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Edited by J. R. PATADIA

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2. general survey articles, popular articles, expository papers, Book-Reviews.
3. problems and solutions of the problems,
4. new, clever proofs of theorems that graduate / undergraduate students might see in their course work,
5. research papers (not highly technical, but of interest to larger readership) and
6. articles that arouse curiosity and interest for learning mathematics among readers and motivate them for doing mathematics.

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EDITORIAL NOTE

The publication of the Mathematics Student began from 1933 as a quarterly to serve the students and teachers of mathematics. In fact, one observes the following sentence clearly printed on the front cover page of each volume of The Mathematics Student till 1952

A Quarterly Dedicated to the Service of Students and
Teachers of Mathematics in India

and notes that it used to have in its contents sections on *Questions and solutions, Mathematical notes, Classroom notes, Book-Reviews*, etc. right from the beginning till around mid-fifties, but it appears to have gradually disappeared then after.

During the past couple of years, it was widely felt among some of the well-wishers of the Indian Mathematical Society (the IMS) that the Mathematics Student has become a research level journal with no appeal to students; and it was suggested that since the IMS has no such journal for students nor perhaps is there any such journal published in India, the IMS seriously should address this deficiency. The thread was picked up by the Council of the Society, an Editorial Board for the Mathematics Student is now formed and, as a humble beginning, we are happy to resume the Problem Section beginning with this issue.

The Council is aware that the success to make this periodical more relevant, accessible and accptable to the larger section of mathematical community, our students and teachers of mathematics, lies in the active participation from the entire mathematical community of the motherland and earnestly appeals and invites one and all for the active participation to serve the cause of mathematics in this way. It may be noted here that for making it feasible for everyone to participate, the soft copy of the Mathematics Student is made available, **for free open access to all**, on Society's website:

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PRIZES, RECOGNITIONS AND PROMOTION OF MATHEMATICS*

S. G. DANI

Manjul Bhargava was awarded the Fields Medal this year, at the inaugural function of the International Congress of Mathematicians at Seoul, South Korea, on 13 August 2014. Joyous messages were exchanged, following the announcement, by the Indian mathematical community and the Indian people in general. Many mathematicians from India present at the function wrote home enthusiastically from the venue with whatever additional information they could gather. Many newspapers and magazines came out with special write-ups with different levels of outreach. It was an historic occasion, with a mathematician of Indian origin receiving the medal for the first time. Let me begin this talk by recording my heartiest congratulations to Prof. Manjul Bhargava for the achievement.

In fact the occasion was historic in more ways. Maryam Mirzakhani, another concurrent recipient of the medal, is the first woman to receive the honour. She is also the first mathematician from Iran to be awarded the Fields Medal. My ardent felicitations to her for the accomplishment; may she inspire our daughters and granddaughters to excel in mathematics.

The Fields Medal had been the unique coveted Prize in mathematics for a long time**, and has often been described loosely as the "Nobel Prize in mathematics" (the Nobel Prize not being awarded in mathematics), notwithstanding the vast difference between the prize amounts of the two.

In recent years however there have been other significant new prizes (including some with large monetary components). It would be worthwhile to recall the names of some of the Indian awardees in this respect. The first and foremost name that comes to mind is that of S. R. S. Varadhan, born and educated in India, getting

* The text of the Presidential Address (general) delivered at the 80th Annual Conference of the Indian Mathematical Society held at the Indian School of Mines, Dhanbad-826 004, Jharkhand, during the period December 27 - 30, 2014.

** The Fields Medal was first awarded in 1936, to Lars Alhfors and Jessy Douglas, but the next set of awards came only in 1950, on account of the disruption caused by the second world war; since then however it has been awarded every four years, and the number of awards was raised from two to four in 1966.

the Abel Prize in 2007 [1]. M. S. Narasimhan received the King Faisal International Prize in 2006. Among the prizes with an age restriction (such as not exceeding 40 years), like the Fields Medal, one may recall the names of Madhu Sudan and Subhash Khot, both born and educated in India, who were awarded the Nevanlinna Prize for Computer science in 2002 and 2014 respectively (the latter at the latest ICM at Seoul, along with Bhargava).

There are other major international honours received by Indian mathematicians. I may especially mention the Fellowship of the Royal Society of London; the legendary genius Srinivasa Ramanujan was the first Indian mathematician to be conferred the award, and in recent years M. S. Narasimhan, C. S. Seshadri, and M. S. Raghunathan have also been honoured with it. C. R. Rao has been another internationally celebrated Indian name, in the closely related field of Statistics; in 2002 he was honoured with the National Medal of Science, the highest award in U.S.A. in the scientific field, as a "prophet of new age". In yet another form of recognition I may mention the invitations to R. Balasubramanian and R. Parimala to deliver plenary talks at the International Congress of Mathematicians in 2010, held at Hyderabad. There are also many other international prizes, Fellowships and other honours, some with geographical conditions for eligibility - such as being from a developing country - that have come the way of Indian mathematicians. For young achievers there are the International Mathematical Olympiads every year, and so far 11 Gold Medals, 61 Silver Medals and 57 Bronze Medals have been awarded to Indian students, since India started participating in 1989.

As you would all know, there are also national prizes and recognitions, such as the Shanti Swarup Bhatnagar Prize, Infosys Prize, Srinivasa Ramanujan Medal of the Indian National Science Academy, B. M. Birla Prizes, Swarnajayanti Fellowships and J. C. Bose Fellowships of the Department of Science and Technology, Fellowships of the Indian Academies, Young Scientist awards, Young Associateships of the National Academies, recognition in National Mathematical Olympiads etc. It is not my purpose to go into a comprehensive account of the numerous honours and recognitions which mathematicians, of different ages, can look forward to earning in India, which you would be quite familiar with. My aim is to reflect on the role of the prizes and recognitions in promoting mathematics, and some issues around it, with the overall context as above in the background.

The role of a prize in motivating and inspiring individuals to excel in various activities needs no elaboration. Right from our infanthood we crave for appreciation and if it comes in kind that is all the better. We focus on developing those skills, from the assortment that we come endowed with, that have greater

[1]The Abel Prize was instituted in 2003 and Varadhan is one of the early recipients of the award.

potential for being appreciated. The measure and intensity of this may vary with the individual, but it remains a characteristic feature of our being. It is a notable aspect of human societies that this basic instinct in the individual is harnessed for collective good. The abilities of individuals are encouraged through spontaneous responses as well as systematic provisions that would work as inducements to perform. Prizes are but an instance of this phenomenon, focussing on a particular activity, such as sports, science, mathematics etc.

To be sure, people do not quite enter an area on account of the institutionalized prizes that it may have on offer. The benefits of instituting prizes and such other inducements, to the area and the society at large, accrue in a much more complicated fashion than that. Stephen Hawking is quoted to have said, on the occasion of being awarded the Milner Prize, "No one undertakes research in physics with the intention of winning a prize. It is the joy of discovering something no one knew before. Nevertheless prizes like these play an important role in giving public recognition for achievement in physics. They increase the stature of physics and interest in it.". The quote no doubt applies equally well to mathematics in place of physics.

Once we are in the area, the prizes and recognitions generate an interest in us, and make us aware of the intricacies of the subject, even when we do not quite envisage ourselves as being contenders for them. We exult over prize-winning ideas, exchange notes on prize-winners' achievements, argue over whether an award was well-deserved or not, and why someone else did not get it; (why was Harish-Chandra not awarded the Fields Medal?). Involvement of the community, a large section of it at any rate, gives the subject an identity, a sense of what is worthy of respect, worthy of being keenly pursued with all the resources that one may be able to muster.

Prizes and recognitions come in various forms. You would all be well aware of the million dollar prizes instituted by the Clay Mathematical Institute for specific problems identified for the purpose. Long before these, there were also other problems with a "prize-tag", such as Fermat's last theorem, which created popular interest in mathematics.[2] There are some prizes applicable to specific areas, such as Number Theory. Some are exclusively for women. Many like the Fields Medal, the Shanti Swarup Bhatnagar Prize and the more recent Infosys Prize have age restrictions. Some are designed to be in recognition of peak achievements, while others reward cumulative contributions. Then there are prizes for best papers, like our A. Narasinga Rao Medal, and also best thesis awards at various universities and institutions. These variations reflect the variety of objectives cherished by the

[2]Unfortunately the value of the prize amount for Fermat's last theorem dwindled to very little due to inflation by the time the theorem was proved!

community, or one or other section of it, while instituting the prizes. There are also many recognitions open to mathematicians of different ages, parallel to those in other sciences.

For the young there are the International Mathematical Olympiads (IMO), providing an avenue to compete at an international level. The selection of a team of 6 to represent the country each year is organized by the National Board for Higher Mathematics (NBHM) and the process involves two steps, the Regional Mathematical Olympiad (RMO), and the Indian National Mathematical Olympiad for those selected from RMO. The process exposes a large number of students to competitive problem-solving, thus creating in them an interest in mathematics. In the course of discussions in NBHM over Mathematical Olympiads many of us felt an acute need for a similar avenue at the college level. Rather fortuitously a project proposal in consonance with the perceived need was proposed by S. P. College, Pune, under the leadership of Prof. V. M. Sholapurkar. That led to the genesis, in 2009, of the "Madhava Mathematical Competition (MMC)", named after the 14 th century mathematician from Kerala, who produced innovative results and a new mathematical framework akin to Calculus as it developed in Europe over two hundred years later. The participation in MMC has grown exponentially over the years, giving a major fillip to quality in undergraduate studies in mathematics in the country, and I would like to record my congratulations here to Prof. Sholapurkar and the team around him for the laudable success.

Prizes, recognitions and competitions thus have a great potential in promoting mathematics. I sometimes wonder however whether enough care goes into formulating and instituting due processes for their execution, to make them effective instruments. In recent years the amounts associated with prizes have also grown enormously and I wonder if this is really a good thing. When someone who stands tall among the rest, by common consensus, is awarded a handsome prize, it becomes a joyous moment for the community as a whole. But such peaks are rare. Very often several others in competition are not far behind, if behind at all, by general agreement. Showering a windfall on a chosen one seems incongruous in such a situation. It could become a source of negative feeling about the process itself, and loss of collegiality in our institutions. To my mind prizes need to be attractive enough to be coveted, but limited enough not to seem like lottery winnings. Of course the precise line can not be drawn by an individual such as myself, but I do hope as a society we begin to take cognizance of this aspect.

The process for selection of awardees also involves various pitfalls. Firstly many prizes and recognitions involve nomination. This would be fine in an ideal academic set up where the leaders are alert to the developments all around, and would nominate candidates at the appropriate time. In today's world however, especially in mathematics, it is not easy to keep track of achievements of people and their relative merits. There are also prejudgements and inertia on the part of the nominators (especially in the face of the somewhat arduous nomination procedures) that take their toll. As a result many deserving candidates do not get the awards due to them, simply because they do not get nominated at the appropriate time.

It is also hard to associate a high degree of infallibility with the evaluation schemes for selection of the awardees. Firstly there are inherent uncertainties involved in the comparisons that need to be made, when the nominees are from a variety of branches, backgrounds etc. Then there are practical issues faced by the decision-making bodies, and members of committees they appoint, with regard to investing resources and time in arriving at the most appropriate candidate for the prize. I have had some occasions to be part of international committees in this respect (not the Fields Medal, I may clarify). Sometimes there were no meetings to discuss the merits of the individual candidates, and the decisions were arrived at by pooling together the rankings allotted by the individual committee members; one slight redeeming feature was that there was a short-listing round and then a concluding round. As the nominations were in varied fields it was not easy to make comparative judgements, and not surprisingly the pattern of relative scores of candidates varied widely with the members allotting them. Major prizes and recognitions in India do involve meetings of the committees involved, which is helpful, but in the overall context involving a variety of complex issues their role in the process of selection is often inadequate. In my cumulative assessment, it can be reasonably said that in general the awardee is *one of the best* candidates, and perhaps the best candidate *with reasonable probability* but typically the processes do not convincingly produce a unique candidate for the award. I do hope more thought will go into improving these systems.

These observations do not diminish in any way the superior merits of prize-winners. The point is to emphasize that some others around would also be worthy of our respect in the same way as the prize-winners. For the community as a whole the awards are an important instrument for promotion of the field, and for it to be effective, it needs to be ensured that no significant negative feeling is generated in the process. In this respect we need to inculcate a sense of rejoicing not only over the winners, but also over creativity coming from others, and over the whole process. At the level of the individual, as I often like to emphasize at prize-distribution events for young students, the journey towards excellence is as much a reward as the destination in the form of a prize, and we should cherish the experience dearly as well.[0.2 cm]

I will conclude after touching briefly upon one more issue in this context. Amid the large number of euphoric messages exchanged in the aftermath of Manjul Bhargava's award, I found one with a pensive, contemplative note. A mathematician with whom I have had good acquaintance and occasion to work with in the public sphere, wrote "Wonder why our talents sprout when we are abroad. Is there insufficient encouragement in our country or / and are we unable to spot talent (despite all our Olympiads and so on) and nurture it early on?" There are no easy answers to this, but think about it we must. While we may have a fondness for the international prizes and prize-winners, that we like to express from a safe distance, so to speak, we somehow do not seem to get into the game. How many of us can claim reasonable familiarity with works of the winners of international prizes like the Abel prize or the Fields Medal, or even the leading lights in a reasonably broad area of mathematics? Unfortunately the honest answer is very few. We are focused mostly on short-term gains and unfortunately our systems tend to encourage writing worthless papers in journals with no standing, rather than engaging with profound ideas and contributions with a deep commitment to knowledge. We have put in place various institutions and instruments to encourage talent, which would indeed help to an extent, but unless and until as a community we learn to involve ourselves seriously in mathematics, our efforts may have limited effect.

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DYNAMICS OF NUMBERS*

S. G. DANI

1. Introduction

Hermann Minkowski published in 1896 his book *Geometrie der Zählen* (Geometry of Numbers), which introduced in effect a profoundly new way of looking at problems of finding integral solutions to a large class of inequalities. This revolutionized the subject, and gave rise to the topic that now goes by the title of his book, which has thrived ever since. One of Minkowski's major achievements was to show that given an inequality of the form $Q(x_1, x_2, \ldots, x_n) < a$, where a is a positive real number and Q is a positive definite quadratic form in $n \geq 2$ variables (with real coefficients), integers $x_1, \ldots, x_n$, not all 0, can be found whenever a is large enough so that the volume of the set $\{v = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid Q(v_1, \ldots, v_n) < a\}$ exceeds 2^n. The convexity of the latter set in the Euclidean space, which is a geometric feature of the set of real solutions, was thus harnessed to derive a purely arithmetical conclusion about integral solutions; the bound one gets is not quite optimal, but it may be noted that the corresponding statement would obviously not hold if 2^n above is replaced by the volume of the unit ball in $\mathbb{R}^n$, viz. $\pi^{n/2}/\Gamma(\frac{n}{2}+1)$, where Γ is Euler's Gamma function. The Minkowski theory was later developed further by several mathematicians and applied to a host of problems in number theory, bolstering it also with analytic techniques.

Our story now moves in another direction. Let us consider an analogous inequality $|Q(x_1, x_2, \ldots, x_n)| < a$ where Q is a quadratic form which is indefinite, viz. neither positive definite, nor negative definite, or equivalently one that takes all real values over the set of n-vectors with real coordinates. Can we say that there must exist nontrivial integral solutions? In analogy with the above one may consider the sets $\{v \in \mathbb{R}^n \mid |Q(v)| < a\}$. We note however that these sets are not convex and the Minkowski theory does not provide any serious insights into the question. Indeed, one could consider the volumes of convex *subsets* of the sets as above and it can be seen that when a is large enough existence of a solution can be concluded for any quadratic form with a given discriminant, as in the preceding case. However, seen in the overall context of the problem this yields only a weak result, and the question remains for smaller values of a, for which also the volume of the set $\{v \in \mathbb{R}^n \mid |Q(v)| < a\}$ is infinite.

Such a question was considered by A. Oppenheim in his papers around 1930, where he proposed a conjecture that if Q is a nondegenerate indefinite quadratic

* The text of the Presidential address (technical) delivered at the 80th Annual Conference of the Indian Mathematical Society held at the Indian School of Mines, Dhanbad-826 004, Jharkhand, during the period December 27-30, 2014.

form in $n \geq 5$ variables which is not a scalar multiple of a form with integral coefficients, then the set $Q(\mathbb{Z}^n) = \{Q(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{Z}\}$, is dense in $\mathbb{R}$, namely for any $t \in \mathbb{R}$ and $\epsilon > 0$ there exist $x_1, \ldots, x_n \in \mathbb{Z}$ such that $0 < |Q(x_1, \ldots, x_n) - t| < \epsilon$. Some discussion on the significance of the conditions in the conjecture may be in order. The condition of Q being indefinite is a basic assumption with which we started, and is also evidently needed if $Q(\mathbb{Z}^n)$ is to be dense in $\mathbb{R}$. The nondegeneracy assumption is made for simplicity; without it the problem would be of a different nature and may be addressed separately (see [12] for some details). We note that if the coefficients of Q are integers then $Q(\mathbb{Z}^n)$ would consist of integers, and would not be dense. The analogous statement would also apply for quadratic forms which are scalar multiples of these forms, and as such they all need to be excluded from consideration. The condition $n \geq 5$ may be compared with the fact that by Meyer's theorem for a positive definite quadratic form with *integer* coefficients in $n \geq 5$ variables the equation $Q(x_1, \ldots, x_n) = 0$ admits nontrivial solutions, whereas the same is not true for $n \leq 4$; to see the analogy observe that for $t = 0$ the problem under consideration is an "approximate version" of that, with equality replaced by an "approximate equality" allowing for a small "error". Also, for $n \geq 5$ Oppenheim could prove the conjecture under the additional condition that Q admits nontrivial integral solutions, i.e. there exist $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \backslash \{0\}$ such that $Q(x_1, \ldots, x_n) = 0$. However, notwithstanding the analogy and the partial result of Oppenheim, with better understanding of the issues involved it was later realized that the condition $n \geq 5$ in Oppenheim's conjecture should be replaced by $n \geq 3$.

For the case $n = 2$ however the analogue of the statement in the Oppenheim conjecture indeed does not hold. Counterexamples for this arise from what are called badly approximable numbers. A real number α is said to be *badly approximable* if there exists a $\delta > 0$ such that for all $p, q \in \mathbb{Z}$, $q \neq 0$, $|\alpha - \frac{p}{q}| > \delta q^{-2}$. The concept has its origins in the theory of approximation of irrational numbers by rational numbers, and it is known that while for almost all real numbers α in fact there exist infinitely many rationals $\frac{p}{q}$, in reduced form, such that $|\alpha - \frac{p}{q}| \leq \frac{1}{\sqrt{5}} q^{-2}$, there exist α for which such an inequality does not hold even with a smaller positive constant in place of $\frac{1}{\sqrt{5}}$, viz. badly approximable numbers; here "almost all" is meant with respect to the Lebesgue measure, and in particular the set of badly approximable numbers has Lebesgue measure 0; moreover, a number α is badly approximable if and only if the partial quotients in its continued fraction expansion are bounded by a fixed constant (see [3], [22] for these results). Now if α is a badly approximable number and $\delta > 0$ is as above then $|q^2\alpha - pq| > \delta$ except when it is 0, and thus for the quadratic form $Q(x_1, x_2) = x_1x_2 - \alpha x_2^2$, $Q(\mathbb{Z}^2)$ is disjoint from $(0, \delta)$, and in particular is not dense. It also turns out that for a quadratic form Q given by $Q(x_1, x_2) = (x_1 - \alpha x_2)(x_1 - \beta x_2)$, $\alpha, \beta \in \mathbb{R}$, with α, β nonzero and $\alpha \neq \beta$, $Q(\mathbb{Z}^2)$ is not dense when α and β are both badly approximable (see for instance [12]).

The Oppenheim conjecture was proved, for all $n \geq 3$, by G.A. Margulis in 1986, thus establishing the following:

Theorem 1.1. *Let Q be a nondegenerate indefinite quadratic form in $n \geq 3$ variables, which is not a scalar multiple of a form with integral coefficients. Then*

$Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$, viz. for all $t \in \mathbb{R}$ and $\epsilon > 0$ there exist integers $x_1, \dots, x_n$ such that

$$|Q(x_1, \dots, x_n) - t| < \epsilon.$$

For over half a century since the conjecture was formulated, partial results were obtained on the conjecture by various authors. A 1956 paper of H. Davenport settled the conjecture for $n \geq 74$. Through subsequent works of Birch, Davenport and Ridout the conjecture was established by 1968 for $n \geq 21$, together with partial results for smaller values of n under additional conditions on the quadratic form, such as diagonalizibility, signature etc. (see [25] for a detailed account). The number theoretic methods however have an inherent limitation for smaller number of variables; in [1], which appeared as late as 1987, while making some further contributions on the problem via these methods, it was remarked about the conjecture that it "does not seem likely to be settled soon at the present rate of progress".

Unlike the earlier papers on the problem, Margulis's proof is based on a different strategy, involving study of a class of dynamical systems on the space of lattices (see below). The strategy itself is involved also in some earlier papers of the present author on values of systems of linear forms over integral tuples. Motivated by certain results in the thesis of Jyotsna Dani [5], M.S. Raghunathan introduced a strategy for attacking the Oppenheim conjecture via dynamics on homogeneous spaces, which was later adopted by Margulis for solving the conjecture. Raghunathan also proposed a stronger a conjecture about a class of flows on homogeneous spaces (see following sections) which was to have far reaching consequences. The Raghunathan conjecture was proved by Marina Ratner around 1990 and since then it has had a large number of applications in number theory and geometry.

The topic is now generally referred as "Homogeneous Dynamics", which involves the study of actions on homogeneous spaces induced by subgroups of the ambient group, and its applications. I believe the number theoretic aspect in fact qualifies to be called "Dynamics of numbers", in analogy with "Geometry of numbers". The aim of this article is to give an overall introduction to this topic. The exposition is intended for a broad readership and I shall not strive for Lie group-theoretic generality while describing the results. I may also mention that the focus will be on how the material can be viewed in hindsight, and not how it evolved historically; for a historical perspective the reader is referred to [26], [27] and [11].

In the next section we begin by introducing the "space of lattices", which is at the heart of the topic, and the dynamics on it. We shall then give applications of the theme, made prior to and after Margulis's proof of the Oppenheim conjecture, and in particular indicate some details of the proof of the latter.

2. Spaces of lattices

We denote by $\mathbb{R}^n$, $n \geq 2$, the usual n-dimensional Euclidean space. We shall identify it as the space of column vectors with n rows, and consider it equipped with the usual Euclidean norm, denoted by $\|\cdot\|$. We denote by $\{e_1, \dots, e_n\}$ the standard basis of $\mathbb{R}^n$. By a lattice Λ in $\mathbb{R}^n$ we mean a subgroup of $\mathbb{R}^n$ generated by n linearly independent vectors, viz. a basis of $\mathbb{R}^n$. Thus in particular $\mathbb{Z}^n$, consisting of all column vectors with integer entries, is a lattice, generated by $\{e_1, \dots, e_n\}$. A lattice Λ is a discrete subgroup and the quotient $\mathbb{R}^n/\Lambda$ is compact. Conversely, a discrete subgroup Λ such that $\mathbb{R}^n/\Lambda$ is compact is a lattice. Let Λ be a lattice in $\mathbb{R}^n$, say generated by a basis $\{v_1, \dots, v_n\}$. Consider the parallelopiped in $\mathbb{R}^n$ with the 2^n vertices at $\sum_{i=1}^n \delta_i v_i$, with $\delta_i = 0$ or 1 for all i. The volume of such a parallelopiped is independent of the choice of the basis $\{v_1, \dots, v_n\}$ and depends only on the lattice Λ, and it is called the *discriminant* of Λ. We shall denote by $\mathfrak{L}$ the space of all lattices in $\mathbb{R}^n$ with discriminant 1.

Let $G = SL(n, \mathbb{R})$, the group of $n \times n$ matrices with determinant 1, under matrix multiplication, equipped with the usual topology. Then we have a natural action of G on $\mathbb{R}^n$ by matrix multiplication on the left; for $g \in G$ and $v \in \mathbb{R}^n$ we shall denote by gv the outcome of the action of g on v. Given a lattice Λ in $\mathbb{R}^n$ and $g \in G$ we get a new lattice, consisting of vectors $\{gv \mid v \in \Lambda\}$, which we shall denote by $g\Lambda$; we note that the discriminant of $g\Lambda$ is the same as that of Λ, and thus if $\Lambda \in \mathfrak{L}$ then $g\Lambda \in \mathfrak{L}$. In other words, this gives us an action of G on $\mathfrak{L}$. Moreover, the action is *transitive* which means that given $\Lambda_1, \Lambda_2 \in \mathfrak{L}$ there exists $g \in G$ such that $g\Lambda_1 = \Lambda_2$, and in particular for every $\Lambda \in \mathfrak{L}$ there exists $g \in G$ such that $\Lambda = g\mathbb{Z}^n$.

On $\mathfrak{L}$ we define a topology as follows: Given a lattice $\Lambda \in \mathfrak{L}$ generated a basis $\{v_1, \dots, v_n\}$ and any $\epsilon > 0$ the set of lattices Λ' generated by a basis $\{w_1, \dots, w_n\}$ where $w_1, \dots, w_n \in \mathbb{R}^n$ are such that $\|w_i - v_i\| < \epsilon$ for all i is considered an open neighbourhood of Λ. It can be verified that this system of neighbourhoods defines a topology on $\mathfrak{L}$, with respect to which $\mathfrak{L}$ is a locally compact second countable Hausdorff topological space. Also it can be seen that the G-action on $\mathfrak{L}$ as above, namely the map $(g, \Lambda) \mapsto g\Lambda$, is continuous.

Let $\Gamma = SL(n, \mathbb{Z})$, the subgroup consisting of all matrices in G with integer entries. We note that Γ is precisely the subgroup consisting of elements of G whose action leaves $\mathbb{Z}^n$ invariant, viz. $\Gamma = \{g \in G \mid g\mathbb{Z}^n = \mathbb{Z}^n\}$. We denote by G/Γ the quotient space, consisting of the cosets of the form $x\Gamma$, $x \in G$, equipped with its quotient topology. Since the G-action on $\mathfrak{L}$ is transitive, it follows that $\mathfrak{L}$ is in canonical bijective correspondence with the quotient space G/Γ, given by $g\mathbb{Z}^n \leftrightarrow g\Gamma$. It can further be verified that this correspondence is a homeomorphism of $\mathfrak{L}$ and G/Γ. Thus a sequence $\{\Lambda_j\}$ in $\mathfrak{L}$ converges to a $\Lambda \in \mathfrak{L}$ if and only if there exists a sequence $\{g_j\}$ in G such that $g_j \to e$, the identity element in G, and $\Lambda_j = g_j\Lambda$ for all j.

We now recall some properties which enable us to relate $\mathfrak{L}$ to questions in Diophantine approximation. Firstly here is a simple fact which is easy to prove; the proof is left to the reader (note that it suffices to prove the statement for dense *sequences* in place of general dense subsets, in view of the separability of $\mathfrak{L}$).

Proposition 2.1. *Let $\mathfrak{S}$ be a dense subset of $\mathfrak{L}$. Then $\cup_{\Lambda \in \mathfrak{S}} \Lambda$ is a dense subset of $\mathbb{R}^n$.*

A second, deeper connection is given by what is called Mahler's criterion, asserting the following.

Theorem 2.2. *A sequence $\{\Lambda_j\}$ in $\mathfrak{L}$ is contained in a compact subset of $\mathfrak{L}$ if and only if there is a neighbourhood Ω of 0 in $\mathbb{R}^n$ such that $\Lambda_j \cap \Omega = \{0\}$ for all j.*

The reader is urged to get an intuitive feel for what this means. Another interpretation of the statement is that a sequence $\{\Lambda_j\}$ diverges (i.e. has no limit point in $\mathfrak{L}$) if and only if there exist $x_j \in \Lambda_j$, $x_j \neq 0$ for all j, such that $x_j \to 0$ as $j \to \infty$. Since one can easily construct sequences of lattices in $\mathfrak{L}$ containing nonzero vectors arbitrarily close to 0, this points to $\mathfrak{L}$ being a noncompact space; this fact however can be proved more directly and is not meant to be deduced from Theorem 2.2, which is a deeper result (see, for instance [14] for a proof).

Let me now illustrate how the above results are used, taking the example of the Oppenheim conjecture proved by Margulis, described in the last section. Let Q be a nondegenerate indefinite quadratic form, viewed as a function on $\mathbb{R}^n$. We are interested in the set of values of Q over integer points, viz. $Q(\mathbb{Z}^n)$. Let $H = \{g \in G \mid Q(gv) = Q(v) \text{ for all } v \in \mathbb{R}^n\}$, which is called the *special orthogonal group* corresponding to Q, and is denoted by $SO(Q)$. Suppose we show that the set of lattices $\{h\mathbb{Z}^n \mid h \in H\}$ is dense in $\mathfrak{L}$. Then by Proposition 2.1 $\cup_{h\in H} h\mathbb{Z}^n$ is dense in $\mathbb{R}^n$, and since $Q(\mathbb{Z}^n) = Q(\cup_{h\in H} h\mathbb{Z}^n)$ it follows that $Q(\mathbb{Z}^n)$ coincides with the set of values of Q over a dense subset of $\mathbb{R}^n$ and since $Q(\mathbb{R}^n) = \mathbb{R}$, this implies that $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$. It turns out that when $n \geq 3$ and Q is not a scalar multiple of a form with integer coefficients, as in the hypothesis of the Oppenheim conjecture, the set of lattices $\{h\mathbb{Z}^n \mid h \in H\}$ is in fact dense in $\mathfrak{L}$ and the above strategy would prove the conjecture.

Actually the conjecture was first proved by Margulis, by establishing a weaker result, and using Theorem 2.2 rather than Proposition 2.1, as follows: Firstly, it is straightforward to see that it suffices to prove the assertion for $n = 3$; the higher-dimensional case can be dealt with by substituting for some of the extra variables in terms of the others (the choice for this needs to be made carefully however). Margulis proved that when $n = 3$ and Q is not a scalar multiple of a form with integral coefficients, $\{h\mathbb{Z}^n \mid h \in H\}$ is not contained in a compact subset of $\mathfrak{L}$. By Theorem 2.2 this implies that $\cup_{h\in H} h\mathbb{Z}^n$ contains a sequence of nonzero vectors converging to 0. Thus for any $\epsilon > 0$ there exist $h \in H$ and $p \in \mathbb{Z}^3$ such that $|Q(p)| = |Q(hp)| < \epsilon$. For Q for which there are no nonzero integral points p such that $Q(p) = 0$ a (relatively easy) theorem of Oppenheim asserts that the preceding conclusion implies that $Q(\mathbb{Z}^3)$ is in fact dense in $\mathbb{R}$; so we are through in this case. Via a somewhat technical argument Margulis upheld the conclusion also to forms which may admit integral solutions, thus completing the proof of the conjecture.

Soon after the Oppenheim conjecture was proved by Margulis, along the lines indicated above, I had an opportunity to work jointly with him and we showed, in [15], that for $n = 3$ the set of lattices $\{h\mathbb{Z}^n \mid h \in H\}$ is in fact dense in $\mathfrak{L}$ whenever it is not closed, and it is closed only when Q is a scalar multiple of a form with integer coefficients. This result shows also that for the forms as in the Oppenheim conjecture the set of values of Q over the set of primitive integral vectors (viz. those in which the entries are not all divisible by any integer $d \geq 2$) is also dense in $\mathbb{R}$; the argument for this is similar to the above but we shall not go into the details here (see [15]). A further simplified proof of this was presented in [16]; see also [13].

Since these results were proved in the 1980's there has been considerable development in the topic. Later I shall give a brief exposition of the results and the current directions. Before that it would be appropriate to discuss the Raghunathan conjecture mentioned in the introduction.

3. The Raghunathan conjecture

A $n \times n$ matrix ν is said to be nilpotent if there exists a natural number r such that $\nu^r = 0$, the zero matrix; we recall from linear algebra that if there exists such an integer r then $\nu^n = 0$. Let ν be a nilpotent $n \times n$ matrix and for any $t \in \mathbb{R}$ set $\exp t\nu = \sum_{j=0}^{\infty}(t^j/j!)\nu^j = \sum_{j=0}^{n}(t^j/j!)\nu^j$; ($\nu^0$ is understood to be the identity matrix). Then $\{\exp t\nu\}$ is a "one-parameter subgroup" of G, in the sense that $\exp(s+t)\nu = \exp s\nu \exp t\nu$ for all $s, t \in \mathbb{R}$, and $t \mapsto \exp t\nu$ is a continuous map. Also, for any t all eigenvalues of $\exp t\nu$ are 1; such a matrix is said to be *unipotent*. We call a subgroup U of G a *unipotent one-parameter subgroup* if there exists a nilpotent $n \times n$ matrix ν such that $U = \{\exp t\nu \mid t \in \mathbb{R}\}$.

Raghunathan's conjecture, restricted to the context of the G-action on $\mathfrak{L}$ as above, is that if $\Lambda \in \mathfrak{L}$ and U is a unipotent one-parameter subgroup, then there exists a closed subgroup F_U of G (which would depend also on Λ, but we shall not involve Λ in the notation) such that the closure $\{u\Lambda \mid u \in U\}$ in $\mathfrak{L}$ is precisely $\{f\Lambda \mid f \in F_U\}$. Once this is established it can be deduced, via some technical arguments, that if H is a closed subgroup of G which is generated by the unipotent one-parameter subgroups contained in it, then the closure of $\{h\Lambda \mid h \in H\}$ also has the form $\{f\Lambda \mid f \in F_H\}$ for some closed subgroup F_H of G; for every unipotent one-parameter subgroup U contained in H we have $F_U \subset F_H$ and there exist U for which $F_H = F_U$.

Now consider a nondegenerate indefinite quadratic form Q on $\mathbb{R}^n$. The special orthogonal group associated with Q (see §2) has two connected components and we denote by H the connected component of the identity. It turns out that when $n \geq 3$, H is generated by the unipotent one-parameter subgroups contained in it. (For instance, if $Q_0(x_1, x_2, x_3) = x_1x_3 - 2x_2^2$, then with $\nu^+ = E_{12} + E_{23}$ and $\nu^- = E_{21} + E_{32}$, the one-parameter subgroups $\{\exp t\nu^+\}$ and $\{\exp t\nu^-\}$ can be seen to generate the subgroup H as defined above; the general case follows from equivalence of quadratic forms.) Thus from what was noted above there exists a closed subgroup F_H such that the closure of $\{h\Lambda \mid h \in H\}$ in $\mathfrak{L}$ is precisely $\{f\Lambda \mid f \in F_H\}$. Such a subgroup F_H has to be connected, and in the present instance this further implies that F_H is either H itself, or the whole of G. Now suppose the Raghunathan conjecture is established. For $n \geq 3$, we choose $\Lambda = \mathbb{Z}^n$ and get that $F_H = H$ or G; if $F_H = G$, it means that $\{h\Lambda \mid h \in H\}$ is dense in $\mathfrak{L}$, which as seen above implies that $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$, and if $F_H = H$ it can be proved (with techniques that were known earlier) that Q is a scalar multiple of a form with integer coefficients. Thus Raghunathan's conjecture implies the Oppenheim conjecture. This was a strategy proposed by Raghunathan to attack the Oppenheim conjecture. Margulis was inspired by the strategy, though he actually proved the conjecture by establishing a weaker result as noted above.

4. Ergodic theory on $\mathfrak{L}$

The discussion in the preceding sections highlights the importance of understanding the closures of the orbits $\{h\Lambda \mid h \in H\}$, where $\Lambda \in \mathfrak{L}$ and H is a subgroup of $G = SL(n, \mathbb{R})$; without loss of generality we may assume H to be closed. If H is compact then each orbit is closed; it follows in particular that it is not dense in $\mathfrak{L}$, since in that case it would have to be the whole of $\mathfrak{L}$ and simple geometrical considerations show that no compact subgroup of G acts transitively on $\mathfrak{L}$. In the light of this we shall consider actions of only noncompact closed subgroups.

One crucial fact that comes in handy in studying the orbits, at least the "typical orbits" is that the space $\mathfrak{L}$ admits a Borel measure (viz. a measure defined on the σ-algebra consisting of all Borel subsets with respect to the locally compact topology) m such that $m(\mathfrak{L}) = 1$ and $m(gE) = m(E)$ for all $g \in G$ and Borel subsets E, viz. m is invariant under the action of G on $\mathfrak{L}$; here and in the sequel gE denotes the set $\{gx \mid x \in E\}$.

A subset E of $\mathfrak{L}$ is said to be H-invariant, where H is a subgroup of G, if $hE = E$ for all $h \in H$. Also, a sequence $\{h_j\}$ in a noncompact locally compact group H is said to be *divergent* if it has no limit point in H.

Theorem 4.1. *Let H be a noncompact closed subgroup of G. Then we have the following:*

i) if E is a Borel subset of $\mathfrak{L}$ which is H-invariant then $m(E) = 0$ or $m(E) = 1$;

ii) if E and F are two Borel subsets of $\mathfrak{L}$ and $\{h_j\}$ is a divergent sequence in H then $m(h_jE \cap F) \to m(E)m(F)$ as $j \to \infty$.

In terms of the nomenclature in Ergodic theory assertion (i) states that the H-action on $\mathfrak{L}$ is *ergodic* and (ii) states that the action is *mixing*. It is easy to see that (ii) implies (i), and it may seem redundant to state (i) separately. However (ii) is substantially more difficult to prove than (i). Also (i) generalises to a larger class of actions, in a broader context, than (i). Thus it would be fruitful to think of them independently, in spite of one being stronger than the other. The first property is a special case of "Moore's ergodicity theorem" and the second is a special case of a theorem of Howe and Moore (see [2] for details).

The theorem has the following topological consequence:

Corollary 4.2. *Let H be a noncompact closed subgroup of G. Then the H-orbits of almost all $\Lambda \in \mathfrak{L}$ are dense in $\mathfrak{L}$; i.e.*

$$m(\{\Lambda \in \mathfrak{L} \mid \overline{H\Lambda} \neq \mathfrak{L}\}) = 0.$$

Moreover, if $\{h_j\}$ is a divergent sequence in H then for almost all Λ, $\{h_j\Lambda\}$ is dense in $\mathfrak{L}$; i.e.

$$m(\{\Lambda \mid \overline{\{h_j\Lambda \mid j \in \mathbb{N}\}} \neq \mathfrak{L}\}) = 0.$$

It is a simple exercise to deduce the Corollary from the theorem, keeping in mind that $\mathfrak{L}$ is a second countable space. The general form of the first assertion, for a more general ergodic action on a second countable space, is called Hedlund's lemma, after G.A. Hedlund who first formulated and applied it (see [2]). The second assertion is proved via an analogous argument using the mixing property.

Corollary 4.2 readily implies the assertion as in the Oppenheim conjecture for "almost all" quadratic forms Q, via the argument indicated in the last section; here by "almost all quadratic forms" we mean quadratic forms $v \mapsto Q_0(gv)$, with Q_0 a fixed nondegenerate indefinite quadratic form and g in a subset of G whose complement has zero Haar measure in G. To prove the conjecture for *all* quadratic forms one needs to know about closures of *all* orbits of $SO(Q)$.

Ergodic theory also concerns distribution of orbits of dynamical systems in the ambient space.

Definition 4.3. Let φ be a homeomorphism of a locally compact space X and μ be a probability measure on X. The orbit $\{\varphi^j(x) \mid j = 1, 2, \dots\}$ is said to be uniformly distributed with respect to μ if for every bounded continuous function f on X,

$$\frac{1}{n}\sum_{j=0}^{n} f(\varphi^j(x)) \to \int f d\mu \quad \text{as } n \to \infty.$$

When the above condition holds for all bounded continuous functions it holds also for characteristic functions of Borel sets E such that the (topological) boundary of E has μ-measure zero, and the statement then signifies that for such sets the orbit visits the set E with asymptotic frequency equal to the "proportion of E" in the whole space viz. $m(E)$; this is referred to as time averages converging to space averages. By Birkhoff's ergodic theorem when the action is ergodic with respect to m, the orbits of *almost all* orbits are uniformly distributed.

An analogous notion of uniform distribution of orbits can be defined for one-parameter flows $\{\varphi_t\}$ on X as above, in place of single homeomorphisms, by requiring that $\frac{1}{T}\int_0^T f(\varphi_t(x))dt \to \int f dm$ as $T \to \infty$, for all bounded continuous functions f. Also, by Birkhoff's ergodic theorem ergodicity of a flow implies that orbits of almost all points are uniformly distributed. We note that when an orbit is uniformly distributed with respect to a probability measure that assigns positive measure to all nonempty open subsets, it is necessarily dense. The converse is not true however - certain orbits which are dense, may visit some sets with greater, or lesser, frequency than the measure of the set.

While for every closed noncompact subgroup H of G *almost all* orbits of H on $\mathfrak{L}$ are dense in $\mathfrak{L}$, it turns out that the nature of the set of Λ in $\mathfrak{L}$ for which it holds depends intricately on the subgroup H. Broadly two types of behaviour, of opposite kinds, are worth highlighting. For a class of subgroups H the set of exceptional orbits, viz. those for which the orbit is not dense, is "small"; specifically, in these cases the set is a countable union of lower dimensional submanifolds of $\mathfrak{L}$, and sometimes even empty (viz. all orbits are dense). Furthermore in this case the closures of the individual orbits are nice - they are submanifolds, and even homogeneous spaces. This phenomenon is an aspect of what is generally termed as "rigidity" of the action. Such a behaviour exemplified by actions, on $\mathfrak{L}$, by unipotent one-parameter subgroups, and subgroups that are generated by such subgroups; these actions we shall briefly refer to as unipotent flows. On the other hand for one-parameter subgroups of the form $\{\delta_t = \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \mid t \in \mathbb{R}\}$, where $\lambda_1, \dots \lambda_n \in \mathbb{R}$ and $\lambda_1 + \cdots + \lambda_n = 0$ (so that $\delta_t \in G$ for all $t \in \mathbb{R}$), while the action is ergodic and almost all orbits are dense, the set of Λ for which it is not true is quite complicated, and it is "large" by various criteria such as the Hausdorff dimension (even though it is of measure 0). In a far deeper sense that we shall not go into, the action is "chaotic". In the next two sections we shall discuss these two classes of examples in some respects, focussing especially on their connection with problems in Diophantine approximation.

5. Behaviour of orbits of unipotent flows

To begin with let us consider the case with $n = 2$; thus $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$, and H the unipotent one-parameter subgroup $\{\exp tE_{12} \mid t \in \mathbb{R}\}$. Then it can be seen that any $\Lambda \in \mathfrak{L}$ (now a lattice in $\mathbb{R}^2$, with discriminant 1) the H-orbit of Λ is periodic if Λ contains λe_1 for some $\lambda \neq 0$, and such an orbit is not dense. It was proved by G.A. Hedlund, way back in 1936, that these are the only orbits that are not dense. In matrix form $\Lambda \in \mathfrak{L}$ satisfies the above condition if and only if $\Lambda = p\mathbb{Z}^2$, with p an upper triangular matrix in G. It may be seen that these lattices form a 2-dimensional cylinder, densely embedded in $\mathfrak{L}$ (which in this case is 3-dimensional).

The situation is analogous with regard to the issue of the orbits being uniformly distributed. In [7] I proved that all the non-periodic orbits as above are in fact uniformly distributed over $\mathfrak{L}$, with respect to the canonical invariant probability measure, viz. m as in § 4.

Raghunathan's conjecture, and its extension due to Margulis mentioned earlier, involve in particular that for any unipotent flow the set of exceptional orbits form a countable union of lower dimensional submanifolds - the lower dimensional manifolds in question are in fact homogeneous spaces, namely orbits of closed subgroups. Though in the above example the exceptional orbits lay on a single cylinder, in general more submanifolds may be involved, and they need not be mutually disjoint; e.g. consider $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$, where $n \geq 3$, and N the subgroup of G consisting of all the elements which fix each of the vectors $e_1, \dots, e_{n-1}$; for any $k = 1, \dots n-1$ let P_k be the subgroup of G consisting of all elements which leave invariant the subspace spanned by $\{e_1, \dots, e_k\}$; then for any k, and $\Lambda \in P_k\mathbb{Z}^n$ the N-orbit of Λ is not dense, and these form mutually intersecting submanifolds of exceptional orbits of N. We mention also that in general the number of submanifolds involved can be (countably) infinite. Orbit-closures of N as above and other analogous "horospherical" subgroups in $SL(n, \mathbb{R})$, acting on $\mathfrak{L}$ were studied in a paper of S. Raghavan and the present author [19], generalizing the above mentioned result of Hedlund for $n = 2$; the results confirmed the Raghunathan-Margulis conjecture for the subgroups in question; of course the results are now subsumed by Ratner's general results. The following simple application of the results of [19] to Diophantine approximation may be of broader interest.

Corollary 5.1. *Let $1 \leq p \leq n-1$, $V = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (p copies) and consider the action of $\Gamma = SL(n, \mathbb{Z})$ on V, defined by the natural action on each component.*

Then for $v = (v_1, \ldots, v_p) \in V$, where $v_1, \ldots, v_p \in \mathbb{R}^n$, the Γ-orbit of v is dense if and only if no linear combination $\sum_{i=1}^{p} \lambda_i v_i$, with $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$, not all 0, is a vector with rational coordinates.

The conjectures of Raghunathan and Margulis, and also a conjecture about classification of invariant measures of the actions that I formulated in [6] to serve as a route towards understanding orbit closures, were all proved by Marina Ratner around 1990. The general results are in terms of Lie groups and homogeneous spaces with finite invariant measure, and we shall not go into the details. For actions of unipotent one-parameter subgroups on $\mathfrak{L}$ the theorem yields the following. The reader is referred to [29] for an exposition of Ratner's theory, and [11] for a survey of the area.

Theorem 5.2. *Let $U = \{u_t\}$ be a unipotent one-parameter subgroup of $G = SL(n, \mathbb{R})$ and consider the action of $\{u_t\}$ on $\mathfrak{L}$. Let $\Lambda \in \mathfrak{L}$. Then there exists a closed subgroup F of G such that $\overline{U\Lambda} = F\Lambda$. The set $F\Lambda$ admits an F-invariant probability measure λ, and the orbit of Λ under $U = \{u_t\}$ is uniformly distributed with respect to λ. In particular, if Λ is such that there is no proper subgroup F containing U for which $F\Lambda$ is closed, then the U-orbit of Λ is uniformly distributed over $\mathfrak{L}$.*

Just as study of orbits being dense enables proving existence of solutions of inequalities as in Oppenheim's conjecture, study of distribution of orbits (which however involves stronger and more technical results than in the above discussion) leads to results on asymptotic growth of the number of integral solutions of the inequalities. We shall not go into the technical details of this but mention the following results arising from the study.

Theorem 5.3. *Let Q be a quadratic form as in Theorem 1.1. Let $a, b \in \mathbb{R}$, $a < b$, and $\epsilon > 0$ be given. For $r > 0$ let $B(r)$ denote the closed ball in $\mathbb{R}^n$ with center at 0 and radius r. Then we have the following (here # stands for "cardinality of"):*

i) there exists r_0 such that for all $r \geq r_0$,

$$\#\{x \in \mathbb{Z}^n \cap B(r) \mid Q(x) \in (a, b)\} \geq (1 - \epsilon)\mathrm{vol}(\{v \in B(r) \mid Q(v) \in (a, b)\});$$

ii) if $n \geq 5$, then $\dfrac{\#\{x \in \mathbb{Z}^n \cap B(r) \mid a < Q(x) < b\}}{\mathrm{vol}(\{v \in B(r) \mid a < Q(v) < b\})} \to 1$ *as $r \to \infty$.*

Assertion (i) was proved in a paper of Margulis and myself, and (ii) was proved by Eskin, Margulis and Mozes. The reader is referred to [26], [27] and [11] for details, and further results along the theme. It may be noted that the volumes involved in the above expressions are asymptotically of the order cr^{n-2} for a constant c; see [26] for more details.

6. Behaviour of diagonal flows

As before let us consider first the case when $n = 2$; upto scaling the diagonal one-parameter subgroup in G is given by $\{\delta_t\} = \{\mathrm{diag}(e^t, e^{-t})\}$. This subgroup is of importance in hyperbolic geometry; it represents the "geodesic flow" associated with what is called the modular surface. We briefly recall the connection. In the last section this will also be involved in relating the dynamics of the action of $\{\delta_t\}$ as above with values of binary quadratic forms.

Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, called the Poincaré upper half plane. It is one of the standard models for hyperbolic geometry, equipped with the Poincaré metric. While we shall not go into the details of the metric (see, for instance, [2] for details), it may be recalled that the geodesics in this geometry are precisely semicircles with endpoints on $\mathbb{R} \cup \{\infty\}$ (the endpoints themselves are not part of $\mathbb{H}$, and hence of the geodesics); here, as usual we interpret vertical lines in $\mathbb{H}$ as semicircles with one endpoint at ∞. We have an action of the group $G = SL(2,\mathbb{R})$ on $\mathbb{H}$, in which $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ acts by $z \mapsto \dfrac{az+b}{cz+d}$, for all $z \in \mathbb{H}$. The action of each element is an isometry of $\mathbb{H}$ with respect to the Poincaré metric (in fact G is the group of all orientation-preserving isometries of $\mathbb{H}$). The "modular surface" is the quotient space $M = \mathbb{H}/\Gamma$, where $\Gamma = SL(2,\mathbb{Z})$, under the action as above. The metric on $\mathbb{H}$ defines a metric on the quotient (except at some "singular" points which shall not concern us) and the corresponding geodesics are images of the geodesics in $\mathbb{H}$ described above.

Using the G-action as above the geodesic flow associated with the modular surface (which is defined on set of pairs (z,ζ), with $z \in M$ and ζ a unit tangent direction at z) may be identified with the flow on $\mathfrak{L}$ given by $\Lambda \mapsto \delta_t\Lambda$, with δ_t as above (see for instance [2], [14]). Via this correspondence, for each trajectory (viz. the "forward" orbit) $\{\delta_t\Lambda \mid t \geq 0\}$, $\Lambda \in \mathfrak{L}$, we have the forward segment of the geodesic over which it lies, and in turn the endpoint of the forward segment of the semicircle in $\mathbb{R}\cup\{\infty\}$; for $\Lambda \in \mathfrak{L}$ let $e(\Lambda)$ denote this endpoint. Analogously there is an endpoint corresponding to the "backward" orbit $\{\delta_t\Lambda \mid t \leq 0\}$, which we shall denote by $e^-(\Lambda)$. Then we have the following interesting interconnections; note that in each of the cases below the orbit of Λ under $\{\delta_t\}$ is not dense in $\mathfrak{L}$.

Theorem 6.1. *i) $\{\delta_t\Lambda \mid t \geq 0\}$ is divergent (viz. has no limit point in $\mathfrak{L}$ as $t \to \infty$), if and only if $e(\Lambda)$ is either ∞ or rational number in $\mathbb{R}$.*

ii) $\{\delta_t\Lambda \mid t \geq 0\}$ is asymptotic to a periodic orbit if and only if $e(\Lambda)$ is a quadratic irrational, viz. the root of an irreducible quadratic polynomial with rational coefficients; it is itself a periodic orbit if and only if $e^-(\Lambda)$ is the conjugate of $e(\Lambda)$, namely the other root of the irreducible quadratic polynomial.

iii) $\{\delta_t\Lambda \mid t \geq 0\}$ is bounded (has compact closure) in $\mathfrak{L}$ if and only if $e(\Lambda)$ is badly approximable (see Section 1 for definition); the (full) orbit $\{\delta_t\Lambda \mid t \in \mathbb{R}\}$ is bounded if and only if $e(\Lambda)$ and $e^-(\Lambda)$ are both badly approximable.

It may be noticed that the Λ's appearing in (i) and (ii) above form a small subset; the corresponding sets of endpoints are countable, and for each of them there is a two-dimensional submanifold consisting of Λ's with that as the endpoint of the trajectory. On the other hand, the set of badly approximable numbers, though a set of Lebesgue measure 0, is large in other respects. For instance it is uncountable, and has Hausdorff dimension 1, the maximum possible for a subset of $\mathbb{R}$. It was shown by W.M. Schmidt, by realising it as the winning set of a class of games (see [30] and [10] for details) that if $\{f_j\}$ is a sequence of continuously differentiable functions on $\mathbb{R}$ with nowhere-vanishing derivatives then there exists an uncountable subset E of $\mathbb{R}$ such that for all $t \in E$, $f_j(t)$ is badly approximable for all j. In a recent work of Hemangi Shah and the present author [20] it was shown that given a sequence $\{f_j\}$ as above and a "Cantor-like" set (a compact totally disconnected subset satisfying certain regularity conditions, such as the usual Cantor ternary set) C there exists an uncountable subset E of C such that for any $t \in E$, $f_j(t)$ is badly approximable for all j.

Let us now consider a general $n \geq 2$. Let $1 \leq p \leq n-1$ and $\kappa = p/(n-p)$. For $t \in \mathbb{R}$ let $\delta_p(t) = \mathrm{diag}(e^t, \dots, e^t, e^{\kappa t}, \dots, e^{\kappa t})$; then $\delta_t(t) \in G = SL(n, \mathbb{R})$ for all $t \in \mathbb{R}$. In my paper [8] a correspondence analogous to the one noted above in the case $n = 2$, was brought out between the behaviour of trajectories $\{\delta_p(t)\Lambda \mid t \geq 0\}$, $\Lambda \in \mathfrak{L}$, and Diophantine properties of certain systems of linear forms associated with Λ, which I shall now briefly indicate.

Consider a system $L_1, \dots, L_k$ of k linear forms in l variables. For $x \in \mathbb{Z}^l$ let $|x|$ denote the maximum of the absolute values of the coordinate entries of x. Also, for $t \in \mathbb{R}$ let $\|t\|$ denote the distance of t from the integer nearest to it. The system $L_1, \dots, L_k$ is said to be *singular* if for every $\epsilon > 0$ there exists N_0 such that for all $N \geq N_0$ there exists $x \in \mathbb{Z}^l$ with $|x| \leq N$ such that $\|L_j(x)\| < \epsilon N^{-\frac{1}{k}}$, and it is said to be *badly approximable* if there exists $c > 0$ such that for all nonzero $x \in \mathbb{Z}^l$ such that $\max_{1 \leq j \leq k} \|L_j(x)\| > c|x|^{-\frac{1}{k}}$.

Theorem 6.2. *Let $1 \leq p \leq n-1$. Let L be a $(n-p)\times p$ matrix and let $L_1, \dots, L_{n-p}$ be the corresponding system of $n-p$ linear forms in p variables. Let Λ be the lattice in $\mathbb{R}^n$ consisting of the vectors $\begin{pmatrix} u \\ Lu+v \end{pmatrix}$, with $u \in \mathbb{Z}^p$ and $v \in \mathbb{Z}^{n-p}$. Then we have the following:*

i) $\{\delta_p(t)\Lambda \mid t \geq 0\}$ is a divergent trajectory if any only if the system $L_1, \dots, L_{n-p}$ is singular.

ii) $\{\delta_p(t)\Lambda \mid t \geq 0\}$ is a bounded trajectory if any only if the system $L_1, \dots, L_{n-p}$ is badly approximable.

These correspondences have inspired a broad framework for connecting various Diophantine problems to dynamical questions about flows, and has led to resolution of some longstanding problems, in the work of Kleinbock, Margulis and many other authors, and continues to be a flourishing topic. We shall not go into the details here.

As mentioned earlier, the behaviour of the diagonal flows is chaotic from a dynamical point of view and in particular there is no neat description of the set of points whose orbits are dense/not dense. The orbits discussed above are some special types. Behaviour of some more types is covered in the generalizations mentioned above.

Curiously, however, when in place of a single diagonal one-parameter subgroup as above we consider the action of a subgroup generated two or more of such (distinct) subgroups, a kind of "rigidity" as in the case of unipotent flows is found to kick in. This also turns out to be important for Diophantine approximation and there is a conjecture of Margulis for these, analogous to the one for subgroups generated by unipotent one-parameter subgroups, but more technical in nature;

(cf. [27] and [31] for details). It may be mentioned here that proving the conjecture would also establish another well-known conjecture in Number Theory, due to Littlewood, that for any two irrational numbers $\alpha, \beta \in \mathbb{R}$, $\liminf_{n\to\infty} n\|\alpha\|\|\beta\| = 0$, where $\|t\|$, for $t \in \mathbb{R}$, denotes the distance of t from the integer nearest to it. The understanding in this respect is quite incomplete as of now. However it has been proved, using dynamical methods, that the set of pairs (α, β) for which the statement does not hold is a "negligible" set, viz. a set of Hausdorff dimension 0; this is a result of Einsiedler, Katok and Lindenstrauss, which featured in the citation for the Fields Medal awarded to Lindenstrauss in 2010 (see [31]).

7. Application of diagonal flows to values of binary quadratic forms

We now consider again the case of the diagonal flow in the case of $n = 2$; thus let $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$, $\mathfrak{L}$ the space of lattices in $\mathbb{R}^2$ with discriminant 1, and consider the action of $\{\delta_t\} = \{\mathrm{diag}(e^t, e^{-t})\}$ on $\mathfrak{L}$. Though, as was mentioned earlier, the orbit behaviour is chaotic, one can nevertheless benefit from studying the trajectories, for understanding values of binary quadratic forms on integer pairs. This is notable particularly since binary quadratic forms do not fall in the ambit of the theory around the Oppenheim conjecture (precisely because the special orthogonal group in this case is not generated by unipotent one-parameter subgroups - in fact it contains no nontrivial unipotent elements). Here I shall briefly indicate some of the directions followed in this respect.

Let Q be an indefinite binary quadratic form, which upto a scalar multiple we may consider to be in the form $(x - \alpha y)(x - \beta y)$, where $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$. It turns out that the question of the closure of $Q(\mathbb{Z}^2)$ depends crucially on α, β, and specifically one can conclude various things about $Q(\mathbb{Z}^2)$ by examining the continued fraction expansions of α and β.

For $\rho \in \mathbb{R}$ let $\rho = [a_0, a_1, \dots]$ denote the usual simple continued fraction expansion of ρ, with $a_0 \in \mathbb{Z}$ and $a_j \in \mathbb{N}$ for $j \geq 1$ (see [22], for instance). We say that $\rho \in \mathbb{R}$ is *generic* if given any finite block $(b_1, \dots, b_l)$, $b_1, \dots, b_l \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $(a_k, \dots, a_{k+l-1}) = (b_1, \dots, b_l)$. It can be seen that generic real numbers form a set of full Lebesgue measure (viz. the complement has 0 measure). It follows from a classical result of E. Artin that given $Q(x, y) = (x - \alpha y)(x - \beta y)$, where $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, if either α or β is generic then $Q(\mathbb{Z}^2)$ is dense in $\mathbb{R}$. For details the reader is referred to [18], where we proved a strengthening of the result, concerning conditions for $Q(\mathbb{N}^2)$, the set of values of Q on positive integral pairs, to be dense; the technique there is however involves a different point of view.

In a recent joint paper of Manoj Chaudhuri and the present author [4] the connection between the flow given by $\{\delta_t\}$ on $\mathfrak{L}$ with values of quadratic forms binary quadratic forms is explored in another respect. Here we consider another continued fraction expansion, known as the Hurwitz expansion; the latter is constructed analogously to the simple continued fraction, where at each stage one takes out the integer nearest and inverts the remainder, whenever it is nonzero (see [4] for further details). Given a quadratic form $Q(x, y) = (x - \alpha y)(x - \beta y)$, where $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, and $\epsilon > 0$ we discuss how $\#\{z \in \mathbb{Z}^2 \mid |Q(z)| < \epsilon, \|z\| \leq r\}$ grows, as r tends to infinity. (Of course the numbers can be 0, when α, β are badly approximable for instance, but typically they are not, and one can consider their growth.) We show that various pattens of growth, including arbitrarily slow to reasonably fast, are possible depending on the asymptotics of the partial quotients of the Hurwitz expansions of α and β. Since real numbers can be produced meeting various requirements that arise on the Hurwitz expansions in this respect, this enables us to get quadratic forms with various patterns of growth for the numbers as above. We omit the details.

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ON SOME RECENT APPLICATIONS OF CIRCLE METHOD*

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Abstract. This is a survey of some of my recent works where I applied variations of the circle method to produce subconvex bounds for L-functions and asymptotics for the count of rational points on varieties.

1. Introduction

Consider the function $\delta : \mathbb{Z} \to \mathbb{C}$ which is defined by $\delta(0) = 1$ and $\delta(n) = 0$ for $n \neq 0$. This simple function appears in several problems in analytic number theory, e.g. Diophantine equations (see [8], [10]). Often the solution of the problem lies in suitably extending the domain of the function to $\mathbb{R}$. Of course there are infinitely many ways one can extend the definition of δ to all real numbers, but one seeks for an extension which is well behaved under Fourier analysis. For example it will be not at all beneficial to extend it by a smooth bump function supported in $[-1/2, 1/2]$ and taking value 1 at 0. The Fourier transform of this function will be too 'spread out'. (This is the uncertainty principle in Fourier analysis.)

A useful extension of the δ function was first conceived by Ramanujan and then developed by Hardy, Littlewood and Ramanujan (see [7]). This approach is based on the simple integral representation

$$\delta(n) = \int_0^1 e(nx)\mathrm{d}x,$$

where $e(z) = e^{2\pi i z}$. The right hand side can also be realized as an integral over a circle, which is precisely the reason for the name 'circle method'. This is already an extension of δ, but it is too simple to be of much help in any arithmetic problem. The crux of the method lies in treating the points on the range of the integral according to their 'arithmetic complexity'. This is achieved through

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Farey segmentation (see [10]) of the interval $[0,1]$. The main input is the Dirichlet approximation theorem (see [10]), which says that every $x \in [0,1]$ satisfies

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{qQ},$$

with $1 \leq q \leq Q$, $(a,q) = 1$, where Q is a fixed positive number. One of the high points of the circle method of Hardy-Littlewood-Ramanujan is the following formula of Kloosterman. (A proof of this formula is given in [10].) For $C \geq 1$ we have

$$\delta(n) = 2\mathrm{Re} \int_0^1 \sum\sum_{\substack{1 \leq c \leq C < d \leq c+C \\ (c,d)=1}} \frac{1}{cd}\, e\left(\frac{n\bar{d}}{c} - \frac{nx}{cd} \right) \mathrm{d}x, \tag{1.1}$$

where $\bar{d}$ is an integer such that $d\bar{d} \equiv 1 \bmod c$. Though nice and completely explicit this formula has its own drawbacks - one of them being the appearance of d in the 'analytic part'.

Let us compare Kloosterman's formula with a more modern version of the 'circle method' introduced by Duke, Friedlander, Heath-Brown and Iwaniec (see [4] and [8]). Let w be a smooth bump function supported in $[1/2, 1]$ such that

$$\int w(x)\mathrm{d}x = 1$$

and for $x > 0$, set

$$h(x,y) = \sum_{j=1}^{\infty} \frac{1}{xj} \left(w(xj) - w(|y|/xj) \right).$$

For $C \geq 1$ we define

$$a_C = \frac{1}{C} \sum_{c=1}^{\infty} w\left(\frac{c}{C} \right),$$

by Poisson summation one can show that $a_C = 1 + O_N(C^{-N})$. Then we have the following formula

$$\delta(n) = \frac{a_C}{C^2} \sum_{c=1}^{\infty} \left[\sum_{\substack{d \bmod c \\ (d,c)=1}} e\left(\frac{dn}{c} \right) \right] h\left(\frac{c}{C}, \frac{n}{C^2} \right). \tag{1.2}$$

Observe that d now appears only in the 'arithmetic part' of the formula. It does not interfere with the analytic weight. This is the main advantage of this formula over Kloosterman's formula (1.1). However the 'analytic part' is now more involved, as the weight function h is not completely well-behaved. Nevertheless the analytic properties, e.g. size of the derivatives, of the weight h can be derived quite easily, and in most applications this is all that is required.

Before proceeding further I would like to draw the attention of the reader to the fact that (1.2) is derived from a completely different observation. Instead of starting with the integral representation of δ, one starts with the observation that an integer n is zero if and only if all integers divide it. Moreover if n is a non-zero integer then it has very few divisors, and among the two complementary divisors d and n/d, one does not exceed $n^{1/2}$. For details regarding this formula see Chapter 20 of Iwaniec and Kowalski [10] and the paper of Heath-Brown [8]. The formula of Kloosterman also appears in [10].

From another point of view the formulae (1.1) and (1.2) are similar, as they are both based on $GL(1)$ harmonics, viz. the additive character $e(nd/c) = e^{2\pi i nd/c}$. In some cases this is too restrictive. The reason being that there are only C^2 many harmonics of modulus c up to C. One can acquire more 'elbow room' by allowing higher rank harmonics in the formula. One such formula can be derived from the well-known trace formula of Petersson (see [10]). Let p be a prime number and let $k \equiv 3 \bmod 4$ be a positive integer. Let ψ be a character of $\mathbb{F}_p^\times$ satisfying $\psi(-1) = -1 = (-1)^k$. So in particular ψ is primitive modulo p. The collection of Hecke cusp forms of level p, weight k and nebentypus ψ is denoted by $H_k(p,\psi)$, and they form an orthogonal basis of the space of cusp forms $S_k(p,\psi)$. Let

$$\omega_f^{-1} = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2}$$

be the spectral weights. The Petersson formula gives

$$\sum_{f\in H_k(p,\psi)} \omega_f^{-1}\lambda_f(m)\overline{\lambda_f(r)} = \delta(m-r) + 2\pi i \sum_{c=1}^{\infty} \frac{S_\psi(r,m;cp)}{cp} J_{k-1}\left(\frac{4\pi\sqrt{mr}}{cp}\right).$$

This gives an expansion of the delta function in terms of the Kloosterman sums

$$S_\psi(a,b;c) = \sideset{}{^\star}\sum_{\alpha \bmod c} \psi(\alpha) e\left(\frac{\alpha a + \bar{\alpha} b}{c}\right)$$

and the (Hecke normalized) Fourier coefficients $\lambda_f(m)$ of holomorphic forms f, if pk is taken to be sufficiently large (so that the space $S_k(p,\psi)$ is non-trivial). Here J_ℓ stands for the J-Bessel function of order ℓ.

Let P be a parameter and let

$$P^\star = \sum_{\substack{P<p<2P\\ p \text{ prime}}} \sum_{\psi \bmod p} (1-\psi(-1)) = \sum_{\substack{P<p<2P\\ p \text{ prime}}} \phi(p) \asymp \frac{P^2}{\log P}.$$

Then for $Pk \gg 1$ (sufficiently large), we have

$$\begin{aligned}\delta(n) = &\frac{1}{P^\star} \sum_{\substack{P<p<2P\\ p \text{ prime}}} \sum_{\psi \bmod p} (1-\psi(-1)) \sum_{f\in H_k(p,\psi)} \omega_f^{-1}\lambda_f(m)\overline{\lambda_f(r)} \\ &- \frac{2\pi i}{P^\star} \sum_{\substack{P<p<2P\\ p \text{ prime}}} \sum_{c=1}^{\infty} \frac{1}{cp} \sum_{\psi \bmod p} (1-\psi(-1))\, S_\psi(r,m;cp) J_{k-1}\left(\frac{4\pi\sqrt{mr}}{cp}\right), \qquad (1.3)\end{aligned}$$

where we pick any splitting $n = m - r$ of n as a difference of two positive integers. (There is also a possibility of summing over all such splitting.) In order to compare this formula with (1.1) and (1.2), let us look at the part of the last sum where $p \nmid c$. Then the Kloosterman sum splits as

$$S_\psi(r, m; cp) = S_\psi(\bar{c}r, \bar{c}m; p) S(\bar{p}r, \bar{p}m; c),$$

and we get

$$\sum_{\psi \bmod p} (1 - \psi(-1))\, S_\psi(r, m; cp) \\ = \phi(p)\, S(\bar{p}r, \bar{p}m; c) \left(e\left(\frac{\bar{c}(r+m)}{p}\right) - e\left(-\frac{\bar{c}(r+m)}{p}\right)\right).$$

Then applying reciprocity we now relate the last sum in (1.3) to sums of the form

$$\frac{2\pi i}{P^\star} \sum_{\substack{P<p<2P \\ p \text{ prime}}} \sum_{c=1}^{\infty} \frac{\phi(p)}{cp} \left[S(\bar{p}r, \bar{p}m; c) e\left(\frac{\bar{p}(r+m)}{c}\right)\right] \\ \times e\left(-\frac{(r+m)}{cp}\right) J_{k-1}\left(\frac{4\pi\sqrt{mr}}{cp}\right).$$

This can now be compared with the formulae given in (1.1) and (1.2). However there are some advantages in the present formula. First here we need to take modulus $c \ll N^{1+\varepsilon}/P$ where N is the 'size of the equation' $n = 0$. So the size of the modulus can be regulated by choosing the level p. Of course for larger p the first sum in (1.3) becomes more complicated as the conductor rises. The second apparent advantage in the above sum is the extra sum over p. But let me put a word of caution here. If the size of the parameter P is so chosen that the first sum in (1.3) is trivial due to small size of the conductor of one of the involved L-functions, then after an application of a summation formula of Voronoi type, the inner sum in the second sum in (1.3) behaves independently of p. Or in other words the above formula can not yield a circle method of classical form with smaller modulus.

In the following sections we will give some applications of the above expansions of the δ function. For details see [2], [11], [12], [13], [14], [15], [16] and [17].

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2. Rational points on varieties

Let V be a projective variety defined by a system of r homogeneous polynomials in n variables with integer coefficients. So the affine cone on V is given by

$$\hat{V} = \{\mathbf{x} \in \mathbb{C}^n : f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0\}.$$

The points on the variety V correspond to the lines in $\hat{V}$ passing through the origin. Number theorists are interested in the set of rational points on V. These are in one to one correspondence with reduced integral vectors in $\hat{V}$. An integral vector $\mathbf{m} \in \mathbb{Z}^n$ is said to be reduced if $\mathbf{m} \neq \mathbf{0}$, the first non-zero entry in $\mathbf{m}$ is positive and $(\mathbf{m}) = 1$ (i.e. gcd of the entries of $\mathbf{m}$ is 1). Of course the statistics of the reduced integral vectors on $\hat{V}$ can be easily derived from that of $\tilde{V} = \hat{V} \cap \mathbb{Z}^n$, the so called lattice points on $\hat{V}$. It is not even clear whether $\tilde{V}$ contains any non-trivial point (i.e. other than the origin). Rough heuristics indicate that if the number of variable is large compared to the sum of the degrees of the polynomials f_i then there are quite a few points. Let $\mathcal{B}$ be a box in the n-dimensional space $\mathbb{R}^n$, and let $Y\mathcal{B}$ denote the box that is obtained by multiplying each point in $\mathcal{B}$ by a factor Y. Then we define the counting function

$$N_{V,\mathcal{B}}(Y) = \# \tilde{V} \cap Y\mathcal{B}. \tag{2.1}$$

One expects that

$$N_{V,\mathcal{B}}(Y) \asymp Y^{n-\sum d_i}$$

where $d_i = \deg f_i$, unless there is reason why it should not. On our support we have the following very general result of Birch [1]. Suppose $d_i = d$ for all i, so that all the defining homogeneous polynomials have same degree. Let $V^\star$ denote the locus of singularities in the sense of Birch. Then for

$$n > 2^{d-1}(d-1)r(r+1) + \dim V^\star$$

one has the asymptotic

$$N_{V,\mathcal{B}}(Y) \sim c_{V,\mathcal{B}} Y^{n-rd} \tag{2.2}$$

for some constant $c_{V,\mathcal{B}}$ which depends on the variety and the box, but not on Y. In fact Birch proves much more. He gives an exact asymptotic with a power saving in the error term and shows that the leading constant is a product of the local densities times the singular integral. In particular this proves the Hasse principle for such varieties.

Birch's result is a culmination of decades of intensive research in circle method as developed by Hardy, Littlewood, Vinogradov, Kloosterman, Davenport and others. The result stands as a benchmark in the subject of Diophantine analysis. For hypersurfaces (i.e. varieties defined by one equation) the result has been improved in some cases. Notably for cubic hypersurfaces, Heath-Brown [9] has reduced the required number of variables from 17 to 14 under certain conditions. But for varieties given by more than one equation, Birch's result has not been improved without imposing serious restrictions on the nature of the forms.

Let us focus on the case of smooth intersection of two quadrics, i.e. $r = 2$ and f_1, f_2 are quadratic forms. Let M_i denote the underlying symmetric matrices of size $n \times n$. The singular locus, in the sense of Birch, is given by

$$V^\star = \{\mathbf{x} \in \mathbb{C}^n : \operatorname{rank}(M_1\mathbf{x}, M_2\mathbf{x}) \leq 1\},$$

which coincides with the union of the eigenspaces of $M_2^{-1}M_1$. Let Δ be the dimension of the singular locus, which is same as the maximum dimension of the eigenspaces. One can show that the smoothness implies that $\Delta = 1$ (see [15]). The main result of Birch in this situation gives

$$\sum\cdots\sum_{\substack{\mathbf{m}\in\mathbb{Z}^n\cap Y\mathcal{B}\\ f_1(\mathbf{m})=f_2(\mathbf{m})=0}} 1 = \mathfrak{S}J_0(\mathcal{B})Y^{n-4} + O\left(Y^{n-4-\delta}\right) \tag{2.3}$$

where $\delta > 0$ if the number of variables $n \geq 14$. Here $\mathfrak{S}$ is the standard singular series and $J_0(\mathcal{B})$ is the singular integral. One seeks to improve this and reduce the required number of variables (at least) to $n \geq 9$. Indeed from the work of Demyanov it is known that for $n \geq 9$ local solutions exist for any finite prime. On the other hand from the work of Colliot-Thélène, Sansuc and Swinnerton-Dyer, we know that the Hasse principle holds for $n \geq 9$. The main result in [15] gives a substantial improvement over Birch.

Theorem 2.1. *Suppose $\mathcal{B}$ is a box in $\mathbb{R}^n$, such that $\mathcal{B} \cap V^\star = \emptyset$. Then we have*

$$\sum\cdots\sum_{\substack{\mathbf{m}\in Y\mathcal{B}\\ f_1(\mathbf{m})=f_2(\mathbf{m})=0}} 1 = \mathfrak{S}J_0(\mathcal{B})Y^{n-4} + O\left(Y^{n-4-\delta}\right)$$

with some $\delta > 0$, as long as $n \geq 11$. Here the singular integral is given by

$$J_0(\mathcal{B}) = \lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{1}{4\varepsilon_1\varepsilon_2}\int_{\substack{|f_1(\mathbf{y})|<\varepsilon_1\\ |f_2(\mathbf{y})|<\varepsilon_2\\ \mathbf{y}\in\mathcal{B}}} \mathrm{d}\mathbf{y}.$$

The singular series is given by

$$\mathfrak{S} = \sum_{c_1=1}^{\infty}\sum_{c_2=1}^{\infty}\frac{C_{c_1,c_2}}{c_1^{n-1}c_2^n},$$

where

$$C_{c_1,c_2} = \sideset{}{^\star}\sum_{d_1 \bmod c_1}\ \sideset{}{^\star}\sum_{d_2 \bmod c_2}\ \sum\cdots\sum_{\substack{\mathbf{b} \bmod c_1c_2\\ f_2(\mathbf{b})\equiv 0 \bmod c_1}} e\left(\frac{d_1f_1(\mathbf{b})c_2 + d_2f_2(\mathbf{b})}{c_1c_2}\right).$$

The proof of this theorem rests on the formula (1.2). But there are two equations and so the formula has to be used twice. If it is applied simultaneously (and independently) for both the equations, then one is left with a deadlock situation which one does not know how to tackle. The main idea of the proof was to use the formula (1.2) twice in a repeated fashion such that in the second application one uses the information that one has already employed the formula once. Let us make it little more precise. We first prove a version of the theorem with a smooth weight function. Then one removes the weight using approximation.

Let

$$N = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ f_1(\mathbf{m})=f_2(\mathbf{m})=0}} \cdots \sum W\left(\frac{\mathbf{m}}{Y}\right)$$

where W is a nice bump function, say with $W^{(\mathbf{j})} \ll H^{j_1+\cdots+j_n}$, approximating the indicator function for the box $\mathcal{B}$. Using (1.2) and choosing C to be Y, we get

$$N = \frac{a_Y}{Y^2}\sum_{c_1=1}^{\infty} \sideset{}{^\star}\sum_{d_1 \bmod c_1} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ f_2(\mathbf{m})=0}} \cdots \sum e\left(\frac{d_1 f_1(\mathbf{m})}{c_1}\right) h\left(\frac{c_1}{Y}, \frac{f_1(\mathbf{m})}{Y^2}\right) W\left(\frac{\mathbf{m}}{Y}\right),$$

which we rewrite as

$$\frac{a_Y}{Y^2}\sum_{c_1=1}^{\infty} \sideset{}{^\star}\sum_{d_1 \bmod c_1} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ f_2(\mathbf{m})\equiv 0 \bmod c_1}} \cdots \sum e\left(\frac{d_1 f_1(\mathbf{m})}{c_1}\right) h\left(\frac{c_1}{Y}, \frac{f_1(\mathbf{m})}{Y^2}\right) W\left(\frac{\mathbf{m}}{Y}\right) \delta\left(\frac{f_2(\mathbf{m})}{c_1}\right).$$

This seemingly trivial step is most vital in this work. We are building up the second application of the circle method using the existing modulus from the first application of the circle method. This 'nesting' process acts like a conductor lowering mechanism and we end up having sums over $\mathbf{m} \in \mathbb{Z}^n$ with modulus of size $Y^{3/2}$ instead of Y^2, which is what one has without nesting.

Now applying (1.2) again, choosing $C = \sqrt{Y}$ in this application of the circle method, and rearranging the sums, we get

$$N = \frac{1}{Y^3}\sum_{c_1=1}^{\infty}\sum_{c_2=1}^{\infty} \sideset{}{^\star}\sum_{d_1 \bmod c_1} \sideset{}{^\star}\sum_{d_2 \bmod c_2} N(\mathbf{a},\mathbf{c};Y) + O(Y^{-2015})$$

where

$$N(\mathbf{a},\mathbf{c};Y) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ f_2(\mathbf{m})\equiv 0 \bmod c_1}} \cdots \sum e\left(\frac{d_1 f_1(\mathbf{m})c_2 + d_2 f_2(\mathbf{m})}{c_1 c_2}\right)$$
$$\times\, h\left(\frac{c_1}{Y}, \frac{f_1(\mathbf{m})}{Y^2}\right) h\left(\frac{c_2}{\sqrt{Y}}, \frac{f_2(\mathbf{m})}{c_1 Y}\right) W\left(\frac{\mathbf{m}}{Y}\right).$$

The error term of size $O(Y^{-2015})$ is introduced as a_Y is being replaced by 1. Applying the Poisson summation formula with modulus c_1c_2 we deduce that

$$
\begin{aligned}
N(\mathbf{a},\mathbf{c};Y) =& \frac{Y^n}{(c_1c_2)^n}\sum\cdots\sum_{\mathbf{m}\in\mathbb{Z}^n} I_{c_1,c_2}(\mathbf{m})\\
&\times\left[\sum\cdots\sum_{\substack{\mathbf{b} \bmod c_1c_2\\ f_2(\mathbf{b})\equiv 0 \bmod c_1}} e\left(\frac{d_1f_1(\mathbf{b})c_2+d_2f_2(\mathbf{b})+\mathbf{b}.\mathbf{m}}{c_1c_2}\right)\right],
\end{aligned}
$$

where

$$
I_{c_1,c_2}(\mathbf{m}) = \int_{\mathbb{R}^n} h\left(\frac{c_1}{Y}, f_1(\mathbf{y})\right) h\left(\frac{c_2}{\sqrt{Y}}, \frac{Yf_2(\mathbf{y})}{c_1}\right) W(\mathbf{y})\, e\left(-\frac{Y\mathbf{m}.\mathbf{y}}{c_1c_2}\right) \mathrm{d}\mathbf{y}.
$$

We set

$$
C_{c_1,c_2}(\mathbf{m}) = \sideset{}{^\star}\sum_{d_1 \bmod c_1}\ \sideset{}{^\star}\sum_{d_2 \bmod c_2}\ \sum\cdots\sum_{\substack{\mathbf{b} \bmod c_1c_2\\ f_2(\mathbf{b})\equiv 0 \bmod c_1}} e\left(\frac{d_1f_1(\mathbf{b})c_2+d_2f_2(\mathbf{b})+\mathbf{b}.\mathbf{m}}{c_1c_2}\right)
$$

so that

$$
N = Y^{n-3}\sum_{c_1=1}^{\infty}\sum_{c_2=1}^{\infty}\frac{1}{(c_1c_2)^n}\sum\cdots\sum_{\mathbf{m}\in\mathbb{Z}^n} C_{c_1,c_2}(\mathbf{m})\, I_{c_1,c_2}(\mathbf{m}) + O(Y^{-2015}).
$$

It is expected that the non-zero frequencies $\mathbf{m}\neq\mathbf{0}$ contribute to the error term, and the zero frequency gives rise to the main term. As usual the estimation of the zero frequency is a relatively easy task. Though there are certain concerns that one needs to take care of. For details readers are referred to the paper [15]. For the non-zero frequencies one needs to get precise estimates for the character sums. Also one needs to get cancellation in the sum of character sums when summed over c_2. The analysis is rather delicate and instead of rewriting it here I prefer to refer the reader to the original paper [15]. Further details can also be found in the paper [2].

3. Bounds for L-functions

In this section we will apply the circle method to study analytic properties of certain L-functions. It will take us too far afield if we try to define L-functions here. (Interested readers are advised to read the standard references e.g. [5] or [10].) Instead we look at the prototype - the Riemann zeta function. This is defined by

$$
\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p \text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1}
$$

which converges absolutely for $\sigma = \mathrm{Re}(s) > 1$. This is a common property of all L-functions. In the half plane $\sigma > 1$ they are given by an absolutely convergent Dirichlet series and an absolutely convergent Euler product. Moreover like $\zeta(s)$ all L-functions have analytic continuation to all of $\mathbb{C}$. They are holomorphic except for a possible pole at $s = 1$, and they satisfy a functional equation where the L-value $L(s)$ is related to $\overline{L(1-\bar{s})}$. This implies that in the left half plane $\sigma < 0$, the L-function can be expressed in terms of an absolutely convergent Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{1-s}}.$$

A slight generalization of the Riemann zeta function is the Dirichlet L-function. Suppose χ is a primitive character of the group $(\mathbb{Z}/M\mathbb{Z})^{\star}$. It is extended to a function on $\mathbb{Z}$ using the recipe - $\chi(n) = 0$ if $(n, M) \neq 1$, otherwise $\chi(n)$ is defined to be value taken by χ on the residue class of n modulo M. Such a multiplicative function on the $\mathbb{Z}$ is called a Dirichlet character modulo M. The corresponding L-function is defined as

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Again the series and the product converges absolutely in the half plane $\sigma > 1$, and the function can be extended to an entire function (except for $M = 1$ and χ trivial, when $L(s,\chi) = \zeta(s)$) and there is a functional equation relating $L(s,\chi)$ with $L(1-s,\bar{\chi})$. A far reaching generalization of the Riemann zeta function is the automorphic L-functions $L(s,\pi)$ attached to automorphic representations π of $SL(d,\mathbb{Z})$. In this case the Euler product is given by

$$L(s,\pi) = \prod_{p \text{ prime}} \prod_{i=1}^{d} \left(1 - \frac{a_{i,\pi}(p)}{p^s}\right)^{-1}$$

which is of degree d. For χ as above we can now define the twisted L-function

$$L(s,\pi\otimes\chi) = \prod_{p \text{ prime}} \prod_{i=1}^{d} \left(1 - \frac{a_{i,\pi}(p)\chi(p)}{p^s}\right)^{-1},$$

which is associated to an automorphic representation of a discrete subgroup of $SL(d,\mathbb{R})$.

Understanding the size of the L-function is an important problem with far reaching consequences. Let

$$Q = Q(s,\chi) = \{(2+|t|)M\}^d.$$

This is basically the size of the conductor of the L-function $L(s,\pi\otimes\chi)$ if π is kept fixed. From the absolutely convergent Dirichlet series representation in $\sigma > 1$ it follows that in this domain one has $L(s,\pi\otimes\chi) \ll Q^{\varepsilon}$. From the explicit form of the functional equation one gets that $L(s,\pi\otimes\chi) \ll Q^{\frac{1}{2}-\sigma+\varepsilon}$ for $\sigma < 0$. In the critical strip $0 \leq \sigma \leq 1$ one can use the convexity principle from complex analysis to derive a bound. It is clear that it is enough to study the size of the function on the critical line $\sigma = 1/2$. On this line the convexity bound is given by

$$L(1/2 + it, \pi \otimes \chi) \ll Q^{\frac{1}{4}+\varepsilon}, \tag{3.1}$$

whereas the Lindelöf hypothesis which is a consequence of the grand Riemann hypothesis implies that one should have

$$L(1/2 + it, \pi \otimes \chi) \ll Q^{\varepsilon}.$$

This hypothesis is completely out of reach given the current state of technology. However in most applications one just needs a bound which is just slightly stronger than the convexity bound. One seeks a bound of the form

$$L(1/2 + it, \pi \otimes \chi) \ll Q^{\frac{1}{4}-\eta+\varepsilon}, \tag{3.2}$$

with some $\eta > 0$. This problem is known as the subconvexity problem for automorphic L-functions.

The subconvexity problem has a long history. The first instance of subconvexity was established by Weyl in 1920's (see [19]). He showed that

$$\zeta(1/2 + it) \ll (2 + |t|)^{\frac{1}{6}+\varepsilon}.$$

To prove this result he introduced a new technique of estimating exponential sums - the Weyl differencing. In the last 90 years or so, after Weyl's work, the exponent has been improved only slightly. In 1960's Burgess [3] proved

$$L(1/2, \chi) \ll M^{\frac{3}{16}+\varepsilon},$$

for Dirichlet L-functions. This bound still remains unsurpassed. A crucial input in Burgess' work is the Riemann hypothesis for algebraic curves over finite fields. Whereas Weyl's bound gives the subconvexity in the t-aspect, Burgess' bound furnishes subconvexity in the twist-aspect (or M-aspect). Heath-Brown combined them and produced the hybrid subconvex bound

$$L(1/2 + it, \chi) \ll Q^{\frac{3}{16}+\varepsilon}.$$

This completely solved (up to the value of the exponent) the subconvexity problem for Dirichlet L-functions or the degree one L-functions on $\mathbb{Q}$.

There are two sources for degree two L-functions - the holomorphic modular forms and the real analytic Maass forms. A prototype for modular forms is the Ramanujan delta function

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

where $q = e(z) = e^{2\pi i z}$. This is a holomorphic function on the upper half plane $\mathrm{Im}(z) > 0$, and satisfies the automorphy relation

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Ramanujan introduced the τ function by writing

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)e(nz)$$

and made deep conjectures regarding the nature of the numbers $\tau(n)$. We set $\tau_0(n) = \tau(n)/n^{11/2}$ and define the associated L-function

$$L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^s}.$$

This is a typical example of degree two L-function which is associated to an automorphic representation for $SL(2,\mathbb{Z})$. The Euler product representation of this L-function was conjectured by Ramanujan, and was proved by Mordell and Hecke. (An excellent reference is Serre's book [18].) In [6] Good proved the t-aspect subconvexity for such L-functions. It follows from his work that

$$L(1/2+it,\Delta) \ll (2+|t|)^{\frac{1}{3}+\varepsilon},$$

which is the analogue of the Weyl bound in the context of degree two L-functions. Good established this bound by computing the second moment

$$\int_0^T |L(1/2+it,\Delta)|^2 \,\mathrm{d}t.$$

In [4] Duke, Friedlander and Iwaniec produced the analogue of the Burgess bound (with a weaker exponent) for these L-functions. They showed that

$$L(1/2,\Delta\otimes\chi) \ll M^{\frac{1}{2}-\frac{1}{22}+\varepsilon}.$$

This was derived by computing an amplified second moment

$$\sum_{\chi \bmod M} |L(1/2,\chi)|^2 |\sum_{\ell\sim L} a_\ell \chi(\ell)|^2.$$

Developing on this circle of ideas and strengthening the method of Bykovskii, a hybrid bound was established by Blomer and Harcos in 2008. Few years later, I introduced a different approach in [13] and was able to conclude the stronger bound

$$L(1/2+it,\Delta\otimes\chi) \ll Q^{\frac{1}{4}-\frac{1}{18}+\varepsilon}.$$

Recall that here $Q = (2+|t|)^2M^2$.

The main contribution of [13] was not just in strengthening of the exponent, but an introduction of a new route for producing subconvexity. In this work the circle method was used right from the start to separate the oscillation of the Fourier coefficients $\tau_0(n)$ from that of the character (i.e. $\chi(n)n^{-it}$). This turned out to be a powerful alternative for the moment method, which had been the stumbling block for establishing subconvexity for higher degree L-functions. However the circle method all by itself is not that powerful, and one needs to invent a conductor lowering trick. The idea was to use the oscillatory factors already present in the sum to construct a circle method with smaller modulus than usual. This same idea was used in the previous section. There we used the oscillatory factors already present after the first application of the circle method to lower the modulus of the second application of the circle method. This was first systematically utilized in [14], and an archimedean analogue of this trick was used in [16] to establish the following result. This is the degree three analogue (with a weaker exponent) of Weyl's bound for zeta function and Good's bound for degree two L-functions.

Theorem 3.1. *Let π be a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$. Then we have*

$$L\left(\tfrac{1}{2}+it,\pi\right) \ll_{\pi,\varepsilon} (1+|t|)^{\frac{3}{4}-\frac{1}{16}+\varepsilon}.$$

The proof of this result is based on Kloosterman's formula (1.1). Below I will now quickly sketch the proof of the theorem. The readers are again advised to refer to the main paper for details. Suppose $t>2$, then by approximate functional equation we have

$$L\left(\tfrac{1}{2}+it,\pi\right) \ll t^{\varepsilon} \sup_{N\leq t^{3/2+\varepsilon}} \frac{|S(N)|}{N^{1/2}} + t^{-2012} \tag{3.3}$$

where $S(N)$ is a sum of type

$$S(N) := \sum_{n=1}^{\infty} \lambda(1,n) n^{-it} V\left(\frac{n}{N}\right)$$

for some smooth function V (allowed to depend on t) supported in $[1,2]$ and satisfying $V^{(j)}(x) \ll_j 1$. Here $\lambda(m,n)$ are the Fourier coefficients of the form π. To establish subconvexity we need to show cancellation in the sum $S(N)$ for N roughly of size $t^{3/2}$. We can and shall further normalize V, for convenience, so that $\int V(y)\mathrm{d}y = 1$.

We apply (1.1) directly to $S(N)$ as a device for separation of the oscillation of the Fourier coefficients $\lambda(1,n)$ and n^{-it}. This by itself does not seem very effective, and we need to introduce an extra integral namely

$$S(N) = \frac{1}{K}\int_{\mathbb{R}} V\left(\frac{v}{K}\right) \mathop{\sum\sum}_{\substack{n,m=1\\ n=m}}^{\infty} \lambda(1,n) m^{-it} \left(\frac{n}{m}\right)^{iv} V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) \mathrm{d}v,$$

where $t^\varepsilon < K < t$ is a parameter which will be chosen optimally later, and U is a smooth function supported in $[1/2, 5/2]$, with $U(x) = 1$ for $x \in [1,2]$ and $U^{(j)} \ll_j 1$. For $n, m \in [N, 2N]$, the integral

$$\frac{1}{K}\int_{\mathbb{R}} V\left(\frac{v}{K}\right)\left(\frac{n}{m}\right)^{iv} \mathrm{d}v$$

is small if $|n-m| \gg Nt^\varepsilon/K$. This step is the most crucial 'trick' in this paper. The reader should realize that this is the analytic analogue of the arithmetic condition $q_1|f_2(\mathbf{m})$, that we had in the previous section. There we used this to replace $\delta(f_2(\mathbf{m}))$ by $\delta(f_2(\mathbf{m})/q_1)$, and hence the optimal choice of the size of the modulus in the circle method reduced from Y to $Y^{1/2}$. Here also it reduces the optimal size of the modulus in the circle method from $N^{1/2}$ to $(N/K)^{1/2}$. This is probably not completely obvious at this stage, but it should become clear later.

So the 'natural choice' for C in (1.1) to detect the event $n - m = 0$ is $C = \left(\frac{N}{K}\right)^{1/2}$ and we get

$$S(N) = S^+(N) + S^-(N)$$

where

$$\begin{aligned} S^\pm(N) =& \frac{1}{K}\int_0^1 \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{1\leq c\leq C}\ \sideset{}{^\star}\sum_{C<d\leq c+C} \frac{1}{cd} \\ &\times \sum\sum_{n,m=1}^{\infty} \lambda(1,n) n^{iv} m^{-i(t+v)} e\left(\pm\frac{(n-m)\bar{d}}{c} \mp \frac{(n-m)x}{cd}\right) V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) \mathrm{d}v\mathrm{d}x. \end{aligned} \tag{3.4}$$

We will analyse $S^+(N)$ (the same analysis holds for $S^-(N)$). Temporarily assume the Ramanujan conjecture $\lambda(1,n) \ll n^\varepsilon$. This is not very serious, as at any step where it is required one can use Cauchy inequality and use Ramanujan bound on average. The circle method has been used to separate the sums on n and m and we have arrived at (3.4). Trivially estimating the sum we get $S^+(N) \ll N^{2+\varepsilon}$. (For simplicity assume that $N = t^{3/2}$ and $c \asymp C$.) So we are required to save N (and a little more) in a sum of the form

$$\int_K^{2K} \sum_{c\asymp C}\ \sideset{}{^\star}\sum_{C<d\leq c+C} \sum_{n\asymp N} \lambda(1,n) n^{iv} e\left(\frac{n\bar{d}}{c} - \frac{nx}{cd}\right) \sum_{m\asymp N} m^{-i(t+v)} e\left(-\frac{m\bar{d}}{c} + \frac{mx}{cd}\right) \mathrm{d}v.$$

The sum over m has 'conductor' $Ct \asymp N^{1/2}t/K^{1/2}$. Roughly speaking the conductor takes into account both the arithmetic modulus, which is c, and the amplitude of oscillation in the analytic weight function, which is of size t. Note that both $m^{-i(t+v)}$ and m^{-it} have same amplitude if $|v| \ll t^{1-\varepsilon}$. So the extra oscillating term, namely m^{-iv} which we are inserting is not hurting us here. On the other hand larger is the K smaller is the arithmetic modulus. So the overall conductor in the sum over m is reduced. Applying Poisson summation (and 'executing' the sum over d) we are able to save $N/(Ct)^{1/2} \times C^{1/2} = N/t^{1/2}$. To this end we need the second derivative bound for the resulting exponential integral. In fact we need to use the stationary phase method. Observe that the saving so far is independent of K. Now we need to save $t^{1/2}$ in a sum of the form

$$\int_K^{2K} \sum_{c\asymp C} \sum_{\substack{(m,c)=1\\|m|\asymp Ct/N}} \left(\frac{(t+v)cd}{(x-md)}\right)^{-i(t+v)} \sum_{n\asymp N} \lambda(1,n) e\left(\frac{nm}{c}\right) n^{iv} e\left(-\frac{nx}{cd}\right) \mathrm{d}v,$$

where d is the unique multiplicative inverse of m modulo c in the range $(C, c+C]$.

Consider the sum over n, which involves the Fourier coefficients, and has 'conductor' $(CK)^3$. Observe that larger values of K is taking us to a worse situation. But applying Voronoi summation formula we are able to save $N/(CK)^{3/2} = N^{1/4}/K^{3/4}$. To this end we need Weil bound for Kloosterman sums and second derivative bound for certain exponential integrals that arise in the integral transform resulting from Voronoi. Moreover we are able to save $K^{1/2}$ in the integral over v. We now need to save $K^{1/4}t^{1/8}$ in a sum of the form

$$\sum_{n\asymp N^{1/2}K^{3/2}} \lambda(n,1) \sum_{\substack{c\asymp C,\\|m|\asymp ct/N}} \sum_{(m,c)=1} S(\bar{m}, n; c) \int_{-K}^{K} n^{-i\tau} g(c,m,\tau) \mathrm{d}\tau$$

where the function g is of size $O(1)$ but highly oscillatory. It is basically a product of a smooth bump function and

$$\left(\frac{N}{c^3}\right)^{-i\tau} \left(-\frac{(t+\tau)c}{Nm}\right)^{-i(t+\tau)}.$$

The role of the v integral and the parameter K is not yet clear. At this moment it seems to be hurting us more rather than helping.

The next step involves taking Cauchy to get rid of the Fourier coefficients, but this process also squares the amount we need to save. So now we face with the task of saving $K^{1/2}t^{1/4}$ in a sum which roughly looks like

$$\sum_{n\asymp N^{1/2}K^{3/2}} \Big| \sum_{\substack{c\asymp C\\|m|\asymp Ct/N\\(m,c)=1}} \sum S(\bar{m}, n; c) \int_{-K}^{K} n^{-i\tau} g(c,m,\tau) \mathrm{d}\tau \Big|^2.$$

One should note that we need to save $K^{1/2}t^{1/4}$ together with square root saving in the Kloosterman sum and $K^{1/2}$ saving in the integral (which is the second derivative bound). The idea is to open the absolute square and execute the sum over n using Poisson summation. The resulting diagonal contribution or the zero frequency contribution is satisfactory for our purpose if

$$\text{the number of terms inside the absolute value } = \frac{C^2 t}{N} > K^{1/2}t^{1/4}$$

or equivalently $t^{1/2} > K$. On the other hand by Poisson we make a saving (ignoring the zero frequency) of size $N^{1/2}K^{3/2}/CK^{1/2} = K^{3/2}$, as the length of the sum is

$N^{1/2}K^{3/2}$ and the conductor is of size $C^2K = N$. This is where we are getting help from the parameter K. The conductor is independent of K, but the length of the sum increases with K. So effectively we have a drop in the conductor of the sum. So the contribution of the non-zero frequencies is satisfactory if

$$K^{3/2} > K^{1/2}t^{1/4}$$

or $K > t^{1/4}$. In particular by choosing K in the range $t^{1/4} < K < t^{1/2}$ we can get a bound which breaks the convexity barrier.

So far we have seen two applications of the circle method in two different contexts. In the previous section we used the second formula (1.2) to count rational points on certain variety and in this section we used the first formula (1.1) to obtain subconvex bound for certain automorphic L-function. Again let me stress that main similarity between these two applications was the conductor lowering trick. In both cases one does not obtain anything without this trick. However in both cases a circle method based on $GL(1)$ harmonics was sufficient. But to understand higher degree L-functions it seems crucial that one utilizes the higher rank harmonics. In [17] I used (1.3) to establish the analogue of Burgess' bound for Dirichlet L-function in the context of degree three L-functions.

Theorem 3.2. *Let π be a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$ satisfying the Ramanujan conjecture and the Ramanujan-Selberg conjecture. Let χ be a primitive Dirichlet character modulo M. Suppose M is a prime number. Then there is a computable absolute constant $\delta > 0$ such that*

$$L\left(\tfrac{1}{2}, \pi \otimes \chi\right) \ll_\pi M^{\frac{3}{4}-\delta}.$$

The proof of this result is really involved, and I am afraid even a short sketch will take us too far afield. I will conclude this article by noting that as of now there are no subconvexity results for generic L-functions of degree four and higher. Of course for Rankin-Selberg convolution there is a large volume of work. But Rankin-Selberg convolutions can be treated using moment method based on $GL(2)$ trace formula, which of course does not work for generic forms. Perhaps a circle method based on $GL(3)$ Kuznetsov formula holds the key for this important problem. However one will not expect it to happen in near future as a $GL(3)$ Kuznetsov formula for general congruence group is not yet written down.

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HOW PERMUTATIONS, PARTITIONS, PROJECTIVE REPRESENTATIONS AND POSITIVE MAPS ENTANGLE WELL FOR QUANTUM INFORMATION THEORY

AJIT IQBAL SINGH

1. A PERSONAL ACCOUNT

I thank the organisers, particularly, Prof. Geetha S. Rao and Prof. N. K. Thakare for inviting me to give this lecture in memory of late Prof. Hansraj Gupta.

1.1. **Cherished memories.** Like many of you, I am lucky to have enjoyed his encouragement and inspiration for my research and teaching ever since I met him soon after my post-graduation in 1965. And when after completing his wonderful life of creativity and concern for education, in 1988 he started living instead in the hearts and minds of many, I decided to go to the very place he nurtured with his thoughts and actions. Yes, I was fortunate to get an opportunity in 1989, thanks to his right-hand man Prof. R. P. Bambah and his colleagues like Prof. H. L. Vasudeva and Prof. R. J. Hans-Gill, to work in the excellent research atmosphere at the Centre for Advanced Study in Mathematics, Department of Mathematics, Panjab University, Chandigarh, of which he was the Founder-Head. In his fond memory, the Department Complex is now called Hansraj Gupta Hall. In addition to my three joint research papers in Harmonic Analysis with H. L. Vasudeva and his Ph.D. student, Savita Kalra (now Prof. Savita Bhatnagar), I was able to do a

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Part II consisting of "Polytopes and Corners" and "Applications to Quantum Information Theory" will follow soon.

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substantial amount of work on "Completely positive hypergroup actions", Memoir Amer Math Soc, Vol. 124, No. 593(1996), which I dedicated to him, in fact.

1.2. **Tables.** His illustrious meaningful life ran parallel to that of my maternal grandfather, S. Hakam Singh Kalra, a rigorous yet practical school teacher of English. He usually asked me questions and I made him happy with my answers. Here is a combined impact.

Table

Does every table have legs?
    No, a Time-table with clock around.
Gupta's talent produced a Perpetual calendar,
    Knowing absolutely no time-bound.
And Tables of Partitions with wings wide-spread,
    Flying on their own strength all the way,
Bringing immediate recognition and big awards,
    Being used by many till today!
Nothing else in this expository article,
    is really new or truly mine,
Except for the choice of the material,
    its partitioning or the way to re-align!

2. A GLIMPSE OF GUPTA GANIT

Gg	Ua	Pn	Ti	At
Un	Pi	Tt	Ag	Ga
Pt	Tg	Aa	Gn	Ui
Ta	An	Gi	Ut	Pg
Ai	Gt	Ug	Pa	Tn

This orthogonal latin square is symbolic of Mathematics created by Gupta. If we look at it carefully we will find more GUPTA or ganit hidden.

2.1. **Latin squares.** Latin squares are useful in Designs of Experiments, Quantum Information Theory and much more. A latin square of order n is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column. Euler used latin characters. Two latin squares of the same order are said to be orthogonal to each other, if, when any of the squares is superimposed on the other, every ordered pair of symbols appears exactly once. Like we can use Colours and letters, or Greek and Latin letters, or Capital and small letters ... In 1782, Euler conjectured that there does not exist a pair of orthogonal latin squares of order $4t + 2$, $t \geq 1$. Euler's conjecture is false for $t > 1$. And it was proved by R.C. Bose, S. S. Shrikhande and E.T. Parker in 1960. For a readable account one can see Aloke Dey's article on Orthogonal latin squares and the falsity of Euler's conjecture in the book entitled "Connected at infinity-II" edited by Rajendra Bhatia, C. S. Rajan and myself, Hindustan Publishing House 2013. It also contains expository articles on Craemer-Rao lower bound and Information Geometry, Frobenius splittings, String equation of Narasimha, Representations of complex semi-simple Lie groups and Lie algebras, Parthasarathy Dirac operators and discrete series representations, The role of Verma in representation theory, and, The work of Wiener and Masani on Prediction theory and Harmonic analysis.

2.2. **Perpetual Calender.** In fact, Table 2 in Gupta's Perpetual Calendar consisting of three tables is also a Latin square after a little bit of clubbing of months as follows.

January and October	H	A	N	S	R.	J.	G.
February, March and November	R.	J.	G.	H	A	N	S
April and July	A	N	S	R.	J.	G.	H
May	G.	H	A	N	S	R.	J.
June	S	R.	J.	G.	H	A	N
August	J.	G.	H	A	N	S	R.
September and December	N	S	R.	J.	G.	H	A

To know more about this we can see Gupta's first paper, Gupta's perpetual calendar for finding the day of the week, J. Indian Math. Soc. 15(1923), 65–68.

2.3. **Panorama of papers.** To get a glimpse of his panorama of 190 papers from 1923 to 1985 giving him an average of 3 papers per year displaying his ingenuity, hardwork, patience and, at times, his good sense of humour, we can, of course, see AMS MathSciNet and then find individual papers; or, we can refer "Collected Papers of Professor Hansraj Gupta" published recently by the Ramanujan Mathematical Society, India, in two volumes together with two CDs to go with them and where in the Editors, R. J. Hans-Gill and Madhu Raka, have given excellent account of all his papers with thoughtful Editors' notes updating the reader on historical details or recent progress.

For my lecture, I chose the following five papers of Hansraj Gupta just like that. I gave some details in the talk, but here I will only give the titles and details of one of them.

(i) Hansraj Gupta, Partitions into distinct primes, Proc. Nat. Inst. Sci. India Part A. 21 (1955), 185-187.

We may note that partitions unite Algebra and Topology. The set of cycle type of a permutation on n symbols is a partition of n, and, therefore, the set of conjugacy classes of the permutation group S_n is naturally bijective with the set of all partitions of n. On the other hand, for a regular topology on a set with n

elements, closures of singletons give rise to a partition of n, and, this partition also serves as a basis for the topology. Therefore, the number of (non-homeomorphic) regular topologies on a set with n elements coincides with that of partitions of n.

(ii) Hansraj Gupta, Combinatorial proof of a theorem on partitions into an even or odd number of parts, J. Comb. Theory Ser. A 21 (1978), No. 1, 100–103.

(iii) We can all see that

$$2^0 = 1 = 3^0, \quad 2^2 = 3 + 1, \quad 2^8 = 3^5 + 3^2 + 3 + 1.$$

In a letter to Hansraj Gupta, the great Paul Erdös mentioned that these might be the only powers of 2 representable as a sum of distinct powers of 3.

Hansraj Gupta, in his paper entitled, "Powers of 2 and sums of distinct powers of 3", in Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602–633 (1978), 151–158 (1979), verifies that, for $8 < n < 4374$, 2^n cannot be represented as a sum of distinct powers of 3.

(iv) Hansraj Gupta, The rank vector of a partition, Fibo. Quarterly, 1978, 548–552.

To indicate the importance of this paper, we give below a few excerpts from some papers by George E. Andrews.

(a) In his paper, Basis Partition Polynomials, Overpartitions And The Rogers-Ramanujan Identities, J. Approximation Theory, available online 20 May, 2014, George E. Andrews says:

The late Hansraj Gupta (1978) introduced the concept of the rank vector of a partition and the basis partition for a rank vector. In this paper a common generalization of the Rogers-Ramanujan series and the generating function for basis partitions is studied.

(b) Rogers-Ramanujan Identities is too well-known a bunch of words even more so after world-wide celebration of the Birth Centenary of the legendary mathematician Srinivasa Ramanujan in 1987 and, more recently, 125th Anniversary of his birth in 2012. How Ramanujan, always engrossed in calculations ("ganan mein rama hua") could fit in infinite amount of work in his short life-span of nearly π decades - is beyond anybody's imagination. It is like applying the transformation arc tan which takes the interval $(-\infty, \infty)$ to the interval $(-\pi/2, \pi/2)$.

(c) It might be appropriate to give some relevant excerpts from George E. Andrews, An Analytic Generalization of the Rogers-Ramanujan Identities for Odd Moduli, Proc. Nat. Acad. Sci. USA. Vol. 71, No. 10, October 1974, pp. 4082-4085.

Reader can see the paper or other sources referred to therein and elsewhere for more details.

In 1894, L.J. Rogers proved

$$\sum_{n\geqq 0}\frac{q^{n^2}}{(q)_n}=\prod_{\substack{n=1\\ n\neq 0,\pm 2(\mathrm{mod}\ 5)}}^{\infty}(1-q^n)^{-1},$$

and

$$\sum_{n\geqq 0}\frac{q^{n^2+n}}{(q)_n}=\prod_{\substack{n=1\\ n\neq 0,\pm 1(\mathrm{mod}\ 5)}}^{\infty}(1-q^n)^{-1},$$

where $(a)_0=1$, $(a)_n=(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$, and $|q|<1$. These are known as the Rogers-Ramanujan or Rogers-Schur Identities.

In 1917 L.J. Rogers also discovered further identities related to the modulus 7; for example, he proved that

$$\prod_{m=1}^{\infty}(1+q^m)\sum_{n\geqq 0}\frac{q^{2n^2}}{(q^2;q^2)_n(-q)_{2n}}=\prod_{\substack{n=1\\ n\neq 0,\pm 3(\mathrm{mod}\ 7)}}^{\infty}(1-q^n)^{-1}.$$

In the same vein, in 1947, W.N. Bailey proved three identities related to the modulus 9; for example,

$$\prod_{m=1}^{\infty}(1+q^m+q^{2m})\sum_{m\geqq 0}\frac{q^{3m^2}(q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}}=\prod_{\substack{n=1\\ n\neq 0,\pm 4(mod\ 9)}}(1-q^n)^{-1}.$$

A large amount of work was done on identities of this type, and a collection of some 130 such identities was published by L.J. Slater in 1952. In each of the identities given by Slater, the modulus arising in the infinite product is always $\leqq 64$ and always has its prime factors among $\{2,3,5,7\}$.

Five double series identities were obtained related to the modulus 11; by G.E. Andrews in 1974. For instance,

$$\prod_{m=1}^{\infty}(1+q^m+q^{2m}+q^{3m})\sum_{n\geqq 0}\sum_{j\geqq 0}\frac{q^{4n^2+12nj+8j^2+j}(-1)^j(q)_{4n+2j}}{(q^4;q^4)_n(q^2;q^2)_j(q^4;q^4)_{2n+2j}}$$

$$=\prod_{\substack{n=1\\ n\neq 0,\pm 5(mod\ 11)}}^{\infty}(1-q^n)^{-1}.$$

Unfortunately the methods used to prove these generalisations seem limited to special cases and do not provide general theorems for an infinite family of moduli.

In 1954, H.L. Alder was able to show that there exist polynomials $G_{k,i}(n,q)$ such that

$$\sum_{n\geqq 0}\frac{G_{k,i}(n;q)}{(q)_n}=\prod_{\substack{n=1\\ n\neq 0,\pm i(mod\ 2k+1)}}^{\infty}(1-q^n)^{-1}.$$

However neither his work nor subsequent papers on the Alder polynomials by himself, V. N. Singh, G. E. Andrews, L. Carlitz, yielded analytic identities of the elegance and simplicity of the Rogers-Ramanujan Identities.

Andrews' object in this paper is to prove the following generalization of the Rogers-Ramanujan identities.

Theorem. Let $1 \leqq i \leqq k$ be integers; then

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots++N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$

$$= \prod_{\substack{n=1 \\ n \neq 0, \pm i (mod\ 2k+1)}}^{\infty} (1-q^n)^{-1},$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

After this interesting account, from Andrews paper, which might have made my contemporaries recall the 1994 Hansraj Gupta Memorial Award lecture by Arun Verma a bit, the next item tells why I gave it!

(v) Hansraj Gupta, Diophantine Equations in Partitions, Mathematics of Computation, Volume 42, Number 165, January 1984, pages 225-229.

Given $m > 1$, and integers $r_1, r_2, r_3, \ldots, r_j$ such that

$$0 < r_1 < r_2 < \ldots < r_j < m;$$

we have to find the number $P(n, m; R)$ of partitions of a given number n into parts belonging to the set R of residue classes $r_1 (mod\, m)$, $r_2 (mod\, m)$, $\ldots$, $r_j (mod\, m)$, if any such exist.

This is done in two different ways.

This leads to an identity which is more general than, but not as elegant as, the well-known Rogers-Ramanujan identities.

To give an idea, consider the set of those partitions of n which have

$$\begin{aligned} &a_1 \text{ parts each congruent to } r_1 (mod\, m),\\ &a_2 \text{ parts each congruent to } r_2 (mod\, m),\\ &\quad\vdots\\ &a_j \text{ parts each congruent to } r_j (mod\, m). \end{aligned}$$

For n to have such a partition, it is necessary that

$$n = r_1 a_1 + r_2 a_2 + \cdots + r_j a_j + Cm \text{ for some integer } C \geq 0.$$

We write $X(a_i)$ for $1/\{(1-x)(1-x^2)\cdots(1-x^{a_i})\}$ with $X(0) = 1$.

Elaborate, yet simple computations give

$$P(n, m; R) = \sum_{C=0}^{[n/m]} \sum\nolimits_{(a_i)} \text{ coefficient of } x^C \text{ in } X(a_1)X(a_2)\cdots X(a_j),$$

where each $a_i > C$ can be replaced by C, and the j-tuple (a_i) varies over solutions of equation talked about above.

It is a well-known fact that $P(n,m;R)$ is the coefficient of x^n in the expansion of

$$\prod_{q=0}^{[n/m]} \left\{ \prod_{i=1}^{j} (1 - x^{r_i+qm}) \right\}^{-1}.$$

Equating the two expressions,we get what Andrews asks for!

Little wonder that Andrews contributed his lucid article, My Friend, Hansraj Gupta, for the Collected papers of Professor Hansraj Gupta, Volume I.

3. PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

We now come to the remarkable *projective representations* of finite groups, introduced and fully solved by I. Schur in a series of three papers in German. The last one in 1911 has been translated into English in *International J. of Theoretical Physics*, Vol. 40, No. 1, 2001, p. 413-458. Good expositions are available, amongst others, in books by K.R. Parthasarathy (1992, 2005 to 2013), V.S. Varadarajan (1985), M. Hamermesh (1962, 1989), Chapter 12; G. Karpilovsky (1985), Chapters 3 and 4, and most begin with special cases of Weyl operators. Some recent ones come from Alexander Kleshchev et al.

3.1. **Definition.** Let G be a finite group, V be a finite dimensional vector space over a field F and $GL(V)$, the group of invertible linear transformations of V to itself.

Definition 3.1. A mapping $\rho : G \to GL(V)$ is called a projective representation of G over F if there exists a mapping $\alpha : G \times G \to F \setminus \{0\}$ such that

(i) $\rho(x)\rho(y) = \alpha(x,y)\rho(xy)$, $x, y \in G$,

(ii) $\rho(1_G) = I_V$.

In case we are interested in a complex inner product space and unitary transformations/operators instead of just invertible ones we need $\alpha(x,y) \in \mathbf{T}$, the unit circle. We say ρ is a projective unitary representation of G. The word projective is related to projective geometry and fractional linear transformation associated to an invertible matrix, and, is replaced by ray or factor in literature.

Thus a linear (unitary) or a unitary representation is a *projective (unitary) representation* with $\alpha(x,y) = 1$ for all x, y in G. Projective unitary representations have applications to Quantum Information Theory.

3.2. **Projective representations as developed by Schur.** We give relevant extracts from Schur's papers as presented in §12-1 and §12-2 of M. Hamermesh's excellent book entitled, "Group Theory and its Application to Physical Problems" (Addison-Wesley 1962, Dover 1989). This serves the purpose of keeping the article accessible to a graduate student well. The same is true of the next two subsections.

Schur gave the general method for finding the irreducible projective representations of a finite group in terms of *fractional linear transformation* (*projective transformation, collineations*).

We all know fractional transformation or Möbius transformation in the context of complex numbers given by a function that takes z to $(az+b)/(cz+d)$, a, b, c, d being complex numbers satisfying $ad-bc \neq 0$. We also associated such a bijective map of the extended complex plane to itself to the non-singular complex matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and any non-zero multiple of this matrix gives rise to the same function again.

Schur considered a generalisation to $n \times n$ complex matrices.

We associate to each element $A, B, \ldots$ of a finite group a *fractional linear transformation*

$$A \to \{A\} : \quad x_i' = \frac{a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{i,n-1}x_{n-1} + a_{in}}{a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1} + a_{nn}} \qquad (i = 1, \ldots, n-1) \tag{3.1}$$

where the matrix $D(A)$ of the coefficients a_{ij} is nonsingular. The fractional linear transformations give a representation of the group H if

$$\{A\}\{B\} = \{AB\}. \tag{3.2}$$

The degree of the representation is the degree n of the matrices $D(A)$.

The transformations (3.1) can be regarded as linear transformations

$$y_i' = \sum_{j=1}^{n} a_{ij} y_j \quad (i = 1, \ldots, n), \tag{3.3a}$$

where

$$x_i = \frac{y_i}{y_n} \quad (i = 1, \ldots, n-1), \tag{3.3b}$$

are homogeneous coordinates.

From (3.1) and (3.3b) we see that the multiplication of the elements a_{ij} by a common factor does not change the transformation $\{A\}$. If we write out the matrices corresponding to (3.2), we find

$$D(A)D(B) = \omega_{A,B} D(AB), \tag{3.4}$$

where the $\omega_{A,B}$ are a set of constants depending on the choice of $D(A)$ and $D(B)$. Conversely, if we are given a set of nonsingular matrices satisfying (3.4), we can find a corresponding representation (3.2) in terms of fractional linear transformations. Thus the problem of finding representations of a group by fractional linear transformations is equivalent to finding representations by linear transformations $D(A)$ which satisfy (3.4).

We shall call the representation (3.4) a *projective representation* belonging to the *factor system* $\omega_{A,B}$. If, in particular, all factors $\omega_{A,B}$ are equal to unity, Eq. (3.4) defines the usual representations, which we shall call *vector representations*.

The concepts of equivalence and reducibility are the same as usual. The (projective) representation $D'(A)$ is equivalent to the representation $D(A)$ of (3.4) if there exists a nonsingular matrix S such that $D'(A) = SD(A)S^{-1}$ for all A in H. Then

$$D'(A)D'(B) = \omega_{A,B}D'(AB) \tag{3.5}$$

so that equivalent representations belong to the same factor system. If we can find a matrix S which gives the representations $D'(A)$ the reduced form

$$D'(A) = \begin{bmatrix} D_1'(A) & 0 \\ 0 & D_2'(A) \end{bmatrix} \tag{3.6}$$

the representation is reducible to the direct sum of the (projective representations) $D_1' + D_2'$. From (3.5) and (3.6) we see that the factor system $\omega_{A,B}$ is the same for $D(A)$, $D'(A)$, $D_1'(A)$, and $D_2'(A)$.

The projective representation (3.4) is *irreducible* if there is no equivalent representation having the form (3.6).

The element $\omega_{A,B}$ of the factor system in (3.4) cannot be chosen arbitrarily. Suppose that the finite group H consists of h elements $H_0 = E, H_1, \ldots, H_{h-1}$. If the matrices $D(H_i)$ give a projective representation of H belonging to the factor system $\omega_{A,B}$, then for any three elements P, Q, R of the group H,

$$\begin{aligned} &D(P)D(Q) = \omega_{P,Q}D(PQ) \\ &D(P)D(Q)D(R) = \omega_{P,Q}D(PQ)D(R) = \omega_{P,Q}\omega_{PQ,R}D(PQR); \qquad (3.7a) \\ &D(Q)D(R) = \omega_{QR}(QR), \\ &D(P)D(Q)D(R) = \omega_{Q,R}D(P)D(QR) = \omega_{Q,R}\omega_{P,QR}D(PQR); \qquad (3.7b) \end{aligned}$$

So,

$$\omega_{P,Q}\omega_{PQ,R} = \omega_{P,QR}\omega_{Q,R} \quad (P, Q, R = H_0, \ldots, H_{h-1}) \tag{3.8}$$

Thus the associative law for group multiplication forces the h^2 constants $\omega_{A,B}$ to satisfy (3.8).

Conversely, if an h^2-tuple of constants $w_{A,B}$ satisfy (3.8) then there exists a projective representation of H belonging to the factor system $\omega_{A,B}$.

In fact, the details of the proof given show that such a representation of degree h can be chosen even to satisfy that for each R, the matrix D(R) contains only one nonzero element in each row and each column.

The equations (3.8) have an infinite number of solutions. If $\omega_{A,B}$ is a solution and $c_{H_0}, c_{H_1}, \dots, c_{H_{h-1}}$ are any constants not equal to zero, then

$$\omega'_{A,B} = \frac{c_A c_B}{c_{AB}} \omega_{A,B} \tag{3.9}$$

is also a solution. Solutions $\omega_{A,B}$ and $\omega'_{A,B}$ for which one can find constants c_A so that (3.9) is satisfied, are called *associated* or *equivalent* factor systems. If we find a representation $D(A)$ belonging to the factor system $\omega_{A,B}$, the matrices

$$D'(A) = c_A D(A) \tag{3.10}$$

satisfy

$$D'(A)D'(B) = c_A c_B D(AB) = \frac{c_A c_B}{c_{AB}} \omega_{A,B} D'(AB) = \omega'_{A,B} D'(AB) \tag{3.11}$$

Thus, if two factor systems are equivalent, the representations belonging to one are immediately determined from the representations belonging to the other by means of (3.9). Representations belonging to nonequivalent factor systems are said to be of *different type*.

In view of (3.4), taking n=h and letting $\det D(R) = d_R$, we have

$$(\omega_{R,S})^h = \frac{d_R d_S}{d_{RS}}. \tag{3.12}$$

If we choose the constant c_A in (3.9) to be $c_A = d_A^{-1/h}$ (so that c_A is one of the h solutions of the equation $c_A^{-h} = d_A$), we can obtain from the factor system $\omega_{R,S}$ an equivalent system

$$\omega'_{R,S} = \frac{c_R c_S}{c_{RS}} \omega_{R,S} \tag{3.13}$$

for which, according to (3.23)

$$(\omega'_{R,S})^h = 1. \tag{3.14}$$

Thus a class of equivalent factor systems necessarily contains a factor system in which the h^2 quantities $\omega_{A,B}$ are hth roots of unity. Therefore the number of classes cannot exceed h^{h^2}, and hence the number of classes is finite.

Let us denote the m classes by $K_0, K_1, \dots, K_{m-1}$, where K_0 is the class containing the factor system $\omega_{A,B} \equiv 1$. If the factor systems $\omega^{(\lambda)}_{A,B}$ and $\omega^{(\mu)}_{A,B}$ are solutions of (3.9) and belong to the class K_λ and K_μ, their products $\omega^{(\lambda)}_{A,B} \cdot \omega^{(\mu)}_{A,B}$ also satisfy (3.9). The class K_ν, to which this new solution belongs, does not depend on the particular choice of the solutions $\omega^{(\lambda)}_{A,B}$ and $\omega^{(\mu)}_{A,B}$ within the classes K_λ and K_μ, but it is determined by the classes themselves. We write $K_\lambda K_\mu = K_\nu$, and note that $K_\mu K_\lambda = K_\lambda K_\mu = K_\nu$. We thus have defined a multiplication of classes which is commutative. Furthermore, we easily see that if $K_\alpha K_\beta = K_\alpha K_\gamma$, then $K_\beta = K_\gamma$. The class K_0 serves as the unit element. Finally, it is clear that this class multiplication is associative, and the classes $K_0, K_1, \dots, K_{m-1}$ therefore form an abelian group of order m, which is called the *multiplicator* of the group H.

Schur developed general methods for finding the multiplicator and for the construction of the irreducible projective representations of large classes of finite groups. It is too long to be given here. Instead we illustrate the procedure for some simple cases.

3.3. **Examples.**

(i) (a) For cyclic groups, the multiplicator consists of the unit element K_0, and all projective representations are equivalent to the usual vector representations. The irreducible representations are all one-dimensional.

(b) Similar situation occurs for the dihedral group D_n for odd n.

(ii) (a) For the four group $\{I, A, B, C\}$ we have the usual one-dimensional representations and projective representations given by anticommuting matrices $D(A)$, $D(B)$, $D(C)$ having their square equal to the identity. Thus the irreducible projective representation is given in terms of the Pauli matrices:

$$D(A) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D(C) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

(b) Similar situation occurs for the dihedral groups D_n of higher even orders.

3.4. **Relation of projective representations with partitions.** In line with a glimpse of GUPTA GANIT given above, the most interesting case is of the symmetric group S_n of permutations on a set of n elements. It is related to partitions!

The symmetric group S_n is generated by the $n-1$ transpositions

$$T_1 = (12),\ T_2 = (23),\ \ldots,\ T_{n-1} = (n-1, n)$$

which satisfy the equations

$$T_i^2 = E,\ (T_jT_{j+1})^3 = E,\ T_rT_s = T_sT_r \begin{bmatrix} i = 1, \ldots, n-1 \\ j = 1, \ldots, n-2 \\ r = 1, 2, \ldots, n-3 \\ s = r+2, \ldots, n-1 \end{bmatrix} \tag{3.15}$$

Conversely, one can show that the abstract group defined by the equations (3.15) is isomorphic to S_n.

Let us assume that we have a projective representation of S_n in which the elements T_i are represented by matrices A_i. The operator equations corresponding to (3.15) are

$$A_i^2 = a_i\mathbf{1}, \tag{3.16a}$$

$$(A_jA_{j+1})^3 = b_j\mathbf{1}, \tag{3.16b}$$

$$A_rA_s = c_{rs}A_sA_r, \tag{3.16c}$$

where the a_i, b_j, c_{rs} are constants different from zero. The constants c_{rs} occur only for $n \geq 4$ and, as we see from (3.16c), cannot be altered by multiplying the matrices A_r by factors. Thus the constants c_{rs} are determined by the fractional linear transformations.

From (3.16c), we have $A_rA_sA_r^{-1} = c_{rs}A_s$. Squaring and using (3.16a), we find

$$c_{rs}^2 = 1\,. \tag{3.17}$$

The indices r, $r+1$, s, $s+1$, which appear in the permutations T_r and T_s of (3.15), are all different. If we take another pair of permutations $T_{r'}$, $T_{s'}$ for which r', $r'+1$, s', $s'+1$ are all different, there exists a permutation T which takes the indices $r, r+1, s, s+1$ into $r', r'+1, s', s'+1$, so that

$$TT_rT^{-1} = T_{r'}, \quad TT_sT^{-1} = T_{s'}\,. \tag{3.18}$$

If the permutation T is represented by the operator A, the corresponding equations for the operators of the representation are

$$AA_rA^{-1} = uA_{r'}, \quad AA_sA^{-1} = vA_{s'}\,. \tag{3.19}$$

where u and v are constants different from zero. From (3.16c),

$$AA_rA^{-1}AA_sA^{-1} = c_{rs}A_sA^{-1}AA_rA^{-1}$$

and, using (3.19), we have

$$uv\,A_{r'}A_{s'} = uv\,c_{rs}A_{s'}A_{r'} \quad \text{or} \quad A_{r'}A_{s'} = c_{rs}A_{s'}A_{r'}$$

Comparing with (3.16c), that is,

$$A_{r'}A_{s'} = c_{r's'}A_{s'}A_{r'}$$

we find that $c_{rs} = c_{r's'}$. Thus according to (3.17) all constant c_{rs} are the same and equal to ±1. Letting $c = \pm1$, we have $c_{rs} = c$ for all r, s.

From (3.16b) we have

$$A_jA_{j+1}A_j = b_jA_{j+1}^{-1}A_j^{-1}A_{j+1}^{-1}\,.$$

Squaring, we get

$$\begin{aligned} A_jA_{j+1}A_j^2A_{j+1}A_j &= b_j^2A_{j+1}^{-1}A_j^{-1}A_{j+1}^{-2}A_j^{-1}A_{j+1}^{-1}\,, \\ a_j^2a_{j+1} &= \frac{b_j^2}{a_ja_{j+1}^2}\,, \\ b_j^2 &= a_j^3a_{j+1}^3\,. \end{aligned} \tag{3.20}$$

Since the matrices A_i can be multiplied by arbitrary constants, we can choose the constants a_i in (3.16a) arbitrarily. We illustrate a method of construction with an easy choice.

First we set

$$a_1 = a_2 = \ldots = a_{n-1} = 1\,, \tag{3.21}$$

and introduce matrices $C_1, \ldots, C_{n-1}$:

$$C_1 = A_1,\ C_2 = b_1A_2,\ C_3 = b_1b_2A_3,\ C_4 = b_1b_2b_3A_4, \ldots \tag{3.22}$$

which satisfy

$$C_i^2 = 1,\ (C_jC_{j+1})^3 = 1,\ C_rC_s = cC_sC_r\,. \tag{3.23}$$

For $c = 1$, the system reduces to (3.16a), and we obtain the class K_0 of vector representations of the permutation group. For $c = -1$, we obtain the class K_1 of projective representations, with operator equations

$$C_i^2 = 1,\ (C_jC_{j+1})^3 = 1,\ C_rC_s = -C_sC_r\,. \tag{3.24}$$

For $n < 4$, the indices r, s do not occur in (3.15) nor in (3.16). Thus for $n < 4$, the symmetric group has only the usual irreducible vector representations. For $n \geq 4$, there are additional irreducible representations. Schur showed that for $n \geq 4$ the number of inequivalent irreducible representations of this second type is equal to the number of partitions

$$n = \nu_1 + \nu_2 + \ldots + +\nu_m \quad (\nu_1 > \nu_2 > \cdots > \nu_m > 0) \tag{3.25}$$

of n into unequal integers. To the partition (3.25) there corresponds an irreducible representation of degree

$$f_{\nu_1,\ldots,\nu_m} = 2^{[(n-m)/2]}\frac{n!}{\nu_1!\nu_2!\cdots\nu_m!}\prod_{\alpha<\beta}\frac{\nu_\alpha - \nu_\beta}{\nu_\alpha + \nu_\beta} \tag{3.26}$$

where $[(n-m)/2]$ denotes the largest integer which is $\leq (n-m)/2$. For $n = 4$, we have two new irreducible representations of degree $f_4 = 2$ and $f_{3,1} = 4$. For $n = 4$, the equations (3.24) are:

$$C_1^2 = C_2^2 = C_3^2 = 1,\ (C_1C_2)^3 = (C_2C_3)^3 = 1,\ C_1C_3 = -C_3C_1. \tag{3.27}$$

The representation of degree two is obtained by setting

$$C_1 = \sigma_x,\ C_2 = \frac{\sqrt{3}}{2}\sigma_y - \frac{1}{2}\sigma_x,\ C_3 = \sqrt{\frac{2}{3}}\sigma_z - \sqrt{\frac{1}{3}}\sigma_y\,, \tag{3.28}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. Even verification of this is a learning experience.

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A GENERATING FUNCTION RELATED TO π

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Abstract. The generating function (in the variable t) given by

$$\int_0^1\int_0^1\int_0^1 \frac{v\sqrt{d}}{(d+x^2)\sqrt{1-v^2}\sqrt{1-u^2}}\frac{(1+tu^2)^2}{\sqrt{\{1+tu^2(2-4x^2u^2v^2+tu^2)\}^3}}dudvdx$$

is shown to be related to certain algebraic approximations to π.

1. Introduction and main result.

The aim of this note is to present a new generating function given by formula (2) which (in some sense) is related to certain algebraic approximations to π. For a source book on π we refer the reader to [3]. Our result is not meant to compete with other existing results. We prove the following theorem.

Theorem 1.1. *Let $0<d, \sqrt{d}$. For $n=1,2,3,\dots,$ set*

$$A_n(d) := \frac{(-1)^n}{4^n}\sum_{j=0}^{n}(-1)^j\binom{2n+2j}{2n}\binom{2n}{n-j}\Big\{\frac{1}{2j-1}-\frac{d}{2j-3}+\frac{d^2}{2j-5}-\cdots+(-d)^{j-1}\Big\},$$

$$B_n(d) := \frac{(-1)^n}{4^n}\sum_{j=0}^{n}\binom{2n+2j}{2n}\binom{2n}{n-j}d^j, \tag{1}$$

and $A_0(d)=0, B_0(d)=1$. Note that in $A_n(d)$, the bracket $\{\}=0$ if $j=0$.

i) *For t in a neighbourhood of zero one has*

$$\sum_{n=0}^{\infty}\Big\{\sqrt{d}A_n(d)+B_n(d)\arctan\Big(\frac{1}{\sqrt{d}}\Big)\Big\}t^n = \tag{2}$$

$$\frac{2}{\pi}\int_0^1\int_0^1\int_0^1 \frac{v\sqrt{d}}{(d+x^2)\sqrt{1-v^2}\sqrt{1-u^2}}\frac{(1+tu^2)^2}{\sqrt{\{1+tu^2(2-4x^2u^2v^2+tu^2)\}^3}}dudvdx.$$

ii) *If $d\ge 3$ then $B_n(d), A_n(d)\neq 0$ for all $n\ge 1$. Also $A_n(d)4^n lcm\{1,\dots,2n\}$*

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and $B_n(d)4^n$ *are polynomials of degree* $n-1$ *and* n *respectively in* $Z(d)$. *Let* d *be any of the numbers* $d_0 = 3$, $d_1 = 7+4\sqrt{3}$, $d_2 = 15+8\sqrt{3}+2\sqrt{104+60\sqrt{3}}, \ldots,$ *where* d_k *is obtained recursively from the formula*

$$d_k = 2d_{k-1} + 1 + 2\sqrt{d_{k-1}(d_{k-1}+1)}.$$

Then for $n \geq 1, k \geq 0$ *one has*

$$0 < |\sqrt{d_k} A_n(d_k) + B_n(d_k)\frac{\pi}{6.2^k}| \leq \frac{(\sqrt{d_k+1} - \sqrt{d_k})^{2n}}{\sqrt{d_k}}, \tag{3}$$

and $\frac{\pi}{6.2^k} = \arctan(\frac{1}{\sqrt{d_k}})$.

2. Proof.

i) Set

$$F_n(x) := \sum_{j=0}^{n} \frac{(-1)^{n+j}}{4^n} \binom{2n+2j}{2n} \binom{2n}{n-j} x^{2j},$$

and considering $i = +\sqrt{-1}$ observe that

$$\begin{aligned} A_n(d) &= \int_0^1 \frac{F_n(x) - F_n(i\sqrt{d})}{d+x^2} dx, \\ B_n(d) &= F_n(i\sqrt{d}), \\ \arctan\left(\frac{1}{\sqrt{d}}\right) &= \sqrt{d} \int_0^1 \frac{dx}{d+x^2} \end{aligned}$$

$$\textit{and} \qquad \sqrt{d} \int_0^1 \frac{F_n(x)dx}{d+x^2} = A_n(d)\sqrt{d} + B_n(d) \arctan\left(\frac{1}{\sqrt{d}}\right). \tag{4}$$

To prove the formula (2), recall the notations $(2m-1)!! = 1.3.5\cdots(2m-1)$, $(2m)!! = 2.4\cdots(2m)$ (see [1]) and notice that

$$\frac{\pi}{2}\frac{(2m-1)!!}{(2m)!!} = \int_0^1 \frac{x^{2m}}{\sqrt{1-x^2}} dx,$$

and

$$\frac{(2m)!!}{(2m+1)!!} = \int_0^1 \frac{x^{2m+1}}{\sqrt{1-x^2}} dx,$$

so that for $0 \leq j, 1 \leq n$

$$\begin{aligned} &\frac{[2(j+n)-1][2(j+n)-3]\cdots(2j+3)(2j+1)}{(j+n)(j+n-1)\cdots(j+1)} \\ &= \frac{(2j+1)2^{n+1}}{\pi} \int_0^1 \frac{u^{2(j+n)}}{\sqrt{1-u^2}} du \int_0^1 \frac{v^{2j+1}}{\sqrt{1-v^2}} dv, \end{aligned}$$

and

$$\begin{aligned} &\frac{[2(j+n)-1][2(j+n)-3]\cdots(2j+3)(2j+1)}{(j+n)(j+n-1)\cdots(j+1)} \binom{n}{j}\binom{n+j}{j} \\ &= \frac{1}{2^n}\binom{2n+2j}{2n}\binom{2n}{n-j}. \end{aligned}$$

Thus

$$F_n(x) = \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j}\binom{n+j}{j}(2j+1)\Big(\frac{2}{\pi}\int_0^1 \frac{u^{2(j+n)}}{\sqrt{1-u^2}}du \int_0^1 \frac{v^{2j+1}}{\sqrt{1-v^2}}dv\Big)x^{2j}$$

$$= \frac{2}{\pi}\int_0^1\int_0^1 G_n(xuv)\frac{u^{2n}}{\sqrt{1-u^2}}\frac{v}{\sqrt{1-v^2}}dudv, \tag{5}$$

where

$$G_n(w) := \sum_{j=0}^{n} (-1)^{n+j}\binom{n}{j}\binom{n+j}{j}(2j+1)w^{2j}.$$

The definition of Legendre polynomials (see [2]) gives

$$\frac{1}{\sqrt{1+2y(1-2w^2)+y^2}} = \sum_{n=0}^{\infty} y^n \Big\{\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\binom{n+j}{j}w^{2j}\Big\},$$

and therefore

$$\sum_{n=0}^{\infty} G_n(w)y^n = \frac{d}{dw}\Big\{\frac{w}{\sqrt{1+2y(1-2w^2)+y^2}}\Big\} = \frac{(1+y)^2}{(1+y(2-4w^2+y))^{3/2}}. \tag{6}$$

The formula (2) now follows from (4), (5) and (6).

ii) From formula (1) it follows that $B_n(d) \neq 0$ if $d > 0$, $n = 1, 2, 3, \ldots$. Also notice that $\frac{1}{2j-1} - \frac{d}{2j-3} + \frac{d^2}{2j-5} - \ldots + (-d)^{j-1}$ is equal to $(-d)^{j-1}$ multiplied by something positive if $d \geq 3$; which gives, using formula (1), that $A_n(d) \neq 0$ as stated.

That $A_n(d)4^n lcm\{1,\ldots,2n\}$ and $B_n(d)4^n$ are polynomials of degree $n-1$ and n respectively in $Z(d)$ is trivial from the definitions.

The following formula holds: if $1 \leq d, n$; $d \in \mathbb{R}$ then

$$0 < |\sqrt{d}A_n(d) + B_n(d)\arctan\Big(\frac{1}{\sqrt{d}}\Big)| \leq \frac{(\sqrt{d+1}-\sqrt{d})^{2n}}{\sqrt{d}}.$$

This yields the inequality (3). The above formula is a special case of Proposition 1 in [4] and we refer the reader to that paper.

Finally the recursive formula for the d_k is obtained from the relation

$$\arctan(z) = 2\arctan\Big(\frac{z}{\sqrt{z^2+1}+1}\Big).$$

Thus putting $z = \frac{1}{\sqrt{d}}$ in this last formula gives

$$\arctan\frac{1}{\sqrt{d}} = 2\arctan\frac{1}{\sqrt{d'}}, \qquad \text{if} \qquad d' = 2d+1+2\sqrt{d(d+1)}.$$

It follows that

$$\arctan\frac{1}{\sqrt{d_0}} = 2\arctan\frac{1}{\sqrt{d_1}} = \cdots = 2^k\arctan\frac{1}{\sqrt{d_k}}$$

For the sequence starting with $d_0 = 3$, notice that $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. This completes the proof of the formula (3). $\square$

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ON A TYPE OF LP-SASAKIAN MANIFOLDS ADMITTING A QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. The object of the present paper is to study a quarter symmetric non metric connection ∇ in an LP-Sasakian manifold and find some geometrical properties.

1. Introduction

The idea of metric connection with torsion tensor in Riemannian manifold was introduced by Hayden [5]. Later, Yano [18] studied some properties of semi symmetric metric connection on Riemannian manifold. The semi symmetric metric connection has important physical application such as the displacement on the earth surface following a fixed point is metric and semi symmetric. Golab [4] introduced and studied quarter symmetric connection in Riemannian manifold with an affine connection which generalizes the idea of semi symmetric connection. After Golab, Rastogi ([14], [15]) continued the systematic study of quarter symmetric metric connection. Pandey and Mishra [8] studied quarter symmetric metric connection in Riemannian, Kahlerian and Sasakian manifolds. It is also studied by many geometers like as Yano et al. [19], De and Biswas [2], Mukhopadhya [10], Mondal et al. [9] and many others.

On the other hand Matsumoto [6] introduced the notion of LP-Sasakian manifolds.Then Mihai and Rosoca [7] introduced the same notion independently and obtained several results on this manifold. LP-Sasakian manifolds are also studied by De et al. [3], Singh [16] and others.

In the present paper we studied a quarter symmetric non metric connection in an LP- Sasakian manifold. The paper is organized as follows: In section 2, we give a brief account of an LP- Sasakian manifold, a quarter symmetric non metric connection and find the expression for curvature tensor of this connection. Section 3 deals with locally ϕ-symmetric LP-Sasakian manifolds with respect to a

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quarter symmetric non metric connection. In section 4 locally pseudo projective ϕ-symmetric LP-Sasakian manifolds admitting a quarter symmetric non metric connection is studied. ϕ-pseudo projectively flat and $\phi - W_2$ projectively flat LP-Sasakian manifolds with respect to a quarter symmetric non metric connection are studied in sections 5 and 6 respectively and it is proved that if M^n $(n > 3)$ be a $\phi{-}W_2$ projectively flat LP-Sasakian manifolds with respect to a quarter symmetric non metric connection then M^n is an η-Einstein manifold with zero scalar curvature. In the last section, we study $R.\overline{R} = 0$ and obtain that M^n is an η-Einstein manifold.

2. Preliminaries

An n-dimensional differentiable manifold M^n is a Lorentzian Para-Sasakian (briefly LP-Sasakian) manifolds if it admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a covariant vector field η, and a Riemannian metric g, which satisfy

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\phi\xi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$(D_X\phi)Y = g(X, Y)\xi + \eta(Y)X + 2\,\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$D_X\xi = \phi X, \tag{2.6}$$

for any vector fields X and Y, where D denotes covariant differentiation with respect to g ([7], [6]).

It can be easily seen that in an LP-Sasakian manifold the following relations hold

$$(a)\ \ \eta(\xi) = -1 \qquad\qquad (b)\ \ \eta(\phi X) = 0, \tag{2.7}$$

$$rank(\phi) = (n-1). \tag{2.8}$$

Further in an LP-Sasakian manifold M^n with the structure (ϕ, ξ, η, g) the following relations hold:

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{2.9}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.11}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.13}$$

$$(D_X\eta)(Y) = g(\phi X, Y) = g(\phi Y, X), \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

for any vector fields X, Y, Z, where R and S are the Riemannian curvature tensor and Ricci tensor of the manifold respectively ([7], [6]).

An LP-Sasakian manifold M^n is said to be η- Einstein if its Ricci tensor S is of the form

$$S(X,Y) = \alpha\, g(X,Y) + \beta\, \eta(X)\eta(Y), \tag{2.16}$$

for any vector fields X and Y, where $\alpha,\ \beta$ are smooth functions on M^n [1].

Here we consider a quarter symmetric non metric connection ∇ on LP-Sasakian manifolds

$$\nabla_X Y = D_X Y - \eta(X)\phi Y \tag{2.17}$$

given by Mishra and Pandey [8] which satisfies

$$(\nabla_X g)(Y,Z) = 2\, \eta(X)\, g(\phi Y, Z). \tag{2.18}$$

The curvature tensor $\overline{R}$ with respect to a quarter symmetric non metric connection ∇ and the curvature tensor R with respect to Riemannian connection D in LP-Sasakian manifold are related as

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\, \eta(X)\xi - g(X,Z)\, \eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \end{aligned} \tag{2.19}$$

Contracting (2.19) with respect to X we get

$$\overline{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\, \eta(Y)\eta(Z), \tag{2.20}$$

where $\overline{S}$ is Ricci tensor of M^n with respect to quarter symmetric non metric connection.
Contracting (2.20) , we get

$$\overline{r} = r, \tag{2.21}$$

where $\overline{r}$ and r are the scalar curvature tensor of the connection ∇ and D respectively.

3. Locally ϕ-Symmetric LP-Sasakian Manifolds admitting a Quarter-Symmetric non Metric Connection

An LP-Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((D_W R)(X,Y)Z) = 0, \tag{3.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi [17] for Sasakian manifolds.

Analogous to the definition of locally ϕ-symmetric LP-Sasakian manifolds with respect to Levi-Civita connection, we define a locally ϕ -symmetric LP-Sasakian manifolds with respect to the quarter-symmetric non metric connection by

$$\phi^2((\nabla_W \overline{R})(X,Y)Z) = 0, \tag{3.2}$$

for all vector fields X, Y, Z, W orthogonal to ξ.
By virtue of (2.17) we can write

$$((\nabla_W \overline{R})(X,Y)Z = ((D_W \overline{R})(X,Y)Z) - \eta(W)\phi\overline{R}(X,Y)Z. \tag{3.3}$$

Now differentiating (2.19) covarintaly with respect to W, we have

$$\begin{aligned}(D_W \overline{R})(X,Y)Z &= (D_W R)(X,Y)Z + \{(D_W \eta)(X)\, g(Y,Z)\\ &- (D_W \eta)(Y)\, g(X,Z)\}\xi\\ &+ \{\eta(X)\, g(Y,Z) - \eta(Y)\, g(X,Z)\}\, (D_W \xi)\\ &+ \{(D_W \eta)(X)Y - (D_W \eta)(Y)X\}\, \eta(Z)\\ &+ \{\eta(X)Y - \eta(Y)X\}\, (D_W \eta)(Z),\end{aligned} \tag{3.4}$$

which on using equations (2.6) and (2.14) reduces to

$$\begin{aligned}(D_W \overline{R})(X,Y)Z &= (D_W R)(X,Y)Z + \{\eta(X)Y - \eta(Y)X\}g(\phi Z, W)\\ &+ \{g(\phi X, W)\, g(Y,Z) - g(\phi Y, W)\, g(X,Z)\}\xi\\ &+ \{\eta(X)\, g(Y,Z) - \eta(Y)\, g(X,Z)\}\phi W\\ &+ \{g(\phi X, W)Y - g(\phi Y, W)X\}\eta(Z).\end{aligned} \tag{3.5}$$

By virtue of equations (3.5) and (2.7), equation (3.3) takes the form

$$\begin{aligned}\phi^2((\nabla_W \overline{R})(X,Y)Z) &= \phi^2((D_W R)(X,Y)Z)\\ &+ \{\eta(X)\, g(Y,Z) - \eta(Y)\, g(X,Z)\}\phi^2(\phi W)\\ &+ \{g(\phi X, W)(\phi^2 Y) - g(\phi Y, W)(\phi^2 X)\}\, \eta(Z)\\ &+ \{\eta(X)(\phi^2 Y) - \eta(Y)(\phi^2 X)\}g(\phi Z, W)\\ &- \eta(W)\, \phi^2(\phi\overline{R}(X,Y)Z.\end{aligned} \tag{3.6}$$

If X, Y, Z, W are orthogonal to ξ then above equation yields

$$\phi^2((\nabla_W \overline{R})(X,Y)Z) = \phi^2((D_W R)(X,Y)Z). \tag{3.7}$$

This leads to the following theorem.

Theorem 3.1. *In an LP-Sasakian manifold the quarter-symmetric non metric connection ∇ is locally ϕ-symmetric if and only if so is the Riemannian connection.*

4. Locally pseudo projective ϕ -symmetric LP-Sasakian Manifolds admitting a Quarter-Symmetric non Metric Connection

An n-dimensional LP-Sasakian manifold M^n is said to be locally pseudo projective ϕ -symmetric if

$$\phi^2((D_W P^*)(X,Y)Z) = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where P^* is the pseudo projective curvature tensor given by [13]

$$\begin{aligned} P^*(X,Y,Z) &= a\,R(X,Y,Z) + b[(S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}(\frac{a}{n-1} + b)\,\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{4.2}$$

where a, b are constant such that $a, b \neq 0$.

Analogous to this an n-dimensional LP-Sasakian manifolds M^n is said to be locally pseudo projective ϕ-symmetric with respect to the quarter-symmetric non metric connection if

$$\phi^2((\nabla_W \overline{P}^*)(X,Y)Z) = 0, \tag{4.3}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where $\overline{P}^*$ is the pseudo projective curvature with respect to a quarter-symmetric non metric connection given by

$$\begin{aligned} \overline{P^*}(X,Y,Z) &= a\,\overline{R}(X,Y,Z) + b[(\overline{S}(Y,Z)X - \overline{S}(X,Z)Y] \\ &- \frac{\overline{r}}{n}(\frac{a}{n-1} + b)\,\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{4.4}$$

Using (2.17) we can write

$$(\nabla_W \overline{P^*})(X,Y,Z) = (D_W \overline{P^*})(X,Y,Z) - \eta(W)\phi\overline{P^*}(X,Y,Z). \tag{4.5}$$

Now, differentiating (4.4) covariantly with respect to W, we obtain

$$\begin{aligned} (D_W \overline{P^*}(X,Y,Z) &= a\,(D_W \overline{R})(X,Y,Z) + b[(D_W \overline{S})(Y,Z)X - (D_W \overline{S})(X,Z)Y] \\ &- \frac{(D_W \overline{r})}{n}(\frac{a}{n-1} + b)\,\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{4.6}$$

In view of (3.5), (4.5),(2.19),(2.20) and (2.21) the equation (4.6) takes the form

$$\begin{aligned} (\nabla_W \overline{P}^*(X,Y,Z) &= a\,(D_W R)(X,Y,Z) - \eta(W)\phi P^*(X,Y)Z \\ &+ \{g(\phi X,W)\,g(Y,Z) - g(\phi Y,W)\,g(X,Z)\}\xi \\ &+ \{\eta(X)\,g(Y,Z) - \eta(Y)\,g(X,Z)\}\phi W \\ &+ \{g(\phi X,W)Y - g(\phi Y,W)X\}\eta(Z) \\ &+ \{\eta(X)Y - \eta(Y)X\}g(\phi Z,W) \\ &- \frac{(D_W r)}{n}(\frac{a}{n-1} + b)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ b\{(D_W S)(Y,Z)X - (D_W S)(X,Z)Y\} \\ &+ b\,n\,[g(\phi Y,W)\eta(Z)X + g(\phi Z,W)\eta(Y)X \\ &- g(\phi Z,W)\eta(X)Y - g(\phi X,W)\eta(Z)Y] \\ &- \eta(W)\phi\overline{P^*}(X,Y,Z). \end{aligned} \tag{4.7}$$

Taking account of (4.2) above equation can be written as

$$\begin{aligned}(\nabla_W \overline{P}^*)(X,Y,Z) &= a\,(D_W P^*)(X,Y,Z) - \eta(W)\phi P^*(X,Y)Z \\ &+ (1-b\,n)\{g(\phi X,W)Y - g(\phi Y,W)X\}\eta(Z) \\ &+ \{\eta(X)\,g(Y,Z) - \eta(Y)\,g(X,Z)\}\phi W \\ &+ \{g(\phi X,W)g(Y,Z) - g(\phi Y,W)g(X,Z)\}\xi \\ &+ (1-b\,n)\{\eta(X)Y - \eta(Y)X\}\,g(\phi Z,W) \\ &- \eta(W)\phi\overline{P^*}(X,Y,Z). \qquad (4.8)\end{aligned}$$

Operating ϕ^2 both sides and using (2.2) in above, we obtain

$$\begin{aligned}\phi^2((\nabla_W \overline{P}^*)(X,Y)Z) &= \phi^2((D_W P^*)(X,Y)Z) - \eta(W)\phi^2(\phi P^*(X,Y)Z) \\ &+ \{\eta(X)\,g(Y,Z) - \eta(Y)\,g(X,Z)\}\phi^2(\phi W) \\ &+ (1-b\,n)\{g(\phi X,W)\phi^2 Y - g(\phi Y,W)\phi^2 X\}\eta(Z) \\ &+ (1-b\,n)\{\eta(X)\phi^2 Y - \eta(Y)\phi^2 X\}\,g(\phi Z,W) \\ &- \eta(W)\phi^3\overline{P^*}(X,Y,Z). \qquad (4.9)\end{aligned}$$

If we consider X, Y, Z, W orthogonal to ξ, (4.9) reduces to

$$\phi^2((\nabla_W \overline{P}^*)(X,Y)Z) = \phi^2((D_W P^*)(X,Y)Z). \qquad (4.10)$$

Thus, we can state the following:

Theorem 4.1. *An n-dimensional LP-Sasakian manifold is locally pseudo projective ϕ - symmetric with respect to ∇ if and only if it is locally projective ϕ - symmetric with respect to the Riemannian connection D.*

5. ϕ - pseudo Projectively Flat LP-Sasakian Manifolds admitting a Quarter-Symmetric non Metric Connection

Definition 5.1. An n-dimensional differentiable manifold M^n satisfying the equation

$$\phi^2(P^*(\phi X,\phi Y)\phi Z) = 0, \qquad (5.1)$$

is called ϕ-pseudo projectively flat.

Analogous to the above an n-dimensional LP-Sasakian manifold is said to be ϕ pseudo projectively flat with respect to quarter-symmetric non metric connection if it satisfies

$$\phi^2(\overline{P}^*(\phi X,\phi Y)\phi Z) = 0, \qquad (5.2)$$

where $\overline{P}^*$ is the Pseudo projective curvature tensor of the manifold with respect to quarter symmetric non metric connection.

Theorem 5.1. *Let M^n be an n-dimensional, $n > 3$, ϕ pseudo projectively flat LP-Sasakian manifolds admitting a quarter-symmetric non metric connection. Then M^n is an η-Einstein manifold with the scalar curvature $r = \frac{(n-1)}{b}\,(b\,n - a + 1)$.*

Proof. Suppose M^n is ϕ pseudo projectively flat LP-Sasakian manifolds with respect to quarter symmetric non metric connection. It is easy to see that $\phi^2(\overline{P}^*(\phi X, \phi Y)\phi Z) = 0$ holds if and only if

$$g(\phi(\overline{P}^*(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{5.3}$$

for any vector fields X, Y, Z, W . Taking account of the above the equation (4.4) assume the form as

$$\begin{aligned} a\, g(\overline{R}(\phi X, \phi Y)\phi Z, \phi W) &+ b\,[\overline{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \overline{S}(\phi X, \phi Z)g(\phi Y, \phi W)] \\ &= \frac{\overline{r}}{n}(\frac{a}{n-1} + b)\{g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &- g(\phi X, \phi Z)g(\phi Y, \phi W)\}. \end{aligned} \tag{5.4}$$

By making use of (2.19), (2.20) and (2.21) in above, we get

$$\begin{aligned} a\, g(R(\phi X, \phi Y)\phi Z, \phi W) &+ b\,[\{S(\phi Y, \phi Z) - g(\phi Y, \phi Z)\}g(\phi X, \phi W) \\ &- \{S(\phi X, \phi Z) - g(\phi X, \phi Z)\}g(\phi Y, \phi W)] \\ &= \frac{r}{n}(\frac{a}{n-1} + b)\{g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &- g(\phi X, \phi Z)g(\phi Y, \phi W)\}. \end{aligned} \tag{5.5}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in M^n. It is obvious that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also local orthonormal basis. Now, putting $X = W = e_i$ in equation (5.5) and summing over i, we get

$$\begin{aligned} a \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &+ b \sum_{i=1}^{n-1}[\{S(\phi Y, \phi Z) - g(\phi Y, \phi Z)\}g(\phi e_i, \phi e_i) \\ &- \{S(\phi e_i, \phi Z) - g(\phi e_i, \phi Z)\}g(\phi Y, \phi e_i)] \\ &= \frac{r}{n}(\frac{a}{n-1} + b)\sum_{i=1}^{n-1}\{g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}. \end{aligned} \tag{5.6}$$

Also, it can be seen that [11]

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{5.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n + 1, \tag{5.8}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1, \tag{5.9}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) S(\phi Y, \phi e_i) = S(\phi Y, \phi Z) \tag{5.10}$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{5.11}$$

Hence by virtue of equations (5.7) to (5.11) equation (5.6) takes the form

$$(a + b\,n) S(\phi Y, \phi Z) = \{r\,(\frac{a}{n-1} + b) + b\,n - a\} g(\phi Y, \phi Z). \tag{5.12}$$

Now, using equations (2.3) and (2.15) in above equation, we get

$$\begin{aligned} S(Y,Z) &= \frac{1}{(a+b\,n)}\{r\,(\frac{a}{n-1} + b) + b\,n - a - (n-1)\}\ \eta(Y)\eta(Z) \\ &+ \frac{1}{(a+b\,n)}\{r\,(\frac{a}{n-1} + b) + b\,n - a\} g(Y,Z), \end{aligned} \tag{5.13}$$

provided $a + b\,n \neq 0$. Contracting the above equation, we obtain

$$r = \frac{(n-1)}{b}\,(b\,n - a + 1), \tag{5.14}$$

which implies M^n is an η- Einstein manifold with the scalar curvature $r = \frac{(n-1)}{b}\,(b\,n - a + 1)$. This completes the proof of the theorem. □

6. ϕ - W_2 Projectively Flat LP-Sasakian Manifolds admitting a Quarter-Symmetric non Metric Connection

An n-dimensional differentiable manifold M^n satisfying the equation

$$\phi^2(W_2(\phi X, \phi Y)\phi Z) = 0, \tag{6.1}$$

is called ϕ, W_2 projectively flat where W_2 projective curvature tensor is given by [12]

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}\{g(Y,Z)QX - g(X,Z)QY\}, \tag{6.2}$$

where Q is Ricci operator.

Analogous to the above an n-dimensional LP-Sasakian manifolds is said to be ϕ pseudo projectively flat with respect to quarter-symmetric non metric connection if it satisfies

$$\phi^2(\overline{W}_2(\phi X, \phi Y)\phi Z) = 0, \tag{6.3}$$

where $\overline{W}_2$ is the W_2 projective curvature tensor of the manifold with respect to quarter symmetric non metric connection.

Theorem 6.1. *Let M^n be an n-dimensional, $n > 3$, ϕ pseudo projectively flat LP-Sasakian manifolds admitting a quarter-symmetric non metric connection. Then M^n is an η-Einstein manifold with zero scalar curvature .*

Proof. Suppose M^n is ϕ-pseudo projectively flat LP-Sasakian manifolds with respect to quarter symmetric non metric connection. It is easy to see that $\phi^2(\overline{P}^*(\phi X, \phi Y)\phi Z) = 0$ holds if and only if

$$g(\phi(\overline{W}_2(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{6.4}$$

for any vector fields X, Y, Z, W . Taking account of the above equation the (6.2) assume the form as

$$\begin{aligned} g(\overline{R}(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{(n-1)}\{\overline{S}(\phi X, \phi W)g(\phi Y, \phi Z) \\ &- \overline{S}(\phi Y, \phi W)g(\phi X, \phi Z)\}. \end{aligned} \tag{6.5}$$

By making use of (2.19) and (2.20) in (6.5), we have

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{(n-1)}[\{S(\phi X, \phi W) - g(\phi X, \phi W)\}g(\phi Y, \phi Z) \\ &- \{S(\phi Y, \phi W) - g(\phi Y, \phi W)\}g(\phi X, \phi Z)\}]. \end{aligned} \tag{6.6}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in M^n. Using the fact that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also local orthonormal basis. Putting $X = W = e_i$ in equation (6.6) and summing over i, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \frac{1}{(n-1)}\sum_{i=1}^{n-1}[\{S(\phi e_i, \phi e_i) - g(\phi e_i, \phi e_i)\}g(\phi Y, \phi Z) \\ &- \{S(\phi Y, \phi e_i) - g(\phi Y, \phi e_i)\}g(\phi e_i, \phi Z)]. \end{aligned} \tag{6.7}$$

By virtue of equations (5.7) to (5.11) the equation (6.7) yields

$$S(\phi Y, \phi Z) = \{\frac{r}{n} - 1\}g(\phi Y, \phi Z). \tag{6.8}$$

Making use of the equations (2.3) and (2.15) the above equation becomes

$$S(Y, Z) = \{\frac{r}{n} - 1\}g(Y, Z) + \{\frac{r}{n} - n\}\eta(Y)\eta(Z). \tag{6.9}$$

Contracting the above equation, we get $r = 0$. This shows that M^n is an η - Einstein manifold with respect to Riemannian connection of zero curvature tensor. Hence our theorem is proved. □

7. An LP-Sasakian Manifold admitting a Quarter-Symmetric non Metric Connection satisfying $R.\overline{R}=0$

Let us suppose that the LP-Sasakian manifold admitting a Quarter-symmetric non metric connection M^n satisfying $(R(\xi,X).\overline{R})(Y,Z)U=0$. Then it can be written as

$$\begin{aligned} & g(\overline{R}(Y,Z)U,X)\xi-\eta(\overline{R}(Y,Z)U)X-g(X,Z)\overline{R}(\xi,Z)U \\ + \quad & \eta(Y)\overline{R}(X,Z)U-g(X,Z)\overline{R}(Y,\xi)U+\eta(Z)\overline{R}(Y,X)U \\ - \quad & g(X,U)\overline{R}(Y,Z)\xi+\eta(U)\overline{R}(Y,Z)X=0. \end{aligned} \tag{7.1}$$

Making use of the equations (2.19),(2.10),(2.11),(2.13) and (2.16) in (6.1), we obtain

$$\begin{aligned} & R(Y,Z,U,X)\xi+\{\eta(X)\eta(Y)g(Z,U)-\eta(X)\eta(Z)g(Y,U)+\eta(Y)\eta(U)g(X,Z) \\ - \quad & \eta(Z)\eta(U)g(X,Y)\}\xi+2\,g(X,Y)\{\eta(U)Z+\eta(U)\eta(U)\xi\} \\ - \quad & 2\,g(X,Z)\{\eta(U)Y-\eta(U)\eta(Y)\xi\}-2\,g(X,U)\{\eta(Z)Y-\eta(Y)Z\} \\ + \quad & \eta(Y)R(X,Z)U-\eta(Z)R(Y,X)U+\eta(U)R(Y,Z)X \\ + \quad & \eta(X)\eta(Y)g(Z,U)\xi-\eta(Y)\eta(Z)g(X,U)\xi+\eta(X)\eta(Y)\eta(U)Z \\ - \quad & \eta(Y)\eta(Z)\eta(U)X+\eta(Z)\eta(Y)g(X,U)\xi-\eta(Z)\eta(X)g(Y,U)\xi \\ + \quad & \eta(Z)\eta(Y)\eta(U)X-\eta(X)\eta(Z)\eta(U)Y+\eta(U)\eta(Y)g(X,U)\xi \\ - \quad & \eta(Z)\eta(U)g(X,Y)\xi+\eta(X)\eta(Y)\eta(U)Z-\eta(X)\eta(Z)\eta(U)Y=0. \end{aligned} \tag{7.2}$$

Now, taking the inner product of above equation with ξ and using the equations (2.1) and (2.4), we get

$$\begin{aligned} R(Y,Z,U,X) \quad + \quad & \eta(X)\eta(Y)g(Z,U)+5\,\eta(Y)\eta(U)g(X,Z) \\ - \quad & \eta(X)\eta(Z)g(Y,U)-\eta(U)\eta(Z)g(X,Y)=0. \end{aligned} \tag{7.3}$$

Putting $Y=X=e_i$ in the above equation and taking summation over i, we get

$$S(Z,U)=g(Z,U)+(4-n)\,\eta(U)\eta(Z). \tag{7.4}$$

Thus, we can state the following theorem.

Theorem 7.1. *An n-dimensional LP-Sasakian manifold admitting quarter-symmetric non metric connection satisfying $R(\xi,X).\overline{R}=0$ is an η-Einstein manifold.*

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DERIVING FORSYTH-GLAISHER TYPE SERIES FOR $\frac{1}{\pi}$ AND CATALAN'S CONSTANT BY AN ELEMENTARY METHOD

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Abstract. This paper reviews Glaisher's series for $\frac{1}{\pi}$ and contains many series derived by the author through an elementary algebraic approach. The author further supplements the list of series involving Catalan's constant and evaluates $\int_0^{\pi/2} u\cot(u)\ln(\cot(u/2))\,du$, $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2}$ and $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)^4}$.

1. Introduction

A number of series involving the binomial coefficients $\binom{2n}{n}$ that form the central column of Pascal's triangle, are known whose sums are of the form $\frac{p\pi^m}{q}$, $m = \pm1, \pm2, 3, 4$ and p and q are integers. Evidently, $\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{1\cdot3\cdot5\cdots(2n-1)2^{2n}}{2\cdot4\cdot6\cdots(2n)}$. Further, $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$; $\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$. The related Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$, denoted by C_n, are generated by $\frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} C_n x^n$. In terms of Gamma and Beta functions, $\binom{2n}{n} = \frac{\Gamma(2n+1)}{(\Gamma(n+1))^2} = B(n+1,n+1)$, where for $m, n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$ and $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt$.

Ramanujan's amazing formulae for $\frac{1}{\pi}$ contained in his 1914 paper [10] are not unprecedented. The German mathematician G. Bauer (1820-1906) had obtained in 1859 the following alternating series involving $\frac{\binom{2n}{n}^3}{2^{6n}}$ by employing Legendre polynomials [3]

$$1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot3}{2\cdot4}\right)^3 - 13\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3 + 17\left(\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\right)^3 - \cdots = \frac{2}{\pi}. \quad (1)$$

The series (1) is noted on p.92, Chapter XII, of the Manuscript Book 1 and on p.118, Chapter X, of the Manuscript Book 2 of Ramanujan. It was communicated by him to Hardy [4] in a letter of January 16, 1913. By the way, it also appears

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on p.114 of Todhunter's *'An Elementary Treatise on Laplace's Functions, Lame's Functions and Bessel's Functions'* (1875).

The British mathematician A. R. Forsyth (1858-1942) derived the following series (2) in 1883 [6] through Legendre polynomials

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}} = \frac{4}{\pi}, \tag{2}$$

where the Legendre polynomials, denoted by $P_n(x)$, are defined by the generating function $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n,\ |x| \le 1, |t| < 1.$

Glaisher [7] obtained the following similar series (3) from the expansions of Complete Elliptic integrals in 1905

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-3)^2(2n-1)^2 2^{4n}} = \frac{32}{27\pi}. \tag{3}$$

He used expansions given by Legendre

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}} k^{2n},$$

and

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta = -\frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}} \frac{k^{2n}}{2n-1}.$$

along with two combinations of these, namely, $I = E - K$ and $G = E - k^2K$. It may be appropriate to bring in the following hypergeometric representation here

$$P_n(x) = F(-n, n+1; 1; (1-x)/2),$$

where

$$F(a, b; c; z) = 1 + \frac{a \cdot b \cdot z}{c \cdot 1!} + \frac{a(a+1) \cdot b(b+1) \cdot z^2}{c(c+1) \cdot 2!} + \cdots.$$

For $c \neq 0, -1, -2, \cdots$ and $c - a - b > 0$, we have $F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. Further, $K(k) = \frac{\pi}{2}F(1/2, 1/2; 1; k^2)$ and $E(k) = \frac{\pi}{2}F(-1/2, 1/2; 1; k^2)$. Glaisher also derived eighteen series (falling into four types) for $\frac{p}{q\pi}$:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n+2j)}, \quad k = 1, 2, 3, 4;$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)\prod_{j=1}^{k}(2n+2j)}, \quad k = 1, 2, 3, 4;$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)^2\prod_{j=1}^{k}(2n+2j)}, \quad k = 1, 2, 3, 4;$$

and

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(4n+k)(4n+k+d)\prod_{j=1}^{k}(n+j)}, k=1,3;\ d=4,8,12.$$

Besides these, using Legendre polynomials, he derived seven Bauer-type series for $\frac{p}{q\pi}$ involving $\frac{\binom{2n}{n}^3}{2^{6n}}$ and four series for $\frac{4}{\pi^2}$ & $\frac{8}{\pi^2}$ involving $\frac{\binom{2n}{n}^4}{2^{8n}}$. Gert Almkvist [2] and Paul Levrie [8] use this approach for deriving such formulae. I intend to derive six types of series with enhanced convergence

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{1\le j\le k}(2n-2j+1)}, \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n+2j)},$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n-2j+1)(2n+2j)}, \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n-2j+1)^2},$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n+2j)^2}, \text{ and } \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n-2j+1)^2(2n+2j)^2}.$$

I follow an elementary algebraic approach and build the whole edifice on the next result (4) alone using intra-class recurrence relations and inter-class relations.

As Glaisher observed, when n is large, the quantity $\frac{1^2 3^2 5^2 \cdots (2n-1)^2}{2^2 4^2 6^2 \cdots (2n)^2}$ is of the order $\frac{1}{n}$ and hence $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}}$ diverges. So to make this series converge, we need either to remove some factor(s) from the numerator or to supply additional factor(s) to the denominator. We shall execute both the operations - first using linear factors and then square factors.

2. Linear factors

We remove one of the last factors, i.e., (2n-1), from the numerator and obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{1}{2^{4n}(2n-1)} &= -1 + \frac{1}{2^2} + \frac{1^2 3}{2^2 4^2} + \frac{1^2 3^2 5}{2^2 4^2 6^2} + \cdots \\ &= -\frac{2}{\pi}. \end{aligned} \tag{4}$$

Proof. To establish (4), it is sufficient to prove

$$\frac{2}{\pi} = 1 - \frac{1}{2^2} - \frac{1^2 3}{2^2 4^2} - \frac{1^2 3^2 5}{2^2 4^2 6^2} - \frac{1^2 3^2 5^2 7}{2^2 4^2 6^2 8^2} - \cdots.$$

Since $(n+1)(n-1) = n^2 - 1$, we may write the right hand side as

$$
\begin{aligned}
&= 1 - \frac{1}{2^2} - \frac{2^2-1}{2^2 4^2} - \frac{(2^2-1)(4^2-1)}{2^2 4^2 6^2} - \frac{(2^2-1)(4^2-1)(6^2-1)}{2^2 4^2 6^2 8^2} - \cdots \\
&= \left(1-\frac{1}{2^2}\right) - \left(1-\frac{1}{2^2}\right)\frac{1}{4^2} - \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\frac{1}{6^2} \\
&\qquad - \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\frac{1}{8^2} - \cdots \\
&= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2} - \left(1-\frac{1}{4^2}\right)\frac{1}{6^2} - \left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\frac{1}{8^2} - \cdots\right) \\
&= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2} - \left(1-\frac{1}{6^2}\right)\frac{1}{8^2} - \cdots\right) \\
&= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{8^2} - \cdots\right) \\
&= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{8^2}\right)\cdots \\
&= \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi},
\end{aligned}
$$

which is Euler's version of Wallis's famous product formula derivable from the infinite product expansion of the sine function. $\square$

Let us now supply an additional factor $(2n+2)$ to the denominator. We then obtain the following formula

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n+2)2^{4n}} &= \frac{1}{2} + \frac{1^2}{2^2 4} + \frac{1^2 3^2}{2^2 4^2 6} + \frac{1^2 3^2 5^2}{2^2 4^2 6^2 8} + \cdots \\
&= \frac{2}{\pi}
\end{aligned}
\tag{5}
$$

Proof. We shall prove (5) by applying mathematical induction on the partial sums of (4) and (5) and using the result (4). Let S_k denotes the kth partial sum of (4) and s_k that of (5). Then we see that $S_1 = -1$ and $s_1 = \frac{1}{2}$. We can easily establish by induction that for $k > 0$

$$
S_{k+1} = -\frac{1^2 3^2 5^2 \cdots (2k-1)^2(2k+1)}{2^2 4^2 6^2 \cdots (2k)^2}; \;\; s_{k+1} = \frac{1^2 3^2 5^2 \cdots (2k+1)^2}{2^2 4^2 6^2 \cdots (2k)^2 (2k+2)}.
$$

Then $\frac{s_{k+1}}{S_{k+1}} = -\frac{2k+1}{2k+2}$. Hence, $\lim_{k\to\infty} \frac{s_k}{S_k} = -1$, establishing (5). $\square$

We shall now derive two useful recurrence relations.

$$
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k-1)} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 (2n+1)^2}{2^{4n}(2n-2k+1)(2n+2)^2}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)(n+1)} \\
&\quad + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)(2n+2)^2} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} - \frac{1}{2k+1}\left(\sum_{n=0}^{\infty} \frac{2\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)}\right) \\
&\quad + \frac{1}{2k+1}\left(\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)(2n+2)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2}\right) \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} - \frac{1}{2k+1}\left(\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)}\right) \\
&\quad + \left[\frac{1}{(2k+1)^2}\left(\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)}\right)\right. \\
&\quad \left. - \frac{1}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2}\right] \\
&= \left(1 - \frac{2}{2k+1} + \frac{1}{(2k+1)^2}\right)\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} \\
&\quad + \left(\frac{1}{2k+1} - \frac{1}{2(2k+1)^2}\right)\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} - \frac{1}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2} \\
&= \frac{((2k+1)-1)^2}{(2k+1)^2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} + \frac{4k+1}{(2k+1)^2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)} \\
&\quad - \frac{1}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2}.
\end{aligned}
$$

So,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k-1)} &= \frac{(2k)^2}{(2k+1)^2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} \\
&\quad + \frac{4k+1}{(2k+1)^2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)} - \frac{1}{(2k+1)}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2}
\end{aligned}
\tag{6}
$$

If we set $k = 0$ in (6), we get

$$
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2}. \tag{7}
$$

Now we have already evaluated the first two series. Using their values in (7), we thus obtain

$$
\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2} = \frac{4}{\pi} - 1. \tag{8}
$$

Now put the derived values in (6) deducing this recurrence relation

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k-1)} = \frac{(2k)^2}{(2k+1)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)} - \frac{2}{\pi(2k+1)^2}. \tag{9}$$

Similarly, we derive this recurrence relation for $k = 0, 1, 2, \ldots$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+k+1)} &= \frac{(2k)^2}{(2k+1)^2} \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+k)} \\ +\frac{4k+1}{(2k+1)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} &- \frac{k}{(2k+1)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)^2}. \end{aligned} \tag{10}$$

Putting the known values in (10) and dividing both sides by 2, we obtain

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+2)} = \frac{(2k)^2}{(2k+1)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k)} + \frac{2}{\pi(2k+1)^2}. \tag{11}$$

Since the recurrence relations (9) and (11) have the same form, let us denote them respectively by $a_{k+1} = \frac{(2k)^2}{(2k+1)^2} a_k - \frac{2}{\pi(2k+1)^2}$ and $b_{k+1} = \frac{(2k)^2}{(2k+1)^2} b_k + \frac{2}{\pi(2k+1)^2}$ with initial values being $\frac{-2}{\pi}$ and $\frac{2}{\pi}$ — the R.H.S. of (4) and of (5), hence we have:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k-1)} = -\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+2)}. \tag{12}$$

So we now have an infinite number of such formulae

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-3)} = -\frac{10}{9\pi} = -\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+4)}. \tag{13}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+4)(2n-3)} = -\frac{20}{7 \cdot 9\pi}. \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-5)} = -\frac{178}{225\pi} = -\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+6)}. \tag{15}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+6)(2n-5)} = -\frac{356}{11 \cdot 225\pi}. \tag{16}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-7)} = -\frac{762}{1225\pi} = -\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+8)}. \tag{17}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+8)(2n-7)} = -\frac{1524}{15 \cdot 1225\pi}. \tag{18}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-9)} = -\frac{51218}{99225\pi} = -\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+10)}. \tag{19}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+10)(2n-9)} = -\frac{102436}{19 \cdot 99225\pi}. \tag{20}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-11)} = -\frac{5320250}{12006225\pi} = -\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+12)}. \tag{21}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+12)(2n-11)} = -\frac{10640500}{23 \cdot 12006225\pi}. \tag{22}$$

In general, we have the following simple and direct formula

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k)(2n-2k+1)} = \frac{2}{4k-1} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-2k+1)}. \tag{23}$$

We deduce this relation for $k = 0, 1, 2, 3, \cdots$ from $2 \cdot (9) - (11)$

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+k+1)(2n-2k-1)} \\ &\quad = \frac{(4k+1)(2k)^2}{(4k+3)(2k+1)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+k)(2n-2k+1)2^{4n}} \\ &\qquad - \frac{8}{\pi(4k+3)(2k+1)^2}. \end{aligned} \tag{24}$$

Let $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n-2j+1)} = \frac{L_k}{\pi}$ and $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(n+j)} = \frac{M_k}{\pi}$. Then, by combining the individual results, we obtain

$$L_2 = \frac{2^2}{3^2} = \frac{M_2}{2^2}. \tag{25}$$

$$L_3 = -\frac{2^4}{3^2 5^2} = -\frac{M_3}{2^3}. \tag{26}$$

$$L_4 = \frac{2^5}{3^2 5^2 7^2} = \frac{M_4}{2^4}. \tag{27}$$

$$L_5 = -\frac{2^8}{3^2 5^2 7^2 9^2} = -\frac{M_5}{2^5}. \tag{28}$$

$$L_6 = \frac{2^9}{3^2 5^2 7^2 9^2 11^2} = \frac{M_6}{2^6}. \tag{29}$$

Note that $M_k = (-1)^k 2^k L_k$. Prof. Levrie recognized these series (when communicated to him) to be special cases of the Gauss-evaluation of ${}_2F_1$ and pointed out that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n-2j+1)} = C_k \, {}_2F_1\left(\frac{1}{2}, -\frac{2k-1}{2}; 1; 1\right), \quad k = 1, 2, 3, \cdots.$$

We now combine the two classes of formulae to get mixed formulae

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)(2n+2)2^{4n}} = -\frac{2^2}{3\pi}. \tag{30}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)(2n-3)(2n+2)(2n+4)2^{4n}} = \frac{2^5}{315\pi}. \tag{31}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)(2n-3)(2n-5)(2n+2)(2n+4)(2n+6)2^{4n}} = -\frac{2^9}{155925\pi}. \tag{32}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)(2n-3)(2n-5)(2n-7)(2n+2)(2n+4)(2n+6)(2n+8)2^{4n}} = \frac{2^{12}}{70945875\pi}. \tag{33}$$

We can go on indefinitely like this.

3. Quadratic factors

In this section, we shall take away the last square factor $(2n-1)^2$ from the numerator or append square factor $(2n+2)^2$ to the denominator.

Compare the terms of the two series:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n+2)^2 2^{4n}} = \frac{1}{2^2} + \frac{1^2}{2^2 4^2} + \frac{1^2 3^2}{2^2 4^2 6^2} + \frac{1^2 3^2 5^2}{2^2 4^2 6^2 8^2} + \cdots,$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}} = 1 + \frac{1}{2^2} + \frac{1^2}{2^2 4^2} + \frac{1^2 3^2}{2^2 4^2 6^2} + \frac{1^2 3^2 5^2}{2^2 4^2 6^2 8^2} + \cdots.$$

The comparison makes it obvious that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}} = 1 + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n+2)^2 2^{4n}}. \tag{34}$$

In fact, the two classes of series are related to each other in a strikingly lovely way for $k = 1, 2, 3, \cdots$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-2k+1)^2 2^{4n}} = \frac{2^{4k}}{(2k)^2 \binom{2k}{k}^2} + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n+2k)^2 2^{4n}}. \tag{35}$$

To establish (35), we use the notation $\alpha_k = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-2k+1)^2 2^{4n}}$ and $\beta_k = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n+2k+1)^2 2^{4n}}$. The method used for a_k and b_k eventually leads to

$$\alpha_{k+1} = \frac{4k^2}{(1+2k)^2}\alpha_k + \frac{4k}{(1+2k)^3}a_k + \frac{4}{(1+2k)^3\pi}$$

and

$$\beta_{k+1} = \frac{4k^2}{(1+2k)^2}\beta_k - \frac{4k}{(1+2k)^3}b_k + \frac{4}{(1+2k)^3\pi}.$$

Since $a_k = -b_k$, we see that the two recurrence relations are equivalent with initial values deduced in (8) and (34). Hence, $\alpha_{k+1} - \beta_{k+1} = \frac{(2k)^2}{(2k+1)^2}(\alpha_k - \beta_k)$.

Let $\alpha_k - \beta_k = \delta_k$. Then $\delta_1 = 1$ and

$$\begin{aligned}
\delta_{k+1} &= \frac{(2k)^2}{(2k+1)^2}\delta_k \\
&= \frac{(2k)^2(2k-2)^2}{(2k+1)^2(2k-1)^2}\delta_{k-1} \\
&= \cdots \\
&= \frac{(2k)^2(2k-2)^2\cdots 4^2 2^2}{(2k+1)^2(2k-1)^2\cdots 5^2 3^2}1 \\
&= \frac{2^{4k}}{(2k)^2\binom{2k}{k}^2}.
\end{aligned}$$

We thus obtain

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n-1)^2} = \frac{4\cdot 1}{\pi} = 1 + \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2}. \tag{36}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n-3)^2} = \frac{4\cdot 11}{27\pi} = \frac{2^2}{3^2} + \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n+4)^2}. \tag{37}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n-5)^2} = \frac{4\cdot 847}{3375\pi} = \frac{2^2 4^2}{3^2 5^2} + \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n+6)^2}. \tag{38}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n-7)^2} = \frac{4\cdot 23201}{128625\pi} = \frac{2^2 4^2 6^2}{3^2 5^2 7^2} + \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n+8)^2}. \tag{39}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n-9)^2} = \frac{4\cdot 4390787}{31255875\pi} = \frac{2^2 4^2 6^2 8^2}{3^2 5^2 7^2 9^2} + \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n+10)^2}. \tag{40}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n-11)^2} = \frac{4\cdot 191218129}{1664062785\pi} = \frac{2^2 4^2 6^2 8^2 10^2}{3^2 5^2 7^2 9^2 11^2} + \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}(2n+12)^2}. \tag{41}$$

We now evaluate their combinations. Let $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{2^{4n}\prod_{j=1}^{k}(2n-2j+1)^2} = \frac{A_k}{\pi}$. Then we have these beautiful formulae

$$A_1 = \frac{2^2}{1^3}. \tag{42}$$

$$A_2 = \frac{2^5}{3^3}. \tag{43}$$

$$A_3 = \frac{2^9}{3^3 5^3}. \tag{44}$$

$$A_4 = \frac{2^{12}}{3^2 5^3 7^3}. \tag{45}$$

$$A_5 = \frac{2^{17}}{3^2 5^3 7^3 9^3}. \tag{46}$$

$$A_6 = \frac{2^{20}}{3^2 5^3 7^3 9^3 11^3}. \tag{47}$$

$$A_7 = \frac{2^{24}}{3^2 5^3 7^3 9^3 11^3 13^3}. \tag{48}$$

$$A_8 = \frac{2^{27}}{3 \cdot 5^3 7^3 9^3 11^3 13^3 15^3}. \tag{49}$$

$$A_9 = \frac{2^{33}}{3 \cdot 5^2 7^2 9^3 11^3 13^3 15^3 17^3}. \tag{50}$$

$$A_{10} = \frac{2^{36}}{3 \cdot 5^2 7^2 9^2 11^3 13^3 15^3 17^3 19^3}. \tag{51}$$

$$A_{11} = \frac{2^{40}}{3 \cdot 5 \cdot 7^2 9^2 11^3 13^3 15^3 17^3 19^3 21^3}. \tag{52}$$

$$A_{12} = \frac{2^{43}}{3 \cdot 5 \cdot 7^2 9^2 11^3 13^3 15^3 17^3 19^3 21^3 23^3}. \tag{53}$$

Prof. Levrie commented on these series (when communicated to him) that they too are special cases of the Gauss-evaluation of ${}_2F_1$ and that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n} \prod_{1 \le j \le k} (2n-2j+1)^2} = c_k \; {}_2F_1\left(-\frac{(2k-1)}{2}, -\frac{(2k-1)}{2}; 1; 1\right),$$

$k = 1, 2, 3, \cdots$. We can deduce such formulae for the associated series also. Let me illustrate this by an example

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)^2 2^{4n}} &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 (2n+1)^2}{(n+1)^2 (n+2)^2 2^{4n}} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+2)^2 2^{4n}} - \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)(n+2)^2 2^{4n}} \\
&\quad + \frac{1}{4} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)^2 (n+2)^2 2^{4n}} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+2)^2 2^{4n}} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)(n+2) 2^{4n}} \\
&\quad + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+2)^2 2^{4n}} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)^2 (n+2)^2 2^{4n}} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+2)^2 2^{4n}} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1) 2^{4n}} + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1) 2^{4n}} \\
&\quad + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+2)^2 2^{4n}} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)^2 (n+2)^2 2^{4n}}.
\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2(n+2)^2 2^{4n}} \quad = \quad & 4\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2 2^{4n}} - 8\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+2)^2 2^{4n}} \\ & +4\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2 2^{4n}} - 4\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+2) 2^{4n}}.\end{aligned}$$

Putting the values of the results derived earlier, we get the desired result

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2(n+2)^2 2^{4n}} = \frac{512}{27\pi} - \frac{52}{9}. \tag{54}$$

Proceeding in this way, we can derive further results also. The first five results, including (8) and (54), are listed below

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2 2^{4n}} = \frac{2^4}{\pi} - 1 = \ {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; 2, 2; 1\right). \tag{55}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2(n+2)^2 2^{4n}} = \frac{2^9}{3^3\pi} - \frac{52}{3^2} = \ {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; 3, 3; 1\right). \tag{56}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2(n+2)^2(n+3)^2 2^{4n}} = \frac{2^{15}}{3^3 5^3\pi} - \frac{689}{3^2 5^2}. \tag{57}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2(n+2)^2(n+3)^2(n+4)^2 2^{4n}} = \frac{2^{20}}{3^2 5^3 7^3\pi} - \frac{9517}{3^2 5^2 7^2}. \tag{58}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(n+1)^2(n+2)^2(n+3)^2(n+4)^2(n+5)^2 2^{4n}} = \frac{2^{27}}{3^2 5^3 7^3 9^3\pi} - \frac{2169049}{3^2 4^2 5^2 7^2 9^2}. \tag{59}$$

Note that the multiple of $\frac{1}{\pi}$ in this category $= 2^{2k}A_k$. But the quantity subtracted does not seem to follow any perceptible pattern.

The combination of the two classes result in the following formulae

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(2n-1)^2(2n+2)^2 2^{4n}} = \frac{2^5}{3^3\pi} - \frac{1}{3^2}. \tag{60}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(2n-3)^2(2n-1)^2(2n+2)^2 2^{4n}} = \frac{2^9}{3^3 5^3\pi} - \frac{1}{3^2 5^2}. \tag{61}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(2n-1)^2(2n+2)^2(2n+4)^2 2^{4n}} = \frac{2^9}{3^3 5^3\pi} - \frac{29}{2^2 3^2 5^2}. \tag{62}$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2}{(2n-3)^2(2n-1)^2(2n+2)^2(2n+4)^2 2^{4n}} = \frac{2^{12}}{3^2 5^3 7^3\pi} - \frac{53}{2^2 3^2 5^2 7^2}. \tag{63}$$

We can go on indefinitely like this.

4. Catalan's constant

We have investigated up to now series with denominators having odd factors of the type $2n+2k+1$ and $(2n+2k+1)^2$, where k was a negative integer. We shall now take up the case of k being non-negative integer. Let us begin with derivation of a recurrence relation.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+1)} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(2n+1)^2\binom{2n}{n}^2}{2^{4n}(n+1)^2(2n+2k+3)} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)(n+1)} \\
&\qquad\qquad + \frac{1}{4}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)(n+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} - \frac{1}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} \\
&\qquad + \frac{1}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} + \frac{1}{4(2k+1)}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)^2} \\
&\qquad\qquad - \frac{1}{2(2k+1)}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)(n+1)} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} - \frac{1}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} \\
&\qquad + \frac{2}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} + \frac{1}{4(2k+1)}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)^2} \\
&\qquad\qquad - \frac{1}{2(2k+1)}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)(n+1)} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} - \frac{1}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} \\
&\qquad + \frac{2}{2k+1}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} + \frac{1}{4(2k+1)}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)^2} \\
&\qquad - \frac{1}{2(2k+1)^2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} + \frac{1}{(2k+1)^2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} \\
&= \left(1 + \frac{2}{2k+1} + \frac{1}{(2k+1)^2}\right)\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} \\
&\qquad - \left(\frac{1}{2k+1} + \frac{1}{2(2k+1)^2}\right)\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} + \frac{1}{4(2k+1)}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)^2}.
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+1)} - \frac{1}{2k+1} = \frac{(2k+2)^2}{(2k+1)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} - \frac{4k+3}{2(2k+1)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)} + \frac{1}{4(2k+1)} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(n+1)^2}.$$

Simplifying and putting the derived values of the series in the equation, we finally obtain the recurrence relation:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+3)} = \frac{(2k+1)^2}{(2k+2)^2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+1)} + \frac{1}{2(k+1)^2\pi} \tag{64}$$

for $k = 0, 1, 2, 3, \ldots$. Ramanujan gave this formula [4, 11]

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; 1\right) = \frac{4G}{\pi}. \tag{65}$$

Here $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190\ldots$ is known as Catalan's constant.

Using (64) and (65), we get

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+3)} = \frac{2G+1}{2\pi}. \tag{66}$$

Interestingly, $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+3)} = -\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)(2n+1)}$.

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+5)} = \frac{18G+13}{32\pi}. \tag{67}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+7)} = \frac{450G+389}{1152\pi}. \tag{68}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+9)} = \frac{22050G+21365}{73728\pi}. \tag{69}$$

Combining the above results, we obtain

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)(2n+3)} = \frac{6G-1}{4\pi}. \tag{70}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+3)(2n+5)} = \frac{14G+3}{64\pi}. \tag{71}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)(2n+3)(2n+5)} = \frac{82G-19}{256\pi}. \tag{72}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+5)(2n+7)} = \frac{198G+79}{2304\pi}. \tag{73}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+3)(2n+5)(2n+7)} = \frac{306G+29}{9216\pi}. \tag{74}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)(2n+3)(2n+5)(2n+7)} = \frac{2646G-713}{55296\pi}. \tag{75}$$

These formulae suggest two general formulas:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k+1)} = \frac{aG+b}{\pi c}, \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}\prod_{1\le j\le k}(2n+2j+1)} = \frac{AG-B}{\pi C}$$

where $a, c, A, C \in \mathbb{N}$, $b, B \in \{0\} \cup \mathbb{N}$, $k = 0, 1, 2, \ldots$

Now we proceed to evaluate a related sum.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)^3} &= \sum_{n=0}^{\infty} \frac{(2n+2)^2(2n+1)^2\binom{2n}{n}^2}{2^{4n+4}(2n+1)^3(n+1)^4} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)(2n+2)^2} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)(2n+2)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2}.
\end{aligned}$$

Putting the derived values, we obtain

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)^3} = \frac{4G-6}{\pi}. \tag{76}$$

This series has better rate of convergence as compared to Ramanujan's series. If we eliminate G using (65) and (66), we get

$$\sum_{n=0}^{\infty} \frac{(6n+1)\binom{2n}{n}^2}{2^{4n}(2n+1)(2n+3)} = \frac{2}{\pi}. \tag{77}$$

In fact, it is only a special case of the general theorem:

$$\sum_{n=0}^{\infty} \frac{(2n(4k-1)+(2k-1))\binom{2n}{n}^2}{2^{4n}(2n+2k-1)(2n+2k+1)} = \frac{2}{\pi}, k = 1, 2, 3, \cdots \tag{78}$$

Proof. From (64), with $G_k = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k-1)}$ and $G_1 = \frac{4G}{\pi}$ for brevity, we have the recurrence relation

$$G_{k+2} = \frac{(2k+1)^2}{(2k+2)^2} G_{k+1} + \frac{1}{(k+1)^2\pi}; \text{that is, } (2k)^2 G_{k+1} - (2k-1)^2 G_k = \frac{2}{\pi}.$$

This is equivalent to

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{4n}} \left[\frac{(2k)^2}{2n+2k+1} - \frac{(2k-1)^2}{2n+2k-1} \right] = \frac{2}{\pi}.$$

That is, $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2k-1)(2n+2k+1)} = \frac{2}{\pi}$. □

Combining the first two pairs, we deduce

$$\sum_{n=0}^{\infty} \frac{(41n+6)\binom{2n}{n}^2}{2^{4n}(2n+1)(2n+3)(2n+5)} = \frac{2}{\pi}. \tag{79}$$

5. New series and integral

Squaring the factor (2n+1) in Ramanujan's formula, we obtain

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2} = {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}, \frac{3}{2}; 1\right). \tag{80}$$

Setting $k = x$ in Legendre's expansion $K(k)$ recorded earlier, one obtains

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}} x^{2n}. \tag{81}$$

Integrating both sides of this with respect to x, we get

$$\int_0^t dx \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}} \frac{t^{2n+1}}{2n+1}. \tag{82}$$

Change of the order of integration then yields

$$L.H.S. = \int_0^{\pi/2} d\theta \int_0^t \frac{dx}{\sqrt{1-k^2}\sin^2\theta} = \int_0^{\pi/2} \frac{\sin^{-1}(t\sin\theta)}{\sin\theta} d\theta.$$

Thus we have

$$\int_0^{\pi/2} \frac{\sin^{-1}(t\sin\theta)}{\sin\theta} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}} \frac{t^{2n+1}}{2n+1}. \tag{83}$$

Upon dividing both sides by t and then integrating, this gives

$$\int_0^1 dt \int_0^{\pi/2} \frac{\sin^{-1}(t\sin\theta)}{t\sin\theta} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2}. \tag{84}$$

Set $t\sin\theta = \sin u$, that is, $\theta = \sin^{-1}\frac{\sin u}{t}$, in the left hand side of (84). Then, $d\theta = \frac{1}{\sqrt{1-\frac{\sin^2 u}{t^2}}} \frac{\cos u}{t} du$. Also, if $\theta = 0, u = 0$ and if $\theta = \frac{\pi}{2}, u = \sin^{-1} t$. Hence, the left hand side transforms into

$$\int_0^1 dt \int_0^{\sin^{-1} t} \frac{u}{\sin u} \frac{1}{\sqrt{1-\frac{\sin^2 u}{t^2}}} \frac{\cos u}{t} du = \int_0^1 dt \int_0^{\sin^{-1} t} \frac{u\cot u}{\sqrt{t^2-\sin^2 u}} du.$$

Change of the order of integration then gives

$$\begin{aligned} L.H.S. &= \int_0^{\pi/2} du \int_{\sin u}^{1} \frac{u \cot u}{\sqrt{t^2 - \sin^2 u}}\, dt \\ &= \int_0^{\pi/2} u \cot u \left[\ln\left(1 + \cos u\right) - \ln \sin u\right] du \\ &= \int_0^{\pi/2} u \cot u \ln\left(\cot \frac{u}{2}\right) du. \end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2} = \frac{2}{\pi} \int_0^{\pi/2} u \cot u \ln \cot\left(\frac{u}{2}\right) du. \tag{85}$$

On author's request, Hwang evaluated the right hand side of (85) using the software MATHEMATICA to get its

$$L.H.S. = \frac{(\pi + 2i \ln 2)(9\pi^2 - 8i\pi \ln 2 - 4(\ln 2)^2) + 6i\left(64 Li_3\left(\frac{1+i}{2}\right) - 35\zeta(3)\right)}{24\pi}, \tag{86}$$

where $Li_3(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^3}$ is the tri-logarithm function and $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$. As suggested by Levrie, this can be simplified to a neat form

$$L.H.S.of(85) = \frac{3\pi^2}{8} + \frac{(\ln 2)^2}{2} + \frac{8i}{\pi}\left[Li_3\left(\frac{1+i}{2}\right) - Li_3\left(\frac{1-i}{2}\right)\right]. \tag{87}$$

Simplifying Hwangs result and comparing it with the form suggested by Levrie we can deduce

$$\begin{aligned} &5\pi^2 \ln 2 - 4(\ln 2)^3 + 3\left(64 Li_3\left(\frac{1+i}{2}\right) - 35\zeta(3)\right) \\ &= 96\left(Li_3\left(\frac{1+i}{2}\right) - Li_3\left(\frac{1-i}{2}\right)\right). \end{aligned}$$

That is,

$$\frac{1}{24}(\ln 2)^3 - \frac{5}{96}\pi^2 \ln 2 + \frac{35}{32}\zeta(3) = Li_3\left(\frac{1+i}{2}\right) + Li_3\left(\frac{1-i}{2}\right). \tag{88}$$

Since $\frac{1+i}{2} = \frac{1}{\sqrt{2}} e^{i\pi/4}$, therefore $Li_3(z) = \sum_{n=1}^{\infty} \frac{\left(\cos\frac{\pi}{4} + i \sin pi4\right)^n}{(\sqrt{2})^n n^3}$ and Adegoke [1] derives his formula (9)

$$\sum_{n=1}^{\infty} \frac{(\cos\frac{\pi}{4})^n}{(\sqrt{2})^n n^3} = \frac{1}{48}(ln2)^3 - \frac{5}{192}\pi^2 \ln 2 + \frac{35}{64}\zeta(3). \tag{89}$$

This goes to establish the validity of our deduction from the two forms. This could be done directly by putting $\theta = \frac{\pi}{4}$ in formula (46) given in Appendix A.2 in Lewin [9] on which Adegoke bases his derivation.

The first 100 digits of the sum computed by Hwang are
1.0379477643055853174823229566147139504766704386502568421879451331484012102164216065727872745773035 40.

It is easy to see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)^4} &= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 (2n+2)^2(2n+1)^2}{2^{4n}(2n+1)^4(n+1)^4} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2(2n+2)^2} \\
&= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2} + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)^2} - 2\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)} \\
&\quad +2\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+2)}.
\end{aligned}$$

Hence, putting the values we determined earlier, we obtain the sum

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n-1)^4} = \frac{(\ln 2)^2}{2} + \frac{3\pi^2}{8} + \frac{8}{\pi}\left[1 - G + i\left(Li_3\left(\frac{1+i}{2}\right) - Li_3\left(\frac{1-i}{2}\right)\right)\right]. \tag{90}$$

I conclude with a series for $\frac{1}{\pi^2}$ that Glaisher missed

$$\frac{16}{\pi^2} = \sum_{n=0}^{\infty} \frac{(4(3n+1)(2n-1)+3)\binom{2n}{n}^4}{2^{8n}(n+1)(2n-1)^3}. \tag{91}$$

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ON ESCAPING SETS OF SOME FAMILIES OF ENTIRE FUNCTIONS AND DYNAMICS OF COMPOSITE ENTIRE FUNCTIONS

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ABSTRACT. We consider two families of functions $\mathcal{F} = \{f_{\lambda,\xi}(z) = e^{-z+\lambda} + \xi : \lambda, \xi \in \mathbb{C}, \operatorname{Re}\lambda < 0, \operatorname{Re}\xi \geq 1\}$ and $\mathcal{F}' = \{f_{\mu,\zeta}(z) = e^{z+\mu} + \zeta : \mu, \zeta \in \mathbb{C}, \operatorname{Re}\mu < 0, \operatorname{Re}\zeta \leq -1\}$ and investigate the escaping sets of members of the family $\mathcal{F}$ and $\mathcal{F}'$. We also consider the dynamics of composite entire functions and provide conditions for equality of escaping sets of two transcendental entire functions.

1. INTRODUCTION

Let f be a transcendental entire function. For $n \in \mathbb{N}$, let f^n denote the n-th iterate of f. The set $F(f) = \{z \in \mathbb{C} : \{f^n\}_{n\in\mathbb{N}}$ is normal in some neighborhood of $z\}$ is called the Fatou set of f or the set of normality of f and its complement $J(f)$ is called the Julia set of f. For an introduction to the properties of these sets see [3]. The escaping set of f denoted by $I(f)$ is the set of points in the complex plane that tend to infinity under iteration of f. In general, it is neither an open nor a closed subset of $\mathbb{C}$ and has interesting topological properties. The escaping set for a transcendental entire function f was studied for the first time by Eremenko [7] who established that

(1). $I(f) \neq \emptyset$; (2). $J(f) = \partial I(f)$;
(3). $I(f) \cap J(f) \neq \emptyset$; (4). $\overline{I(f)}$

has no bounded components.

In the same paper he stated the following conjectures:
(i). Every component of $I(f)$ is unbounded;
(ii). Every point of $I(f)$ can be connected to ∞ by a curve consisting of escaping points.

For the exponential maps of the form $f(z) = e^z + \lambda$ with $\lambda > -1$, it is known, by Rempe [17], that the escaping set is a connected subset of the plane, and for $\lambda < -1$, it

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is the disjoint union of uncountably many curves to infinity, each of which is connected component of $I(f)$ [18]. (These maps have no critical points and exactly one asymptotic value which is the omitted value λ).

In [20], it was shown that every escaping point of every exponential map can be connected to ∞ by a curve consisting of escaping points. Furthermore, it was also shown in [20] that if f is an exponential map, that is, $f = e^{\lambda z}$, $\lambda \in \mathbb{C} \setminus \{0\}$, then all components of $I(f)$ are unbounded, that is, Eremenko's conjecture [7] holds for exponential maps.

A complex number $w \in \mathbb{C}$ is a critical value of a transcendental entire function f if there exist some $w_0 \in \mathbb{C}$ with $f(w_0) = w$ and $f'(w_0) = 0$. Here w_0 is called a critical point of f. The image of a critical point of f is critical value of f. Also $\zeta \in \mathbb{C}$ is an asymptotic value of a transcendental entire function f if there exist a curve Γ tending to infinity such that $f(z) \to \zeta$ as $z \to \infty$ along Γ. Recall the Eremenko-Lyubich class

$$\mathcal{B} = \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental entire : Sing}(f^{-1}) \text{ is bounded}\},$$

where Singf^{-1} is the set of critical values and asymptotic values of f and their finite limit points. Each $f \in \mathcal{B}$ is said to be of bounded type. A transcendental entire function f is of finite type if Singf^{-1} is a finite set. Furthermore, if the transcendental entire functions f and g are of bounded type then so is $f \circ g$ as Sing $((f \circ g)^{-1}) \subset$ Sing $f^{-1} \cup f(\text{Sing}(g^{-1}))$, [5]. Singularities of a transcendental map plays an important role in its dynamics. They are closely related to periodic components of the Fatou set [15]. For any transcendental entire function Sing$f^{-1} \neq \emptyset$, [10, p. 66]. It is well known [8, 9], if f is of finite type then it has no wandering domains. Recently Bishop [6] has constructed an example of a function of bounded type having a wandering domain. In [16], it was shown that if f is an entire function of bounded type for which all singular orbits are bounded (that is, f is postsingularly bounded), then each connected component of $I(f)$ is unbounded, providing a partial answer to a conjecture of Eremenko [7].

Two functions f and g are called permutable if $f \circ g = g \circ f$. Fatou [2] proved that if f and g are two permutable rational functions then $F(f) = F(g)$. This was an important result that motivated the dynamics of composition of complex functions. Similar results for transcendental entire functions is still not known, though it holds in some very special cases [1, Lemma 4.5]. If f and g are transcendental entire functions, then so is $f \circ g$ and $g \circ f$ and the dynamics of one composite entire function helps in the study of the dynamics of the other and vice-versa. In [12], the authors considered the relationship between Fatou sets and singular values of transcendental entire functions f, g and $f \circ g$. They gave various conditions under which Fatou sets of f and $f \circ g$ coincide and also considered relation between the singular values of f, g and their compositions. In [13], the authors have constructed several examples where the dynamical behavior of f and g vary greatly from the dynamical behavior of $f \circ g$ and $g \circ f$. Using approximation theory of entire functions, the authors have shown the existence of entire functions f and g having infinite number of domains satisfying various properties and relating it to their compositions. They explored and enlarged all the maximum possible ways of the solution in comparison to the past result worked out. Recall that if g and h are transcendental entire functions and f is a continuous map of the complex plane into itself with $f \circ g = h \circ f$, then g and h are said to be semiconjugated by f and f is called a semiconjugacy [4]. In [14], the author considered the dynamics of semiconjugated entire functions and provided several conditions under which the semiconjugacy carries Fatou set of one entire function into Fatou set of other entire function appearing in the semiconjugation. Furthermore, it was shown that under certain conditions on the growth of entire functions appearing in the semiconjugation, the set of asymptotic values of the derivative of composition of the entire functions is bounded.

In this paper, we shall consider the two families of functions $\mathcal{F} = \{f_{\lambda,\xi}(z) = e^{-z+\lambda} + \xi : \lambda, \xi \in \mathbb{C}, \operatorname{Re}\lambda < 0, \operatorname{Re}\xi \geq 1\}$ and $\mathcal{F}' = \{f_{\mu,\zeta}(z) = e^{z+\mu} + \zeta : \mu, \zeta \in \mathbb{C}, \operatorname{Re}\mu < 0, \operatorname{Re}\zeta \leq -1\}$. We have given an explicit description of escaping sets of members of the families $\mathcal{F}$ and $\mathcal{F}'$. For the family $\mathcal{F}$, we have repeatedly used the fact that those points of the complex plane which land into the right half plane under iteration of functions in this family are not going to escape. Also, for the family $\mathcal{F}'$, we have repeatedly used the fact that those points of the complex plane which land into the left half plane under iteration of functions in this family are not going to escape. We have shown that for each $f \in \mathcal{F}$, $I(f) \subset \{z = x + iy : x < 0, (4k-3)\frac{\pi}{2} < y < (4k-1)\frac{\pi}{2}, k \in \mathbb{Z}\}$ and for each $f \in \mathcal{F}'$, $I(f) \subset \{z = x + iy : x > 0, (4k-1)\frac{\pi}{2} < y < (4k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$. We shall see that for each $f \in \mathcal{F}$ and for each $g \in \mathcal{F}'$, $I(f) \cap I(g) = \emptyset$. Moreover, we shall consider the dynamics of composite entire functions and provide conditions for equality of escaping sets of two transcendental entire functions. We have also investigated the relation between escaping sets of two conjugate entire functions.

2. Theorems and their proofs

Theorem 2.1. *For each* $f \in \mathcal{F}, I(f)$ *is contained in* $\{z = x + iy : x < 0, (4k-3)\frac{\pi}{2} < y < (4k-1)\frac{\pi}{2}, k \in \mathbb{Z}\}$.

Theorem 2.2. *For each* $f \in \mathcal{F}'$, $I(f) \subset \{z = x + iy : x > 0, (4k-1)\frac{\pi}{2} < y < (4k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$.

The proof of Theorem 2.1 is elementary. It is divided in several lemmas. The main objective is to show that none of the points in the right half plane belongs to $I(f), f \in \mathcal{F}$. Using this notion, we try to find out the points in the left half plane, which do not belong to $I(f)$, $f \in \mathcal{F}$. We observe that the right half plane is invariant under each $f \in \mathcal{F}$. In similar spirit, the proof of Theorem 2.2 is also elementary. It is also divided in several lemmas. Here too, the main objective is to show that none of the points in the left half plane belongs to $I(g), g \in \mathcal{F}'$. Using this notion, we try to find out the points in the right half plane, which do not belong to $I(g)$, $g \in \mathcal{F}'$. We observe that the left half plane is invariant under each $g \in \mathcal{F}'$.

Observe that Theorem 2.1 follows from the following three lemmas we prove.

Lemma 2.3. *The set* $I_1 = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Re}(e^{-z+\lambda}) > 0,\ \lambda \in \mathbb{C},$ *with* $\operatorname{Re}\lambda < 0\}$ *does not intersect* $I(f)$ *for each* $f \in \mathcal{F}$.

Proof. Observe that $\operatorname{Re}(e^{-z+\lambda}) > 0$, implies $\dfrac{(4k-1)\pi}{2} + \operatorname{Im}\lambda < y < \dfrac{(4k+1)\pi}{2} + \operatorname{Im}\lambda$, where $k \in \mathbb{Z}$. Therefore, the set I_1 is the entire right half plane $\{z : \operatorname{Re} z > 0\}$. We show that no point in I_1 escapes to ∞ under iteration of each $f \in \mathcal{F}$. For this we show $|f^k(z)| \leq 1 + |\xi|$ for all $k \in \mathbb{N}$, $z \in I_1$. Suppose on the contrary there exist $n \in \mathbb{N}$ and $z \in I_1$ such that $|f^n(z)| > 1 + |\xi|$. Then $|f(f^{n-1}(z))| > 1 + |\xi|$, implies $1 + |\xi| < |e^{-f^{n-1}(z)+\lambda} + \xi|$. This shows that $e^{-\operatorname{Re} f^{n-1}(z) + \operatorname{Re}\lambda} > 1$, and as $\operatorname{Re}\lambda < 0$, we obtain $-\operatorname{Re} f^{n-1}(z) > 0$, that is, $\operatorname{Re} f^{n-1}(z) < 0$. Further this implies that $\operatorname{Re}(f(f^{n-2}(z))) < 0$, that is, $\operatorname{Re}(e^{-f^{n-2}(z)+\lambda} + \xi) < 0$, which implies $-\operatorname{Re}(e^{-f^{n-2}(z)+\lambda}) > \operatorname{Re}\xi \geq 1$. Since $|z| \geq -\operatorname{Re} z$ for all $z \in \mathbb{C}$, we get $|(e^{-f^{n-2}(z)+\lambda}| > 1$, that is, $e^{-\operatorname{Re} f^{n-2}(z)+\operatorname{Re}\lambda} > 1$ and so $\operatorname{Re} f^{n-2}(z) < 0$. By induction we will get $\operatorname{Re} f(z) < 0$. But $\operatorname{Re} f(z) = \operatorname{Re}(e^{-z+\lambda}) + \operatorname{Re}\xi > 0 + 1 = 1$, so we arrive at a contradiction and therefore proves the assertion. □

Lemma 2.4. *The set* $I_2 = \{z \in \mathbb{C} : \operatorname{Re} z = 0\}$ *does not intersect* $I(f)$ *for each* $f \in \mathcal{F}$.

Proof. Any $z \in I_2$ has the form $z = iy$ for some $y \in \mathbb{R}$. Now $\operatorname{Re} f(z) = \operatorname{Re}(e^{-iy+\lambda} + \xi) = e^{\operatorname{Re}\lambda}\cos(y - \operatorname{Im}\lambda) + \operatorname{Re}\xi$, and as $\operatorname{Re}\xi \geq 1$ we get $\operatorname{Re} f(z) > 0$. From above lemma, $I_2 \cap I(f) = \emptyset$, for each $f \in \mathcal{F}$ and hence the result. □

Lemma 2.5. *The set* $I_3 = \{z \in \mathbb{C} : \operatorname{Re} z < 0, \operatorname{Re}(e^{-z+\lambda}) > 0\}$ *does not intersect* $I(f)$ *for each* $f \in \mathcal{F}$.

Proof. For each $f \in \mathcal{F}$ and for each $z \in I_3$, $f(z)$ belongs to the right half plane and hence cannot escape to ∞ using Lemma 2.3. □

Remark 2.6. The right half plane is invariant under each $f \in \mathcal{F}$.

Next, Theorem 2.2 follows from the following three lemmas we prove.

Lemma 2.7. *The set* $I_1' = \{z \in \mathbb{C} : \operatorname{Re} z < 0, \operatorname{Re}(e^{z+\mu}) < 0,\ \mu \in \mathbb{C}$ *with* $\operatorname{Re}\mu < 0\}$ *does not intersect* $I(f)$ *for each* $f \in \mathcal{F}'$.

Proof. Observe that $\operatorname{Re}(e^{z+\mu}) < 0$, implies $\dfrac{(4k-3)\pi}{2} - \operatorname{Im}\mu < y < \dfrac{(4k-1)\pi}{2} - \operatorname{Im}\mu$, where $k \in \mathbb{Z}$. Therefore, the set I_1' is the entire left half plane $\{z : \operatorname{Re} z < 0\}$. We show no point in I_1' escapes to ∞ under iteration of each $f \in \mathcal{F}'$. For this we show $|f^k(z)| \leq 1 + |\zeta|$ for all $k \in \mathbb{N}$, $z \in I_1'$. Suppose on the contrary there exist $n \in \mathbb{N}$ and $z \in I_1'$ such that $|f^n(z)| > 1 + |\zeta|$. Now $|f(f^{n-1}(z))| > 1 + |\zeta|$, implies $1 + |\zeta| < |e^{f^{n-1}(z)+\mu} + \zeta|$. This shows that $e^{\operatorname{Re} f^{n-1}(z)+\operatorname{Re}\mu} > 1$, and as $\operatorname{Re}\mu < 0$, we obtain $\operatorname{Re} f^{n-1}(z) > 0$. Further this implies that $\operatorname{Re}(f(f^{n-2}(z))) > 0$, that is, $\operatorname{Re}(e^{f^{n-2}(z)+\mu} + \zeta) > 0$, which implies $\operatorname{Re}(e^{f^{n-2}(z)+\mu}) > -\operatorname{Re}\zeta \geq 1$. Since $|z| \geq \operatorname{Re} z$ for all $z \in \mathbb{C}$, we get $|(e^{f^{n-2}(z)+\mu})| > 1$, that is, $e^{\operatorname{Re} f^{n-2}(z)+\operatorname{Re}\mu} > 1$ and so $\operatorname{Re} f^{n-2}(z) > 0$. By induction we will get $\operatorname{Re} f(z) > 0$. But $\operatorname{Re} f(z) = \operatorname{Re}(e^{z+\mu}) + \operatorname{Re}\zeta < 0 - 1 = -1$, so we arrive at a contradiction and therefore proves the assertion. $\square$

Lemma 2.8. *The set $I_2' = \{z \in \mathbb{C} : \operatorname{Re} z = 0\}$ does not intersect $I(f)$ for each $f \in \mathcal{F}'$.*

Proof. Any $z \in I_2'$ has the form $z = iy$ for some $y \in \mathbb{R}$. Now $\operatorname{Re} f(z) = \operatorname{Re}(e^{iy+\mu}+\zeta) = e^{\operatorname{Re}\mu}\cos(y + \operatorname{Im}\mu) + \operatorname{Re}\zeta$, and as $\operatorname{Re}\zeta \leq -1$ we get $\operatorname{Re} f(z) < 0$. From above lemma, $I_2' \cap I(f) = \emptyset$, for each $f \in \mathcal{F}'$ and hence the result. $\square$

Lemma 2.9. *The set $I_3' = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Re}(e^{z+\mu}) < 0\}$ does not intersect $I(f)$ for each $f \in \mathcal{F}'$.*

Proof. For each $f \in \mathcal{F}'$ and for each $z \in I_3'$, $f(z)$ belongs to the left half plane and hence cannot escape to ∞ using Lemma 2.7. $\square$

Remark 2.10. The left half plane is invariant under each $f \in \mathcal{F}'$.

The following corollary is immediate

Corollary 2.11. *For each $f \in \mathcal{F}$ and for each $g \in \mathcal{F}'$, $I(f) \cap I(g) = \emptyset$.*

However, if the entire functions f and g share some relation, then their escaping sets do intersect. For instance, we have

Theorem 2.12. *Let f be a transcendental entire function of period c, and let $g = f^s + c$, $s \in \mathbb{N}$. Then $I(f) = I(g)$.*

Proof. For $n \in \mathbb{N}$, $g^n = f^{ns} + c$ and so $I(g) = I(f)$. $\square$

We illustrate this with an example.

Example 2.13. Let $f = e^{\lambda z}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $g = f^s + \frac{2\pi i}{\lambda}$, $s \in \mathbb{N}$. For $n \in \mathbb{N}$, $g^n = f^{ns} + \frac{2\pi i}{\lambda}$ and so $I(g) = I(f)$.

3. Composite entire functions and their dynamics

In this section, we prove some results related to escaping sets of composite entire functions. Recall that if a transcendental entire function f is of bounded type, then $I(f) \subset J(f)$ and $J(f) = \overline{I(f)}$ [8].

Theorem 3.1. *If f and g are permutable transcendental entire functions of bounded type, then $\overline{I(f)}$ and $\overline{I(g)}$ are completely invariant under $f \circ g$.*

Proof. From [1], we have $g(J(f)) \subset J(f)$ and so $g(\overline{I(f)}) \subset \overline{I(f)}$. From [4], $g^{-1}(\overline{I(f)}) \subset \overline{I(f)}$. Hence $\overline{I(f)}$ is completely invariant under g. On similar lines, $\overline{I(g)}$ is completely invariant under f. As $J(f) = \overline{I(f)}$ is completely invariant under f and $J(g) = \overline{I(g)}$ is completely invariant under g, we have $\overline{I(f)}$ and $\overline{I(g)}$ are both completely invariant under f and g respectively and this completes the proof of the theorem. $\square$

We next prove an important lemma which will be used heavily in the results to follow.

Lemma 3.2. *Let f and g be transcendental entire functions satisfying $f \circ g = g \circ f$. Then $F(f \circ g) \subset F(f) \cap F(g)$.*

Proof. In [5], it was shown that $z \in F(f \circ g)$ if and only if $f(z) \in F(g \circ f)$. Since $f \circ g = g \circ f$, $F(f \circ g)$ is completely invariant under f and by symmetry, under g respectively and so, in particular, it is forward invariant under them. So $f(F(f \circ g)) \subset F(f \circ g)$ and $g(F(f \circ g)) \subset F(f \circ g)$, which by Montel's Normality Criterion implies $F(f \circ g) \subset F(f)$ and $F(f \circ g) \subset F(g)$ and hence the result. $\square$

Theorem 3.3. *Let f and g be transcendental entire functions of bounded type satisfying $f \circ g = g \circ f$. Then $\overline{I(f)} \cup \overline{I(g)} \subset \overline{I(f \circ g)}$.*

Proof. Let $z_0 \notin \overline{I(f \circ g)}$. Then there exist a neighborhood U of z_0 such that $U \cap I(f \circ g) = \emptyset$. As $f \circ g$ is of bounded type, we get $U \subset F(f \circ g)$. From Lemma 3.2, $U \subset F(f)$ and $U \subset F(g)$. Therefore, $U \cap I(f) = \emptyset$ and $U \cap I(g) = \emptyset$. Thus $z_0 \notin \overline{I(f)} \cup \overline{I(g)}$ and this proves the result. $\square$

Theorem 3.4. *Let f and g be transcendental entire functions satisfying $f \circ g = g \circ f$. Then*
(i) $I(f \circ g)$ is completely invariant under f and g respectively;
(ii) $I(f \circ g) \subset I(f) \cup I(g)$;
(iii) For any two positive integers i and j, $I(f^i \circ g^j) = I(f \circ g)$.

Proof. (i) We first show that $z \in I(f \circ g)$ if and only if $g(z) \in I(g \circ f)$. Let $z \in I(f \circ g)$. Then $(f \circ g)^n(z) \to \infty$ as $n \to \infty$, that is, $f((g \circ f)^{n-1} g(z)) \to \infty$ as $n \to \infty$. As f is an entire function, this implies that $(g \circ f)^{n-1} g(z) \to \infty$ as $n \to \infty$, that is, $g(z) \in I(g \circ f)$. On the other hand, let $g(z) \in I(g \circ f)$. Then $(g \circ f)^n(g(z)) \to \infty$ as $n \to \infty$, that is, $g((f \circ g)^n(z)) \to \infty$ as $n \to \infty$. Again, as g is entire, this forces $(f \circ g)^n(z) \to \infty$ as $n \to \infty$. So, $z \in I(f \circ g)$ which proves the claim. As $f \circ g = g \circ f$, we obtain $z \in I(f \circ g)$ if and only if $g(z) \in I(f \circ g)$ which implies $I(f \circ g)$ is completely invariant under g, and by symmetry, under f respectively.

(ii) Suppose $z_0 \notin I(f) \cup I(g)$. Then both $f^n(z_0)$ and $g^n(z_0)$ are bounded as $n \to \infty$, which in turn implies $(f \circ g)^n(z_0)$ is bounded as $n \to \infty$ and hence the result.

(iii) For $i, j \in \mathbb{N}$, assume $i \geq j$. We first show that $I(f^i \circ g^j) \subset I(f \circ g)$. To this end, let $w \notin I(f \circ g)$. Then $(f \circ g)^n(w)$ is bounded as $n \to \infty$, which in turn (using a diagonal sequence argument) implies that $(f^i \circ g^j)^n(w)$ is bounded as $n \to \infty$. On similar lines, we get $I(f \circ g) \subset I(f^i \circ g^j)$ and hence $I(f^i \circ g^j) = I(f \circ g)$ for all $i, j \in \mathbb{N}$. $\square$

Theorem 3.5. *Let f and g be transcendental entire functions satisfying $f \circ g = g \circ f$. Then $g(I(f)) \supset I(f)$.*

Proof. Let $w \notin I(f)$. Then $f^n(w)$ is bounded and so $g(f^n(w))$ is bounded, which implies $g(w) \notin I(f)$ which proves the result. □

We now provide an important criterion for the equality of escaping sets for two entire functions.

Theorem 3.6. *Let f and g be two transcendental entire functions of bounded type satisfying $f \circ g = g \circ f$. Assume for each $w \in \overline{I(g)}$ and for a sequence $\{w_n\} \subset I(g)$ converging to w, $\lim_{n\to\infty} \lim_{k\to\infty} g^k(w_n) = \lim_{k\to\infty} \lim_{n\to\infty} g^k(w_n)$. Then $I(f) = I(g)$.*

Proof. From [11, Lemma 5.8], $F(f) = F(g)$ and so $J(f) = J(g)$ which implies $\overline{I(f)} = \overline{I(g)}$. Let $w \in I(f)$. Then there exist a sequence $\{w_n\} \subset I(g)$ such that $w_n \to w$ as $n \to \infty$. For each $n \in \mathbb{N}, g^k(w_n) \to \infty$ as $k \to \infty$. Now taking limit as n tends to ∞, and interchanging the two limits (by hypothesis) we obtain, $g^k(w) \to \infty$ as $k \to \infty$ which implies $w \in I(g)$ and so $I(f) \subset I(g)$. On similar lines, one obtains $I(g) \subset I(f)$ and this completes the proof of the result. □

Remark 3.7. The result, in particular, establishes one of Eremenko's conjecture [7] that every component of $I(f)$ is unbounded.

We now provide some conditions under which $\overline{I(f)}$ equals $\overline{I(f \circ g)}$.

Theorem 3.8. *Let f and g be two transcendental entire functions. Then the following holds:*
(i) If f and g are permutable and of bounded type then $\overline{I(f)} = \overline{I(f \circ g)}$;
(ii) If f is of period c and $g = f^m + c$ for some $m \in \mathbb{N}$, then $\overline{I(f)} = \overline{I(f \circ g)}$.

Proof. (i) In view of Theorem 3.4(ii), it suffices to show that $\overline{I(f)} \subset \overline{I(f \circ g)}$. To this end, let $w \notin \overline{I(f \circ g)}$. Then there exist a neighborhood U of w such that $U \cap I(f \circ g) = \emptyset$. As $f \circ g$ is of bounded type, it follows that $U \subset F(f \circ g)$ and so from Lemma 3.2, $U \subset F(f)$. Therefore, $U \cap I(f) = \emptyset$ which implies $w \notin \overline{I(f)}$ and this proves the result.

(ii) Observe that $f \circ g(z) = f^{m+1}(z)$ and hence the result. □

Remark 3.9. Combining Theorem 3.3 and Theorem 3.8(i), and using Theorem 3.4(ii) we get that if f and g are permutable and of bounded type, then $\overline{I(f \circ g)} = \overline{I(f)} \cup \overline{I(g)}$.

Finally, we discuss the relation between the escaping sets of two conjugate entire functions. Recall that two entire functions f and g are conjugate if there exist a conformal map $\phi : \mathbb{C} \to \mathbb{C}$ with $\phi \circ f = g \circ \phi$. By a conformal map $\phi : \mathbb{C} \to \mathbb{C}$ we mean an analytic and univalent map of the complex plane $\mathbb{C}$ that is exactly of the form $az + b$, for some non zero a. If f and g are two rational functions which are conjugate under some Mobius transformation $\phi : \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$, then it is well known [2, p. 50], $\phi(J(f)) = J(g)$.

This gets easily carried over to transcendental entire funtions which are conjugate under a conformal map $\phi : \mathbb{C} \to \mathbb{C}$. Moreover, if f is of bounded type which is conjugate under the conformal map ϕ to an entire function g, then g is also of bounded type and $\phi(\overline{I(f)}) = \overline{I(g)}$. More generally, if transcendental entire functions f and g are conjugate by conformal map ϕ, then $\phi(I(f)) = I(g)$.

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EXISTENCE OF REDUCTION OF IDEALS OVER SEMI LOCAL RINGS

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Abstract. Let S be a Noetherian semi local ring, $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the maximal ideals of S and $S/\mathfrak{m}_i := k_i$ be infinite residue fields, $i = 1, \ldots, t$. Suppose $I \subset S$ is an ideal and $\mu(I^n) < \binom{n+r}{r}$ for some positive integers n and r, where μ denotes the minimal number of generators. Then there exists ideal $J = (a_1, \ldots, a_r) \subset I$ such that $JI^n = I^{n+1}$.

1. Introduction

Let I be an ideal of a ring R and t be a variable. The Rees algebra of I defined as $R[It] = \{a_0 + a_1 t + \cdots + a_n t^n \mid n \in \mathbb{N}, a_i \in I^i \text{ and } I^0 = R\} = \oplus_{n \geq 0} I^n t^n$. The Rees algebra of an ideal is a classical object that has been studied by the mathematicians ([2], [4], [7], [9], [10], [8], [11], [12]). The Rees algebra of an ideal is a subring of the polynomial ring $R[t]$, which encodes several analytic properties of the variety of the ideal I. It is an algebraic realization of the blow-up of a variety along a subvariety. Though, blow-up is a fundamental operation in the birational study of algebraic varieties, in the process of desingularization and one of the key operations in the birational algebraic geometry, it can be replaced by considerations involving the Rees algebra [2]. Thus the Rees algebra is an analogue of the blow-up operation in algebraic geometry.

The Rees algebra of an ideal in Noetherian ring captures a great deal of information about how the ideal is contained in the ring R and how its powers change. These structures give rise to other numerical measures of the ideal such as, reduction of an ideal, integral closure of ideals and reduction number. Reduction number gives the largest degree of an element in a homogeneous minimal generating set of the Rees algebra of an ideal ([11], Theorem 8.2.1).

Integral closure has important role in number theory & algebraic geometry and began with the work of Krull and Zariski in the 1930's. Integral closures of ideals, rings and modules overlap many important topics, including the core of ideals,

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Hilbert functions, homological algebra, multiplicities (mixed and otherwise), singularity theory, Rees algebras, resolution of singularities, tight closure, and valuation theory. The integral closure of ideals arise in commutative algebra for understanding the growth of ideals. A classic example of this phenomenon is the Hilbert-Samuel polynomial for an m-primary ideal I in a local Noetherian ring (R, m) of Krull dimension d. Then the multiplicity of I=$e(I) = lim_{n\to\infty} \dfrac{\lambda(R/I^n)}{n^d} d!$, where $\lambda(.)$ is the length function. The Hilbert function of the ideal is $\lambda(R/I^n)$ which is given by a polynomial in n of degree d for large $n > 0$. If the ideal is a monomial, then the limit can be interpreted as a Riemann sum of volume (normalized by the factor $d\,!$). The coefficients of this polynomial are necessarily rational and are numerical invariants which are important for study of the pair (R, I). Note that $e(I)$ is called the multiplicity of I on R (which is the multiplicity of R if $I = m$). A key question is what elements r can be added to an m-primary ideal I so that I and $I + rR$ exhibit the same power-growth or have the same multiplicity? First approach can be that if there is an n such that $r^n \in I^n$, i.e. powers of r grow as powers of I. This has merit, but allows too few elements r.

There are other natural approaches. One approach through its relationship to tight closure is to ask whether there is an element c such that for all large n, $cr^n \in I^n$. Taking n^{th} roots and letting $n \to \infty$ somehow captures the sense that r is almost in I. A next approach due to Rees is an asymptotic version. If the $lim_{n\to\infty} \dfrac{v_n(r)}{n}$ exists and is at least 1, then r grows at least as fast as I. Another approach is to use valuations to measure the relative sizes of I and r. One can simply require that $v(r) \geq v(I)$ for every valuation v. This a priori leads to a new and different idea of growth, but it turns out to be the same as Rees asymptotic approach above. All of these different approaches lead to the same concept, which is called the integral closure of an ideal I.

The following theorem of Eakin-Sathaye gives the reduction of an ideal over a Noetherian local ring.

Theorem 1.1 (Eakin-Sathaye [3]). *Let $(R, \mathfrak{m})$ be Noetherian local ring with infinite residue field k and $I \subset R$ be an ideal. Suppose there exist integers n and r such that $\mu(I^n) < \binom{n+r}{r}$. Then there exist $y_1, \ldots, y_r \in I$ such that $(y_1, \ldots, y_r)I^n = I^{n+1}$.*

In fact, if J is a reduction of I generated by r elements and I^n can be generated by less than $\binom{n+r}{r}$ elements, then reduction number of I with respect to J is at most n.

Carroll [1] gave a generalization of Eakin-Sathaye theorem as follows: Let A be a standard graded algebra over an infinite field k and if for positive integers n

and r, $dim_k(A_n) < \binom{n+r}{r}$. Then there exists $\{y_1, \ldots, y_r\} \subset A_1$ such that $A_n = (y_1, \ldots, y_r)A_{n-1}$. Moreover, if $(x_1, \ldots, x_p)$ is such that $(x_1, \ldots, x_p)^n = A_n$, then generic elements $y_1, \ldots, y_r$ are linear combinations of $x_1, \ldots, x_p$. Since reductions of ideals are generated by the elements which are linear forms in the fiber cone, one may obtain the original statement of Eakin-Sathaye by applying the above result to the fiber cone. In this paper we generalize Eakin-Sathaye theorem for the semi local rings.

2. Preliminaries

Definition 2.1. A reduction of I is an ideal $J \subseteq I$ such that $JI^n = I^{n+1}$, for some non-negative integer n. The least such integer n is called the reduction number of I relative to J. An ideal $J \subset I$ is a minimal reduction if it is not contained in any reduction of I.

There is a close relation between the integral dependence and the reductions of an ideal. An ideal I is said to be integral dependent over J if each element $x \in I$ satisfies some polynomial

$$x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where $a_i \in J^i$ for all $1 \leq i \leq n$.

Definition 2.2. Let $R = k[X_1, \ldots, X_d]$ be the polynomial ring, where k is a field. An element of the form $X_1^{n_1}X_2^{n_2}\ldots X_d^{n_d}$ in $k[X_1, \ldots, X_d]$, where n_i are non-negative integers, is called a monomial and $(n_1, \ldots, n_d) \in N^d$ is its exponent vector. An ideal is said to be monomial if it is generated by monomials.

The following useful result due to Swanson and Huneke deserves to be more well known. For the convenience of the reader we give here.

Proposition 2.3 ([11], Proposition 1.4.2)**.** *The Integral closure of a monomial ideal I in a polynomial ring $k[X_1, \ldots, X_d]$ is a monomial ideal.*

Proof. Suppose to the contrary that $f \in \bar{I}$ is not a monomial and that no homogeneous component of f is in $\bar{I}$. Write $f = \sum_{l\in\wedge} f_l$, where $\wedge$ is a finite subset of $\mathbb{N}^d$ and f_l is the component of f of degree l. Let $L \in \wedge$ with $f_L \neq 0$. First assume that k is algebraically closed. Any ring automorphism ϕ of $k[X_1, \ldots, X_d]$ maps an integral equation of f over I into an integral equation of $\phi(f)$ over $\phi(I)$. In particular, for any units $u_1, \ldots, u_d$ in k, if ϕ_u is the ring automorphism taking X_i to u_iX_i, then $\phi_u(I) = I$ and $\phi_u(f)$ is integral over I. Furthermore, each ϕ_u has the property that f is non-zero in degree l. As k is algebraically closed and f is not a scalar multiple of a monomial, there exist $u_1, \ldots, u_d \in k$ such that $\phi_u(f)$ is not a polynomial multiple of f. As both f and $\phi_u(f)$ are integral over I, so is

$g = {u_1}^{L_1} \dots {u_d}^{L_d} f - \phi_u(f)$. As $\phi_u(f)$ is not a polynomial multiple of f, then g is not zero. Note that the component of g in degree L is 0 and that whenever g is non-zero in degree l, then f is also non-zero in degree l, so that $l \in \wedge$. Thus g has strictly fewer non-zero homogeneous components than f, so by induction on the number of components, each component of g lies in the integral closure of I. But each component of g is a scalar multiple of a component of f, which proves that some homogeneous components of f are in the integral closure of I, contradicting the choice of f.

Now let k be an arbitrary field and $\bar{k}$ its algebraic closure. By the previous case we know that each monomial appearing in f with a non-zero coefficient is integral over $I\bar{k}[X_1 \dots X_d]$. Thus each monomial r appearing in f satisfies an integral equation of the form $r^i - a_i = 0$ for some a_i that is a product of i monomials of $I\bar{k}[X_1 \dots X_d]$. Hence a_i is also a product of i monomials of I, so that r is integral over I. □

Proposition 2.4 ([11], Proposition 1.4.6). *The exponent set of the integral closure of a monomial ideal I is equal to all the integer lattice points in the convex hull of the exponent set of I.*

Proof. Let $k[X_1, \dots, X_d]$ be a polynomial ring and let I be a monomial ideal. Let its generators be $m_j = {X_1}^{n_{j_1}} {X_2}^{n_{j_2}} \dots {X_d}^{n_{j_d}}$, $j = 1, 2, \dots, s$. Let a monomial $r = {X_1}^{n_1} {X_2}^{n_2} \dots {X_d}^{n_d}$ be integral over I. Then there exists a positive integer i and a product a_i of i monomials of I such that $r^i - a_i = 0$. An arbitrary product a_i of i monomials in I is of the form $b{m_1}^{k_1} {m_2}^{k_2} \dots {m_s}^{k_s}$, where b is another monomial, and the k_j are non-negative integers which sum up to i. Then $r^i = b{m_1}^{k_1} {m_2}^{k_2} \dots {m_s}^{k_s}$, and for each $l = 1, \dots, d$, $i.n_l \geq \sum_j k_j n_{jl}$, or in other words $n_l \geq \sum_j \dfrac{k_j}{i} n_{jl}$. Thus the problem of finding monomials $r = {X_1}^{n_1} {X_2}^{n_2} \dots {X_d}^{n_d}$ that are integral over I reduces to finding non-negative rational numbers $c_1, \dots, c_d$ with $\sum_{j=1}^{d} c_j = 1$, such that component wise

$$(n_1, n_2, \dots, n_d) \geq \sum_j c_j (n_{j1}, n_{j2}, \dots, n_{jd}). \tag{2.1}$$

Conversely, suppose there are non-negative rational numbers $c_1, \dots, c_d$ with $\sum_{j=1}^{d} c_j = 1$, and such that the inequality in (1) holds component wise. Write $c_j = \dfrac{k_j}{i}$ for some $k_j, i \in \mathbb{N}$, $i \neq 0$. Then $r^i = b{m_1}^{k_1} {m_2}^{k_2} \dots {m_d}^{k_d}$, for some monomial $b \in k[X_1, \dots, X_d]$, so that $r \in \bar{I}$.

Geometrically the construction of $(n_1, n_2, \dots, n_d)$ satisfying the inequality (1) is the same as finding the integer lattice points in the convex hull of the exponent set of the ideal I. □

Proposition 2.5. *Let R be a ring and $J \subseteq I$ be ideals of R. If I is a finitely generated ideal then J is a reduction of I iff every element of I is integral over J.*

Proof. Suppose J is a reduction of I. Then $JI^n = I^{n+1}$ for some $n \geq 0$. Since I^n is a finitely generated ideal, suppose $I^n = (x_1, \ldots, x_t)$. Let $a \in I$. Then $ax_i \in I^{n+1} = JI^n$ for $i = 1, \ldots, t$. Therefore,

$$ax_i = \sum_{j=1}^{t} b_{ij}\ x_j, \text{where } b_{ij} \in J \text{ for } i = 1, \ldots, t \qquad \text{and} \qquad \sum_{j=1}^{t}(a\delta_{ij} - b_{ij})x_j = 0,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then by multiplying adjoint of the matrix $(a\delta_{ij} - b_{ij})$, we have $\triangle x_j = 0$, for all $j = 1, \ldots, t$, where $\triangle$ is the determinant of the matrix $(a\delta_{ij} - b_{ij})$. This implies that $\triangle I^n = 0$. Since $a \in I$, $a^n \in I^n$. Hence $\triangle a^n = 0$. Now expansion of $\triangle$ gives an equation which shows that a is integral over J.

Conversely, note that $JI^n \subseteq I^{n+1}$, for $J \subseteq I$. Since $x \in I$ is integral over J, $x^{n+1} + a_1x^n + a_2x^{n-1} + \cdots + a_{n+1} = 0$, where $a_i \in J^i$. It follows that $x^{n+1} = -a_1x^n - a_2x^{n-1} - \cdots - a_{n+1}$. Since $a_1 \in J$ and $x^n \in I^n$, $a_1x^n \in JI^n$. Similarly $a_2x^{n-1} \in J^2I^{n-1} = JJI^{n-1} \subseteq JI^n, \ldots, a_n \in J^n \subseteq JI^n$ for, $JI^{n-1} \subseteq I^n$. Therefore, $I^{n+1} \subseteq JI^n$. Hence $JI^n = I^{n+1}$ and J is a reduction of I. □

Example 2.6. *Let $R = k[X_1, X_2]$ be a ring and $I = \langle {X_1}^3, X_1X_2, {X_2}^4\rangle$ be an ideal of R. Then $J = < X_1X_2, {X_1}^3 + {X_2}^4 >$ is a reduction of I with reduction number* 1. *First note that $JI \subseteq I^2$, for $J \subseteq I$. To show that $I^2 \subseteq JI$, we observe that $I^2 = < {X_1}^6, {X_1}^2{X_2}^2, {X_2}^8, {X_1}^4X_2, X_1{X_2}^5 >$ and $JI = < {X_1}^4X_2, {X_1}^6 + {X_2}^4{X_1}^3, {X_1}^2{X_2}^2, {X_1}^4X_2 + {X_2}^5X_1, {X_2}^4{X_1}^3 + {X_2}^8 >$. Then clearly $I^2 \subseteq JI$ and $JI = I^2$.*

Proposition 2.7 ([11], Lemma 8.1.10)**.** *Let R be a Noetherian ring, I and J ideals of R. Suppose J is a reduction of I. Then*

$$(1)\ \ rad(I) = rad(J) \quad \textit{and} \quad (2)\ \ ht(I) = ht(J).$$

Note that the converse of the above proposition may not be true.

Example 2.8. *Let $R = k[[X, Y]]$ be a ring over an infinite field k. Suppose $J = (X^2, Y)$ and $I = (X, Y)$ are ideals in R. Note that $J \subset I$, $rad(I) = rad(J) = (X, Y)$ and $ht(I) = ht(J) = 2$ but J is not a reduction of I.*

Proof. Suppose to the contrary that J is a reduction of I. Then there exists $n \geq 0$ such that $JI^n = I^{n+1}$, where I^{n+1} is generated by the monomials $X^{n+1}, X^nY, \ldots, XY^n, Y^{n+1}$ and JI^n is generated by the monomials $X^{n+2}, Y^{n+1}, X^{n+1}Y, \ldots, Y^{n+1}X$. But $X^{n+1} \notin JI^n$ for all $n \geq 0$, which is a contradiction. Therefore J is not a reduction of I. □

If J_1 and J_2 are two reductions of I, then $J_1 \cap J_2$ may not be reduction of I:

Example 2.9. *Take $R = k[[X,Y]]$ over an infinite field k, $J_1 = (X^2,(X+Y)^2) \subset (X^2,Y^2,XY) = I$ and $J_2 = (X^2,Y^2) \subset (X^2,Y^2,XY) = I$. Then $J_1I = J_2I = I^2$ with reduction number 1, where $J_1I = (X^4, X^2(X^2+Y^2+XY), X^2Y^2, Y^2(X^2+Y^2+XY), X^3Y, XY(X^2+Y^2+XY))$, $J_2I = (X^4, X^2Y^2, Y^4, X^3Y, XY^3)$ and $I^2 = (X^4, Y^4, X^2Y^2, X^3Y, XY^3)$. But $J = J_1 \cap J_2 = (X^2, XY, Y^3) \subset (X^2, Y^2, XY)$ is not a reduction of I. Suppose contrary that J is a reduction of I. Then there exists $n \geq 0$ such that $JI^n = I^{n+1}$, where I^{n+1} is generated by the monomials X^{2n+2}, $X^{n+1}\,Y^{n+1}$, Y^{2n+2}, $\ldots$, XY^{2n+1} and JI^n is generated by the monomials $X^{2n+2}, X^{2n+1}Y, Y^{n+1}, X^{2n-1}Y^2, \ldots, Y^{2n+2}X$. But $Y^{2n+2} \notin JI^n$ for all $n \geq 0$, which is a contradiction.*

Remark 2.1. If I is a nilpotent ideal, then any ideal contained in I is a reduction.

Remark 2.2. (1) Reduction defines a relationship between two ideals which is preserved under homomorphisms and ring extensions.
(2) Reduction process gets rid of the superfluous elements of an ideal without disturbing the algebraic multiplicities associated with it.
(3) Reductions play a role in the theory of finite morphisms of the blow-up $Blow_{V(I)}(Spec(R))$, with the reduction number being a control element. In case R is a local ring (of infinite residue field), the minimal reductions are particularly valuable because they help to control the co-homology of the blow-up.

An exposition on the Rees algebra by using the method of moving curves and surfaces, the defining equations of the Rees algebra has been given by Cox [2]. The Rees algebra of an ideal is the quotient of polynomial ring by its ideal of the defining equations as follows: Given the ideal $I = (v_1, v_2, \ldots, v_r)$ of R, its Rees algebra is a graded R-algebra $R[It] = R \oplus It \oplus I^2t^2 \oplus \ldots$, where the elements in the summand I^n are assigned degree n. Therefore, Rees algebra $R[It]$ can be described as the image of R-algebra homomorphism $h : R[X_1, \ldots, X_r] \to R[It]$ defined by $X_i \to v_it$. Note that the map h is onto. If K is the kernel of the map h, then by the fundamental theorem of R-algebra homomorphism $R[It] \cong \dfrac{R[X_1,\ldots,X_r]}{K}$. The generators of K are the defining equations of Rees algebra.

The Rees algebra of $I = \langle v_1, v_2, \ldots, v_r \rangle \subset R$ is closely related to the symmetric algebra $Sym_R(I) = R \oplus I \oplus Sym^2(I) \oplus Sym^3(I) \oplus \ldots$ via the canonical surjection $\alpha : Sym_R(I) \to R[It]$. If K_1 is a graded piece of K in degree 1 with respect to X_i, then $K_1 = \{a_1X_1 + a_2X_2 + \cdots + a_rX_r \mid a_i \in R, a_1v_1 + a_2v_2 + \cdots + a_rv_r = 0\} \cong Syz(v_1, v_2, \ldots, v_r)$. Since $< K_1 > \subset K$, we get a commutative diagram

$$\begin{array}{ccc} \dfrac{R[X_1,\ldots,X_r]}{K_1} & \longrightarrow & \dfrac{R[X_1,\ldots,X_r]}{K} \\ \downarrow & & \downarrow \\ Sym_R(I) & \longrightarrow & R[It] \end{array}$$

The vertical map on the right is an isomorphism and vertical map on the left is known to be an isomorphism. Thus syzygies of the v_i define the symmetric algebra and give relations of the Rees algebra of degree 1.

Now we show by an example that the defining equations of the syzygies can be of higher degree if the generators of the ideal fail to be a regular sequence. In this example "LIB "reesclos.lib" is a library in the singular software to compute the integral closure of an ideal I in a polynomial ring $R = K[X_1, \ldots, X_n]$ using the Rees Algebra $R[It]$ of I.

Example 2.10. *Let $R = k[X_1, X_2]$, where k is a field and $I = \langle {X_1}^2, X_1X_2, {X_2}^2\rangle$. Then by using singular software, we can compute $R[It]$ as follows:*
LIB "reesclos.lib";
ring $R = 0, (X_1, X_2)$, dp;
ideal $I = X_1^2, X_1X_2, X_2^2$;
def $Rees = L[1]$;
setring Rees;
Rees;
ker;
$ker[1] = X_2T_1 - X_1T_2$,
$ker[2] = X_2T_2 - X_1T_3$,
$ker[3] = -{T_2}^2 + T_1T_3$.
Therefore,

$$R[It] \cong \frac{k[X_1, X_2, T_1, T_2, T_3]}{(X_2T_1 - X_1T_2, X_2T_2 - X_1T_3, -{T_2}^2 + T_1T_3)}.$$

3. The Existence of reductions

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a local ring and $J \subset I$ ideals of R. Then J is a reduction of I if and only if $J + \mathfrak{m}I$ is a reduction of I.*

Proof. Note that $J \subseteq J + \mathfrak{m}I \subseteq I$. Let J be a reduction of I. Then there exists $n \geq 0$ such that $I^{n+1} = J\ I^n \subseteq (J + \mathfrak{m}I)I^n \subseteq JI^n + \mathfrak{m}I^{n+1} \subseteq I^{n+1}$. So equality holds throughout and $J + \mathfrak{m}I$ is a reduction of I. Conversely, if $J + \mathfrak{m}I$ is a reduction of I, then $I^{n+1} = (J + \mathfrak{m}I)I^n \subseteq J\ I^n + \mathfrak{m}I^{n+1}$ for some $n \geq 0$ and going through modulo $J\ I^n$, we have $\frac{I^{n+1}}{J\ I^n} \subseteq \frac{J\ I^n + \mathfrak{m}I^{n+1}}{J\ I^n} = \mathfrak{m}\left(\frac{I^{n+1}}{J\ I^n}\right)$. Then by Nakayama lemma, $\frac{I^{n+1}}{J\ I^n} = 0$. Therefore, $I^{n+1} = J\ I^n$. $\square$

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a local ring and I an ideal of R. Suppose $a_i - b_i \in \mathfrak{m}I$ for $i = 1, \ldots, r$. Then $(a_1, \ldots, a_r)$ is a reduction of I if and only if $(b_1, \ldots, b_r)$ is a reduction of I.*

Proof. Let $J = (a_1, \ldots, a_r)$ be a reduction of I. Then by Lemma 3.1, $J + \mathfrak{m}I$ is a reduction of I. Since $a_i + \mathfrak{m}I = b_i + \mathfrak{m}I$ for $i = 1, \ldots, r$, $(a_1, \ldots, a_r) + \mathfrak{m}I = (b_1, \ldots, b_r) + \mathfrak{m}I$. Therefore, $(b_1, \ldots, b_r) + \mathfrak{m}I$ is a reduction of I with reduction number at most n. Again by Lemma 3.1, $(b_1, \ldots, b_r)$ is a reduction of I. $\square$

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and I is an ideal of R. Suppose $\{\overline{x_1}, \ldots, \overline{x_r}\}$ is a basis of $I/\mathfrak{m}I$, where $\overline{x_i} = x_i + \mathfrak{m}I$, $x_i \in I$ and $\overline{a_j} = \sum_{i=1}^{r} \overline{b_{ij}}\ \overline{x_i}$, $j = 1, \ldots, r$, $\overline{b_{ij}} \in k$. Then $\{\overline{a_1}, \ldots, \overline{a_r}\}$ is a basis of $I/\mathfrak{m}I$ if and only if the matrix $[\overline{b_{ij}}]_{r\times r}$ is an invertible matrix.*

Proof. Proof is trivial. $\square$

Now we prove the existence of the minimal number of generators of I through a Zariski-open set as follows:

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field k. Suppose I is an ideal of R and $dim_k(I/\mathfrak{m}I) = r$. Then there exists a Zariski-open subset U of $\left(\dfrac{I}{\mathfrak{m}I}\right)^r$ such that if $(\overline{a_1}, \ldots, \overline{a_r}) \in U$, where $\overline{a_i}$ denotes image of $a_i \in I$ in $I/\mathfrak{m}I$, then $I = (a_1, \ldots, a_r)$.*

Proof. Suppose $\{\overline{x_1}, \ldots, \overline{x_r}\}$ is a basis of $I/\mathfrak{m}I$, where $\overline{x_i} = x_i + \mathfrak{m}I$, $x_i \in I$ and $\overline{a_j} = \sum_{i=1}^{r} \overline{b_{ij}}\ \overline{x_i}$, for $j = 1, \ldots, r$, $\overline{b_{ij}} \in k$. Since $k^r \cong I/\mathfrak{m}I$, the isomorphism $(k^r)^r \xrightarrow{T} (I/\mathfrak{m}I)^r$ is given by $T([\overline{b_{ij}}]_{r\times r}) = (\overline{b_{11}}\ \overline{x_1} + \cdots + \overline{b_{1r}}\ \overline{x_r}, \ldots, \overline{b_{r1}}\ \overline{x_1} + \cdots + \overline{b_{rr}}\ \overline{x_r})$. Note that there is a one to one correspondence of open sets of the topological space $(k^r)^r$ to open sets of the topological space $(I/\mathfrak{m}I)^r$, for $(k^r)^r \cong (I/\mathfrak{m}I)^r$. Suppose U is a Zariski-open set in $(I/\mathfrak{m}I)^r$ corresponding to the matrix $[\overline{b_{ij}}]_{r\times r}$ such that $det([\overline{b_{ij}}]_{r\times r}) \neq 0$.

Then $U = \{(\overline{a_1}, \ldots, \overline{a_r}) \in \left(\dfrac{I}{\mathfrak{m}I}\right)^r \mid det([\overline{b_{ij}}]_{r\times r}) \neq 0\}$.

$U = \{(\overline{a_1}, \ldots, \overline{a_r}) \in \left(\dfrac{I}{\mathfrak{m}I}\right)^r \mid \{\overline{a_1}, \ldots, \overline{a_r}\}$ *is a basis of* $I/\mathfrak{m}I\}$ *by Lemma* 3.3.

$U = \{(\overline{a_1}, \ldots, \overline{a_r}) \in \left(\dfrac{I}{\mathfrak{m}I}\right)^r \mid (a_1, \ldots, a_r) = I\}$ ([5], Page 172).

Note that $U \neq \varnothing$. Since $det([\overline{b_{ij}}]_{r\times r}) \neq 0$, $\{\overline{a_1}, \ldots, \overline{a_r}\}$ is a linearly independent set of $I/\mathfrak{m}I$ by Lemma 3.3. This implies that $(\overline{a_1}, \ldots, \overline{a_r}) \in U$. $\square$

Proposition 3.5 (Chinese Remainder Theorem)**.** *Let I be an ideal of a ring R and $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ be set of distinct maximal ideals of R. Suppose $a_1, \ldots, a_r \in I$. Then there exists $a \in I$ such that $a - a_i \in \mathfrak{m}_i I$.*

More general form of the Chinese remainder theorem for the modules is the following:

Proposition 3.6. *Let R be a commutative ring with 1 and $I_1, \ldots, I_r$ be ideals in R and M be an R-module. Then*

(1) the map $M \longrightarrow \dfrac{M}{I_1 M} \oplus \ldots \oplus \dfrac{M}{I_r M}$, $x \to (x + I_1 M, \ldots, x + I_r M)$ *is an R-module homomorphism with kernel* $I_1 M \cap I_2 M \cap \ldots \cap I_r M$.

(2) if the ideals $I_1, \ldots, I_r$ *are pairwise comaximal i.e.* ($I_i + I_j = R$, *for all* $i \neq j$), *then* $\dfrac{M}{(I_1 \ldots I_r) M} \cong \dfrac{M}{I_1 M} \oplus \ldots \oplus \dfrac{M}{I_r M}$.

Now we prove the main result of this paper, which is a generalization of the Theorem 1.1 for the semi local rings.

Theorem 3.7. *Let S be a Noetherian semi local ring, $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the maximal ideals of S and $S/\mathfrak{m}_i := k_i$ be infinite residue fields, $i = 1, \ldots, t$. Suppose $I \subset S$ is an ideal and $\mu(I^n) < \binom{n+r}{r}$ for some positive integers n and r. Then there exists ideal $J = (a_1, \ldots, a_r) \subset I$ such that $JI^n = I^{n+1}$.*

Proof. Note that $IS_{\mathfrak{m}_i} = I_{\mathfrak{m}_i}$ and $n_i = \mathfrak{m}_i S_{\mathfrak{m}_i}$ are maximal ideals of $S_{\mathfrak{m}_i}$. Therefore $(S_{\mathfrak{m}_i}, n_i)$ is a Noetherian local ring having infinite residue field k_i and $I_{\mathfrak{m}_i} \subset S_{\mathfrak{m}_i}$ is an ideal with $\mu(I^n_{\mathfrak{m}_i}) < \binom{n+r}{r}$. Then by Theorem 1.1 there exists $J_{\mathfrak{m}_i} = (b_1^{(i)}, \ldots, b_r^{(i)}) \subset I_{\mathfrak{m}_i}$ such that $J_{\mathfrak{m}_i}(I_{\mathfrak{m}_i})^n = (I_{\mathfrak{m}_i})^{n+1}$. We show that

$$(b_1^{(i)}, \ldots, b_r^{(i)}) I_{\mathfrak{m}_i} = (c_1^{(i)}, \ldots, c_r^{(i)}) IS_{\mathfrak{m}_i}, \qquad \text{where} \qquad c_j^{(i)} \in I.$$

Indeed, if $x_i \in (b_1^{(i)}, \ldots, b_r^{(i)}) I_{\mathfrak{m}_i}$, then $x_i = y_i\, z_i$, where $y_i \in (b_1^{(i)}, \ldots, b_r^{(i)})$ and $z_i \in I_{\mathfrak{m}_i}$. Therefore $y_i = b_1.s_1/s_1^{(i)} + b_2.s_2/s_2^{(i)} + \cdots + b_r.s_r/s_r^{(i)}$, where $s_j^{(i)} \notin \mathfrak{m}_i$, $b_i \in I$ and $s_i \in S$. So

$$y_i = \frac{b_1 s_1 s_2^{(i)} \ldots s_r^{(i)} + b_2 s_2 s_1^{(i)} s_3^{(i)} \ldots s_r^{(i)} + \cdots + b_r s_1^{(i)} \ldots s_{r-1}^{(i)}}{s_1^{(i)} \ldots s_r^{(i)}}.$$

Then $c_1^{(i)} = b_1 s_1 s_2^{(i)} \ldots s_r^{(i)}$, $c_2^{(i)} = b_2 s_2 s_3^{(i)} \ldots s_r^{(i)}, \ldots, c_r^{(i)} = b_r s_r s_1^{(i)} \ldots s_{r-1}^{(i)}$ and hence $x \in (c_1^{(i)}, \ldots, c_r^{(i)}) IS_{\mathfrak{m}_i}$. Similarly $(c_1^{(i)}, \ldots, c_r^{(i)}) IS_{\mathfrak{m}_i} \subseteq (b_1^{(i)}, \ldots, b_r^{(i)}) I_{\mathfrak{m}_i}$. Therefore, $(c_1^{(i)}, \ldots, c_r^{(i)}) I^n S_{\mathfrak{m}_i} = I^{n+1} S_{\mathfrak{m}_i}$. By Proposition 3.5, there exists $a_1 \in I$ such that $a_1 - c_1^{(1)} \in \mathfrak{m}_1 I$, $a_1 - c_1^{(2)} \in \mathfrak{m}_2 I, \ldots, a_1 - c_r^{(t)} \in \mathfrak{m}_t I$. Similarly we get $a_2, \ldots, a_r \in I$ such that $a_2 - c_1^{(2)} \in \mathfrak{m}_1 I, \ldots, a_2 - c_2^{(t)} \in \mathfrak{m}_t I$ and $a_r - c_r^{(1)} \in \mathfrak{m}_1 I, \ldots, a_r - c_r^{(t)} \in \mathfrak{m}_t I$. By Proposition 3.2, $(a_1, \ldots, a_r) I^n S_{\mathfrak{m}_i} = I^{n+1} S_{\mathfrak{m}_i}$ for each maximal ideal. Therefore, $(a_1, \ldots, a_r) I^n = I^{n+1}$ ([11], Proposition 8.1.1). $\square$

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A STUDY OF THE CRANK FUNCTION IN RAMANUJAN'S LOST NOTEBOOK

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Abstract. In this note, we shall give a brief survey of the results that are found in Ramanujan's Lost Notebook related to cranks. Recent work by B. C. Berndt, H. H. Chan, S. H. Chan and W. -C. Liaw have shown conclusively that cranks was the last mathematical object that Ramanujan studied. We shall closely follow the work of Berndt, Chan, Chan and Liaw and give a brief description of their work.

1. Introduction

This note is divided into three sections. This section is a brief introduction to the results that we discuss in the remainder of the note and to set the notations and preliminaries that we shall be needing in the rest of our study. In the second section, we formally study the crank statistic for the general partition function which is defined as follows.

Definition 1.1 (Partition Function)**.** *If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function.*

We state a few results of George E. Andrews and Frank G. Garvan [4] and then show conclusively that Ramanujan had also been studying the crank function much earlier in another guise. We follow the works of Bruce C. Berndt and his collaboraters ([9, 10] and [11]) to study the work of Ramnaujan dealing with cranks. In the final section, we give some concluding remarks.

Although no originality is claimed with regards to the results proved here, in some cases, the arguments may have been modified to give an easier justification.

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1.1. **Motivation.** The motivation for the note comes from the work of Freeman Dyson [17], in which he gave combinatorial explanations of the following famous congruences given by Ramanujan

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}, \tag{1.3}$$

where $p(n)$ is the ordinary partition function defined as above.

In order to give combinatorial explanations of the above, Dyson defined the *rank* of a partition to be the largest part minus the number of parts. Let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Then Dyson conjectured that

$$N(k,5,5n+4) = \frac{p(5n+4)}{5},\ 0 \leq k \leq 4,$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{7},\ 0 \leq k \leq 6,$$

which yield combinatorial interpretations of (1.1) and (1.2).

These conjectures were later proved by Atkin and Swinnerton-Dyer in [5]. The generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)a^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(aq;q)_n(q/a;q)_n}.$$

Here $|q|<1,\ |q|<|a|<1/|q|$. For each nonnegative integer n, we set

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\ (a)_\infty := (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, |q|<1.$$

We also set

$$(a_1,\ldots,a_m;q)_n := (a_1;q)_n\cdots(a_m;q)_n$$

and

$$(a_1,\ldots,a_m;q)_\infty := (a_1;q)_\infty\cdots(a_m;q)_\infty.$$

However, the corresponding analogue of the rank doesn't hold for (1.3), and so Dyson conjectured the existence of another statistic which he called the *crank*. In his doctoral dissertation F. G. Garvan defined vector partitions which became the forerunner of the true crank.

Let $\mathcal{P}$ denote the set of partitions and $\mathcal{D}$ denote the set of partitions into distinct parts. Following Garvan, [19] we denote the set of vector partitions V to be defined by

$$V = \mathcal{D}\times\mathcal{P}\times\mathcal{P}.$$

For $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}$, the crank$(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$, and $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$, where $|\pi|$ is the sum of the parts of π. The number of vector partitions of n with crank m counted according to the weight ω is denoted by

$$N_V(m,n) = \sum_{\vec{\pi} \in V, |\vec{\pi}|=n, crank(\vec{\pi})=m} \omega(\vec{\pi}).$$

We then have

$$\sum_m N_V(m,n) = p(n)$$

where the summation is taken over all possible cranks m.

Let $N_V(m,t,n)$ denote the number of vector partitions of n with crank congruent to m modulo t counted according to the weights ω. Then we have

Theorem 1.2 (Garvan, [19])**.**

$$N_V(k,5,5n+4) = \frac{p(5n+4)}{5}, \ 0 \le k \le 4,$$

$$N_V(k,7,7n+5) = \frac{p(7n+5)}{7}, \ 0 \le k \le 6,$$

$$N_V(k,11,11n+6) = \frac{p(11n+6)}{11}, \ 0 \le k \le 10.$$

On June 6, 1987 at a student dormitory in the University of Illinois, Urbana-Champaign, G. E. Andrews and F. G. Garvan [4] found the true crank.

Definition 1.3 (Crank)**.** *For a partition π, let $\lambda(\pi)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be*

$$c(\pi) = \begin{cases} \lambda(\pi) & if \ \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi) & if \ \mu(\pi) > 0. \end{cases}$$

Let $M(m,n)$ denote the number of partitions of n with crank m, and let $M(m,t,n)$ denote the number of partitions of n with crank congruent to m modulo t. For $n \le 1$ we set $M(0,0) = 1, M(m,0) = 0$, otherwise $M(0,1) = -1, M(1,1) = M(-1,1) = 1$ and $M(m,1) = 0$. The generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n) a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{1.4}$$

The crank not only leads to combinatorial interpretations of (1.1) and (1.2), but also of (1.3). In fact we have the following result.

Theorem 1.4 (Andrews-Gravan [4]). *With $M(m,t,n)$ defined as above,*

$$M(k,5,5n+4)=\frac{p(5n+4)}{5},\ 0\le k\le 4,$$
$$M(k,7,7n+5)=\frac{p(7n+5)}{7},\ 0\le k\le 6,$$
$$M(k,11,11n+6)=\frac{p(11n+6)}{11},\ 0\le k\le 10.$$

Thus, we see that an observation by Dyson lead to concrete mathematical objects almost 40 years after it was first conjectured. Following the work of Andrews and Garvan, there have been a plethora of results by various authors including Garvan himself, where cranks have been found for many different congruence relations. We see that, the crank turns out to be an interesting object to study.

1.2. **Notations and Preliminaries.** We record here some of the notations and important results that we shall be using throughout our study. Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b):=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2},\ |ab|<1.$$

Our aim in Section 2 is to study the following more general function

$$F_a(q)=\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty}.$$

To understand the theorems mentioned above and also the results of Ramanujan stated in his Lost Notebook and the other results to be discussed here, we need some well-known as well as lesser known results which we shall discuss briefly here. A more detailed discussion can be found in [2, 8] and [14].

One of the most common tools in dealing with q-series identities is the Jacobi Triple Product Identity, given by the following.

Theorem 1.5 (Jacobi's Triple Product Identity). *For $z\neq 0$ and $|x|<1$, we have*

$$\prod_{n=0}^{\infty}\{(1+x^{2n+2})(1+x^{2n+1}z)(1+x^{2n+1}z^{-1})\}=\sum_{n=-\infty}^{\infty}x^{n^2}z^n.$$

For a very well known proof of this result by Andrews, the reader is referred to [1].

It is now easy to verify that Ramanujan's theta function $f(a,b)$ satisfies Jacobi triple product identity

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

and also the elementary identity

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}), \tag{1.6}$$

for any integer n.

We now state a few results that are used to prove some of the results discussed in the next section.

Theorem 1.6 (Ramanujan)**.** *If*

$$A_n := a^n + a^{-n},$$

then

$$\frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} = \frac{1 - \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn} (A_{n+1} - A_n)}{(q;q)_\infty}.$$

It is seen that the above theorem is equivalent to the following theorem.

Theorem 1.7 (Kăc and Wakimoto)**.** *Let* $a_k = (-1)^k q^{k(k+1)/2}$*, then*

$$\frac{(q;q)_\infty^2}{(q/x;q)_\infty (qx;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{a_k(1-x)}{1-xq^k}.$$

Several times in the sequel we shall also employ an addition theorem found in Chapter 16 of Ramanujan's second notebook [6, p. 48, Entry 31].

Lemma 1.8. *If* $U_n = \alpha^{n(n+1)/2} \beta^{n(n-1)/2}$ *and* $V_n = \alpha^{n(n-1)/2} \beta^{n(n=1)/2}$ *for each integer* n*, then*

$$f(U_1, V_1) = \sum_{k=0}^{N-1} U_k f\left(\frac{U_{N+k}}{U_k}, \frac{V_{N-k}}{U_k} \right).$$

Also useful to us are the following famous results, called the quintuple product identity and Winquist's Identity.

Lemma 1.9 (Quintuple Product Identity)**.** *Let* $f(a,b)$ *be defined as above, and let*

$$f(-q) := f(-q, -q^2) = (q;q)_\infty,$$

then

$$f(P^3Q, Q^4/P^3) - P^2 f(Q/P^3, P^3Q^5) = f(-Q^2) \frac{f(-P^2, -Q^2/P^2)}{f(PQ, Q/P)}.$$

In [16], Shawn Cooper studied the history and all the known proofs of the above result till 2006 systematically. In particular, mention may be made of Andrews' nifty proof where he uses the ${}_6\psi_6$ summation formula to derive the above result. It should be mentioned that Sun Kim has given the first combinatorial proof of the quintuple product identity in [21]. Zhu Cao in [12] has developed a new technique using which he proves the quintuple product identity.

Lemma 1.10 (Winquist's Identity)**.** *Following the notations given earlier, we have*

$$\begin{aligned}(a, q/a, b, q/b, ab, q/(ab), a/b, bq/a, q, q; q)_\infty \\ = f(-a^3, -q^3/a^3)\{f(-b^3q, -q^2/b^3) - bf(-b^2q^2, -q/b^3)\} \\ - ab^{-1}f(-b^3, -q^3/b^3)\{f(-a^3q, -q^2/a^3) \\ - af(-a^3q^2, -q/a^3)\}.\end{aligned}$$

Like Lemma 1.9, this result is also very well known and was first used to prove (1.3) by Winquist. Recently Cao, [13] gave a new proof of this result using complex analysis and basic facts about q-series.

We now have almost all the details that we need for a detailed study of various results related to cranks in the following section.

2. Crank for the Partition Function

In this section, we shall systematically study the claims related to cranks, that are found in Ramanujan's Lost Notebook. We closely follow the wonderful exposition of Bruce C. Berndt, Heng Huat Chan, Song Heng Chan and Wen-Chin Liaw [9, 10] and [11]. We also take the help of the wonderful exposition of Andrews and Berndt in [3]. This section is an expansion of [23].

2.1. **Cranks and Dissections in Ramanujan's Lost Notebook.** As mentioned in Section 1, the generating function for $M(m, n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{2.7}$$

Here $|q| < 1$, $|q| < |a| < 1/|q|$. Ramanujan has recorded several entries about cranks, mostly about (2.7). At the top of page 179 in his lost notebook [22], Ramanujan defines a function $F(q)$ and coefficient λ_n, $n \geq 0$ by

$$F(q) := F_a(q) = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n. \tag{2.8}$$

Thus, by (2.7), for $n > 1$,

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n)a^m.$$

Then Ramanujan offers two congruences for $F(q)$. These, like others that are to follow are to be regarded as congruences in the ring of formal power series in the two variables a and q. Before giving these congruences, we give the following definition.

Definition 2.1 (Dissections)**.** *If $P(q) := \sum_{n=0}^{\infty} a_n q^n$ is any power series, then the m-dissection of $P(q)$ is given by*

$$P(q) = \sum_{j=0}^{m-1} \sum_{n_j=0}^{\infty} a_{n_j m+j} q^{n_j m+j}.$$

For congruences we have

$$P(q) \equiv \sum_{j=0}^{m-1} \sum_{n_j=0}^{\infty} a_{n_j m+j,r} q^{n_j m+j} \pmod{r}$$

where r is any positive integer and $a_{n_j m+j} \equiv a_{n_j m+j,r} \pmod{r}$.

We now state the two congruences that were given by Ramanujan below.

Theorem 2.2 (2-dissection)**.** *We have*

$$F_a(\sqrt{q}) \equiv \frac{f(-q^3;-q^5)}{(-q^2;q^2)_\infty} + \left(a - 1 + \frac{1}{a}\right)\sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \left(\operatorname{mod} a^2 + \frac{1}{a^2}\right). \quad (2.9)$$

We note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \pmod{\lambda_2}$ and $a^8 \equiv 1 \pmod{\lambda_2}$. Thus, in (2.9) a behaves like a primitive 8th root of unity modulo λ_2. If we let $a = \exp(2\pi i/8)$ and replace q by q^2 in the definition of a dissection, (2.9) will give the 2-dissection of $F_a(q)$.

Theorem 2.3 (3-dissection)**.** *We have*

$$\begin{aligned}
&F_q(q^{1/3}) \equiv \\
&\frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty} + (a - 1 + 1/a)\, q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\
&\quad + \left(a^2 + \frac{1}{a^2}\right) q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \left(\operatorname{mod} a^3 + 1 + \frac{1}{a^3}\right). \quad (2.10)
\end{aligned}$$

Again we note that, $\lambda_3 = a^3 + 1 + \frac{1}{a^3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1 \pmod{\lambda_3}$. So in (2.10), a behaves like a primitive 9th root of unity. While if we let $a = \exp(2\pi i/9)$ and replace q by q^3 in the definition of a dissection, (2.10) will give the 3-dissection of $F_a(q)$.

In contrast to (2.9) and (2.10), Ramanujan offered the 5-dissection in terms of an equality.

Theorem 2.4 (5-dissection)**.** *We have*

$$\begin{aligned}
F_a(q) = \frac{f(-q^2,-q^3)}{f^2(-q,-q^4)} f^2(-q^5) &- 4\cos^2(2n\pi/5) q^{1/5} \frac{f^2(-q^5)}{f(-q,-q^4)} \\
&+ 2\cos(4n\pi/5) q^{2/5} \frac{f^2(-q^5)}{f(-q^2,-q^3)} \\
&- 2\cos(2n\pi/5) q^{3/5} \frac{f(-q,-q^4)}{f^2(-q^2,-q^3)} f^2(-q^5). \quad (2.11)
\end{aligned}$$

We observe that (2.13) has no term with $q^{4/5}$, which is a reflection of (1.1). In fact, one can replace (2.13) by a congruence and in turn (2.9) and (2.10) by equalities. This is done in [9]. Ramanujan did not specifically give the 7- and 11-dissections of $F_a(q)$ in [22]. However, he vaugely gives some of the coefficients occuring in those dissections. Uniform proofs of these dissections and the others already stated earlier are given in [9].

Ramanujan gives the 5-dissection of $F(q)$ on page 20 of his lost notebook [22]. It is interesting to note that he does not give the alternate form analogous to those of (2.9) amd (2.10), from which the 5-dissection will follow if we set a to be a primitive fifth root of unity. On page 59 in his lost notebook [22], Ramanujan has recorded a quotient of two power series, with the highest power of the numerator being q^{21} and the highest power of the denominator being q^{22}. Underneath he records another power series with the highest power being q^5. Although not claimed by him, the two expressions are equal. This claim was stated in the previous section as Theorem 1.6.

In the following, we shall use the results and notations that we discussed in Section 1. It must be noted that some of these results have been proved by numerous authors, but we follow the exposition of Berndt, Chan, Chan and Liaw [9].

Theorem 2.5. *We have*

$$F_a(q) \equiv \frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty} + \left(a - 1 + \frac{1}{a}\right) q \frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty} \pmod{A_2}. \tag{2.12}$$

We note that (2.12) is equivalent to (2.9) if we replace $\sqrt{q}$ by q. We shall give a proof of this result using a method of *rationalization*. This method does not work in general, but only for those n-dissections where n is *small*.

Proof. Throughout the proof, we assume that $|q| < |a| < 1/|q|$ and we shall also frequently use the facts that $a^4 \equiv -1$ modulo A_2 and that $a^8 \equiv 1$ modulo A_2. We write

$$\frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} = (q;q)_\infty \prod_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} (aq^n)^k \right) \left(\sum_{k=0}^{\infty} (q^n/a)^k \right). \tag{2.13}$$

We now subdivide the series under product sign into residue classes modulo 8 and then sum the series. Using repeatedly congruences modulo 8 for the powers of a, we shall obtain from (2.13) the following

$$\frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} \equiv \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} \prod_{n=1}^{\infty} (1+aq^n)(1+q^n/a) \pmod{A_2}, \tag{2.14}$$

upon multiplying out the polynomials in the product that we obtain and using the congruences for powers of a modulo A_2.

Now, using Lemma 1.8 with $\alpha = a, \beta = q/a$ amd congruences for powers of a modulo A_2, we shall find after some simple manipulations

$$(q;q)_\infty(-aq;q)_\infty(-q/a;q)_\infty \equiv f(-q^6,-q^{10})+(A_1-1)qf(-q^2,-q^{14}) \pmod{A_2}.$$

Using (2.14) in the above, we shall get the desired result. □

Using similar techniques, but with more complicated manipulations, we shall be able to find the following theorem.

Theorem 2.6. *We have*

$$\begin{aligned} F_a(q) \equiv \frac{f(-q^6,-q^{21})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty} + (A_1-1)q\frac{f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty} \\ + A_2q^3\frac{f(-q^3,-q^{24})f(-q^6,-q^{21})}{(q^{27};q^{27})_\infty} \pmod{A_3+1}. \end{aligned}$$

When $a = e^{2\pi i/9}$, the above theorem gives us Theorem 2.3. For the remainder of this section, we shall use the following notation

$$S_n(a) := \sum_{k=-n}^{n} a^k. \tag{2.15}$$

We note that, when p is an odd prime then $S_{(p-1)/2}(a) = a^{(1-p)/2}\Phi_p(a)$, where $\Phi_n(a)$ is the minimal, monic polynomial for a primitive nth root of unity. We now give the 5-dissection in terms of a congruence.

Theorem 2.7. *With $f(-q)$, S_2 and A_n as defined earlier, we have*

$$\begin{aligned} F_a(q) \equiv \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})}f^2(-q^{25}) + (A_1-1)q\frac{f^2(-q^{25})}{f(-q^5,-q^{20})} \\ + A_2q^2\frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} - A_1q^3\frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})}f^2(-q^{25}) \pmod{S_2}. \end{aligned}$$

In his lost notebook [22, p. 58, 59, 182], Ramanujan factored the coefficients of $F_a(q)$ as functions of a. In particular, he sought factors of S_2 in the coefficients. The proof of Theorem 2.7 uses the famous Rogers-Ramanujan continued fraction identities, which we shall omit here.

We now state and prove the following 7-dissection of $F_a(q)$.

Theorem 2.8. *With usual notations defined earlier, we have*

$$\begin{aligned} \frac{(q;q)_\infty}{(qa;q)_\infty(q/a;q)_\infty} \equiv \\ \frac{1}{f(-q^7)}(A^2 + (A-1-1)qAB + A_2q^2B^3 + (A_3+1)q^3AC \\ - A_1q^4BC - (A_2+1)q^6C^2) \pmod{S_3}, \end{aligned}$$

where $A = f(-q^{21},-q^{28})$, $B = f(-q^{35},-q^{14})$ and $C = f(-q^{42},-q^7)$.

Proof. Rationalizing and using Theorem 1.5 we find

$$\begin{aligned}\frac{(q;q)_\infty}{(qa;q)_\infty(q/a;q)_\infty} &\equiv \frac{(q;q)_\infty^2(qa^2;q)_\infty(q/a^2;q)_\infty(qa^3;q)_\infty(q/a^3;q)_\infty}{(q^7;q^7)_\infty} \qquad (2.16)\\ &\equiv \frac{1}{f(-q^7)}\frac{f(-a^2,-q/a^2)}{(1-a^2)}\frac{f(-a^3,-q/a^3}{(1-a^3)} \pmod{S_3}.\end{aligned}$$

Now using Lemma 1.8 with $(\alpha,\beta,N)=(-a^2,-q/a^2,7)$ and $(-a^3,-q/a^3,7)$ respectively, we find that

$$\frac{f(-a^2,-q/a^2)}{(1-a^2)} \equiv A - q\frac{(a^5-a^4)}{(1-a^2)}B + q^3\frac{(a^3-a^6)}{(1-a^2}C \pmod{S_3} \qquad (2.17)$$

and

$$\frac{f(-a^3,-q/a^3)}{(1-a^3)} \equiv A - q\frac{(a^4-a^6)}{(1-a^3)}B + q^3\frac{(a-a^2)}{(1-a^3}C \pmod{S_3}. \qquad (2.18)$$

Now substituting (2.17) and (2.18) in (2.16) and simplifying, we shall finish the proof of the desired result. □

Although there is a result for the 11 dissection as well, we do not discuss it here. We just state the 11-disection of $F_a(q)$ below.

Theorem 2.9. *With A_m and S_5 defined as earlier, we have*

$$\begin{aligned}F_a(q) \equiv \frac{1}{(q^{11};q^{11})_\infty(q^{121};q^{121})_\infty^2}&(ABCD+\{A_1-1\}qA^2BE\\ &+A_2q^2AC^2D+\{A_3+1\}q^3ABD^2\\ &+\{A_2+A_4+1\}q^4ABCE-\{A_2+A_4\}q^5B^2CE\\ &+\{A_1+A_4\}q^7ABDE-\{A_2+A_5+1\}q^{19}CDE^2\\ &-\{A_4+1\}q^9ACDE-\{A_3\}q^{10}BCDE) \pmod{S_5},\end{aligned}$$

where $A=f(-q^{55},-q^{66})$, $B=f(-q^{77},-q^{44})$, $C=f(-q^{88},-q^{33})$, $D=f(-q^{99},-q^{22})$, and $E=f(-q^{110},-q^{11})$.

Although the proof of Theorem 2.9 is not difficult, it is very tedious involving various routine q-series identities and results mentioned in Section 1, so we shall omit it here.

2.2. **Other results from the Lost Notebook.** Apart from the results that we have discussed so far in this section, Ramanujan also recorded many more entires is his lost notebook which pertain to cranks. For example, on page 58 in his lost notebook [22], Ramanujan has written out the first 21 coefficients in the power series representation of the crank $F_a(q)$, where he had incorrectly written the coefficient of q^{21}. On the following page, beginning with the coefficient of q^{13}, Ramanujan listed some (but not necessarily all) of the factors of the coefficients up to q^{26}. He did not indicate why he recorded an incomplete list of such factors. However, it can be noted that in each case he recorded linear factors only when the leading index is ≤ 5.

On pages 179 and 180 in his lost notebook [22], Ramanujan offered ten tables of indices of coefficients λ_n satisfying certain congruences. On page 61 in [22], he offers rougher drafts of nine of these ten tables, where Table 6 is missing. Unlike the tables on pages 179 and 180, no explanations are given on page 61. It is clear that Ramanujan had calculated factors beyond those he has recorded on pages 58 and 59 of his lost notebook as mentioned in the earlier paragraph. In [10], the authors have verified these claims using a computer algebra software. Among other results, O.–Y. Chan [15] has verified all the tables of Ramanujan. We explain below these tables following Berndt, Chan, Chan and Liaw [10].

Table 1. $\lambda_n \equiv 0 \pmod{a^2 + 1/a^2}$

Here Ramanujan indicates which coefficients of λ_n have a_2 as a factor. If we replace q by q^2 in (2.9), we see that Table 1 contains the degree of q for those terms with zero coefficients for both $\dfrac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty}$ and $q\dfrac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty}$. There are 47 such values.

Table 2. $\lambda_n \equiv 1 \pmod{a^2 + 1/a^2}$

Returning again to (2.9) and replacing q by q^2, we see that Ramanujan recorded all the indices of the coefficients that are equal to 1 in the power series expansion of $\dfrac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty}$. There are 27 such values.

Table 3. $\lambda_n \equiv -1 \pmod{a^2 + 1/a^2}$

This table can be understood in a similar way as the previous table. There are 27 such values.

Table 4. $\lambda_n \equiv a - 1 + \frac{1}{a} \pmod{a^2 + 1/a^2}$

Again looking at (2.9), we note that $a - 1 + \frac{1}{a}$ occurs as a factor of the second expression on the right side. Thus replacing q by q^2, Ramanujan records the indices of all coefficients of $q\dfrac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty}$ which are equal to 1. There are 22 such values.

Table 5. $\lambda_n \equiv -\left(a - 1 + \dfrac{1}{a}\right) \pmod{a^2 + 1/a^2}$

This table can also be interpreted in a manner similar to the previous one. There are 23 such values.

Table 6. $\lambda_n \equiv 0 \pmod{a + 1/a}$

Ramanujan here gives those coefficients which have a_1 as a factor. There are only three such values and these values can be discerned from the table on page 59 of the lost notebook.

From the calculation
$$\frac{(q;q)_\infty}{(aq;q)_\infty)(q/a;q)_\infty} \equiv \frac{(q;q)_\infty}{(-q^2;q^2)_\infty} = \frac{f(-q)f(-q^2)}{f(-q^4)} \pmod{a+1/a},$$
we see that in this table Ramanujan has recorded the degree of q for the terms with zero coefficients in the power series expansion of $\frac{f(-q)f(-q^2)}{f(-q^4)}$.

From the next three tables, it is clear from the calculation
$$\frac{(q;q)_\infty}{(aq;q)_\infty)(q/a;q)_\infty} \equiv \frac{(q^2;q^2)_\infty}{(-q^3;q^3)_\infty} = \frac{f(-q^2)f(-q^3)}{f(-q^6)} \pmod{a+1/a},$$
that Ramanujan recorded the degree of q for the terms with coefficients 0, 1 and -1 respectively in the power series expansion of $\frac{f(-q^2)f(-q^3)}{f(-q^6)}$.

Table 7. $\lambda_n \equiv 0 \pmod{a-1+1/a}$

There are 19 such values.

Table 8. $\lambda_n \equiv 1 \pmod{a-1+1/a}$

There are 26 such values.

Table 9. $\lambda_n \equiv -1 \pmod{a-1+1/a}$

There are 26 such values.

Table 10. $\lambda_n \equiv 0 \pmod{a+1+1/a}$

Ramanujan put 2 such values. From the calculation
$$\frac{(q;q)_\infty}{(aq;q)_\infty)(q/a;q)_\infty} \equiv \frac{(q;q)^2_\infty}{(-q^3;q^3)_\infty} = \frac{f^2(-q)}{f(-q^3)} \pmod{a+1+1/a},$$
it is clear that Ramanujan had recorded the degree of q for the terms with zero coefficients in the power series expansion of $\frac{f^2(-q)}{f(-q^3)}$.

The infinite products $\frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty}$, $\quad q\frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty}$, $\quad \frac{f(-q)f(-q^2)}{f(-q^4)}$, $\frac{f(-q^2)f(-q^3)}{f(-q^6)}$ and $\frac{f^2(-q)}{f(-q^3)}$ do not appear to have monotonic coefficients for sufficiently large n. However, if these products are dissected, then we have the following conjectures by Berndt, Chan, Chan and Liaw [10].

Conjecture 2.10. Each component in each of the dissections for the five products given above has monotonic coefficients for powers of q above 600.

The authors have checked this conjecture for $n = 2000$.

Conjecture 2.11. For any positive integers α and β, each component of the $(\alpha+\beta+1)$-dissection of the product
$$\frac{f(-q^\alpha)f(-q^\beta)}{f(-q^{\alpha+\beta+1})}$$
has monotonic coefficients for sufficiently large powers of q.

It is clear that Conjecture 2.10 is a special case of Conjecture 2.11 for the last three infinite products given above when we set $(\alpha, \beta) = (1,2), (2,3)$, and $(1,1)$ respectively. Using the Hardy-Ramanujan circle method, these conjectures have been verified by O. -Y. Chan [15].

On page 182 in his lost notebook [22], Ramanujan returns to the coefficients λ_n in (2.7). He factors λ_n for $1 \leq n \leq n$ as before, but singles out nine particular factors by giving them special notation. These are the factors which occured more than once. Ramanujan uses these factors to compute $p(n)$ which is a special case of (2.7) with $a = 1$. It is possible that through this, Ramanujan may have been searching for results through which he would have been able to give some divisibility criterion of $p(n)$. Ramanujan has left no results related to these factors, and it is up to speculation as to his motives for doing this.

Again on page 59, Ramanujan lists two factors one of which is Theorem 1.6. Further below this he records two series, namely,

$$S_1(a,q) := \frac{1}{1+a} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} + \frac{(-1)^n q^{n(n+1)/2}}{a+q^n} \right) \tag{2.19}$$

and

$$S_2(a,q) := 1 + \sum_{m=1, n=0}^{\infty} (-1)^{m+n} q^{m(m+1)/2+nm} (a_{n+1} + a_n), \tag{2.20}$$

where here $a_0 := 1$. Although no result has been written by Ramanujan, the authors in [10] have found the following theorem.

Theorem 2.12. *With the notations as described above we have*

$$(1+a)S_1(a,q) = S_2(a,q) = F_{-a}(q).$$

Proof. We multiply (2.19) by $(1+a)$ to get

$$\begin{aligned}
(1+a)S_1(a,q) &= 1 + (1+a) \sum_{n=1}^{\infty} \left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} + \frac{(-1)^n q^{n(n+1)/2}}{a+q^n} \right) \\
&= 1 + (1+a) \sum_{n=1}^{\infty} \left(\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} + \frac{(-1)^n q^{n(n-1)/2}}{1+aq^{-n}} \right) \\
&= 1 + (1+a) \sum_{n \neq 0} \frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(1+a)}{1+aq^n} \\
&= \frac{(q;q)_\infty^2}{(-aq;q)_\infty(-q/a;q)_\infty} \qquad (2.21)
\end{aligned}$$

by Theorem 1.7.

Next, by Theorem 1.6 we have

$$S_2(a,q) = 1+ \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+nm}(-(-a)^{n+1} - (-a)^{-n-1} + (-a)^n + (-a)^{-n})$$
$$= \frac{(q;q)_\infty^2}{(-aq;q)_\infty(-q/a;q)_\infty}. \tag{2.22}$$

(2.21) and (2.22) now completes the proof. □

From the preceding discussion, it is clear that Ramanujan was very much interested in finding some general results with the possible intention of determining arithmetical properties of $p(n)$ from them by setting $a = 1$. Although he found many beautiful results, his goal eluded him. The kind of general theorems on the divisibility of λ_n by sums of powers of a appear to be very difficult. Also, a challenging problem is to show that Ramanujan's Table 6 is complete.

2.3. **Cranks – The final problem.** In his last letter to G. H. Hardy, Ramanujan announced a new class of functions which he called mock theta functions, and gave several examples and theorems related to them. The letter was dated 20th January, 1920, a little more than three months before his death. Thus for a long time, it was widely believed that the last problem on which Ramanujan worked on was mock theta functions. But the wide range of topics that are covered in his lost notebook [22] suggests that he had worked on several problems in his death bed. Of course, this is only a speculation and some educated guess that we can make. However, the work of Berndt, Chan, Chan and Liaw [11] has provided evidence that the last problem on which Ramanujan worked on was cranks, although he would not have used this terminology.

We have already seen the various dissections of the crank generating function that Ramanujan had provided. In the preceeding section, we saw various other results of Ramanujan related to the coefficients λ_n. In order to calculate those tables, Ramanujan had to calculate with hand various series up to hundreds of coefficients. Even for Ramanujan, this must have been a tremendous task and we can only wonder what might have led him to such a task. Clearly, there was very little chance that he might have found some nice congruences for these values like (1.1) for example.

In the foregoing discussion of the crank, pages 20, 59–59, 61, 53–64, 70–71, and 179–181 from [22] are cited. Ramanujan's 5-dissection for the crank is given on page 20. The remaining ten pages are devoted to the crank. In fact in pages 58–89, there is some scratch work from which it is very difficult to see where Ramanujan was aiming them at. But it is likely that all these pages were related to the crank. In [11], the authors have remarked that pages 65 (same as page 73), 66, 72, 77, 80–81, and 83–85 are almost surely related to the crank, while they were unable to determine conclusively if the remaining pages pertain to cranks.

In 1983, Ramanujan's widow Janaki told Berndt that there were more pages of Ramanujan's work than the 138 pages of the lost notebook. She claimed that during her husband's funeral service, some gentlemen came and took away some of her husband's papers. The remaining papers of Ramanujan were donated to the University of Madras. It is possible that Ramanujan had two stacks of paper, one for scratch work or work which he did not think complete, and the other where he put down the results in a more complete form. The pages that we have analysed most certainly belonged to the first stack and it was Ramanujan's intention to return to them later. In the time before his death, it is certainly clear that most of Ramanujan's mathematical thoughts had been only on one topic - cranks.

However, one thing is clear that Ramanujan probably didn't think of the crank or rank as we think of it now. We do not know with certainty whether or not Ramanujan thought combinatorially about the crank. Since his notebooks contain very little words and also since he was in his death bed so he didn't waste his time on definitions and observations which might have been obvious to him; so we are not certain about the extent to which Ramanujan thought about these objects. It is clear from some of his published papers that Ramanujan was an excellent combinatorial thinker and it would not be surprising if he had many combinatorial insights about the crank. But, since there is not much recorded history from this period of his life, we can at best only speculate.

3. Concluding Remarks

Although we have focused here on only one aspect of Ramanujan's work related to crank, this is by no means the complete picture. Recent work by many distinguished mathematicians have shed light on many different aspects of Ramanujan's work related to cranks. We plan to address those issues in a subsequent article. The interested reader wanting to know more about Ramanujan's mathematics can look at [7]. For more information on cranks and its story, a good place is [18]. For some more work of Garvan related to cranks, the interested reader can look into [19] and [20]. There are some more results in the Lost Notebook related to cranks, Andrews and Berndt have provided an excellent exposition of those in Chapters 2, 3 and 4 of [3].

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INTEGRABILITY OF CONTINUOUS FUNCTIONS

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Abstract. A new concise proof of the theorem on integrability of continuous functions is presented. The proof utilizes only elementary notions of continuity and lower and upper sums. Uniform continuity of a continuous function is proved by the same technique.

1. Introduction

Treatment of integrability of continuous functions varies from one calculus book to another. The theorem in [1, Ch.5.6, Th.5.6.3] is stated without proof and in the remark to [1, Ch.5.6, Def.5.6.1] the readers are referred to more advanced calculus, thereby avoiding the discussion. For proving the integrability, other texts, such as [3], use uniform continuity results. Tracing this development backwards, we need to prove that a function continuous on a finite closed interval is uniformly continuous. This proof, in turn, relies on the Bolzano-Weierstrass theorem which guarantees that every bounded sequence possesses a convergent subsequence. Of course, subsequences need to be introduced prior to this theorem.

Another approach is employed by [4]. While the theorem on integrability of continuous functions is presented in Chapter 13, the proof is postponed until the end of Chapter 14. In the interim, many definitions and theorems are presented in order to facilitate the required proof. Upper and lower integrals are introduced as functions of upper limits. Two key results are taken for granted, namely that continuous functions are bounded on closed intervals, and that two functions with equal derivatives must differ by a constant. In each of these standard approaches all of the preliminary theorems take a significant amount of time to understand and digest.

It is interesting to find out that the proof of integrability of continuous functions in [2] takes only a couple of pages. Instead of discussing why students have to struggle with so many pages today, we present a different concise proof of integrability of a function continuous on a finite closed interval.

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2. Integrability of continuous functions

We assume that the reader is familiar with the basic definitions, but, nevertheless, we review them to establish notation.

Definition 2.1. A *partition* of a bounded interval $[a, b]$, where $a < b$, is a finite sequence of values $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

Each interval $[x_{i-1}, x_i]$ is called a subinterval of the partition. Let $\xi_i \in [x_{i-1}, x_i]$ for each $i = 1, 2, \ldots, n$. We investigate the sum $\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$. Similarly, we could pick some other $\xi_i' \in [x_{i-1}, x_i]$ and form the sum $\sum_{i=1}^{n} f(\xi_i')(x_i - x_{i-1})$. These sums are called *integral sums.* We note that the most frequently used definition in the Riemann sense uses the norm of a partition $\max_{i=1,\ldots,n}(x_i - x_{i-1})$. If there is a number I such that for every $\epsilon_r > 0$ there is a δ for which the norm of the partition being less than this δ implies $\left|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - I\right| < \epsilon_r$, where $\xi_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, we call I the Riemann integral of f over $[a, b]$.

But a more convenient approach for our purposes uses the *upper* and *lower sums* that are due to [2].

Definition 2.2. Let f be a function defined on a bounded interval $[a, b]$, $a < b$. Let $a = x_0, x_1, \ldots, x_n = b$ be a partition of $[a, b]$. If f is bounded on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, we denote by S_i the supremum of $f(x)$ over $[x_{i-1}, x_i]$ and by s_i the infimum of $f(x)$ over $[x_{i-1}, x_i]$. The *upper* and *lower sums* of f are

$$\sum_{i=1}^{n} S_i(x_i - x_{i-1}) \qquad \text{and} \qquad \sum_{i=1}^{n} s_i(x_i - x_{i-1}), \qquad \text{respectively.}$$

Obviously $\sum_{i=1}^{n} S_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} s_i(x_i - x_{i-1})$. It is also intuitively easy to show that an upper sum for any partition is greater than or equal to a lower sum for any other partition. Nevertheless, we refer to [4, Ch.13, Th.1] or [3, Ch.6, Th.32.4].

Definition 2.3. Let f be a function defined on a bounded interval $[a, b]$, $a < b$. If the infimum of the upper sums taken over all possible partitions of $[a, b]$ is equal to the supremum of all the lower sums over all possible partitions, we denote their common value by I and call it a *Darboux integral.*

So far, the definitions have been general, from now on we restrict ourselves to continuous functions. The proof of the following theorem may seem complex but we use only basic notions, such as continuity, limit of an increasing sequence, supremum, and upper and lower sums. Drawing an x-axis and plotting the points of the increasing sequence on it will make the proof easier to understand.

Theorem 2.1. *If f is continuous on $[a, b]$, $a < b$, then for any $\epsilon > 0$ there is a partition $a = x_0, x_1, \ldots, x_n = b$ of $[a, b]$ such that for any $i = 1, 2, \ldots, n$ if $x \in [x_{i-1}, x_i]$ then $|f(x) - f(x_{i-1})| < \epsilon$.*

Proof. Let $\epsilon > 0$. We construct an increasing sequence $a = x_0, x_1, \dots$ inductively. Let $a \le x_{i-1} < b$. Starting with $i = 1$, we define

$$y_i = \sup\{x \le b : |f(t) - f(x_{i-1})| < \epsilon \quad \text{for all} \quad t \in [x_{i-1}, x]\}.$$

Since $|f(t) - f(x_{i-1})|$ is continuous at $t = x_{i-1}$, there is a $\delta_i > 0$ such that $x_{i-1} + \delta_i < b$, $|f(t) - f(x_{i-1})| < \epsilon$ for $t \in [x_{i-1}, x_{i-1} + \delta_i]$. It follows that $x_{i-1} < y_i$. If $y_i < b$ or $y_i = b$ and $|f(b) - f(x_{i-1})| = \epsilon$, we take the midpoint $x_i = (x_{i-1} + y_i)/2$ (to avoid the use of the Heine theorem). If $y_i = b$ and $|f(b) - f(x_{i-1})| < \epsilon$, we set $x_i = b$ and the sequence stops.

We want to show that the sequence $a = x_0, x_1, \dots$ is finite. If the sequence $a = x_0, x_1, \dots$ is infinite, it is increasing and bounded by b, it converges to some limit L. We show it leads to a contradiction.

First we assume $L < b$. Since f is continuous at L, there is a $\delta > 0$ such that $L + \delta < b$ and $|f(t) - f(L)| < \epsilon/2$ for all t for which $|t - L| < \delta$, $t \in [a, b]$. Since $x_0, x_1, x_2, \dots$ converges to L, there is an N such that $0 < |L - x_N| < \delta/2$. If $t \in [L - \delta/2, L + \delta]$, then $|f(t) - f(L)| < \epsilon/2$ for such t and for x_N specifically. The next point of the sequence, x_{N+1}, is greater than or equal to the midpoint $(x_N + L + \delta)/2 > (L - \delta/2 + L + \delta)/2 = L + \delta/4 > L$, which is a contradiction because

$$\begin{aligned}|f(t) - f(x_N)| &= |f(t) - f(L) + f(L) - f(x_N)| \\ &\le |f(t) - f(L)| + |f(L) - f(x_N)| < \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

for $t < x_{N+1}$. Thus L is not the limit.

Similarly, if $L = b$, there is a $\delta > 0$ such that $|f(t) - f(b)| < \epsilon/2$ for all t for which $b - t < \delta$, $t \in [a, b]$. Since $x_0, x_1, x_2, \dots$ converges to b, there is an N such that $0 < b - x_N < \delta/2$. The next point is $x_{N+1} = b$ because

$$|f(t) - f(b)| = |f(t) - f(x_N)| + |f(x_N) - f(b)| < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $t \in [x_N, b]$. It means the sequence $x_0, x_1, \dots$ is finite. $\square$

Theorem 2.2. *If f is continuous on $[a, b]$, $a < b$, then for any $\epsilon_d > 0$ there is a partition of $[a, b]$ for which the difference between the upper and lower sums is less than ϵ_d.*

Proof. Let $\epsilon_d > 0$. By Theorem 2.1 with $\epsilon = \epsilon_d/(4(b-a))$ we determine a partition $a = x_0, x_1, \dots, x_n = b$. We recall Theorem 2.1 again and the definition of x_i, given x_{i-1}, and see that, if $x \in [x_{i-1}, x_i]$, then $|f(x) - f(x_{i-1})| \le \epsilon_d/(4(b-a))$. Thus $f(x)$ is bounded on $[x_{i-1}, x_i]$. Therefore both the supremum S_i and infimum s_i of $f(x)$ over $[x_{i-1}, x_i]$ are finite.

Since the number of subintervals n in the partition is finite we can pick a point $\xi_i \in [x_{i-1}, x_i]$ in each subinterval such that $S_i < f(\xi_i) + \frac{\epsilon_d}{4(b-a)}$ and a point $\xi_i' \in [x_{i-1}, x_i]$ in each subinterval such that $s_i > f(\xi_i') - \frac{\epsilon_d}{4(b-a)}$. Then

$$\begin{aligned}
&\sum_{i=1}^{n} S_i(x_i - x_{i-1}) - \sum_{i=1}^{n} s_i(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} (f(\xi_i) + \frac{\epsilon_d}{4(b-a)})(x_i - x_{i-1}) - \sum_{i=1}^{n} (f(\xi_i') - \frac{\epsilon_d}{4(b-a)})(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \big(f(\xi_i) - f(\xi_i')\big)(x_i - x_{i-1}) + 2\sum_{i=1}^{n} \frac{\epsilon_d}{4(b-a)})(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} \big|f(\xi_i) - f(x_{i-1}) + f(x_{i-1}) - f(\xi_i')\big|(x_i - x_{i-1}) + \frac{\epsilon_d}{2} \\
&\leq \sum_{i=1}^{n} \big|f(\xi_i) - f(x_{i-1})\big|(x_i - x_{i-1}) + \sum_{i=1}^{n} \big|f(x_{i-1}) - f(\xi_i')\big|(x_i - x_{i-1}) + \frac{\epsilon_d}{2} \\
&\leq 2\frac{\epsilon_d}{4(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1}) + \frac{\epsilon_d}{2} \leq \frac{\epsilon_d}{2} + \frac{\epsilon_d}{2} = \epsilon_d,
\end{aligned}$$

thus completing the proof. □

3. Uniform continuity

Definition 3.1. Let f be a function defined on a bounded interval $[a, b]$, $a < b$. The function f is called *uniformly continuous* on $[a, b]$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $u_1, u_2 \in [a, b]$ if $|u_2 - u_1| < \delta$ holds then $|f(u_2) - f(u_1)| < \epsilon$.

Theorem 3.1. *A continuous function defined on a closed finite interval* $[a, b]$ *is uniformly continuous.*

Proof. Let $\epsilon_u > 0$. We set $\epsilon = \epsilon_u/3$ and use Theorem 2.1 to form a partition $a = x_0, x_1, \ldots, x_n = b$ of $[a, b]$.

We can, therefore, pick a smallest subinterval, say $[x_{k-1}, x_k]$, and we can set $\delta = x_k - x_{k-1}$. If we take any interval $[c, d] \subset [a, b]$ of length less than δ, it is either contained in just one subinterval $[x_{i-1}, x_i]$ of the partition or in the union $[x_{i-1}, x_{i+1}] = [x_{i-1}, x_i] \cup [x_i, x_{i+1}]$ of two contiguous subintervals of the partition.

If u_1, u_2 are given with $a \leq u_1 < u_2 \leq b$ and $u_2 - u_1 < \delta$, then either both $u_1, u_2 \in [x_{i-1}, x_i]$ for some i or $u_1 \in [x_{i-1}, x_i]$, $u_2 \in [x_i, x_{i+1}]$ for some i. In the former case we have

$$|f(u_2) - f(u_1)| \leq |f(u_2) - f(x_{i-1})| + |f(x_{i-1}) - f(u_1)| \leq 2\epsilon < \epsilon_u,$$

while in the latter case we get

$$\begin{aligned}
&|f(u_2) - f(u_1)| \leq |f(u_2) - f(x_i)| + |f(x_i) - f(u_1)| \\
&\leq \epsilon + |f(x_i) - f(x_{i-1})| + |f(x_{i-1}) - f(u_1)| \leq 3\epsilon = \epsilon_u,
\end{aligned}$$

and hence the theorem is proved. □

The uniform continuity provides a standard way of proving that a function continuous on a closed finite interval is Riemann integrable.

The technique in Theorem 2.1 can be used to prove some well known results as an exercise.

Exercise 1. If f is continuous on $[a, b]$, $a < b$, it is bounded on $[a, b]$.

Proof. Form a partition as in Theorem 2.1, find a bound on each of the n subintervals and take the maximum of all the n bounds.

Exercise 2. One obtains larger subintervals if one uses Heine theorem and defines $x_i = \sup\{x \le b : |f(t) - f(x_{i-1})| < \epsilon_1 \text{ for all } t \in [x_{i-1}, x]\}$ without using the midpoints of intervals.

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A VERY SIMPLE PROOF OF STIRLING'S FORMULA

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Abstract. We present a short proof of Stirling's formula that uses only a basic knowledge of calculus.

1. Introduction

Numerous proofs [1, 2, 3, 6, 7, 8] of Stirling's formula of varying levels of sophistication have appeared in the literature. For instance, the proof in [6] invokes Lebesgue's dominated convergence theorem while [7] uses Poisson distribution from probability theory; familiarity with these ideas cannot be expected of students in first year calculus courses. While the ideas in [8] are elementary, the work done in achieving a slight improvement to the barebones Stirling's formula does not clearly convey the basic idea behind this asymptotic formula. In this note, we present a short proof of Stirling's formula that uses only a basic knowledge of calculus.

We shall first prove that

$$n! \sim C\sqrt{n}\left(\frac{n}{e}\right)^n \tag{1.1}$$

for some constant C and subsequently prove that $C = \sqrt{2\pi}$. It is a curious fact that the discovery of (1.1) is due to de Moivre and Stirling's contribution to Stirling's formula is recognising that $C = \sqrt{2\pi}$. The first rigorous proof that the constant is $\sqrt{2\pi}$ is to be found in de Moivre's monograph "Miscellanea Analytica" [5] consisting of results about summation of series. This proof is also the widely known proof that uses Wallis's product formula. While Stirling offers no proof of his claim, it is likely that Stirling's own reasoning involves Wallis's formula. In his extensive analyses of Stirling's works, I. Tweddle [9] suggests that the digits of $\sqrt{\pi}$ may have been known to Stirling; Stirling computes the first nine places of $\sqrt{\pi}$ using Bessel's interpolation formula [9, p. 244] but "he certainly offers no proof here for the introduction of π".

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Key words and phrases: Stirling's formula, de Moivre's formula, Taylor series

2. Proof of Stirling's Formula

We begin with the following easy observation about the logarithm function

2.1. **Observation.** For $|x| < 1$, we have the following convergent Taylor expansion centered at 0

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

This tells us that, for $|x| < 1$, we have

$$|-\log(1-x) - x| \leqslant \sum_{k=2}^{\infty} |x|^k = \frac{|x|^2}{1-|x|}.$$

Changing x to $-x$ gives us

$$|\log(1+x) - x| \leqslant \frac{|x|^2}{1-|x|}. \tag{2.1}$$

2.2. **The proof.** We now study the function $\log n!$ which is

$$\sum_{k=1}^{n} \log k = \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log t \,\mathrm{d}t + \sum_{k=1}^{n} \left(\log k - \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \log t \,\mathrm{d}t \right). \tag{2.2}$$

The first integral is evaluated easily

$$I_1 := \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log t \,\mathrm{d}t = \left(n + \frac{1}{2}\right) \log\left(n + \frac{1}{2}\right) - n + \frac{\log 2}{2}. \tag{2.3}$$

Now, note that

$$\log\left(n + \frac{1}{2}\right) = \log n + \log\left(1 + \frac{1}{2n}\right)$$

and our observation (2.1) gives

$$\left|\log\left(1 + \frac{1}{2n}\right) - \frac{1}{2n}\right| \leqslant \frac{\left(\frac{1}{2n}\right)^2}{1 - \frac{1}{2n}} \leqslant \frac{1}{2n^2}$$

where the last inequality holds since $1 - (2n)^{-1} \geqslant \frac{1}{2}$ for $n \geqslant 1$. Hence we have

$$\log\left(n + \frac{1}{2}\right) = \log n + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

so that the integral (2.3) becomes

$$I_1 = \left(n + \frac{1}{2}\right) \log n - n + \frac{1 + \log 2}{2} + O\left(\frac{1}{n}\right). \tag{2.4}$$

Let us now consider the sum on the right hand side of (2.2). The kth summand is

$$\begin{aligned}
&\log k - \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \log t \, \mathrm{d}t \\
&= \log k - \left\{ \left(k+\frac{1}{2}\right) \log\left(k+\frac{1}{2}\right) - \left(k-\frac{1}{2}\right)\log\left(k-\frac{1}{2}\right) - 1 \right\} \\
&= \log k - \left\{ k \log\left(\frac{k+1/2}{k-1/2}\right) + \frac{1}{2}\log\left(k^2 - \frac{1}{4}\right) - 1 \right\}. \qquad (2.5)
\end{aligned}$$

Now, when $|x| < 1$, considering the Taylor expansions of the functions $\log(1+x)$ and $\log(1-x)$, we have

$$\log\left(\frac{1+x}{1-x}\right) = 2x + O(x^3) \qquad (2.6)$$

so that we get

$$\log\left(\frac{k+1/2}{k-1/2}\right) = \log\left(\frac{1+1/2k}{1-1/2k}\right) = \frac{1}{k} + O(k^{-3}). \qquad (2.7)$$

Finally, writing

$$\log\left(k^2 - \frac{1}{4}\right) = \log k^2 + \log\left(1 - \frac{1}{4k^2}\right), \qquad (2.8)$$

for $k \geqslant 1$, we have from our observation (2.1) that

$$\left|\log\left(1 - \frac{1}{4k^2}\right) + \frac{1}{4k^2}\right| \leqslant \frac{\frac{1}{16k^4}}{1 - \frac{1}{4k^2}} \leqslant \frac{1}{12k^4} \qquad (2.9)$$

so that $\log(1 - \frac{1}{4k^2}) = O(k^{-2})$. Thus, (2.8) becomes

$$\log\left(k^2 - \frac{1}{4}\right) = 2\log k + O(k^{-2}). \qquad (2.10)$$

Putting (2.7) and (2.10) together into (2.5), our summand now becomes

$$\begin{aligned}
&\log k - \left\{ k\left(\frac{1}{k} + O(k^{-3})\right) + \frac{1}{2}\left(2\log k + O(k^{-2})\right) - 1 \right\} \\
&= O(k^{-2}).
\end{aligned}$$

In particular, the sum converges as $n \to \infty$ by the comparison test. Thus, we estimate the sum on the right hand side of (2.2) by passing to the infinite sum: writing S_k for the kth summand, we get

$$\begin{aligned}
\sum_{k=1}^{n} S_k &= \sum_{k>0} S_k - \sum_{k>n} S_k \\
&= A + O(n^{-1})
\end{aligned}$$

for some constant A, since $\sum_{k>n} k^{-2} = O(n^{-1})$ by comparison with the integral. Putting all this together, we have proved that, for some constant A',

$$\log n! = \left(n + \frac{1}{2}\right)\log(n) - n + A' + r_n \qquad (2.11)$$

where $r_n = O(n^{-1})$. Exponentiating, we get

$$n! \sim C\sqrt{n}\left(\frac{n}{e}\right)^n \tag{2.12}$$

for some constant C.

2.3. **The constant.** We describe a method to arrive at the constant C in (1.1). For every non-negative integer n, consider the integral

$$I_n := \int_0^{\pi/2} \sin^n \theta \, \mathrm{d}\theta. \tag{2.13}$$

By integrating by parts, we have the following recurrence

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{2.14}$$

In particular, as $n \to \infty$, we must have

$$\frac{I_n}{I_{n-2}} \to 1.$$

It is immediate from (2.13) that $I_0 = \frac{\pi}{2}$ and $I_1 = 1$. Thus, using (2.14), an induction on n shows that

$$I_{2n} = \binom{2n}{n}\frac{\pi}{2^{2n+1}} \text{ and } I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Now, since $0 \leqslant \sin\theta \leqslant 1$ when $\theta \in [0, \frac{\pi}{2}]$, it follows that $I_{n-2} \geqslant I_{n-1} \geqslant I_n$ and so

$$\lim_{n\to\infty} \frac{I_n}{I_{n-1}} = 1.$$

Computing along the even subsequence, we are immediately led to a limit due to de Moivre and Stirling [4, pp. 243–254]

$$1 = \lim_{n\to\infty} \frac{I_{2n}}{I_{2n-1}} = \lim_{n\to\infty} \frac{\pi n}{2^{4n}}\binom{2n}{n}^2. \tag{2.15}$$

Now, if the asymptotic formula (1.1) holds, then, we must have that $C = \sqrt{2\pi}$ in (1.1). To see this, we use the de Moivre's formula (2.15)

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2^{2n}}\binom{2n}{n} = \frac{1}{\sqrt{\pi}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2^{2n}} \frac{C\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}{\left(C\sqrt{n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{1}{\sqrt{\pi}}$$

$$\lim_{n\to\infty} \frac{\sqrt{2}}{C} = \frac{1}{\sqrt{\pi}}$$

thus proving our claim.

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DIOPHANTINE EQUATIONS OF THE FORM $F(X) = G(Y)$ - AN EXPOSITION

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Abstract. In this article, we discuss some of the present-day methods used in dealing with Diophantine equations of the form $f(x) = g(y)$ for solutions in integers x, y, where f, g are polynomials with integer coefficients. We mention mainly results proved in the last two decades and, point out in the last section some unsolved questions arising from an observation of Ramanujan. This is a survey meant for non-experts and students. The last section has some new results for which we give complete proofs.

1. Introduction

The theory of Diophantine equations is a branch of number theory which deals with the solutions of integer polynomial equations in integers; more generally, it is customary to also consider solutions in rational numbers. A unique aspect of this subject is that it is most often very easy to state the problem but it is very difficult to guess if the problem is trivial to solve or, if it needs deep ideas from other branches of mathematics. A well-known example is Fermat's Last theorem which was solved more than 350 years after it was stated, using very deep ideas from various branches of mathematics.

Questions on counting often involve finding integer solutions of equations of the form $f(x) = g(y)$ for integral polynomials f, g. An example comes from counting lattice points in generalized octahedra. The number of integral points on the n-dimensional octahedron $|x_1| + |x_2| + \cdots + |x_n| \leq r$ is given by the expression $p_n(r) = \sum_{i=0}^{n} 2^i \binom{n}{i}\binom{r}{i}$ and the question of whether two octahedra of different dimensions m, n can contain the same number of integral points becomes equivalent to the solvability of $p_m(x) = p_n(y)$ in integers x, y. Another natural question is to determine, for fixed distinct natural numbers m, n, how often $\binom{x}{m} = \binom{y}{n}$ in integers x and y. Bilu, Rakaczki, Stoll and Tichy ([3]) have established precise finiteness

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results for these equations. One more result of this kind proved by Stoll & Tichy is that for the sequences of classical orthogonal polynomials $p_m(x)$ like the Laguerre, Legendre and Hermite polynomials, an equation of the form $ap_m(x) + bp_n(y) = c$ with $a, b, c \in Q$ and $ab \neq 0$ and $m > n \geq 4$ has only finitely many solutions in integers x, y. Yet another classical problem is to find products of consecutive integers which are perfect powers. Erdős and Selfridge ([9]) proved in 1975 that any finite product of consecutive integers can never be a perfect power. In other words, the Diophantine equation

$$x(x+1)(x+2)\cdots(x+m-1) = y^n$$

does not have any nontrivial solution in integers when $m,\ n > 1$. In a later section, we will outline a proof of this classical result. Another example is the question as to which natural numbers have all their digits to be 1 with respect to two different bases. This is equivalent to solving

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}$$

in natural numbers $x, y > 1$ for some $m, n > 2$. For example, it was Observed by Goormaghtigh nearly a century ago that 31 and 8191 have this property

$$(11111)_2 = (111)_5\ ,\ (111)_{90} = 2^{13} - 1.$$

However, it is still unknown whether there are only finitely many solutions in all variables x, y, m, n. In fact, no other solutions are known.

Let us begin by discussing a beautiful theorem.

2. Erdős-Selfridge theorem

As mentioned in the introduction, Erdős and Selfridge ([9]) proved in 1975 that *any finite product of consecutive integers can never be a perfect r-th power for any* $r > 1$. They used a classical theorem due to Sylvester which asserts (see also [25]):

Theorem 2.1 (Sylvester)**.** *Any set of k consecutive positive integers, with the smallest $> k$, contains a multiple of a prime $> k$.*

Note that the special case of this, when the numbers are $k+1, \cdots, 2k$, is known as Bertrand's postulate. We outline the proof of the Erdős-Selfridge theorem for the case of squares.

Suppose $(n+1)(n+2)\cdots(n+k) = y^2$ in positive integers n, y where $k \geq 2$. Write $n + i = a_i x_i^2$ with a_i square-free. Clearly each prime factor of each a_i is less than k. *The key point is to show that all these a_i's must be distinct.*

Now, if $n < k$ then Bertrand's postulate gives a prime p between $[(n+k)/2]$ and $n + k$. As $n < (n+k)/2$, the prime p is one of the terms $n + i (1 \leq i \leq k)$ and, therefore, p^2 cannot divide the product $(n+1)(n+2)\cdots(n+k)$ which is a contradiction.

If $n \geq k$ then Sylvester's theorem gives a prime number $q > k$ which divides the product $(n+1)(n+2)\cdots(n+k) = y^2$. So, q^2 divides some $n+i$ and hence $n+i \geq q^2 \geq (k+1)^2$. Therefore, $n \geq k^2+1$; that is, $n > k^2$.
But, if $a_i = a_j$ for some $i > j$, then

$$k > (n+i) - (n+j) = a_j(x_i^2 - x_j^2) > 2a_jx_j \geq 2\sqrt{a_jx_j^2} = 2\sqrt{n+j} > \sqrt{n},$$

a contradiction.

Finally, one easily bounds the product $a_1a_2\cdots a_k$ below by the product of the first k s quare-free numbers and one uses the fact that each prime divisor of each a_i is $< k$ to bound the product of the a_i's from above to get a contradiction.

3. Modus operandi-Siegel's theorem

Many results appearing in the last ten years have been made possible by a beautiful theorem of Bilu & Tichy which was built out of deep ideas of Michael Fried (see [11], [12], [13]). To motivate these results, we first need to recall the basic classical theorem due to C. L. Siegel. More generally, consider a polynomial $F \in \mathbb{Z}[X,Y]$. Recall that F is said to be absolutely irreducible if it is irreducible over the field $\overline{\mathbb{Q}}$ of algebraic numbers. Then, the celebrated 1929 theorem due to Siegel ([26]) asserts:

Theorem 3.1 (Siegel, 1929)**.** *If $F \in \mathbb{Z}[X,Y]$ is absolutely irreducible and the curve $F = 0$ has genus > 0, then the number of integral points on the curve is finite. Further, the finiteness of the number of integer points holds good except when the (projective completion of the) curve defined by $F = 0$ has genus 0 and at most 2 points at infinity.*

For the rational solutions, a finiteness theorem is the following one due to Faltings ([10]).

Theorem 3.2 (Faltings, 1983)**.** *If $F \in \mathbb{Q}[X,Y]$ is irreducible, and if the curve $F = 0$ has genus > 1, then the equation $F(x,y) = 0$ has only finitely many solutiuons $x, y \in \mathbb{Q}$.*

As this article is meant for students also, we will explain the notions involved and how one computes the relevant things. Before that, we explain how the theorem is used in proving finiteness of solutions of Diophantine equations. It should be pointed out that Siegel's theorem is, unfortunately, ineffective - in other words, no explicit bound for x, y can be determined beyond which there are no solutions.
To determine finiteness or otherwise of the integral solutions of a given equation $F(x,y) = 0$ using Siegel's theorem, one splits $F(x,y)$ into irreducible factors in $\mathbb{Q}[x,y]$, and for each factor which is irreducible over $\bar{\mathbb{Q}}$ one finds the genus and the number of points at infinity. Then, for each of those factors which have genus 0

and ≤ 2 points at infinity, one can try to determine whether the number of integral solutions is finite or not.

4. Genus of the curves $f(X) = g(Y)$

In order to keep the discussion as elementary as possible, we do not go into general definitions of genus but merely recall how the genus is determined. As mentioned at the outset, we will stick to curves of the form $f(X) = g(Y)$. Recall that a complex zero of f' is called a stationary point of the polynomial f; we denote by S_f, the set of stationary points of f. The classical Riemann-Hurwitz formula can be applied to the function field extension $\mathbb{C}(X,Y)/\mathbb{C}(Y)$ given by $f(X) - g(Y)$ to deduce ([4]) the following explicit result.

Lemma 4.1. *Let $f, g \in \mathbb{C}[X]$ be two polynomials such that the polynomial $f(X) - g(Y) \in \mathbb{C}[X,Y]$ in two variables is irreducible. Suppose the stationary points of f and g are simple. For each stationary point $a \in S_f$, define*

$$r_a := |\{b \in S_g : f(a) = g(b)\}|.$$

Then, the genus g of the curve $f(X) = g(Y)$ is given by

$$2g = \sum_{a \in S_f} (\deg(g) - 2r_a) - \deg(f) + 2 - GCD(\deg(f), \deg(g)).$$

Let us see how useful this is by means of the following simple example.

Example 4.2. *For any $\lambda \in \mathbb{C}^*$, consider the equation*

$$x(x+1) = \lambda y(y+1)(y+2). \tag{4.1}$$

If $f(X) = X(x+1)$ and $g(Y) = Y(Y+1)(Y+2)$ then $S_f = \{-1/2\}$ and $S_g = \{2\sqrt{3}/9, -2\sqrt{3}/9\}$. Now $g(\pm 2\sqrt{3}/9) = f(-1/2)$ if, and only if, $\lambda = \pm 3\sqrt{3}/8$, and in this case $r_{-1/2} = 1$. Therefore, the genus is 0 for $\lambda = \pm 3\sqrt{3}/8$, and 1 for other λ. Hence, it follows from Siegel's theorem that the equation (4.1) has only finitely many integral solutions unless $\lambda = \pm 3\sqrt{3}/8$.

Definition 4.3 (Points at infinity)**.** *Given $F \in \mathbb{Q}[X,Y]$, one may homogenize this polynomial to a homogeneous polynomial of three variables X, Y, Z. Then, the points in the projective space corresponding to the solutions of $F(x,y,0) = 0$ are called the points at infinity of the curve $F = 0$.*

In Siegel's theorem, finiteness of integral solutions follows if either genus of the curve is positive or, if the genus is 0 but there are at least three points at infinity. Therefore, to check for finiteness of integral solutions, one computes the genus and, when it is 0 and there are at most two points at infinity, check separately for finiteness of integral solutions.

5. Congruent numbers

In this section, we discuss the classical congruent number problem which leads to an equation of the form $f(x) = g(y)$; more precisely, an equation of the form $y^2 = x(x+n)(x-n)$ where n is a positive integer. However, in this case, we will be interested in solutions in *rational* numbers. The curve $Y^2 - X(X+n)(X-n) = 0$ arising in this manner has genus 1 as we can easily check from the above lemma on genus computation. Therefore, Siegel's theorem shows finiteness of the number of integer solutions. However, Faltings's theorem quoted above for finiteness of rational solutions does not apply as the genus is 1.

A positive integer d is said to be a *congruent number* if there is a right-angled triangle with rational sides and area d.

It is an ancient Greek problem to determine which positive integers are congruent numbers and which are not. Firstly, the property of being a congruent number for a positive integer d turns out to be equivalent to the existence of an arithmetic progression of three terms which are all squares of rational numbers having the common difference d. Here is an argument to show the equivalence.

Indeed, let $u \leq v < w$ be the sides of a right triangle with sides rational. Then $x = w/2$ is such that $(v-u)^2/4, w^2/4, (u+v)^2/4$ form an arithmetic progression. Conversely, if $x^2 - d = y^2, x^2, x^2 + d = z^2$ are three rational squares in arithmetic progression, then $z-y, z+y$ are the legs of a right angled triangle with legs rational, area $(z^2 - y^2)/2 = d$ and rational hypotenuse $2x$ because $2(y^2 + z^2) = 4x^2$.

For example, $5, 6$ and 7 are congruent numbers. This can be seen by considering the following three right-angled triangles:

with sides $3/2, 20/3$ and $41/6$ having area 5;

with sides $3, 4$ and 5 having area 6;

and, with sides $35/12, 24/5$ and $337/60$ having area 7.

On the other hand, $1, 2$ and 3 are not congruent numbers. The fact that $1, 2$ are not congruent numbers is essentially equivalent to Fermat's last theorem for the exponent 4.

Indeed, if $a^2 + b^2 = c^2$, $\frac{1}{2}ab = 1$ for some rational numbers a, b, c then $x = c/2, y = |a^2 - b^2|/4$ are rational numbers satisfying $y^2 = x^4 - 1$. Similarly, if $a^2 + b^2 = c^2, \frac{1}{2}ab = 2$ for rational numbers a, b, c, then $x = a/2, y = ac/4$ are rational numbers satisfying $y^2 = x^4 + 1$.

These equations reduce to the equation $x^4 \pm z^4 = y^2$ over integers which was proved by Fermat, using the method of descent, not to have nontrivial solutions. The unsolvability of $y^2 = x^4 \pm 1$ in rational numbers is exactly equivalent to showing $1, 2$ are not congruent.

In fact $y^2 = x^4 - 1$ for rational x, y gives a right-angled triangle with sides $y/x, 2x/y, (x^4+1)/xy$ and area 1. Similarly, $y^2 = x^4 + 1$ for rational x, y gives a right-angled triangle with sides $2x, 2/x, 2y/x$ and area 2.

Though it is an ancient problem to determine which natural numbers are congruent, it is only in late 20th century that substantial progress has been made.

The rephrasing in terms of arithmetic progressions of squares emphasizes a connection of the problem with rational solutions of the equation $y^2 = x^3 - d^2x$. Such equations define *elliptic curves.* It turns out that

d is a congruent number if, and only if, the elliptic curve $E_d : y^2 = x^3 - d^2x$ has a solution with $y \neq 0$.

In fact, $a^2+b^2 = c^2, \frac{1}{2}ab = d$ implies $bd/(c-a), 2d^2/(c-a)$ is a rational solution of $y^2 = x^3 - d^2x$. Conversely, a rational solution of $y^2 = x^3 - d^2x$ with $y \neq 0$ gives the rational, right-angled triangle with sides $(x^2 - d^2)/y, 2xd/y, (x^2 + d^2)/y$ and area d.

The set of rational solutions of an elliptic curve over $\mathbb{Q}$ forms a group and it is an easy fact, from the way the group law is defined, that there is a solution with $y \neq 0$ if and only if there are infinitely many rational solutions ([15]). Therefore, if d is a congruent number, there are infinitely many rational-sided right-angled triangles with area d.

A point to note is that even for an equation with integral coefficients as the one above, it is the set of rational solutions which has a nice (group) structure. Thus, from two rational solutions, one can produce another rational solution by 'composition'. So, it is inevitable that in general one needs to understand rational solutions even if we are interested only in integral solutions. For example, the equation $y^2 = x^3 + 54$ has only *two integral solutions* $(3, \pm 9)$ but the set of rational solutions is the *infinite cyclic* group generated by $(3, 9)$.

The connection with elliptic curves has been used to show that numbers which are $1, 2$ or $3 \bmod 8$ are not congruent.

Further, assuming the truth of the weak Birch & Swinnerton-Dyer conjecture ([15]), Stephens showed this provides a complete characterization of congruent numbers.

6. Irreducibility, indecomposability and Fried's work

It was Michael Fried who realized the peculiarities of factorizing a polynomial of the form $f(X) - g(Y)$. He used geometric methods - particularly, the theory of monodromy groups - to deduce several striking results. Historically, one of the earliest irreducibility results was proved by Ehrenfeucht ([8]) in 1958; it asserts :

Theorem 6.1 (Ehrenfeucht)**.** *If $\deg f, \deg g$ are relatively prime, then $f(X) - g(Y)$ is irreducible.*

The proof is not difficult and can be worked out by defining a weighted degree of polynomials in $\mathbb{C}[X, Y]$ with weight deg(f) for X and deg(g) for Y and comparing the top degree terms in an expression of the form

$$f(X) - g(Y) = F(X, Y)G(X, Y).$$

We mention in passing that Davenport, Fried, Leveque, Lewis, Runge, and Schinzel made fundamental contributions to the question of irreducibility of $f(X) - g(Y)$; see [16], [7], [23], [24], [27], [28].

There are some cases when one can observe that $f(X) - g(Y)$ is reducible. For instance, over $\mathbf{C}$, $T_n(X) + T_n(Y)$ is a product of quadratic factors (and a linear factor if n is odd) where $T_n(X)$ is the Chebychev polynomial $T_n(X + X^{-1}) = X^n + X^{-n}$. Another simple observation is that if f_1, g_1, F are polynomials with deg $F > 0$, then $f_1(X) - g_1(Y)$ is a factor of $F(f_1(X)) - F(g_1(Y))$. Thus, the possibility of decomposing two given polynomials f, g in the form $f = F \circ f_1, g = F \circ g_1$ for a single, nonconstant polynomial F becomes interesting and is of relevance. A remarkable result due to Fried & MacRae (1969) ([14]) is that the converse also holds.

Proposition 6.2. *$f(X) - g(Y)$ has a factor of the form $f_1(X) - g_1(Y)$ if (and only if), there is $F(T) \in \mathbf{C}[T]$ such that*

$$f(T) = F(f_1(T)) \ , \ g(T) = F(g_1(T)).$$

Fried had made a deep study of the factors of $f(X) - g(Y)$ and proved in 1973 the following theorem([11]).

Theorem 6.3. *Given $f, g \in \mathbb{Z}[X]$, there exist f_1, f_2, g_1, g_2 in $\mathbf{Z}[X]$ such that*

(1) $f(X) = f_1(f_2(X)), g(X) = g_1(g_2(X))$,
(2) Splitting fields of $f_1(X) - t$ and of $g_1(X) - t$ over $\mathbb{Q}(t)$ (where t is a new indeterminate) are the same, and
(3) the irreducible factors of $f(X) - g(Y)$ are in bijection with those of $f_1(X) - g_1(Y)$.

6.1. **Arithmetic monodromy groups.** The above results follow by considerations of monodromy groups. Firstly, we call $f = f_1 \circ f_2$ a proper decomposition of the polynomial f if f_1, f_2 are both nonconstant polynomials. If a proper decomposition exists, one says f is *decomposable*; otherwise, it is said to be *indecomposable.* As this notion is insensitive to linear changes of the variable, there is an evident notion of equivalent decompositions.

Now, if $f \in \mathbb{Z}[X]$ is a nonconstant monic polynomial, then consider a splitting field K of the polynomial $f(X) - t$ over $\mathbb{Q}(t)$; the Galois group $G :=$ Gal $(K/\mathbb{Q}(t))$ viewed as a group of permutations of the roots of $f(X) - t$ in K, is known as the *arithmetic monodromy group of f*. If $K = \mathbb{Q}(t)(\alpha)$, then $f(\alpha) = t$. Consider the stabilizer subgroup H of α in G. The theorem of Luröth helps us derive the following reformulation of decomposability of f.

Proposition 6.4 ([17], Lemma 2.7). *Let f, K, α, G, H be as above. Let $f = f_1(f_2(\cdots(f_r))\cdots)$ be a decomposition of f into indecomposable polynomials f_i over $\mathbb{Q}$. Let H_i denote the stabilizer in G of $f_i(f_{i+1}(\cdots(f_r(\alpha)))\cdots)$ for $1 \leq i \leq r$. Then,*

$$G \supset H_1 \supset H_2 \cdots \supset H_r \supset H$$

is a maximal strictly decreasing chain of intermediate subgroups between G and H. Conversely, any such strictly decreasing maximal chain of subgroups corresponds to a decomposition of f into indecomposables.

Implicit in the above proposition are two classical theorems of Ritt ([22]) from 1922. The first theorem says that any two proper decompositions have the same length and the second one finds all solutions of $f_1 \circ f_2 = g_1 \circ g_2$ when f_1, g_1 have coprime degrees and g_1, g_2 have coprime degrees.
As we mentioned earlier, applying Siegel's theorem involves checking irreducibility of $f(X) - g(Y)$; so, a related question is to determine when a decomposable polynomial $f_1 \circ f_2$ is irreducible. This is addressed by the following elementary classical lemma.

Lemma 6.5 (Capelli). *Let $f_1, f_2 \in K[X]$, where K is an arbitrary field. Let α be a root of f_1 in a fixed algebraic closure $\bar{K}$ of K. Then, $f_1 \circ f_2$ is irreducible over K if, and only if, f_1 is irreducible over K and $f_2 - \alpha$ is irreducible over $K(\alpha)$.*

Proof. Let $f_2(\beta) = \alpha$ where $\beta \in \bar{K}$. Then $f_1(f_2(\beta)) = 0$ which means that

$$[K(\beta) : K] \leq \deg(f_1 \circ f_2) = \deg(f_1)\deg(f_2).$$

But, $[K(\beta) : K(\alpha)] \leq \deg(f_2 - \alpha) = \deg(f_2)$. As $[K(\alpha) : K] \leq \deg(f_1)$, it follows that

$$[K(\beta) : K] = [K(\beta) : K(\alpha)][K(\alpha) : K] \leq \deg(f_1)\deg(f_2)$$

with equality holding in the last inequality if, and only if, $[K(\beta) : K(\alpha)] = \deg(f_2)$, and $[K(\alpha) : K] = \deg(f_1)$. In other words, $f_1 \circ f_2$ is irreducible if, and only if, f_1 is irreducible over K and $f_2 - \alpha$ is irreducible over $K(\alpha)$. □

In various special cases, the following elementary observation can be used to prove indecomposability of a polynomial. First, we recall a definition.

For a polynomial $P(x) \in \mathbb{C}[x]$, a complex number c is said to be an *extremum*, if $P(x) - c$ has multiple roots. The *type of c* (with respect to P) is defined to be the tuple $(\mu_1, \cdots, \mu_s)$ of the multiplicities of the distinct roots of $P(x) - c$.

Lemma 6.6. *Let f be any complex polynomial and suppose $f = g \circ h$ for complex polynomials g, h of degrees ≥ 2. Then, if $\alpha \in \mathbb{C}$ is such that $g'(\alpha) = 0$, then the polynomial $h(x)-\alpha$ divides both $f(x)-g(\alpha)$ and $f'(x)$. In particular, if $f(x) \in \mathbb{C}[x]$ satisfies the condition that any extremum $\lambda \in \mathbb{C}$ has the type $(1, 1, \cdots, 1, 2)$, then f is indecomposable over $\mathbb{C}$.*

Proof. The former statement implies the later one. For, it implies that if $f(x) = G_1(G_2(x))$ is a decomposition of $f(x)$ with deg $G_1, G_2 > 1$, then there exists $\lambda \in \mathbb{C}$ such that deg gcd $(f(x)-\lambda, f'(x)) \geq$ deg G_2. But, then the type of λ (with respect to f) cannot be $(1, 1, \cdots, 1, 2)$. So, we prove the former statement. Evidently, for any $\alpha \in \mathbb{C}$, the polynomial $h(x) - \alpha$ divides $f(x) - g(\alpha)$. Moreover, if α is such that $g'(\alpha) = 0$, then consider any root θ of $h(x) - \alpha$. Suppose its multiplicity is a. Then, since the multiplicity of θ in $h'(x)$ is $a-1$ and since $g'(h(\theta)) = g'(\alpha) = 0$, it follows that $(x - \theta)^a$ divides $f'(x) = g'(h(x))h'(x)$. This completes the proof. $\square$

We remark that the proof shows the following refined version of above lemma holds for polynomials over $\mathbb{Q}$. If $f(x) \in \mathbb{Q}[x]$ is such that each extremum $\lambda \in \bar{\mathbb{Q}}$ of degree $\leq \frac{\deg f}{2} - 1$ has type $(1, 1, \cdots, 1, 2)$, then f is indecomposable over $\mathbb{Q}$. The above observation can be used to verify, for instance, that the polynomials $1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!}$ are indecomposable (see section 9).

7. Baker and Schinzel-Tijdeman theorems

In general, using Siegel's theorem yields ineffective results. However, for special cases, one might hope for effective results. In 1969, Alan Baker ([1]) considers the equation $f(x) = y^n$ when $n \in \mathbb{Z}$ is fixed. He proves the following theorem,

Theorem 7.1. *(Baker.) Assume that $f(x) \in \mathbb{Q}[x]$ has at least 3 simple roots and $n > 1$, or $f(x)$ has at least 2 simple roots and $n > 2$. Then, the equation $f(x) = y^n$ has only finitely many solutions in $x \in \mathbb{Z}$ and $y \in \mathbb{Q}$; further, the solutions can be effectively computed.*

In 1976, Schinzel & Tijdeman ([27]) proved the following beautiful effective result where n also varies.

Theorem 7.2. *Schinzel-Tijdeman Let $f \in \mathbb{Q}[X]$ have at least two simple roots. Then, there exists an effectively computable constant $N(f)$ such that for any solution of $f(x) = y^n$ in integers x, n and $y \in \mathbb{Q}$, we have $n \leq N(f)$.*

We note that in the above theorems we are sometimes looking more generally allowing some variables to take rational values rather than just integer values.

8. The Bilu-Tichy theorem

As Siegel's theorem is ineffective, it is particularly difficult to use when polynomials depending on parameters are involved in the Diophantine equation. Building on

Fried's work, in 2000, Bilu and Tichy proved a remarkable theorem in which they obtained an explicit finiteness criterion for the equation $f(x) = g(y)$. Before going to the statement of the theorem first let us recall some concepts.
Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are called *equivalent* if there exist a linear polynomial $l(x) \in \mathbf{C}[x]$ such that $G_1(x) = H_1(l(x))$ and $H_2(x) = l(G_2(x))$.
The *Dickson polynomial* $D_n(x, a)$ is defined as

$$D_n(x, a) = \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}.$$

It has degree n.
In what follows a and b are nonzero elements of some field, m and n are positive integers, and $p(x)$ is a nonzero polynomial (which may be constant).

Definition 8.1. *By a standard pair over a field k, we mean that $a, b \in k$, and $p(x) \in k[x]$. A standard pair*

(1) *of the first kind is $(x^u, ax^v p(x)^u)$ or $(ax^v p(x)^u, x^u)$, where $0 \leq v < u$, $(v, u) = 1$ and $v + degp(x) > 0$.*
(2) *of the second kind is $(x^2, (ax^2 + b)p(x)^2)$ or $(ax^2 + b)p(x)^2, x^2)$.*
(3) *of the third kind is $(D_k(x, a^l), D_l(x, a^k))$ where $(k, l) = 1$. Here D_l is the l-th Dickson polynomial.*
(4) *of the fourth kind is $(a^{-l/2} D_l(x, a), b^{-k/2} D_k(x, a))$ where $(k, l) = 2$.*
(5) *of the fifth kind is $((ax^2 - 1)^3, 3x^4 - 4x^3)$ or $(3x^4 - 4x^3, (ax^2 - 1)^3)$.*

Bilu & Tichy produced five families of pairs of polynomials such that a general pair (f, g) of polynomials satisfying the two properties: (i) $f(x) - g(y)$ is absolutely irreducible and (ii) $f(X) - g(Y) = 0$ is a curve of genus zero, is a standard pair up to linear changes of the variable. The theorem allows us to consider solutions in rational numbers with bounded denominator. If $F(X, Y) = 0$ is a polynomial in $\mathbb{Q}[X, Y]$, then the equation $F(x, y) = 0$ is said to have infinitely rational solutions with bounded denominators if there exists a constant $C(F)$ such that there are infinitely many rational numbers $x,\ y$ such that $F(x, y) = 0$ and $x, y \in \frac{1}{C(F)}\mathbb{Z}$. Moreover, the Bilu-Tichy theorem showed that each pair (f, g) for which $f(x) = g(y)$ has infinitely many solutions with bounded denominator can be determined from standard pairs. The precise statement of their theorem is as follows.

Theorem 8.2 (Bilu-Tichy)**.** *For non-constant polynomials $f(x)$ and $g(x) \in \mathbf{Q}[x]$, the following are equivalent*

(a) The equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator.

(b) We have $f = \phi(f_1(\lambda))$ and $g = \phi(g_1(\mu))$ where $\lambda(x), \mu(x) \in \mathbf{Q}[X]$ are linear polynomials, $\phi(x) \in \mathbf{Q}[X]$, and $(f_1(x), g_1(x))$ is a standard pair over $\mathbf{Q}$ such that the equation $f_1(x) = g_1(y)$ has infinitely many rational solutions with a bounded denominator.

9. Some applications of the Bilu-Tichy theorem

The theorem is particularly useful when dealing with equations of the form $f(x) = g(y)$ where f and/or g run through families of polynomials depending on certain parameters. We mention some results proved using this theorem and the theorems of Baker and of Schinzel-Tijdeman.

Theorem 9.1 ([5])**.** *Let r be a nonzero rational number which is not a perfect power in $\mathbb{Q}$. Then, the equation $x(x+1)(x+2)\cdots(x+m-1)+r = y^n$ has at most finitely many solutions $(x,\ y,\ m,\ ,n)$ satisfying $(x,\ m,\ n) \in \mathbb{Z}$ and $y \in \mathbb{Q}$, $m,\ n \geq 2$. Moreover, all the solutions can be calculated effectively.*

We may use the Bilu-Tichy theorem to study the finiteness question of solutions of the equation $f_m(x) = g(y)$ for the polynomials $f_m = X(X+1)(X+2)...(X+(m-1))$ where, $m > 2$ and g is a polynomial of degree $n \geq 2$ over $\mathbb{Q}$. We obtain the following precise result.

Theorem 9.2 ([18])**.** *The following holds true.*

(i) Fix $m \geq 3$ such that $m \neq 4$ and let g be an irreducible polynomial in $\mathbb{Q}[X]$. Then, there are only finitely many solutions of the equation $x(x+1)\cdots(x+m-1) = g(y)$ in rational numbers x, y with any bounded denominator.

(ii) If $m = 4$ and g is irreducible in $Q[y]$ then the equation $x(x+1)\cdots(x+m-1) = g(y)$ has infinitely many solutions precisely when $g = \frac{9}{16}+bX^2(X+c) \in \mathbb{Q}[X]$ where $b \in \mathbb{Q}^*, c \in \mathbb{Q}$. Besides these, the above equation has only finitely many solutions.

Interestingly, it turns out that we may bound m by using a simple consequence of the Chebotarev density theorem. The simple consequence we need asserts that an irreducible polynomial over $\mathbb{Q}$ has no roots modulo infinitely many primes. For the sake of completeness, we recall the following general statement.

Chebotarev's density theorem. *Let L/K be a Galois extension of algebraic number fields. Let C be a conjugacy class in the Galois group G. Then, the set of prime ideals P of O_K which are unramified in L and whose Frobenius automorphism Fr_P is in the conjugacy class C, has density $|C|/|G|$.*

We prove the following theorem.

Theorem 9.3. *Assume that $g \in \mathbb{Q}[X]$ is irreducible and that Δ is a positive integer. Then, there exists a constant $C = C(\Delta, g)$ such that for any $m \geq C$, the* *equation $x(x+1)\cdots(x+m-1)=g(y)$ does not have any rational solutions with bounded denominator Δ. Moreover, C can be calculated effectively.*

The information about possible decompositions of the polynomial is given by the following computation due to Bilu et al ([2]).

Proposition 9.4. *Let $m \geq 3$ and $f_m(X) = X(X+1)...(X+(m-1))$. Then*
(1). $f_m(X)$ is indecomposable if m is odd, and
(2). if $m = 2k$, then any nontrivial decomposition of $f_m(X)$ is equivalent to $f_m(X) = R_k((X-\frac{m-1}{2})^2)$, where

$$R_k = (X-\frac{1}{4})(X-\frac{9}{4})\cdots(X-\frac{(2k-1)^2}{4}).$$

In particular, the polynomial R_k is indecomposable.

Let us now consider the family of Bernoulli polynomials $B_m(x)$ defined by the generating series

$$\frac{te^{tx}}{e^t-1} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}.$$

Then $B_m(x) = \sum_{i=0}^{m}\binom{m}{i}B_{m-i}x^i$, where $B_r = B_r(0)$ is called the r-th Bernoulli number. They are rational numbers and can be computed from the recursive relation $\sum_{i=0}^{n-1}\binom{n}{i}B_i = 0$. The sum of the n^{th} powers of the first k natural numbers can be expressed as

$$1^n + 2^n + \cdots + x^n = S_n(x) = \frac{B_{n+1}(x+1) - B_{n+1}}{n+1}.$$

Bernoulli polynomials have following very interesting properties ([6]).
(i) $B_n(x) = x^n - \frac{n}{2}x^{n-1} + \frac{n(n-1)}{12}x^{n-2} + \cdots$
(ii) $B'_{n+1}(x) = (n+1)B_n(x)$.
(iii) $B_n(x) = (-1)^n B_n(1-x)$.
(iv) $f(x+1) - f(x) = nx^n$ $f(x) = B_n(x)$+Constant.
The decomposability of Bernoulli polynomials was investigated in a paper by Bilu, Brindza, Kirschenhofer, Pinter, Schinzel & Tichy with an appendix by Schinzel ([2]). They proved the following proposition.

Proposition 9.5 ((BBKPT))**.** *Let $m \geq 2$. Then*
(i) B_m is indecomposable over $\mathbb{Q}$ if m is odd, and
(ii) if $m = 2k$, then any nontrivial decomposition of B_m is equivalent to $B_m(x) = \phi((x-\frac{1}{2})^2)$ for a unique polynomial ϕ over $\mathbb{Q}$.

For a general $g \in \mathbb{Q}[X]$ we obtain ([19]) the following theorem.

Theorem 9.6. *Let $g \in \mathbb{Q}[X]$ have degree $n \geq 3$ and let $m \geq 3$. The equation $B_m(x) = g(y)$ has only finitely many rational solutions x, y with any bounded denominator apart from the following exceptions.*

(i) $g(y) = B_m(h(y))$ where h is a polynomial over $\mathbb{Q}$;
(ii) m is even and $g(y) = \phi(h(y))$, where h is a polynomial over $\mathbb{Q}$, whose square-free part has at most two zeroes, such that h takes infinitely many square values in $\mathbb{Z}$ and ϕ is the unique polynomial satisfying $B_m(x) = \phi((x - \frac{1}{2})^2)$;
(iii) $m = 3, n \geq 3$ odd and $g(x) = \frac{1}{8(3^{3(n+1)/2})} D_n(\delta(x), 3^3)$;
(iv) $m = 4, n \geq 3$ odd and $g(x) = \frac{1}{2^{2(n+3)}} D_n(\delta(x), 2^4) - \frac{1}{480}$;
(v) $m = 4, n \equiv 2 \bmod 4$ and $g(x) = \frac{-\beta^{-n/2}}{64} D_n(\delta(x), \beta) - \frac{1}{480}$.
Here δ is a linear polynomial over $\mathbb{Q}$ and $\beta \in \mathbb{Q}^*$. Furthermore, in each of the exceptional cases, there are infinitely many solutions with a bounded denominator.

Although the above theorem treats a general equation of the form $B_n(x) = g(y)$, it is often possible to obtain more precise results for special g for a more general equation. For instance, consider a polynomial C over $\mathbb{Q}$, and the Diophantine equations of the form

$$aB_m(x) = bB_n(y) + C(y)$$

with $m \geq n >$deg $C + 2$, or of the forms

$$ax(x+1)\cdots(x+m-1) = bB_n(y) + C(y)$$

and

$$ax(x+1)\cdots(x+m-1) + C(x) = bB_n(y)$$

for solutions in integers x, y when a, b are non-zero rational numbers. In these cases, we have the following more precise theorems ([19], [20]).

Theorem 9.7. *For nonzero rational numbers a, b and a polynomial $C \in \mathbb{Q}[X]$, if $m \geq n >$ deg $C + 2$ then the equation*

$$aB_m(x) = bB_n(y) + C(y)$$

has only finitely many rational solutions with bounded denominators except when $m = n, a = \pm b$ and $C(y) \equiv 0$. In these exceptional cases, there are infinitely many rational solutions with bounded denominators if, and only if, $a = b$ or $a = -b$ and $m = n$ is odd.
In particular, if c is a nonzero constant, then the equation

$$aB_m(x) = bB_n(y) + c$$

has only finitely many solutions for all $m, n > 2$.

Theorem 9.8. *Let a, b be nonzero rational numbers. For $m \geq n > deg(C) + 2$, the equation*

$$aB_m(x) = by(y+1)\cdots(y+n-1) + C(y)$$

has only finitely many rational solutions with bounded denominator except in the following situations.

(i) $m = n, m+1$ *is a perfect square,* $a = b(\sqrt{m+1})^m$,

(ii) $m = 2n, \frac{n+1}{3}$ *is a perfect square,* $a = b(\frac{n}{2}\sqrt{\frac{n+1}{3}})^n$.

In each case, there is a uniquely determined polynomial C for which the equation has infinitely many rational solutions with a bounded denominator. Further, C is identically zero when $m = n = 3$ and has degree $n-4$ when $n > 3$.

Theorem 9.9. *Let a, b be nonzero rational numbers. For $m \geq n > deg(C) + 2$, the equation*

$$ax(x+1)\cdots(x+m-1) = bB_n(y) + C(y)$$

has only finitely many rational solutions with bounded denominator excepting the following situations when it has infinitely many.

$m = n$, $m+1$ is a perfect square, $b = a(\sqrt{m+1})^m$.

In these exceptional situations, the polynomial C is also uniquely determined to be

$$C(x) = af_m((\pm\sqrt{m+1})x + \frac{1-m\mp\sqrt{m+1}}{2}) - bB_m(x)$$

and has degree $m-4$.

The next theorem addresses the equations of the form $E_n(x) = g(y)$ where $E_n = 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!}$. Recall lemma 6.5 on the indecomposability of polynomials f whose extrema have the type $(1, 1, \cdots, 1, 2)$. We need to show that E_n has this property for each n. Here is how that is verified.

Proposition 9.10 ([21])**.** *Each extremum of the polynomial*

$$E_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

has the type $(1, 1, \cdots, 1, 2)$. In particular, $E_n(x)$ is indecomposable for all n. Moreover, E_n has only simple roots for any n.

Proof. Note that $E'_{n+1} = E_n$ for any $n \geq 0$. Therefore, it is clear that, for each $n \geq 0$, the roots of E_n are simple, for $E_{n+1}(\alpha) = 0$ implies

$$E'_{n+1}(\alpha) = E_n(\alpha) = E_{n+1}(\alpha) - \alpha^{n+1}/(n+1)! = -\alpha^{n+1}/(n+1)! \neq 0.$$

Now, let λ be a complex number such that $E_{n+1}(x) - \lambda$ has a multiple root α. Then $E_n(\alpha) = 0$ and $\lambda = E_{n+1}(\alpha) = \alpha^{n+1}/(n+1)!$. If β is another multiple root of $E_{n+1}(x) - \lambda$, then $\alpha^{n+1} = \beta^{n+1}$. This implies that there exists $\theta \neq 1$ with $\theta^{n+1} = 1$ such that E_n has two roots $\alpha, \alpha\theta$. We show that this is impossible.

Note that n must be > 1. Let ζ be a primitive $(n+1)$-th root of unity. Then $\theta = \zeta^i$ for some $0 < i \leq n$. It is a well-known result of Schur that E_n is irreducible over $\mathbf{Q}$ and that the Galois group of its splitting field K is A_n or S_n according as to whether 4 divides n or not.

Now, write $K = \mathbf{Q}(\alpha, \alpha\theta, \alpha_3, \cdots, \alpha_n)$ for the splitting field of E_n.

First, let $n \not\equiv 0 \bmod 4$. We shall use the fact that the Galois group contains the n-cycle $\sigma = (\alpha, \alpha\zeta^i, \alpha_3, \cdots, \alpha_n)$.
Since $\sigma(\zeta^i)$ must be a power of ζ, it follows that each α_j with $3 \leq j \leq n$ must be $\alpha\zeta^k$ for some k. Thus, the set $\{\alpha, \alpha\zeta^i, \alpha_3, \cdots, \alpha_n\}$ of all the roots of E_n is the set of all $\alpha\zeta^r$ $(0 \leq r \leq n)$ with one $\alpha\zeta^m$ missing for some $1 \leq m \leq n$.
Now, the sum of the roots of E_n gives

$$-n = \sum_{r \neq m} \alpha\zeta^r = -\alpha\zeta^m.$$

Therefore, $\alpha = n\zeta^{-m}$. The product of all roots of E_n gives

$$(-1)^n n! = \alpha^n \zeta^{n(n+1)/2-m} = n^n \zeta^{n(n+1)/2-m-mn} = n^n \zeta^{n(n+1)/2}.$$

Hence $1 = |\zeta^{n(n+1)/2}| = n!/n^n$, which is impossible for $n > 1$.
Finally, let $4|n$. Then, the Galois group, which is A_n, contains each $(n-1)$-cycle of the form $(\alpha, \alpha\zeta^i, \alpha_{i_1}, \cdots, \alpha_{i_{n-3}})$ where $\alpha_{i_1}, \cdots, \alpha_{i_{n-3}}$ are any $n-3$ among $\alpha_3, \cdots, \alpha_n$. Therefore, each α_j with $3 \leq j \leq n$ is of the form $\alpha\zeta^k$ for some k and, the argument above goes through as it is. This proves the proposition. □

In view of this proposition, we have the following finiteness results for equations of the form $E_n(x) = g(y)$ as an application of the Bilu-Tichy theorem. As noted above, it works slightly more generally for f which have each extremum to be of the type $(1, 1, \cdots, 1, 2)$.

Theorem 9.11 ([21]). *Let f, g be polynomials of degrees n, m respectively, with rational coefficients. Suppose each extremum (with respect to f) has type $(1, 1, \cdots, 1, 2)$. Then, for $n, m \geq 3$, the equation $f(x) = g(y)$ has only finitely many rational solutions (x, y) with a bounded denominator except in the following two cases.*
(i) $g(x) = f(h(x))$ for some nonzero polynomial $h(x) \in \mathbb{Q}(x)$,
(ii) $n = 3, m \geq 3$ and $f(x) = c_0 + c_1 D_3(\lambda(x), c^m)$, $g(x) = c_0 + c_1 D_m(\mu(x), c^3)$ for linear polynomials λ and μ over $\mathbb{Q}$ and $c_i \in \mathbb{Q}$ with $c_1, c \neq 0$.
In each exceptional case, there are infinitely many solutions.

10. An observation of Ramanujan

In this last section, we begin with the following wonderful observation due to Ramanujan.

$$2 + (1/2)^2, 2.3 + (1/2)^2, 2.3.5 + (1/2)^2, 2.3.5.7 + (1/2)^2, 2.3.5.7.11.13.17 + (1/2)^2$$

are, respectively, the perfect squares

$$(3/2)^2, (5/2)^2, (11/2)^2, (29/2)^2, (1429/2)^2$$

of rational numbers. The natural question arises as to whether there are other solutions to the equation $p_1 p_2 \cdots p_k + r^2 = y^2$ where r, y are non-zero rational numbers and p_i's are primes. Diophantine equations seeking prime number solutions are notoriously difficult to solve compared to solutions in arbitrary integers. Wadim Zudilin asked whether for a given non-zero rational number r, the equation

$$1.3.5\cdots(2m-1)+r=y^2$$

has only finitely many solutions in m and y. Here, we show that a stronger finiteness result holds for any arithmetic progression under some conditions on r. Indeed, it can be deduced from an earlier work ([5]) with some modifications; at the time of writing that paper, this question was not asked; otherwise, we could have added the assertions made here. More precisely, we prove the following theorem.

Theorem 10.1. *Let c, d be positive integers and r be a rational number which is not a perfect power such that $v_p(r) = 0$ for each prime p dividing d. The number of 4-tuples (x, y, m, n) with $m > 2$ $(m \neq 4)$, $n > 1$, $x \in \mathbb{Z}$, $y \in \mathbb{Q}$ such that*

$$cx(cx+d)(cx+2d)\cdots(cx+(m-1)d)+r=y^n$$

is finite.

Here, $v_p(r)$ denote the integer power of p dividing the rational number r.

10.1. **A lemma on stationary points.** For positive integers c, d and $m > 2$, consider the polynomial

$$f_m = cX(cX+d)(cX+2d)\cdots(cX+(m-1)d).$$

For a rational number r, we look at the polynomial $f_m + r$ and, for $n > 1$, consider the equation

$$f_m(x)+r=y^n$$

for solutions in integers x and rational numbers y.
Now

$$f_m(x) = cX(cX+d)(cX+2d)\cdots(cX+(m-1)d)$$

clearly satisfies

$$f_m(x) = d^m g_m(\frac{cx}{d}),$$

where

$$g_m(x) = x(x+1)(x+2)\cdots(x+m-1).$$

Therefore, the derivative satisfies

$$f'_m(x) = cd^{m-1}g'_m(\frac{cx}{d}).$$

The following lemma on stationary points of the polynomial g_m is elementary.

Lemma 10.2. *For any complex number a, the set*

$$S(g_m, a) := \{\alpha : g'_m(\alpha) = 0, g_m(\alpha) = a\}$$

has cardinality at most 2 (respectively, at most 1) if m is even (respectively, if m is odd).

Remark. As $f_m(x) = d^m g_m(\frac{cx}{d})$ and $f'_m(x) = cd^{m-1} g'_m(cx/d)$, we have

$$\frac{dS(g_m, a)}{c} = S(f_m, d^m a) \ \forall \ a \in \mathbf{C}.$$

Therefore, cardinality of $S(f_m, b)$ is at the most 2 when m is even and at the most 1 when m is odd.
From this, following can be deduced immediately.

Corollary 10.3. *For each $r \in \mathbb{Q}$, the polynomial $f_m(x) + r$ has at least three simple roots when $m \geq 5$ is odd or if $m > 6$ is even.*

For the smaller cases $m = 3, 4, 6$, we have

Lemma 10.4. *(i) If $m = 3$, then for any $r \in \mathbb{Q}$, the polynomial $f_3(x) + r$ has simple roots. (ii) If $m = 4$, then for any rational number $r \neq d^4, \frac{-9d^4}{16}$, the polynomial $f_4(x) + r$ has distinct roots.*

The polynomial $f_4(x) + d^4$ has two double roots $\frac{d(-3\pm\sqrt{5})}{2c}$.

The polynomial $f_4(x) - 9d^4/16$ has one double root $\frac{3d}{2c}$ and two simple roots $(-3 + \sqrt{10})d/2c, (-3 - \sqrt{10})d/2c$.

(iii) If $m = 6$, the polynomial $f_6(x) + r$ has simple roots if $r \neq (\frac{15d^3}{8})^2$.

The polynomial $f_6(x) + 15^2 d^6/8^2$ has one double root $-5d/2c$ and four simple roots $\frac{-5d}{2c} \pm \frac{d}{c}\sqrt{\frac{35\pm 8\sqrt{7}}{2\sqrt{3}}}$.

The Schinzel-Tijdeman theorem recalled above implies for the polynomial $f_m(x) = cX(cX + d)(cX + 2d) \cdots (cX + (m - 1)d)$ the following result.

Proposition 10.5. *(i) Let $m > 2$ $(m \neq 4)$, $r \in \mathbb{Q}$, and let c, d be positive integers. Then, the equation*

$$f_m(x) + r = y^n$$

has only finitely many solutions in (x, y, n) for $n > 1$, integers x, and rational numbers y.

(ii) If $m = 4$, then for $r \neq d^4, = 9d^4/16$, the equation $f_4(x) + r = y^n$ has only finitely many solutions in x, y, n for $n > 1$.

The equation $f_4(x) - \frac{9d^4}{16} = y^n$ has only finitely many solutions in x, y, n with $n > 2$.

10.2. **Bounding n for given r.** For given $m > 2$ (with $m \neq 4$), $r \in \mathbb{Q}$, we have already seen that there are only finitely many n, x, y such that

$$cx(cx+d)(cx+2d)\cdots(cx+(m-1)d)+r=y^n.$$

Now, we show that n can be bounded in terms of r alone.

Proposition 10.6. *Let $m > 2$ (with $m \neq 4$), let c, d be positive integers and let $r \in \mathbb{Q}$ with $r \neq 0, 1, -1$ and $v_p(r) = 0$ for each prime $p|d$. Then, there exists a constant $C(r)$ depending only on r such that for any integers x and rational number y satisfying*

$$cx(cx+d)(cx+2d)\cdots(cx+(m-1)d)+r=y^n,$$

we have $n \leq C(r)$.

Proof. As $r \neq \pm 1$, there exists a prime p such that $v_p(r) = t \neq 0$. Note by hypothesis that $(p, d) = 1$. If $m \geq p$, one of the integers

$$cx, cx+d, \cdots, cx+(m-1)d$$

is a multiple of p since $(p, d) = 1$. In fact, if $m \geq (t+1)p$ then p^{t+1} divides $f_m(x)$. Thus, $v_p(f_m(x)) \geq t+1$ which means (since $v_p(r) = t$) that

$$v_p(f_m(x)+r) = t = v_p(y^n) = nv_p(y).$$

Hence, for $m \geq (t+1)p$, we have $n \leq |t|$.

Now, the proposition 10.5 shows that there is some constant $C(m)$ depending on m so that for any solution (x, y) of the equation, $n \leq C(m)$. Now, take $C_0(r) = max(C(m) : m < (t+1)p)$ if $t > 0$ and take $C_0(r) = 0$ if $t < 0$. Then, for any solution of

$$cx(cx+d)(cx+2d)\cdots(cx+(m-1)d)+r=y^n$$

we have $n \leq C(r)$ where $C(r) = max(C_0(r), |t|)$. This completes the proof. □

10.3. **Bounding m absolutely.** In this section, we show that m itself is bounded for a solution to exist.

Proposition 10.7. *Let r be any rational number such that $v_p(r) = 0$ for all primes $p|d$ and such that r is not an n-th power. Then, there are only finitely many x, y, m (for $m > 2$, $m \neq 4$) with $f_m(x) + r = y^n$.*

Proof. As r is not an n-th power and $v_p(r) = 0$ for each prime $p|d$, either n is even and $r = -s^n$ or there is a prime p divisor of n such that $v_p(r)$ is not a multiple of n. In the latter case, note that $(p, d) = 1$ by hypothesis. In the former case when n is even and $r = -s^n$, choose a prime number $p \equiv 3 \bmod 4$ such that $v_p(r) = 0$. Then, choosing $m \geq p$, the equality

$$f_m(x) - s^n = y^n$$

shows that $-1 \equiv (y^{n/2})^2 \bmod p$. This is a contradiction. Hence $m < p$ and proposition 10.5 shows finiteness of solutions x, y.

In the latter case, there is a prime p such that $v_p(r) \neq 0$ and is not a multiple of n. Choose $m \geq (v_p(r)+1)p$ if $v_p(r) > 0$ and let m be arbitrary if $v_p(r) < 0$. Then

$$v_p(cx(cx+d)(cx+2d)\cdots(cx+(m-1)d)) \geq v_p(r)+1.$$

Here, we have used the fact that $(p, d) = 1$. This gives

$$v_p(f_m(x)+r) = v_p(r) = nv_p(y)$$

which is a multiple of n, a contradiction to the choice of p.

Hence, $m \leq v_p(r)$ and, once again, proposition 10.5 implies that there are only finitely many solutions in x, y. This completes the proof. $\square$

Combining the propositions 10.6 and 10.7, we have the following main theorem.

Theorem 10.8. *Let r be any rational number such that $v_p(r) = 0$ for all primes $p|d$ and such that r is not a perfect power. Then, there exist only finitely many tuples (m, n, x, y) for $m > 2 (m \neq 4), n > 1, x \in \mathbb{Z}, y \in \mathbb{Q}$ such that*

$$cx(cx+d)(cx+2d)\cdots+(cx+(m-1)d)+r = y^n$$

Further, for $m = 4$ and $r \neq d^4, -9d^4/16$, the number of solutions in x, y, n is finite.

Proof. For r as above, the number of $n > 1$'s admitting a solution is bounded by a constant $C(r)$ by proposition 10.6. For each of these finitely many n, proposition 10.7 shows that there are only finitely many (x, y, m) for $m > 2$ (with $m \neq 4$). The last assertion was already noted above. $\square$

We observe here that finiteness of the number of solutions of the equation $f(x)+r = y^n$ for a given rational number r and an integral polynomial f implies finiteness of the number of solutions of a related equation with integer coefficients. We observe:

Proposition 10.9. *Let $f \in \mathbf{Q}[X]$ be a polynomial which takes integer values at all integer points. Let a/b be a non-zero rational number and $n > 1$ be a positive integer. Then, for each solution $x \in \mathbf{Z}, y \in \mathbf{Q}$ of the equation $f(x) + \frac{a}{b} = y^n$ satisfies $by^n = u^n$ for some $u \in \mathbf{Z}$ and $x, u \in \mathbf{Z}$ satisfy the equation $bf(x)+a = u^n$.*

Proof. Start with $x \in \mathbf{Z}, y = \frac{u}{v} \in \mathbf{Q}$ such that

$$f(x) + \frac{a}{b} = y^n = \frac{u^n}{v^n}.$$

While writing the rational numbers a/b and u/v, we may assume that $b, v > 0$. Then

$$bf(x) + a = \frac{bu^n}{v^n}.$$

As v^n divides bu^n, and $(v^n, u^n) = 1$, we have that v^n divides b. Write $c = \frac{b}{v^n}$; then $bf(x) + a = cu^n$. Note that $c > 0$. If $c \neq 1$, look at any prime p dividing c; then $p|b$. So, the equality $bf(x) + a = cu^n$ implies that p divides a, which is a contradiction. Therefore, $b = v^n$ and we have $bf(x) + a = u^n$. □

Remarks. (i) It is not clear if there are only finitely many solutions in integers x, y and $n > 1$ for the equation $bf(x) + a = u^n$ if it is known that there are only finitely many integers x, rationals y and $n > 1$.

(ii) The question as to whether Ramanujan's observations above are the only solutions is a difficult open question. It is hard to tackle equations of the form $f(x) + r = y^n$ when r is a perfect power of a rational number.

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PROBLEM SECTION

The Editors are pleased to announce resuming the Problem Section with this issue of the Mathematics Student. The Problem Section of the Mathematics Student was a prominent and very popular feature right from the beginning till it gradually disappeared. We are happy to mention that some of the mathematicians the editors contacted responded enthusiastically and the list below is a first fruit of this initiative. We invite the readers to submit their solutions to

msindianmathsociety@gmail.com

mentioning the precise problem number for which the solution is submitted. Correct solutions among those received will be published in the forthcoming issues.

We also invite the readers to submit fresh problem proposals and would like the proposers to also submit solutions to help the editors to arrive at a decision on their inclusion.

That brings us to talking about the crucial role of problems in general, and their importance in engaging a prospective solver in a genuine creative endeavour in particular. Many of us have heard this urban legend* A student arrives late to a class and manages to solve a particularly difficult problem posed on the blackboard which later turns to be an unsolved problem. This story appears to have actually happened at least once in the not too distant past. It involves George Bernard Dantzig, famous for the simplex algorithm and also one of the founders of linear programming. Below are Dantzig's own words**

> During my first year at Berkeley I arrived late one day to one of [Jerzy] Neyman's classes. On the blackboard were two problems which I assumed had been assigned for homework. I copied them down. A few days later I apologized to Neyman for taking so long to do the homework - the problems seemed to be a little harder to do than usual. I asked him if he still wanted it. He told me to throw it on his desk. I did so reluctantly because his desk was covered with such a heap of papers that I feared my homework would be lost there forever. About six weeks later, one Sunday morning about eight o'clock, [my wife] Anne and I were awakened by someone banging on our front door. It was Neyman. He rushed in with papers in hand, all excited: "I've just written an introduction to one of your papers. Read it so I can send it out right away for publication." For a minute I had no idea what he was talking about. To make a long story short, the problems on the blackboard that I had solved

*courtesy C. S. Aravinda

**http://rev-inv-ope.univ-paris1.fr/files/26305//IO-26305-1.pdf.

> thinking they were homework were in fact two famous unsolved problems in statistics. That was the first inkling I had that there was anything special about them.
>
> A year later, when I began to worry about a thesis topic, Neyman just shrugged and told me to wrap the two problems in a binder and he would accept them as my thesis.

The above story is related with the caveat that some of the problems posed in this section may actually be either open or published solutions may be too technical or, sometimes, solution may be not be known to the proposer. It is to entice a smart reader to try and come up with a novel or an easier solution!

EDITORS

MS-2015, Nos.1-2: Problem-1: Proposed by Satya Deo.

The Brouwer fixed point theorem says that any continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$, where $\mathbb{D}^n$ is the usual n-disk, has a fixed point. The Borsuk-Ulam theorem says that there is no continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ which is antipodal preserving on the boundary S^{n-1} of $\mathbb{D}^n$, i.e., f satisfies the condition $f(-x) = -f(x)$ for all $x \in S^{n-1}$. One can quickly see that the Borsuk-Ulam theorem implies the Brouwer fixed point theorem: suppose $f : \mathbb{D}^n \to \mathbb{D}^n$ is a continuous map which has no fixed point, then we can define a continuous map $g : \mathbb{D}^n \to S^{n-1}$ which is identity on S^{n-1}, which clearly preserves the antipodal points, a contradiction to the Borsuk-Ulam theorem.

What about the converse? Is it true that the Brouwer fixed point theorem also implies the Borsuk-Ulam theorem or is there some counterexample?

MS-2015, Nos.1-2: Problem-2: Proposed by George E. Andrews.

Prove that, for $|q| < 1$ and b not a negative power of q,

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(1-bq^n)} = \sum_{n=1}^{\infty} nq^n \prod_{m=n}^{\infty} \frac{(1-q^{m+1})}{(1-bq^m)}. \tag{0.1}$$

Fokkink, Fokkink, and Wang [1] proved that the number of divisors of n equals

$$-\sum_{\pi \in \mathcal{D}_n} (-1)^{\#(\pi)} \sigma(\pi),$$

where $\mathcal{D}_n$ is the set of integer partitions of n into distinct parts, $\#(\pi)$ is the number of parts of π and $\sigma(\pi)$ is the smallest part of π.

Ex. When $n = 6$, $\mathcal{D}_6 = \{6, 5+1, 4+2, 3+2+1\}$, so the above sum is $6 - 1 - 2 + 1 = 4$, and there are four divisors of 6, namely 1,2,3, and 6.

Deduce the Fokkink, Fokkink, and Wang theorem from (0.1).

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MS-2015, Nos.1-2: Problem-3: Proposed by Kannappan Sampath, Graduate Student, Queen's University, Canada; submitted through M. Ram Murty.

Let $\mathbf{H}$ denote the division algebra of quaternions over the field $\mathbf{R}$ of real numbers. Let n be a positive integer. Suppose that A and B are two $n \times n$ matrices such that $AB = I_{n\times n}$. Show that $BA = I_{n\times n}$.

MS-2015, Nos.1-2: Problem-4: Proposed by M. Ram Murty.

> Show that $n \mid (2^n - 1)$ over $\mathbf{Z}$ if and only if $n = 1$.

MS-2015, Nos.1-2: Problem-5: Proposed by Atul Dixit.

> Prove that
> $$\int_{\pi/4}^{\pi/3} e^{x\sin x+\cos x}\left(\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right)dx$$
> $$= \frac{1}{12\pi}\left(e^{\frac{4+\pi}{4\sqrt{2}}}(48\sqrt{2}-3\pi^2) + e^{\frac{3+\pi\sqrt{3}}{6}}(4\pi^2-72)\right).$$

MS-2015, Nos.1-2: Problem-6: Proposed by Amritanshu Prasad, IMSc, Chennai; submitted through C. S. Aravinda.

> Let X be the set of all subsets of $\{1,\ldots,n\}$ of size 2 (so X has $\binom{n}{2}$ elements). Let V be the complex vector space
> $$V = \Big\{f : X \to \mathbf{C} \mid \sum_{\{s\in X \mid i\in X\}} f(s) = 0\Big\}.$$
> Show that V is a vector space of dimension $n(n-3)/2$ over $\mathbf{C}$.

MS-2015, Nos.1-2: Problem-07: Proposed by Ashay Burungale, UCLA, USA; submitted through B. Sury.

> Consider a kingdom made up of an odd number $2k+1$ of islands. There are bridges connecting some of the islands. Suppose each set S of k islands has the property that there is an island not among S which is connected by a bridge to each island in S. Show that some island of the kingdom has bridges connecting it to all the other islands.

MS-2015, Nos.1-2: Problem-8: Proposed by Abhijin Adiga, Network Dynamics and Simulation Science Laboratory, Virginia Bioinformatic Institute, USA; (this is an old problem, can be seen in 'EXtremal Graph Theory' by Jukna), submitted through L. Sunil Chandran.

> Consider the set $S = \{1, 2, ..., n\}$. Let S_i be n distinct non-empty subsets of S. Prove that there exists an element $k \in S$, such that $S_i \setminus \{k\}$ are still distinct.

MS-2015, Nos.1-2: Problem-9: Proposed by B. Sury.

Consider the polynomial $f = X^3 - 15X^2 + 75X - 120$. Let $f_1 = f, f_2 = f \circ f$ and, in general, $f_n = f \circ f_{n-1}$. Determine the zeroes of the polynomial f_{10}. Generalize.

MS-2015, Nos.1-2: Problem-10: Proposed by Mathew Francis, ISI Chennai; submitted through L. Sunil Chandran.

Given a t-regular graph that is properly t-edge coloured with colours $1, 2, \cdots t$. Consider any cut of the graph and let k_i be the number of edges coloured i crossing the cut. Show that $k_1 = k_2 = \cdots = k_t \ mod\ 2$.

MS-2015, Nos.1-2: Problem-11: Proposed by Mahender Singh, IISER, Mohali; submitted through C. S. Aravinda.

Let G be a finite p-group of order p^n and $Aut(G)$ be the group of all automorphisms of G. It is easy to see that if $n = 1$ then $|Aut(G)| = (p-1)$, and if $n = 2$ then $|Aut(G)| = (p^2 - p)$ or $(p^2 - 1)(p^2 - p)$.

Now suppose that $n \geq 3$ and G is abelian. Then show that $|G|$ divides $|Aut(G)|$. It is an open conjecture that, in this case, $|G|$ divides $|Aut(G)|$ even if G is non-abelian.

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