotient map $M \to M/\Gamma$ is an orbifold covering map. See [4, Proposition 5.2.6]. Furthermore, if H is a subgroup of Γ, then the map $M/H \to M/\Gamma$ is an orbifold covering map. Therefore, the quotient map of a finite group action on an orbifold is always an orbifold covering map.

In general, an orbifold need not have a manifold as an orbifold covering space.

Definition 1.3 ([4])**.** If an orbifold has an orbifold covering space which is a manifold, then the orbifold is called *good* or *developable.*

In the above example M/Γ is a good orbifold.

Remark 1.1. One can show that a good compact 2-dimensional orbifold has a finite sheeted orbifold covering space, which is a manifold [3, Theorem 2.5]. In the case of closed (that is, compact and the underlying space has empty boundary) 2-dimensional orbifolds, only the sphere with one cone point and the sphere with two cone points of different orders are not good orbifolds. See the figure below. Also, see [4, Theorem 5.5.3].

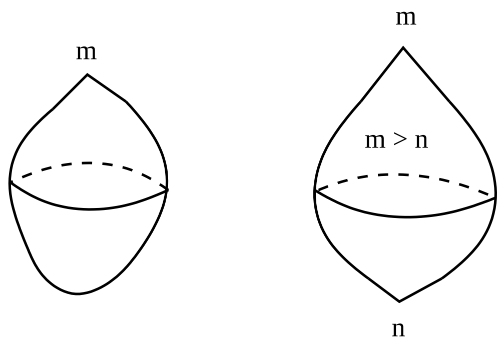

Figure 2: Bad orbifolds

Definition 1.4. Let $f : \hat{M} \to M$ be an orbifold covering map of connected orbifolds. Let $x \in M$ be a regular point and $\hat{x} \in \hat{M}$ with $f(\hat{x}) = x$. Then $\hat{M}$ is called the *universal orbifold cover* of M if given any other connected orbifold covering $g : N \to M$ with $g(y) = x$, for $y \in N$, there is a unique $h : \hat{M} \to N$ so that $h(\hat{x}) = y$, h is an orbifold covering map and $h \circ g = f$.

Definition 1.5. The universal orbifold cover always exists for a connected orbifold M [4, Proposition 5.3.3] and its group of Deck transformations is defined as the *orbifold fundamental group* of M. It is denoted by $\pi_1^{orb}(M)$.

In Remark 1.1 we have already seen a classification of closed good 2-dimensional orbifolds. But in the literature I did not find any condition which will say when an orbifold (compact or not) is good.

In this article we use the above classification of good closed orbifolds, to prove the following theorem which gives a sufficient condition for an arbitrary 2-dimensional orbifold to be good.

Theorem 1.1. *Let S be a connected 2-dimensional orbifold with finitely generated and infinite orbifold fundamental group. Then, S is good.*

Remark 1.2. There are 2-dimensional good orbifolds with finite orbifold fundamental groups. The simplest example is the 2-dimensional disc quotiented out by the action of a finite cyclic group acting by rotation around the center. See Remark 2.1 for a classification of such orbifolds.

2. Proof of Theorem 1.1

For the proof of Theorem 1.1 we need the following two lemmas, which follow from standard techniques and covering space theory of orbifolds.

Lemma 2.1. *Let M be a connected orbifold and $f : \tilde{M} \to M$ be a connected covering space of the underlying space of M. Then, $\tilde{M}$ has an orbifold structure induced from M by f, so that f is an orbifold covering map.*

Proof. Choose an open covering of $\tilde{M}$ which consists of inverse images under f of evenly covered open subsets of M which are associated to orbifold charts of M, and then pull back the finite group actions using f. That gives the required orbifold structure on $\tilde{M}$. □

Lemma 2.2. *Let $q : \tilde{M} \to M$ be a finite sheeted orbifold covering map between two connected orbifolds. Then, the induced map $q_* : \pi_1^{orb}(\tilde{M}) \to \pi_1^{orb}(M)$ is an injection, and the image $q_*(\pi_1^{orb}(\tilde{M}))$ is a finite index subgroup of $\pi_1^{orb}(M)$.*

Proof. See [1, Corollary 2.4.5]. □

Proof of Theorem 1.1. Let S be a 2-dimensional orbifold as in the statement. Note that, the underlying space of a 2-dimensional orbifold is a 2-dimensional (smooth) manifold. (This follows easily from the discussion below on the possible types of singular points on a 2-dimensional orbifold and the fact that manifolds of dimension ≤ 3 has unique smooth structure. Also see [3, p. 422, last paragraph]). Therefore, after going to a two sheeted cover of the underlying space of S and using Lemma 2.1, we can assume that the underlying space of S is an orientable 2-dimensional manifold. (In such a situation, it is standard to call the orbifold *orientable*).

For the remaining part of the proof we refer the reader to the discussion on pages 422-424 of [3].

It is known that, S has three types of singularities: cone points, reflector lines and corner reflectors [3, p. 422].

Note that, other than the cone points the remaining singular set contributes to the orbifold boundary components of the orbifold. We take double of S along the orbifold boundary, then we get an orientable orbifold $\tilde{S}$ which is a 2-sheeted orbifold covering of S and $\tilde{S}$ has only cone singularities [3, p. 423]. Furthermore, since $\pi_1^{orb}(\tilde{S})$ is a finite index subgroup of $\pi_1^{orb}(S)$ (Lemma 2.2), $\pi_1^{orb}(\tilde{S})$ is again infinite and finitely generated.

So, to prove the Theorem we only have to show that $\tilde{S}$ is a good orbifold.

Note that, $\tilde{S}$ has no orbifold boundary component. If there is any manifold boundary of $\tilde{S}$, then we take the double of $\tilde{S}$ along these boundary components and denote it by $D\tilde{S}$. Therefore, $D\tilde{S}$ is an orientable orbifold with no boundary in its underlying space and only has cone singularities. Also, if $D\tilde{S}$ is good then so is $\tilde{S}$.

We now prove that $D\tilde{S}$ is good.

First, we consider the case when $D\tilde{S}$ is noncompact. The cone points form a discrete subset of $D\tilde{S}$ and hence we can write the underlying space of $D\tilde{S}$ as an increasing union of compact orientable sub-manifolds S_i, $i \in \mathbb{N}$ with no singular points on the boundary of S_i. (This can be done by taking a proper smooth map from the underlying space of $D\tilde{S}$ to $\mathbb{R}$). Let DS_i be the double of S_i. DS_i is an orbifold with twice as many cone points as S_i. Suppose DS_i is of genus g_i and has k_i cone points with orders $p_1, p_2, \ldots, p_{k_i}$. Then, DS_i is a closed orientable orbifold with only cone singularities, and by [3, p. 424] the orbifold fundamental group of DS_i is given by the following presentation.

$$\pi_1^{orb}(DS_i) \simeq \langle a_1, b_1, \ldots, a_{g_i}, b_{g_i}, x_1, \ldots, x_{k_i} \mid x_j^{p_j} = 1,\ j = 1, 2, \ldots, k_i,$$
$$\Pi_{j=1}^{g_i}[a_j, b_j]x_1 x_2 \cdots x_{k_i} = 1\rangle.$$

Hence, the abelianization of $\pi_1^{orb}(DS_i)$ is isomorphic to $\mathbb{Z}^{2g_i} \oplus K_{k_i-1}$. Where, K_{k_i-1} is a finite abelian group with $k_i - 1$ number of generators. Furthermore, $g_1 \leq g_2 \leq \cdots$ and $k_1 \leq k_2 \leq \cdots$.

This shows that, since $\pi_1^{orb}(D\tilde{S})$ is finitely generated (as $\pi_1^{orb}(\tilde{S})$ is finitely generated), there is an i_0 such that $g_j = g_l$ and $k_j = k_l$ for all $j, l \geq i_0$.

Therefore, $D\tilde{S}$ has finitely many cone points, all contained in S_{i_0}, and outside S_{i_0}, $D\tilde{S}$ is a finite union of components, each homeomorphic to the infinite cylinder $\mathbb{S}^1 \times (0, \infty)$. Next, we cut the infinite ends of $D\tilde{S}$ at some finite stage and denote the resulting compact orbifold again by S_{i_0}.

Note that, here $\pi_1^{orb}(S_{i_0})$ is a subgroup of $\pi_1^{orb}(DS_{i_0})$ and hence $\pi_1^{orb}(DS_{i_0})$ is infinite. Hence, from the classification of closed 2-dimensional orbifolds using geometry (see [4, Theorem 5.5.3]), it follows that DS_{i_0} is a good orbifold. Since, DS_{i_0} has even number of cone points, and in the case of two cone points they have the same orders. See Figure 2. Clearly, then S_{i_0}, and hence $D\tilde{S}$ (which is homeomorphic to the interior of S_{i_0}) has an orbifold covering space, which is a manifold.

Next, if $D\tilde{S}$ is compact then the same argument as in the above paragraph shows that it is good. This completes the proof of the Theorem. □

Remark 2.1. We end the paper with a remark on 2-dimensional orbifolds with finite orbifold fundamental group. As in the proof of the Theorem, after going to a finite sheeted covering and then doubling along manifold boundary components, we can assume that the orbifold (say, S) is orientable and has finitely many cone singularities. Ignoring the cone points we have a surjective homomorphism $\pi_1^{orb}(S) \to \pi_1(S)$. Hence, the fundamental group of the underlying space of S is finite and therefore, the underlying space is $\mathbb{S}^2$ or $\mathbb{R}^2$. It is easy to see that in the last case S is good and in the $\mathbb{S}^2$ case we already have a classification (Remark 1.1).

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S. K. Roushon
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai - 400005, India
E-mail: *roushon@math.tifr.res.in*
URL http://www.math.tifr.res.in/~roushon/

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ON THE LAW OF THE ITERATED LOGARITHM FOR THE ITERATED BROWNIAN SEQUENCE

R. P. PAKSHIRAJAN
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Abstract. The robustness of the choice of the norming constants in the Hartman-Wintner law of the iterated logarithm was brought out in [3]. We present here a parallel result for the iterated Brownian sequence.

1. Introduction

All random variables (*rv*s) considered below are assumed to be defined on a common probability space $(\Omega,\ \mathscr{S},\ P)$, where Ω is an abstract set of points ω, $\mathscr{S}$ a σ-field of subsets of Ω and P a probability measure on $\mathscr{S}$.

Let $Y(t),\ t \in [0,\ \infty)$ be a standard Brownian Motion Process (BMP) defined on Ω. By that we understand that the family $Y(t)$ of *rv*s has the following properties: $Y(0,\omega) = 0$ for all $\omega \in \Omega$ and for every $0 < t_1 < t_2 < \dots < t_p$, the vector $(Y(t_1),\ Y(t_2),\ \dots,\ Y(t_p))$ has the multivariate normal distribution, each component having zero mean and cov$(Y(t_j),\ Y(t_{j+k}))$ $= t_j$. In such a situation it is known that the sample functions (i.e., the real functions of t: $Y(t,\omega)$) are continuous for all ω except possibly for the ωs forming a set of P-measure zero. It is possible to construct a new probability space on which the $Y(t)$s are defined and *all* of whose sample functions are continuous.

Let $X_1,\ X_2$ be two independent copies of Y. It is assumed that the basic probability space is versatile enough to support these processes but this can always be arranged.

Define, for $-\infty < t < \infty$,

$$X(t) = \begin{cases} X_1(t) & \text{if } t \geq 0, \\ X_2(-t) & \text{if } t \leq 0. \end{cases} \tag{1}$$

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The standard iterated Brownian motion process (IBMP) Z is defined on $[0,\ \infty)$ thus:

$$Z(t) = X(Y(t)),\ t \geq 0. \tag{2}$$

More explicitly, $Z(t,\omega) = X(Y(t,\omega),\omega)$. Our interest is in the iterated Brownian sequence $(Z(n))$.

Hu et al [1] have shown that, with probability 1

$$\overline{\lim_{n\to\infty}}\, n^{-\frac{1}{4}}(\log\log n)^{-\frac{3}{4}}|Z(n)| = 2^{\frac{1}{2}}3^{-\frac{3}{4}}. \tag{3}$$

Hence,

$$P(|Z(n)| > n^{\frac{1}{4}}b_\delta(n)\,\text{i.o.}) = 0 \quad \text{for every } \delta > 0, \tag{4}$$

where

$$b_\delta(n) = (1+\delta)2^{\frac{1}{2}}3^{-\frac{3}{4}}(\log\log n)^{\frac{3}{4}}. \tag{5}$$

[At an early stage in tne development of the subject, the result $\overline{\lim_{n\to\infty}}\, \frac{S_n}{\sqrt{2nlog\log n}} = 1$ was called the the *law of the iterated logarithm* (LIL) for the sequence (S_n) becacuse of the factor $\log\log n$. Here, for $n = 1,\ 2,\ ...,\ S_n$ is the sum of n independent copies of a variable X with mean $\mathbb{E}X = 0$ and variance $\mathbb{E}X^2 = 1$. Even though $Z(n)$ is not the sum of variables, (3) is called the LIL for the sequence $(Z(n))$.]

Define $V_n(\delta) = 1$ if $|Z(n)| > n^{\frac{1}{4}}b_\delta(n)$ and $V_n(\delta) = 0$ otherwise; $N_0(\delta) = 0$, $N_m(\delta) = \sum_{n=1}^m V_n(\delta)$ and $N_\infty(\delta) = \lim_{m\to\infty} N_m(\delta) = \sum_{n=1}^\infty V_n(\delta)$. We note that, by (4), the integer valued variable $N_\infty(\delta)$ is, with probability 1, finite valued. We further note $N_m(\delta) \leq N_{m+1}(\delta) \leq N_\infty(\delta)$.

Theorem 1. $\mathbb{E}N_\infty^\lambda(\delta) = \infty$ *for every* $\lambda > 0$.

(Note. For a parallel result relating to the Hartman-Wintner law of the iterated logarithm (ref. [2, p. 444]), refer to [3] or to [2, Theorem 6.9.6, p 451].

Proof. Clearly it is enough to establish the theorem for $\lambda \in (0,\ 1)$ and $\delta \in (0,\ \sqrt{2}-1)$. Let $\lambda,\ \delta$ be as above. Hence,

$$(1+\delta)^2 \times 2 \times 3^{-\frac{3}{2}} < 1. \tag{6}$$

$$\begin{aligned}
\mathbb{E}N_\infty^\lambda(\delta) &\geq \lim_{m\to\infty} \mathbb{E}N_m^\lambda(\delta) = \lim_{m\to\infty} \sum\nolimits_{n=1}^m \mathbb{E}\{N_n^\lambda(\delta) - N_{n-1}^\lambda(\delta)\} \\
&= \lim_{m\to\infty} \sum\nolimits_{n=1}^m \mathbb{E}\{(N_{n-1}(\delta) + V_n(\delta))^\lambda - N_{n-1}^\lambda(\delta)\} \\
&= \lim_{m\to\infty} \sum\nolimits_{n=1}^m \mathbb{E}\Big[\mathbb{E}\{(N_{n-1}(\delta) + V_n(\delta))^\lambda - N_{n-1}^\lambda(\delta)\}\Big|V_n(\delta)\Big] \\
&= \lim_{m\to\infty} \sum\nolimits_{n=1}^m \mathbb{E}\Big[\mathbb{E}\{(N_{n-1}(\delta)+V_n(\delta))^\lambda - N_{n-1}^\lambda(\delta)\}\Big|V_n(\delta)=1\Big] \times P(V_n(\delta)=1) \\
&= \lim_{m\to\infty} \sum\nolimits_{n=1}^m \mathbb{E}\Big[\mathbb{E}\{(N_{n-1}(\delta)+1)^\lambda - N_{n-1}^\lambda(\delta)\}\Big|V_n(\delta)=1\Big] \times P(V_n(\delta)=1).
\end{aligned}$$

When $0 < \lambda < 1$ and $x > 0$,

$$\text{the function } (x+1)^{\lambda} - x^{\lambda} \quad \text{is monotonic decreasing.} \tag{7}$$

Since $N_{n-1}(\delta) \le n-1$, we conclude

$$\begin{aligned}
\mathbb{E}N_{\infty}^{\lambda} &\ge \lim_{m\to\infty} \sum\nolimits_{n=1}^{m} \mathbb{E}\Big[\mathbb{E}\{(n-1+1)^{\lambda} - (n-1)^{\lambda}\}\Big|V_n(\delta)\Big] \times P(V_n(\delta){=}1) \\
&\ge \lim_{m\to\infty} \sum\nolimits_{n=1}^{m} \{n^{\lambda} - (n-1)^{\lambda}\} P(|Z(n)| > n^{\frac{1}{4}} b_{\delta}(n)) \\
&\ge \lim_{m\to\infty} \sum\nolimits_{n=1}^{m} \{(n+1)^{\lambda} - (n)^{\lambda}\} P(|Z(n)| > n^{\frac{1}{4}} b_{\delta}(n)). \quad \text{by (5)} \\
&\ge \lim_{m\to\infty} \sum\nolimits_{n=1}^{m} \{\lambda \int_{n}^{n+1} x^{\lambda-1}\, \mathrm{d}x\} P(|Z(n)| > n^{\frac{1}{4}} b_{\delta}(n)) \\
&\ge \lambda \sum\nolimits_{n=1}^{\infty} n^{\lambda-1} P(|Z(n)| > n^{\frac{1}{4}} b_{\delta}(n)).
\end{aligned} \tag{8}$$

Now, the characteristic function (*cf*) ψ_n of $Z(n)$ is :

$$\psi_n(u) = \mathbb{E}e^{iuZ(n)} = \mathbb{E}e^{iuX(Y(n))},$$

where $\mathbb{E}e^{iuX(Y(n))}$ stands for the expected value of the variable $e^{iuX(Y(n))}$. In other words, it stands for the integral $\int_{\Omega} e^{iuX(Y(n,\,\omega),\,\omega)}\, \mathrm{d}P(\omega)$. To evaluate this expected value we use the tool of conditional expectation. The tool is almost same as or similar to iterated integral: treat $Y(n)$ as fixed, integrate and then integrate the resulting function of $Y(n)$. By the rules of conditional expectation, and noting

$$\mathbb{E}e^{iuX(t)} = e^{-\frac{u^2}{2}|t|}, \tag{9}$$

we have

$$\begin{aligned}
\mathbb{E}e^{iuX(Y(n))} &= \mathbb{E}[\mathbb{E}\{e^{iuX(Y(n))}\Big|Y(n)\}] \\
&= \mathbb{E}e^{-\frac{u^2}{2}|Y(n)|} \quad \text{using independence of } X \text{ and } Y \text{and (9)} \\
&= \mathbb{E}e^{-\frac{u^2}{2}\sqrt{n}\ |\xi|},
\end{aligned}$$

where ξ stands for a standard normal variable.
The *cf* φ of $Z(n)n^{-1/4}$ would be: $\varphi(u) = \mathbb{E}e^{-\frac{u^2}{2}|\xi|}$. We have

$$\int_{-\infty}^{\infty} |\varphi(u)|\, \mathrm{d}u = \mathbb{E}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2|\xi|}\, \mathrm{d}u,$$

wherein the change of order of integration is justified as the integrand is non-negative. It follows that

$$\int_{-\infty}^{\infty} |\varphi(u)|\, \mathrm{d}u = \mathbb{E}\sqrt{2\pi}(1/|\xi|^{1/2}) = \sqrt{2\pi}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} u^{-\frac{1}{2}} e^{-\frac{1}{2}u^2}\, \mathrm{d}u = 2^{\frac{1}{4}}\Gamma(\frac{1}{4}).$$

Hence the frequency function g of $Z(n)/n^{-1/4}$ exists, is bounded and continuous [2, p. 153]. We have

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\varphi(u)\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\mathbb{E}e^{-\frac{u^2}{2}|\xi|}\,\mathrm{d}u$$
$$= \mathbb{E}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}e^{-\frac{u^2}{2}\,|\xi|}\,\mathrm{d}u,$$

by Fubini's Theorem which is applicable since the double integral exists finitely when the integrand is replaced by its absolute value. It follows that

$$g(x) = \mathbb{E}\frac{1}{2\pi}|\xi|^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-ix|\xi|^{-\frac{1}{2}}v}e^{-\frac{v^2}{2}}\,\mathrm{d}v = \frac{1}{\sqrt{2\pi}}\mathbb{E}|\xi|^{-\frac{1}{2}}e^{-\frac{1}{2}\frac{x^2}{|\xi|}}.$$

This leads to: writing $A_n = 1 < |\xi| < b_\delta(n)$ and letting χ_{A_n} stand for its indicator function,

$$P\left(|\frac{Z(n)}{n^{\frac{1}{4}}}| > b_\delta(n)\right) = \frac{2}{\sqrt{2\pi}}\int_{b_\delta(n)}^{\infty}\mathbb{E}\frac{1}{|\xi|^{\frac{1}{2}}}e^{-\frac{x^2}{2|\xi|}}\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\mathbb{E}\{1-\Phi(\frac{b_\delta(n)}{|\xi|^{\frac{1}{2}}})\}$$
$$\geq \sqrt{\frac{2}{\pi}}\mathbb{E}\{\Big(1-\Phi(\frac{b_\delta(n)}{|\xi|^{\frac{1}{2}}})\Big)\chi_{A_n}\}$$
$$\geq \frac{1}{2}\mathbb{E}\Big\{(\frac{|\xi|^{\frac{1}{2}}}{b_\delta(n)} - \frac{|\xi|^{\frac{3}{2}}}{(b_\delta(n))^3})e^{-\frac{1}{2}\frac{b_\delta^2(n)}{|\xi|}}\chi_{A_n}\Big\}\ (ref.\ [2, p.252])$$
$$\geq \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_1^{b_\delta(n)}\frac{y^{\frac{1}{2}}}{b_\delta(n)}\{1-\frac{y}{(b_\delta(n))^2}\}e^{-\frac{1}{2}\frac{b_\delta^2(n)}{y}}e^{-\frac{1}{2}y^2}\,\mathrm{d}y$$
$$\geq \frac{1}{4}\frac{1}{b_\delta(n)}e^{-\frac{1}{2}b_\delta^2(n)}\int_1^{b_\delta(n)}\{1-\frac{1}{b_\delta(n)}\}e^{-\frac{1}{2}y^2}\,\mathrm{d}y$$
$$\geq \frac{1}{8}\frac{1}{b_\delta(n)}e^{-\frac{1}{2}b_\delta^2(n)}\int_1^{b_\delta(n)}e^{-\frac{1}{2}y^2}\,\mathrm{d}y \quad \text{(for all large } n)$$
$$\geq \frac{1}{8}e^{-b_\delta^2(n)}\frac{b_{\delta_n}-1}{b_{\delta_n}} \geq \frac{1}{16}e^{-b_\delta^2(n)} \quad \text{(for all large } n)$$
$$\geq \frac{1}{16}e^{-(1+\delta)^2\times 2\times\ 3^{-\frac{3}{2}}(\log\log n)^{\frac{3}{2}}}$$
$$\geq \frac{1}{16}e^{-(\log\log n)^{\frac{3}{2}}} \quad \text{(by (6))}$$
$$\geq \frac{1}{16}e^{-\lambda\log n} \quad \text{(for all large n).}$$

Hence $\sum_n n^{\lambda-1}P(|Z(n)| \geq n^{\frac{1}{4}}b_\delta(n)) = \infty$. It now follows from (8) that $\mathbb{E}N_\infty^\lambda(\delta) = \infty$. □

Acknowledgement. I am thankful to the unknown referee for his suggestions which led to the removal of typos and the correction of a minor error, for pointing out to me my less than careful phrasing at places and for his words, after reading the first revision of the paper, "Now the manuscript looks better".

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R. P. Pakshirajan
227, 18th Main, 6th Block, Koramangala
Bangalore-560 095, Karnataka, India
E-mail: vainatheyarajan@yahoo.in

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PROBLEM SECTION

In the last issue of the Math. Student Vol. **87**, Nos. 3-4, July-December (2018), we had invited solutions from the floor to the remaining five problems *1, 2, 4, 5* and *6* of the MS, 87, 1-2, 2018 as well as to the six new problems *7, 8, 9, 10, 11* and *12* presented therein till April 30, 2019.

I. The status of the remaining five problems of MS, 87, 1-2, 2018:
No solution is received from the floor to any of these problems and hence we provide in this issue the Proposer's solution to these problems.

II. The status of the six new problems of MS 87, 3-4, 2018:
1. We received from the floor **three correct** solutions to the problem *7*, **two correct** solutions to problem*10* and **one correct** solution to each of the problems *8, 9* and *11.* we publish these here (in case of the problem *7*, the one which gives a solution to a more general version of the problem and in case of the problem *10* the rigourous one). Also, we received one incorrect and incomplete solution to the problem *9* with some correct ideas.
2. We received one incomplte solution lacking rigour to the problem*12*, but **no complete and correct solution** is received from the floor to the problem *12*. Readers can try their hand on this till October 31, 2019.

In this issue we first present **eight new problems**. Solutions to these problems as also to the remaining problem *12* of MS 87, 3-4, 2018, received from the floor till October 31, 2019, will be published in the MS 88, 3-4, 2019 if approved by the Editorial Board.

MS-2019, Nos. 1-2: Problem-1:
(proposed by R. P. Pakshirajan, formerly of the Mysore University, through J. R. Patadia).

Let g be an arbitrary real continuous function defined on $[0, 1]$. Let μ be the standard Wiener measure on the Borel σ-field $\mathscr{C}$ of the space $\mathbf{C}$ of real continuous functions on $[0, 1]$ all vanishing at zero. Let $\alpha_n = \sum_{j=1}^{2^n} g(t_{j,n})(x(\frac{j}{2^n}) - x(\frac{j-1}{2^n}))$, $t_{j,n}$ being an arbitrary point in $(\frac{j-1}{2^n}, \frac{j}{2^n})$. Show that α_n is a Cauchy sequesnce in $L^2(\mu)$.
Desribe the distribution of the limit function α ($=\lim_n \alpha_n$) under μ.

MS-2019, Nos. 1-2: Problem-2:
(proposed by M. Ram Murty).

Prove that $\limsup_{n\to\infty} \sin n = 1$ and $\liminf_{n\to\infty} \sin n = -1, \quad n \in \mathbb{N}$.

The two problems proposed by George E. Andrews.

MS-2019, Nos. 1-2: Problem-3:

Prove that the number of partitions of n with no part congruent to 3 mod 4 equals the number of partitions of n in which the difference between any two parts can not be 1 if the larger part is even.
For example, for $n = 7$, the nine partitions of 7 with no part congruent to 3 mod 4 are: $6+1,\ 5+2,\ 5+1+1,\ 4+2+1,\ 4+1+1+1,\ 2+2+2+1,\ 2+2+1+1+1,\ 2+1+1+1+1+1,\ 1+1+1+1+1+1+1$.
The nine partitions in which no even part is followed by a part that is 1 less are: $7,\ 6+1,\ 5+2,\ 5+1+1,\ 4+1+1+1,\ 3+3+1,\ 3+2+2,\ 3+1+1+1+1,\ 1+1+1+1+1+1+1$.

MS-2019, Nos. 1-2: Problem-4:

Prove that $xe^x = \sum_0^\infty \left(e^x - \sum_{j=0}^n (x^j/j!)\right)$.

The four problems proposed by B. Sury

MS-2019, Nos. 1-2: Problem-5:

Let θ be a non-zero real number and let N be a positive integer. Consider the fractional parts of $n\theta$ for $0 \le n < N$. This creates a maximum of N possible intervals in $[0,1]$. Prove that there are only three possible lengths for these intervals.

MS-2019, Nos. 1-2: Problem-6:

If V is a vector space of finite dimension over a finite field F, find the minimum number of proper subspaces of V whose union is the whole of V. Answer the same question when the proper subspaces have dimension at most $\dim V - k$ for some fixed $k < \dim V$.

MS-2019, Nos. 1-2: Problem-7:

For a positive integer n, a prime $p \le n$, and any $0 \le k < p$, find that the power of p dividing $\sum_{r \equiv k mod\ p} \binom{n}{r}(-1)^r$ is at least the smallest integer $\ge (n-1)/(p-1)$.

MS-2019, Nos. 1-2: Problem-8:

Given any positive integer n, characterize all positive integers m such that there is an invertible $n \times n$ matrix with rational entries whose order is m.

Solution from the floor: MS-2018, Nos. 3-4: Prob-7: Prove that

$$\sum\nolimits_{m,n=1}^{\infty} \frac{(m,n)}{m^2n^2} = (5/2)\sum\nolimits_{n=1}^{\infty} \frac{1}{n^3},$$

where (m,n) denotes the greatest common divisor of m and n.

(Solution submitted on 02-04-2019 by **Dasari Naga Vijay Krishna**, Machilipatnam, Andhra Pradesh-521001; *Vijay9290009015@gmail.com*).

Solution. We prove the following *general statement*:

$$\sum\nolimits_{m,n} \frac{(m,n)^r}{m^sn^t} = \frac{\zeta(s)\zeta(t)\zeta(s+t-r)}{\zeta(s+t)}, \tag{0.1}$$

where (m,n) denotes the greatest common divisor of m and n and $\zeta(\cdot)$ represents the zeta function. Observe that

$$\sum_{m,n} \frac{(m,n)^r}{m^sn^t} = \sum_{d=1}^{\infty}\sum_{(m,n)=d} \frac{(m,n)^r}{m^sn^t} = \sum_{d=1}^{\infty}\sum_{(m,n)=1} \frac{d^r}{(dm)^s(dn)^t} = \zeta(s+t-r)A_1,$$

if we let $A_d = \sum_{(m,n)=d} \frac{1}{m^sn^t}$. Note that $\sum_{r|d} A_d = \frac{1}{r^{s+t}}\zeta(s)\zeta(t)$. Therefore

$$A_1 = \sum\nolimits_{r=1}^{\infty} \mu(r) \sum\nolimits_{r|d} A_d = \zeta(s)\zeta(t)\sum\nolimits_{r=1}^{\infty} \frac{\mu(r)}{r^{s+t}} = \frac{\zeta(s)\zeta(t)}{\zeta(s+t)}.$$

Substituting this value of A_1 in the above equation, we get (0.1). Taking $r = 1$ and $s = t = 2$ in (0.1) and recalling the well known facts that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$, we obtain

$$\sum\nolimits_{m,n} \frac{(m,n)^r}{m^sn^t} = \frac{\zeta(2)^2\zeta(3)}{\zeta(4)} = (5/2)\sum\nolimits_{n=1}^{\infty} \frac{1}{n^3}.$$

Correct solution was also received from the floor from:

Siddhi Pathak; (Graduate Student, Queens University, Kingston, Ontario, Canada; *siddhi@mast.queensu.ca*, received on 31-03-2019).

Prajnanaswaroopa S.; (Amrita University, Coimbatore-641 112, Tamilnadu, India; *sntrm4@rediffmail.com*, received on 04-04-2019).

Solution from the floor: MS-2018, Nos. 3-4: Prob-8: If $\lceil a \rceil$ denotes the smallest integer greater than or equal to a, prove:

(i) the closest integer to a is $\lceil a - 1/2 \rceil$.

(ii) Further, prove

$$\sum\nolimits_{k=1}^{n} d(2k-1) = \sum\nolimits_{r=1}^{n} \lceil n/(2r-1) - 1/2 \rceil.$$

(Solution submitted on 04-04-2019 by **Prajnanaswaroopa S.**, Amrita University, Coimbatore, Tamilnadu, India; *sntrm4@rediffmail.com*).

Solution. (i) Let b be the closest integer to a. Then, we have $|b-a| \le 1/2$. Now, $|b-a| \le \frac{1}{2} \implies -\frac{1}{2} \le b-a \le \frac{1}{2} \implies a - \frac{1}{2} \le b - a \le a + \frac{1}{2} \implies b = \lceil a - \frac{1}{2} \rceil = \lfloor a + \frac{1}{2} \rfloor$. Note that when $b = a \pm \frac{1}{2}$, that is, when a is exactly half greater than integer, either $a - \frac{1}{2}$ or $a + \frac{1}{2}$ can be taken as the closest integer. This is why the inequality at $a = b \pm \frac{1}{2}$.

(ii) Now, the sum of divisors of first n odd numbers can be seen to be in bijection with the number of times the first n odd numbers divide n, or the closest integer to $\frac{n}{2k-1}$, where $2k-1$ is the corresponding odd number. This can be seen by observing that all divisors of any odd number will always be odd, and whenever we sum the divisors the number of times an odd number is repeated is exactly the quotient of $\frac{n}{2k-1}$, where $2k-1$ is the corresponding odd number, like, for example, 1 is repeated n times, for 1 is the divisor of all numbers, 3 appears closest to $\frac{n}{3}$ times, for it would be repeated whenever we sum the divisors of a multiple of 3, which appears closest to $\frac{n}{3}$ times, and so on. Since, by the previous part, the integer closest to any number is $\lceil a - \frac{1}{2} \rceil$, where a be the number, so the proposition is proved.

Solution from the floor: MS-2018, Nos. 3-4: Prob-9: Determine with proof the set of all polynomials with complex coefficients that take rational values at all rational numbers and irrational values at all irrational numbers.

(Solution submitted on 31-03-2019 by **Anup Dixit**, Postdoctoral fellow, Queens University, Kingston, Ontario, Canada.; *anup.dixit@queensu.ca*).

Solution. We claim that

$$\{f \in \mathbb{C}[x] : f \text{ takes rationals to rationals and irrationals to irrationals}\}$$
$$= \{ax + b : a, b \in \mathbb{Q}, a \neq 0\}.$$

It is clear that linear polynomials with rational coefficients take rational numbers to rational numbers and irrational numbers to irrational numbers. We will now prove the reverse containment.

Suppose $f(x) \in \mathbb{C}[x]$ is a polynomial that takes rational numbers to rational numbers and irrational numbers to irrational numbers. Suppose α is a root of f. Since $f(\alpha) = 0 \in \mathbb{Q}$, $\alpha \in \mathbb{Q}$. Since all the coefficients of f can be expressed as symmetric polynomials in the roots of f therefore f must have rational coefficients. We may assume without loss of generality that $f(x) = a_n x^n + \cdots + a_0$, $a_i \in \mathbb{Z}\ \ \forall i, a_n > 0$. We will show that $n = 1$.

Consider any $n \in \mathbb{N}$. By the rational root theorem, any rational solution p/q of $f(x) = N$ must satisfy $q | a_n$. Therefore, for any two distinct rational solutions p_1/q_1 and p_2/q_2 of $f(x) = N$, we have

$$|p_1/q_1 - p_2/q_2| > 1/a_n. \tag{0.2}$$

Now suppose $n > 1$. clearly the degree of the derivative $f'(x)$ is at least 1. Hence $f'(x)$ increases as $x \to \infty$. Choose $j > 0$ such that $f'(x) > 3a_n$ for all $x > j$. Then

$$f(j + 1/a_n) - f(j) = f'(c)1/a_n \text{ for some } c \in (j, j + 1/a_n),$$

which implies $f(j+1/a_n)-f(j) > 3$ in view of the choice of j. Therefore, by the intermediate value theorem, there are at least two solutions to $f(x) = N$, for some $N \in \mathbb{N}$, in $[j, j + 1/a_n]$. These two solutions will be rational by the special property of $f(x)$ and will have a difference less than $1/a_n$, which contradicts (0.2). Hence, $n = 1$ and the proof is completed.

Solution from the floor: MS-2018, Nos. 3-4: Prob-10: Let $a_1 < a_2 < \cdots$ be an infinite sequence of pairwise coprime positive integers which are all composites. Prove that $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$.

(Solution submitted on 01-04-2019 by **Siddhi Pathak**; (Graduate Student, Queens University, Kingston, Ontario, Canada; *siddhi@mast.queensu.ca*).

Solution. Let q_n denote the smallest prime factor of a_n, for $n > 1$. Since a_n are pairwise coprime, $q_n \neq q_m$ for any $n \neq m$. Since the numbers a_n are composite, $q_n^2 \leq a_n$. Hence

$$\sum_{n=1}^{\infty}(1/a_n) \leq a_1 + \sum_{n=1}^{\infty}(1/q_n^2) \leq \sum_{n=1}^{\infty}(1/n^2) < \infty.$$

Correct solution was also received from the floor from:
Prajnanaswaroopa S.; (Amrita University, Coimbatore-641 112, Tamilnadu, India; *sntrm4@rediffmail.com*, received on 04-04-2019).

Solution from the floor: MS-2018, Nos. 3-4: Prob-11: For any positive integer n and odd prime p, prove that the following congruence holds modulo p:

$$(n^p - n)/p \equiv -\sum_{r=1}^{p-1} \frac{1^r + 2^r + \cdots + n^r}{r}.$$

Here, two rational numbers s and t are said to be congruent modulo p if their difference $\frac{u}{v}$, written in the lowest terms, satisfies $p|u$.

(Solution submitted on 16-04-2019 by **Dasari Naga Vijay Krishna**, Machilipatnam, Andhra Pradesh-521001; *Vijay9290009015@gmail.com*).

Solution. We require the folowing identity which is easy to prove.

When p is an odd prime $(1/k) \equiv (-1)^{k-1}\ (1/p)\binom{p}{k}(mod\ p)$.

Now, observe that

$$-\sum_{r=1}^{p-1}\left(\frac{1^r+2^r+\cdots+n^r}{r}\right)=-\left(\left(\frac{1^1+2^1+\cdots+n^1}{1}\right)+\left(\frac{1^2+2^2+\cdots+n^2}{2}\right)+\right.$$

$$\left(\frac{1^3+2^3+\cdots+n^3}{3}\right)+\cdots+\left(\frac{1^{p-1}+2^{p-1}+\cdots+n^{p-1}}{p-1}\right)\Bigg)$$

$$= -\sum_{x=1}^{n}\left(\frac{x^1}{1}+\frac{x^2}{2}+\cdots+\frac{x^{p-1}}{p-1}\right).$$

But

$$\left(\frac{x^1}{1}+\frac{x^2}{2}+\cdots+\frac{x^{p-1}}{p-1}\right)$$

$$\equiv \frac{1}{p}\left(x\binom{p}{1}-x^2\binom{p}{2}+x^3\binom{p}{3}-\cdots x^{p-1}\binom{p}{p-1}\right)\right)(mod\ p)$$

$$= (1/p)\big((x-1)^p-x^p+1\big)\ (mod\ p),\quad \text{and hence}$$

$$-\sum_{x=1}^{n}\left(\frac{x^1}{1}+\frac{x^2}{2}+\cdots+\frac{x^{p-1}}{p-1}\right)=-\frac{1}{p}\sum_{x=1}^{n}((x-1)^p-x^p+1)\,(mod\ p)$$

$= (1/p)(n^p-n)(mod\ p)$ which completes the solution.

Solution by the Proposer B. Sury: MS-2018, Nos. 1-2: Prob-1: Find the largest possible diameter of a circular hole that can be covered by two square boards of unit area.

Solution. Firstly, the Figure 1 below shows how to cover a hole of radius at most $2-\sqrt{2}$ by two unit squares.

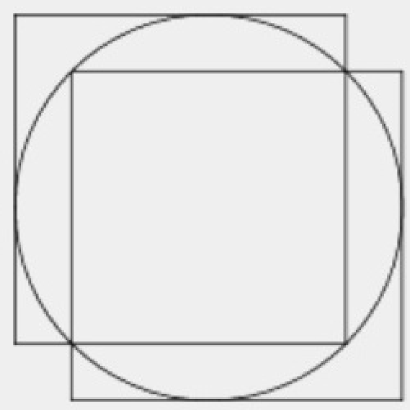

Figure 1

We show that the answer to our question on the diameter is $2(2-\sqrt{2})$ by proving that if a unit square covers at least half the perimeter of a circle, then its radius $r \le 2-\sqrt{2}$. So, consider a circle C of radius r and a unit square S that covers at least half of C. If r is at least $1/2$, the square S can be translated so that each side of S or its extension intersects C in at least one point and the covered part of the curve increases (as in the following figure 2). Therefore, we may assume $r > 1/2$ and that the extension of each side of S intersects C. Now, we first show that one vertex of the S intersects C. Now, we first show that one vertex of the square, say A, cannot be inside the circle (as depicted in figure 3 below). To see this, write

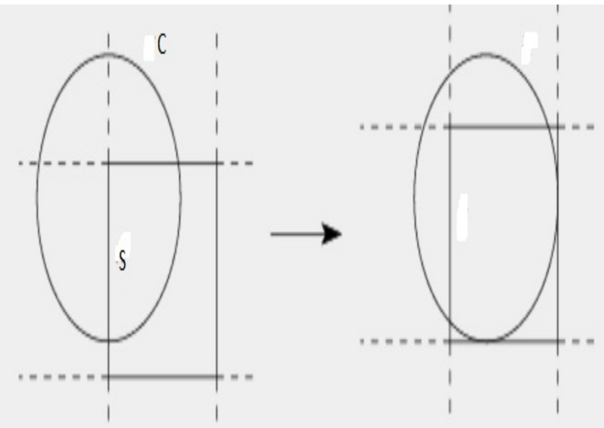

Figure 2

P, Q for the points of intersection of the sides AB and AD with the circle. Now, the angle at A being a right angle, the vertex A and the center of the circle must be on the same side of PQ. But then the arc PQ not covered by the square has more than half the perimeter, a contradiction. Thus, each

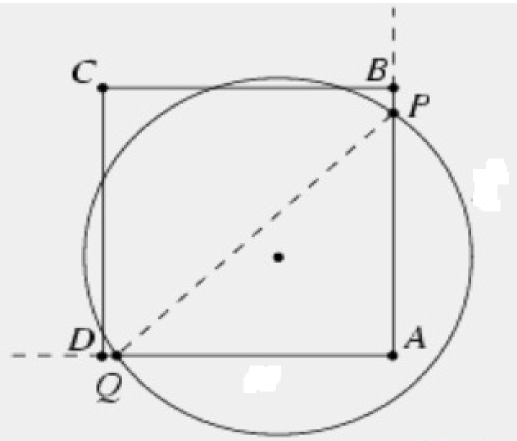

Figure 3

vertex is outside the circle and each side intersects the circle at points on the perimeter of the square as in figure 4 on the next page. The angles $\alpha, \beta, \gamma, \delta$ must add to less than 90 degrees since at most half of the arcs of the circle is uncovered. Without loss of generality, we assume $\alpha + \gamma < 45$ degrees. Now $r(\cos\alpha + \cos\gamma) = 1 = r(\cos\beta + \cos\delta)$. Then, the concavity

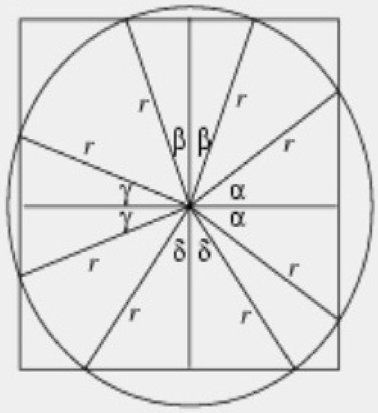

Figure 4

property of the cosine function gives $1 \geq r(1+\cos(\alpha+\gamma)) \geq r(1+\cos 45)$ which immediately yields $r \leq 2-\sqrt{2}$.

Solution by the Proposer B. Sury: MS-2018, Nos. 1-2: Prob-2: Find a characterization of the positive integers m such that S_n has an element of order m.

Solution. Any permutation is a product of disjoint cycles; so, its order is the LCM of the lengths of the cycles. There exists a permutation in S_n having order m if, and only if, either $m = 1$ or $m = \prod_{i=1}^{k} q_i^{e_i}$ and $\sum_{i=1}^{k} q_i^{e_i} \leq n$. This is a consequence of the following observation on LCMs:

If $LCM(a_1, \cdots, a_k) = m > 1$ *and* $m = \prod_{i=1}^{r} q_i^{e_i}$*, then* $\sum_i q_i^{e_i} \leq \sum_j a_j$.

This is easy to see as follows.

Given m, choose $a_1, \cdots, a_k$ with $\mathrm{LCM}(a_1, \cdots, a_k) = m$ and $\sum_i a_i$ minimum possible. Then each $a_i > 1$; else, we may drop such an a_i and get a smaller sum. Also, each a_i must be a prime power; else, writing $a_i = uv$ with u, v coprime and $u > v > 1$ and so a_i may be replaced by two terms u, v whose sum $u + v < a_i = uv$. Finally, the primes occurring in a_i and a_j are different for $i \neq j$; else, we may simply drop the smaller prime power. Hence, $m = \prod_i a_i$ is the prime power factorization of m.

Interestingly, the maximum order of an element of S_n is related to the prime number theorem (as already pointed out by E.Landau) - see a nice exposition by W. Miller, "The Maximum Order of an Element of a Finite Symmetric Group" in the *Amer. Math. Monthly*, Vol. **94** (1987), 497–506.

Solution by the Proposer B. Sury: MS-2018, Nos. 1-2: Prob-4: Find with proof, the unique solution to the following multiplication table where each digit 0 to 9 is repeated twice:

$$\begin{array}{rrrrr} & & * & * & * \\ & \times & * & * & * \\ \hline & & * & * & * \\ & * & * & * & \\ * & * & * & & \\ \hline * & * & * & * & * \\ \hline \end{array}$$

Solution. This problem can be solved logically considering several cases. It leads to the unique solution below but the explanation of each step, though elementary, becomes messy. We just give the solution. The reader is invited to argue why each digit occurring is forced.

$$\begin{array}{rrrrr} & & 1 & 7 & 9 \\ & \times & 2 & 2 & 4 \\ \hline & & 7 & 1 & 6 \\ & 3 & 5 & 8 & \\ 3 & 5 & 8 & & \\ \hline 4 & 0 & 0 & 9 & 6 \\ \hline \end{array}$$

Solution by the Proposer B. Sury: MS-2018, Nos. 1-2: Prob-5: A spider starts crawling along an elastic rope at the rate of 1 foot per second. Initially, the elastic rope is f feet long and at the end of each second, it is stretched uniformly by f feet. Does the spider ever reach the other end of the rope?

Solution. After the n-th stretching, the elastic rope is $f + fn = (n+1)f$ feet long. If the spider does not ever reach the other end, let it be at a distance of d_n feet from its starting point after n seconds. After one more second, it should be $d_n + f$ feet away whereas, because of stretching, it will be at a distance of d_{n+1}. Thus, the uniformness of stretching means that

$$\frac{d_{n+1}}{d_n + f} = \frac{(n+2)f}{(n+1)f} = \frac{n+2}{n+1}.$$

Therefore,

$$\frac{d_{n+1}}{n+2} = \frac{d_n}{n+1} + \frac{f}{n+1}.$$

Summing from $n = 0$ to k, we have

$$\sum_{n=0}^{k}\frac{d_{n+1}}{n+2}=\sum_{n=0}^{k}\frac{d_n}{n+1}+\sum_{n=0}^{k}\frac{f}{n+1}.$$

Therefore, $\dfrac{d_{k+1}}{k+2}=f\sum_{n=0}^{k}\dfrac{1}{n+1}$; that is,

$$d_{k+1}=f(k+2)\sum_{n=0}^{k}\frac{1}{n+1}.$$

If the spider does not reach the other end after the $(k+1)$-th second, d_{k+1} must be less than $(k+2)f$. That is, if the spider never reaches the other end, we must have $\sum_{n=0}^{k}\frac{1}{n+1}<1$ for all k. This is impossible as the harmonic series diverges.

Solution by the Proposer George E. Andrews: MS-2018, Nos. 1-2: Problem 6: Prove that $PO_2(n)$, the number of partitions of n in which each odd part appears at most twice, must be even if $n\equiv 2$ *or* $3(mod 4)$. For example, there are 8 such partitions of 6, namely, $6,\ 5+1,\ 4+2,\ 4+1+1,\ 3+3,\ 3+2+1,\ 2+2+2,\ 2+2+1+1;$ and $6\equiv 2(mod 4)$..

Solution. Let $\omega=e^{2\pi i/3}$, then

$$\begin{aligned}\sum_{n\geq 0}PO_2(n)\ q^n &= \prod_{n=1}^{\infty}\frac{1+q^{2n-1}+q^{4n-2}}{1-q^{2n}}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-\omega q^{2n-1})(1-\omega^{-1}q^{2n-1})}{(1-q^{2n})^2}\\ &= \frac{\sum_{\infty}^{\infty}\omega^n q^{n^2}}{\prod_{n=1}^{\infty}(1-q^{2n})^2}\quad\text{by Jacobi triple product}\\ &\equiv \frac{\sum_{-\infty}^{\infty}\omega^{2n}q^{4n^2}+q\sum_{-\infty}^{\infty}\omega^{2n+1}q^{4n^2+4n}}{\prod_{n=1}^{\infty}(1-q^{4n})}\cdot(mod\ 2).\end{aligned}$$

Clearly, no powers of q congruent to 2 or 3 modulo 4 can appear in the expansion of the last expression. Hence $PO_2(n)$ is even if $n\equiv$ 2 or 3 modulo 4.

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