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THE MATHEMATICS STUDENT

Edited by J. R. PATADIA

In keeping with the current periodical policy, THE MATHEMATICS STUDENT will seek to publish material of interest not just to mathematicians with specialized interest but to the postgraduate students and teachers of mathematics in India. With this in view, it will ordinarily publish material of the following type:

1. research papers,
2. the texts (written in a way accessible to students) of the Presidential Addresses, the Plenary talks and the Award Lectures delivered at the Annual Conferences.
3. general survey articles, popular articles, expository papers, Book-Reviews.
4. problems and solutions of the problems,
5. new, clever proofs of theorems that graduate / undergraduate students might see in their course work, and
6. articles that arouse curiosity and interest for learning mathematics among readers and motivate them for doing mathematics.

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PROFESSOR V. M. SHAH - A REMEMBRANCE (25-09-1929 - 03-12-2017)

SATYA DEO

Professor Vadilal Maganlal Shah (popularly known as Prof. V. M. Shah) breathed his last very recently on Dec 3, 2017 at Vadodara. He was suffering from chronic kidney disease and had been with his eldest daughter at Ahmedabad where he had a mild stroke and was treated there. He did recover briefly, but other complications cropped up leading to the end. The entire family of the Indian Mathematical Society and his other friends came to know about this sad news with great shock and sorrow. Prof Shah had been a great source of encouragement for all of us in the Indian Math. Society. We lost a great educationist, a noted mathematician, a great teacher and an extraordinary human being. He is survived by his wife and four daughters.

A Mathematician and an Administrator

Prof Shah started his academic career as an Assistant Lecturer in the Department of Mathematics, Faculty of Science, the M.S.University of Baroda, Vadodara, Gujarat. He worked under the supervision of Prof U. N. Singh for his doctoral degree at the M. S. University, became Reader and then Professor at the same university. His field of research was Fourier Analysis and related areas, which was a very popular topic during those days. He served the M. S. University of Baroda with great distinction as Head of the Mathematics Department, as Dean, Faculty of Science and with numerous administrative responsibilities. Prof V. M. Shah was elected Fellow of the National Academy of Sciences, India in the year 1980. He was also elected as the Sectional President (Mathematics including Statistics) of the Indian Science Congress. He enjoyed the reputation of an excellent teacher, a sound scholar, and a visionary educationist not only in Gujarat, but in the whole country. He has been on several UGC committees as well as the Gujarat State level committees to serve the cause of mathematics teaching and research. He wrote extremely popular elementary mathematics books entitled "*Majedar Ganit*" for the benefit of students as well as teachers. This must be the case because of his fascination for mathematics. He joined the Council of the Indian Mathematical Society first as a Treasurer, but slowly moved upward as its General Secretary, the most important position of the Society by its constitution.

Prof Shah and the IMS

As noted earlier, Prof Shah served the Indian Mathematical Society as Treasurer, then as the editor of the Journal of Indian Mathematical Society and Mathematics Student both, also as its President, and most importantly, as its General Secretary from 2005 until 2013. As General Secretary, he had to face a big challenge when there were some attempts by a group of people to run the Society in their arbitrary style; it surfaced when without taking the Academic Secretary or the General Secretary of IMS in confidence, a number of academic programmes were formulated and even announced. This was a great surprise to the elected office bearers of the Society who did not know what to do in a situation like this. Prof Shah faced this challenge with great patience, took all the Council members in confidence and proceeded to neutralize all the moves very quickly. Everything afterwards became normal, but it took some time and determination to act the way Prof Shah did. Later on as the General Secretary of the IMS, Prof Shah took the Society to a great level of prestige, popularity, and academic standard. He was not only the eldest of all but also most experienced, most patient and highly sympathetic towards his colleagues. He would always listen to each one of us, will appreciate opinion of everybody and will go by the consensus. I never saw him insisting on any point or forcing his opinion on any one.The team of senior mathematicians like Profs N. K. Thakare, Satya Deo, J. R. Patadia and other Council members of the IMS worked under his guidance with a noble mission and motto to strengthen the Society in the same spirit as the founding fathers of the Society had imagined. During his tenure as the General Secretary, the working style of the Society became more democratic; more dedicated to the development of Mathematics in India and established remarkable traditions in its functioning. Every Council member was there only to give his/her best to the Society and not to get anything for himself/herself. He left the Society only when most of the things were ideally streamlined. We never wanted that Prof Shah should retire from the Society, but because of his health and unshakable principles he declined our request and took retirement from the IMS in the year 2013. Nevertheless, his style of functioning, his important actions and visionary decisions in the IMS are permanent foot-prints for us to follow.

The Centenary Year 2007 of the IMS

I came in contact with Prof V. M. Shah in December 2005 when he was the General Secretary of IMS and I was nominated as the Academic Secretary. It took no time for me to realize that Prof. Shah was a great leader and totally committed to see that the IMS becomes one of the best mathematical societies of the world as soon as possible. He made plans, wrote letters to several government organizations and leading mathematicians to seek their cooperation. The year 2007 was the Centenary year of the Indian Mathematical Society, and under his leadership the society celebrated its centenary year in a big way. We invited almost all of the Indian mathematicians within the country and abroad to come and attend the Centenary Conference organized by the Department of Mathematics at the Pune University, Pune. Even though Prof Shah remained in the M. S. University of Baroda throughout his life, he knew almost all the leading mathematicians working in universities, research institutes, IITs and NITs really well. Prof N. K. Thakare, myself and J. R. Patadia were always there to offer suggestions which he accepted with great confidence. Most of the leading mathematicians were invited to give plenary talks, invited talks and to organize symposia on contemporary topics during that annual session. A large number of research scholars presented their works during the paper-reading session. Prof Thakare, working as the editor of the JIMS put all his efforts to make the JIMS up to date with no backlog. During the centenary year, special editions of the JIMS and Mathematics Student were published and a great academic programme was chalked out and implemented. Field medallist Prof. S. T. Yau, Abel Laureate Prof. S. R. S. Varadhan, Richard Hamilton (famous for his Ricci Flow) and many other mathematicians of very high stature came and participated in the centenary conference. Everyone who came expressed the opinion that the conference was really a great success.

IMS stamp in 2009

When we were preparing to celebrate the centenary year of the IMS, the Council of the IMS, under the leadership of Prof Shah, approached the Govt. of India to issue a postal stamp honouring 100 years of the IMS. Prof Shah wrote to various authorities giving all details about the work done by the Society. After some correspondences and phone calls, the Central Government agreed to issue a "*five rupee postal stamp*" bearing the name "*Indian Mathematical Society*" in Hindi as well as English. It was during the inaugural function of the Platinum Jubilee conference of the IMS on Dec 27, 2009 that a special commemorative stamp release function was organized. The Department of Posts, Govt. of India issued the stamp on completion of 100 years of IMS. It was released by Shri K. Ramchandiran, Post Master General, Madurai and was received by the Chancellor of Kalasalingam University - the host of the IMS session. This was indeed a historic event in the annals of the Indian Mathematical Society and a moment of pride for all of us in the IMS when the stamp appeared in the postal mails among the general masses.

IMS and the ICM-2010

Then came the International Congress of Mathematicians (ICM) organized by the NBHM at Hyderabad (for the first time in India). Prof M. S. Raghunathan, who was a member of the Executive Committee of the International Mathematical Union, had been trying to have an ICM in India and this finally materialized in 2010. He roped in all the mathematical societies of India including the oldest Indian Mathematical Society to join together and work for the success of the ICM

2010. Prof Shah lent all support to the organizers of the ICM and himself prepared a booklet for the IMS to be distributed to the participants of the ICM. Prof Shah had been trying hard through the IMS that the birth day of our legendary Indian Mathematical genius Srinivasa Ramanujan should be made the National Mathematics Day. Prof Shah regularly asked each of us, whoever became the sectional president of the Indian Science Congress, to push for such a decision in the Council of ISCA. We continued doing it year after year. However, it became a reality only when the NBHM, under Prof M. S. Raghunathan, made the final decisive effort. When the country was celebrating the 125th birthday of Srinvasa Ramanujan in Madras University, Chennai, in 2012, the then Prime Minister of India, Dr Manmohan Singh declared *December 22* as *The National Mathematics Day* and the year 2012 as the National Mathematics Year. It was a matter of great joy for all mathematicians of India when this announcement was made. I must remark here that the IMS, under the leadership of Prof V. M. Shah, had already prepared a good background amongst the scientists and the govt. authorities for this noble decision to be taken.

Digitization of IMS periodicals

Everyone in Indian Mathematical world knows that the Journal of the Indian Mathematical Society has a glorious past in terms of its reputation ever since late V. Ramswamy Aiyar started the Journal way back in 1907. The great genius Srinivasa Ramanujan published five of his important research papers in this journal besides many modern mathematicians of the world like Andre Weil, M. H. Stone and many others of that stature. Prof Shah, along with all of us, was very clear that our back volumes of the JIMS, which are lying in the Ramanujan Institute of Advance Studies in Mathematics, Madras University, Chennai, must be digitized as soon as possible. A large number of mathematicians had been requesting for the old papers published in the JIMS. He appointed a committee under Prof Thakare to go Chennai, see the situation of IMS library there, meet the Chairman, NBHM and suggest as to how this can be done. We then embarked upon this great task by contacting several publishing houses to do this job for us. This job has now been accomplished through the Informatics Private Ltd (IPL for short) people. Thanks to the efforts of Prof Shikare and the University of Pune, this digitization work has moved toward its completion. Yet another important achievement in this effort has been to publish our journal JIMS online in cooperation with IPL. I am happy to note here that the online publication of the JIMS has been regularly going on positively on January 1 and July 1 every year since 2013 onward.

IMS got its permanent home

In the beginning days of its establishment, the IMS had its headquarter and library at the Fergusson College, Pune, but the Society was registered afterwards at Delhi. Some people pointed out that the IMS does not have even its registration number which is indeed against the rule of societies. Prof. Shah dug out the registration number of IMS at Delhi and from that point onward made it a point to mention the IMS registration number clearly in all our correspondences, periodicals, newsletters etc. Prof V. M. Shah was of the firm opinion that the IMS can become an international Mathematical Society only if we have our own Headquarters. He worked on this mission also. The suggestions were given that we should try to have our own land and then construct an IMS building. Since a suitable piece of land in Pune could not be found immediately, we decided to buy a flat in Pune until we get such a land. For this purpose alone Prof Shah made exclusive trips to Pune a couple of times, arranged meetings with the revenue registrar and an advocate and finalized the purchase deal of an IMS flat. He got it registered in the name of the Indian Mathematical Society. This work was completed during his tenure as the general secretary of IMS. Of course, the Pune Council members extended all possible help in achieving this deal. As a result now the IMS has its permanent headquarter in Pune.

As a human being

Once I visited his residence in Baroda long back when I was not in the IMS. It was a beautiful house named "*Anant*" which means "*infinity*". I am sure this was named so because of his genuine love for mathematics. I was too junior to him, but he and his wife both treated me so nicely that it became an unforgettable meeting. I had heard about Prof. Shah at Jabalpur from my senior colleague Prof H. P. Dikshit, but was fortunate to experience it personally. Prof Shah had a unique way of mixing with people; he/she may be a high profile mathematician, may be an official, a colleague, a student or a research scholar. He had absolutely no ego of any kind, and had lot of patience to hear everyone carefully. His huge experience, always in forefront, will win over anyone he met. He treated his students in a very friendly manner and used to help them not only in academics but also in their personal problems. Having a photographic memory, he was always the center of attraction in the annual sessions of the IMS. In his addresses during the annual sessions, he always spoke with perfect clarity. He was highly organized and will hardly forget anything at all. He would always carry a bunch of small shorthand papers making sure that all he wanted to say has really gone out completely. Even during the Council meetings, those papers will always be in his hands so that the decisions are taken with full facts.

By nature he was always very fair to everyone. He took special care to make sure that the IMS remains equally represented by all quarters of the country, and the identity of the IMS remains highly visible. He emphasised that the annual sessions of the IMS should be held in the remote north, south, east and west corners of the country. He was very concerned that the colleges and universities should be given more importance, but was well aware of the quality research being done by the people working in research institutes, IITs, NITs etc. He never lost his temper except only once when someone had exceeded all limits of non-performance, and still continued to be an office bearer of the IMS.

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IMS, PUNE UNIVERSITY AND PROFESSOR V. M. SHAH

N. K. THAKARE

It was deeply sad news that Professor Vadilal Maganlal Shah of Vodadara is no more. It is an irreparable and great loss to the Indian Mathematical Society which he built with the proactive support from stalwarts such as Prof. R. P. Agarwal, H. C. Khare, M. K. Singal, D. N. Verma to name a very few. When he recused himself from the responsibilities of the General Secretaryship of IMS after Surat Annual Conference in December 2010, I was asked to carry out on his behalf the duties of general secretaryship of IMS. I was happy to act like a Laxmana (of Ramayan Epic) till 31st March 2013 when his term was to be completed. Though, he was not physically present in the subsequent three annual conferences held respectively at Nanded (2011-12), Varanasi (2012-13) and Cochin (2013-14), I abided by his suggestions and desires on almost daily basis.

I have written about him earlier [see Math. Student, 82 (2013) p. 9-11] in the context of his association with IMS while he was very much around us. The purpose of this tribute is rather to remember his involvement in building up the Department of Mathematics of Pune University (now Savitribai Phule Pune University) from outside and his genuine and intense desire to have a permanent headquarters of IMS at Pune.

In 1985, I joined the Department of Mathematics of Pune University as Professor of Mathematics (Lokmanya Tilak Chair) which was then known only as a very good teaching department. The research component was yet to attain its rightful place expected of a university department. Prof. V. M. Shah, Prof. Vasanti Bhat-Nayak were a very few good mathematicians who helped our department to rise to that expectation in the real sense of the term. They helped the university in selection of good and talented faculty under the DSA (Department for Special Assistance), COSIST (Department of Science and Technology, Government. of India) programmes leading finally to the status of Centre for Advanced Studies in Mathematics (CAS) in 2010. They also helped the young faculty in modernizing the syllabi. Professor V. M. Shah was a bacon light for the faculty in reaching to that distinct position of (CAS). So much so that with positive attitudes of a few good mathematicians like Prof. V. M. Shah, V. Kannan, S. Arumugam, the

Department of Mathematics of Pune University received the maximum grants from the University Grants Commission in 2016 amongst the only three Advanced Centres of Studies in Mathematics in the Universities in India. It virtually means that the said department is at the top of the ladder of mathematics departments in the universities in India. It is a unique achievement, Indeed! And Prof. V. M. Shah was directly or indirectly instrumental in that achievement.

When the Indian Mathematical Society (the original name was the Analytical Club) came into existence in 1907, its headquarters were at Pune; and it continued to be so till 1950. Later on the headquarters of IMS became almost mobile with no permanent place of its own. Professor Shah was very serious about having permanent premises of IMS at Pune. Being a former Treasurer of IMS he knew how to raise and handle the finances truthfully and judiciously. It was his personal dream that became a reality when he signed the purchase documents for the 900 sq.ft. permanent premises on 9th September 2010 with N. K. Thakare, J. R. Patadia, S. K. Nimbhorkar, M. M. Shikare and B. N. Waphare as witnesses. That is how, IMS has its own permanent premises at Pune. Earlier Prof. Shah, the author of five volumes of *Mazedar Ganit* in Gujarathi went into reclusion after December 2010 Surat Conference and now he went in reclusion for his permanent heavenly abode. His passing away is a great loss to all of us Indeed!

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PROFESSOR V. M. SHAH - MY TEACHER AND COLLEAGUE

J. R. PATADIA

It was at about 11.30 am on December 03, 2017 when I received a call giving the shocking news about the sad demise of Prof. V. M. Shah and informing that his mortal remains will be taken to "*Shmashān*" for the last rituals at about 12.15 p.m. I rushed to his residence and upon reaching there learned that it will take little more time as "*his eyes are being donated*". The family had brought him to Vadodara at about 9.00 am that morning itself from the Shalby hospital at Ahmedabad where he was in ICU under the care of his eldest daughter Shruti Vaidya, and he breathed his last at 10.10 am that morning at his residence in Vadodara; and they had immediately called the doctors for donating his eyes.

Prof. V. M. Shah had been undergoing kidney dialysis, initially twice a week and later on thrice a week. He usually was staying with one of his daughter's place. In that course, lastly he had been to Ahmedabad to stay with his eldest daughter before Diwali, on October 07, 2017 and everything was going on smoothly. On October 28, due to mild brain hemorrhage, he was admitted to ICU of Shalby Hospitals at Ahmedabad. He recovered briefly, but other complications cropped up and was re-admitted to ICU.

The sad news created a feeling of sort of an emptiness in me and my colleagues, and I remembered the warmness, personal guidence and inspiration received from him during my long association of more than 47 years as his student, colleague and a family friend. In fact, it was during 1969-70 that I first came in contact with him when I was in M.Sc. part-1 and he taught us a course on complex analysis leaving a lasting impression as a sincere and excellent teacher always stressing on the basic conceptual clarity.

I worked for my doctoral researches under his guidance. He indicated a broad direction, suggested books and then some research papers to read. The best thing was I never felt any pressure from him and he appreciated my pace of research progress. The care, concern, freedom, independence and the moral support needed at times received from him was so impressive that my contemporary research students from other disciplines considered it extraordinary as some of them were not that comfortable with their supervisors to frankly express their views.

Since 1978 onwards, I was his colleague in the same department and I have noticed in him some rare qualities which I wish to mention.

A visionary. He had a strong intuition and was able to judge persons and unfolding events well in advance. Knowing well that not every master's student opts for research career and being aware of the need of industries around Vadodara in particular and Gujarat in general, he could forsee, in early seventies, the importance of job oriented courses like Linear Programming, Operations Research, Computer Programming and Numerical Analysis and he introduced these courses at graduate as well as post graduate level - our University was the first in Gujarat for such initiative. His way of thinking used to be so balanced and objective that often his opinion was sought on many occasions by members of even higher authorities of the university. I remember very well he telling us once: "I suggested the Vice-Chancellor not to open several fronts (of confrontations) simultaneously when I met but he seems to be firm in his decisions"; and indeed the Vice-Chancellor suffered severe consequences.

Very practical, never demanding and committed to cause. As an office bearer of IMS, of late, he naturally had lots of on line work and correspondences to make. He neither was well versed with using computer for it nor had necessary computer facilities at his place. On the other hand, I had all the necessary facilities at my place. Under the circumstances, though I always felt embarrassed as I was much younger and was his student, for such IMS work, *it was he, at the age above 80, with left leg knee problem, who used to come to my place twice a week regularly on a two wheeler, driving almost 7 km. one way, for almost six years till he took retirement from IMS in 2013!!* He didn't prefer autorikshaw, once he told me when I asked, because one has to walk for some distance & often one has to wait also for some time to pick one, and if one fixes one for regular travel then flexibility will be lost; moreover, he continued, "*I am confident I can drive scooter*!" And, indeed he never had any driving problem during those years. Of course, I very often used to make a precautionary call on his youngest daughter to convey that 'Pappa has left my place' or 'how long back Pappa has left your home' if he is little late in reaching my place! All these hard work and risks he took at that age because he wanted to set things right for IMS before he retires from IMS!

From his own observation and analysis, he felt convinced that students should be catched very young before they develop "Math Phobia" if cause of mathematics is to be served well by making them feel "mathematics too is very interesting". Out of this conviction only, he wrote five volumes of "*Majedar Ganit*" in Gujarati which became very popular and several editions are already printed. However, even I was surprised when I learned from my friend Prof. V. D. Pathak that as late as the last week of November 2018, when he called on him to inquire about his health (Prof. Shah was then in ICU and undergoing dialysis alternate day), he said to him,"Pathak, the book '*Pako Payo Ganitno*' (Strong foundation of (primary) mathematics) is complete and I am giving final touch". What a determination and dedication for the task on hand for young students!!

He worked for achieving goal even if he has to go all alone. Being very optimistic, he always believed that there will certanly be some positive outcome if sustained efforts are made.

I have never noticed him in indulging into "*self praise*" - another appreciable characteristic of him.

Wishful Donor. When I learned about the donation of eyes of Prof. Shah, I immediately remembered that he often talked to me about Shri Deepchand Gardi, a silent donor basically from Padadhari, Rajkot, Gujarat (later based in Bombay, died in 2014, at the age of 99) fondly called *Modern day Daanveer Bhamasha* who has donated crores for the cause of education and health with number of educational institutions and hospitals comming up from his donations. (In fact, Gardiji had started donating at least Rs. 1000/- per day from the age of 45). I understand Prof. Shah greatly appreciated Gardiji because he himself strongly belived in donating for social cause and had donated, as per his financial ability, for the cause of education (refer my article on him: The Math. Student, **82**, 1-4 (2013)). Apart from this, I am unable to resist my temptation to mention the following of his *unfulfilled desires*.

Since 2012, he repeatedly told me and conveyed to the Head, Dept. of Mathematics as well that he wishes to donate a corpus fund of Rs. 15 lakh, through the "*Prof. (Dr.) V. M. Shah Charitable Trust*", to institute two scholarships in each semester (totally ten scholarships), in the Department of Mathematics, Faculty of Science of his alma mater, for F.Y.B.Sc. through M.Sc.(Final) students taking mathematics as principal subject - one for the topper from general category students and one for the topper from reserved category students. He several times reminded me whether I have framed the rules as requested by him. Initially, I did prepare rough draft of rules, then didn't get time and later on knowing that he has to spend a lot every month towards his dialysis and not aware of his real financial position, I remained cool. I am not sure whether I did right or wrong, but sometimes I do feel sorry that his desire remained unfulfilled may be because of me.

Also, in 2014, he purchased a piece of land in the vicinity of his native village in Vadodara district and invited us all there for an inaugural party (insisting that I must go, though I could not). He later declared and got the resolution passed in the Trust Meeting about two years before to spend Rs. 50 lakh to build a hospital building on this purchased land with an intension to start an eye hospital in due course for the benefit of the rural people of the area there with management under the control of the Trust. The land is already purchased, fund is there, the Trust (and all his family members) in its last meeting on December 31, 2017 did resolve to make all the efforts for this, but all the concerned people are full time busy and the absence of the driving force was strongly felt!!

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PRESIDENTIAL GENERAL ADDRESS*

MANJUL GUPTA

Hon'ble Chief Guest and Vice-Chancellor Prof. A. Damodaram of Sri Venkateswara University, Prof. S. Sreenath, Local Organizing Secretary, Prof. N. K. Thakare, General Secretary, IMS, other dignataries sitting on the dias, fellow mathematicians, young researchers, ladies and gentlemen:

It gives me a great pleasure to welcome all of you to the 83rd Annual Conference of the Indian Mathematical Society which has grown to be an International Meet from this year. This conference is being held at Tirupati which is referred to as the "Spiritual Capital of Andhra Pardesh" and is the holiest Hindu privileged city because of Tirumala Venkateswara temple. Indeed, "Tiru" means "sacred" in Dravidian translation and also referred to Goddess Lakshmi and "Pati" means "husband" and so it is known as the sacred abode of Lord Vishnu. The idols of the incarnations of Lord Vishnu, namely Shri Rama and Krishna find their place in the sanctum sanctorium of the temple. According to Varha Purana, during Treta Yugam, Shri Rama resided here along with Sita Devi and Lakshmana on his return from Lankapuri.

I also take this opportunity to express my gratitude to the members of the Indian Mathematical Society for electing me the President of the Society for the year 2017-2018 in which I have such a distinguished line of predecessors. It is indeed a great honor and privilege for me to address this august gathering of mathematicians and invited guests today.

From distinguished line of my predecessors, I would like to name late Prof. H. P. Dikshit whom I met when I was a research scholar and later also when he visited IIT Kanpur. We have lost him this year in April 2017. He was a distinguished mathematician having experience in the formal and distance education system, besides being an able administrator. I am also deeply pained to add one more name of late Prof. V. M. Shah who breathed his last almost a week back on December 3, 2017. He had served the Society in different capacities for 26 years. He was a soft spoken person with cognitive personality.

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* The text of the Presidential Address (general) delivered at the 83^{rd} Annual Conference of the Indian Mathematical Society - An International Meet, held at Sri Venkateswara University, Tirupati - 517 502, Andhra Pradesh, India during December 12 - 15, 2017.

Being a lady, I can not stop recalling the name of the woman who won the 'Nobel Prize' of mathematics, that is, 'The Field Medal Award' in 2014. This award is given in every four years since 1936 to mathematicians under the age of 40 for their outstanding contribution in the subject. Her name is Maryam Mirzakhani from Iran, whose untimely demise this year at the age of forty has created a vaccum in the realm of mathematics. She was professor at the Stanford University, USA and was given this award for her contribution to "Dynamics and Geometry of Riemann Surfaces and Modulii Spaces" along with three other mathematicians including Prof. Manjul Bhargava. As the number of women scientists and mathematicians are limited, she said, "I will be happy if this award encourages young female scientists and mathematicians. I am sure there will be many more women winning this kind of award in coming years."

I pay my homage to these mathematicians - Prof. H. P. Dikshit, Prof. V. M. Shah and Prof. Maryam Mirzakhani.

As Tirupati is an ancient city having a vibrant historical heritage, similar is the case with mathematics. Its historical development in India is as old as the civilization of its people. Indeed, history of mathematics is the common heritage of mankind, irrespective of any particular nation, race or country. However, mathematics being the mirror of civilization, its development from prehistoric time has been affected by place , time and sociological requirements. Accordingly, we have prehistoric mathematics (time before the invention of writing system, stone age etc.), Sumerian mathematics (2000-1500 BC), Babylonian mathematics (3000 BC), Egyptian mathematics (2000 to 1800 BC), Greek mathematics (700-450 BC), Roman mathematics (middle of first century BC), Mayan mathematics (250 to 900 CE), Chinese mathematics (2000 BC), Indian mathematics (sutras from early vedic period 1000 BC), Islamic mathematics (9th to 15th century), Medieval Europian mathematics (12th to 15th century) and then came the period of 16th, 17th, 18th 19th and 20th century mathematics. I won't go into the details of evolution of mathematics as many articles and books are available on this topic. Besides many societies, like - Indian Society of History of Mathematics (ISHM), The British Society of History of Mathematics, Canadian Society of the History and Philosophy of Mathematics, International Commission of the History of Mathematics etc., have been formed in order to promote this area and are holding conferences all over the world. Recently, "International Conference on Exploring the History of Indian Mathematics" was held about a week back at IIT Gandhinagar from December 04 to December 06, 2017.

With the women empowerment in all fields of our society at present, I was curious to know whether women had a role to play in the development of mathematics in olden days. While tracing the history of Indian mathematics, I came across with the name "Lilavati" which means playful or one possessing play (from Sanskrit, 'Lila' means 'play' and 'vati' means 'female'). Indeed, it is the book written by the great Indian mathematician Bhaskaracharya II, around 1150 AD, in the name of his daughter who must have been a very bright young woman. Quoting [2], about Lilavati, it is said that she was destined to pass her life unmarried and without children. To avoid this fate, Bhaskaracharya II ascertained a lucky hour for contracting her in a marriage, but the destiny had a role to play. Inspite of his best efforts, the lucky hour passed away and marriage could not be performed. Disappointed father said to his daughter that he would write a book which shall remain to the latest times - for a good name is a second life, and the ground work of the eternal existence. Thus the motivation for writing this book came from a young girl who happened to be the daughter of a great mathematician.

This book deals with arithmetic and geometry through sutras (aphormisms), many of these are addressed to Lilavati. The sutras are original thoughts spoken or written in a concise and memorable form and come from our civilization. Indeed, our sages passed on their collected work orally from one generation to other using codes which unlock various layers of meaning. This system was lost to the world until it was discovered by a scholar "Swami Trithaji" (Jagadguru Swāmī̄ Śrī̄ Bhāratī̄ Krsna Tī̄rthajī̄ Mahāraja, 1884 -1960) in the last century and so it can be thought as twentieth century mathematics. The mathematics sutras hold the power to speed up your calculation, give you confidence and make mathematics fun and interesting, as suggested by the name Lilavati.

Let us now consider two sutras with their applications.

1. **Nikhilam Navataščaraman Dašatah** (All from nine and the last from ten): For applying this sutra, powers of 10 are taken as base. Let us apply this sutra to multiplication and division.

(a) **Multiplication**:

Let us multiply 997 by 985 and 115 by 107 in (i) and (ii) respectively. For (i) the nearest base is 1000 whereas for (ii) it is 100. Subtract base from these numbers and write the remainders on the right of the corresponding numbers with appropriate sign and separated by a vertical line. Cross add and put them below these numbers (as shown in the diagram). Multiply the remainders vertically and write below on the right side of the vertical line with 0 preceding the multiplication (having the same number of digits as the number of zeros in the base). In (ii) the multiplication is three digit number, so 1 is carried to the left and added as shown.

$$(i)\quad \begin{array}{r|r} 1000 & \\ 997 & -3 \\ 985 & -15 \\ \hline 982 & 045 \end{array} \qquad\qquad (ii)\quad \begin{array}{r|r} 100 & \\ 115 & 15 \\ 107 & 07 \\ \hline 122 & 05 \\ 1 & \\ \hline 123 & 05 \end{array}$$

Thus for (i) and (ii), the answers are 982045 and 12305 respectively.

In case both the multiplicand and multiplier are not near to a convenient base, we apply the subsutra (or subformula) "Ānurūpyena" which means "proportionality", e.g., in examples (iii) and (iv) our bases are $10 \times 5 = 50$ and $\frac{1000}{4} = 250$ respectively.

Let us now begin as above with the following diagrams for the multiplication of the numbers 48 with 47 and 249 with 245.

$$(iii)\quad \begin{array}{r|r} 50 & \\ 48 & -2 \\ 47 & -3 \\ \hline 45 & 6 \\ 5 & \\ \hline 225 & 6 \end{array} \qquad\qquad (iv)\quad \begin{array}{r|r} 250 & \\ 249 & -1 \\ 245 & -5 \\ \hline 4)244 & 005 \\ 4 & \\ \hline 61 & 005 \end{array}$$

In case of (iii) we multiply 45 by 5 in order to get the answer 2256, whereas in case of (iv) we divide 244 by 4 and we get the answer as 61005.

(b) **Division**

When we divide 115 by 58, our working base is 100, which we write as in the diagram (i) below. Here divisor is 58, dividend is 115, quotient is 1 and remainder is 57. Indeed, we subtract 58 from 100 to get 42 and put a line between 1 and 15 as shown. We multiply 42 with 1 and put the product below 15 and add to get the remainder as 57. We follow the same procedure for (ii), where we divide 20137 by 9819. Here the working base is 10000, quotient is 2 and remainder is 499. While multiplying 8 with 2, we get 16 which is written and added as shown in (ii).

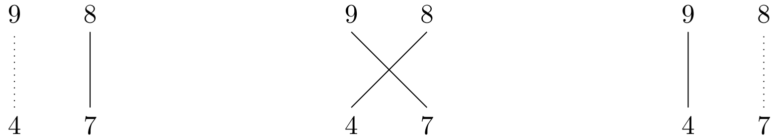
$$(i)\quad \begin{array}{r|r} 100 & \\ 58 & 1/15 \\ 42 & 42 \\ \hline & 1/57 \end{array} \qquad\qquad (ii)\quad \begin{array}{r|r} 10000 & \\ 9819 & 2/0137 \\ 0181 & 02_162 \\ \hline & 2/0499 \end{array}$$

2. **Ürdhva Tiryagbhyäm (vertically and crosswise)**

As the name suggests, we multiply vertically and crosswise while applying this sutra, illustrated in the following examples:

(I) Let us multiply 98 by 47 using the following diagram:

Let us place the digits of the numbers to be multiplied at the points on the upper and lower line. Now perform the multiplication of the corresponding numbers following the dark lines. In case, after multiplication, we get one digit number, put it below that particular vertical line and if the number is more than one digit leave the extreme right digit below the vertical line and carry over the remaining digits to the next line, which appear on the left of its previous digit. Adding these digits to the multiplication of the last column, we get the final answer.

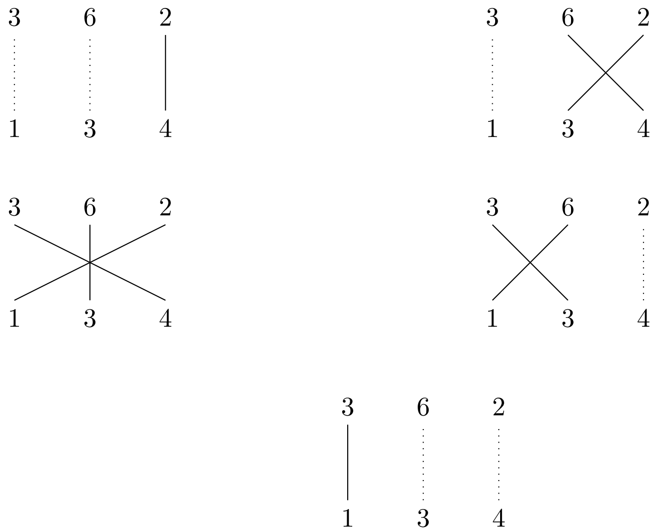


Indeed, according to figures from left to right, we get (i) $8 \times 7 = 56$, (ii) $9 \times 7 + 8 \times 4 = 95$, (iii) $9 \times 4 = 36$.

Now put 6 of 56 in the column of 8 & 7, 5 of 95 in column of 9 &4 and 36 preceding 56 below the horizontal dark line, and carry 5 of 56 to the column of 9 &4, and 9 of 95 below 6 of 36 in the next line, as shown below in the diagram. Then add to get the answer as 4606 (indeed, whenever one of the results contains more than one digit, the right hand most digit is to be put down there and the preceding digit or digits should be carried over to the next line and placed under the previous digit or digits of the upper row)

$$\begin{array}{r} 98 \\ 47 \\ \hline 3656 \\ 95 \\ \hline 4606 \end{array}$$

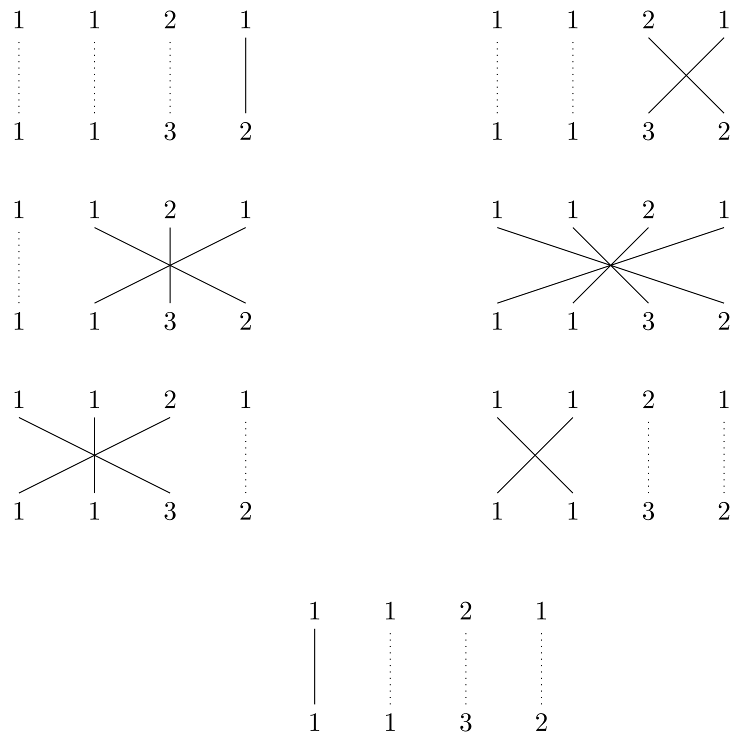
(II) Similarly using the following diagram and placing the numbers appropriately as above, we can multiply 3 digit numbers, e.g. for 362×134, we get 48508 as the answer



Indeed, figures, (i) $2\times4=8$, (ii) $6\times4+2\times3=30$, (iii) $2\times1+4\times3+6\times3=32$, (iv) $3\times3+6\times1=15$ and (v) $3\times1=3$ yield

$$\begin{array}{r} 362 \\ 134 \\ \hline 35208 \\ 133 \\ \hline 48508 \end{array}$$

(III) For four digit numbers we follow the following diagram, for example, we get $1121\times1132=1268972$.



One may proceed like this for multiplying higher digit numbers using these sutras.

There are sixteen sutras and thirteen subsutras discovered by Swami Tirthaji which deal with basic arithmetic operations, decimal/fraction, conversions etc. The purpose of these sutras is to evolve a method which makes calculations easy and short. In case one sutra does not help , other one is used; for instance Nikhilam does not help when divisors are small digit numbers, then sutras like- Parāvartya Yojayet (transpose and adjust), Dhvajāṅka (flag method) may apply. I urge the audience to enjoy the basic mathematics with sutras and teach children to learn the same. One may refer to [1, 3].

Finally, I sum up with the note that mathematics has shaped our lives and moulded our civilization through ages and still the process is on with more vigour and passion.

Thank You.

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SOME TOPICS IN FUNCTIONAL ANALYSIS: OUR CONTRIBUTIONS WITH EMPHASIS ON HOLOMORPHY

MANJUL GUPTA

Abstract. In this talk, I shall give a brief description of our work carried out by me with my group of co-researchers during a period of almost forty five years. I shall pay more attention to the study of the spaces of holomorphic functions defined on open subsets of a finite/infinite dimensional locally convex space.

1. INTRODUCTION

After the birth of a basis in a Banach space by J. Schauder around the year 1927, notion of a general topological space by F. Hausdorff in 1904, and that of a locally convex space by John von Neumann in the year 1935, mathematicians working in Functional Analysis got interested to consider all spaces of importance for finding the structural properties of these spaces vis-á-vis basis and their relationships. Among such spaces, the class of entire functions occupy a privileged position because of its vast applications in the theory of distributions, theory of locally convex spaces etc, as well as providing examples for various concepts being studied in several off-shoots of Functional Analysis, for instance, nuclear locally convex spaces, hypercyclicity of linear operators, different types of bases etc.

I started my research career with the spaces of analytic/entire functions of two variables, which played an important role in motivating me to carry out researches in several topics of Functional Analysis and Operator Theory, namely, Schauder basis and decompositions, scalar and vector valued sequence spaces, nuclearity of spaces and operators, bi-locally convex spaces/mixed convergence, ordered generalized sequence spaces and ordered decompositions, approximation quantities and ideals of linear operators, dynamics of linear operators and currently pursuing researches in infinite dimensional holomorphy. All these topics which are interrelated in one way or the other involve the spaces of entire and analytic functions except possibly our study of order structures; for this study is usually carried out for the vector spaces defined over the real field.

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Before passing on to describing the salient features of our contributions in these topics, let us familiarize ourselves with the basics of locally convex spaces and the spaces of entire functions of one complex variable.

1.1. **Locally Convex Spaces.** For a subset A of a vector space X over a vector field $\mathbb{K} = \mathbb{R}$ (reals) or $\mathbb{C}$ (complex numbers), its Minkowski's functional or gauge p_A is defined as

$$p_A(x) = \inf\{\rho > 0 : x \in \rho A\},\ x \in X.$$

If A is balanced, convex and absorbing subset of X, then p_A is a semi-norm on X.

A topological vector space (TVS) is a vector space equipped with a topology T for which the vector operations are jointly continuous. A locally convex space(LCS) (X, T) is a TVS for which there exists a fundamental neighborhood system $\mathcal{U}_X$ at origin consisting of absorbing, balanced and convex sets or equivalently the topology T is generated by the family $D_T = \{p_U : U \in \mathcal{U}_X\}$ of continuous semi-norms. We assume throughout that (X, T) is a Hausdorff LCS, which means that $p(x) = 0, \forall\ p \in D_T$ yields $x = 0$. A subset B of X is *bounded* if it is absorbed by each member $U \in \mathcal{U}_X$, that is, there exists a $\lambda > 0$ such that $B \subset \lambda U$; and is said to be *bornivorous* if it absorbs every bounded subset of X. A net $\{x_\alpha\}$ in X tends to $x \in X$ with respect to the topology T if and only if $p(x_\alpha - x) \to 0, \forall\ p \in D_T$. A LCS (X, T) is said to be *complete* (*sequentially complete*) if every net $\{x_\alpha\}$ (sequence $\{x_n\}$) converges in (X, T); and it is called a *barrelled* (*infrabarrelled*) space if every barrel (bornivorous barrel) in (X, T) is a neighborhood of origin in (X, T) where a *barrel* in (X, T) means an absorbing, balanced, convex and T-closed set. A complete metrizable locally convex space is called a *Fréchet space*. A LCS (X, T) is said to be a *semi-Montel* space if every bounded subset of X is relatively compact. An infrabarrelled semi-Montel space is said to be a *Montel* space.

Let us now recall polar topologies defined corresponding to a dual pair $< E, F >$ of vector spaces E and F with respect to a bilinear form $< \cdot, \cdot >$. For a given subset A of E, the polar of A is the set $A^\circ = \{y \in F : |< x, y >| \leq 1,\ \forall x \in A\}$. The *weak, Mackey and strong topologies* on E are respectively the topologies $\sigma(E, F)$, $\tau(E, F)$ and $\beta(E, F)$ generated by the family $\{p_{A^\circ} : A \in \mathfrak{S}\}$, where $\mathfrak{S}$ is respectively the collection of finite or singleton subsets of F; all balanced, convex, $\sigma(F, E)$-compact subsets of F; and all $\sigma(F, E)$-bounded subsets of F. Similarly, one can have the topologies $\sigma(F, E)$, $\tau(F, E)$ and $\beta(F, E)$ on F. A set $B \subset X^*$ is said to be *equicontinuous* if $B \subset U^\circ$ for some $U \in \mathcal{U}_X$.

For a LCS (X, T), $< X, X^* >$ forms a dual pair with respect to the bi-linear form $< x, f >= f(x)$, $x \in X$, $f \in X^*$, topological dual of X. We call a LCS (X, T) a *Mackey space* if $T \approx \tau(X, X^*)$ and an *S-space* if $(X^*, \sigma(X^*, X))$ is sequentially complete. It is known that every barrelled space is a Mackey S-space.

The generalization of normed linear spaces (nls) is two fold, namely, locally convex spaces and locally bounded spaces (a TVS is *locally bounded* if there exists a bounded neighborhood at 0). Among locally convex spaces, the class which is much closer to finite dimensional space as compared to the Hilbert spaces, is the class of nuclear locally convex spaces. These spaces were introduced by A. Grothendieck [39] around the year 1953, where he studied these spaces using the technique of tensor products. To have an appreciative idea of what these spaces are, let us recall that a linear map $A : (X, T_X) \to (Y, T_Y)$ where (X, T_X) and (Y, T_Y) are two LCS, is said to be *nuclear* if there exist $\{\alpha_n\}$ in ℓ_1 (the class of all absolutely summable sequences), an equicontinuous sequence $\{f_n\}$ in X^* and a bounded sequence $\{y_n\}$ in Y with

$$A(x) = \sum_{n\geq 1} \alpha_n f_n(x) y_n, \ \forall\ x \in X.$$

A LCS (X, T_X) is said to be *nuclear* (resp. *Schwartz*) if for each $U \in \mathcal{U}_X$, there exits $V \in \mathcal{U}_X$ such that V is absorbed by U, i.e. $V \subset \alpha U$ for some $\alpha > 0$, and the mapping $K_U^V : X_V \to X_U$ with $K_U^V(x_V) = x_U$ is nuclear (precompact), where $X_V = X/kerp_V$ and $X_U = X/kerp_U$ are the normed spaces w.r.to the quotient norm $\hat{p_V}$ and $\hat{p_U}$ respectively, and $kerp_V = \{x \in X : p_V(x) = 0\}$, $kerp_U = \{x \in X : p_U(x) = 0\}$. The references for the detailed theory of locally convex as well as nuclear spaces are [83, 114].

1.2. **Spaces of Entire Functions of One Variable.** The rudimentary theory of the spaces of entire and analytic functions dates back to eighteenth century in the investigations of L. Euler and J.R. d'Alembert; however, the full following of the subject was possible only after the publications of the fundamental works of A. L. Cauchy, B. Riemann and K. Weierstrass in the nineteenth century. Its methods are widely used in many branches of mathematics and engineering. The literature on the structural study of spaces of analytic/entire functions of one variable is too vast to be elaborated here; but for our work I would like to mention the names of early contributors, namely, A. I. Markusevich, V. G. Iyer, M. G. Arsove, V. Krishnamurthy etc.

All of us know that a function $f : \mathbb{C} \to \mathbb{C}$ is said to be *entire* if it has the expansion

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \ z \in \mathbb{C}, \tag{1.1}$$

where $|\hat{f}(n)|^{\frac{1}{n}} \to 0$ as $n \to \infty$.

Let us denote by $\mathcal{H}(\mathbb{C})$ the class of all entire functions and equip this space with the *compact open topology* τ generated by the family of semi-norms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ where

$$\|f\|_n = \sup_{|z|\leq n} |f(z)|, \ f \in \mathcal{H}(\mathbb{C}).$$

The translation invariant metric d on $\mathcal{H}(\mathbb{C})$, which also generates τ, is defined as

$$d(f,g) = \sum_{n\geq 1} \frac{1}{2^n} \frac{\|f-g\|_n}{1+\|f-g\|_n},\ f,g \in \mathcal{H}(\mathbb{C}).$$

The space $(\mathcal{H}(\mathbb{C}), \tau)$ is a nuclear Fréchet space, cf. [114]; and a sequence $\{f_n\}$ converges to f in $(\mathcal{H}(\mathbb{C}), \tau)$ if and only if $f_n(z) \to f(z)$ uniformly on all compact subsets of $\mathbb{C}$.

For an entire function f and $r > 0$, define $M(r) = \sup_{|z|=r} |f(z)|$. Then f is said to be of *finite order* if there exists a positive number μ such that

$$M(r) < e^{r^{\mu}} \tag{1.2}$$

for all sufficiently large r. The number $\rho = \inf \mu$, infimum being considered over such μ's for which (1.2) holds, is called the *order* of an entire function f. Given an entire function f of order ρ, suppose there exists a positive number k such that

$$M(r) < e^{kr^{\rho}} \tag{1.3}$$

for all sufficiently large r. Then f is said to be of *finite type* and $\sigma = \inf k$, over those values of k for which (1.3) holds, is called the *type* of f.

An element γ of $\mathcal{H}(\mathbb{C})$ with $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n,\ \gamma_n > 0$ for each $n \in \mathbb{N}_0$ is called a *comparison function* if the sequence of ratios $\frac{\gamma_{n+1}}{\gamma_n}$ decreases to zero as n increases to ∞, where $\mathbb{N}$ is the set of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In addition, if the sequence $\{w_n\}$, $w_n = \frac{n\gamma_n}{\gamma_{n-1}}$ is also monotonically decreasing, we call γ an *admissible comparison function*.

The differential and translation operators on the space $\mathcal{H}(\mathbb{C})$, which are respectively denoted by D and T_a, $0 \neq a \in \mathbb{C}$, are defined as $Df(z) = f'(z)$ and $T_a f(z) = f(z+a)$, for $f \in \mathcal{H}(\mathbb{C})$ and $z \in \mathbb{C}$.

Using the function γ and results on sequence spaces, various subspaces of the space $\mathcal{H}(\mathbb{C})$ have been defined, which we shall consider in due course.

Hardy space defined on the open unit disc $\mathbb{D}$ of $\mathbb{C}$ as

$$\mathcal{H}^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{n\geq 0} a_n z^n : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

is a Hilbert space with orthonormal basis $\delta_n(z) = z^n$, $n \in \mathbb{N}_0$ and the inner product is given by $< f, g >= \sum_{n\geq 0} a_n \overline{b_n}$ for $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$. For the theory of holomorphic/entire functions, we refer to [103].

2. Salient Features of Our Research Contributions

In this section we mention briefly, the salient features of our contributions. Let us begin with

(a) Spaces of Analytic Functions of Two Variables: Let us denote by X the space of all entire functions of two variables $z_1,\ z_2 \in \mathbb{C}$, equipped with the topology T of uniform convergence on compact sets in $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. In other words, $f \in X$ if and only if

$$f(z_1,z_2)=\sum_{\substack{m\\ m+n\geq 0}}\sum_{n} a_{mn}z_1^m z_2^n,\ \ z_1,z_2\in\mathbb{C}$$

with

$$\limsup_{m+n\to\infty}|a_{mn}|^{\frac{1}{m+n}}=0.$$

If $M(f;r_1,r_2)=\sup_{\substack{|z|\leq r_i\\ i=1,2}}|f(z_1,z_2)|$ for $r_1,\ r_2>0$, then the topology T is generated by the family of semi-norms $\{M(f;r_1,r_2):r_1>0,r_2>0\}$.

The space (X,T) is known to be a non-normed nuclear Fréchet space, cf. [114]. In the papers [41, 85, 87], we were interested in studying the topological aspects of certain subspaces X_1 and X_2 of the space X and the problem of characterizing the proper bases (those bases equivalent or similar to $\delta_{mn}(z_1,z_2)=z_1^m z_2^n$) in them. Indeed, for given finite positive numbers $\rho_1,\rho_2;\sigma_1,\sigma_2$, we define $X_1\equiv X(\rho_1,\rho_2)$ and $X_2\equiv X(\rho_1,\rho_2;\sigma_1,\sigma_2)$ as follows

$$\begin{aligned}X_1\equiv\{f\in X:\ \text{for } \epsilon>0,\ M(f;r_1,r_2)\leq \exp[r_1^{\rho_1+\epsilon}+r_2^{\rho_2+\epsilon}],\\ \text{for all sufficiently large } r_1,\ r_2\geq 0\}\end{aligned}$$

and

$$\begin{aligned}X_2\equiv\{f\in X:\ \text{for } \epsilon>0,\ M(f;r_1,r_2;\sigma_1,\sigma_2)\leq \exp[(\sigma_1+\epsilon)r_1^{\rho_1}\\ +(\sigma_2+\epsilon)r_2^{\rho_2}]\ \text{valid for all large } r_1,r_2\}.\end{aligned}\tag{2.1}$$

In the terminology of entire functions of several complex variables, an f in X_1 is said to possess an *order point at best equal to* (ρ_1,ρ_2) and an $f\in X_2$ is said to possess *a type at best equal to* (σ_1,σ_2) *relative to the order point* (ρ_1,ρ_2). Clearly, $X_2\subset X_1\subset X$.

Making use of a characterization of Fred Gross [29] for members of X_2, namely- "an f in X belongs to X_2 if and only if

$$|a_{mn}|<[\frac{e\rho_1(\sigma_1+\epsilon)}{m}]^{\frac{m}{\rho_1}}[\frac{e\rho_2(\sigma_2+\epsilon)}{n}]^{\frac{n}{\rho_2}}$$

for all sufficiently large $m+n$ and arbitrary $\epsilon>0$; in [85], we topologize the space X_2 by the family of semi-norms $\{\|\cdot;\sigma_1+\delta,\sigma_2+\delta\|:\delta>0\}$, where

$$\|f;\sigma_1+\delta,\sigma_2+\delta\|=|a_{00}|+\sum_{\substack{n\\ m+n\geq 1}}\sum_{m}|a_{mn}|[\frac{m}{(\sigma_1+\delta)e\rho_1}]^{\frac{m}{\rho_1}}[\frac{n}{(\sigma_2+\delta)e\rho_2}]^{\frac{n}{\rho_2}}$$

and show that the sequence $\{\delta_{mn}\}$, $\delta_{mn}(z_1,z_2)=z_1^m z_2^n$, $z_1,\ z_2\in\mathbb{C}$, $m,n\in\mathbb{N}$, forms a Schauder base for X_2 with respect to the topology T_2 which makes X_2 a Fréchet space. Further, we use the series representation of members of X_2 to characterize its topological dual and continuous linear operators from X_2 into itself. We also deal with the problem of proper bases in X_2.

However, for members of X_1, the characterization of Fred Gross in terms of Taylor coefficients doesn't help in getting the topological structure of X_1; and so in [87] we first characterize the members of X_1 in the form which may be suitable to topologize this space; indeed, we prove

Theorem 2.1. *An f in X belongs to X_1 if and only if for each $\delta > 0$, there exists an integer $N_0 \equiv N_0(\delta)$ such that*

$$|a_{mn}| \leq m^{-\frac{m}{\rho_1+\delta}} \; n^{-\frac{n}{\rho_2+\delta}}$$

for all $m + n \geq N_0$.

This characterization gives rise to a family of norms $\{\|\cdot; \rho_1+\delta, \rho_2+\delta\| : \delta > 0\}$ on X_1, where for $f \in X_1$

$$\|f; \sigma_1 + \delta, \sigma_2 + \delta\| = |a_{00}| + \sum_{\substack{m \\ m+n\geq 1}} \sum_{n} |a_{mn}| m^{\frac{m}{\rho_1+\delta}} n^{\frac{n}{\rho_2+\delta}}.$$

The topology T_1 generated by this family of norms is a Fréchet topology on X_1, which is stronger than the induced topology from X. The problems tackled in this paper [87], as well as in the papers [86, 88] where we consider functions analytic in a bicylinder, are similar in nature as mentioned above and so we shall not elaborate them. In [41], we show that the space (X_1, T_1) is a Montel space. In subsequent work [44], certain simultaneous automorphism on X and a subspace Y of it, bearing a topology stronger than the induced one, have been constructed (*simultaneous automorphism* on X and Y means a mapping f such that f is an automorphism on X and $f|Y$ is an automorphism on Y).

Our paper [60] deals with the space $K < x_1, x_2 >$ of entire functions of two variables $x_1, x_2 \in K$ which is a non-archimedean valued field complete under the metric of valuation of rank 1 or real valuation (*a non-archimedean valuation of rank* 1 is a real-valued function ϕ defined on the field satisfying the properties: $\phi(a) > 0$ for $a \neq 0$, $\phi(0) = 0$; $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a + b) \leq \max\{\phi(a), \phi(b)\}$ for $a, b \in K$, cf. [16]). We introduce a complete linear metrizable topology on $K < x_1, x_2 >$, which makes it a totally disconnected space. We characterize continuous linear functionals on $K < x_1, x_2 >$ and show that its dual is not a topological vector space. For the subject matter of this section, one may refer to any elementary book on functions of several complex variables, for instance, [30].

(b) **Schauder Bases and Decompositions:**

The theory of Schauder bases plays an important role for various investigations in the theory of sequence spaces, theory of nuclear spaces, theory of ordered topological vector spaces etc. For the study of these topics in Banach spaces, we refer to the books [101, 104, 127]. However, our major contributions in this direction are the monographs [91, 92, 93] which contain almost all our researches on this topic carried out till the time of their publications. To have the glimpse of some important contributions and later publications, let us first consider Schauder bases and then Schauder decompositions.

(I) Schauder Bases: A *topological base* or just a base in a TVS (X,T) is a pair $\{x_n, f_n\}$ of sequences $\{x_n\} \subset X$, $\{f_n\} \subset X'$ (algebraic dual of X) with $f_i(x_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta such that each $x \in X$ has the form

$$x = \sum_{i=1}^{\infty} f_i(x)x_i,$$

where the convergence of the series is considered in the linear topology T of X. If $\{f_n\} \subset X^*$, the base $\{x_n, f_n\}$ is said to be a *Schauder base* for X. A base $\{x_n, f_n\}$ forming a Schauder base for the topology $\sigma(X, X^*)$ of a TVS is called a *weak Schauder base.*

Depending on the types of convergence and properties of the components comprising a base, one can define various types of Schauder bases in a TVS. Before we pass on to various notions of Schauder bases, let us consider the following from [91].

Definition 2.2. A formal series $\sum_{n\geq 1} x_n$ in a LCS (X,T) is said to be (i) *convergent* if the sequence $\{\sum\limits_{i=1}^{n} x_i\}$ converges in X, (ii) *absolutely convergent* if for each $p \in D_T$, $\sum_{n\geq 1} p(x_n) < \infty$, (iii) *weakly convergent* if there exists $x \in X$ such that $\sum_{i\geq 1} f(x_i)$ converges to $f(x)$, for each $f \in X^*$, (iv) *bounded multiplier convergent* if for each $b = \{b_i\} \in \ell^\infty$ (the space of all scalar bounded sequences), the series $\sum_{i\geq 1} b_i x_i$ converges in (X,T), (v) *subseries convergent* if for any increasing sequence J in $\mathbb{N}$, the series $\sum_{i\in J} x_i$ converges in (X,T), and (vi) *unconditionally or reordered convergent* if for any $\sigma \in \Pi$ (the set of all permutations of $\mathbb{N}$), the series $\sum_{i\geq 1} x_{\sigma(i)}$ converges to the same element x in X.

It is known that in a sequentially complete LCS (X,T), every absolutely convergent series is convergent, but converse holds in a metrizable LCS (X,T), i.e., in that case (X,T) becomes a Fréchet space, cf. [91], p.142.

In our work, the following notions of Schauder bases are of particular interest.

Definition 2.3. A Schauder base $\{x_n, f_n\}$ in a LCS (X,T) is said to be *unconditional* (*subseries, bounded multiplier*) convergent if the series $\sum\limits_{n\geq 1} f_n(x)x_n$ is convergent *unconditionally* (*subseries, bounded multiplier*) in (X,T) for each $x \in X$, (ii) *semi-absolute* if the series $\sum_{n\geq 1} f_n(x)x_n$ is absolutely convergent or equivalently, $\sum_{n\geq 1} |f_n(x)|p(x_n) < \infty$, $\forall x \in X$ and $p \in D_T$, (iii) *absolute* if it is semi-absolute and whenever $\sum_{n\geq 1} |\alpha_n|p(x_n) < \infty$ for some sequence $\{\alpha_n\}$ of scalars and $\forall\, p \in D_T$, then $\sum_{n\geq 1} \alpha_n x_n$ converges in (X,T), (iv) *fully absolute* if it is semi-absolute and $T \approx T_Q$ where T_Q is the locally convex topology generated by the family Q of semi-norms Q_p on X, where $Q_p(x) = \sum_{n\geq 1} |f_n(x)|p(x_n)$, (v) *shrinking (almost shrinking)* if $\{f_n, \Psi(x_n)\}$ is a Schauder basis for $(X^*, \beta(X^*, X))$ (resp. $(X^*, \tau(X^*, X))$), where Ψ is the usual canonical embedding from X into $X^{**} = (X^*, \beta(X^*, X))^*$, (vi) *boundedly complete* if the series $\sum_{i\geq 1} \alpha_i x_i$ converges in X whenever the sequence $\{\sum_{i=1}^n \alpha_i x_i\}$ is bounded in X, where $\{\alpha_n\} \subset \mathbb{K}$, and (vii) *symmetric* if for each pair (ρ, σ) in $\Pi \times \Pi$ and $x \in X$, the series $\sum_{n\geq 1} f_{\rho(n)}(x) x_{\sigma(n)}$ converges in (X, T).

For the historical briefing of the developments of these types of Schauder bases, we refer to [93], p.35, and the references given therein. Shrinking and boundedly complete bases/decompositions have been found useful by R. C. James and T. A. Cook in characterizing the semi-reflexivity of the space in which they are present, cf. [92], p.133, 144 and [93], p.190. The notions of shrinking and almost shrinking bases are enveloped in the definition of $\mathfrak{S}$-uniform basis which were introduced in [89]; indeed

Definition 2.4. Corresponding to a collection $\mathfrak{S}$ of bounded subsets in a LCS (X, T) such that $\mathfrak{S}$ covers X, a Schauder base $\{x_n, f_n\}$ is said to be $\mathfrak{S}$-*uniform* if $\lim\limits_{n\to\infty} \sup\limits_{x\in B} |f(x - S_n(x))| = 0$ for each $f \in X^*$ and each $B \in \mathfrak{S}$, where $S_n(x) = \sum\limits_{i=1}^{n} f_i(x) x_i$.

Concerning such bases, we prove in [89]

Theorem 2.5. *In a LCS (X, T), a Schauder base $\{x_n, f_n\}$ is $\mathfrak{S}$-uniform if and only if $\{f_n, \psi(x_n)\}$ is a base for X^* for the topology of $\mathfrak{S}$-convergence. In addition, if every $\mathfrak{S}$-bounded sequence in X^* is equicontinuous and if $\{x_n, f_n\}$ is $\mathfrak{S}$-uniform, then $\{f_n, \psi(x_n)\}$ is a boundedly complete base for X^* in the topology of $\mathfrak{S}$-convergence.*

Note that $\mathfrak{S}$-uniform base is a shrinking base (almost shrinking base) if $\mathfrak{S}$ is the collection of all bounded subsets of X (of all balanced convex and $\sigma(X, X^*)$-compact subsets of X).

The importance of a weak Schauder base in asserting the completeness of the space or its dual containing the base was also proved by us, cf. [92], p.39.

The notion of an absolute base was used for the first time by Karlin [96] in proving a result that a Banach space having an absolute Schauder base is isometrically isomorphic to l^1. This led to many generalizations, especially replacing the Banach space by more general LCS and dissecting an absolute Schauder base into various components in order to see their effect on the structure of the underlying LCS (some of these have already been defined in Definition 2.3). We show the independence of several notions of absolute Schauder bases defined by Kalton [95] in a LCS, which lie in the lumped form of absolute Schauder base in a Fréchet space, cf. [93], p.63. We also obtain some interesting applications of fully absolute base in the form of

Theorem 2.6. *Let (X, T) be a locally convex space containing a fully absolute Schauder base $\{x_n, f_n\}$. Then T and $\sigma(X, X^*)$ define the same convergent and*

Cauchy sequences. Also, each f in X^* is precisely of the form

$$f(x) = \sum_{n\geq 1} \alpha_n f_n(x),\ x = \sum_{n\geq 1} f_n(x)x_n,$$

where $\{\frac{|\alpha_n|}{p(x_n)}\} \in \ell^\infty$ for some $p \in D_T$, it being understood that $\frac{0}{0}$ is to be regarded as 0.

The above results envelop the results of Newns [110] proved for Fréchet spaces.

A close relationship of absolute Schauder bases with a nuclear locally convex spaces is well known to the experts working in this area. I would like to mention the following two results due to A. Dynin-B.S. Mityagin and W. Wojtyński; and A. Pietsch, cf. [93], p. 401.

Theorem 2.7. *Let a Fréchet space X have a Schauder basis. Then X is nuclear if and only if all its Schauder basis are absolute.*

Theorem 2.8. *A Fréchet space X with an absolute basis $\{x_n, f_n\}$ is nuclear if and only if $\{f_n, \psi(x_n)\}$ is an absolute base for $(X^*, \beta(X^*, X))$.*

The concept of absolute bases and nuclearity have further been extended to λ-bases and λ-nuclearity corresponding to a sequence space λ and we refer to [94, 128] and references given therein, for various results and historical development of this topic. Besides, λ-similarity between Schauder bases present in LCS (X, T) and (Y, S) have been explored in [63], cf. also [93].

We have also found necessary and sufficient conditions for a given sequence to form a Schauder base and also give examples in support of hypothesis of results related to the characterization of continuous linear functionals, cf. [92], p.41,66. The stability theorems of Paley-Wiener type have been dealt in [64]; indeed if $\{x_n\}$ is a Schauder base in a LCS X and $\{y_n\}$ is a sequence in X chosen sufficiently near to $\{x_n\}$, one may ask whether $\{y_n\}$ is a Schauder base for X, or $\{y_n\}$ is stable with respect to Schauder base character?

The presence of a Schauder base $\{x_n, f_n\}$ in a real LCS (X, T) yields the order structure induced by the cone $K = \{x : f_n(x) \geq 0, \forall\ n \geq 1\}$. This study has been carried out in [62], where it has been shown that X is a vector lattice if the Schauder base is unconditional.

(II) Schauder Decompositions:

There are situations when separable Banach spaces don't possess Schauder bases; but it is known that every Banach space has a decomposition, not necessarily a Schauder decomposition- a concept which generalizes the notion of a Schauder base and is defined as follows

Definition 2.9. A pair $\{M_i, P_i\}$ where $\{P_i\}$ is a sequence of orthogonal projections on a LCS (X, T) such that $P_i(X) = M_i$, is said to be a *decomposition* of X if to each $x \in X$, there corresponds a unique sequence $\{x_i\}$, $x_i \in M_i$, such that $x_i = P_i(x)$ and

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} x_i = \sum_{i\geq 1} x_i = \sum_{i\geq 1} P_i(x),$$

where the convergence of the infinite series being with respect to topology T of X. In case, each P_i is continuous, the decomposition $\{M_i, P_i\}$ is known as a *Schauder decomposition.*

Clearly, if each M_i is one dimensional, we get the notion of a Schauder base. As in the case of Schauder basis, we have different types of Schauder decompositions, like, *unconditional, absolute, bounded multiplier, subseries, boundedly complete, shrinking* etc. A notion useful for our study on infinite dimensional holomorphy is $\mathcal{S}$-absolute decomposition which is defined as

Definition 2.10. Corresponding to a sequence space $\mathcal{S} = \{(\beta_n)\}_{n=1}^{\infty} : \{\beta_n\} \subset \mathbb{C}$ with $\limsup_{n\to\infty} |\beta_n|^{\frac{1}{n}} \leq 1\}$, a Schauder decomposition $\{M_i, P_i\}$ is said to be $\mathcal{S}$-*absolute* if for each $\beta = (\beta_j) \in \mathcal{S}$, $x = \sum_{j=1}^{\infty} x_j \in X$, $x_j \in M_j$, $\beta \cdot x = \sum_{j=1}^{\infty} \beta_j x_j \in X$; and $p_\beta(x) = \sum_{j=1}^{\infty} |\beta_j| p_\beta(x_j)$ defines a continuous semi-norm on X for each $p \in D_T$.

Like Schauder basis, various type of Schauder decompositions help characterizing the structure of space in which they are present, cf. [42] and reference given therein. In our first contribution [80] in this direction, we have interrelated Schauder decompositions in X, X^* and X^{**} when they are equipped with various polar topologies. The applications of Schauder decompositions in asserting the continuity of linear operators and semi-norms have been obtained, cf. [92], p.43. The problem concerning the existence of a Schauder decomposition for a subspace of the whole space has also been tackled, cf [92], p.115.

For the historical developments of Schauder bases and decompositions, we refer to the theses [40, 121] and the references given therein.

We will see in the next subsection that whereas Schauder bases have close connections with scalar valued sequence spaces, Schauder decompositions correspond to vector valued sequence spaces. Both these concepts play an important role in the structural study of locally convex spaces via their relationships with sequence spaces.

(c) Scalar and Vector Valued Sequence Spaces:

The theory of scalar valued sequence spaces (SVSS) was founded by G. Köthe and O. Toeplitz around the year 1935 and the vector valued sequence spaces (VVSS) emerged as generalizations of SVSS by Phoung Các [119], though these were earlier used by A. Grothendieck in the form of tensor product of ℓ^p with a Banach space for developing the theory of nuclear spaces and later studied extensively by A. Pietsch in his book [113]. For historical development of these two theories, we refer to [97, 111].

Corresponding to a vector space X, let us write $\Omega(X)$ and $\Phi(X)$ respectively for the vector space of all sequences formed by the elements of X and the subspace of $\Omega(X)$ consisting of all finitely non-zero sequences. A *vector valued sequence space* (VVSS) or *a generalized sequence space* $\Lambda(X)$ is a subspace of $\Omega(X)$ containing $\Phi(X)$. Members of $\Lambda(X)$ are denoted by $\overline{x} = \{x_n\}$, $\overline{y} = \{y_n\}$ etc. where $x_n,\ y_n \in X$ for each $n \in \mathbb{N}$. The symbols δ_n^x for $x \in X$ and $n \in \mathbb{N}$ stands for the sequence $\{0, 0, \ldots, x, 0, \ldots\}$, where x is placed at the n^{th}-coordinate. The n^{th}-section of $\overline{x} = \{x_n\}$ in $\Lambda(X)$ is written as $\bar{x}^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\} = \sum\limits_{i=1}^{n} \delta_i^{x_i}$. A VVSS $\Lambda(X)$ is said to be (i) *symmetric* if $\bar{x}_\sigma = \{x_{\sigma(n)}\} \in \Lambda(X)$ whenever $\bar{x} = \{x_n\} \in \Lambda(X)$ and $\sigma \in \Pi$, the set of all permutations of $\mathbb{N}$; (ii)*monotone* if $m_0\Lambda(X) \subset \Lambda(X)$, where $m_0 = spanA$, A being the set of all sequences of zeros and ones; (iii) *normal* if for $\overline{x} = \{x_n\} \in \Lambda(X)$ and $\bar{\alpha} = \{\alpha_n\} \subset \mathbb{K}$ with $|\alpha_n| \leq 1$ for every $n \in \mathbb{N}$, the sequence $\bar{\alpha}\bar{x} = \{\alpha_n x_n\} \in \Lambda(X)$, or equivalently $\ell_\infty\Lambda(X) \subset \Lambda(X)$, where ℓ_∞ is the sequence space of all bounded sequences in $\mathbb{K}$.

Corresponding to a dual pair $< X, Y >$ of vector spaces, the *generalized Köthe dual* $\Lambda^\times(Y)$, *the generalized* β*-dual* $\Lambda^\beta(Y)$ and the *generalized* δ*-dual* $\Lambda^\delta(Y)$ of $\Lambda(X)$ are the spaces defined as

$$\Lambda^\times(Y) = \{\bar{y} = \{y_n\} \subset Y : \sum_{n\geq 1} | < x_n, y_n > | < \infty,\ \forall\ \bar{x} = \{x_n\} \in \Lambda(X)\},$$

$$\Lambda^\beta(Y) = \{\bar{y} = \{y_n\} \subset Y : \sum_{n\geq 1} < x_n, y_n > \text{ converges } \forall\ \bar{x} = \{x_n\} \in \Lambda(X)\},\ \textit{and}$$

$$\Lambda^\delta(Y)=\{\bar{y} = \{y_n\} \subset Y : \sum_{n\geq 1} | < x_n, y_{\sigma_n} > | < \infty,\ \forall\ \bar{x} = \{x_n\} \in \Lambda(X)\ \&\ \sigma \in \Pi\}.$$

Clearly, $\Lambda^\delta(Y) \subset \Lambda^\times(Y) \subset \Lambda^\beta(Y)$.

A VVSS is said to be *perfect* if $\Lambda(X) = \Lambda^{\times\times}(X) = (\Lambda^\times(Y))^\times$. Every perfect VVSS is normal and every normal VVSS is monotone.

In case $X = \mathbb{K}$, the field of scalars, we get results on SVSS. We write $\Omega(\mathbb{K}) = \omega$, $\Phi(\mathbb{K}) = \phi$ and $\Lambda(\mathbb{K}) = \lambda$, a SVSS containing ϕ. If $e^n = \{\delta_{nj}\}$, δ_{nj} denoting the Kronecker delta, is the n-th unit vector in ω, one can easily see that $\phi = span\{e^n : n \geq 1\}$. The members of λ are written as $\bar{\alpha} = \{\alpha_n\}$, $\bar{\beta} = \{\beta_n\}$ etc. and the n^{th}-section of $\bar{\alpha} = \{\alpha_n\} \in \lambda$ as $\bar{\alpha}^{(n)} = \sum_{i=1}^{n} \alpha_i e^i$. A sequence $\bar{\alpha} > 0$ means $\alpha_n > 0$ for each $n \in \mathbb{N}$. Symmetric, monotone, normal or solid SVSS are defined analogously. Also, $\alpha-$ or Köthe, $\beta-$ and $\delta-$ duals in case of SVSS λ are denoted by $\lambda^\times$ or λ^α, λ^β and λ^δ respectively. A SVSS λ is *perfect* if $\lambda = \lambda^{\times\times}$.

Let (X, T) be a Hausdorff LCS. Then a VVSS $\Lambda(X)$ equipped with a locally convex topology τ is said to be a *GK-space* if the maps $P_i : \Lambda(X) \to X$, $P_i(\bar{x}) = x_i$, $i \geq 1$ are continuous. A GK-space is said to be a *GAD-space* if $\Phi(X)$ is dense in $\Lambda(X)$; and a *GAK-space* if for each $\bar{x} = \{x_i\} \in \Lambda(X)$, $\bar{x}^{(n)} \to \bar{x}$ in τ. A VVSS $(\Lambda(X), \tau)$ is said to be a *GC -space (GSC-space)* if the maps $R_i : X \to \Lambda(X)$,

$R_i(x) = \delta_i^x$, $i \geq 1$ and $x \in X$ are continuous (sequentially) continuous.

For SVSS, the concepts GK-, GAK- and GAD- defined above are referred in the literature as K-, AK- and AD- respectively. Relating the topological dual with the Köthe dual of a sequence space, we have [91], p.59, 60.

Theorem 2.11. *Let (λ, τ) be an AK-space. Then λ^* can be identified as $\lambda_s = \{\{f(e^n)\} : f \in \lambda^*\}$ and $\lambda^* \subset \lambda^\beta$. If (λ, τ) is also barrelled, then $\lambda^* = \lambda^\beta$ and additionally if λ is monotone, then $\lambda^* = \lambda^\times$.*

If λ, μ are sequence spaces such that $\mu \subset \lambda^\beta$, then $< \lambda, \mu >$ forms a dual system and as such we can define topologies $\sigma(\lambda, \mu)$, $\tau(\lambda, \mu)$ and $\beta(\lambda, \mu)$ on λ (similarly on μ by interchanging the roles of λ and μ). In addition, there is a natural locally convex topology on λ, the *normal or solid topology* generated by the family $\{p_{\bar{\beta}} : \bar{\beta} \in \lambda^\times\}$ of semi-norms defined as $\{p_{\bar{\beta}}(\bar{\alpha}) = \sum_{i \geq 1} |\alpha_i \beta_i|$, $\forall\ \bar{\alpha} = \{\alpha_i\} \in \lambda$. Note that the topology $\sigma(\lambda, \lambda^\times)$ is generated by $\{q_{\bar{\beta}} : \bar{\beta} \in \lambda^\times\}$, where $q_{\bar{\beta}}(\bar{\alpha}) = |\sum_{i \geq 1} \alpha_i \beta_i|$. A subset B of a SVSS λ is said to be *normal* if $\bar{\alpha} = \{\alpha_n\} \in B$ whenever $|\alpha_n| \leq |\beta_n|$, $n \geq 1$, for some $\bar{\beta} = \{\beta_n\} \in B$; and a semi-norm p on λ is called *solid* if $p(\bar{\alpha}) \leq p(\bar{\beta})$ for $\bar{\alpha}$, $\bar{\beta} \in \lambda$ with $|\alpha_n| \leq |\beta_n|$ for $n \geq 1$. Thus the normal topology is generated by the family of solid semi-norms. Relating these topologies, we have

Proposition 2.12. *For any sequence space λ, $\sigma(\lambda, \lambda^\times) \subset \eta(\lambda, \lambda^\times) \subset \tau(\lambda, \lambda^\times)$; in particular*

$$(\lambda, \sigma(\lambda, \lambda^\times))^* = (\lambda, \eta(\lambda, \lambda^\times))^* = (\lambda, \tau(\lambda, \lambda^\times))^* = \lambda^*.$$

Proposition 2.13. *If λ is monotone, then $(\lambda^\times, \sigma(\lambda^\times, \lambda))$ is sequentially complete and $\bar{\alpha}^{(n)} \to \bar{\alpha}$ in $\tau(\lambda, \lambda^\times)$ as $n \to \infty$, $\forall\ \bar{\alpha} \in \lambda$.*

Thus we have

Proposition 2.14. *Let λ be a sequence space and μ be a subspace of λ containing ϕ. Then $\{e^n, e^n\}$ is a Schauder basis for $(\lambda, \sigma(\lambda, \mu))$ and $(\lambda, \eta(\lambda, \mu))$. If λ is monotone, then $\{e^n, e^n\}$ is a Schauder basis for $(\lambda, \tau(\lambda, \lambda^\times))$.*

Proposition 2.15. *A sequence space $(\lambda, \eta(\lambda, \lambda^\times))$ is nuclear (resp. Schwartz) space if for $0 < \bar{\alpha} \in \lambda^\times$, there corresponds a sequence $0 < \bar{\beta} \in \lambda^\times$ such that $\bar{\alpha} \leq \bar{\beta}$, that is $\alpha_n \leq \beta_n$ for each $n \in \mathbb{N}$ and $\{\frac{\alpha_n}{\beta_n}\} \in \ell_1$ (resp. c_0).*

A sequence space (λ, τ) is said to be *simple* if every bounded set is dominated by a member in λ. It is shown in [84] that each perfect nuclear space is simple.

Particular type of sequence spaces which have played a vital role in our researches are Orlicz, Lorentz and Musielak-Orlicz sequence spaces. I shall present here only Orlicz sequence spaces. An *Orlicz function M* is a continuous convex function from $[0, \infty)$ to itself such that $M(0) = 0$, $M(r) > 0$ for $r > 0$. An Orlicz function M is always increasing and $M(r) \to \infty$ as $r \to \infty$. Such function M has the integral representation

$$M(r) = \int_0^r p(t)dt,$$

where $p(t)$, known as the kernel of M, is right continuous, non-decreasing function for $t > 0$. Also, $tp(t) \to \infty$ as $t \to \infty$ and $tp(t) = 0$ for $t = 0$. However $p(t) > 0$ for $t = 0$ is equivalent to the fact that the Orlicz sequence space ℓ_M is isomorphic to ℓ_1, cf. [91], p.309. Hence we assume that $p(0) = 0$ and $p(t) \to \infty$ as $t \to \infty$.

For a given Orlicz function M with kernel p, the function N defined as $N(s) = \int_0^s q(t)dt$, where $q(s) = \sup\{t : p(t) \leq s\}$ for $s \geq 0$ is also an Orlicz function. The functions M and N are called *mutually complementary Orlicz functions* and satisfy the following conditions

(i) For $s,\ t \geq 0$, $st \leq M(s) + N(t)$ (Young's inequality), (ii) For $s \geq 0$, $sp(s) = M(s) + N(p(s))$; and (iii) $M(\alpha s) \leq \alpha M(s)$ for $s \geq 0$ and $0 < \alpha < 1$.

An Orlicz function M *is said to satisfy the* Δ_2*-condition for small* t or at '0' if for each $k > 0$, there exists $R_k > 1$ and $t_k > 0$ such that $M(kt) \leq R_k M(t)$, $\forall t \in (0, t_k]$, or equivalently, $\lim\limits_{t\to 0} \frac{M(2t)}{M(t)} < \infty$. The Orlicz sequence space ℓ_M defined by the Orlicz function M is given by

$$\ell_M = \{\bar{\alpha} = \{\alpha_n\} : \sum_{n\geq 1} M(\frac{|\alpha_n|}{\rho}) < \infty, \text{ for some } \rho > 0\}.$$

An equivalent definition of this space is also given in terms of complementary Orlicz function. In this case if we denote by $\tilde{\ell_N}$ the sequence class

$$\tilde{\ell_N} = \{\bar{\alpha} = \{\alpha_n\} : \delta(\bar{\alpha}, N) = \sum_{N\geq 1} N(|\alpha_n|) < \infty\}$$

we have

$$\ell_M = \{\bar{\alpha} = \{\alpha_n\} : \sum_{n\geq 1} \alpha_n\beta_n \text{ converges } \forall\ \bar{\beta} = \{\beta_n\} \in \tilde{\ell_N}\}$$

Thus on ℓ_M, we have two equivalent norms given by

$$\|\bar{\alpha}\|_M = \inf\{\rho > 0 : \sum_{n\geq 1} M(\frac{|\alpha_n|}{\rho}) \leq 1\} \ \&\ \|\bar{\alpha}\|_{(M)} = \sup\{|\sum_{n\geq 1} \alpha_n\beta_n| : \delta(\bar{\beta}, N) \leq 1\};$$

indeed, $\|\bar{\alpha}\|_M \leq \|\bar{\alpha}\|_{(M)} \leq 2\|\bar{\alpha}\|_M$, for $\bar{\alpha} = \{\alpha_n\} \in \ell_M$.

The space ℓ_M is BK (Banach K-) space when it is equipped with either of the above two norms. An AK-, BK- subspace of ℓ_M is h_M defined as

$$h_M = \{\bar{\alpha} = \{\alpha_n\} : \sum_{n\geq 1} M(\frac{|\alpha_n|}{\rho}) < \infty, \forall \rho > 0\}.$$

We have

Theorem 2.16. *For an Orlicz function* M*, following is true:*
M *satisfies* Δ_2*-condition at '0'* $\Leftrightarrow$ $h_M = \ell_M \Leftrightarrow$ ℓ_M *is an AK space.*

Theorem 2.17. *If* M *and* N *are mutually complimentary Orlicz functions and* M *satisfies* Δ_2*-condition at 0, then* $\ell_M^\times = \ell_M^\beta = \ell_M^* = \ell_N$*. Indeed, if both* M *and* N *satisfy* Δ_2*-condition at 0, then the spaces* ℓ_M *and* ℓ_N *are perfect.*

All the above results on SVSS are contained in [91] and I refer to this book for the detailed theory of SVSS. However, for brief history on Orlicz and modular sequence spaces, the reference is [118].

In the following few pages, we shall see that there is a close connection between spaces having Schauder basis (Schauder decompositions) and the theory of SVSS (VVSS), cf. [91, 92, 93, 120]. Let us first consider Schauder basis. In case (X, T) is a LCS having a Schauder basis $\{x_n, f_n\}$, let us define the following sequence spaces

$$\delta \equiv \delta_{\{x_n, f_n\}} = \{\{f_n(x)\} : x \in X\}$$

and

$$\mu \equiv \mu_{\{x_n, f_n\}} = \{\{f_n(x)\} : f \in X^*\}.$$

The spaces δ and μ form a dual system corresponding to the bilinear form $< \cdot, \cdot >$ on $\delta \times \mu$ given by

$$< \{f_n(x)\}, \{f(x_n)\} > = \sum_{n \geq 1} f_n(x) f(x_n), \; x \in X, f \in X^*.$$

Let $F : \delta \to X$ and $G : X^* \to \mu$ be defined by $F(\{f_n(x)\}) = x$ and $G(f) = \{f(x_n)\}$, then both F and G are bijective and one has

Proposition 2.18. *$(X, \sigma(X, X^*)) \cong (\delta, \sigma(\delta, \mu))$ under F^{-1} and $(X^*, \sigma(X^*, X)) \cong (\mu, \sigma(\mu, \delta))$ under G, where the symbol $\cong$ denotes the topological isomorphism.*

Let us note that

Proposition 2.19. *If a LCS (X, T) has an absolute base $\{x_n, f_n\}$, then δ is perfect. In addition, if (X, T) is an S-space, then $\mu = \delta^\times$.*

All the above results are used in proving [90]

Theorem 2.20. *If (X, T) is a barrelled space possessing an absolute Schauder base $\{x_n, f_n\}$, then (X, T) is complete.*

Various other connections between types of a Schauder base in a LCS and the structure of δ have been established in [62, 81] etc. For instance, we have

Proposition 2.21. *A Schauder base $\{x_n, f_n\}$ for a LCS (X, T) is bounded multiplier (resp. subseries) base if and only if δ is normal (resp. monotone).*

Theorem 2.22. *A Schauder base $\{x_n, f_n\}$ for a barrelled space (X, T) is symmetric if and only δ is symmetric.*

As mentioned in the beginning of this subsection, there are basically two directions to study the topological properties of a VVSS $\Lambda(X)$, namely, (i) when no restriction is placed on the structure of $\Lambda(X)$ relative to a given SVSS λ and (ii) when members of $\Lambda(X)$ are chosen corresponding to a given SVSS λ. In case (i) the topology on $\Lambda(X)$ is obtained as a consequence of its dual relationship with $\Lambda^\times(X^*)$, and in case (ii) the topology of $\Lambda(X)$ is introduced with the help of the topologies of the parent SVSS λ and the topology of LCS (X, T). For (i) we refer to [35, 111, 119, 120] and the references for (ii) are [98, 113, 123]. Whereas the importance of VVSS of type (ii) lies in the study of nuclear spaces, λ-compact operators etc. [10, 114], the methods developed by the duality of $\Lambda(X)$ and $\Lambda^{\times}(X^*)$ has immediate applications in Schauder decomposition theory; indeed, it is shown that every VVSS equipped with the weak topology $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$ has an unconditional Schauder decomposition, and various results interrelating the types of Schauder decompositions with the properties of the space $\Lambda(X)$ have also been proved, cf. [42, 120]. Generalizing the Schur's property of ℓ_1 following has been proved in [43]

Theorem 2.23. *Let (X, T) be a reflexive LCS. Then $\sigma(\ell^1(X), \ell^\infty(X^*))-$ and $\beta(\ell^1(X), \ell^\infty(X^*))$-convergent sequences in $\ell^1(X)$ are the same if and only if $\sigma(X, X^*)$ and $\beta(X, X^*)$-convergent sequences are same, where $\ell^1(X) = \{\{x_i\} : x_i \in X \ \forall i \geq 1$ and $\sum_{i\geq 1} p(x_i) < \infty, \ \forall \ p \in D_T\}$ and $\ell^\infty(X^*) = \{\{f_i\} : f_i \in X^*, \ \forall i \geq 1$ and $\{f_i\}$ is $\beta(X^*, X) -$ bounded$\}$.*

In the paper [79], we deal with the generalized μ-dual of a VVSS $\Lambda(X)$, defined corresponding to an SVSS μ and the dual pair $< X, Y >$ of vector spaces as follows

$$\Lambda(X)^\mu = \Lambda^\mu(Y) = \{\bar{y} \in \Omega(Y) : \{< x_i, y_i >\} \in \mu, \ \forall \ \bar{x} \in \Lambda(X)\}.$$

For $\mu = \ell_1$, cs (the class of scalar sequences for which series converges) and bs (the class of scalar sequences for which partial sums of the series are bounded), cf. [91, 100], the corresponding μ-duals are generalized Köthe, $\beta-$ and γ-duals. Clearly

$$\Lambda^\delta(Y) \subset \Lambda^\times(Y) \subset \Lambda^\beta(Y) \subset \Lambda^\gamma(Y).$$

It is shown in [79] that (i) $\Lambda^\beta(Y) = \Lambda^\times(Y)$ if $\Lambda(X)$ is monotone, (ii) $\Lambda^\gamma(Y) = \Lambda^\times(Y)$ if $\Lambda(X)$ is normal, (iii) $\Lambda^\delta(Y) = \Lambda^\times(Y)$ if $\Lambda(X)$ is symmetric. Also, the topological dual of a barrelled GAK- and GC- vector valued sequence space $(\Lambda(X), \tau)$ is $\Lambda^\beta(X^*)$, i.e. $\Lambda^\beta(X^*) = \Lambda(X)^*$. This generalizes Theorem 2.11. If μ is a SVSS equipped with a Hausdorff locally convex topology T_μ generated by the family D_μ of T_μ continuous semi-norms, then we can topologize the space $\Lambda(X)$ with a locally convex topology , which we refer to σ_μ-topology and is generated by the family of semi-norms $\{p_{\bar{y}} : p \in D_\mu, \bar{y} \in \Lambda(X)^\mu\}$, where $p_{\bar{y}}(\bar{x}) = p(\{< x_i, y_i > \})$ for $\bar{x} = \{x_i\} \in \Lambda(X)$ and $p \in D_\mu$. Various results involving the structural properties of (μ, T_μ) and $(\Lambda(X), \sigma_\mu)$ have been proved in [79].

In case of SVSS, for a given sequence $\alpha = \{\alpha_n\}$ in ω and a sequence space μ; in [78], we consider $\alpha\mu$-duals which as a particular case includes the notions of Köthe, $\beta-$, γ- and the other duals studied in [15, 91]. Indeed, we have

Definition 2.24. For a given sequence $\alpha = \{\alpha_n\}$ and a sequence space μ, the $\alpha\mu$-dual of a sequence space λ is the subspace λ^μ_α of ω defined as

$$\lambda^\mu_\alpha = \{b = \{b_n\} \in \omega : \alpha ab = \{\alpha_n a_n b_n\} \in \mu, \ \forall \ a = \{a_n\} \in \lambda\}.$$

If (μ, T_μ) is a locally convex sequence space with T_μ being generated by a family D_μ of semi-norms, then we can topologize either of the spaces λ and λ^μ_α with the corresponding locally convex topologies $T_{\alpha,\mu}$ and $T^*_{\alpha,\mu}$, respectively generated by the families $\{p_b^\alpha : p \in D_\mu,\ b \in \lambda_\alpha^\mu\}$ and $\{p_a^\alpha : p \in D_\mu,\ a \in \lambda\}$ of semi-norms where for $a \in \lambda$, $b \in \lambda_\alpha^\mu$ and $p \in D_\mu$,

$$p_b^\alpha(a) = p_a^\alpha(b) = p(\{\alpha_n a_n b_n\}).$$

A SVSS λ is said to be $\alpha\mu$-perfect if $\lambda = \lambda_{\alpha\alpha}^{\mu\mu}$, where $\lambda_{\alpha\alpha}^{\mu\mu} = (\lambda_\alpha^\mu)_\alpha^\mu = \{c = \{c_n\} \in \omega : \alpha bc \in \mu, \text{for each } b = \{b_n\} \in \lambda_\alpha^\mu\}$, the $\alpha\mu$-dual of λ_α^μ. Invoking the structural properties of these spaces in relation to μ, we prove several results which have been used in the structural study of several subspaces of the space of holomorphic mappings, cf. [78].

Coming back to our study of VVSS, A. Pietsch, R. C. Rosier, N. De Grande - De Kimpe and others studied the spaces of the following type

$$\lambda[X] = \{\{x_i\} : x_i \in X, i \geq 1,\ \textit{and}\ \{f(x_i)\} \in \lambda,\ \forall\ f \in X^*\},$$

$$\lambda(X) = \{\{x_i\} : x_i \in X, i \geq 1,\ \textit{and}\ \{p(x_i)\} \in \lambda,\ \forall\ p \in D_T\} \quad \text{and}$$

$$\lambda < X > = \{\{x_i\} : x_i \in X, i \geq 1,\ \textit{and}\ \{\|x_i\|_B\} \in \lambda, \quad \text{for some closed absolutely convex, bounded subset B of } X\},$$

where λ is a perfect SVSS, (X, T) is a LCS and $\|\cdot\|_B$ denotes the norm generated by the Minkowski functional of B on the space $X_B = \cup_{n\in\mathbb{N}} nB$. Pietsch [114] proved that a LCS X is nuclear if and only if $\ell^1(X) = \ell^1[X]$. It was shown by Kimpe that a Fréchet space X is nuclear if and only if $\ell^p(X) = \ell^p[X]$ for some p, $1 \leq p < \infty$, cf. [98], p.144, and a barrelled space X possessing an absolute base is nuclear if and only if $\ell^1 < X_\beta^* >= \ell^1[X_\beta^*]$, where $X_\beta^* = (X^*, \beta(X^*, X))$. On the other hand, it is proved in [57] that for a Banach space X, $\ell^1[X]$ is a GAK-space if and only if X does not contain a copy of c_0. Also, for $1 < p < \infty$ and a reflexive Banach space X, $\ell^p[X^*]$ is a GAK-space if and only if each member of $\mathcal{L}(X^*, \ell_p)$ is a compact linear operator.

As in the case of Schauder basis, we can define a VVSS through a Schauder decomposition $\{M_n, P_n\}$ present in a LCS (X, T), indeed, we have, cf. [42, 120]

$$\Lambda(X) = \{\bar{x} : \bar{x} = \{P_n(x)\},\ x \in X\} \quad \text{and}$$

$$\Delta(X^*) = \{\bar{f} : \bar{f} = \{P_n^*(f)\},\ f \in X^*\}.$$

Then using the duality between $\Lambda(X)$ and $\Delta(X^*)$, we have proved

Theorem 2.25. *If $\Lambda(X)$ and $\Delta(X^*)$ are as defined above for a LCS (X, T) with a Schauder decomposition $\{M_n, P_n\}$ and if $\Lambda^\times(X) = \Delta(X^*)$, then $\Lambda(X)$ is perfect if $\{M_n\}$ is a boundedly complete Schauder decomposition and it is normal if $\{M_n\}$ is bounded multiplier Schauder decomposition.*

Matrix Transformations between Sequence Spaces:

The theory of matrix transformations on scalar-valued sequence spaces dates back to the beginning of the last century; indeed it was shown by G. Köthe and O. Toeplitz that a linear transformation A from a SSVS λ to another SSVS μ, where λ is monotone, is a matrix transformation if and only if it is $\sigma(\lambda, \lambda^\times) - \sigma(\mu, \mu^\times)$

continuous. This matrix representation of a continuous linear map on SVSS finds numerous applications in the study of various aspects of Functional Analysis, for instance, nuclearity theory, theory of locally convex spaces, summability domains etc., cf.[91, 100, 114] and several references given therein. In case of VVSS, the problem of representing a linear operator in terms of linear maps between underlying spaces was first tackled in [35] and further investigations were carried out in [45, 47, 71, 72], cf. also [1, 111]. Indeed, we have

Definition 2.26. Let (X, T_X) and (Y, T_Y) be two Hausdorff LCS with corresponding duals X^* and Y^* respectively. Assume that $< \Lambda(X), \Lambda^\times(X^*) >$ and $< \Lambda(Y), \Lambda^\times(Y^*) >$ form dual pairs of VVSS. A linear map $Z : \Lambda(X) \to \Lambda(Y)$ is said to be a T_Y-*matrix transformation* or just a matrix transformation from $\Lambda(X)$ to $\Lambda(Y)$ relative to T_Y if there exists a matrix $[Z_{ij}]$ of linear maps $Z_{ij} : X \to Y$, $i \in \mathbb{N}$, such that for $\bar{x} = \{x_i\} \in \Lambda(X)$, the series $\sum_{j\geq 1} Z_{ij}(x_j)$ converges to y_i in (Y, T_Y) for each $i \geq 1$ and $Z(\bar{x}) = \{y_i\} = \bar{y} \in \Lambda(Y)$. The transpose $Z^\perp$ of a matrix $Z \equiv [Z_{ij}]$ of linear maps $Z_{ij} : X \to Y$, $i, j \in \mathbb{N}$ is defined as the transpose of the matrix of the adjoint maps of Z_{ij}, i.e., $Z^\perp = [Z^*_{ji}]$. A matrix transformation $Z \equiv [Z_{ij}]$ is said to be a *diagonal transformation* if $Z_{ij} = 0$ for $i \neq j$ and if $Z_{ii} = Z_i$, $i \geq 1$, and for $\bar{x} = \{x_i\} \in \Lambda(X)$, $Z(\bar{x}) = \{Z_i(x_i)\} \in \Lambda(Y)$. In case $X = Y = \mathbb{K}$, the field of scalars, we get matrix transformations on SVSS.

For two SVSS λ and μ, writing (λ, μ) for the class of all matrix transformations $A = [a_{ij}]$ from λ to μ, and $A^\perp = [a_{ji}]$ for the transpose of $A = [a_{ij}]$, following has been proved in [61]

Theorem 2.27. *For an infinite matrix $A = [a_{ij}]$, the following statements are equivalent:*

(i) $A \in (\ell^1, \delta)$;

(ii) $A^\perp \in (d, \ell^\infty)$,

(iii) for every α, $0 < \alpha < \infty$, *there exists an element* $\bar{\beta} = \{\beta_i\} \in \ell^\infty$ *such that* $|a_{ij}|\alpha^i \leq \beta_j$, $i, j \geq 1$; *and*

(iv) $|a_{ij}|^{\frac{1}{i}} \to 0$ *as* $i \to \infty$, *uniformly in* $j \geq 1$, *where* $\delta = \{\{\alpha_n\} : |\alpha_n|^{\frac{1}{n}} \to 0\}$ *and* $d = \{\{\beta_n\} : \sup_n |\beta_n|^{\frac{1}{n}} < \infty\}$.

For a SVSS λ and the matrix $A = [a_{ij}]$, we write $\lambda_A = \{x \in \lambda : Ax$ exists and $Ax \in \lambda\}$, known as the *summability domain* of A, cf. [91], p.208. It has been proved in [76]

Proposition 2.28. *Let (λ, τ) be a Fréchet K-space and $A = [a_{ij}]$ be a bijective map from λ_A to λ. Let M and N be Orlicz functions satisfying Δ_2-condition at '0' and* $\ell^M_\alpha = \{x \in \omega : \{\frac{x_n}{\alpha_n}\} \in \ell_M\}$, $\ell^\alpha_N = \{x \in \omega : \{\alpha_n x_n\} \in \ell_N\}$ *for fixed* $\alpha = \{\alpha_n\} > 0$. *Then* $A \in (\ell^M_\alpha, \lambda)$ *if and only if* $\{f(a^i)\} \in \ell^\alpha_N$ *for all* $f \in \lambda^*$ *and* $i \in \mathbb{N}$, *where* a^i *is the* i^{th} *row of the matrix* A.

Besides proving the general representation results on matrix transformations, we initiate the study of precompactness and nuclearity of diagonal operators on VVSS in [72]. In our subsequent work [47, 71], we introduce the notion of generalized simple sequence spaces and characterize matrix transformations on these as well as nuclear VVSS.

In [45, 47], we consider VVSS of type $\lambda(X) = \{\bar{x} = \{x_i\} : x_i \in X,\ \forall\ i \in \mathbb{N}\ and\ \{\|x_i\|_X\} \in \lambda\}$, where $(X, \|\cdot\|_X)$ is a Banach space and $(\lambda, \|\cdot\|_\lambda)$ is a Banach sequence space, and characterize the compact and approximable diagonal operators defined on $\lambda(X)$ using Kolomogrov and approximation numbers respectively. Indeed, for Banach spaces X, Y with closed unit balls B_X and B_Y respectively and $S \in \mathcal{L}(X, Y)$ and $n \in \mathbb{N}$, the n^{th}-Kolomogrov numbers and n^{th}-approximation numbers of S are respectively given by

$$d_n(S) = \inf\{\delta > 0 : S(B_X) \subset N_\delta + \delta B_Y;\ N_\delta \text{ is a subspace of Y with } dim N_\delta < n\}$$

and

$$a_n(S) = \inf\{\|S - A\| : A \in \mathcal{L}(X, Y),\ rank A < n\}.$$

An operator $S \in \mathcal{L}(X, Y)$ is said to be *approximable* if $a_n(S) \to 0$ as $n \to \infty$. It has been shown in [47] (resp. in [45]) that for a suitably restricted Banach SVSS λ, the diagonal operator $Z : \lambda(X) \to \lambda(Y)$ is approximable (resp. compact) if and only if Z_{ii} is approximable (resp. compact) for each $i \in \mathbb{N}$ and $\|Z_{ii}\| \to 0$ as $i \to \infty$. Also, in [47], the approximation numbers of the inclusion maps between the spaces of the type $\ell^p(X)$, $1 \leq p \leq \infty$, have been computed. In case of the inclusion map $I : \ell^1(X) \to \ell^\infty(X)$, it is shown that $\frac{1}{2} \leq a_n(I) \leq 1,\ n \geq 2,\ n \in \mathbb{N}$; further $a_n(I) = 1,\ \forall n \in \mathbb{N}$ if dim $X = \infty$; and if dim $X = k$, then $a_n(I) = 1,\ 1 \leq n \leq k$ and $a_n(I) = \frac{1}{2},\ \forall n > k$. This shows how the dimension of the underlying space affects the value of the approximation number of the inclusion map.

For the historical development of the theories of SVSS and VVSS, I would like to refer to the theses [1, 111, 120] and the books [91, 100, 124] on SVSS.

(d) Bi-locally Convex Spaces/Mixed Structure:

While studying topological vector spaces, one usually encounters with several linear topologies on a vector space and sometimes two Hausdorff linear topologies give rise to a mixed structure which finds its applications in the theory of summability, theory of sequence spaces, theory of Schauder decompositions etc. Motivated by the earlier work of G. Fichtenholz, A. Alexiewicz introduced the concept of a two-norm space which was systematically developed by himself in collaboration with Z. Semadeni and later utilized by P. K. Subramanian and R. Rothman for Banach spaces having Schauder decompositions, cf. [22] for references. Having observed these contributions, an initiative to develop the structural study for vector spaces equipped with two Hausdorff locally convex topologies was taken in [58]; indeed, a *bi-locally convex space* (bi-LCS) is a triplet (X, T_1, T_2)

denoted by X_b where T_1 and T_2 are Hausdorff locally convex topologies on X such that T_1 is finer than T_2; and a net $\{x_\alpha\}$ in a bi-LCS X_b is said to be γ-*convergent* if $\{x_\alpha\}$ is T_1-bounded and converges to x in T_2. Depending on γ-*convergence*, various notions like γ-completeness, γ-boundedness, γ-compactness, γ-reflexivity etc. have been defined and studied in this paper. In our subsequent work [77], it has been shown that a LCS with a Schauder decomposition yields a bi-LCS which is termed as *canonical* bi-LCS and of which the structural properties could be related with types of Schauder decompositions. This study is continued in [59], where k-reflexive bi-LCS are introduced and characterized in terms of the subspaces forming the Schauder decomposition. Also, the γ-completion of such a bi-LCS is identified as a VVSS and the relationship between k- and γ-reflexivity of a canonical bi-LCS has been established. For chronological development of this theory and results, I refer to [21, 22].

(e) Ordered Generalized Sequence Spaces:

The theory of VVSS of which we have seen the importance in earlier subsections, poses a natural question regarding the impact of ordering of the underlying space X on $\Lambda(X)$ and vice-versa, in case X is assumed to be an ordered vector space (OVS). Indeed, the real SVSS are OVS in a natural way with respect to the co-ordinatewise ordering. Motivated by the work of J. Dieduonne on function spaces which are clearly partially ordered spaces, A. L. Peressini tried to relate the ordering and topological structure of SVSS, and proved several results, cf. [112] and references given therein. For detailed developments of results on OVS, Riesz spaces, VVSS and linear operators on Riesz spaces, one is referred to Chapter 2 of the doctoral dissertation [97].

With the rich literature of ordered vector spaces and locally convex lattices in hand [2, 102, 112, 129], one may possibly think of tackling problem in the following three directions involving VVSS, namely (i) the impact of ordering and duality of underlying space with its order dual, on the VVSS and the matrix transformations defined thereof, (ii) combined effect of ordering and topological properties of the underlying space on VVSS; and (iii) the applications of the ordered VVSS in the study of positive absolutely summing and majoring operators with domain and range contained in vector or locally convex vector lattices. An attempt in the direction of attacking problems related to (ii) and (iii) was made by Walsh [130] who introduced ordered VVSS analogous to c_0, ℓ_1 and ℓ_∞ and used them in the study of K-absolutely summing and majorizing operators introduced for Banach lattices, cf. [2, 126]. However, the problems related to (i) have been considered by us , cf. [97]. Besides, various polar topologies which involve the concepts of ordering as well as duality theory of LCS, have been considered in [65]. It is known that every order complete Riesz space has an order decomposition, cf. [129], but the indexing set for the family of bands (order closed ideals) constituting the order decomposition is not necessarily countable. However, in case of ordered VVSS which are Riesz spaces, it is possible to find countable ordered decompositions in a natural way, cf. [97]. In [66], the concept of o-similarity between countable order decompositions has been introduced and the results proved for ordered VVSS associated with countable order decomposition in a Riesz space and its sequential order dual, have been employed to characterize the o-similarity of countable order decompositions.

(f) Operator Ideals and Approximation Quantities:

Sequence spaces have played a significant role in the study of operator ideals which has its origin in the work of J. W. Calkin, R. Schatten and J. von Neumann on the rings of operators defined on Hilbert spaces. On the other hand, approximation quantities which yield the measure of compactness for linear operators defined on Banach or Hilbert spaces, appeared for the first time in the work of E. Schmidt, L. S. Pontrjagin and L. G. Schnirelman. Indeed, motivated by the work of Pontrjagin and Schnirelman for classifying massiveness of sets in metric spaces, involving the least number of coverings with radii tending to zero, A. N. Kolomogorov invented the idea of metric entropy or ϵ-entropy to characterize compact sets according to their massivity. Besides, eigenvalues of a compact operator defined on a Hilbert space form a member of c_0. This observation led researchers to introduce the concept of s-numbers of operators on Hilbert spaces; indeed, these are defined as the eigenvalues of the positive operator $(T^*T)^{\frac{1}{2}}$, arranged in decreasing order, for a bounded linear operator T defined on a Hilbert space H into itself. In 1957, D. Eh. Allakhverdiev proved that the n^{th} s-number of such an operator T is given by $s_n(T) = \inf\{\|T - S\| : \ S$ is a finite rank operator on H with rank(T) $< n\}$. Motivated by this result, A. Pietsch introduced the concept of approximation numbers $\{a_n(T)\}$ for a bounded linear operator defined on a Banach space in 1963. Indeed, various kinds of geometric quantities like Kolomogrov numbers, Gelfand numbers for such operators were presented by B. S. Mitiagin and A. Peleczynski in "Proc. Internat. Cong. Math (Moscow) 1966; Izdat 'Mir', Moscow, 51, (1968), 366–372" of which the unified study was carried out by A. Pietsch [117] under the nomenclature of s-numbers of operators. Using these numbers and sequence spaces, different types of operator ideals have been introduced and studied. Let us now consider

Definition 2.29. Let $\mathcal{L}$ be the class of bounded linear operators between any pair of Banach spaces and $\mathcal{A} \subset \mathcal{L}$ and $\mathcal{A}(X,Y) = \mathcal{A} \cap \mathcal{L}(X,Y)$ for Banach spaces X and Y. Then the collection $\mathcal{A}$ is said to be an *operator ideal* if it satisfies the conditions (i) all finite rank operators are contained in $\mathcal{A}$; (ii) $T + S \in \mathcal{A}(X,Y)$ for $T, S \in \mathcal{A}(X,Y)$ and (iii) if $T \in \mathcal{A}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$, then $ST \in \mathcal{A}(X,Z)$ and also if $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{A}(Y,Z)$, then $ST \in \mathcal{A}(X,Z)$.

For a given pair of Banach spaces X and Y, the collection $\mathcal{A}(X,Y)$ is called a *component* of A. An *ideal quasi-norm* is a real-valued function $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}^+$(positive reals) satisfying the properties (i) $0 \leq \|T\|_{\mathcal{A}} < \infty$, for each $T \in \mathcal{A}$ and $\|T\|_{\mathcal{A}} = 0$ if and only if $T = 0$, (ii) there exists a constant $\sigma_{\mathcal{A}} \geq 1$ such that $\|S+T\|_{\mathcal{A}} \leq \sigma_{\mathcal{A}}(\|S\|_{\mathcal{A}} + \|T\|_{\mathcal{A}})$ for $S,\ T \in \mathcal{A}(X,Y)$ and (iii) (a) $\|RS\|_{\mathcal{A}} \leq \|R\|\|S\|_{\mathcal{A}},\ S \in \mathcal{A}(X,Z),\ R \in \mathcal{L}(Z,Y)$; (b) $\|RS\|_{\mathcal{A}} \leq \|S\|\|R\|_{\mathcal{A}}$ for $S \in \mathcal{L}(X,Z)$, $R \in \mathcal{A}(Z,Y)$.

A *quasi-normed operator ideal* is an operator ideal equipped with an ideal quasi-norm; and a quasi-Banach operator ideal is a quasi-normed operator ideal of which each component is complete with respect to the ideal quasi-norm. We refer to [115] for the detailed theory of this topic.

Approximation quantities we deal with, are essentially s-numbers and entropy numbers. These quantities give rise to several examples of operator ideals, for which we refer to [1, 10, 18, 115, 116]. Due to Pietsch [117], we have

Definition 2.30. The s-number function is defined from $\mathcal{L}$ to ω^+, the class of sequences of non-negative real numbers, satisfying the conditions:
(i) $\|T\| = s_1(T) \geq s_2(T) \geq s_3(T) \geq \cdots \geq$;
(ii) $s_n(S+T) \leq s_n(S) + \|T\|$, for $n \in \mathbb{N}$ and $S, T \in \mathcal{L}(X,Y)$;
(iii) $s_n(RST) \leq \|R\| s_n(S) \|T\|$ for $n \in \mathbb{N}$, $R \in \mathcal{L}(Y, Y_0)$, $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(X_0, X)$;
(iv) $s_n(T) = 0$ if rank $(T) < n$ and
(v) $s_n(I_X) = 1$ if dim $(X) \geq n$, where I_X denotes the identity map on X.

Approximation numbers and Kolomogrov numbers are particular type of s-numbers which have already been defined. Another type of approximation quantities which are not s-numbers, is given in

Definition 2.31. For $T \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$, the nth- entropy number of T is defined as

$$\epsilon_n(T) = \inf\{\delta > 0 : T(B_X) \subset \cup_{i=1}^{k}(y_i + \delta B_Y),$$
$$\text{for some } y_i \in Y, i = 1, 2, \ldots, k,\ k \leq n\}.$$

The entropy numbers also satisfy the properties (i), (ii) and (iii) of the above definition. However, the entropy numbers, approximation numbers and Kolmogrov numbers satisfy the general version of the properties (ii) and (iii) in the following form:
For each $k, n \in \mathbb{N}$ and $S, T \in \mathcal{L}(X,Y)$, (ii) (a) $\epsilon_{kn}(S+T) \leq \epsilon_k(S) + \epsilon_n(T)$,
(b) $a_{k+n-1}(S+T) \leq a_k(S) + a_n(T)$, and (c) $d_{k+n-1}(S+T) \leq d_k(S) + d_n(T)$
(iii) For $k, n \in \mathbb{N}$ and $T \in \mathcal{L}(X,Z)$, $S \in \mathcal{L}(Z,Y)$, (d) $\epsilon_{kn}(ST) \leq \epsilon_k(S)\epsilon_n(T)$;
(e) $a_{k+n-1}(ST) \leq a_k(S) a_n(T)$, and (f) $d_{k+n-1}(ST) \leq d_k(S) d_n(T)$.

For a class λ of scalar sequences, a mapping $T \in \mathcal{L}(X,Y)$, where X and Y are Banach spaces, is said to be of *type* λ if $\{a_n(T)\} \in \lambda$, and it is said to be *pseudo*

λ-*nuclear* if there exist sequences $\{\lambda_n\} \in \lambda$, $\{f_n\} \subset X^*$ and $\{y_n\} \subset Y$ such that $\|f_n\| \leq 1$, $\|y_n\| \leq 1$, $n \in \mathbb{N}$ and T has the representation: $T(x) = \sum_{n\geq 1} \lambda_n f_n(x) y_n$, $x \in X$, cf. [93].

If the class λ contains ϕ, finite rank operators are of type λ. If $\lambda = \ell^p$, $0 < p < \infty$, mappings of type λ are known as the *mappings of type* ℓ^p, and for $\lambda = \ell^1$, these are nuclear mappings, cf. [114]. These mappings are compact and characterize nuclear spaces. Moreover, all such operators of the same type constitute an ideal in the class of all bounded linear operators. Whereas in [46] representation theorems for operators of type $\ell^{v,w,\psi}_{p,q}$ and $S_{w,\psi}$ in terms of finite rank operators have been obtained, where $0 < p < q \leq \infty$, $\psi : [0, \infty) \to [0, \infty)$ a continuous, strictly sub-additive function satisfying $\psi(0) = 0$, and $v = \{v_n\}$, $w = \{w_n\}$ sequences of positive numbers; the paper [48] incorporates results on the Orlicz type operators ($\lambda = \ell_M$), indeed, it is shown that the space $\mathcal{L}^M$ of all such operators T between arbitrary Banach spaces such that $\{a_n(T)\} \in \ell_M$, for a fixed Orlicz function M, is an operator ideal; and also its component $\ell_M(X, Y)$ is a Köthe sequence space of linear operators. Further generalizations of these operators which form an operator ideals have been studied in [55, 56].

In the paper [54], the concept of λ-*compact operator* has been introduced and it is shown that such operators form a quasi-normed operator ideal under suitable restriction on λ; a set K is said to be λ-*compact* if there exists $\{x_n\} \subset X$ with $\{\|x_n\|\} \in \lambda$ such that $K \subset \{\sum_{n\geq 1} \alpha_n x_n : \bar{\alpha} = \{\alpha_n\} \in B_{\lambda^\times}\}$, and $T \in \mathcal{L}(X, Y)$ is a λ-*compact operator* if T maps bounded sets to λ-compact sets, where X, Y and λ are Banach spaces. Equipping λ with the normal topology $\eta(\lambda, \lambda^\times)$, the notion of λ_η-compact sets has been introduced and studied in [10] and it is proved that the collection of all λ_η-compact operators form an operator ideal.

For the theories and historical development of these topics, I refer to [1, 10, 18, 115, 116].

(g) Dynamics of Linear Operators:

Linear Dynamics or dynamics of linear operators deals with chaos and related properties of linear systems. Indeed, it is mainly concerned with the behavior of orbits generated by operators defined on topological vector spaces. This subject lies at the intersection of several domains of mathematics such as operator theory, topological dynamics, universality and ergodic theory. The main theme in linear dynamics is hypercyclicity which comes from the operator theory via cyclicity- a notion related to the famous invariant subspace problem. In topological dynamics, hypercyclicity is related to the topological transitivity via the Birkhoff's transitivity theorem which states that if f is continuous map on a separable complete metric space X with no isolated points, then there exists an $x \in X$ such that the *orbit* $\{x, f(x), f^2(x), \dots\}$ is dense in X if and only if for every pair of non-empty open sets U and V, we have $f^n(U) \cap V \neq \phi$ for some $n \in \mathbb{N}$. Also, hypercyclicity is a particular case of universality, where a family $\{f_\alpha\}$ of maps between topological spaces X and Y is said to be *universal* if $\{f_\alpha(x)\}$ is dense in Y for some $x \in X$. An operator f from a TVS X into itself is said to be *hypercyclic* if its orbit is dense in X It was S. Rolewicz [122] who initiated the study of hypercyclic operators in the Banach space setting, though the translation and differentiation operators were known to be hypercyclic on the space $(\mathcal{H}(\mathbb{C}), \tau)$ of entire functions in the work of G. D. Birkhoff and G. R. MacLane . The concept of supercyclicity of operators was introduced by H. M. Hilden and L. J. Wallen, but the researches on these topics gained momentum after C. Kitai presented a detailed and systematic study of hypercyclic and supercyclic operators in her Ph.D thesis [99]. On the other hand, motivated by the contributions in ergodic theory, especially Birkhoff's ergodic theorem, cf. [38]; in 2006, F. Bayart and S. Grivaux [7] introduced the concept of frequent-hypercyclicity of operators, a notion stronger than hypercyclicity. A good account of these results are given in survey articles [26, 36, 37] and the theses [108, 118].

Motivated by the study of the frequently hypercyclic operators, for a given natural number q, we have introduced the concept of q-frequent hypercyclic operator in [68]; indeed, we define the *q-lower density* of a subset A of $\mathbb{N}$ as

$$q - \underline{den} A = \liminf_{N \to \infty} (\{n \in A : n \leq N^q\}/N^q).$$

A continuous linear operator T on a separable topological vector space X is said to be *q-frequent hypercyclic* if there exist $x \in X$ such that for any non-empty open subset U of X, the set $N(x, U) = \{x \in \mathbb{N} : T^n(x) \in U\}$ has positive q-lower density. Such a vector x is called a *q-frequent hypercyclic vector* for T. Since the notion of 1-lower density and lower density coincides, frequent hypercyclicity of an operator is the same as its 1-frequent hypercyclicity. In general the following implications are true:

$$frequent\ hypercyclicity \Rightarrow q - frequent\ hypercyclicity \Rightarrow hypercyclicity,$$

for all $q \in \mathbb{N}$. In [68], examples have been given in order to show that none of the converse implications is true. Besides obtaining the sufficient criterion, it is shown that the non- convolution operator T_μ on $(\mathcal{H}(\mathbb{C}), \tau)$, $T_\mu(f)(z) = f'(\mu z)$ is frequently hypercyclic for $|\mu| \geq 1$. In [67], we modify the sufficient criterion for q-hypercyclicity and study the conditions for a linear map of the form $C_{R,T}(S) = RST$ to be q-frequently hypercyclic defined on algebras of operators on separable Banach spaces. Also, the frequent hypercyclicity of the operators $C_{M_\phi^*, M_\psi}$ on the spaces $S_p(\mathcal{H}^2(\mathbb{D})), \|\cdot\|_p)$ and $K(\mathcal{H}^2(\mathbb{D}), \|\cdot\|_p)$ have been characterized, where ϕ, ψ are non-zero, bounded, analytic functions on $\mathbb{D}$ (open unit disc in $\mathbb{C}$) with one of them being non-constant and M_ϕ is the multiplication operator associated to ϕ, $S_p(\mathcal{H}^2(\mathbb{D})) = \{T \in \mathcal{L}(\mathcal{H}^2(\mathbb{D})) : \{\alpha_n(T)\} \in \ell_p\}$, $1 \leq p < \infty$ (the pth Schatten von-Neumann class) equipped with the norm $\|T\|_p = (\sum a_n(T)^p)^{\frac{1}{p}}$ and

$K(\mathcal{H}^2(\mathbb{D}))$, is the algebra of compact operators on Hardy space $\mathcal{H}^2(\mathbb{D})$. The paper [69] incorporates the study of the supercyclicity of the operators of the form $C_{R,S}$ defined on different spaces of operators, where a continuous operator T defined on a topological vector space X is said to be *supercyclic* on X if $\{\lambda T^n x : \lambda \in \mathbb{K}, n \geq 0\}$ is dense in X for some $x \in X$. A closed infinite dimensional subspace Y of a topological vector space X is said to be *hypercyclic*(*q-frequently hypercyclic*) *subspace* if every non-zero vector in Y is hypercyclic (q-frequently hypercyclic) vector for T. Study of such spaces for conjugate operators defined on $S_p(\mathcal{H}^2(\mathbb{D}))$ and $K(\mathcal{H}^2(\mathbb{D}))$ have been carried out in [70]. For the literature of this subject, we refer to the books [6, 38]. We shall continue our study of hypercyclicity on spaces of entire functions in the next section.

3. Functional Analytic Study of Spaces of Holomorphic Mappings in Finite and Infinite Dimensional Banach Spaces

We divide this section into two parts, namely,

(a) holomorphic mappings in finite dimensional spaces; and

(b) holomorphic mappings in infinite dimensional spaces.

Let us begin with

(a) Holomorphic Mappings in Finite Dimensional Spaces:
We have observed in the subsection (a) of the previous section that growth conditions play a vital role in deciding the subclasses of the space of entire functions. In case of $\mathcal{H}(\mathbb{C})$, we know that the exponential functions are the simplest rapidly growing transcendental functions, whereas polynomials are entire functions of order zero. On the other hand, it has been shown in [27] that for any preassigned transcendental growth rate $\sigma > 0$, there is a hypercyclic vector with slower growth for the translation operator T_a for given $a \neq 0$ on $\mathcal{H}(\mathbb{C})$. This observation led K.C. Chan and J.H. Shapiro [20] to raise the problem, "can the Birkhoff's result on the hypercyclicity of the translation operator on $(\mathcal{H}(\mathbb{C}), \tau)$ be extended to Hilbert space of entire functions having arbitrary slow growth?" Indeed, they pointed out that the phrase "arbitrary slow growth" has to be limited in view of a result of Kitai [99] that no finite dimensional Hilbert space (indeed, Banach space) supports a hypercyclic operator; cf. also the observation of S. Rolewicz in [122]. Besides, motivated by the work of D.A. Herrero and Z.Y. Wang on compact perturbations of hypercyclic operator, Chan and Shapiro raised an operator-theoretic twist of the problem on hypercyclicity, namely, " On Hilbert spaces, can arbitrarily compact infinite rank operator T perturb the identity I to a hypercyclic operator, i.e. $I + T$ is hypercyclic," for identity is the most non-hypercyclic operator. Indeed, they proved that a bounded linear operator on a Hilbert space is not hypercyclic if its adjoint has an eigenvalue and so no perturbation of the identity on the Hilbert space by a finite rank operator is hypercyclic. Having observed the connections between complex analysis and operator theory via hypercyclicity [20], we thought

of carrying out researches on the subspaces of the space of entire functions, which are defined with the help of Orlicz sequence spaces and the comparison function γ, thus involving the sequence space theory and growth properties. Indeed, in the paper [73], using the representation (1.1) for $f \in \mathcal{H}(\mathbb{C})$ and recalling the definitions of M, N, $\tilde{\ell_M}$ and γ, we introduce

$$E^M(\gamma) = \{f \in \mathcal{H}(\mathbb{C}) : \sum_{n=0}^{\infty} \frac{\hat{f}(n)}{\gamma_n}\beta_n \text{ converges for all } \bar{\beta} = \{\beta_n\} \in \tilde{\ell_N}\}.$$

For $f \in E^M(\gamma)$, we write

$$\|f\|_{M,\gamma} = \sup\{|\sum_{n=0}^{\infty} \frac{\hat{f}(n)}{\gamma_n}\beta_n| : \ \delta(\bar{\beta}; N) \leq 1\} \qquad \text{and}$$

$$\|f\|_{(M),\gamma} = \inf\{\rho > 0 : \sum_{n=0}^{\infty} M(\frac{\hat{f}(n)}{\gamma_n \rho}) \leq 1\}.$$

The space $E^M(\gamma)$ equipped with the any of the above equivalent norms, is a Banach space and the subspace topology on $E^M(\gamma)$ induced from the compact open topology τ is weaker than the norm topology. If we write $u_n(z) = \gamma_n z^n$, for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we prove

Theorem 3.1. *Let the Orlicz function M satisfy Δ_2-condition at '0' and the complimentary Orlicz function N be such that $N(1) = 1$. Then the sequence $\{u_n\}$ forms a normalized Schauder base for $E^M(\gamma)$ with $\|u_n\|_{M,\gamma} = 1$, for each $n \in \mathbb{N}$; and a fortiori, the space $E^M(\gamma)$ is separable.*

We also introduce the Banach subspaces $\ell^\infty(\gamma)$ and $E^\infty(\gamma)$ as follows:

$$\ell^\infty(\gamma) = \{f \in \mathcal{H}(\mathbb{C}) : \|f\|_{\infty,\gamma} = \sup |(\hat{f}(n)/\gamma_n)| < \infty\};$$

$$E^\infty(\gamma) = \{f \in \mathcal{H}(\mathbb{C}) : \|f\|_\gamma^\infty = \sup\{(|f(z)|/\gamma(|z|) : z \in \mathbb{C}\} < \infty\}$$

and prove

Proposition 3.2. *If M is an Orlicz function such that $M(1) = 1$, then $E^M(\gamma) \subset \ell^\infty(\gamma) \subset E^\infty(\gamma)$ and the inclusion map from $E^M(\gamma)$ to $\ell^\infty(\gamma)$, and to $E^\infty(\gamma)$ are continuous.*

In the process of finding whether the inclusions in the above proposition are strict, surprisingly it turns out that these spaces are almost the same as shown in

Proposition 3.3. *If the comparison function γ is such that the sequence $\{\frac{n\gamma_n}{\gamma_{n-1}}\}$ is bounded, and $\tilde{\gamma}(z) = a_0 + a_1 z + a_2 z^2 + z^3\gamma(z)$ (where a_0, a_1 and a_2 are chosen appropriately so that $\tilde{\gamma}(z)$ becomes a comparison function), then $E^\infty(\gamma) \subset E^M(\tilde{\gamma})$ and $\|f\|_{M,\tilde{\gamma}} \leq B\|f\|_\gamma^\infty$ for some $B > 0$.*

Corresponding to two Orlicz functions M and N, we define *multiplier* between the spaces $E^M(\gamma)$ and $E^N(\gamma)$ as a sequence $\bar{\alpha} = \{\alpha_n\}$ of scalars such that for each $f \in E^M(\gamma)$, $\bar{\alpha}f \in E^N(\gamma)$ where $(\bar{\alpha}f)(z) = \sum_{n=0}^\infty \hat{f}(n)\alpha_n z^n$. Clearly, for $\{\alpha_n\} \in \ell_\infty$ and $f \in E^M(\gamma)$, the function g defined on $\mathbb{C}$ as $g(z) = \sum_{n=0}^\infty \hat{f}(n)\alpha_n z^n$, is an entire function. Using this fact and the Orlicz function $M_N^*(s)$ defined with the help of two Orlicz funtions M and N as

$$M_N^*(s) = \max\{0, \sup\{N(st) - M(t) :\ 0 \le t \le 1\}\},\ s \ge 0,$$

we prove

Theorem 3.4. *Let M and N be Orlicz function such that $M(1) = 1$. Then for any $\phi \in E^{M_N^*}(\gamma)$, the sequence $\{\frac{\hat{\phi}(n)}{\gamma_n}\}$ is a multiplier between the spaces $E^M(\gamma)$ and $E^N(\gamma)$.*

For Hilbert spaces of functions, Godefroy and Shapiro [34] define the multiplier as follows:

Definition 3.5. Let H be a Hilbert space of functions defined on an open connected set G. A function $\phi : G \to \mathbb{C}$ is said to be a *multiplier* if $\phi f \in H$ for every $f \in H$.

Since for $M(x) = x^2 = N(x)$, the space $E^2(\gamma) = \{f \in \mathcal{H}(\mathbb{C}) : \{\frac{\hat{f}(n)}{\gamma_n}\} \in \ell^2\}$ equipped with the norm $\|f\|_{2,\gamma} = (\sum_{n=0}^{\infty}(\frac{|\hat{f}(n)|}{\gamma_n})^2)^{\frac{1}{2}}$, is a reproducing kernel Hilbert space, using Liouville's theorem, we show

Theorem 3.6. *Every multiplier on $E^2(\gamma)$ is constant.*

We also consider the sequential analogue of the space $E^M(\gamma)$ and its Köthe dual. Indeed, corresponding to complimentary Orlicz functions M and N with $M(1) = 1$ and $N(1) = 1$, we consider the spaces

$$\lambda_{\gamma,M} = \{\{\alpha_n\} \in \omega : \{\ \alpha_n/\gamma_n\} \in \ell_M\} \qquad \text{and}$$

$$\mu_N^\gamma = \{\{\beta_n\} \in \omega : \{\gamma_n\beta_n\} \in \ell_N\}.$$

Clearly, $\lambda_{\gamma,M}$ corresponds to the space $E^M(\gamma)$ and it is a proper subspace of the space $\delta = \{\{\alpha_n\} : |\alpha_n|^{\frac{1}{n}} \to 0 \ as\ n \to \infty\}$ of entire sequences, since $\{\gamma_n\} \notin \lambda_{\gamma,M}$. Equipping $\lambda_{\gamma,M}$ and μ_N^γ with the norms $\|\cdot\|_\lambda$ and $\|\cdot\|_\mu$ respectively, where

$$\|\bar{\alpha}\|_\lambda = \inf\{\rho > 0 : \sum_{k=0}^{\infty} M(\frac{|\alpha_k|}{\gamma_k\rho}) \le 1\},\ \bar{\alpha} = \{\alpha_n\} \in \lambda_{\gamma,M} \qquad \text{and}$$

$$\|\bar{\beta}\|_\mu = \inf\{\rho > 0 : \sum_{k=0}^{\infty} M(\frac{\gamma_k|\beta_k|}{\rho}) \le 1\},\ \bar{\beta} = \{\beta_n\}\ \ in\ \ \mu_N^\gamma,$$

we show that these spaces are $AK-, BK-$, perfect sequence spaces if M and N satisfy Δ_2-condition at zero. By Theorem 2.11, it follows that the spaces $\lambda_{\gamma,M}$ and μ_N^γ are the topological duals of each other. As $\lambda_{\gamma,M}$ can be identified with the space $E^M(\gamma)$, we conclude that the topological dual of $E^M(\gamma)$ must correspond to the sequence space μ_N^γ , i.e., $(E^M(\gamma))^*$ must contain those analytic functions f which have the representation of the form $f(z) = \sum_{n\ge0}\alpha_n z^n$, where $\{\alpha_n\gamma_n\} \in \ell_N$. Clearly, this space is not contained in $\mathcal{H}(\mathbb{C})$; indeed $\lambda_{\gamma,M} \subsetneq \ell^\infty \subsetneq \mu_N^\gamma$ for $\{n\} \in \mu_N^\gamma$.

More general versions of these sequence spaces have been studied in [74].

In our subsequent work [75], we consider the restriction of the differential and translation operators on the space $E^M(\gamma)$. Recalling the differential operator D on $\mathcal{H}(\mathbb{C})$ as

$$Df(z) = \sum_{n=1}^{\infty} n\hat{f}(n)z^{n-1}, \ z \in \mathbb{C}$$

for $f(z) = \sum_{n=1}^{\infty} \hat{f}(n)z^n$; when it is restricted to $E^M(\gamma)$, we prove

Theorem 3.7. *Let M and N be complementary Orlicz functions such that M satisfies Δ_2-condition at '0' and $N(1) = 1$. If γ is a comparison function with $w_n = \frac{n\gamma_n}{\gamma_{n-1}}$, then we have (i) The operator D is a bounded linear operator from $(E^M(\gamma), \|\cdot\|_{M,\gamma})$ to itself if and only if $\{w_n\}$ is a bounded sequence.*

(ii) D is compact on $(E^M(\gamma), \|\cdot\|_{M,\gamma})$ if $w_n \to 0$ as $n \to \infty$. Converse holds if γ is an admissible comparison function, and both the Orlicz functions M and N satisfy Δ_2-condition at '0'.

(iii) D from $E^M(\gamma)$ to itself is nuclear if $\{w_n\} \in \ell_1$.

Relating the approximation number $\{\alpha_n(D)\}$ of the differential operator D with the sequence $\{w_n\}$, we have

Proposition 3.8. *Let γ be an admissible comparison function. Then $\alpha_n(D) \leq w_{n+1}$.*

The above result immediately yields that for an admissible comparison function γ, D is of type ℓ^p if $\{w_n\} \in \ell^p$, $p \geq 1$. Let us note that an admissible comparison function γ with $\lim_{n\to\infty} w_n = \tau$, is an entire function of order 1 and type τ, cf. [20]. For such a comparison function, we prove that the spectrum $\sigma(D)$ of $D|E^M(\gamma)$ is the closed unit disc $\bar{B}_\tau = \{z : |z| \leq \tau\}$. Also, we show that if $G(z) = \sum_{n=0}^{\infty} a_n z^n$, is an analytic function in a neighborhood of $\bar{B}_\tau$, then the series $\sum_{n=0}^{\infty} a_n D^n$ converges to a bounded linear operator, say $G(D)$, in the Banach space $\mathcal{L}(E^M(\gamma), E^M(\gamma))$ equipped with the usual operator norm; further, if $a_0 \neq 0$, then $G(D)$ is invertible. Concerning the restriction of the translation operator T_a, $a \neq 0$ on $E^M(\gamma)$, we prove

Proposition 3.9. *If the sequence $\{\frac{n\gamma_n}{\gamma_{n-1}}\} = \{w_n\}$ is bounded, then the operator T_a, $a \neq 0$, is a bounded linear operator on $E^M(\gamma)$ and has the following representation*

$$T_a = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n$$

in the operator norm topology of $\mathcal{L}(E^M(\gamma), E^M(\gamma))$, where D is the differential operator.

Theorem 3.10. *Let γ be an admissible comparison function and $a(\neq 0) \in \mathbb{C}$, then the translation operator T_a on $E^M(\gamma)$ is hypercyclic.*

Generalizations of some of the above results will be considered in the next subsection.

(b) Holomorhic Functions in Infinite Dimensional Spaces:

Infinite dimensional holomorphy is a branch of Functional Analysis which deals with the study of holomorphic functions defined on open, or compact or more general subsets of complex locally convex spaces. This theory has its origin in the work of V. Volterra around the year 1887; but Gâteaux's work [32, 33] published posthumously, in 1919 and 1922 comprised the study of holomorphic mappings defined through the complex polynomials or Gâteaux differentials. However, it was L. Nachbin [109] who developed the present theory of infinite dimensional holomorphy. In 1968, he noticed the existence of a holomorphic function on an infinite dimensional space, which takes a bounded set to unbounded set (a distinctive feature from the finite dimensional case). which motivated him to define the class $\mathcal{H}_b(E,F)$ of holomorphic mappings of bounded type, i.e., the holomorphic mappings taking bounded subsets of E to bounded subsets of F, where E and F are complex Banach spaces, cf. [19], p.218. This class has an analogy with the class of holomorphic mappings defined on an open subset of a finite dimensional space. This theory is now developed extensively and we have several books available on this topic, cf. [5, 19, 24, 25, 106, 109]. For historical development of this subject, I refer to the doctoral thesis [23] and books [24, 25, 106].

Before proceeding further for our contributions on this topic, we first consider some basic notions and results in infinite dimensional holomorphy.

For each $m \in \mathbb{N}, \mathcal{L}(^mE;F)$ is the Banach space of all continuous m-linear mappings from E to F endowed with its natural sup norm, namely

$$\| A \| = \sup_{\substack{x_i \neq 0 \\ i=1,2,\dots,m}} \frac{\| A(x_1,\dots,x_m) \|}{\| x_1 \| \dots \| x_m \|}.$$

The closed subspace of $\mathcal{L}(^mE;F)$ consisting of all continuous symmetric m-linear mappings A, i.e., $A(x_1,\dots,x_m) = A(x_{\sigma(1)},\dots,x_{\sigma(m)})$ for each $\sigma \in \pi_m$, the class of all permutations of $\{1,2,\dots,m\}$, is denoted by $\mathcal{L}_s(^mE;F)$. For all $x,y \in E$, each A in $\mathcal{L}_s(^mE;F)$ satisfies the *Newton Binomial formula*

$$A(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} A(x^{m-k}, y^k).$$

A continuous m-homogeneous polynomial P is a mapping from E to F defined as

$$P(x) = A(x,\dots,x), \quad x \in E,$$

where $A \in \mathcal{L}_s(^mE;F)$. In this case, we also write $P = \hat{A}$. The space of all m-homogeneous continuous polynomials from E to F is denoted by $\mathcal{P}(^mE;F)$ which is a Banach space endowed with the norm

$$\| P \| = \sup_{x \neq 0} \frac{\| P(x) \|}{\| x \|^m}.$$

A continuous polynomial P is a mapping from E into F which can be represented as a sum $P = P_0 + P_1 + \cdots + P_k$ with $P_m \in \mathcal{P}(^mE;F)$, $m = 0,1,\dots,k$. The vector space of continuous polynomials from E into F is denoted by $\mathcal{P}(E;F)$. The subspace of $\mathcal{P}(^mE) \equiv \mathcal{P}(^mE;\mathbb{C})$ spanned by the collection $\{\phi^m : \phi \in E^*\}$, is denoted by $\mathcal{P}_f({}^mE)$ and each member of $\mathcal{P}_f({}^mE)$ is known as a *polynomial of finite type*. Other than the subspace norm, we have a stronger norm $\|\cdot\|_N$ known as the *nuclear norm* defined as

$$\|P\|_N = \inf\{\sum_{i=1}^{m} \|\phi_i\|^m : P = \sum_{i=1}^{m} \phi_i^m,\ \phi \in E^*\},$$

where the infimum is taken over all possible representations of P.

Definition 3.11. A mapping $f : U \to F$ is said to be *holomorphic* at $\xi \in U$, if there exists a sequence of polynomials $P_m \in \mathcal{P}({}^mE; F)$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - \xi), \tag{3.1}$$

where the series converges uniformly in some ball $B_\rho(\xi) = \{x : \|x - \xi\| < \rho\} \subset U$. The series (3.1) is called the Taylor series of f at ξ and in analogy with the complex case, P_m is written as $\frac{1}{m!}\ \hat{d}^m f(\xi)$, *i.e.,*

$$f(x) = \sum_{m=0}^{\infty} P_m(x - \xi) = \sum_{m=0}^{\infty} \frac{1}{m!}\ \hat{d}^m f(\xi)(x - \xi). \tag{3.2}$$

If f is holomorphic at each $\xi \in U$, then f is said to be *holomorphic* from U to F. The space of all holomorphic mappings from U to F is denoted by $\mathcal{H}(U; F)$. In case $U = E$, the class $\mathcal{H}(E, F)$ is the *space of entire mappings* from E into F. An entire mapping $f : E \to F$ is said to be of *bounded type* if f is bounded on all bounded subsets of E. The space of entire mappings of bounded type is denoted by $\mathcal{H}_b(E; F)$ which is a Fréchet space when endowed with the topology τ_b of the uniform convergence on bounded sets. We write $\mathcal{H}(E)$, $\mathcal{H}_b(E)$, $\mathcal{P}(E)$, $\mathcal{P}({}^mE)$, $\mathcal{L}({}^mE)$ and $\mathcal{L}_s({}^mE)$ respectively for $\mathcal{H}(E;\mathbb{C})$, $\mathcal{H}_b(E;\mathbb{C})$, $\mathcal{P}(E;\mathbb{C})$, $\mathcal{P}\ ({}^mE;\mathbb{C})$, $\mathcal{L}({}^mE;\mathbb{C})$ and $\mathcal{L}_s({}^mE;\mathbb{C})$.

If (3.2) is the Taylor series expansion of $f \in \mathcal{H}(U; F)$ at ξ, then the Taylor series expansion of $\hat{d}^i f(x)$ at ξ is

$$\hat{d}^i f(x) = \sum_{k=0}^{\infty} \hat{d}^i P_{k+i}(x - \xi). \tag{3.3}$$

For $f \in \mathcal{H}(U; F)$ and $\xi \in U$, the inequalities

$$\|\frac{1}{i!}\ \hat{d}^i f(\xi)\| \le \frac{1}{\rho^i} \sup_{\|x-\xi\|=\rho} \|f(x)\|, \quad i = 0, 1, 2, ... \tag{3.4}$$

are known as the *Cauchy inequalities*, where $\bar{B}_\rho(\xi) = \{x : \|x - \xi\| \le \rho\} \subset U$, for some $\rho > 0$. Following [13, 14], in case of a Hausdorff TVS (X, T) and a sequentially complete LCS (Y, S), let us write $\mathcal{P}_a({}^mX; Y)$ and $\mathcal{P}({}^mX; Y)$ for the class of algebraic and continuous m-homogeneous polynomials from X to Y. For an open subset U of (X, T), we have

Definition 3.12. (i) A function $f : U \to Y$ is said to be *G-holomorphic* (holomorphic) if for every $x \in U$, there exists a series $\sum_{n\geq 1} f_n$, $f_n \in \mathcal{P}_a(^m X; Y)$ (resp. $\mathcal{P}(^m X; Y)$) such that

$$f(x+h) = \sum_{n\geq 1} f_n(h)$$

for all h in a neighborhood of 0 in X.

(ii) A mapping $f : U \to Y$ is known as *hypoanalytic* if it is G-holomorphic and is continuous on compact sets.

As mentioned above $\mathcal{H}(U)$ denotes the class of all holomorphic mappings from U to $\mathbb{C}$ and by $\mathcal{H}_{hy}(U)$ we denote the class of all hypoanalytic mappings from U to $\mathbb{C}$. Clearly $\mathcal{H}(U) \subset \mathcal{H}_{hy}(U)$.

For considering subspaces of $\mathcal{H}(E)$, let us denote by μ a normal sequence space equipped with a Hausdorff locally convex topology T_μ generated by the family D_μ of solid semi-norms; $\alpha = \{\alpha_n\}$, a sequence in ω with $\alpha_n \neq 0$, $n \geq 0$, and λ a sequence space. In [78], we introduce the spaces

$$\mathcal{H}^\mu(E) = \{f \in \mathcal{H}(E) : \hat{d}^n f(0) \in \mathcal{P}(^n E) \text{ with } \{(\frac{\|\hat{d}^n f(0)\|}{n!})^{\frac{1}{n}}\} \in \mu\} \quad (3.5)$$

$$\mathcal{H}^\mu_N(E) = \{f \in \mathcal{H}(E) : \hat{d}^n f(0) \in \mathcal{P}(^m E) \text{ with } \{(\frac{\|\hat{d}^n f(0)\|_N}{n!})^{\frac{1}{n}}\} \in \mu\} \quad (3.6)$$

$$\mathcal{H}^\mu_\alpha(E,\lambda) = \{f \in \mathcal{H}^\mu(E) : \ \{\|\hat{d}^n f(0)\|^{\frac{1}{n}}\} \in \lambda^\mu_\alpha\} \quad \text{and} \quad (3.7)$$

$$\mathcal{H}^\mu_{N,\alpha}(E,\lambda) = \{f \in \mathcal{H}^\mu_N(E) : \ \{\|\hat{d}^n f(0)\|_N^{\frac{1}{n}}\} \in \lambda^\mu_\alpha\} \quad (3.8)$$

Clearly $\mathcal{H}^\mu_N(E) \subset \mathcal{H}^\mu(E)$ and $\mathcal{H}^\mu_{N,\alpha}(E,\lambda) = \mathcal{H}^\mu_\alpha(E,\lambda) \cap \mathcal{H}^\mu_N(E)$. These spaces (3.5-3.8) are respectively equipped with the Hausdorff locally convex topologies T_h, T_h^N, $T_{h,\alpha}$ and $T^N_{h,\alpha}$ generated by the families $D_h = \{Q_p : p \in D_\mu\}$, $D_h^N = \{Q_p^N : p \in D_\mu\}$, $D_h^\lambda = \{Q_p^a : p \in D_\mu, a \in \lambda\}$ and $D_h^{N\ \lambda} = \{Q^N_{p,a} : p \in D_\mu, a \in \lambda\}$, where for $p \in D_\mu$ and $a \in \lambda$

$$Q_p(f) = p((\frac{\|\hat{d}^n f(0)\|}{n!})^{\frac{1}{n}}),\ f \in \mathcal{H}^\mu(E)$$
$$Q_p^N(f) = p((\frac{\|\hat{d}^n f(0)\|_N}{n!})^{\frac{1}{n}}),\ f \in \mathcal{H}^\mu_N(E)$$
$$Q_{p,a}(f) = p(\{\|\hat{d}^n f(0)\|^{\frac{1}{n}} \alpha_n a_n\}),\ f \in \mathcal{H}^\mu_\alpha(E;\lambda) \qquad \text{and}$$
$$Q^N_{p,a}(f) = p(\{\|\hat{d}^n f(0)\|_N^{\frac{1}{n}} \alpha_n a_n\}),\ f \in \mathcal{H}^\mu_{N,\alpha}(E;\lambda).$$

We prove

Proposition 3.13. *Let (μ, T_μ) be a complete K-space such that $p_0(e^n) = 1$, for each $n \geq 0$ and some $p_0 \in D_\mu$. Then the spaces $(\mathcal{H}^\mu(E), T_h)$ and $(\mathcal{H}^\mu_N(E), T_h^N)$ are quasi-complete. In addition, if λ contains an element $b = \{b_n\}$ satisfying the condition $|b_n| > \frac{1}{|\alpha_n| n!^{\frac{1}{n}}}$, $\forall n \geq 0$, then the spaces $(\mathcal{H}^\mu_{N,\alpha}(E;\lambda), T^N_{h,\alpha})$ and $(\mathcal{H}^\mu_\alpha(E;\lambda), T_{h,\alpha})$ are quasi-complete.*

For $\alpha_n = (1/n)$, $n \geq 1$, $\mu = c_0$ and $\lambda = \{a \in \omega : (a_n/n) \to 0 \ as \ n \to \infty\}$, the spaces $\mathcal{H}^{\mu}_{N,\alpha}(E;\lambda)$ and $\mathcal{H}^{\mu}_{\alpha}(E;\lambda)$ have been studied by Boland [15], and the above proposition includes his result (Proposition 2.1, p.49). We have also characterized the bounded and relatively compact sets in $\mathcal{H}^{\mu}_{N,\alpha}(E;\lambda)$ and $\mathcal{H}^{\mu}_{\alpha}(E;\lambda)$. The main result of the paper is the following

Theorem 3.14. *Let $(\lambda, \eta(\lambda, \lambda^{\times}))$ be a barrelled sequence space such that $(\lambda^{\times}, \beta(\lambda^{\times}, \lambda))$ is an AK-space. Then the class $\mathcal{H}_{hy}(U)$ of hypoanalytic functions defined on a normal open subset U of $(\lambda^{\times}, \beta(\lambda^{\times}, \lambda))$ equipped with the topology τ_0 of uniform convergence on compact subsets of U, is a complete nuclear space and the set $\{f^m : m \in \mathbb{N}^{\mathbb{N}}\}$ of monomials forms a fully ℓ^1-base for $(\mathcal{H}_{hy}(U), \tau_0)$, where*

$$\mathbb{N}^{\mathbb{N}} = \cup \lim_{r \geq 1} \mathbb{N}^r,$$

$\mathbb{N}^r = \{m = \{m_1, m_2, m_3, \ldots, m_r, 0, 0, \ldots\} : m_i \in \mathbb{N}_0, \ i = 1, 2, \ldots, r\}$ and the mappings $f^m : \omega \to \mathbb{C}$, given by $f^m(a) = \prod \lim_{n=1}^{\infty} a_n^{m_n} = a^m$ for $m = \{m_1, m_2, m_3, \ldots, m_r, 0, 0, \ldots\} \in \mathbb{N}^{\mathbb{N}}$, are the monomials.

In [53], we have introduced and studied the subspaces $\mathcal{H}^{\gamma}_p(E;F)$, $1 \leq p \leq \infty$, of the space $\mathcal{H}_b(E;F)$ defined with the help of the comparison function γ as

$$\mathcal{H}^{\gamma}_p(E;F) = \left\{ f \in \mathcal{H}(E;F) : \left\{ \frac{\|\frac{1}{m!}\ \hat{d}^m f(0)\|}{\gamma_m} \right\} \in l_p \right\}, \quad 1 \leq p < \infty \quad \text{and}$$

$$\mathcal{H}^{\gamma}_{\infty}(E;F) = \left\{ f \in \mathcal{H}(E,F) : \left\{ \frac{\|\frac{1}{m!}\ \hat{d}^m f(0)\|}{\gamma_m} \right\} \in l_{\infty} \right\}, \quad p = \infty.$$

This study was initiated in [20] for the case $E = F = \mathbb{C}$ and $p = 2$ and further generalized in [73]. The spaces $(\mathcal{H}^{\gamma}_p(E;F), \|\cdot\|^{\gamma}_p)$ are Banach spaces and are properly contained in the space of entire mappings of bounded type. Indeed, we have

Proposition 3.15. *Let E and F be complex Banach spaces and γ be a comparison function.Then for $1 \leq p < q < \infty$, $\mathcal{H}^{\gamma}_p(E;F) \subsetneq \mathcal{H}^{\gamma}_q(E;F) \subsetneq \mathcal{H}^{\gamma}_{\infty}(E;F) \subsetneq \mathcal{H}_b(E;F)$.*

Using the vector-valued sequence space theory, we have established

Proposition 3.16. *The spaces $\{\mathcal{P}(^nE;F)\}_{n\geq 0}$ form a monotone shrinking Schauder decomposition for the space $(\mathcal{H}^{\gamma}_p(E;F), \|\cdot\|^{\gamma}_p)$, $1 \leq p < \infty$.*

In case of infinite dimension, the differentiation operators $D^i_a : \mathcal{H}(E) \to \mathcal{H}(E)$ for $a \in E$, $a \neq 0$ and $i \in \mathbb{N}_0$, are defined as

$$D^i_a(f)(x) = (1/i!)\ \hat{d}^i f(x)(a), \quad x \in E, \tag{3.9}$$

and the translation operator $T_a : \mathcal{H}(E) \to \mathcal{H}(E)$ is defined as

$$T_a f(x) = f(x+a) = \sum_{i \geq 0} \frac{1}{i!}\ \hat{d}^i f(x) a, \quad \forall x \in E. \tag{3.10}$$

These operators are bounded on $\mathcal{H}(E)$ with respect to the the topology of uniform convergence on compact subsets of E, cf. [82]. In [53], we have proved that the restriction of these operators on the spaces $\mathcal{H}_p^\gamma(E),\ 1 \le p < \infty$, are continuous for suitably restricted comparison function γ. We have also considered the continuity and the compactness of the composition operator C_ϕ defined corresponding to a holomorphic function ϕ from F to E, where E and F are Banach spaces; indeed, if $f \circ \phi \in \mathcal{H}_p^\gamma(F)$ for each $f \in \mathcal{H}_p^\gamma(E)$, then $C_\phi : \mathcal{H}_p^\gamma(E) \to \mathcal{H}_p^\gamma(F)$ is defined as $C_\phi(f) = f \circ \phi,\ f \in \mathcal{H}_p^\gamma(E)$.

(II) Weighted Spaces of Holomorphic Mappings: The study of growth conditions of holomorphic functions defined on a balanced open subset of a finite/infinite dimensional space, leads to the class of weighted spaces of such functions which have been investigated extensively after the appearance of D. L. Williams'work. There are two fundamental directions of researches on these spaces, namely (A) the study of the structural properties, and (B) the study of different operators acting between them. In [49, 50, 51, 52], we have confined our researches in direction (A) and have studied approximation property and some of its variants, namely-the bounded approximation property, the compact approximation property and the metric approximation property.

A **weight** w defined on an open subset U of a Banach space E is a continuous and strictly positive function satisfying

$$0 < \inf_A w(x) \le \sup_A w(x) < \infty \tag{3.11}$$

for each U-bounded set A (a subset A of U is *U-bounded* if A is bounded and $\mathrm{dist}(A, \partial U) > 0$, where ∂U denotes the boundary of U).

If U is also balanced, a weight w is said to be **radial** if $w(tx) = w(x)$ for all $x \in U$ and $t \in \mathbb{C}$ with $|t| = 1$. In case $U = E$ and $\sup_{x \in E} w(x)\|x\|^m < \infty$ for each $m \in \mathbb{N}_0$, the weight w is said to be *rapidly decreasing.* A function $g : U \to \mathbb{R}$ *vanishes at infinity outside U-bounded sets* if for each $\epsilon > 0$, there exists a U-bounded set $A \subset U$ such that $|g(x)| < \epsilon$ for all $x \in U \setminus A$.

Corresponding to a weight function w, the weighted spaces of holomorphic functions are defined as

$$\mathcal{H}_w(U, F) = \{f \in \mathcal{H}(U; F) : \|f\|_w = \sup_{x \in U} w(x)\|f(x)\| < \infty\};$$

and

$\mathcal{H}_w^0(U, F) = \{f \in \mathcal{H}_w(U; F) : w\|f\|$ vanishes at infinity outside U-bounded sets$\}$, where $w\|f\|(x) = w(x)\|f(x)\|$ for any $x \in U$.

The space $(\mathcal{H}_w(U, F), \|\cdot\|_w)$ is a Banach space and $\mathcal{H}_w^0(U, F)$ is a closed subspace of $\mathcal{H}_w(U, F)$, cf. [125]. For $w \equiv 1$, $(\mathcal{H}_w(U, F), \|\cdot\|_w) \equiv (\mathcal{H}^\infty(U, F), \|\cdot\|_\infty)$, the space of all bounded holomorphic mappings from U to F, which has been studied in [107]. For $F = \mathbb{C}$, we write $\mathcal{H}_w(U)$ and $\mathcal{H}_w^0(U)$ for $\mathcal{H}_w(U, \mathbb{C})$ and $\mathcal{H}_w^0(U, \mathbb{C})$ respectively. In case U is an open subset of $\mathbb{C}^n$, $n \ge 1$, $\mathcal{H}_w^0(U)$ is defined as $\mathcal{H}_w^0(U) = \{f \in \mathcal{H}(U) : w|f|$ vanishes at infinity on $U\}$. We denote by B_w and B_w^0, the closed unit balls of $(\mathcal{H}_w(U), \|\cdot\|_w)$ and $(\mathcal{H}_w^0(U), \|\cdot\|_w)$ respectively. For the weight w given by an entire function γ with positive coefficients, *i.e.*, $w(x) = \frac{1}{\gamma(\|x\|)}$, we write $\mathcal{H}_\gamma$ for $\mathcal{H}_w$. Since each compact subset K of U is U-bounded, and for each U-bounded set A we have $\sup_{x\in A}\|f(x)\| \le \frac{1}{\inf\limits_{x\in A} w(x)}\|f\|_w < \infty$, it follows that $\tau_0 \le \tau_{\|\cdot\|_w} \le \tau_b$ and so the following inclusions are continuous

$$(\mathcal{H}_w(U;F), \|\cdot\|_w) \hookrightarrow (\mathcal{H}_b(U;F), \tau_b) \hookrightarrow (\mathcal{H}(U;F), \tau_0).$$

The evaluation map $\delta_x : \mathcal{H}(U) \to \mathbb{C}$ defined as $\delta_x(f) = f(x)$, $f \in \mathcal{H}_w(U)$ is τ_0-continuous. Consequently, the closed unit ball B_w of $\mathcal{H}_w(U)$ which can be written as $B_w = \bigcap \overline{\delta_x^{-1}(B(0, \frac{1}{w(x)})}$, is τ_0-closed. Hence B_w is τ_0-compact and by Ng's Theorem (namely-if $\tilde{\tau}$ is a Hausdorff locally convex topology on a Banach space E weaker than the norm topology, such that B_E is $\tilde{\tau}$-compact, then E is isometrically isomorphic to a dual space), we have

Proposition 3.17. *For a weight w on an open subset U of a Banach space E, $\mathcal{H}_w(U)$ is a dual Banach space and its predual is given by*

$$\mathcal{G}_w(U) = \{\phi \in \mathcal{H}_w(U)' : \phi|B_w \text{ is } \tau_0 - \text{continuous}\}.$$

Further, the evaluation mapping $J_U^w : \mathcal{H}_w(U) \to \mathcal{G}_w(U)^*$, $J_U^w(f) = \hat{f}$ with $\hat{f}(h) = h(f)$, $f \in \mathcal{H}_w(U)$ and $h \in \mathcal{G}_w(U)$, is an isometric isomorphism.

In our paper [49], we have proved the following linearization theorem for the space $\mathcal{H}_w(U;F)$.

Theorem 3.18. *(Linearization Theorem) For an open subset U of a Banach space E and a weight w on U, there exists a Banach space $\mathcal{G}_w(U)$ and a mapping $\Delta_w \in \mathcal{H}_w(U;\mathcal{G}_w(U))$ with the following property: for each Banach space F and each mapping $f \in \mathcal{H}_w(U;F)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}_w(U);F)$ such that $T_f \circ \Delta_w = f$. The correspondence Ψ between $\mathcal{H}_w(U;F)$ and $\mathcal{L}(\mathcal{G}_w(U);F)$ given by $\Psi(f) = T_f$, is an isometric isomorphism. The space $\mathcal{G}_w(U)$ is uniquely determined up to an isometric isomorphism by these properties.*

In case, the weight w being given by an entire function γ with positive coefficients, *i.e.*, $w(x) = \frac{1}{\gamma(\|x\|)}$, we write $\mathcal{H}_\gamma$ for $\mathcal{H}_w$, $\mathcal{G}_\gamma$ for $\mathcal{G}_w$, $\Delta_w = \Delta_\gamma$, and in that case $\|\Delta_\gamma\| = 1$. As an application of linearization theorem, we have

Proposition 3.19. *Let w be a weight defined on an open subset U of a Banach space E such that $\mathcal{P}(E) \subset \mathcal{H}_w(U)$. Then E is topologically isomorphic to a complemented subspace of $\mathcal{G}_w(U)$. In case $U = E$ and w is a radial rapidly decreasing weight on E, E is topologically isomorphic to a 1-complemented subspace of $\mathcal{G}_w(U)$.*

Proposition 3.20. *The space $Q(^mE)$ is topologically isomorphic to a complemented subspace of $\mathcal{G}_w(U)$, where $Q(^mE)$ is the predual of $\mathcal{P}(^mE)$ defined as $\mathcal{Q}(^mE) = \{\phi \in \mathcal{P}(^mE)' : \phi|B_m \text{ is } \tau_0 - \text{continuous}\}$.*

For our study of the approximation property in weighted spaces of holomorphic mappings, linearization theorems play a vital role. This property, appeared for the first time in Banach's book [4] in 1932, has been found useful in the structural study of Banach spaces. A systematic treatment of this property along with its variants was carried out by Grothendieck [39] in 1955. As a Banach space with a Schauder basis has the approximation property and the spaces of holomorphic mappings on infinite dimensional spaces do not possess Schauder basis, it becomes all the more interesting to study this property in these spaces. This study is possible either by using the technique of linearization or a representation of holomorphic mappings as an infinite sum of homogeneous polynomials. The approximation property for spaces of holomorphic mappings was first considered by R. Aron and M. Schottenloher [3] in the year 1976 and further continued by several mathematicians for various classes of holomorphic mappings.

A locally convex space X is said to have (i) the *approximation property* (AP) if for every compact set K of X, p a continuous semi-norm on X and $\epsilon > 0$, there exists a finite rank operator $T = T_{\epsilon,K}$ such that $\sup_{x\in K} p(T(x)-x) < \epsilon$ and (ii) the *compact approximation property* (CAP) if there is a compact linear operator T such that $\sup\limits_{x\in K} p(T(x)-x) < \epsilon$. If $\{T_{\epsilon,K} : \epsilon > 0 \textit{ and } K$ varies over compact subsets of $X\}$ is an equicontinuous subset of $\mathcal{F}(X,X)$, class of all finite rank operators on X, X is said to have the *bounded approximation property* (BAP). When X is normed and above T can be chosen with $\|T\| \leq \lambda$, we say X has the λ-*bounded approximation property.* In case $\lambda = 1$, X is said to have the *metric approximation property*(MAP).

In order to study the approximation property for the weighted spaces, in [49] we have introduced a locally convex topology $\tau_{\mathcal{M}}$ on $\mathcal{H}_w(U;F)$ of which the particular cases have been considered in [107, 125]. For a finite set A and $r > 0$, we define $N(A,r) = \{f \in \mathcal{H}_w(U;F) : \inf\limits_{x\in A} w(x) \sup\limits_{y\in A} \|f(y)\| \leq r\}$ and consider the class

$$\mathcal{U} = \{\cap \lim_{j=1}^{\infty} N(A_j, r_j) : \ (A_j) \text{ varies over all sequences of finite subsets of } U \text{and}$$
$$(r_j) \text{ varies over all positive sequences diverging to infinity}\}.$$

Each member of $\mathcal{U}$ is balanced, convex and absorbing. Thus it forms a fundamental neighborhood system at 0 for a locally convex topology which is denoted by $\tau_{\mathcal{M}}$. Equivalently, this topology is generated by the family $\{p_{\overline{\alpha},\overline{A}} : \overline{\alpha} = (\alpha_j) \in c_0^+,\ \overline{A} = (A_j),\ A_j$being finite subset of U for each $j\}$ of semi-norms given by

$$p_{\overline{\alpha},\overline{A}}(f) = \sup_{j\in\mathbb{N}}(\alpha_j \inf_{x\in A_j} w(x) \sup_{y\in A_j} \|f(y)\|)$$

These are the Minkowski functionals of members in $\mathcal{U}$. For $F = \mathbb{C}$, $\tau_{\mathcal{M}} = \tau_{bc}$, cf. [125].

Relating $\tau_{\mathcal{M}}$ with τ_0 and $\tau_{\|.\|_w}$, and bounded sets with respect to these topologies, we have

Proposition 3.21. *For a weight w on an open subset U of a Banach space E, the following hold:*
(i) $\tau_0 \leq \tau_{\mathcal{M}} \leq \tau_{\|\cdot\|_w}$ *on* $\mathcal{H}_w(U;F)$.
(ii) $\tau_{\mathcal{M}}$ *and* $\|.\|_w$*-bounded sets are the same.*
(iii) $\tau_{\mathcal{M}}|\mathcal{B} = \tau_0|\mathcal{B}$ *for any* $\|\cdot\|_w$*- bounded set* $\mathcal{B}$.

Proposition 3.22. *Let E and F be Banach spaces. For a radial weight w on a balanced open subset U of E with $\mathcal{P}(E) \subset \mathcal{H}_w(U)$, the space $\mathcal{P}(E;F)$ is $\tau_{\mathcal{M}}$-dense in $\mathcal{H}_w(U;F)$.*

The following result which identifies the class of holomorphic mappings with the class of linear operators is useful for our study on approximation properties.

Theorem 3.23. *Let E and F be Banach spaces, and w be a weight on an open subset U of E. Then the mapping $\Psi : (\mathcal{H}_w(U;F), \tau_{\mathcal{M}}) \to (\mathcal{L}(\mathcal{G}_w(U);F), \tau_c)$ is a topological isomorphism.*

We write

$$\mathcal{H}_w(U) \otimes F = \{f \in \mathcal{H}_w(U;F) : f \text{ has finite dimensional range}\}$$

and

$$\mathcal{H}^c_w(U;F) = \{f \in \mathcal{H}_w(U;F) : wf \text{ has a relatively compact range}\}.$$

The interplay between the properties of a mapping $f \in \mathcal{H}_w(U;F)$ and the corresponding operator $T_f \in \mathcal{L}(\mathcal{G}_w(U);F)$ is established as follows

Proposition 3.24. *Let U be an open subset of a Banach space E and w be a weight on U. Then for any Banach space F,*
(a) $f \in \mathcal{H}_w(U) \otimes F$ *if and only if* $T_f \in \mathcal{F}(\mathcal{G}_w(U);F)$, *and*
(b) $f \in \mathcal{H}^c_w(U;F)$ *if and only if* $T_f \in \mathcal{K}(\mathcal{G}_w(U);F)$.

Proposition 3.25. *Let w be a weight on an open subset U of a Banach space E. Then $\overline{\mathcal{F}(\mathcal{G}_w(U);F)}^{\|\cdot\|} = \mathcal{K}(\mathcal{G}_w(U);F)$ if and only if $\overline{\mathcal{H}_w(U) \otimes F}^{\|\cdot\|_w} = \mathcal{H}^c_w(U;F)$ for each Banach space F.*

Proposition 3.26. *Let w be a weight on an open subset U of a Banach space E. Then $\overline{\mathcal{F}(\mathcal{G}_w(U);F)}^{\tau_c} = \mathcal{L}(\mathcal{G}_w(U);F)$ if and only if $\overline{\mathcal{H}_w(U) \otimes F}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U;F)$ for each Banach space F.*

Using the above results, we have established the following holomorphic analogues of Grothendieck's characterization of the approximation property for a Banach space E.

Theorem 3.27. *Let E be a Banach space. Then for each Banach space F, the following are equivalent:*
(i) E has the approximation property.
(ii) $\overline{\mathcal{H}_w(V) \otimes E}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(V;E)$, *for each open subset V of F and weight w on V.*
(iii) $\overline{\mathcal{H}_w(V) \otimes E}^{\|\cdot\|_w} = \mathcal{H}^c_w(V;E)$, *for each open subset V of F and weight w on V.*

Next, we characterize the approximation property for the weighted space $\mathcal{H}_w(U)$.

Theorem 3.28. *For an open subset U of a Banach space E, $\mathcal{H}_w(U)$ has the approximation property if and only if $\mathcal{H}_w(U) \otimes F$ is $\|\cdot\|_w$-dense in $\mathcal{H}_w^c(U;F)$ for each Banach space F.*

Theorem 3.29. *Let E be a Banach space and w be a radial weight on a balanced open subset U of E such that $H_w(U)$ contains all the polynomials. Then the following assertions are equivalent:*
(i) E has the approximation property.
(ii) $\overline{\mathcal{P}_f(E;F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U;F)$ for each Banach space F.
(iii) $\overline{\mathcal{H}_w(U) \otimes F}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U;F)$, for each Banach space F.
(iv) $\mathcal{G}_w(U)$ has the approximation property.

Every Banach space with the approximation property also has the compact approximation property. However, the converse implication is not true, cf. [131]. In [51], we have studied the compact approximation property for the weighted spaces and their preduals and have obtained the analogous results.

In [51], we have also defined the locally convex topology $\tau'_{\mathcal{M}}$ on $\mathcal{H}_w(U,F)$ generated by the family $\{q_{\overline{\alpha},\overline{A}} : \overline{\alpha} = (\alpha_j) \in c_0^+,\ \overline{A} = (A_j),\ A_j$ being finite subset of U for each $j\}$ of semi-norms given by

$$q_{\overline{\alpha},\overline{A}}(f) = \sup_{j\in\mathbb{N}}(\alpha_j \sup_{x\in A_j} w(x)\|f(x)\|),$$

Clearly $\tau_{\mathcal{M}} \leq \tau'_{\mathcal{M}}$. Equipping $\mathcal{H}_w(U,F)$ with $\tau'_{\mathcal{M}}$, we prove

Theorem 3.30. *Let w be a radial weight on a balanced open subset U of a Banach space E such that $\mathcal{P}(E) \subset \mathcal{H}_w(U)$. Then for each Banach space F, the following assertions are equivalent:*

(a) $w^{\frac{1}{i}-1} I_U \in \overline{\mathcal{H}_w^c(U;E)}^{\tau'_{\mathcal{M}}}$ for each $i \in \mathbb{N}$, where $I_U : U \to E$ is the inclusion mapping.

(b) $\overline{\mathcal{H}_w^c(U;F)}^{\tau'_{\mathcal{M}}} = \mathcal{H}_w(U,F)$.

For studying the metric approximation property in [50], we have proved the following representation theorem for members of $\mathcal{G}_w(U)$.

Theorem 3.31. *Let w be a weight on an open subset U of a Banach space E. Then each u in $\mathcal{G}_w(U)$ has a representation of the form*

$$u = \sum_{n\geq 1} \alpha_n w(x_n)\Delta_w(x_n) \tag{3.12}$$

for $\bar{\alpha} = (\alpha_n) \in l_1$ and $(x_n) \subset U$. Also,

$$\|u\| = \inf\{\|\bar{\alpha}\|_1 : \bar{\alpha} \text{ varies over all represtations of } u \text{ in (3.12)}\}.$$

It is proved in [50]

Theorem 3.32. *Let w be a radial rapidly decreasing weight on a Banach space E satisfying $w(x) \leq w(y)$ whenever $\|x\| \geq \|y\|$, $x, y \in E$. Then the following assertions are equivalent:*
(a) E has the MAP.
(b) $\overline{\{P \in \mathcal{P}_f(E, F) : \|P\|_w \leq 1\}}^{\tau_c} = B_{\mathcal{H}_w(E,F)}$ for any Banach space F.
(c) $\overline{B_{\mathcal{H}_w(E) \bigotimes F}}^{\tau_c} = B_{\mathcal{H}_w(E,F)}$ for any Banach space F.
(d) $\Delta_w \in \overline{B_{\mathcal{H}_w(E) \bigotimes \mathcal{G}_w(E)}}^{\tau_c}$.
(e) $\mathcal{G}_w(E)$ has the MAP.

Since a reflexive Banach space E having AP is equivalent to having MAP, cf.[101], a reflexive Banach space E has BAP if and only if $\mathcal{G}_w(E)$ has BAP for a weight w satisfying the conditions of Theorem 3.32.

E. Çalişkan [17] proved that for $w = 1$, a separable Banach space E has the BAP if and only if $\mathcal{G}_w(U_E)$ has the BAP. It would be interesting to know the solution of the the following problem for the case when w is a radial rapidly decreasing weight.

Problem 3.33. *Let E be a separable Banach space and w be a radial rapidly decreasing weight on E such that $w(x) \leq w(y)$ whenever $\|x\| \geq \|y\|$, $x, y \in E$. Then E has the BAP if and only if $\mathcal{G}_w(E)$ has the BAP.*

A Banach space having the bounded approximation property also has the approximation property but the converse is not true in general, cf. [28]. However, the answer to the above problem for a particular type of weight given by η, where η is a continuous decreasing function from $[0, \infty)$ to $(0, \infty)$, satisfying the condition $\eta(r)r^k \to 0$ for each $k \in \mathbb{N}_0$ and $w(x) = \eta(\|x\|)$ is given as follows, cf. [52]

Theorem 3.34. *A separable Banach space E has the BAP if and only if $\mathcal{G}_\eta(E)$ has the BAP*

In [50, 52], we have considered the approximation property for the weighted Fréchet and (LB)-spaces of holomorphic functions on Banach spaces. Whereas weighted Fréchet spaces were introduced by D. Garcia, M. Maestre and P. Rueda in [31], weighted (LB)-spaces were considered by M. J. Beltrán [8]. These spaces are defined as follows

Definition 3.35. Let $\mathcal{V} = \{v_n\}$ be a countable family of positive continuous functions on U such that for each $x \in U$, there is $n \in \mathbb{N}$ such that $v_n(x) > 0$. The weighted Fréchet space of holomorphic functions is defined as

$$\mathcal{HV}(U; F) = \{f \in \mathcal{H}(U; F) : p_{v_n}(f) = \sup_{x \in U} v_n(x)\|f(x)\| < \infty \text{ for each } n \in \mathbb{N}\}$$

and

$$\mathcal{HV}_0(U; F) = \{f \in \mathcal{HV}(U; F) : \text{for given } \epsilon > 0 \text{ and } n \in \mathbb{N}, \text{ there exists a U-boundedset } A \text{ such that } \sup_{x \in U \setminus A} v_n(x)\|f(x)\| < \epsilon\}.$$

For $F = \mathbb{C}$, we write $\mathcal{HV}(U)$ and $\mathcal{HV}_0(U)$ instead of $\mathcal{HV}(U, F)$ and $\mathcal{HV}_0(U, F)$. The space $\mathcal{HV}(U)$ and $\mathcal{HV}_0(U)$ are endowed with the topology τ_V generated by the family of semi-norms $\{p_{v_n} : n \in \mathbb{N}\}$.

A family $\mathcal{V}$ of weights satisfies *condition I* if for each U-bounded set A there exists $n \in \mathbb{N}$ such that $\inf_{x \in A} v_n(x) > 0$ and, $\mathcal{V}$ satisfies *condition II* if for each $n \in \mathbb{N}$, there exist $R > 1$ and $m \in \mathbb{N}$ such that

$$p_{v_n}(P^j f(0)) \le \frac{1}{R^j} p_{v_m}(f)$$

for each $j \in \mathbb{N}$ and $f \in \mathcal{HV}(U)$.

For $\alpha = \{\alpha_n\}_{n=1}^{\infty}$, a sequence of strictly positive numbers, define $D_\alpha = \{f \in \mathcal{HV}(U) : p_{v_n}(f) \le \alpha_n$ for each n$\}$. Clearly D_α is a τ_V-bounded set. Also, $\{D_\alpha\}_\alpha$ is a fundamental system of absolutely convex τ_V-bounded sets which are τ_0-compact. Further, it can be easily seen that $\{V_{n,\epsilon} : n \in \mathbb{N}, \epsilon > 0\}$, where $V_{n,\epsilon} = \{f \in \mathcal{HV}(U) : p_{v_n}(f) \le \epsilon\}$, is a fundamental 0-neighborhood system consisting of absolutely convex τ_0-closed sets. Then the space

$$\mathcal{GV}(U) = \{\phi \in \mathcal{HV}(U)' : \phi | D_\alpha \text{ is } \tau_0 - \text{continuous}\}$$

is a complete barrelled DF-space(see [100], p.396 for definition of DF-space) whose strong dual is topologically isomorphic to $\mathcal{HV}(U)$, cf. [11].

In the finite dimensional case, K.D. Bierstedt, J. Bonet and A. Galbis in [12] considered the bounded approximation property(BAP) for the weighted spaces of holomorphic mappings containing polynomials; indeed, in this case Cesáro mean operators C_n are the finite rank operators converging uniformly on compact sets to the identity operator. For weighted spaces of holomorphic mappings defined on an open balanced subset of a Banach space, the techniques involving $\mathcal{S}$-absolute decompositions come to our rescue to establish results for the BAP. In our paper [50], we have shown

Theorem 3.36. *Let $\mathcal{V}$ be a family of weights satisfying the conditions I and II, and $\mathcal{HV}(U)$ contain all the polynomials. Then*
(i) the sequence of spaces $\{Q(^nE)\}\}_{n=0}^{\infty}$ is an $\mathcal{S}-$absolute decomposition for $\mathcal{GV}(U)$, and
(ii) the sequence $\{\mathcal{P}(^nE)\}_{n=0}^{\infty}$ is an $\mathcal{S}-$absolute decomposition for $\mathcal{HV}(U)$.

Consequently, using a general result of E. Çalişkan, namely- "if $\{X_n\}_{n=1}^{\infty}$ is an $\mathcal{S}$-absolute decomposition of a locally convex space X, then X has the BAP if and only if each X_n has the BAP" and the above result, we derive

Theorem 3.37. *Let $\mathcal{V}$ be a family of weights defined on an open balanced subset U of a Banach space E. Assume that $\mathcal{V}$ satisfies the conditions I and II, and $\mathcal{HV}(U)$ contains all the polynomials. Then the following are equivalent:*
(a) E has the BAP.

(b) $\mathcal{Q}({}^mE)$ *has the BAP for each* $m \in \mathbb{N}$.
(c) $\mathcal{GV}(U)$ *has the BAP.*

We now consider weighted (LB)-spaces defined corresponding to a countable decreasing family $V = \{v_n\}$ of weights $(v_{n+1} \leq v_n)$ for each $n \in \mathbb{N}$ as

$$VH(E) = \bigcup_{n\geq 1} \mathcal{H}_{v_n}(X) \textit{ and } VH_0(E) = \bigcup_{n\geq 1} \mathcal{H}^0_{v_n}(X).$$

These spaces are endowed with the locally convex inductive topologies $\tau_{\mathcal{I}}$ and $\tau'_{\mathcal{I}}$ respectively. Since the closed unit ball B_{v_n} of each $\mathcal{H}_{v_n}(E)$ is τ_0-compact, it follows by Mujica's completeness Theorem, cf. [105], $VH(E)$ is complete and its predual

$$VG(E) = \{\phi \in VH(E)' : \phi|B_{v_n} \text{ is } \tau_0 - \text{continuous for each } n \in \mathbb{N}\}$$

is endowed with the topology of uniform convergence on the sets B_{v_n}. In case V consists of a single weight v, $VG(E)$ is written as $\mathcal{G}_v(E)$.

Analogous to condition (II), Beltrán [8] introduced a condition for a decreasing family V; indeed, a decreasing family V of weights satisfies *condition* (A) if for each $m \in \mathbb{N}$, there exist $D > 0$, $R > 1$ and $n \in \mathbb{N}$, $n \geq m$ such that

$$\|P^j f(0)\|_{v_n} \leq \frac{D}{R^j}\|f\|_{v_m}$$

for each $j \in \mathbb{N}$ and $f \in \mathcal{H}_{v_m}(E)$.

In [52], we have proved results analogous to Theorem 3.36 for a a decreasing family of radial rapidly decreasing weights satisfying condition (A) and derive

Theorem 3.38. *Let* V *be a decreasing family of radial rapidly decreasing weights satisfying condition (A). Then the following are equivalent:*

(a). E *has the BAP(AP).*
(b). $\mathcal{Q}({}^mE)$ *has the BAP(AP)for each* $m \in \mathbb{N}$.
(c). $VG(E)$ *has the BAP(AP).*

Concerning the BAP for the dual of E, we prove

Theorem 3.39. *Let* E *be a Banach space with* $\mathcal{P}({}^mE) = \mathcal{P}_w({}^mE)$*, for each* $m \in \mathbb{N}$*. Then, for a decreasing family* V *of radial rapidly decreasing weights on* E *satisfying condition (A), the following are equivalent:*

(a). E^* *has the BAP.*
(b). $\mathcal{P}({}^mE)$ *has the BAP for each* $m \in \mathbb{N}$.
(c). $VH(E)$ *has the BAP.*

Remark 3.40. We would like to point out that Theorem 3.34 is not a particular case of Theorem 3.38 as the family V consisting of a single weight v never satisfies condition (A); indeed, if there exists some $D > 0$ and $R > 1$ such that for each $j \in \mathbb{N}$ and $f \in \mathcal{H}_v(U)$

$$\|P^j f(0)\|_v \leq \frac{D}{R^j}\|P^j f(0)\|_v$$

would yield $R^j \leq D$ for each j, which is not possible.

Illustrating the above results, we consider the Hörmander algebra $A_p(E)$ of entire functions defined on a Banach space E, cf. [9]. This algebra is defined with the help of the family $V = \{v_n\}$ of weights being given by $v_n(x) = e^{-np(\|x\|)}$, $x \in E$, where a growth condition p is an increasing continuous function $p : [0, \infty) \to [0, \infty)$ with the following properties:

(a) $\phi : r \to p(e^r)$ is convex.

(b) $log(1 + r^2) = o(p(r))$ as $r \to \infty$.

(c) there exists $\lambda \geq 0$ such that $p(2r) \leq \lambda(p(r) + 1)$ for each $r \geq 0$.

Then $V = \{v_n\}$ is a family of rapidly decreasing weights satisfying condition (A), cf. [9]. Consequently

Example 3.41. Under the hypothesis of Theorem 3.39 for E, *i.e.*, $\mathcal{P}(^mE) = \mathcal{P}_w(^mE)$, for each $m \in \mathbb{N}$, $A_p(E)$ has the BAP whenever E^* has the BAP. In particular, $A_p(\mathbb{C}^N)$ has the BAP for each growth condition p and $N \in \mathbb{N}$.

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RANGES OF FUNCTORS IN ALGEBRA*

FRIEDRICH WEHRUNG

Abstract. The problem of determining the range of a given functor arises in various parts of mathematics. We present a sample of such problems, with focus on various functors, arising in the contexts of nonstable K_0-theory of rings, congruence lattices of universal algebras, spectral spaces of ring-like objects. We also sketch some of the ideas involved in the solutions of those problems.

1. Introduction

The present short survey paper deals with the following kind of problem. We are given classes $\mathcal{A}$ and $\mathcal{B}$, together with a map $\Psi\colon \mathcal{A} \to \mathcal{B}$. We would like to find a way to recognize the members of the range $\Psi[\mathcal{A}] \underset{\text{def}}{=} \{\Psi(A) \mid A \in \mathcal{A}\}$ of Ψ. That is, we would like to find simple criteria for a given $B \in \mathcal{B}$ to be $\Psi(A)$ for some $A \in \mathcal{A}$.

Our main line of argument is that in many situations, if there exists a solution A, then there exists a "nice" solution A_{nice}, where "nice" means that part of the structure of B can be read on the construction of A_{nice}.

In most of those situations, $\mathcal{A}$ and $\mathcal{B}$ are classes of *structures*: groups, rings, modules, lattices (which are all particular instances of so-called *universal algebras*), but also topological spaces. In all those instances, both $\mathcal{A}$ and $\mathcal{B}$ afford a notion of *isomorphism*, which enables us to state that two structures, although not identical in the set-theoretical sense, share all the properties that matter (e.g., isomorphic groups, homeomorphic topological spaces).

The relevant context, for this kind of problem, is thus *category theory*. Now $\mathcal{A}$ and $\mathcal{B}$ are both categories and Ψ is a *functor* from $\mathcal{A}$ to $\mathcal{B}$. Moreover, instead of

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asking for $\Psi(A) = B$, we only ask for $\Psi(A) \cong B$ (say that A *lifts* B *with respect to* Ψ), and the "range" of Ψ is the class of all B that are isomorphic to $\Psi(A)$ for some object A of $\mathcal{A}$: in formula,

$$\Psi[\mathcal{A}] \underset{\text{def}}{=} \{B \in \mathcal{B} \mid (\exists A \in \mathcal{A})(\Psi(A) \cong B)\}\,. \tag{1.1}$$

Observe right away that (1.1) already involves a quite common abuse of notation, used there for simplicity's sake: "$B \in \mathcal{B}$" should be "B is an object of $\mathcal{B}$", and, similarly, "$A \in \mathcal{A}$" should be "A is an object of $\mathcal{A}$". Calling the right hand side of (1.1) the "range" of Ψ is a similar abuse of terminology, as the $\Psi[\mathcal{A}]$ of (1.1) usually properly contains the set-theoretical range of Ψ.

"Niceness" of the solution A means that if B is obtained as a colimit of (usually simpler) structures B_i, then A is also the colimit of suitably chosen structures A_i with each $\Psi(A_i) \cong B_i$. Thinking of the B_i as all the "finitely generated" substructures of B, we would like to lift (with respect to Ψ) not only the *object* B, but the *diagram* formed by all the B_i and the inclusion maps between them. We are thus trying to lift the *slice category* $\mathcal{B}\downarrow B$ of all arrows from some object of $\mathcal{B}$ to B.

Now it is often the case that we are only trying to lift a *subdiagram* of $\mathcal{B}\downarrow B$. A "diagram" in $\mathcal{B}$ is, really, a functor $\Phi\colon \mathcal{I} \to \mathcal{B}$, for some category $\mathcal{I}$. Our initial problem can thus be recast as follows: we are given categories $\mathcal{I}$, $\mathcal{A}$, $\mathcal{B}$ together with functors $\Phi\colon \mathcal{I} \to \mathcal{B}$ and $\Psi\colon \mathcal{A} \to \mathcal{B}$. We are asking whether there exists a functor $\Gamma\colon \mathcal{I} \to \mathcal{A}$ such that the functors Φ and $\Psi \circ \Gamma$ are isomorphic, in notation $\Phi \cong \Psi \circ \Gamma$ (two functors are *isomorphic* if there exists a natural transformation from one to the other all of whose components are isomorphisms; the relation $\Phi \cong \Psi \circ \Gamma$ could be paraphrased as "$\Phi = \Psi \circ \Gamma$ up to isomorphism").

The monograph Gillibert and Wehrung [12] deals extensively with this kind of problem, in situations arising from algebra. It is articulated around a technical statement called there the *Condensate Lifting Lemma* (CLL for short, Lemma 3.4.2 in [12]). This statement makes it possible, in many situations, to reduce the liftability of a *diagram* $\vec{B}$ of $\mathcal{B}$ to the liftability of a suitable *object* B of $\mathcal{B}$, called a *condensate* of $\vec{B}$. Even for diagrams indexed by a finite partially ordered set (*poset* for short) P, the construction of a condensate can be a difficult matter, relying on objects called *lifters* of P and whose existence follows from statements of infinite combinatorics based on *Kuratowski's Free Set Theorem* (cf. Kuratowski [21], Erdős *et al.* [10, Theorem 46.1]). The "size" (often defined as cardinality) of a condensate may be (strictly) larger than the size of the diagram $\vec{B}$: even in case $\vec{B}$ is a diagram of finite structures, indexed by a finite poset P, the condensate B may have transfinite cardinality, often $\aleph_{n-1}$ where n is nothing else than the order-dimension of the poset P (cf. Gillibert and Wehrung [13]). In addition, this construction works if P is a finite lattice, but may fail for general finite posets.

Many readers would object that this is sounding a bit abstract. Hence, let us now inch away from category theory and present some more specific situations.

2. Spectral spaces: Stone duality and Hochster's Theorem

For more details and references about the present section, we refer the reader to Grätzer [15, § 2.5], Hochster [17], Johnstone [18, § II.3 and Ch. V], Stone [27].

2.1. **Hochster's Theorem.** The *Zariski spectrum* of a commutative unital ring A, is a topological space, fundamental in algebraic geometry. Let us recall the definition of that space. An ideal P of A is *prime* if it is proper (i.e., $P \neq A$, equivalently $1 \notin P$) and the quotient ring A/P is a domain (equivalently, $xy \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in A$). We denote by $\operatorname{Spec} A$ the set of all prime ideals of A, endowed with the topology whose *closed* sets are exactly those of the form

$$\operatorname{Spec}(A, X) \underset{\text{def}}{=} \{P \in \operatorname{Spec} A \mid X \subseteq P\}\,, \tag{2.1}$$

for $X \subseteq A$. This is a so-called *hull-kernel topology.*

The assignment $A \mapsto \operatorname{Spec} A$ can be extended to a (contravariant) *functor*, by sending any homomorphism $f\colon A \to B$ of commutative unital rings to the map $\operatorname{Spec} f\colon \operatorname{Spec} B \to \operatorname{Spec} A$, $Q \mapsto f^{-1}[Q] \underset{\text{def}}{=} \{x \in A \mid f(x) \in Q\}$. This map is easily seen to be *continuous* (this is not the best possible guess, but it will do for now). Hence, Spec defines a contravariant functor, from the category of all commutative unital rings, with unital ring homomorphisms, to the category of all topological spaces, with continuous maps.

What kind of topological space is $\operatorname{Spec} A$? Easy examples show that it may not be Hausdorff. However, other easy examples show that not every topological space is the Zariski spectrum of a commutative unital ring, even in the finite case. Let us now prepare for the definition of the relevant topological spaces.

Definition 2.1. A nonempty closed set F in a topological space X is *irreducible* if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all *closed* sets A and B. We say that X is *sober* if every irreducible closed set is the closure of a unique singleton[1].

Definition 2.1 can be paraphrased by saying that a topological space is sober if it has as few points as possible with respect to the lattice structure of its collection of open sets.

For a topological space X, we shall denote by $\overset{\circ}{\mathcal{K}}(X)$ the set of all open and compact[2] subsets of X, partially ordered under set inclusion. In general, for arbitrary $U, V \in \overset{\circ}{\mathcal{K}}(X)$, the union $U \cup V$ always belongs to $\overset{\circ}{\mathcal{K}}(X)$. However, the intersection $U \cap V$ may not belong to $\overset{\circ}{\mathcal{K}}(X)$.

[1]Due to the uniqueness, every sober space is T_0 (not all references assume this).

[2]Throughout the paper, "compact" means what some other references call "quasicompact" (i.e., every open cover has a finite subcover); in particular, it does not imply Hausdorff.

Definition 2.2. A topological space X is *spectral* if it is sober and $\overset{\circ}{\mathcal{K}}(X)$ is a basis of the topology of X, closed under finite intersection.

Defining the empty intersection as the whole space X, this shows that a spectral space is always compact.

It is well know, and easy to verify, that the Zariski spectrum of any commutative unital ring is a spectral space. The converse is given by the following theorem of Hochster [17].

Theorem 2.3 (Hochster)**.** *Every spectral space is homeomorphic to the Zariski spectrum of some commutative unital ring.*

Hochster's construction, that assigns, to any spectral topological space X, a commutative unital ring A_X such that $X \cong \operatorname{Spec} A_X$, is *functorial*, from the category of all commutative unital rings with unital ring *embeddings*, to spectral spaces with *spectral maps*. By definition, for spectral spaces X and Y, a map $\varphi\colon X \to Y$ is *spectral* if $\varphi^{-1}[V]$ is compact open whenever V is a compact open subset of Y; every spectral map is continuous, but not every continuous map is spectral.

In particular, Hochster's Theorem implies that the range of the Zariski spectrum functor is completely described: it is the class of all spectral topological spaces.

2.2. **Stone duality.** The construction of the Zariski spectrum, for commutative unital rings, can be extended to bounded distributive lattices, in the following way. Recall that a *lattice* is a structure $(L, \vee, \wedge)$, where $\vee$ and $\wedge$ are both binary operations on a set L such that there is a partial ordering $\leq$ for which $x \vee y = \sup(x, y)$ (the *join* of $\{x, y\}$) and $x \wedge y = \inf(x, y)$ (the *meet* of $\{x, y\}$) whenever $x, y \in L$. Necessarily, $x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow x \wedge y = x$ whenever $x, y \in L$, so the partial ordering $\leq$ is uniquely determined by either $\vee$ or $\wedge$. We say that L is

- *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ whenever $x, y, z \in L$;
- *bounded* if $\leq$ has a smallest element (then denoted by 0) and a largest element (then denoted by 1).

An ideal P, in a bounded distributive lattice D, is *prime* if it is proper (i.e., $P \neq D$, equivalently $1 \notin P$) and $x \wedge y \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in D$.

The remainder of the definition of the spectrum, of a bounded distributive lattice, follows the one for commutative unital rings. We denote by $\operatorname{Spec} D$ the set of all prime ideals of D, endowed with the topology whose closed sets are exactly those of the form

$$\operatorname{Spec}(D, X) \underset{\text{def}}{=} \{P \in \operatorname{Spec} D \mid X \subseteq P\},$$

for $X \subseteq D$. It can be easily seen that $\operatorname{Spec} D$ is a spectral space.

The assignment $D \mapsto \operatorname{Spec} D$ can be extended to a (contravariant) *functor*, by sending any $0,1$-lattice homomorphism $f\colon D \to E$ to the map $\operatorname{Spec} f\colon \operatorname{Spec} E \to \operatorname{Spec} D$, $Q \mapsto f^{-1}[Q]$. This map is easily seen to be *continuous*, and even *spectral* (cf. Section 2.1). Hence, Spec defines a contravariant functor, from the category of all bounded distributive lattices, with $0,1$-lattice homomorphisms, to spectral spaces, with spectral maps.

Stone duality [27] is contained in the following theorem and the subsequent comments.

Theorem 2.4 (Stone)**.** *The pair* (Spec, $\overset{\circ}{\mathcal{K}}$) *induces a* (*categorical*) duality*, between bounded distributive lattices with* $0,1$*-lattice homomorphisms and spectral spaces with spectral maps.*

The dual of a bounded distributive lattice D is its spectrum $\operatorname{Spec} D$, and the dual of a spectral space X is the lattice $\overset{\circ}{\mathcal{K}}(X)$ of all its compact open subsets. The dual of a spectral map $\varphi\colon X \to Y$, between spectral spaces, is the $0,1$-lattice homomorphism $\overset{\circ}{\mathcal{K}}(\varphi)\colon \overset{\circ}{\mathcal{K}}(Y) \to \overset{\circ}{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$. For a bounded distributive lattice D, the assignment $a \mapsto \{P \in \operatorname{Spec} D \mid a \notin P\}$ defines a lattice isomorphism $\varepsilon_D\colon D \to \overset{\circ}{\mathcal{K}}(\operatorname{Spec} D)$, and the assignment $D \mapsto \varepsilon_D$ defines a natural transformation from the identity functor, on bounded distributive lattices, to the functor $\overset{\circ}{\mathcal{K}} \circ \operatorname{Spec}$. For a spectral space X, the assignment $x \mapsto \{U \in \overset{\circ}{\mathcal{K}}(X) \mid x \notin U\}$ defines a homeomorphism $\eta_X\colon X \to \operatorname{Spec} \overset{\circ}{\mathcal{K}}(X)$, and the assignment $X \mapsto \eta_X$ is a natural transformation from the identity functor, on spectral spaces, to the functor $\operatorname{Spec} \circ \overset{\circ}{\mathcal{K}}$.

The duality described above restricts to the classical Stone duality between Boolean algebras and zero-dimensional compact Hausdorff spaces. It can also be extended, *mutatis mutandis*, to what should be called Stone duality between distributive lattices with zero, with cofinal[3] zero-preserving lattice homomorphisms, and so-called "generalized spectral spaces" with spectral maps (cf. Rump and Yang [26, page 63], Johnstone [18, § II.3], Grätzer [15, § II.5]).

While Stone's Theorem and Hochster's Theorem both characterize spectral spaces as spectra of commutative rings and bounded distributive lattices, respectively, the former result achieves one more feature: since we are dealing with a *duality*, every spectral space is the spectrum of a *unique* (up to isomorphism) bounded distributive lattice, while a commutative unital ring is usually not determined by its Zariski spectrum (e.g., the Zariski spectrum of any field is the one-point topological space).

[3]For posets P and Q, a map $f\colon P \to Q$ is *cofinal* if for every $q \in Q$ there exists $p \in P$ such that $q \le f(p)$.

Something can still be said about the interaction between Stone duality and Hochster's Theorem. For a subset X in a commutative unital ring A, the Zariski-closed set $\operatorname{Spec}(A, X)$ introduced in (2.1) depends only on the ideal I generated by X: namely, $\operatorname{Spec}(A, X) = \operatorname{Spec}(A, I)$. This can be even made more precise, by observing that $\operatorname{Spec}(A, I) = \operatorname{Spec}(A, \sqrt{I})$, where $\sqrt{I}$, the *radical* of I, is defined as $\{x \in A \mid x^n \in I \text{ for some positive integer } n\}$. Now this is as precise as it can get, as by Krull's Theorem, every radical ideal (an ideal I is *radical* if $I = \sqrt{I}$) is the intersection of all the prime ideals containing it. Building on that observation, it is easy to verify that the Stone dual of $\operatorname{Spec} A$ is isomorphic to the (bounded, distributive) lattice $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ of all radicals of finitely generated ideals[4] (beware the ambiguity about the "finitely generated radical ideal" terminology!) of A. Hence, by applying Stone duality to Theorem 2.3, we obtain the following.

Corollary 2.5. *Every bounded distributive lattice with zero is isomorphic to* $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ *for some commutative unital ring* A.

3. Congruence lattices

For more detail and references about this section, we refer the reader to Wehrung [36, 37, 38].

3.1. **The Congruence Lattice Problem.** A *congruence* of a lattice $(L, \vee, \wedge)$ is an equivalence relation θ on L such that $x_1 \equiv_\theta y_1$ and $x_2 \equiv_\theta y_2$ implies both $x_1 \vee x_2 \equiv_\theta y_1 \vee y_2$ and $x_1 \wedge x_2 \equiv_\theta y_1 \wedge y_2$, whenever $x_1, x_2, y_1, y_2 \in L$ (we say that θ is *compatible* with the operations $\vee$ and $\wedge$). Here, $x \equiv_\theta y$ is short for $(x, y) \in \theta$.

The set $\operatorname{Con} L$ of all congruences of a lattice L, partially ordered under $\subseteq$, is a *complete lattice*, in which

$$\bigwedge_{i\in I} \theta_i = \bigcap_{i\in I} \theta_i \quad \text{and} \quad \bigvee_{i\in I} \theta_i = \text{congruence generated by } \bigcup_{i\in I} \theta_i$$

for every collection $\{\theta_i \mid i \in I\}$ of congruences of L.

A congruence θ is *finitely generated* if it is the least one such that $x_1 \equiv_\theta y_1$ and $\cdots$ and $x_n \equiv_\theta y_n$, for some $x_i, y_i \in L$ $(1 \le i \le n)$. A congruence θ is finitely generated iff it is a *compact* element of $\operatorname{Con} L$, that is, whenever $\theta \subseteq \bigvee_{i\in I} \theta_i$, there exists a finite subset J of I such that $\theta \subseteq \bigvee_{i\in J} \theta_i$.

The lattice $\operatorname{Con} L$ is *algebraic*, that is, it is complete and every congruence can be written in the form $\bigvee_{i\in I} \theta_i$ with all θ_i *compact*.

Up to this point, there is nothing special about the structure of lattice and the concept of congruence can be extended to any "universal algebra" (i.e., nonempty set A with a [possibly infinite] collection of operations $A^n \to A$ for various n). For example, the congruences of a *group* G are in one-to-one correspondence with the *normal subgroups* of G. However, the congruences of a lattice L are, usually, *not* in any natural one-to-one correspondence with subsets of L.

[4] The letter "r" stands for "radical" while the letter "c" stands for "compact", which is the lattice-theoretical counterpart of "finitely generated".

The congruence lattice of any universal algebra is an algebraic lattice. The converse, stating that every algebraic lattice arises as the congruence lattice of some universal algebra, is a deep theorem of Grätzer and Schmidt [16].

A large part of what makes lattices so special is the following fundamental result, established in Funayama and Nakayama [11]. We show a proof for convenience.

Theorem 3.1 (Funayama and Nakayama)**.** *The congruence lattice of every lattice is distributive.*

Proof. Define the *lower median* operation on a lattice L by setting $m(x,y,z) \underset{\text{def}}{=} (x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$, whenever $x,y,z\in L$. The map m is a *majority operation*, that is, $m(x,x,y)=m(x,y,x)=m(y,x,x)=x$ whenever $x,y\in L$. Now let $\alpha,\beta,\gamma\in\operatorname{Con}L$ and let $(x,y)\in\alpha\cap(\beta\vee\gamma)$. Since (x,y) belongs to $\beta\vee\gamma$, which is the transitive closure of $\beta\cup\gamma$, there exists a finite sequence $(z_0,\dots,z_n)$ of elements of L such that $z_0=x$, $z_n=y$, and each $(z_i,z_{i+1})\in\beta\cup\gamma$. Since m is a composition of the fundamental operations $\vee$ and $\wedge$, every congruence of L is compatible with m. Setting $t_i \underset{\text{def}}{=} m(x,y,z_i)$ for each i, it follows that each $(t_i,t_{i+1})\in\beta\cup\gamma$. Moreover, since m is a majority operation, $t_0=x$ and $t_n=y$. From $x\equiv_\alpha y$ it follows that each $t_i\equiv_\alpha m(x,x,z_i)=x$, thus each $(t_i,t_{i+1})\in\alpha\cap(\beta\cup\gamma)=(\alpha\cap\beta)\cup(\alpha\cap\gamma)$. Therefore, $(x,y)\in(\alpha\cap\beta)\vee(\alpha\cap\gamma)$, thus completing the proof that $\alpha\cap(\beta\vee\gamma)\subseteq(\alpha\cap\beta)\vee(\alpha\cap\gamma)$. The converse containment is trivial. □

Theorem 3.1 is quite lattice-specific, in the sense that it does *not* extend to groups, modules, rings... For example, usually $A\cap(B+C)\neq(A\cap B)+(A\cap C)$ for submodules A, B, C of a given module.

In the 1940's, Dilworth proved that conversely, every *finite* distributive lattice is the congruence lattice of a (finite) lattice. Then he asked whether this could be extended to the infinite case:

The Congruence Lattice Problem (CLP), $\sim$ 1940**.** *Is every distributive algebraic lattice the congruence lattice of a lattice?*

CLP initiated a considerable amount of work, leading to a host of *positive* results. All those results are more conveniently stated in terms of the set $\operatorname{Con_c}L$ of all compact (i.e., finitely generated) congruences of L, partially ordered under set inclusion. It should be noted that $\operatorname{Con_c}L$ is *not* a lattice as a rule: for compact congruences α and β, the join $\alpha\vee\beta$ is compact, but the meet $\alpha\cap\beta$ may not be compact. Hence, $\operatorname{Con_c}L$ is a $(\vee,0)$-*semilattice*[5]. It is *distributive*, that is, whenever $\alpha\subseteq\beta_1\vee\beta_2$ in $\operatorname{Con_c}L$, there are $\alpha_i\subseteq\beta_i$ in $\operatorname{Con_c}L$ such that $\alpha=\alpha_1\vee\alpha_2$. Moreover, one can go naturally from $\operatorname{Con}L$ to $\operatorname{Con_c}L$ (the latter is the semilattice

[5]Or, equivalently, a commutative, idempotent monoid, endowed with the partial ordering $\leq$ given by $x\leq y\Leftrightarrow x\vee y=y$.

of all compact elements of the former) and back (the former is isomorphic to the ideal lattice of the latter), and this *functorially.* We thus reach the following

Semilattice formulation of CLP. *Is every distributive* $(\vee, 0)$*-semilattice* representable, *that is, isomorphic to* $\mathrm{Con_c}\, L$ *for some lattice* L*?*

Some known positive instances of CLP are given by the following result:

Theorem 3.2. *Let* S *be a distributive* $(\vee, 0)$*-semilattice. In each of the following cases,* S *is representable:*

(i) S *is countable* (*Bauer* $\sim$ 1980)*;*
(ii) $\operatorname{card} S \leq \aleph_1$ (*Huhn* 1989)*;*
(iii) S *is a lattice* (*Schmidt* 1981)*;*
(iv) $S = \varinjlim_{n<\omega} S_n$*, with all transition maps* $S_n \to S_{n+1}$ $(\vee, 0)$*-homomorphisms and all* S_n *distributive lattices* (*Wehrung* 2003).

In any of those cases, a representing lattice L (such that $\mathrm{Con_c}\, L \cong S$) can be taken *sectionally complemented* (a lattice L with zero is sectionally complemented if whenever $a \leq b$ in L, there exists $x \in L$ such that $a \vee x = b$ and $a \wedge x = 0$). The following result was established in Wehrung [30, 31].

Theorem 3.3. *For every cardinal number* $\kappa \geq \aleph_2$*, there exists a distributive* $(\vee, 0, 1)$*-semilattice* S_κ*, of cardinality* κ*, not isomorphic to* $\mathrm{Con_c}\, L$ *for any* sectionally complemented *lattice* L*.*

The task of removing "sectionally complemented" from the statement of Theorem 3.3 was completed nearly ten years later in Wehrung [32], thus yielding a negative solution of CLP, using the class of counterexamples constructed for Theorem 3.3 (with a far more elaborate proof):

Theorem 3.4. *The distributive* $(\vee, 0, 1)$*-semilattice* $S_{\aleph_{\omega+1}}$ *is not representable.*

The optimal cardinality bound was subsequently obtained by Růžička in [24]:

Theorem 3.5 (Růžička)**.** *The distributive* $(\vee, 0, 1)$*-semilattice* $S_{\aleph_2}$ *is not representable.*

Those results, together with the proof of Theorem 3.1, shift the original Congruence Lattice Problem to the following (still unsolved) problem: *is every distributive algebraic lattice the congruence lattice of a majority algebra*? By definition, a *majority algebra* is a pair (A, m), where $m\colon A^3 \to A$ is a majority operation. The proof of Theorem 3.1 shows that the congruence lattice of any majority algebra is distributive.

3.2. **A heavy cube, and congruence-permutable algebras.** Referring to the terminology "lifting" (with respect to a functor) from Section 1, "heavy" is intended to mean "hard to lift".

We consider the diagram $\mathcal{D}_c$ of $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms represented in Figure 3.1, where $e(x) = (x, x)$, $p(x, y) = x \vee y$, and $s(x, y) = (y, x)$ whenever $x, y \in \{0, 1\}$. This diagram is obviously commutative.

The following result was established in Tůma and Wehrung [28].

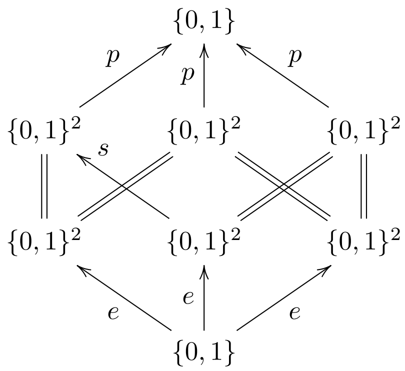


FIGURE 3.1. The heavy cube $\mathcal{D}_c$

Theorem 3.6. *The cube* $\mathcal{D}_c$ *cannot be lifted* (*with respect to the functor* $\mathrm{Con_c}$)*, by any cube of* sectionally complemented *lattices and lattice homomorphisms.*

In fact, it turns out that Theorem 3.6 can be extended to a much broader algebraic context; in particular, it is not lattice-specific. For binary relations α and β on a set A, we set $\alpha \circ \beta \underset{\mathrm{def}}{=} \{(x, y) \in A \times A \mid (\exists z \in A)((x, z) \in \alpha \text{ and } (z, y) \in \beta)\}$. We say that an algebra A is *congruence-permutable* if $\alpha \circ \beta = \beta \circ \alpha$ for all congruences α and β of A. For example, *groups*, *modules*, *rings* are all congruence-permutable (e.g., $HK = KH$ for *normal* subgroups in a group). However, *not every lattice is congruence-permutable* (e.g., consider the three-element chain). The following result was established in Růžička, Tůma, and Wehrung [25].

Theorem 3.7. *The cube* $\mathcal{D}_c$ *cannot be lifted* (*with respect to the functor* $\mathrm{Con_c}$)*, by any cube of* congruence-permutable (*universal*) *algebras. In particular, it cannot be lifted by groups, rings, or modules.*

It is still unknown whether every diagram of finite Boolean $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms, indexed by a finite lattice, can be lifted with respect to the functor $\mathrm{Con_c}$ on lattices.

The proof of Theorem 3.7 (i.e., non-liftability of $\mathcal{D}_c$ by congruence-permutable algebras), yielded the construction of a distributive $(\vee, 0)$-semilattice that cannot be lifted by congruence-permutable algebras, in Růžička, Tůma, and Wehrung [25].

Theorem 3.8. *For every cardinal number* $\kappa \geq \aleph_2$*, the distributive* $(\vee, 0, 1)$*-semilattice* S_κ *is not isomorphic to* $\mathrm{Con_c}\, A$ *for any* congruence-permutable *algebra* A*. In particular,* $S_{\aleph_2}$ *is not isomorphic to* $\mathrm{Con_c}\, A$ *whenever* A *is a* sectionally complemented lattice*, a* group*, a* module*, or a* ring*. Moreover, in the case of sectionally complemented lattices, groups, modules, rings, the cardinality bound* $\aleph_2$ *is* optimal*.*

Since congruences of a module are identified with submodules, it follows, for example, that $S_{\aleph_2}$ is not isomorphic to the submodule lattice of any module. Similarly, it is not isomorphic to the normal subgroup lattice of any group.

In all the results described in this section, the unliftability statements about cubes (of order-dimension $n = 3$) always parallel unliftability statements about objects of cardinality $\aleph_{n-1} = \aleph_2$. General principles explaining this correspondence are developed in the monograph Gillibert and Wehrung [12]. Nonetheless, the constructions described in the present section predate the monograph.

4. Nonstable K_0-theory

For further details and references about this section, see Ara [2], Goodearl [14], Wehrung [33].

Two idempotent matrices a and b over a (not necessarily commutative or unital) ring R are *Murray - von Neumann equivalent*, in symbol $a \sim b$, if there are matrices x and y such that $a = xy$ and $b = yx$. For square matrices x and y over R, we set $x \oplus y \underset{\text{def}}{=} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. If $x_1 \sim y_1$ and $x_2 \sim y_2$, then $x_1 \oplus x_2 \sim y_1 \oplus y_2$, for all square matrices (not necessarily of the same dimension) x_1, x_2, y_1, y_2. Hence, Murray - von Neumann equivalence classes $[a] \underset{\text{def}}{=} \{x \mid a \sim x\}$, for idempotent matrices a over R, can be added, *via* the rule $[a] + [b] \underset{\text{def}}{=} [a \oplus b]$. The *monoid* $\mathrm{V}(R) \underset{\text{def}}{=} \{[a] \mid a \text{ idempotent matrix on } R\}$ is *commutative* $(x + y = y + x)$ and *conical* $(x + y = 0 \Rightarrow x = y = 0)$. It encodes the *nonstable* K_0*-theory* of R.

If R is *unital*, then $\mathrm{V}(R)$ is isomorphic to the monoid of isomorphism classes of all finitely generated projective right (or, equivalently, left) R-modules, and the enveloping group (also called *Grothendieck group*) of $\mathrm{V}(R)$ is the group usually denoted by $\mathrm{K}_0(R)$.

For example, if R is a field, or more generally a division ring, then the finitely generated right R-modules are exactly the finite-dimensional right R-vector spaces, which are classified by dimension. Hence, $\mathrm{V}(R)$ is isomorphic to the monoid $\mathbb{N}_0 \underset{\text{def}}{=} \{0, 1, 2, \dots\}$ of nonnegative integers.

The following result was established in Bergman [4], Bergman and Dicks [5]:

Theorem 4.1 (Bergman and Dicks)**.** *Every commutative conical monoid M is isomorphic to $\mathrm{V}(R_M)$ for some ring R_M.*

To my knowledge, the functoriality of the assignment $M \mapsto R_M$ has not yet been studied.

Further restrictions on the ring R are usually met with restrictions on the nonstable K_0-theory. An important class of rings for which this occurs is the following. It was introduced by Warfield [29] in the unital case, Ara [1] in the general case.

Definition 4.2. A ring R is an *exchange ring* if for all $x \in R$, there are an idempotent $e \in R$ and $r, s \in R$ such that $e = rx = x + s - sx$.

It can be established (and this is not trivial) that this condition is left-right symmetric. Every *von Neumann regular ring* (i.e., satisfying $(\forall x)(\exists y)(xyx = x)$)

is an exchange ring, and a C*-algebra is an exchange ring iff it has *real rank zero* (Ara *et al.* [3], Ara [1]).

Extending earlier results about von Neumann regular rings and C*-algebras of real rank zero, Ara established in [1] the following result, in full generality.

Theorem 4.3 (Ara)**.** *Let R be an exchange ring. Then* $\mathrm{V}(R)$ *is a* refinement monoid*, that is, for all $a_0, a_1, b_0, b_1 \in \mathrm{V}(R)$ such that $a_0 + a_1 = b_0 + b_1$, there are $c_{i,j} \in \mathrm{V}(R)$, for $i, j \in \{0, 1\}$, such that each $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$.*

It is not known whether every conical refinement monoid M arises in this way, that is, $M \cong \mathrm{V}(R)$ for some exchange ring R. Nevertheless, it is known since Wehrung [30] that the answer to that problem is negative for von Neumann regular rings (with counterexamples of cardinality $\aleph_2$), and it is known since Wehrung [33] that the answer is negative for C*-algebras of real rank zero (with counterexamples of cardinality $\aleph_3$, using the CLL tool from Gillibert and Wehrung [12]).

Say that a monoid is *simplicial* if it is isomorphic to a finite power of the additive monoid $\mathbb{N}_0$ of all nonnegative integers. The following result was established in Wehrung [33]:

Theorem 4.4. *There is a commutative cube, of simplicial monoids and monoid homomorphisms, that can be lifted, with respect to the functor* V*, by exchange rings and by C*-algebras of real rank* 1*, but not by semiprimitive exchange rings, thus neither by von Neumann regular rings nor by C*-algebras of real rank* 0*.*

The cube of Theorem 4.4 is obtained from the one of Figure 3.1 by replacing $\{0, 1\}$ by $\mathbb{N}_0$ and setting $e(x) = (x, x)$, $p(x, y) = x + y$, and $s(x, y) = (y, x)$ whenever $x, y \in \mathbb{N}_0$.

Theorem 4.4 could be paraphrased by saying that *at diagram level*, the nonstable K_0-theory of exchange rings *properly* contains both the one of von Neumann regular rings and the one of C*-algebras of real rank zero. By using again the tools of Gillibert and Wehrung [12], this result is extended to *objects* in [33].

Theorem 4.5. *There exists a unital exchange ring of cardinality* $\aleph_3$ (*resp., an* $\aleph_3$*-separable unital C*-algebra of real rank* 1) *R such that* $\mathrm{V}(R)$ *is not isomorphic to* $\mathrm{V}(B)$ *for any ring B which is either a C*-algebra of real rank* 0 *or a von Neumann regular ring.*

5. Back to spectral spaces

For more details and references about the present section, we refer the reader to Delzell and Madden [8], Grätzer [15, § 2.5], Johnstone [18, § II.3 and Ch. V], Keimel [20], Coste and Roy [6], Dickmann [9, Ch. 6], Wehrung [34, 35].

The Zariski spectrum construction can be extended to various contexts, such as *Abelian ℓ-groups* (yielding the *ℓ-spectrum*) and *partially ordered, commutative unital rings* (yielding the *real spectrum*). Tailoring the methods above (*in particular, CLL*) to that new context, further results can be obtained on ℓ-spectra and real spectra. Let us give a short summary of such results.

5.1. **The ℓ-spectrum of an Abelian ℓ-group.** An *ℓ-group* is a group G endowed with a translation-invariant lattice ordering. An *ℓ-ideal* of G is an order-convex normal subgroup of G, closed under the lattice operations. An ℓ-ideal P of an *Abelian* ℓ-group G is *prime* if it is proper (i.e., $P \neq G$) and $x \wedge y \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in G$. The *ℓ-spectrum* of G, denoted $\operatorname{Spec}_\ell G$, is the set of all prime ℓ-ideals of G, endowed with the topology whose closed sets are exactly those of the form

$$\operatorname{Spec}_\ell(G, X) \underset{\text{def}}{=} \{P \in \operatorname{Spec}_\ell G \mid X \subseteq P\}, \tag{5.1}$$

for $X \subseteq G$. An *order-unit* of G is an element e of the positive cone G^+ of G such that every element of G^+ lies below an integer multiple of e. To ease the presentation, let us deal only with Abelian ℓ-groups with order-unit. For those ℓ-groups, $\operatorname{Spec}_\ell G$ is a spectral space. (If G has no order-unit, then $\operatorname{Spec}_\ell G$ is only a *generalized spectral space*.)

A spectral space X is *completely normal* if for all $z \in X$ and all x, y in the closure of $\{z\}$, either x belongs to the closure of $\{y\}$ or y belongs to the closure of $\{x\}$. It is known since Keimel [19] that the ℓ-spectrum of any Abelian ℓ-group with unit is completely normal. The characterization problem of all ℓ-spectra of Abelian ℓ-groups with unit has been open since then, sometimes under equivalent names (mostly Mundici's *MV-spectrum problem*, see [23]). The countable case was recently solved in Wehrung [34]:

Theorem 5.1. *Every second countable, completely normal spectral space is homeomorphic to the ℓ-spectrum of an Abelian ℓ-group with order-unit.*

Due to a counterexample by Delzell and Madden [7], Theorem 5.1 cannot be extended to spectral spaces that fail to be second countable. In fact, due to the results of [34], the class of all Stone duals of ℓ-spectra of Abelian ℓ-groups with unit cannot be characterized by any class of $\mathscr{L}_{\infty,\omega}$-formulas (thus, in particular, it is not first-order).

5.2. **The real spectrum of a commutative unital ring.** Let A be a commutative unital ring. A subset C of A is a *cone* if it is both an additive and a multiplicative submonoid of A, containing all squares in A. A cone P of A is *prime* if $A = P \cup (-P)$ and the *support* $P \cap (-P)$ is a prime ideal of A. For a prime cone P, $-1 \notin P$ (otherwise $1 \in P \cap (-P)$, thus $P \cap (-P) = A$, a contradiction). We denote by $\operatorname{Spec}_{\mathrm{r}} A$ the set of all prime cones of A, endowed with the topology generated by all subsets of the form $\{P \in \operatorname{Spec}_{\mathrm{r}} A \mid a \notin P\}$, for $a \in A$, and we call $\operatorname{Spec}_{\mathrm{r}} A$ the *real spectrum of A*. The real spectrum of A is always a completely normal spectral space. Delzell and Madden [7] established that not every completely normal spectral space arises in this way. Mellor and Tressl [22] extended that result by proving that *for any infinite cardinal number λ, the class of Stone duals of all real spectra cannot be defined by any class of $\mathscr{L}_{\infty,\lambda}$-formulas.*

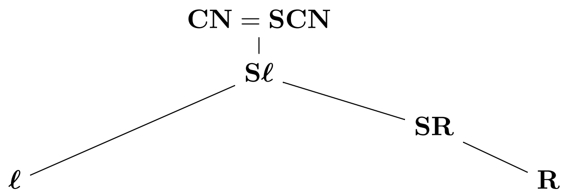
It is not known whether the result of Theorem 5.1 can be extended to real spectra: *is every second countable completely normal spectral space the real spectrum of some commutative unital ring*?

It is still possible to compare ℓ-spectra, real spectra, and their spectral subspaces, with respect to inclusion. Let

$$\begin{aligned}\mathbf{CN} &\underset{\text{def}}{=} \{\text{completely normal spectral spaces}\}\,,\\ \boldsymbol{\ell} &\underset{\text{def}}{=} \{\ell\text{-spectra of Abelian }\ell\text{-groups with unit}\}\,,\\ \mathbf{R} &\underset{\text{def}}{=} \{\text{real spectra of commutative unital rings}\}\,,\\ \mathbf{SX} &\underset{\text{def}}{=} \{\text{spectral subspaces of members of }\mathbf{X}\}\,,\end{aligned}$$

for any class $\mathbf{X}$ of spectral spaces. (By definition, a spectral space X is a *spectral subspace* of a spectral space Y if $X \subseteq Y$ and the inclusion map from X into Y is a spectral map.) The following result was established in Wehrung [35].

Theorem 5.2. *All containments and non-containments between the classes* $\mathbf{CN}$*,* $\boldsymbol{\ell}$*,* $\mathbf{S}\boldsymbol{\ell}$*,* $\mathbf{R}$*,* $\mathbf{SR}$ *can be read on the following diagram:*



Moreover, the topological counterexamples proving the various non-containments in that diagram all have bases of cardinality $\aleph_1$ *, except for the one proving* $\mathbf{S}\boldsymbol{\ell} \subsetneqq \mathbf{CN}$*, which has a basis of cardinality* $\aleph_2$ *.*

The Stone duals of the topological counterexamples proving the relations $\boldsymbol{\ell} \subsetneqq \mathbf{S}\boldsymbol{\ell}$, $\mathbf{R} \subsetneqq \mathbf{SR}$, $\mathbf{R} \nsubseteq \boldsymbol{\ell}$ are all constructed as condensates of one-arrow diagrams. The counterexample illustrating $\boldsymbol{\ell} \nsubseteq \mathbf{SR}$ is obtained *via* a direct construction. The counterexample illustrating $\mathbf{S}\boldsymbol{\ell} \subsetneqq \mathbf{CN}$ is obtained *via* a “free construction” similar, in spirit, to the one introduced in Wehrung [30].

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APPROXIMATION BY SPLINE FUNCTIONS AND ITS APPLICATIONS*

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Abstract. In this article, the historical background and motivation for the definition of spline functions initiated by I. J. Schoenberg have been discussed along with the basis functions for linear and higher degree spline space. The concept of discrete splines introduced by Mangasarian and Schumaker has been also presented.

After bivariate splines, the idea of Finite Elements has been elaborated. Using an interesting result of geometry, Wachspress has initiated the rational basis functions on convex polygons and polycons. The constructive aspects of these basis functions have been introduced.

The applications of spline functions to Civil Engineering and One-Dimensional Cutting Stock Problems are highlighted in the concluding part of the article.

1. Introduction and Motivation

The numerical solution to a problem is in general expressed in terms of an approximation $U(x)$ to the true solution $u(x)$ for x in some prescribed region D. A norm $||.||$ may be defined and U may be chosen as function which best approximates u over some approximation space A in the sense that $U(x) = U(x, a_0)$, where

$$H = ||U(x, a_0) - u(x)|| = \min_{U(x,a)\in A} ||U(x, a) - u(x)||.$$

Questions of existence and uniqueness of a_0, and of convergence of H to zero as the dimension of A is increased some how, are considered in many research papers. This is a central problem in approximation theory (cf. [68]).

With the advancement of computer technology, function which can be easily stored in computer and which have nice mathematical properties such as:

- Strong approximation power
- Computational convenience

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- Sufficient degree of smoothness

play a significant role in the field of approximation theory.

At the early stage, the process of approximation was quite crude; like if two values are available corresponding to two distinct points of the domain, the approximate value in between the two points used to be a constant function obtained by computing the average of two values (refer Figure 1.1).

Later, it was noticed that the polynomials are most suitable functions for approximation as they can be evaluated, differentiated and integrated easily. The simple theorem of polynomial interpolation upon which much practical numerical analysis rests states as:

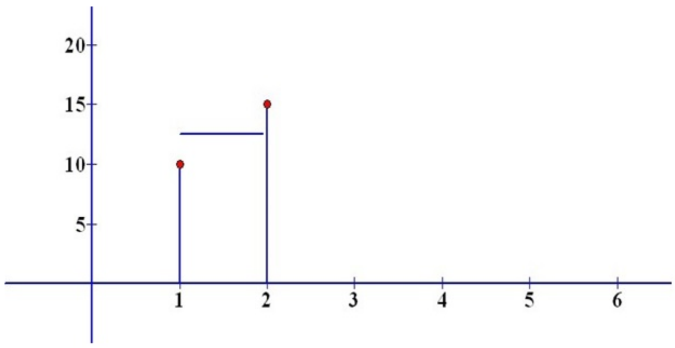

FIGURE 1.1. Crude Approximation

Theorem 1.1 ([8], p.24). *Given* $(n+1)$ *distinct points (real or complex)* $z_0, z_1, \ldots,$ z_n *and* $n+1$ *(real or complex) values* $w_0, w_1, \ldots, w_n$. *There exists polynomial* $p_n(z)$ *(of degree* n *or less)* $\in P_n$ *for which*

$$p_n(z_i) = w_i \qquad i = 0, 1, \ldots, n$$

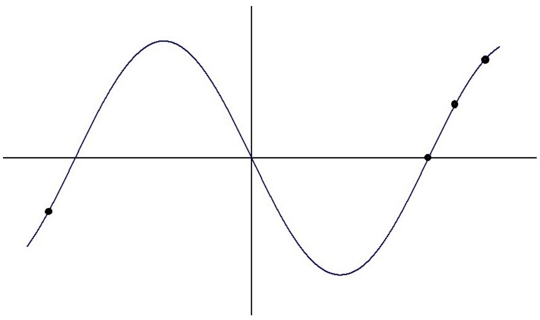

FIGURE 1.2. Polynomial interpolation of degree 3

Please refer Figure 1.2 for polynomial interpolation of degree 3.

The major problem with the polynomial approximation was the complexity of computation for sufficiently large number of data.

We now state the second fundamental theorem due to Weierstrass [8] which was very strong and popular during 1885.

Theorem 1.2. *Let $f(x) \in C[a,b]$, given an $\epsilon > 0$, we can find a polynomial $p_n(x)$ (of sufficiently high degree) for which*

$$|f(x) - p_n(x)| \leq \epsilon \qquad a \leq x \leq b$$

Weierstrass theorem asserts the possibility of uniform approximation by polynomials to continuous functions over a closed interval (see Figure 1.3).

Multivariate analogue of the foregoing theorem is also known for a continuous function defined on a compact subset of R^n (the n-dimensional Euclidean space).

A constructive and elegant proof of this theorem depends on the Bernstein polynomials $B_n(f)$ associated with f, which are defined in terms of Bernstein basis polynomials (see [11]);

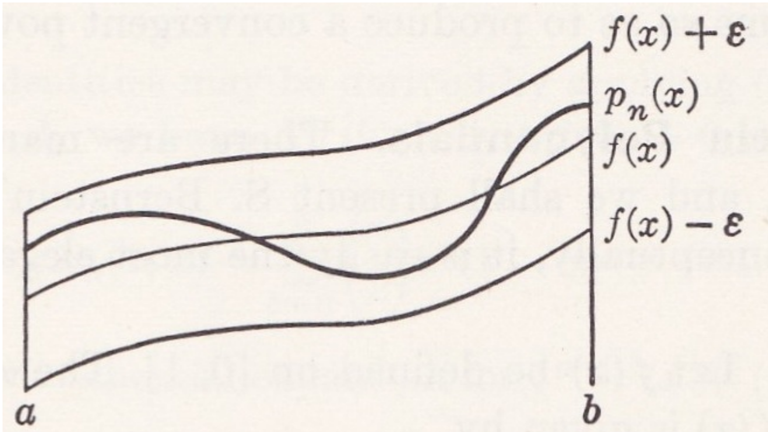

FIGURE 1.3. Polynomial Approximation of continuous function

For a given positive integer n and $x \in [0,1]$

$$\begin{aligned} B_j^n(x) &= \binom{n}{j}(1-x)^{n-j}x^j \quad j = 0,1,\ldots,n; \\ B_0^0 &\equiv 1, \\ B_j^n &\equiv 0 \text{ for } n < j \text{ or } j < 0. \end{aligned}$$

The Bernstein polynomial associated with a given function f defined over $[0,1]$ is given by

$$B_n(f)(x) = \sum_{k=0}^{n} f(\binom{k}{n})B_k^n(x)$$

In fact for each $f \in C[a,b]$ (space of all continuous functions defined over $[a,b]$), we have

$$||f - B_n(f)|| \to 0 \text{ as } n \to \infty$$

One of the most surprising facts in the theory of interpolation and approximation is that the simplest and most natural approach to synthesis leads to failure or rather to an impossibility. Referring Weierstrass theorem of 1885, given a function of class $C[a,b]$, what is more natural than to think that if a sequence of polynomials $p_n(f,x)$ is set up that duplicates the function at $n+1$ points of interval, then as $n \to \infty$, the question is whether $p_n(f,x)$ will converge to $f(x)$ or not? The answer is quite disappointing, $p_n(f,x)$ may not converge to f in general.

Runge ([64], p. 102) looks at the function $f(x) = \frac{1}{1+x^2}$ out of several meromorphic functions and found the following to be true:

If $p_n(f,x)$ interpolates to f at $n+1$ equidistant points of the interval $|x| \leq 5$, p_n converges to f only in the interval $|x| \leq 3.63...$ and diverges out side this interval (see Figure 1.4). In 1912, Bernstein proved that equidistant interpolation over $|x| \leq 1$ to the function $y = |x|$ diverges for $0 < |x| < 1$.

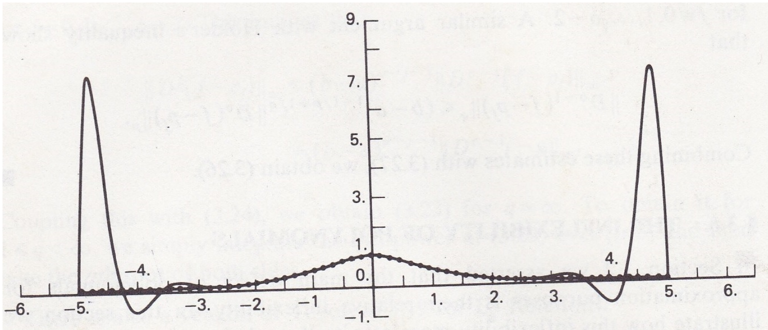

FIGURE 1.4. Lagrange Interpolation of $\frac{1}{1+x^2}$ at 15 equidistant points on $[-5,5]$

These results relate to equidistant points of interpolation. The space was still open for different choices of interpolatory points. The hopes for this idea vanished when Bernstein and Faber in 1914 simultaneously discovered the following:

Theorem 1.3 (cf. [8]). *Let*

$$-1 \leq x_{0n} < x_{1n} < \cdots < x_{nn} \leq 1 \quad n = 1,2,\ldots \tag{1.1}$$

be any prescribed triangular system of interpolation points. Then for each sequence of interpolation sets (1.1), there exists $f \in C[-1,1]$ for which its $n-th$ degree interpolating polynomials $p_n f$, matching f at x_{in} $i = 0,1,\ldots,n$ do not converge uniformly to f.

In view of these results, it was quite natural to search for some other approximants which can overcome these deficiencies.

The beginning of one such major approach could be traced back to the pioneering work of I. J. Schoenberg [62](see also [63]). The basic idea is to employ piecewise polynomial functions (pp functions) with certain degree of smoothness at the joints which are termed as splines. Basically, SPLINE is a French word. Its meaning is a device of small pieces of curves which are used by architects to design a railway track. One apparent advantage in the use of pp functions instead of polynomials is that a higher degree of flexibility is achieved through splines without paying for additional computational complexities inherent in dealing with polynomials of higher degree. A formal definition of spline is as follows [10]:

Consider a closed interval $[a, b]$ on $\mathbb{R}$ and define a partition P as

$$P \ : \ a = \tau_0 < \tau_1 < \cdots < \tau_n = b$$

where τ_i's are distinct points of the interval $[a, b]$ and are called as knots or nodal points. A class of m^{th} degree splines with respect to partition P is denoted by $S_m(p)$; $s \in S_m(p)$ if

- its restriction on each subinterval $[\tau_{i-1}, \tau_i]$ $(i = 1, 2, \ldots, n)$ is a polynomial of degree m or less,
- at each nodal point τ_i $(i = 1, 2, \ldots, n-1)$, $s \in C^{m-1}[a, b]$.

We next consider interpolation of derivatives also. Let $\tau_1, \tau_2, \ldots, \tau_r$ be given distinct real points which are associated with multiplicities m_j (positive integers), $j = 1, 2, \ldots, r$, such that $m_1 + m_2 + \cdots + m_r = n + 1$. Under Hermite interpolation by m^{th} degree polynomial p_m, we expect it to satisfy the following conditions:

$$p^l_m(\tau_j) \ = \ c_{j,l}, \quad l = 0, 1, \ldots, m_j - 1; \quad j = 1, 2, \ldots, r. \tag{1.2}$$

If $c_{j,l} = f^l(\tau_j)$, we say that polynomial $p_m = p_m f$ is the Hermite interpolant of the function f. If $m_j = 1$ for every j, $r = n + 1$, we call p_m the Lagrange interpolant of f at $\{\tau_0, \tau_1, \ldots, \tau_n\}$. It may be checked easily that there is a unique polynomial satisfying the interpolatory conditions (1.2).

A common technique of computing approximants is to use the approximation space

$$A = U(x; a) = \sum_{i=1}^{n} a_i w_i(x), \tag{1.3}$$

where w_i are known basis functions and the a_i's are combining coefficients.

2. Univariate Splines

Basis for Linear Approximation. We denote the piecewise linear or broken line interpolant (cf. [10]) to g at points $\tau_1, \ldots, \tau_n$ with $a = \tau_1 < \tau_2 < \cdots < \tau_n = b$ by $\mathbb{I}_2 g$. Let $\mathbb{S}_2$ denotes the linear space of all continuous broken lines on $[\tau_1, \tau_n]$ with breaks at $\tau_2, \ldots, \tau_{n-1}$.

The convenient basis for $\mathbb{S}_2$ is

$$H_i(x) = \begin{cases} (x - \tau_{i-1})/(\tau_i - \tau_{i-1}), & \tau_{i-1} < x \leq \tau_i; \\ (\tau_{i+1} - x)/(\tau_{i+1} - \tau_i), & \tau_i \leq x < \tau_{i+1}; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that $H_i(x)$ are linearly independent functions. $H_i(x)$ are called Hat functions (cf. Figure 2.1 on the next page).

We now define the space of splines of degree k and smoothness of order r.

Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of an interval $[a, b]$ and $k > r \geq 0$ be given integers. Then define

$$\begin{aligned} S^r_k(p) \ &= \ \{s \in C^r[a, b] \ : \ s \text{ is a polynomial of degree} \leq k \\ &\qquad \text{on each subinterval } [x_{i-1}, x_i], \ i = 1, 2, \ldots, n\} \end{aligned}$$

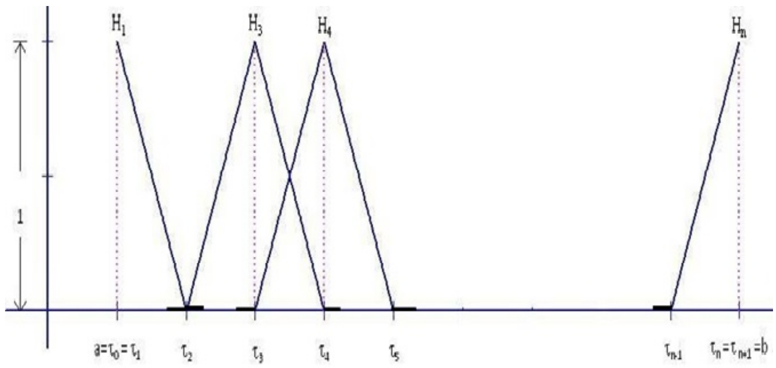

FIGURE 2.1. Hat Functions

When $k = 0$, the space $S_0^{-1}(p)$ consists of step functions with possible jumps at the points x_i $(i = 1, 2, \ldots, n-1)$.

The following example of spline functions is based on the truncated power function:

$$x_+^k = \begin{cases} x^k, & \text{for } x \geq 0; \\ 0, & \text{for } x < 0, \end{cases}$$

where $0^0 := 1$ by convention. Then the functions

$$s_j(x) = (x - x_j)_+^k \quad j = 1, 2, \ldots, n-1$$

are examples of splines of the space $S_k^{k-1}(p)$. It is well known that the dimension of the space $S_k^r(p)$ is $n(k-r) + r + 1$ and a basis in terms of truncated power function has been defined in ([10], p. 108).

Definition 2.1. *Let $t := (t_i)$ be non-decreasing sequence (which may be finite, infinite or bi-infinite). The $i-th$ (normalized) B-splines of order k for the knot sequence t is denoted by $B_{i,k,t}$ and is defined as*

$$B_{i,k,t}(x) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](. - x)_+^{k-1} \quad \text{all } x \in \mathbb{R}$$

Quadratic Splines. The dimension of the space $S(2, p)$ of quadratic splines is $n + 2$ and in order to compute quadratic spline interpolating at the nodal points, one boundary condition is required. Existence and uniqueness of such quadratic splines have been studied by de Boor [10]. Marsden [29] in 1972 has considered periodic quadratic spline interpolation at mid points between the two consecutive nodal points and estimated the error bounds. This result has been further extended by many mathematicians (see Kammerer, Reddien [25] and Demko [9] etc.). Later Neuman [32] in 1985 had established an interesting result which guarantee uniform convergence of s to given function f as $p \to 0$ (see also Sakai [61], Meinardus and Taylor [30]).

Considering the mean averaging condition viz. $\int_{x_{i-1}}^{x_i} (f - s)dg = 0$,

$i = 1, 2, \ldots, n$. Sharma and Tzimbalario [66] have established the existence and uniqueness theorems for 1-periodic function which is locally integrable along with non-negative measure dg.

Some interesting studies concerning monotonicity of spline interpolant of monotone data have been also made (see [33]).

Piecewise Cubic Interpolation. Broken lines are neither very smooth nor very efficient approximators. Both for a smoother approximation and for more efficient approximation, one has to go to piecewise polynomial approximation with higher order pieces. The most popular choice continue to be a piecewise cubic approximating function.

We shall be highlighting some major contribution. Applying the technique of matching of integral means of Sharma and Tzimbalario [66], Dikshit [12] in 1978 established the following result:

Theorem 2.1. *Let $f \in C^2[0,1]$ be a 1-periodic locally integrable function with respect to a non-negative measure $d\mu$ satisfying $\mu(x+h) - \mu(x) = \kappa$ Suppose further that either $\int_0^h \alpha(x)d\mu > 0$ or $\int_0^h \alpha(h-x)d\mu > 0$, where $\alpha(x) = 3x^3 - 6hx^2 + h^3$ and $h = x_{i+1} - x_i$. Then there exists a unique $s(x) \in S_p^0(3,\Delta)$ satisfying the following conditions:*

$$\int_{x_{i-1}}^{x_i} \{f(x) - s(x)\}d\mu = 0 \quad i = 1, 2, \ldots, n$$

$$S^r(0) = S^r(1) \qquad r = 0, 1, 2$$

[Dimension : $4n - (n-1)3 = n + 3$ n interpolatory conditions and 3 boundary conditions.]

The above theorem has been proved for two different cases viz. (i) n even (ii) n odd with different choice of $\alpha(x)$

Theorem 2.2 (cf. [12])**.** *The following error bound holds:*

$$||e''|| \le [1 + \{(9H + 12B(1) + 4B(3))/\delta\}]\omega(f; h)$$

where $\delta = \int_0^h \alpha(x)d\mu$ or $\int_0^h \alpha(h-x)d\mu$ according as

$$\int_0^h \alpha(x)d\mu > 0 \quad or \quad \int_0^h \alpha(h-x)d\mu > 0$$

where $e(x) = s(x) - f(x)$.

By making less restrictive continuity requirements at the joints and having two interpolatory conditions, Dikshit and Powar [13] (see also [15]) had posed the following problem:

Problem 2.1. Let f be a 1-periodic locally integrable function with respect to a positive measure dg, where g satisfies:

- $g(x+h) - g(x) =$ a constant k'
- $g(x) = k$ $(0 \le x < mh,\ 2/9 \le m < 1/2)$

- $g(x) = k \ (mh \leq x \leq h,\ 1/2 \leq m \leq 7/9)$

Find 1-periodic $s(x) \in S(3,\Delta)$ satisfying the conditions:

$$f(\theta_i) \ = \ s(\theta_i), \quad \int_{x_{i-1}}^{x_i} (f-s)(x)dg \ = \ 0, \quad i = 1, 2, \ldots, n,$$

where $\theta_i \ = \ x_{i-1} + mh$.

The answer to this Problem was given in the following Theorem;

Theorem 2.3. *There exists a unique $s(x) \in S(3,\Delta)$ which satisfies the conditions of Problem 2.1 provided that $g(x)$ does not have just one jump in $[0,h]$ at the point mh, $0 \leq m \leq 1$.*

The following error estimate was also computed in [13].

Theorem 2.4. *Suppose $s(x)$ is the deficient cubic spline of Theorem 2.3 interpolating $f(x)$ and $f(x) \in C^1[0,1]$. Then*

$$\max_x |s(x) - f(x)| \leq \kappa(m)\omega(f;h) \qquad \text{and}$$

$$\max_x |s'(x) - f'(x)| \leq \kappa'(m)\omega(f;h) + \kappa''(m)\omega_1(f;h),$$

where $\kappa(m)$, $\kappa'(m)$ and $\kappa''(m)$ are functions only depending on m.

Several such results on existence and uniqueness had been explored (see [61], [62], [63], [64] etc.). The problems of computation of error bounds and the refinement of the bounds which are already available have been also studied extensively (see [6], [11]).

Discrete Splines. Mangasarian and Schumaker [28] in 1973 introduced discrete splines as solutions of certain minimization problems involving differences. Existence, uniqueness and error bounds for discrete cubic splines interpolating at the knots are due to Lyche [27].

In order to achieve the desired order of continuity, instead of derivatives, the following differences have been matched at the nodal points.

$$D_h^1 f(x) = \frac{f(x+h) - f(x)}{h}, \quad D_h^2 f(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}, \quad \text{etc.}$$

By simply letting $h \to 0$ in the above formulae, Classical splines as a special case of discrete splines are obtained.

The existence, uniqueness and convergence of discrete cubic splines which interpolate to a given function at one interior point of each mesh interval had been studied in [14] (see also [37]).

Considering $\alpha_i \ = \ x_{i+1} + \alpha p \quad (p \ = \ x_i - x_{i-1})$ with $0 \ \leq \ \alpha \ \leq \ 1$ and the interpolatory condition

$$(f-s)(\alpha_i) = 0 \quad i = 1, 2, \ldots, n \tag{2.1}$$

for given functional value, following problem was proposed in [14]:

Problem 2.2. Given $h > 0$ for what choice of α, does the interpolatory problem (2.1) admit unique solutions for splines of class $D_1(3,p,h)$ or class $D_2(3,p,h)$?

The answer to the above problem was given in the form of following theorem:

Theorem 2.5. [14] *Suppose* $0 < h \leq p$ *and (i)* $0 \leq \alpha \leq \frac{1}{3}$ *or (ii)* $\frac{2}{3} \leq \alpha \leq 1$. *The interpolation problem (2.1) admits a unique solution in* $D_1(3,p,h)$, *if* f *is periodic with period* $b-a$. *(2.1) admits a unique solution in* $D_2(3,p,h)$ *if* $(s-f)(b) = 0$ *when (i) holds or if* $(s-f)(a) = 0$ *when (ii) holds.*

Many more results dealing with error bounds have been explored in [14].

3. Bivariate Splines

The domain under consideration is simply connected region for which a discretization is considered (see Figure 3.1).

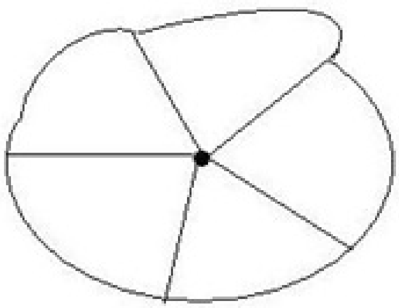

Figure 3.1. Discretization of the domain

On each member of the partition, a bivariate polynomial of degree m is defined and the inter-element continuity is imposed. Polynomial of degree m in n variables have $\binom{m+n}{n} = \frac{(m+n)!}{m!n!}$ free parameters. Many of the mathematicians (see [1], [2], [3], [7] etc.) worked on bivariate splines. Due to discretization of the domain and complexity in bivariate polynomial pieces, it was difficult to handle and settle the dimension problem, ultimately the field reached to the stagnation.

4. Finite Element Method

Given a simply connected domain, the discretization of the domain into finite number of sub-domains (polygons/polycons) is considered. Each sub-domain is called the FINITE ELEMENT (cf. [5]). The discretization (see Figure 4.1 on the next page) has been done in such a way that between two adjacent elements, only two vertices and a side is common.

Triangular Element.

- **Basis for the linear approximation over triangular element**

  For the triangle in Figure 4.2 on the next page, a linear approximation $U(x,y)$ (cf. [68]) is given by

$$U(x,y) = \sum_{i=1}^{3} u(x_i,y_i)\omega_i(x,y) \tag{4.1}$$

  where $\omega_i(x,y)$ $(i = 1,2,3)$ are defined as follows:

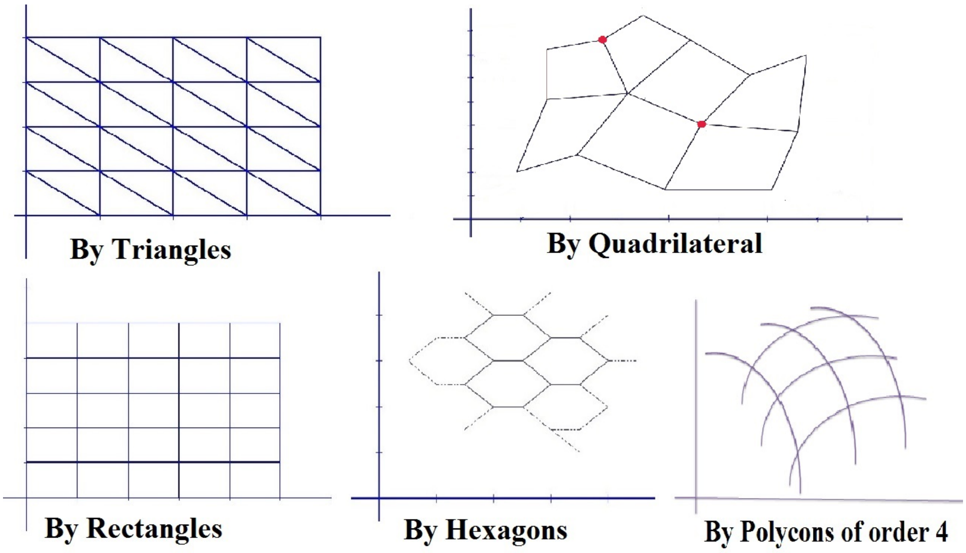

FIGURE 4.1. Discretizations

Let $L_1(x, y)$ be the linear function (polynomial of degree one) which vanishes on side (2,3) of the triangle (Figure 4.2). Then, we have

$$\omega_1(x, y) \;=\; L_1(x, y)/L_1(x_1, y_1),$$

where (x_1, y_1) is a coordinate of the vertex 1.

FIGURE 4.2. Triangular Finite Element

This wedge basis function is associated with the node 1 of the triangle (1,2,3) (see Figure 4.3).

$$\omega_1(1) \;=\; \begin{cases} 1, & \text{at the vertex 1;} \\ \text{linear,} & \text{on the sides (1,2) and (1,3);} \\ 0, & \text{on (2,3) and otherwise} \end{cases}$$

The other wedges are defined similarly. It follows directly by construction that the collection $\{\omega_i(x, y)\}_{i=1}^{3}$ is linearly independent. $U(x, y)$ (cf. relation (4.1)) is a linear approximation by Finite Element Method.

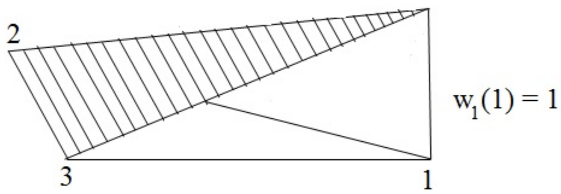

FIGURE 4.3. Wedge construction at the vertex 1

- **Higher order approximation over triangular element**

  $U(x, y)$ is said to be the finite element approximation of degree n or less over the triangular discretization of the domain if (cf. [4], see also [3])

  – Its restriction on one triangle is a bivariate polynomial of degree n or less.

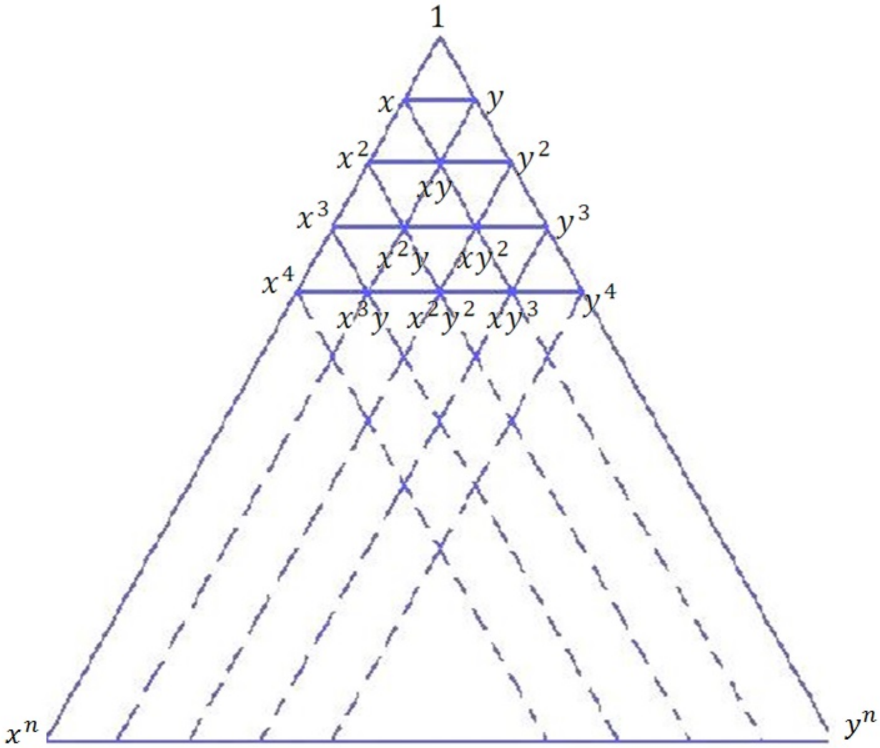

FIGURE 4.4. Pascal's Triangle

  – Inter-element continuity across the common edge follows by construction.

We now describe a famous Pascal's triangle (cf. Figure 4.4). The basis of bivariate polynomial has been prescribed on a triangular finite element for general degree n approximation.

Approximation over Rectangular element.

- **Linear approximation**

  The basis function associated with the vertex 1 of the rectangle (1,2,3,4) is given by (refer Figure 4.5 on the next page)

$$\omega_1(x, y) = (a - x)(b - y)/ab$$

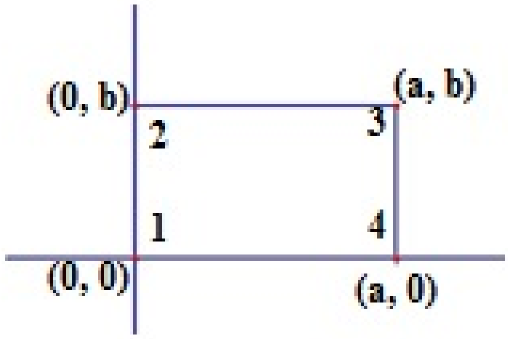

FIGURE 4.5. Rectangular element

$$\omega_1(1) = \begin{cases} 1, & \text{at the vertex 1;} \\ \text{Linear,} & \text{on the adjacent sides (1,2) and (1,4);} \\ 0, & \text{on (3,4) and (2,3).} \end{cases}$$

Other wedges may be defined similarly. It is easy to verify that the collection $\{\omega_i\}_{i=1}^4$ is linearly independent and $U(x,y) = \sum_{i=1}^4 u(x_i, y_i)\omega_i(x,y)$ is a linear approximation over a rectangular element.

- **Higher order approximation over rectangular element**

  The basis of bivariate monomial has been prescribed on a rectangular element for general $n-$degree approximation (cf. Figure 4.6).

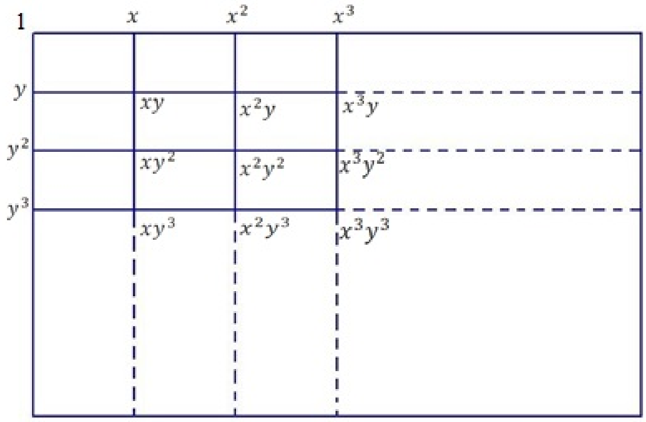

FIGURE 4.6. Monomials

Approximation over parallelogram element. Consider the parallelogram $(1,2,3,4)$ (cf. Figure 4.7 on the next page). The basis functions for nodes 1 and 2 are given by

$$\omega_1(x,y) = (2;3)(3;4)/[(2;3)(3;4)|_1]; \qquad \omega_2(x,y) = (3;4)(4;1)/[(3;4)(4;1)|_2]$$

FIGURE 4.7. parallelogram wedges

The other wedges at nodes 3 and 4 are defined similarly. It is a direct consequence of the construction that $\{\omega_i\}_{i=1}^{4}$ are linearly independent and $U(x,y) = \sum_{i=1}^{4} u(x_i, y_i)\omega_i$ is a linear approximation over parallelogram-element.

A composite finite element of class C^1. Hsieh-Clough-Tocher triangle which is abbreviated as the HCT [5] triangle is defined as follows (cf. Figure 4.8):

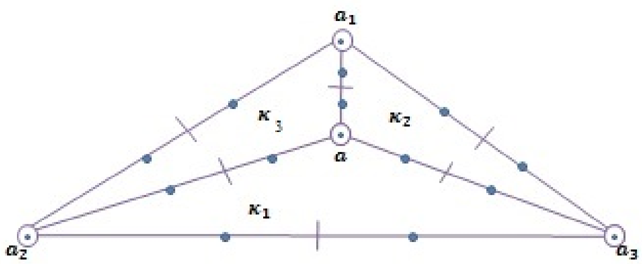

FIGURE 4.8. C^1- triangle due to HCT

The set κ is a triangle subdivided into three triangles κ_i with vertices $a,\ a_{i+1}$, $a_{i+2}, 1 \ \leq i \leq \ 3$ (counted modulo 3) $\dim P_3(\kappa_i) = 10$, it is necessary to find 30 equations to define the three polynomials $P|\kappa_i$ $(1 \leq i \leq 3)$. It has been established that the 30×30 matrix of the corresponding linear system is invertible (cf. Theorem 6.1.2 p. 341 of [5]).

In 1991, Wachspress [69] had described the construction of rational basis functions for patchwork C^1-approximation over a regular algebraic mesh of planar region. It was a new concept of clubbing rational wedge functions with non-rational wedges. Following the rational construction demonstrated by Wachspress, we (cf. [34]) had evaluated interpolants corresponding to the adjacent triangular elements and explored that the inter-element C^0-continuity did not hold (see [38]-[49]).

The Quadrilateral Element. The analysis of the Quadrilateral element was initiated in Dundee, Scotland while Prof. E.L. Wachspress (see [68]) was a visiting fellow, participating in one year symposium on numerical analysis sponsored by the Science Research Council of Great Britain. With the motivation of Prof. A.R.

Mitchell, the general quadrilateral element was reported at the Dundee conference of Application of Numerical Analysis held in April, 1971.

Having demonstrated the inadequacy of polynomials as quadrilateral wedges, the rational form of the basis function was investigated after going through several exercises. The choice of the denominator function is based on a following lemma which plays a key role in defining the rational form.

Lemma 4.1 (Lemma 2.1, p.34 [68])**.** *If three lines intersect at a point, then the ratio of linear forms which vanish on any two of these lines is constant on the third line.*

- **Wedge construction for linear approximation**

  It has been noticed that the polynomials and the monomials are inadequate to define wedge functions over quadrilateral element.

  Thus, the rational function of least degree in numerator and denominator which can be a candidate for the wedge is (cf. Figure 4.9):

$$\omega_1(x,y) = \frac{Q_1}{(2,3)(3,4)}|_1 \quad \frac{(2,3)(3,4)}{Q_1} \tag{4.2}$$

  Hence, the linear form Q_1 should be such that

  (a): $Q_1 \neq 0$ within the quadrilateral.

  (b): $\frac{(2,3)(3,4)}{Q_1}$ should be linear on both $(4,1)$ and $(1,2)$.

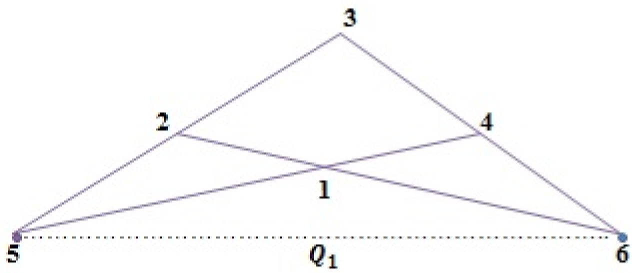

FIGURE 4.9. Quadrilateral element

In view of Lemma 4.1, the exact choice of Q_1 is the exterior diagonal $(5,6)$ obtained by joining exterior intersection points 5 and 6 (cf. Figure 4.9) which satisfies both the conditions (a) and (b).

Wedge functions ω_2, ω_3 and ω_4 at the vertices 2, 3, 4 may be defined similarly.

Thus, $U(x,y) = \sum_{i=1}^{4} u(x_i,y_i)\omega_i$ is a rational linear approximation over the quadrilateral element.

- **Wedge construction for quadratic approximation**

  Consider a quadrilateral $(1,2,3,4)$ (cf. Figure 4.10 on the next page) with mid points $5,6,7,8$ on the sides $(1,4),(1,2),(2,3)$ and $(3,4)$ respectively. The linear forms l_i $(i = 1,2,\ldots,8)$ joining two points have been displayed in Figure 4.10. The basis functions associated with nodes 1, 2 and 5 are as follows:

$$\omega_1 = \kappa_1(l_2 l_3 l_7/Q),\ \omega_2 = \kappa_2(l_3 l_4 l_6/Q), \ldots, \omega_5 = \kappa_5(l_1 l_2 l_3/Q), \ldots, \omega_8$$

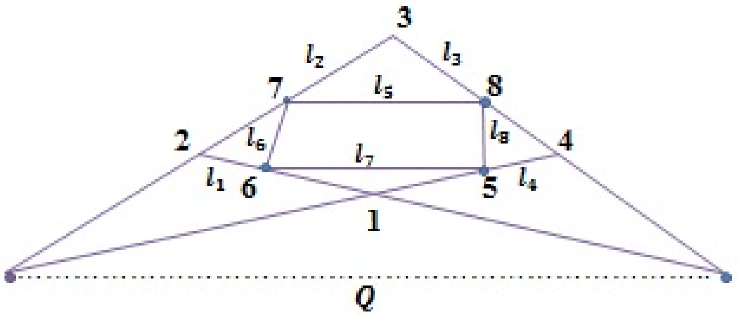

FIGURE 4.10. Quadratic approximation over quadrilateral element

The other wedges may be defined similarly. Applying Lemma 4.1, it may be verified easily that the restriction of each wedge on the adjacent sides is quadratic and zero on the opposite sides. Thus, $U(x, y) = \sum_{i=1}^{8} u(x_i, y_i)\omega_i$ is a rational quadratic approximation over the quadrilateral element.

Similar wedges can be constructed for higher degree approximation over convex quadrilateral.

Wedge construction on convex polygons of order n and some polycons has been discussed in details in [68] (see also [38]-[50])

5. TRIANGULATION METHODS

The Local Optimization Procedure (LOP) for constructing a locally optimal triangulation is widely used in many cases to get an optimal triangulation of a general point set $U = \{u_i = (x_i, y_i)\}_1^N$. In the local procedure, we need to consider the sets of four points (see Figure 5.1) that form convex quadrilaterals. The only two possible triangulations of convex quadrilateral are those which are

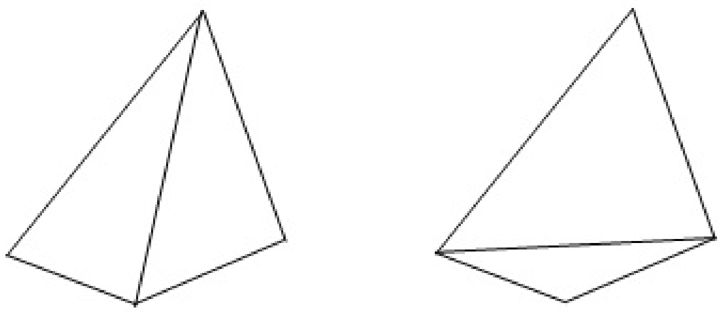

FIGURE 5.1

obtained by adjoining two different diagonals of it. In order to satisfy the criterion, choosing a diagonal is called the Local Optimization Procedure.

The problem of Minimal roughness property of the Delaunay triangulation has been discussed in [60] which is based on a lemma involving the circle criterion. The proof of the lemma was quite lengthy and complex.

We [35] have redefined the circle criterion in terms of power of a point and established a very short and elegant proof of the lemma which states as follows:

Lemma 5.1. *Let $s_T \in S_1^0(T)$ and $s_{T^*} \in S_1^0(T)$ be the interpolators to the data vector f on triangulations T and T^* respectively. Then*

$$|s_{T^*}|^2 - |s_T|^2 = \{\phi(f,r,s)\mu(r,s)\}/\beta(r,s) \tag{5.1}$$

where

$$\begin{aligned}\phi(f,r,s) &= \{f_2(rq-ps)+sf_4-qf_3\}^2,\\ \mu(r,s) &= \{qr^2+qs^2-rq+(p-p^2-q^2)s\}/q,\\ \beta(r,s) &= s(rq-ps)\{q(r-1)-s(p-1)\}\end{aligned}$$

Hansford [24] has investigated an interesting construction of a neutral set for the min-max angle criterion. The technique described by her requires three data points to obtain the locus of the fourth point. We [36] have noticed that the min-max neutral set may be obtain even with two data set points and a given angle.

6. Applications

- Civil Engineering
- One-Dimensional Cutting Stock Problem

Civil Engineering. Watershed management is one of the key branch of Civil Engineering. Soil, water and vegetation are natural resources, and their management is fundamental to the maintenance of ecosystem and economic development. In India, about 175 M/ha area which constitutes about 53 % of the aggregate geographical area, is subjected to land degradation due to soil erosion. It is estimated that 5334 million tons of soil is eroded annually, which is a matter of serious concern for agricultural productivity (cf. [65]).

Our study is focused on soil erosion from watershed. This work was carried over in collaboration with Prof. V. P. Singh (see [67]), Department of Biological and Agricultural Engineering, Texas A & M University, College Station, USA and Dr. Sarita Gajbhiye Meshram, Dr. D.S. Kothari Fellow.

The method is illustrated with a case study of one watershed of Narmada Basin located in Mandala district, Madhya Pradesh (cf. [31], see also [56]).

On the basis of the data extracted from satellite, the Sediment Yield Index (SYI) for fourteen sub-watersheds (corresponding to their Curve Number (CN)) have been computed by using following formula:

$$SYI = \frac{\sum_{i=1}^{n} A_i * W_i * DR}{A_w} * 100$$

where

A_i	–	The area of the $i-$th unit in km^2
W_i	–	The weightage value of the $i-$th unit (Dimension less unit)
n	–	The number of mapping units
DR	–	Delivery ratio (Dimension less) (without units)
A_w	–	The sub watershed area in km^2

For developing relationship between SYI and CN, the proposed procedure is as follows:

(1) Estimation of sub watershed wise Sediment Yield Index (SYI) and curve number
(2) CN of different watersheds are arranged in increasing order with respect to the magnitude which will determine the interval of CN as the domain of cubic spline interpolation.
(3) For our analysis fourteen watersheds were considered out of which seven were selected for the construction of cubic splines (see Figure 6.1) and the remaining were used for the validation of the result.
(4) Referring the cubic spline curve, compute Sediment Yield Index corresponding to CN's of remaining seven watersheds (see Figure 6.1).
(5) The computed SYI (SYIc) was compared with the observed SYI (SYIo) computed as per the guide lines of All India Soil and Land Use Survey (AISLUS, 1991).

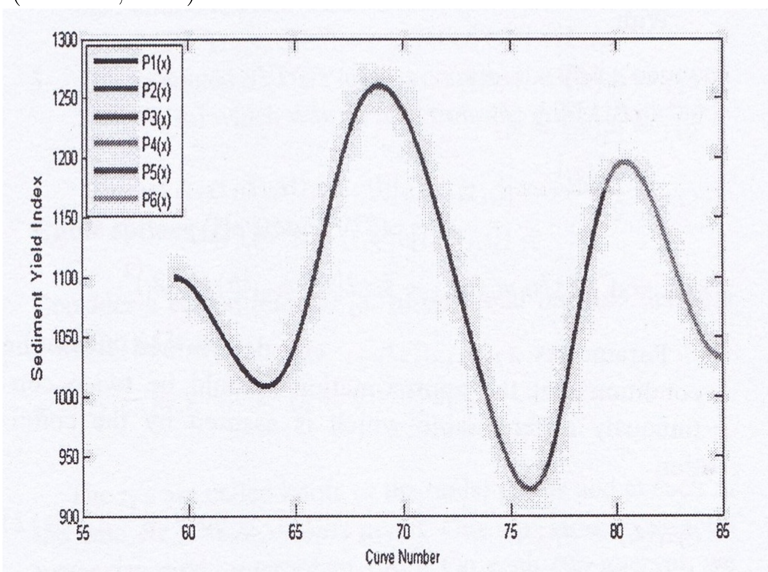

FIGURE 6.1. Cubic spline approximation for soil erosion

Remark: It has been observed that the approximate SYI's were quite close to the actual SYI's. Computation of SYI is quite tedious and lengthy process. With the information of curve number (which is easily available) for any watershed, the approximate SYI can be predicted immediately and accordingly some preventive steps can be taken to control the soil erosion. For those curve numbers lying out side the domain, some extrapolation methods can be applied.

One-Dimensional Cutting Stock Problem. Kantarovich [26] is the first known Soviet Mathematician and economist who formulated the cutting stock problem in the year 1939. In 1961, Gilmore and Gomory [16] (see also [17] - [19]) proposed the first solution methodology for cutting stock problem by applying some techniques of Operations Research (see [20]-[23]). During last two decade this problem become very popular and many mathematicians, economists, computer scientists and engineers suggested several cutting plans by taking in consideration space constraint, time constraint, cost minimization, trim loss minimization etc.

Our study is focused on transmission tower manufacturing industry. One of the major component for the viability of any industry dealing with Cutting Stock Problem (CSP) is based on the scrap or the trim loss.

We have suggested (cf. [51]-[57]) the environment friendly cutting plan which may be easily acceptable by any industry as it resolves the following problems up-to some extent:

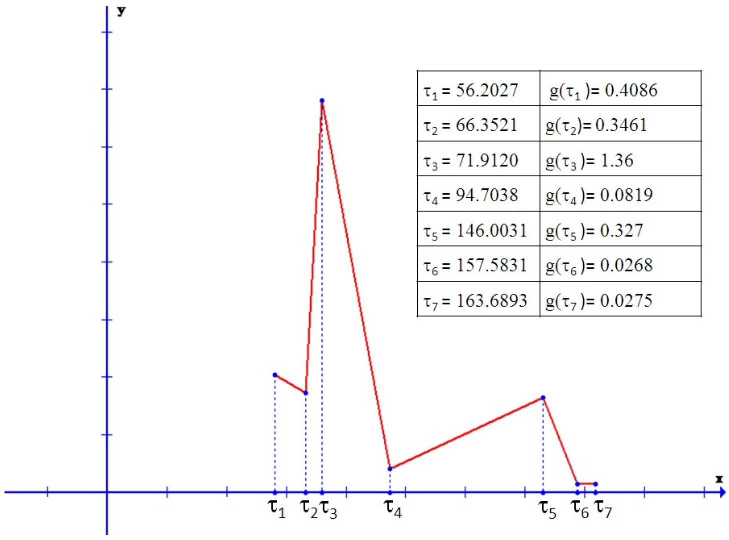

FIGURE 6.2. Linear spline interpolation for 1D-CSP

- The cutting plan consists of cutting of at most two order lengths at a time out of the required n order lengths from a given set of m stock lengths. This consideration requires minimum working space, even one person can carry out the sorting process conveniently.
- The next set of two order lengths has been considered for cutting, only when the demand of first set of two order lengths is completed.

- Using the techniques of summability methods, sustainable trims of different orders have been defined in terms of order lengths and corresponding required number of pieces.
- Corresponding to sustainable trim as nodal point on the interval of the domain and data as total trim loss, the linear (cubic) (cf. Figure 6.2 (on previous page) and Figure 6.3) approximation has been computed (cf. [54]).
- Extracting the data from KEC international (transmission tower manufacturing industry), the complete data analysis has been carried out.
- The mathematical model for the cutting plan has been designed by considering the nature of the data.
- Writing algorithm and the programming was the most tedious job of this work (see also [57]-[59]).

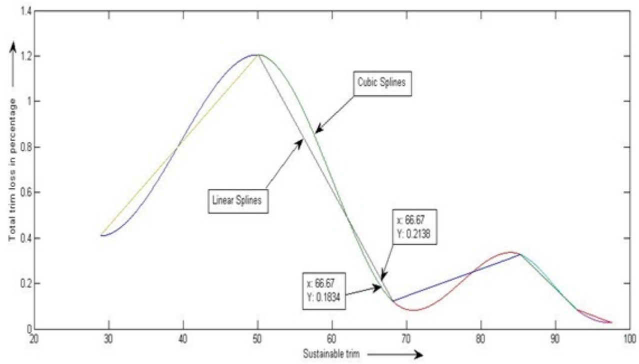

FIGURE 6.3. Cubic spline interpolation for 1D-CSP

Acknowledgement. On this occasion of my professional carrier, I propose my indebtedness to my teacher Late Prof. H. P. Dikshit, (former Professor at R. D. University, Jabalpur and Jiwaji University, Gwalior; former Vice-Chancellor, R. D. University, Jabalpur; Himachal Pradesh University, Shimla; IGNOU, New Delhi and Bhoj University, Bhopal) for initiating me to the interesting area of research in Spline Theory.

Prof. E. L. Wachspress took me one step ahead and very kindly guided me to understand the concept of Finite Element Methods. I shall never forget his generosity and selfless help.

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FLUID MATHEMATICS*

RODDAM NARASIMHA
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Man lives on land in a planet with a life-supporting atmosphere and many rivers and oceans. And we wonder at the mind-boggling diversity of the way in which air and water flow: clouds, tornadoes, dust devils, waterfalls, river bores, placid lakes, ocean waves, tsunamis, whirlpools,... sometimes soothing, sometimes beautiful, sometimes fearsome, often turbulent. And the equations governing these fluid flows have been known for nearly two centuries. But today we still do not understand, and cannot satisfactorily predict, any turbulent flow – from the pressure drop in the pipe circulating water in our kitchen (the answer is known by experience, not by reasoning) to that wonderful, majestic cumulus cloud that can be called the queen of the tropical sky (after Riehl). As Feynman said, turbulence is the last great unsolved problem of classical physics. If the equations are known but the solutions are not, it is clearly a problem in mathematics – one case where mathematics has been singularly ineffective till now. So it is not such a great surprise perhaps that the Clay Foundation in the US has offered a prize of a million dollars for finding the answers to a fundamental mathematical problem in fluid mechanics. (By the way, how is it that we talk of fluid mechanics, fluid dynamics, fluid physics and fluids engineering but never of fluid mathematics, when the subject is really closely tied to mathematics, as I have just pointed out?)

Figure-1 : The Clay Prize in Fluid Mathematics

Figure 1a : THE CLAY PRIZE PROBLEMS: THEIR NATURE
Among the seven Clay Million Dollar Prize Problems considered to be most important unsolved problems in mathematics (including the Riemann Hypothesis) are two problems on the **Navier-Stokes equations**. The basic question underlying the Navier-Stokes problems is whether smooth, physically reasonable solutions of the three dimensional Navier-Stokes equations exist or not. (The two dimensional problem was solved by the Soviet mathematician Ladyzhenskaya (1922-2004) in the 1960s.)

* The article is based on the convocation address delivered at the Chennai Mathematical Institute, Chennai, on August 1, 2008, with some minor editorial changes by the author.

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Key words and phrases: Fluid dynamics, Turbulence, Navier-Stokes equations, Nonlinearity.

Figure 1b: THE CLAY PRIZE PROBLEMS: THEIR ESSENCE

As solution of the problem, proof is demanded of one of four statements. The flavour of these statements is conveyed by the following question. If the initial velocity field is sufficiently smooth everywhere, and the forcing function $\mathbf{F}(\mathbf{x}, \mathbf{t})$ is also similarly sufficiently smooth everywhere and at all times, does the Navier-Stokes solution for the velocity field remain smooth with finite energy, or can it blow up?

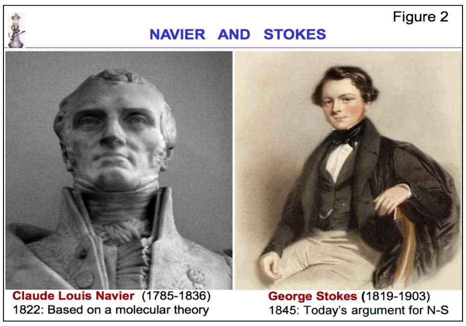
Figure 1c: THE CLAY PRIZE PROBLEMS:MORE PRECISELY STATED

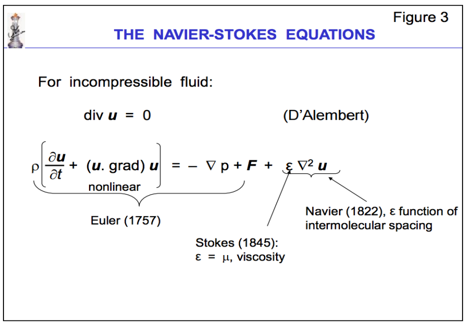
1. Let $\nu > 0$ and $n = 3$. Let $\mathbf{u_o}(\mathbf{x})$ be a given smooth vector-valued function with div$\mathbf{u_o} = 0$, satisfying a given decay relation at ∞. Let $\mathbf{F}(\mathbf{x}, \mathbf{t}) \equiv \mathbf{0}$. Prove that there exists a physically reasonable solution to the Navier-Stokes equations such that $\mathbf{u}(\mathbf{x}, \mathbf{0}) = \mathbf{u_o}(\mathbf{x})$.

2. Let $\nu > 0$ and $n = 3$. Find a smooth vector-valued function $\mathbf{u_o}$, with $div\mathbf{u_o} = 0$, and a smooth function $\mathbf{F}(\mathbf{x}, \mathbf{t})$ satisfying (specified) decay conditions for which there is no physically reasonable solution to the Navier-Stokes equations satisfying $\mathbf{u}(\mathbf{x}, \mathbf{0}) = \mathbf{u_o}(\mathbf{x})$.

The Prize is about limiting solutions of the Navier-Stokes equations (Figure 1abc), which govern the flow of common fluids like air and water: mundane, classical, perhaps prosaic, you may think. Not so – as I hope to show you: for behind the mundane in this case lurks the arcane, in Michael Berry's phrase about a certain kind of science.

Figures 2, 3





The equations were first written down by the French engineer Louis Navier (Figure 2) in 1822, with a molecular argument that led to a momentum diffusion term. According to Navier this had nothing to do with viscosity. But in 1845 George Stokes (also in Figure 2) derived the same partial differential equations as Navier did but appealing explicitly to viscosity, and to linear relations between stress and strain rate in an isotropic medium – very much as we would do it today. Between Navier and Stokes there were others who contributed; at any rate the equations had a fairly long gestation period of 23 years before acceptance (Figure 3). Finding the right boundary conditions took a little longer.

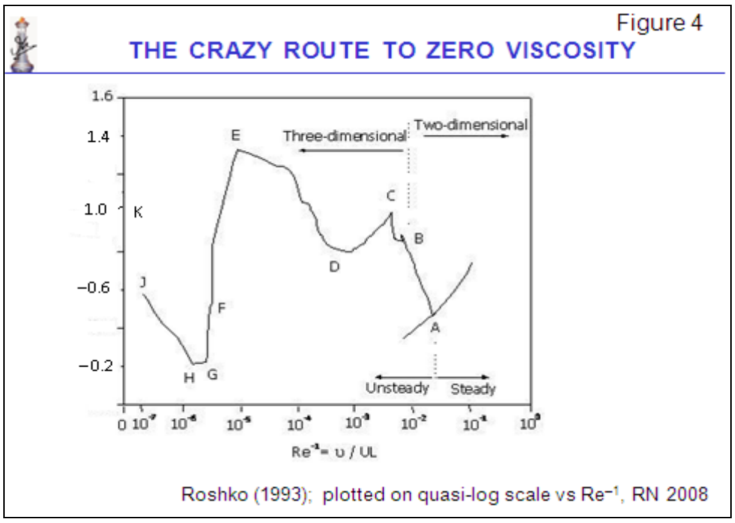
A solution to the Clay mathematical problem is actually going to help in engineering at some stage or the other. The reason is that the limit $\nu \to 0$ (vanishing viscosity) is very important for engineers. To see this, we must first realize that the relevant value of ν is not what you get for any fluid in the hand books of physics: it is the value relevant to the flow problem, i.e., it should be expressed in the natural *flow units* velocity (U, say) $\times$ length (L, say), not in meters and seconds. That is, as the British engineer Osborne Reynolds figured out in 1883, we really want the handbook value divided by UL, which gives us the reciprocal of what came to be called the Reynolds number. Using a flow-scaled ν, a mainline Boeing aircraft would fly around $\nu \sim 10^{-7}$, birds at about 10^{-4}, the smallest insects at around 1. It is legitimate to ask: Is 10^{-7}, or 10^{-4} (or even 1) close to zero? (Equivalently, is the reciprocal, the Reynolds number, close to the limit of infinity at 10^7?)

But the answer is not obvious. To see this, look at how crazy and long the crooked road to $\nu = 0$ is for one particular fluid-dynamical parameter, namely the base pressure (more precisely, the nondimensionalized pressure coefficient at the rear stagnation point) on a perfect, smooth circular cylinder in a stream of given velocity. Who would have thought that the path from 10^{-1} to 10^{-7} (meticulously surveyed by Anatol Roshko in 1993) could be so full of ups and downs, zigs and zags (Figure 4)?

Why is this so? The chief reason is: Because the non-linearity in the Navier-Stokes equations becomes more important as $\nu \to 0$ (the other limit $\nu \to \infty$ is linear and boring (I am exaggerating)), and because the road is strewn with rocks known as regime transitions, some of them big boulders causing huge changes, and some little pebbles causing tiny kinks. Almost all of the nine transitions shown in the figure (labelled A to J) come from experimental data (acquired in several different labs across the world). Apart from the relatively smooth curve at the boring end of ν there are no theories, and only one or two transitions at the same end of ν can be computed at present. Here, incidentally, is the challenge to Fluid Mathematics and to computer science.

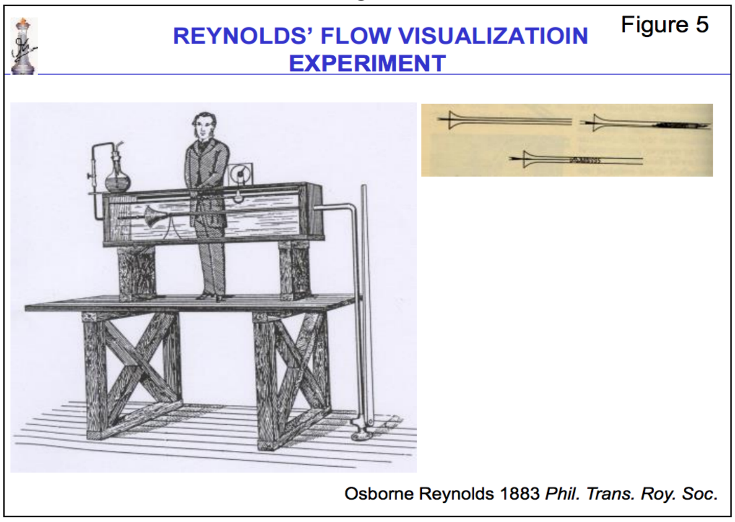
The most famous transition is from laminar to turbulent flow. This, by the way, was the main subject of that 1883 paper by Reynolds: transition to turbulence in pipe flow. (His paper on the subject began by pointing out that the problem was of both *practical* and *philosophical interest*; we could say the same thing today, more than a century later.) He showed that transition – and, as it turned out, the whole character of any flow – depended only on the Reynolds number *combination*, not on the fluid itself; thus a dust particle in air may behave like a marble in honey, for the Reynolds number can be the same in both cases, although (or because) honey is far, far more viscous than air, and the marble is much, much bigger than a speck of dust.

Figure 4



Ordinate in the above figure (Figure 4): Base pressure coefficient C_{pb}

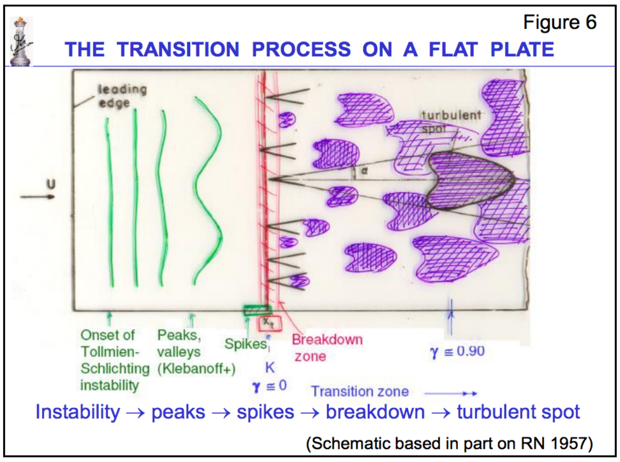
Figure 5



Notice those flashes in Reynolds's diagram (at the top right in Figure 5)? they are slugs or islands of turbulence. Some 70 years later similar slugs were found in flow past a flat plate (a proxy for an aircraft wing), where they got named as turbulent spots by Howard Emmons. I started my career in fluid dynamics looking at spots and transition on a plate – for practical reasons connected with the limitations of a wind tunnel that we were using at the time at the Indian

Institute of Science. (Mundane and arcane again!) I had the great good fortune of having as my adviser Satish Dhawan, who combined many worlds in himself beginning with degrees in Mathematics and Physics and in Engineering (not to mention English literature). I got some results that pleased me (Figure 6), and in 1956/57 had an unexpected opportunity to tell Kurt Friedrichs about them when he visited Bangalore. He was the first real mathematician I had ever met. (I was familiar with his name as co-author, with Courant, of the famous book *Supersonic Flow and Shock Waves*, a mathematical treatise that engineers of the time used to decorate their book-shelves with.) For the first 15 minutes I told him what I was doing, going through all the different stages of the transition process: smooth laminar flow near the leading edge, appearance of instability, breakdown into turbulence at points along an 'onset' line in the form of tiny spots of turbulence, and their eventual growth and merger leading to fully turbulent flow (Figure 6). Friedrichs seemed fascinated (he had clearly not heard anything about 'spots' in boundary layer transition before), and I was pleased. For the next half-hour however he kept shooting ideas at me about why I might be seeing what I saw, and I had great difficulty keeping up with his agile mind as he kept firing new proposals at me, quickly modifying them if I should so much as utter a mild word of doubt about any of them. So I saw at first-hand how a great mathematician thinks about fluid flow problems.

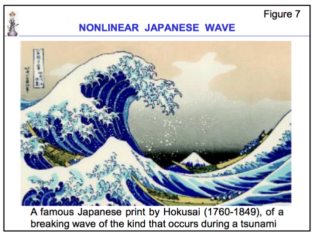
Figure 6

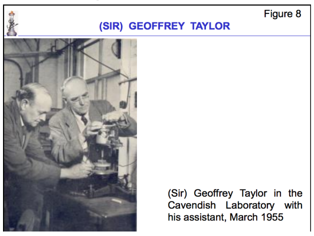


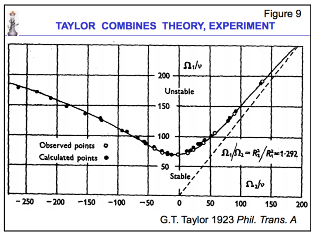
We return to the long sequence of transitions seen in the base pressure on the cylinder, ranging over some six decades in the Reynolds number. Such transitions are due not only to the non-linearity of the Navier-Stokes equations, but also to their three-dimensionality. Nonlinearity is writ large over most interesting fluid dynamics: let us look at some other examples.

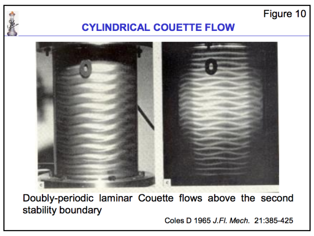
One consequence of nonlinearity is non-uniqueness of a certain kind in the solutions. When a water wave has a huge amplitude (as in a tsunami), the height of the water surface can be a three-valued function of distance along propagation (Figure 7); in this celebrated print, the great Japanese artist Hokusai chose to catch one of his *Eighteen Views of Mount Fuji* by framing it within a curling wave just before it crashes. Another instance is the flow between two rotating concentric cylinders with fluid in the gap, usually called cylindrical Couette flow. G. I. Taylor, one of the great leaders of fluid dynamics in the 20th century (Figure 8), showed theoretically, and demonstrated experimentally, that there was an instability in this flow involving vortical cells, and there was excellent agreement between his theory and his experiments on the onset of instability (Figure 9). But as we move further away from onset the number of possible flow regimes – defined say by the number of cells and the number of the azimuthal waves that can accompany the cells (Figure 10) – can be numerous even at given cylinder speeds, as many as around 25 in one case (Figure 11, due to Donald Coles). Here is non-uniqueness run riot, so to speak (although, for a given temporal history in rotation speed of the cylinder the end state is unique).

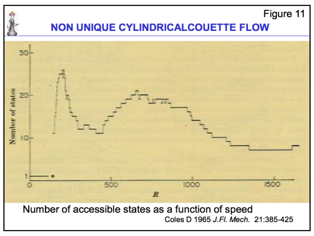
Figures 7, 8, 9, 10, 11











Taylor was a Cambridge mathematician who not only made experiments (as Figure 9 indicates) and solved equations, but also sailed boats and flew airplanes. Fluid dynamics during the 20th century had very porous boundaries. Among the people I learnt most from at Caltech were Hans Liepmann (a physicist by education), Paco Lagerstrom (a mathematician who was working at Douglas Aircraft before Clark Millikan (Robert's son and Director of the Aero Lab) took him to Caltech) and Anatol Roshko (an engineer).

Let us now look at Nonlinearity, Act I. In the late 19th and early 20th c. nonlinearity was still new, and the subject of hydrodynamics abounded in paradoxes. Most of these were considered by Garrett Birkhoff in a book published in 1950

(Figure 12; his definition of paradox was that it represented an apparent inconsistency between fact and plausible argument). The subtitle of Birkhoffs book tells us about the mathematical mystery surrounding fluid flows. The frontispiece to the book (Figure 13) showed a supersonic flow with shock waves, which Birkhoff considered at the time an instance of common sense contradicted, because it involved so many virtually discontinuous solutions. But today such pictures are daily fare for space scientists and missile engineers.

The grand-father of all these paradoxes is the one due to d'Alembert, who in 1768 proved (assuming zero viscosity, among other things) that the drag of a finite body in steady motion was zero. (His 'proof' implied that at $\nu = 0$ the flow should plot at $+1$ (point K) on the ordinate in Figure 4 - i. e., steeply up the scale in the diagram.) Of course we know that drag is not zero in real life, hence the paradox. D'Alembert knew this too, but he left it 'to future Geometers for elucidation' (he must have been a Greek at heart, for mathematics meant geometry in classical Greece). The prize that was promised by the Berlin Academy on the subject was not awarded, because they had (wisely) demanded experimental evidence in support of the analytical 'proof'. (So we can wonder whether mathematics is here seeking to provide the 'proof' for the experiment, or experiment to provide the 'proof' for the mathematics, but at any rate the Clay Prize of today continues an old fluid-dynamical tradition of problem-solving awards.) This 'd'Alembert paradox' was one of the many instances that led to the Hydraulics-Hydrodynamics wars, fought because it was said that the former (created and exploited by engineers) observed things which could not be explained, whereas the latter (pursued by 'geometers', i.e. mathematicians) explained things that could not be observed. (The word hydrodynamics was invented by Daniel Bernoulli, who confused the issue by using it as the title of a book he wrote on hydraulics.)

Figures 12, 13

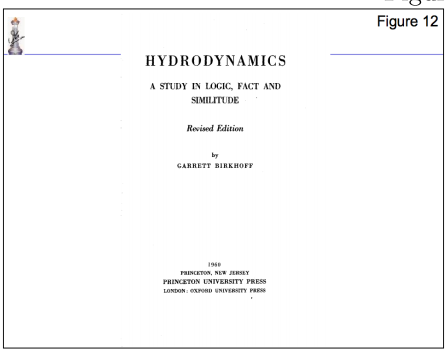


Figure 12

HYDRODYNAMICS

A STUDY IN LOGIC, FACT AND SIMILITUDE

Revised Edition

by

GARRETT BIRKHOFF

1960

PRINCETON, NEW JERSEY

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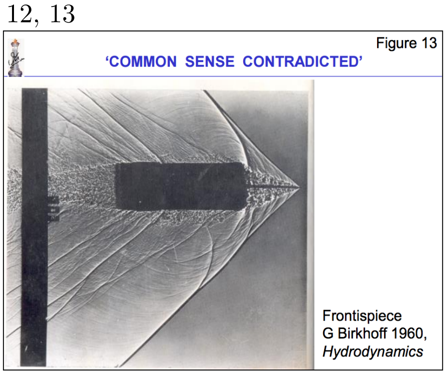
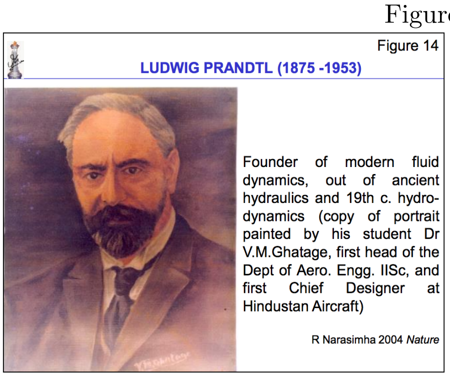


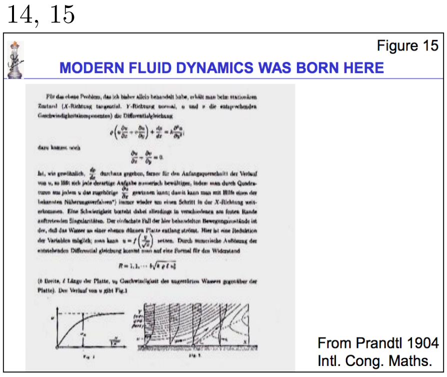
The paradoxes slowly got resolved in the first half of the 20th century, largely through the change in thinking wrought by one man, the German engineer Ludwig Prandtl (Figure 14), who successfully bridged the epistemological gulf with the mathematicians. (The portrait of Prandtl is a copy of one painted by his student V.

M. Ghatage, who got his PhD on an atmospheric problem, but went on to become India's first professor in aeronautics – at IISc – and India's first aircraft designer as well, retiring as Managing Director at Hindustan Aircraft (now Aeronautics), Bangalore.) Prandtl was working in industry when the famous mathematician Felix Klein persuaded him to move to Göttingen as a professor of applied mechanics – surely one of the most inspired appointments ever made in the history of fluid dynamics. In 1903 Prandtl read a paper at the 3rd International Congress of Mathematics at Heidelberg (porous borders again!), and introduced a new mathematical method of handling nonlinearity – chiefly in laminar flow problems – by creating the concept of boundary or singular layers – small regions within which the nonlinearity and viscous effects were concentrated and could by clever approximations be managed. This approach yielded interesting asymptotic solutions in the $\nu \to 0$ limit. We could in fact truthfully say that this particular page in Prandtl's paper of 1904 (Figure 15) gave birth to modern fluid dynamics, and resolved d'Alembert's paradox.

Prandtl's method was later codified into singular perturbation theory by (among others) the same Friedrichs I had met in Bangalore, and in particular by Paco Lagerstrom, developing what came to be called the method of matched asymptotic expansions (MAX for short). Many paradoxes dissolved under the appropriate MAX attack. One of the most dramatic uses of MAX (not so called at the time) was the demonstration of the instability of Prandtl's boundary layer by his student Tollmien (1929). It took more than a decade before this instability was observed experimentally, in contrast with Taylor's almost instantaneously successful comparison of theory and experiment (shown in Figure 9). This led to some controversies between the two leaders in 1938, although each of them respected the other greatly. (G I Taylor proposed Prandtl for the Nobel Prize, even calling Prandtl his 'chief'; but next year their two countries were at war with each other, and the Nobel Prize was forgotten.)

Figures 14, 15

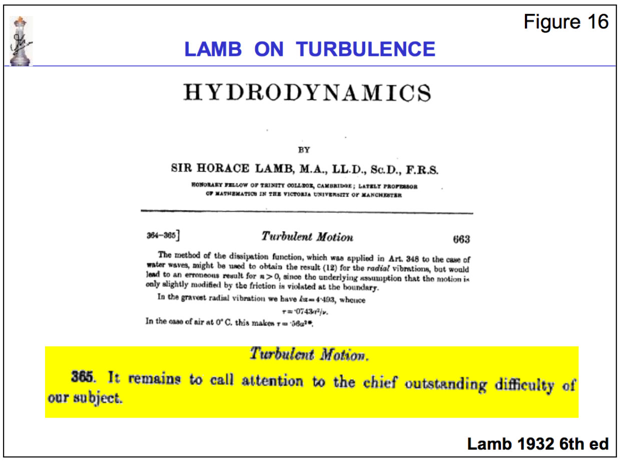




The whole area of hydrodynamic instability has been the joint enterprise of physicists (Sommerfeld, Rayleigh, Heisenberg), mathematicians (Orr, Taylor), engineers (Reynolds, Prandtl, Tollmien, Schubauer) (porous borders again). The subject is important because many flows crumple into some form of instability or other at the slightest excuse, making them prone to turbulence.

But even MAX cannot handle what (Sir) Horace Lamb, in his ever-enduring treatise called *Hydrodynamics* that summarised pre-modern fluid dynamics, called 'the chief outstanding difficulty of our subject', namely turbulence – the apparently random/irregular/chaotic/intermittent vortical motion that is the natural state of most air and water flows we observe.

Figure 16



That brings us back to turbulence and pipe flow, and how our Indus valley ancestors who pushed water through their cities more than 4000 years ago clearly knew things that as mathematicians we still cannot explain. In nearly 135 years since Reynolds made his famous experiments we have learnt a lot about turbulent flows. For example, engineers can predict not only the pressure loss in a pipe, but also the drag of an aircraft or the trajectory of a satellite launch vehicle with considerable confidence – but not starting from first principles, and not without appealing to experimental data at some stage or the other. All we can say is that a lot of clever analysis has gone into reducing the amount of data that needs to be acquired, but the problem of getting it all from first principles remains unsolved. So the chief outstanding difficulty of Horace Lamb remains just that to this day.

Before I close however I want to show you some flow pictures, for fluid dynamics is nothing if it is not photogenic. (As John Ruskin said, nature produces its most beautiful inorganic forms in water flow.) Prandtl's path-breaking paper at the International Congress of Mathematics contained numerous flow pictures at the end. The pictures I now show (following his example) were all taken in Bangalore. The first (Figure 17 on the next page) shows how flows can be generated in the laboratory to look like clouds in the atmosphere. As the fake clouds are generated in a water tank the great resemblance we see with real clouds shows the extraordinary power of the idea of dynamical similarity: isn't it amazing that a cloud made by the motion of moist air over a height of several kilometers in the atmosphere can be mimicked in a plume generated in a little water tank that is a cube of about 1 m side if only we put the right amount of heat at the right place and right time into the laboratory plume?

Figure 17

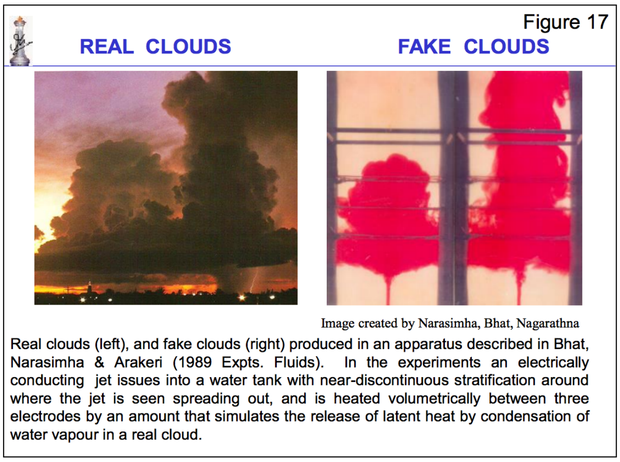
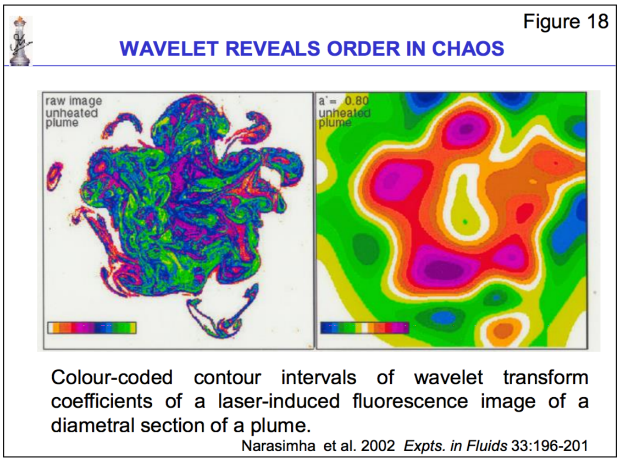


Figure 18

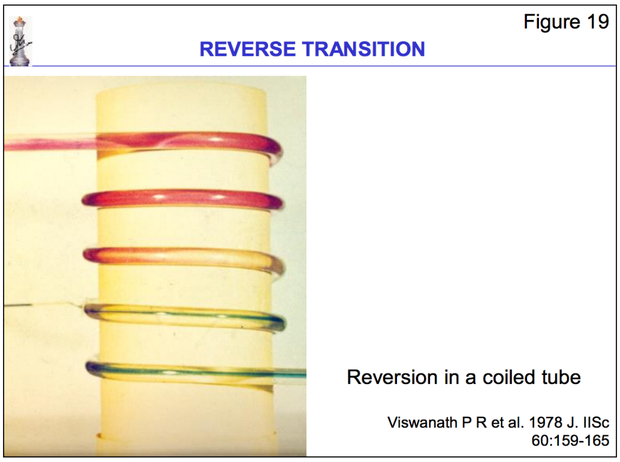


(Left: Raw image. Right: Wavelet transform, revealing a fluted vortex ring at a particular value of the wavelet length scale.)

One of the intriguing things about turbulence is that a flow we consider random or chaotic may actually possess concealed order. This was convincingly demonstrated by Brown and Roshko in the 1970s. Figure 18 shows an instance where the order concealed in a plume can be revealed by its wavelet transform.

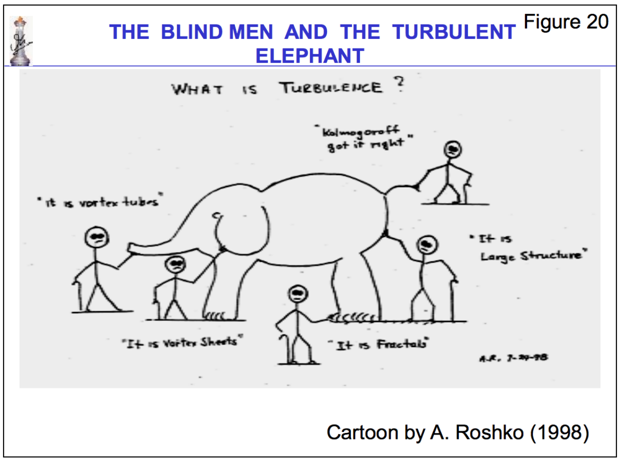
I have already mentioned the phenomenon of transitions from laminar to turbulent flow but reverse transitions from turbulent to laminar flow are also possible. Here is one instance (Figure 19) where that happens in a rather dramatic way, by the simple expedient of bending the tube carrying turbulent flow (red, diffused in the picture) into a coil (leading to the green, non-diffusing filament of laminar flow at the third coil). This phenomenon can be seen as the anti of that studied in Reynolds's experiment.

Figure 19



So the big question is: *What is the mathematics that can handle all these phenomena?* In a well known lecture given at the Courant Institute, Eugene Wigner once spoke eloquently about the unreasonable effectiveness of mathematics. But for all those who have been collectively wrestling with turbulence for more than a century, what seems striking is the singular *ineffectiveness* of mathematics in handling the highly nonlinear Navier-Stokes equations (hence the Clay prize). Fluid flow has always been a complex problem. The *Yoga-Vāsiṣṭha* (6th / 7th c.?) drew parallels between the richly diverse ways in which water flows and the human mind works. Galileo said he knew more about the motion of heavenly bodies than about the flow of water under his very eyes. Echoing this thought more than three centuries later, Feynman pointed out that we can follow the evolution of a star till convection starts, but thereafter we can no longer deduce what should happen. 'A few million years later the star explodes, but we cannot figure out the reason' – thus his conclusion that turbulence was the greatest unsolved problem in classical physics. After writing down his equations in 1757 (basically Navier-Stokes without the viscous terms), Euler declared that the difficulty with fluid flows was not the sufficiency of the principles of mechanics but the fact that 'analysis abandons us here'. But the man who saw most clearly the fundamentally mathematical nature of the problem was von Neumann. He thought that if the turbulence problem were solved, its impact on pure mathematics would be even greater than on fluid dynamics (this thought is tucked away in a report he wrote for the US Office of Naval Research that has not been widely read, in part because he thought it was not ready for publication). That is why the greatest advances in fluid dynamics have for long generally depended on deep physical insight, clever experiments, smart approximations, accurate numerical solutions and so on, rather than a grand solution to the central problem. Simply put, if we know the equations but cannot find their crazy solutions, we must conclude that we need some new mathematics. May be some young person some day will create it! As of now, we are like the blind men and the elephant in the universally famous Buddhist story (Figure 20, with many thanks to Prof. Anatol Roshko for his wonderful cartoon).

Figure 20



Good luck to all those who are still fighting the 'turbulence problem' with Fluid Mathematics.

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DEFINING AND COMPUTING CUBIC SPLINES

T. AUGUSTA MESQUITA
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Abstract. In this note we address the students' usual misperception of the cubic spline definition and construction within an undergraduate numerical course in order to present an alternative initial approach. We describe succinctly the standard path from the interpolating polynomial towards a smooth piecewise interpolation model and we focus on a simpler, though unstable, cubic spline structure. This less common choice aims to promote a more comprehensible dialogue between the notion of cubic spline functions and the numerical (in)adequacy of different methods for computing such functions.

1. Introduction

The construction of real valued functions working as models of empirical data is a classical topic in numerical analysis courses. In such first approach to data fitting, we typically face the solution given by polynomial interpolation if we are searching for a model that totally replicates the numerical observations. If we assume differently by aiming to find a function not reproducing the entire set of data but reasonably adjusting into it, the solution is usually provided by the least-square method. Persevering on the interpolation option, we commonly point to the direction of a piecewise polynomial function, more precisely, we address the construction of the spline functions of order m. Their definition is motivated by the rigidity of polynomials phenomenon and can be accompanied with enlightening sketches for the first, second and third order ($m = 1$, $m = 2$ and $m = 3$). Nonetheless, the prompt construction of such functions - provided by several branches - constitutes many times a great challenge to students, in particular for the cubic case $m = 3$. The cubic spline is a versatile and smooth function that appears often in a diverse number of engineering applications, though its derivation starts usually with two integrations. This initial part transmits an extra level of difficulty from which emerges the questions: why do we need to follow that algorithm? Are there other possibilities? In order to answer these questions and simultaneously provide students with means to exercise their numerical sensibility while working with unstable solutions, we explore here an alternative approach to the presentation of the spline functions in an undergraduate numerical analysis classroom.

2010 Mathematics Subject Classification: 97D70; 97N50; 97D80.
Key words and phrases : Curve fitting; polynomial rigidity; piecewise interpolation; numerical stability; cubic splines; Octave 4.0.0.

2. From the interpolating polynomial to the cubic spline

The polynomial interpolation is a classical method which aim is to create a continuous function acting as a precise model of a given set of numerical data. Although easy to construct and unique when the given numerical information has the form

$$(x_i, y_i),\ i = 0, \ldots, n, \qquad a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \tag{2.1}$$

it becomes easily an unsuitable solution, such as when the dependent variable y has sudden changes. A data set (2.1) with $n+1$ points allows the definition of a polynomial $p(x)$ of degree n (or lower) such that $p(x_i) = y_i,\ i = 0, \ldots, n$. Even knowing that theoretically the increase of data points, and thus the increase of the degree of the correspondent polynomial, shall give place to a better description of the underlying function $y(x)$, the implementation shows us that the provided model has undesirable fluctuations. In fact, a larger number of interpolating points implies a higher degree of the interpolating polynomial which augments the absolute values of the coefficients of its derivatives, indicating bigger oscillations. As an example, let us take the numerical data of Table 1, where the variable y equals the mean variation of the annual temperature in different latitudes (x), when we

x	65	55	45	35	25	15		
y	-3.1	-3.22	-3.3	-3.32	-3.17	-3.07		
	x	5	-5	-15	-25	-35	-45	-55
	y	-3.02	-3.02	-3.12	-3.2	-3.35	-3.37	-3.25

Table 1. Data from *Philosophical Magazine* 41, 237 (1896) referred in [1, p. 76]

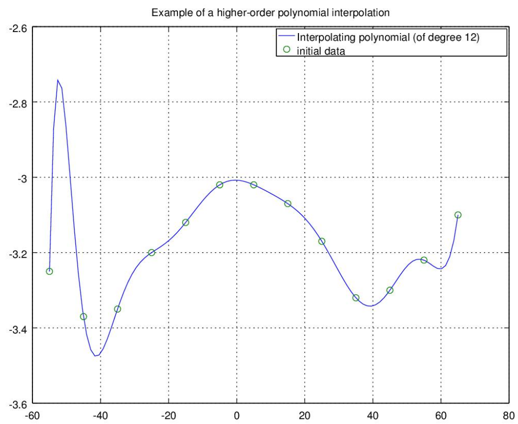

Figure 1. An example of the rigidity of an higher-order interpolatory polynomial.

consider a fixed concentration of carbonic acid [1]. The unique polynomial (of degree 12) that interpolates the entire table is depicted in Figure 1.

The very well documented rigidity or inflexibility [2] of the interpolating polynomials can be overcome by diverse numerical methods regarding curve fitting. Nonetheless, when the interpolation feature must be maintained, the piecewise polynomial interpolation accomplished by the spline functions is a natural and very common response in engineering applications. The use of polynomial solutions in approximation problems is nicely analysed in [2]. Several historical notes can be read in this last reference as well as a wide list of attractive properties of the polynomial splines. Even though a mathematical review on this topic goes beyond the scope of this paper, some definitions and further material must be presented and will be included in the text gradually.

Definition 2.1. [3, 4] A function S is a polynomial spline of degree $m \in \mathbb{N}$ with regard to the abscissa set $a = x_0 < x_1 < \cdots < x_n = b$ if it fulfils the following properties.

(1) S coincides in each interval $[x_{i-1}, x_i)$, $i = 1, \ldots, n$, with a polynomial of degree $\leq m$;
(2) S has continuous derivatives up to the order $m - 1$.

The abscissas $x_0, \ldots, x_n$ are called the knots or nodes of the spline, and from this point on the amplitude of each subinterval will be denoted by h_i, where $h_i = x_i - x_{i-1}$.

Example 2.1.1. Spline of degree one.
The function $S(x)$ matches on each interval $[x_{i-1}, x_i)$ a polynomial of degree ≤ 1 and $S(x)$ is continuous on $[x_0, x_n] = [a, b]$. This linear spline is then formed by straightlines as illustrated in Figure 2 for the data of Table 1.

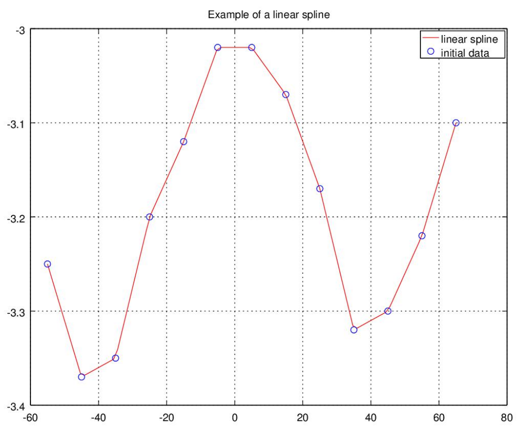

FIGURE 2. A linear spline.

Let us notate by $S_i(x)$ the polynomial of degree ≤ 1 corresponding to the definition of $S(x)$ on each interval $[x_{i-1}, x_i)$. Thus,

$$S_i(x) = y_{i-1}\frac{x_i - x}{h_i} + y_i\frac{x - x_{i-1}}{h_i}, \quad x_{i-1} \leq x < x_i,$$

where $y_i = f(x_i)$. It is obvious that $S(x)$ is continuous on $[a, b]$.

Example 2.1.2. Spline of second degree.

If $m = 2$, then $S(x)$ coincides with a polynomial of degree ≤ 2 on each subinterval $[x_{i-1}, x_i)$ and $S(x)$ is a continuous function with derivative function $S'(x)$ also continuous on $[a, b]$.

3. Cubic Splines

When we choose $m = 3$ in Definition 2.1, we get a function $S(x)$ that on each subinterval $[x_{i-1}, x_i)$ coincides with a polynomial of degree ≤ 3 and has derivative functions of first and second order, $S'(x)$ and $S''(x)$, continuous on $[a, b]$.

For a given set of numerical data, the construction of an interpolating cubic spline usually starts in the numerical methods textbooks by considering the second derivative of each branch of $S(x)$ (e.g. [5, 3, 4]). Let us denote the i-th branch of $S(x)$ by $S_i(x)$, with $i = 1, \ldots, n$. It is then easy to note that the second derivative of each branch must be a polynomial of degree ≤ 1 defined on an interval $[x_{i-1}, x_i)$. The construction of $S(x)$ is therefore redirected to the computation of the values of $S''(x)$ in the knots $x_0, x_1, \ldots, x_n$, notated as $M_i,\ i = 1, \ldots, n$. In other words, when we take the linear interpolation of the two points (x_{i-1}, M_{i-1}) and (x_i, M_i), where $M_i = S_i''(x_i) = S''(x_i)$, as follows

$$S_i''(x) = M_{i-1}\frac{x_i - x}{h_i} + M_i\frac{x - x_{i-1}}{h_i}\ , \tag{3.1}$$

it becomes immediately assured the continuity of the second derivative. Then, we have to integrate twice both sides of the equation (3.1) to obtain the expression of $S_i(x)$, yielding:

$$S_i(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + \alpha_i\frac{x_i - x}{h_i} + \beta_i\frac{x - x_{i-1}}{h_i}, \quad \alpha_i, \beta_i \in \mathbb{R}.$$

Taking into account the interpolating conditions $S_i(x_{i-1}) = y_{i-1},\ S_i(x_i) = y_i,\ \ i = 0, \ldots, n$, the constants of integration α_i, β_i are easily calculated and we finally get:

$$\begin{aligned} S_i(x) &= M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} \\ &+ \left(y_{i-1} - M_{i-1}\frac{(h_i)^2}{6}\right)\frac{x_i - x}{h_i} + \left(y_i - M_i\frac{(h_i)^2}{6}\right)\frac{x - x_{i-1}}{h_i}. \end{aligned} \tag{3.2}$$

The full numerical definition of $S(x)$ is now reduced to the computation of the constants $M_0, M_1, \ldots, M_n$. Differentiating (3.2) we get $S_i'(x)$ and the continuity of $S'(x)$ at the interior knots $x_1, \ldots, x_{n-1}$ justify the next list of $n-1$ linear equations $(i = 1, 2, \ldots, n-1)$.

$$\frac{h_i}{6}M_{i-1} + \frac{h_i + h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}$$

Recalling that we have $n + 1$ unknowns $(M_0, \ldots, M_n)$, we get two degrees of freedom requiring additional input. Depending on the context of application, we encounter more frequently the following designations or definitions [6, 7].

- Clamped cubic spline : $S'(x_0) = y'_0$ and $S'(x_n) = y'_n$, for supplementary given values y'_0 and y'_n.
- Natural cubic spline: $M_0 = 0$ and $M_n = 0$.
- not-a-knot cubic spline: the third derivative of $S(x)$ is continuous at the second and the next-to-last knots, that is, $S_1(x) = S_2(x)$ and $S_{n-1}(x) = S_n(x)$.

The parameter set $M_0, \ldots, M_n$ of a cubic spline is then computed by means of the following linear system, where two further conditions, regarding b_0 and b_n, shall be required.

$$\begin{bmatrix} 2 & \lambda_0 & 0 & \cdots & 0 \\ \mu_1 & 2 & \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & \cdots & 0 & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}, \qquad (3.3)$$

$$\lambda_0 = 1, \quad \mu_n = 1, \text{ and for } i = 1, ..., n-1 :$$

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad \mu_i = 1 - \lambda_i,$$

$$b_i = \frac{6}{h_i + h_{i+1}} \left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \right).$$

Several years of teaching experience allowed us to notice that the computation of a cubic spline is not taken by a set of different groups of students as a straightforward task, mainly because they frequently get absorbed in the writing and solving of system (3.3). The final definition of the precise algebraic expression of each branch of $S(x)$ seems to be neglected by students together with the forthcoming and most important applications of $S(x)$.

4. A SIMPLE BASIS

Given a set of points (2.1) let us now consider the next step-type function which allows us to recall a simpler basis for the entire space of cubic splines defined on a interval $[x_0, x_n] = [a, b]$, denoted by $S_{[a,b]}$, as explained in [3].

$$(x - x_i)_+^3 = \begin{cases} (x - x_i)^3 & \text{if } x_i \le x \\ 0 & \text{otherwise} \end{cases} \qquad (4.1)$$

Taking into account that $\{1, x, x^2, x^3, (x - x_1)_+^3, \ldots, (x - x_{n-1})_+^3\}$ spans $S_{[a,b]}$, we may derive any cubic spline by calculating the $n+3$ coefficients b_0, b_1, b_2, b_3, c_1, $\ldots, c_{n-1}$ of the generic expression

$$S(x) = \sum_{i=0}^{3} b_i x^i + c_1 (x - x_1)_+^3 + \cdots + c_{n-1} (x - x_{n-1})_+^3 . \qquad (4.2)$$

It is important to remark that this structure naturally assures the continuity of $S(x)$, $S'(x)$ and $S''(x)$ on $[a, b]$. Furthermore, from the interpolatory conditions

$S(x_i) = y_i,\ i = 0, \ldots, n$, we obtain $n+1$ linear equations for the unknowns b_0, b_1, b_2, b_3, c_1, $\ldots$, c_{n-1}. The additional two conditions again reflect an option as before recalled through the clamped or natural cubic spline, among other possible choices. For example, the coefficients of a natural cubic spline can be obtained by means of the following linear system.

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & & & \\ 1 & x_1 & x_1^2 & x_1^3 & & & \\ 1 & x_2 & x_2^2 & x_2^3 & (x_2-x_1)^3 & & \\ 1 & x_3 & x_3^2 & x_3^3 & (x_3-x_1)^3 & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ 1 & x_n & x_n^2 & x_n^3 & (x_n-x_1)^3 & \cdots & (x_n-x_{n-1})^3 \\ 0 & 0 & 2 & 6x_0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 6x_n & 6(x_n-x_1) & \cdots & 6(x_n-x_{n-1}) \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \\ 0 \\ 0 \end{bmatrix} \tag{4.3}$$

Replacing in (4.2) the complete polynomial of degree three $\sum_{i=0}^{3} b_i x^i$ by $y_0 + d_1 (x - x_0)_+^3$, we can define the following particular model.

$$S(x) = y_0 + d_1 (x - x_0)_+^3 + \cdots + d_n (x - x_{n-1})_+^3 \tag{4.4}$$

We have lost the generality of function (4.2) in the sense that we cannot derive all functions of $S_{[a,b]}$ through expression (4.4), nevertheless the interpolatory conditions and the continuity of $S(x)$ allow an iterative computation of $d_1, \ldots, d_n$, and a straightforward verification clarifies that $S'(x)$ and $S''(x)$ are continuous. In other words, function (4.4) allows us to quickly confirm all items of Definition 2.1 with $m = 3$ and furnishes a simple instrument for constructing analytically a cubic spline function which interpolates a small set of points.

The next section proposes a small classroom project focused on the computation of (4.4), when given a numerical table, with potential to be resized if other related numerical discussions are to be included.

5. A short assignment

The particular structure of a cubic spline given by equation (4.4) was proposed in a self contained worksheet to a classroom of a numerical methods short course attended by eight Civil Engineering students to be solved with the help of the software Octave [8], in its 4.0.0 version. We remark that the definition of the function $(x - x_i)_+^3$ demands a step function. The Octave's symbolic package has a built-in function Heaviside [9]. Let us recall that the Heaviside function is defined as

$$\text{Heaviside}(x) = \begin{cases} 1\,, & x > 0 \\ 0.5\,, & x = 0 \\ 0\,, & x < 0 \end{cases}.$$

From our standpoint we should use the syntax `heaviside(x,zero_value)` with `zero_value=1`, so that the value at $x = 0$ is changed to 1. Since the symbolic package does not come within the basic installation, a similar function can be easily defined and furnished or even proposed to students as a way of testing the Octave's `function` environment. A simple suggestion is issued in Table 2 and corresponds to the one applied on the development of the outlined tasks.

```
function[y] = step(x,a)
[r , c] = size(x);
y = zeros(1,c);
for i = 1 : r
    for j = 1 : c
      if ( x(i,j) >=a ) ==1;
      y(i,j) = 1;
      endif
    end
end
endfunction
```

TABLE 2. A step function

5.1. **Worksheet on Cubic Splines.** Let us consider the following set of five points.

x	x_0	x_1	x_2	x_3	x_4
y	y_0	y_1	y_2	y_3	y_4

1. Confirm carefully that the function $S(x)$ defined below has all the defining properties of a cubic spline interpolating the given five points.

$$S(x) = y_0 + d_1\,(x-x_0)_+^3 + d_2\,(x-x_1)_+^3 + d_3\,(x-x_2)_+^3 + d_4\,(x-x_3)_+^3\,, \quad (5.1)$$

or

$$S(x) = \begin{cases} y_0 + d_1\,(x-x_0)^3\,, & \text{if } x_0 \le x < x_1 \\ y_0 + d_1\,(x-x_0)^3 + d_2\,(x-x_1)^3\,, & \text{if } x_1 \le x < x_2 \\ y_0 + d_1\,(x-x_0)^3 + d_2\,(x-x_1)^3 + d_3\,(x-x_2)^3\,, & \text{if } x_2 \le x < x_3 \\ y_0 + d_1\,(x-x_0)^3 + d_2\,(x-x_1)^3 + d_3\,(x-x_2)^3 + d_4\,(x-x_3)^3\,, & \text{if } x_3 \le x \le x_4 \\ 0 & \text{otherwise} \end{cases}$$

2. Let us now consider the next data table.

x	1	2	3	5	6
y	7	4	5.5	40	82

Construct the cubic spline with the previous form (5.1) which interpolates the given points, by calculating the constants d_1, d_2, d_3 and d_4 therein involved. Elaborate further a script where the constants d_1, d_2, d_3 and d_4 are calculated and the function $S(x)$ is presented graphically together with the not-a-knot cubic spline given by the built-in function `spline`. You may use the provided function which performs a step behaviour called `step`.

5.2. **Discussion and Final Remarks.** A possible layout for a straightforward resolution of the proposed second task of subsection 5.1 can be read in Table 3.

```
clear all
x = [1 2 3 5 6];
y = [7 4 5.5 40 82];
d1 = (y(2)-y(1))/(x(2)-x(1))^3;
d2 = (y(3)-y(1)-d1*(x(3)-x(1))^3) /(x(3)-x(2))^3;
d3 = (y(4)-y(1)-d1(x(4)-x(1))^3-d2*(x(4)-x(2)) ^ 3)/(x(4)-x(3))^3;
d4 = (y(5)-y(1)-d1*(x(5)-x(1))^ 3-d2 *(x(5)-x(2))^ 3-d3 *(x(5)-x(3))^ 3)/(x(5)-x(4)) ^ 3;
xx = linspace(x(1),x(end));
yy = y(1)+d1 .*(xx-x(1)).^3.* step(xx,x(1))+d2.*(xx-x(2)) .^ 3 .*step(xx,x(2))
     + d3.*(xx-x(3)).^ 3 .* step(xx,x(3))+d4.*(xx-x(4)).^ 3 .* step(xx,x(4));
zz = spline (x, y, xx);
plot(xx,yy, x,y,'o', xx, zz, '- -')
grid on
```

TABLE 3. A possible script for item 2. of the proposed worksheet.

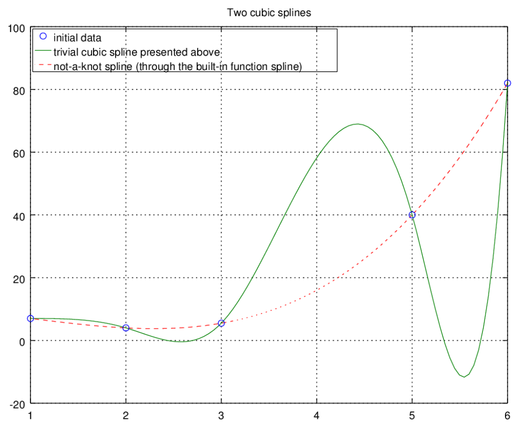

FIGURE 3. The not-a-knot cubic spline *versus* a trivial cubic spline of Table of item 2. of the proposed worksheet

The software Octave [8], as any other algebra manipulator, has built-in functions for many interpolation methods. In particular, the function `spline` implements by default a not-a-knot cubic spline based on the first described method of derivation whose main expression is given by (3.2). By means of a few instructions, a student can visualize the graph of a smooth piecewise polynomial fit.

The graphical comparison between both cubic spline functions is shown in Fig.3 and is the output of the latest steps of the script of Table 3. In other words, the very simple cubic structure chosen does not provide a suitable model, showing instability, whereas the more theoretically demanding option avoids unstable operations.

Other concepts strictly related to linear algebra can be explored, as for instance, the computational analysis of the stability of the system (4.3), when compared with the one defined in (3.3) with a tridiagonal matrix. With respect to the system solved in item 1. of subsection 5.1, by forward substitution, such analysis can also be done taking into consideration that it can be written as next.

$$\begin{bmatrix} (x_1-x_0)^3 & 0 & 0 & 0 \\ (x_2-x_0)^3 & (x_2-x_1)^3 & 0 & 0 \\ (x_3-x_0)^3 & (x_3-x_1)^3 & (x_3-x_2)^3 & 0 \\ (x_4-x_0)^3 & (x_4-x_1)^3 & (x_4-x_2)^3 & (x_4-x_3)^3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} y_1-y_0 \\ y_2-y_0 \\ y_3-y_0 \\ y_4-y_0 \end{bmatrix} \tag{5.2}$$

Moreover, there is some potential to explore different strategies for dealing with the particular algebra manipulator and its syntax.

The group of students reacted positively to this new orientation on the subject, having revealed during the evaluations a clearer notion of the structure of a cubic spline of a given numerical table. They have also applied cubic splines in estimating derivatives and integrals. In spite of the small number of students working under this approach, the work developed suggests that the introduction of a particular structure for the derivation of a cubic spline together with the more traditional scheme help students to give an extensive account of the characteristics of such mathematical models and to become familiar with different derivation proposals endowed with distinct numerical performances in a computational environment.

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A DIOPHANTINE PROBLEM ON BIQUADRATES REVISITED

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Abstract. In this paper we obtain a new parametric solution of the problem of finding two triads of biquadrates with equal sums and equal products.

This paper is a sequel to my earlier paper [2] concerning the problem of finding two triads of biquadrates with equal sums and equal products. The problem is mentioned in [6, p. 214], and thirty numerical solutions, obtained by computer search, were presented in [2]. Subsequently, a method of obtaining infinitely many numerical examples of such triads was given in [4, pp. 346–47]. In this paper we obtain a parametric solution of the problem. The triads of biquadrates are given by the following theorem.

Theorem: A parametric solution of the simultaneous diophantine equations,

$$X_1^4 + X_2^4 + X_3^4 = Y_1^4 + Y_2^4 + Y_3^4, \qquad X_1X_2X_3 = Y_1Y_2Y_3, \tag{1}$$

is given by

$$\begin{aligned} X_1 &= (2a-b)(a^2-ab+b^2)(5a^3-7a^2b+4ab^2-b^3),\\ X_2 &= (a^3-3a^2b+2ab^2-b^3)(5a^3-8a^2b+5ab^2-b^3),\\ X_3 &= (a^3-ab^2+b^3)(3a^2-3ab+b^2)b,\\ Y_1 &= (2a-b)(a^2-ab+b^2)(5a^3-8a^2b+5ab^2-b^3),\\ Y_2 &= (a^3-3a^2b+2ab^2-b^3)(3a^2-3ab+b^2)b,\\ Y_3 &= (a^3-ab^2+b^3)(5a^3-7a^2b+4ab^2-b^3), \end{aligned} \tag{2}$$

where a and b are arbitrary parameters.

Proof: To obtain a solution of the simultaneous equations (1), we will first obtain a parametric solution of the simultaneous diophantine equations,

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2, \tag{3}$$

$$x_1x_2x_3 = y_1y_2y_3, \tag{4}$$

and then choose the parameters such that $x_i,\ y_i,\ i = 1, 2, 3$, become perfect squares. The complete solution of Eqs. (3) and (4) is already known ([1], [7]).

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Since we need to ensure that $x_i,\ y_i,\ i = 1, 2, 3$ eventually become perfect squares, instead of using the solutions given in ([1], [7]), we will solve Eqs. (3) and (4) directly by writing,

$$\begin{aligned} x_1 &= (tu+1)p^2, & x_2 &= (u+1)q^2, & x_3 &= r^2, \\ y_1 &= (u+1)p^2, & y_2 &= q^2, & y_3 &= (tu+1)r^2, \end{aligned} \tag{5}$$

where p, q, r, t and u are arbitrary nonzero parameters.

With the values of $x_i,\ y_i,\ i = 1, 2, 3$ given by (5), Eq. (4) is identically satisfied while Eq. (3) reduces to a linear equation in u which is readily solved, and we thus obtain a parametric solution of the simultaneous equations (3) and (4). Since both these equations are homogeneous in the variables $x_i,\ y_i$, we obtain the following solution of Eqs. (3) and (4) after appropriate scaling:

$$\begin{aligned} x_1 &= \{(p^4-r^4)t^2 - 2(p^4-q^4)t + p^4 - q^4\}p^2k, \\ x_2 &= \{(p^4-r^4)t^2 - 2(p^4-r^4)t + p^4 - q^4\}q^2k, \\ x_3 &= \{(p^4-r^4)t^2 - (p^4-q^4)\}r^2k, \\ y_1 &= \{(p^4-r^4)t^2 - 2(p^4-r^4)t + p^4 - q^4\}p^2k, \\ y_2 &= \{(p^4-r^4)t^2 - (p^4-q^4)\}q^2k, \\ y_3 &= \{(p^4-r^4)t^2 - 2(p^4-q^4)t + p^4 - q^4\}r^2k, \end{aligned} \tag{6}$$

where p, q, r, t and k are arbitrary parameters.

To ensure that $x_i,\ y_i,\ i = 1, 2, 3$, are all perfect squares, we note that since the ratios y_1/x_2, y_2/x_3 and y_3/x_1 are perfect squares, it suffices to ensure that $x_i,\ i = 1, 2, 3$, are all perfect squares. We will first make the products x_1x_3 and x_2x_3 simultaneously perfect squares by choice of the parameters p, q, r and then choose k suitably such that x_1, x_2 and x_3 become perfect squares simultaneously.

Now x_1x_3 and x_2x_3 may be considered as two quartic polynomials in t with coefficients in terms of the parameters p, q, r, and we need to choose a suitable value of t such that both of these quartic polynomials become perfect squares. The problem of making two quartic polynomials perfect squares simultaneously is a very difficult diophantine problem and there is no general method of finding a solution. Here we will solve the problem by finding different values of t that make each quartic a perfect square separately, and we will equate two such values of t to obtain a condition that must be satisfied by the parameters p, q, r. Finally, to obtain a solution, we will find suitable values of p, q, r that satisfy the condition just obtained.

We note that the coefficient of t^4 in both of our quartic polynomials are perfect squares. Accordingly, we may divide each quartic polynomial by the respective coefficient of t^4 to obtain two monic quartic polynomials both of which must be made perfect squares simultaneously. Thus, we must find a common value of t that satisfies both of the following two diophantine equations for suitably chosen values of p, q and r:

$$\xi_1^2 = t^4 - 2(p^4-q^4)t^3/(p^4-r^4) + 2(p^4-q^4)^2 t/(p^4-r^4)^2 - (p^4-q^4)^2/(p^4-r^4)^2, \quad (7)$$

$$\xi_2^2 = t^4 - 2t^3 + 2(p^4-q^4)t/(p^4-r^4) - (p^4-q^4)^2/(p^4-r^4)^2. \quad (8)$$

A method of obtaining values of t that make a monic quartic polynomial of t a perfect square has been given by Fermat [5, p. 639]. This method, however, leads to very cumbersome solutions of (7) and (8). To obtain relatively simpler solutions of (7) and (8), we will apply a theorem given in [3, Theorem 4.1, p. 789-90] by which we can use two known solutions of a quartic equation of type (7) to obtain a new solution.

It is readily verified that two solutions of Eq. (7) are given by

$$(t,\,\xi_1) = \big(1,\ (q^4-r^4)/(p^4-r^4)\big) \quad \text{and}$$

$$(t,\,\xi_1) = \big((p^2+q^2)/(p^2+r^2),\ 2pq(p^2+q^2)(q^2-r^2)/\{(p^2-r^2)(p^2+r^2)^2\}\big),$$

and on applying the aforementioned theorem, we get the following value of t that provides a solution of Eq. (7):

$$\frac{(p^2+q^2)(p^4+2p^3q-2p^2r^2-2pq^3-4pqr^2+q^4+2q^2r^2+2r^4)}{(p^2-r^2)(p^4+2p^3q-2p^2r^2+2pq^3+q^4-2q^2r^2-2r^4)}. \quad (9)$$

Similarly, two solutions of Eq. (8) are given by

$$(t,\,\xi_2) = \big((p^4-q^4)/(p^4-r^4),\ -(p^4-q^4)(q^4-r^4)/(p^4-r^4)^2\big)$$

and $(t,\,\xi_2) = \big((p^2-q^2)/(p^2-r^2),\ 2pr(p^2-q^2)(q^2-r^2)/\{(p^2-r^2)^2(p^2+r^2)\}\big)$,

and using these solutions, we get the following value of t that provides a solution of Eq. (8):

$$\frac{(p^2+q^2)(p^4-2p^3r-2p^2q^2+4pq^2r+2pr^3+2q^4+2q^2r^2+r^4)}{(p^2-r^2)(p^4-2p^3r-2p^2q^2-2pr^3-2q^4-2q^2r^2+r^4)}. \quad (10)$$

On equating the two values of t given by (9) and (10), we get the following condition:

$$(q+r)(p^2+q^2)(q^2+r^2)(p^2+pq-pr+q^2-qr+r^2) \times (p^3-pq^2+pqr-pr^2+q^3-r^3) = 0. \quad (11)$$

On equating to 0 any of the factors on the left-hand side of Eq. (11), except the last, we get a trivial result. When, however, we equate to 0 the last factor, we get a cubic equation in p, q, r which represents a curve of genus 0 in the projective plane with a singularity at $(-1, -1, 1)$. On drawing an arbitrary straight line through the singular point and taking its intersection with the cubic curve, we get a rational point which immediately yields the following solution for p, q, r:

$$p = 2a^3-3a^2b+3ab^2-b^3,\quad q = -a^3+3a^2b-2ab^2+b^3,\quad r = a^3-ab^2+b^3, \quad (12)$$

where a and b are arbitrary parameters. With these values of p, q, r, the two values of t given by (9) and (10) coincide and are given by

$$-(5a^4 - 8a^3b + 8a^2b^2 - 4ab^3 + b^4)/\{a(a-b)(5a^2 - 5ab + 2b^2)\}. \qquad (13)$$

On substituting the values of p, q, r given by (12) and the value of t given by (13) in (6) and choosing k suitably, we obtain a parametric solution of Eqs. (3) and (4) in which x_i, y_i, $i = 1, 2, 3$, are all perfect squares. This immediately gives the solution (2) of the simultaneous equations (1) stated in the theorem.

As a numerical example, when $a = 1$, $b = -1$, we get the two triads of biquadrates $\{7^4, 133^4, 153^4\}$ and $\{17^4, 49^4, 171^4\}$ with equal sums and equal products.

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SOME FACTS ABOUT FIBONACCI NUMBERS, LUCAS NUMBERS AND GOLDEN RATIO

M. NARAYAN MURTY

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Abstract. In this article, some facts about Fibonacci numbers, Lucas numbers and golden ratio are presented. Fibonacci numbers and Fibonacci curve occur in nature. Lucas numbers are related to Fibonacci numbers and both sequences are related. Electrical energy can be produced with more efficiency using solar panels when arranged in Fibonacci sequence. The golden ratio comes from Fibonacci numbers. The concept of golden ratio finds application in architecture.

1. Introduction

The sequence of numbers introduced by famous Italian mathematician Leonardo Pisano (1170 -1250) is known as Fibonacci sequence. Fibonacci was his nickname. He was born in Pisa, Italy. Fibonacci sequence first appeared in his book 'Liber Abaci' in the year 1202. Fibonacci day is observed on November 23, as it has digits '$1, 1, 2, 3'$, which is a part of Fibonacci sequence. Fibonacci introduced the sequence of numbers by considering the growth of an idealized rabbit population, shown in Figure 1. Starting with a male and female rabbit, he has calculated the number of pairs of rabbits born in a year as per the following steps.

- Begin with one male rabbit and one female rabbit that have just been born. After one month, this pair is not at sexual maturity and cant mate.
- Let the rabbits reach sexual maturity after one month. So, at the end of two months, the rabbits have mated after reaching sexual maturity but not yet given birth. That is, at the end of two months, there is only one pair of rabbits.
- Let the gestation period of a female rabbit is one month. At the end of three months, the female rabbit gave birth to another pair. Now there are two pairs of rabbits.
- At the end of four months, the original pair gives birth to another pair. The second pair reaches sexual maturity and mates, but does not give

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birth. That is, there are three pairs of rabbits at the end of four months. Continuing the above steps, there will be 144 pairs of rabbits at the end of one year. The pairs of rabbits at the end of every month form the Fibonacci sequence.

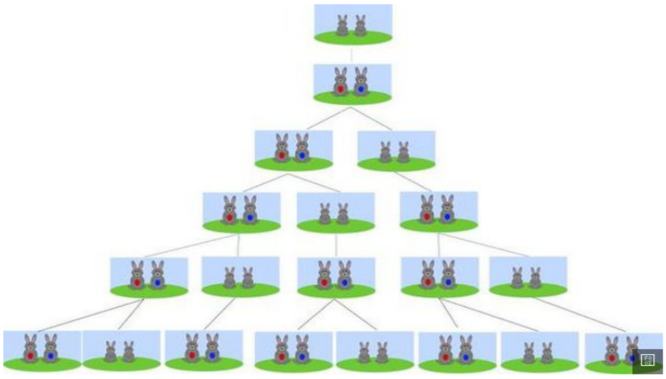


Figure 1. Growth of rabbit population

The Fibonacci sequence is given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots \tag{1.1}$$

Each term in the sequence after the first two is the sum of the immediately preceeding two terms. The n^{th} term, denoted by F_n, is called the n^{th} Fibonacci number. Note that $F_3 = F_2 + F_1 = 1 + 1 = 2$; $F_6 = F_5 + F_4 = 5 + 3 = 8$. In general we can write

$$F_n = F_{n-1} + F_{n-2}, \quad (n \geq 3), \tag{1.2}$$

which is known as the recurrence relation for the Fibonacci sequence. We observe the following properties of the Fibonacci sequence.

(1) Every third number, $(2, 8, 34, 144, 610, ...)$, is a multiple of 2. Every fourth number, $(3, 21, 144, 987, ...)$, is a multiple of 3. Also, every fifth number, $(5, 55, 610, 6765, ...)$, is a multiple of 5 and so on. These multiples, $2, 3, 5, ...$, again form Fibonacci sequence.

(2) Fibonacci numbers grow rapidly. Their order in the sequence indicates that

$$F_{5n+2} > 10^n \quad \text{for} \quad n \geq 1. \tag{1.3}$$

Thus, $F_7 > 10, \quad F_{12} > 100, \quad F_{17} > 1000, \cdots$

(3) Any positive integer can be written as a sum of Fibonacci numbers. For example, $4 = 1+1+2$ or $4 = 2+2$, $6 = 1+2+3$ or $6 = 3+3$, $7 = 2+5$, $9 = 1+8$.

There is another sequence of natural numbers, called Lucas numbers. This sequence was introduced by Edouard Lucas (1842-1891). He was a French mathematician who taught at the Lycee Charlemagne (a secondary school) in Paris. The sequence of Lucas numbers is given by

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \cdots \tag{1.4}$$

In the above sequence, from the third term onwards, each term is the sum of its immediate two predecessors as in Fibonacci sequence (1.1). Denoting the n^{th} term, called the n^{th} Lucas number, by L_n, from (1.4) we have its recurrence relation given by

$$L_n = L_{n-1} + L_{n-2}, \quad (n \geq 3). \tag{1.5}$$

Thus, for example, $L_7 = L_6 + L_5 = 18 + 11 = 29$. The Lucas numbers L_4, L_8, L_{16}, L_{32}, ..all have 7 as the final digit.

Now to define golden ratio, consider the quadratic equation $x^2 - x - 1 = 0$. The roots of this equation are $\phi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$. The constant ϕ, having approximate value 1.618, is known as golden ratio. In the twentieth century, American mathematician, Mark Barr, began using the Greek letter ϕ to represent the golden ratio.

The rest of the article is organized as follows. Some facts about Fibonacci numbers are given in Section 2 and some facts about Lucas numbers are described in Section 3. The Section 4 deals golden ratio.

2. Some facts about Fibonacci numbers

2.1. **Terms below zero.** In modern usage, 0 is taken as the initial number of Fibonacci sequence. This sequence also works below zero as shown in the following Table 1. The sequence below zero has the same numbers as the sequence above zero, except they follow + - + - + -....pattern.

Table 1: Fibonacci numbers below zero.

n	...	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	...
F_n	...	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	...

It can be written that

$$F_{-n} = (-1)^{n+1} F_n \tag{2.1}$$

Any number below zero is the sum of two integers immediately following it in the sequence. For example, $2 = -3 + 5$ and $-3 = 5 + (-8)$.

2.2. **Fibonacci primes.** A prime number which occurs in Fibonacci sequence is a Fibonacci prime. The first few such primes in the sequence are $2, 3, 5, 13, 89, 233$, 1597.... Till date Fibonacci primes with thousands of digits have been found.

2.3. **Generating function.** The generating function $g(x)$ of Fibonacci sequence is a power series given by

$$g(x) = \sum_{n \geq 1} F_n x^n. \tag{2.2}$$

Taking $x = 0.1$, this gives
$g(0.1) = \sum_{n \geq 1} F_n (0.1)^n = 0.1 + 0.01 + 0.002 + 0.0003 + 0.00005 + 0.000008 + 0.0000013 + 0.000000021 + ... = 0.11235955... = 10/89$. Hence

$$\sum_{n \geq 1} (F_n / 10^{n+1}) = 1/89. \tag{2.3}$$

2.4. **Fibonacci curve.** Now we shall draw the Fibonacci curve. Draw a square of 1×1 unit. Then draw a square with same dimensions above this square. Now draw a 2×2 unit square to the left of 2×1 rectangle. Then draw a 3×3 square below the 3×2 rectangle. Now draw a 5×5 square to the right of 5×3 rectangle. Proceeding in this manner in the anticlockwise direction, we get squares of the order $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ etc., which is the Fibonacci sequence. Now, if we draw a curve satisfying the equation of logarithmic spiral in polar coordinates starting from the left bottom corner of the first 1×1 square, passing through the corners of other squares in anticlockwise direction, we get Fibonacci curve or spiral as shown in Figure 2.

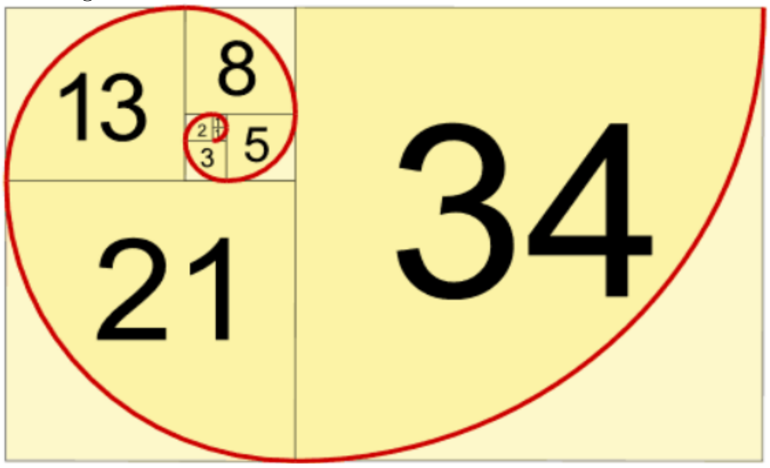


Figure 2. Fibonacci curve

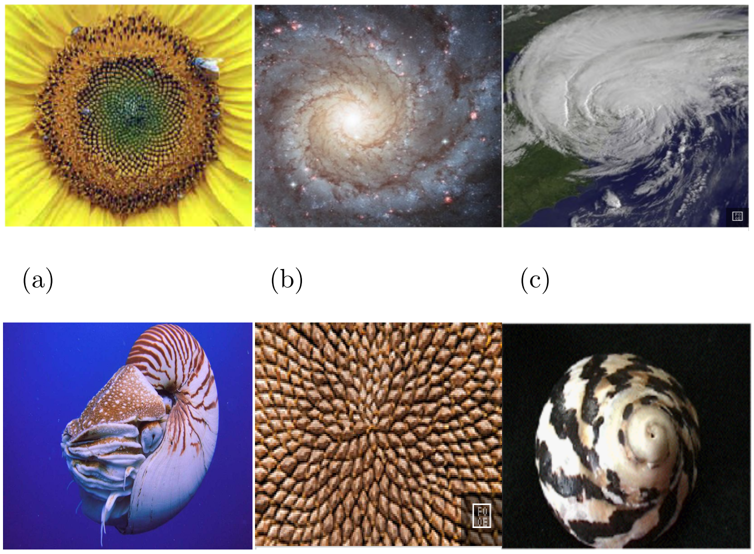
2.5. **Fibonacci numbers and spiral in nature.** The numbers of petals in many flowers are equal to Fibonacci numbers. A large number of common flowers have $1, 3, 5, 8, 13, 21$ petals. Some giant sunflowers have $34, 55$ or 89 petals Fibonacci spiral occur in nature. For instance, sunflower seeds are arranged in Fibonacci spiral and the fruitlets of a pine apple are arranged in a spiral, same as the Fibonacci spiral, about the trunk of the pine apple. Some other examples of Fibonacci spiral are chambered nautilus of deep sea, hurricanes, galaxies and shells as shown in Figure 3 on the next page.

2.6. **Identities involving Fibonacci numbers.** There are a number of identities involving Fibonacci numbers. Ten identities are mentioned below.

$$\sum\nolimits_{1 \leq n} F_k = F_{n+2} - 1, \tag{2.4}$$

$$\sum\nolimits_{1 \leq n} F_{2k} = F_{2n+1} - 1, \tag{2.5}$$

$$\sum\nolimits_{0 \leq n} F_{2k+1} = F_{2n+2}, \tag{2.6}$$



(a) (b) (c)

(d) (e) (f)

Figure 3. Fibonacci spiral in nature
(a) Sunflower seeds (b) Spiral galaxy (c) Hurricane
(d) Nautilus (e) Fruit lets of pine apple (f) Shell.

$$\sum_{1\leq n} F_{3k} = (F_{3n+2} - 1)/2, \tag{2.7}$$

$$F_{m+n} = F_{m+1}F_n + F_m F_{n-1}, \tag{2.8}$$

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \tag{2.9}$$

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \quad (n \geq 2) \tag{2.10}$$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{2.11}$$

$$F_{2k+1}F_{2k-1} - 1 = F_{2k}^2. \tag{2.12}$$

The last identity (2.12) can be described geometrically as per the following example. Taking $k = 3$, we obtain $F_{2k} = F_6 = 8$, $F_{2k+1} = F_7 = 13$ and $F_{2k-1} = F_5 = 5$. A square is shown in Figure 4 with it side equal to the Fibonacci number $F_{2k} = 8$. The area of this square is $F_{2k}^2 = 64$. This Square is partitioned into two right angled triangles (A and B) of equal area and two quadrilaterals (C and D) of equal area. A rectangle with Fibonacci numbers $F_{2k+1} = 13$ and $F_{2k-1} = 5$ as its sides is shown in Figure 5. The area of this rectangle is $F_{2k+1}F_{2k-1} = 65$. Within this rectangle, the geometrical figures A, B, C and D along with one parallelogram E of unit area can be adjusted. Because, according to the identity (2.12), $65 - 1 = 64$. Hence, this identity indicates that the area of the rectangle minus the area of the parallelogram is equal to the area of the square.

2.7. **Expression for finding Fibonacci numbers.** Fibonacci numbers can be found using the expression

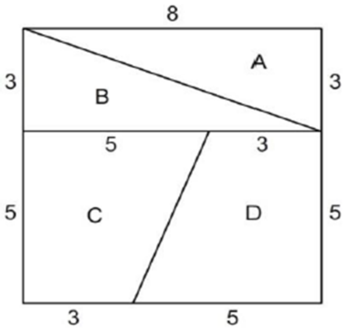


Figure4. Square with side $F_{2k} = 8$.

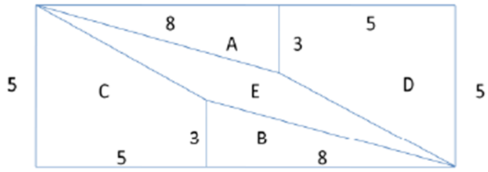


Figure5. Rectangle with length $F_{2k+1} = 13$ and breadth $F_{2k-1} = 5$.

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{2}{1+\sqrt{5}}\right)^n cosn\pi\right\}. \tag{2.13}$$

For example, since $cos3\pi = -1$, we get $F_3 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{2}{1+\sqrt{5}}\right)^3(-1)\right\} = 2$, as can be easily shown.

2.8. **Pascal's triangle.** This triangle is named after French mathematician Blaise Pascal (1623 -1662). The description about the triangle is found in his mathematical treatise Traite du triangle arithmatique (1653). Pascal's triangle is formed by the binomial coefficients in binomial expansions. If we expand algebraic expression $(x+y)^n$ for $n = 1, 2, 3, \cdots$, etc., we obtain

$$(x+y)^1 = 1x + 1y,$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2,$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3,$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4,$$

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5,$$

$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6,$$

$$(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7,$$

....

Taking 1 at the top and arranging the binomial coefficients in RHS of the above expressions in a triangular pattern, we obtain the Pascal's triangle, as shown Figure 6. Diagonals are drawn across the Pascal's triangle. The first Fibonacci number 1 is along the first diagonal. The second number 1 is along the second diagonal. The sum of the integers along the third diagonal is equal to 2, third Fibonacci number. The sum of the integers along the fourth diagonal is equal to 3, the fourth Fibonacci number. Hence in general, the sum of the entries in the n^{th} diagonal of the Pascal's triangle is equal to n^{th} Fibonacci number.

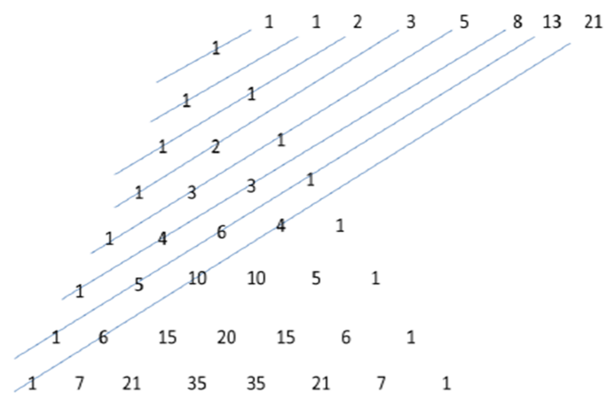


Figure 6. Pascal's triangle

2.9. **Applications.**

- Recent study by 13 years old Aiden Dwyer from New York shows that if the solar panels, following Fibonacci sequence, are arranged in the form of a tree, then the efficiency is improved very much. This artificial tree contains solar panels instead of leaves. This Fibonacci tree design performed better than flat-solar-panel model. It produced 20 percent more electricity and collected 2.5 more hours of sunlight during the day compared to flat-panel model. More interestingly, this tree design made 50 percent more electricity and collection time of sunlight was up to 50 percent longer in comparison to flat-panel model during winter. So, the distribution of solar panels according to Fibonacci sequence improves energy generation.
- Fibonacci numbers are used in the computational runtime analysis of Euclid's algorithm to determine the greatest common divisor of two integers.
- Things can be packed together tightly using Fibonacci sequence as shown in Figure 2.

3. Some facts about Lucas numbers

3.1. **Few identities satisfied by Lucas numbers.**

$$\sum_{1\leq n} L_k = L_{n+2} - 3, \tag{3.1}$$

$$\sum\nolimits_{1\leq n} L_{2k} = L_{2n+1} - 1, \tag{3.2}$$

$$\sum\nolimits_{0\leq n-1} L_{2k+1} = L_{2n} - 2, \tag{3.3}$$

$$L_{2n} = L_n^2 - 2(-1)^n, \tag{3.4}$$

$$L_{2n+1} = L_n L_{n+1} + (-1)^{n+1}, \tag{3.5}$$

$$L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n+1}, \;\; (n \geq 2) \tag{3.6}$$

$$L_{m+n} + (-1)^n L_{m-n} = L_m L_n. \tag{3.7}$$

$$L_n^2 = L_{2n} + 2(-1)^n. \tag{3.8}$$

3.2. **Some relations between Lucas numbers and Fibonacci numbers.** Lucas numbers are related to Fibonacci numbers. Some relations between Lucas numbers and Fibonacci numbers are given below.

$$L_n = F_{n+1} + F_{n-1} = F_n + 2F_{n-1}, \; (n \geq 2) \tag{3.9}$$

$$L_{n+1} + L_{n-1} = 5F_n, \; (n \geq 2) \tag{3.10}$$

$$F_{2n} = F_n L_n, \; (n \geq 1) \tag{3.11}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n, \; (n \geq 1) \tag{3.12}$$

$$2L_{m+n} = L_m L_n + 5F_m F_n, \; (m \geq 1, n \geq 1) \tag{3.13}$$

$$2F_{m+n} = F_m L_n + F_n L_m, \; (m \geq 1, n \geq 1) \tag{3.14}$$

$$L_n = F_{n+2} - F_{n-2}, \; (n \geq 3) \tag{3.15}$$

$$L_n^2 = F_n^2 + 4F_{n+1}F_{n-1}, \; (n \geq 2) \tag{3.16}$$

$$L_{2n+1} = 5F_n F_{n+1} + (-1)^n, \; (n \geq 1) \tag{3.17}$$

$$L_n^2 - F_n^2 = 4F_{n-1}F_{n+1}, \; (n \geq 2) \tag{3.18}$$

4. Golden ratio

4.1. **Relation between Fibonacci numbers and golden ratio.** We know the roots of the quadratic equation $x^2 = x+1$ are $\phi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$, where the constant ϕ, having approximate value 1.618, is the golden ratio. We can write the Fibonacci numbers in terms of golden ratio. In fact, we observe that $F_1 = 1 = \frac{\phi-\psi}{\phi-\psi}$, $F_2 = 1 = (\phi+\psi) = \frac{\phi^2-\psi^2}{\phi-\psi}$, and similarly, the n^{th} Fibonacci number can be expressed as

$$F_n = (\phi^n - \psi^n)/(\phi - \psi). \tag{4.1}$$

Since (i) $\psi = 1-\phi$, (ii) $\phi - \psi = \sqrt{5}$ and using the value of $\phi = 1.61803398\ldots$ in ψ, one gets (iii) $\psi = 1-\phi = -0.61803398\ldots \approx -\frac{1}{\phi}$, it follows that

$$F_n = (\phi^n - (1-\phi)^n)/\sqrt{5} = (\phi^n - (-\phi)^{-n})/\sqrt{5}. \tag{4.2}$$

Using this any Fibonacci number can be calculated from the golden ratio. For example, for $n = 6$, one gets $F_6 = ((1.618034)^6 - (-0.618034)^6)/\sqrt{5} = 8.000001073 \approx 8$. Any two successive Fibonacci numbers have a ratio close to the golden ratio.

The larger the pair of Fibonacci numbers, the closer the approximation as can be seen in the following Table 2. It is indeed the fact that $\lim_{n\to\infty}(F_n/F_{n-1}) = \phi$.

Table 2: Ratios of successive Fibonacci numbers

F_{n-1}	F_n	F_n/F_{n-1}
$F_3 = 2$	$F_4 = 3$	1.5
$F_4 = 3$	$F_5 = 5$	$1.6666\ldots$
$F_5 = 5$	$F_6 = 8$	1.6
$F_6 = 8$	$F_7 = 13$	1.625
...	...	...
$F_{12} = 144$	$F_{13} = 233$	$1.61805555\ldots$
$F_{13} = 233$	$F_{14} = 377$	$1.61802575\ldots$
$F_{14} = 377$	$F_{15} = 610$	$1.61803713\ldots$
...	...	...
$F_{49} = 7778742049$	$F_{50} = 12586269025$	$1.61803398\ldots$

4.2. **Golden ratio as infinite continued fraction.** Consider the infinite continued fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \tag{4.3}$$

It is clear that x satisfies the equation $x = 1 + 1/x$, i.e., $x^2 - x - 1 = 0$, the solutions of which are $(1 \pm \sqrt{5})/2$. Since $x > 1$, the solution is the golden ratio ϕ. Thus, $\phi = x$, the infinite continued fraction we started with.

4.3. **Golden ratio from golden Section.** Geometrically, the golden ratio can be defined by considering the golden section (Figure 7). If a (straight) line segment is divided into two parts, such that the ratio of the length of the whole segment to the larger part is equal to the ratio of the larger part to the smaller part, then that division of the line segment is called *golden section.* The golden ratio is equal to the reciprocal of the larger part of the line segment. Consider a line segment AB of unit length (Figure 7) and a point C on it with AC=y, say, as larger part. If the point C divides AB in the golden section, then AB:AC = AC:BC, i.e. $1 : y = y : 1 - y$. This gives $y^2 + y - 1 = 0$, the solutions of which are $y = (-1 \pm \sqrt{5})/2$. Since y is positive, $y = (-1 + \sqrt{5})/2 = 0.61803398\ldots$, and hence the golden ratio ϕ is the reciprocal of y, i.e. $\phi = 1/y = 1.61803\ldots$.

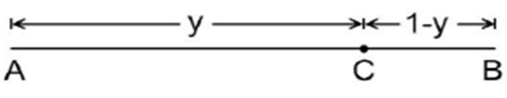


Figure 7. Golden section

4.4. **Golden rectangle.** A rectangle whose length and breadth are in the ratio $\phi : 1$ is known as the golden rectangle. For any positive x, the rectangle with length ϕx and breadth x is clearly a golden rectangle. Given a golden rectangle (see Figure 8), if a square A of side x is cut off from it, the length and breadth of a new rectangle B formed will be respectively x and $(\phi x - x) = x(\phi - 1) = x/\phi$, as $1 - \phi \approx -(1/\phi)$. It follows that B too is a Golden rectangle. Within this golden rectangle B, another smaller golden rectangle of length $(\phi x - x) = x/\phi$ and breadth

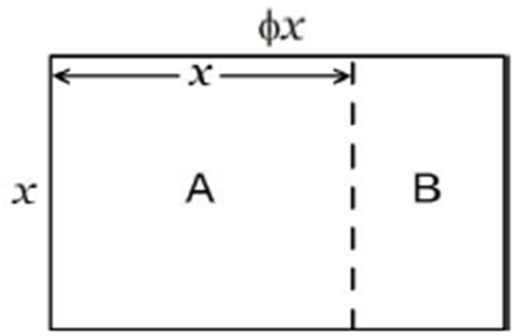


Figure 8. Golden rectangle

$(x - x/\phi) = (\phi - 1)x/\phi = x/(\phi)^2$ can be constructed. In this way, starting with a golden rectangle, we can obtain large number of smaller and smaller golden rectangles within the given golden rectangle whose areas gradually tend to zero. Observe that the rectangles of dimensions $55 \times 34, 34 \times 21, 21 \times 13, 13 \times 8, 8 \times 5, 5 \times 3$ and 3×2 shown in Figure 2 are golden rectangles.

4.5. **Powers of golden ratio.** An interesting pattern will arise when the golden ratio is raised to consecutive powers. In fact, $\phi = 1.618$.
$\phi^2 = 2.618 = 1 + \phi$
$\phi^3 = \phi\phi^2 = \phi(1 + \phi) = \phi + \phi^2 = \phi + (1 + \phi) = 1 + 2\phi$
$\phi^4 = \phi.\phi^3 = \phi(1 + 2\phi) = \phi + 2\phi^2 = \phi + 2(1 + \phi) = 2 + 3\phi,$
$\phi^5 = \phi.\phi^4 = \phi(2 + 3\phi) = 2\phi + 3\phi^2 = 2\phi + 3(1 + \phi) = 3 + 5\phi,$
$\phi^6 = \phi.\phi^5 = \phi(3 + 5\phi) = 3\phi + 5\phi^2 = 3\phi + 5(1 + \phi) = 5 + 8\phi.$

Observe that the powers of golden ratio follow the Fibonacci numbers. In general, it can be shown that

$$\phi^n = F_{n-1} + F_n\phi. \tag{4.4}$$

4.6. **Golden ratio in a regular pentagon.** The ratio of the diagonal of a regular pentagon (Figure 9 on the next page) to its side is equal to the golden ratio ϕ; that is, $(AC/BC) = \phi$.

4.7. **Golden ratio in architecture.** It has been observed that the Greeks have

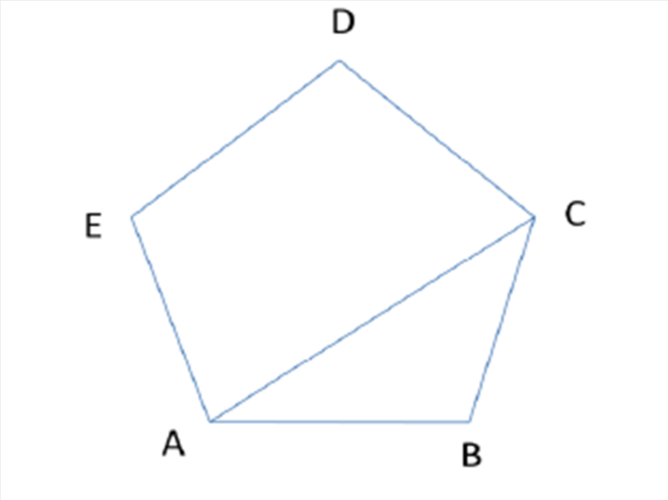


Figure 9. Regular Pentagon

used the concept of golden ratio in the construction of temples. The Parthenon in Athens is the classic example of the golden ratio being used in architecture. It was constructed between 448-432 BC as a temple for the Goddess Athena. This structure contains a multitude of golden rectangles. The great pyramid in Giza, Egypt, is another example of an ancient structure where golden ratio is used in its design. The ratio of the height of its triangular face to half of the side of its square base approximates to golden ratio ϕ.

4.8. **Illustrations of golden ratio in human body.** According to the book by Samuel Obara [6], the following ratios in the human body are very close to golden ratio.

- Height of a person to the distance between the naval point and the foot.
- Separation between the pupils of the eyes to the average length between the eyebrows.
- Width of the nose to the length between the nostrils.
- Length of the shoulder line to the length of the head.
- Distance from the ground to the naval point to the distance from the ground to the knee.
- Length of the hand to the distance from wrist to the elbow.

If the above ratios in human body are close to golden ratio, the body appears beautiful.

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A UNIFYING MULTIPLICATION ON CARTESIAN PRODUCT OF BANACH ALGEBRAS*

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Abstract. Let $\mathcal{C}$ and $\mathcal{B}$ be Banach algebras, $\mathcal{A}$ be a closed subalgebra of $\mathcal{C}$, $T : \mathcal{B} \to \mathcal{C}$ be an algebra homomorphism with $\|T\| \leq 1$, and let $P : \mathcal{C} \to \mathcal{A}$ be linear with $\|P\| \leq 1$. We define a Banach algebra product on the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, which we shall denote by $\mathcal{A} \times_T^P \mathcal{B}$. This product simultaneously generalizes several multiplications on $\mathcal{A} \times \mathcal{B}$ like θ-Lau product, T- product, semidirect product, generalized semidirect product and the pointwise product. When $\mathcal{A} \times_T^P \mathcal{B}$ is a commutative Banach algebra, we characterize the Gelfand space and multipliers.

1. Introduction

Given commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$, the Cartesian product space $\mathcal{A} \times \mathcal{B}$ can be endowed with several multiplications making it a linear associative algebra with suitable norms. These include the standard coordinatewise product, θ- Lau product [7, 8], defined with the help of a multiplicative linear functional θ on $\mathcal{B}$, the semidirect product and its generalization [1, 2, 3, 10], a product recently introduced in [9]. There is also a product introduced in [4] that eventually turns out to be isomorphic to the pointwise product [5]. Computing Gelfand spaces and multipliers of a Banach algebra have been an important aspect of Banach algebra theory. For the product considered above, some aspects of these have been discussed in [6, 8, 9].

In the present note, given linear associative algebras $\mathcal{A}$ and $\mathcal{B}$, we consider a fairly general framework to define an associative product on $\mathcal{A}\times\mathcal{B}$ incorporating all these different multiplications; and when $\mathcal{A} \times \mathcal{B}$ is a commutative Banach algebra with this multiplication, we compute the Gelfand space and multipliers.

2. Generalized Product $\mathcal{A} \times_T^P \mathcal{B}$

Let $\mathcal{C}$ and $\mathcal{B}$ be linear associative algebras, and let $\mathcal{A}$ be a subalgebra of $\mathcal{C}$. Let $T : \mathcal{B} \to \mathcal{C}$ be an algebra homomorphism, and $P : \mathcal{C} \to \mathcal{A}$ be a linear map. For

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$(a,b),(c,d)\in\mathcal{A}\times\mathcal{B}$, we introduce the multiplication

$$(a,b)(c,d) \quad = \quad (P(ac+T(b)c+aT(d)),bd). \tag{2.1}$$

Lemma 2.1. *The product defined by equation (2.1) on $\mathcal{A}\times\mathcal{B}$ is associative if and only if the following equations hold for all $a,c,e\in\mathcal{A}$ and $b,d,f\in\mathcal{B}$.*

(i) $P(aP(ce))=P(P(ac)e)$ *(v)* $P(T(b)P(cT(f)))=P(P(T(b)c)T(f))$

(ii) $P(aP(T(d)e))=P(P(aT(d))e)$ *(vi)* $P(T(b)P(T(d)e))=P(T(b)T(d)e)$

(iii) $P(aP(cT(f)))=P(P(ac)T(f))$ *(vii)* $P(P(aT(d))T(f))=P(aT(d)T(f))$

(iv) $P(T(b)P(ce))=P(P(T(b)c)e)$

Proof. Let $(a,b),(c,d),(e,f)\in\mathcal{A}\times\mathcal{B}$. Then $(a,b)((c,d)(e,f))$

$$\begin{aligned}
&= (a,b)(P(ce+T(d)e+cT(f)),df)\\
&= (P(aP(ce+T(d)e+cT(f))+aT(df)+T(b)P(ce+T(d)e+cT(f)),bdf)\\
&= \begin{array}{l}(P(aP(ce))+aP(T(d)e)+aP(cT(f))+aT(df)+T(b)P(ce)\\ \qquad +T(b)P(T(d)e)+T(b)P(cT(f)),bdf),\end{array}\\
&= \begin{array}{l}(P(aP(ce))+P(aP(T(d)e))+P(aP(cT(f)))+P(aT(df))\\ +P(T(b)P(ce))+P(T(b)P(T(d)e))+P(T(b)P(cT(f))),bdf).\end{array}
\end{aligned} \tag{2.2}$$

Also, $((a,b)(c,d))(e,f)$

$$\begin{aligned}
&= (P(ac+T(b)c+aT(d)),bd)(e,f)\\
&= (P(P(ac+T(b)c+aT(d))e+P(ac+T(b)c+aT(d))T(f)+T(bd)e),bdf)\\
&= \begin{array}{l}(P(P(ac)e)+P(T(b)c)e+P(aT(d))e+P(ac)T(f)\\ \quad +P(T(b)c)T(f)+P(aT(d))T(f)+T(bd)e),bdf)\end{array}\\
&= \begin{array}{l}(P(P(ac)e)+P(P(T(b)c)e)+P(P(aT(d))e)+P(P(ac)T(f))\\ \quad +P(P(T(b)c)T(f))+P(P(aT(d))T(f))+P(T(bd)e),bdf).\end{array}
\end{aligned} \tag{2.3}$$

It follows that the product is associative if and only if (i)-(vii) hold. □

We shall denote the multiplication in (2.1) by $\times_T^P$, and $\mathcal{A}\times\mathcal{B}$ by $\mathcal{A}\times_T^P\mathcal{B}$. Throughout we assume that the map $P:\mathcal{A}\to\mathcal{C}$ is such that $\mathcal{A}\times_T^P\mathcal{B}$ is associative.

Now, let $\mathcal{C}$ and $\mathcal{B}$ be a Banach algebras, $\mathcal{A}$ be a closed subalgebra of $\mathcal{C}$, $T:\mathcal{B}\to\mathcal{C}$ be an algebra homomorphism with $\|T\|\le 1$, and let $P:\mathcal{C}\to\mathcal{A}$ be a linear map with $\|P\|\le 1$. For $(a,b)\in\mathcal{A}\times_T^P\mathcal{B}$, let

$$\|(a,b)\|_1=\|a\|+\|b\|.$$

If $(a,b),(c,d)\in\mathcal{A}\times\mathcal{B}$, then

$$\begin{aligned}
\|(a,b)(c,d)\|_1=\|(P(ac+aTd+T(b)c),bd)\|_1 &= \|P(ac+aTd+T(b)c)\|+\|bd\|\\
&\le \|ac+aTd+T(b)c\|+\|bd\|\\
&\le \|a\|\|c\|+\|a\|\|d\|+\|b\|\|c\|+\|b\|\|d\|\\
&= (\|a\|+\|b\|)(\|c\|+\|d\|)\\
&= \|(a,b)\|_1\|(c,d)\|_1.
\end{aligned}$$

Therefore $\mathcal{A} \times_T^P \mathcal{B}$ is a Banach algebra.

We show below that certain standard multiplications considered in the literature are special cases of $\mathcal{A} \times_T^P \mathcal{B}$.

(1) T**- product** [4]**:** Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Let $P : \mathcal{A} \to \mathcal{A}$ be the identity map, i.e., $P(a) = a$ $(a \in \mathcal{A})$, and let $T : \mathcal{B} \to \mathcal{A}$ be a homomorphism with $\|T\| \leq 1$. If $(a, b), (c, d) \in \mathcal{A} \times \mathcal{B}$, then

$$(a, b)(c, d) = (P(ac + T(b)c + aT(d)), bd) = (ac + T(b)c + aT(d), bd).$$

It follows that $\mathcal{A} \times_T^P \mathcal{B} = \mathcal{A} \times_T \mathcal{B}$.

In particular, if T is a zero map, then we get the usual Cartesian product of $\mathcal{A}$ and $\mathcal{B}$. It should be noted that as recently shown in [5] that $\mathcal{A} \times_T \mathcal{B}$ is isomorphic to the Cartesian product space $\mathcal{A} \times \mathcal{B}$ via the map $(a, b) \mapsto (a + T(b), b)$ from $\mathcal{A} \times_T \mathcal{B}$ to $\mathcal{A} \times \mathcal{B}$.

(2) **Semidirect product** [1, 2, 3, 6, 10]**:** Suppose that $\mathcal{A}$ is a Banach algebra such that it can be written as a direct sum of a closed ideal I and a closed subalgebra $\mathcal{B}$, i.e., $\mathcal{A} = I \oplus \mathcal{B}$, and that $\|a + b\| = \|a\| + \|b\|$ for all $a + b \in \mathcal{A}$. Let $T : \mathcal{B} \to \mathcal{A}$ be an algebra homomorphism with $\|T\| \leq 1$, and let $P : \mathcal{A} \to I$ be the projection on I, i.e., $P(i + b) = i$ $(i + b \in \mathcal{A})$. If $(i, b), (j, d) \in I \oplus \mathcal{B} = \mathcal{A}$, then

$$(i, b)(j, d) = (P(ij + T(b)j + iT(d)), bd) = (ij + T(b)j + iT(d), bd).$$

In particular, if we take $T(b) = b$ $(b \in \mathcal{B})$, then we get the semidirect product. Therefore generalized semidirect product is a particular case of $\mathcal{I} \times_T^P \mathcal{B}$.

(3) θ**- Lau Product** [7]**:** Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebra, and let θ be a multiplicative linear functional on $\mathcal{B}$. For $(a, b), (c, d) \in \mathcal{A} \times \mathcal{B}$, let

$$(a, b)(c, d) = (ac + \theta(b)c + \theta(d)a, bd).$$

Then $\mathcal{A} \times \mathcal{B}$ is a Banach algebra with this multiplication, called the θ- Lau product, and the norm $\|(a, b)\| = \|a\| + \|b\|$ $((a, b) \in \mathcal{A} \times \mathcal{B})$. The Banach algebra $\mathcal{A} \times \mathcal{B}$ with this multiplication is denoted by $\mathcal{A} \times_\theta \mathcal{B}$.

Take $\mathcal{C} = \mathcal{A}_e$, the unitization of $\mathcal{A}$. Let $T : \mathcal{B} \to \mathcal{C}$ be $T(b) = \theta(b)e$ $(b \in \mathcal{B})$, where e is the identity in $\mathcal{C}$. Let $P : \mathcal{C} \to \mathcal{A}$ be $P(a + \lambda e) = a$ $(a + \lambda \in \mathcal{C})$.

If $(a, b), (c, d) \in \mathcal{A} \times \mathcal{B}$, then

$$\begin{aligned} (a, b)(c, d) &= (P(ac + T(b)c + aT(d)), bd) \\ &= (P(ac + \theta(b)c + a\theta(d)), bd) = (ac + \theta(b)c + \theta(d)a, bd). \end{aligned}$$

It follows that $\mathcal{A} \times_T^P \mathcal{B} = \mathcal{A} \times_\theta \mathcal{B}$.

This product was introduced by Lau [7] for certain class of Banach algebras and was extended by Sangani Monfared [8] for the general case.

(4) [9] Let $\mathcal{A}$ be a commutative Banach algebra. A bounded linear map $T : \mathcal{A} \to \mathcal{A}$ is a *multiplier* if $T(ab) = aT(b) = T(a)b$ $(a, b \in \mathcal{A})$. Let $M(\mathcal{A})$ be the collection of all multipliers on $\mathcal{A}$. Then $M(\mathcal{A})$ is a unital commutative Banach algebra with composition as multiplication and with the norm $\|T\| = \sup\{\|Ta\| : \|a\| \leq 1\}$.

Let $\mathcal{C} = \mathcal{A} \oplus M(\mathcal{A})$. Then $\mathcal{C}$ is a commutative Banach algebra with the product

$$(a,F)(b,G) = (ab + G(a) + F(b), FG) \quad ((a,F),(b,G) \in \mathcal{C})$$

and the norm

$$\|(a,F)\| = \|a\| + \|F\| \quad ((a,F) \in \mathcal{C}).$$

Then both $\mathcal{A}$ and $M(\mathcal{A})$ are closed subalgebras of $\mathcal{C}$. We shall identify element of $\mathcal{A}$ and element of $M(\mathcal{A})$ with elements of $\mathcal{C}$ by $a \leftrightarrow (a,0)$ and $F \leftrightarrow (0,F)$ respectively. Thus if $a \in \mathcal{A}$ and $F \in M(\mathcal{A})$, then $Fa = aF = F(a) = (F(a),0)$.

Let $\mathcal{B}$ be a Banach algebra, and let $T' : \mathcal{B} \to M(\mathcal{A})$ be a homomorphism with $\|T'\| \le 1$. Define $T : \mathcal{B} \to \mathcal{C}$ by $T(b) = (0, T'(b))$ $(b \in \mathcal{B})$. Then T is a homomorphism with $\|T\| = \|T'\| \le 1$. Let $P : \mathcal{C} \to \mathcal{A}$ be $P((a,F)) = a$ $((a,F) \in \mathcal{C})$. Then P is linear and $\|P\| \le 1$. If $(a,b),(c,d) \in \mathcal{A} \times \mathcal{B}$, then

$$\begin{aligned}(a,b)(c,d) = ((a,0),b)((c,0),d) &= (P((a,0)(c,0) + (a,0)T(d) + T(b)(c,0)), bd)\\ &= (P((ac,0) + (a,0)(0,T'(d)) + (0,T'(b))(c,0)), bd)\\ &= (P((ac,0) + (T'(d)(a),0) + (T'(b)(c),0)), bd)\\ &= (ac + T'(d)(a) + T'(b)c, bd).\end{aligned}$$

Thus in this case the product on $\mathcal{A} \times_T^P B$ is product defined in [9].

All these make the multiplication $\times_T^P$ on $\mathcal{A} \times \mathcal{B}$ exciting and interesting. The present note is aimed at describing the Gelfand space and multipliers of $\mathcal{A} \times_T^P \mathcal{B}$ in the commutative case. Notice that recently there has been considerable interest in special cases discussed above [1, 2, 4, 6, 7, 9].

A Banach algebra $\mathcal{A}$ is *without order* if $a = 0$ whenever $a \in \mathcal{A}$ and $a\mathcal{A} = \{0\}$. It is easy to see that θ- product (T-product, semidirect product) of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is without order if and only if both $\mathcal{A}$ and $\mathcal{B}$ are without order. The product $\times_T^P$ defined using P and T described below is not covered in the particular cases described above.

Example 2.2. *Let $\mathcal{A} = \mathcal{C} = C_b([0,1) \cup (1,2])$ = the set of all bounded continuous functions on $[0,1) \cup (1,2]$, $\mathcal{B} = C[0,2]$, both with the sup norm $\|\cdot\|_\infty$. Define $T : \mathcal{B} \to \mathcal{A}$ and $P : \mathcal{A} \to \mathcal{A}$ by $T(f) = f\chi_{[0,1)}$ for all $f \in \mathcal{B}$ and $P(f) = f\chi_{[0,1)}$ for all $f \in \mathcal{A}$, where $\chi_{[0,1)}$ is the characteristic function on $[0,1)$. If $(f_1,g_1),(f_2,g_2) \in \mathcal{A} \times_T^P \mathcal{B}$, then*

$$\begin{aligned}(f_1,g_1)(f_2,g_2) &= (P(f_1f_2 + T(g_1)f_2 + f_1T(g_2)), g_1g_2)\\ &= (P(f_1f_2 + g_1\chi_{[0,1)}f_2 + f_1g_2\chi_{[0,1)}), g_1g_2)\\ &= (f_1f_2\chi_{[0,1)} + g_1f_2\chi_{[0,1)} + f_1g_2\chi_{[0,1)}, g_1g_1).\end{aligned}$$

Take an element $(\chi_{(1,2]},0) \in \mathcal{A} \times_T^P \mathcal{B}$. Then $(\chi_{(1,2]},0) \neq 0$. If $(f,g) \in \mathcal{A} \times_T^P \mathcal{B}$, then

$$\begin{aligned}(\chi_{(1,2]},0)(f,g) &= (P(\chi_{(1,2]}f + \chi_{(1,2]}T(g)),0) = \big((\chi_{(1,2]}f + \chi_{(1,2]}T(g))\chi_{[1,0)},0\big)\\ &= \big((f + T(g))\chi_{(1,2]}\chi_{[1,0)},0\big) = (0,0).\end{aligned}$$

Hence $\mathcal{A} \times_T^P \mathcal{B}$ is not without order even though both $\mathcal{A}$ and $\mathcal{B}$ are without order.

3. Gelfand space and multipliers of commutative $\mathcal{A} \times_T^P \mathcal{B}$

Theorem 3.1. *Let $\mathcal{A}$ be a closed subalgebra of a Banach algebra $\mathcal{C}$, let $\mathcal{B}$ be a Banach algebra, let $T : \mathcal{B} \to \mathcal{C}$ be a homomorphism with $\|T\| \leq 1$, and let $P : \mathcal{C} \to \mathcal{A}$ be linear with $\|P\| \leq 1$. Then $\mathcal{A} \times_T^P \mathcal{B}$ is commutative if and only if $\mathcal{B}$ is commutative and $P(ca) = P(ac), P(aT(b)) = P(T(b)a)$ for all $a, c \in \mathcal{A}, b \in \mathcal{B}$.*

Proof. The algebra $\mathcal{A} \times_T^P \mathcal{B}$ commutative if and only if $(a,b)(c,d) = (c,d)(a,b)$ for all $(a,b),(c,d) \in \mathcal{A} \times_T^P \mathcal{B}$ if and only if $(P(ac + aT(d) + T(b)c), bd) = (P(ca + T(d)a + cT(b), db)$ for all $(a,b),(c,d) \in \mathcal{A} \times_T^P \mathcal{B}$ if and only if $bd = db, P(ca) = P(ac), P(aT(b)) = P(T(b)a)$ for all $a, c \in \mathcal{A}, b, d \in \mathcal{B}$. □

Let $\mathcal{A}$ be a commutative Banach algebra. The collection of all nonzero multiplicative linear functionals on $\mathcal{A}$ is called the *Gelfand space* of $\mathcal{A}$. We shall now identify the Gelfand space of the commutative Banach algebra $\mathcal{A} \times_T^P \mathcal{B}$. For an element $a \in \mathcal{A}$, let $L_a : \mathcal{C} \to \mathcal{C}$ be $L_a(c) = ac$ $(c \in \mathcal{C})$.

Theorem 3.2. *Let $\mathcal{A} \times_T^P \mathcal{B}$ be a commutative Banach algebra. Then the Gelfand space $\Delta(\mathcal{A} \times_T^P \mathcal{B})$ of $\mathcal{A} \times_T^P \mathcal{B}$ is the union of the sets $E = \{(0,\psi) : \psi \in \Delta(\mathcal{B})\}$ and $F = \{(\phi, \phi \circ P \circ L_a \circ T) : \phi \in \mathcal{A}^*,\ \phi(P(xy)) = \phi(x)\phi(y)\ (x, y \in \mathcal{A}),\ \phi(a) = 1\}$.*

Proof. Let $\Phi \in \Delta(\mathcal{A} \times_T^P \mathcal{B})$. Since $\Phi \in (\mathcal{A} \times_T^P \mathcal{B})^*$, $\Phi = (\phi, \psi)$ for some $\phi \in \mathcal{A}^*$ and $\psi \in \mathcal{B}^*$ such that $\Phi(a,b) = (\phi,\psi)(a,b) = \phi(a) + \psi(b)$ for all $(a,b) \in \mathcal{A} \times_T^P \mathcal{B}$. Let $(a,b),(c,d) \in \mathcal{A} \times_T^P \mathcal{B}$. Then

$$\begin{aligned} & \phi(a)\phi(c) + \phi(a)\psi(c) + \psi(b)\phi(c) + \psi(b)\psi(d) \\ = \; & (\phi(a) + \psi(b))(\phi(c) + \psi(d)) \\ = \; & (\phi,\psi)(a,b)(\phi,\psi)(c,d) \\ = \; & (\phi,\psi)(P(ac + aT(d) + T(b)c), bd) \\ = \; & \phi(P(ac + aT(d) + T(b)c)) + \psi(bd) \qquad (3.1) \end{aligned}$$

Taking $a = c = 0$, $b = d = 0$ and $b = c = 0$ in the equation (3.1) respectively we get $\psi(bd) = \psi(b)\psi(d)$, $\phi(P(ac)) = \phi(a)\phi(c)$ and $\phi(P(aT(d))) = \phi(a)\psi(d)$.

If $\phi = 0$, then $\Phi = (0,\psi)$ and $\psi \in \Delta(\mathcal{B})$.

If $\phi \neq 0$, then there is $a \in \mathcal{A}$ such that $\phi(a) = 1$. Then $\psi(d) = \phi\left(P\left(aT(d)\right)\right)$ for all $d \in \mathcal{B}$.

We now show that $\psi(d)$ is independent of an element $a \in \mathcal{A}$ with $\phi(a) = 1$. Let $c \in \mathcal{A}$ with $\phi(c) = 1$. If $d \in \mathcal{B}$, then

$$\begin{aligned} \phi\left(P\left(aT(d)\right)\right) &= \phi(c)\phi\left(P\left(aT(d)\right)\right) = \phi\left(P\left(cP\left(aT(d)\right)\right)\right) \\ &= \phi\left(P\left(P\left(ca\right)T(d)\right)\right) = \phi\left(P\left(P\left(ac\right)T(d)\right)\right) \\ &= \phi\left(P\left(aP\left(cT(d)\right)\right)\right) = \phi(a)\phi\left(P\left(cT(d)\right)\right) \\ &= \phi\left(P\left(cT(d)\right)\right). \end{aligned}$$

Conversely, if $\Phi \in F$, then it is a complex homomorphism on $\mathcal{A} \times_T^P \mathcal{B}$. Suppose that $\Phi = (\phi, \phi \circ P \circ L_x \circ T)$ for some $x \in \mathcal{A}$ with $\phi(x) = 1$. If $c \in \mathcal{A}$ with $\phi(c) = 1$ and $d \in \mathcal{B}$, then

$$\begin{aligned}
\phi(P(xT(d))) &= \phi(P(xT(d)))\phi(c) = \phi(P(P(xT(d))c)) \\
&= \phi(P(xP(T(d)c))) = \phi(P(xP(cT(d)))) = \phi(P(P(xc)T(d))).
\end{aligned}$$

If $(a,b),(c,d) \in \mathcal{A} \times_T^P \mathcal{B}$, then $\Phi((a,b)(c,d))$

$$\begin{aligned}
&= \Phi(P(ac + aT(d) + T(b)c, bd)) \\
&= \phi(P(ac + aT(d) + T(b)c)) + \phi\left(P\left(xT(bd)\right)\right) \\
&= \phi(P(ac)) + \phi(P(aT(d))) + \phi(P(T(b)c)) + \phi\left(P\left(P\left(x^2\right)T(b)T(d)\right)\right) \\
&= \phi(P(ac)) + \phi(x)\phi(P(aT(d))) + \phi\left(P\left(T(b)c\right)\right)\phi(x) \\
&\quad + \phi\left(P\left(P\left(x^2 T(b)\right)T(d)\right)\right) \\
&= \phi(P(ac)) + \phi\left(P\left(xP(aT(d))\right)\right) + \phi\left(P\left(P(T(b)c)x\right)\right) \\
&\quad + \phi\left(P\left(P\left(T(b)x^2\right)T(d)\right)\right) \\
&= \phi(P(ac)) + \phi\left(P\left(P\left(xa\right)T(d)\right)\right) + \phi\left(P\left(T(b)P\left(cx\right)\right)\right) \\
&\quad + \phi\left(P\left(P\left(P\left(T(b)x\right)x\right)T(d)\right)\right) \\
&= \phi(P(ac)) + \phi\left(P\left(P\left(ax\right)T(d)\right)\right) + \phi\left(P\left(T(b)P\left(xc\right)\right)\right) \\
&\quad + \phi\left(P\left(P\left(T(b)x\right)P\left(xT(d)\right)\right)\right) \\
&= \phi(P(ac)) + \phi\left(P\left(P\left(ax\right)T(d)\right)\right) + \phi\left(P\left(P\left(T(b)x\right)c\right)\right) \\
&\quad + \phi\left(P\left(P\left(xT(b)\right)P\left(xT(d)\right)\right)\right) \\
&= \phi(P(ac)) + \phi\left(P\left(P\left(ax\right)T(d)\right)\right) + \phi\left(P\left(\left(P\left(T(b)x\right)\right)c\right)\right) \\
&\quad + \phi\left(P\left(P\left(xT(b)\right)P\left(xT(d)\right)\right)\right) \\
&= \phi(P(ac)) + \phi\left(P\left(a\left(P\left(xT(d)\right)\right)\right)\right) + \phi\left(P\left(\left(P\left(xT(b)\right)\right)c\right)\right) \\
&\quad + \phi\left(P\left(P\left(xT(b)\right)P\left(xT(d)\right)\right)\right) \\
&= \phi(a)\phi(c) + \phi(a)\phi\left(P\left(xT(d)\right)\right) + \phi\left(P\left(xT(b)\right)\right)\phi(c) \\
&\quad + \phi\left(P\left(xT(b)\right)\right)\phi\left(P\left(xT(d)\right)\right) \\
&= \left(\phi(a) + \phi\left(P\left(xT(b)\right)\right)\right)\left(\phi(c) + \phi\left(P\left(xT(d)\right)\right)\right) = \Phi(a,b)\Phi(c,d).
\end{aligned}$$

Therefore $\Phi \in \Delta(\mathcal{A} \times_T^P \mathcal{B})$. □

Theorem 3.3. *Suppose that $\mathcal{A} \times_T^P \mathcal{B}$ is a commutative Banach algebra and both $\mathcal{A}$ and $\mathcal{B}$ are without order. Then a bounded linear map S is a multiplier on $\mathcal{A} \times_T^P \mathcal{B}$ if and only if there exist bounded linear maps $S_1 : \mathcal{A} \times_T^P \mathcal{B} \to \mathcal{A}$ and $S_2 : \mathcal{A} \times_T^P \mathcal{B} \to \mathcal{B}$ such that $S = (S_1, S_2)$, $S_2|_{\{0\}\times\mathcal{B}}$ is a multiplier on $\mathcal{B}$, $S_2|_{\mathcal{A}\times\{0\}} = 0$, $P(aS_1(c,0)) = P(S_1(a,0)c)$, $P(T(b)S_1(0,d)) = P(S_1(0,b)T(d))$ and $P(T(b)S_1(c,0)) = P(S_1(0,b)c + T(S_2(0,b))c)$ for all $(a,b),(c,d) \in \mathcal{A} \times_T^P \mathcal{B}$.*

Proof. Let $S = (S_1, S_2) : \mathcal{A} \times_T^P \mathcal{B} \to \mathcal{A} \times_T^P \mathcal{B}$ be a multiplier. Then both $S_1 : \mathcal{A} \times_T^P \mathcal{B} \to \mathcal{A}$ and $S_2 : \mathcal{A} \times_T^P \mathcal{B} \to \mathcal{B}$ are bounded linear maps. If $(a, b), (c, d) \in \mathcal{A} \times_T^P \mathcal{B}$, then $(a, b)(S_1, S_2)((c, d)) = (S_1, S_2)((a, b))(c, d)$ implies $P(aS_1(c, d) + aT(S_2(c, d)) + T(b)S_1(c, d)) = P(S_1(a, b)c + S_1(a, b)T(d) + T(S_2(a, b))c$ and $bS_2(c, d) = S_2(a, b)d$.

Taking $b = d = 0$, $a = c = 0$, $a = d = 0$ and $b = c = 0$ in above equations, respectively, we get

$$\begin{aligned}
P(aS_1(c,0) + aT(S_2(c,0))) &= P(S_1(a,0)c + T(S_2(a,0))c), && (3.2)\\
P(T(b)S_1(0,d)) &= P(S_1(0,b)T(d)), && (3.3)\\
bS_2(0,d) &= S_2(0,b)d,, && (3.4)\\
P(T(b)S_1(c,0)) &= P(S_1(0,b)c + T(S_2(0,b))c) && (3.5)\\
bS_2(c,0) &= 0 && (3.6)\\
P(aS_1(0,d) + aT(S_2(0,d))) &= P(S_1(a,0)T(d)) && (3.7)\\
0 &= S_2(a,0)d. && (3.8)
\end{aligned}$$

Since $\mathcal{B}$ is faithful, it follows from equation (3.6) and equation (3.8) that $S_2(a, 0) = 0$ for all $a \in \mathcal{A}$. Therefore the required conditions are satisfied.

Conversely, if S_1 and S_2 satisfy the given conditions, then $S = (S_1, S_2)$ is a multiplier on $\mathcal{A} \times_T^P \mathcal{B}$. □

It should be noted that several conditions on P enumerated in Lemma 2.1 are all needed to ensure the associativity of the unifying multiplication (2.1). We conclude with a remark that one may extend these framework to non associative algebras and search for conditions ensuring the resulting algebra $\mathcal{A} \times_T^P \mathcal{B}$ to be a Jordan algebra, an Alternating algebra or a Lie algebra.

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ON THE ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES

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Abstract. In this paper we study the absolute Nörlund summability of Fourier series of functions of the classes $\Lambda BV([0,2\pi])$ and $\Lambda BV^{(r)}([0,2\pi])$.

1. Introduction

Absolute summability is a generalization of the concept of the absolute convergence just as the summability is an extension of the concept of the convergence. The absolute convergence of Fourier series is closely related to the variation and the modulus of continuity of functions. Since there are several classical criteria for the absolute convergence of Fourier series (see e.g., [8, 9]), one can extend the concept of the absolute convergence of Fourier series to absolute summability just as we can do with the convergence of Fourier series by the summability. Several results have been obtained for the absolute summability of Fourier series by Nörlund means (see, [1, 2, 3, 4, 5]). Hyslop [2] extended Bernstein's theorem by using absolute Cesàro summability. This theorem was extended by McFadden [6] applying absolute Nörlund summability, but the final result in the same direction is due to M. Izumi and I. Izumi [3] and S. N. Lal [4]. In 1970, by applying absolute Nörlund summability, S. M. Shah [7] obtained an analog of Zygmund's result for the class of functions of bounded variation. Shah observes that his theorem supplements an earlier result of S. N. Lal [4]. Later in 1977, the study of the absolute Nörlund summability of Fourier series of functions of $r-$bounded variation ($r \geq 1$) was done by S. N. Lal and S. Ram [5]. Here, generalizing these results, we study the absolute Nörlund summability of Fourier series of functions of the classes $\Lambda BV([0,2\pi])$ and $\Lambda BV^{(r)}([0,2\pi])$ ($r \geq 1$).

2. Notations and Definitions

Let $\{p_n\}$ be a sequence of nonnegative numbers and write

$$P_n = \sum_{v=0}^{n} p_v \ (P_{-1} = p_{-1} = 0).$$

The sequence to sequence transformation

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$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v}\ S_v\ (P_n > 0)$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$ generated by the sequence of coefficients $\{p_n\}$.

Definition 2.1. *The series $\sum a_n$ with the sequence of partial sums S_n is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if*

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty.$$

In particular, when $p_0 = 1$ and $p_n = 0$, for $n > 0$, the summability $|N, p_n|$ reduces to the absolute convergence of the series $\sum a_n$.
The notion of bounded variation is generalized in many ways and many different notions of generalized bounded variations are introduced. One of such generalization is as follows.
In the sequel $\mathbb{T} = [0, 2\pi)$, $\mathbb{L}$ is a class of non-decreasing sequences $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_n \frac{1}{\lambda_n}$ diverges, $\Lambda_n = \sum_{k=1}^{n} \frac{1}{\lambda_k}$, $P(t) = P_{[t]}$, where $[t]$ denotes the integer part of t, and C denotes a positive constant not necessarily the same at each occurrence.

Definition 2.2. *Given sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty} \in \mathbb{L}$ and $r \geq 1$, a function f defined on the interval $\overline{\mathbb{T}}$ is said to be of $r-\Lambda-$bounded variation (that is, $f \in \Lambda BV^{(r)}(\overline{\mathbb{T}})$) if*

$$V_{\Lambda_r}(f, \overline{\mathbb{T}}) = \sup_T \left\{ \left(\sum_i \frac{|\Delta f(x_i)|^r}{\lambda_i} \right)^{\frac{1}{r}} \right\} < \infty,$$

where $T : 0 = x_1 < x_2 < \cdots < x_n = 2\pi$ and $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$.

In the above Definition 2.2, for $\Lambda = \{1\}$ (that is, $\lambda_n = 1$, for all n) and $r = 1$ one gets the class $BV(\overline{\mathbb{T}})$ of functions of bounded variation; for $r = 1$ one gets the class $\Lambda BV(\overline{\mathbb{T}})$ of functions of $\Lambda-$bounded variation; and for $\Lambda = \{1\}$ one gets the class $BV^{(r)}(\overline{\mathbb{T}})$ of functions of $r-$bounded variation.
Note that for any $f \in \Lambda BV^{(r)}(\overline{\mathbb{T}})$, we have

$$|f(x)| \leq (\lambda_1)^{\frac{1}{r}} \left(\frac{|f(x) - f(0)|^r}{\lambda_1} \right)^{\frac{1}{r}} + |f(0)| \leq (\lambda_1)^{\frac{1}{r}}\, V_{\Lambda_r}(f, \overline{\mathbb{T}}) + |f(0)| < \infty.$$

Thus, $f \in \Lambda BV^{(r)}(\overline{\mathbb{T}})$ is bounded on $\overline{\mathbb{T}}$. In particular, when $r = 1$, $f \in \Lambda BV(\overline{\mathbb{T}})$ is bounded on $\overline{\mathbb{T}}$.

Definition 2.3. *The modulus of continuity of a function f is defined as*

$$\omega(\delta) \equiv \omega(f; \delta) = \sup\{|f(x+h) - f(x)| :\ 0 \leq h < \delta\}$$

and the $p-$integral modulus of continuity of an $f \in L^p(\overline{\mathbb{T}})$ $(p \geq 1)$ is defined as

$$\omega^{(p)}(\delta) \equiv \omega^{(p)}(f; \delta) = \sup\left\{ \left(\int_{\overline{\mathbb{T}}} |f(x+h) - f(x)|^p\ dx \right)^{\frac{1}{p}} :\ 0 \leq h < \delta \right\}.$$

For a $2\pi-$periodic function $f \in L^1(\overline{\mathbb{T}})$, its Fourier series is defined as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \; cos \; nt + b_n \; sin \; nt) \equiv \sum_{n=0}^{\infty} A_n(t),$$

where a_n and b_n are given by the usual Euler-Fourier formulae.

3. Statement of the results

Theorem 3.1. *Let $\{p_n\}$ be a nonnegative, nonincreasing sequence such that* $\lim_{n\to\infty} p_n = 0$, $\{p_n - p_{n+1}\}$ *is nonincreasing and*

$$\sum_{k=n}^{\infty} \frac{P_k^2}{k^2} \leq C\frac{P_n^2}{n}. \tag{3.1}$$

If $f \in \Lambda BV(\overline{\mathbb{T}})$,

$$\sum_{n=1}^{\infty} \frac{1}{P_n}\left(\frac{\omega\left(f;\frac{\pi}{n}\right)}{n \; \Lambda_{2n}}\right)^{1/2} < \infty \tag{3.2}$$

and

$$\sum_{n=1}^{\infty} \frac{\omega\left(f;\frac{1}{n}\right)}{n} < \infty, \tag{3.3}$$

then the Fourier series of f is summable $|N, p_n|$.

Remark 3.2. *Observe that if $\Lambda = \{1\}$ then the class $\Lambda BV(\overline{\mathbb{T}})$ reduces to the class $BV(\overline{\mathbb{T}})$ and one gets the result of S. M. Shah* [7, Theorem 1, p. 301] *as a particular case of our Theorem 3.1.*

Theorem 3.3. *Let $\{p_n\}$ be a sequence as in Theorem 3.1 and*

$$\sum_{k=n}^{\infty} \frac{P_k^{2p}}{k^2} \leq C\frac{P_n^{2p}}{n} \;\; (1 < p < \infty). \tag{3.4}$$

If $f \in \Lambda BV^{(r)}(\overline{\mathbb{T}})$,

$$\sum_{n=1}^{\infty} \frac{1}{P_n}\left(\frac{\left(\omega^{(r+(2-r)q)}\left(f;\frac{\pi}{n}\right)\right)^{2-r/p}}{n \; \left(\Lambda_{2n}\right)^{1/p}}\right)^{1/2} < \infty, \tag{3.5}$$

where $1 \leq r < 2p, \; 1 < p < \infty, \; \frac{1}{p} + \frac{1}{q} = 1,$ and

$$\sum_{n=1}^{\infty} \frac{\omega\left(f;\frac{1}{n}\right)}{n} < \infty \tag{3.6}$$

then the Fourier series of f is summable $|N, p_n|$.

Remark 3.4. *Note that if $\Lambda = \{1\}$ then the class $\Lambda BV^{(r)}(\overline{\mathbb{T}})$ reduces to the class $BV^{(r)}(\overline{\mathbb{T}})$ and one gets the result of S. N. Lal and S. Ram* [5, Theorem, p. 88] *as a particular case of our Theorem 3.3.*

4. Proof of the results

We need the following lemmas from [5, p. 89] to prove the results.

Lemma 4.1. *If $\{p_n\}$ is nonnegative and nonincreasing, then for $0 \leq a \leq b \leq \infty$, $0 < t \leq \pi$ and for any n, we have*

$$\left|\sum_{k=a}^{b} p_k \ e^{i(n-k)t}\right| \leq CP(t^{-1}) \tag{4.1}$$

and

$$P(2^\gamma) \leq CP(2^{\gamma-1}) \ as \ \gamma \to \infty. \tag{4.2}$$

Lemma 4.2. *If $\{p_n\}$ is nonnegative and nonincreasing and if we take*

$$\rho(t) = \sum_{u=0}^{\infty} p_u \ e^{iut},$$

then for t in (h, π) we have $\quad |\rho(t+2h) - \rho(t)| \leq Cht^{-1}P(h^{-1}).$

Lemma 4.3. *Under the hypothesis (3.4) of the Theorem 3.3 there exists a C such that*

$$\sum_{k=n}^{\infty} \frac{P_k^2}{k^2} \leq C\frac{P_n^2}{n} \qquad and \qquad \sum_{k=n}^{\infty} \frac{P_k}{k^2} \leq C\frac{P_n}{n}.$$

Proof of the Theorem 3.1. Let

$$\phi(t) = f(x+t) + f(x-t) - 2f(x), \ \alpha(t) = \sum_{u=0}^{\infty} p_u \ cos \ ut, \ \beta(t) = \sum_{u=0}^{\infty} p_u \ sin \ ut,$$

$$\alpha_n = \int_0^\pi \phi(t) \ \alpha(t) \ cos \ nt \ dt \quad and \quad \beta_n = \int_0^\pi \phi(t) \ \beta(t) \ sin \ nt \ dt.$$

We have

$$A_n(x) = \frac{1}{\pi}\int_0^\pi \phi(t) \ cos \ nt \ dt.$$

Proceeding as in [5] we get

$$\begin{aligned}
\pi \ |t_n - t_{n-1}| &= \frac{1}{P_n \ P_{n-1}} \left|\int_0^\pi \phi(t) \sum_{k=0}^{n-1} (p_k \ P_n - p_n \ P_k) \ cos(n-k)t \ dt\right| \\
&\leq \frac{|\alpha_n|}{P_{n-1}} + \frac{|\beta_n|}{P_{n-1}} + \frac{1}{P_{n-1}} \left|\int_0^{1/n} \phi(t) \left(\sum_{k=n}^{\infty} p_k \ cos(n-k)t\right) dt\right| \\
&+ \frac{p_n}{P_n \ P_{n-1}} \left|\int_0^{1/n} \phi(t) \left(\sum_{k=0}^{n-1} P_k \ cos(n-k)t\right) dt\right| \\
&+ \frac{1}{P_{n-1}} \left|\int_{1/n}^{\pi} \frac{\phi(t)}{2 \ sin(t/2)} \left(\sum_{k=n}^{\infty} (p_k - p_{k+1}) \ sin(n-k+1/2)t\right) dt\right| \\
&+ \frac{p_n}{P_n \ P_{n-1}} \left|\int_{1/n}^{\pi} \frac{\phi(t)}{2 \ sin(t/2)} \left(\sum_{k=0}^{n-1} p_k \ sin(n-k+1/2)t\right) dt\right| \\
&+ \frac{p_n}{2 \ P_{n-1}} \left(1 - \frac{P_{n-1}}{P_n}\right) \left|\int_{1/n}^{\pi} \phi(t) \ dt\right| = \sum_{j=1}^{7} I_j(n), \ say.
\end{aligned}$$

In order to prove the theorem we have to show that

$$\sum_{n=2}^{\infty} I_j(n) < \infty, \ \ for\ j = 1\ to\ 7.$$

Under the hypotheses of the theorem $\alpha(t) \in L^2$ and as $f \in \Lambda BV(\overline{\mathbb{T}})$ is bounded, ϕ is bounded, and hence $\phi(t)\alpha(t) \in L^2$.
Since the Fourier series of $\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)$ is

$$-\frac{4}{\pi}\sum_{n=1}^{\infty} \alpha_n\ sin\ nt\ sin\ nh,$$

in view of Parseval's formula we get

$$\begin{aligned}
\sum_{n=1}^{\infty} |\alpha_n\ sin\ nh|^2 &\le C\int_0^{\pi} |\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)|^2 dt \\
&\le C\int_0^{\pi} \alpha^2(t+h)|\phi(t+h) - \phi(t-h)|^2 dt + C\int_{-h}^{h} \phi^2(t)\alpha^2(t+2h)\ dt \\
&\quad + C\int_{-h}^{h} \phi^2(t)\alpha^2(t)\ dt + C\int_{h}^{\pi} \phi^2(t)|\alpha(t+2h) - \alpha(t)|^2 dt \\
&= \sum_{j=1}^{4} T_j(h),\ say. \qquad (4.3)
\end{aligned}$$

Take

$$\Delta\phi_k = \phi\left(t + \frac{k\pi}{N}\right) - \phi\left(t + \frac{(k-1)\pi}{N}\right).$$

Now (cf, [5, p. 92]) we get

$$\begin{aligned}
&\int_0^{\pi} \alpha^2\left(t + \frac{\pi}{2N}\right)\left|\phi\left(t + \frac{\pi}{2N}\right) - \phi\left(t - \frac{\pi}{2N}\right)\right|^2 dt \\
&= \int_0^{\pi} \alpha^2\left(t + \frac{\pi}{N}\right)\left|\phi\left(t + \frac{\pi}{N}\right) - \phi\left(t\right)\right|^2\ dt \\
&\le \omega\left(f; \frac{\pi}{N}\right)\int_0^{\pi} \alpha^2\left(t + \frac{\pi}{N}\right)\left|\phi\left(t + \frac{\pi}{N}\right) - \phi\left(t\right)\right|\ dt \\
&= \left(\frac{\omega\left(f; \frac{\pi}{N}\right)}{\Lambda_{2N}}\right)\sum_{k=1}^{2N}\frac{1}{\lambda_k}\int_0^{\pi} \alpha^2\left(t + \frac{k\pi}{N}\right)|\Delta\phi_k|\ dt \\
&\le C\left(\frac{\omega\left(f; \frac{\pi}{N}\right)}{\Lambda_{2N}}\right)\left[1 + \sum_{k=1}^{\infty}\frac{P_k^2}{k^2}\right].
\end{aligned}$$

Thus,

$$T_1\left(\frac{\pi}{2N}\right) \le C\left(\frac{\omega\left(f; \frac{\pi}{N}\right)}{\Lambda_{2N}}\right).$$

By Lemma 4.1,

$$T_2(h) \le C\int_{-h}^{h} \omega^2(t)P^2\left(\frac{1}{t+2h}\right) dt \le Ch\omega^2(h)P^2(h^{-1}), \qquad (4.4)$$

and

$$T_3(h) \le C\omega^2(h)\int_{1/h}^{\infty}\frac{P^2(t)}{t^2}dt \le Ch\omega^2(h)P^2(h^{-1}), \tag{4.5}$$

in view of (3.1). By Lemma 4.2,

$$T_4(h) \le Ch^2P^2(h^{-1})\int_{1/\pi}^{1/h}\omega^2(t^{-1})dt. \tag{4.6}$$

Taking $h = \frac{\pi}{2N}$ in (4.3), we have

$$\sum_{n=1}^{\infty}|\alpha_n|^2 \le C\Big[\left(\frac{\omega\left(f;\frac{\pi}{N}\right)}{\Lambda_{2N}}\right) + N^{-1}\omega^2\left(\frac{\pi}{2N}\right)P^2\left(\frac{2N}{\pi}\right)$$

$$+N^{-2}P^2\left(\frac{2N}{\pi}\right)\int_{1/\pi}^{2N/\pi}\omega^2(t^{-1})\ dt\Big].$$

Putting $N = 2^u$, we get

$$\left(\sum_{n=2^{u-1}+1}^{2^u}|\alpha_n|^2\right)^{1/2} \le C\left[\left(\frac{\omega\left(f;\frac{\pi}{2^u}\right)}{\Lambda_{2^{u+1}}}\right)^{1/2}\right]$$

$$+C\left[2^{-u}P\left(\frac{2^{u+1}}{\pi}\right)\left(\int_{1/\pi}^{2^{u+1}/\pi}\omega^2(t^{-1})\ dt\right)^{1/2}\right].$$

Now, by Hölder's inequality and making use of (4.2) of Lemma 4.1, we get

$$\sum_{n=2^{u-1}+1}^{2^u}\frac{|\alpha_n|}{P_{n-1}} \le \left(\sum_{n=2^{u-1}+1}^{2^u}|\alpha_n|^2\right)^{1/2}\left(\sum_{n=2^{u-1}+1}^{2^u}P_{n-1}^{-2}\right)^{1/2}$$

$$\le C\left[\frac{2^{u/2}}{P(2^u)}\left(\frac{\omega\left(f;\frac{\pi}{2^u}\right)}{\Lambda_{2^{u+1}}}\right)^{1/2} + 2^{-u/2}\left(\int_{1/\pi}^{2^{u+1}/\pi}\omega^2(t^{-1})\ dt\right)^{1/2}\right]$$

and therefore

$$\sum_{n=2}^{\infty}I_1(n) = \sum_{u=1}^{\infty}\sum_{n=2^{u-1}+1}^{2^u}\frac{|\alpha_n|}{P_{n-1}}$$

$$\le C\sum_{u=1}^{\infty}\frac{2^{u/2}}{P(2^u)}\left(\frac{\omega\left(f;\frac{\pi}{2^u}\right)}{\Lambda_{2^{u+1}}}\right)^{1/2} + C\sum_{u=1}^{\infty}2^{-u/2}\left(\int_{1/\pi}^{2^{u+1}/\pi}\omega^2(t^{-1})\ dt\right)^{1/2}$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{P_n}\left(\frac{\omega\left(f;\frac{\pi}{n}\right)}{n\ \Lambda_{2n}}\right)^{1/2} + C\sum_{u=1}^{\infty}2^{-u/2} + C\sum_{u=1}^{\infty}2^{-u/2}\left(\sum_{k=2}^{2^u}\omega^2(k^{-1})\right)^{1/2}$$

$$\le C + C\sum_{u=1}^{\infty}2^{-u/2}\sum_{m=1}^{u}\left(\sum_{n=2^{m-1}+1}^{2^m}\omega^2(k^{-1})\right)^{1/2}$$

$$\le C + C\sum_{m=1}^{\infty}2^{m/2}\omega(1/2^m)\sum_{u=m}^{\infty}2^{-u/2} \le C + C\sum_{m=1}^{\infty}\omega(1/2^m) < \infty,$$

by the hypothesis (3.2) and (3.3) of the theorem. Similarly we can prove that $\sum_{n=2}^{\infty} I_2(n) < \infty$. Finally,

$$\sum_{n=2}^{\infty} I_j(n) < \infty \ (j = 3 \ to \ 7)$$

follows from the proof of [5, Theorem] and hence the result follows.

Proof of the Theorem 3.3: Since $f \in \Lambda BV^{(r)}(\overline{\mathbb{T}})$ is bounded implies ϕ is bounded and hence $\phi \in L^2$. Proceeding as in the proof of the Theorem 3.1, we get (4.3). Take

$$\Delta\phi_k = \phi(t + k\pi/N) - \phi(t + (k-1)\pi/N).$$

Since

$$2 = ((2-r)q + r)/q + r/p,$$

by using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^{\pi} \alpha^2 \left(t + \frac{\pi}{2N}\right) \left|\phi\left(t + \frac{\pi}{2N}\right) - \phi\left(t - \frac{\pi}{2N}\right)\right|^2 dt \\
&= \int_0^{\pi} \alpha^2 \left(t + \frac{\pi}{N}\right) \left|\phi\left(t + \frac{\pi}{N}\right) - \phi\left(t\right)\right|^2 dt \\
&= \int_0^{\pi} \alpha^2 \left(t + \frac{k\pi}{N}\right) |\Delta\phi_k|^2 \, dt \\
&\le \left(\int_0^{\pi} \alpha^{2p} \left(t + \frac{k\pi}{N}\right) |\Delta\phi_k|^r \, dt\right)^{1/p} \times \left(\int_0^{\pi} |\Delta\phi_k|^{(2-r)q+r} \, dt\right)^{1/q} \\
&\le \left(\omega^{((2-r)q+r)}\left(f; \frac{\pi}{N}\right)\right)^{2-r/p} \left(\int_0^{\pi} \alpha^{2p} \left(t + \frac{k\pi}{N}\right) |\Delta\phi_k|^r \, dt\right)^{1/p} \\
&= \left(\omega^{((2-r)q+r)}\left(f; \frac{\pi}{N}\right)\right)^{2-r/p} \left[\frac{\sum_{k=1}^{2N} \frac{1}{\lambda_k} \left(\int_0^{\pi} \alpha^{2p}\left(t + \frac{k\pi}{N}\right) |\Delta\phi_k|^r \, dt\right)^{1/p}}{\Lambda_{2N}}\right] \\
&\le \left(\omega^{((2-r)q+r)}\left(f; \frac{\pi}{N}\right)\right)^{2-r/p} \times \\
&\qquad \left[\frac{\left(\sum_{k=1}^{2N} \frac{1}{\lambda_k} \int_0^{\pi} \alpha^{2p}\left(t + \frac{k\pi}{N}\right) |\Delta\phi_k|^r \, dt\right)^{1/p} \ (\Lambda_{2N})^{1-1/p}}{\Lambda_{2N}}\right] \\
&\le \frac{\left(\omega^{((2-r)q+r)}\left(f; \frac{\pi}{N}\right)\right)^{2-r/p}}{(\Lambda_{2N})^{1/p}} \left[\left(\sum_{k=1}^{2N} \frac{1}{\lambda_k} \int_0^{\pi} \alpha^{2p}\left(t + \frac{k\pi}{N}\right) |\Delta\phi_k|^r \, dt\right)^{1/p}\right] \\
&\le C \frac{\left(\omega^{((2-r)q+r)}\left(f; \frac{\pi}{N}\right)\right)^{2-r/p}}{(\Lambda_{2N})^{1/p}} \left[1 + \sum_{k=1}^{\infty} \frac{P_k^{2p}}{k^2}\right]^{1/p}.
\end{aligned}$$

Thus,

$$I_1\left(\frac{\pi}{2N}\right) \le C \frac{\left(\omega^{((2-r)q+r)}\left(f; \frac{\pi}{N}\right)\right)^{2-r/p}}{(\Lambda_{2N})^{1/p}}.$$

Now, proceeding as in the proof of the Theorem 3.1 (from (4.4) onward), we obtain the theorem.

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THE LOG-BALANCEDNESS OF THE GENERALIZED MOTZKIN NUMBERS

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Abstract. In this paper, we discuss the log-balancedness of the generalized Motzkin numbers $M_n(b,c)$. For $b \geq 1$, we prove that the sequence $\{M_n(b,1)\}_{n\geq 0}$ is log-balanced. In addition, we discuss the log-balancedness of some sequences related to $M_n(b,c)$. For example, we investigate the log-balancedness of $\{\frac{M_n(b,1)}{n}\}_{n\geq 1}$ and $\{M_n^2(b,1)\}$ when $b \geq 6$ is fixed.

1. Introduction

For $n \geq 0$, let M_n denote the n^{th} term of the Motzkin numbers. It is well known that $\{M_n\}_{n\geq 0}$ satisfies the following recurrence relation

$$(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1}, \quad n \geq 1,$$

where $M_0 = M_1 = 1$. The Motzkin numbers M_n are related to the number of lattice paths from $(0,0)$ to $(n,0)$. Some values of $\{M_n\}_{n\geq 0}$ are as follow:

n	0	1	2	3	4	5	6	7	8
M_n	1	1	2	4	9	21	51	127	323

Sun [12] introduced the generalized Motzkin numbers $M_n(b,c)$:

$$M_n(b,c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k} c^k,$$

where b and c are positive integers and $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} term of the Catalan numbers. For $\{M_n(b,c)\}$, Sun [12] gave the following recurrence relation:

$$(n+3)M_{n+1}(b,c) = b(2n+3)M_n(b,c) - (b^2-4c)nM_{n-1}(b,c), \quad n \geq 1. \quad (1.1)$$

It is clear that $M_n(1,1) = M_n$. In this paper, we mainly discuss the log-behavior of $\{M_n(b,c)\}_{n\geq 0}$.

Now we recall some definitions in combinatorics. For a positive sequence $\{z_n\}_{n\geq 0}$, we say it *log-convex* (*log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ ($z_n^2 \geq z_{n-1}z_{n+1}$) for $n \geq 1$. For a log-convex sequence $\{z_n\}_{n\geq 0}$, we say it *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$

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is log-concave (Došlić [3] gave this definition). It is well known that $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (nonincreasing) and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $\frac{(n+1)z_n}{z_{n-1}} \geq \frac{nz_{n+1}}{z_n}$ for each $n \geq 1$. It is clear that log-balancedness is a special case of log-convexity and the quotient sequence of a log-balanced sequence does not grow too fast. Log-behavior is not only one of the sources for inequalities, but also plays an important role in many disciplines. Log-concave sequences often occur in combinatorics, algebra, and geometry. For instance, the binomial coefficient $\binom{n}{k}$ is log-concave for k when $n \geq 2$ is fixed. See [2] for various results of log-concavity. Log-convexity of sequences and functions is relevant in many subjects such as combinatorics, economics, mathematical biology, and physics. Many famous combinatorial sequences including Catalan numbers, Motzkin numbers, Fine numbers, large Schröder numbers, and central Delannoy numbers, are log-convex. Log-convexity of a bond price means that the relative decline is decreasing when the interest rate grows. In finance, it was found that log-convexity of the discount function is equivalent to the decreased impatience. Log-convex sequences arise as the enumerating sequences of secondary structures of certain biopolymers. For various applications of log-behavior, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11]. We note that log-balanced sequences are suitable for providing important examples in white noise theory (See [7] for more details). In addition, since log-balancedness is related to log-convexity and log-concavity, it can help us to obtain more inequalities. Hence, log-balancedness of sequences deserves to be studied. Došlić [3] gave several sufficient conditions for log-balancedness and proved log-balancedness of many sequences including Motzkin numbers, Fine numbers, large Schröder numbers, central Delannoy numbers, and so on.

Some recurrences of general secondary structure numbers are analogous to those for the Motzkin numbers (See Došlić [6] for more details). From the log-balancedness of Motzkin numbers, we can know properties for the growth rate of secondary structure numbers. Hence, it is useful studying log-balancedness of Motzkin numbers. Došlić [3] showed that $\{M_n\}_{n\geq 0}$ is log-balanced. Motivated by the article Došlić [3], we consider log-balancedness of $\{M_n(b,c)\}_{n\geq 0}$. In the next section, we discuss log-balancedness of $\{M_n(b,1)\}_{n\geq 0}$. In addition, we also investigate log-balancedness of some sequences related to $\{M_n(b,1)\}_{n\geq 0}$.

2. The log-balancedness of $\{M_n(b,1)\}_{n\geq 0}$

Došlić [3] proved that the sequence $\{M_n\}_{n\geq 0}$ is log-balanced. In this section, we discuss the log-balancedness of $\{M_n(b,1)\}_{n\geq 0}$. The following lemmas will be used.

Lemma 2.1. *Assume that $b \geq 1$. For $n \geq 0$, let $x_n = \frac{M_{n+1}(b,1)}{M_n(b,1)}$. When $b \geq 3$, we have*

$$\lambda_n \leq x_n \leq \mu_n, \quad n \geq 0$$

where $\lambda_n = \frac{(b+2)n+2b+1}{n+4}$ *and* $\mu_n = \frac{(b+2)n+5b+3}{n+5}$.

Proof. We prove by induction that $\lambda_n \leq x_n \leq \mu_n$ for $n \geq 0$. It follows from (1.1) that

$$x_k = \frac{b(2k+3)}{k+3} - \frac{(b^2-4)k}{(k+3)x_{k-1}}, \quad k \geq 1. \tag{2.1}$$

By calculus, we have $x_0 = b$ and $x_1 = b + \frac{1}{b}$. It is clear that $\lambda_0 < x_0 < \mu_0$ and $\lambda_1 < x_1 < \mu_1$. Assume that $\lambda_k \leq x_k \leq \mu_k$ for $k \geq 1$. By applying (2.1), we get

$$x_{k+1} - \lambda_{k+1} = \frac{b(2k+5)}{k+4} - \frac{(b^2-4)(k+1)}{(k+4)x_k} - \lambda_{k+1}$$

and

$$x_{k+1} - \mu_{k+1} = \frac{b(2k+5)}{k+4} - \frac{(b^2-4)(k+1)}{(k+4)x_k} - \mu_{k+1}.$$

Due to $\lambda_k \leq x_k \leq \mu_k$ for $k \geq 1$, we derive

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &\geq \frac{b(2k+5)}{k+4} - \frac{(b^2-4)(k+1)}{(k+4)\lambda_k} - \lambda_{k+1} \\ &= \frac{(2b+16)k^2 + 81k + 6b^2 - 11b + 68}{(k+4)(k+5)[(b+2)k+2b+1]} \qquad > 0 \end{aligned}$$

and

$$x_{k+1} - \mu_{k+1} \leq -\frac{(6b-16)k^2 + (52b-85)k + 82b - 60}{(k+4)(k+6)[(b+2)k+5b+3]} \qquad < 0.$$

Hence $\lambda_n \leq x_n \leq \mu_n$ for $n \geq 0$. □

Lemma 2.2. [14] *Suppose that the sequences* $\{a_n\}_{n\geq 0}$ *and* $\{b_n\}_{n\geq 0}$ *are both log-convex. Let* $s_n = \frac{a_{n+1}b_{n+1}}{(n+1)a_n b_n}$ *for* $n \geq 0$. *If* $\{s_n\}_{n\geq 0}$ *is decreasing, then* $\{a_n b_n\}_{n\geq 0}$ *is log-balanced.*

Theorem 2.3. *For* $b \geq 1$, *the sequence* $\{M_n(b,1)\}_{n\geq 0}$ *is log-balanced.*

Proof. For $n \geq 0$, put $x_n = \frac{M_{n+1}(b,1)}{M_n(b,1)}$. Since Wang and Zhang [13] proved the sequence $\{M_n(b,c)\}_{n\geq 0}$ is log-convex when $b^2 \geq c$, $\{M_n(b,1)\}_{n\geq 0}$ is log-convex. In order to prove the log-balancedness of $\{M_n(b,1)\}_{n\geq 0}$, we only need to show that $(n+2)x_n - (n+1)x_{n+1} \geq 0$ for $n \geq 0$.

(i) For $b = 1$, $M_n(b,1) = M_n$. Došlić [3] proved that the Motzkin sequence $\{M_n\}_{n\geq 0}$ is log-balanced.

(ii) For $b = 2$, It follows from (2.1) that $x_n = \frac{2(2n+3)}{n+3}$. Since

$$\begin{aligned} (n+2)x_n - (n+1)x_{n+1} &= \frac{2(n+2)(2n+3)}{n+3} - \frac{2(n+1)(2n+5)}{n+4} \\ &= \frac{2(2n^2+8n+9)}{(n+3)(n+4)} \quad > \ 0, \quad (n \geq 0), \end{aligned}$$

the sequence $\{M_n(2,1)\}_{n\geq 0}$ is log-balanced.

(iii) We consider the case for $b \geq 3$.

It follows from (2.1) that

$$\begin{aligned}&(n+2)x_n-(n+1)x_{n+1}\\&=\frac{(n+2)(n+4)x_n^2-b(n+1)(2n+5)x_n+(b^2-4)(n+1)^2}{(n+4)x_n}.\end{aligned}$$

For any real number t, define a function
$f(t)=(n+2)(n+4)t^2-b(n+1)(2n+5)t+(b^2-4)(n+1)^2$.
It is evident that

$$(n+2)x_n-(n+1)x_{n+1}=f(x_n)/((n+2)x_n).$$

Since $f'(t)>0$ for $t>\rho_n$, where $\rho_n=\frac{b(n+1)(2n+5)}{2(n+2)(n+4)}$, $f(t)$ is increasing on $[\rho_n,+\infty)$. It follows from Lemma 2.1 that $x_n\geq\lambda_n$, where $\lambda_n=\frac{(b+2)n+2b+1}{n+4}$. Since $\lambda_n\geq\rho_n$, $f(t)$ is increasing on $[\lambda_n,+\infty)$. We note that

$$\begin{aligned}f(\lambda_n)&=\frac{(n+2)[(b+2)n+2b+1]^2-b(n+1)(2n+5)[(b+2)n+2b+1]}{n+4}\\&\quad+(b^2-4)(n+1)^2\\&=\frac{(b^2+2b-12)n^2+(2b^2+7b-27)n+2b^2+3b-14}{n+4}\;>\;0.\end{aligned}$$

Since $f(x_n)\geq f(\lambda_n)$, $f(x_n)>0$. This implies that $(n+2)x_n-(n+1)x_{n+1}>0$. Hence the sequence $\{M_n(b,1)\}_{n\geq 0}$ is log-balanced. □

Zhao [15] proved that the sequence $\{\frac{M_n}{n}\}_{n\geq 4}$ is log-balanced. Zhang and Zhao [14] showed that $\{M_n^2\}_{n\geq 2}$ is also log-balanced. Now we discuss the log-balancedness of $\{\frac{M_n(b,1)}{n}\}_{n\geq 1}$ and $\{M_n^2(b,1)\}_{n\geq 0}$ by Lemma 2.2.

Theorem 2.4. *Suppose that $b\geq 1$ is fixed.*

(i) For $b\geq 3$, there exists a positive integer L_b such that $\{\frac{M_n(b,1)}{n}\}_{n\geq L_b}$ is log-balanced.

(ii) For $b\geq 6$, there exists a positive integer N_b such that the sequence $\{M_n^2(b,1)\}_{n\geq N_b}$ is log-balanced.

Proof. For $n\geq 0$, let $x_n=\frac{M_{n+1}(b,1)}{M_n(b,1)}$ and $y_n=\frac{nx_n}{(n+1)^2}$ $(n\geq 1)$. It is clear that the sequences $\{\frac{1}{n}\}_{n\geq 1}$ and $\{M_n(b,1)\}_{n\geq 0}$ $(b\geq 1)$ are both log-convex. In order to prove the log-concavity of $\{\frac{M_n(b,1)}{nn!}\}_{n\geq 1}$ and $\{\frac{M_n^2(b,1)}{n!}\}_{n\geq 0}$, it is sufficient to show that $\{y_n\}_{n\geq 1}$ is decreasing and $(n+2)x_n^2-(n+1)x_{n+1}^2\geq 0$ holds for $n\geq 0$.

(i) It is clear that

$$y_n-y_{n+1}=\frac{n(n+2)^2x_n-(n+1)^3x_{n+1}}{(n+1)^2(n+2)^2}.$$

It follows from (2.1) that

$$\begin{aligned}&n(n+2)^2x_n-(n+1)^3x_{n+1}\\&=n(n+2)^2x_n-(n+1)^3\frac{b(2n+5)x_n-(b^2-4)(n+1)}{(n+4)x_n}\\&=\frac{n(n+4)(n+2)^2x_n^2-b(2n+5)(n+1)^3x_n+(b^2-4)(n+1)^4}{(n+4)x_n}.\end{aligned}$$

For any real number t, define a function

$$g(t) = n(n+4)(n+2)^2t^2 - b(2n+5)(n+1)^3t + (b^2-4)(n+1)^4.$$

It is obvious that

$$4n(n+2)^2x_n - (n+1)^3x_{n+1} = g(x_n)/((n+4)x_n)$$

and

$$g'(t) = 2n(n+4)(n+2)^2t - b(2n+5)(n+1)^3.$$

Since $g'(t) > 0$ for $t > \sigma_n$, where $\sigma_n = \frac{b(2n+5)(n+1)^3}{2n(n+4)(n+2)^2}$, $g(t)$ is increasing on $[\sigma_n, +\infty)$. We note that $x_n \geq \lambda_n$ and $\lambda_n \geq \sigma_n$, where $\lambda_n = \frac{(b+2)n+2b+1}{n+4}$. Then $g(t)$ is increasing on $[\lambda_n, +\infty)$. This implies that $g(x_n) \geq g(\lambda_n)$. We note that

$$\begin{aligned} g(\lambda_n) =& \frac{n(n+2)^2[(b+2)n+2b+1]^2 - b(2n+5)(n+1)^3[(b+2)n+2b+1]}{n+4} \\ &+(b^2-4)(n+1)^4 \\ =& \frac{1}{n+4}[(b^2+2b-12)n^4 + (3b^2+7b-55)n^3 + (b^2+b-92)n^2 \\ &-(6b^2+11b+64)n - 6b^2 - 5b - 16]. \end{aligned}$$

For fixed b and $n \geq 1$, put

$$\begin{aligned} \varphi(n) \quad = \quad & (b^2+2b-12)n^4 + (3b^2+7b-55)n^3 + (b^2+b-92)n^2 \\ & -(6b^2+11b+64)n - 6b^2 - 5b - 16. \end{aligned}$$

Since $\lim_{n\to+\infty} \varphi(n) = +\infty$, there exists a positive integer L_b such that $\varphi(n) > 0$ for $n \geq L_b$. This means that $y_n - y_{n+1} > 0$ for $n \geq L_b$. Hence, the sequence $\{y_n\}_{n\geq L_b}$ is decreasing.

(ii) It follows from (2.1) that

$$\begin{aligned} &(n+2)x_n^2 - (n+1)x_{n+1}^2 \\ &= (n+2)x_n^2 - (n+1)\left[\frac{b(2n+5)}{n+4} - \frac{(b^2-4)(n+1)}{(n+4)x_n}\right]^2 \\ &= \frac{1}{(n+4)^2x_n^2}\Bigg[(n+2)(n+4)^2x_n^4 - b^2(n+1)(2n+5)^2x_n^2 \\ &\quad +2b(b^2-4)(n+1)^2(2n+5)x_n - (b^2-4)^2(n+1)^3\Bigg]. \end{aligned}$$

For any real number t, define a function

$$\begin{aligned} h(t) \quad = \quad & (n+2)(n+4)^2t^4 - b^2(n+1)(2n+5)^2t^2 \\ & +2b(b^2-4)(n+1)^2(2n+5)t - (b^2-4)^2(n+1)^3. \end{aligned}$$

It is clear that

$$\begin{aligned} (n+2)x_n^2 \quad - \quad (n+1)x_{n+1}^2 \;&=\; h(x_n)/((n+4)^2x_n), \\ h'(t) \quad = \quad & 4(n+2)(n+4)^2t^3 - 2b^2(n+1)(2n+5)^2t \\ & +2b(b^2-4)(n+1)^2(2n+5), \end{aligned}$$

and

$$h''(t) = 12(n+2)(n+4)^2t^2 - 2b^2(n+1)(2n+5)^2.$$

Since $h''(t) > 0$ for $t > \tau_n$, where $\tau_n = \frac{b(2n+5)}{n+4}\sqrt{\frac{n+1}{6(n+2)}}$, $h'(t)$ is increasing on $[\tau_n, +\infty)$. We note that $\lambda_n > \tau_n$, where $\lambda_n = \frac{(b+2)n+2b+1}{n+4}$. Then $h'(t)$ is increasing on $[\lambda_n, +\infty)$. Since

$$\begin{aligned} h'(\lambda_n) &= \frac{2}{n+4}[(4b^2+16b+16)n^4+(b^3+26b^2+52b+56)n^3 \\ &\quad +(3b^3+66b^2+6b+60)n^2+(2b^3+73b^2-92b+26)n \\ &\quad +2b^3+23b^2-56b+4] \ > \ 0, \end{aligned}$$

$h(t)$ is increasing on $[\lambda_n, +\infty)$. Then $h(x_n) \geq h(\lambda_n)$. Upon computation, we get

$$\begin{aligned} & h(\lambda_n) \\ &= (n+2)\frac{[(b+2)n+2b+1]^4}{(n+4)^2} - b^2(n+1)(2n+5)^2\frac{[(b+2)n+2b+1]^2}{(n+4)^2} \\ &\quad +2b(b^2-4)(n+1)^2(2n+5)\frac{(b+2)n+2b+1}{n+4} - (b^2-4)^2(n+1)^3 \\ &= \frac{1}{(n+4)^2}\Big[(b^4+4b^3-24b^2-112b-112)n^4 \\ &\quad +(4b^4+14b^3-154b^2-560b-600)n^3 \\ &\quad +(4b^4+14b^3-396b^2-940b-1112)n^2 \\ &\quad +(-4b^4+12b^3-441b^2-632b-879)n \\ &\quad -4b^4+4b^3-169b^2-144b-254\Big]. \end{aligned}$$

Since $\lim_{n\to+\infty} h(\lambda_n) = +\infty$ when b is fixed, there exists a positive integer N_b such that $h(\lambda_n) > 0$ for $n \geq N_b$. This means that $(n+2)x_n^2 - (n+1)x_{n+1}^2 \geq 0$ for $n \geq N_b$. Hence, the sequence $\{M_n^2(b,1)\}_{n\geq N_b}$ is log-balanced. □

For Theorem 2.4, we find that the values of L_b and N_b are related to b. For example, $\{\frac{M_n(b,1)}{n}\}_{n\geq 2}$ and $\{M_n^2(6,1)\}_{n\geq 5}$ are log-balanced.

3. Conclusions

For generalized Motzkin numbers $M_n(b,c)$, we have shown that $\{M_n(b,1)\}_{n\geq 0}$ $(b \geq 2)$ is log-balanced. In addition, we also have proved that sequence $\{M_n^2(b,1)\}$ is log-balanced under suitable conditions. In our future work, we will study applications of generalized Motzkin numbers and log-balancedness of various nonlinear recurrence sequences in combinatorics.

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PROBLEM SECTION

In the last issue of the Math. Student Vol. **86**, Nos. 3-4, July-December (2017), we had invited solutions from the floor to the remaining problems *2* and *5* of the MS, 86, 1-2, 2017 as well as to the seven new problems *6, 7, 8, 9, 10, 11* and *12* presented therein till April 30, 2018.

No solution was received from the floor to the remaining problems *2* and *5* of the MS, 86, 1-2, 2017 and hence we provide in this issue the Proposer's solution to these problems.

The status regarding the problems of MS, 86, 3-4, 2017 is as under.

1. We received from the floor **four correct** solutions to problem *7* and publish here the solution from the floor which also states and proves a generalized version of the problem;

2. We received from the floor **one incorrect** solution to the Problems *8 and 10*; and two comments *correctly pointing out* that "*some extra condition is required in Pr-10*", and hence we now provide here the corrected version of Pr-10. Readers can try their hand on both these problems till April 30, 2018.

3. We received from the floor **one incomplete** solution to problem *11.* Readers can try their hand on this till April 30, 2018.

4. We also received **two correct** solutions to the problem *12* and publish here one of them.

5. **No solutions** were received from the floor to the remaining problems *6 and 9.* Readers can try their hand on these till April 30, 2018.

In this issue we first present **six new problems**. Solutions to these problems as also to the remaining problems *6, 8, 9, 10, and 11* of MS, 86, 3-4, 2017, received from the floor till October 31, 2018, if approved by the Editorial Board, will be published in the MS, 87, 3-4, 2018.

Problems proposed by B. Sury

MS-2018, Nos. 1-2: Problem-1:

Find the largest possible diameter of a circular hole that can be covered by two square boards of unit area.

MS-2018, Nos. 1-2: Problem-2:

Find a characterization of the positive integers m such that S_n has an element of order m.

MS-2018, Nos. 1-2: Problem-3:

Consider a convex polygon of n sides. Draw $n-3$ diagonals which do not intersect inside. Then, the polygon is broken into $n-2$ triangles. Let a_i be the number of those triangles formed which have P_i as vertex. For instance, for $n=6$, one may draw diagonals P_1P_3, P_1P_4 and P_1P_5. in this case, $(a_1, a_2, a_3, a_4, a_5, a_6) = (4, 1, 2, 2, 2, 1)$. Notice that $4 - \frac{1}{1-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}} = 0$.

Prove that in general,

$$a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\cdots - \frac{1}{a_{n-1}}}}} = 0.$$

MS-2018, Nos. 1-2: Problem-4:

Find with proof, the unique solution to the following multiplication table where each digit 0 to 9 is repeated twice :

```
      * * *
    × * * *
  ---------
      * * *
    * * *
  * * *
  ---------
  * * * * *
  ---------
```

MS-2018, Nos. 1-2: Problem-5:

A spider starts crawling along an elastic rope at the rate of 1 foot per second. Initially, the elastic rope is f feet long and at the end of each second, it is stretched uniformly by f feet. Does the spider ever reach the other end of the rope?

MS-2018, Nos. 1-2: Problem-6: Proposed by **George E. Andrews.**

Prove that $PO_2(n)$, the number of partitions of n in which each odd part appears at most twice, must be even if $n \equiv 2$ *or* $3 (mod 4)$. For example, there are 8 such partitions of 6, namely, 6, $5+1$, $4+2$, $4+1+1$, $3+3$, $3+2+1$, $2+2+2$, $2+2+1+1$; and $6 \equiv 2 (mod 4)$.

MS-2017, Nos. 3-4: Problem-10. (corrected version):

Show that there does not exist a symmetric matrix over rational numbers whose characteristic polynomial is $X^2 - 3$.

Solution from the floor: MS-2017, Nos. 3-4: Problem 7: Evaluate

$$\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{3}}}.$$

(Solution submitted on 30-11-2017 by **Dasari Naga Vijay Krishna**, Machilipatnam, Andhra Pradesh-521001; *Vijay9290009015@gmail.com*).

Solution. Denote the given integral by I. Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$$I = \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{3}}dx}{1+(\tan x)^{\sqrt{3}}}.$$

Adding this to given integral we obtain $2I = \pi/2$. Hence $I = \pi/4$.

Actually the given integral is a particular case of the following general fact:

Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be a continuous fucntion such that $f(a+b-x) = 1/f(x), f(x) \neq 0$, for all x in (a,b). Then for all real exponents k, we have

$$\int_a^b \frac{dx}{1+f(x)^k} = \frac{b-a}{2}.$$

Here $a = 0, b = \pi/2, f(x) = tanx$ and $k = \sqrt{3}$; hence I=$\pi/4$.

Correct solution was also received from the floor from:

Subhash Chand Bhoriya.; (Village & Post Office - Bakali-136132, near Ladwa, Kurukshetra, Haryana, India; E-mail: *scbhoria89@gmail.com*, received on 29-11-2017)

R. Mabel Lizzy.; (Research Scholar, Dept. of Mathematics, Bharathiar University, Coimbatore-641046, India; E-mail: *mabel.math.bu@gmail.com*, received on 02-12-2017).

Prajnanaswroopa S. (Bangalore, Karnataka, India; *sntrm4@rediffmail.com*, received on 30-04-18).

Solution from the floor: MS-2017, Nos. 3-4: Problem 12: Show that

$$\prod_{n\geq 2}\left(\left(1-\frac{1}{n^2}\right)^{2(n^2-1)}\left(\frac{1+1/n}{1-1/n}\right)^n\right) = \pi.$$

(Solution submitted on 01-12-2017 by **Subhash Chand Bhoriya.**; Village & Post Office - Bakali-136132, near Ladwa, Kurukshetra, Haryana, India; E-mail: *scbhoria89@gmail.com*).

Solution. Let FP=$\prod_{n\geq 2}^m \left(\left(1-\frac{1}{n^2}\right)^{2(n^2-1)}\left(\frac{1+1/n}{1-1/n}\right)^n\right)$, the finite product. Then

$$\begin{aligned} FP &= \prod_{n\geq 2}^m \left(\left(1-\frac{1}{n}\right)^{2n^2-n-2}\left(1+\frac{1}{n}\right)^{2n^2+n-2}\right) \\ &= \prod_{n\geq 2}^m \left(1-\frac{1}{n}\right)^{2n^2-n-2} \prod_{n\geq 2}^m \left(1+\frac{1}{n}\right)^{2n^2+n-2}. \end{aligned}$$

We re-index the first product in the last step by changing n into $n+1$ and obtain

$$\begin{aligned}
FP &= \prod_{n\geq 1}^{m-1}\left(\frac{n}{n+1}\right)^{2n^2+3n-1}\prod_{n\geq 2}^{m}\left(\frac{n+1}{n}\right)^{2n^2+n-2}\\
&= \frac{1}{2}\left(\frac{m+1}{m}\right)^{2m^2+3m-1}\prod_{n\geq 1}^{m}\left(\left(\frac{n}{n+1}\right)^{2n^2+3n-1}\left(\frac{n+1}{n}\right)^{2n^2+n-2}\right)\\
&= \frac{1}{2}\left(\frac{m+1}{m}\right)^{2m^2+3m-1}\prod_{n\geq 1}^{m}\left(\frac{n}{n+1}\right)^{2n+1}\\
&= \frac{1}{2}\left(\frac{m+1}{m}\right)^{2m^2+3m-1}\frac{1^3}{2^3}.\frac{2^5}{3^5}.\frac{3^7}{4^7}\cdots\frac{(m-1)^{2m-1}}{m^{2m-1}}.\frac{m^{2m+1}}{(m+1)^{2m+1}},
\end{aligned}$$

and after some cancellation in the last step we obtain

$$FP = \frac{1}{2}\left(\frac{m+1}{m}\right)^{2m^2+3m-1}\frac{(m!)^2}{(m+1)^{2m+1}} = \frac{1}{2}\left(\frac{m+1}{m}\right)^{2m^2+m-2}\frac{(m!)^2}{m^{2m+1}}.$$

Multiplying and dividing the last expression of right side by $\frac{e^{2m}}{2\pi}$ we arrive at

$$FP = \frac{1}{2}\left(\frac{m+1}{m}\right)^{2m^2+m-2}\frac{(m!)^2e^{2m}}{2\pi m^{2m+1}}\times\frac{2\pi}{e^{2m}}.$$

Now taking limit as $m\to\infty$ on both sides we see that given product $= \pi\lim_{m\to\infty}A_m \times 1$, where $A_m = \frac{1}{e^{2m}}.\left(\frac{m+1}{m}\right)^{2m^2+m-2}$, as $\lim_{m\to\infty}\frac{(m!)^2e^{2m}}{2\pi m^{2m+1}} = 1$ in view of the well known Stirling Approximation. Let us now show that $\lim_{m\to\infty}A_m = 1$.

Observe that $\log A_m = (2m^2+m-2)\big(\log(m+1)-\log m\big)-2m$. Putting $m = 1/p$ in the last equality and taking $p\to 0$ we get,

$$\begin{aligned}
\lim_{p\to 0}\log A_{1/p} &= \lim_{p\to 0}\left(\frac{2}{p^2}+\frac{1}{p}-2\right)\left(\log\left(\frac{1}{p}+1\right)-\log\frac{1}{p}\right)-\frac{2}{p}\\
&= \lim_{p\to 0}\frac{(2+p-2p^2)\log(p+1)-2p}{p^2}. \qquad (0/0 \text{ form})
\end{aligned}$$

Applying L'Hospital's Rule we get,

$$\log\lim_{p\to 0}A_{1/p} = \lim_{p\to 0}\frac{-p-2p^2+(1-4p)(p+1)\log(p+1)}{2p(p+1)}. \qquad (0/0 \text{ form})$$

Again applying L'Hospital's Rule we obtain,

$$\log\lim_{p\to 0}A_{1/p} = \lim_{p\to 0}\frac{-8p+(1-4p)\log(p+1)-4(p+1)\log(p+1)}{2(2p+1)} = 0.$$

It follows that $\lim_{p\to 0}A_{1/p} = 1$, that is, $\lim_{m\to\infty}A_m = 1$. This completes the proof of our desired equality.

Correct solution was also received from the floor from:
Dasari Naga Vijay Krishna (Machilipatnam-521 001, Andhra Pradesh, India; E-mail: *Vijay9290009015@gmail.com*, received on 01-12-2017)

Solution by the Proposer B. Sury: MS-2017, Nos. 1-2: Problem 2: Let n be a positive integer and $d_1, \cdots, d_r$ be certain divisors of n (not necessarily all and not necessarily distinct). Let $a_1, \cdots, a_r, b$ be arbitrary integers. Determine the number of solutions of the congruence

$$a_1x_1 + \cdots + a_rx_r \equiv b \ mod \ n$$

for integers x_i satisfying $(x_i, n) = d_i$.

Solution. Attempted answers to this question over the years had gaps or they proved partial cases. Finally, now this has appeared as a theorem by Bibak, Khodakhast; Kapron, Bruce M.; Srinivasan, Venkatesh; Tauraso, Roberto; Töth, Läszlo in a paper titled "Restricted linear congruences", J. Number Theory 171 (2017), 128-144. The readers are invited to look at that paper for a wealth of results of this kind. We just recall the expression that gives the number we seek. Recall the Ramanujan sums defined by $c_n(m) = \sum_{(r,n)=1} exp(2ir\pi/n)$ which can also be expressed as $\sum_{d|(m,n)} d\mu(n/d)$. Then, the number of solutions of the congruence is

$$\frac{1}{n}(\prod_i \phi(n/d_i)) \sum_{d|n} c_d(b) \prod_i \frac{\mu(d/(a_id_i, d))}{\phi(d/(a_id_i, d))}.$$

Solution by the Proposer B. Sury: MS-2017, Nos. 3-4: Problem 5: Prove that $\sum_{k=1}^n \frac{1}{k^r} = \sum_I \binom{n}{i_r} \frac{(-1)^{r-1}}{i_1 i_2 \cdots i_r}$ where $I = (i_1, \cdots, i_r)$ runs through r-tuples satisfying $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$.

Solution. For each $k \geq 1$, consider

$$I(k,r) := \int_0^1 \frac{1}{x_1} \int_0^{x_1} \frac{1}{x_2} \cdots \int_0^{x_{r-1}} x_r^{k-1} dx_r \cdots dx_1$$

Note that only the last integrand is x_r^{k-1}; others are $\frac{1}{x_i}$. Simplification leads to $I(k,r) = \frac{1}{k^r}$ easily which gives $\sum_{k=1}^n I(k,r) = \sum_{k=1}^n \frac{1}{k^r}$.

On the other hand, rewrite $I(k,r)$ with the variable x_r changed to $y_r = 1 - x_r$. Summing over all k from 1 to n, we have

$$\sum_k I(k,r) = \int_0^1 \frac{1}{x_1} \cdots \int_{1-x_{r-1}}^1 \frac{1-(1-y_r)^n}{y_r} dy_r \cdots dx_1$$

$$= \sum_{l_r \geq 1} (-1)^{l_r - 1} \binom{n}{l_r} \int_0^1 \frac{1}{x_1} \cdots \int_{1-x_{r-1}}^1 y_r^{l_r-1} dy_r \cdots dx_1$$

Simplification gives us the RHS $\sum_{1 \leq i_1 \leq \leq \cdots \leq i_r \leq n} \binom{n}{i_r} \frac{(-1)^{i_r - 1}}{i_1 i_2 \cdots i_r}$ of our assertion.

FORM IV
(See Rule 8)

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